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Author(s):
Bakshi, Rhea Palak; Kim, Seongjeong; Wang, Xiao

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ON THE KAUFFMAN BRACKET SKEIN MODULE OF $(S^1 \times S^2) \# (S^1 \times S^2)$

RHEA PALAK BAKSHI, SEONGJEONG KIM, AND XIAO WANG

Abstract. Determining the structure of the Kauffman bracket skein module of all 3-manifolds over the ring of Laurent polynomials $\mathbb{Z}[A^{\pm 1}]$ is a big open problem in skein theory. Very little is known about the skein module of non-prime manifolds over this ring. In this paper, we compute the Kauffman bracket skein module of the 3-manifold $(S^1 \times S^2) \# (S^1 \times S^2)$ over the ring $\mathbb{Z}[A^{\pm 1}]$. We do this by analysing the submodule of handle sliding relations, for which we provide a suitable basis. Along the way we also compute the Kauffman bracket skein module of $(S^1 \times S^2) \# (S^1 \times D^2)$.

Contents

1. Introduction

Skein modules are 3-manifold invariants that generalise the skein theory of polynomial link invariants in S^3 to any arbitrary 3-manifold. They were introduced independently by Przytycki [Prz1] and Turaev [Tur] in the late 1980's, and have since become indispensable in bridging the fields of quantum topology, knot theory, algebraic geometry, hyperbolic geometry, and physics. The Kauffman bracket skein module, which serves as a generalisation of the Kauffman bracket polynomial to arbitrary 3-manifolds, is conceivably the best understood skein module of all. In this paper, we determine the Kauffman bracket skein module of $(S^1 \times S^2) \# (S^1 \times S^2)$. One motivation for our work comes from constructing traces, such as the Yang-Mills measure, on the

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Kauffman bracket skein module of a thickened surface. In [BFK], the Yang-Mills measure is defined away from roots of unity using the Kauffman bracket skein module, henceforth known simply as the skein module, of $\#_k(S^1 \times S^2)$ over the field $\mathbb{C}$. To construct other traces on the skein module of a surface at roots of unity, it is imperative to know the structure of the skein module of $\#_k(S^1 \times S^2)$ over $\mathbb{Z}[A^{\pm 1}]$. Knowing the traces on a skein module will aid us in the construction of the 3-manifold invariants that may be defined as traces. For example, the Yang-Mills measure may be used to define the Turaev-Viro invariant at roots of unity. We expect that our computation of the skein module of $(S^1 \times S^2) \# (S^1 \times S^2)$ will prove to be effective in this regard.

From the perspective of algebraic geometry, it is known that, modulo the nilradical, the Kauffman bracket skein module of an oriented 3-manifold over $\mathbb{C}[A^{\pm 1}]$ with $A = -1$ has an algebra structure that is isomorphic to the coordinate ring of the $SL(2, \mathbb{C})$ character variety of the fundamental group of that manifold [Bul, PS]. Furthermore, if the underlying algebraic set $X(M)$ of the $SL(2, \mathbb{C})$ character variety of the fundamental group of a closed oriented 3-manifold M is infinite, then the skein module of M is wild, that is, it is not tame [DKS].[1] For example, the underlying algebraic set $X(M)$ of the $SL(2, \mathbb{C})$ character variety of $\pi_1(S^1 \times S^2)$ is infinite, and hence, the Kauffman bracket skein module of $S^1 \times S^2$ is not tame as has been proved by Hoste and Przytycki in [HP2]. In fact, until now, $S^1 \times S^2$ is the only closed 3-manifold with infinite $X(M)$ whose skein module has been computed. The manifold $(S^1 \times S^2) \# (S^1 \times S^2)$ is the next example of the skein module of a closed 3-manifold with this property (see [GM]).

Furthermore, the resolution of Witten's finiteness conjecture for Kauffman bracket skein modules in [GJS] implies that over $\mathbb{Q}(A)$, the Kauffman bracket skein module of any closed oriented 3-manifold is always finite dimensional. However, over $\mathbb{Z}[A^{\pm 1}]$, the structure of the skein module is more complicated. For example, the skein module of $S^1 \times S^2$ is infinitely generated over $\mathbb{Z}[A^{\pm 1}]$ [HP2]. Recently, the first author [Bak] disproved a conjecture posited by Marché (see [DW]), which stated that the skein module of any closed oriented 3-manifold can be decomposed into free and torsion modules. The counterexample to this conjecture was given by the skein module of the connected sum of two copies of the real projective space (see [Mro]). This emphasises the fact that save for a handful of manifolds, the structure of the skein module is not as well understood over $\mathbb{Z}[A^{\pm 1}]$ as it is over $\mathbb{Q}(A)$. To better understand the structure of the skein module of oriented 3-manifolds over $\mathbb{Z}[A^{\pm 1}]$, we study the skein module of the connected sums of 3-manifolds. Thus, with these motivations, we compute the skein module of $(S^1 \times S^2) \# (S^1 \times S^2)$.

The paper is organised as follows. In Section 2 we define absolute and relative Kauffman bracket skein modules and discuss some of their properties. We include a description of the module using generators and relations. We then compute the skein module of $(S^1 \times S^2) \# (S^1 \times S^2)$ in Section 3. Our technique employs the relative Kauffman bracket skein module in determining the complete set of handle sliding relations. We include all our calculations towards this computation in the Appendix. Furthermore, in Section 4 we provide an obstruction for the decomposition of the skein module of $(S^1 \times S^2) \# (S^1 \times S^2)$ into the direct sum of free and $(A^k - A^{-k})$-torsion modules, for each $k \geq 1$ and in Section 5, we discuss future directions.

[1] A $\mathbb{Z}[A^{\pm 1}]$-module is said to be tame if it is a direct sum of cyclic $\mathbb{Z}[A^{\pm 1}]$-modules and it does not contain $\mathbb{Z}[A^{\pm 1}]/(\phi_{2N})$ as a submodule, for at least one odd N, where ϕ_{2N} is the $2N$-th cyclotomic polynomial.

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2. Basic definitions and properties

We begin with introducing the Kauffman bracket skein module and the relative Kauffman bracket skein module.

Definition 2.1. *Let M be an oriented 3-manifold, R a commutative ring with unity, and $A \in R$ a fixed invertible element. Consider the set of ambient isotopy classes of unoriented framed links (including the empty link $\varnothing$) in M, which we denote by $\mathcal{L}^{fr}$, and the free R-module with basis $\mathcal{L}^{fr}$, denoted by $R\mathcal{L}^{fr}$. Let $S_{2,\infty}^{sub}$ be the submodule of $R\mathcal{L}^{fr}$ generated by the following expressions:*

(1) the Kauffman bracket skein expression: $L_+ - AL_0 - A^{-1}L_\infty$ and
(2) the trivial component expression: $L \sqcup \bigcirc + (A^2 + A^{-2})L$,

where $\bigcirc$ denotes the trivial framed knot in M and the skein triple (L_+, L_0, L_∞) denotes three framed links in M, which are identical except in a small 3-ball in M where they differ as illustrated in Figure 2.1.

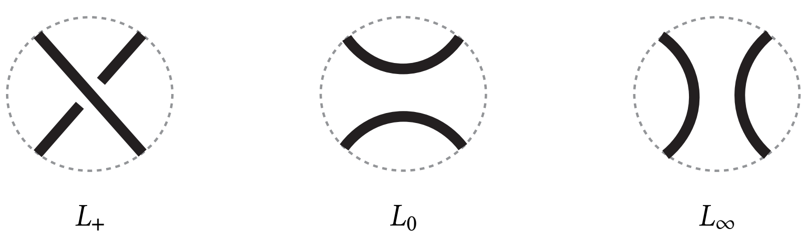


Figure 2.1. Skein triple for the Kauffman bracket skein module.

The **Kauffman bracket skein module** *of M is defined as the quotient:*

$$\mathcal{S}_{2,\infty}(M; R, A) = \frac{R\mathcal{L}^{fr}}{S_{2,\infty}^{sub}}.$$

Computations of the Kauffman bracket skein module for various 3-manifolds have been carried out over several rings, such as $\mathbb{Z}[A^{\pm1}]$, $\mathbb{Q}(A)$, or a ring R in which $A^k - 1$ is invertible for all k. In our paper, we work over $\mathbb{Z}[A^{\pm1}]$ and use the notation $\mathcal{S}_{2,\infty}(M)$ in this case. The existence of the Kauffman bracket polynomial can be interpreted in the language of skein modules as follows.

Example 2.2. [Kau] $\mathcal{S}_{2,\infty}(S^3) = \mathbb{Z}[A^{\pm1}]\varnothing$. *More precisely, $\varnothing$ is the basis element of the module and $L = [L]\varnothing = (-A^2 - A^{-2})\langle L \rangle\varnothing$, where $[L]$ is the unreduced Kauffman bracket polynomial of a framed link L. Moreover, $\mathcal{S}_{2,\infty}(B^3) \cong \mathcal{S}_{2,\infty}(\mathbb{R}^3) = \mathbb{Z}[A^{\pm1}]\varnothing$.*

We can also define a relative version of the Kauffman bracket skein module for oriented 3-manifolds that have framed (or marked) points on their boundaries (see [Prz1, Prz3]).

Definition 2.3. *Let $(M, \partial M)$ be an oriented 3-manifold, $\{x_i\}_1^{2n}$ be a set of $2n$ oriented framed*[2] *points on ∂M, and R be a commutative ring with unity with a fixed invertible element A. Let $\mathcal{L}^{fr}(2n)$ be the set of all relative framed links in $(M, \partial M)$ considered up to ambient isotopy keeping ∂M fixed, such that $L \cap \partial M = \partial L = \{x_i\}_1^{2n}$. Consider the submodule $S_{2,\infty}^{sub}(2n)$ of the free R-module $R\mathcal{L}^{fr}(2n)$ generated by the Kauffman bracket skein expressions. Then the* **relative Kauffman bracket skein module**, *henceforth known as the relative skein module, of M is the quotient:*

$$\mathcal{S}_{2,\infty}(M, \{x_i\}_1^{2n}; R, A) = \frac{R\mathcal{L}^{fr}(2n)}{S_{2,\infty}^{sub}(2n)}.$$

We will use the notation $\mathcal{S}_{2,\infty}(M, \{x_i\}_1^{2n})$ when $R = \mathbb{Z}[A^{\pm 1}]$. The following results about the skein module and its relative version of the product of an oriented surface with the unit interval are pertinent to our work.

Theorem 2.4. [Prz1, Prz3]
Let Σ be an oriented surface in which each link is equipped with blackboard framing and let I denote the unit interval $[0, 1]$. Then $\mathcal{S}_{2,\infty}(\Sigma \times I; R, A)$ is a free R-module whose basis consists of the empty link $\varnothing$ and simple closed curves in Σ that have no trivial components. This applies in particular to handlebodies, since $H_n = \Sigma_{0,n+1} \times I$, where H_n is a handlebody of genus n and $\Sigma_{g,b}$ denotes a genus g surface with b boundary components.

The following example discusses the skein module of the thickened annulus.

Example 2.5. *$\mathcal{S}_{2,\infty}(\Sigma_{0,2} \times I; R, A)$ is free and infinitely generated by the curves $\{x^i\}_0^{\infty}$, where x denotes the homotopically nontrivial simple closed curve on the annulus and x^0 denotes the empty link $\varnothing$. Note that, $\mathcal{S}_{2,\infty}(S^1 \times D^2; R, A) \cong \mathcal{S}_{2,\infty}(\Sigma_{0,2} \times I; R, A)$.*

A result similar to Theorem 2.4 also holds for relative skein modules.

Theorem 2.6. [Prz3]
Let Σ be an oriented surface, where $\partial\Sigma \neq \emptyset$, and let $\{x_i\}_1^{2n}$ be $2n$ oriented framed points centred at $\partial\Sigma \times \{\frac{1}{2}\}$. Then $\mathcal{S}_{2,\infty}(\Sigma \times I, \{x_i\}_1^{2n}; R, A)$ is a free R-module whose basis is composed of relative links[3] *in $\Sigma \times \{\frac{1}{2}\}$ without trivial components.*

The skein module of a surface times an interval may be equipped with an algebra structure for which the multiplication operation is defined as follows.

Definition 2.7. *Consider two framed links L_1 and L_2 in $\Sigma \times I$. Define their product $\cdot$ by placing L_1 over L_2 in $\Sigma \times I$, that is, $L_1 \cdot L_2 = L_1 \sqcup L_2$ such that $L_1 \subset \Sigma \times (\frac{1}{2}, 1)$ and $L_2 \subset \Sigma \times (0, \frac{1}{2})$. The empty link $\varnothing$ serves as the multiplicative identity. This multiplication endows the skein module of a thickened surface $\Sigma \times I$ with a natural algebra structure. The Kauffman bracket skein module equipped with this algebra structure is called the Kauffman bracket skein algebra.*

We denote the Kauffman bracket skein algebra, henceforth known simply as the skein algebra, by $\mathcal{S}^{alg}(\Sigma; R, A)$. This new notation emphasises the fact that the skein algebra depends on the surface and its product structure. For brevity, we use the notation $\mathcal{S}^{alg}(\Sigma)$ when $R = \mathbb{Z}[A^{\pm 1}]$.

[2] A framed point in ∂M is an interval in ∂M. Thus, a relative framed link intersects ∂M at framed points.

[3] Relative links in Σ are families of properly embedded arcs and closed curves in $\Sigma \times \{\frac{1}{2}\}$.

Remark 2.8. *$\mathcal{S}_{2,\infty}(\Sigma \times I, \{x_i\}_1^{2n}; R, A)$ is a bimodule over the algebra $\mathcal{S}^{alg}(\Sigma; R, A)$, which contains the ring R. Let L_1 be a relative framed link in $\Sigma \times I$ and L_2 be a framed link in $\Sigma \times I$. Then, $L_1 \cdot L_2$ is defined by placing L_1 above L_2, that is, $L_1 \subset \Sigma \times (\frac{1}{3}, 1)$ and $L_2 \subset \Sigma \times (0, \frac{1}{3})$. Similarly, $L_2 \cdot L_1$ is defined by placing L_2 over L_1, that is, $L_2 \subset \Sigma \times (\frac{2}{3}, 1)$ and $L_1 \subset \Sigma \times (0, \frac{2}{3})$.*

We now state some properties of the skein module required for proving our main results.

