A generalization of the Davis–Moussong complex for Dyer groups

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A generalization of the Davis–Moussong complex for Dyer groups

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Abstract. A common feature of Coxeter groups and right-angled Artin groups is their solution to the word problem. Matthew Dyer introduced a class of groups, which we call Dyer groups, sharing this feature. This class includes, but is not limited to, Coxeter groups, right-angled Artin groups, and graph products of cyclic groups. We introduce Dyer groups by giving their standard presentation and show that they are finite-index subgroups of Coxeter groups. We then introduce a piecewise Euclidean cell complex Σ which generalizes the Davis–Moussong complex and the Salvetti complex. The construction of Σ uses simple categories without loops and complexes of groups. We conclude by proving that the cell complex Σ is CAT(0).

1. Introduction

There is extensive literature on Coxeter groups as well as on right-angled Artin groups and more generally graph products of cyclic groups. One common feature of these two families of groups is their solution to the word problem. It was given by Tits for Coxeter groups [16] and by Green for graph products of cyclic groups [9]. The algorithm does not only give a solution to the word problem but also allows to detect whether a word is reduced or not. In his study of reflection subgroups of Coxeter groups, Dyer introduces a family of groups which contains both Coxeter groups and graph products of cyclic groups. A close study of [7] also implies that this class of groups, which we call Dyer groups, has the same solution to the word problem as Coxeter groups and graph products of cyclic groups. A complete and explicit proof is given in [15].

Similar to Coxeter groups and right-angled Artin groups, the presentation of a Dyer group can be encoded in a graph. Consider a simplicial graph Γ with vertices $V = V(\Gamma)$ and edges $E = E(\Gamma)$, a vertex labeling $f : V \to \mathbb{N}_{\geq 2} \cup \{\infty\}$, and an edge labeling $m : E \to \mathbb{N}_{\geq 2}$. We say that the triple (Γ, f, m) is a Dyer graph if for every edge $e = \{v, w\}$ with $f(v) \geq 3$ we have $m(e) = 2$. The associated Dyer group

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$D = D(\Gamma, f, m)$ is given by the following presentation:

$$D = \langle x_v, v \in V \mid x_v^{f(v)} = \mathbf{e} \text{ if } f(v) \neq \infty, \\ [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} \text{ for all } e = \{u, v\} \in E \rangle,$$

where $[a, b]_k = \underbrace{aba\dots}_{k}$ for any $a, b \in D$, $k \in \mathbb{N}$, and we denote the identity with $\mathbf{e}$.

It is natural to ask the following question. Consider a property P satisfied by both Coxeter groups and graph products of cyclic groups. Do Dyer groups also satisfy P?

In [6], Davis and Januszkiewicz show that right-angled Artin groups are finite-index subgroups of right-angled Coxeter groups. For a right-angled Artin group A, they give right-angled Coxeter groups W and W', where W' and A are both finite-index subgroups of W and moreover the cubical complexes corresponding to A and W' are identical. Inspired by this work, we consider the following question: are Dyer groups finite-index subgroups of Coxeter groups? Out of a Dyer graph (Γ, f, m), we build a labeled simplicial graph Λ and prove the following statement.

Theorem 1.1 (Theorem 2.8). *We have $W(\Lambda) \cong D(\Gamma, f, m) \rtimes_\xi (\mathbb{Z}/2\mathbb{Z})^k$ for some determined $k \in \mathbb{N}$.*

The next corollary is a direct consequence.

Corollary 1.2 (Corollary 2.9). *Every Dyer group is a finite-index subgroup of some Coxeter group.*

This corollary has many interesting consequences; among others, it implies that Dyer groups are CAT(0) [5, Theorem 12.3.3], linear [1, Corollary 2], and biautomatic [14]. This is the starting point for a more precise study of their geometry. Coxeter groups are known to act properly and cocompactly by isometries on the Davis–Moussong complex, while right-angled Artin groups are known to act properly and cocompactly by isometries on the Salvetti complex. Moreover, graph products of cyclic groups are known to be CAT(0) by [8, Theorem 8.20]. The aim is to construct an analog of the Davis–Moussong and Salvetti complexes for Dyer groups, and by way of construction, give a unified description of them. The piecewise Euclidean cell complex Σ associated to a Dyer group D is constructed as follows. One considers a simple category without loops $\mathcal{X}$ and a complex of groups $\mathfrak{D}(\mathcal{X})$. The development $\mathcal{C}$ of $\mathfrak{D}(\mathcal{X})$ will then encode the necessary information to build Σ. In Section 4.1, this is done for the case of spherical Dyer groups, where a Dyer group D is spherical if it decomposes as a direct product $D_2 \times D_\infty \times D_p$ with D_2 a finite Coxeter group, $D_\infty = \mathbb{Z}^n$ for some $n \in \mathbb{N}$, and D_p a direct product of finite cyclic groups. In Section 4.2, the construction of Section 4.1 is extended to any Dyer group. The complexes $\mathfrak{D}(\mathcal{X})$ and $\mathcal{C}$ are analogs to the poset of spherical subsets $\mathcal{S}$ and the poset of spherical cosets $W\mathcal{S}$ for Coxeter groups, which are recalled in Section 3.2. Finally, Section 4.3 is devoted to the construction of the piecewise Euclidean cell complex Σ, on which the Dyer group D acts properly and cocompactly, and culminates with the proof of the following statement.

Theorem 1.3 (Theorem 4.21). *The cell complex* Σ *is* CAT(0).

As we do not assume the reader to be familiar with simple categories without loops (scwols), Section 3.1 recalls the definitions and statements needed for the construction of the scwol $\mathcal{C}$. In Sections 3.2 and 3.3, we recall the constructions of the Davis–Moussong complex and of the Salvetti complex.

2. Dyer groups

We recall the definition of Dyer groups as given in the introduction. These groups were introduced by Dyer in [7]. It follows from Dyer's work that Dyer groups have the same solution to the word problem as Coxeter groups and right-angled Artin groups; this aspect of Dyer groups is discussed in detail in [15].

Definition 2.1. Let Γ be a simplicial graph with set of vertices $V = V(\Gamma)$ and set of edges $E = E(\Gamma)$. Consider maps $f : V \to \mathbb{N}_{\geq 2} \cup \{\infty\}$ and $m : E \to \mathbb{N}_{\geq 2}$ such that for every edge $e = \{v, w\}$ with $f(v) \geq 3$ we have $m(e) = 2$. We call the triple (Γ, f, m) a *Dyer graph.*

Definition 2.2. Let (Γ, f, m) be a Dyer graph. The *Dyer group* $D = D(\Gamma, f, m)$ associated to the Dyer graph (Γ, f, m) is given by the following presentation:

$$\begin{aligned} D = \langle x_v, v \in V \mid x_v^{f(v)} = \mathbf{e} \text{ if } f(v) \neq \infty, \\ [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} \text{ for all } e = \{u, v\} \in E \rangle, \end{aligned}$$

where $[a, b]_k = \underbrace{aba\dots}_{k}$ for any $a, b \in D$, $k \in \mathbb{N}$, and we denote the identity with $\mathbf{e}$. The pair $(D, \{x_v, v \in V\})$ is called a *Dyer system.*

Example 2.3. As mentioned in the introduction, Coxeter groups, right-angled Artin groups, and graph products of cyclic groups are examples of Dyer groups.

Remark 2.4. For a subset $W \subset V$, we can consider Γ_W the full subgraph of Γ spanned by W and the restrictions

$$f_W = f|_W \quad \text{and} \quad m_W = m_{|E(\Gamma_W)}.$$

The triple (Γ_W, f_W, m_W) is again a Dyer graph. We denote the associated Dyer group by D_W. From [7], we know that the homomorphism $D_W \to D$ induced by the inclusion $W \hookrightarrow V$ is injective, hence, D_W can be regarded as a subgroup of D.

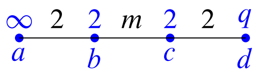


Figure 1. Dyer graph $\Gamma_{m,q}$ for some $m, q \in \mathbb{N}_{\geq 2}$.

Definition 2.5. Let $V_2 = \{v \in V \mid f(v) = 2\}$, $V_\infty = \{v \in V \mid f(v) = \infty\}$, and $V_p = V \setminus \{V_2 \cup V_\infty\}$. For $i \in \{2, p, \infty\}$, let Γ_i be the full subgraph spanned by V_i and D_i the Dyer group associated to the triple $(\Gamma_i, f_{V_i}, m_{V_i})$. Note that D_2 is a Coxeter group, D_∞ a right angled Artin group, and D_p a graph product of finite cyclic groups.

Example 2.6. Let $m, q \in \mathbb{N}_{\geq 2}$. Consider the Dyer graph $\Gamma_{m,q}$ given in Figure 1. The associated Dyer group is

$$D_{m,q} = \langle a, b, c, d \mid b^2 = c^2 = d^q = \mathbf{e},\ ab = ba,\ (bc)^m = \mathbf{e},\ cd = dc\rangle.$$

We recall the definition of Coxeter groups.

Definition 2.7. Let Λ be a simplicial graph with set of vertices $V = V(\Lambda)$ and set of edges $E = E(\Lambda)$. Let $m : E \to \mathbb{N}_{\geq 2}$ be an edge labeling of Λ. The Coxeter group $W = W(\Lambda)$ associated to the graph Λ is given by the following presentation:

$$\begin{aligned} W = \langle x_v, v \in V \mid x_v^2 = \mathbf{e} \text{ for all } v \in V,& \\ [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} &\text{ for all } e = \{u, v\} \in E\rangle, \end{aligned}$$

where $[a, b]_k = \underbrace{aba\ldots}_{k}$ for any $a, b \in W$, $k \in \mathbb{N}$, and we denote the identity with $\mathbf{e}$.
Note that for an edge $e = \{u, v\} \in E$ the relation

$$[x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)}$$

is equivalent to the relation $(x_v x_u)^{m(e)} = \mathbf{e}$, since $x_u^2 = x_v^2 = \mathbf{e}$.

Dyer groups are finite-index subgroups of Coxeter groups. The aim is now to show that every Dyer group is a finite-index subgroup of a Coxeter group. From a given Dyer graph (Γ, f, m), we build a graph Λ with edge labeling m'. We then show that $D(\Gamma, f, m)$ is a finite-index subgroup of $W(\Lambda)$. See Example 2.11 for a simple case. We define the undirected labeled simplicial graph Λ. Its set of vertices is the disjoint union $V(\Lambda) = V \amalg (V_p \cup V_\infty)$. We will refer to the elements of the disjoint copy of $V_p \cup V_\infty$ as v' for $v \in V_p \cup V_\infty$. Two vertices $u, v \in V \subset V(\Lambda)$ span an edge in Λ if and only if they span an edge $e = \{u, v\}$ in Γ, and we set the label of the edge $e = \{u, v\} \in E(\Lambda)$ to be $m'(e) = m(e)$. For all $u \in V_p \cup V_\infty$ and $v \in V(\Lambda) \setminus \{u, u'\}$, there is an edge $e = \{u', v\} \in E(\Lambda)$ labeled by $m'(e) = 2$. Finally, for all $u \in V_p$, there is an edge $e = \{u, u'\} \in E(\Lambda)$ labeled by $m'(e) = f(u)$. So, $V \subset V(\Lambda)$ spans a copy of Γ in Λ and the disjoint copy $V_p \cup V_\infty \subset V(\Lambda)$ spans a complete graph in Λ. Let $W = W(\Lambda)$ be the Coxeter group associated to the graph Λ. We give an action of $(\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$ on D. For $v \in V_p \cup V_\infty$, let $\xi_v : \{x_u, u \in V\} \to D$ with $\xi_v(x_u) = x_u$ for any $u \in V \setminus \{v\}$ and $\xi_v(x_v) = x_v^{-1}$. For all $v \in V_p \cup V_\infty$, the map ξ_v extends to a homomorphism $\xi_v : D \to D$. Moreover, for all $u, v \in V_p \cup V_\infty$, $\xi_v \circ \xi_u = \xi_u \circ \xi_v$, and $(\xi_v)^2 = \mathbf{e}$. Hence, we have an action $\xi : (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty} \times D \to D$.

Theorem 2.8. *We have* $W \cong D \rtimes_\xi (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$.

Proof. Let us first recall the presentations of W, D, and $U = D \rtimes_\xi (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$:

$$W = \langle y_v, v \in V(\Lambda) \mid \forall v \in V(\Lambda), (y_v)^2 = \mathbf{e} \text{ and } \forall e = \{u,v\} \in E(\Lambda), (y_u y_v)^{m'(e)} = \mathbf{e} \rangle,$$

$$D = \langle x_v, v \in V \mid x_v^{f(v)} = \mathbf{e} \text{ if } f(v) \neq \infty, [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} \text{ for all } e = \{u,v\} \in E \rangle,$$

$$\begin{aligned} U = \langle \{x_u, u \in V\} \cup \{\xi_v, v \in V_p \cup V_\infty\} \mid\ & x_u^{f(u)} = \mathbf{e} \text{ for all } u \in V \text{ with } f(u) \neq \infty, \\ & [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} \text{ for all } e = \{u,v\} \in E, \\ & \xi_v^2 = \mathbf{e} \text{ for all } v \in V_p \cup V_\infty,\ \xi_v \xi_u = \xi_u \xi_v \text{ for all } u, v \in V_p \cup V_\infty, \\ & \xi_u x_v = x_v \xi_u \text{ for all } u \in V_p \cup V_\infty, v \in V \setminus \{u\}, \\ & \xi_u x_u \xi_u = x_u^{-1} \text{ for all } u \in V_p \cup V_\infty \rangle. \end{aligned}$$

We show that U is isomorphic to W by giving explicit homomorphisms $\phi : W \to U$ and $\psi : U \to W$ satisfying $\phi \circ \psi = \mathrm{Id}_U$ and $\psi \circ \phi = \mathrm{Id}_W$.

First, consider the map $\phi : \{y_v, v \in V(\Lambda)\} \to U$ defined as follows: for $u \in V_2$, $\phi(y_u) = x_u$ and for $u \in V_p \cup V_\infty$, $\phi(y_u) = \xi_u x_u$, and $\phi(y_{u'}) = \xi_u$. We show that ϕ extends to a homomorphism $\phi : W \to U$.

(1) For $u \in V_2$, $\phi(y_u)^2 = x_u^2 = \mathbf{e}$. For $u \in V_p \cup V_\infty$, $\phi(y_{u'})^2 = \xi_u^2 = \mathbf{e}$ and $\phi(y_u)^2 = \xi_u x_u \xi_u x_u = x_u^{-1} x_u = \mathbf{e}$. So, $\phi(y_u)^2 = \mathbf{e}$ for all $u \in V(\Lambda)$.

(2) Let $u, v \in V \subset V(\Lambda)$ with $e = \{u,v\} \in E(\Lambda)$, so $e \in E$ and $m'(e) = m(e)$. If $u, v \in V_2$, we have

$$(\phi(y_u)\phi(y_v))^{m'(e)} = (x_u x_v)^{m(e)} = \mathbf{e}$$

since $x_u^2 = x_v^2 = \mathbf{e}$, and hence, $[x_u, x_v]_{m(e)} = [x_v, x_u]_{m(e)}$ is equivalent to $(x_u x_v)^{m(e)} = \mathbf{e}$. If $u \in V_2$ and $v \in V_p \cup V_\infty$, we have $m'(e) = 2$, and so, the relations in U give the equality

$$\phi(y_u)\phi(y_v) = x_u \xi_v x_v = \xi_v x_u x_v = \xi_v x_v x_u = \phi(y_v)\phi(y_u).$$

If $u, v \in V_p \cup V_\infty$, $m'(e) = 2$, and we have

$$\phi(y_u)\phi(y_v) = \xi_u x_u \xi_v x_v = \xi_u \xi_v x_u x_v = \xi_v \xi_u x_v x_u = \xi_v x_v \xi_u x_u = \phi(y_v)\phi(y_u).$$

(3) Let $u \in V_p \cup V_\infty$ and $v \in V \setminus \{u\}$. Then, there is an edge $\{u', v\} \in E(\Lambda)$ with $m'(\{u', v\}) = 2$. If $v \in V_2$, $\phi(y_{u'})\phi(y_v) = \xi_u x_v = x_v \xi_u = \phi(y_v)\phi(y_{u'})$. If $v \in V_p \cup V_\infty \setminus \{u\}$, we have

$$\phi(y_{u'})\phi(y_v) = \xi_u \xi_v x_v = \xi_v \xi_u x_v = \xi_v x_v \xi_u = \phi(y_v)\phi(y_{u'}).$$

(4) Let $u \in V_p \cup V_\infty$ and $v \in (V_p \cup V_\infty) \setminus \{u\}$, then there is an edge $\{u', v'\} \in E(\Lambda)$ with $m'(\{u', v'\}) = 2$, and we have

$$\phi(y_{u'})\phi(y_{v'}) = \xi_u \xi_v = \xi_v \xi_u = \phi(y_{v'})\phi(y_{u'}).$$

(5) For every $u \in V_p$, there is an edge $\{u', u\} \in E(\Lambda)$ labeled by $m'(\{u', u\}) = f(u)$, and we have

$$(\phi(y_{u'})\phi(y_u))^{f(u)} = (\xi_u \xi_u x_u)^{f(u)} = x_u^{f(u)} = \mathbf{e}.$$

So, the map ϕ extends to a homomorphism $\phi : W \to U$.

