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June 1995

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Voronoi Diagrams of Moving Points*

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Abstract

Consider a set of n points in d-dimensional Euclidean space, $d \geq 2$, each of which is continuously moving along a given individual trajectory. At each instant in time, the points define a Voronoi diagram. As the points move, the Voronoi diagram changes continuously, but at certain critical instants in time, topological *events* occur that cause a change in the Voronoi diagram. In this paper, we present a method of *maintaining* the Voronoi diagram over time, while showing that the number of topological events has an upper bound of $O(n^d \lambda_s(n))$, where $\lambda_s(n)$ is the maximum length of a (n,s)-Davenport-Schinzel sequence [AgShSh 89, DaSc 65] and s is a constant depending on the motions of the point sites. Our results are a linear-factor improvement over the naive $O(n^{d+2})$ upper bound on the number of topological events.

In addition, we show that if only k points are moving (while leaving the other $n-k$ points fixed), there is an upper bound of $O(kn^{d-1}\lambda_s(n) + (n-k)^d \lambda_s(k))$ on the number of topological events.

We give a numerically stable algorithm for the update of the topological structure of the Voronoi diagram, using only $O(\log n)$ time per event (which is worst-case optimal per event).

Keywords: combinatorial complexity, dynamic computational geometry, Delaunay triangulation, Davenport-Schinzel theory, geometric data structure, moving objects, proximity, Voronoi diagram.

1 Introduction

Voronoi diagrams are a fundamental tool expressing the proximity of geometric objects. So, it is not surprising that they appear in many variations in computational geometry as well as other related scientific areas (see [Au 90] for a survey on this topic).

*Preliminary versions of this paper appeared at the 17^{th} Workshop on Graph-Theoretic Concepts in Computer Science WG'91 [GuMiRo 91] and at the 3^{rd} Scandinavian Workshop on Algorithm Theory SWAT'92 [AlRo 92].

†Work on this paper was partially supported by the Deutsche Forschungsgemeinschaft (DFG) under contract No 88/10-1.

‡Partially supported by a grant from Hughes Research Laboratories, Malibu, CA, and by NSF Grant ECSE-8857642.

§The author acknowledges the partial support by NSF grant CCR-9215219.

¶Partially supported by the Swiss National Science Foundation (SNF) under grant 21-39328.93.

A problem of recent interest has been that of allowing the set of objects S to vary continuously over time. This "dynamic" version has been studied in the case of points in the Euclidean plane by [AoImImTo 90, ImSuIm 89, Ro 90]. Most recently, [Al 91, Ro 91] generalized these ideas with respect to the dimension ($d = 3$) and the order of the Voronoi diagram.

In this paper, we consider the following problem: We are given a set S of n points in d-dimensional Euclidean space, $d \geq 2$, each of which is continuously moving along a given trajectory. At each instant in time, the points define a Voronoi diagram. As the points move, the Voronoi diagram changes continuously, but at certain critical instants in time, topological *events* occur that cause a change in the dual graph, the Delaunay diagram. Our goal is to characterize the elementary topological events in order to *maintain* the Voronoi diagram over time in some useful data structure.

The main result is to prove a new $O(n^d \, \lambda_s(n))$ upper bound on the number of topological events, where $\lambda_s(n)$ denotes the maximum length of a (n, s)-Davenport-Schinzel sequence and s is a constant depending on the motions of the point sites. In the special case of points moving along polynomial curves of degree q (so-called polynomial q-motions), we get $s = (d+2)\, q$. As we will see, our results are a linear-factor improvement over the naive $O(n^{d+2})$ upper bound.

In the case that only k of the n points of S are moving (while the remaining $n - k$ stay fixed), our bound on the number of events becomes $O(k\, n^{d-1}\, \lambda_s(n) + (n-k)^d\, \lambda_s(k))$, which is approximately $O(n^d)$ for fixed k. In addition to that, very recently [Ro 93a] proved that there is a tighter bound of $O(k^{d+1}(n-k)^{\lceil d/2 \rceil} + k^d\, \lambda_s(k))$ in the case of $k \in O(\sqrt{n})$. This should be contrasted with the best known lower worst-case bound of $\Theta(k\,(n-k)^{\lceil d/2 \rceil})$. Thus, the major open problem in this area is to close the gap between the upper and lower worst-case bounds, i.e. to give tight worst-case bounds.

Finally, there are recent results by [HuKeKl 92, To 88] for dynamic Voronoi diagrams of rigidly moving sets of points. For g groups of each n points in the plane, they could prove an upper bound of $O(g^2 n^2 \lambda_s(g))$ events.

We also present a numerically stable algorithm for the update over time of the topological structure of the Voronoi diagram, using only $O(\log n)$ time for each topological change. It is known [Ro 91, Ro 93b] that this update time is worst-case optimal (even in the planar case).

