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Abstract

We introduce banks that issue liquid deposits backed by bonds and capital into an otherwise standard cash-in-advance economy. Liquidity transformation by banks increases aggregate consumption and investment relative to a cash-only economy but can also lead to inefficient overinvestment. Furthermore, liquidity transformation can lead to multiple steady-state equilibria with different interest rates and real outcomes. Whenever multiple equilibria exist, one of them constitutes a 'liquidity trap', in which nominal bond rates equal zero and banks are indifferent between holding bonds and reserves. Whether economic activity is higher in a liquidity trap or in a (coexisting) equilibrium with positive interest rates is ambiguous, but the liquidity trap equilibrium is more likely to go in hand with overinvestment.

Keywords: *Banks, Liquidity, Monetary Policy, Zero-Lower Bound.*
JEL codes: E4, E5.

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1 Introduction

In the years following the financial crisis of 2007/08, many advanced economies spent long periods in a liquidity trap – a situation in which returns on government bonds and other similar assets are at the zero lower bound, and agents become indifferent between holding these assets and fiat money. This episode has inspired numerous papers aimed at better understanding the interaction between the financial system and the macroeconomy in general, and liquidity traps more specifically. The present paper is closely related, in particular, to the New Monetarist literature on the topic, which has significantly contributed to our understanding of liquidity traps (e.g. Williamson (2012, 2016), Andolfatto and Williamson (2015), Rocheteau et al. (2018) and Altermatt (2022)). These papers highlight that interest rates on different assets have a liquidity premium component, with liquidity traps describing situations in which nominal rates on the most liquid assets are at the zero lower bound. A key insight from this literature is that liquidity traps can arise endogenously as a result of a 'shortage' of liquid assets, in which case the way out of the trap is to increase the supply of these assets, for instance, through open-market bond sales (Rocheteau et al. (2018)).

In this paper, we first replicate some of the main results from the New Monetarist literature on liquidity traps within a cash-in-advance (CIA) model, which demonstrates that these results do not depend on a particular modelling choice. In particular, we highlight that a liquidity trap equilibrium is generally not equivalent with the Friedman rule.[1] Then, we go beyond the existing literature by showing that even if policy is such that the supply of liquid (government-) bonds is relatively scarce, a liquidity trap may not be the only equilibrium; rather, it may coexist with equilibria where nominal bond rates are strictly positive. Therefore, a liquidity trap may not only arise due to a policy-induced shortage of liquid assets, but may also be a result of equilibrium selection.[2]

We start with a CIA economy as in Cooley and Hansen (1989), where firms use capital and labour to produce a consumption good sold (only) against fiat money.

[1] While the former describes any situation where the nominal interest rate on bonds is equal to zero, the latter only describes the special case where that is true *and* bonds pay no liquidity premium. Thus, in a liquidity trap, the opportunity cost of holding money instead of bonds is zero, but liquidity may still be scarce overall as both money and bonds pay a liquidity premium.

[2] Benhabib et al. (2001) show that active monetary policy adhering to the Taylor principle can also lead to multiple steady-state equilibria, with the liquidity trap equilibrium being one of them. In Benhabib et al. (2001), the root cause of equilibrium multiplicity are self-fulfilling changes in inflation, which is different from the mechanism in our paper. Importantly, the liquidity trap equilibrium in our paper is not associated with deflation.

To this setup, we introduce banks, which can issue liquid deposits backed by government bonds and capital (think of the latter as loans to firms).[3] We assume some households still have to pay with fiat money even in the presence of banks, which ensures that fiat money issued by the government continues to have value. While only deposits and fiat money are directly liquid – meaning that households can use them to pay for consumption – bonds and capital may still carry a liquidity premium due to their 'indirect' liquidity value, which results from the fact that banks can finance these assets by issuing deposits.[4] Liquidity transformation by banks increases aggregate consumption and investment relative to a cash-only economy à-la Cooley and Hansen (1989) for two reasons. First, as usual in economies with a liquidity-in-advance constraint, aggregate demand on goods markets depends on the average opportunity cost of carrying liquid assets, which decreases when (some) households can pay with interest-bearing deposits. Second, when banks can finance capital (partly) with deposits, the resulting liquidity premium on capital increases investment, which in some cases leads to inefficient overinvestment.

A key result is that liquidity transformation can lead to multiple steady-state equilibria, which is due to the fact that households and firms interact both on goods markets and on the financial side of the economy (through banks).[5] This dual interaction means that liquidity premia and aggregate demand influence each other. On the one hand, liquidity premia affect the deposit rate, which determines households' opportunity cost of carrying liquidity and thereby influences aggregate demand. On the other hand, changes in aggregate demand impact investment and thus the supply of indirectly liquid assets, which, in turn, affects liquidity premia. This mechanism enables self-fulfilling prophecies on goods markets that are accompanied by changes in interest rates. Consider, for instance, a situation where interest rates are high. If firms expect high demand on goods markets, they may be willing to invest a lot despite the high interest rates, which make investment more expensive. In turn, high investment by firms implies a high supply of indirectly liquid capital to banks, which in turn implies that liquidity premia on assets with liquidity value are low (i.e., interest rates on these assets are high). Finally, these low liquidity premia

[3] Aruoba et al. (2011) develop a New Monetarist model that shares some of the basic ideas with Cooley and Hansen's model. Altermatt et al. (2022) introduce banks into the Aruoba et al. (2011) setup, and their model has many similarities with the one in the present paper. Besides using a different modelling approach, the two papers have a different focus: while Altermatt et al. (2022) study bank runs, the present paper abstracts from runs and focuses on steady-state equilibria.

[4] The liquidity premium on bonds is higher since banks can finance their entire bond holdings with deposits, while capital holdings can only be partly funded with deposits.

[5] Similar mechanisms would be at play if households and firms interacted with each other directly on financial markets rather than via banks.

allow banks to pay high interest on deposits, which means that households' opportunity cost of carrying liquidity are low. Hence, households are willing to carry a lot of liquid assets which implies a high demand on goods markets, validating firms' expectations. If instead firms expect low aggregate demand, this similarly allows for a low-interest equilibrium to exist.[6]

In general, we show that there can be three types of steady-state equilibria in our economy. First, if the supply of indirectly liquid bonds and capital is plentiful relative to banks' demand for these assets, the economy is in a *fundamental equilibrium* (FE), in which bonds and capital do not exhibit a liquidity premium, meaning their real return equals the discount rate. The deposit return equals the discount rate as well, such that carrying deposits entails no opportunity cost. Second, when banks' demand for indirectly liquid assets exceeds the supply even when nominal bond rates are zero, the economy is in a *zero-lower bound equilibrium* (ZE). Bonds and deposits pay no interest in a ZE, such that the opportunity cost of carrying deposits and money is the same; nevertheless, as long as capital has indirect liquidity value, the resulting liquidity premium on capital means that investment and consumption will be higher than in a cash-only economy. Lastly, when the asset market clears at some intermediate interest rate, the economy is in an *interior equilibrium* (IE), where interest rates on bonds and deposits are in between the zero-lower bound and the discount rate.

After analysing under what conditions which type of equilibrium exists, we show that all three equilibria may coexist for the same fundamental and policy parameters, and we discuss what makes equilibrium multiplicity more likely. We also show that in the case of multiple steady-state equilibria, it is ambiguous in which equilibrium aggregate consumption and investment are highest. To see why, let us compare the FE with the ZE. On the one hand, the opportunity cost of carrying liquidity is lower in the FE due to the higher deposit rates which, taken by itself, would imply higher economic activity in the FE. On the other hand, when capital has liquidity value, the liquidity premium on capital in a ZE implies that investment and real wages in the ZE are higher, which, however, can also be a sign of inefficient overinvestment.

We also study the transitional dynamics of the model to determine whether multiple equilibria exist. Note that the existence of multiple steady-state equilibria does not imply equilibrium multiplicity; for that, we further need that for some initial level of capital, multiple stable saddle paths exist. To answer this question, we apply a variation of a backward-shooting algorithm and find that while the multiplicity

[6] This is a slight simplification of the mechanism since demand on goods markets also depends on real wages, which, in turn, depend on firms' investment choices.

of steady states does not always imply the existence of multiple equilibria in our model, there are indeed parameters for which multiple equilibria exist, such that from some initial level of capital, the economy may either transition to an FE or a ZE equilibrium.

In our model, liquidity transformation by banks can cause bonds and capital to carry a liquidity premium, which has repercussions on aggregate economy activity. This relates our paper to a broader macro-financial literature studying models in which assets other than fiat money carry a liquidity premium, either because they can be used directly to settle transactions (Lagos and Rocheteau (2008), Andolfatto and Williamson (2015), Rocheteau et al. (2018), Altermatt et al. (2023)), they can be sold against money when needed (Geromichalos and Herrenbrueck (2022)) or, as in our model, banks can finance them by issuing liquid deposits (Williamson (2012, 2016), Altermatt (2022), Keister and Sanches (2023)). In all of these papers, aggregate demand on goods markets is positively related to the aggregate supply of assets with liquidity value: the more abundant (directly or indirectly) liquid assets, the lower liquidity premia and thus the lower the opportunity cost of holding liquid assets required to settle transactions. What distinguishes our paper is that the interaction between asset supply and aggregate demand goes in both directions. Specifically, in our model, an increase in aggregate demand spurs capital investment, which then increases the aggregate supply of interest-paying assets that can be used to back deposit issuance. This mutual interaction between aggregate asset supply and aggregate demand is crucial for the multiplicity result we obtain.[7]

Outline. The rest of this paper is structured as follows: Section 2 presents the environment; Section 3 discusses the equilibrium in an economy without banks; Section 4 introduces banks; Section 5 discusses the banking equilibrium, including cases where multiple steady-state equilibria coexist; Section 6 discusses the transitional dynamics of the model to determine whether multiple equilibria are possible; finally, Section 7 concludes.

[7] The only other paper featuring this two-way interaction we are aware of is Geromichalos and Herrenbrueck (2022). Similar mechanisms are also present in Altermatt et al. (2022) but are not the subject of analysis in that paper.

2 Environment

Time is discrete, indexed by $t = 0, 1, 2, ..$, and continues forever.[8] The economy is populated by a unit mass of infinitely-lived households. There is a single good in the economy, which can be consumed by households or converted into capital one for one. The good is produced according to

$$Y = F(K_{-1}, L), \tag{1}$$

where K_{-1} is capital brought into the current period, L is current aggregate labour supply, and F is a constant returns to scale (CRS) production function exhibiting the usual neoclassical properties. The aggregate resource constraint is

$$C + K = Y + (1 - \delta)K_{-1}, \tag{2}$$

where C is aggregate consumption by households, and $\delta \in (0, 1]$ is the depreciation rate of capital. Both C and K are subject to nonnegativity constraints. Output is produced by a representative firm, which rents capital and labour from households at real prices ψ and w, respectively. We define the capital-labour ratio as $\kappa \equiv K_{-1}/L$ and $f(\kappa) \equiv F(\kappa, 1)$. Then, CRS of (1) implies

$$\psi = f'(\kappa), \tag{3}$$

$$w = f(\kappa) - \kappa f'(\kappa), \tag{4}$$

and zero profits for the firm.

Besides capital K, two other storable objects exist in the economy: fiat money M and one-period nominal bonds B, both issued by the government. Households may store wealth in M, B, or K. Let ϕ denote the value of money (in terms of numeraire Y), and let i denote the net nominal interest rate on bonds.[9] We denote the gross inflation rate by $1 + \pi \equiv \phi_{-1}/\phi$.

Households' lifetime preferences are

$$U = \sum_{t=0}^{\infty} \beta^t \Big[u(c_t) - l_t\Big], \tag{5}$$

[8]To reduce notational clutter, we will mostly omit time subscripts. We use subscripts -1 and $+1$ to denote previous-period and next-period variables, respectively.

[9]The value of money ϕ denotes how many units of numeraire Y one unit of money buys, meaning that $\frac{1}{\phi}$ is the price level. The nominal rate i_{+1} denotes the net return in terms of money a bondholder earns when holding a bond from the current to the next period.

where $\beta \in (0,1)$ denotes the households' fundamental discount factor, while c_t and l_t denote period-t household consumption and labour supply, respectively.[10] We assume $u'(c) > 0 > u''(c)$ and $\lim_{c\to 0} u'(c) = \infty$. Households are subject to the budget constraint

$$c + k + \phi(m + b) \leq \big[\psi + (1-\delta)\big]k_{-1} + wl + \phi\big(m_{-1} + (1+i)b_{-1} + \tau\big),$$

where (m, b, k) denotes a household's portfolio of money, bonds, and capital, and τ is a nominal lump-sum transfer from the government (or tax if negative). Further, as is standard in the CIA literature, we assume that households are subject to the constraint

$$c \leq \phi m_{-1}, \tag{6}$$

i.e. consumption has to be financed with money carried over from the last period.[11] For now, households can only pay with fiat money M. In Section 4, we will introduce banks and assume that some households can pay with bank deposits, which themselves need to be backed by bonds B and capital K.

There is a government issuing money M and nominal government bonds B. We denote by $\mathcal{M} \equiv \phi M$ and $\mathcal{B} \equiv \phi B$ the real supply of money and government bonds, respectively. The government lets the money supply grow at a constant rate γ,

$$M = (1+\gamma)M_{-1}, \tag{7}$$

where we assume $\gamma > \beta - 1$, which implies that steady-state inflation will be above the Friedman rule. Next, the government determines the real quantity of bonds as a function of output, the real money supply and the real interest rate,

$$\phi B = \mathcal{B}(\mathcal{M}, Y, i_{+1}, \pi_{+1}), \tag{8}$$

where $\mathcal{B}$ is some nonnegative, differentiable function. We leave the exact fiscal rule $\mathcal{B}(\cdot)$ open for the moment. The government budget constraint then writes

$$\phi(M + B) = \phi(M_{-1} + (1+i)B_{-1} + \tau), \tag{9}$$

[10] We use this quasilinear utility function in order to keep the model tractable in the sense that portfolio decisions are independent of beginning-of-period wealth.

[11] We adopt the timing convention that the goods market opens before the asset market. That is, after production has taken place at the start of a given period, households first use their liquid assets brought over from last period to purchase the produced consumption goods and then decide on the asset portfolio which they carry over to next period. An illiquid asset purchased at date t pays out in the asset market of date $t+1$. Date-t wages and transfers are paid out in the asset market of date t.

where the transfer τ adjusts such that (9) holds given (7) and (8).

For reasons that will become clear later on, we sometimes call $w/(\beta w_{-1})$ the (gross) discount rate, and we define

$$1 + \iota \equiv \frac{1+\pi}{\beta}\frac{w}{w_{-1}} \tag{10}$$

as the 'Fisher rate', which, loosely speaking, is the nominal interest rate that compensates for inflation and discounting.

The first-best allocation. As a benchmark, consider the first-best allocation, which maximises households' lifetime utility (5) subject to the economy's resource constraint (1)-(2), $c_t = C_t$, $l_t = L_t$, and given some initial capital stock K_{-1}. The first-best allocation satisfies

$$u'(c_t) = \frac{1}{F'_L(K_{t-1}, L_t)} \quad \text{and} \quad u'(c_t) = \beta u'(c_{t+1})(F_K(K_{t+1}, L_t) + 1 - \delta).$$

Denoting c^* and κ^* as the first-best steady-state values of c and κ, we find that

$$u'(c^*) = \frac{1}{w(\kappa^*)} \quad \text{and} \quad f'(\kappa^*) + 1 - \delta = \frac{1}{\beta}, \tag{11}$$

where $w(\kappa^*) = f(\kappa^*) - f'(\kappa^*)\kappa^*$ is the real wage when $\kappa = \kappa^*$.

