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Abstract

In this report a *sequent calculus with restricted weakening rules* is investigated. (Weakening rules admit adding new formulas to derivable sequents.) This calculus produces the fragment of classical logic, which is *"robust against loss of information"*. Thus, the proposed logic could be helpful in the field of databases.
The interpretation of the calculus is very simple: The calculus is sound and complete with respect to a *three-valued semantics*, where the third truth-value has the intention "neutral". Furthermore, this semantics corresponds to a special case of Girard's phase semantics (for linear logic) and is related to the semantics of RM_3, the strongest logic in the family of relevance logics.

Zusammenfassung

In diesem Bericht wird ein *Sequenzenkalkül mit eingeschränkten Abschwächungsregeln* untersucht. (Die Abschwächungsregeln erlauben es, zu einer herleitbaren Sequenz neue Formeln hinzuzufügen.) Der Kalkül erzeugt genau dasjenige Fragment der klassischen Logik, welches *"robust ist gegenüber Informationsverlust"*. Somit kann diese Logik auch im Gebiet der Datenbanken sehr nützlich sein.
Die Interpretation des Kalküls ist sehr einfach: Der Kalkül ist korrekt und vollständig bezüglich einer *3-wertigen Semantik*, wobei der dritte Wahrheitswert die Bedeutung "neutral" hat. Ferner entspricht diese Semantik einem Spezialfall der Phasensemantik von Girards linearen Logik und es besteht auch eine Beziehung zur Semantik von RM_3, der strengsten Logik in der Familie der Relevanzlogiken.

Contents

1 Motivation

In recent years, some variations of Gentzen's sequent calculus [Ta87] have been investigated; in particular, by restricting the structural rules. Sequent calculi without contraction rules (these rules reduce two occurrences of a formula into one occurrence) were studied by [Gr82], [KW84] and [Me92]. Girard's linear logic [Gi87] is also based on a sequent calculus without contraction rules and without weakening rules (these rules allow the addition of new formulas). The weakening rules are the focus of this report.

The sequent calculus formalizes the concept of a proof. This means that the sequent $A, B \supset A$ expresses the fact that A is a logical consequence of the specified knowledge A, B. Therefore, the weakening rule on the left hand side (lW) is related to the *monotonicity* of the logic, whereas the weakening rule on the right (rW) is related to the *paraconsistency* of the logic. The most interesting case is a sequent calculus where *only the rule* (rW) *is missing.* The family of derivable sequents then becomes smaller than in classical logic since it is impossible to weaken, or to water down, the consequence of a specified knowledge. For example, the sequent $A \supset A, B$ is no longer derivable.

The derivable sequents can be described as follows:
A sequent $\gamma \supset \delta$ *is derivable without the rule* (rW) *if and only if every sequent* $\gamma' \supset \delta'$ *which arises from* $\gamma \supset \delta$ *by discarding prime formulas is derivable in classical logic.*
This property could be helpful in the field of databases. For example, if the basic information (prime formulas) together with some of the conclusions obtained by the restricted calculus are stored, then these conclusions remain correct even if some of the information is lost.

Furthermore, the interpretation of the calculus is very simple. *The calculus is sound and complete w.r.t. a three-valued semantics, where the third truth-value has the intention* neutral *and where the values true and neutral are distinguished.* If the value of a formula A is neutral then the value of $A \wedge B$, as well as the value of $A \vee B$, is the same as the value of the formula B. As usual the existential quantifier is interpreted as a (possibly infinite) disjunction and the universal quantifier as a (possibly infinite) conjunction.

This three-valued semantics can be interpreted as a counterpart to Bočvar's semantics (or to Kleene's weak semantics, respectively). It is also possible to transfer this semantics into the three-valued semantics of Łukasiewicz. But the most interesting relation is the following: The proposed three-valued semantics is *a special case of Girard's phase semantics* (for linear logic). Specifically, this three-valued semantics can be obtained from the multiplicative connectives by a suitable restriction. Moreover, Kleene's strong semantics can be obtained from the additive connectives by the *same* restriction.

Note that *every logic* can be made robust against loss of information by forbidding the weakening of a logical consequence. For example, placing such a restriction on Kleene's strong 3-valued logic leads to a four-valued logic, where the fourth value also has the intention neutral (see [Ho93]). But in this paper, we concentrate on this restriction applied to classical logic only.

2 Calculus $\mathbf{LC}_R$

Let $\mathcal{L}$ be an arbitrary first order language with free variables $u, v, u_1, v_1, \ldots$, bound variables $x, y, x_1, y_1, \ldots$, function and relation symbols and $\neg, \wedge, \vee, \exists, \forall$ as logical connectives and quantifiers.

The terms $r, s, r_1, s_1, \ldots$ and formulas $A, B, C, A_1, B_1, \ldots$ are defined as usual where the negation of a formula is defined by $\neg\neg A :\equiv A$, $\neg(A \wedge B) :\equiv \neg A \vee \neg B$, $\neg(A \vee B) :\equiv \neg A \wedge \neg B$, $\neg(\exists x A(X)) :\equiv \forall x \neg A(x)$ and $\neg(\forall x A(x)) :\equiv \exists x \neg A(x)$.

Small Greek letters $\gamma, \delta, \sigma, \pi, \gamma_1, \delta_1, \sigma_1, \pi_1, \ldots$ denote finite sequences of $\mathcal{L}$-formulas, capital Greek letters $\Gamma, \Delta, \Sigma, \Pi, \Gamma_1, \Delta_1, \Sigma_1, \Pi_1, \ldots$ denote finite sets of $\mathcal{L}$-formulas and expressions of the form $\gamma \supset \delta$ are called $\mathcal{L}$-sequents.

Defining a calculus without the weakening rule on the right side, we have to prevent that the other rules imply weakening on the right hand side or transfer the weakening from the left to the right hand side. Therefore, the rules

$$\frac{\gamma \supset \delta}{\gamma \supset \delta, A}\ (rW)\,, \qquad \frac{\gamma \supset \delta, A}{\gamma \supset \delta, A \vee B}\ (r\vee)'\,, \qquad \frac{\gamma \supset \delta, A(s)}{\gamma \supset \delta, \exists x A(x)}\ (r\exists)'\,,$$

which weaken the right side of a sequent as well as the rules

$$\frac{\gamma, A \supset \delta}{\gamma \supset \delta, \neg A}\ (r\neg) \quad \text{and} \quad \frac{\gamma \supset \delta, A \qquad \sigma, A \supset \pi}{\gamma, \sigma \supset \delta, \pi}\ (cut)'$$

which could transfer the weakening from the left to the right side are not allowed. On the other side, the discarding rules of LC_R are missing in Gentzen's sequent calculus of classical logic [Ta87], but yet they are sound with respect to classical semantics. These rules enable a controlled exchange between both sides of a sequent and they are necessary in order to reduce the right side of a sequent.
Note that the axiom in Gentzen's sequent calculus is formulated as $A \supset A$. Because we can not increase the right side of a sequent in the restricted calculus, we have to replace the axiom by $\sigma \supset \sigma$.