2 Preliminaries

This section briefly summarizes the elementary definitions and properties of d-dimensional Euclidean Voronoi diagrams, $d \geq 2$, of point sets. As usual, we let $d(.,.)$ denote Euclidean distance. At the beginning, we are given a finite set

$$S \;:=\; \{P_1, \ldots, P_n\}$$

of $n \geq d+2$ sites in d-dimensional Euclidean space $\mathbb{E}^d$, $d \geq 2$. (As usual, the dimension d is assumed to be a constant.) The perpendicular *bisector of* P_i *and* P_j is defined to be the hyperplane

$$B_{ij} \;:=\; \{x \in \mathbb{R}^d \mid d(x, P_i) \;=\; d(x, P_j)\}.$$

The (convex) *Voronoi polygon/polyhedron of* P_i is given by

$$v(P_i) \;:=\; \{x \in \mathbb{R}^d \mid \forall_{j \neq i} \;\; d(x, P_i) \;\leq\; d(x, P_j)\}.$$

The vertices of the Voronoi polyhedrons are called *Voronoi points* and the bisector portions on the boundary are called *Voronoi edges/k-faces* (according to their affine dimension k). Finally the *Voronoi diagram of* S is defined by

$$VD(S) \;:=\; \{v(P_i) \mid P_i \in S\}.$$

The embedding of the Voronoi diagram into d-dimensional real space provides a graph that we call the *geometrical structure* of the underlying Voronoi diagram.

Now we turn our attention to the dual graph of the Voronoi diagram, the so-called *Delaunay triangulation/graph* $DT(S)$. If S is in general position — i.e. no $d+2$ points of S lie on a common hypersphere and no $d+1$ points of S lie on a common hyperplane — every Voronoi $(d-i)$-face in $VD(S)$ corresponds to an i-face in $DT(S)$, for $i = 0, \ldots, d$.

In the following, we use a *one-point-compactification* to simplify our discussion. We augment set S by adding the "point at infinity" ∞, yielding a new set of sites

$$S' := S \cup \{\infty\}.$$

The extended Delaunay graph is given by

$$DT(S') = DT(S) \cup \{(P_i, \infty) \mid P_i \in S \cap \partial CH(S)\}.$$

In addition to the Delaunay graph $DT(S)$, every point on the boundary of the convex hull $\partial CH(S)$ is connected to ∞. We call the underlying graph of the extended Delaunay graph $DT(S')$ the *topological structure* of the Voronoi diagram. In contrast with $DT(S)$, $DT(S')$ has the nice property that there are exactly $d+1$ $(d+1)$-tuples adjacent to each $(d+1)$-tuple in $DT(S')$. This will significantly simplify the description of the algorithm presented below.

Next, we adopt two functions[1] from [GuSt 85] providing a nice classification of the $(d+1)$-tuples of the extended Delaunay graph $DT(S')$. In particular, let $v(P_0, \ldots, P_d)$ denote the center of the hyperball $C(P_0, \ldots, P_d)$ of $d+1$ sites $P_0, \ldots, P_d \in S$, we have:

$$\begin{aligned}
\{P_0, \ldots, P_d\} \in DT(S') \quad &\Longleftrightarrow \quad v(P_0, \ldots, P_d) \text{ is a Voronoi point in } VD(S). \\
&\Longleftrightarrow \quad C(P_0, \ldots, P_d) \text{ contains no point of } S \text{ in its interior.} \\
&\Longleftrightarrow \quad \forall_{P' \in S \setminus \{P_0, \ldots P_d\}} \ \mathrm{OUTSIDE}(P_0, \ldots, P_d, P') := \\
&\qquad\qquad \mathrm{sign}\left[\mathrm{VOL}(P_0, \ldots, P_d) * \mathrm{INS}(P_0, \ldots, P_d, P')\right] = 1.
\end{aligned}$$

Naturally, an analogous statement can be given for the extended $(d+1)$-tuples. If $\{P_0, \ldots, P_d\}$ and $\{P_0, \ldots, P_{d-1}, \infty\}$ are adjacent $(d+1)$-tuples in $DT(S')$ with $VOL(P_0, \ldots, P_d) > 0$, we have:

$$\begin{aligned}
\{P_0, \ldots, P_{d-1}, \infty\} \in DT(S') \quad &\Longleftrightarrow \quad P_0, \ldots, P_{d-1} \text{ are the vertices of a } (d-1)\text{-face on} \\
&\qquad \text{the boundary of the convex hull } \partial CH(S). \\
&\Longleftrightarrow \quad \forall_{P' \in S \setminus \{P_0, \ldots P_{d-1}\}} \ \mathrm{OUTSIDE}(P_0, \ldots, P_{d-1}, \infty, P') := \\
&\qquad\qquad \mathrm{sign}\left[\mathrm{VOL}(P_0, \ldots, P_{d-1}, P')\right] = 1.
\end{aligned}$$

The proof of these statements is straightforward. In the following, these classifications will be very useful characterizing the elementary topological events of two- and higher dimensional dynamic Voronoi diagrams.

