User-centric Data Traffic Engineering with Karma

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USER-CENTRIC DATA TRAFFIC ENGINEERING WITH KARMA

Semester Project Report

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Abstract

Imagine this: you are hosting a movie night with friends, but the video keeps buffering due to limited bandwidth. You may consider purchasing a high-speed bundle, but prioritizing Internet access based on monetary payment can undermine Net Neutrality. To date, there are no known fair means to elicit truthful demand information, leading to a polarization of existing bandwidth allocation schemes that either assume truthfulness or are fully agnostic to demand. In this project, we design a non-monetary karma economy for bandwidth allocation to fill this gap. A significant challenge is to choose a social welfare function that is suitable for the infinitely divisible nature of bandwidth, for which we propose and axiomatically justify weighted proportional fairness. With this function, we prove the well-posedness of our karma-based model and demonstrate our allocation system's efficiency and fairness in various numerical experiments.

Contents

1 Introduction

Data traffic engineering refers to the problem of finding a bandwidth allocation in a data network that optimally satisfies user demands without exceeding data link capacities. Generally, data traffic engineering algorithms could be classified as centralized or decentralized. Centralized algorithms typically involve solving a centralized optimization problem and are most commonly used in isolated data networks including wide-area backbone networks (WANs) of big companies such as Microsoft, Google, etc [1, 2]. On the other hand, decentralized algorithms, which most famously include the Transmission Control Protocol (TCP) and User Datagram Protocol (UDP), are used in large-scale networks such as the Internet since they operate in a real-time, distributed fashion by imposing bandwidth control protocols on the end-devices. Notably, both the centralized and decentralized paradigms are designed for a cooperative environment: in the former, it is assumed that end-users report their demands to the central optimization truthfully, while in the latter it is assumed that end-users truthfully follow the bandwidth control protocol.

However, recent years have seen a rapid increase in the data intensity of common Internet tasks including streaming high-definition media and playing graphics-intensive games online [3]. Such simultaneous high bandwidth demands often cannot be satisfied by the scarce network resources, leading to a competitive environment in which all end-users try to gain as much bandwidth as possible. To demonstrate this point, we conducted a simple experiment for the centralized case where multiple clients share a single bottleneck link. As shown in Figure 1, a client can easily 'cheat' by reporting fake demand to get more bandwidth. This sort of selfish behavior is naturally expected to occur in any medium to large-scale network and leads to severe performance deterioration when everyone cheats. Similarly, it is well known that decentralized schemes such as TCP can also be exploited for selfish gain at the cost of social welfare [4].

Therefore, it is more appropriate to analyze and derive robust bandwidth allocation algorithms for this scarcity situation based on game theory, which explicitly accounts for the behavior of rational players in competitive environments and how that behavior influences social welfare. As a starting point for the analysis, we will focus on centralized traffic engineering algorithms in this study.

From the perspective of game theory, one way to approach the competitive nature of the bandwidth allocation problem is to exchange priority with money i.e. purchasing a high-speed bundle to acquire higher bandwidth. However, monetary approaches could lead to unfairness where wealthy users persist to gain a systematic advantage [5]. This manifested in the recent debate over *net neutrality* where the public expressed severe concerns that the Internet will lose its integrity as an open and equal resource for everyone if Internet service providers (ISPs) discriminate Internet data and users based on their willingness to pay[6, 7, 8, 9]. In contrast, recent works have proposed the concept of karma economies, which use non-monetary credits called karma to allocate shared resources[10, 11]. This approach was demonstrated to achieve high efficiency without requiring private information about users' true demand and to solve the unfairness issue since karma is not exchangeable for money. In this study, we focus on devising a karma economy for the data traffic engineering problem, which requires modeling such a problem precisely and ensuring it has a tractable solution.

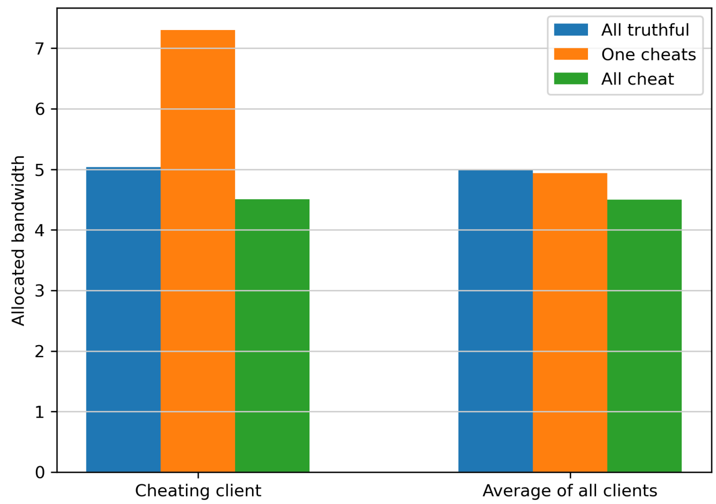


Figure 1: Effect of competitive behavior in centralized traffic engineering. The detailed analysis is included in Appendix A.

1.1 Related work

As mentioned above, there are two categories of data traffic engineering algorithms and we will focus on the case of centralized algorithms in this study. Two well-studied centralized algorithms in the literature are *forward fault correction* (FFC) [1] and *Traffic Engineering Applying Value at Risk* (TEAVAR) [2]. Notably, the main focus of these and other works is to be robust against multiple concurrent link failures, rather than being robust against selfish behavior. In particular, FFC focuses on handling multiple concurrent failures, while TEAVAR focuses on handling failure cases with high probability. Both of these algorithms take the truthful reporting of demand for granted, which as discussed above leads to severe performance degradation in scarce, competitive environments. We consider this study to be complementary to these approaches, focusing on strategic behavior while neglecting the important problem of link failure handling for simplicity.

On the other hand, the core concept of a karma economy is: *If I give in now, I will be rewarded in the future.* While, to the extent of our knowledge, this concept has not been applied yet to the traffic engineering problem, it has been previously explored in the context of peer-to-peer file sharing.

It is well known that peer-to-peer file sharing relies on spontaneous seeding. Similar to the aforementioned cheating behavior in traffic engineering, if most clients attempt to free-ride by downloading without seeding, the resource will be lost on the Internet [12]. Therefore, some websites introduce *private trackers* (PT) to track the download and upload data of individual clients. Once the ratio of download to upload exceeds a certain threshold, the client is identified as a free-rider and disallowed further downloads [13].

Notably, this and similar approaches are largely heuristic and lack rigorous game-theoretic justification. For example, in the case of PT it is difficult to rigorously identify the free riding threshold in a centralized manner. In contrast, this study builds on the rigorous game-theoretic formulation of karma economies proposed in [10], which is reviewed formally in Section 2.1.

1.2 Main contributions

In this study, we initiate the analysis of centralized data traffic engineering in competitive environments by considering a simple single-link network topology contested by a large number of clients. Our analysis first focuses on the important problem of choosing the social welfare function of centralized optimization, for which we propose the weighted proportional fairness function as it satisfies several natural properties. We then adapt the original karma dynamic population game formulation by choosing the immediate reward function carefully. Under the criterion of weighted proportional fairness, we demonstrate that our bandwidth allocation algorithm achieves both efficiency and fairness (see Section 2.3) without requiring access to the private information of clients' true demand. We instead incentivize clients to reveal this private information by the actions they made.

The main contributions could be summarized as:

- We formalize the data traffic engineering problem in a competitive environment as a dynamic population game [14], adapt the karma economy to this game, and prove the existence of a Stationary Nash Equilibrium so that the dynamic game could be studied from a static perspective, see Section 3.3.
- We propose weighted proportional fairness function as the centralized social welfare function, prove that it satisfies a set of natural axioms, and contrast it to other common choices that do not satisfy the axioms, see Section 3.1.1.
- We choose a specific immediate reward function that is consistent with the social objective of weighted proportional fairness and satisfies other desirable properties, see Section 3.2.2.
- Through numerical experiments, we demonstrate that the karma economy for data traffic engineering achieves both efficiency and fairness in some simple networks.

1.3 Notation

Let $a, d \in D \subseteq \mathbb{N}$ and $c \in C \subseteq \mathbb{R}^n$, then for a vector-valued function $f : C \mapsto \mathbb{R}^{|D|}$, we use $f[d](c)$ to denote the d^{th} element of vector $f(c)$. Similarly, $g[a \mid d](c)$ denotes the conditional probability of a given d and c.

2 Preliminaries

2.1 General karma economy

Here, we revisit the *karma economy* introduced in [10], which is modeled as a *dynamic population game* [14].

We consider a large number of clients N who repeatedly compete for access to a scarce shared resource and approximate them by a continuum of mass. At each time step, a client has some karma $k \in \mathbb{N}$ and can submit an integer bid $b \in \mathcal{B}^k := \{b \in \mathbb{N} | b \leq k\}$ which should not exceed the karma. Then the resource is allocated through an auction-like mechanism and karma is transferred among the clients. Apart from karma, each client has an urgency state $u \in \mathcal{U} = \{u_0, u_1, \ldots, u_M\} \subset \mathbb{R}_{>0}$ that decribes the urgency to acquire the resource.

Jointly the urgency and karma form the time-varying state of a client, denoted by $x = [u, k] \in \mathcal{X} = \mathcal{U} \times \mathbb{N}$. In addition to client states, we consider a finite number of client types $\tau \in \Gamma \subset \mathbb{N}$, and the urgency of a client of type τ evolves according to an exogenous Markov chain, denoted by $\phi_\tau [u^+|u]$. The distribution of client types is denoted by $g \in \Delta(\Gamma)$, where $g_\tau \in [0, 1]$ represents the mass of client with type $\tau \in \Gamma$. Accordingly, the joint *type-state distribution* can be written as

$$d \in \mathcal{D} = \left\{ d \in \mathbb{R}_+^{|\Gamma| \times |\mathcal{X}|} \middle| \sum_{u,k} d_\tau [u, k] = g_\tau, \forall \tau \in \Gamma \right\}.$$

At each time step, the action of a client is the bid b, which is limited by its karma k. Clients of the same type τ follow the homogeneous randomized policy

$$\pi_\tau : \mathcal{X} \to \Delta(\mathcal{B}^k) := \left\{ \sigma \in \mathbb{R}_+^{k+1} \middle| \sum_b \sigma[b] = 1 \right\},$$

where $\pi_\tau[b|u, k]$ denotes the probabilistic weight that these clients place on bid b when in state $[u, k]$. The concatenation of the policies of all types $\pi = (\pi_\tau)_{\tau \in \Gamma}$ is simply referred to as the *policy*. The tuple of type-state distribution and policy (d, π) is defined as the *social state*.

Let $\kappa[k^+|k, b](d, \pi)$ be the karma transition function that describes how a client's karma changes between two consecutive time steps given its current karma k, bid b, and the social state (d, π). Then, together with the urgency transition function $\phi_\tau [u^+|u]$, the joint state transition function is given by

$$\rho_\tau[u^+, k^+|u, k, b](d, \pi) = \phi_\tau \left[u^+|u\right] \kappa[k^+|k, b](d, \pi).$$

Moreover, we define $\zeta [u, b] (d, \pi)$ as the *immediate reward function* of each client in urgency u and taking bid b. Both the karma transition function and immediate reward function will be discussed thoroughly in Section 3.2.

Given the social state, each client faces a Markov decision process. Specifically, the expected reward of the clients of type τ is given by

$$R_\tau [u, b] (d, \pi) = \sum_b \pi_\tau[b|u, k]\zeta [u, b] (d, \pi)$$

and the state transition follows

$$\mathbb{P}_\tau[u^+, k^+|u, k](d, \pi) = \sum_b \pi_\tau[b|u, k]\rho_\tau[u^+, k^+|u, k, b](d, \pi).$$

Accordingly, the value function in the infinite horizon is derived as

$$V_\tau\left[u, b\right](d, \pi) = R_\tau\left[u, b\right](d, \pi) + \delta \sum_{u^+, k^+} \mathbb{P}_\tau[u^+, k^+|u, k](d, \pi) V_\tau\left[u^+, k^+\right](d, \pi),$$

where $\delta \in (0, 1]$ is the discount factor.

To describe the rational decision of each client, we also need to define the Q-function as

$$Q_\tau\left[u, k, b\right](d, \pi) = \zeta\left[u, b\right](d, \pi) + \delta \sum_{u^+, k^+} \rho_\tau[u^+, k^+|u, k, b](d, \pi) V_\tau\left[u^+, k^+\right](d, \pi).$$

Then, to maximize the long-term return, each client chooses a policy based on the *best response* correspondence, given by

$$B_\tau\left[u, k\right](d, \pi) = \left\{ \sigma \in \Delta(\mathcal{B}^k) \middle| \forall \sigma' \in \Delta(\mathcal{B}^k), \sum_b (\sigma[b] - \sigma'[b]) Q_\tau\left[u, k, b\right](d, \pi) \geq 0 \right\}.$$

Before defining the equilibrium of this game, we need two more conditions:

Condition 2.1. Continuity
The immediate reward function $\zeta\left[u, b\right](d, \pi)$ and the karma transition function $\kappa[k^+|k, b](d, \pi)$ are continuous in the social state (d, π).

Condition 2.2. Preservation of Karma
Karma is preserved in expectation for all social state (d, π), i.e., $\mathbb{E}[k^+] = \mathbb{E}[k]$, which expands to

$$\sum_{\tau, u, k} d_\tau\left[u, k\right] \sum_b \pi_\tau[b|u, k] \sum_{k^+} \kappa[k^+|k, b](d, \pi) k^+ = \sum_{\tau, u, k} d_\tau\left[u, k\right] k.$$

The readers could refer to [10] for a detailed discussion on the conditions and functions mentioned above. Then, we can finally define the equilibrium state in a karma economy.

Definition 2.3. Stationary Nash Equilibrium
A Stationary Nash Equilibrium is a social state $(d^*, \pi^*) \in \mathcal{D} \times \Pi$ such that, $\forall(\tau, u, k) \in \Gamma \times \mathcal{U} \times \mathbb{N}$,

$$d^*_\tau\left[u, k\right] = \sum_{u^-, k^-} d^*_\tau\left[u^-, k^-\right] \mathbb{P}_\tau[u, k|u^-, k^-](d^*, \pi^*)$$

$$\pi^*_\tau[\cdot|u, k] \in B_\tau\left[u, k\right](d^*, \pi^*)$$

We conclude this section by stating that the existence of Stationary Nash Equilibrium is guaranteed in the above setting. The detailed proof of existence could be found in [10].

Theorem 2.4. Existence of Stationary Nash Equilibrium
If Conditions 2.1 and 2.2 hold, then given $\bar{k} \in \mathbb{N}$, there always exists a Stationary Nash Equilibrium (d^*, π^*) such that $\sum_{\tau,u,k} d^*_\tau[u,k]\, k = \bar{k}$, and $\bar{k}$ is the average amount of karma per client.

2.2 Data traffic engineering with competition

Following the traffic engineering literature [1, 2], we represent the network topology with a directed graph $G = (V, A)$, where vertex set V represents switches and arc set A represents links between switches. The flow set $F \subseteq V \times V$ contains all source-target switch pairs with the demand of transmitting data. Each flow $f \in F$ is associated with a tunnel set $T_f \subseteq \text{Pow}(2^A)$ which contains tunnels $t \subseteq 2^A$ connecting the source and the target of flow f. The tunnel set T_f represents all tunnels/paths that a flow $f \in F$ is allowed to access. Let $C[a]$ denotes the capacity of link $a \in A$ in Gbits/s.