3 Equilibrium without Banks

In this section, we briefly discuss the model without banks, which serves as a useful reference point. In the unbanked economy, households choose their asset portfolio (m, b, k) each period. The representative household's problem is given by

$$\begin{aligned} &V(m_{-1}, b_{-1}, k_{-1}) = \max_{c,l,m,b,k\geq 0} u(c) - l + \beta V(m, b, k) \\ &\text{s.t.} \quad c + k + \phi(m+b) \leq \big[\psi + (1-\delta)\big]k_{-1} + wl + \phi\big(m_{-1} + (1+i)b_{-1} + \tau\big) \\ &\text{s.t.} \quad c \leq \phi m_{-1}. \end{aligned}$$

We ignore the non-negativity constraints on the household's choice variables, and we denote the Lagrange multipliers on the budget constraint and the liquidity constraint by λ and μ, respectively.[12] The first-order conditions of this problem are:

$$l: \qquad \lambda = \frac{1}{w}$$

[12]It is safe to ignore the non-negativity constraints as they never bind in equilibrium: if either m or k were zero, consumption would necessarily be zero as well, but this is at odds with the Inada conditions. For bonds, demand cannot be negative in equilibrium as supply is positive, and prices adjust such that the market clears.

$$
\begin{aligned}
c: &\quad u'(c) = \lambda + \mu \\
m: &\quad \phi\lambda = \beta\phi_{+1}[\lambda_{+1} + \mu_{+1}] \\
b: &\quad \phi\lambda = \beta\phi_{+1}(1 + i_{+1})\lambda_{+1} \\
k: &\quad \lambda = \beta\big[\psi_{+1} + (1-\delta)\big]\lambda_{+1}
\end{aligned}
$$

Rearranging these yields:

$$\frac{1+i}{1+\pi} = \psi + 1 - \delta = \frac{1}{\beta}\frac{w}{w_{-1}} \tag{12}$$

$$u'(c) = \frac{1+i}{w} \tag{13}$$

$$\phi m \geq (1+\pi_{+1})c_{+1} \quad \text{with equality if } i_{+1} > 0 \tag{14}$$

Condition (12) contains a no-arbitrage condition stating that the real return on government bonds must equal the return on capital net of depreciation, and it states that both these rates must equal the discount rate. This implies $i = \iota$, i.e. the equilibrium bond rate equals the Fisher rate.[13] Next, condition (13) pins down equilibrium consumption as a function of i and w; it shows that the nominal rate i (the opportunity cost of holding money) creates a wedge between the marginal utility of consumption and the opportunity cost of leisure. Finally, condition (14) simply states that real money balances need to be sufficient to cover consumption expenses.

For the remainder of this section, we impose steady state where all real variables are constant. Constant real money supply $\mathcal{M}$ implies $\gamma = \pi$, i.e. steady-state inflation equals the money growth rate. Since real wages are constant in the steady-state, we get from (12) that

$$1 + i = 1 + \iota = \frac{1+\gamma}{\beta}. \tag{15}$$

Given our assumption that $\gamma > \beta - 1$, we get from (15) that $i > 0$, which implies that the CIA constraint (14) binds. Next, from (3) and (12), we get that κ is pinned down by

$$\frac{1}{\beta} = f'(\kappa) + 1 - \delta. \tag{16}$$

Finally, from (4) and (13), we get that steady-state consumption is determined by

$$u'(c) = \frac{1+i}{w} = \frac{1+i}{f(\kappa) - \kappa f'(\kappa)}. \tag{17}$$

[13] Quasilinear preferences imply that households are willing to hold an arbitrary amount of assets paying the Fisher rate. They would not be willing to hold an asset paying less than the Fisher rate, and they would choose to hold an infinite amount of assets paying more than the Fisher rate.

Definition 1. *A steady state in the economy without banking is given by* i, κ, c *that solve equations* (15)-(17).

Proposition 1. *There exists a unique steady state in the economy without banking.*[14]

We get from (11) and (16)-(17) that in the steady-state equilibrium of the unbanked economy, $\kappa = \kappa^*$ and $c < c^*$. Steady-state consumption is below first-best as a result of the CIA friction. Notice that, while κ is at its first-best level, the fact that aggregate consumption C (and thus aggregate output Y) is below first-best implies that the aggregate capital stock K is below first-best as well. Note also that the real bond supply $\mathcal{B}$ has no effect on the steady-state equilibrium in the unbanked economy. Policy only matters through the money growth rate, which determines the opportunity cost of holding money and thus affects equilibrium consumption, output, and the aggregate capital stock.

The version of the model presented in this section represents the standard way of thinking about monetary policy in much of the monetary literature. In particular, there is no difference between the bond rate i and the Fisher rate ι. This, in turn, implies that a zero-lower bound equilibrium ($i = 0$) is equivalent to running the Friedman rule ($\iota = 0$) and delivers the first-best. In the next section, we demonstrate why this way of thinking may be misleading when bonds (and possibly capital) carry a liquidity premium.

4 Banking Equilibrium

We now introduce a representative bank that issues nominal bank deposits D, which can be backed by government bonds B, capital K, and money M.[15] In any given period some, but not all, households can pay for their consumption expenditures with bank deposits. Specifically, with banking, households' liquidity constraint is given by

$$c \leq (1 - \Theta)\phi m_{-1} + \Theta \phi d_{-1},$$

[14] Unless mentioned otherwise, all proofs are given in Appendix B.

[15] The role of banking introduced here, namely to transform illiquid assets B and K into liquid deposits, is the same as in Altermatt (2022). While our results hinge on bonds and capital having liquidity value in the sense that they can be used (directly or indirectly) to pay for consumption, the precise way how these assets become liquid is not crucial. While we believe that banks providing liquidity transformation is a natural way to model this, similar results would obtain if instead capital and bonds were directly liquid as, e.g. in Altermatt et al. (2023), or if they could be traded on secondary asset markets after idiosyncratic shocks are realised as in Geromichalos and Herrenbrueck (2022).

where d_{-1} are bank deposits brought over from last period, and $\Theta \in \{0,1\}$ is an i.i.d. idiosyncratic shock. In any given period, an individual household finds itself in state $\Theta = 1$ with probability $\theta \in (0,1)$, in which case the household can pay for consumption by transferring bank deposits.[16] With probability $1-\theta$, a household finds itself in state $\Theta = 0$ and can only pay for consumption with fiat money. Let c_Θ denote consumption of households in state $\Theta \in \{0,1\}$, such that aggregate consumption satisfies

$$C = (1-\theta)c_0 + \theta c_1. \tag{18}$$

Each period, all households contribute an identical amount of funds to the bank, which the bank then invests into money, bonds and capital. At the start of the next period when households' idiosyncratic states are realised, the bank provides a certain amount of money to households in state $\Theta = 0$ (think of this as households in state $\Theta = 0$ withdrawing money from the bank) while it provides a certain amount of bank deposits (backed by bonds, capital, and possibly money) to households in state $\Theta = 1$. Since there is no aggregate uncertainty, by a law of large numbers, in any given period a fraction θ of the bank's depositors will be in state $\Theta = 1$ while a fraction $1-\theta$ will be in state $\Theta = 0$. To abstract from bank run equilibria, we assume the bank observes the realisation of the states Θ.[17]

In each period, the bank chooses its asset portfolio (m, b, k) and the payouts given to households (depositors) in state $\Theta \in \{0,1\}$ so as to maximise the expected utility of its depositors. The bank's problem can be expressed as

$$V(m_{-1}, b_{-1}, k_{-1}) = \max_{c_0, c_1, l, m, b, k \geq 0} \theta u(c_1) + (1-\theta)u(c_0) - l + \beta V(m, b, k)$$

subject to the constraints:

$$\begin{aligned} &\theta c_1 + (1-\theta)c_0 + k + \phi(m+b) \\ &\qquad \leq \big[\psi + (1-\delta)\big]k_{-1} + wl + \phi\big(m_{-1} + (1+i)b_{-1} + \tau\big) && (19) \\ &(1-\theta)c_0 \leq \phi m_{-1} && (20) \\ &\theta c_1 \leq \phi d_{-1} = (1+i)\phi b_{-1} + \chi(\psi + 1 - \delta)k_{-1} + [\phi m_{-1} - (1-\theta)c_0] && (21) \end{aligned}$$

Condition (19) is the budget constraint, which states that total consumption expenditure plus investments in money, bonds and capital cannot exceed the revenue from the previous asset portfolio plus households' income from wage payments and

[16]The assumption that households in state $\Theta = 1$ can only pay with deposits simplifies the exposition and is without loss of generality. As long as banks are fully competitive, which we assume throughout the paper, using deposits to pay for consumption always weakly dominates using money as banks may use money to back deposits.

[17]Altermatt et al. (2022) show how self-fulfilling panics can occur in a similar model.

transfers.[18] Condition (20) is the liquidity constraint for households in state 0, saying that their consumption needs to be financed entirely with money brought over from last period. Finally, (21) is the liquidity constraint for households in state 1, saying that their consumption needs to be financed with bank deposits brought over from last period; the deposits, in turn, need to be backed by bonds, capital, and money not paid out to households in state 0 (think of the latter as excess reserves). The parameter $\chi \in [0, 1]$ denotes the fraction of the bank's capital holdings that can be used to back deposits.[19] In what follows, we will sometimes say that capital is 'illiquid' when $\chi = 0$, i.e. when capital cannot be used to back deposit issuance. Conversely, we will say that capital is 'liquid' whenever $\chi > 0$, keeping in mind that, in our model, capital is only indirectly liquid via liquidity transformation by banks. Notice that the deposit return equals the return on the bank's asset portfolio used to back deposits. In this manner, the return from the bonds and (a fraction χ of) the capital brought into period t by the bank becomes (indirectly) available to finance period-t consumption.[20]

Denoting λ as the Lagrange multiplier for the budget constraint, μ_Θ as the multipliers for the liquidity constraints of households in state $\Theta \in \{0, 1\}$, and using our definition of the Fisher rate from (10), the first-order conditions of the bank's problem are:

$$l: \quad \lambda = \frac{1}{w} \tag{22}$$

$$c_0: \quad u'(c_0) = \lambda + \mu_0 + \mu_1 \tag{23}$$

$$c_1: \quad u'(c_1) = \lambda + \mu_1 \tag{24}$$

$$m: \quad \iota = w(\mu_0 + \mu_1) \tag{25}$$

$$b: \quad 1 + \iota = (1 + i)(1 + \mu_1 w) \tag{26}$$

$$k: \quad 1 + \iota = (1 + \pi)(\psi + 1 - \delta)(1 + \chi\mu_1 w) \tag{27}$$

From (25) and (26), we find that

$$\mu_0 = \frac{1}{w}\left(\frac{i(1 + \iota)}{1 + i}\right) \quad \text{and} \quad \mu_1 = \frac{1}{w}\left(\frac{\iota - i}{1 + i}\right). \tag{28}$$

[18] Given that the bank maximises the expected utility of households, it is without loss of generality to assume that households contribute their entire income to the bank.

[19] Put differently, when deposit issuance is backed by capital, the bank needs to hold $(1/\chi)$ units of capital per issued deposit, where we take the value of χ as exogenous. The share of the capital held by the bank that is not used to back deposits can be regarded as (illiquid) bank equity held in equal proportion by all households.

[20] Since there is no discounting within a period, the purchasing power of deposits in the goods market does not depend on when exactly within the period the interest payments on deposits are made (i.e. whether interest payments occur before or after the goods market is open).

This shows that the liquidity constraint for households in state 0 binds whenever $i > 0$, while the liquidity constraint for households in state 1 binds whenever $i < \iota$. If $i = 0$, the opportunity cost of carrying money and bonds is the same, such that providing liquidity to households in state 0 and state 1 is equally costly. The higher i, the lower the opportunity cost of carrying interest-paying assets and thus the lower the cost of providing liquidity to households in state 1. If $i = \iota$, then carrying bonds entails no opportunity cost, meaning that providing liquidity to households in state 1 is costless and the associated liquidity constraint is slack. Note the difference between the Fisher rate ι and the bond rate i: while ι denotes the interest rate on a perfectly illiquid asset, i denotes the rate on bonds that have liquidity value since banks can use them to back issuance of bank deposits.

Next, combining (28) with (23) and (24), we get

$$u'(c_0) = \frac{1+\iota}{w} \quad \text{and} \quad u'(c_1) = \frac{1}{w}\frac{1+\iota}{1+i}, \tag{29}$$

i.e. equilibrium consumption of households in state 0 is determined by the real wage and the Fisher rate, while equilibrium consumption of households in state 1 is determined by the real wage and the ratio of the bond rate to the Fisher rate.

Next, by combining (27) with our expression for μ_1 from (28) , we get that the return on capital net of depreciation satisfies

$$\psi + 1 - \delta = \frac{1}{\beta}\frac{w}{w_{-1}}\frac{1+i}{1+i+\chi(\iota - i)}. \tag{30}$$

Condition (30) shows that if $i < \iota$ and $\chi > 0$, then capital carries a liquidity premium in the sense that the return on capital is below the discount rate. The liquidity premium on capital is increasing in χ and decreasing in i.

Finally, from the liquidity constraint for households in state 0, (20), we get that the bank's demand for money satisfies

$$\phi m \geq (1+\pi_{+1})(1-\theta)c_{0,+1} \quad \text{with equality if } i_{+1} > 0, \tag{31}$$

and from the liquidity constraint for households in state 1, (21), we get that the bank's demand for money, bonds and capital satisfies

$$\begin{aligned}\phi m + (1+i_{+1})\phi b + (1+\pi_{+1})\chi(\psi_{+1}+1-\delta)k \\ \geq (1+\pi_{+1})[(1-\theta)c_{0,+1} + \theta c_{1,+1}] \quad \text{with equality if } i_{+1} < \iota_{+1}.\end{aligned} \tag{32}$$

We now impose steady state for the remainder of this section, which, as in Section 3, implies $\pi = \gamma$ and

$$1 + \iota = \frac{1+\gamma}{\beta}. \tag{33}$$

Combining (29)-(30) with (3)-(4), we get that in steady-state

$$u'(c_0) = \frac{1+\iota}{f(\kappa) - f'(\kappa)\kappa}, \tag{34}$$

$$u'(c_1) = \frac{1+\iota}{1+i}\frac{1}{f(\kappa) - f'(\kappa)\kappa}, \tag{35}$$

$$\frac{1}{\beta} = (f'(\kappa) + 1 - \delta)\left(1 + \chi\left(\frac{\iota - i}{1+i}\right)\right). \tag{36}$$

Consider first equation (36), which pins down κ. If $\chi = 0$ (capital cannot be used to back deposits), then there is no liquidity premium on capital, i.e. the return on capital is pinned down by fundamental parameters β and δ, and we have $\kappa = \kappa^*$. If instead $\chi > 0$ but $i = \iota$, this still holds, since in this case, the liquidity constraint for households in state 1 is slack, implying that there is no scarcity of investment opportunities for banks to invest in. Finally, if $\chi > 0$ and $i < \iota$, then investment opportunities are scarce, capital carries a liquidity premium, and we have $\kappa > \kappa^*$. In this case, κ is strictly decreasing in i, which results from a no-arbitrage condition: when i increases, the real return to bonds increases, which means the real return to capital must increase as well. Consider next equations (34)-(35), which show that c_0 decreases in ι, while c_1 decreases in the ratio $(1+\iota)/(1+i)$. Note that if $\chi > 0$, then changes in i indirectly affect both c_0 and c_1 by affecting κ and thus the real wage w. Specifically, an increase in i will decrease κ, which then reduces w, thus negatively affecting both c_0 and c_1. An increase in i will therefore negatively affect c_0, while the effect on c_1 is ambiguous: on the one hand, the real wage falls, which as such has a negative effect on c_1, but on the other hand, the opportunity cost of providing liquidity to households in state 1 falls as well, which as such has a positive effect on c_1.

Next, the aggregate resource constraint (2) together with our expression for aggregate consumption (18) implies that

$$L = \frac{(1-\theta)c_0 + \theta c_1}{f(\kappa) - \delta\kappa}. \tag{37}$$

From (31), we get that the bank's demand for real money holdings in the steady state satisfies

$$\mathcal{M} \geq \beta(1+\iota)(1-\theta)c_0 \quad \text{with equality if } i > 0, \tag{38}$$

and from (32), we get that the bank's demand for real assets satisfies

$$\frac{1}{\beta(1+\iota)}\Big[\mathcal{M} + (1+i)\mathcal{B}\Big] + \chi\big(f'(\kappa) + 1 - \delta\big)\kappa L$$
$$\geq \theta c_1 + (1-\theta)c_0 \quad \text{with equality if } i < \iota. \tag{39}$$

Finally, recall that the real bond supply satisfies (8).