[1]These functions VOL and INS (mnemonic for "volume" and "insphere") are defined as follows:

$$\mathrm{VOL}(P_0, \ldots, P_d) := \begin{vmatrix} 1 & P_{01} & \ldots & P_{0d} \\ \vdots & \vdots & & \vdots \\ 1 & P_{d1} & \ldots & P_{dd} \end{vmatrix}, \quad \mathrm{INS}(P_0, \ldots, P_{d+1}) := \begin{vmatrix} 1 & P_{01} & \ldots & P_{0d} & P_{01}^2 + \cdots + P_{0d}^2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & P_{d1} & \ldots & P_{dd} & P_{d1}^2 + \cdots + P_{dd}^2 \\ 1 & P_{d+11} & \ldots & P_{d+1d} & P_{d+11}^2 + \cdots + P_{d+1d}^2 \end{vmatrix}$$

3 Voronoi Diagrams of Moving Points

The contents of this section is to describe the changes in the topological structure of a set of continuously moving points in d-dimensional Euclidean space $\mathbb{E}^d$, $d \geq 2$. For that, we are given a finite set of $n \geq d+2$ continuous trajectory curves in d-dimensional Euclidean space $\mathbb{E}^d$, $S := S(t) := \{P_1(t), \dots, P_n(t)\}$. Thereby the points are allowed to appear or disappear according to a specific life cycle. We make the following assumptions about the trajectories: First, we assume that the points move without collisions, or in other words:

$$\forall_{i \neq j} \; \forall_{t \in \mathbb{R}} \;\; P_i(t) \neq P_j(t).$$

In addition, we demand the existence of an instant $t_0 \in \mathbb{R}$ when $S(t_0)$ is in general position; this is necessary to obtain a definite topological structure at the starting position t_0.

Now, consider the situation at a moment $t \in \mathbb{R}$ when all points in $S(t)$ are in general position. On the one hand, by investigating the continuity of a suitable product of determinants, it is easy to see that a sufficiently small continuous motion of the points does not change the fact that the points are in general position. On the other hand, the topological structure $DT(S')$ is completely determined by the *active* Voronoi points which currently appear in $VD(S)$ and by the d-tuples of sites forming the boundary of the convex hull $\partial CH(S)$. Therefore, the topological structure can only change in the following two different situations:

Case (1) The appearance (disappearance) of an inactive (active) Voronoi point.

Case (2) The appearance (disappearance) of a point on the boundary of the convex hull.

However, in both cases the loss of general position of the points $S(t)$ is necessary for changing the topological structure $DT(S'(t))$. This proves that the topological structure $DT(S')$ is *locally stable* as long as the points are in general position.

In order to address the question of sufficient conditions, we proceed with an investigation of the *elementary changes* of the topological structure of a Voronoi diagram. In the two-dimensional case, it is well-known (see, e.g., [Ro 90]) that such elementary changes can be described as "swaps" of adjacent triangles in $DT(S')$ (cf. Figure 1). However, in higher dimensions these transitions turn out to be more complex.

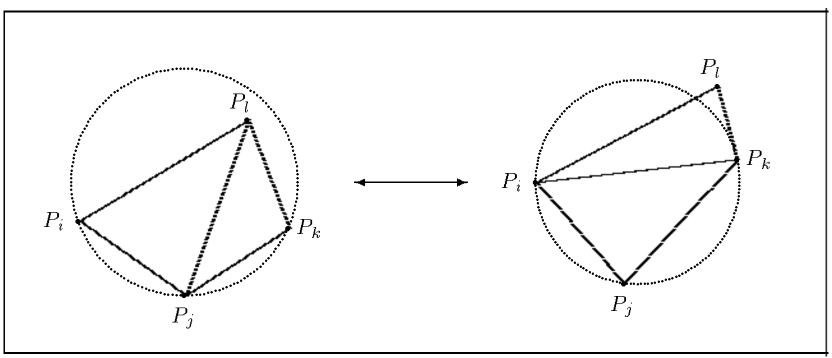

Figure 1: A reversible swap, i.e. a (2,2)-transition of neighboring Delaunay triangles in $\mathbb{R}^2$.

In our first case above, an inactive Voronoi point $v(P_0, \dots, P_d)$ becomes activated, if the last point $P_{d+1} \in S$ leaves the variable circumsphere $C(P_0, \dots, P_d)$. As well, an active Voronoi point $v(P_0, \dots, P_d)$ becomes inactivated, if a point $P_{d+1} \in S$ enters this variable circumsphere. Additionally, we assume at that instant t' (when $d+2$ points lie on a common hypersphere) that no further point of S lies on the boundary of the circumsphere $C(P_0, \ldots, P_d)$. If we select $\varepsilon > 0$ sufficiently small, the entrance of the point P_{d+1} can be described as follows:

$$\begin{aligned} & \mathrm{OUTSIDE}(P_0, \ldots, P_d, P_{d+1})(t' - \varepsilon) \;=\; 1, \\ & \mathrm{OUTSIDE}(P_0, \ldots, P_d, P_{d+1})(t' + \varepsilon) \;=\; -1, \\ \forall_{P' \in S \setminus \{P_0, \ldots, P_{d+1}\}} \; \forall_{t \in [t'-\varepsilon, t'+\varepsilon]} \; & \mathrm{OUTSIDE}(P_0, \ldots, P_d, P')(t) \;=\; 1. \end{aligned}$$

In fact, this corresponds to a real *zero-crossing* of the function $\mathrm{OUTSIDE}(P_0, \ldots, P_{d+1})$ because the point P_{d+1} changes on what side of the sphere $C(P_0, \ldots, P_d)$ it lies at the instant t'.