Since we are interested in medium to large-scale networks, and to be compatible with the karma dynamic population game model, we assume that the number of clients N is large enough that they can be approximated by a continuum of mass. We further assume that all clients can be classified in a finite number of types $\tau \in \Gamma$, with $N_\tau = g_\tau N$. Each type $\tau \in \Gamma$ associates the client to a flow f_τ and a demand/urgency pattern $\phi_\tau[u^+ \mid u]$, where u denotes the true demand of the client in Gbit/s and belongs to a finite set $\mathcal{U}$. In a competitive environment, clients report a potentially non-truthful demand b, and we let $N_\tau[b]$ denote the number of clients in type τ reporting demand b.

Given the network topology and the reported demands, the centralized traffic engineering optimization decides on the bandwidth $s_{f_\tau}[b,t]$ (in Gbit/s) on tunnel $t \in T_{f_\tau}$ to allocate to a client of type $\tau \in \Gamma$ reporting b. Let $s_{f_\tau}[b] = \sum_{t \in T_{f_\tau}} s_{f_\tau}[b,t]$ be the bandwidth allocated to a client of type $\tau \in \Gamma$ reporting b. We should notice that we use $s_{f_\tau}[b,t]$ instead of $s_\tau[b,t]$ since the system can only access a client's source and target represented by f_τ instead of both f_τ and its urgency pattern ϕ_τ. Let $S_\tau[t] = \sum_b N_\tau[b] s_{f_\tau}[b,t]$ be the total supply to type $\tau \in \Gamma$ on tunnel $t \in T_{f_\tau}$, $S_\tau = \sum_{t \in T_{f_\tau}} S_\tau[t]$ be the total supply to type $\tau \in \Gamma$, and $S[a] = \sum_{\tau \in \Gamma} \sum_{t \in T_{f_\tau} :\, a \in t} S_\tau[t]$ be the total supply on link a. Then we must have $S[a] \le C[a]$, $\forall a \in A$ to not exceed the capacity of any link. In addition, let $S_f = \sum_{\tau \in \Gamma :\, f_\tau = f} S_\tau$ be the total supply to all types τ with the same flow $f_\tau \in F$.

On a normalized scale, let instead $c[a] = \frac{C[a]}{N}$ be the normalized capacity of link $a \in A$, $\xi_\tau[b]$ be the proportion of clients of type τ reporting demand b such that $N_\tau[b] = \xi_\tau[b] N_\tau$, $\bar{s}_\tau[t] = \sum_b \xi_\tau[b] s_{f_\tau}[b,t]$ be the average bandwidth allocated to type $\tau \in \Gamma$ on tunnel $t \in T_{f_\tau}$, $\bar{s}_\tau = \sum_{t \in T_{f_\tau}} \bar{s}_\tau[t]$ be the average bandwidth allocated to type $\tau \in \Gamma$, and $\bar{s}[a] = \sum_{\tau \in \Gamma} g_\tau \sum_{t \in T_{f_\tau} :\, a \in t} \bar{s}_\tau[t]$ be the normalized supply on link $a \in A$ which must satisfy $\bar{s}[a] \le c[a]$. In addition, let $\bar{s}_f[t] = \sum_{\tau \in \Gamma :\, f_\tau = f} g_\tau \bar{s}_\tau[t]$ be the normalized supply to all clients with flow $f \in F$ on tunnel $t \in T_f$, and $\bar{s}_f = \sum_{\tau \in \Gamma :\, f_\tau = f} \bar{s}_\tau$ be the normalized supply to all types τ with the same flow $f_\tau \in F$.

After defining some rules for clients to report demand, the centralized traffic engineering optimization could optimize the allocation scheme $s_f[b,t]$ towards a social welfare function and output an optimal solution $s^*_f[b,t]$. In this study, such rules are defined by the karma economy and will be discussed in Section 3.

2.3 Common social welfare functions

In this section, we will review several common social welfare functions $\mathcal{F}$ appearing in the literature associated with traffic engineering. In the literature, it is common to associate individual clients to the flows f, and therefore we will present the social welfare measures defined with respect to the flow allocations S_f rather than the allocations s_{f_τ} to the individual clients. Correspondingly, we will use the total true demand U_f of a flow f instead of the true demand u of individual clients. Concretely, $U_f = N \sum_{\tau\in\Gamma:\, f_\tau = f} \sum_{u,k} d_\tau[u,k]u$. In order to simplify the expression of S_f, let $\mathcal{S} = \left\{ \mathbf{S} \,\middle|\, \mathbf{S} = \left(S_{f_1}, S_{f_2}, \ldots, S_{f_{|F|}}\right)^T \text{ and } \mathbf{S} \text{ feasible} \right\}$ containing all feasible total supply to flows F. Since all constraints for $\mathbf{S}$ feasible are linear, $\mathcal{S}$ is a polyhedral and thus convex. Similarly to $\mathbf{S}$, let $\mathbf{U} = \left(U_{f_1}, U_{f_2}, \ldots, U_{f_{|F|}}\right)^T$ be the compact representation of $U_f, f \in F$. Then the social welfare function is a function of $\mathbf{S}$ and $\mathbf{U}$: $\mathcal{F} = \mathcal{F}(\mathbf{S}, \mathbf{U})$.

We will first introduce social welfare functions without truncation of overflowed bandwidth, which means that we assume there is no waste of bandwidth: $S_f \le U_f$, $\forall f \in F$. We will derive the version with truncation at the end of this section.

For FFC [1], they use the *sum of supply* as the social welfare function, which could be written as

$$\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f\in F} S_f$$

For TEAVAR [2], they use the *minimum availability* as the social welfare function, which could be written as

$$\mathcal{F}(\mathbf{S}, \mathbf{U}) = \min_{f\in F} \left\{ \frac{S_f}{U_f} \right\}$$

Proportional fairness [15] is a kind of fairness such that any possible aggregate percentage increase of supply for a group of clients results in an equal or greater aggregate percentage decrease of supply. Formally, the definition is:

Definition 2.5. Proportional fairness
A specific allocation $\mathbf{S}^*$ is proportionally fair if:

$$\forall \mathbf{S} \in \mathcal{S},\ \sum_{f\in F} \frac{S_f - S^*_f}{S^*_f} \le 0$$

Such fairness could be achieved by maximizing the following *proportional fairness function*:

$$\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f\in F} \ln\left(S_f\right)$$

More concretely, we have the following theorem

Theorem 2.6. If the feasible allocation set $\mathcal{S}$ is convex, then there exists a unique allocation $\mathbf{S}^*$ that maximizes the following social welfare function

$$\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f \in F} \ln(S_f)$$

and $\mathbf{S}^*$ is also proportionally fair.

The proof of theorem 2.6 could be found in Appendix B.

However, such a kind of fairness doesn't consider the different demands of each flow. Therefore, we could derive a modified version of proportional fairness, which takes the different demands of each client as weight and could be named per-unit-demand proportional fairness or just weighted proportional fairness.

Definition 2.7. Weighted proportional fairness (Per-unit-demand proportional fairness)
A specific allocation $\mathbf{S}^*$ is weighted proportionally fair if:

$$\forall \mathbf{S} \in \mathcal{S}, \sum_{f \in F} U_f \frac{S_f - S_f^*}{S_f^*} \leq 0$$

The modified version with weight could be regarded as splitting both the supply and the demand of a flow f into U_f sub-flows evenly if the demand U_f is integral. Each sub-flows created from f has a demand of 1 and a supply of S_f/U_f. The weighted proportional fairness is indeed the proportional fairness after substituting all flows $f \in F$ with sub-flows with unit demand.

Similarly, we could achieve weighted proportional fairness by maximizing the following *weighted proportional fairness function*:

$$\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f \in F} U_f \ln(S_f)$$

More concretely, we have the following theorem:

Theorem 2.8. If the feasible allocation set $\mathcal{S}$ is convex, then there exists a unique allocation $\mathbf{S}^*$ that maximizes the following social welfare function

$$\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f \in F} U_f \ln(S_f)$$

and $\mathbf{S}^*$ is also weighted proportionally fair.

The proof of theorem 2.8 is given in Appendix C.

Besides these social welfare functions related to proportional fairness, we also propose the *fairness with quadratic loss*:

$$\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f \in F} -\left(S_f - U_f\right)^2$$

and the *sum of availability*:

$$\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f \in F} \frac{S_f}{U_f}$$

We should notice that the assumption: $S_f \leq U_f,\ \forall f \in F$ made at the beginning of this section is not guaranteed in all feasible allocations $\mathcal{S}$. Therefore, we need to truncate the overflowed bandwidth by substituting S_f with $\min\{S_f, U_f\}$ as shown in Table 1.

Table 1: Different social welfare functions with and without truncation.

Social welfare function	**Without truncation**	**With truncation**
Sum of supply	$\sum_{f \in F} S_f$	$\sum_{f \in F} \min\{S_f, U_f\}$
Minimum availability	$\min_{f \in F} \left\{\frac{S_f}{U_f}\right\}$	$\min_{f \in F} \left\{\min\left\{\frac{S_f}{U_f}, 1\right\}\right\}$
Proportional fairness	$\sum_{f \in F} \ln(S_f)$	$\sum_{f \in F} \ln(\min\{S_f, U_f\})$
Weighted proportional fairness	$\sum_{f \in F} U_f \ln(S_f)$	$\sum_{f \in F} U_f \ln(\min\{S_f, U_f\})$
Fairness with quadratic loss	$\sum_{f \in F} -\left(S_f - U_f\right)^2$	$\sum_{f \in F} -\left(\min\{S_f - U_f, 0\}\right)^2$
Sum of availability	$\sum_{f \in F} \frac{S_f}{U_f}$	$\sum_{f \in F} \min\left\{\frac{S_f}{U_f}, 1\right\}$

We will discuss the performance of these social welfare functions thoroughly and also show how to assess the individual allocations in Section 3.1.1.

3 Data traffic engineering with karma

To adapt the general karma economy to traffic engineering with competition, we will describe the behavior of the karma mechanism from two perspectives: system and client. From the perspective of the system, we will mainly discuss how bandwidth is allocated to different clients according to their bids. From the perspective of the client, we will discuss how clients act based on the rules of the bandwidth allocation system.

3.1 Karma-based bandwidth allocation system

3.1.1 Choice of social welfare function

Like other data traffic engineering algorithms, we need to first determine our social welfare function as the objective to be optimized. Therefore, we compare the different social welfare functions introduced in Section 2.3. In line with Section 2.3, we will first discuss the social welfare function at the level of flows f, before deriving an expression of the chosen function at the client level at the end of the section.

To guide our choice of social welfare function, we introduce the following two basic axioms that the social welfare function *without truncation* should satisfy.

Axiom 3.1. Uniqueness

$$\forall \mathbf{S}^*, \mathbf{S}^{**} \in \mathcal{S} \text{ s.t. } \mathcal{F}(\mathbf{S}^*, \mathbf{U}) = \mathcal{F}(\mathbf{S}^{**}, \mathbf{U}) = \max_{\mathbf{S} \in \mathcal{S}} \mathcal{F}(\mathbf{S}, \mathbf{U}),\ S_f^* = S_f^{**}\ \forall f \in F.$$

Axiom 3.2. Client-splitting invariance

Let $(F, \mathbf{U})$ be a set of flows and the corresponding demand vector, and let $\mathbf{S}^*$ be an optimal allocation under $\mathcal{F}$ for $(F, \mathbf{U})$. Let $\tilde{f} \in F$ be an arbitrary flow. Let $(F', \mathbf{U}')$ be another set of flows and demands such that $F' = \left(F \setminus \left\{\tilde{f}\right\}\right) \cup \left\{\tilde{f}_1, \tilde{f}_2\right\}$, $\tilde{f}_1 = \tilde{f}_2 = \tilde{f}$, $U'_{\tilde{f}_1} + U'_{\tilde{f}_2} = U_{\tilde{f}}$, $U'_{f'} = U_{f'} \forall f' \in F \setminus \tilde{f}$. Then the following allocation $\mathbf{S}'^*$ is optimal under $\mathcal{F}$ for $(F', \mathbf{U}')$:

$$\forall f' \in F', S'^*_{f'} = \begin{cases} \frac{U'_{\tilde{f}_1}}{U_{\tilde{f}}} S^*_{\tilde{f}}, & f' = \tilde{f}_1; \\ \frac{U'_{\tilde{f}_2}}{U_{\tilde{f}}} S^*_{\tilde{f}}, & f' = \tilde{f}_2; \\ S^*_{f'}, & \text{else.} \end{cases}$$

The first axiom is *Uniqueness*, which means that the optimal allocation scheme $\mathbf{S}^* \in \mathcal{S}$ with respect to the social welfare selected is unique at the level of flows. With such an axiom, it is possible to represent the optimal allocation by a variable s^* instead of a set containing all possible optimal allocations. The second axiom is *Client-splitting invariance*, which means that the splitting of any flow f will not affect the optimal allocation of the other flows $f' \in F \setminus \{f\}$. If we combine both axioms, the aggregation of any flow f will also not affect the optimal allocation of the other flows.

Table 2 compares the social welfare functions introduced in Section 2.3 with respect to Axioms 3.1 and 3.2. We find that *weighted proportional fairness function* is the only one that satisfies both axioms according to Theorem 3.3, which has important consequences for the proposed karma scheme and will be discussed in detail in Sections 3.1.1 and 3.2.2. In addition, optimizing towards the *weighted proportional fairness function* could result in weighted proportional fairness defined in Section 2.3. Therefore, we select *weighted proportional fairness function* as the social welfare function in the upcoming analysis. The proof of Table 2 is in Appendix D.

Table 2: Comparison of social welfare functions with respect to Axioms 3.1 and 3.2.

Fairness function	**Uniqueness**	**Client-splitting invariance**
Sum of supply	×	✓
Minimum availability	×	✓
Proportional fairness	✓	×
Weighted proportional fairness	**✓**	**✓**
Fairness with quadratic loss	✓	×
Sum of availability	×	×

Theorem 3.3. *Weighted proportional fairness function* meets both Axioms 3.1 and 3.2.

Proof of Theorem 3.3:

First, let's prove that Axioms 3.1 holds for the *weighted proportional fairness function*.