Now, consider the map $\psi : \{x_u, u \in V\} \cup \{\xi_v, v \in V_p \cup V_\infty\} \to W$ defined as follows: for $u \in V_2$, $\psi(x_u) = y_u$ and for $u \in V_p \cup V_\infty$, $\psi(x_u) = y_{u'} y_u$, and $\psi(\xi_u) = y_{u'}$. We show that ψ extends to a homomorphism from U to W.

(1) For all $v \in V_2$, $f(v) = 2$, and so, $\psi(x_v)^{f(v)} = y_v^2 = \mathbf{e}$, and for all $v \in V_p$, there is an edge $e = \{v, v'\} \in E(\Lambda)$ with $m'(e) = f(v)$, so

$$\psi(x_v)^{f(v)} = (y_{v'} y_v)^{f(v)} = \mathbf{e}.$$

(2) For all $e = \{u, v\} \in E$, there is an edge $e = \{u, v\} \in E(\Lambda)$ with $m'(e) = m(e)$. If $u, v \in V_2$, we have

$$[\psi(x_u), \psi(x_v)]_{m(e)} = [y_u, y_v]_{m(e)} = [y_v, y_u]_{m(e)} = [\psi(x_v), \psi(x_u)]_{m(e)}.$$

If $u \in V_2$ and $v \in V_p \cup V_\infty$, we have $m(e) = 2$ and

$$\psi(x_u)\psi(x_v) = y_u y_{v'} y_v = y_{v'} y_u y_v = y_{v'} y_v y_u = \psi(x_v)\psi(x_u).$$

If $u, v \in V_p \cup V_\infty$, $m(e) = 2$, and

$$\psi(x_u)\psi(x_v) = y_{u'} y_u y_{v'} y_v = y_{v'} y_v y_{u'} y_u = \psi(x_v)\psi(x_u),$$

as $y_u y_{v'} = y_{v'} y_u$, $y_v y_{u'} = y_{u'} y_v$, and $y_{u'} y_{v'} = y_{v'} y_{u'}$.

(3) For all $v \in V_p \cup V_\infty$, we have $\psi(\xi_v)^2 = y_{v'}^2 = \mathbf{e}$.

(4) For all $u, v \in V_p \cup V_\infty$ distinct, we have $e = \{u', v'\} \in E(\Lambda)$ with $m'(e) = 2$, so $\psi(\xi_u)\psi(\xi_v) = y_{u'} y_{v'} = y_{v'} y_{u'} = \psi(\xi_v)\psi(\xi_u)$.

(5) For all $u \in V_p \cup V_\infty$ and $v \in V \setminus \{u\}$, we have $\{u', v\}, \{u', v'\} \in E(\Lambda)$ with labels $m'(\{u', v\}) = 2$ and $m'(\{u', v'\}) = 2$. If $v \in V_2$, we have

$$\psi(\xi_u)\psi(x_v) = y_{u'} y_v = y_v y_{u'} = \psi(x_v)\psi(\xi_u).$$

If $v \in V_p \cup V_\infty$, we have

$$\psi(\xi_u)\psi(x_v) = y_{u'} y_{v'} y_v = y_{v'} y_{u'} y_v = y_{v'} y_v y_{u'} = \psi(x_v)\psi(\xi_u).$$

(6) For all $u \in V_p \cup V_\infty$,

$$\psi(\xi_u)\psi(x_u)\psi(\xi_u) = y_{u'} y_{u'} y_u y_{u'} = (y_{u'} y_u)^{-1} = \psi(x_u)^{-1}.$$

So, the map ψ extends to a homomorphism $\psi : U \to W$.

We now check that $\phi \circ \psi = \mathrm{Id}_U$ and $\psi \circ \phi = \mathrm{Id}_W$ by showing that these maps are the identity on the generators. For $v \in V_2$, we have $\phi(\psi(x_v)) = \phi(y_v) = x_v$ and $\psi(\phi(y_v)) = \psi(x_v) = y_v$. For $v \in V_p \cup V_\infty$,

$$\phi(\psi(x_v)) = \phi(y_{v'} y_v) = \xi_v \xi_v x_v = x_v$$

and $\phi(\psi(\xi_v)) = \phi(y_{v'}) = \xi_v$. For $v \in V_p \cup V_\infty$,

$$\psi(\phi(y_v)) = \psi(\xi_v x_v) = y_{v'} y_{v'} y_v = y_v$$

and $\psi(\phi(y_{v'})) = \psi(\xi_v) = y_{v'}$. ∎

Corollary 2.9. *Every Dyer group is a finite-index subgroup of some Coxeter group.*

Remark 2.10. As mentioned in the introduction, Corollary 2.9 has many interesting consequences. It implies that Dyer groups are CAT(0) [5, Theorem 12.3.3], linear [1], and biautomatic [14] and that they satisfy the Baum–Connes conjecture, the Farrell–Jones conjecture, the Haagerup property, and the strong Tits alternative. They also admit a proper and virtually special action on a CAT(0) cube complex.

Example 2.11. We apply the previous theorem to Example 2.6. The corresponding graph Λ is given in Figure 2. So, by Theorem 2.8, the Dyer group $D_{m,q}$ is an index 4 subgroup of the Coxeter group

$$\begin{aligned} W = \big\langle a, b, c, d, a', d' \mid\ & a^2 = b^2 = c^2 = d^2 = a'^2 = d'^2 = \mathbf{e}, \\ & (ab)^2 = (bc)^m = (cd)^2 = \mathbf{e}, \\ & (a'b)^2 = (a'c)^2 = (a'd)^2 = (a'd')^2 = \mathbf{e}, \\ & (d'a)^2 = (d'b)^2 = (d'c)^2 = \mathbf{e},\ (d'd)^q = \mathbf{e} \big\rangle. \end{aligned}$$

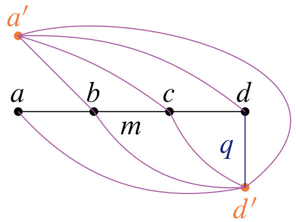


Figure 2. The graph $\Lambda_{m,q}$ built out of the Dyer graph $\Gamma_{m,q}$ for some $m, q \in \mathbb{N}_{\geq 2}$. We color coded the vertices $V \subset V(\Lambda)$ and $\{v' \mid v \in V_p \cup V_\infty\}$. For the edges: for edges of the form $e = \{u, u'\}$ and for edges of the form $e = \{u', v\}$, $v \in V(\Lambda) \setminus \{u, u'\}$, and $u' \in \{v' \mid v \in V_p \cup V_\infty\}$. Every edge is labeled by 2, if not specified otherwise.

The Dyer group D is not the only Dyer group, up to isomorphism, which is a finite-index subgroup of W. We describe another such Dyer group $D' = D(\Omega, g, n)$ by giving the Dyer graph (Ω, g, n). The vertices of Ω are the disjoint union

$$V(\Omega) = V \amalg V_\infty.$$

We will refer to the elements of the disjoint copy of V_∞ as v' for $v \in V_\infty$. The labeling of the vertices is defined as follows: $g_{|(V_2 \cup V_\infty) \amalg V_\infty} = 2$ and $g_{|V_p} = f_{|V_p}$. The subsets $V_2 \cup V_p \cup V_\infty$ and $(V_2 \cup V_p) \amalg V_\infty$ both span copies of Γ, with the same labeling of edges, and for $u, v \in V_\infty$, the vertices v, u' span an edge labeled by 2 in Ω if and only if v and u span an edge in Γ. Let D' be the Dyer group associated to Ω. Note that every generator x_v, $v \in V(\Omega)$ of D' has finite order. We now give an action of $(\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$ on D'. For $v \in V_p$, let $\kappa_v : \{x_u, u \in V(\Omega)\} \to D'$ with $\kappa_v(x_u) = x_u$ for any $u \in V(\Omega) \setminus \{v\}$ and $\kappa_v(x_v) = x_v^{-1}$. For $v \in V_\infty$, let $\kappa_v : \{x_u, u \in V(\Omega)\} \to D'$ with $\kappa_v(x_u) = x_u$ for any $u \in V(\Omega) \setminus \{v, v'\}$ and $\kappa_v(x_v) = x_{v'}$ and $\kappa_v(x_{v'}) = x_v$. For all $v \in V_p \cup V_\infty$, the map κ_v extends to a homomorphism $\kappa_v : D' \to D'$. Moreover, for all $u, v \in V_p \cup V_\infty$, $\kappa_v \circ \kappa_u = \kappa_u \circ \kappa_v$, and $(\kappa_v)^2 = \mathbf{e}$. Hence, we have an action $\kappa : (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty} \times D' \to D'$.

Theorem 2.12. *We have $W \cong D' \rtimes_\kappa (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$.*

Proof. Let us first recall the presentations of W, D', and $U = D' \rtimes_\kappa (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$.

$$\begin{aligned} W = \langle y_v, v \in V(\Lambda) \mid \forall v \in V(\Lambda), (y_v)^2 = \mathbf{e} \\ \text{and } \forall e = \{u, v\} \in E(\Lambda), (y_u y_v)^{m'(e)} = \mathbf{e} \rangle, \end{aligned}$$

$$\begin{aligned} D' = \langle x_v, v \in V(\Omega) \mid x_v^{g(v)} = \mathbf{e} \text{ for all } v \in V(\Omega), \\ [x_v, x_u]_{n(e)} = [x_u, x_v]_{n(e)} \text{ for all } e = \{u, v\} \in E(\Omega) \rangle, \end{aligned}$$

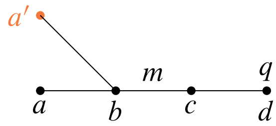


Figure 3. The graph $\Omega_{m,q}$ built out of the Dyer graph $\Gamma_{m,q}$ for some $m, q \in \mathbb{N}_{\geq 2}$. There are two types of vertices: $V \subset V(\Lambda_{m,q})$ and $\{v' \mid v \in V_\infty\}$. Every vertex and every edge is labeled by 2, if not specified otherwise.

$$\begin{aligned} U = \langle \{x_u, u \in V(\Omega)\} \cup \{\kappa_v, v \in V_p \cup V_\infty\} \mid\ & x_u^{g(u)} = \mathbf{e} \text{ for all } u \in V(\Omega), \\ & [x_v, x_u]_{n(e)} = [x_u, x_v]_{n(e)} \text{ for all } e = \{u, v\} \in E(\Omega), \\ & \kappa_v^2 = \mathbf{e} \text{ for all } v \in V_p \cup V_\infty,\ \kappa_v \kappa_u = \kappa_u \kappa_v \text{ for all } u, v \in V_p \cup V_\infty, \\ & \kappa_u x_v = x_v \kappa_u \text{ for all } u \in V_p, v \in V(\Omega) \setminus \{u\}, \\ & \kappa_u x_v = x_v \kappa_u \text{ for all } u \in V_\infty, v \in V(\Omega) \setminus \{u, u'\}, \\ & \kappa_u x_u \kappa_u = x_u^{-1} \text{ for all } u \in V_p,\ \kappa_u x_u \kappa_u = x_{u'} \text{ for all } u \in V_\infty \rangle. \end{aligned}$$

As in Theorem 2.8, we can check that U is isomorphic to W by considering explicit homomorphisms $\phi : W \to U$ and $\psi : U \to W$ satisfying $\phi \circ \psi = \mathrm{Id}_U$ and $\psi \circ \phi = \mathrm{Id}_W$.

The map $\phi : \{y_v, v \in V(\Lambda)\} \to U$ is given as follows: for $u \in V_2$, $\phi(y_u) = x_u$; for $u \in V_p$, $\phi(y_u) = \kappa_u x_u$, and $\phi(y_{u'}) = \kappa_u$, and for $u \in V_\infty$, $\phi(y_u) = x_u$, and $\phi(y_{u'}) = \kappa_u$. One can easily check, using methods which are similar to those used in the proof of Theorem 2.8, that the map ϕ induces a homomorphism

$$\phi : W \to U.$$

The map $\psi : \{x_u, u \in V(\Omega)\} \cup \{\kappa_v, v \in V_p \cup V_\infty\} \to W$ is given as follows: for $u \in V_2$, $\psi(x_u) = y_u$; for $u \in V_p$, $\psi(x_u) = y_{u'} y_u$ and $\psi(\kappa_u) = y_{u'}$; and finally, for $u \in V_\infty$, $\psi(x_u) = y_u$, $\psi(x_{u'}) = y_{u'} y_u y_{u'}$, and $\phi(\kappa_u) = y_{u'}$. Again, one can easily check, using methods which are similar to those used in the proof of Theorem 2.8, that the map ψ induces a homomorphism

$$\psi : U \to W,$$

that $\phi \circ \psi = \mathrm{Id}_U$, and that $\psi \circ \phi = \mathrm{Id}_W$. ∎

Example 2.13. We apply the previous theorem to Example 2.6. The corresponding graph $\Omega_{m,q}$ is given in Figure 3. The associated Dyer group is

$$\begin{aligned} D'_{m,q} = \langle a, b, c, d, a' \mid\ & a^2 = a'^2 = b^2 = c^2 = d^q = \mathbf{e}, \\ & ab = ba, a'b = ba', (bc)^m = \mathbf{e}, cd = dc \rangle. \end{aligned}$$

It is an index 4 subgroup of the Coxeter group W associated to the graph $\Lambda_{m,q}$ given in Figure 2.

Remark 2.14. If the Dyer group D is a right-angled Artin group, i.e., $V = V_\infty$, the constructions described here are those given in [6]. In particular, if D is a right-angled Artin group, the groups W and D' are right-angled Coxeter groups. So, there is a decomposition of W as a semi-direct product of a right-angled Artin group and the right-angled Coxeter group $(\mathbb{Z}/2\mathbb{Z})^V$.

Remark 2.15. There is a Coxeter group W' associated to the Dyer group D' such that $W' = D' \rtimes (\mathbb{Z}/2\mathbb{Z})^{V_p}$. The following questions arise: do W and W' relate in any (meaningful) way? What can we say about their Davis–Moussong complexes? How do D and D' relate to each other? What are all the Dyer subgroups of a given Coxeter group?

3. Complexes of groups

This section introduces the tools needed in Section 4. We first recall necessary definitions and results about small categories without loops. We then recall the constructions of the Davis–Moussong complex and of the Salvetti complex.

3.1. Introduction to scwols

Small categories without loops (scwols) and complexes of groups were introduced by Haefliger in [10, 11]. Based on [2], we would like to recall some notions about scwols and complexes of groups, as we do not assume the reader to be familiar with these constructions. We hope to give all necessary definitions and results; details can be found in [2]. The reader familiar with scwols might only consider the specific examples developed in this section as they will be used in the construction of the cell complex Σ.