How can we describe the resulting change of the topological structure? For that, we investigate the active $(d+1)$-tuples of $DT(S'(t'-\varepsilon))$ at an instant $t'-\varepsilon$, with $\varepsilon > 0$ sufficiently small. At first, it is apparent that the *local* topological structure in the neighborhood of $v(P_0, \ldots, P_d)$ is completely determined by the points $S_d := \{P_0, \ldots, P_{d+1}\}$. Thus, we only have to consider all $d+2$ subsets of points of S_d of size $d+1$. These subsets can be generated, for example, by eliminating the i-th element for $i = 0, \ldots, d+1$, respectively. So, let

$$\pi_i \;:=\; \begin{cases} (P_1, \ldots, P_{d+1}) & i = 0, \\ (P_0, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{d+1}) \quad \text{iff} & 1 \le i \le d, \\ (P_0, \ldots, P_d) & i = d+1, \end{cases}$$

denote the sequence which has been obtained after eliminating the i-th element. Using the fact, that the determinants considered are alternating forms (i.e. transposing two rows in any determinant changes its sign), we'll prove now that there exists a complete, disjoint *partition* of the π_i's into two subsets A and B, with $2 \le |A|, |B| \le d$, such that:

$$(*) \begin{cases} \forall_{\pi_i \in A} \quad \mathrm{OUTSIDE}(\pi_i, P_i)(t' - \varepsilon) \;=\; 1 \quad \text{and} \quad \mathrm{OUTSIDE}(\pi_i, P_i)(t' + \varepsilon) \;=\; -1, \\ \forall_{\pi_i \in B} \quad \mathrm{OUTSIDE}(\pi_i, P_i)(t' - \varepsilon) \;=\; -1 \quad \text{and} \quad \mathrm{OUTSIDE}(\pi_i, P_i)(t' + \varepsilon) \;=\; 1, \\ \forall_{P' \notin \pi_i \cup P_i} \; \forall_{t \in [t'-\varepsilon, t'+\varepsilon]} \; \mathrm{OUTSIDE}(\pi_i, P')(t) \;=\; 1. \end{cases}$$

These equations are obviously equivalent (due to the classification above) to the following so-called (i, j)-*transition*[2] of the local topological structure:

$$\boxed{\{\pi_i \in DT(S'(t' - \varepsilon)) \mid \pi_i \in A\} \quad \longleftrightarrow \quad \{\pi_i \in DT(S'(t' + \varepsilon)) \mid \pi_i \in B\}.}$$

Next, we proceed by constructing the announced sets A and B:

$$\begin{aligned} A &:= \{\pi_i \mid \operatorname{sign}[\mathrm{VOL}(P_0, \ldots, P_d)] = -\operatorname{sign}[\mathrm{VOL}(P_0, \ldots, P_{i-1}, P_{d+1}, P_{i+1}, \ldots, P_d)]\}, \\ B &:= \{\pi_i \mid \operatorname{sign}[\mathrm{VOL}(P_0, \ldots, P_d)] = \operatorname{sign}[\mathrm{VOL}(P_0, \ldots, P_{i-1}, P_{d+1}, P_{i+1}, \ldots, P_d)]\}. \end{aligned}$$

In other words, set A and set B include all π_i's where P_i and P_{d+1} lie on different sides or on the same side of the hyperplane spanned by the sites $P_0, \ldots, P_{i-1}, P_{i+1}, \ldots, P_d$, respectively. Thereby, we assume that the sites of π_i do not change their orientation at the instant t'. Now, if we use the fact that the sequence (π_i, P_i) can be obtained from the sequence $(P_0, \ldots, P_{d+1})$ by $d - i + 1$ transpositions, we have for any $\pi_i \in A$:

$$\begin{aligned} & \mathrm{OUTSIDE}(\pi_i, P_i) \\ &= \operatorname{sign}[\mathrm{VOL}(\pi_i)] \,*\, \operatorname{sign}[\mathrm{INS}(\pi_i, P_i)] \\ &= (-1)^{d-i} \operatorname{sign}[\mathrm{VOL}(P_0, \ldots, P_{i-1}, P_{d+1}, P_{i+1}, \ldots, P_d)] \,*\, (-1)^{d-i+1} \operatorname{sign}[\mathrm{INS}(P_0, \ldots, P_{d+1})] \\ &= (-1)^{d-i+1} \operatorname{sign}[\mathrm{VOL}(P_0, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_d)] \,*\, (-1)^{d-i+1} \operatorname{sign}[\mathrm{INS}(P_0, \ldots, P_{d+1})] \\ &= \mathrm{OUTSIDE}(P_0, \ldots, P_{d+1}). \end{aligned}$$

[2] Thereby, i and j denote the cardinality of set A and B, respectively.