Let $\mathbf{S}^*, \mathbf{S}^{**} \in \mathcal{S}$ be two optimal solutions of $\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f \in F} U_f \ln(S_f)$: $\mathcal{F}(\mathbf{S}^*, \mathbf{U}) = \mathcal{F}(\mathbf{S}^{**}, \mathbf{U}) = \max_{\mathbf{S} \in \mathcal{S}} \mathcal{F}(\mathbf{S}, \mathbf{U})$. Since the set $\mathcal{S}$ that contains all feasible allocations is convex, $\frac{\mathbf{S}^*+\mathbf{S}^{**}}{2} \in \mathcal{S}$. Then we have

$$
\begin{aligned}
\mathcal{F}(\frac{\mathbf{S}^* + \mathbf{S}^{**}}{2}, \mathbf{U}) &= \sum_{f \in F} U_f \ln\left(\frac{S_f^* + S_f^{**}}{2}\right) \\
&\geq \sum_{f \in F} U_f \frac{\ln\left(S_f^*\right) + \ln\left(S_f^{**}\right)}{2} \\
&= \frac{1}{2}\left(\sum_{f \in F} U_f \ln\left(S_f^*\right) + \sum_{f \in F} U_f \ln\left(S_f^{**}\right)\right) \\
&= \frac{1}{2}\left(\mathcal{F}(\mathbf{S}^*, \mathbf{U}) + \mathcal{F}(\mathbf{S}^{**}, \mathbf{U})\right) \\
&= \frac{1}{2}\left(\max_{\mathbf{S} \in \mathcal{S}} \mathcal{F}(\mathbf{S}, \mathbf{U}) + \max_{\mathbf{S} \in \mathcal{S}} \mathcal{F}(\mathbf{S}, \mathbf{U})\right) \\
&= \max_{\mathbf{S} \in \mathcal{S}} \mathcal{F}(\mathbf{S}, \mathbf{U}).
\end{aligned}
$$

On the other hand, we know that $\mathcal{F}(\frac{\mathbf{S}^*+\mathbf{S}^{**}}{2}, \mathbf{U}) \leq \max_{\mathbf{S} \in \mathcal{S}} \mathcal{F}(\mathbf{S}, \mathbf{U})$. Therefore, $\mathcal{F}(\frac{\mathbf{S}^*+\mathbf{S}^{**}}{2}, \mathbf{U}) = \max_{\mathbf{S} \in \mathcal{S}} \mathcal{F}(\mathbf{S}, \mathbf{U})$. The equality of $\sum_{f \in F} U_f \ln\left(\frac{S_f^*+S_f^{**}}{2}\right) \geq \sum_{f \in F} U_f \frac{\ln\left(S_f^*\right)+\ln\left(S_f^{**}\right)}{2}$ is achieved when

$S_f^* = S_f^{**}$, $\forall f \in F$. Therefore, $\mathbf{S}^* = \mathbf{S}^{**}$. The Axiom 3.1 holds for the *weighted proportional fairness function*.

Then, let's prove that Axioms 3.2 holds.

By contradiction, we assume that Axioms 3.2 does not hold for the *weighted proportional fairness function*, which is equivalent to assume $\exists \mathbf{S}'^{**} \in \mathcal{S}'$ such that $\mathbf{S}'^{**}$ is optimal under $\mathcal{F}$ for $(F', \mathbf{U}')$, so $\mathcal{F}(\mathbf{S}'^{**}, \mathbf{U}'; F') > \mathcal{F}(\mathbf{S}'^{*}, \mathbf{U}'; F')$. We can expand $\mathcal{F}(\mathbf{S}'^{**}, \mathbf{U}'; F')$ as

$$\mathcal{F}(\mathbf{S}'^{**}, \mathbf{U}'; F') = U'_{\tilde{f}_1} \ln\left(S'^{**}_{\tilde{f}_1}\right) + U'_{\tilde{f}_2} \ln\left(S'^{**}_{\tilde{f}_2}\right) + \sum_{f \in F \setminus \{\tilde{f}_1, \tilde{f}_2\}} U_f \ln\left(S_f^{**}\right).$$

We assert $\frac{S'^{**}_{\tilde{f}_1}}{S'^{**}_{\tilde{f}_2}} = \frac{U'_{\tilde{f}_1}}{U'_{\tilde{f}_2}}$. To prove this assertion, we can construct $S'^{***} \in \mathcal{S}'$:

$$\forall f' \in F', S'^{***}_{f'} = \begin{cases} \frac{U'_{\tilde{f}_1}}{U'_{\tilde{f}_1} + U'_{\tilde{f}_2}} \left(S'^{**}_{\tilde{f}_1} + S'^{**}_{\tilde{f}_2}\right), & f' = \tilde{f}_1; \\ \frac{U'_{\tilde{f}_2}}{U'_{\tilde{f}_1} + U'_{\tilde{f}_2}} \left(S'^{**}_{\tilde{f}_1} + S'^{**}_{\tilde{f}_2}\right), & f' = \tilde{f}_2; \\ S'^{**}_{f'}, & \text{else.} \end{cases}$$

which redistribute the bandwidth of flows $\tilde{f}_1$ and $\tilde{f}_2$ proportionally to their demand $U'_{\tilde{f}_1}$ and $U'_{\tilde{f}_2}$. We have that

$$\begin{aligned}
&\mathcal{F}(\mathbf{S}'^{**}, \mathbf{U}'; F') - \mathcal{F}(\mathbf{S}'^{***}, \mathbf{U}'; F') \\
&= U'_{\tilde{f}_1} \ln\left(S'^{**}_{\tilde{f}_1}\right) + U'_{\tilde{f}_2} \ln\left(S'^{**}_{\tilde{f}_2}\right) - \left(U'_{\tilde{f}_1} \ln\left(S'^{***}_{\tilde{f}_1}\right) + U'_{\tilde{f}_2} \ln\left(S'^{***}_{\tilde{f}_2}\right)\right) \\
&= U'_{\tilde{f}_1} \ln\left(\frac{S'^{**}_{\tilde{f}_1}}{S'^{***}_{\tilde{f}_1}}\right) + U'_{\tilde{f}_2} \ln\left(\frac{S'^{**}_{\tilde{f}_2}}{S'^{***}_{\tilde{f}_2}}\right) \\
&= \left(U'_{\tilde{f}_1} + U'_{\tilde{f}_2}\right) \left[\frac{U'_{\tilde{f}_1}}{U'_{\tilde{f}_1} + U'_{\tilde{f}_2}} \ln\left(\frac{S'^{**}_{\tilde{f}_1}}{S'^{***}_{\tilde{f}_1}}\right) + \frac{U'_{\tilde{f}_2}}{U'_{\tilde{f}_1} + U'_{\tilde{f}_2}} \ln\left(\frac{S'^{**}_{\tilde{f}_2}}{S'^{***}_{\tilde{f}_2}}\right)\right] \\
&\leq \left(U'_{\tilde{f}_1} + U'_{\tilde{f}_2}\right) \ln\left(\frac{U'_{\tilde{f}_1}}{U'_{\tilde{f}_1} + U'_{\tilde{f}_2}} \frac{S'^{**}_{\tilde{f}_1}}{S'^{***}_{\tilde{f}_1}} + \frac{U'_{\tilde{f}_2}}{U'_{\tilde{f}_1} + U'_{\tilde{f}_2}} \frac{S'^{**}_{\tilde{f}_2}}{S'^{***}_{\tilde{f}_2}}\right) \quad \text{(Jensen's inequality)} \\
&= \left(U'_{\tilde{f}_1} + U'_{\tilde{f}_2}\right) \ln\left(\frac{S'^{**}_{\tilde{f}_1}}{S'^{**}_{\tilde{f}_1} + S'^{**}_{\tilde{f}_2}} + \frac{S'^{**}_{\tilde{f}_2}}{S'^{**}_{\tilde{f}_1} + S'^{**}_{\tilde{f}_2}}\right) \quad \text{(Substitute } S'^{***} \text{ with } S'^{**}\text{)} \\
&= \left(U'_{\tilde{f}_1} + U'_{\tilde{f}_2}\right) \ln(1) \\
&= 0
\end{aligned}$$

Since $\mathbf{S}'^{**}$ is an optimal solution, we have that $\mathcal{F}(\mathbf{S}'^{**}, \mathbf{U}'; F') \geq \mathcal{F}(\mathbf{S}'^{***}, \mathbf{U}'; F')$. Therefore, $\mathcal{F}(\mathbf{S}'^{**}, \mathbf{U}'; F') = \mathcal{F}(\mathbf{S}'^{***}, \mathbf{U}'; F')$. Since the equality in the Jensen's inequality above holds if and only if $\frac{S'^{**}_{\tilde{f}_1}}{S'^{***}_{\tilde{f}_1}} = \frac{S'^{**}_{\tilde{f}_2}}{S'^{***}_{\tilde{f}_2}}$, which is equivalent to $\mathbf{S}'^{**} = \mathbf{S}'^{***}$, $\frac{S'^{**}_{\tilde{f}_1}}{S'^{**}_{\tilde{f}_2}} = \frac{U'_{\tilde{f}_1}}{U'_{\tilde{f}_2}}$, the assertion is proved.

Then, we can construct an allocation $\mathbf{S}^{**}$ for $(F, \mathbf{U})$:

$$\forall f \in F, S_f^{**} = \begin{cases} S'^{**}_{\tilde{f}_1} + S'^{**}_{\tilde{f}_2}, & f = \tilde{f}; \\ S'^{**}_f, & \text{else.} \end{cases}$$

where we merge the bandwidth of the flows $\tilde{f}_1, \tilde{f}_1 \in F'$ into the bandwidth of the flow $\tilde{f} \in F$ and keep the bandwidth of other flows unchanged. From the definition of $\mathbf{S}^{**}$ we should notice that $S'^{**}_{\tilde{f}_1} = \frac{U'_{\tilde{f}_1}}{U'_{\tilde{f}}} S^{**}_{\tilde{f}}$ and $S'^{**}_{\tilde{f}_2} = \frac{U'_{\tilde{f}_2}}{U'_{\tilde{f}}} S^{**}_{\tilde{f}}$. Since $\mathbf{S}'^{**} \in \mathcal{S}'$, we know that $\mathbf{S}^{**} \in \mathcal{S}$.

Since $\mathcal{F}(\mathbf{S}'^{*}, \mathbf{U}'; F') < \mathcal{F}(\mathbf{S}'^{**}, \mathbf{U}'; F')$, we could expand it as

$$
\begin{aligned}
&\sum_{f' \in F'} U'_{f'} \ln\left(S'^{*}_{f'}\right) < \sum_{f' \in F'} U'_{f'} \ln\left(S'^{**}_{f'}\right) \\
\Rightarrow\ & U'_{\tilde{f}_1} \ln\left(S'^{*}_{\tilde{f}_1}\right) + U'_{\tilde{f}_2} \ln\left(S'^{*}_{\tilde{f}_2}\right) + \sum_{f' \in F' \setminus \{\tilde{f}_1, \tilde{f}_2\}} U'_{f'} \ln\left(S'^{*}_{f'}\right) \\
&< U'_{\tilde{f}_1} \ln\left(S'^{**}_{\tilde{f}_1}\right) + U'_{\tilde{f}_2} \ln\left(S'^{**}_{\tilde{f}_2}\right) + \sum_{f' \in F' \setminus \{\tilde{f}_1, \tilde{f}_2\}} U'_{f'} \ln\left(S'^{**}_{f'}\right) \\
\Rightarrow\ & U'_{\tilde{f}_1} \ln\left(\frac{U'_{\tilde{f}_1}}{U_{\tilde{f}}} S^{*}_{\tilde{f}}\right) + U'_{\tilde{f}_2} \ln\left(\frac{U'_{\tilde{f}_2}}{U_{\tilde{f}}} S^{*}_{\tilde{f}}\right) + \sum_{f' \in F' \setminus \{\tilde{f}_1, \tilde{f}_2\}} U'_{f'} \ln\left(S'^{*}_{f'}\right) \\
&< U'_{\tilde{f}_1} \ln\left(\frac{U'_{\tilde{f}_1}}{U_{\tilde{f}}} S^{**}_{\tilde{f}}\right) + U'_{\tilde{f}_2} \ln\left(\frac{U'_{\tilde{f}_2}}{U_{\tilde{f}}} S^{**}_{\tilde{f}}\right) + \sum_{f' \in F' \setminus \{\tilde{f}_1, \tilde{f}_2\}} U'_{f'} \ln\left(S'^{**}_{f'}\right) \\
\Rightarrow\ & \left(U'_{\tilde{f}_1} + U'_{\tilde{f}_2}\right) \ln\left(S^{*}_{\tilde{f}}\right) + \sum_{f' \in F' \setminus \{\tilde{f}_1, \tilde{f}_2\}} U'_{f'} \ln\left(S'^{*}_{f'}\right) \\
&< \left(U'_{\tilde{f}_1} + U'_{\tilde{f}_2}\right) \ln\left(S^{**}_{\tilde{f}}\right) + \sum_{f' \in F' \setminus \{\tilde{f}_1, \tilde{f}_2\}} U'_{f'} \ln\left(S'^{**}_{f'}\right) \\
\Rightarrow\ & U_{\tilde{f}} \ln\left(S^{*}_{\tilde{f}}\right) + \sum_{f \in F \setminus \{\tilde{f}\}} U_f \ln\left(S^{*}_{f}\right) \\
&< U_{\tilde{f}} \ln\left(S^{**}_{\tilde{f}}\right) + \sum_{f \in F \setminus \{\tilde{f}\}} U_f \ln\left(S^{**}_{f}\right) \\
\Rightarrow\ & \sum_{f \in F} U_f \ln\left(S^{*}_{f}\right) < \sum_{f \in F} U_f \ln\left(S^{**}_{f}\right) \\
& \mathcal{F}(\mathbf{S}^{**}, \mathbf{U}; F) < \mathcal{F}(\mathbf{S}^{*}, \mathbf{U}; F)
\end{aligned}
$$

Also, since $\mathbf{S}^{*}$ is an optimal solution of $\mathcal{F}$ under $(F, \mathbf{U})$, we have that $\mathcal{F}(\mathbf{S}^{**}, \mathbf{U}; F) \geq \mathcal{F}(\mathbf{S}^{*}, \mathbf{U}; F)$, which contradicts the inequality between $\mathcal{F}(\mathbf{S}^{**}, \mathbf{U}; F)$ and $\mathcal{F}(\mathbf{S}^{*}, \mathbf{U}; F)$ derived above.

Therefore, we prove that Axioms 3.2 holds for *weighted proportional fairness function* by contradiction. Together with the proof above for Axioms 3.1, we prove Theorem 3.3.

Given our choice of *weighted proportional fairness function* on the flow level, we could rely on Axiom 3.2 to derive a consistent *weighted proportional fairness function* on the client level. What we are doing is splitting flows $f \in F$ infinitesimally to all clients on f.

The original *weighted proportional fairness function* at the flow level is

$$\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f \in F} U_f \ln\left(S_f\right).$$

By splitting each flow into types we have

$$\begin{aligned}
\mathcal{F} &= \sum_{\tau\in\Gamma} U_\tau \ln\left(\bar{S}_\tau\right) \\
&= \sum_{\tau\in\Gamma} N_\tau \bar{u}_\tau \ln\left(N_\tau \bar{s}_\tau\right) \\
&= N\sum_{\tau\in\Gamma} g_\tau \bar{u}_\tau \ln\left(\bar{s}_\tau\right) + N\sum_{\tau\in\Gamma} g_\tau \bar{u}_\tau \ln\left(N_\tau\right),
\end{aligned}$$

where U_τ is the total true demand of clients in type τ and $\bar{u}_\tau$ is the average true demand of clients in type τ. Since $\mathcal{F}$ is the objective function to be optimized, we could discard terms N and $\ln(N_\tau)$ without changing the optimal solution. Thus, *weighted proportional fairness function* at the type level could be written as

$$\mathcal{F}(\bar{s}_\tau, \bar{u}_\tau) = \sum_{\tau\in\Gamma} g_\tau \bar{u}_\tau \ln\left(\bar{s}_\tau\right).$$

Similarly, from the perspective of clients we have

$$\begin{aligned}
\mathcal{F}(s_f[b,t], d_\tau) &= \sum_{\tau\in\Gamma}\sum_{u,k,b} \pi_\tau[b|u,k] d_\tau[u,k] u \ln\left(\sum_{t\in T_{f_\tau}} s_{f_\tau}[b,t]\right) \\
&= \sum_{f\in F}\sum_{b}\left[\ln\left(\sum_{t\in T_f} s_f[b,t]\right) \sum_{\substack{u,k \\ \tau\in\Gamma:\, f_\tau=f}} \pi_\tau[b|u,k] d_\tau[u,k] u\right], \\
\mathcal{F}(s_f[b], d_\tau) &= \sum_{\tau\in\Gamma}\sum_{u,k,b} \pi_\tau[b|u,k] d_\tau[u,k] u \ln\left(s_{f_\tau}[b]\right) \\
&= \sum_{f\in F}\sum_{b}\left[\ln\left(s_f[b]\right) \sum_{\substack{u,k \\ \tau\in\Gamma:\, f_\tau=f}} \pi_\tau[b|u,k] d_\tau[u,k] u\right].
\end{aligned}$$

Therefore, an allocation that maximizes client-level weighted proportional fairness will maximize flow-level weighted proportional fairness once clients are aggregated to flows (and vice-versa).