A small category without loops (scwol) $\mathcal{X}$ consists of a set $V(\mathcal{X})$, called the vertex set of $\mathcal{X}$, and a set $E(\mathcal{X})$, called the set of edges of $\mathcal{X}$. Additionally, two maps $i : E(\mathcal{X}) \to V(\mathcal{X})$ and $t : E(\mathcal{X}) \to V(\mathcal{X})$ are given. We call $i(\alpha)$ the initial vertex of $\alpha \in E(\mathcal{X})$ and $t(\alpha)$ the terminal vertex of $\alpha \in E(\mathcal{X})$. The set $E^{(2)}(\mathcal{X})$ denotes the pairs $(\alpha, \beta) \in E(\mathcal{X}) \times E(\mathcal{X})$ with $i(\alpha) = t(\beta)$. A third map $\circ : E^{(2)}(\mathcal{X}) \to E(\mathcal{X})$ associates to each pair (α, β) an edge $\alpha\beta$, called their composition. These sets and maps need to satisfy the following conditions:

(1) for all $(\alpha, \beta) \in E^{(2)}$, we have $i(\alpha\beta) = i(\beta)$ and $t(\alpha\beta) = t(\alpha)$;

(2) for all $\alpha, \beta, \gamma \in E(\mathcal{X})$, if $i(\alpha) = t(\beta)$ and $i(\beta) = t(\gamma)$, then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;

(3) for each $\alpha \in E(\mathcal{X})$, we have $i(\alpha) \neq t(\alpha)$.

A subscwol $\mathcal{X}'$ of $\mathcal{X}$ is given by subsets $V(\mathcal{X}') \subset V(\mathcal{X})$ and $E(\mathcal{X}') \subset E(\mathcal{X})$ such that if $\alpha \in E(\mathcal{X}')$, then $i(\alpha), t(\alpha) \in V(\mathcal{X}')$, and if $\alpha, \beta \in E(\mathcal{X}')$ with $i(\alpha) = t(\beta)$, then $\alpha\beta \in E(\mathcal{X}')$.

Remark 3.1. To any poset $(\mathcal{P}, <)$ we can associate a scwol $\mathcal{X}$, where the set of vertices is $\mathcal{P}$ and the set of edges are pairs $(a, b) \in \mathcal{P} \times \mathcal{P}$ such that $b < a$, $t((a, b)) = a$, and $i((a, b)) = b$.

Definition 3.2 (Product of scwols). Given two scwols $\mathcal{X}$ and $\mathcal{Y}$, their product $\mathcal{X} \times \mathcal{Y}$ is the scwol defined as follows: $V(\mathcal{X} \times \mathcal{Y}) = V(\mathcal{X}) \times V(\mathcal{Y})$ and

$$E(\mathcal{X} \times \mathcal{Y}) = (E(\mathcal{X}) \times V(\mathcal{Y})) \sqcup (E(\mathcal{X}) \times E(\mathcal{Y})) \sqcup (V(\mathcal{X}) \times E(\mathcal{Y})).$$

The maps $i, t : E(\mathcal{X} \times \mathcal{Y}) \to V(\mathcal{X} \times \mathcal{Y})$ are defined by $i(\alpha, \beta) = (i(\alpha), i(\beta))$ and $t(\alpha, \beta) = (t(\alpha), t(\beta))$ (we consider $i(v) = t(v) = v$ for any $v \in V(\mathcal{X}) \sqcup V(\mathcal{Y})$) and the composition $(\alpha, \alpha')(\beta, \beta') = (\alpha\beta, \alpha'\beta')$ whenever $t(\beta, \beta') = i(\alpha, \alpha')$ (we consider $\alpha\beta = \alpha$ whenever $\alpha \in E(\mathcal{X}) \sqcup E(\mathcal{Y})$ and $\beta = i(\alpha) \in V(\mathcal{X}) \sqcup V(\mathcal{Y})$, and $\alpha\beta = \beta$ whenever $\beta \in E(\mathcal{X}) \sqcup E(\mathcal{Y})$ and $\alpha = t(\beta) \in V(\mathcal{X}) \sqcup V(\mathcal{Y})$).

Remark 3.3. Let $[n] = \{1, \dots, n\}$. One can inductively define the product of n scwols $\mathcal{X}_1, \dots, \mathcal{X}_n$. Then, the product $\mathcal{X} = \prod_{j \in [n]} \mathcal{X}_j$ is the scwol with vertices $V(\mathcal{X}) = \prod_{j \in [n]} V(\mathcal{X}_j)$ and edges

$$E(\mathcal{X}) = \bigsqcup_{S \subset [n], S \neq \emptyset} \Big(\prod_{j \in S^c} V(\mathcal{X}_j) \Big) \times \Big(\prod_{j \in S} E(\mathcal{X}_j) \Big).$$

The maps $i, t : E(\mathcal{X}) \to V(\mathcal{X})$ are defined by

$$i(\alpha) = i((\alpha_j)_{j \in [n]}) = (i(\alpha_j))_{j \in [n]} \quad \text{and} \quad t(\alpha) = t((\alpha_j)_{j \in [n]}) = (t(\alpha_j))_{j \in [n]}$$

(we consider $i(v) = t(v) = v$ for any $v \in \bigsqcup_{j \in [n]} V(\mathcal{X}_j)$) and the composition $\alpha\beta = (\alpha_j)_{j \in [n]} (\beta_j)_{j \in [n]} = (\alpha_j \beta_j)_{j \in [n]}$ whenever defined.

Example 3.4. Consider a finite set S. Let $\mathcal{P}(S)$ be the set of subsets of S. Consider the poset $(\mathcal{P}(S), \subset)$ and its associated scwol $\mathcal{Y}_S$. Then, $\mathcal{Y}_S = \prod_{v \in S} \mathcal{Y}_{\{v\}}$. Moreover, for any $v \in S$, the scwol $\mathcal{Y}_{\{v\}}$, also denoted by $\mathcal{Y}_v$, has two vertices $\emptyset$ and $\{v\}$ and a single edge e_v with $i(e_v) = \emptyset$ and $t(e_v) = \{v\}$.

Example 3.5. Consider a finite set S. For $v \in S$, let $\mathcal{Z}_{\{v\}} = \mathcal{Z}_v$ be the scwol consisting of two vertices $\emptyset$ and $\{v\}$ and of two edges denoted by $(\emptyset, \{v\}, \emptyset)$ and $(\emptyset, \{v\}, \{v\})$ with $i(\emptyset, \{v\}, \emptyset) = i(\emptyset, \{v\}, \{v\}) = \emptyset$ and $t(\emptyset, \{v\}, \emptyset) = t(\emptyset, \{v\}, \{v\}) = \{v\}$. Let $\mathcal{Z}_S = \prod_{v \in S} \mathcal{Z}_{\{v\}}$. Note that $V(\mathcal{Z}_S) = \mathcal{P}(S)$ and the edges can be described as

$$E(\mathcal{Z}_S) = \{(A, B, \lambda) \mid A \subsetneq B \subset S, \lambda \subset B \setminus A\},$$

where $i(A, B, \lambda) = A$ and $t(A, B, \lambda) = B$. This example seems artificial at this point but will be quite useful later as the geometric realization of $\mathcal{Z}_S$ is a torus $\mathbb{T}^S$ and its fundamental group is $\mathbb{Z}^S$. Indeed, we are particularly interested in the case where S is the vertex set of a complete Dyer graph Γ for which all vertices are labeled by ∞.

A *simple complex of groups* $\mathcal{G}(X) = (G_v, \psi_\alpha)$ over a scwol $\mathcal{X}$ is given by the following data:

(1) for each $v \in V(\mathcal{X})$, a group G_v called the local group at v;

(2) for each $\alpha \in E(\mathcal{X})$, an injective homomorphism $\psi_\alpha : G_{i(\alpha)} \to G_{t(\alpha)}$, with the following compatibility condition: $\psi_{\alpha\beta} = \psi_\alpha \psi_\beta$ whenever defined.

A simple complex of groups $\mathcal{G}(X) = (G_v, \psi_\alpha)$ over a scwol $\mathcal{X}$ is called *inclusive* if it additionally satisfies the following condition: for each $\alpha \in E(\mathcal{X})$, we have $G_{i(\alpha)} < G_{t(\alpha)}$ and $\psi_\alpha(g) = g$ for all $g \in G_{i(\alpha)}$. We will only be considering simple inclusive complexes of groups. These are restrictions on the more general definition of complexes of groups, which can be found in [2].

Definition 3.6. The product $\mathcal{G}(X) \times G(Y)$ of two simple complexes of groups $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ is the simple complex of groups over the scwol $\mathcal{X} \times \mathcal{Y}$ given by the following data:

(1) for each $v = (v_1, v_2) \in V(\mathcal{X} \times \mathcal{Y})$, the local group $G_v = G_{v_1} \times G_{v_2}$ is the direct product of the corresponding local groups in $\mathcal{G}(X)$ and $\mathcal{G}(Y)$;

(2) for each $\alpha = (\alpha_1, \alpha_2) \in E(\mathcal{X} \times \mathcal{Y})$, the injective homomorphism is $\psi_\alpha = \psi_{\alpha_1} \times \psi_{\alpha_2}$ (if one α_i is a vertex, we set ψ_{α_i} to be the identity on G_{α_i}).

As $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ are simple complexes of groups, so is $\mathcal{G}(X) \times G(Y)$.

Similar to the definition of products of scwols, this definition can be extended to finite products of simple inclusive complexes of groups. The product $\prod_{i \in [n]} \mathcal{G}(\mathcal{X}_i)$ of simple complexes of groups $\mathcal{G}(\mathcal{X}_i)$, $i \in [n]$, is the simple complex of groups over the scwol $\prod_{i \in [n]} \mathcal{X}_i$ given by the following data:

(1) for each $v = (v_i)_{i \in [n]} \in V(\prod_{i \in [n]} \mathcal{X}_i)$, the local group $G_v = \prod_{i \in [n]} G_{v_i}$ is the direct product of the corresponding local groups in $\mathcal{G}(\mathcal{X}_i)$;

(2) for each $\alpha = (\alpha_i)_{i \in [n]} \in E(\prod_{i \in [n]} \mathcal{X}_i)$, the injective homomorphism is $\psi_\alpha = \prod_{i \in [n]} \psi_{\alpha_i}$ (if an α_i is a vertex, we set ψ_{α_i} to be the identity on G_{α_i}).

We will now give fundamental examples of complexes of groups and products of complexes of groups over the scwols introduced in Examples 3.4 and 3.5.

Example 3.7. We consider the scwols defined in Example 3.4. For every $v \in S$, we choose a positive integer p_v. Let C_v be the finite cyclic group of order p_v. For $v \in S$, let $\mathfrak{D}(\mathcal{Y}_v)$ be a simple complex of groups over $\mathcal{Y}_v$ by choosing $G_\emptyset = \langle \mathbf{e} \rangle$ the trivial group, $G_{\{v\}} = C_v$, and ψ_{e_v} the trivial map. We define a simple complex of groups $\mathfrak{D}(\mathcal{Y}_S)$ over $\mathcal{Y}_S$ as follows:

(1) for $A \in V(\mathcal{Y}_S)$, we set $G_A = \prod_{v \in A} C_v$;

(2) for $e \in E(\mathcal{Y}_S)$ with $i(e) = A$ and $t(e) = B$, we have $A \subset B$, so $G_A < G_B$, and so, there is a canonical inclusion $\psi_e : G_A \to G_B$. These inclusions satisfy the compatibility condition.

We indeed have $\mathfrak{D}(\mathcal{Y}_S) = \prod_{v \in S} \mathfrak{D}(\mathcal{Y}_{\{v\}})$.

Example 3.8. For a finite Coxeter system (W, S), we can define $\mathfrak{W}(\mathcal{Y}_S)$, a simple complex of groups over $\mathcal{Y}_S$, as follows:

(1) for $A \in V(\mathcal{Y}_S)$, we choose the local group to be W_A, the subgroup of W generated by A;

(2) for $e \in E(\mathcal{Y}_S)$ with $i(e) = A$ and $t(e) = B$, we have $A \subset B$, so there is a canonical inclusion $\psi_e : W_A \to W_B$. These inclusions satisfy the compatibility condition.

In general, in this case, we have $\mathfrak{W}(\mathcal{Y}_S) \neq \prod_{v \in S} \mathfrak{W}(\mathcal{Y}_{\{v\}})$ even though the scwols satisfy $\mathcal{Y}_S = \prod_{v \in S} \mathcal{Y}_{\{v\}}$.

Example 3.9. We consider the scwols defined in Example 3.5. We can always define a trivial complex of groups over a scwol. The product of trivial complexes of groups will again be trivial. This leads to the following notation. For every $v \in S$, we define a simple complex of groups $\mathfrak{D}(\mathcal{Z}_v)$ over each scwol $\mathcal{Z}_v$ by choosing

$$G_\emptyset = G_{\{v\}} = \langle \mathbf{e} \rangle,$$

the trivial group, and $\psi_{(\emptyset,\{v\},\emptyset)} = \psi_{(\emptyset,\{v\},\{v\})}$, the trivial map. Similarly, we define a simple complex of groups $\mathfrak{D}(\mathcal{Z}_S)$ by choosing $G_A = \langle \mathbf{e} \rangle$, the trivial group, for every $A \in V(\mathcal{Z}_S)$ and $\psi_{(A,B,\lambda)}$, the trivial map, for every $(A, B, \lambda) \in E(\mathcal{Z}_S)$. We have

$$\mathfrak{D}(\mathcal{Z}_S) = \prod_{v \in S} \mathfrak{D}(\mathcal{Z}_v).$$

Assume that the scwol $\mathcal{X}$ is connected; i.e., there is only one equivalence class on $V(\mathcal{X})$ for the equivalence relation generated by $(i(\alpha) \sim t(\alpha)$ for every edge $\alpha \in E(\mathcal{X}))$. One can define the fundamental group of a complex of groups $\mathcal{G}(X)$ over a scwol $\mathcal{X}$. For simplicity and as it suffices for the cases we consider, we give the following characterization.

Definition 3.10. Consider a simple complex of groups $\mathcal{G}(X)$ over a connected scwol $\mathcal{X}$. Assume that each group G_v is finitely presented with $G_v = \langle S_v \mid R_v \rangle$. Choose a maximal tree T in the underlying graph. Let $E(\mathcal{X})^{\pm} = \{\alpha^+, \alpha^- \mid \alpha \in E(\mathcal{X})\}$. Then, the fundamental group $\pi_1(\mathcal{G}(X), T)$ is generated by the set

$$\Big(\bigsqcup_{v\in V(\mathcal{X})} S_v\Big) \sqcup E(\mathcal{X})^{\pm}$$

subject to the following relations:

$$\begin{aligned}
&\text{all the relations } R_v \text{ in the groups } G_v,\\
&(\alpha^+)^{-1} = \alpha^- \text{ for all edges } \alpha \in E(\mathcal{X}),\\
&\alpha^+\beta^+ = (\alpha\beta)^+ \text{ for } \alpha,\beta \in E(\mathcal{X}), \text{ whenever } \alpha\beta \in E(\mathcal{X}) \text{ is defined},\\
&\psi_\alpha(s) = \alpha^+ s\alpha^-, \ \forall \alpha \in E(\mathcal{X}), \ \forall s \in S_{i(\alpha)},\\
&\alpha^+ = \mathbf{e}, \ \forall \alpha \in T.
\end{aligned}$$

Different choices of T will give isomorphic fundamental groups. So, we can consider $\pi_1(\mathcal{G}(X)) = \pi_1(\mathcal{G}(X), T)$ for any choice of maximal tree T. Moreover, the subgroup of $\pi_1(\mathcal{G}(X), T)$ generated by $\{\alpha^+, \alpha \in E(\mathcal{X})\}$ corresponds to the fundamental group of the scwol $\mathcal{X}$.

Remark 3.11. The fundamental group of a complex of groups can also be characterized in more categorical terms. By [11, Definition 2.7.1], the fundamental group of a complex of groups $\mathcal{G}(X)$ is also given by the fundamental group of the geometric realization $B\mathcal{G}(X)$ of the nerve $N(C\mathcal{G}(X))$ of the associated category $C\mathcal{G}(X)$. The corresponding definitions are given in [11].