LC$_R$ denotes a Logic Calculus without weakening on the Right hand side.

Definition 2.1 *LC_R is defined by the following axiom and rules:*

Axiom: $\sigma \supset \sigma$ *(σ : finite, non empty sequence of negated or unnegated prime formulas)*

Rules:

Structural Rules

$$\frac{\gamma \supset \delta}{A,\gamma \supset \delta} \quad (lW)$$

$$\frac{\gamma,A,B,\vartheta \supset \delta}{\gamma,B,A,\vartheta \supset \delta} \quad (lE) \qquad \frac{\gamma \supset \delta,A,B,\pi}{\gamma \supset \delta,B,A,\pi} \quad (rE)$$

$$\frac{\gamma,A,A \supset \delta}{\gamma,A \supset \delta} \quad (lC) \qquad \frac{\gamma \supset \delta,A,A}{\gamma \supset \delta,A} \quad (rC)$$

Logical Rules

$$\frac{\gamma,A \supset \delta \quad \vartheta,B \supset \delta}{\gamma,\vartheta,A \vee B \supset \delta} \quad (l\vee) \qquad \frac{\gamma \supset \delta,A,B}{\gamma \supset \delta,A \vee B} \quad (r\vee)$$

$$\frac{\gamma,A,B \supset \delta}{\gamma,A \wedge B \supset \delta} \quad (l\wedge) \qquad \frac{\gamma \supset \delta,A \quad \vartheta \supset \pi,B}{\gamma,\vartheta \supset \delta,\pi,A \wedge B} \quad (r\wedge)$$

$$\frac{\gamma,A(r) \supset \delta}{\gamma,\forall x A(x) \supset \delta} \quad (l\forall) \qquad \frac{\gamma \supset \delta,A(u) \quad \vartheta \supset \pi,A(s)}{\gamma,\vartheta \supset \delta,\pi,\forall x A(x)} \quad (r\forall)^*$$

$$\frac{\gamma,A(u) \supset \delta}{\gamma,\exists x A(x) \supset \delta} \quad (l\exists)^* \qquad \frac{\gamma \supset \delta,A(u)}{\gamma \supset \delta,\exists x A(x)} \quad (r\exists)^*$$

Discarding Rules

$$\frac{\gamma,\neg A \supset \delta,A}{\gamma \supset \delta,A} \quad (lD) \qquad \frac{\gamma,A \supset \delta,\neg A}{\gamma,A \supset \delta} \quad (rD)$$

Cut Rule

$$\frac{\gamma,A \supset \delta \quad \vartheta \supset A}{\gamma,\vartheta \supset \delta} \quad (cut)$$

*: *The free variable u must not occur in the conclusion.*

Definition 2.2 *A sequent* $\gamma \supset \delta$ $(\delta \neq \emptyset)$ *is* derivable in n steps *if either*

- $\gamma \supset \delta$ *is an axiom*
- $\gamma \supset \delta$ *is the conclusion of a structural rule and the premise* $\gamma' \supset \delta'$ *of this rule is derivable in* n *steps.*
- $\gamma \supset \delta$ *is the conclusion of a logical rule, a discarding rule or the cut rule and every premise* $\gamma_i \supset \delta_i$ *of this rule is derivable in* n_i *steps and* $n > \max_i n_i$.

A sequent $\gamma \supset \delta$ *is called* derivable *if there exists a natural number* n *such that* $\gamma \supset \delta$ *is derivable in* n *steps.*

Theorem 2.1 *By adding the rule* (rW) *to* LC_R*, we obtain a sound and complete sequent calculus of classical predicate logic.*

Proof: Note that the modified rules $(r\forall)$, $(r\exists)$ and (cut) are sound w.r.t. the classical semantics. To proof completeness, we confine to show that the rule $(r\exists)'$ is derivable in the calculus $\mathrm{LC}_R + (rW)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma \supset \delta, A(r)}{\gamma, \neg A(r) \supset \delta, A(r)}\,(lW)}{\gamma, \neg A(r) \supset \delta, A(r), A(u)}\,(rW)}{\gamma, \neg A(r) \supset \delta, A(r), \exists x A(x)}\,(r\exists)}{\gamma, \neg A(r) \supset \delta, \exists x A(x)}\,(rD)}{\gamma, \forall x \neg A(x) \supset \delta, \exists x A(x)}\,(l\forall)}{\gamma \supset \delta, \exists x A(x)}\,(lD)$$

q.e.d.

The following lemma shows that the sequent $\gamma \supset \delta, X, X$ is derivable if we can derive the sequent $\gamma \supset \delta, X$.

Lemma 2.2 *If* $\gamma \supset \delta$ *and* $\pi \supset \sigma$ *are derivable then* $\gamma, \pi \supset \delta, \sigma$ *is derivable, too.*

Proof: The proof directly follows by induction on the sum of the lengths of the derivations of $\gamma \supset \delta$ and $\pi \supset \sigma$.

Lemma 2.3 *The sequent $A \supset A$ is derivable for any formula A.*

Proof: Recall that $A \supset A$ is an axiom only if A is a negated or unnegated prime formula. Thus, the lemma is proved by induction on the structure of A:

$A \equiv B \vee C$: By the induction hypothesis and lemma 2.2, the sequents $B, \neg B \supset B, \neg B$ and $C, \neg C \supset C, \neg C$ are derivable. Hence:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c}\vdots\\ B,\neg B \supset B,\neg B\end{array}}{B \supset B,\neg B}\,(lD)}{\supset B,\neg B}\,(lD) \qquad \cfrac{\cfrac{\begin{array}{c}\vdots\\ C,\neg C \supset C,\neg C\end{array}}{C \supset C,\neg C}\,(lD)}{\supset C,\neg C}\,(lD)}{\supset B, C, \neg B \wedge \neg C}\,(r\wedge)}{B \vee C \supset B, C, \neg B \wedge \neg C}\,(lW)}{B \vee C \supset B, C}\,(rD)}{B \vee C \supset B \vee C}\,(r\vee)$$

$A \equiv B \wedge C$: analogous to $B \vee C$

$A \equiv \forall x B(x)$: By the induction hypothesis, it follows that the sequent $B(u) \supset B(u)$ is derivable. Hence:

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ B(u) \supset B(u)\end{array}}{\forall x B(x) \supset B(u)}\,(l\forall) \qquad \cfrac{\begin{array}{c}\vdots\\ B(u) \supset B(u)\end{array}}{\forall x B(x) \supset B(u)}\,(l\forall)}{\forall x B(x) \supset \forall x B(x)}\,(r\forall)$$

$A \equiv \exists x B(x)$: By the induction hypothesis and lemma 2.2, the sequents $\neg B(u) \supset \neg B(u)$ and $B(v), \neg B(v) \supset B(v), \neg B(v)$ are derivable. In virtue of the rule (lD), the sequent $\supset B(v), \neg B(v)$ is derivable, too.