Theorem 2.9. [Prz3]

(1) Let $i : M \hookrightarrow N$ be an orientation preserving embedding of 3-manifolds. This yields a homomorphism $i_* : \mathcal{S}_{2,\infty}(M; R, A) \longrightarrow \mathcal{S}_{2,\infty}(N; R, A)$ of skein modules. This correspondence leads to a functor from the category of 3-manifolds and orientation preserving embeddings (up to ambient isotopy) to the category of R-modules with a specified invertible element $A \in R$.

(2) Let $M = M_1 \sqcup M_2$ be the disjoint union of oriented 3-manifolds M_1 and M_2. Then

$$\mathcal{S}_{2,\infty}(M; R, A) \cong \mathcal{S}_{2,\infty}(M_1; R, A) \otimes_R \mathcal{S}_{2,\infty}(M_2; R, A).$$

(3) (The Universal Coefficient Property) Let R and R' be commutative rings with unity and $r : R \longrightarrow R'$ be a homomorphism. Then the identity map on $\mathcal{L}^{fr}$ induces the following isomorphism of R' (and R) modules:

$$\bar{r} : \mathcal{S}_{2,\infty}(M; R, A) \otimes_R R' \longrightarrow \mathcal{S}_{2,\infty}(M; R', r(A)).$$

The following theorem determines how the Kauffman bracket skein module behaves under handle addition, thereby giving its presentation in terms of generators and relations.

Theorem 2.10. [Prz3, HP1]

(1) If N is obtained from M by adding a 3-handle to M and $i : M \hookrightarrow N$ is the associated embedding, then the induced homomorphism $i_* : \mathcal{S}_{2,\infty}(M; R, A) \longrightarrow \mathcal{S}_{2,\infty}(N; R, A)$ is an isomorphism.

(2) (Handle Sliding Lemma) Let $(M, \partial M)$ be a 3-manifold with boundary and γ be a simple closed curve on ∂M. Additionally, let $N = M_\gamma$ be the 3-manifold obtained from M by adding a 2-handle along γ and $i : M \hookrightarrow N$ be the associated embedding. Then the induced homomorphism $i_* : \mathcal{S}_{2,\infty}(M; R, A) \longrightarrow \mathcal{S}_{2,\infty}(N; R, A)$ is an epimorphism. Furthermore, the kernel of i_* is generated by the relations yielded by 2-handle slidings. In particular, if $\mathcal{L}^{fr}_{gen}$ is a set of framed links in M that generates $\mathcal{S}_{2,\infty}(M; R, A)$, then $\mathcal{S}_{2,\infty}(N; R, A) \cong \mathcal{S}_{2,\infty}(M; R, A)/\mathcal{J}$, where $\mathcal{J}$ is the submodule of $\mathcal{S}_{2,\infty}(M; R, A)$ generated by the expressions $L - sl_\gamma(L)$. Here $L \in \mathcal{L}^{fr}_{gen}$ and $sl_\gamma(L)$ is obtained from L by sliding it along γ.

The handle sliding lemma can be generalised to the case where a manifold is obtained by attaching more than one 2-handle to the 3-manifold M. The following result by McLendon states this precisely.

Proposition 2.11. [McL]

Let $(M, \partial M)$ be a 3-manifold with boundary and β and η be disjoint simple closed curves in ∂M. Glue two 2-handles to M, one each along the curves β and η, and denote the resultant 3-manifold by N. If $\mathcal{J}_1$ is the submodule of $\mathcal{S}_{2,\infty}(M; R, A)$ generated by handle slides along β and

$\mathcal{J}_2$ is the submodule of $\mathcal{S}_{2,\infty}(M;R,A)$ generated by handle slides along η, then $\mathcal{S}_{2,\infty}(N;R,A) \cong \mathcal{S}_{2,\infty}(M;R,A)/(\mathcal{J}_1 + \mathcal{J}_2)$.

Thus, Theorem 2.10.1, the handle sliding lemma, and Proposition 2.11 together lead to the following result.

Theorem 2.12. [Prz3]
Let M be a compact oriented 3-manifold. Then M is obtained from a genus m handlebody H_m by adding 2- and 3-handles to it and the generators of $\mathcal{S}_{2,\infty}(M;R,A)$ are generators of $\mathcal{S}_{2,\infty}(H_m;R,A)$, while the relations of $\mathcal{S}_{2,\infty}(M;R,A)$ are yielded by 2-handle slidings.

The handle sliding lemma and Theorem 2.12 reduce the problem of computing the skein module of any compact oriented 3-manifold to that of determining all the 2-handle sliding relations. We now prove the following small result about how the Kauffman bracket skein module behaves under 0-handle addition.

Proposition 2.13. *Let N be a 3-manifold obtained by adding a 0-handle to M and $i : M \hookrightarrow N$ be the associated embedding. Then $\mathcal{S}_{2,\infty}(N;R,A) \cong \mathcal{S}_{2,\infty}(M;R,A) \otimes_R R$.*

Proof. Gluing a 0-handle to ∂M is the same as taking the disjoint union of M with the 3-ball B^3. Thus, from Theorem 2.9.2, we get that $\mathcal{S}_{2,\infty}(N;R,A) \cong \mathcal{S}_{2,\infty}(M;R,A) \otimes_R \mathcal{S}_{2,\infty}(B^3;R,A)$. An application of Example 2.2 gives us the result.

□

We remark that there is no definitive result for the skein module under 1-handle addition. For example, gluing a 1-handle to a 3-ball results in a solid torus. Their skein modules are discussed in Examples 2.2 and 2.5 and should be compared. In the next section we compute the Kauffman bracket skein module of $(S^1 \times S^2) \# (S^1 \times S^2)$.

3. The Kauffman bracket skein module of $(S^1 \times S^2) \# (S^1 \times S^2)$

Let β and η be two simple closed curves in the boundary of the genus two handlebody, H_2, as illustrated in Figure 3.1. Glue a 2-handle along each of these curves and then cap off the holes with two 3-handles. The resultant 3-manifold is $(S^1 \times S^2) \# (S^1 \times S^2)$. From Theorems 2.10.1 and 2.10.2, it follows that the natural embedding $i : H_2 \hookrightarrow (S^1 \times S^2) \# (S^1 \times S^2)$ yields the epimorphism $i_* : \mathcal{S}_{2,\infty}(H_2) \longrightarrow \mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2))$ of skein modules. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the submodules of $\mathcal{S}_{2,\infty}(H_2)$ generated by the handle sliding relations obtained from 2-handle sliding along β and η, respectively. From Theorems 2.11 and 2.12 we get that $\mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2)) \cong \mathcal{S}_{2,\infty}(H_2)/(\mathcal{J}_1 + \mathcal{J}_2)$. Thus, our main problem reduces to determining the submodules $\mathcal{J}_1$ and $\mathcal{J}_2$.

Note that, $H_2 \cong \Sigma_{0,3} \times I$. Since all framed links in $(S^1 \times S^2) \# (S^1 \times S^2)$ have representatives in H_2, framed links in $(S^1 \times S^2) \# (S^1 \times S^2)$ may be presented by link diagrams on $\Sigma_{0,3}$; see Theorem 2.4. We illustrate the curves β and η as line segments in $\Sigma_{0,3}$; see Figure 3.2b. Thus, we first describe the skein module and algebra of $\Sigma_{0,3} \times I$.

3.1. **The skein module of $\Sigma_{0,3} \times I$.**

From Theorem 2.4 we obtain the following result about the skein module of $\Sigma_{0,3} \times I$.

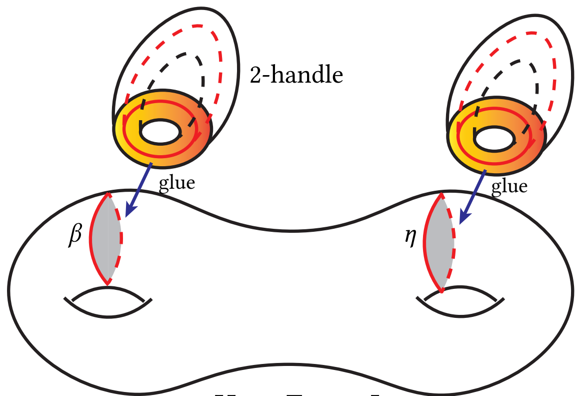
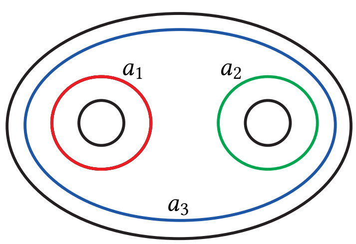


FIGURE 3.1. The 3-manifold $(S^1 \times S^2) \# (S^1 \times S^2)$ is obtained by gluing a 2-handle each to ∂H_2 along the curves β and η.

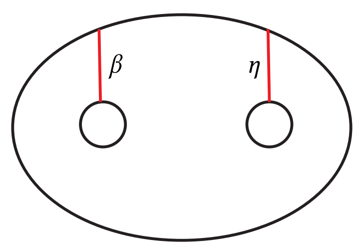
Theorem 3.1. [Prz1, Prz3]
$\mathcal{S}_{2,\infty}(\Sigma_{0,3} \times I)$ is a free and infinitely generated $\mathbb{Z}[A^{\pm 1}]$-module whose standard basis consists of monomials of the form $\{a_1^i a_2^j a_3^k\}_{i,j,k \geq 0}$, where a_1, a_2, and a_3 represent the homotopically nontrivial curves on $\Sigma_{0,3}$ as illustrated in Figure 3.2a. The empty link is represented by $a_1^0 a_2^0 a_3^0$.

Note that the set $\{S_i(a_1)S_j(a_2)S_k(a_3)\}_{i,j,k \geq 0}$ also forms a basis for $\mathcal{S}_{2,\infty}(\Sigma_{0,3} \times I)$. Here, S_q denotes the Chebyshev polynomials of the second kind, which satisfy the recurrence relation $S_{q+1}(x) = xS_q(x) - S_{q-1}(x)$, with the initial conditions $S_0(x) = 1$ and $S_1(x) = x$. We also have the following result due to Bullock and Przytycki about the skein algebra of $\Sigma_{0,3}$.

Theorem 3.2. [BuPr]
$\mathcal{S}^{alg}(\Sigma_{0,3})$ is a commutative algebra and is isomorphic to $\mathbb{Z}[A^{\pm 1}][a_1, a_2, a_3]$.



(A) The generators of $\mathcal{S}^{alg}\Sigma_{0,3}$.



(B) Projection of $(S^1 \times S^2) \# (S^2 \times S^2)$ onto $\Sigma_{0,3}$.

FIGURE 3.2.

Henceforth, we will use the notation a_1, a_2, and a_3 for the boundary parallel curves interchangeably with the boundary components they surround. We first compute the submodule $\mathcal{J}_1$ of $\mathcal{S}_{2,\infty}(H_2)$. The submodule $\mathcal{J}_2$ may be obtained symmetrically.

3.2. **Handle sliding relations from relative skein modules.** We appeal to relative Kauffman bracket skein modules to compute the submodule $\mathcal{J}_1$ of handle sliding relations. Consider the two marked points u and v, such that they lie on the simple closed curve β in ∂H_2 and they divide the curve β into two curves β_1 and β_2 (see Figure 3.3).

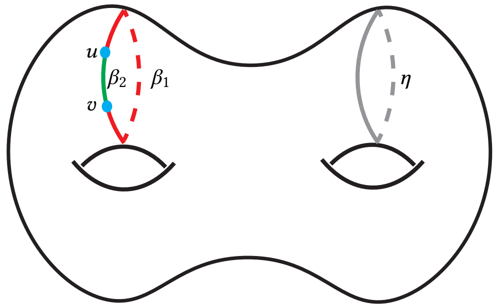


FIGURE 3.3. Marked points u and v on the simple closed curve β in ∂H_2 that divide it into curves β_1 and β_2.

Consider any relative curve α in $(H_2; u, v)$. Now, handle slidings in $(H_2)_\beta$ take place locally in the neighbourhood of the curve β. Consider fixed tangents at the points u and v and let the relative curve α approach these points along the tangents. For every relative curve α, handle sliding in $(H_2)_\beta$ replaces the curve $\alpha \cup \beta_2$ with the curve $\alpha \cup \beta_1$. This gives the handle sliding relation, $\alpha \cup \beta_2 \equiv \alpha \cup \beta_1$. By introducing the $\mathbb{Z}[A^{\pm 1}]$-linear homomorphism $\omega : \mathcal{S}_{2,\infty}(H_2; u, v) \longrightarrow \mathcal{S}_{2,\infty}(H_2)$, defined by $\omega(\alpha) = \alpha \cup \beta_2 - \alpha \cup \beta_1$, we see that $\omega(\mathcal{S}_{2,\infty}(H_2; u, v)) = \mathcal{J}_1$. Hence, the image of any basis of $\mathcal{S}_{2,\infty}(H_2; u, v)$ generates $\mathcal{J}_1$. See Figure 3.4 for a visual explanation. We note that this method of describing 2-handle sliding relations was pioneered by Bullock and Lo Faro in [BuLo]. See also [BLP].

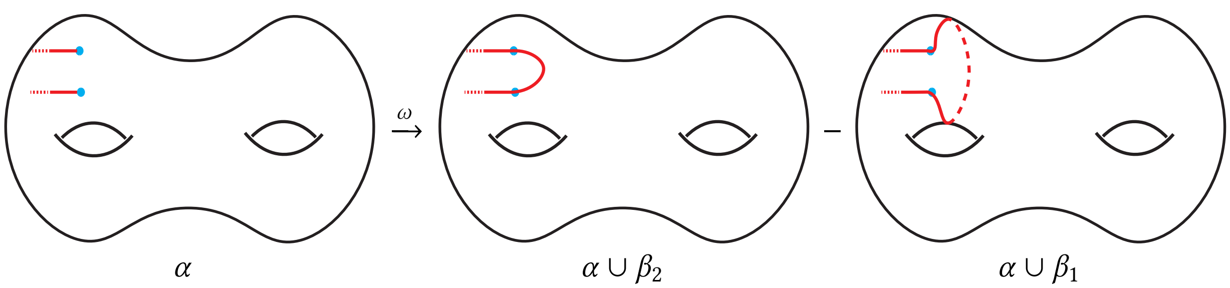


FIGURE 3.4. Illustration of $\omega(\alpha)$.