Lemma 3.12. *For two simple inclusive complexes of groups $\mathcal{G}(X)$ and $\mathcal{G}(Y)$, we have*

$$\pi_1(\mathcal{G}(X)) \times \pi_1(\mathcal{G}(Y)) = \pi_1(\mathcal{G}(X) \times G(Y)).$$

Proof. We use the characterization of the fundamental group given in Remark 3.11. Let $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ be two simple inclusive complexes of groups. Then, the category $C(\mathcal{G}(X) \times G(Y))$ associated to the product of complexes of groups $\mathcal{G}(X) \times G(Y)$ is isomorphic to the product category $C\mathcal{G}(X) \times C\mathcal{G}(Y)$ of the categories associated to $\mathcal{G}(X)$ and $\mathcal{G}(Y)$. For the nerves, we have that $N(C(\mathcal{G}(X) \times G(Y)))$ is isomorphic to the cartesian product $N(C\mathcal{G}(X)) \times N(C\mathcal{G}(Y))$ (where the k-skeleton $(N(C\mathcal{G}(X)) \times N(C\mathcal{G}(Y)))_k$ is $(N(C\mathcal{G}(X)))_k \times (N(C\mathcal{G}(Y)))_k$). So, by [12, Theorem 2], the geometric realization $B(C(\mathcal{G}(X) \times G(Y)))$ of the nerve $N(C(\mathcal{G}(X) \times G(Y)))$ is homeomorphic to the product $B(C\mathcal{G}(X)) \times B(C\mathcal{G}(Y))$ of the geometric realizations of $N(C\mathcal{G}(X))$ and $N(C\mathcal{G}(Y))$. So, the fundamental group $\pi_1(B(C(\mathcal{G}(X) \times G(Y))))$ is isomorphic to the product $\pi_1(B(C\mathcal{G}(X))) \times \pi_1(B(C\mathcal{G}(Y)))$. ∎

Example 3.13. In Example 3.7, the fundamental group of $\mathfrak{D}(\mathcal{Y}_S)$ is $\times_{v\in S} C_v$. In Example 3.9, the fundamental group of $\mathfrak{D}(\mathcal{Z}_S)$ is $\mathbb{Z}^S$. In Example 3.8, the fundamental group of $\mathfrak{W}(\mathcal{Y}_S)$ is the Coxeter group W.

We will now consider morphisms. Consider two scwols $\mathcal{X}$ and $\mathcal{Y}$. A morphism of scwols $f : \mathcal{X} \to \mathcal{Y}$ is a map that sends $V(\mathcal{X})$ to $V(\mathcal{Y})$, and sends $E(\mathcal{X})$ to $E(\mathcal{Y})$ such that

(1) for every $\alpha \in E(\mathcal{X})$, $f(i(\alpha)) = i(f(\alpha))$ and $f(t(\alpha)) = t(f(\alpha))$,

(2) for composable edges $\alpha, \beta \in E(\mathcal{X})$, $f(\alpha\beta) = f(\alpha)f(\beta)$.

Let $\mathcal{G}(\mathcal{X}) = (G_v, \psi_\alpha)$ and $\mathcal{H}(\mathcal{Y}) = (H_v, \xi_\alpha)$ be two simple complexes of groups. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of scwols. A morphism of complexes of groups $\phi = (\phi_v, \phi(\alpha)) : \mathcal{G}(\mathcal{X}) \to \mathcal{H}(\mathcal{Y})$ over f consists of

(1) a homomorphism $\phi_v : G_v \to H_{f(v)}$ for every $v \in V(\mathcal{X})$,

(2) an element $\phi(\alpha) \in H_{t(f(\alpha))}$ for every edge $\alpha \in E(\mathcal{X})$ such that we have $\mathrm{Ad}(\phi(\alpha))\xi_{f(\alpha)}\phi_{i(\alpha)} = \phi_{t(\alpha)}\psi_\alpha$ and $\phi(\alpha\beta) = \phi(\alpha)\xi_{f(\alpha)}(\phi(\beta))$ for all composable edges $\alpha, \beta \in E(\mathcal{X})$,

where $\mathrm{Ad}(\phi(\alpha))$ is the conjugation by $\phi(\alpha)$ (where $\mathrm{Ad}(\phi(\alpha))(g) = \phi(\alpha)g\phi(\alpha^{-1})$ for $g \in H_{t(f(\alpha))}$).

Finally, let G be a group. A morphism $\phi = (\phi_v, \phi(\alpha)) : \mathcal{G}(\mathcal{X}) \to G$ consists of a homomorphism $\phi_v : G_v \to G$ for each $v \in V(\mathcal{X})$ and an element $\phi(\alpha) \in G$ for each $\alpha \in E(\mathcal{X})$ such that $\mathrm{Ad}(\phi(\alpha))\phi_{i(\alpha)} = \phi_{t(\alpha)}\psi_\alpha$ and $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ whenever composable. There is always a morphism from the complex of groups to the fundamental group of the complex of groups $\phi = (\phi_v, \phi(\alpha)) : \mathcal{G}(\mathcal{X}) \to \pi_1(\mathcal{G}(\mathcal{X}))$ with $\phi(\alpha) = \alpha^+ \in \pi_1(\mathcal{G}(\mathcal{X}))$ for every edge $\alpha \in E(\mathcal{X})$.

Example 3.14. Consider the complex of groups $\mathfrak{D}(\mathcal{Y}_S)$ given in Example 3.7. Its fundamental group is the product $\pi_1(\mathfrak{D}(\mathcal{Y}_S)) = \times_{v\in S} C_v$. For $A \in V(\mathcal{Y}_S)$, let $\phi^S_A : \times_{v\in A} C_v \to \times_{v\in S} C_v$ with $\phi^S(g) = g$, and for $\alpha \in E(\mathcal{Y}_S)$, let $\phi^S(\alpha) = \mathbf{e}$. The morphism $\phi^S = (\phi^S_A, \phi^S(\alpha)) : \mathfrak{D}(\mathcal{Y}_S) \to \times_{v\in S} C_v$ is injective on the local groups.

Example 3.15. Consider the complex of groups $\mathfrak{W}(\mathcal{Y}_S)$ given in Example 3.8. Its fundamental group is $\pi_1(\mathfrak{W}(\mathcal{Y}_S)) = W$. For $A \in V(\mathcal{Y}_S)$, let $\phi_A : W_A \to W$ be the inclusion with $\phi(g) = g$, and for $\alpha \in E(\mathcal{Y}_S)$, let $\phi(\alpha) = \mathbf{e} \in W$. The morphism

$$\phi = (\phi_A, \phi(\alpha)) : \mathfrak{W}(\mathcal{Y}_S) \to W$$

is injective on the local groups.

Example 3.16. Consider the complex of groups $\mathfrak{D}(\mathcal{Z}_S)$ given in Example 3.9. Its fundamental group is $\pi_1(\mathfrak{D}(\mathcal{Z}_S)) = \mathbb{Z}^S$. For the notation, let $\mathbf{e}$ be the trivial element in $\mathbb{Z}^S$, and let x_s, $s \in S$ be the standard generators of $\mathbb{Z}^S$. For $A \in V(\mathcal{Z}_S)$, let

$$\phi^S_A : \langle \mathbf{e} \rangle \to \mathbb{Z}^S$$

with $\phi^S(\mathbf{e}) = \mathbf{e}$, and for $(A, B, \lambda) \in E(\mathcal{Z}_S)$, let $\phi^S((A, B, \lambda)) = \prod_{s\in\lambda} x_s$. The morphism $\phi^S = (\phi^S_A, \phi^S(\alpha)) : \mathfrak{D}(\mathcal{Z}_S) \to \mathbb{Z}^S$ is injective on the local groups.

Definition 3.17. A complex of groups $\mathcal{G}(\mathcal{X})$ over a scwol $\mathcal{X}$ is developable if there exists a morphism ϕ from $\mathcal{G}(\mathcal{X})$ to some group G which is injective on the local groups.

Remark 3.18. This definition is not the original definition given in [2, Corollary III.$\mathcal{C}$.2.11] but it is equivalent to it by Corollary III.$\mathcal{C}$.2.15 in [2] and better suited to our use.

Let $\mathcal{G}(\mathcal{X})$ be a complex of groups over a scwol $\mathcal{X}$; let G be a group and $\phi : \mathcal{G}(\mathcal{X}) \to G$ a morphism. The development of $\mathcal{X}$ with respect to ϕ is the scwol $\mathcal{C}(\mathcal{X}, \phi)$ given as follows:

(1) its vertices are

$$V(\mathcal{C}(\mathcal{X},\phi)) = \{(g\phi_v(G_v), v) \mid v \in V(\mathcal{X}),\ g\phi_v(G_v) \in G/\phi_v(G_v)\};$$

(2) its edges are

$$\begin{aligned} &E(\mathcal{C}(\mathcal{X},\phi)) \\ &\quad = \{(g\phi_{i(\alpha)}(G_{i(\alpha)}), \alpha) \mid \alpha \in E(\mathcal{X}),\ g\phi_{i(\alpha)}(G_{i(\alpha)}) \in G/\phi_{i(\alpha)}(G_{i(\alpha)})\}, \end{aligned}$$

(3) the maps $i, t : E(\mathcal{C}(\mathcal{X},\phi)) \to V(\mathcal{C}(\mathcal{X},\phi))$ are

$$i(g\phi_{i(\alpha)}(G_{i(\alpha)}), \alpha) = (g\phi_{i(\alpha)}(G_{i(\alpha)}), i(\alpha))$$

and

$$t(g\phi_{i(\alpha)}(G_{i(\alpha)}), \alpha) = (g\phi(\alpha)^{-1}\phi_{t(\alpha)}(G_{t(\alpha)}), t(\alpha));$$

(4) the composition is

$$(g\phi_{i(\alpha)}(G_{i(\alpha)}), \alpha)(h\phi_{i(\beta)}(G_{i(\beta)}), \beta) = (h\phi_{i(\beta)}(G_{i(\beta)}), \alpha\beta),$$

where α, β are composable and $g\phi_{i(\alpha)}(G_{i(\alpha)}) = h\phi(\beta)^{-1}\phi_{i(\alpha)}(G_{i(\alpha)})$.

Note that, by [2, Theorem III.$\mathcal{C}$.2.13], $\mathcal{C}(\mathcal{X},\phi)$ is indeed well defined. Moreover, there is an action of G on $\mathcal{C}(\mathcal{X},\phi)$, where $G \setminus \mathcal{C}(\mathcal{X},\phi) = \mathcal{X}$.

As for simplicial complexes, we can define geometric realizations of scwols. For a scwol $\mathcal{X}$, denote its geometric realization by $|\mathcal{X}|$. If a scwol does not have multiple edges, this construction coincides with the geometric realization of simplicial complexes. This is the only case we will need in this article, and details on the general construction can be found in [2, Chapter III.$\mathcal{C}$.1]. The action of G on $\mathcal{C}(\mathcal{X},\phi)$ induces an action of G on $|\mathcal{C}(\mathcal{X},\phi)|$. If we require the action of G on $|\mathcal{C}(\mathcal{X},\phi)|$ to be by isometries, putting a metric on $|\mathcal{C}(\mathcal{X},\phi)|$ corresponds to putting a metric on $|\mathcal{X}|$ as $G \setminus |\mathcal{C}(\mathcal{X},\phi)| = |\mathcal{X}|$.

Example 3.19. Consider the complex of groups $\mathfrak{D}(\mathcal{Y}_S)$ and the associated morphism $\phi^S : \mathfrak{D}(\mathcal{Y}_S) \to \prod_{v \in S} C_v$ from Example 3.14. One can check that the development $\mathcal{C}(\mathcal{Y}_S, \phi^S)$ is the product $\prod_{v \in S} \mathcal{C}(\mathcal{Y}_{\{v\}}, \phi^{\{v\}})$. Each $\mathcal{C}(\mathcal{Y}_{\{v\}}, \phi^{\{v\}})$ is a scwol with set of vertices $\{(g, \emptyset) \mid g \in C_v\} \cup \{(C_v, \{v\})\}$ and set of edges $\{(g, e_v) \mid g \in C_v\}$ with $i(g, e_v) = (g, \emptyset)$ and $t(g, e_v) = (C_v, \{v\})$. So, it is a star on $|C_v|$ branches, the leaves correspond to the vertices $\{(g, \emptyset) \mid g \in C_v\}$, the central vertex is $(C_v, \{v\})$, and the edges are oriented from the leaves to the center. The group C_v acts by rotation and stabilizes the central vertex. For each $v \in S$, choose $l_v > 0$. Let $\mathrm{Stern}(v)$ be the geometric realization of $\mathcal{C}(\mathcal{Y}_v, \phi^{\{v\}})$ as follows: for $g \in C_v$, consider the interval $I_g = [0, l_v]$, then $\mathrm{Stern}(v) = \bigcup_{g \in C_v} I_g / \sim$ where $0 \in I_g \sim 0 \in I_{\mathbf{e}}$. Note that C_v acts by isometries on $\mathrm{Stern}(v)$. The space $\mathrm{Stern}(S) = \prod_{v \in S} \mathrm{Stern}(v)$ with the product metric is a geometric realization of $\mathcal{C}(\mathcal{Y}_S, \phi^S)$, due to the product structure of $\mathcal{C}(\mathcal{Y}_S, \phi^S)$. Moreover, $\prod_{v \in S} C_v$ acts by isometries on $\mathrm{Stern}(S)$.

Example 3.20. Consider the complex $\mathfrak{W}(\mathcal{Y}_S)$ and the morphism $\phi : \mathfrak{W}(\mathcal{Y}_S) \to W$ from Example 3.15. The development $\mathcal{C}(\mathcal{Y}_S, \phi)$ is a scwol with $\{(gW_A, A) \mid A \subset S, gW_A \in W/W_A\}$ as set of vertices and $\{(gW_A, (A, B)) \mid A \subsetneq B,\ gW_A \in W/W_A\}$ as set of edges where $i(gW_A, (A, B)) = (gW_A, A)$ and $t(gW_A, (A, B)) = (gW_B, B)$. It is the scwol associated to the poset $W\mathcal{P}(S) = \bigcup_{T \subset S} W/W_T$, the poset of parabolic cosets ordered by inclusion. In Section 3.2, we will introduce the Coxeter polytope $\mathrm{Cox}(W)$ of W, which is a geometric realization of $\mathcal{C}(\mathcal{Y}_S, \phi)$.

Example 3.21. Consider the complex $\mathfrak{D}(\mathcal{Z}_S)$ and the morphism $\phi^S : \mathfrak{D}(\mathcal{Z}_S) \to \mathbb{Z}^S$ from Example 3.16. One can check that the development $\mathcal{C}(\mathcal{Z}_S, \phi^S)$ is the product $\times_{v \in S} \mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})$. Each $\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})$ is a scwol with vertices

$$V(\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})) = \{(g, \emptyset) \mid g \in \langle x_v \rangle\} \cup \{(g, \{v\}) \mid g \in \langle x_v \rangle\}$$

and edges

$$E(\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})) = \{(g, (\emptyset, \{v\}, \emptyset)) \mid g \in \langle x_v \rangle\} \cup \{(g, (\emptyset, \{v\}, \{v\})) \mid g \in \langle x_v \rangle\},$$

where $i(g, (\emptyset, \{v\}, \emptyset)) = (g, \emptyset)$, $t(g, (\emptyset, \{v\}, \emptyset)) = (g, \{v\})$, $i(g, (\emptyset, \{v\}, \{v\})) = (g, \emptyset)$, and $t(g, (\emptyset, \{v\}, \{v\})) = (gx_v^{-1}, \{v\})$. So, it is a line; each vertex has either two incoming or two outgoing edges. The group $\mathbb{Z} = \langle x_v \rangle$ acts by translation. There are two orbits: one corresponding to the vertices with incoming edges and one to the vertices with outgoing edges. A geometric realization of $\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})$ is the real line $\mathbb{R}$, where we identify $(\mathbf{e}, \emptyset) \in \mathcal{C}(\mathbb{Z}_{\{v\}}, \phi^{\{v\}})$ with $0 \in \mathbb{R}$, and $(\mathbf{e}, \{v\})$ with 0.5, and $(x_v^{-1}, \{v\})$ with -0.5. Since we want $\mathbb{Z}$ to act by isometries, this means that for every $x_v^k \in \mathbb{Z}$ we identify the vertex $(x_v^k, \emptyset)$ with $k \in \mathbb{R}$ and $(x_v^k, \{v\})$ with $k + 0.5 \in \mathbb{R}$. Using the product structure with the ℓ_2 -metric, we get that $\mathbb{R}^S$ with the Euclidean metric is a geometric realization of $\mathcal{C}(\mathcal{Z}_S, \phi^S)$ on which $\mathbb{Z}^S$ acts by translation.