Hence:

$$
\dfrac{
\begin{array}{c} \vdots \\ \supset\; B(v), \neg B(v) \end{array}
\qquad
\dfrac{\begin{array}{c} \vdots \\ \neg B(u) \supset \neg B(u) \end{array}}{\forall x \neg B(x) \supset \neg B(u)}\,(l\forall)
}{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\forall x \neg B(x) \supset \forall x \neg B(x), B(v)}{\forall x \neg B(x), \exists x B(x) \supset \forall x \neg B(x), B(v)}\,(lW)
}{\forall x \neg B(x), \exists x B(x) \supset B(v)}\,(rD)
}{\forall x \neg B(x), \exists x B(x) \supset \exists x B(x)}\,(r\exists)
}{\exists x B(x) \supset \exists x B(x)}\,(lD)
}\,(r\forall)
$$

q.e.d.

By virtue of the lemmata 2.2 and 2.3, the sequent $\gamma \supset \gamma$ is derivable for every finite sequence γ.

3 Robust Semantics

We define a three-valued semantics where a third truth-value n is added to t (true) and f (false). The third truth-value can be described as *neutral.* If the truth-value of a formula A is n, then the value of the formulas $A \wedge B$ and $A \vee B$ corresponds to the truth-value of B. The truth-value n never succeeds in presence of t or f. I.e. the value neutral is "less important" than the values true or false.

A colloquial interpretation of the truth-values could be:
true $\approx$ "to accept a proposal", *false* $\approx$ "to turn down a proposal" and *neutral* $\approx$ "an indifferent opinion, e.g. neither to accept nor to turn down a proposal".

Definition 3.1 *A* robust valuation *for $\mathcal{L}$ is a function V which assigns a truth-value $V(A) \in \{w, f, n\}$ to all $\mathcal{L}$-formulas A and satisfies the following conditions:*

$$V(\neg A) = \begin{cases} t & : \textit{if } V(A) = f \\ n & : \textit{if } V(A) = n \\ f & : \textit{otherwise} \end{cases}$$

$$V(A \vee B) = \begin{cases} t & : \textit{if } V(A) = t \textit{ or } V(B) = t \\ n & : \textit{if } V(A) = V(B) = n \\ f & : \textit{otherwise} \end{cases}$$

$$V(A \wedge B) = \begin{cases} f & : \textit{if } V(A) = f \textit{ or } V(B) = f \\ n & : \textit{if } V(A) = V(B) = n \\ t & : \textit{otherwise} \end{cases}$$

$$V(\exists x A(x)) = \begin{cases} t & : \textit{if } V(A(s)) = t \textit{ for some term } s \textit{ of } \mathcal{L} \\ n & : \textit{if } V(A(s)) = n \textit{ for all terms } s \textit{ of } \mathcal{L} \\ f & : \textit{otherwise} \end{cases}$$

$$V(\forall x A(x)) = \begin{cases} f & : \textit{if } V(A(s)) = f \textit{ for some term } s \textit{ of } \mathcal{L} \\ n & : \textit{if } V(A(s)) = n \textit{ for all terms } s \textit{ of } \mathcal{L} \\ t & : \textit{otherwise} \end{cases}$$

Examples: Let $V(A(s)) = t$ and $V(B(s)) = n$.
Then $V(A(s) \wedge B(s)) = t$ and $V(\forall x(\neg A(x) \wedge B(x))) = f$.
For all robust valuations V holds:
$V(\exists x \neg A(x)) = V(\neg(\forall x A(x)))$ and $V(\neg A \wedge \neg B) = V(\neg(A \vee B))$.

Remark: In this semantics we can express that a formula A dominates a formula B. We define this new connective as follows.

$$A \lhd B :\equiv A \vee (A \wedge B)$$

The truth-value of $A \lhd B$ corresponds to the value of A if the value of A is not neutral, else it corresponds to the value of B. Thus, we can describe a hierarchy of formulas. In the above example, the formula expresses that A can decide at first to accept or turn down a proposal. Only if A has a indifferent opinion, then B can decide.

Definition 3.2 *A non empty set of formulas* Δ *is called a* robust consequence *of the set of formulas* Γ *if for all robust valuations* V *holds:* $V(A) = f$ *for some* A *in* Γ *or* $V(B) = t$ *for some* B *in* Δ *or* $V(B) = n$ *for all* B *in* Δ.
In this case we write $\Gamma \models_R \Delta$.

A formula A *is called a* robust tautology *if* $\models_R A$ *holds.*

Remark: For all Γ and Δ holds:

$$\Gamma \models_R \Delta \implies \Gamma \models_{classical} \Delta$$

Examples: For all formulas A, B holds

$$\forall x(A(x) \wedge B(x)) \models_R \forall x A(x) \wedge \forall x B(x)$$

$$\forall x A(x) \wedge \forall x B(x) \models_R \forall x(A(x) \wedge B(x))$$

$$\models_R A \vee \neg A$$

But not every classical consequence is a robust consequence. E.g.

$$\forall x A(x) \vee \forall x B(x) \not\models_R \forall x(A(x) \vee B(x))$$

Let $V(A(s_0)) = n$, $V(B(s_0)) = f$ and $V(A(s)) = V(B(s)) = t$ for all terms $s \neq s_0$. Then $V(\forall x A(x)) = t$ and $V(\forall x(A(x) \vee B(x))) = f$.
Note the formula $A(s_0)$ is dominated by $A(s)$ in the formula $\forall x A(x)$ and by $B(s_0)$ in the formula $\forall x(A(x) \vee B(x))$.

4 Soundness and Completeness

We will prove that the calculus LC_R is sound and complete with respect to the robust semantics. The proof of the soundness is easy in contrast to the proof of the completeness, where we need some further properties of LC_R. See the lemmata 4.2 - 4.7.

Definition 4.1 *We define* $\mathrm{set}(\gamma)$ *as the set of the components of* γ.

Theorem 4.1 (Soundness)
If $\gamma \supset \delta$ *is derivable, then* $set(\delta)$ *is a robust consequence of* $set(\gamma)$.

Proof: The proof follows by induction on the length of the derivation.

The following properties of LC_R will be used in the proof of the completeness:

Lemma 4.2 *The sequent* $\gamma, A, \neg A, B, \neg B \supset \delta$ *is derivable if one of the following sequents is derivable:*

a1) $\gamma, A, \neg A, B, \neg B, A \vee B \supset \delta$
a2) $\gamma, A, \neg A, B, \neg B, A \vee B \supset \delta, \neg A \wedge \neg B$
a3) $\gamma, A, \neg A, B, \neg B \supset \delta, \neg A \wedge \neg B$
b1) $\gamma, A, \neg A, B, \neg B, A \wedge B \supset \delta$
b2) $\gamma, A, \neg A, B, \neg B, A \wedge B \supset \delta, \neg A \vee \neg B$
b3) $\gamma, A, \neg A, B, \neg B \supset \delta, \neg A \vee \neg B$

Proof: We modify the calculus LC_R by replacing the axiom by a new axiom of the form $A, \neg A, B, \neg B, \sigma \supset \sigma$. By virtue of the rules (lW) and (lC), the sequent $\gamma, A, \neg A, B, \neg B \supset \delta$ is derivable in LC_R if and only if this sequent is derivable in the modified calculus.