Definition 2. *A steady-state equilibrium with banking is given by* c_1, c_0, κ, ι, $i \in [0, \iota]$, $\mathcal{M}$, $\mathcal{B}$, *and* L *that satisfy* (8) *and* (33)-(39).

Compared to the allocation without banking discussed in Section 3, introducing banking affects the economy via two main channels. First, the transformation of illiquid bonds and capital into liquid deposits relaxes the liquidity constraint of households who pay with deposits, which has a positive effect on equilibrium consumption of these households. Second, when capital carries a liquidity premium, the capital-labour ratio κ increases, which in turn increases real wages and positively affects consumption of all households, including those who continue to pay with money. For these two reasons, aggregate consumption C, aggregate output Y and the aggregate capital stock K are are all higher in the steady-state equilibrium with banking compared to the unbanked economy. Note, however, that from (36), we get that $f'(\kappa) < \delta$ when χ is sufficiently high and i is sufficiently low, meaning that capital accumulation can be inefficiently high in the economy with banking.[21]

In the following, we group steady-state equilibria into three equilibrium cases, where the Lagrange multipliers identified in (28) determine these equilibrium cases.

Fundamental equilibrium (FE). An FE is a steady-state equilibrium in which $i = \iota$, which implies $\mu_0 > 0$ and $\mu_1 = 0$, i.e. the liquidity constraint for households in state 0 binds while the one for households in state 1 is slack. The liquidity constraint for households in state 1 is slack since acquiring bonds to back deposits entails no opportunity cost for the bank. From (36) we get that, in an FE, $\frac{1}{\beta} = f'(\kappa) + 1 - \delta$. This shows that capital is fundamentally priced (it does not carry a liquidity premium), and $\kappa = \kappa^*$. Denoting c^{NB} as steady-state consumption in the unbanked economy, we get from (34)-(35) that consumption levels c_0 and c_1 in an FE satisfy $c^{NB} = c_0 < c_1 = c^*$.

Zero-lower bound equilibrium (ZE). A ZE is a steady-state equilibrium in which $i = 0$, which implies $\mu_0 = 0$ and $\mu_1 > 0$. From (36), we get that, in a ZE, $\frac{1}{\beta} = (f'(\kappa) + 1 - \delta)(1 + \chi\iota)$. The term $(1 + \chi\iota)$ reflects that, as long as $\chi > 0$, the bank overinvests in capital to provide liquidity to households, implying that capital is priced above its fundamental value, and we have $\kappa > \kappa^*$. From (34)-(35), we find that in a ZE $c_0 = c_1$ since the cost of providing liquidity is the same for all households. For a given κ, consumption levels in the ZE are the same as in the

[21] The result that there can be overinvestment when capital has liquidity value is well known in the New Monetarist literature (Lagos and Rocheteau (2008)). To the best of our knowledge, we are the first to show this result in a CIA model.

unbanked economy. However, as long as $\chi > 0$, κ is higher than in the unbanked economy, which implies that the real wage w and thus consumption levels c_0 and c_1 are higher as well.[22] Note also that the ZE is the only equilibrium in which the bank may choose to hold excess reserves, i.e. it may hold more money than necessary to pay for consumption of households in state 0.

Interior equilibrium (IE). Finally, an IE is a steady-state equilibrium in which $i \in (0, \iota)$, which implies $\mu_1 > 0$ and $\mu_0 > 0$, meaning that the liquidity constraint binds both for households in state 0 and in state 1. Providing liquidity to households in state 1 is costly for the bank since the bond rate i does not fully compensate for inflation and discounting; providing liquidity to households in state 0 (by accumulating non-interest bearing money) is even more costly. From (34)-(36), we get that $c_1 > c_0$ in an IE, which reflects that providing liquidity to households in state 0 is more costly than to those in state 1. Furthermore, as long as $\chi > 0$, we have $\kappa > \kappa^*$ and $c_0 > c^{NB}$, meaning that (as a result of higher real wages) consumption of all households is higher than in the unbanked economy.

In the following section, we discuss under which conditions different equilibrium cases coexist. But before doing so, we want to relate the results we have just presented to the discussion at the end of Section 3. As shown, once banking is introduced, which in turn allows for bonds and capital to attain a liquidity premium, the bond rate i may differ from the Fisher rate ι. This implies in particular that a zero-lower bound equilibrium ($i = 0$) is generally not equivalent to a Friedman rule equilibrium ($\iota = 0$).[23] As the next section will show, i can be varied by changing the real amount of bonds in circulation, $\mathcal{B}$. Since we can think of $\mathcal{B}$ as being affected by monetary policy interventions such as open market operations, we view i as the policy rate, while ι depends on the long-run inflation target γ.[24] For a more in-depth discussion of how to interpret i and ι in models similar to the one presented here and what this implies for various puzzles in the literature, see Herrenbrueck and

[22] For $\chi = 0$, we have $\kappa = \kappa^*$ and $c_0 = c_1 = c^{NB}$, i.e. the allocation in a ZE is identical to the steady-state equilibrium of the unbanked economy.

[23] While the Friedman rule implies that the economy must be at the zero-lower bound, the converse is not true: a zero-lower bound equilibrium is any situation where $i = 0$, while ι may be strictly positive. Note also that, as in the unbanked economy, the Friedman rule would implement the first-best allocation in the economy with banking.

[24] Additional reasons to regard i and not ι as the rate set by monetary policy are that: (i) i is observable in reality while ι typically is not, since almost all assets have some degree of liquidity; (ii) while bond interest rates are not exactly identical to policy rates in reality, actual policy rates such as the Fed funds rate behave similarly to bond rates.

Wang (2023).

5 (Co-)Existence of Equilibrium Cases

We now discuss the conditions under which the different equilibrium cases exist, and whether there is a unique equilibrium. Uniqueness here means that given the economy's fundamental parameters as well as fiscal and monetary policies, we can determine which of the three equilibrium cases occurs. If there is coexistence of equilibrium cases, there is no clear mapping from a given set of policies and parameters to equilibrium cases, and thus several real outcomes are possible for the same underlying economic conditions.

To derive existence conditions for the different equilibrium cases, it will be useful to define

$$\mathcal{A}^s(i) \equiv \mathcal{B}(i) + \frac{\chi(1+\iota)}{1+i+\chi(\iota - i)} K(i) \tag{40}$$

as the 'liquidity adjusted' real asset supply, which, loosely speaking, equals the sum of the real bond supply plus the share of the capital stock that can be used to back deposit issuance. For $\chi = 0$ (capital has no liquidity value), we have $\mathcal{A}^s(i) = \mathcal{B}(i)$, and for $\chi = 1$ (capital has the same liquidity value as bonds), we have $\mathcal{A}^s(i) = \mathcal{B}(i) + K(i)$. The dependence of all variables on i has been made explicit in (40). With regard to the real bond supply, the only thing that will matter for our purposes is how it changes with i; we thus express the real bond supply as a function of i only, capturing both direct and indirect (e.g., via $\mathcal{M}$ or Y) effects of i on $\mathcal{B}$. Further, by using $K = \kappa L$ together with condition (37), we get

$$K(i) = q(\kappa(i))\, C(i), \quad \text{where} \quad q(\kappa) \equiv \frac{\kappa}{f(\kappa) - \delta\kappa}. \tag{41}$$

Notice that $q'(\kappa) > 0$.[25] Expression (41) highlights that the aggregate capital stock K depends both on the capital-labour ratio κ as well as on aggregate consumption C. Changes in i affect K both via their effect on κ and via their effect on C. For instance, as long as $\chi > 0$, an increase in i reduces κ (see equation (36)), which, taken by itself, has a negative effect on K. Whether C increases or decreases in i is ambiguous. On the one hand, an increase in i reduces the real wage w (via the effect of i on κ), which negatively affects both c_0 and c_1. On the other hand, an increase in i reduces the cost of providing liquidity to households paying with deposits, which has a positive effect on c_1.

[25] Denoting $\hat{\alpha}(\kappa) \equiv [\kappa f'(\kappa)]/f(\kappa)$ as the capital share (the fraction of output going to capital owners), we have $q(\kappa) = [\hat{\alpha}(\kappa)]/[f'(\kappa) - \delta\hat{\alpha}(\kappa)]$, i.e. q is increasing in the capital share.

Next, define

$$\mathcal{A}^d(i) \equiv \frac{\beta(1+\iota)}{1+i}\theta c_1(i) \tag{42}$$

as the real asset demand resulting from the bank's deposit issuance required to grant households in state 1 a given consumption level $c_1(i)$. Expression (42) highlights that, as usual, changes in i affect the asset demand $\mathcal{A}^d(i)$ both via a substitution effect and an income effect. On the one hand, an increase in i reduces the cost of providing liquidity to households in state 1 relative to those in state 0, which effectively makes consumption in state 1 cheaper relative to consumption in state 0 and will increase the optimal ratio c_1/c_0; taken by itself, this has a positive effect on $\mathcal{A}^d(i)$. On the other hand, an increase in i means the bank has to purchase less assets to provide households in state 1 with a given consumption level c_1, which, taken by itself, has a negative effect on $\mathcal{A}^d(i)$. Which effect dominates depends on the curvature of the utility function $u(c)$. Note that, in addition to these standard substitution- and income effects, changes in i affect $\mathcal{A}^d(i)$ also via their effect on the real wage, which in turn affects the desired consumption level c_1.

From (38) and (39), we get that asset market clearing requires:

$$\mathcal{A}^s(i) \begin{cases} \geq \mathcal{A}^d(i) & \text{if } i = \iota \quad \text{(FE)} \\ = \mathcal{A}^d(i) & \text{if } i \in (0, \iota) \quad \text{(IE)} \\ \leq \mathcal{A}^d(i) & \text{if } i = 0 \quad \text{(ZE)} \end{cases} \tag{43}$$

Condition (43) shows that the economy will be in an FE if the asset supply is plentiful relative to the asset demand, it will be in a ZE if the asset supply is scarce relative to the asset demand, and it will be in an IE in an intermediate case.

We can see immediately from expression (43) that a necessary condition for multiple steady-state equilibria is that the difference $\mathcal{A}^s(i) - \mathcal{A}^d(i)$ (weakly) increases in i over at least part of the interval $i \in [0, \iota]$. Intuitively, if an increase in i (i.e. a decrease in asset prices) leads to an increase in the asset supply $\mathcal{A}^s$ relative to the asset demand $\mathcal{A}^d$, then changes in asset prices can be self-fulfilling and multiple equilibria are possible.

Note that the economy can always be put in an FE by saturating it with bonds. Furthermore, multiple equilibria become more likely when the real bond supply $\mathcal{B}(i)$ increases in i, which makes it more likely that $\mathcal{A}^s(i) - \mathcal{A}^d(i)$ increases in i. Our main interest is to study how the liquidity premium on capital can generate multiple equilibria. To focus on this, we will assume for most of the following analysis that the government keeps the real amount of bonds in circulation at some constant level. Additionally, we will also consider the case where the government keeps the ratio of money to total government debt constant, which, as we will show, implies that $\mathcal{B}(i)$

decreases in i. We will now briefly discuss these two policies.

Fixed real bond supply. Suppose the government keeps the real quantity of bonds at some (exogenous) constant level $\overline{\mathcal{B}} \geq 0$. With a constant real bond supply, the aggregate asset supply $\mathcal{A}^s(i)$ is fully driven by changes in the aggregate capital stock $K(i)$. It will be helpful to define

$$Q(i) \equiv \mathcal{A}^d(i) - \mathcal{A}^s(i) + \mathcal{B}(i) = \mathcal{A}^d(i) - \frac{\chi(1+\iota)}{1+i+\chi(\iota - i)} K(i) \tag{44}$$

as the difference between the bank's demand for assets to back deposit issuance and the total capital stock that can be used to back deposits. In other words, $Q(i)$ is the difference between asset demand and the supply of real assets (i.e. without bonds). With a fixed real bond supply, our asset market clearing condition from (43) can be rewritten as:

$$\overline{\mathcal{B}} \begin{cases} \geq Q(i) & \text{if } i = \iota \quad \text{(FE)} \\ = Q(i) & \text{if } i \in (0, \iota) \quad \text{(IE)} \\ \leq Q(i) & \text{if } i = 0 \quad \text{(ZE)} \end{cases} \tag{45}$$

Lemma 1. *Suppose $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$. A sufficient condition for an FE, an IE and a ZE to coexist for some $\overline{\mathcal{B}} > 0$ is that $Q(0) > 0$ and $Q'(i) < 0$ for $i \in [0, \iota]$.*

Intuitively, $Q'(i) < 0$ means that the aggregate capital stock (more precisely, the liquidity value of the aggregate capital stock) increases by more (or decreases by less) than the bank's demand for assets when i increases. Put differently, a fall in asset prices (an increase in i) increases asset supply relative to asset demand, such that the fall in asset prices can be self-fulfilling. Note that if $Q(0) < 0$, then capital is so plentiful that the economy cannot be in a ZE for any nonnegative bond supply.

Constant money-to-debt ratio. Suppose now that, instead of fixing the real bond supply, the government fixes the ratio of money to total nominal government debt. Denoting $\eta \in (0, 1]$ as the constant money-to-debt ratio, we then have $M = \eta(M+B)$ and $\mathcal{B} = \frac{1-\eta}{\eta}\mathcal{M}$. The key difference to before is that the real bond supply now changes with i since equilibrium real money balances depend on i. To see this, note that whenever $i > 0$, the CIA constraint for households in state 0 binds, and steady-state real money balances $\mathcal{M}$ are strictly increasing in c_0 (see (38)). From (34) we get that c_0, in turn, is strictly increasing in the real wage w, which itself is decreasing in i (via the effect of i on κ). Therefore, a higher steady-state interest rate i is associated with lower $\mathcal{M}$. Given that $\mathcal{B}$ is a fixed multiple of $\mathcal{M}$, a higher steady-state interest rate is thus associated with a lower $\mathcal{B}$. Via this channel, an increase in i exerts a negative effect on the aggregate asset supply $\mathcal{A}^s(i)$ under a fixed money-to-debt ratio, which makes it less likely that $\mathcal{A}^s(i) - \mathcal{A}^d(i)$ increases in i. For this reason, multiple equilibria are generally easier to obtain with a fixed real bond supply than with a fixed money-to-debt ratio.

Lemma 2. *Suppose an FE and a ZE coexist for some fixed money-to-debt ratio* $\eta \in (0, 1)$. *Then there exists some* $\overline{\mathcal{B}} > 0$ *such that an FE and a ZE coexist in the same economy when the real bond supply is held constant at* $\overline{\mathcal{B}}$.

5.1 Illiquid Capital

In this subsection, we discuss briefly the case where capital cannot be used to back issuance of bank deposits ($\chi = 0$). The return to capital is then pinned down by β and δ (see condition (36)), and we have $\kappa = \kappa^*$.

Proposition 2. *Suppose* $\chi = 0$, $\frac{\partial \mathcal{B}(i)}{\partial i} \leq 0$ *and* $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. *Then a sufficient condition for uniqueness of the steady-state equilibrium is* $\sigma < 1$.

The intuition for the result in Proposition 2 is as follows. As described above, multiple equilibria are only possible if the asset supply $\mathcal{A}^s(i)$ increases relative to the asset demand $\mathcal{A}^d(i)$ when i increases. With $\chi = 0$, we have $\mathcal{A}^s(i) = \mathcal{B}(i)$. Given $\mathcal{B}'(i) \leq 0$, a necessary condition for multiplicity is then that asset demand $\mathcal{A}^d(i)$ (weakly) decreases in i. When $\chi = 0$, κ is fixed at κ^*, which means real wages do not move with i. Therefore, whether $\mathcal{A}^d(i)$ increases or decreases in i depends only on whether the substitution- or the income effect, as described above, dominates.[26] If σ is low, then households have a high willingness to shift consumption away from state 0 towards state 1 when i increases. This implies that the substitution effect dominates and $\mathcal{A}^d(i)$ increases in i, which precludes multiple equilibria.