The construction of the future cell complex Σ relies on the understanding of the local combinatorial structure of the development $\mathcal{C}(\mathcal{X}, \phi)$ of a complex of groups $\mathcal{G}(X)$. We first define links and stars of vertices for scwols.

Definition 3.22 (Links and stars of vertices). Let $\mathcal{X}$ be a scwol. Let $v \in V(\mathcal{X})$. The incoming link $\mathrm{Lk}_{in}(v, \mathcal{X})$ is the scwol with vertex set

$$V(\mathrm{Lk}_{in}(v, \mathcal{X})) = \{\alpha \in E(\mathcal{X}) \mid t(\alpha) = v\},$$

edges

$$E(\mathrm{Lk}_{in}(v, \mathcal{X})) = \{(\alpha, \beta) \in E^{(2)}(\mathcal{X}) \mid t(\alpha) = v\},$$

maps $i, t : E(\mathrm{Lk}_{in}(v, \mathcal{X})) \to V(\mathrm{Lk}_{in}(v, \mathcal{X}))$, with $i((\alpha, \beta)) = \alpha\beta$, $t((\alpha, \beta)) = \alpha$, and composition $(\alpha, \beta)(\alpha', \beta') = (\alpha, \beta\beta')$ when $\alpha\beta = \alpha'$. Similarly, we define the outgoing link $\mathrm{Lk}_{\mathrm{out}}(v, \mathcal{X})$ as the scwol with vertex set

$$V(\mathrm{Lk}_{\mathrm{out}}(v, \mathcal{X})) = \{\alpha \in E(\mathcal{X}) \mid i(\alpha) = v\},$$

edges

$$E(\mathrm{Lk}_{\mathrm{out}}(v, \mathcal{X})) = \{(\alpha, \beta) \in E^{(2)}(\mathcal{X}) \mid i(\beta) = v\},$$

maps $i, t : E(\mathrm{Lk}_{\mathrm{out}}(v, \mathcal{X})) \to V(\mathrm{Lk}_{\mathrm{out}}(v, \mathcal{X}))$, with $i((\alpha, \beta)) = \beta$, $t((\alpha, \beta)) = \alpha\beta$, and composition $(\alpha, \beta)(\alpha', \beta') = (\alpha\alpha', \beta')$ when $\beta = \alpha'\beta'$. The incoming star is the oriented combinatorial join

$$\mathrm{St}_{in}(v, \mathcal{X}) = \mathrm{Lk}_{in}(v, \mathcal{X}) \star \{v\},$$

and the outgoing star is defined similarly:

$$\mathrm{St}_{\mathrm{out}}(v, \mathcal{X}) = \{v\} \star \mathrm{Lk}_{\mathrm{out}}(v, \mathcal{X}).$$

If the scwol $\mathcal{X}$ does not have multiple edges between vertices, the links and stars of vertices can be regarded as subscwols of $\mathcal{X}$. Let $v \in V(\mathcal{X})$. The incoming link $\mathrm{Lk}_{in}(v, \mathcal{X})$ is then the subscwol of $\mathcal{X}$ spanned by the vertex set

$$V(\mathrm{Lk}_{in}(v, \mathcal{X})) = \{u \in V(\mathcal{X}) \mid \exists e \in E(\mathcal{X}) : i(e) = u \text{ and } t(e) = v\}.$$

The incoming star $\mathrm{St}_{in}(v, \mathcal{X})$ is the subscwol spanned by

$$V(\mathrm{St}_{in}(v, \mathcal{X})) = \{u \in V(\mathcal{X}) \mid \exists e \in E(\mathcal{X}) : i(e) = u \text{ and } t(e) = v\} \cup \{v\}.$$

Similarly, the outgoing link $\mathrm{Lk}_{\mathrm{out}}(v, \mathcal{X})$ is then the subscwol of $\mathcal{X}$ spanned by the vertex set

$$V(\mathrm{Lk}_{\mathrm{out}}(v, \mathcal{X})) = \{u \in V(\mathcal{X}) \mid \exists e \in E(\mathcal{X}) : t(e) = u \text{ and } i(e) = v\}.$$

The outgoing star $\mathrm{St}_{\mathrm{out}}(v, \mathcal{X})$ is the subscwol spanned by

$$V(\mathrm{St}_{\mathrm{out}}(v, \mathcal{X})) = \{u \in V(\mathcal{X}) \mid \exists e \in E(\mathcal{X}) : t(e) = u \text{ and } i(e) = v\} \cup \{v\}.$$

We will only study links of star of developments $\mathcal{C}(\mathcal{X}, \phi)$ of a connected scwol $\mathcal{X}$ over the canonical morphism $\phi : \mathcal{G}(\mathcal{X}) \to \pi_1(\mathcal{G}(\mathcal{X}))$. In this case, the development $\mathcal{C}(\mathcal{X}, \phi)$ is simply connected and in particular does not have multiple edges between vertices (see [2, Theorem 3.13]).

Example 3.23. Consider the development $\mathcal{C}(\mathcal{Y}_{\{v\}}, \phi^{\{v\}})$ from Example 3.19. The incoming star of the vertex $(g, \emptyset)$, $g \in C_v$ only contains the vertex $(g, \emptyset)$, but the incoming star of the vertex $(C_v, \{v\})$ is $\mathcal{C}(\mathcal{Y}_{\{v\}}, \phi^{\{v\}})$.

Example 3.24. Consider the development $\mathcal{C}(\mathcal{Y}_S, \phi)$ from Example 3.20. The incoming link of a vertex $(gW_A, A) \in \mathcal{C}(\mathcal{Y}_S, \phi)$ is spanned by the vertices

$$\{(hW_B, B) \mid B \subset A, gh^{-1} \in W_A\}.$$

Example 3.25. Consider the development $\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})$ from Example 3.21. Let $g \in \langle x_v \rangle$. The incoming link of the vertex $(g, \emptyset)$ is empty, and its outgoing link consists of the two disjoint vertices $(g, \{v\})$ and $(gx_v^{-1}, \{v\})$. The incoming link of the vertex $(g, \{v\})$ consists of the two disjoint vertices $(g, \emptyset)$ and $(gx_v, \emptyset)$, and its outgoing link is empty. As $\mathcal{C}(\mathcal{Z}_S, \phi^S)$ is the product $\times_{v \in S} \mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})$, the incoming (resp., outgoing) link of a vertex in $\mathcal{C}(\mathcal{Z}_S, \phi^S)$ can also be expressed as a product of incoming (resp., outgoing) links.

3.2. Coxeter groups and the Davis–Moussong complex

The discussion of the Davis–Moussong complex is based primarily on [5]. We will omit most proofs as they can be found in the literature, in particular in [1,5].

Let S be a finite set. Let $M = (m(s,t))_{s,t \in S}$ be a symmetric matrix with $m(s,t) \in \mathbb{N} \cup \{\infty\}$, $m(s,s) = 1$, and $m(s,t) = m(t,s) \geq 2$ if $s \neq t$. Such a matrix is called a Coxeter matrix. The Coxeter group associated to M is given by the following presentation:

$$W = \langle s \in S \mid (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \rangle,$$

where $m(s,t) = \infty$ means that there is no relation given between s and t. The pair (W, S) is called a Coxeter system. Consider the $S \times S$ matrix c defined by

$$c_{st} = \cos(\pi - \pi/m(s,t)),$$

the matrix c is called the cosine matrix of the Coxeter matrix M. When $m(s,t) = \infty$, we interpret π/∞ to be 0 and $\cos(\pi - \pi/\infty) = -1$. The following fact states a classical result, giving a necessary and sufficient condition for a Coxeter group to be finite.

Fact 3.26 ([5, Theorem 6.12.9]). A Coxeter group W is finite if and only if the cosine matrix c is positive definite.

For $T \subset S$, let W_T be the subgroup of W generated by T. Consider the poset of spherical subsets $\mathcal{S} = \{T \subset S \mid W_T \text{ is finite}\}$ ordered by inclusion. In an abuse of notation, let us also write $\mathcal{S}$ for the scwol associated to the poset $\mathcal{S}$. Similarly, to Examples 3.8, 3.15, and 3.20, let $\mathfrak{W}(\mathcal{S})$ be the complex of groups over $\mathcal{S}$, where the local group at $T \in \mathcal{S}$ is W_T, and for an edge (R, T), the associated map $\psi_{(R,T)} : W_R \to W_T$ is the inclusion $\psi_{(R,T)}(r) = r$ for every $r \in R$. The fundamental group of $\mathfrak{W}(\mathcal{S})$ is W, and there is an injective morphism $\phi = (\phi_T, \phi((R,T)))$, where $\phi_T : W_T \to W$ is the inclusion and $\phi((R,T)) = \mathbf{e}$ for every edge (R,T). Let $\mathcal{C}(\mathcal{S}, \phi)$ be the development of $\mathfrak{W}(\mathcal{S})$ with respect to ϕ. Let us also consider the poset $W\mathcal{S} = \bigcup_{T \in \mathcal{S}} W/W_T$, called the poset of spherical cosets. In a similar abuse of notation, let us also write $W\mathcal{S}$ for the scwol associated to the poset $W\mathcal{S}$.

Remark 3.27. The set of vertices of $\mathcal{C}(\mathcal{S}, \phi)$ is $\{(wW_T, T) \mid T \in \mathcal{S}, wW_T \in W/W_T\}$. The set of edges of $\mathcal{C}(\mathcal{S}, \phi)$ is $\{(wW_R, (R,T)) \mid R, T \in \mathcal{S},\ R \subset T,\ wW_R \in W/W_R\}$, where $(wW_R, (R,T))$ is an edge from the vertex (wW_R, R) to the vertex (wW_T, T). In particular, there is an edge from a vertex (wW_R, R) to a vertex $(w'W_T, T)$ if and only if $R \subset T$ and $wW_T = w'W_T$ (i.e., $w'^{-1}w \in W_T$).

Lemma 3.28. *The scwols $\mathcal{C}(\mathcal{S}, \phi)$ and $W\mathcal{S}$ are equal.*

Proof. It follows from Remark 3.27 that the two scwols have the same set of vertices. For the edges, note that, for $wW_R, w'W_T \in W\mathcal{S}$, we have $wW_R \subset w'W_T$ if and only if $R \subset T$ and $w'^{-1}w \in W_T$. So, using Remark 3.27, the sets of edges also coincide. ∎

Coxeter polytope. From now on, assume that W is finite. Let us recall the canonical representation of W. Consider $\Pi = \{\alpha_s \mid s \in S\}$ and $V = \bigoplus_{s \in S} \mathbb{R}\alpha_s$. Let

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

be the scalar product on V given by $\langle \alpha_s, \alpha_t \rangle = -\cos(\frac{\pi}{m(s,t)})$. The canonical representation of W on $GL(V)$ is given by $\rho : W \to GL(V)$ with $\rho(s)(x) = x - 2\langle \alpha_s, x\rangle \alpha_s$ for $s \in S$, $x \in V$. The scalar product on V is $\rho(W)$-invariant. There is a dual basis $\prod^* = \{\alpha_s^* \mid s \in S\}$, satisfying $\langle \alpha_s^*, \alpha_t \rangle = 0$ if $s \neq t$ and $\langle \alpha_s^*, \alpha_s \rangle = 1$. We choose $x_0 = \sum_{s \in S} \alpha_s^* \in V$. The Coxeter polytope of (W, S), denoted by $\mathrm{Cox}(W)$, is the convex hull of $\{\rho(w)(x_0) \in V \mid w \in W\}$. It is endowed with the Euclidean metric. Note that its interior is nonempty. For a subset $T \subset S$, let $\prod_T = \{\alpha_s \mid s \in T\}$ and V_T be the subvector space of V spanned by $\prod_T$. Let $\prod_T^* = \{\alpha_{s,T}^* \mid s \in T\}$ be the dual basis of $\prod_T$ in V_T. Fix $x_{0,T} = \sum_{s\in T} \alpha^*_{s,T}$. Let $\mathrm{Cox}(W_T)$ be the convex hull of $\{\rho(w)(x_{0,T}) \in V_T \mid w \in W_T\}$. Moreover, let $\mathrm{Cox}_T(W)$ be the convex hull of $\{\rho(w)(x_0) \in V \mid w \in W_T\}$. Let $u = x_0 - x_{0,T}$ and $t_u : V \to V$ be the translation by the vector u. This translation sends $\rho(w)(x_{0,T})$ to $\rho(w)(x_0)$ for every $w \in W_T$. Specifically, it is an isometry from $\mathrm{Cox}(W_T)$ to $\mathrm{Cox}_T(W)$.

Lemma 3.29 ([5, Lemma 7.3.3]). *The poset $W\mathcal{S}$ and the face poset $\mathcal{F}(\mathrm{Cox}(W))$ of $\mathrm{Cox}(W)$ are isomorphic. Specifically, the correspondence $wW_T \to w\,\mathrm{Cox}_T(W)$ induces an isomorphism of posets.*

So, we can identify $W\mathcal{S}$ and hence $\mathcal{C}(\mathcal{S},\phi)$ with the barycentric subdivision of the Coxeter polytope $\mathrm{Cox}(W)$, thus identifying $|\mathcal{C}(\mathcal{S},\phi)|$ isometrically with $\mathrm{Cox}(W)$. The metric on $|\mathcal{C}(\mathcal{S},\phi)|$ induced by the identification with $\mathrm{Cox}(W)$ is called the Moussong metric. In particular, for $wW_T \in W\mathcal{S}$, the geometric realization

$$|W\mathcal{S}_{\leq wW_T}| \subset |W\mathcal{S}|$$

is identified with the face $w\,\mathrm{Cox}_T(W)$.

The general case. We now consider any Coxeter group W, so W need not necessarily be finite. We put a coarser cell structure on $W\mathcal{S}$ (or equivalently on $\mathcal{C}(\mathcal{S},\phi)$) to build the Davis–Moussong complex Σ by identifying each subposet $(W\mathcal{S})_{\leq wW_T, T\in\mathcal{S}}$, which is isomorphic to the poset $W_T(\mathcal{S}_{\leq T})$, with a Coxeter polytope $\mathrm{Cox}(W_T)$. So, we can give the following description of Σ.

Theorem 3.30 ([5, Proposition 7.3.4]). *There is a natural cell structure on Σ so that*

(1) *its vertex set is W, its 1-skeleton is the Cayley graph* $\mathrm{Cay}(W,S)$*, and its 2-skeleton is the Cayley 2-complex over* $\mathrm{Cay}(W,S)$ *with the relations*

$$(st)^{m(s,t)} = \mathbf{e} \quad \textit{for all } s,t \in S,\ s \neq t;$$

(2) *each cell is a Coxeter polytope;*

(3) *the link* $\mathrm{Lk}(v,\Sigma)$ *of each vertex is isomorphic to the abstract simplicial complex $\mathcal{S}_{>\emptyset}$;*

(4) *a subset of W is the vertex set of a cell if and only if it is a spherical coset;*

(5) *the poset of cells in Σ is $W\mathcal{S}$.*

Note that the Cayley graph $\mathrm{Cay}(W,S)$ is considered to be undirected; hence, there are no double edges between vertices, even though all elements of S have order 2 in W. Furthermore, all edges in $\mathrm{Cay}(W,S)$ are labeled; hence, the edges of Σ are labeled. This labeling coincides with the labeling of vertices in $\mathrm{Lk}(v,\Sigma)$. By [5, Lemma 12.1.1], the piecewise Euclidean structure on Σ induces a piecewise spherical structure on the link $\mathrm{Lk}(v,\Sigma)$ of a vertex and as such on the abstract simplicial complex $\mathcal{S}_{>\emptyset}$ with edge length $d(u, v) = \pi - \pi/m(u, v)$ for two adjacent vertices $u, v \in S$.