Now, the claims follow by simultaneous induction of the length of the derivation in the modified calculus.

q.e.d.

Lemma 4.3 *If the sequent* $\gamma, A \vee B, \neg A, B \supset \delta$ *is derivable then* $\gamma, A \vee B, \neg A \supset \delta$ *is also derivable.*

Proof: As we can see in the proof of lemma 2.3, $A \vee B \supset A, B$ is derivable. Thus, we can derive the sequent $\gamma, A \vee B, \neg A \supset B$ using the rules (lW) and (rD). Now, obtain a proof as follows.

$$\frac{\gamma, A \vee B, \neg A, B \supset \delta \qquad \gamma, A \vee B, \neg A \supset B}{\gamma, A \vee B, \neg A \supset \delta}\ (cut)$$

q.e.d.

Lemma 4.4 *If the sequent* $\gamma \supset \delta, \exists x A(x), A(s)$ *is derivable then* $\gamma \supset \delta, \exists x A(x)$ *is derivable, too.*

Proof: Now, obtain a proof as follows.

$$\cfrac{\cfrac{\cfrac{\cfrac{\gamma \supset \delta, \exists x A(x), A(s)}{\gamma, \neg A(s) \supset \delta, \exists x A(x), A(s)}\ (lW)}{\gamma, \neg A(s) \supset \delta, \exists x A(x)}\ (rD)}{\gamma, \forall x \neg A(x) \supset \delta, \exists x A(x)}\ (l\forall)}{\gamma \supset \delta, \exists x A(x)}\ (lD)$$

q.e.d.

Lemma 4.5 *The sequent* $\gamma, \forall x \neg A(x), \forall x A(x) \supset \delta$ *can be derived if one of the following sequents is derivable:*

a) $\gamma, \forall x \neg A(x), \forall x A(x), \exists x A(x) \supset \delta$
b) $\gamma, \forall x \neg A(x), \forall x A(x), \exists x A(x) \supset \delta, \forall x \neg A(x)$
c) $\gamma, \forall x \neg A(x), \forall x A(x) \supset \delta, \forall x \neg A(x)$

Proof: As before, it suffices to consider derivations which start with an axiom of the form $\forall x \neg A(x), \forall x A(x), \sigma \supset \sigma$. By simultaneous induction on the length of the derivation, the proposition is easy to prove. For example:

a) If the last inference is

$$\frac{\gamma, \forall x \neg A(x), \forall x A(x), A(u) \supset \delta}{\gamma, \forall x \neg A(x), \forall x A(x), \exists x A(x) \supset \delta} \; (l\exists)$$

then the sequent $\gamma, \forall x \neg A(x), \forall x A(x), A(u) \supset \delta$ is derivable. The proposition follows using the rules $(l\forall)$ and (lC).

If the last inference is

$$\frac{\gamma, \forall x \neg A(x), \forall x A(x), \exists x A(x) \supset \delta, \forall x \neg A(x)}{\gamma, \forall x \neg A(x), \forall x A(x), \exists x A(x) \supset \delta} \; (rD)$$

then the induction hypothesis of *b)* yields the proposition.

b) Define $\sigma_1 :\equiv \gamma, \forall x \neg A(x), \forall x A(x), \exists x A(x)$ and $\sigma_2 :\equiv \gamma, \forall x \neg A(x), \forall x A(x)$.
Let δ_1, δ_2 be such that $\delta_1, \delta_2 \equiv \delta$. If the last inference is

$$\frac{\sigma_1 \supset \delta_1, \neg A(u) \qquad \sigma_1 \supset \delta_2, \neg A(s)}{\sigma_1 \supset \delta, \forall x \neg A(x)} \; (r\forall)$$

then the induction hypothesis of *a)* yields that $\sigma_2 \supset \delta_1, \neg A(u)$ and $\sigma_2 \supset \delta_2, \neg A(s)$ are derivable. By virtue of the rule $(r\exists)$, we can also derive the sequent $\sigma_2 \supset \delta_1, \exists x \neg A(x)$. From lemma 2.2 we obtain that $\sigma_2 \supset \delta, \exists x \neg A(x), \neg A(s)$ is derivable and from lemma 4.4 $\sigma_2 \supset \delta, \exists x \neg A(x)$. Thus, the proposition follows using the rule (rD).

If the last inference is

$$\frac{\gamma, \forall x \neg A(x), \forall x A(x), A(u) \supset \delta, \forall x \neg A(x)}{\gamma, \forall x \neg A(x), \forall x A(x), \exists x A(x) \supset \delta, \forall x \neg A(x)} \; (l\exists)$$

then the induction hypothesis of *c)* yields that the sequent $\gamma, \forall x \neg A(x), \forall x A(x), A(u) \supset \delta$ is derivable. The proposition follows using the rules $(l\forall)$ and (lC).

q.e.d.

Lemma 4.6 *The sequent $\gamma, \forall x A(x) \supset \delta$ is derivable if $\gamma, \forall x A(x), \forall x \neg A(x), A(u) \supset \delta$ and $\gamma, \forall x A(x), A(u) \supset \neg A(u)$ are derivable (where u is a free variable which does not occur in $\gamma, \forall x A(x) \supset \delta$).*

Proof: The sequent $\gamma, \forall x A(x) \supset \forall x \neg A(x)$ can be derived from $\gamma, \forall x A(x), A(u) \supset \neg A(u)$ using the rules (lD) and $(r\forall)$.
The sequent $\gamma, \forall x A(x), \forall x \neg A(x) \supset \delta$ can be derived from the sequent $\gamma, \forall x A(x), \forall x \neg A(x), A(u) \supset \delta$ using the rule $(l\forall)$. Thus, the proposition follows using the (cut).
q.e.d.

Lemma 4.7 *The sequent $\gamma, \exists x A(x) \supset \delta$ is derivable if $\gamma, \exists x A(x), \forall x A(x), A(u) \supset \delta$ and $\gamma, \exists x A(x), A(u) \supset \neg A(u)$ are derivable*
(where u is a free variable which does not occur in $\gamma, \exists x A(x) \supset \delta$).