3.1.2 Karma-based bandwidth allocation algorithm

In the previous section, we use the the *weighted proportional fairness function*:

$$\begin{aligned}
\mathcal{F}(\mathbf{S}, \mathbf{U}) &= \sum_{f\in F} U_f \ln\left(S_f\right) \\
\mathcal{F}(\bar{s}_\tau, \bar{u}_\tau) &= \sum_{\tau\in\Gamma} g_\tau \bar{u}_\tau \ln\left(\bar{s}_\tau\right) \\
\mathcal{F}(s_f[b], d_\tau) &= \sum_{f\in F}\sum_{b}\left[\ln\left(s_f[b]\right) \sum_{\substack{u,k \\ \tau\in\Gamma:\, f_\tau=f}} \pi_\tau[b|u,k] d_\tau[u,k] u\right]
\end{aligned}$$

to describe the social welfare at the flow, the type and the client level respectively. We should notice that from the perspective of the bandwidth allocation system, the true demand of each client is not accessible since true demand is private information which clients tend not to report truthfully as shown in Figure 1. Therefore, we need to find a way to approximate the true demand. Here, we use the bid b of each client to substitute the true demand u for the bandwidth allocation system. In such a way, the *weighted proportional fairness function* could be modified as:

$$\mathcal{F}(\mathbf{S},\mathbf{B}) = \sum_{f\in F} B_f \ln\left(S_f\right)$$

$$\mathcal{F}(\bar{s}_\tau,\bar{b}_\tau) = \sum_{\tau\in\Gamma} g_\tau \bar{b}_\tau \ln\left(\bar{s}_\tau\right)$$

$$\mathcal{F}(s_f[b],d_\tau) = \sum_{f\in F}\sum_{b}\left[\ln\left(s_f[b]\right)\sum_{\substack{u,k\\ \tau\in\Gamma:\, f_\tau=f}} \pi_\tau[b|u,k]d_\tau[u,k]b\right],$$

where $\bar{b}_\tau = \frac{1}{g_\tau}\sum_b \left[\sum_{u,k}\pi_\tau[b|u,k]d_\tau[u,k]b\right]$ represents the average bid of all clients with type τ, $B_f = N\sum_{\tau\in\Gamma:\, f_\tau=f}\sum_{u,k,b}\pi_\tau[b|u,k]d_\tau[u,k]b$ represents the total bid of all clients in the flow f and $\mathbf{B}$ is the vector obtained by stacking B_f of all flows $f\in F$.

In such a way, the centralized system could optimize the allocation of bandwidth towards the modified *weighted proportional fairness function* just like other traffic engineering algorithms, as shown in Algorithm 1.

Then, we have a proposition for the output $s_f^*[b,t]$ of Algorithm 1:

> **Proposition 3.4.** The output $s_f^*[b,t]$ of Algorithm 1 optimizes $\mathcal{F}(\mathbf{S},\mathbf{B})$, $\mathcal{F}(\bar{s}_\tau,\bar{b}_\tau)$ and $\mathcal{F}(s_f[b],d_\tau)$ simultaneously.

Proposition 3.4 guarantees that the optimal social welfare value at three different levels (flows, types and clients) could be achieved simultaneously. By the hierarchical algorithm, the optimal allocation is tractable with a flow-level optimization instead of a more complex and intractable client-level optimization. Proposition 3.4 could be derived from Axioms 3.2 and 3.1 obviously.

3.2 Clients' strategic problem

After defining the rules of the bandwidth allocation system, we could analyze the behavior of clients by defining the two most important functions in karma mechanisms – karma transition function $\kappa[k^+|k,b](d,\pi)$ and immediate reward function $\zeta\left[u,b\right]\left(d,\pi\right)$.

3.2.1 Karma transition function

The karma transition function $\kappa[k^+|k,b](d,\pi)$ encodes the rules of how karma is transferred between clients. Here we suppose a simple scheme in which every client pays their

Algorithm 1: Karma-based bandwidth allocation algorithm

Input: Social state (d, π)

Output: allocations $s_f^*[b, t]$

```
/* Aggregate bids                                                    */
for f in F do
    B_f ← 0
    for τ in Γ do
        if f_τ = f then
            for u in U do
                for k in 0, 1, 2, ..., k_max do
                    for b in {0, 1, 2, ..., k} do
                        B_f ← B_f + π_τ[b|u, k] d_τ[u, k] b
/* Solve optimization on level of flows                              */
```

$$\bar{s}_f^*[t] \leftarrow \underset{\bar{s}_f[t]}{\arg\max} \sum_{f \in F} B_f \ln \left(\sum_{t \in T_f} \bar{s}_f[t] \right)$$

```
/* Get allocation of bandwidth for different bids                    */
for f in F do
    g_f ← 0
    for τ in Γ do
        if f_τ = f then
            g_f ← g_f + g_τ
    b̄_f = B/g_f
    for b in {0, 1, 2, ..., k_max} do
        s*_f[b, t] ← b/b̄_f * s̄*_f[t]
```

bid. Let $p[b]$ be the karma payment made by a client who bids b, then we have

$$p[b] = b$$

Accordingly, the *average payment* over all clients is

$$\bar{p}(d,\pi) = \sum_{\tau,u,k} \left(d_\tau[u,k] \sum_b b \; \pi_\tau[b|u,k] \right)$$

To preserve the integer value of karma, $\lceil \bar{p}(d,\pi) \rceil$ karma is randomly distributed to some clients and $\lfloor \bar{p}(d,\pi) \rfloor$ karma to the others. To make sure that all karma paid is distributed completely, we should keep $\mathbb{E}[karma\ received] = \bar{p}(d,\pi)$, from which we get

$$\mathbb{P}(karma\ received = \lceil \bar{p}(d,\pi) \rceil) = \bar{p}(d,\pi) - \lfloor \bar{p}(d,\pi) \rfloor$$
$$\mathbb{P}(karma\ received = \lfloor \bar{p}(d,\pi) \rfloor) = \lceil \bar{p}(d,\pi) \rceil - \bar{p}(d,\pi)$$

This yields the following karma transition function

$$\kappa[k^+|k,b](d,\pi) = \begin{cases} \bar{p}(d,\pi) - \lfloor \bar{p}(d,\pi) \rfloor, & k^+ = k - b + \lceil \bar{p}(d,\pi) \rceil, \\ \lceil \bar{p}(d,\pi) \rceil - \bar{p}(d,\pi), & k^+ = k - b + \lfloor \bar{p}(d,\pi) \rfloor, \\ 0, & otherwise. \end{cases}$$

3.2.2 Immediate reward function

In this section, we define the immediate reward function $\zeta_\tau[u,b](d,\pi)$ as a function $r(s,u)$ of the bandwidth allocated to a specific client $s_{f_\tau}[b]$ and its demand u, which depends on the output $s^*_f[b,t]$ of Algorithm 1 since $s_{f_\tau}[b] = \sum_{t \in T_{f_\tau}} s^*_f[b,t]$.

Since we are using the *weighted proportional fairness function* which achieves per-unit-demand proportional fairness, we want to make every client bid and thus gain bandwidth proportionally to their demand/urgency. Therefore, we want a kind of function $r(s,u)$ such that

Condition 3.5. For a specific immediate reward function $r(s,u)$,

$$\forall m \in \mathbb{Z}_{>0}, \forall i \in \{1,2,\ldots,m\}, \forall u_i > 0, \forall s_{total} \in \left[0, \sum_{i=1}^{m} u_i\right]$$

$$\left(\frac{u_1}{\sum_{i=1}^{m} u_i} s_{total}, \frac{u_2}{\sum_{i=1}^{m} u_i} s_{total}, \ldots, \frac{u_m}{\sum_{i=1}^{m} u_i} s_{total} \right)^T = \underset{\substack{s_1,s_2,\ldots,s_m \geq 0 \\ \sum_{i=1}^{m} s_i = s_{total}}}{\operatorname{argmax}} \sum_{i=1}^{m} r(s_i, u_i)$$

In Condition 3.5, we are assuming that one client is playing against its future self. The array $u_1, u_2, \ldots, u_m$ represents the true demand of the same client at different time steps and s_{total} is a total budget of the client over the time horizon $t = 1, 2, \ldots, m$. Being strategic over the m time steps, a client's optimal strategy shoud be splitting the bandwidth proportionally if it knows the true demand of the whole time period. To construct $r(s, u)$ that meets condition 3.5, we have the following theorem:

Theorem 3.6. Function $r(s, u)$ meets condition 3.5 if

$$\exists C(\cdot) : \mathbb{R} \mapsto \mathbb{R}, \exists f(\cdot) : \mathbb{R} \mapsto \mathbb{R} \; s.t. \; \forall x \in [0, \frac{s_{total}}{\min\limits_{i \in 1,2,\ldots,n} u_i}],$$
$$f''(x) < 0, r(s, u) = uf(\frac{s}{u}) + C(u)$$

The proof of Theorem 3.6 could be found in Appendix E.

From Theorem 3.6, we can construct $r(s, u)$ by choosing a specific $f(\cdot)$ and a specific $C(\cdot)$. For example, if we choose $f(x) = \ln(x)$ and $C(u) = \ln(u)$, which meet the condition 3.5, then $r(s, u) = u \ln(s)$. To discourage waste of bandwidth, we could truncate the overflowed supply by substituting s with $\min\{s, u\}$. Then, we could formulate the immediate reward function as:

$$\zeta_\tau[u, b](d, \pi) = u \ln\left(\min\left\{s_{f_\tau}[b], u\right\}\right), \tau \in \Gamma$$

Where $s_{f_\tau}[b]$ could be calculated by Algorithm 1.

After defining the immediate reward function, we will discuss the inherent relevance between the immediate reward function and the *weighted proportional fairness function* to demonstrate the rationality of such an immediate reward function.

Recall the *weighted proportional fairness function* at the level of clients defined in section 3.1.1:

$$\mathcal{F}(s_f[b], d_\tau) = \sum_{\tau \in \Gamma} \sum_{u,k,b} \pi_\tau[b|u, k] d_\tau[u, k] u \ln\left(s_{f_\tau}[b]\right).$$

If we truncate the overflowed bandwidth, then the *weighted proportional fairness function* is

$$\mathcal{F}(s_f[b], d_\tau) = \sum_{\tau \in \Gamma} \sum_{u,k,b} \pi_\tau[b|u, k] d_\tau[u, k] u \ln\left(\min\left\{s_{f_\tau}[b], u\right\}\right).$$

We should notice the inner part '$u \ln\left(\min\left\{s_{f_\tau}[b], u\right\}\right)$' of the *weighted proportional fairness function* with truncation coincides with the immediate reward function $\zeta_\tau[u, b](d, \pi) = u \ln\left(\min\{s_{f_\tau}[b], u\}\right)$, which builds consistency between social welfare and individual Utilitarianism.

3.3 At the equilibrium

After defining the rules of the bandwidth allocation system and the reward for individual clients, we can study the behavior of clients and their interaction with the system. However, it is intractable to analyze them from the perspective of dynamic games since the evolution of the dynamic process is too complicated. If we could satisfy all prerequisites of Theorem 2.4, then we could analyze the system and clients at the Stationary Nash Equilibrium, where the social state (d, π) becomes stationary. With such a static social state, the system could also be studied statically.

The two prerequisites of Theorem 2.4 are Conditions 2.1 and 2.2. It is obvious that Condition 2.2 holds since the invariance of the average karma is guaranteed by the karma transition function in Section 3.2.1. Therefore, we only need to discuss Condition 2.1.

For the immediate reward function introduced in Section 3.2.2, we could observe that when the bid is 0, the reward is negative infinity: $\zeta_\tau[u, 0](d, \pi) = u \ln(\min\{0, u\}) \to -\infty$, which violates Condition 2.1. Therefore, we can apply small trick to fit the condition. We could add a term $\epsilon \ll 1$ to the immediate reward function: $\zeta_\tau[u, 0](d, \pi) = u \ln(\min\{0, u\} + \epsilon)$, which is equivalent to

$$\zeta_\tau[u, b](d, \pi) = \begin{cases} u \ln(\min\{s_\tau[b], u\}) & , b > 0 \\ -C & , b = 0 \end{cases}$$

Where $C \gg 1$ is a positive number large enough, i.e. 1e10. In other words, when the bid is not 0, we keep the immediate reward function unchanged; when the bid is 0, we set the immediate reward to a large negative constant.

On the other hand, for the case that all clients bid zero, the behavior of the bandwidth allocation system is still undefined. Therefore, we need to modifiy Algorithm 1. At the beginning of the algorithm, we could increase the bids of all clients with a infinitely small value $\epsilon \ll 1$: $b \leftarrow b + \epsilon$. With this modified bids, the algorithm will allocate bandwidth assuming the same bid of all clients. Otherwise, the algorithm will allocate zero bandwidth to the client who bids zero, just as the original one.

With the approximation introduced above, the immediate reward function satisfies Condition 2.1. Therefore, a Stationary Nash Equilibrium is guaranteed by Theorem 2.4, and we could use numerical methods to solve this equilibrium.

4 Numerical experiments

In this section, we will test the performance of the karma economy for the traffic engineering problem in competitive environments. First, we will test with different network topologies. Then, we will test with different parameters of the karma economy. Since it is intractable to solve the Stationary Nash Equilibrium analytically, all experiments will be conducted by numerical methods. We use the *weighted proportional fairness function* with truncation mentioned in Section 3.2.2

$$\mathcal{F}(s_f[b], d_\tau) = \sum_{\tau \in \Gamma} \sum_{u,k,b} \pi_\tau[b|u,k] d_\tau[u,k] u \ln \left(\min \{ s_{f_\tau}[b], u \} \right)$$

to evaluate the performance.

4.1 Simple network topology

The simple network topology could be represented as $G = (V, A)$ with $V = 1, 2$ and $A = (1, 2)$. This simple network contains only one arc with the normalized capacity $c[(1, 2)] = 5$. Therefore, the only possible flow is $f = (1, 2)$.

4.1.1 Single type of clients

Let all N clients share the same type τ and the true demand follows a uniform distribution

$$\phi_\tau[u \mid u^-] = \begin{cases} 0.05 & , u \in \{1, 2, \dots, 20\} \\ 0 & , \text{else} \end{cases}.$$

Then the Stationary Nash Equilibrium calculated with discount factor $\delta = 0.99$ and average karma of 10 is shown in Figure 2.

Figures 2a, 2b, 2c, 2d, 2e and 2f show the equilibrium bidding policy π^* of the clients with the true demand of 1, 5, 9, 12, 16 and 20 respectively, where for a given level of karma (x-axis) the intensity of the red color in each cell denotes the probabilistic weight placed on the bids (y-axis) and invalid bids that exceed the one's karma are displayed black. These two figures exhibit multiple intuitive behaviors. We could notice that the client with higher true demand tends to bid more, which means clients could look into the future and bid rationally. In addition, if the amount of karma is large enough, the client will not tend to bid all its karma at once to save for further usage, which reflects the core concept of karma economies: *If I give in now, I will be rewarded in the future.*

Figure 2g shows the distribution of karma at the equilibrium. Combined with the distribution of true demand $\phi_\tau[u \mid u^-]$, the figure could represent the equilibrium joint distribution of client state d^*. Figure 2h shows the distribution of bid at the equilibrium, which could be calculated by $\mathbb{P}[b] = \sum_{k,u} \pi^*_\tau[b|u,k] d^*_\tau[u,k] b$.