We emphasise that the submodule $\mathcal{J}_2$ of handle sliding relations that correspond to the 2-handle glued to ∂H_2 along the curve η may be obtained in a symmetric manner. We now discuss a basis of the relative skein module of $(H_2; u, v)$.

3.3. Basis of the relative skein module of $(H_2; u, v)$.

Consider $\Sigma_{0,3}$ with marked points u and v on its boundary as illustrated in Figure 3.5.

Proposition 3.3. *Any relative curve connecting the points u and v is of the form $c_{k,m}$, $k, m \in \mathbb{Z}$. The curves $c_{k,m}$ for small k and m are illustrated in Figure 3.6.*[4]

[4]The mapping class group Mod^+ of the twice punctured disc is isomorphic to the braid group on two strands, which is isomorphic to $\mathbb{Z}$. Furthermore, $Mod^+(\Sigma_{0,3}) \cong PB_2 \times \mathbb{Z} \times \mathbb{Z}$, where PB_2 is the pure braid group on two strands, which is again isomorphic to $\mathbb{Z}$. Hence, $Mod^+(\Sigma_{0,3}) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

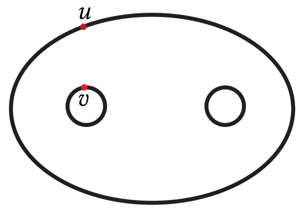


FIGURE 3.5. Marked points u and v on $\partial\Sigma_{0,3}$, where u lies on the boundary component a_3 and v lies on the boundary component a_1.

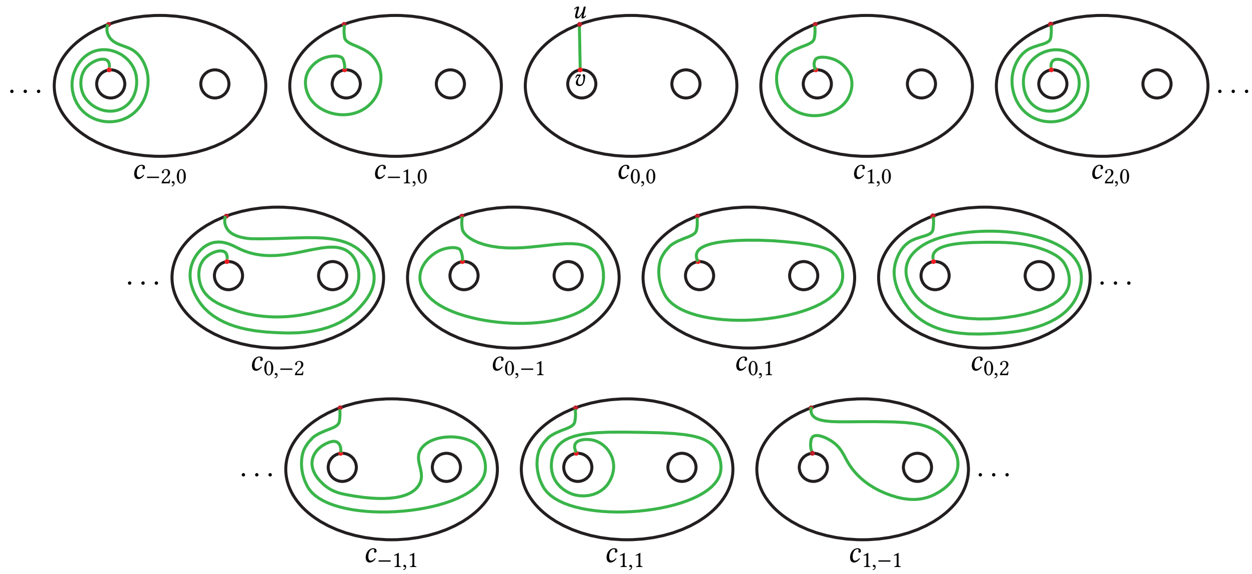


FIGURE 3.6. Relative curves in $(\Sigma_{0,3} \times I; u, v)$.

Note that the boundary curve a_2 multiplicatively commutes with $c_{k,m}$, for all k and m, while the boundary curves a_1 and a_3 do not commute with any $c_{k,m}$. Thus, from Theorem 2.6, Remark 2.8, and Proposition 3.3 we get the following result.

Corollary 3.4. *The elements $c_{k,m}S_i(a_2)$, $i \in \mathbb{Z}^+ \cup \{0\}$ form a basis for $\mathcal{S}_{2,\infty}(H_2; u, v)$.*

3.4. **Generators of the submodule $\mathcal{J}_1$.** From Corollary 3.4 it follows that $\mathcal{J}_1$ is generated by $\omega(c_{m,n}a_2^k) = \omega(c_{m,n})a_2^k$. To find the generators of $\mathcal{J}_1$, let us consider $C(m,n) := \omega(c_{m,n})$ for all $m, n \in \mathbb{Z}$. We have the following three cases:

(1) $m, n \geq 0$,
(2) $m \geq 1, n \leq -1$,
(3) remaining cases.

Case I: $C(m,n)$ for $m, n \geq 0$.
For $m, n \geq 0$, $C(m,n) = \omega(c_{m,n}) = c_{m,n} \cup \beta_2 - c_{m,n} \cup \beta_1$. As an example, $C(2,1)$ is illustrated in Figure 3.7. Since $c_{2,1} \cup \beta_1$ has a negative kink and $c_{2,1} \cup \beta_2$ has a positive kink, we obtain two diagrams without kinks with coefficients $-A^{-3}$ and $-A^3$, respectively. Let us denote the resultant diagrams without kinks by $N(2,1)$ and $P(2,1)$, respectively. In general, $C(m,n) = \omega(c_{m,n})$ can be written as the linear combination

$$C(m,n) = -A^3 P(m,n) + A^{-3} N(m,n), m, n \geq 0,$$

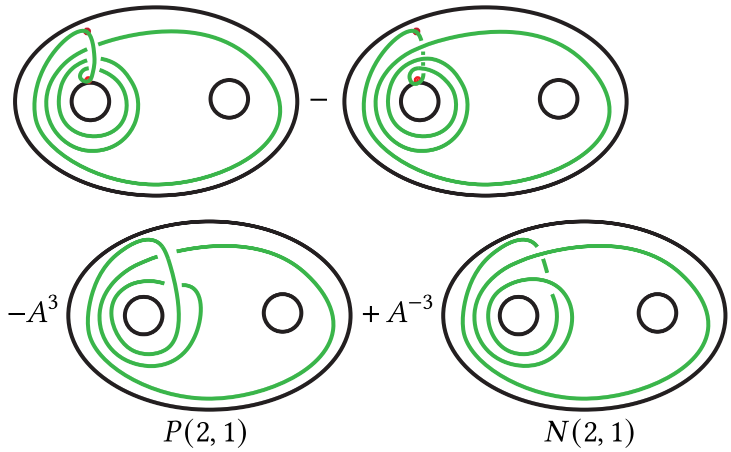


FIGURE 3.7. Illustration of $C(2, 1)$.

where $N(m, n)$ and $P(m, n)$ are obtained from $c_{m,n} \cup \beta_1$ and $c_{m,n} \cup \beta_2$, respectively, by removing kinks as described in Figure 3.8.

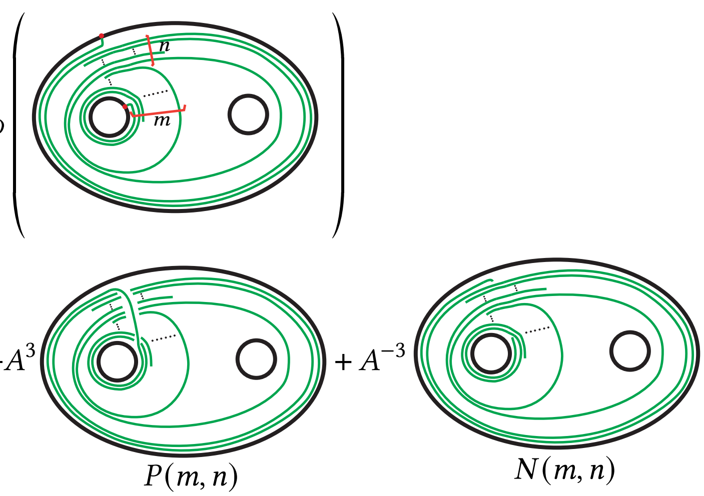


FIGURE 3.8. Illustration of $C(m, n)$.

Our goal is to find closed formulae for $P(m, n)$ and $N(m, n)$, which we achieve through the following series of lemmas and corollaries. Their proofs will be provided in the Appendix. Since the diagram for $N(m, n)$ is the mirror image of the diagram for $P(m, n)$, it is sufficient to find the formula for $P(m, n)$. We obtain the following lemma for $P(m, n)$.

Lemma 3.5. *There exists a sequence $PP(m, n)$, $m, n \geq 0$ such that*

$$P(m, n) = A^{m+n-1} PP(m, n) - A^{m+n-5} PP(m-2, n),$$

satisfying the following relations:

$$\begin{aligned} PP(0, 0) &= 1,\ PP(1, 0) = a_1,\ PP(0, 1) = a_3,\ PP(1, 1) = a_1 a_3, \\ PP(m, 0) &= PP(m-1, 0) a_1 - PP(m-2, 0),\ m \geq 2, \end{aligned}$$

$$\begin{aligned} PP(m,1) &= PP(m,0)a_3 + A^{-2}PP(m-1,0)a_2 + A^{-4}PP(m-2,1),\ m \geq 2, \textit{ and} \\ PP(m,n) &= PP(m,n-1)a_3 - PP(m,n-2), m \geq 0, n \geq 2. \end{aligned}$$

Since $N(m,n)$ is the mirror image of $P(m,n)$, the equality $N(m,n)(A) = P(m,n)(A^{-1})$ follows and we obtain the following lemma.

Lemma 3.6. *There exists a sequence $NN(m,n)$, $m,n \geq 0$ such that*

$$N(m,n) = A^{-m-n+1}NN(m,n) - A^{-m-n+5}NN(m-2,n),$$

satisfying the following relations:

$$\begin{aligned} NN(0,0) &= 1,\ NN(1,0) = a_1,\ NN(0,1) = a_3,\ NN(1,1) = a_1a_3, \\ NN(m,0) &= NN(m-1,0)a_1 - NN(m-2,0), m \geq 2, \\ NN(m,1) &= NN(m,0)a_3 + A^2NN(m-1,0)a_2 + A^4NN(m-2,1), m \geq 2, \textit{ and} \\ NN(m,n) &= NN(m,n-1)a_3 - NN(m,n-2), m \geq 0, n \geq 2. \end{aligned}$$

Therefore, we obtain

$$C(m,n) = A^{m+n+2}PP(m,n) - A^{m+n-2}PP(m-2,n) - A^{-m-n-2}NN(m,n) + A^{-m-n+2}NN(m-2,n).$$

Notice that $PP(m,n)$ and $NN(m,n)$ satisfy the Chebyshev recurrence relation in the variable a_3. From this observation we get the following lemma.

Lemma 3.7. *The sequence $PP(m,n)$ in Lemma 3.5 satisfies*

$$PP(m,n) = PP(m,1)S_{n-1}(a_3) - PP(m,0)S_{n-2}(a_3),$$

where $S_n(a_3)$ is the Chebyshev polynomial of the second kind satisfying

$$\begin{aligned} S_{-2}(a_3) &= -1,\ S_{-1}(a_3) = 0, S_0(a_3) = 1,\ S_1(a_3) = a_3, \\ S_n(a_3) &= S_{n-1}(a_3)a_3 - S_{n-2}(a_3) \textit{ for } n \geq 2. \end{aligned}$$

Analogously, the sequence $NN(m,n)$ in Lemma 3.6 satisfies

$$NN(m,n) = NN(m,1)S_{n-1}(a_3) - NN(m,0)S_{n-2}(a_3).$$

Since $NN(m,0)$ and $PP(m,0)$ have the same recurrence relation and initial conditions, we obtain $PP(m,0) = NN(m,0) = S_m(a_1)$ for $m \geq 0$, where $S_m(a_1)$ is the Chebyshev polynomial of the second kind in the variable a_1. From the equalities

$$\begin{aligned} & PP(m,1) = PP(m,0)a_3 + A^{-2}PP(m-1,0)a_2 + A^{-4}PP(m-2,1), \\ \Leftrightarrow\quad & PP(m,1) - A^{-4}PP(m-2,1) = PP(m,0)a_3 + A^{-2}PP(m-1,0)a_2, m \geq 2 \end{aligned}$$

and

$$\begin{aligned} & NN(m,1) = NN(m,0)a_3 + A^2NN(m-1,0)a_2 + A^4NN(m-2,1), \\ \Leftrightarrow\quad & NN(m,1) - A^4NN(m-2,1) = NN(m,0)a_3 + A^2NN(m-1,0)a_2, m \geq 2, \end{aligned}$$

we can prove the following statement.

Lemma 3.8. *For $m,n \geq 0$,*

$$\begin{aligned} C(m,n) &= (-A^{m+n+2} + A^{-m-n-2})S_m(a_1)S_n(a_3) + \\ &\quad +(-A^{m+n} + A^{-m-n})S_{m-1}(a_1)S_{n-1}(a_3)a_2 + \end{aligned}$$

$$+(-A^{m+n-2}+A^{-m-n+2})S_{m-2}(a_1)S_{n-2}(a_3),$$

where $S_m(a_1)$ *is the Chebyshev polynomial of the second kind satisfying*

$$\begin{aligned} S_{-2}(a_1) &= -1,\ S_{-1}(a_1) = 0, S_0(a_1) = 1,\ S_1(a_1) = a_1, \\ S_m(a_1) &= S_{m-1}(a_1)a_1 - S_{m-2}(a_1) \text{ for } m \geq 2, \end{aligned}$$

and $S_n(a_3)$ *is the Chebyshev polynomial of the second kind satisfying*

$$\begin{aligned} S_{-2}(a_3) &= -1,\ S_{-1}(a_3) = 0, S_0(a_3) = 1,\ S_1(a_3) = a_3, \\ S_n(a_3) &= S_{n-1}(a_3)a_3 - S_{n-2}(a_3) \text{ for } n \geq 2. \end{aligned}$$

Case II: $C(m,-n)$ for $m,n \geq 1$.
For $m,n \geq 1$, $C(m,-n) = \omega(c_{m,-n}) = c_{m,-n} \cup \beta_2 - c_{m,-n} \cup \beta_1$. Since $c_{m,-n} \cup \beta_1$ and $c_{m,-n} \cup \beta_2$ each have both a negative and a positive kink, we obtain two diagrams without any kinks and coefficients. We denote them by $N(m,-n)$ and $P(m,-n)$, respectively. Hence, we obtain

$$C(m,-n) = P(m,-n) - N(m,-n), m,n \geq 1,$$

as illustrated in Figure 3.9.