Proof: As we can see in the proof of lemma 2.3, the following sequent $\forall x \neg A(x), \exists x A(x) \supset A(u)$ (where u is a free variable) is derivable. Therefore, the sequent $\forall x \neg A(x), \exists x A(x) \supset \forall x A(x)$ can also be derived.
Furthermore, using the rule $(l\forall)$ we obtain $\gamma, \exists x A(x), \forall x A(x) \supset \delta$ from the sequent $\gamma, \exists x A(x), \forall x A(x), A(u) \supset \delta$.
Using the rule (lD) we obtain $\gamma, \exists x A(x) \supset \neg A(u)$ from the sequent $\gamma, \exists x A(x), A(u) \supset \neg A(u)$. Thus, the sequent $\gamma, \exists x A(x) \supset \forall x \neg A(x)$ can be derived by the rules (lD) and $(r\forall)$. Now, construct a proof as follows.

$$\frac{\gamma, \exists x A(x), \forall x A(x) \supset \delta \qquad \forall x \neg A(x), \exists x A(x) \supset \forall x A(x)}{\forall x \neg A(x), \exists x A(x), \gamma \supset \delta} \ (cut)$$

$$\frac{\gamma, \exists x A(x) \supset \forall x \neg A(x) \qquad \forall x \neg A(x), \exists x A(x), \gamma \supset \delta}{\exists x A(x), \gamma \supset \delta} \ (cut)$$

q.e.d.

Theorem 4.8 (Completeness)
If $set(\delta)$ is a robust consequence of $set(\gamma)$, then $\gamma \supset \delta$ is derivable.

Proof: Let $\gamma \supset \delta$ a non derivable sequent. We prove that $set(\delta)$ is not a robust consequence of $set(\gamma)$. Note that the order and multiplicity of the formulas of γ and δ is insignificant, because the calculus LC_R contains the rules (lE), (rE), (lC), (rC), (lW) and because lemma 2.2 holds.

The proof is based on the concept of deduction chains and it is divided in the following four steps:
1) Definition of a deduction chain
2) Introduction of a robust valuation V
3) Verification of some properties of the valuation V
4) Conclusions

1) *Definition of a deduction chain*

We define by induction on i an infinite sequence of sequents $\gamma_0 \supset \delta_0$, $\gamma_1 \supset \delta_1$, $\gamma_2 \supset \delta_2$, ... such that $\gamma_i \supset \delta_i$ is not derivable for all $i \in \mathbb{N}$. This sequence is called deduction chain.

$\delta_0 := \delta$ and $\gamma_0 := \gamma, \neg\delta$ ($\neg\delta$ denotes the sequence $\neg A_1, ..., \neg A_k$, where δ is $A_1, ..., A_k$.)
Since $\gamma \supset \delta$ is not derivable and since LC_R contains the rule (lD), the sequent $\gamma_0 \supset \delta_0$ is not derivable, too.

Let $u_1, u_2, ...$ and $s_1, s_2, ...$ arbitrary enumerations of the free variables and the terms.
Let $\gamma_i \equiv A_1, ..., A_{n_i}$. Dependent on A_1, we define $\gamma_{i+1} \supset \delta_{i+1}$ as follows.

$A_1 \equiv L$ where L is a negated or unnegated prime formula.

$$\gamma_{i+1} := A_2, ..., A_{n_i}, A_1 \text{ and } \delta_{i+1} := \delta_i$$

$A_1 \equiv B \wedge C$ We define

$$\gamma'_{i+1} := A_2, ..., A_{n_i}, B, C, A_1 \tag{D.1.1}$$

By virtue of the rule $(l\wedge)$, the sequent $\gamma'_{i+1} \supset \delta_i$ is not derivable.

$$\gamma_{i+1} := \begin{cases} \gamma'_{i+1}, \neg B \vee \neg C & : \text{if } \neg B, \neg C \in set(\gamma'_{i+1}) \\ \gamma'_{i+1} & : \text{otherwise} \end{cases}$$
$$\delta_{i+1} := \delta_i \tag{D.1.2}$$

The sequent $\gamma_{i+1} \supset \delta_{i+1}$ is not derivable. See lemma 4.2 *a1)*.

$A_1 \equiv B \vee C$ We define

$$\gamma'_{i+1} := \begin{cases} A_2, ..., A_{n_i}, B, A_1 & : \text{if } A_2, ..., A_{n_i}, B, A_1 \supset \delta_i \text{ is not derivable} \\ A_2, ..., A_{n_i}, C, A_1 & : \text{otherwise} \end{cases} \tag{D.2.1}$$

By virtue of the rule $(l\vee)$, the sequent $\gamma'_{i+1} \supset \delta_i$ is not derivable.

Define

$$\gamma''_{i+1} := \begin{cases} \gamma'_{i+1}, B, C: & \text{if } \neg B, \neg C \in set(\gamma'_{i+1}) \\ \gamma'_{i+1}, C \ : & \text{if } \neg B \in set(\gamma'_{i+1}) \text{ and } \neg C \notin set(\gamma'_{i+1}) \\ \gamma'_{i+1}, B \ : & \text{if } \neg C \in set(\gamma'_{i+1}) \text{ and } \neg B \notin set(\gamma'_{i+1}) \\ \gamma'_{i+1} \ : & \text{otherwise} \end{cases} \tag{D.2.2}$$

By virtue of the lemma 4.3, the sequent $\gamma''_{i+1} \supset \delta_i$ is not derivable.

Define

$$\gamma_{i+1} := \begin{cases} \gamma''_{i+1}, \neg B \wedge \neg C: & \text{if } B, \neg B, C, \neg C \in set(\gamma''_{i+1}) \\ \gamma''_{i+1} \ : & \text{otherwise} \end{cases}$$

$$\delta_{i+1} := \delta_i \tag{D.2.3}$$

By virtue of lemma 4.2 *b1)*, the sequent $\gamma_{i+1} \supset \delta_{i+1}$ is not derivable.

$A_1 \equiv \exists x B(x)$ We define

$$\gamma'_{i+1} := A_2, ..., A_{n_i}, B(u_k), A_1 \tag{D.3.1}$$

where k is the least number such that u_k does not occur in $\gamma_i \supset \delta_i$.

By virtue of the rule $(l\exists)$, the sequent $\gamma'_{i+1} \supset \delta_i$ is not derivable.

Define

$$\gamma_{i+1} := \gamma'_{i+1}, \forall x B(x), B(u_l) \text{ and } \delta_{i+1} := \delta_i$$

or

$$\gamma_{i+1} := \gamma'_{i+1}, B(u_l) \text{ and } \delta_{i+1} := \neg B(u_l) \tag{D.3.2}$$

where l is the least number such that u_l does not occur in $\gamma'_{i+1} \supset \delta_i$
Note we can choose $\gamma_{i+1}, \delta_{i+1}$ such that $\gamma_{i+1} \supset \delta_{i+1}$ is not derivable. (See lemma 4.7.) Furthermore, $\gamma_{i+1} \supset \delta_i$ is not derivable, too.

$A_1 \equiv \forall x B(x)$ We define

$$\gamma'_{i+1} := A_2, ..., A_{n_i}, B(s_k), A_1 \tag{D.4.1}$$

where k is the least number such that $B(s_k)$ is not an element of $set(\gamma_i)$.
By virtue of the rule $(l\forall)$, the sequent $\gamma'_{i+1} \supset \delta_i$ is not derivable.