The evaluation of performance is shown in Figure 3. The experiment is conducted twice with the Monte Carlo method and analytical method respectively. It is shown that the gap between the two methods is very close, therefore, we will only use the Monte Carlo method hereafter. From Figure 3 we could notice that the value of social welfare function of the karma economy is close to the global optimum where all clients reveal their true demand

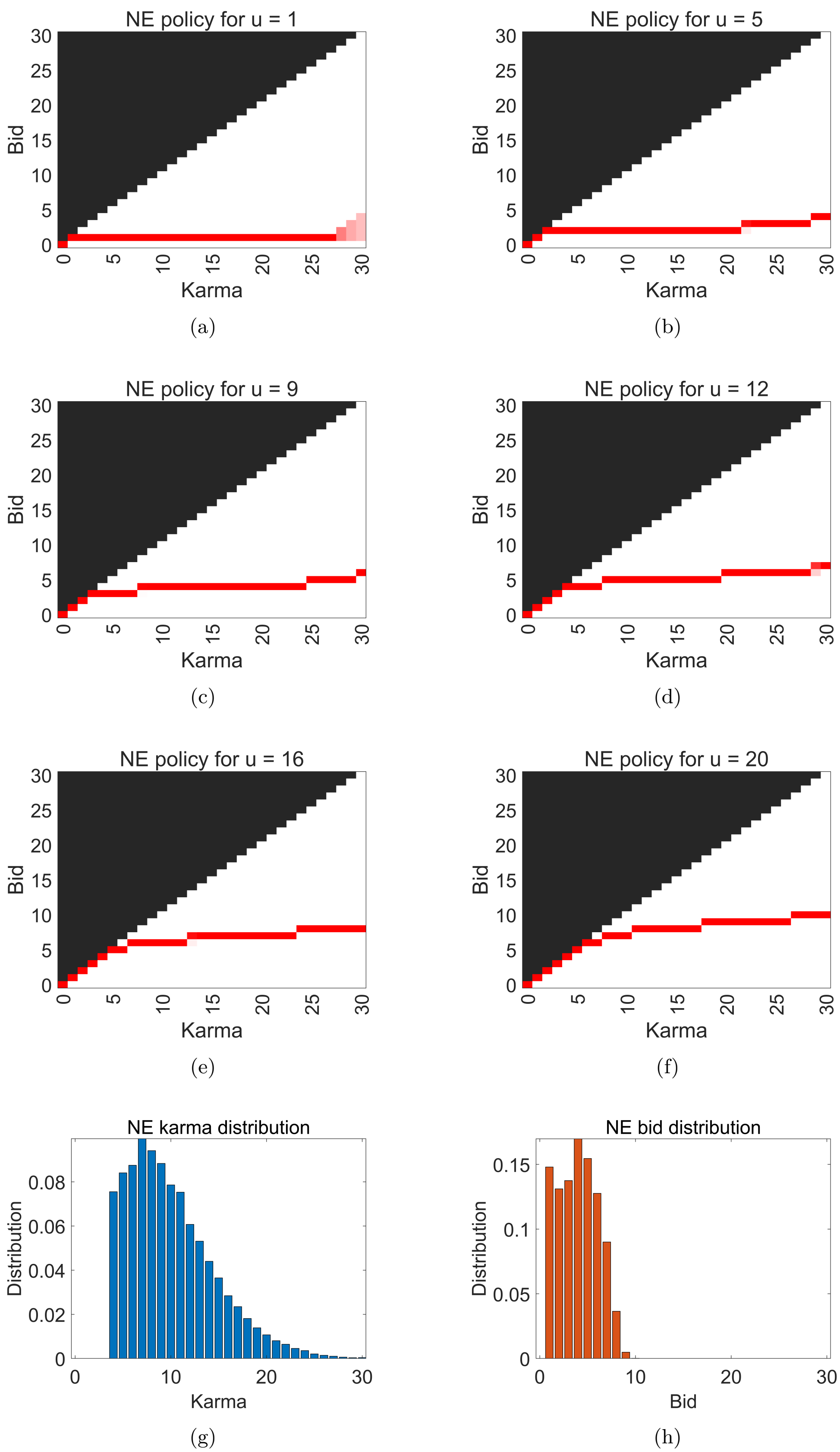


Figure 2: The Stationary Nash Equilibrium for the case with a single type of clients

and the karma economy is closer to the global optimum than the case that all clients cheat, which shows the efficiency of the karma economy.

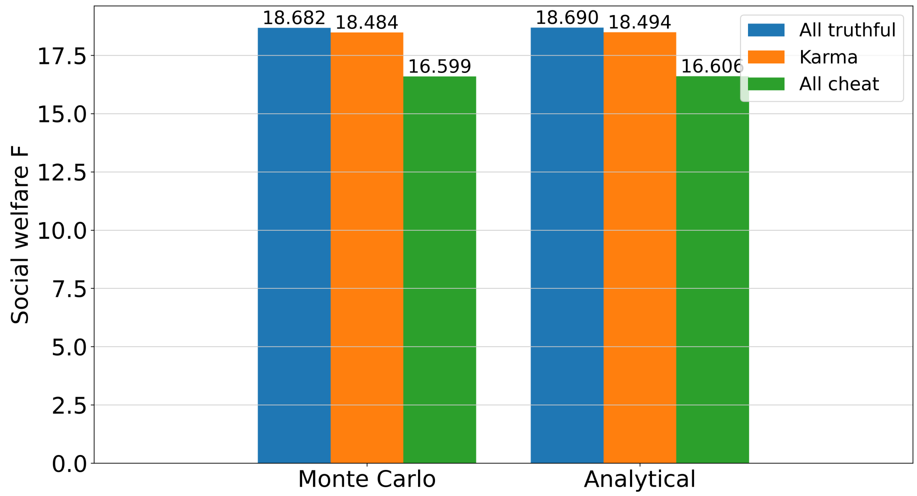


Figure 3: Evaluation of the karma economy with a single type of clients

Although the allocation system is designed for *weighted proportional fairness function* in this study, the experiment could still give a positive result if we change the social welfare function and the corresponding immediate reward function. For more details, please refer to Appendix F.

4.1.2 Multiple types of clients

Let $\frac{1}{2}N$ clients share the type τ_1 and the other $\frac{1}{2}N$ clients share the type τ_2. The true demand of τ_1 and τ_2 follows different uniform distribution

$$\phi_{\tau_1}[u \mid u^-] = \begin{cases} 0.05 & , u \in \{1, 2, \dots, 20\} \\ 0 & , \text{else} \end{cases}$$

$$\phi_{\tau_2}[u \mid u^-] = \begin{cases} 0.1 & , u \in \{1, 2, \dots, 10\} \\ 0 & , \text{else} \end{cases}.$$

Then the Stationary Nash Equilibrium calculated with discount factor $\delta = 0.99$ and average karma of 10 is shown in Figure 4.

Figures 4a and 4b show the equilibrium bidding policy π^* of the clients with the true demand of 10 and the types τ_1 and τ_2 respectively. Since the average true demand of τ_1 is higher than τ_2, τ_1 prefers saving karma for the future while τ_2 prefers spending karma instantly.

Figures 4c and 4d show the distribution of karma at the equilibrium of τ_1 and τ_2 respectively. Figures 4e and 4f show the distribution of bids. We could notice that the variance of bid of τ_1 is wider than τ_2, so that τ_1 could deal with the wider range of true demand more flexibly.

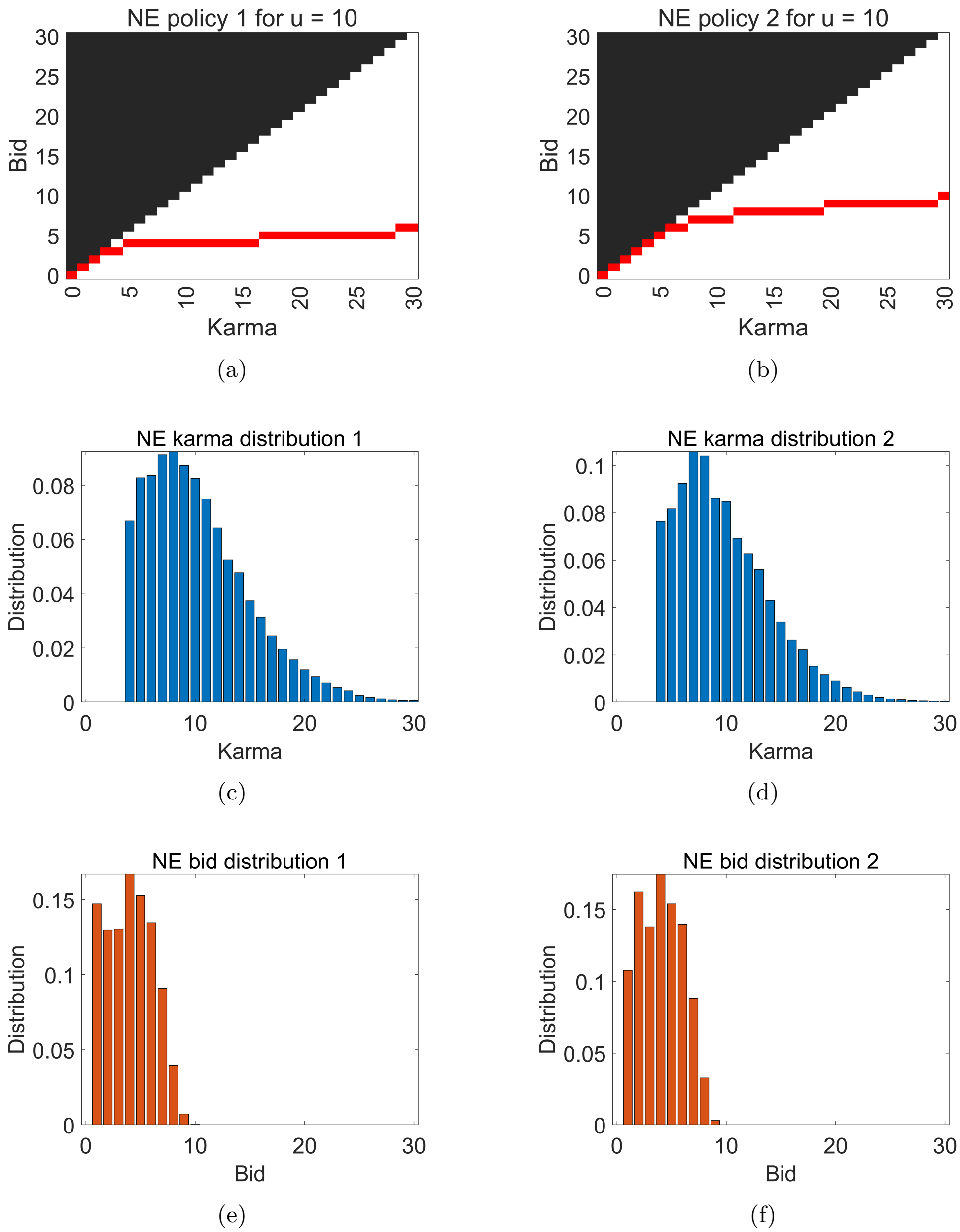


Figure 4: The Stationary Nash Equilibrium for the case with multiple types of clients

The evaluation of performance is shown in Figure 5. From Figure 5 we could notice that the karma economy is not as good as the one in Section 4.1.1 due to the invisible heterogeneity of the true demand of different types of clients. However, the karma economy could still improve social welfare compared to the case that all clients report fake demands.

4.2 General network topology

In this section, we will test the karma economy with a more complex network as shown in Figure 6. The network could be represented as $G = (V, A)$ with $V = 1, 2, 3$ and $A =$

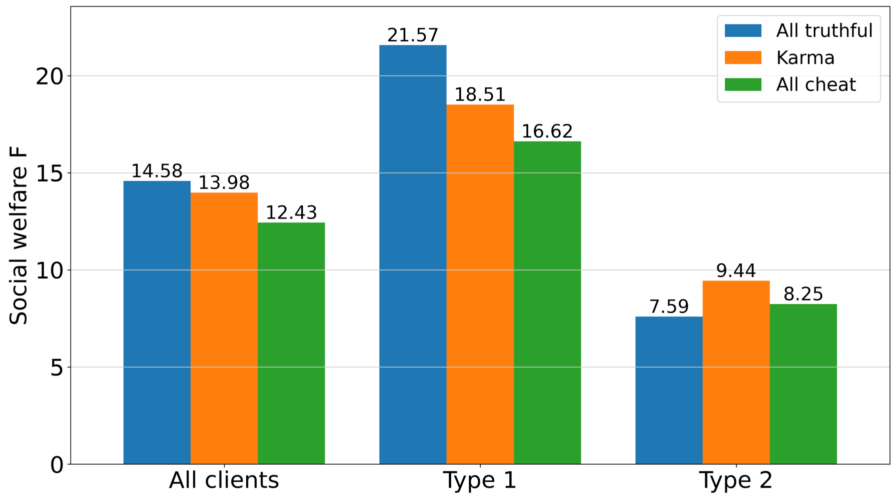


Figure 5: Evaluation of the karma economy with multiple types of clients

$(1,2),(2,3)$. This network contains two arcs with the normalized capacity $c[(1,2)] = 2.5$ and $c[(2,3)] = 4$. There are 3 flows in the flow set $F = \{(1,2),(2,3),(1,3)\}$.

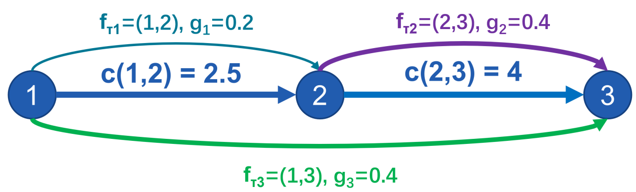


Figure 6: A more complex network topology

We set 3 types of clients $\Gamma = \{\tau_1, \tau_2, \tau_3\}$. For each type, the correponding flow is $f_{\tau_1} = f_1 = (1,2)$, $f_{\tau_2} = f_2 = (2,3)$ and $f_{\tau_3} = f_3 = (1,3)$. Let the proportion of population with different type as $g_{\tau_1} = 0.2$, $g_{\tau_2} = 0.4$, $g_{\tau_3} = 0.4$. The true demand of all three types follows the same uniform distribution

$$\phi_{\tau_1}[u \,|\, u^-] = \phi_{\tau_2}[u \,|\, u^-] = \phi_{\tau_3}[u \,|\, u^-] = \begin{cases} 0.05 & , u \in \{1, 2, \ldots, 20\} \\ 0 & , else \end{cases}.$$

Then the Stationary Nash Equilibrium calculated with discount factor $\delta = 0.99$ and average karma of 10 is shown in Figure 7.