The result of Proposition 2 evidently applies to the case where the real bond supply $\mathcal{B}$ is held constant. Furthermore, it is not hard to see that the result also applies to the case where the money-to-debt ratio is kept constant. The reason is that, as described above, a constant money-to-debt ratio implies that a higher steady-state interest rate i is associated with a lower real bond supply $\mathcal{B}$.

Proposition 3. *Suppose* $\chi = 0$, $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$ *and* $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ *with* $\sigma \geq 1$. *Then there exist debt levels* $\overline{\mathcal{B}} > 0$ *such that an FE, an IE and a ZE coexist.*

[26] We show in the proof of Proposition 2 that a sufficient condition for uniqueness with a general utility function is $\frac{c_1'(i)}{c_1(i)} > \frac{1}{1+i}$ over $i \in [0, \iota]$.

Proposition 3 shows that with illiquid capital, multiple equilibria can occur when σ is high. With high σ, households have a low willingness to shift consumption away from state 0 towards state 1 when i increases, implying that the real asset demand $\mathcal{A}^d(i)$ falls in i. In this case, an increase in i (i.e. a decrease in bond prices) goes together with a decrease in the real amount of bonds demanded, which can in turn justify the lower bond prices and opens up the possibility of multiple equilibria.[27]

While we are not aware of a reference which makes this point explicitly, we believe the result that multiple steady-state equilibria may exist when $\sigma \geq 1$ is well understood among economists working on banking and macro-finance since several papers from this literature assume $\sigma < 1$ (examples include Haslag and Martin (2007), Williamson (2012), and Altermatt (2022)). We also focus on $\sigma < 1$ for the remainder of the paper because (i) we want to highlight how a liquidity premium on capital can be a novel source of equilibrium multiplicity, over and above the multiplicity that may result from a strong income effect due to high σ; and (ii) as pointed out in Footnote 27, assuming $\sigma > 1$ implies that money demand increases with inflation, which goes against the empirical evidence.

With illiquid capital and $\sigma < 1$, our economy behaves very similarly to the one in Williamson (2012).[28] Marginal exogenous changes in the real bond supply $\mathcal{B}$ have no effect on the steady-state equilibrium if bonds are very scarce (in which case the economy will be in a ZE) or plentiful (in which case the economy will be in an FE). If the economy is in an IE, a marginal exogenous increase in $\mathcal{B}$ will lead to an increase in i, thereby relaxing the liquidity constraint of households in state 1 and increasing c_1; this in turn will increase output Y and the aggregate capital stock K.

5.2 Liquid Capital

We now consider the case where $\chi > 0$, meaning that capital can be used to back issuance of bank deposits. This implies that aggregate consumption affects the supply of assets that can be used to back deposit issuance, which in turn affects aggregate consumption. This feedback loop can expand the set of parameters for which multiple equilibria are possible – in particular, multiple equilibria can occur for $\sigma < 1$:

[27] In a similar vein, an increase in inflation π can lead to an increase in the amount of real money $\mathcal{M}$ demanded when σ is high.

[28] The main difference is that in Williamson (2012), capital is not used to produce the consumption good sold against money and deposits. However, with illiquid capital, this difference does not materially affect the behavior of the economy.

Proposition 4. *Consider an economy with a utility function*

$$u(c) = D\frac{c^{1-\sigma}}{1-\sigma} \text{ with } \sigma < 1 \tag{46}$$

and a CES production function

$$Y = A\left(\alpha K^{\frac{\rho-1}{\rho}} + (1-\alpha)L^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}} \tag{47}$$

with $A, D > 0$, $\alpha \in (0,1)$ and $\rho \geq 0$. Suppose $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$. There exist parameters and debt levels $\overline{\mathcal{B}} > 0$ for which an FE, an IE, and a ZE coexist. Denoting C^j as aggregate consumption in equilibrium case $j \in \{FE, IE, ZE\}$, there exist cases with multiple equilibria where $C^{FE} > C^{IE} > C^{ZE}$, and there exist cases with multiple equilibria where $C^{FE} < C^{IE} < C^{ZE}$.

We prove Proposition 4 by examples. Below, we provide an example of an economy with multiple equilibria where $C^{FE} > C^{IE} > C^{ZE}$, and in Appendix A.1, we provide an example with $C^{FE} < C^{IE} < C^{ZE}$.

To gain some intuition about the result in Proposition 4, consider first the case where $C^{FE} > C^{IE} > C^{ZE}$, i.e. the FE is the *high-activity equilibrium* and the ZE the *low-activity equilibrium*. To illustrate why multiple equilibria with this property can occur, suppose we start from a ZE ($i = 0$) and consider what happens when the interest rate i increases. Recall that an increase in i has counteracting effects on C: on the one hand, an increase in i implies lower opportunity costs of carrying liquidity for households who can pay with deposits, but on the other hand, an increase in i leads to a decrease in the capital-labour ratio κ and thus to a decrease in the real wage w. Suppose the first effect dominates, such that an increase in i goes together with an increase in C. The increase in consumption demand resulting from a higher i then leads firms to invest more despite the higher interest rate. If the resulting increase in the aggregate capital stock $K(i)$ outpaces (to a sufficient degree) the increase in the bank's demand for assets $\mathcal{A}^d(i)$, then $Q(i)$ decreases, and the fall in asset prices (i.e. the increase in i) can be self-fulfilling.

Consider next the case of multiple equilibria where $C^{FE} < C^{IE} < C^{ZE}$, i.e. the FE is the low-activity equilibrium and the ZE the high-activity equilibrium. To illustrate why multiple equilibria with this property can occur, suppose again we start from a ZE ($i = 0$) and consider what happens when the interest rate i increases. Suppose that, different to the previous example, the negative effect of an increase in i on C (via the fall in real wages) dominates, such that C falls in i. The fall in consumption demand then implies a decrease in the bank's demand for assets, $\mathcal{A}^d(i)$. If the decrease in $\mathcal{A}^d(i)$ is sufficiently large relative to the fall in $K(i)$, then $Q(i)$ decreases, and the fall in asset prices can again be self-fulfilling.

In general, multiple equilibria are more likely to exist when the capital-labour ratio κ is relatively insensitive to changes in i. To see this, note that when κ decreases strongly in i, it is less likely that $K(i)$, and hence $\mathcal{A}^s(i)$, increases relative to $\mathcal{A}^d(i)$ when i increases.

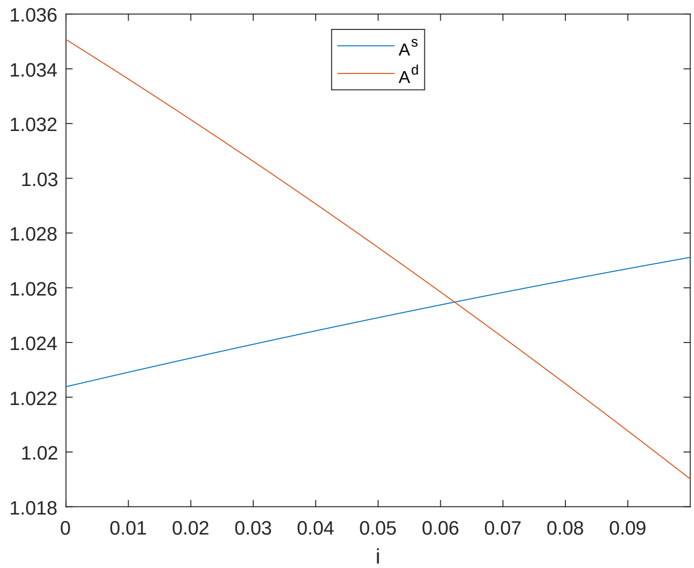


Figure 1: Example of equilibrium multiplicity with $C^{FE} > C^{IE} > C^{ZE}$.

Table 1: Parameter values for Figure 1.

β	γ	θ	σ	D	A	α	ρ	χ	δ	$\mathcal{B}$
0.95	0.045	0.8	0.7	1	2	0.1	1/3	1	1	0.5

Example of multiple equilibria with $C^{FE} > C^{IE} > C^{ZE}$. Figure 1 shows $\mathcal{A}^d(i)$ and $\mathcal{A}^s(i)$ for the parameter values given in Table 1. Comparing the figure with (43) reveals that the conditions for all types of equilibria are satisfied simultaneously: $\mathcal{A}^d(0) > \mathcal{A}^s(0)$, which constitutes a ZE since banks can make up the shortfall in asset supply by holding excess reserves at the zero-lower bound; we also have $\mathcal{A}^d(\iota) < \mathcal{A}^s(\iota)$, which constitutes an FE since banks are happy to hold the excess supply of assets as it is costless to do so in a fundamental equilibrium; finally, we have $\mathcal{A}^d(0.063) = \mathcal{A}^s(0.063)$, implying that there is an IE at a bond rate of 6.3%. Aggregate consumption levels in the three equilibria equal $C^{FE} = 1.307$, $C^{IE} = 1.282$ and $C^{ZE} = 1.238$.

An explicit condition for multiplicity. It turns out that deriving an explicit parameter condition guaranteeing the existence of multiple equilibria is easier for a fixed money-to-debt ratio than for a fixed real bond supply:

Proposition 5. *Consider an economy with a CRRA utility function* (46) *with* $D = 1$*, a CES production function* (47)*, and* $\chi > 0$*. Suppose* $\mathcal{B} = \frac{1-\eta}{\eta}\mathcal{M}$*, and define* $q^* \equiv q(\kappa^*)$*. If*

$$q^* < \frac{\beta\theta}{\chi}, \tag{48}$$

and

$$\rho < \frac{\beta f'(\kappa^*)}{q^*(1+\delta q^*)}\left[\frac{\theta}{\chi^2}\left(\beta - \frac{1}{\sigma}\Big(\beta - \chi q^*\Big)\right) - \left(\frac{1}{\chi} - 1\right)q^*\right], \tag{49}$$

and ι *is sufficiently close to 0, then there exist money-to-debt ratios* $\eta \in (0,1)$ *such that an FE, an IE and a ZE coexist.*

Condition (48) ensures that the first-best capital stock (more precisely, the share of the capital stock that can be used to back deposit issuance) is not too high, since otherwise the economy will be in an FE even if the bond supply is zero ($\eta = 1$). Condition (49) is equivalent to the condition that $\mathcal{A}^s(i) - \mathcal{A}^d(i)$ be strictly increasing in i in an ϵ-neighbourhood of the Friedman rule. Notice that (49) can only be fulfilled if the elasticity of substitution between labour and capital, ρ, is not too high, which ensures that κ does not fall too strongly when i increases. While assuming ι to be close to zero simplifies considerably the derivation of a parameter condition guaranteeing existence of multiple equilibria, this is not a necessary condition for equilibrium multiplicity: in Appendix A.2, we provide an example where multiple equilibria exist for a constant money-to-debt ratio with $\iota = 0.1$.

The result in Proposition 5 also allows to show formally that multiple equilibria can occur for $\sigma < 1$ under a constant money-to-debt ratio. To see this, note that for $\chi = 1$, condition (49) becomes

$$\rho < \frac{\beta\theta f'(\kappa^*)}{q^*(1+\delta q^*)}\left(\beta - \frac{1}{\sigma}(\beta - q^*)\right). \tag{50}$$

The right-hand side of condition (50) is strictly positive whenever $\sigma > 1 - (q^*/\beta)$. It follows that for any σ satisfying $1 - (q^*/\beta) < \sigma < 1$, condition (50) will be fulfilled as long as ρ is sufficiently low.

Finally, it follows immediately from Lemma 2 that the parameter conditions (48)-(49) are also sufficient to obtain multiplicity (in the vicinity of the Friedman rule) for some constant real bond supply. But they are not necessary: in fact, for the parameter values in Table 1, condition (49) is not satisfied, yet multiple equilibria exist.

Comparing the FE and the ZE. Given that multiple steady-state equilibria may coexist, it is natural to ask how, in case of equilibrium multiplicity, the different equilibria compare to the first-best allocation. We focus our discussion of this issue on the FE and the ZE, and we assume $\chi > 0$ and $u(c) = c^{1-\sigma}/(1-\sigma)$.[29] Notice that there are three margins to consider: the amount of output produced, the combination of capital and labour with which a given output is produced, and how a given output is distributed among households.[30]

We denote steady-state values in the FE (ZE) by an FE- (ZE-) superscript. As already shown further above, we get from (36) that $\kappa^{FE} = \kappa^*$ and $\kappa^{ZE} > \kappa^*$. From (2) and (11), we get that steady-state labour supply in the first-best allocation satisfies

$$l^*(\kappa) = \frac{c}{f(\kappa) - \delta\kappa} = \frac{[w(k)]^\sigma}{f(\kappa) - \delta\kappa}.$$

From (34)-(35) and (37), we get that labour-supply in the ZE and the FE, respectively, satisfies

$$l^{ZE}(\kappa) = \left(\frac{1}{1+\iota}\right)^\sigma l^*(\kappa) \text{ and } l^{FE}(\kappa) = \left[(1-\theta)\left(\frac{1}{1+\iota}\right)^\sigma + \theta\right] l^*(\kappa).$$

This shows that $l^{ZE}(\kappa) < l^{FE}(\kappa) < l^*(\kappa)$, i.e., for a given κ, labour supply is lower than optimal both in the FE and in the ZE, and it is lower in the ZE than in the FE. The latter reflects that the liquidity-in-advance friction is mitigated in the FE since part of households can pay with interest-bearing deposits.

From (11), we have that consumption in the first-best allocation satisfies $c^*(\kappa) = [w(\kappa)]^\sigma$. From (34)-(35), we get that consumption levels c_0 and c_1 in the FE and the ZE satisfy

$$c_0^{FE}(\kappa) = c_0^{ZE}(\kappa) = c_1^{ZE}(\kappa) = \left(\frac{1}{1+\iota}\right)^\sigma c^*(\kappa) \quad \text{and} \quad c_1^{FE}(\kappa) = c^*(\kappa).$$

This shows that, given κ, consumption levels are again closer to the first-best in the FE. Nevertheless, since $l(\kappa)$ is inefficiently low in both equilibria, we know from the theory of the second best that an unambiguous welfare ranking of the two equilibria is not possible. Specifically, correcting the inefficiently low $l(\kappa)$ with an inefficiently

[29]Whenever both an FE and a ZE exist, then an IE (with κ in between the FE and the ZE) exists as well.

[30]The fact that there are three relevant margins is an important difference to Geromichalos and Herrenbrueck (2022), where there is only one relevant margin. In Geromichalos and Herrenbrueck (2022), as in our model, output and investment can be inefficiently low when $\iota > 0$. In their model, this wedge can be fully closed by setting $i < \iota$, which spurs investment and allows to attain the first-best allocation even when $\iota > 0$. Different to Geromichalos and Herrenbrueck (2022), who assume an inelastic labour supply, it is in general not possible to achieve the first-best allocation with an inefficiently large capital stock in our model. While it may in principle be possible to bring output to its first-best level by setting $\kappa > \kappa^*$, the output will be produced with an inefficient mix of capital and labour.

high κ can in principle increase welfare.[31] Note in particular that, since $\kappa^{FE} < \kappa^{ZE}$, we have $c_0^{FE} < c_0^{ZE}$ independent of whether the FE or the ZE is the high-activity equilibrium, and c_0^{ZE} may be closer to c^* than c_0^{FE}.

6 Transitional Dynamics

The coexistence of multiple steady states in itself does not imply that multiple equilibria exist in the economy we have described. Additionally, what is required for multiple equilibria is that for certain values of the state variables, multiple saddle paths with different real allocations exist. Since the amount of capital K_{-1} is the only state variable in this economy, what is thus required for equilibrium multiplicity is that the same initial amount of capital may lead to different steady-state equilibria. In this section, we investigate the transitional dynamics of the model to determine whether multiple equilibria do exist. Because of the highly non-linear nature of our model, we focus on global dynamics for parametrised examples.