Define

$$\gamma''_{i+1} := \begin{cases} \gamma'_{i+1}, \exists x B(x) & : \text{ if } \forall x \neg B(x) \in set(\gamma'_{i+1}) \\ \gamma'_{i+1} & : \text{ otherwise} \end{cases} \tag{D.4.2}$$

By virtue of lemma 4.5, the sequent $\gamma''_{i+1} \supset \delta_i$ is not derivable.

Define

$$\gamma_{i+1} := \gamma''_{i+1}, \forall x \neg B(x), B(u_l) \text{ and } \delta_{i+1} := \delta_i$$

or

$$\gamma_{i+1} := \gamma''_{i+1}, B(u_l) \text{ and } \delta_{i+1} := \neg B(u_l) \tag{D.4.3}$$

where l is the least number such that u_l does not occur in $\gamma''_{i+1} \supset \delta_i$
Note we can choose $\gamma_{i+1}, \delta_{i+1}$ such that $\gamma_{i+1} \supset \delta_{i+1}$ is not derivable. (See lemma 4.6.) Furthermore, $\gamma_{i+1} \supset \delta_i$ is not derivable, too.

The definition of the deduction chain yields

$$\gamma_k \supset \delta_i \text{ is not derivable for all } i \leq k. \tag{*}$$

2) *Introduction of a robust valuation V*

Let $\Pi := \cup_{i\in\mathbb{N}} set(\gamma_i)$. Let P a n-ary $\mathcal{L}$-relation symbol and $s_1, ..., s_n$ $\mathcal{L}$-terms. We define V as follows.

$$V(P(s_1, ..., s_n)) := \begin{cases} t & \text{:if } P(s_1, ..., s_n) \in \Pi \text{ and } \neg P(s_1, ..., s_n) \notin \Pi \\ n & \text{:if } P(s_1, ..., s_n), \neg P(s_1, ..., s_n) \in \Pi \\ f & \text{:otherwise} \end{cases}$$

Thus, V is well defined for all $\mathcal{L}$-formulas. (See Definition 3.1)

3) *Verification of the following two properties of the valuation V*

$$\text{If } A \in \Pi, \text{ then } V(A) \neq f \qquad \text{(V.1)}$$

$$\text{If } A \in \Pi \text{ and } V(A) = n, \text{ then } \neg A \in \Pi \qquad \text{(V.2)}$$

We prove the properties by simultaneous induction on the structure of A.

$A \equiv P(s_1, ..., s_n)$: The propositions immediately follow from the definition of V.

$A \equiv \neg P(s_1, ..., s_n)$: see above.

$A \equiv B \wedge C$: Since A is an element of Π, it follows by (D.1.1) that B as well as C are elements of Π.

(V.1): By the induction hypothesis of (V.1), $V(B)$ and $V(C)$ are not false. Thus, the value of A is not false.

(V.2): Since $V(A) = n$, it holds $V(B) = V(C) = n$. By the induction hypothesis of (V.2), $\neg B$ and $\neg C$ are elements of Π. By (D.1.2), $\neg A \in \Pi$.

$A \equiv B \vee C$: Since A is an element of Π, it follows by (D.2.1) that B or C is an element of Π. Let $B \in \Pi$.

(V.1): By the induction hypothesis of (V.1), $V(B) \neq f$. If $V(B) = t$ then $V(A) \neq f$. If $V(B) = n$ then the induction hypothesis of (V.2) yields that $\neg B$ is an element of Π. By (D.2.2), $C \in \Pi$. By the induction hypothesis of (V.1), $V(C) \neq f$. Thus, $V(A) \neq f$.

(V.2): Let $V(A) = n$. Thus $V(B) = V(C) = n$. The induction hypothesis of (V.2) yields that $\neg B$ is an element of Π. By (D.2.2), $C \in \Pi$. The induction hypothesis of (V.2) yields that $\neg C$ is an element of Π. (D.2.3) implies $\neg A \in \Pi$.

$A \equiv \exists x B(x)$: Since $A \in \Pi$, there exists a $k \in \mathbb{N}$ such that holds $\gamma_k \equiv A_1, ..., A_{n_k}$ and $A \equiv A_1$. By (D.3.1) and (D.3.2), it follows $B(u) \in set(\gamma_{k+1})$ for a free variable u and $B(v) \in set(\gamma_{k+1})$ for a free variable v. (D.3.2) implies that either $\forall x B(x) \in \gamma_{k+1}$ or $\delta_{k+1} \equiv \neg B(v)$.
Let $\forall x B(x) \in set(\gamma_{k+1})$.

(V.1): Since $\forall x B(x) \in \Pi$ and (D.4.1), $B(s) \in \Pi$ for all terms s. The induction hypothesis of (V.1) yields $V(B(s)) \neq f$ for all terms s. Therefore $V(A) \neq f$.

(V.2): Since $V(A) = n$, $V(B(s)) = n$ for all terms s. Furthermore, there exists $j \in \mathbb{N}$ such that $j > k$, $\gamma_j \equiv A_1, ..., A_m$ and $A_1 \equiv \forall x B(x)$. (D.4.3) implies that $B(u') \in set(\gamma_{j+1})$ for a free variable u' and either $\forall x \neg B(x) \in set(\gamma_{j+1})$ or $\delta_{j+1} \equiv \neg B(u')$.
If $\forall x \neg B(x) \in \Pi$ then the proposition holds.
Assume $\forall x \neg B(x) \notin \Pi$, then $\delta_{j+1} \equiv \neg B(u')$. The property (*) yields that $\gamma_l \supset \delta_{j+1}$ is not derivable for all $l > j$. Thus $\neg B(u') \notin \Pi$. Note that the sequent $\gamma, C \supset C$ is derivable for any formulas C. But $\forall x B(x)$ is an element of Π. This means that $B(s) \in \Pi$ for all terms s, see (D.4.1). Contradiction.

Let $\delta_{k+1} \equiv \neg B(v)$. The property (*) yields that $\gamma_l \supset \delta_{k+1}$ is not derivable for all $l > j$. Thus $\neg B(v) \notin \Pi$.

(V.1): Since $B(v) \in \Pi$, it follows by the induction hypothesis of (V.1) and (V.2) that $V(B(v)) = t$. Thus $V(A) \neq f$.

(V.2): Since $V(A) = n$, $V(B(s)) = n$ for all terms s. By the induction hypothesis of (V.2), it follows $\neg B(v) \in \Pi$. Contradiction.

$A \equiv \forall x B(x)$: Since $A \in \Pi$ and (D.4.1), $B(s) \in \Pi$ for all terms s.

(V.1): By the induction hypothesis of (V.1), $V(B(s)) \neq f$ for all terms s. Thus $V(A) \neq f$.