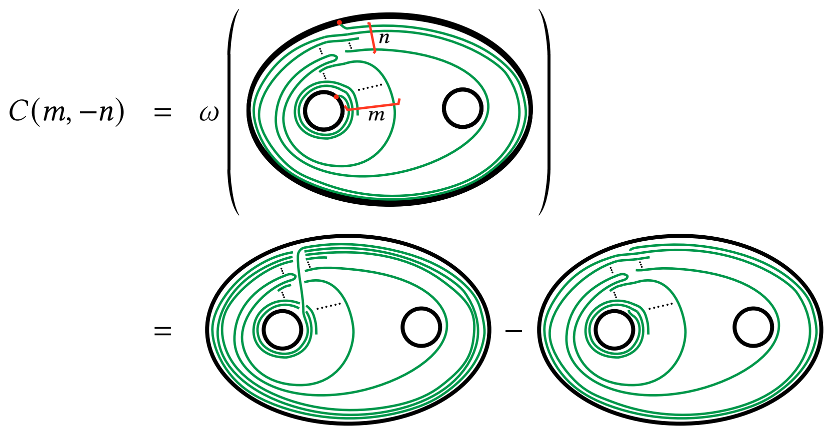


FIGURE 3.9. Illustration of $C(m,-n)$.

Analogous to the previous case, we obtain the following series of lemmas.

Lemma 3.9. *There exists a sequence* $\{PP(m,-n)\}_{m,n\geq 1}$ *such that*

$$P(m,-n) = A^{m-n-1}PP(m,-n) - A^{m-n-5}PP(m-2,-n),$$

satisfying

$$\begin{aligned} PP(1,-1) &= 0, \\ PP(m,-1) &= A^3PP(m,1) - A^3PP(m,0)a_3, m \geq 2, \\ PP(m,-2) &= PP(m,-1)a_3 + A^3PP(m,0), m \geq 1, \text{ and} \\ PP(m,-n) &= PP(m,-n+1)a_3 - PP(m,-n+2), m \geq 1, n \geq 3. \end{aligned}$$

Since $N(m,-n)$ is the mirror image of $P(m,-n)$ we obtain the following lemma.

Lemma 3.10. *There exists a sequence* $\{NN(m,-n)\}_{m,n\geq 1}$ *such that*

$$NN(m,-n) = A^{-m+n+1}NN(m,-n) - A^{-m+n+5}NN(m-2,-n),$$

satisfying

$$\begin{aligned} NN(1,-1) &= 0, \\ NN(m,-1) &= A^{-3}NN(m,1) - A^{-3}NN(m,0)a_3, m \geq 2, \\ NN(m,-2) &= NN(m,-1)a_3 + A^{-3}NN(m,0), m \geq 1, \text{ and} \\ NN(m,-n) &= NN(m,-n+1)a_3 - NN(m,-n+2), m \geq 1, n \geq 3. \end{aligned}$$

Lemma 3.11. *The sequence* $PP(m,-n)$ *in Lemma 3.9 satisfies*

$$\begin{aligned} PP(m,-n) &= S_{n-2}(a_3)PP(m,-2) - S_{n-3}(a_3)PP(m,-1) \\ &= A^3PP(m,1)S_{n-1}(a_3) - A^3PP(m,0)S_n(a_3), \end{aligned}$$

where $S_n(a_3)$ *is the Chebyshev polynomial of the second kind satisfying*

$$\begin{aligned} S_0(a_3) &= 1, \ S_1(a_3) = a_3, \\ S_n(a_3) &= S_{n-1}(a_3)a_3 - S_{n-2}(a_3). \end{aligned}$$

Analogously, the sequence $NN(m,-n)$ *in Lemma 3.10 satisfies*

$$\begin{aligned} NN(m,n) &= S_{n-2}(a_3)NN(m,-2) - S_{n-3}(a_3)NN(m,-1) \\ &= A^{-3}NN(m,1)S_{n-1}(a_3) - A^{-3}NN(m,0)S_n(a_3). \end{aligned}$$

Lemma 3.12. *For all* $m, n \geq 1$,

$$\begin{aligned} C(m,-n) = &-(-A^{m-n+2} + A^{-m+n-2})S_m(a_1)S_{n-2}(a_3) \\ &-(-A^{m-n} + A^{-m+n})S_{m-1}(a_1)S_{n-1}(a_3)a_2 \\ &-(-A^{m-n-2} + A^{-m+n+2})S_{m-2}(a_1)S_n(a_3), \end{aligned}$$

where $S_m(a_1)$ *is the Chebyshev polynomial of the second kind satisfying*

$$\begin{aligned} S_{-2}(a_1) &= -1, \ S_{-1}(a_1) = 0, S_0(a_3) = 1, \ S_1(a_1) = a_1, \\ S_m(a_1) &= S_{m-1}(a_1)a_1 - S_{m-2}(a_1) \text{ for } m \geq 2, \end{aligned}$$

and $S_n(a_3)$ *is the Chebyshev polynomial of the second kind satisfying*

$$\begin{aligned} S_{-2}(a_3) &= -1, \ S_{-1}(a_3) = 0, S_0(a_3) = 1, \ S_1(a_3) = a_3, \\ S_n(a_3) &= S_{n-1}(a_3)a_3 - S_{n-2}(a_3) \text{ for } n \geq 2. \end{aligned}$$

From Lemmas 3.8 and 3.12 we obtain the following theorem.

Theorem 3.13. *For* $m \in \mathbb{N} \cup \{0\}$ *and* $n \in \mathbb{Z}$,

$$\begin{aligned} C(m,n)S_q(a_2) = &\ (-A^{m+n+2} + A^{-m-n-2})S_m(a_1)S_n(a_3)S_q(a_2) \\ &+ (-A^{m+n} + A^{-m-n})S_{m-1}(a_1)S_{n-1}(a_3)S_{q+1}(a_2) \\ &+ (-A^{m+n} + A^{-m-n})S_{m-1}(a_1)S_{n-1}(a_3)S_{q-1}(a_2) \\ &+ (-A^{m+n-2} + A^{-m-n+2})S_{m-2}(a_1)S_{n-2}(a_3)S_q(a_2), \end{aligned}$$

where $S_m(a_1)$ *and* $S_m(a_2)$ *are the Chebyshev polynomials of the second kind satisfying*

$$S_{-2}(a_i) = -1, \ S_{-1}(a_i) = 0, S_0(a_i) = 1, \ S_1(a_i) = a_i,$$

$$S_m(a_i) = S_{m-1}(a_i)a_i - S_{m-2}(a_i) \text{ for } m \geq 2,$$

for $i = 1, 2$ and $S_n(a_3)$ is the Chebyshev polynomial of the second kind satisfying

$$\begin{aligned} S_{-1}(a_3) &= 0,\ S_0(a_3) = 1,\ S_1(a_3) = a_3, \\ S_n(a_3) &= S_{n-1}(a_3)a_3 - S_{n-2}(a_3) \text{ for } n \geq 2, \\ S_n(a_3) &= -S_{-n-2}(a_3) \text{ for } n \leq -2. \end{aligned}$$

Case III: $C(m,n)$ in general.
In the previous cases we calculated $C(m,n)$ for $m,n \in \mathbb{N} \cup \{0\}$ and $C(m,-n)$ for $m,n \in \mathbb{N}$. We now consider $C(-m,-n) = -A^{-3}P(-m,-n)+A^3N(-m,-n)$ for $m,n > 0$, illustrated in Figure 3.10. Note that $P(-m,-n)$ is a diagram of a link L in $\Sigma_{0,3} \times [0,1]$ obtained by the projection of L onto

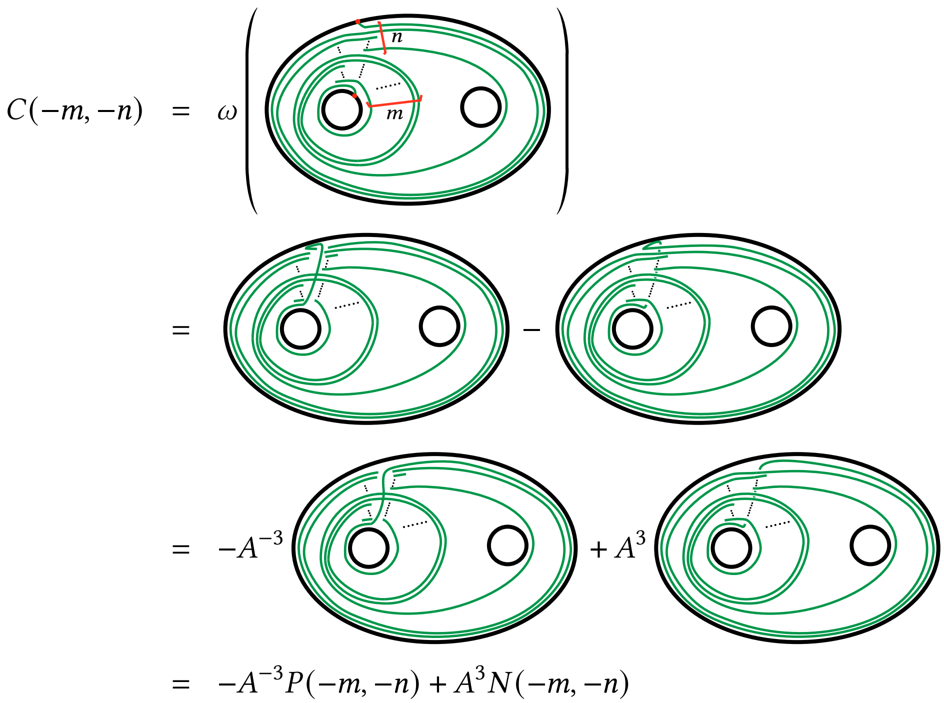


FIGURE 3.10. Illustration of $C(-m,-n)$.

$\Sigma_{0,3} \times \{0\}$. We see that the projection of L onto $\Sigma_{0,3} \times \{1\}$ is the mirror image of $P(-m,-n)$, which is $N(m,n)$. This is illustrated in Figure 3.11. Similarly, the diagrams of $N(-m,-n)$ and $P(m,n)$ are isotopic in $\Sigma_{0,3} \times [0,1]$. Therefore, we obtain

$$C(-m,-n) = -A^{-3}P(-m,-n) + A^3N(-m,-n) = -A^{-3}N(m,n) + A^3P(m,n) = -C(m,n).$$

Analogously, one can show that $C(-m,n) = -C(m,-n)$, for $m,n \in \mathbb{N} \cup \{0\}$. We now obtain the following lemma.

Lemma 3.14. *For $m,n \in \mathbb{Z}$,*

$$C(m,n) = -C(-m,-n).$$

The results of this section lead to the following description of $\mathcal{J}_1$.

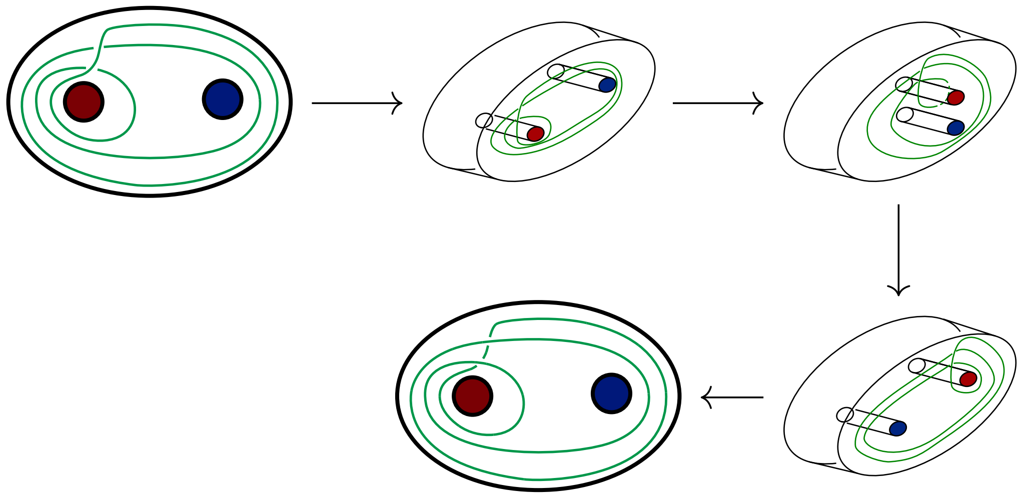


FIGURE 3.11. $P(-1,-2)$ and $N(1,2)$ are diagrams of isotopic links in $\Sigma_{0,3} \times [0,1]$.

Corollary 3.15. *The relations described in Theorem 3.13 generate the submodule $\mathcal{J}_1$ of handle sliding relations of $\mathcal{S}_{2,\infty}(H_2)$.*

We also remark that $\mathcal{S}_{2,\infty}(H_2)/\mathcal{J}_1$ is isomorphic to $\mathcal{S}_{2,\infty}((S^1 \times S^2) \# H_1)$.

3.5. **Generators of the submodule $\mathcal{J}_2$.** Analogous to the case of $\mathcal{J}_1$, we find the exact description of the generators of the submodule $\mathcal{J}_2$ by using the relative Kauffman bracket skein module of $\mathcal{S}_{2,\infty}(H_2; u', v')$. The manifold $\Sigma_{0,3}$ with marked points u' and v' on its boundary is illustrated in Figure 3.12.