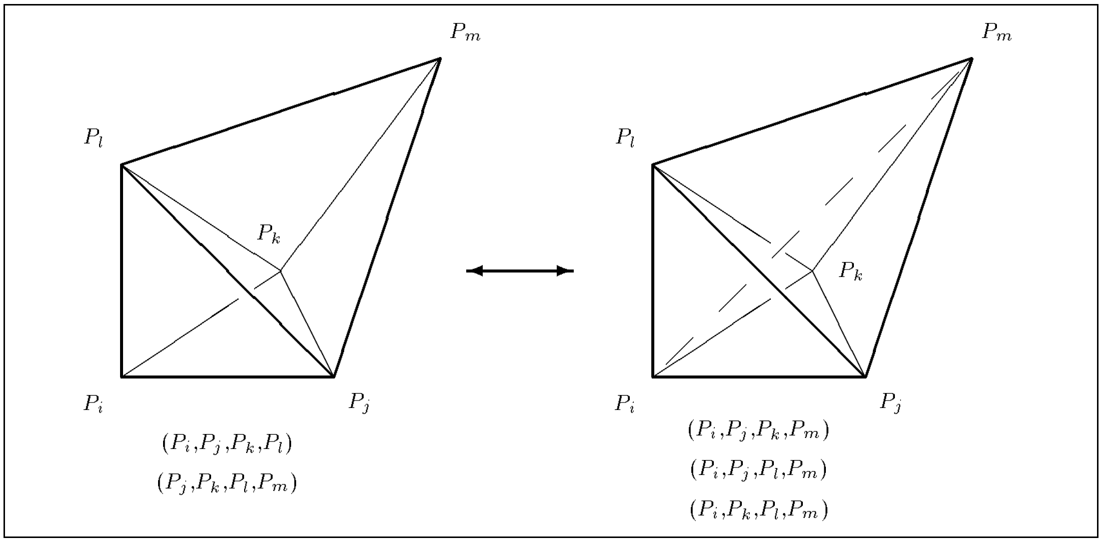

Figure 2: A reversible (2,3)-transition with the active Delaunay $(d+1)$-tuples in $\mathbb{R}^3$.

Analogously, we obtain for any $\pi_i \in B$: $\text{OUTSIDE}(\pi_i, P_i) = -\text{OUTSIDE}(P_0, \ldots, P_{d+1})$. With that, the desired equations $(*)$ hold immediately. Notice that there always exists such a hyperplane spanned by the sites $P_0, \ldots, P_{i-1}, P_{i+1}, \ldots, P_d$ separating P_i and P_{d+1}, which proves $|A| \geq 2$. The proof of $|B| \geq 2$ is also straightforward. A three-dimensional example of a (2,3)-transition is depicted in Figure 2 (from [Al 91]). The transition described is also equivalent to a *fusion* of the corresponding Voronoi points, which come together and disappear at the instant t', and the creation of new (dual) Voronoi points – according to the transition rule above.

Considering our second case, the appearance or disappearance of a point $P' \in S$ on the boundary of the convex hull $\partial CH(S)$ is equivalent to the activation or deactivation of the extended Delaunay edge (P', ∞).

At first, (P', ∞) becomes activated, if P' enters the boundary of the convex hull on a $(d-1)$-face formed by the sites $(P_0, \ldots, P_{d-1})$. According to the classification above, the circumsphere $C(P_0, \ldots, P_{d-1}, P')$ contains no points in its interior already shortly before P' enters the boundary of the convex hull. If we assume at the instant of coplanarity, that no other point of S lies on the hyperplane H formed by these points and if we regard the interior of the infinite sphere through the points $P_0, \ldots, P_{d-1}$ and ∞ as the open halfspace that is bounded by the hyperplane H and lies outside the convex hull $CH(S)$, then we can apply the results of our first case.[3] The deactivation of the edge (P', ∞) can be dealt with analogously. To summarize our results, we present the following theorem.

Theorem 1 *Elementary changes in the topological structure $DT(S')$ of the Voronoi diagram $VD(S)$ are characterized by (i,j)-transitions of adjacent $(d+1)$-tuples in $DT(S')$, except in degenerate cases. Thereby, the indices obey the conditions $i+j = d+2$ and $2 \leq i, j \leq d$.*

According to this, we have solved the question concerning *sufficient* conditions. Roughly speaking, topological events are characterized by non-degenerate loss of *local* general position,

[3] If we replace P_m by ∞, the activation of the extended dual edge (P_i, ∞) can be regarded as a left-to-right transition in Figure 2.

i.e. the loss of general position of adjacent $(d+1)$-tuples in the topological structure. Notice, that the same topological events are generated by several pairs of $(d+1)$-tuples representing the same sites. In this connection, the *original advantage* of the one-point compactification becomes apparent. It allows us the convenience of treating both cases similarly: as simple transitions in the extended dual graph $DT(S')$.

Up to now, we have been ignoring a technicality caused by degeneracies: it may be that more than $d+2$ points in $S(t)$ are lying on a common hypersphere at the same instant or that more than $d+1$ points in $S(t)$ are coplanar at the same instant. In both cases, we *recalculate* the local topological structure of the interior of the convex polygon described by the points at a moment $t+\varepsilon$. However, it is necessary to select $\varepsilon > 0$ in such a way, that the moment of recalculation precedes the next topological event.