(V.2): Since $V(A) = n$, $V(B(s)) = n$ for all terms s. From (D.4.3), it follows either $\forall x \neg B(x)$ is an element of Π or there exists $k \in \mathbb{N}$ such that $\gamma_k \equiv A_1, ..., A_n$ and $A_1 \equiv A$ and $\delta_{k+1} \equiv \neg B(u)$ for a free variable u.
Let $\forall x \neg B(x) \in \Pi$. By (D.4.2), $\exists x \neg B(x) \in \Pi$. This means that $\neg A$ is an element of Π.
Assume $\forall x \neg B(x) \notin \Pi$. Thus $\delta_{k+1} \equiv \neg B(u)$. The property $(*)$ yields that the sequent $\gamma_l \supset \delta_{k+1}$ is not derivable for all $l > k$. Thus $\neg B(u) \notin \Pi$. But $B(u) \in \Pi$ and $V(B(u)) = n$. By the induction hypothesis of (V.2), $\neg B(u) \in \Pi$. Contradiction.

4) *Conclusions*

1. If $A \in set(\gamma)$, then $A \in \Pi$. Therefore, $V(A) \neq f$.
2. If $A \in set(\delta)$, then $\neg A \in \Pi$. Thus, $V(A) \neq w$.
3. There exists a formula $A \in set(\delta)$ where $V(A) \neq n$.
   Assume $V(A) = n$ for all formulas $A \in set(\delta)$. Therefore, it holds $\neg A \in set(\gamma_0)$ for all $A \in set(\delta)$. The property (V.2) yields that A is an element of Π for all $A \in set(\delta)$. Since $set(\gamma_i) \subseteq set(\gamma_{i+1})$ for all $i \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $A \in set(\gamma_j)$ for all $A \in set(\delta)$. Note that $\delta_0 = \delta$ by definition.

   The lemmata 2.2 and 2.3 yield $\gamma_j \supset \delta_0$ is derivable. But the property $(*)$ yields $\gamma_j \supset \delta_0$ is not derivable. Contradiction.

Thus, $set(\delta)$ is not a robust consequence of $set(\gamma)$.
q.e.d.

5 Comparisons

The robust semantics differs from the well known 3-valued semantics of Bočvar, Kleene and Łukasiewicz (see [Av91], [Ep90], [Go89] or [Ur86]), but there are some relations of course. We limit the comparison to the propositional semantics.

We can interpret the robust semantics as a counterpart to *Bočvar's* semantics (or Kleene's weak semantics, respectively). There, the intention of the third truth-value is *paradox*. Thus, this value always succeeds in presence of true and false, in contrast to the robust semantics, where the third truth-value never succeeds.

Furthermore, we can simulate the robust semantics in *Łukasiewicz's* ones. The value of a disjunction in the robust semantics is generated by the following formula in Łukasiewicz's semantics:

$$(A \vee \neg(B \rightarrow \neg B)) \wedge (B \vee \neg(A \rightarrow \neg A))$$

Note that negation is interpreted in the same way and that disjunction together with negation is a base of the connectives in the robust semantics.

The strongest logic in the family of relevance logics, RM_3 [RM82] [Du86], [Fu88], has also a three-valued semantics. Its interpretation of the implication $(A \rightarrow B)$ exactly corresponds to the interpretation of the formula $\neg A \vee B$ in the robust logic. (But the interpretation of conjunction and disjunction correspond to Kleene's interpretations of these connectives.) The weakening is also restricted in RM_3, but in different kind. Note the formula $A \rightarrow A \wedge B$ is a tautology in RM_3, but $A \rightarrow (B \rightarrow A)$ is no tautology in RM_3.

A more detailed analysis of these relations could be found in [Ho92].

The most interesting relation is the following: *The robust semantics is a special case of Girard's phase semantics (for linear logic)* [Gi87], [Tr92]. In particular, it is obtained from the multiplicative connectives by a suitable restriction and Kleene's strong semantics is obtained from the additive connectives by the same restriction. (The modalities will be discussed later.) In order to see the connection, we shortly repeat the most important definitions of the phase semantics:

The language of the propositional linear logic contains the logical connectives $^{\perp}, +, \star, \sqcap$ and $\sqcup$ where $+$ corresponds to the multiplicative disjunction, $\star$ to the multiplicative conjunction, $\sqcap$ to the additive conjunction and $\sqcup$ to the additive disjunction.

A *phase space* $\underline{P}$ consists in a commutative monoid $\langle P, \cdot, 1\rangle$ and a subset $\perp$ of P. $\quad \underline{P} = \langle P, \cdot, 1, \perp\rangle$

If G is a subset of P, then its *dual* $G^{\perp}$ is defined as $\{p \in P \mid \forall q(q \in G \rightarrow p \cdot q \in \perp)\}$.

A *fact* of $\underline{P}$ is a subset G of P such that $G^{\perp\perp} = G$.

The following sets are facts: $\perp$, $\mathbf{1}{:=} \perp^{\perp}$, $\top := \emptyset^{\perp}$, $\mathbf{0}{:=} \top^{\perp}$.

We define:

$$\begin{aligned} G \cdot H &:= \{p \cdot q \mid p \in G \text{ and } q \in H\} \\ G + H &:= (G^{\perp} \cdot H^{\perp})^{\perp} \\ G \star H &:= (G \cdot H)^{\perp\perp} \\ G \sqcap H &:= G \cap H \\ G \sqcup H &:= (G \cup H)^{\perp\perp} \end{aligned}$$

A *phase structure* $\underline{S}$ consists in a phase space $\underline{P}$ and, for each propositional variable B, a fact $S(B)$ of $\underline{P}$. With each formula A we associate its interpretation $S(A)$ in a completely straightforward way.
$\underline{S} = \langle \underline{P}, S\rangle$

A is *valid* in $\underline{S}$ when $1 \in S(A)$.

A is a *linear tautology* when A is valid in any phase structure $\underline{S}$.

Now, we restrict the phase semantics defining $\perp$ as $\{1\}$. We only investigate phase spaces of the form $\langle P, \cdot, 1, \{1\}\rangle$. Hence, $\perp = \mathbf{1} = \{1\}$, $\top = P$ and $\mathbf{0} = \emptyset$. Furthermore, we restrict the range of S as the set $\{\top, \perp, \mathbf{0}\}$. Thus, $\mathbf{1}$ corresponds to the truth-value "neutral", $\top$ to "true" and $\mathbf{0}$ to "false", where true and neutral have to be distinguished, because 1 is an element of $\top$ and $\perp$. The connective

$+$ corresponds to our "or", $\star$ to our "and" and $\sqcap$ corresponds to Kleene's "and", $\sqcup$ to Kleene's "or".
Hence:

Corollary 5.1 *If A is a linear tautology and does not contain additive connectives, then A is a robust tautology.*

Remark: From the same restriction of the phase spaces we also obtain a three-valued interpretation of the modalities ! and ? (of linear logic). The truth-value of $!A$ is defined as neutral if the value of A is not false, else it is defined as false. The truth-value of $?A$ exactly corresponds to the value of $\neg!\neg A$.
Note the value of a formula $?A$ is never false. Hence, $?A$ is a tautology for all formulas A. (The colloquial interpretation of $?A$ is "A is possible". This means that every proposition is possible.)
The following formulas are robust tautologies: $!A \rightarrow A$ and $!A \rightarrow !!A$ whereas the formula $?!A \rightarrow A$ is no tautology.
If A is a tautology, then $!A$ is also a tautology.
A detailed analysis of this extended 3-valued semantics can be found in [Ho93].