Now that we have an appropriate description of Σ, let us state the following geometric property of Σ.

Theorem 3.31 (Moussong's Theorem [13]). *For any Coxeter system, the associated cell complex Σ, equipped with its natural piecewise Euclidean metric, is* CAT(0).

A simplicial complex L with piecewise spherical structure has simplices of size $\geq \pi/2$ if each of its edges has length $\geq \pi/2$. Such a simplicial complex is a metric flag complex if the following condition holds: suppose that $\{v_0, \dots, v_k\}$ is a set of pairwise adjacent vertices of L. Put $c_{ij} = \cos(d(v_i, v_j))$. Then, $\{v_0, \dots, v_k\}$ spans a simplex if and only if the matrix (c_{ij}) is positive definite. Then, Moussong's theorem is the consequence of the following lemmata.

Lemma 3.32 ([5, Lemma 12.3.1]). *Let* Lk *be the link of a vertex in Σ with its natural piecewise spherical structure inherited from Σ. Then,* Lk *is a simplicial complex and has simplices of size $\geq \pi/2$. Moreover, it is a metric flag complex.*

Note that using Fact 3.26 the set of vertices of Lk is S and $T \subset S$ spans a simplex if and only if W_T is finite. Moreover, the distance between two vertices in Lk is given by $d(v, w) = \pi - \pi/m(v, w)$.

Lemma 3.33 (Moussong's lemma [5, Lemma I.7.4]). *Suppose that L is piecewise spherical simplicial cell complex in which all cells are simplices of size $\geq \pi/2$. Then, L is* CAT(1) *if and only if it is a metric flag complex.*

3.3. Right-angled Artin groups and the Salvetti complex

Every Coxeter group has an associated Artin group. We will concentrate on the class of right-angled Artin groups and present their analog to the Davis–Moussong complex, the Salvetti complex S_Γ. An extensive discussion of right-angled Artin groups can be found in Charney's survey [3].

Given a simplicial graph Γ, with vertex set V and edge set E, the associated right-angled Artin group $A(\Gamma)$ is given by the following presentation:

$$A(\Gamma) = \langle x_v, v \in V \mid \text{ for every } e = \{v, w\} \in E,\ x_v x_w = x_w x_v \rangle.$$

If Γ has no edges, $A(\Gamma)$ is the free group of rank $|V|$; if Γ is a complete graph, $A(\Gamma)$ is the free abelian group of rank $|V|$.

Salvetti complex S_Γ. Let Γ be a simplicial graph with vertex set V. For any set of pairwise adjacent vertices $V' = \{v_1, \dots, v_n\}$, consider the corresponding generators

$x_i = x_{v_i}$, and let

$$C(V') = \{x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in A(\Gamma) \mid \varepsilon_i \in \{0,1\} \text{ for every } i \in \{1,\ldots,n\}\}.$$

Note that for two sets $V', V'' \subset V$ of pairwise adjacent vertices we have $V' \neq V''$ if and only if $C(V') \neq C(V'')$. The Salvetti complex S_Γ is the cube complex with vertex set $A(\Gamma)$, and for every $a \in A(\Gamma)$ and every set of pairwise adjacent vertices $V' \subset V$, there is a cube $aC(V')$. The Salvetti complex S_Γ is known to be a CAT(0) cube complex by [4] and $A(\Gamma)$ acts properly and cocompactly on S_Γ.

Recall Examples 3.5, 3.9, and 3.21. Let Γ be a complete simplicial graph with vertex set V. The development $\mathcal{C}(\mathcal{Z}_V, \phi^V)$ is the barycentric subdivision of the Salvetti complex of the Artin group $A(\Gamma)$. Moreover, the geometric realization of $\mathcal{C}(\mathcal{Z}_V, \phi^V)$, described in Example 3.21, is isometric to the Salvetti complex S_Γ endowed with the standard ℓ_2-cubical metric with edge length 1.

4. A piecewise Euclidean cell complex for Dyer groups

The goal of this section is to show geometrically that Dyer groups are CAT(0) by constructing an appropriate Euclidean cell complex Σ. In particular, we want Dyer groups to act properly and cocompactly on the cell complex Σ. The first step is to construct a scwol $\mathcal{C}$ associated to a Dyer group. The scwol $\mathcal{C}$ encodes the necessary information to build Σ. The vertices of $\mathcal{C}$ will correspond to subcomplexes of Σ, and the edges of $\mathcal{C}$ will encode identifications between subcomplexes of Σ. Finally, we will also be able to interpret $\mathcal{C}$ as a simplicial subdivision of the complex Σ. We will first focus on spherical Dyer groups D, which factor as a direct product of a finite Coxeter group and cyclic groups. We start with the construction of a scwol $\mathcal{X}$ associated to a spherical Dyer group D and then define a complex of groups $\mathfrak{D}(\mathcal{X})$. The scwol $\mathcal{C}$ will be the development of the complex of groups $\mathfrak{D}(\mathcal{X})$. The second subsection will discuss this for general Dyer groups. The third subsection will be devoted to the Euclidean cell complex Σ.

4.1. A combinatorial structure for spherical Dyer groups

A Dyer group $D = D(\Gamma, f, m)$ is *spherical* if its underlying graph Γ is complete and the subgroup D_2 is a finite Coxeter group. If $D = D(\Gamma, f, m)$, we also say that $\Gamma = (\Gamma, f, m)$ is a spherical Dyer graph. In this section, we will assume that D is a spherical Dyer group. In particular, we then have $D = D_2 \times D_p \times D_\infty$, where D_2 is a finite Coxeter group, D_p is a direct product of finite cyclic groups, and $D_\infty = \mathbb{Z}^{V_\infty}$. As with Coxeter groups, we can characterize spherical Dyer groups through the cosine matrix. Let (Γ, f, m) be a Dyer graph, and let $V = V(\Gamma)$ and $E = E(\Gamma)$. We extend the map $m : E \to \mathbb{N}_{\geq 2}$ to a map $m : V \times V \to \mathbb{N}_{\geq 2} \cup \{\infty\}$ by setting $m(u, v) = m(\{u, v\})$ if $\{u, v\} \in E$, $m(u, v) = \infty$ if $u \neq v$ and $\{u, v\} \notin E$, and $m(u, u) = 1$. We interpret π/∞ to be 0 and $\cos(\pi - \pi/\infty) = \cos(\pi) = -1$. The cosine matrix associated to a Dyer graph (Γ, f, m) is the $V \times V$ matrix c defined by $c_{uv} = \cos(\pi - \pi/m(u, v))$. The following characterization of spherical Dyer groups follows from Fact 3.26.

Lemma 4.1. *A Dyer group $D(\Gamma, f, m)$ is spherical if and only if the cosine matrix c associated to (Γ, f, m) is positive definite.*

Proof. Assume that D is a spherical Dyer group. Then, the restriction of c to $V_2 \times V_2$ is positive definite. Since additionally Γ is a complete Dyer graph, this implies that the matrix c is positive definite. Now, assume that the cosine matrix c associated to (Γ, f, m) is positive definite. Consider the matrix $M = (m(u, v))_{u,v \in V}$. Then, the cosine matrix c associated to (Γ, f, m) is equal to the cosine matrix of the Coxeter matrix M as defined in Section 3.2. So, by Fact 3.26, the cosine matrix c is positive definite if and only if the Coxeter group associated to M is finite. So, we have $m(u, v) \neq \infty$ for all $u, v \in V$. Moreover, since Γ is a Dyer graph, this also implies that the restriction of c to $V_2 \times V_2$ is positive definite. So, the graph Γ is complete and D_2 is a finite Coxeter group by Fact 3.26. Hence, D is a spherical Dyer group. ∎

Let $\mathcal{X} = \mathcal{X}(\Gamma)$ be the scwol with set of vertices $V(\mathcal{X}) = \{X \subseteq V\}$ and set of edges $E(\mathcal{X}) = \{(X, Y, \omega) \mid X \subsetneq Y \subset V(\Gamma),\ \omega \subseteq (Y \setminus X)_\infty\}$ with $i(X, Y, \omega) = X$ and $t(X, Y, \omega) = Y$ and $(Y, Z, \omega')(X, Y, \omega) = (X, Z, \omega \cup \omega')$. We call $\mathcal{X}$ the scwol associated to the spherical Dyer graph Γ. Similar to the group D, we can also describe $\mathcal{X}$ as a direct product of scwols.

Lemma 4.2. *Let $\mathcal{X}_2 = \mathcal{X}(\Gamma_2)$, $\mathcal{X}_p = \mathcal{X}(\Gamma_p)$ and $\mathcal{X}_\infty = \mathcal{X}(\Gamma_\infty)$. Then, we have the product decomposition $\mathcal{X} = \mathcal{X}_2 \times \mathcal{X}_p \times \mathcal{X}_\infty$. Moreover, $\mathcal{X}_p = \mathcal{Y}_{V_p} = \times_{v \in V_p} \mathcal{Y}_v$ and $\mathcal{X}_\infty = \mathcal{Z}_{V_\infty} = \times_{v \in V_\infty} \mathcal{Z}_v$ as in Examples 3.4 and 3.5.*

Proof. Since $V = V_2 \sqcup V_p \sqcup V_\infty$, every $X \in V(\mathcal{X})$ can be decomposed as a disjoint union $X = X_2 \sqcup X_p \sqcup X_\infty$, so $V(\mathcal{X}) = V(\mathcal{X}_2) \times V(\mathcal{X}_p) \times V(\mathcal{X}_\infty)$. For the edges, note that $(X, Y, \omega) \in E(\mathcal{X})$ if and only if $X_i \subset Y_i$ for every $i \in \{2, p, \infty\}$, and at least, one of those inclusions is strict and $\omega \subset Y_\infty \setminus X_\infty$. ∎

Example 4.3. Let Γ be a complete graph with vertex set $V = \{s, t, u, v\}$. Let

$$f : V \to \mathbb{N}_{\geq 2} \cup \{\infty\}$$

with $f(s) = f(t) = 2$, $f(v) = \infty$, and $f(u) = q$ for some $q \in \mathbb{N}_{>2}$. Let $m : E \to \mathbb{N}_{\geq 2}$ with $m(\{s, t\}) = m$ for some $m \in \mathbb{N}_{>2}$ and $m(e) = 2$ for every other edge $e \in E$. Then, the triple (Γ, f, m) is a Dyer graph and the associated Dyer group D is spherical. The associated scwol $\mathcal{X}$ is given as follows. Its vertex set is $V(\mathcal{X}) = \mathcal{P}(V) = \{X \subseteq V\}$.

For $X, Y \in \mathcal{P}(V)$, there is an edge $(X, Y, \emptyset)$ if and only if $X \subsetneq Y$ and an edge $(X, Y, \{v\})$ if and only if $v \in Y$ and $v \notin X$. Moreover, these are the only edges in $\mathcal{X}$. Since $\mathcal{X}$ has 16 vertices, we do not attempt to draw the corresponding figure here.

We define the simple complex of groups $\mathfrak{D}(\mathcal{X})$ over the scwol $\mathcal{X}$. For each $X \in V(\mathcal{X})$, let the local group be $D_X^f = D_{X_2 \cup X_p}$. As mentioned in Remark 2.4, we know by [7] that if $X \subseteq Y$, $D_X^f < D_Y^f < D$. For each edge $(X, Y, \omega) \in E(\mathcal{X})$, let $\psi_{(X,Y,\omega)} : D_X^f \to D_Y^f$ be the map induced by $\psi(x_v) = x_v$ for every $v \in X_2 \cup X_p$. These maps are all injective. Note that they do not depend on ω. We also introduce the morphism $\phi = \phi^\Gamma : \mathfrak{D}(\mathcal{X}) \to D$, where $\phi_X = \phi_X^\Gamma : D_X^f \to D$ is the natural inclusion and $\phi(X, Y, \omega) = \phi^\Gamma(X, Y, \omega) = \prod_{v \in \omega} x_v$. Note that $\phi(X, Y, \omega)$ is well defined since the subgraph spanned by $\omega \subseteq V_\infty$ is complete. Moreover, $\phi(X, Y, \omega)$ only depends on ω, so we will write

$$\phi(X, Y, \omega) = \phi(\omega) = \prod_{v \in \omega} x_v.$$

Also note that each local group D_X^f is finite.

Lemma 4.4. *The complex of groups $\mathfrak{D}(\mathcal{X})$ is isomorphic to the product of complexes of groups $\mathfrak{D}(\mathcal{X}_2) \times \mathfrak{D}(\mathcal{X}_p) \times \mathfrak{D}(\mathcal{X}_\infty)$. Moreover, $\mathfrak{D}(\mathcal{X}_p)$ is isomorphic to $\prod_{v \in V_p} \mathfrak{D}(\mathcal{Y}_v)$ from Example 3.7 and $\mathfrak{D}(\mathcal{X}_\infty)$ is isomorphic to $\prod_{v \in V_\infty} \mathfrak{D}(\mathcal{Z}_v)$ from Example 3.9.*

Proof. For every $X \in V(\mathcal{X})$, the local group D_X^f can be decomposed as a product $D_X^f = D_{X_2}^f \times D_{X_p}^f$. As $D_{X_\infty}^f$ is the trivial group, we also have

$$D_X^f \cong D_{X_2}^f \times D_{X_p}^f \times D_{X_\infty}^f.$$

Moreover, $D_{X_p} = \prod_{v \in V_p} D_{\{v\}}$ and $D_{X_\infty} = \prod_{v \in V_\infty} D_{\{v\}}$. The morphism $\phi = \phi^\Gamma$ also decomposes as a product $\phi = \phi_2 \times \phi_p \times \phi_\infty$, where $\phi_2 = \phi^{\Gamma_2}$, $\phi_p = \phi^{\Gamma_p}$, and $\phi_\infty = \phi^{\Gamma_\infty}$. So, using Lemma 4.2, we have that the complex $\mathfrak{D}(\mathcal{X})$ is isomorphic to the product $\mathfrak{D}(\mathcal{X}_2) \times \mathfrak{D}(\mathcal{X}_p) \times \mathfrak{D}(\mathcal{X}_\infty)$. ∎

Lemma 4.5. *The fundamental group of $\mathfrak{D}(\mathcal{X})$ is D, and the complex of groups $\mathfrak{D}(\mathcal{X})$ is developable.*

Proof. We use the product decomposition given in Lemma 4.4. The scwol $\mathcal{X}_2$ is associated to the poset $\mathcal{P}(V_2)$, so it is simply connected. Moreover, it contains a unique maximal element V_2, so the fundamental group of $\mathfrak{D}(\mathcal{X}_2)$ is $D_{V_2}^f = D_2$. The same argument implies that the fundamental group of $\mathfrak{D}(\mathcal{X}_p)$ is D_p. Recall that $\mathfrak{D}(\mathcal{X}_\infty)$ is isomorphic to $\prod_{v \in V_\infty} \mathfrak{D}(\mathcal{Z}_v)$. The fundamental group of each $\mathfrak{D}(\mathcal{Z}_v)$ is $\mathbb{Z}$. So, by Lemma 3.12, the fundamental group of $\mathfrak{D}(\mathcal{X}_\infty)$ is $\mathbb{Z}^{V_\infty} = D_\infty$. So, by Lemma 3.12, the fundamental group of $\mathfrak{D}(\mathcal{X})$ is $D_2 \times D_p \times D_\infty = D$. Since the maps ϕ_X are injective for all $X \in V(\mathcal{X})$, the complex $\mathfrak{D}(\mathcal{X})$ is developable. ■

Since the complex $\mathfrak{D}(\mathcal{X})$ is developable, we can describe its development

$$\mathcal{C} = \mathcal{C}(\mathcal{X}, \phi).$$

Since $D_X^f < D$ and the maps ϕ_X are canonical inclusions, we will identify the image $\phi_X(D_X^f)$ with $D_X^f < D$. The set of vertices of $\mathcal{C}$ is

$$V(\mathcal{C}) = \{(gD_X^f, X) \mid X \in V(\mathcal{X}),\ gD_X^f \in D/D_X^f\}.$$

The set of edges of $\mathcal{C}$ is

$$E(\mathcal{C}) = \{(gD_X^f, (X, Y, \omega)) \mid (X, Y, \omega) \in E(\mathcal{X}),\ gD_X^f \in D/D_X^f\},$$

where $i(gD_X^f, (X, Y, \omega)) = (gD_X^f, X)$ and $t(gD_X^f, (X, Y, \omega)) = (g\phi(\omega)^{-1} D_Y^f, Y)$. For a simpler notation, we write gX for a vertex (gD_X^f, X) and $g(X, Y, \omega)$ for an edge $(gD_X^f, (X, Y, \omega))$. Note that $gX = hY$ if and only if $X = Y$ and $g^{-1}h \in D_X^f$. Similarly, $g(X, Y, \omega) = h(X', Y', \omega')$ if and only if $X' = X$, $Y' = Y$, $\omega' = \omega$, and $g^{-1}h \in D_X^f$. In particular, $\mathcal{X}$ is the quotient of $\mathcal{C}$ by the action of the group D.