4 New Upper Bounds

In this section, we present a new upper bound on the number of topological events. As we have seen in the previous section, topological events are characterized by loss of general position. So, it is quite natural to assume that there exist at most $s \in O(1)$ zeros of the functions INS(...) and VOL(...) which are computable in constant time each. Indeed, this additional assumption can be regarded as a certain kind of *non-periodicity* condition, which is achieved, for example, in the case of polynomial curves of bounded degree. This assumption implies that each subset of S' of size $d+2$ generates at most a constant number of topological events and gives a $s\binom{n+1}{d+2} \in O(n^{d+2})$ upper bound on the number of topological events. By a Davenport-Schinzel argument, we improve this naive upper bound by (roughly) a linear factor.

First of all, we have a short look at the construction in the two-dimensional case. The basic observation is that every topological event is related to one quadrilateral, i.e. to one pair of adjacent triangles, leaving the four bounding Delaunay edges of this quadrilateral unchanged. With that, we are able to determine the total number of topological events by adding for every imaginable Delaunay edge (P_i, P_j) the number of adjacent topological events that do not destroy this edge. This provides an $O(\lambda_s(n))$ upper bound on the number of changes for each pair of points and results in an $O(n^2\lambda_s(n))$ upper bound in total [GuMiRo 91].

A similar construction can be done in higher dimensions [AlRo 92]. It is clear that the maximum number of *extended* topological events is bounded by $s\binom{n}{d+1} \in O(n^{d+1})$, since this is the maximum number of instants at which $d+1$ points of S can become coplanar. Therefore we only have to deal with such topological events when $d+2$ points of S lie on a common hypersphere. The basic observation is that every topological event belongs to a local transition of altogether $d+2$ Delaunay $(d+1)$-tuples leaving the bounding Delaunay $(d-1)$-faces unchanged. Thus, we are able to determine the total number of topological events by adding for every imaginable Delaunay $(d-1)$-face $(P_0, \ldots, P_{d-1})$ the number of adjacent topological events that do not destroy this $(d-1)$-face.

With this intention, we consider an arbitrary d-tuple $(P_0, \ldots, P_{d-1})$ of different points and the line $B_{0,\ldots,d-1}(t,\mu)$ which is given by the formulation below:

$$\begin{aligned} B_{0,\ldots,d-1}(t,\mu) &:= m_{0,\ldots,d-1}(t) + \mu\, n_{0,\ldots,d-1}(t) \quad \text{where} \quad \mu \in \mathbb{R}, \\ m_{0,\ldots,d-1}(t) &:= \frac{1}{d}\sum_{i=0}^{d-1} P_i(t) \quad \text{and} \quad n_{0,\ldots,d-1}(t) \perp H_{0,\ldots,d-1}(t). \end{aligned}$$

In other words, $m_{0,\ldots,d-1}(t)$ denotes the *center of gravity* of the d sites and $n_{0,\ldots,d-1}(t)$ a *normal vector* to the affine hyperplane $H_{0,\ldots,d-1}(t)$ spanned by the d points. In addition, let $h^{>}_{0,\ldots,d-1}(t)$ and $h^{<}_{0,\ldots,d-1}(t)$ denote the two open halfspaces bounded by $H_{0,\ldots,d-1}(t)$.

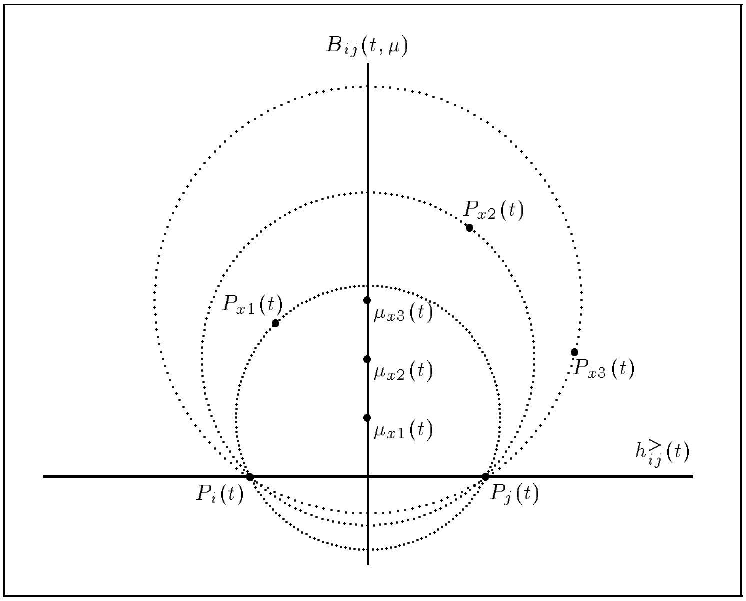

Figure 3: Characterizing the upper triangle $\{P_i, P_j, P_k\}$ in $\mathbb{R}^2$.