6 Cut Rule

Note the cut rule is used in the completeness proof. Indeed, the calculus LC_R without the cut rule is not complete w.r.t. the robust semantics. This means that the cut can not be eliminated. But it is possible to replace the cut by the following three rules.

$$\frac{\gamma \supset \neg A \qquad \gamma, A \vee B, B \supset \delta}{\gamma, A \vee B \supset \delta} \; (D\vee)$$

$$\frac{\gamma \supset \forall x \neg A(x) \qquad \gamma, \forall x A(x), \exists x A(x) \supset \delta}{\gamma, \exists x A(x) \supset \delta} \; (D\exists)$$

$$\frac{\gamma \supset \exists x \neg A(x) \qquad \gamma, \forall x \neg A(x), \forall x A(x) \supset \delta}{\gamma, \forall x A(x) \supset \delta} \; (D\forall)$$

Note that these rules satisfy a "modified" subformula-property and that there only exists a finite number of premises to a given conclusion.

Lemma 6.1 *The cut can not be eliminated in LC_R.*

Proof: At first, we show that $A, \neg A, \neg B, A \vee (B \wedge C) \supset A, \neg A, B$ is derivable in LC_R. As we can see in lemma 2.3, $A, \neg A \supset A, \neg A$ is derivable. We can derive the sequent $A, \neg A, A \vee (B \wedge C) \supset A, \neg A$ using the rule (lW).
Furthermore, every sequent of the form $\supset A, \neg A$ can be derived using the rule (lD). Therefore, by lemma 2.2, the sequent $A, B \supset A, \neg A, B$ is also derivable. Using the rules (lW) and $(l\wedge)$ we obtain the sequent $A, B \wedge C \supset A, \neg A, B$. The following derivation yields the claimed proposition:

$$\dfrac{\dfrac{\dfrac{\dfrac{\supset B \wedge C, \neg B \vee \neg C \qquad A, \neg A, A \vee (B \wedge C) \supset A, \neg A}{A, \neg A, A \vee (B \wedge C) \supset A, \neg A, B \wedge C, \neg A \wedge (\neg B \vee \neg C)} \; (r\wedge)}{A, \neg A, A \vee (B \wedge C) \supset A, \neg A, B \wedge C} \; (rD)}{A, \neg A, A \vee (B \wedge C) \supset \neg A, B \wedge C} \; (rD)}{A, \neg A, A \vee (B \wedge C) \supset B \wedge C} \; (rD)$$

$$\vdots$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A, \neg A, A \vee (B \wedge C) \supset B \wedge C\end{matrix} \qquad A, B \wedge C \supset A, \neg A, B}{A, \neg A, A \vee (B \wedge C) \supset A, \neg A, B}\ (cut)}{A, \neg A, \neg B, A \vee (B \wedge C) \supset A, \neg A, B}\ (lW)$$

Assume the sequent is derivable in LC_R without the cut rule. By induction on the length of the derivation we can show that this assumption leads to a contradiction.
q.e.d.

Theorem 6.2 *Let LC'_R the calculus LC_R, where the cut is replaced by the rules $(D\vee)$, $(D\exists)$ and $(D\forall)$. Then LC'_R is equivalent to LC_R.*

Proof: We show that LC'_R is sound and complete w.r.t. the robust semantics.

It is easy to prove the new rules are sound.
In order to show that LC'_R is complete we modify the completeness proof of LC_R. Note that the lemmata 2.2, 4.2 and 4.5 were proved by induction on the length of the derivation. To proof these lemmata in LC'_R is simple. The cut rule was used in the proofs of the lemmata 4.3, 4.6 and 4.7. We will see that we need the rule $(D\vee)$ for lemma 4.3, the rule $(D\exists)$ for lemma 4.6 and $(D\forall)$ for lemma 4.7.

Lemma 4.3: If the sequent $\gamma, A \vee B, \neg A, B \supset \delta$ is derivable then $\gamma, A \vee B, \neg A \supset \delta$ is derivable, too.

Proof: The proposition immediately follows from the derivability of $\gamma, A \vee B, \neg A \supset \neg A$, the hypothesis and the rule $(D\vee)$.

Lemma 4.6: The sequent $\gamma, \forall x A(x) \supset \delta$ is derivable
if $\gamma, \forall x A(x), \forall x \neg A(x), A(u) \supset \delta$ and $\gamma, \forall x A(x), A(u) \supset \neg A(u)$ are derivable, where u is a free variable which does not occur in the sequent $\gamma, \forall x A(x) \supset \delta$.

Proof: Construct a proof as follows.

$$\dfrac{\dfrac{\gamma, \forall x A(x), \forall x \neg A(x), A(u) \supset \delta}{\gamma, \forall x A(x), \forall x \neg A(x) \supset \delta}\ (l\forall) \qquad \dfrac{\dfrac{\gamma, \forall x A(x), A(u) \supset \neg A(u)}{\gamma, \forall x A(x) \supset \neg A(u)}\ (lD)}{\gamma, \forall x A(x) \supset \exists x \neg A(x)}\ (r\exists)}{\gamma, \forall x A(x) \supset \delta}\ (D\forall)$$

Lemma 4.7: The sequent $\gamma, \exists x A(x) \supset \delta$ is derivable if $\gamma, \exists x A(x), \forall x A(x), A(u) \supset \delta$ and $\gamma, \exists x A(x), A(u) \supset \neg A(u)$ are derivable, where u is a free variable which does not occur in the sequent $\gamma, \exists x A(x) \supset \delta$.

Proof: Construct a proof as follows.

$$\dfrac{\dfrac{\gamma, \exists x A(x), \forall x A(x), A(u) \supset \delta}{\gamma, \exists x A(x), \forall x A(x) \supset \delta}(l\forall) \qquad \dfrac{\dfrac{\dfrac{\gamma, \exists x A(x), A(u) \supset \neg A(u)}{\gamma, \exists x A(x) \supset \neg A(u)}(lD)}{\gamma, \exists x A(x) \supset \forall x \neg A(x)}(r\forall)^{\circ}}{}}{\gamma, \exists x A(x) \supset \delta}(D\exists)$$

$^{\circ}$: It is easy to see that the sequent $\gamma \supset \delta, \forall x A(x)$ is derivable if $\gamma \supset \delta, A(u)$ (where the free variable u does not occur in $\gamma \supset \delta$) is derivable, too.

By virtue of these lemmata the completeness proof of LC'_R exactly corresponds to the proof of LC_R.
q.e.d.

7 References

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