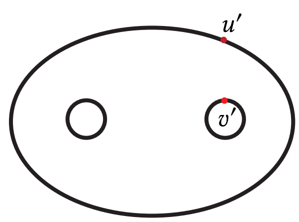


FIGURE 3.12. Marked points u' and v' on $\partial\Sigma_{0,3}$, where u' lies on the boundary component a_3 and v' lies on the boundary component a_2.

Let $\bar{c}_{q,n}$ be a relative curve connecting the points u' and v' such that $\bar{c}_{q,n}$ rotates q-times along a_2 and n-times along a_3 in either the counterclockwise or clockwise directions depending on the signs of q and n, respectively. See Figure 3.13 for an illustration.

Since $S_{2,\infty}(H_2, u', v')$ is generated by $\bar{c}_{q,n}a_1^m$, its submodule $\mathcal{J}_2$ is generated by $\overline{C}(q,n)a_1^m := \omega(\bar{c}_{q,n})a_1^m$. Analogous to the case of $\mathcal{J}_1$, we obtain the following two theorems.

Theorem 3.16. *For $q \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{Z}$,*

$$\begin{aligned}\overline{C}(q,n)S_m(a_1) &= (-A^{q+n+2} + A^{-q-n-2})S_q(a_2)S_n(a_3)S_m(a_1) \\ &+ (-A^{q+n} + A^{-q-n})S_{q-1}(a_2)S_{n-1}(a_3)S_{m+1}(a_1) \\ &+ (-A^{q+n} + A^{-q-n})S_{q-1}(a_2)S_{n-1}(a_3)S_{m-1}(a_1)\end{aligned}$$

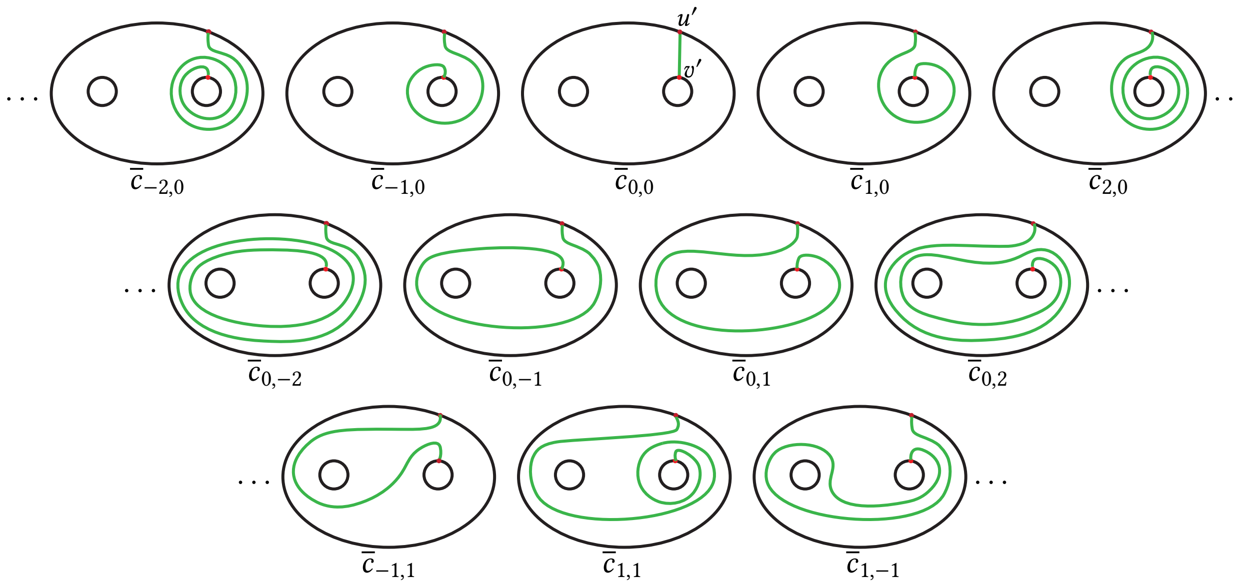


FIGURE 3.13. Relative curves in $(\Sigma_{0,3} \times I; u', v')$.

$$+ \quad (-A^{q+n-2} + A^{-q-n+2})S_{q-2}(a_2)S_{n-2}(a_3)S_m(a_1),$$

where $S_m(a_1)$ and $S_q(a_2)$ are the Chebyshev polynomials of the second kind satisfying

$$\begin{aligned} S_{-2}(a_i) &= -1,\ S_{-1}(a_i) = 0, S_0(a_i) = 1,\ S_1(a_i) = a_i, \\ S_m(a_i) &= S_{m-1}(a_i)a_i - S_{m-2}(a_i) \text{ for } m \geq 2, \end{aligned}$$

for $i = 1, 2$ and $S_n(a_3)$ is the Chebyshev polynomial of the second kind satisfying

$$\begin{aligned} S_{-1}(a_3) &= 0,\ S_0(a_3) = 1,\ S_1(a_3) = a_3, \\ S_n(a_3) &= S_{n-1}(a_3)a_3 - S_{n-2}(a_3) \text{ for } n \geq 2, \\ S_n(a_3) &= -S_{-n-2}(a_3) \text{ for } n \leq -2. \end{aligned}$$

Lemma 3.17. *For $q, n \in \mathbb{Z}$,*

$$\overline{C}(q, n) = -\overline{C}(-q, -n).$$

Combining Corollary 3.15 and Theorem 3.16 with Proposition 2.11 gives us the desired result for the structure of $\mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2))$.

4. An obstruction to Marché's conjecture

In [DW], the following conjecture due to Marché was stated for the the skein module of closed oriented 3-manifolds over the ring $\mathbb{Z}[A^{\pm 1}]$.

Conjecture 4.1. [DW]

Let M be a closed oriented 3-manifold. Then there exists an integer $d > 0$ and finitely generated $\mathbb{Z}[A^{\pm 1}]$-modules N_k so that

$$\mathcal{S}_{2,\infty}(M) \cong \mathbb{Z}[A^{\pm 1}]^d \oplus \bigoplus_{k \geq 1} N_k,$$

where, furthermore, the module N_k is a $(A^k - A^{-k})$-torsion module, for each integer k.

We discuss the splitness of $\mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2))$ into free and $(A^k - A^{-k})$-torsion parts and obtain an obstruction for it. Note that the skein module of $S^1 \times S^2$, which was computed in [HP2], satisfies Conjecture 4.1.

Lemma 4.2. *The empty link $\varnothing \in \mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2))$ is not killed by $(A^k - A^{-k})$ for all integers k.*

Proof. Let R be a ring with $(A^k - 1)$ invertible for any k. Then from the main result of [HP2], we get that $\mathcal{S}_{2,\infty}(S^1 \times S^2; R, A) \cong R$. Combining this with the main result from [Prz4] gives us that $\mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2); R, A) \cong R$. Both these skein modules are all generated by empty links. ◻

Proposition 4.3. *Suppose that the skein module of $(S^1 \times S^2) \# (S^1 \times S^2)$ splits into the sum of free and $(A^k - A^{-k})$-torsion modules, for each k, as in Conjecture 4.1, over the ring $\mathbb{Z}[A^{\pm 1}]$. Then the empty link can not serve as a generator of the free part in any such a decomposition.*

Proof. Supposed there is such a decomposition, then d must be 1 by argument in Lemma 4.2. Assume the free part is generated by $\varnothing$ denoted by y_0. Let $\{y_k^{j_k}\}_{j_k=1}^{J_k}$ denote the set of generators of N_k. From the relations $C(1,1)a_2$ and $\overline{C}(2,0)$, we get the following equations.

$$(-A^4 + A^{-4})a_1a_2a_3 = (A^2 - A^{-2})a_2^2. \tag{1}$$

$$(-A^4 + A^{-4})S_2(a_2) = 0 \tag{2}$$

Consider the equation:

$$\begin{aligned} X &= (A^2 + A^{-2})[(-A^4 + A^{-4})a_1a_2a_3 + (-A^2 + A^{-2})\phi] \\ &\overset{(1)}{=} (A^2 + A^{-2})[(A^2 - A^{-2})a_2^2 - (A^2 - A^{-2})\phi] \\ &= (A^4 - A^{-4})S_2(a_2) \overset{(2)}{=} 0. \end{aligned}$$

On the other hand,

$$0 = X = (-A^4 + A^{-4})[(A^2 + A^{-2})a_1a_2a_3 + \varnothing] = (-A^4 + A^{-4})[(A^2 + A^{-2})\Sigma_{k\geq 0, j_k}\alpha_k^{j_k}y_k^{j_k} + \varnothing],$$

where $\alpha_k^{j_k} \in \mathbb{Z}[A^{\pm 1}]$.

From Lemma 4.2, $\varnothing$ is not a torsion element nor 0, and we have $(A^2+A^{-2})\Sigma_{k\neq 2,4,j_k}\alpha_k^{j_k}y_k^{j_k} + \varnothing = 0$. In particular, $(A^2 + A^{-2})\alpha_0 + 1 = 0$, which leads to a contradiction.

◻

We conjecture the following.

Conjecture 4.4.

(1) *The empty link is a generator of the free part of $\mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2))$. In particular, the skein module of $(S^1 \times S^2) \# (S^1 \times S^2)$ does not split into the sum of free and $(A^k - A^{-k})$-torsion modules.*

(2) *Let M and N be closed oriented 3-manifolds. Then $\mathcal{S}_{2,\infty}(M \# N)$ does not split into the sum of free and $(A^k - A^{-k})$-torsion modules.*

5. Future Directions

Keeping our motivation of constructing traces on skein modules in mind, as a next step it would be beneficial to compute that of $\#_k(S^1 \times S^2)$ over the ring $\mathbb{Z}[A^{\pm 1}]$. To understand the skein module of connected sums of arbitrary 3-manifolds, knowing the skein module of $H_n \# H_m$ would also be beneficial. We note that a result to this end was published in [Prz4] but later proved to be false in [BP]. Characterising the complete set of handle sliding relations is the hardest problem for this manifold. Furthermore, computing the skein module of $\mathbb{R}P^3 \# L(p,q), (p,q) \neq (2,1)$ would be another interesting project because this is one of the few examples of connected sums of 3-manifolds for which it is unknown whether the skein module has torsion or not. See Theorem 4.2 in [Prz3].

6. Appendix

In this section we will provide the details for the proof of Theorem 3.13.

6.1. **Calculation of formulas for $C(m,n)$ for $m,n \geq 0$.** In this section we will prove Lemmas 3.5, 3.7, and 3.8. As described in Figure 3.8, we obtain $C(m,n) = -A^3P(m,n) + A^{-3}N(m,n)$. First, in the following lemma we describe the recurrence relations for $P(m,n)$.

Lemma A. *The sequence $P(m,n)$ for $m,n \in \mathbb{N} \cup \{0\}$ satisfies the following relation:*

$$\begin{aligned}
P(0,0) &= -A^{-3}(-A^2 - A^{-2}),\ P(1,0) = a_1, \\
P(0,1) &= a_3,\ P(1,1) = Aa_1a_3 + A^{-1}a_2, \\
P(m,0) &= AP(m-1,0)a_1 - A^2P(m-2,0), m \geq 2, \\
P(m,1) &= AP(m,0)a_3 + P(m-1,0)a_2 + A^{-2}P(m-2,1), m \geq 2, \\
P(m,n) &= AP(m,n-1)a_3 - A^2P(m,n-2), m \geq 0, n \geq 2.
\end{aligned}$$

Proof. The initial condition $P(0,0)$ is determined from the fact that $C(0,0) = 0$. The initial conditions for $P(1,0)$, $P(0,1)$, and $P(1,1)$ are determined from their diagrams. From a direct calculation on the diagram of $P(m,n)$ in $\Sigma_{0,3}$ we obtain the relations. See Figures 6.1, 6.2, and 6.3.

□

Our strategy is to find another sequence $PP(m,n)$ with Chebyshev recurrence relations in the variables a_1 and a_3 so that the sequence $P(m,n)$ can be presented by a combination of $PP(m,n)$. We first define a sequence $\{Q(m,n)\}_{m,n\in\mathbb{N}\cup\{0\}}$ as follows:

$$\begin{aligned}
Q(0,0) &= -A^{-5},\ Q(1,0) = 0, \\
Q(0,1) &= 0,\ Q(1,1) = -A^{-1}a_2, \\
Q(m,0) &= AQ(m-1,0)a_1 - A^2Q(m-2,0), m \geq 2, \\
Q(m,1) &= AQ(m,0)a_3 + Q(m-1,0)a_2 + A^{-2}Q(m-2,1), m \geq 2, \\
Q(m,n) &= AQ(m,n-1)a_3 - A^2Q(m,n-2), m \geq 0, n \geq 2,
\end{aligned}$$

and define $\{PP(m,n)\}_{m,n\in\mathbb{N}\cup\{0\}}$ by

$$PP(m,n) = A^{-m-n+1}(P(m,n) + Q(m,n)).$$

Lemma 3.5.A. *The sequence $\{PP(m,n)\}_{m,n\in\mathbb{N}\cup\{0\}}$ satisfies*

$$PP(0,0) = 1,\ PP(1,0) = a_1,$$

$$\begin{aligned}
PP(0,1) &= a_3,\ PP(1,1) = a_1a_3,\\
PP(m,0) &= PP(m-1,0)a_1 - PP(m-2,0), m \geq 2,\\
PP(m,1) &= PP(m,0)a_3 + A^{-2}PP(m-1,0)a_2 + A^{-4}PP(m-2,1), m \geq 2,\\
PP(m,n) &= PP(m,n-1)a_3 - PP(m,n-2), m \geq 0, n \geq 2.
\end{aligned}$$