To study the transitional dynamics, we consider the case where $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$, i.e. where the government fixes the real debt supply. The solution technique we apply relies on backward iteration, and in particular on the following proposition:

Proposition 6. *If $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$ and $cu''(c)/u'(c) > -1\ \forall c$, then, given $X = (\kappa, K, Z)$, where $Z \equiv \phi M_{-1}$, there is a unique X_{-1} such that the equilibrium conditions hold.*

The proposition states that there is a unique mapping from $X \mapsto X_{-1}$.[32] We refer to this map as g in what follows, and we use it to backward iterate from some terminal X_T, resembling a backward-shooting algorithm in the spirit of Judd (1999) and Brunner and Strulik (2002).[33] Now, denote $\bar{X}_{FE}$ and $\bar{X}_{ZE}$ as the values of X which describe the FE and ZE steady states, respectively. To derive the transitional dynamics, we use linearisation to characterise sets $\mathscr{E}_{\epsilon,FE} \subseteq \mathscr{B}_{\epsilon}(\bar{X}_{FE})$ and $\mathscr{E}_{\epsilon,ZE} \subseteq \mathscr{B}_{\epsilon}(\bar{X}_{ZE})$, where $\mathscr{B}_{\epsilon}(X)$ is some ϵ-neighbourhood around X, so that if $X \in \mathscr{E}_{\epsilon,FE}$, the economy will transition to $\bar{X}_{FE}$; and if $X \in \mathscr{E}_{\epsilon,ZE}$, the economy will

[31] The matter is further complicated by the fact that, as is well known, the first-best allocation does not maximize steady-state utility, which is maximized when the capital stock is at the golden rule level, $f'(\kappa) = \delta$. For this reason, a capital stock exceeding κ^* (as is the case in the ZE) may be associated with a higher steady-state utility than the first-best allocation.

[32] Importantly, this does not imply that that the inverse is true, i.e., that $X_{-1} \mapsto X$. In fact, equilibrium multiplicity only exists if this is not the case.

[33] Judd (1999) mentions that standard forward-shooting algorithms have limited value in infinite-horizon economic models due to saddle-path stability of equilibrium, entailing that such algorithms are extremely sensitive to small errors in the initial guess.

transition to $\bar{X}_{ZE}$.[34,35] Given this, we can define

$$\mathscr{K}_{\varepsilon,J} = \bigcup_{T=0}^{\infty} g_K^T(\mathscr{E}_{\varepsilon,J}), \quad J \in \{FE, ZE\}. \tag{51}$$

So, if $K_{-1} \in \mathscr{K}_{\varepsilon,FE}$, then $\exists T \in \mathbb{N}_0$ and $\exists X \in \mathscr{E}_{\varepsilon,FE}$ such that $g_K^T(X) = K_{-1}$; for starting value K_{-1}, our numerical approximation suggests existence of an equilibrium path that converges to the FE steady state. Likewise, if $K_{-1} \in \mathscr{K}_{\varepsilon,ZE}$, the approximation suggests existence of an equilibrium path that converges to the ZE steady state. Thus, we have 'approximated' equilibrium multiplicity for some starting value K_{-1} if $K_{-1} \in \mathscr{K}_{\varepsilon,FE} \cap \mathscr{K}_{\varepsilon,ZE}$. In other words, multiple equilibria exist if, by backward iterating both from some $X \in \mathscr{E}_{\epsilon,FE}$ and $X' \in \mathscr{E}_{\epsilon,ZE}$, we find values of K_{-1} from which we can transition to either steady state equilibrium. For all parametrisations we consider, we find that both the FE and ZE equilibrium are saddle-path stable, meaning that $\ln(\mathscr{E}_{\epsilon,J})$, $J \in \{FE, ZE\}$ is a one-dimensional linear subspace, i.e., a straight line. If we therefore define

$$\mathscr{B}_\epsilon(\bar{X}) \equiv \left\{ X \quad \text{s.t.} \quad \left| \frac{X - \bar{X}}{\bar{X}} \right| \leq \epsilon \right\}, \tag{52}$$

where $|\cdot|$ is the Euclidean norm, then we obtain

$$\mathscr{E}_{\varepsilon,J} = \{\bar{X}_J \otimes (1 + \lambda \hat{X}_J), \quad \forall \lambda \in [-\epsilon, \epsilon]\}, \tag{53}$$

where $\hat{X}_J = [\hat{\kappa}_J, \hat{K}_J, \hat{Z}_J]$ is uniquely determined up to some normalisation—we set its Euclidean length to one. Note that it is infeasible to compute $g^T(\mathscr{E}_{\epsilon,J})$ numerically because $\mathscr{E}_{\epsilon,J}$ is an uncountable set. We therefore compute $g^T(\{e_{n,J}\}_{n=0}^N)$ for a discrete sample instead.[36]

Although the direct implication of the procedure sketched above is that we 'only' obtain points in $g^T(\mathscr{E}_{\epsilon,J})$, these points in fact allow us to say more about the set $g^T(\mathscr{E}_{\epsilon,J})$. The reason is that the proof of Proposition 6 implies:

Corollary 1. *The map $g(X)$ is continuous in X.*

The implication is that the set $g^T(\mathscr{E}_{\epsilon,J})$ is continuous, too. Thus, if we find that $K'_{-1}, K''_{-1} \in g_K^T(\mathscr{E}_{\epsilon,J})$, with $K'_{-1} < K''_{-1}$ then we know that $[K'_{-1}, K''_{-1}] \subseteq g^T(\mathscr{E}_{\epsilon,J})$. Thus, given the sample $\{e_{n,J}\}_{n=0}^N \in \mathscr{E}_{\epsilon,J}$, our best approximation of $g^T(\mathscr{E}_{\epsilon,J})$ is

$$\tilde{g}^T(\{e_{n,J}\}_{n=0}^N) \equiv \left[\min_{e \in \{e_{n,J}\}_{n=0}^N} \left\{ g^T(e) \right\}, \max_{e \in \{e_{n,J}\}_{n=0}^N} \left\{ g^T(e) \right\} \right], \tag{54}$$

[34]The linearisation technique is described in Appendix A.3.

[35]Note that we cannot start the backward iteration at the steady-state values since $g(\bar{X}_{FE}) = \bar{X}_{FE}$ and $g(\bar{X}_{ZE}) = \bar{X}_{ZE}$. Note also that the smaller ϵ, the more accurate are the approximations.

[36]In particular, we sample $e_{n,J} = \bar{X}_J \otimes (1 + \lambda_{n,J} \hat{X}_J)$ and $\{\lambda_{n,J}\} \in [-\epsilon, \epsilon]$, and we do this for $T \in \{0, 1, ..., \overline{T}\}$ and use $\lambda_{n,J} = \{-100, -99, ..., -1, 1, 2, ..., 100\} \times e^{-9}$.

Table 2: Parametrisation where equilibrium is unique.

β	γ	θ	σ	D	A	α	ρ	χ	δ	$\mathcal{B}$
0.98	0.02	0.8	0.7	1.0	2.0	0.1	0.3333	1.0	0.6	0.6250

Table 3: Steady state values for parametrisation where equilibrium is unique.

	κ	K	Z	C	Y	L	π	ι	i
FE steady state	0.6413	0.7420	0.3284	1.7192	2.1644	1.1571	2.00	4.08	4.08
IE steady state	0.6542	0.7457	0.3343	1.6937	2.1412	1.1399	2.00	4.08	1.16
ZE steady state	0.6596	0.7473	0.3381	1.6835	2.1319	1.1331	2.00	4.08	0.00

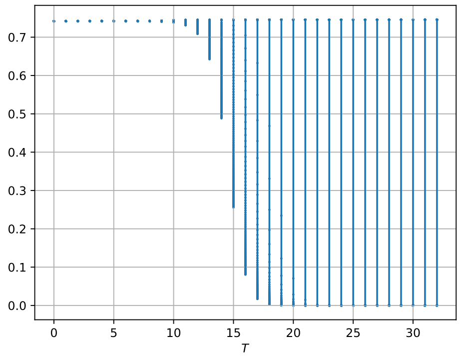
where we know that $\{e_{n,J}\}_{n=0}^{N} \subseteq \mathscr{E}_{\epsilon,J} \;\Rightarrow\; \tilde{g}_K^T(\{e_{n,J}\}_{n=0}^{N}) \subseteq g_K^T(\mathscr{E}_{\epsilon,J})$. In other words, any value of K_{-1} that lies between the most extreme values we find must also be on a saddle-path towards the steady-state equilibrium we backwards-iterate from.

We start by considering the parameterisation described in Table 2, with corresponding steady-state values given in Table 3. Note that for this parametrisation, an FE, an IE and a ZE all exist. Figure 2 plots the results, which show that the values for K_{-1} we obtain by backwards iterating from the FE and ZE steady states do not overlap; instead, backwards iterating from the FE steady state leads K_{-1} to approach the IE steady state from below while backwards iterating from the ZE steady state approaches the IE steady state from above. This shows several things: first, the IE steady state is unstable; second, for $K \in (0, \bar{K}_{IE})$, we transition to the FE steady state, while for $K \in (\bar{K}_{IE}, \infty)$, we transition to the ZE steady state; finally, this implies that although this parameterisation allows for multiple steady states, multiple equilibria do not exist in this case. The existence of multiple steady states may still be relevant for policy, however: It suggests that after an (unexpected) shock to capital, the economy may not transition back to the same steady state, even if policy remains unchanged.

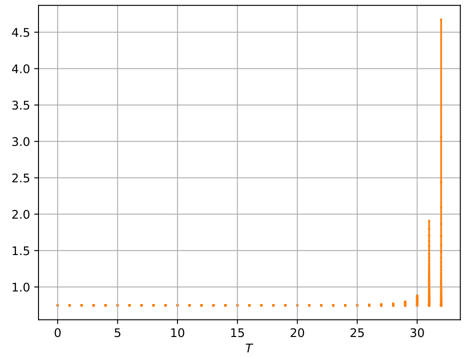
Next, consider the parametrisation characterised in Table 4, with corresponding steady-state values in Table 5. The results from the backward iteration for this

Table 4: Parametrisation with multiple equilibria.

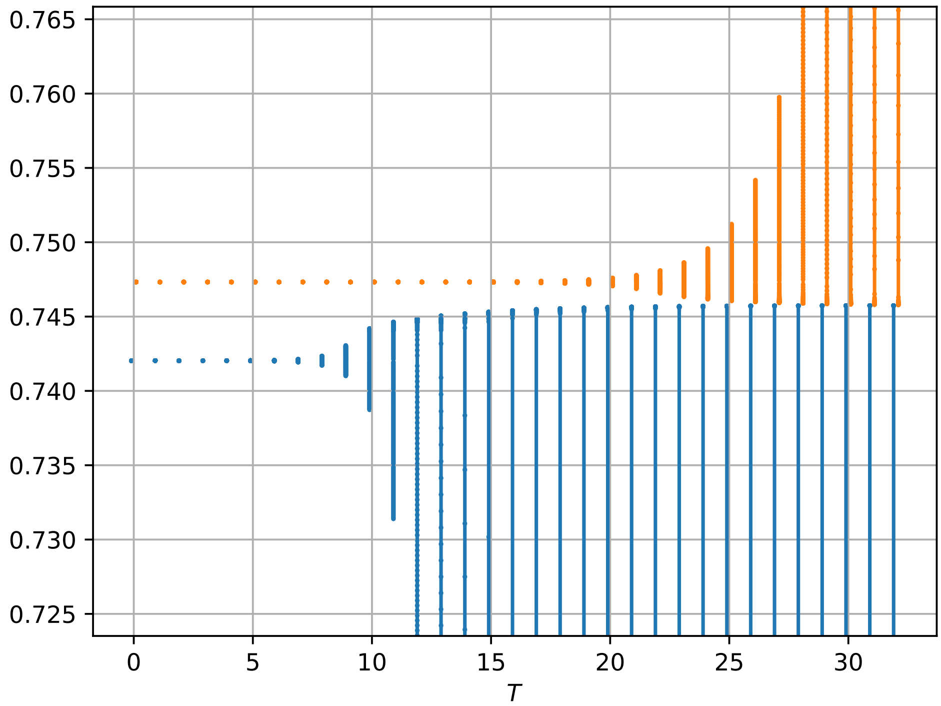
β	γ	θ	σ	D	A	α	ρ	χ	δ	$\mathcal{B}$
0.98	0.02	0.8	0.7	1.0	2.0	0.1	0.3333	1.0	1.0	0.5225



(a) Approximation $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$.



(b) Approximation $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$.



(c) Approximations $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$ (●) and $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$ (●). The range of K is capped from below and from above for enhanced visibility.

Figure 2: Approximations $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$ and $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$ for parametrisation where equilibrium is unique.

Table 5: Steady state values for parametrisation with multiple equilibria.

	κ	K	Z	C	Y	L	π	ι	i
FE steady state	0.5136	0.5531	0.2581	1.3513	1.9045	1.0769	2.00	4.08	4.08
IE steady state	0.5182	0.5522	0.2610	1.3372	1.8895	1.0657	2.00	4.08	2.14
ZE steady state	0.5234	0.5512	0.2688	1.3215	1.8726	1.0531	2.00	4.08	0.00

parametrisation are presented in Figure 3. In this case, we can observe a clear overlap for values of K_{-1} such that the economy may either transition to the FE or the ZE steady state, showing that this parametrisation presents a case of true equilibrium multiplicity. Note further that as T increases, both paths again approach the IE steady state, implying that the IE equilibrium is again unstable. In Figure 4 we plot two equilibrium trajectories starting from $K_{-1} = 0.5522$; one that moves to the FE steady state and one that moves to the ZE steady state. To do so, we first calculate

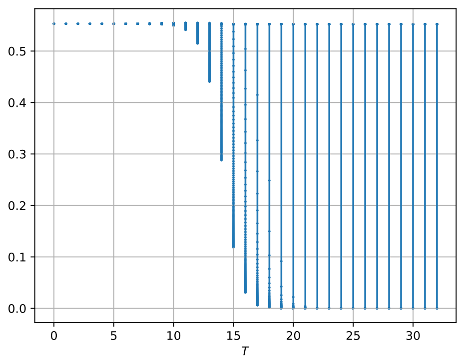
$$\hat{e}_J = \underset{e \in \{e_{n,J}\}_{n=0}^{N}}{\arg\min} \; |g_K^T(e) - K_{-1}|, \quad J \in \{FE, ZE\}, \tag{55}$$

where we set $T = 30$ because Figure 3 suggests that $K_{-1} = 0.5522 \in g_K^T(\mathscr{E}_{\epsilon,FE}) \cap g_K^T(\mathscr{E}_{\epsilon,ZE})$ for $T = 30$. Then, we use as approximate equilibrium trajectories $\{X_{t,J}\}_{t=0}^{T} = \{g^{T-t}(\hat{e}_J)\}_{t=0}^{T}$, $J \in \{FE, ZE\}$. One interpretation is that after a shock to capital such that $K_{-1} = 0.5522$, the economy coordinates on a path that will lead it either to an FE or to a ZE equilibrium, and that output, interest rates, and inflation adjust accordingly immediately. As the figure shows, the real implications of the two equilibria are markedly different both on the transition path as well as in the corresponding steady state. Note, however, that these two trajectories are not necessarily the only two equilibria that exist: Studying Figure 3 shows that the starting value for capital we use also lies at saddle paths where $T > 30$; hence, other saddle paths are possible, and potentially even cyclical equilibria.

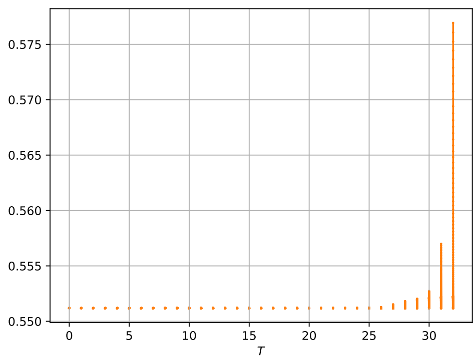
In sum, we have shown by example that multiple equilibria may exist in this economy, but also that the existence of multiple steady states does not generally imply multiple equilibria.