Lemma 4.6. *The development $\mathcal{C}(\mathcal{X}, \phi)$ has a product decomposition*

$$\mathcal{C}(\mathcal{X}_2, \phi_2) \times \mathcal{C}(\mathcal{X}_p, \phi_p) \times \mathcal{C}(\mathcal{X}_\infty, \phi_\infty).$$

Proof. This follows from the product decomposition of $\mathcal{X}$, $\mathfrak{D}(\mathcal{X})$, D, and ϕ. ■

Remark 4.7. For $i \in \{2, p, \infty\}$, Lemma 4.6 implies that we can consider each scwol $\mathcal{C}(\mathcal{X}_i, \phi_i)$ to be a subscwol of $\mathcal{C}(\mathcal{X}, \phi)$. There is a canonical inclusion by identifying a vertex $gX \in \mathcal{C}(\mathcal{X}_i, \phi_i)$ with $gX \in \mathcal{C}(\mathcal{X}, \phi)$. The subscwol $\mathcal{C}(\mathcal{X}_i, \phi_i)$ is then stable under the action of D_i.

Remark 4.8. The incoming star $\mathrm{St}_{in}(gY, \mathcal{C})$ is isomorphic to the incoming star

$$\mathrm{St}_{in}(\mathbf{e}Y, \mathcal{C}(\mathcal{X}(\Gamma_Y), \phi^{\Gamma_Y})).$$

Moreover, the product decomposition of $\mathcal{C}$ induces a product decomposition of the incoming star

$$\mathrm{St}_{in}(\mathbf{e}Y, \mathcal{C}) = \mathrm{St}_{in}(\mathbf{e}Y_2, \mathcal{C}(\mathcal{X}_2, \phi_2)) \times \mathrm{St}_{in}(\mathbf{e}Y_p, \mathcal{C}(\mathcal{X}_p, \phi_p)) \times \mathrm{St}_{in}(\mathbf{e}Y_\infty, \mathcal{C}(\mathcal{X}_\infty, \phi_\infty))$$

and as such also a product decomposition for every $\mathrm{St}_{in}(gY, \mathcal{C})$. Moreover, for a vertex $hZ \in \mathrm{St}_{in}(gY, \mathcal{C})$, the star $\mathrm{St}_{in}(hZ, \mathcal{C})$ is a subscwol of $\mathrm{St}_{in}(gY, \mathcal{C})$.

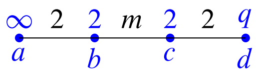


Figure 4. Dyer graph $\Gamma_{m,q}$ for some $m, q \in \mathbb{N}_{\geq 2}$ as given in Figure 1.

4.2. A combinatorial structure for general Dyer groups

Let us now give a similar construction with analogous results for general Dyer groups. Let (Γ, f, m) be a Dyer graph and $D = D(\Gamma)$ the associated Dyer group. We note that $V = V(\Gamma)$. Let $\mathcal{X} = \mathcal{X}(\Gamma)$ be the scwol with set of vertices

$$V(\mathcal{X}) = \{X \subseteq V \mid D(\Gamma_X) \text{ is a spherical Dyer group}\}$$

and edges

$$E(\mathcal{X}) = \{(X, Y, \omega) \mid X, Y \in V(\mathcal{X}),\ X \subsetneq Y,\ \omega \subseteq (Y \setminus X)_\infty\}$$

with $i(X, Y, \omega) = X$ and $t(X, Y, \omega) = Y$ and $(Y, Z, \omega')(X, Y, \omega) = (X, Z, \omega \cup \omega')$. The main difference with the spherical case is the set of vertices of $\mathcal{X}$. Indeed, we do not consider all subsets $X \subseteq V$ but only those for which Γ_X is complete and the group $D_X^f = D_{X_2 \cup X_p}$ is finite. We also define a complex of groups $\mathfrak{D}(\mathcal{X})$ over $\mathcal{X}$. For each $X \in V(\mathcal{X})$, let the local group be $D_X^f = D_{X_2 \cup X_p}$, and for each edge $(X, Y, \omega) \in E(\mathcal{X})$, let $\psi_{(X,Y,\omega)} : D_X^f \to D_Y^f$ be the natural inclusion. By [7], these maps are all injective. The local groups are all finite. We also introduce the morphism $\phi : \mathfrak{D}(\mathcal{X}) \to D$, where $\phi_X : D_X^f \to D$ is the natural inclusion and $\phi(X, Y, \omega) = \phi(\omega) = \prod_{v \in \omega} x_v$ (this element is well defined since $\omega \subseteq V_\infty$ and Γ_ω is complete). As in the spherical case, we can write $\mathfrak{D}(\mathcal{X}(\Gamma))$ and ϕ^Γ when also considering the same construction on a subgraph. As before, we are interested in the development of the complex of groups $\mathfrak{D}(\mathcal{X})$, so we first show that $\mathfrak{D}(\mathcal{X})$ is developable.

Example 4.9. Consider the Dyer graph $\Gamma_{m,q}$, given again in Figure 4, and the Dyer group $D_{m,q}$ from Example 2.6. The associated scwol $\mathcal{X}_{m,q}$ is drawn in Figure 5. Its vertex set is $V(\mathcal{X}_{m,q}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}\}$.

Lemma 4.10. *The scwol $\mathcal{X}$ is isomorphic to the union of scwols $\mathcal{Y} = \bigcup_{Y \in V(\mathcal{X})} \mathcal{X}_Y$, where $\mathcal{X}_Y$ is the scwol associated to the spherical Dyer group D_Y. The fundamental group of $\mathfrak{D}(\mathcal{X})$ is D. In particular, the complex of groups $\mathfrak{D}(\mathcal{X})$ is developable.*

Proof. First, we compare the sets of vertices. If $Y \in V(\mathcal{X})$, then $Y \in V(\mathcal{X}_Y)$, so $Y \in V(\mathcal{Y})$. On the other hand, if $Y \in V(\mathcal{Y})$, we have $Y \in V(\mathcal{X}_Z)$ for some $Z \in V(\mathcal{X})$, so $Y \subseteq Z$, and so, D_Y is spherical. This implies that $V(\mathcal{X}) = V(\mathcal{Y})$. Now, we compare the sets of edges. If $e = (X, Y, \omega) \in E(\mathcal{X})$, then $e \in E(\mathcal{X}_Y)$, so $e \in E(\mathcal{Y})$. On the

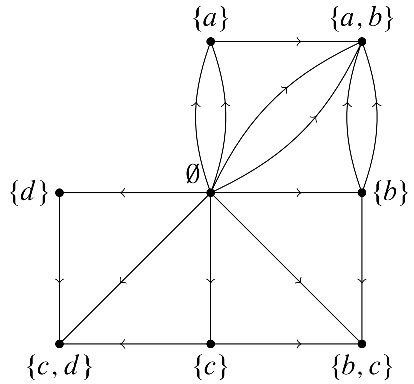


Figure 5. The scwol $\mathcal{X}_{m,q}$ associated to the graph $\Gamma_{m,q}$ given in Figure 4.

other hand, if $e \in E(\mathcal{Y})$, $e \in E(\mathcal{X}_Z)$ for some $Z \in V(\mathcal{X})$, so $e = (X, Y, \omega)$ with $X \subsetneq Y \subseteq Z$ and $\omega \subset (Y \setminus X)_\infty$, so $e \in E(\mathcal{X})$. This implies that $E(\mathcal{X}) = E(\mathcal{Y})$. We can now apply the Seifert–van Kampen theorem for the fundamental group of a complex of groups [2, Chapter III.C, Example 3.11 (5)] to $\mathcal{Y}$. The set $V(\mathcal{X})$ is finite, and each scwol $\mathcal{X}_Y$ is connected. We have $\emptyset \in V(\mathcal{X}_Y)$ for all $Y \in V(\mathcal{X})$ and $\emptyset$ is adjacent to any vertex in any $\mathcal{X}_Y$. So, $\bigcap_{Y \in V(\mathcal{X})} \mathcal{X}_Y$ is nonempty and connected. We can then use the presentations to see that the fundamental group of $\mathfrak{D}(\mathcal{X})$ is D. Finally, by [7], the maps $\phi_X : D_X^f \to D$ are all injective. So, $\mathfrak{D}(\mathcal{X})$ is developable. ∎

Since the complex $\mathfrak{D}(\mathcal{X})$ is developable, we can describe its development

$$\mathcal{C} = \mathcal{C}(\mathcal{X}, \phi).$$

Since $D_X^f < D$ and the maps ϕ_X are canonical inclusions, we will identify the image $\phi_X(D_X^f)$ with D_X^f. The set of vertices of $\mathcal{C}$ is

$$V(\mathcal{C}) = \{(gD_X^f, X) \mid X \in V(\mathcal{X}),\ gD_X^f \in D/D_X^f\}.$$

The set of edges of $\mathcal{C}(\mathcal{X}, \phi)$ is

$$E(\mathcal{C}) = \{(gD_X^f, (X, Y, \omega)) \mid (X, Y, \omega) \in E(\mathcal{X}),\ gD_X^f \in D/D_X^f\},$$

where $i(gD_X^f, (X, Y, \omega)) = (gD_X^f, X)$ and $t(gD_X^f, (X, Y, \omega)) = (g\phi(\omega)^{-1}D_Y^f, Y)$. For a simpler notation, we write gX for a vertex (gD_X^f, X) and $g(X, Y, \omega)$ for an edge $(gD_X^f, (X, Y, \omega))$. Note that $gX = hY$ if and only if $X = Y$ and $g^{-1}h \in D_X^f$. Similarly, $g(X, Y, \omega) = h(X', Y', \omega')$ if and only if $X' = X$, $Y' = Y$, $\omega' = \omega$, and $g^{-1}h \in D_X^f$. As in the spherical case, $\mathcal{C}$ does not have multiple edges between two vertices.

Lemma 4.11. *For every vertex $gY \in V(\mathcal{C})$, the scwols* $\mathrm{St}_{in}(\mathbf{e}Y, \mathcal{C}(\mathcal{X}_Y, \phi^{\Gamma_Y}))$ *and* $\mathrm{St}_{in}(gY, \mathcal{C}(\mathcal{X}, \phi))$ *are isomorphic.*

Proof. It suffices to show this for $g = \mathbf{e}$. Then, the statement is clear as it follows directly from the definitions. ■

4.3. Construction of a piecewise Euclidean cell complex for Dyer groups

The scwol $\mathcal{C}$, which is also a simplicial complex, described in the previous section is a combinatorial object. To build the cell complex Σ, we could try to endow the geometric realization of $\mathcal{C}$ with a CAT(0) metric. This would give a simplicial complex with a non-standard piecewise Euclidean metric. The problem is that Moussong's Lemma 3.33 does not apply directly to simplicial complexes with a piecewise Euclidean metric since dihedral angles should be at least $\pi/2$. The idea is to interpret $\mathcal{C}$ as some generalized face poset of Σ. Indeed, $\mathcal{C}$ does not give us the face structure of Σ but some form of subcomplex structure. Each vertex in $\mathcal{C}$ corresponds to a subcomplex of Σ, and edges give identifications between these subcomplexes. Nevertheless, we will be able to interpret $\mathcal{C}$ as a simplicial subdivision of Σ. We start with the description and study of the subcomplexes associated to vertices, then build Σ, and finally show that Σ is CAT(0) using Moussong's Lemma 3.33.

Let (Γ, f, m) be a Dyer graph, $D = D(\Gamma, f, m)$ the associated Dyer group, $\mathcal{X} = \mathcal{X}(\Gamma)$ the associated scwol, and $\mathfrak{D}(\mathcal{X})$ the associated complex of groups. Consider the injective morphism $\phi : \mathfrak{D}(\mathcal{X}) \to D$ given by the natural inclusion

$$\phi_X : D_X^f \to D \quad \text{and} \quad \phi(X, Y, \omega) = \phi(\omega) = \prod_{v \in \omega} x_v.$$

As in the previous section, we construct the development $\mathcal{C} = \mathcal{C}(\mathcal{X}, \phi)$.

Elementary building blocks. Let $Y \in V(\mathcal{X})$. First, we consider elementary building blocks in the cases $Y = Y_2$, $Y = Y_\infty$, and $Y = Y_p$. For $Y \in V(\mathcal{X})$ with $Y = Y_2$, let $\mathrm{Cox}(Y)$ be the Coxeter polytope associated to the Coxeter group D_Y endowed with its natural Euclidean metric as described in Section 3.2. Its set of vertices is D_Y. For $Y \in V(\mathcal{X})$ with $Y = Y_\infty$, consider $[0,1]^Y \subset \mathbb{R}^Y$ with its standard cubical structure. Its set of vertices is $\mathcal{P}(Y)$, where $0 \in \mathbb{R}^Y$ corresponds to $\emptyset \in \mathcal{P}(Y)$. For $v \in V_p$, let $\mathrm{Stern}(v)$ be the $f(v)$-branched star where each edge of the star is identified with $[0,1]$. Its center is denoted by c_v, and its leaves are identified with the elements of the finite cyclic group $C_{f(v)}$ of order $f(v)$. For $Y \in V(\mathcal{X})$ with $Y = Y_p$, let $\mathrm{Stern}(Y)$ be the product of stars $\prod_{v \in Y} \mathrm{Stern}(v)$ endowed with the ℓ_2 metric. So, its vertex set is $\prod_{v \in Y}(\{c_v\} \cup C_{f(v)})$. Note that

$$V(\mathrm{Stern}(Y)) = \prod_{v \in Y}(\{c_v\} \cup C_{f(v)})$$

can be identified with $\coprod_{Z \subset Y} D_Y/D_Z$. We identify a vertex $(g_v)_{v \in Y} \in \prod_{v \in Y}(\{c_v\} \cup C_{f(v)})$ with $gD_Y/D_Z \in \coprod_{Z \subset Y} D_Y/D_Z$, where

$$Z = \{v \in Y \mid g_v = c_v\} \quad \text{and} \quad g = \prod_{v \in Y \setminus Z} g_v.$$

Since Γ_Y is a complete graph and $Y = Y_p$, the element $g \in D_Y$ is well defined. Let us denote a vertex $gD_Z \in D_Y/D_Z$ in $\mathrm{Stern}(Y)$ with gZ.

The cell complex $\mathrm{Cc}(Y)$. To every $Y \in V(\mathcal{X})$ we associate a Euclidean cell complex $\mathrm{Cc}(Y)$ as follows. Let $\mathrm{Cc}(Y)$ be the product $\mathrm{Cox}(Y_2) \times [0,1]^{Y_\infty} \times \mathrm{Stern}(Y_p)$ endowed with the ℓ_2 metric. Each of its factors is piecewise Euclidean, so it is a piecewise Euclidean cell complex. In particular, $\mathrm{Cc}(Y) = \mathrm{Cc}(Y_2) \times \mathrm{Cc}(Y_\infty) \times \mathrm{Cc}(Y_p)$. The set of vertices of $\mathrm{Cc}(Y)$ is $D_{Y_2} \times \mathcal{P}(Y_\infty) \times \prod_{v \in Y_p}(\{c_v\} \cup C_{f(v)})$. The group D_Y^f acts by isometries on $\mathrm{Cc}(Y)$. Indeed, $D_Y^f = D_{Y_2} \times \prod_{v \in Y_p} C_{f(v)}$. So, D_Y^f acts through D_{Y_2} on $\mathrm{Cox}(Y_2)$ and through $C_{f(v)}$ on $\mathrm{Stern}(v)$. These actions are all isometries.