Figures 7a, 7b and 7c show the equilibrium bidding policy π^* of the clients with the true demand of 20 and the types τ_1, τ_2 and τ_3 respectively. Figures 7d, 7e and 7f show the distribution of karma at the equilibrium of clients with τ_1, τ_2 and τ_3 respectively. From those figures, we could find that clients with type τ_1 or τ_2 are easier to acquire bandwidth while clients with type τ_3 are the most difficult since the average karma of τ_1 and τ_2 are higher than τ_3 and the policy of τ_1 and τ_2 are more conservative than τ_3. Figures 7g, 7h and 7i show the distribution of bid of different types $\tau \in \Gamma$.

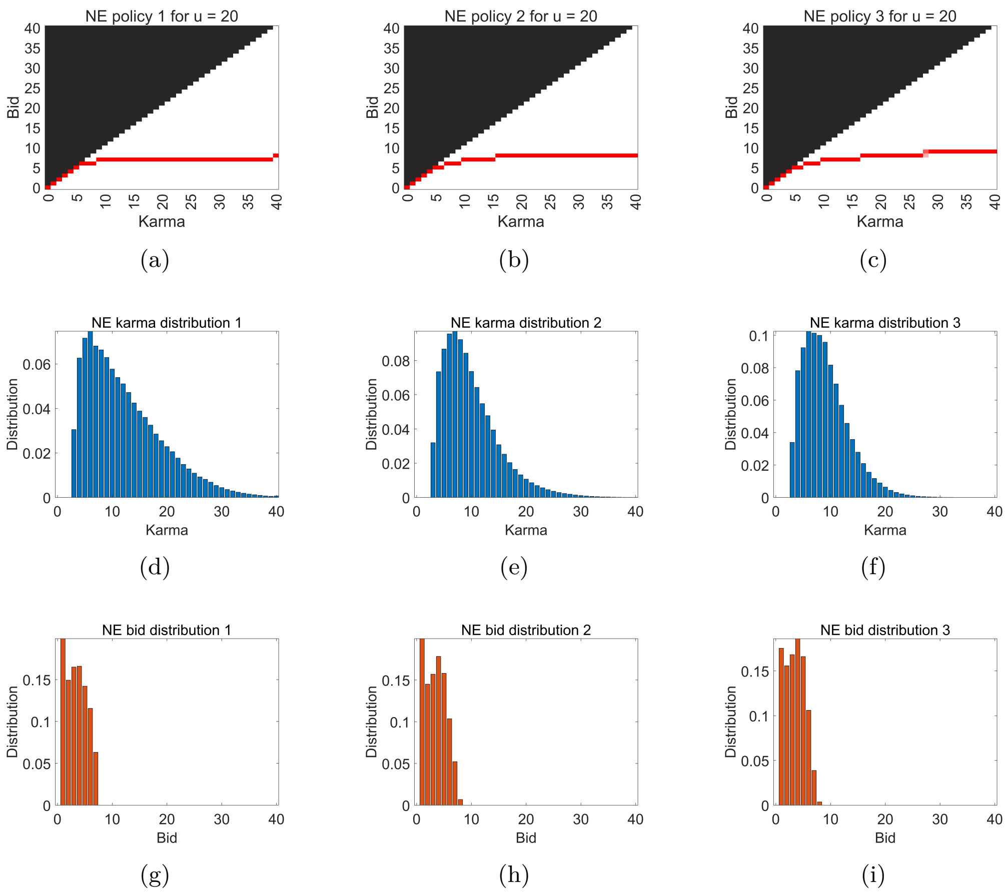


Figure 7: The Stationary Nash Equilibrium for the case with a more complex network

The evaluation of performance is shown in Figure 8. From Figure 8 we could notice that the performance of the karma economy is close to the global optimum. Therefore, this experiment proves that the karma economy could improve social welfare in a more complex and more general network effectively.

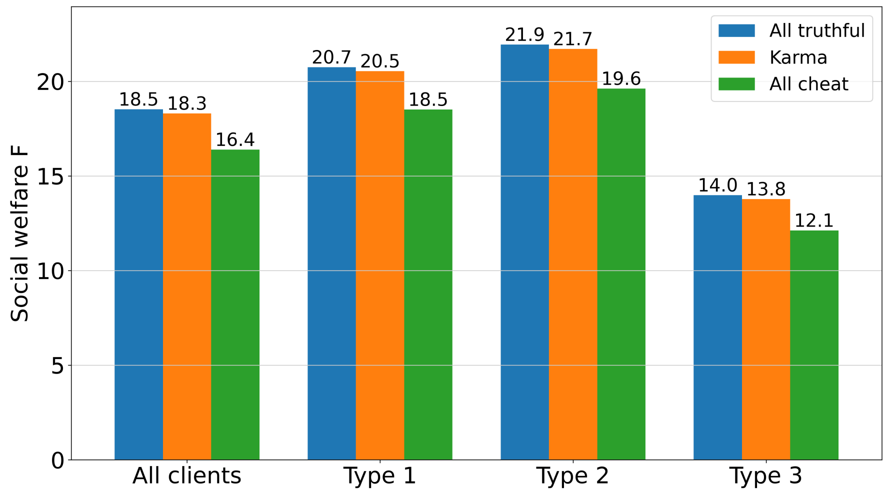


Figure 8: Evaluation of the karma economy with a more complex network

4.3 Different discount factor

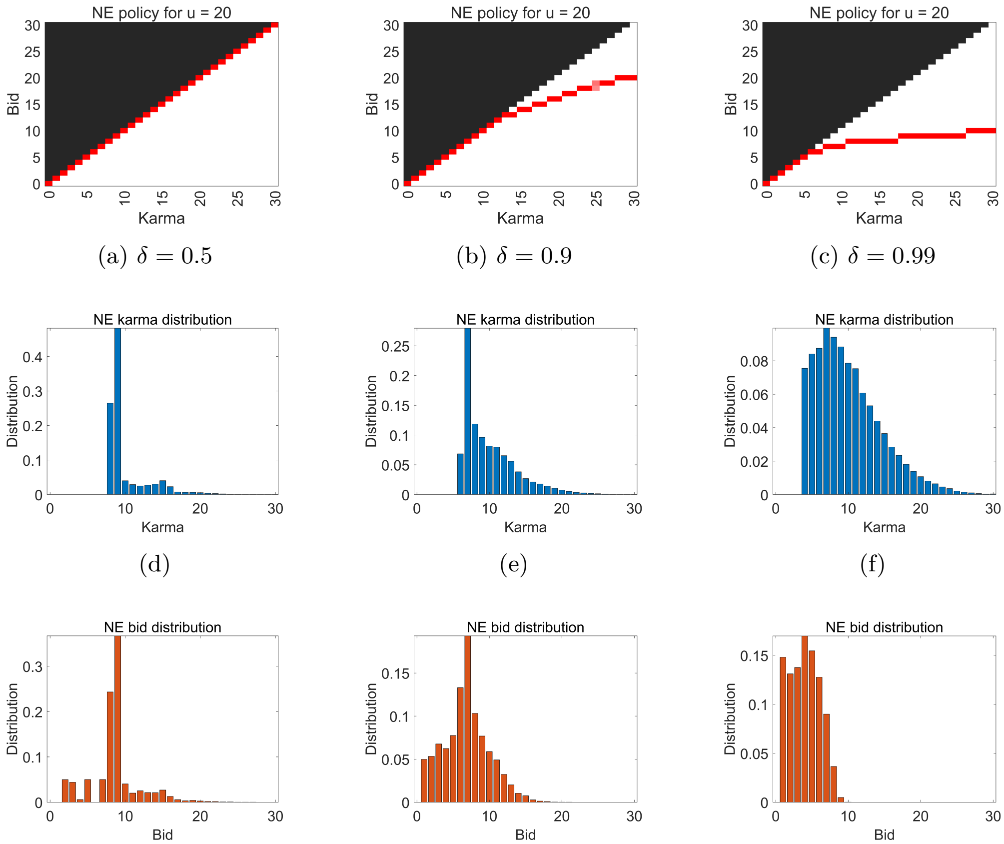
For common dynamic programming with a receding horizon, the closer the discount factor δ is to 0, the more myopia a client could be and vice versa. We will calculate the Stationary Nash Equilibrium according to the same topology and type settings as in Section 4.1.1 and with the different discount factor δ to analyze the impact of the choice of δ.

The Stationary Nash Equilibrium calcualted under discount factor $\delta = 0.5$, 0.9 and 0.99 is shown in Figure 9. From Figures 9a, 9b and 9c we could notice that the closer the discount factor δ is to 1, the more conservative the client is. Figures 9d, 9e and 9f show that the karma distribution of far-sighted clients is smoother than myopia ones, which also means that far-sighted clients tend to save more karma for the future.

The performance of the karma economy under different discount factors is shown in Figure 10. From this figure, it is obvious that a community of far-sighted clients could result in equilibrium with social welfare closer to the global optimum than one of myopia clients, which is consistent with the intuitive interpretation of discount factor δ.

4.4 Different distribution of demand

In all experiments above, we assume that the true demand of each client is sampled from some uniform distribution. However, in daily life, it is more frequent for a client to have a low demand (i.e. chatting online, browsing websites, etc.) rather than a high demand (i.e. downloading software, watching movies online, etc.). Therefore, we will test the karma economy for the case that the true demand of clients tends to be small and seldomly large.



(a) $\delta = 0.5$ (b) $\delta = 0.9$ (c) $\delta = 0.99$

(d) (e) (f)

(g) (h) (i)

Figure 9: The Stationary Nash Equilibrium under different discount factors δ

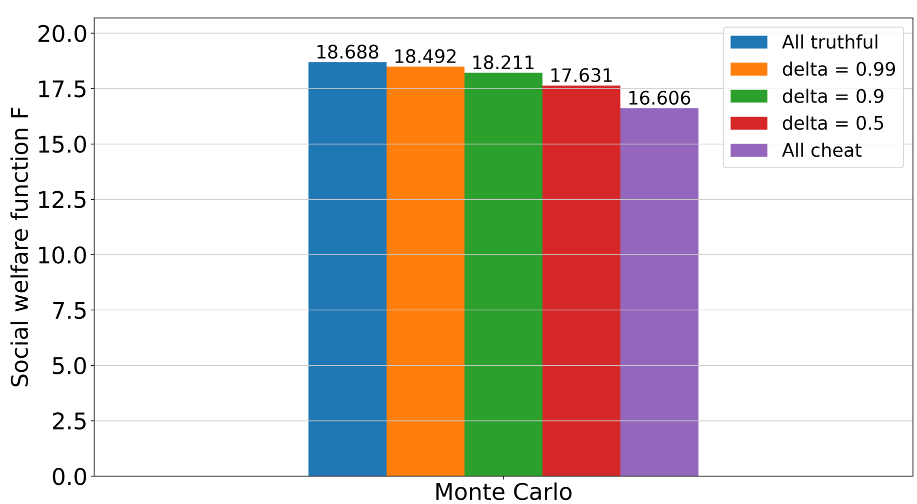


Figure 10: Evaluation of the karma economy under different discount factors *delta*

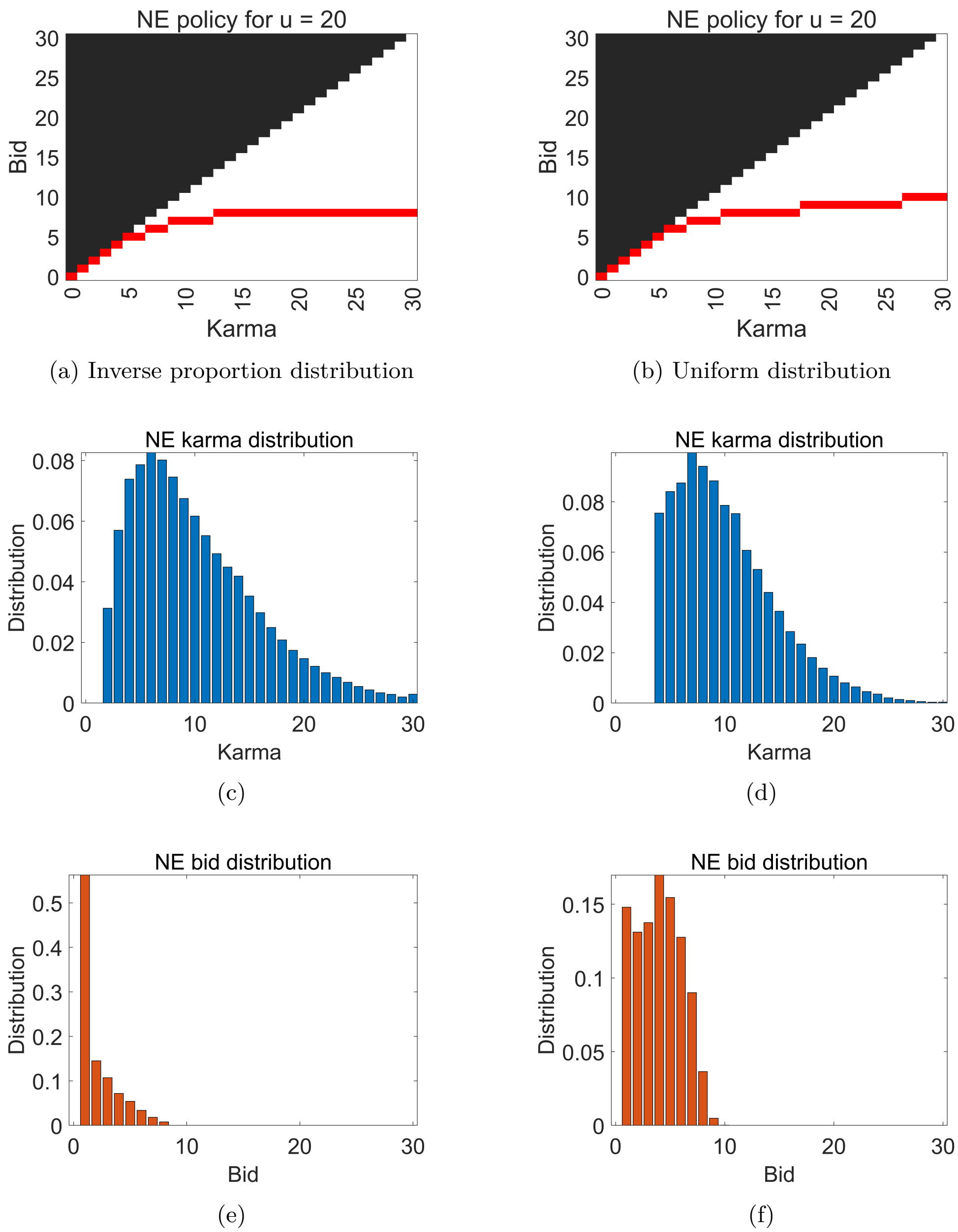
We will use the same topology and type settings as in Section 4.1.1 except for the different distribution of true demand of clients with type τ:

$$\phi_\tau[u \mid u^-] = \begin{cases} \frac{Z}{u} & , u \in \{1, 2, \ldots, 20\} \\ 0 & , \text{else} \end{cases} ,$$

where $Z = \frac{1}{\sum_{i=1}^{20} \frac{1}{i}} \approx 0.278$ is the normalization factor. We name this $\phi_\tau[u \mid u^-]$ as inverse proportion distribution for further discussion.

The Stationary Nash Equilibrium calculated under different distributions of true demand is shown in Figure 11. From Figures 11a, and 11b we could notice that the clients with the inverse proportion distribution of true demand are more conservative than the clients with uniform distribution. Figures 11c and 11d show that the karma distribution of clients with inverse proportion distribution of true demand is smoother, which also means that such clients tend to save more karma for the future. Also, from Figure 11e and 11f it is obvious that the average bid of clients with the inverse proportion distribution of true demand is smaller, which makes the client with high true demand easier to gain enough bandwidth that it needs.