7 Conclusion

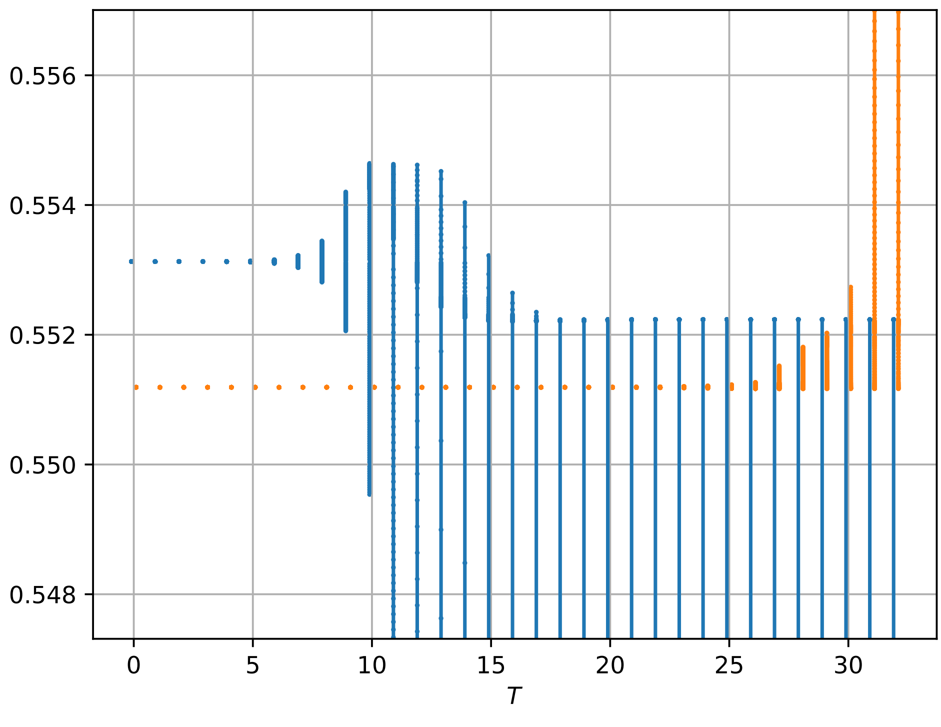
In this paper, we studied liquidity transformation by banks in a monetary model and analysed the channels through which liquidity transformation increases aggregate output and investment. We showed how liquidity transformation can lead to macroeconomic instability in the sense that it can lead to multiple steady-state equilibria with different interest rates in economies that do not exhibit such equilibrium multiplicity without liquidity transformation. By studying the transitional dynamics of the model, we confirm that the coexistence of steady states sometimes allows for equilibrium multiplicity. The paper also makes a methodological contribution by showing how banks can be introduced in a cash-in-advance model. Key



(a) Approximation $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$.

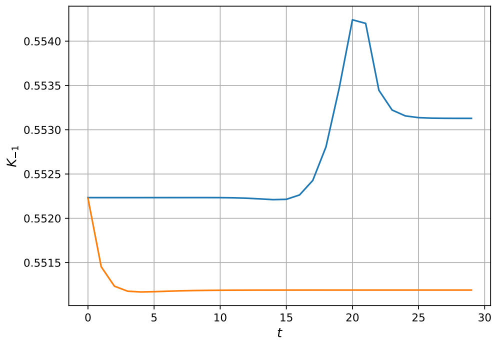


(b) Approximation $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$.

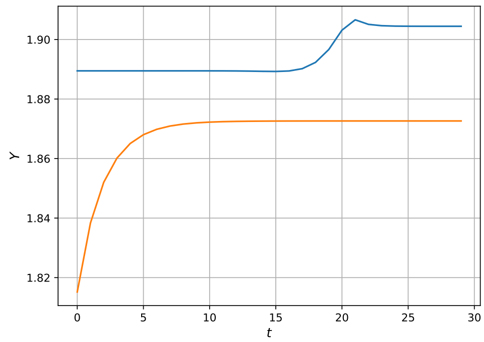


(c) Approximations $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$ (●) and $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$ (●). The range of K is capped from below and from above for enhanced visibility.

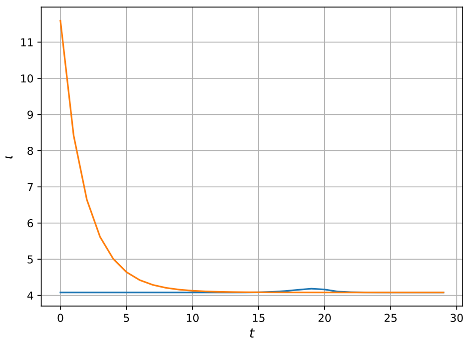
Figure 3: Approximations $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$ and $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$ for parametrisation with multiple equilibria.



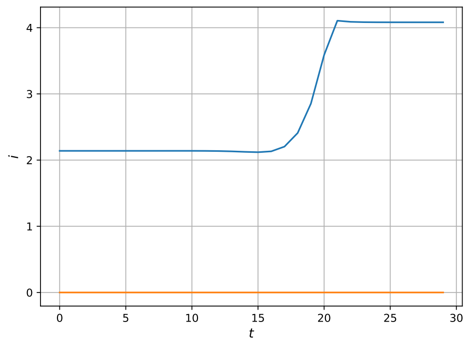
(a) Dynamics for K_{t-1}



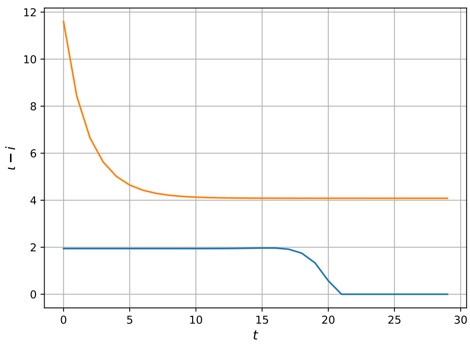
(b) Dynamics for Y_t



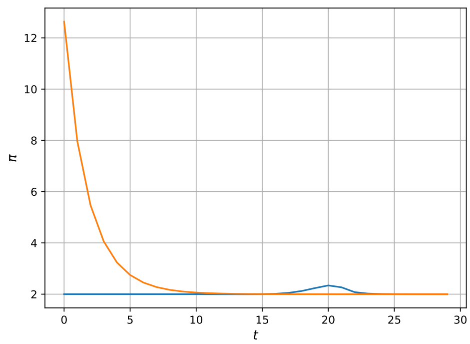
(c) Dynamics for ι_t



(d) Dynamics for i_t



(e) Dynamics for the spread $\iota_t - i_t$



(f) Dynamics for π_t

Figure 4: Equilibrium dynamics with $K_{-1} = 0.5522$ for parametrisation which allows for multiple equilibria. There is an equilibrium path to the FE (●) steady state and an equilibrium path to the ZE (●) steady state.

results from the New Monetarist macro-financial literature, e.g. on the emergence of liquidity traps, can be replicated within the CIA framework.

Bibliography

Altermatt, L. (2022). Inside money, investment, and unconventional monetary policy. *International Economic Review*, 63(4):1527–1560.

Altermatt, L., Iwasaki, K., and Wright, R. (2023). General equilibrium with multiple liquid assets. *Review of Economic Dynamics*, 51:267–291.

Altermatt, L., van Buggenum, H., and Voellmy, L. (2022). Systemic bank runs without aggregate risk: How a misallocation of liquidity may trigger a solvency crisis. *SNB Working Paper no. 10/2022*.

Andolfatto, D. and Williamson, S. (2015). Scarcity of safe assets, inflation, and the policy trap. *Journal of Monetary Economics*, 73:70–92.

Aruoba, S. B., Waller, C., and Wright, R. (2011). Money and capital. *Journal of Monetary Economics*, 58(2):98–116.

Benhabib, J., Schmitt-Grohe, S., and Uribe, M. (2001). The perils of taylor rules. *Journal of Economic Theory*, 96(1-2):40–69.

Brunner, M. and Strulik, H. (2002). Solution of perfect foresight saddlepoint problems: a simple method and applications. *Journal of Economic Dynamics and Control*, 26(5):737–753.

Cooley, T. F. and Hansen, G. D. (1989). The inflation tax in a real business cycle model. *American Economic Review*, 79(4):733–748.

Geromichalos, A. and Herrenbrueck, L. (2022). The liquidity-augmented model of macroeconomic aggregates: A New Monetarist DSGE approach. *Review of Economic Dynamics*, 45:134–167.

Haslag, J. H. and Martin, A. (2007). Optimality of the Friedman rule in an overlapping generations model with spatial separation. *Journal of Money, Credit and Banking*, 39(7):1741–1758.

Herrenbrueck, L. and Wang, Z. (2023). Interest rates, moneyness, and the fisher equation. *mimeo*.

Judd, K. (1999). *Numerical Methods in Economics*. MIT Press.

Keister, T. and Sanches, D. (2023). Should central banks issue digital currency? *The Review of Economic Studies*, 90(1):404–431.

Lagos, R. and Rocheteau, G. (2008). Money and capital as competing media of exchange. *Journal of Economic Theory*, 142(1):247–258.

Rocheteau, G., Wright, R., and Xiao, S. X. (2018). Open market operations. *Journal of Monetary Economics*, 98:114–128.

Williamson, S. (2012). Liquidity, monetary policy, and the financial crisis: A New Monetarist approach. *American Economic Review*, 102(6):2570–2605.

Williamson, S. (2016). Scarce collateral, the term premium, and quantitative easing. *Journal of Economic Theory*, 164:136–165.

Appendix A Additional Material

A.1 Example of Multiple Equilibria with $C^{FE} < C^{IE} < C^{ZE}$

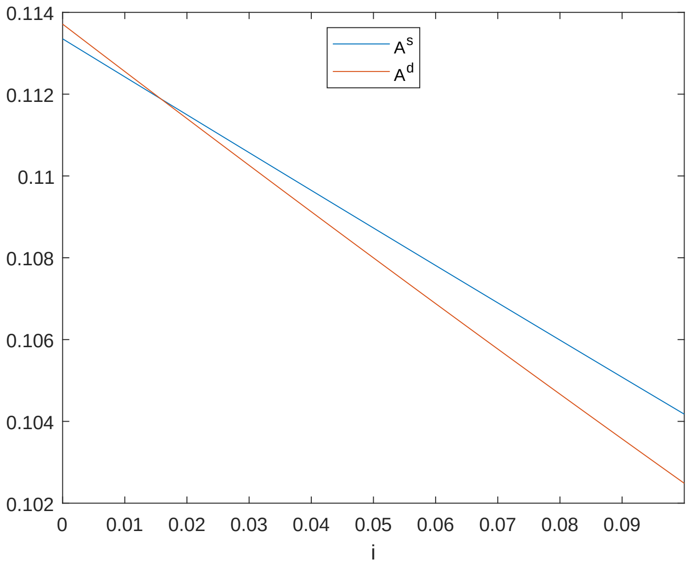


Figure 5: Example of equilibrium multiplicity with $C^{ZE} > C^{IE} > C^{FE}$.

Table 6: Parameter values for Figure 5.

β	γ	θ	σ	D	A	α	ρ	χ	δ	$\mathcal{B}$
0.95	0.045	0.95	0.5	1.1	1.3	0.3	1/11	0.38	1	0.003

Figure 5 shows $\mathcal{A}^d(i)$ and $\mathcal{A}^s(i)$ for the parameter values given in Table 6. From the figure, it is clear that the conditions for all types of equilibria from (43) are satisfied simultaneously, with the IE existing for an interest rate of 1.6%. Aggregate consumption levels in the three equilibria equal $C^{FE} = 0.113$, $C^{IE} = 0.114$ and $C^{ZE} = 0.115$; hence, the ordering of aggregate consumption and welfare is inverse in this example relative to the one presented in the main body of the paper.

A.2 Example of Multiple Equilibria with a Constant Money-to-Debt Ratio

Table 7: Parameter values for Figure 6.

β	γ	θ	σ	D	A	α	ρ	χ	δ	η
0.95	0.045	0.8	0.7	1	2	0.1	1/5	1	1	0.45

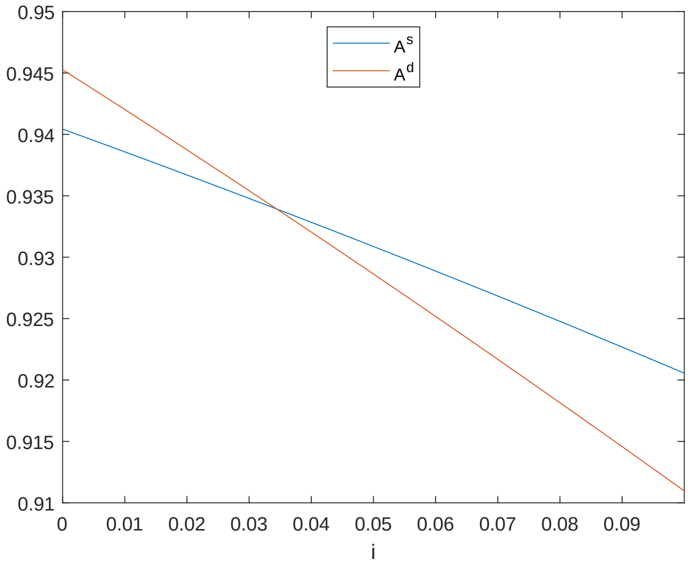


Figure 6: Example of equilibrium multiplicity with a constant money-to-debt ratio.

Figure 6 shows $\mathcal{A}^d(i)$ and $\mathcal{A}^s(i)$ for the parameter values given in Table 7. From the figure, it is clear that the conditions for all types of equilibria from (43) are satisfied simultaneously, with the IE existing for an interest rate of 3.4%. Aggregate consumption levels in the three equilibria equal $C^{FE} = 1.168$, $C^{IE} = 1.145$ and $C^{ZE} = 1.131$, which shows that aggregate consumption in this example is highest in the FE and lowest in the ZE.

A.3 Details on the Linearisation

Given the map $X \to X_{-1} = G(X)$, we can log-linearise around a steady state $\bar{X}$ to obtain

$$\hat{X}_{-1} = \varepsilon_{g,\bar{X}} \hat{X}, \quad \text{where} \quad \hat{X} \equiv (X - \bar{X}) \oslash X, \tag{56}$$

and $\bar{\varepsilon}_{g,X}$ is a 3-by-3 matrix.

To find how $\bar{\varepsilon}_{g,X}$ looks like for either an FE, ZE, or IE, steady state, it is useful to first log-linearise Equations (86), (89), and (91). This yields

$$\hat{\kappa}_{-1} = \frac{\bar{\varepsilon}_{w,\kappa} - \bar{\varepsilon}_{\tilde{\psi},\kappa}}{\bar{\varepsilon}_{w,\kappa}} \hat{\kappa} + \frac{1}{\bar{\varepsilon}_{w,\kappa}} \frac{\chi(1+\bar{\iota})}{(1-\chi)(1+\bar{\iota}) + \chi(1+\bar{i})} \left(\widehat{1+i} - \widehat{1+\iota} \right), \tag{57}$$

$$\hat{K}_{-1} = \frac{1}{\sigma} \frac{1}{\tilde{\varphi}(\bar{\kappa})\bar{K}} \left[\bar{C}(\bar{\varepsilon}_{w,\kappa}\hat{\kappa} - \widehat{1+\iota}) + \theta \bar{c}_1 \widehat{1+i} \right] + \frac{\hat{K}}{\tilde{\varphi}(\bar{\kappa})} - \bar{\varepsilon}_{\tilde{\varphi},\kappa}\hat{\kappa}, \tag{58}$$

$$\hat{Z}_{-1} = \frac{(1-\chi)(1+\bar{i})}{(1-\chi)(1+\bar{i}) + \chi(1+\bar{\iota})} \widehat{1+\iota} + \frac{\chi(1+\bar{\iota})}{(1-\chi)(1+\bar{i}) + \chi(1+\bar{\iota})} \widehat{1+i} + \hat{Z} - \bar{\varepsilon}_{\tilde{\psi},\kappa}\hat{\kappa}, \tag{59}$$

where we defined

$$\bar{\varepsilon}_{\tilde{\psi},\kappa} \equiv \frac{\bar{\kappa} f''(\bar{\kappa})}{f'(\bar{\kappa}) + 1 - \delta}, \quad \bar{\varepsilon}_{\tilde{\varphi},\kappa} \frac{f'(\bar{\kappa}) - f(\bar{\kappa})/\bar{\kappa}}{f(\bar{\kappa})/\bar{\kappa} + 1 - \delta}, \quad \text{and} \quad \bar{\varepsilon}_{w,\kappa} = -\frac{\bar{\kappa}^2 f''(\bar{\kappa})}{f(\bar{\kappa}) - \bar{\kappa} f'(\bar{\kappa})}. \tag{60}$$

Money and asset market clearance themselves pin down (ι, i) as a function of X, so we can think of the log-linearised equations for money and asset market clearance as

$$\widehat{1+\iota} = \bar{\varepsilon}_{\iota,\kappa}\hat{\kappa} + \bar{\varepsilon}_{\iota,K}\hat{K} + \bar{\varepsilon}_{\iota,Z}\hat{Z} \tag{61}$$

$$\widehat{1+i} = \bar{\varepsilon}_{i,\kappa}\hat{\kappa} + \bar{\varepsilon}_{i,K}\hat{K} + \bar{\varepsilon}_{i,Z}\hat{Z} \tag{62}$$

where $\bar{\varepsilon}_{\iota,\kappa}$, $\bar{\varepsilon}_{\iota,K}$, $\bar{\varepsilon}_{\iota,Z}$, $\bar{\varepsilon}_{i,\kappa}$, $\bar{\varepsilon}_{i,K}$, and $\bar{\varepsilon}_{i,Z}$ follow from log-linearisation. Using Equations (61) and (62) in the system (57)-(59) then implicitly gives $\bar{\varepsilon}_{g,X}$.