Proof. We can prove the statement by the following direct calculations:

$$\begin{aligned}
PP(0,0) &= A(P(0,0)+Q(0,0)) & PP(0,1) &= P(0,1)+Q(0,1)\\
&= A(A^{-1}+A^{-5}-A^{-5}) & &= a_3.\\
&= 1. & PP(1,1) &= A^{-1}(P(1,1)+Q(1,1))\\
PP(1,0) &= P(1,0)+Q(1,0) & &= A^{-1}(Aa_1a_3 + A^{-1}a_2 - A^{-1}a_2)\\
&= a_1. & &= a_1a_3.
\end{aligned}$$

$$\begin{aligned}
PP(m,0) &= A^{-m+1}(P(m,0)+Q(m,0))\\
&= A^{-m+1}(AP(m-1,0)a_1 - A^2P(m-2,0) + AQ(m-1,0)a_1 - A^2Q(m-2,0))\\
&= A^{-m+1}(A(P(m-1,0)+Q(m-1,0))a_1 - A^2(P(m-2,0)+Q(m-2,0)))\\
&= A^{-m+1+1}(P(m-1,0)+Q(m-1,0))a_1 - A^{-m+2+1}(P(m-2,0)+Q(m-2,0))\\
&= PP(m-1,0)a_1 - PP(m-2,0).\\
PP(m,1) &= A^{-m}(P(m,1)+Q(m,1))\\
&= A^{-m}(AP(m,0)a_3 + P(m-1,0)a_2 + A^{-2}P(m-2,1)\\
&\quad +AQ(m,0)a_3 + Q(m-1,0)a_2 + A^{-2}Q(m-2,1))\\
&= A^{-m+1}(P(m,0)+Q(m,0))a_3 + A^{-m}(P(m-1,0)+Q(m-1,0))a_2\\
&\quad +A^{-m-2}(P(m-2,1)+Q(m-2,1))\\
&= PP(m,0)a_3 + A^{-2}A^{-m+2}(P(m-1,0)+Q(m-1,0))a_2\\
&\quad +A^{-4}A^{-m+2}(P(m-2,1)+Q(m-2,1))\\
&= PP(m,0)a_3 + A^{-2}PP(m-1,0)a_2 + A^{-4}PP(m-2,1).\\
PP(m,n) &= A^{-m-n+1}(P(m,n)+Q(m,n))\\
&= A^{-m-n+1}(AP(m,n-1)a_3 - A^2P(m,n-2) + AQ(m,n-1)a_3 - A^2Q(m,n-2))\\
&= A^{-m-n+1}(A(P(m-1,0)+Q(m-1,0))a_3 - A^2(P(m-2,0)+Q(m-2,0)))\\
&= A^{-m+1-n+1}(P(m-1,0)+Q(m-1,0))a_3 - A^{-m+2-n+1}(P(m-2,0)+Q(m-2,0))\\
&= PP(m,n-1)a_3 - PP(m,n-2).
\end{aligned}$$

□

Lemma 3.5.B. *The sequence $Q(m,n)$ satisfies $Q(m,n) = A^{m+n-5}PP(m-2,n)$, for $m \geq 2, n \geq 0$. Hence, it follows that*

$$P(m,n) = A^{m+n-1}PP(m,n) - A^{m+n-5}PP(m-2,n),$$

for $m \geq 2, n \geq 0$.

Proof. For $n = 0$

$$Q(2,0) = A^{-3} = A^{-3}PP(0,0),$$

$$Q(3,0) = AQ(2,0)a_1 - A^{-2}Q(1,0) = A^{-2}a_1 = A^{3+0-5}PP(1,0).$$

For $m \geq 4$ and $n = 0$

$$\begin{aligned} Q(m,0) &= AQ(m-1,0)a_1 - A^2Q(m-2,0) \\ &= A \cdot A^{m+n-6}PP(m-1,0)a_1 - A^2 \cdot A^{m+n-7}PP(m-2,0) \\ &= A^{m+n-5}(PP(m-1,0)a_1 - PP(m-2,0)) \\ &= A^{m+n-5}PP(m,0). \end{aligned}$$

For $n = 1$

$$\begin{aligned} Q(2,1) &= AQ(2,0)a_3 + Q(1,0)a_2 + Q(0,1) \\ &= A \cdot A^{-3}a_3 = A^{2+1-5}PP(0,1), \end{aligned}$$

$$\begin{aligned} Q(3,1) &= AQ(3,0)a_3 + Q(2,0)a_2 + A^{-2}Q(1,1) \\ &= A \cdot A^{-1}a_1a_3 + A^{-3}a_2 + A^{-2}(-A^{-1}a_2) \\ &= A^{-1}a_1a_3 = A^{3+1-5}PP(1,1). \end{aligned}$$

For $m \geq 4$ and $n = 1$

$$\begin{aligned} Q(m,1) &= AQ(m,0)a_3 + Q(m-1,0)a_2 + A^{-2}Q(m-2,1) \\ &= A \cdot A^{m-5}PP(m-2,0)a_3 + A^{m-6}PP(m-3,0)a_2 + A^{-2} \cdot A^{m-2+1-5}PP(m-4,1) \\ &= A^{m-4}(PP(m-2,0)a_3 + A^{-2}PP(m-3,0)a_2 + A^{-4}PP(m-4,1)) \\ &= A^{m+1-5}PP(m-2,1). \end{aligned}$$

For $m, n \geq 2$

$$\begin{aligned} Q(m,n) &= AQ(m,n-1)a_3 - A^2Q(m,n-2) \\ &= A \cdot A^{m+n-1-5}PP(m-2,n-1)a_3 - A^2 \cdot A^{m+n-2-5}PP(m-2,n-2) \\ &= A^{m+n-5}(PP(m-2,n-1)a_3 - PP(m-2,n-2)) \\ &= A^{m+n-5}PP(m-2,n). \end{aligned}$$

By the definition of $PP(m,n)$ we obtain

$$P(m,n) = A^{m+n-1}PP(m,n) - A^{m+n-5}PP(m-2,n).$$

□

Since $PP(m,n)$ and $NN(m,n)$ satisfy the recurrence relation for Chebyshev polynomials, we obtain Lemma 3.7.

Lemma 3.7. *For $m, n \in \mathbb{N} \cup \{0\}$*

$$PP(m,n) = PP(m,1)S_{n-1}(a_3) - PP(m,0)S_{n-2}(a_3)$$

where $S_n(a_3)$ is the Chebyshev polynomial satisfying

$$S_{-2}(a_3) = -1, S_{-1}(a_3) = 0, S_0(a_3) = 1, S_1(a_3) = a_3, S_n(a_3) = S_{n-1}(a_3)a_3 - S_{n-2}(a_3).$$

Analogously,

$$NN(m,n) = NN(m,1)S_{n-1}(a_3) - NN(m,0)S_{n-2}(a_3).$$

Proof. We will prove the lemma by mathematical induction on n. Since $S_{-2}(a_3) = -1$, $S_{-1}(a_3) = 0$, and $S_0(a_3) = 1$, the lemma holds for $n = 0$ and 1:

$$\begin{aligned} PP(m,1)S_{-1}(a_3) - PP(m,0)S_{-2}(a_3) &= PP(m,0), \\ PP(m,1)S_0(a_3) - PP(m,0)S_{-1}(a_3) &= PP(m,1). \end{aligned}$$

Let us assume that the result holds for $n = k$. Then

$$\begin{aligned} PP(m,k+1) &= PP(m,k)a_3 - PP(m,k-1) \\ &\overset{Induction}{=} (PP(m,1)S_{k-1}(a_3) - PP(m,0)S_{k-2}(a_3))a_3 \\ &\quad -(PP(m,1)S_{k-2}(a_3) - PP(m,0)S_{k-3}(a_3)) \\ &= (PP(m,1)S_{k-1}(a_3)a_3 - PP(m,1)S_{k-2}(a_3)) \\ &\quad -(PP(m,0)S_{k-2}(a_3)a_3 - PP(m,0)S_{k-3}(a_3)) \\ &= PP(m,1)(S_{k-1}(a_3)a_3 - S_{k-2}(a_3)) - PP(m,0)(S_{k-2}(a_3)a_3 - S_{k-3}(a_3)) \\ &= PP(m,1)S_k(a_3) - PP(m,0)S_{k-1}(a_3). \end{aligned}$$

□

From Lemmas 3.5 and 3.6 we obtain the following equality

(3) $C(m,n) = A^{m+n+2}PP(m,n) - A^{m+n-2}PP(m-2,n) - A^{-m-n-2}NN(m,n) + A^{-m-n+2}NN(m-2,n).$

Lemma 3.8. *For all $m, n \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned} C(m,n) &= (-A^{m+n+2} + A^{-m-n-2})S_m(a_1)S_n(a_3) \\ &\quad +(-A^{m+n} + A^{-m-n})S_{m-1}(a_1)S_{n-1}(a_3)a_2 \\ &\quad +(-A^{m+n-2} + A^{-m-n+2})S_{m-2}(a_1)S_{n-2}(a_3). \end{aligned}$$

Proof. From the equality (3) the following equalities are obtained

$$\begin{aligned} C(m,n) &= -A^{m-n+2}PP(m,n) + A^{-m-n-2}NN(m,n) \\ &\quad +A^{m+n-2}PP(m-2,n) - A^{-m-n+2}NN(m-2,n) \\ &\overset{Lemma\ 3.7}{=} -A^{m-n+2}(S_{n-1}(a_3)PP(m,1) - S_{n-2}(a_3)PP(m,0)) \\ &\quad +A^{-m-n-2}(S_{n-1}(a_3)NN(m,1) - S_{n-2}(a_3)PP(m,0)) \\ &\quad +A^{m+n-2}(S_{n-1}(a_3)PP(m-2,1) - S_{n-2}(a_3)PP(m-2,0)) \\ &\quad -A^{-m-n+2}(S_{n-1}(a_3)NN(m-2,1) - S_{n-2}(a_3)PP(m-2,0)) \\ &= -A^{m+n+2}S_{n-1}(a_3)(PP(m,1) - A^{-4}PP(m-2,1)) \\ &\quad +A^{-m-n-2}S_{n-1}(a_3)(NN(m,1) - A^4NN(m-2,1)) \\ &\quad -(-A^{m+n+2} + A^{-m-n-2})S_{n-2}(a_3)PP(m,0) \\ &\quad +(-A^{m+n-2} + A^{-m-n+2})S_{n-2}(a_3)PP(m-2,0) \\ &= -A^{m+n+2}S_{n-1}(a_3)(PP(m,0)a_3 + A^{-2}PP(m-1,0)a_2) \\ &\quad +A^{-m-n-2}S_{n-1}(a_3)(NN(m,0)a_3 + A^2NN(m-1,0)a_2) \end{aligned}$$

$$
\begin{aligned}
& -(-A^{m+n+2}+A^{-m-n-2})S_{n-2}(a_3)PP(m,0) \\
& +(-A^{m+n-2}+A^{-m-n+2})S_{n-2}(a_3)PP(m-2,0) \\
= \quad & (-A^{m+n+2}+A^{-m-n-2})S_{n-1}(a_3)PP(m,0)a_3 \\
& (-A^{m+n}+A^{-m-n})S_{n-1}(a_3)PP(m-1,0)a_2 \\
& -(-A^{m+n+2}+A^{-m-n-2})S_{n-2}(a_3)PP(m,0) \\
& +(-A^{m+n-2}+A^{-m-n+2})S_{n-2}(a_3)PP(m-2,0) \\
= \quad & (-A^{m+n+2}+A^{-m-n-2})S_m(a_1)S_n(a_3)+ \\
& +(-A^{m+n}+A^{-m-n})S_{m-1}(a_1)S_{n-1}(a_3)a_2+ \\
& +(-A^{m+n-2}+A^{-m-n+2})S_{m-2}(a_1)S_{n-2}(a_3).
\end{aligned}
$$

The fourth equality above is obtained from the following equalities:

$$
\begin{aligned}
& PP(m,1)=PP(m,0)a_3+A^{-2}PP(m-1,0)a_2+A^{-4}PP(m-2,1) \\
\Leftrightarrow \quad & PP(m,1)-A^{-4}PP(m-2,1)=PP(m,0)a_3+A^{-2}PP(m-1,0)a_2
\end{aligned}
$$

and

$$
\begin{aligned}
& NN(m,1)=NN(m,0)a_3+A^2NN(m-1,0)a_2+A^4NN(m-2,1) \\
\Leftrightarrow \quad & NN(m,1)-+A^4NN(m-2,1)=NN(m,0)a_3+A^2NN(m-1,0)a_2.
\end{aligned}
$$

Since $PP(m,0)=S_m(a_1)$, the last equality follows. □

6.2. **Calculation of formulas for $C(m,-n)$ for $m,n \geq 1$.** In this section we will prove Lemmas 3.9, 3.11, and 3.12. As described in Figure 3.9, we obtain $C(m,-n)=P(m,-n)-N(m,-n)$. We first describe the recurrence relations for $P(m,-n)$.

Lemma B. *For $m,n \geq 1$, $P(m,-n)$ satisfies the following:*

$$
\begin{aligned}
P(1,-1) &= a_2, \\
P(m,-1) &= AP(m-1,0)a_2+A^{-1}P(m-1,1)=AP(m,1)-A^2P(m,0)a_3, m \geq 2, \\
P(m,-2) &= AP(m,0)+A^{-1}P(m,-1)a_3, \\
P(m,-n) &= A^{-1}P(m,-n+1)a_3-A^{-2}P(m,-n+2), n \geq 3.
\end{aligned}
$$

Proof. The initial condition $P(1,-1)$ is determined from the diagram of $P(1,-1)$. From a direct calculation on the diagram of $P(m,-n)$ on $\Sigma_{0,3}$, we obtain the relations. See Figures 6.4, 6.5, and 6.6.