The performance of the karma economy under different distributions of true demand is shown in Figure 12. By comparing the relative social welfare of the two cases, we could find that the gap between the global optimum and the worst is larger for the clients with the inverse proportion distribution of true demand. However, the karma economy could still improve social welfare greatly under the inverse proportion distribution of the true demand. This experiment could validate the robustness of the karma economy against different distributions of the true demand.



(a) Inverse proportion distribution

(b) Uniform distribution

(c)

(d)

(e)

(f)

Figure 11: The Stationary Nash Equilibrium under different demand distributions

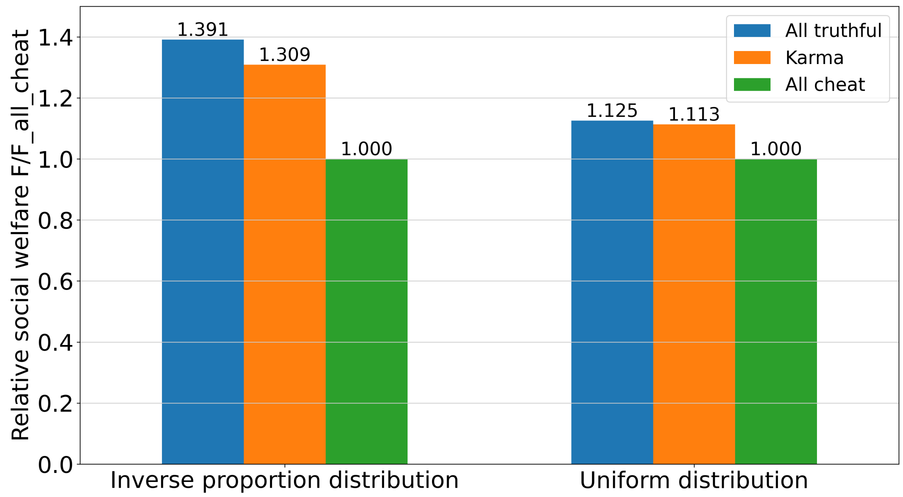


Figure 12: Evaluation of the karma economy under different demand distributions

5 Conclusion

In this study, we propose a dynamic population model for the data traffic engineering problem in a competitive environment from the perspective of game theory, adapt the karma economy to this game, and state the existence of a Stationary Nash Equilibrium of such a game. To evaluate the allocation scheme given by the traffic engineering algorithm, we propose weighted proportional fairness function as the centralized social welfare function and prove the two important axioms the function holds. According to this social welfare function and the two axioms, we derive the bandwidth allocation algorithm hierarchically. By choosing a specific immediate reward function consistent with weighted proportional fairness function, we implement the karma economy for traffic engineering at the level of clients. The numerical experiments conducted in this study show both efficiency and robustness of the karma economy that we introduced for traffic engineering.

The future study includes developing measures against the heterogeneity between different types of clients in traffic engineering with competition i.e. taxing clients according to the karma held, analyzing the behavior of the current karma economy for traffic engineering on all kinds of network topology thoroughly, and also applying the current karma economy to traffic engineering problems in reality. In addition, it is very important to study the properties of the Stationary Nash Equilibrium. If such an equilibrium is not evolutionary stable, then the system will diverge after a small perturbation. It will also be impossible to study both the system and the behavior of clients from a static perspective then.

Appendices

A The inefficiency and unfairness of centralized control caused by dominant strategy

The experiment shown by Figure 1 is a simple traffic engineering problem. A total of 30 clients are sharing the same bottleneck network with a total capacity of bandwidth of 150. Each client has a true demand drawn from the uniform distribution of $\{1, 2, 3, \ldots, 20\}$, which has the same unit as the capacity of the network. The system will allocate the total bandwidth of 150 proportionally to all clients based on the reporting demand. The overflowed bandwidth allocated will be discarded.

The experiment is conducted with three different client strategies: All clients report the true demand, One client reports a fake demand of 20, which is the upper bound of possible true demand while other clients keep truthful and all clients report fake demands as large as possible. The experiment is repeated 1000 times to approximate the expectation.

Figure 1 shows the average bandwidth allocated to the cheating client and among all clients respectively. From the figure we could find that for a specific client, while others are truthful, cheating results in higher bandwidth allocated at the cost of social welfare. Therefore, reporting true demand is not an equilibrium for clients since rational clients tend to cheat. However, if every client is cheating, then the bandwidth that every client gets will be reduced, which demonstrates the importance to set rules for the allocation system and also for clients to control the damage to social welfare.

B Proof of Theorem 2.6

Since $\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum\limits_{f \in F} \ln(S_f)$ is strictly concave with respect to $\mathbf{S}$ and the set $\mathcal{S}$ that contains all feasible allocations $\mathbf{S}$ is convex, an unique maximizer of $\mathcal{F}(\mathbf{S}, \mathbf{U})$ is guaranteed within $\mathcal{S}$. In other words, there exists an unique $\mathbf{S}^* \in \mathcal{S}$ such that $\mathbf{S}^* = \underset{\mathbf{S} \in \mathcal{S}}{\operatorname{argmax}} \sum\limits_{f \in F} \ln(S_f)$.

Next, let's prove that $\mathbf{S}^*$ achieves proportional fairness.

Since $\mathbf{S}^*$ is the unique maximizer of $\mathcal{F}(\mathbf{S}, \mathbf{U})$, $\forall \mathbf{S} \in \mathcal{S} \setminus \{\mathbf{S}^*\}$, $\mathcal{F}(\mathbf{S}, \mathbf{U}) < \mathcal{F}(\mathbf{S}^*, \mathbf{U})$.

Define

$$D(t) = \frac{\mathcal{F}\left(\mathbf{S}^* + t \cdot (\mathbf{S} - \mathbf{S}^*), \mathbf{U}\right) - \mathcal{F}\left(\mathbf{S}^*, \mathbf{U}\right)}{t}$$

Then $\forall t \in (0, 1]$, $D(t)$ is well defined since $\mathcal{S}$ is convex. Also, we know that $\forall t \in (0, 1]$, $D(t) < 0$. Therefore, $\lim\limits_{t \to 0^+} D(t) \leq 0$. By the definition of the directional derivative, we have that

$$\lim_{t \to 0^+} D(t) = \nabla_{\mathbf{S}'} \mathcal{F}(\mathbf{S}', \mathbf{U})|_{\mathbf{S}' = \mathbf{S}^*} \cdot (\mathbf{S} - \mathbf{S}^*) = \sum_{f \in F} \frac{S_f - S_f^*}{S_f^*} \leq 0$$

And when $\mathbf{S} = \mathbf{S}^*$, $\sum\limits_{f \in F} \frac{S_f - S_f^*}{S_f^*} = 0 \leq 0$.

Therefore, $\mathbf{S}^*$ achieves proportional fairness.

C Proof of Theorem 2.8

The proof of theorem 2.8 if very similar to the proof of theorem 2.6.

Since $\mathcal{F}(\mathbf{S},\mathbf{U}) = \sum_{f\in F} U_f \ln(S_f)$ is strictly concave with respect to $\mathbf{S}$ and the set $\mathcal{S}$ that contains all feasible allocations $\mathbf{S}$ is convex, an unique maximizer of $\mathcal{F}(\mathbf{S},\mathbf{U})$ is guaranteed within $\mathcal{S}$. In other words, there exists an unique $\mathbf{S}^* \in \mathcal{S}$ such that $\mathbf{S}^* = \underset{\mathbf{S}\in\mathcal{S}}{\operatorname{argmax}} \sum_{f\in F} U_f \ln(S_f)$.

Next, let's prove that $\mathbf{S}^*$ achieves weighted proportional fairness.

Since $\mathbf{S}^*$ is the unique maximizer of $\mathcal{F}(\mathbf{S},\mathbf{U})$, $\forall \mathbf{S} \in \mathcal{S} \setminus \{\mathbf{S}^*\}$, $\mathcal{F}(\mathbf{S},\mathbf{U}) < \mathcal{F}(\mathbf{S}^*,\mathbf{U})$.

Define

$$D(t) = \frac{\mathcal{F}(\mathbf{S}^* + t\cdot(\mathbf{S}-\mathbf{S}^*),\mathbf{U}) - \mathcal{F}(\mathbf{S}^*,\mathbf{U})}{t}$$

Then $\forall t \in (0,1]$, $D(t)$ is well defined since $\mathcal{S}$ is convex. Also, we know that $\forall t \in (0,1]$, $D(t) < 0$. Therefore, $\lim_{t\to 0^+} D(t) \le 0$. By the definition of the directional derivative, we have that

$$\lim_{t\to 0^+} D(t) = \nabla_{\mathbf{S}'} \mathcal{F}(\mathbf{S}',\mathbf{U})|_{\mathbf{S}'=\mathbf{S}^*} \cdot (\mathbf{S}-\mathbf{S}^*) = \sum_{f\in F} U_f \frac{S_f - S_f^*}{S_f^*} \le 0$$

And when $\mathbf{S} = \mathbf{S}^*$, $\sum_{f\in F} U_f \frac{S_f - S_f^*}{S_f^*} = 0 \le 0$.

Therefore, $\mathbf{S}^*$ achieves weighted proportional fairness.

D Proof of Table 2

We will only prove why certain social welfare functions do not meet Axioms 3.1 and 3.2 by giving counterexamples.

For the *sum of supply*, let $F = \{f_1, f_2\}$, $\mathcal{S} = \{\mathbf{S} | S_{f_1} + S_{f_2} = 1\}$ and $\mathbf{U} = (1,1)^T$. Define $\mathbf{S}^* = (0.3, 0.7)^T$ and $\mathbf{S}^{**} = (0.4, 0.6)^T$, then we could verify that $\mathbf{S}^*, \mathbf{S}^{**} \in \mathcal{S}$ and $\mathcal{F}(\mathbf{S}^*,\mathbf{U}) = \mathcal{F}(\mathbf{S}^{**},\mathbf{U}) = 1 = \max_{\mathbf{S}\in\mathcal{S}} \mathcal{F}(\mathbf{S},\mathbf{U})$. Therefore, the *sum of supply* violates Axiom 3.1.

For the *minimum availability*, let $F = \{f_1, f_2\}$, $\mathcal{S} = \{\mathbf{S} | S_{f_1} < 1, S_{f_2} = 1\}$ and $\mathbf{U} = (1,2)^T$. Define $\mathbf{S}^* = (1,1)^T$ and $\mathbf{S}^{**} = (0.5, 1)^T$, then we could verify that $\mathbf{S}^*, \mathbf{S}^{**} \in \mathcal{S}$ and $\mathcal{F}(\mathbf{S}^*,\mathbf{U}) = \mathcal{F}(\mathbf{S}^{**},\mathbf{U}) = 0.5 = \max_{\mathbf{S}\in\mathcal{S}} \mathcal{F}(\mathbf{S},\mathbf{U})$. Therefore, the *minimum availability* violates Axiom 3.1.

For the *proportional fairness function*, let $F = \{f_1, f_2\}$, $\mathcal{S} = \{\mathbf{S} | S_{f_1} + S_{f_2} = 1\}$ and $\mathbf{U} = (1,1)^T$. We select flow f_1 as $\tilde{f}$ to be splitted. Therefore, $F = \left\{\tilde{f}_1, \tilde{f}_2, f_2\right\}$ and $\mathcal{S}' = \left\{\mathbf{S}' \middle| S'_{\tilde{f}_1} + S'_{\tilde{f}_2} + S'_{f_2} = 1\right\}$. If we distribute the demand of f_1 to $\tilde{f}_1$ and $\tilde{f}_2$ evenly, then $\mathbf{U}' = (0.5, 0.5, 1)^T$. We could verify that $\mathbf{S}^* = (0.5, 0.5)^T$ is the maximizer for $\mathcal{F}(\mathbf{S},\mathbf{U}; F)$ but $\mathbf{S}'^* = (0.25, 0.25, 0.5)^T$ is not the maximizer for $\mathcal{F}(\mathbf{S}',\mathbf{U}'; F')$ since $\mathcal{F}(\mathbf{S}'^*,\mathbf{U}'; F') = -5\ln(2) = -3.47 < -3.30 = -3\ln(3) = \mathcal{F}(\mathbf{S}'^* *, \mathbf{U}'; F')$, where $\mathbf{S}'^* * = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T \in \mathcal{S}'$. Therefore, the *proportional fairness function* violates Axiom 3.2.

For the *fairness with quadratic loss*, let $F = \{f_1, f_2\}$, $\mathcal{S} = \{\mathbf{S} | S_{f_1} + S_{f_2} = 2\}$ and $\mathbf{U} = (4,4)^T$. We select flow f_1 as $\tilde{f}$ to be split. Therefore, $F = \left\{\tilde{f}_1, \tilde{f}_2, f_2\right\}$ and $\mathcal{S}' = \left\{\mathbf{S}' \middle| S'_{\tilde{f}_1} + S'_{\tilde{f}_2} + S'_{f_2} = 1\right\}$. If we distribute the demand of f_1 to $\tilde{f}_1$ and $\tilde{f}_2$ evenly, then $\mathbf{U}' = (2,2,4)^T$. We could verify that $\mathbf{S}^* = (1,1)^T$ is the maximizer for $\mathcal{F}(\mathbf{S},\mathbf{U}; F)$ but $\mathbf{S}'^* = (0.5, 0.5, 1)^T$ is not the maximizer for $\mathcal{F}(\mathbf{S}',\mathbf{U}'; F')$ since $\mathcal{F}(\mathbf{S}'^*,\mathbf{U}'; F') = -13.5 <$

$-12 = \mathcal{F}(\mathbf{S}'^{*}*, \mathbf{U}'; F')$, where $\mathbf{S}'^{*}* = (0,0,2)^T \in \mathcal{S}'$. Therefore, the *fairness with quadratic loss* violates Axiom 3.2.

For the *sum of availability*, let $F = \{f_1, f_2\}$, $\mathcal{S} = \{\mathbf{S} | S_{f_1} + S_{f_2} = 1\}$ and $\mathbf{U} = (1,1)^T$. Define $\mathbf{S}^* = (0.3, 0.7)^T$ and $\mathbf{S}^{**} = (0.4, 0.6)^T$, then we could verify that $\mathbf{S}^*, \mathbf{S}^{**} \in \mathcal{S}$ and $\mathcal{F}(\mathbf{S}^*, \mathbf{U}) = \mathcal{F}(\mathbf{S}^{**}, \mathbf{U}) = 1 = \max_{\mathbf{S} \in \mathcal{S}} \mathcal{F}(\mathbf{S}, \mathbf{U})$. Therefore, the *sum of supply* violates Axiom 3.1.

Still for the *sum of availability*, let $F = \{f_1, f_2\}$, $\mathcal{S} = \{\mathbf{S} | S_{f_1} + S_{f_2} = 1\}$ and $\mathbf{U} = (3,2)^T$. We select flow f_1 as $\tilde{f}$ to be split. Therefore, $F = \left\{ \tilde{f}_1, \tilde{f}_2, f_2 \right\}$ and $\mathcal{S}' = \left\{ \mathbf{S}' \middle| S'_{\tilde{f}_1} + S'_{\tilde{f}_2} + S'_{f_2} = 1 \right\}$. If we distribute the demand of f_1 to $\tilde{f}_1$ and $\tilde{f}_2$ evenly, then $\mathbf{U}' = (1.5, 1.5, 2)^T$. We could verify that $\mathbf{S}^* = (0,1)^T$ is the maximizer for $\mathcal{F}(\mathbf{S}, \mathbf{U}; F)$ but $\mathbf{S}'^* = (0,0,1)^T$ is not the maximizer for $\mathcal{F}(\mathbf{S}', \mathbf{U}'; F')$ since $\mathcal{F}(\mathbf{S}'^*, \mathbf{U}'; F') = 0.5 < \frac{2}{3} = \mathcal{F}(\mathbf{S}'^{*}*, \mathbf{U}'; F')$, where $\mathbf{S}'^{*}* = (1,0,0)^T \in \mathcal{S}'$. Therefore, the *sum of availability* violates Axiom 3.2 as well.