FE steady state. We have that $\bar{i} = \bar{\iota}$ so that $\widehat{1+i} = \widehat{1+\iota}$. In turn, because money market clearance condition (93) must hold with equality, we can log-linearise it to obtain

$$\widehat{1+\iota} = \bar{\varepsilon}_{w,\kappa}\hat{\kappa} - \sigma\hat{Z}. \tag{63}$$

ZE steady state. We have that $\bar{i} = 0$ so that $\widehat{1+i} = 0$. The asset market clearance condition (98) pins down the Fisher rate and it can be log-linearised as

$$\begin{aligned}&\left[\frac{1}{\sigma} - \frac{\varpi_{\mathcal{B}}(1-\chi)(1+\bar{i})}{(1-\chi)(1+\bar{i})+\chi(1+\bar{\iota})}\right]\widehat{1+\iota}\\ &\quad= \left[\frac{\bar{\varepsilon}_{w,\kappa}}{\sigma} + \left(\varpi_K + \frac{\chi\tilde{\psi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa})-\chi\tilde{\psi}(\bar{\kappa})}\right)(\bar{\varepsilon}_{\varphi,\kappa}-\bar{\varepsilon}_{\psi,\kappa}) - \varpi_{\mathcal{B}}\bar{\varepsilon}_{\psi,\kappa}\right]\hat{\kappa} - \varpi_K\hat{K} - \varpi_Z\hat{Z},\end{aligned} \tag{64}$$

where

$$\begin{aligned}\varpi_Z &= \frac{Z}{C}\frac{\tilde{\varphi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa})-\chi\tilde{\psi}(\bar{\kappa})}, \quad \varpi_K = \frac{K}{C}\frac{\chi\tilde{\psi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa})-\chi\tilde{\psi}(\bar{\kappa})}\\ &\text{and} \quad \varpi_{\mathcal{B}} = \frac{\tilde{\psi}(\bar{\kappa})\overline{\mathcal{B}}}{C}\frac{\tilde{\varphi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa})-\chi\tilde{\psi}(\bar{\kappa})}\frac{(1-\chi)(1+\bar{i})+\chi(1+\bar{\iota})}{1+\bar{\iota}}.\end{aligned} \tag{65}$$

IE steady state. We have that $0 < \bar{i} < \bar{\iota}$, so that both Equations (93) and (98) must hold with equality. This gives the log-linearised equations (63) and

$$\begin{aligned}&\left[\frac{1\theta}{\sigma}\frac{\bar{c}_1}{\bar{C}} - \frac{\varpi_{\mathcal{B}}(1-\chi)(1+\bar{i})}{(1-\chi)(1+\bar{i})+\chi(1+\bar{\iota})}\right]\widehat{1+i}\\ &\quad= -\left[\frac{\bar{\varepsilon}_{w,\kappa}}{\sigma} + \left(\varpi_K + \frac{\chi\tilde{\psi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa})-\chi\tilde{\psi}(\bar{\kappa})}\right)(\bar{\varepsilon}_{\varphi,\kappa}-\bar{\varepsilon}_{\psi,\kappa}) - \varpi_{\mathcal{B}}\bar{\varepsilon}_{\psi,\kappa}\right]\hat{\kappa}\\ &\qquad + \varpi_K\hat{K} + \varpi_Z\hat{Z} + \left(\frac{1}{\sigma} - \varpi_{\mathcal{B}}\frac{(1-\chi)(1+\bar{i})}{(1-\chi)(1+\bar{i})+\chi(1+\bar{\iota})}\right)\widehat{1+\iota},\end{aligned} \tag{66}$$

where ϖ_Z, $\varpi_{\mathcal{B}}$, and ϖ_K are as in Equation (65).

Appendix B Proofs

B.1 Proof of Proposition 1

The equilibrium nominal rate i is pinned down by (15), and, since $f(\kappa)$ is strictly concave, condition (16) uniquely determines κ. Since $u(c)$ is strictly concave, condition (17) then uniquely determines c given i and κ. ■

B.2 Proof of Lemma 1

Note first that if $Q(0) > 0$ and $Q'(i) < 0$ for $Q(i) \in [0, \iota]$, then there exists some constant $\overline{\mathcal{B}} > 0$ with $\overline{\mathcal{B}} \in (Q(\iota), Q(0))$. For any such $\overline{\mathcal{B}}$, we have $\overline{\mathcal{B}} > Q(\iota)$ and $\overline{\mathcal{B}} < Q(0)$ such that, by (45), both an FE and a ZE exist. Furthermore, since $Q(i)$ is continuous, we get from the intermediate value theorem that there exists some $i \in (0, \iota)$ such that $\overline{\mathcal{B}} = Q(i)$, which means that an IE exists as well. ■

B.3 Proof of Lemma 2

Denote

$$\mathcal{B}_\eta(i) = \frac{1-\eta}{\eta}\mathcal{M}(i) \tag{67}$$

as the real bond supply under a fixed money-to-debt ratio $\eta \in (0, 1)$.

Step 1: $i_H \geq i_L \Rightarrow c_0(i_H) \leq c_0(i_L)$. This follows from the fact that, by (34), $c_0(i)$ is strictly increasing in w, which itself is strictly increasing in κ, which, in turn, is weakly decreasing in i (see (36)).

Step 2: $i_H \geq i_L \Rightarrow \mathcal{B}_\eta(i_H) \leq \mathcal{B}_\eta(i_L)$. This follows immediately from (38) and (67) together with the result in Step 1.

Suppose an FE and a ZE coexist for some fixed money-to-debt ratio η. By (40) and (43), this implies that

$$\mathcal{B}_\eta(\iota) + \chi K(\iota) \geq \mathcal{A}^d(\iota) \quad \text{and} \quad \mathcal{B}_\eta(0) + \frac{\chi(1+\iota)}{1+\chi\iota}K(0) \leq \mathcal{A}^d(0). \tag{68}$$

From Step 2, we know that there exists a strictly positive constant $\overline{\mathcal{B}}$ with $\overline{\mathcal{B}} \in [\mathcal{B}_\eta(\iota), \mathcal{B}_\eta(0)]$. Since the schedules $K(i)$ and $\mathcal{A}^d(i)$ do not depend on the specification of the bond supply rule $\mathcal{B}(i)$, we have for any such constant $\overline{\mathcal{B}}$ that

$$\overline{\mathcal{B}} + \chi K(\iota) \geq \mathcal{A}^d(\iota) \quad \text{and} \quad \overline{\mathcal{B}} + \frac{\chi(1+\iota)}{1+\chi\iota}K(0) \leq \mathcal{A}^d(0) \tag{69}$$

such that, by (43), both an FE and a ZE exist when $\mathcal{B}(i) = \overline{\mathcal{B}}$. ■

B.4 Proof of Proposition 2

Define

$$Z(i) \equiv \mathcal{A}^s(i) - \mathcal{A}^d(i). \tag{70}$$

It follows immediately from (43) that a sufficient condition to rule out multiple equilibria is that $Z'(i) < 0$ for $i \in [0, \iota]$. With $\chi = 0$, we have

$$Z(i) = \mathcal{B}(i) - \mathcal{A}^d(i) = \mathcal{B}(i) - \frac{\beta(1+\iota)}{1+i}\theta c_1(i),$$

where we used the definitions of $\mathcal{A}^s(i)$ and $\mathcal{A}^d(i)$ in (40) and (42), respectively. Given $\mathcal{B}'(i) \leq 0$, we thus get that

$$Z'(i) < 0 \quad \Leftrightarrow \quad \frac{\partial \mathcal{A}^d(i)}{\partial i} > 0 \quad \Leftrightarrow \quad \frac{\partial \left[\frac{c_1(i)}{1+i}\right]}{\partial i} > 0 \quad \Leftrightarrow \quad \frac{c_1'(i)}{c_1(i)} > \frac{1}{1+i}. \tag{71}$$

With $\chi = 0$, κ does not depend on i (see (36)), which implies that the real wage w does not change with i (see (4)). With CRRA utility, we then get from (35) that

$$c_1 = \left(\frac{1+i}{1+\iota} w\right)^{\frac{1}{\sigma}} \quad \text{and} \quad \frac{c_1'(i)}{c_1(i)} = \frac{1}{\sigma}\frac{1}{1+i}. \tag{72}$$

It follows from (71) and (72) that $Z'(i) < 0$ for $i \in [0, \iota]$ when $\sigma < 1$. ■

B.5 Proof of Proposition 3

Note first that from the definition of $\mathcal{A}^s(i)$ in (40), we get that if $\chi = 0$ and $\mathcal{B}(i) = \overline{\mathcal{B}}$, then $\mathcal{A}^s(i) = \overline{\mathcal{B}}$. Suppose the real bond supply is fixed at $\overline{\mathcal{B}} = \mathcal{A}^d(i_1)$ for some $i_1 \in (0, \iota)$.[37] It then follows from (43) that: (i) an IE exists and (ii) if $\mathcal{A}^d(i)$ is weakly decreasing in i over $i \in [0, \iota]$, then an FE and a ZE exist as well.

Using the definition of $\mathcal{A}^d(i)$ from (42), we get that

$$\frac{\partial \mathcal{A}^d(i)}{\partial i} \leq 0 \quad \Leftrightarrow \quad \frac{\partial \left[\frac{c_1(i)}{1+i}\right]}{\partial i} \leq 0 \quad \Leftrightarrow \quad \frac{c_1'(i)}{c_1(i)} \leq \frac{1}{1+i}. \tag{73}$$

From (72), we get that with CRRA utility and $\chi = 0$, condition (73) is satisfied for all $i \in [0, \iota]$ when $\sigma \geq 1$. Therefore, if $\sigma \geq 1$ and the real bond supply is fixed at $\overline{\mathcal{B}} = \mathcal{A}^d(i_1)$ for some $i_1 \in (0, \iota)$, then an FE, an IE and a ZE coexist.[38] ■

B.6 Proof of Proposition 5

Consider an economy in an IE. From (43), we have that $Z(i) = 0$ in an IE, where $Z(i)$ is defined as in (70). It follows from (43) that in the limit as ι approaches zero, a sufficient condition for a ZE, an IE and an FE to coexist is that $Z'(i) > 0$ at the point $Z(i) = 0$. In the following, we will derive a sufficient condition for the latter to be the case.

With a constant money-to-debt ratio, the real bond supply satisfies

$$\mathcal{B}(i) = \frac{1-\eta}{\eta}\mathcal{M}(i) = \frac{1-\eta}{\eta}\beta(1+\iota)(1-\theta)c_0(i), \tag{74}$$

where, in the second step, we made use of the fact that (38) holds with equality in an IE. Furthermore, with CRRA utility, we get from (18) and (34)-(35) that

$$c_0(i) = \frac{1}{(1-\theta) + \theta(1+i)^{\frac{1}{\sigma}}} C(i) \quad \text{and} \quad c_1(i) = \frac{(1+i)^{\frac{1}{\sigma}}}{(1-\theta) + \theta(1+i)^{\frac{1}{\sigma}}} C(i). \tag{75}$$

[37]Since $c_1 > 0$, which follows from the fact that $\lim_{c \to 0} u'(c) = \infty$, $\mathcal{A}^d$ is always strictly positive.

[38]If $\sigma = 1$, there exist a continuum of equilibria, with any $i \in [0, \iota]$ constituting an equilibrium.

Substituting (41), (42), (74) and (75) into

$$Z(i) = \mathcal{A}^s(i) - \mathcal{A}^d(i) = \mathcal{B}(i) + \frac{\chi(1+\iota)}{1+\chi\iota+(1-\chi)i}K(i) - \mathcal{A}^d(i)$$

yields

$$Z(i) = T(i)C(i), \tag{76}$$

where

$$T(i) \equiv \frac{(1-\eta)\beta(1+\iota)(1-\theta)}{\eta[1-\theta+\theta(1+i)^{\frac{1}{\sigma}}]} + \frac{\chi(1+\iota)q(i)}{1+\chi\iota+(1-\chi)i} - \frac{\beta\theta(1+\iota)}{(1-\theta)(1+i)^{1-\frac{1}{\sigma}}+\theta(1+i)}. \tag{77}$$

In (77), we used the notation $q(i) = q(\kappa(i))$. In an IE, we have $Z(i) = 0$ and hence $T(i) = 0$. Evaluating $T(i)$ at the limit where $\iota = i = 0$, we have that

$$T(i) = 0 \quad \Leftrightarrow \quad \eta = \frac{\beta(1-\theta)}{\beta - \chi q(0)} \equiv \hat{\eta}. \tag{78}$$

An IE with a strictly positive bond supply exists iff $\hat{\eta} \in (0,1)$; since $q(0) = q(\kappa(0)) = q(\kappa^*) = q^*$ when $\iota = i = 0$, this is equivalent to condition (48) in Proposition 5.

Next, from (76), we get that $Z'(i) = T'(i)C(i) + T(i)C'(i)$. In an IE (where $Z(i) = 0$), we therefore have that $Z'(i) > 0 \Leftrightarrow T'(i) > 0$. From (77), we find that

$$\begin{aligned} T'(i) = {} & -\beta(1+\iota)\left(\frac{1-\eta}{\eta}\right)\frac{1}{\sigma}\frac{\theta(1-\theta)(1+i)^{\frac{1}{\sigma}-1}}{\left[1-\theta+\theta(1+i)^{\frac{1}{\sigma}}\right]^2} \\ & + \frac{\chi(1+\iota)}{1+\chi\iota+(1-\chi)i}q'(i) - \frac{\chi(1-\chi)(1+\iota)}{[1+\chi\iota+(1-\chi)i]^2}q(i) \\ & + \frac{\beta\theta(1+\iota)}{\left[(1+i)\theta+(1+i)^{1-\frac{1}{\sigma}}(1-\theta)\right]^2}\left[(1-\theta)\left(1-\frac{1}{\sigma}\right)(1+i)^{-\frac{1}{\sigma}}+\theta\right]. \end{aligned} \tag{79}$$

Evaluating $T'(i)$ at $\iota = i = 0$ and using the fact that $\eta = \hat{\eta}$ in an IE, we find that

$$T'(i) > 0 \quad \Leftrightarrow \quad \chi(1-\chi)q(0) - \chi q'(0) < \theta\left[\beta - \frac{1}{\sigma}(\beta - \chi q(0))\right]. \tag{80}$$

It remains to determine the derivative $q'(0)$. From the definition of $q(i) = q(\kappa(i))$ in (41), we get that

$$q'(i) = \frac{w(\kappa)}{(f(\kappa)-\delta\kappa)^2}\kappa'(i). \tag{81}$$

Next, defining

$$F(\kappa, i) = f'(\kappa) + 1 - \delta - \frac{1}{\beta}\frac{1+i}{1+i+\chi(\iota-i)},$$

we get from (36) that $F(\kappa, i) = 0$ implicitly defines κ as a function of i. From the implicit function theorem, we get

$$\kappa'(i) = -\frac{F'_i(\kappa,i)}{F'_\kappa(\kappa,i)} = \frac{\chi(1+\iota)}{\beta[1+i+\chi(\iota-i)]^2}\frac{1}{f''(\kappa)}. \tag{82}$$

With a CES production function (47), we have

$$f'(\kappa) = \gamma \left(\frac{f(\kappa)}{\kappa} \right)^{\frac{1}{\rho}} \quad \text{and} \quad f''(\kappa) = -\frac{w(\kappa)}{\rho} \frac{f'(\kappa)}{\kappa f(\kappa)}. \tag{83}$$

Combining (81), (82) and (83) and evaluating $q'(i)$ at $\iota = i = 0$, we get

$$-q'(0) = \frac{\chi\rho}{\beta} \frac{\kappa^* f(\kappa^*)}{f'(\kappa^*)(f(\kappa^*) - \delta\kappa^*)^2}. \tag{84}$$

Inserting (84) into the condition in (80), and using $q^* = q(0)$, yields

$$\begin{aligned} & \rho \leq \frac{\beta f'(\kappa^*)(f(\kappa^*) - \delta\kappa^*)^2}{\kappa^* f(\kappa^*)} \left[\frac{\theta}{\chi^2} \left(\beta - \frac{1}{\sigma}(\beta - \chi q^*) \right) - \left(\frac{1}{\chi} - 1 \right) q^* \right] \\ \Leftrightarrow \quad & \rho \leq \frac{\beta f'(\kappa^*)}{q^*(1 + \delta q^*)} \left[\frac{\theta}{\chi^2} \left(\beta - \frac{1}{\sigma}(\beta - \chi q^*) \right) - \left(\frac{1}{\chi} - 1 \right) q^* \right], \end{aligned}$$

which is the same as condition (49) in Proposition 5.