□

Our strategy is the same as that in the previous subsection. Let us define $\{Q(m,-n)\}_{m,n\in\mathbb{N}}$ by

$$
\begin{aligned}
Q(1,-1) &= -a_2, \\
Q(m,-1) &= AQ(m-1,0)a_2+A^{-1}Q(m-1,1)=AQ(m,1)-A^2Q(m,0)a_3, m \geq 2, \\
Q(m,-2) &= AQ(m,0)+A^{-1}Q(m,-1)a_3, \\
Q(m,-n) &= A^{-1}Q(m,-n+1)a_3-A^{-2}Q(m,-n+2), n \geq 3,
\end{aligned}
$$

and define $\{PP(m,-n)\}_{m,n\geq 1}$ by

$$PP(m,n)=A^{-m+n+1}(P(m,n)+Q(m,n)).$$

Lemma 3.9.A. *$\{PP(m,-n)\}_{m,n\geq 1}$ satisfies*

$$\begin{aligned} PP(1,-1) &= 0, \\ PP(m,-1) &= A^3PP(m,1) - A^3PP(m,0)a_3, m \geq 2, \\ PP(m,-2) &= PP(m,-1)a_3 + A^3PP(m,0), \\ PP(m,-n) &= pp(m,-n+1)a_3 - PP(m,-n+2), n \geq 3. \end{aligned}$$

Proof. We prove the statement from the following direct calculations:

$$\begin{aligned} PP(1,-1) &= A^{-1+1+1}(P(1,-1)+Q(1,-1)) = A(a_2-a_2) = 0. \\ PP(m,-1) &= A^{-m+2}(P(m,-1)+Q(m,-1)) \\ &= A^{-m+2}(AP(m,1) - A^2P(m,0)a_3 + AQ(m,1) - A^2Q(m,0)a_3) \\ &= A^{-m+3}(P(m,1)+Q(m,1)) + A^{-m+4}(P(m,0)+Q(m,0))a_3 \\ &= A^3 \cdot A^{-m}(P(m,1)+Q(m,1)) + A^3 \cdot A^{-m+1}(P(m,0)+Q(m,0))a_3 \\ &= A^3PP(m,1) + A^3PP(m,0). \end{aligned}$$

$$\begin{aligned} PP(m,-2) &= A^{-m+3}(P(m,-2)+Q(m,-2)) \\ &= A^{-m+3}(AP(m,1) + A^{-1}P(m,-1)a_3 + AQ(m,1) + A^{-1}Q(m,-1)a_3) \\ &= A^{-m+4}(P(m,1)+Q(m,1)) + A^{-m+2}(P(m,-1)+Q(m,-1))a_3 \\ &= A^3 \cdot A^{-m+1}(P(m,1)+Q(m,1)) + PP(m,-1)a_3 \\ &= A^3PP(m,1) + PP(m,-1)a_3. \end{aligned}$$

$$\begin{aligned} PP(m,-n) &= A^{-m+n+1}(P(m,-n)+Q(m,-n)) \\ &= A^{-m+n+1}(A^{-1}P(m,-n+1)a_3 - A^{-2}P(m,-n+2) \\ &\quad +A^{-1}Q(m,-n+1)a_3 - A^{-2}Q(m,-n+2)) \\ &= A^{-m+n}(P(m,-n+1)+Q(m,-n+1))a_3 + A^{-m+n-1}(P(m,-n+2)+Q(m,-n+2)) \\ &= PP(m,-n+1)a_3 - PP(m,-n+2). \end{aligned}$$

□

Lemma 3.9.B. *The sequence $Q(m,-n)$ satisfies $Q(m,-n) = A^{m-n-5}PP(m-2,-n)$, for $m \geq 2$. It follows that*

$$P(m,-n) = A^{m-n-1}PP(m,-n) - A^{m-n-5}PP(m-2,-n),$$

for $m,n \geq 1$.

Proof. When $n = 1$,

$$\begin{aligned} Q(2,-1) &= AQ(2,1) - A^2Q(2,0)a_3 \\ &= A^{-1}PP(0,1) - A^{-1}PP(0,0)a_3 = 0 = A^{2+1-2}PP(0,-1). \\ Q(m,-1) &= AQ(m,1)A^2Q(m,0)a_3 \\ &= A \cdot A^{m-4}PP(m-2,1) - A^2 \cdot A^{m-5}PP(m-2,0)a_3 \\ &= A^{m-3}(PP(m-2,1) - PP(m-2,0)a_3) \\ &= A^{m-6}(A^3PP(m-2,1) - A^3PP(m-2,0)a_3). \\ &= A^{m-6}PP(m,-1). \end{aligned}$$

$$\begin{aligned}
Q(m,-2) &= AQ(m,0) + A^{-1}Q(m,-1)a_3 \\
&= A \cdot A^{m-5}PP(m-2,0) + A^{-1} \cdot A^{m-6}PP(m-2,-1)a_3 \\
&= A^{m-4}PP(m-2,0) + A^{m-7}PP(m-2,-1)a_3 \\
&= A^{m-7}(A^3PP(m-2,0) + PP(m-2,-1)a_3) = A^{m-7}PP(m,-2). \\
Q(m,-n) &= A^{-1}Q(m,-n+1)a_3 - A^{-2}Q(m,-n+2) \\
&= A^{-1} \cdot A^{m-n+1-5}PP(m-2,-n+1)a_3 - A^{-2}A^{m-n+2-5}PP(m-2,-n+2) \\
&= A^{m-n-5}(PP(m-2,-n+1)a_3 - PP(m-2,-n+2)) \\
&= A^{m-n-5}PP(m-2,-n).
\end{aligned}$$

By definition of $PP(m,-n)$ we obtain

$$P(m,-n) = A^{m-n-1}PP(m,-n) - A^{m-n-5}PP(m-2,-n),$$

for $m, n \geq 1$. □

Since $PP(m,-n)$ and $NN(m,-n)$ satisfy the recurrence relation for Chebyshev polynomials, we obtain Lemma 3.11.

Lemma 3.11. *For $m, n \geq 1$*

$$\begin{aligned}
PP(m,-n) &= S_{n-2}(a_3)PP(m,-2) - S_{n-3}(a_3)PP(m,-1) \\
&= A^3PP(m,1)S_{n-1}(a_3) - A^3PP(m,0)S_n(a_3),
\end{aligned}$$

where $S_n(a_3)$ is the Chebyshev polynomial of the second kind satisfying

$$S_0(a_3) = 1, S_1(a_3) = a_3, S_n(a_3) = S_{n-1}(a_3)a_3 - S_{n-2}(a_3).$$

Analogously,

$$\begin{aligned}
NN(m,n) &= S_{n-2}(a_3)NN(m,-2) - S_{n-3}(a_3)NN(m,-1) \\
&= A^{-3}NN(m,1)S_{n-1}(a_3) - A^{-3}NN(m,0)S_n(a_3).
\end{aligned}$$

Proof. The proof of the first equality is analogous to the proof of Lemma 3.7. The second equality can be obtained as follows:

$$\begin{aligned}
PP(m,-n) &= S_{n-2}(a_3)PP(m,-2) - S_{n-3}(a_3)PP(m,-1) \\
&= S_{n-2}(a_3)(A^3PP(m,0) + PP(m,-1)a_3) - S_{n-3}(a_3)PP(m,-1) \\
&= A^3PP(m,0)S_{n-2}(a_3) + PP(m,-1)S_{n-1}(a_3) \\
&= A^3PP(m,0)S_{n-2}(a_3) + (A^3PP(m,1) - A^3PP(m,0))S_{n-1}(a_3)a_3 \\
&= A^3PP(m,0)S_{n-2}(a_3) + A^3PP(m,1)S_{n-1}(a_3) - A^3PP(m,0)S_{n-1}(a_3)a_3 \\
&= A^3PP(m,1)S_{n-1}(a_3) - A^3PP(m,0)S_n(a_3).
\end{aligned}$$

□

From Lemmas 3.9 and 3.10 we obtain the following equality

$$\begin{aligned}
C(m,-n) &= A^{m-n-1}PP(m,-n) - A^{m-n-5}PP(m-2,-n) \\
&\quad -A^{-m+n+1}NN(m,-n) + A^{-m+n+5}NN(m-2,-n). \qquad (4)
\end{aligned}$$

Theorem 3.13.

$$\begin{aligned} C(m,-n) &= -(-A^{m-n+2}+A^{-m+n-2})S_m(a_1)S_{n-2}(a_3) \\ &\quad -(-A^{m-n}+A^{-m+n})S_{m-1}(a_1)S_{n-1}(a_3)a_2 \\ &\quad -(-A^{m-n-2}+A^{-m+n+2})S_{m-2}(a_1)S_n(a_3), \end{aligned}$$

for all $m, n \geq 1$.

Proof. From equality (4), we obtain the following equalities.

$$\begin{aligned} C(m,-n) &= A^{m-n-1}PP(m,-n) - A^{m-n-5}PP(m-2,-n) \\ &\quad -A^{-m+n+1}NN(m,-n) + A^{-m+n+5}NN(m-2,-n) \\ &\overset{\text{Lemma 3.11}}{=} A^{m-n-1}(A^3PP(m,1)S_{n-1}(a_3) - A^3PP(m,0)S_n(a_3)) \\ &\quad -A^{m-n-5}(A^3PP(m-2,1)S_{n-1}(a_3) - A^3PP(m-2,0)S_n(a_3)) \\ &\quad -A^{-m+n+1}(A^{-3}NN(m,1)S_{n-1}(a_3) - A^{-3}NN(m,0)S_n(a_3)) \\ &\quad +A^{-m+n+5}(A^{-3}NN(m-2,1)S_{n-1}(a_3) - A^{-3}NN(m,0)S_n(a_3)) \\ &= A^{m-n+2}(PP(m,1) - A^{-4}PP(m-2,1))S_{n-1}(a_3) \\ &\quad -A^{-m+n-2}(NN(m,1) - A^4NN(m-2,1))S_{n-1}(a_3) \\ &\quad +(-A^{m-n+2}+A^{-m+n-2})PP(m,0)S_n(a_3) \\ &\quad -(-A^{m-n-2}+A^{-m+n+2})PP(m-2,0)S_n(a_3) \\ &= A^{m-n+2}(PP(m,0)a_3 + A^{-2}PP(m-1,0)a_2)S_{n-1}(a_3) \\ &\quad -A^{-m+n-2}(NN(m,0)a_3 + A^2NN(m-1,0))S_{n-1}(a_3) \\ &\quad +(-A^{m-n+2}+A^{-m+n-2})PP(m,0)S_n(a_3) \\ &\quad -(-A^{m-n-2}+A^{-m+n+2})PP(m-2,0)S_n(a_3) \\ &= -(-A^{m-n+2}+A^{m+n-2})PP(m,0)S_{n-1}(a_3)a_3 \\ &\quad -(-A^{m-n}+A^{-m+n})PP(m-1,0)S_{n-1}(a_3)a_2 \\ &\quad +(-A^{m-n+2}+A^{-m+n-2})PP(m,0)S_n(a_3) \\ &\quad -(-A^{m-n-2}+A^{-m+n+2})PP(m-2,0)S_n(a_3) \\ &= -(-A^{m-n+2}+A^{-m+n-2})PP(m,0)S_{n-2}(a_3) \\ &\quad -(-A^{m-n}+A^{-m+n})PP(m-1,0)S_{n-1}(a_3)a_2 \\ &\quad -(-A^{m-n-2}+A^{-m+n+2})PP(m-2,0)S_n(a_3), \\ &= -(-A^{m-n+2}+A^{-m+n-2})S_m(a_1)S_{n-2}(a_3) \\ &\quad -(-A^{m-n}+A^{-m+n})S_{m-1}(a_1)S_{n-1}(a_3)a_2 \\ &\quad -(-A^{m-n-2}+A^{-m+n+2})S_{m-2}(a_1)S_n(a_3). \end{aligned}$$

The fourth equality is obtained from the following equalites:

$$\begin{aligned} & PP(m,1) = PP(m,0)a_3 + A^{-2}PP(m-1,0)a_2 + A^{-4}PP(m-2,1) \\ \Leftrightarrow\; & PP(m,1) - A^{-4}PP(m-2,1) = PP(m,0)a_3 + A^{-2}PP(m-1,0)a_2 \end{aligned}$$

and

$$NN(m,1) = NN(m,0)a_3 + A^2NN(m-1,0)a_2 + A^4NN(m-2,1)$$

$$\Leftrightarrow \quad NN(m,1) - +A^4 NN(m-2,1) = NN(m,0)a_3 + A^2 NN(m-1,0)a_2.$$

Since $PP(m,0) = S_m(a_1)$, the last equality follows. □

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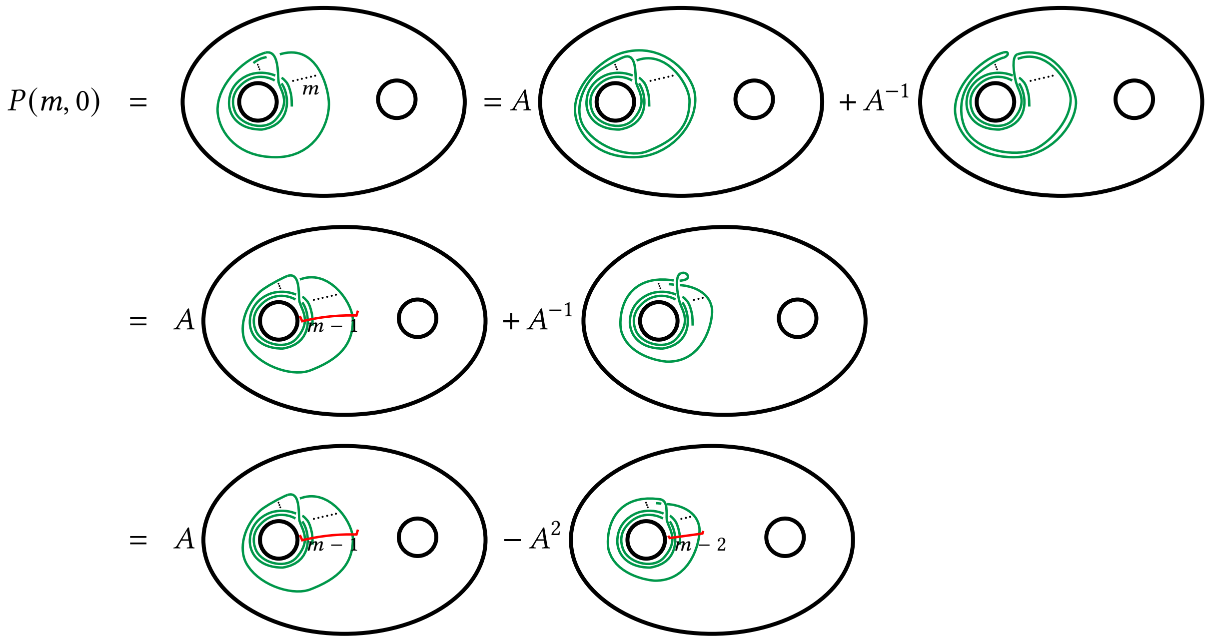
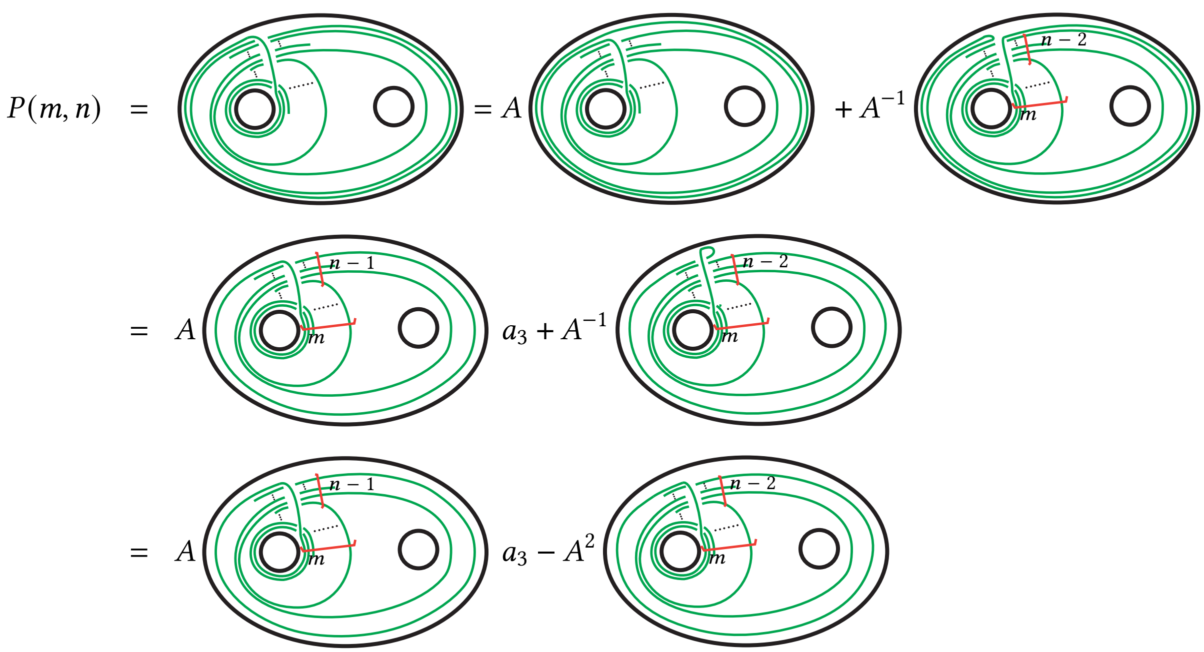