By these counterexamples, we prove all the $\times$ cells in Table 2

E Proof of Theorem 3.6

We could solve $\underset{\substack{s_1, s_2, \ldots, s_m \geq 0 \\ \sum_{i=1}^m s_i = s_{total}}}{\operatorname{argmax}} \sum_{i=1}^m r(s_i, u_i)$ by the *method of Lagrange multipliers.*

Let $\mathcal{L}(\mathbf{s}, \lambda) = \sum_{i=1}^m r(s_i, u_i) - \lambda \left[\left(\sum_{i=1}^m s_i \right) - s_{total} \right]$ be the Lagrangian function. Then at the maximum of $\sum_{i=1}^m r(s_i, u_i)$, we have that

$$
\begin{aligned}
\nabla_{\mathbf{s}} \mathcal{L}(\mathbf{s}, \lambda) &= 0 \\
\frac{\partial \mathcal{L}(\mathbf{s}, \lambda)}{\partial \lambda} &= 0,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
\frac{\partial r(s_i, u_i)}{\partial s_i} - \lambda &= 0, \ \forall i \in \{1, 2, \ldots, m\} \\
\left(\sum_{i=1}^m s_i \right) - s_{total} &= 0.
\end{aligned}
$$

By substituting $r(s,u)$ with $r(s,u) = uf(\frac{s}{u}) + C(u)$, we have that

$$
\begin{aligned}
f'(\frac{s_i}{u_i}) - \lambda &= 0, \ \forall i \in \{1, 2, \ldots, m\} \\
\left(\sum_{i=1}^m s_i \right) - s_{total} &= 0.
\end{aligned}
$$

We could verify that one solution of the equations above is

$$\begin{cases} \mathbf{s}^* & = \left(\frac{u_1}{\sum_{i=1}^{m} u_i} s_{total}, \frac{u_2}{\sum_{i=1}^{m} u_i} s_{total}, \ldots, \frac{u_m}{\sum_{i=1}^{m} u_i} s_{total} \right)^T \\ \lambda^* & = f' \left(\frac{s_{total}}{\sum_{i=1}^{m} u_i} \right) \end{cases}.$$

Since $f''(x) < 0$, we could verify that $\mathbf{s}^*$ is the unique maximizer of $\sum_{i=1}^{m} r(s_i, u_i)$ under the constraints of $s_1, s_2, \ldots, s_m \geq 0$ and $\sum_{i=1}^{m} s_i = s_{total}$, which proves the Theorem 3.6.

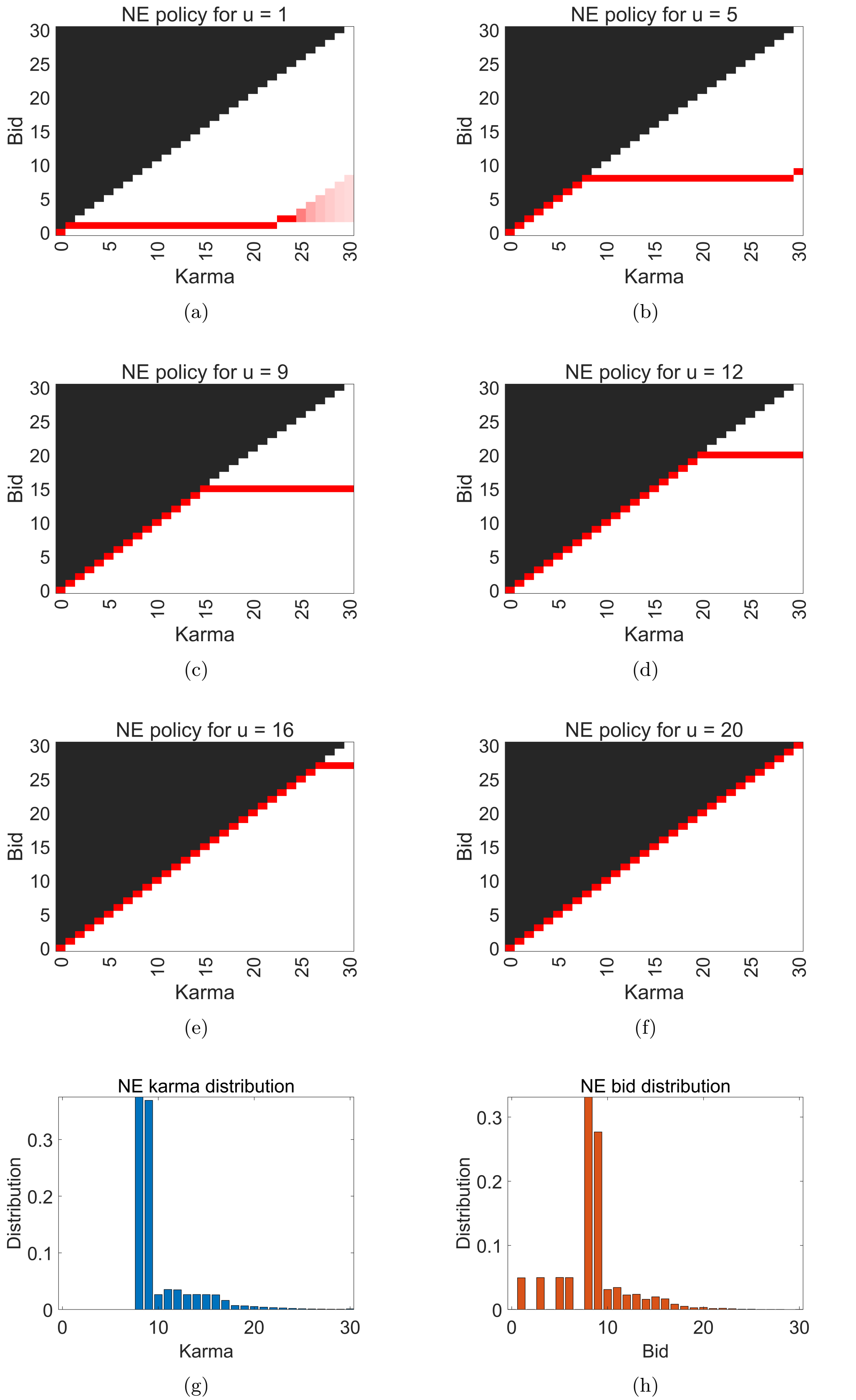
F Experiment with different fairness functions

Here we will use two different social welfare functions to replace the original *weighted proportional fairness function*. First, we choose the *sum of supply* $\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f \in F} S_f$ as the social welfare function. Correspondingly, we choose $\zeta_\tau[u, b](d, \pi) = \min\{s_{f_\tau}[b], u\}$ as the immediate reward function for all clients. If we keeps all other setting the same as in Section 4.1.1, then we could calculate the Stationary Nash Equilibrium as shown in Figure 13.

The evaluation of performance is shown in Figure 14. The experiment is conducted twice with the Monte Carlo method and analytical method respectively. We can see that our karma economy still works under the social welfare function of the *sum of supply*.

Then, we choose the *fairness with quadratic loss* $\mathcal{F}(\mathbf{S}, \mathbf{U}) = \sum_{f \in F} -(S_f - U_f)^2$ as the social welfare function. Correspondingly, we choose $\zeta_\tau[u, b](d, \pi) = -\left(\min\{s_{f_\tau}[b] - u, 0\}\right)^2$ as the immediate reward function for all clients. If we keeps all other setting the same as in Section 4.1.1, then we could calculate the Stationary Nash Equilibrium as shown in Figure 15.

The evaluation of performance is shown in Figure 16. The experiment is conducted twice with the Monte Carlo method and analytical method respectively. We can see that our karma economy also works under the social welfare function of the *fairness with quadratic loss*. The two experiments conducted in this section could reveal the robustness of our karma-based bandwidth allocation system against different social welfare functions.



(a) (b) (c) (d) (e) (f) (g) (h)

Figure 13: The Stationary Nash Equilibrium for the sum of supply as social welfare function

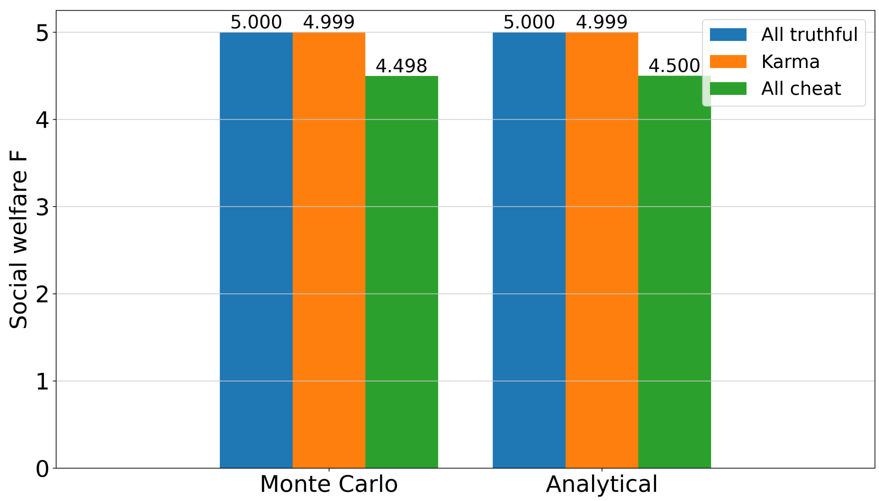


Figure 14: Evaluation of the karma economy for the sum of supply as social welfare function

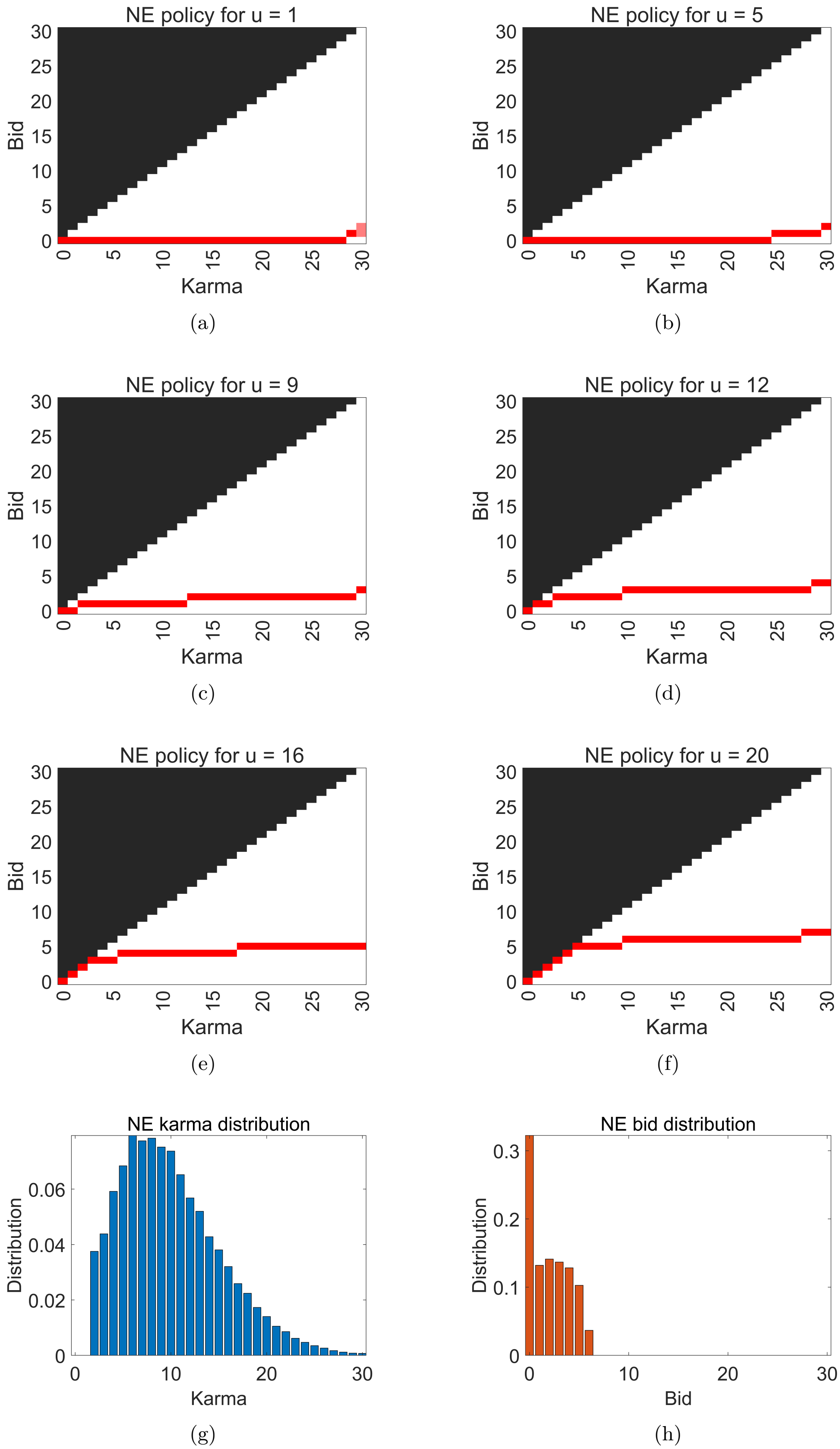


Figure 15: The Stationary Nash Equilibrium for the quadratic loss as social welfare function

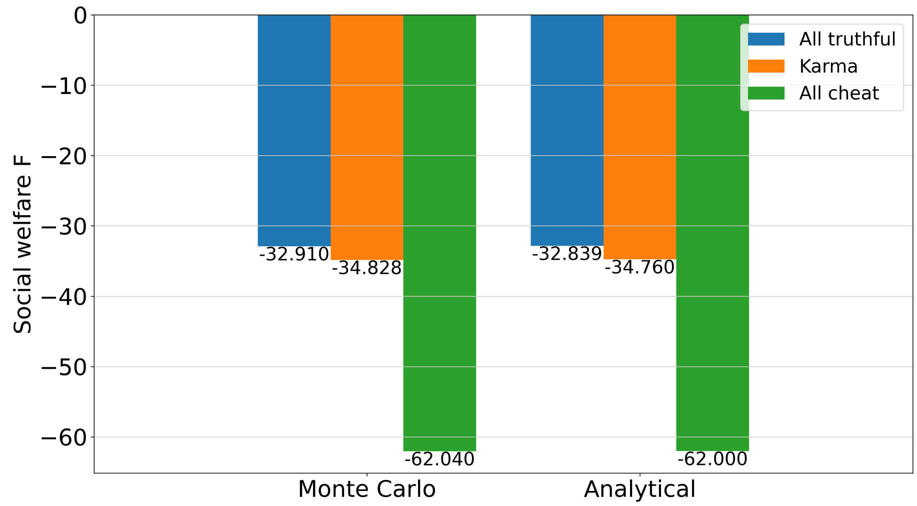


Figure 16: Evaluation of the karma economy for the quadratic loss as social welfare function

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