By continuity, conditions (48) and (49) guarantee coexistence of an IE, a ZE and an FE when $\eta = \hat{\eta}$ and $\iota > 0$ is sufficiently close to zero. ■

B.7 Proof of Proposition 6

The strategy of proof is constructive: we detail the derivation of the map $X \mapsto X_{-1}$, where $X = [\kappa, K, Z]$, with $Z \equiv \phi M_{-1}$, and then show that this map is unique. Of course, we restrict attention to $X \in \mathbb{R}^3_+$. The first step is to derive that there is a unique map $(X, \iota, i) \mapsto X_{-1}$, i.e., once we know the asset market equilibrium, we can backward iterate one period on X. The second step is that given (X, ι, i), we can characterize the money market and asset market clearance conditions. The final step is to show that given X, there is a unique tuple (ι, i) such that $0 \leq i \leq \iota$ (this must always hold to have bounded demand for money and other assets) and the money market and asset market clear.

We show first that we can determine X_{-1} uniquely from (X, ι, i), were we suppose that $0 \leq i \leq \iota$ since this must be true on the equilibrium path. Starting from the resource constraint (2), we have

$$C + K = F(K_{-1}, L) + (1 - \delta)K_{-1}. \tag{85}$$

Using the CRS property of F, that $\kappa \equiv K_{-1}/L$, the fact that $C = (1-\theta)c_0 + \theta c_1$, and the first-order conditions for c_0 and c_1 (see Equation (29)) in the resource constraint gives

$$\begin{aligned} C + K &= L[f(\kappa) + (1 - \delta)\kappa] \\ &= K_{-1}[f(\kappa) + (1 - \delta)\kappa]/\kappa \\ \Rightarrow \quad K_{-1} &= \frac{C + K}{f(\kappa)/\kappa + 1 - \delta} \\ &= \frac{(1 - \theta)c_0 + \theta c_1 + K}{f(\kappa)/\kappa + 1 - \delta} \\ & \frac{(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) + \theta u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota}{1+i}\right) + K}{f(\kappa)/\kappa + 1 - \delta}, \end{aligned} \tag{86}$$

where $w(\kappa)$ is the wage as a function of κ (see Equation (4)).

We then consider the return on capital as in Equation (30):

$$\psi(\kappa) + 1 - \delta = \frac{1}{\beta}\frac{w(\kappa)}{w(\kappa_{-1})}\frac{1+i}{(1-\chi)(1+i)+\chi(1+\iota)}, \tag{87}$$

where $\psi(\kappa)$ is the rental rate for capital as a function of κ (see Equation (3)). Rearranging terms yields

$$w(\kappa_{-1}) = \frac{1}{\beta}\frac{w(\kappa)}{\psi(\kappa)+1-\delta}\frac{1+i}{(1-\chi)(1+i)+\chi(1+\iota)}, \tag{88}$$

Because $w(\kappa)$ is strictly increasing in κ, its inverse exists so that

$$\kappa_{-1} = w^{-1}\left(\frac{1}{\beta}\frac{w(\kappa)}{\psi(\kappa)+1-\delta}\frac{1+i}{(1-\chi)(1+i)+\chi(1+\iota)}\right). \tag{89}$$

We finally recall that $Z \equiv \phi M_{-1}$, so that $Z = Z_{-1}(1+\gamma)/(1+\pi)$. Using the definition of the Fisher rate (10) implies

$$Z_{-1} = Z\frac{\beta(1+\iota)}{1+\gamma}\frac{w(\kappa_{-1})}{w(\kappa)}. \tag{90}$$

Given how $w(\kappa_{-1})$ is determined according to Equation (88), we find Z_{-1} as a function that depends only on (X, ι, i):

$$Z_{-1} = \frac{1+\iota}{1+\gamma}\frac{1+i}{(1-\chi)(1+i)+\chi(1+\iota)}\frac{Z}{\psi(\kappa)+1-\delta}. \tag{91}$$

This last part proves that there is a unique map $(X, \iota, i) \mapsto X_{-1}$.

The second step is to detail how (ι, i) is determined by asset market clearance. To do so, clearance of the market for money requires

$$(1-\theta)c_0 \leq Z, \quad \text{with equality if } i > 0. \tag{92}$$

Using the first-order condition for c_0 (29), we thus have a clearance condition in (X, ι, i):

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) \leq Z, \quad \text{with equality if } i > 0. \tag{93}$$

Clearance for the market for liquid assets requires

$$(1-\theta)c_0 + \theta c_1 \leq Z + (1+i)\phi B_{-1} + \chi[\psi(\kappa) + (1-\delta)]K_{-1}. \tag{94}$$

Here, we focus on $\mathcal{B} = \phi B$ fixed at $\overline{\mathcal{B}}$. This implies $\phi B_{-1} = \overline{\mathcal{B}}/(1+\pi)$. Using the first-order conditions for consumption (29) and the definition of the Fisher rate (10) implies

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) + \theta u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota}{1+i}\right) \leq Z + \frac{\overline{\mathcal{B}}(1+i)}{\beta(1+\iota)}\frac{w(\kappa)}{w(\kappa_{-1})} + \chi[\psi(\kappa)+1-\delta]K_{-1}, \tag{95}$$

with equality if $i < \iota$. We then eliminate K_{-1} and $w(\kappa_{-1})$ by using Equations (86) and (88) from the first step. We thus obtain the asset market clearance condition as function of (X, ι, i):

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right)+\theta u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota}{1+i}\right)\leq Z+[\psi(\kappa)+1-\delta]$$
$$\times\left(\overline{\mathcal{B}}\frac{(1-\chi)(1+i)+\chi(1+\iota)}{1+\iota}+\chi\frac{(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right)+\theta u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota}{1+i}\right)+K}{f(\kappa)/\kappa+1-\delta}\right). \quad (96)$$

Defining

$$\tilde{\psi}(\kappa)\equiv\psi(\kappa)+1-\delta \quad \text{and} \quad \tilde{\varphi}(\kappa)=f(\kappa)/\kappa+1-\delta, \quad (97)$$

we can write Equation (96) more compactly as

$$0\leq Q(X,\iota,i)\equiv Z+\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{(1-\chi)(1+i)+\chi(1+\iota)}{1+\iota}+\frac{\chi\tilde{\psi}(\kappa)K}{\tilde{\varphi}(\kappa)}$$
$$-\left[(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right)+\theta u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota}{1+i}\right)\right]\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}, \quad (98)$$

with equality if $i<\iota$. Note that $\tilde{\psi}(\kappa)<\tilde{\varphi}(\kappa)$ because $f(\kappa)$ is strictly concave. Thus, given X, (ι,i) should be such that Equations (93) and (98) hold, and it should satisfy $0\leq i\leq\iota$.

The third and last step is to show that given X, there is a unique (ι,i) such that $0\leq i\leq\iota$, and Equations (93) and (98) are satisfied. We split this part up by considering two equilibrium cases separately: (a) $0<i\leq\iota$ and (b) $0=i\leq\iota$.

Case (a): $0<i\leq\iota$. With $0<i$, Equation (93) implies

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right)=Z \quad \Rightarrow \quad 1+\iota=w(\kappa)u'\left(\frac{Z}{1-\theta}\right), \quad (99)$$

thus uniquely pinning down ι as a function of X. Because $0<i$ by supposition and $i\leq\iota$, existence of an equilibrium with $0<i\leq\iota$ requires that $0<\iota$. This gives as existence condition

$$Z<(1-\theta)u'^{-1}\left(\frac{1}{w(\kappa)}\right), \quad (100)$$

which depends only on X.

Using Equation (99) allows us to write $Q(X,\iota,i)$ in (98) as

$$Q(X,\iota(X),i)=\frac{\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}Z+\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{(1-\chi)(1+i)+\chi(1+\iota(X))}{1+\iota(X)}+\frac{\chi\tilde{\psi}(\kappa)K}{\tilde{\varphi}(\kappa)}$$
$$-\theta\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota(X)}{1+i}\right), \quad (101)$$

where $\iota(X)$ captures that ι is pinned down as a function of X by Equation (99). Taking the partial derivative of $Q(X,\iota(X),i)$ w.r.t. $1+i$ we find

$$\frac{\partial Q(X,\iota(X),i)}{\partial(1+i)}=\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi}{1+\iota(X)}+\theta\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}\frac{u'(c_1)}{u''(c_1)}\frac{1}{1+i}. \quad (102)$$

Using that $cu''(c)/u'(c) > -1; \forall c$, we thus have

$$\begin{aligned}\frac{\partial Q(X,\iota(X),i)}{\partial(1+i)} &< \tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi}{1+\iota(X)} - \theta\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}\frac{u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota(X)}{1+i}\right)}{1+i} \\ &= -\frac{1}{1+i}\left[\frac{\chi\tilde{\psi}(\kappa)[Z+K]}{\tilde{\varphi}(\kappa)} + \chi\tilde{\psi}(\kappa)\overline{\mathcal{B}} - Q(X,\iota(X),i)\right]\end{aligned} \tag{103}$$

Because $i \leq \iota$ and $Q(X,\iota(X),i) \geq 0$, with equality if $i < \iota$, it follows directly that an $i \leq \iota(X)$ which solves $Q(X,\iota(X),i) \geq 0$ (with equality if $i < \iota(X)$) is unique. Further, it exists and it satisfies $0 < i$ (this must hold by supposition) if and only if $Q(X,\iota(X),0) > 0$, which in turn holds if and only if

$$\frac{[\chi\tilde{\psi}(\kappa)-\theta\tilde{\varphi}(\kappa)]Z + \chi\tilde{\psi}(\kappa)(1-\theta)K}{(1-\theta)\tilde{\varphi}(\kappa)} + \tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi+\chi(1+\iota(X))}{1+\iota(X)} > 0. \tag{104}$$

Using (99), the above reads as

$$\frac{[\chi\tilde{\psi}(\kappa)-\theta\tilde{\varphi}(\kappa)]Z + \chi\tilde{\psi}(\kappa)(1-\theta)K}{(1-\theta)\tilde{\varphi}(\kappa)} + \tilde{\psi}(\kappa)\overline{\mathcal{B}}\left[\frac{1-\chi}{w(\kappa)u'\left(\frac{Z}{1-\theta}\right)} + \chi\right] > 0, \tag{105}$$

which depends only on X. Thus, for the case $0 < i \leq \iota$, given X a tuple (ι, i) that solves Equations (93) and (98) exists if

$$X \in \mathscr{X}_{i>0} \equiv \left\{X \in \mathbb{R}^3_+ \quad \text{s.t. (100) and (105)}\right\}, \tag{106}$$

and this (ι, i) is pinned down uniquely as a function of X.

Case b: $0 = i \leq \iota$. With $i = 0$, Equation (98) implies

$$0 \leq Q(X,\iota,0) = Z + \tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi+\chi(1+\iota)}{1+\iota} + \frac{\chi\tilde{\psi}(\kappa)K}{\tilde{\varphi}(\kappa)} - u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right)\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}, \tag{107}$$

with equality if $0 < \iota$. Taking the partial derivative of $Q(X,\iota,0)$ w.r.t. $1+\iota$ yields

$$\frac{\partial Q(X,\iota,0)}{\partial(1+\iota)} = -\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi}{(1+\iota)^2} - \frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}\frac{u'(C)}{u''(C)}\frac{1}{1+\iota} \tag{108}$$

Using that $cu''(c)/u'(c) > -1; \forall c$, we thus have

$$\begin{aligned}\frac{\partial Q(X,\iota,0)}{\partial(1+\iota)} &> -\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi}{(1+\iota)^2} + \frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right)\frac{1}{1+\iota} \\ &= \frac{Z + \chi\tilde{\psi}(\kappa)K/\tilde{\varphi}(\kappa) + \tilde{\psi}\chi\overline{\mathcal{B}} - Q(X,\iota,0)}{1+\iota}\end{aligned} \tag{109}$$

Because $0 \leq \iota$ and $Q(X,\iota,0) \geq 0$ (with equality if $0 < \iota$), it follows directly that an $0 \leq \iota$ which solves $Q(X,\iota,0) \geq 0$ (with equality if $0 < \iota$) is unique. Equilibrium existence however requires that Equation (93) holds, i.e.,

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) \leq Z \quad \Rightarrow \quad 1+\iota \geq w(\kappa)u'\left(\frac{Z}{1-\theta}\right) \tag{110}$$

Since the $0 \leq \iota$ which solves $Q(X, \iota, 0) \geq 0$ (with equality if $0 < \iota$) is non-negative, Condition (110) is satisfied trivially if

$$Z \geq (1-\theta)u'^{-1}\left(\frac{1}{w(\kappa)}\right), \tag{111}$$

which is the exact opposite of Condition (100). If Condition (111) does not hold, then Condition (110) is satisfied if and only if $Q(X, \underline{\iota}, 0) > 0$, where $1+\underline{\iota} \equiv w(\kappa)u'\left(\frac{Z}{1-\theta}\right)$. This translates into

$$\frac{[\chi\tilde{\psi}(\kappa) - \theta\tilde{\varphi}(\kappa)]Z + \chi\tilde{\psi}(\kappa)(1-\theta)K}{(1-\theta)\tilde{\varphi}(\kappa)} + \tilde{\psi}(\kappa)\overline{\mathcal{B}}\left[\frac{1-\chi}{w(\kappa)u'\left(\frac{Z}{1-\theta}\right)} + \chi\right] \leq 0, \tag{112}$$

which is the exact opposite of Condition (105). Thus, for the case $0 = i \leq \iota$, given X, a tuple (ι, i) that solves Equations (93) and (98) exists if and only if

$$X \in \mathscr{X}_{i=0} \equiv \left\{X \in \mathbb{R}^3_+ \quad \text{s.t. } (111) \text{ or } (112)\right\}, \tag{113}$$

and this (ι, i) is pinned down uniquely as a function of X.

Combining insights from the cases (a) and (b), we see that $\mathscr{X}_{i=0}$ is the complement of $\mathscr{X}_{i>0}$. Hence, given X, we find a unique (ι, i) such that: (i) $0 \leq i \leq \iota$; (ii) the money market clears, i.e., Equation (93) holds; and (iii) the asset market clears, i.e., Equation (98) holds. In other words, there is a unique map $X \mapsto (\iota, i)$. We also established there is a unique map $(X, \iota, i) \mapsto X_{-1}$, thus establishing a unique map $X \mapsto X_{-1}$; there is only one way to backward iterate on X such that the equilibrium conditions are satisfied. ■

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