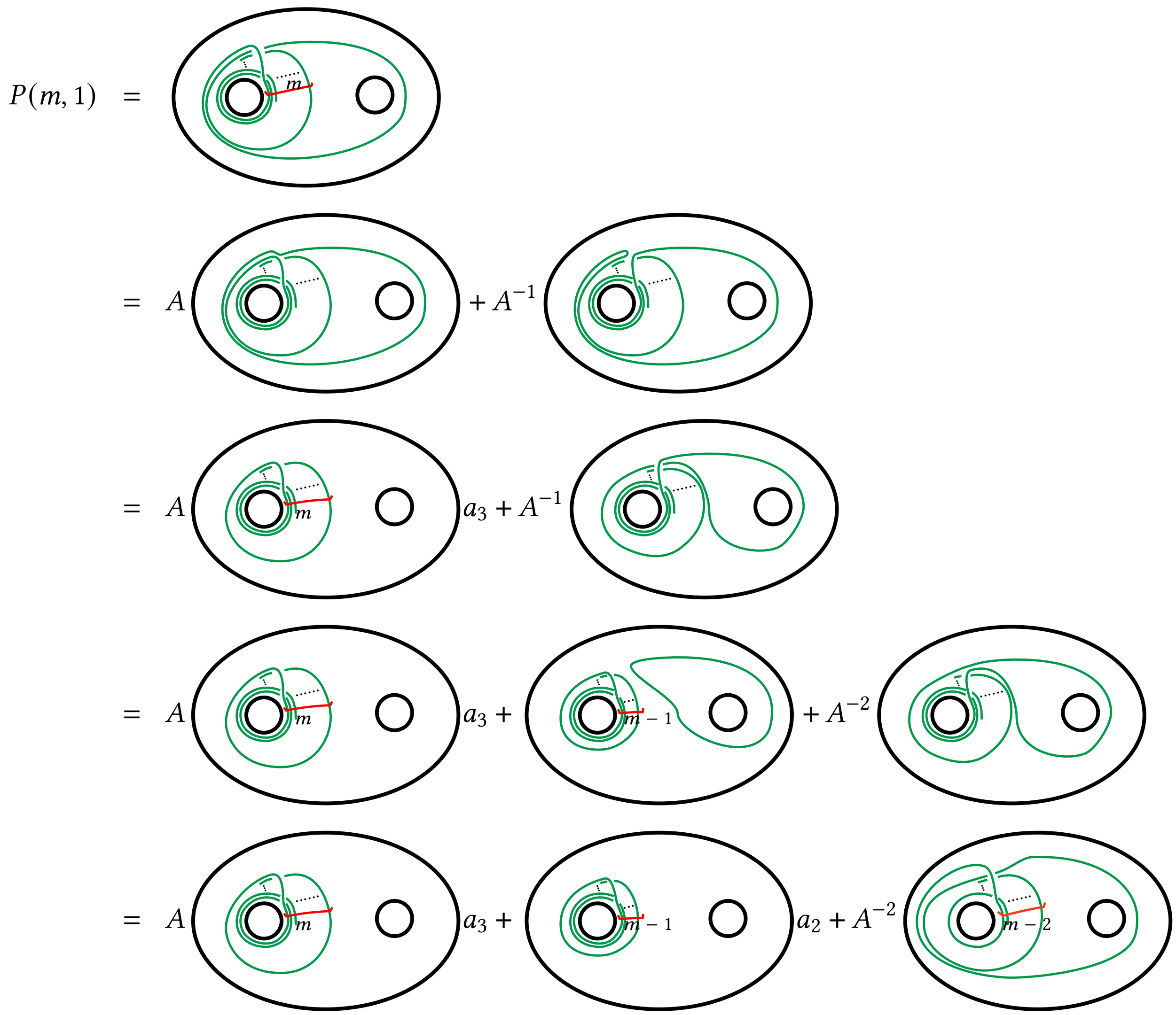
$= AP(m-1,0) - A^2P(m-2,0).$

FIGURE 6.1. $P(m,0) = AP(m-1,0) - A^2P(m-2,0)$ for $m \geq 2$.



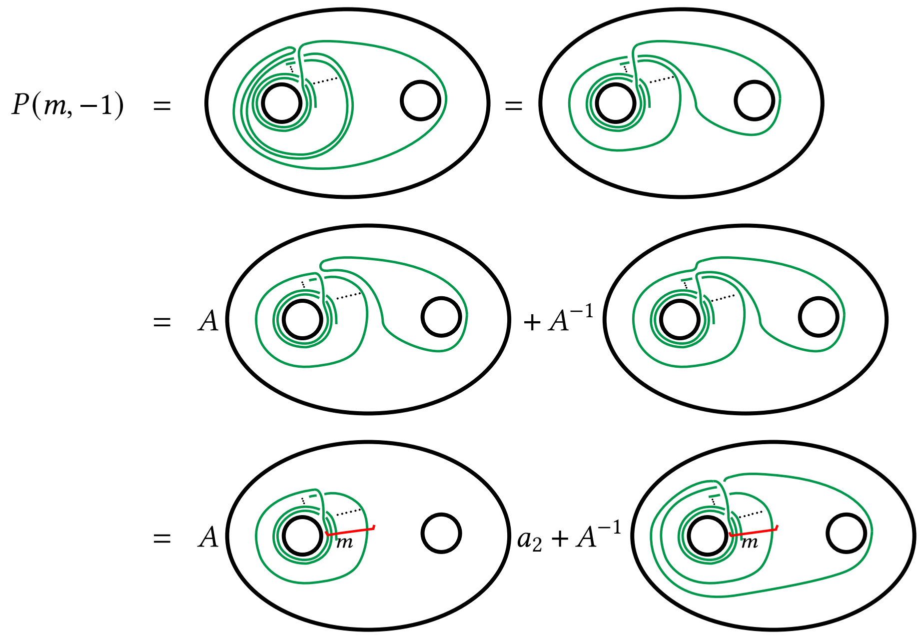
$= AP(m,n-1)a_3 - A^2P(m,n-2).$

FIGURE 6.2. $P(m,n) = AP(m,n-1)a_3 - A^2P(m,n-2)$ for $n \geq 2$.



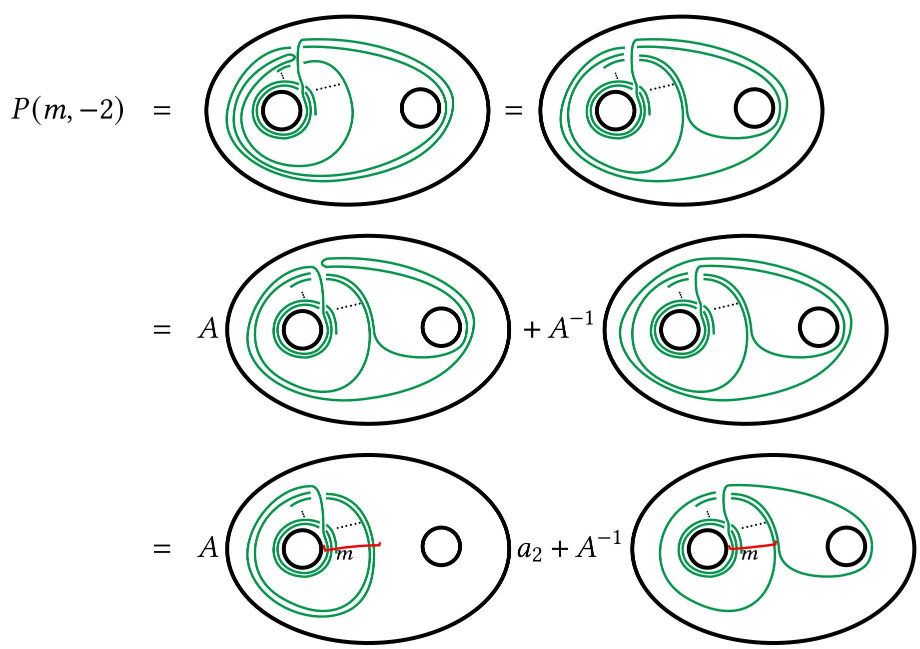
$$= AP(m,0)a_3 + P(m-1)a_2 + A^{-2}P(m-2,1).$$

FIGURE 6.3. $P(m,1) = AP(m,0) + P(m-1,0)a_2 + A^{-2}P(m-2,1)$ for $m \geq 2$.



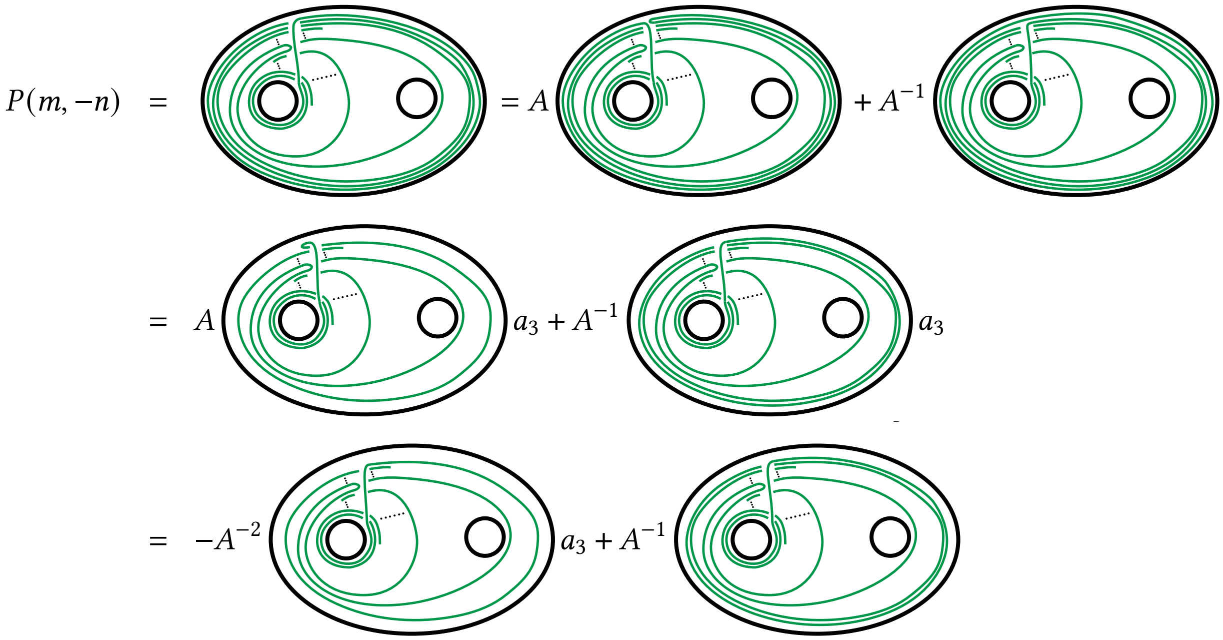
$$= AP(m,0)a_2 + A^{-1}P(m,1).$$

FIGURE 6.4. $AP(m,0)a_2 + A^{-1}P(m,1)$ for $m > 0$.



$$= AP(m,0) + A^{-1}P(m,-1)a_3.$$

FIGURE 6.5. $P(m,-2) = AP(m,0) + A^{-1}P(m,-1)a_3$ for $m > 0$.



$$= \quad -A^{-2}P(m,-n+2) + A^{-1}P(m,-n+1)a_3.$$

FIGURE 6.6. $P(m,-n) = -A^{-2}P(m,-n+2) + A^{-1}P(m,-n+1)a_3$ for $m > 0, n \geq 3$.

INSTITUTE FOR THEORETICAL STUDIES, ETH ZÜRICH, SWITZERLAND
Email address: rheapalak.bakshi@eth-its.ethz.ch | rheapalakbakshi@gmail.com

JILIN UNIVERSITY, CHANGCHUN, CHINA
Email address: kimseongjeong@jlu.edu.cn

JILIN UNIVERSITY, CHANGCHUN, CHINA
Email address: wangxiaotop@jlu.edu.cn