Now, whenever the Delaunay $(d-1)$-face $(P_0, \ldots, P_{d-1})$ exists, there are exactly two $(d+1)$-tuples $\{P_0, \ldots, P_{d-1}, P'\}$ and $\{P_0, \ldots, P_{d-1}, P''\} \in DT(S')$ adjacent to this Delaunay face with

$$P' \in S'_{>} := \left(h^{>}_{0,\ldots,d-1} \cap S\right) \cup \{\infty\} \quad \text{and} \quad P'' \in S'_{<} := \left(h^{<}_{0,\ldots,d-1} \cap S\right) \cup \{\infty\}.$$

If we look at the μ-values $\mu_x(t)$ of the circumcenters of the circumspheres $C(P_0, \ldots, P_{d-1}, P_x)$ on the bisector $B_{0,\ldots,d-1}(t,\mu)$, the upper $(d+1)$-tuple is obviously characterized by the minimum value:

$$\mu_{min}(t) := \min_{P_x \in S'_{>}} \mu_x(t).$$

This can be seen by imaging a point (circumcenter) starting from $m_{0,\ldots,d-1}(t)$ and moving along the line $B_{0,\ldots,d-1}(t,\mu)$ until a first point $P_x \in S'_{>}$ is captured by the variable circumsphere touching the sites $P_0, \ldots, P_{d-1}$. Naturally, an analogous construction can be done for the lower $(d+1)$-tuple. (Figure 3 displays the construction in the planar case.)

Now, if we investigate those moments when the upper $(d+1)$-tuple changes[4] we can restrict ourselves to those intervals in which $h^{>}_{0,\ldots,d-1} \cap S \neq \emptyset$. Next, we look closer at the functions $\mu_x(t)$ and their pairwise points of intersection:

Case (1) $\mu_x(t) = \mu_y(t) < \infty$
Both circumspheres $C(P_0, \ldots, P_{d-1}, P_x)$ and $C(P_0, \ldots, P_{d-1}, P_y)$ are identical, which implies that all $d+2$ points lie on a common hypersphere. By our non-periodicity assumption, this can happen only s times.

Case (2) $\mu_x(t) = \mu_y(t) = \infty$
These moments have no influence on the complexity of the minimum function $\mu_k(t)$, since we have restricted ourselves to intervals where $\mu_k(t) < \infty$.

[4]Notice, that P' can only be replaced by another point of $S'_{>}$, because the Delaunay $(d-1)$-face $(P_0, \ldots, P_{d-1})$ is not destroyed during the topological event.

Finally, we can summarize both cases with the statement that two different functions $\mu_x(t)$ and $\mu_y(t)$ have at most s relevant intersections. Thus, the theory of Davenport-Schinzel sequences implies that the minimum function $\mu_k(t)$ has worst-case complexity $O(\lambda_s(n))$, where $\lambda_s(n)$ is the maximum length of a Davenport-Schinzel sequence of length n and order s. Summing over all $\binom{n}{d}$ tuples of points $(P_0, \ldots, P_{d-1})$, we obtain the following theorem.

Theorem 2 *Given a finite set $S(t)$ of n continuous trajectories in d-dimensional space $\mathbb{R}^d$, the maximum number of topological events over time is $O(n^d\,\lambda_s(n))$. If only $k \le n$ points of S are moving (while the remaining $n-k$ stay fixed), this upper bound goes down to*

$$O(\min\{k^{d+1}(n-k)^{\lceil\frac{d}{2}\rceil} + k^d\lambda_s(k), kn^{d-1}\lambda_s(n) + (n-k)^d\lambda_s(k)\}).$$

To prove the second part of this theorem we consider the $O(k\,n^{d-1})$ moving and $O((n-k)^d)$ fixed d-tuples separately. The crucial fact is that each fixed d-tuple generates only $O(\lambda_s(k))$ instead of $O(\lambda_s(n))$ topological events. To see that, let $\{P_1, \ldots, P_d\}$ be a fixed d-tuple. Now if we investigate the μ_x-functions defined above, any fixed point $P_x \in S_f \setminus \{P_1, \ldots, P_d\}$ leads to a constant μ_x function. From this it follows that

$$\begin{aligned} \mu_k(t) \quad &:= \min_{P_x \in S' \setminus \{P_1, \ldots, P_d\}} \mu_x(t) \\ &= \min\left\{ \min_{P_x \in S_m} \mu_x(t)\ ,\ \mu_{min} \right\}, \end{aligned}$$

where μ_{min} is the minimum function of the constant functions μ_x with $P_x \in S_f \setminus \{P_1, \ldots, P_d\}$. This proves that the function $\mu_k(t)$ has at most $O(\lambda_s(k+1))$ pieces. On the other hand, each of the remaining

$$\binom{n}{d} - \binom{n-k}{d} \in O(k\,n^{d-1})$$

moving d-tuples $(P_1, \ldots, P_d)$ generates at most $O(\lambda_s(n))$ topological events – as we have seen above. Combining these results with the very recent upper bound by [Ro 93a], we obtain the desired bound.

In contrast to that, the known lower worst-case bound is given by the following class of examples (compare [HaDe 59, Kl 80, Se 82]). Imagine $n-k$ points fixed such that the corresponding Voronoi diagram has complexity $O((n-k)^{\lceil d/2\rceil})$ (which is the worst that can happen) and such that the circumspheres of the Delaunay $(d+1)$-tuples can be stabbed by a common line.

After that, we make the k remaining points, one after the other, pass along this line. Using the classification of the Delaunay $(d+1)$-tuples above, all $O((n-k)^{\lceil d/2\rceil})$ Delaunay tuples are destroyed during this movement. If we leave sufficient time between these movements, the topological sub-structure of the static points is destroyed only by the currently crossing point. Therefore every moving point generates $\Omega((n-k)^{\lceil d/2\rceil})$ topological events.

5 Dynamic Scenes

The topological structure of a Voronoi diagram under continuous motions of the points in S can be maintained by the following algorithm:

Algorithm : **Preprocessing :**

1. Compute the topological structure $DT(S'(t_0))$ of the starting position.
2. For every existing pair of $(d+1)$-tuples in $DT(S'(t_0))$ calculate the potential topological events.
3. For the set of the potential topological events create an event queue (priority queue).

Iteration :

1. Determine the next topological event and decide whether it is an (i, j)-transition or a recalculation.
2. Process the topological event and update the event queue.

We look closer at the individual steps of the algorithm and their time and storage requirements. In the first preprocessing step, we compute the initial Delaunay triangulation $DT(S(t_0))$ and augment it with extended dual edges, obtaining $DT(S'(t_0))$ in $O(n^{\lceil \frac{d}{2} \rceil})$ time and and space (e.g., using the optimal algorithm by [Se 90]). In the second preprocessing step, we continue with a flow of the $(d-1)$-faces in $DT(S'(t_0))$ computing the potential topological events. If m denotes the number of $(d+1)$-tuples which appear in the initial topological structure, this step can be done in $O(m)$ time. In the third preprocessing step, we build up the event queue for the set of potential topological events. The topological events are stored in a priority queue according to their temporal appearance, with the corresponding $(d+1)$-tuples stored with each event. This step and therefore the entire preprocessing step requires $O(n^{\lceil \frac{d}{2} \rceil} + m \log m)$ time and $O(n^{\lceil \frac{d}{2} \rceil})$ space.

To determine the next topological event, we simply pop the event queue in time $O(\log n)$. Assuming that the degree of degeneracy remains constant, then one can decide in constant time if the event is an (i, j)-transition or a (local) recalculation. Now, each topological event destroys only a constant number of adjacent $(d+1)$-tuples while creating also a constant number of new ones. Thus, in order to update the event queue, all we have to do is to delete the destroyed pairs of $(d+1)$-tuples and their corresponding topological events in the event queue and to insert the new ones. Thus, we spend time $O(\log n)$ per event (which [Ro 91] shows is worst-case optimal[5] even under linear motions of the points in the plane). In summary, we have:

Theorem 3 *Given a finite set $S(t)$ of n continuous trajectories in d-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 2$. After preprocessing requiring $O(n^{\lceil \frac{d}{2} \rceil} + m \log m)$ time and $O(n^{\lceil \frac{d}{2} \rceil})$ space, we can maintain the topological structure in worst-case optimal $O(\log n)$ time per event. Thereby, m denotes the initial complexity of the Voronoi diagram at the starting position.*

[5] There is a parallel variant of this algorithm using only $O(1)$ time per event [Ro 94].

6 Concluding Remarks and Open Problems

We have presented an algorithm for maintaining Voronoi diagrams of moving points over time. The major open question remaining is to prove that the presented bounds on the number of events are tight.

The algorithm presented here has been implemented in the planar case ($d = 2$) on a SUN workstation, using special methods for numerically stable evaluation of the functions involved [SuIr 89]. Extensive tests suggest that the number of topological events grows with $\Theta(n\sqrt{n})$ in the *average case* under linear motions chosen at random [Ro 93b]. We also expect in higher dimensions that the average number of topological events is significantly smaller than the derived worst-case bounds.

Dynamic Voronoi diagrams can be used for planning the motion of a disk in a dynamic scene of continuously moving points (see [RoNo 91]). Additionally, there are many related geometric structures and problems in computational geometry which can be solved very efficiently if the Voronoi diagram is known in advance (for a survey see, e.g., [Au 90, Ro 91]).

A typical application of higher dimensional dynamic Voronoi diagrams which arises in the area of spatial path planning (such as air-traffic control) is the maintenance of the *closest pair* or the *all-nearest-neighbors* over time. It is also quite interesting to apply the pattern matching methods by [AoImImTo 90, ImSuIm 89] to higher dimensions.

Acknowledgement

The authors wish to thank Peter Widmayer and Peter Remmele for carefully reading the manuscript.

Table of Symbols

S	set of points	$DT(S)$	Delaunay graph of S
S'	$S \cup \{\infty\}$	$DT(S')$	extended Delaunay graph of S
n	number of points	$C(\ldots)$	circumsphere
d	dimension	$v(\ldots)$	circumcenter (Voronoi point)
$\mathbb{E}^d$	d-dimensional Euclidean space	$VOL(\ldots)$	volume determinant
P_i	single point	$INS(\ldots)$	insphere determinant
B_{ij}	bisector of P_i and P_j	$OUTSIDE(\ldots)$	sign of the oriented $INS(\ldots)$ det.
$v(P_i)$	Voronoi polyhedron of P_i	$\lambda_s(n)$	maximum length of a (n, s)-Davenport Schinzel sequence
$VD(S)$	Voronoi diagram of S		

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