Lemma 4.12. *Consider a vertex $l = (w, \lambda, gZ) \in \mathrm{Cc}(Y)$. The link $\mathrm{Lk}(l, \mathrm{Cc}(Y))$ is isometric to the spherical join*

$$\begin{aligned}\mathrm{Lk}(w, \mathrm{Cox}(Y_2)) \star ((\star_{v \in \lambda} \lambda \setminus \{v\}) \star (\star_{v \in Y_\infty \setminus \lambda} \lambda \cup \{v\})) \\ \star (\star_{v \in Z} \mathrm{Lk}(c_v, \mathrm{Stern}(v))) \star (\star_{v \in Y_p \setminus Z} \mathrm{Lk}(e, \mathrm{Stern}(v))),\end{aligned}$$

where the length of an edge between two vertices u, v in two different terms if the decomposition is $d(u,v) = \pi/2$ and $\mathrm{Lk}(w, \mathrm{Cox}(Y_2))$ is identified with the piecewise spherical flag complex with 1-skeleton Γ_{Y_2} and edge length $d(u,v) = \pi - \pi/m(u,v)$ for two vertices $u, v \in Y_2$.

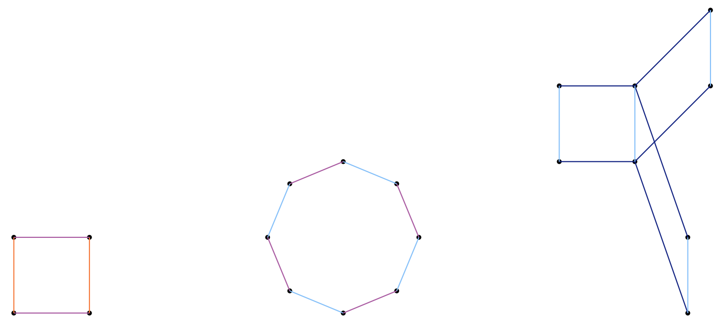
Proof. The link of $l = (w, \lambda, gZ)$ is the spherical join

$$\mathrm{Lk}(w, \mathrm{Cox}(Y_2)) \star \mathrm{Lk}(\lambda, [0,1]^{Y_\infty}) \star \mathrm{Lk}(gZ, \mathrm{Stern}(Y_p)).$$

By Section 3.2, the term $\mathrm{Lk}(w, \mathrm{Cox}(Y_2))$ is identified with the piecewise spherical flag complex with 1-skeleton Γ_{Y_2} and edge length $d(u,v) = \pi - \pi/m(u,v)$ for two vertices $u, v \in Y_2$. The link $\mathrm{Lk}(gZ, \mathrm{Stern}(Y_p))$ is isometric to $\mathrm{Lk}(Z, \mathrm{Stern}(Y_p))$ which is isometric to the spherical join

$$\star_{v \in Z} \mathrm{Lk}(c_v, \mathrm{Stern}(v)) \star_{v \in Y_p \setminus Z} \mathrm{Lk}(\mathbf{e}, \mathrm{Stern}(v)).$$

Each term $\mathrm{Lk}(\mathbf{e}, \mathrm{Stern}(v))$ consists of a single vertex and each term $\mathrm{Lk}(c_v, \mathrm{Stern}(v))$ consists of $|C_{f(v)}|$ disjoint vertices. For every $\lambda \subset Y_\infty$, the term $\mathrm{Lk}(\lambda, [0,1]^{Y_\infty})$ is isometric to the spherical join $(\star_{v \in \lambda} \lambda \setminus \{v\}) \star (\star_{v \in Y_\infty \setminus \lambda} \lambda \cup \{v\})$. So, the link



(a) $\mathrm{Cc}(\{a,b\})$ **(b)** $\mathrm{Cc}(\{b,c\})$ **(c)** $\mathrm{Cc}(\{c,d\})$

Figure 6. The cell complexes associated to some vertices of $\mathcal{X}_{m,q}$.

$\mathrm{Lk}(l,\mathrm{Cc}(Y))$ is isometric to the spherical join

$$\begin{aligned}\mathrm{Lk}(w,\mathrm{Cox}(Y_2)) \star ((\star_{v\in\lambda}\lambda \setminus \{v\}) \star (\star_{v\in Y_\infty\setminus\lambda}\lambda \cup \{v\}))\\ \star\,(\star_{v\in Z}\,\mathrm{Lk}(c_v,\mathrm{Stern}(v))) \star (\star_{v\in Y_p\setminus Z}\,\mathrm{Lk}(e,\mathrm{Stern}(v))).\end{aligned}$$

Note that in particular for two vertices $u,v \in V(\mathrm{Lk}(l,\mathrm{Cc}(Y)))$ in two different terms of the decomposition, we have

$$d(u,v) = \pi/2. \qquad \blacksquare$$

Example 4.13. Let $m,q \in \mathbb{N}_{\geq 2}$. We go back to the example of the Dyer graph $\Gamma_{m,q}$ with associated Dyer group $D_{m,q}$ and scwol $\mathcal{X}_{m,q}$ given in Figure 1, Example 2.6, and Figure 5. Figure 6 shows the cell complexes $\mathrm{Cc}(\{a,b\})$, $\mathrm{Cc}(\{b,c\})$, $\mathrm{Cc}(\{c,d\})$ in the cases $m=4$ and $q=3$.

The cell complex $\Sigma(gY)$. Let $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$. We now describe the subcomplexes of Σ associated to vertices of $\mathcal{C}$. We start by identifying the vertex set of $\mathrm{Cc}(Y)$ with a subset of $V(\mathrm{St}_{in}(Y,\mathcal{C}))$ and more generally with a subset of $V(\mathrm{St}_{in}(gY,\mathcal{C}))$. Let $V_p(gY)$ be the following subset of $V(\mathrm{St}_{in}(gY,\mathcal{C}))$:

$$V_p(gY) = \{kX \in V(\mathcal{C}) \mid X \subseteq V_p \text{ and } kX \in V(\mathrm{St}_{in}(gY,\mathcal{C}))\}.$$

By definition, $kX \in V(\mathrm{St}_{in}(gY,\mathcal{C}))$ if and only if $kX = gY$ or there exists a unique edge $h(X,Y,\omega)$ in $\mathcal{C}$ with initial vertex $kX = hX$ and terminal vertex

$$gY = k\phi(\omega)^{-1}Y.$$

So, $kX \in V(\mathrm{St}_{in}(gY,\mathcal{C}))$ if and only if $X \subseteq Y$, and there exists a unique $\omega \subseteq Y_\infty \setminus X_\infty$ with $k(\prod_{v\in\omega} x_v)^{-1}D_Y^f = gD_Y^f$. So,

$$V_p(gY) = \{kX \in V(\mathcal{C}) \mid X \subseteq V_p \cap Y \text{ and } k\Big(\prod_{v\in\omega} x_v\Big)^{-1} D_Y^f = gD_Y^f \text{ with } \omega \subseteq Y_\infty\}.$$

Lemma 4.14. *The map $j : V(\mathrm{Cc}(Y)) \to V_p(\mathbf{e}Y)$ given by $j(w,\lambda,hZ) = w\phi(\lambda)hZ$ is bijective. Moreover, it induces a bijective map $j_g : V(\mathrm{Cc}(Y)) \to V_p(gY)$ given by*

$$j_g(w,\lambda,hZ) = g \cdot j(w,\lambda,hZ) = gw\phi(\lambda)hZ.$$

Proof. Let $Z \in V(\mathcal{X})$ and $kD_Z^f \in D/D_Z^f$ so that $kZ \in V_p(\mathbf{e}Y)$. So, we have $Z \subseteq V_p \cap Y$ and $k(\prod_{v\in\lambda} x_v)^{-1} D_Y^f = D_Y^f$ for some $\lambda \subseteq Y_\infty$. As $D_Y^f = D_{Y_2} \times D_{Y_p}$, the representative $k(\prod_{v\in\lambda} x_v)^{-1}$ has a unique decomposition

$$k\Big(\prod_{v\in\lambda} x_v\Big)^{-1} = k_2 \prod_{v\in Y_p} k_v,$$

with $k_2 \in D_{Y_2}$ and $k_v \in C_{f(v)}$ for every $v \in Y_p$. This gives a unique decomposition

$$k = k_2\Big(\prod_{v\in\lambda} x_v\Big)\Big(\prod_{v\in Y_p} k_v\Big).$$

In particular, $k \in D_Y$. As Γ_Y is complete and the coset $kD_Z^f \in D/D_Z^f$, we can assume that $k_v = \mathbf{e}$ for every $v \in Z_p$. As D_{Z_2} is a parabolic subgroup of the Coxeter group D_{Y_2}, we can also assume k_2 to be the unique element of minimal length in $k_2 D_{Z_2}$. So, $j(k_2, \lambda, \prod_{v\in Y_p\setminus Z} k_v Z) = kZ$. Hence, the map j is surjective. Such a choice of k_2 and k_v, $v \in Y_p \setminus Z$ is independent of the representative k. Indeed, let k' be another representative, so $k'D_Z^f = kD_Z^f$. Then, again, $k' = k_2'(\prod_{v\in\lambda} x_v)(\prod_{v\in Y_p} k_v')$. As $k^{-1}k' \in D_Z^f$, we have $k_2^{-1}k_2' \in D_{Z_2}$, so $k_2' D_{Z_2} = k_2 D_{Z_2}$, and so, by uniqueness of the minimal representative, $k_2 = k_2'$. Similarly, $k_v = k_v'$ for every $v \in Y_p$. As there is a unique edge from kZ to $\mathbf{e}Y$, the subset $\lambda \subseteq Y_\infty$ is uniquely determined. Hence, the map j is also injective, so it is bijective.

Finally, $kZ \in V_p(gY)$ if and only if $Z \subseteq V_p \cap Y$ and $k(\prod_{v\in\lambda} x_v)^{-1} D_Y^f = gD_Y^f$ for some $\lambda \subseteq Y_\infty$. So, $kZ \in V_p(gY)$ if and only if $g^{-1}k(\prod_{v\in\lambda} x_v)^{-1} D_Y^f = D_Y^f$ for some $\lambda \subseteq Y_\infty$ and $Z \subseteq V_p \cap Y$. So, $kZ \in V_p(gY)$ if and only if $g^{-1}kZ \in V_p(\mathbf{e}Y)$. So, the map j_g is bijective. ∎

For $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$, let $\Sigma(gY)$ be the piecewise Euclidean cell complex given as follows:

(1) The set of vertices (or 0-cells) is $V_p(gY)$.

(2) Every cell in $\Sigma(gY)$ is isometric to a cell in $\mathrm{Cc}(Y)$.

(3) The map $j_g : V(\mathrm{Cc}(Y)) \to V_p(gY)$ extends to an isometry

$$j_g : \mathrm{Cc}(Y) \to \Sigma(gY).$$

Let $hD_Y^f \in D/D_Y^f$ with $hD_Y^f = gD_Y^f$, then $j_g \circ j_h^{-1}$ is an isometry from $\Sigma(hY)$ to $\Sigma(gY)$. So, the cell structure on $\Sigma(gY)$ is well defined.

We now discuss identifications of subcomplexes. Let $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$. Let $Z \in V(\mathcal{X})$ and $hD_Z^f \in D/D_Z^f$ so that

$$hZ \in V(\mathrm{St}_{in}(gY, \mathcal{C})).$$

Then, $\mathrm{St}_{in}(hZ, \mathcal{C})$ is a subscwol of $\mathrm{St}_{in}(gY, \mathcal{C})$, and hence, $V_p(hZ) \subset V_p(gY)$. The following lemma shows that this inclusion induces an isometric embedding of the cell complex $\Sigma(hZ)$ into $\Sigma(gY)$.

Lemma 4.15. *Let $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$. Let $Z \in V(\mathcal{X})$ and $hD_Z^f \in D/D_Z^f$ so that $hZ \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$. The cellular map*

$$\iota : \Sigma(hZ) \to \Sigma(gY)$$

satisfying $\iota(v) = v$ for every vertex $v \in V(\Sigma(hZ))$ is an isometric embedding. In particular, we can identify $\Sigma(hZ)$ with $\iota(\Sigma(hZ))$.

Proof. Since $hZ \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$ if and only if $g^{-1}hZ \in \mathrm{St}_{in}(Y, \mathcal{C})$, it suffices to consider the case $g = \mathbf{e}$. For $hZ \in V(\mathrm{St}_{in}(\mathbf{e}Y, \mathcal{C}))$, we can write $h = h_2 h_\infty h_p$ with $h_2 \in D_{Y_2}$, $h_\infty = \phi(\kappa)$ for a unique $\kappa \subset (Y \setminus Z)_\infty$ and $h_p \in D_{Y_p \setminus Z_p}$. We claim that the cellular map $\iota_h : \mathrm{Cc}(Z) \to \mathrm{Cc}(Y)$ given by $\iota(w, \lambda, mM) = (h_2 w, \lambda \cup \kappa, h_p mM)$ for $(w, \lambda, mM) \in V(\mathrm{Cc}(Z))$ (so $w \in D_{Z_2}$, $\lambda \subseteq Z_\infty$, $M \subseteq Z_p$, and $m \in D_{Z_p \setminus M}$) is an isometric embedding. Both $\mathrm{Cc}(Z) = \mathrm{Cox}(Z_2) \times [0,1]^{Z_\infty} \times \mathrm{Stern}(Z_p)$ and $\mathrm{Cc}(Y) = \mathrm{Cox}(Y_2) \times [0,1]^{Y_\infty} \times \mathrm{Stern}(Y_p)$ are endowed with the ℓ_2 metric. By Section 3.2, the cellular map

$$\iota_2 : \mathrm{Cox}(Z_2) \to \mathrm{Cox}(Y_2)$$

with $\iota(w) = h_2 w$ for $w \in V(\mathrm{Cox}(Z_2))$ is an isometric embedding identifying $\mathrm{Cox}(D_{Z_2})$ with $h_2 \cdot \mathrm{Cox}_{Z_2}(D_{Y_2})$. The cellular map

$$\iota_\infty : [0,1]^{Z_\infty} \to [0,1]^{Y_\infty}$$

with $\iota_\infty(\lambda) = \lambda \cup \kappa$ for $\lambda \in \mathcal{P}(Z_\infty)$ is also an isometric embedding identifying $\prod_{v \in Z_\infty} [0,1]$ with $\prod_{v \in Z_\infty} [0,1] \times \prod_{v \in \kappa} \{1\} \times \prod_{v \in Y_\infty \setminus (\kappa \cup Z_\infty)} \{0\}$ in $\prod_{v \in Y_\infty} [0,1]$. The cellular map

$$\iota_p : \mathrm{Stern}(Z_p) \to \mathrm{Stern}(Y_p)$$

with $\iota_p(mM) = h_p mM$ for $mM \in V(\mathrm{Stern}(Z_p))$ is an isometric embedding identifying $\mathrm{Stern}(Z_p)$ with $\mathrm{Stern}(Z_p) \times \{h_p\} \subset \mathrm{Stern}(Y_p)$. So, the map ι_h decomposes as the product of maps

$$\iota_h = (\iota_2, \iota_\infty, \iota_p) : \mathrm{Cc}(Z) \to \mathrm{Cc}(Y);$$

hence, it is a cellular isometric embedding. Then, the map

$$\iota : \Sigma(hZ) \to \Sigma(\mathbf{e}Y)$$

given by $j_{\mathbf{e}} \circ \iota_h \circ j_h^{-1}$ is a cellular isometric embedding. ∎

Simplicial subdivision. Let $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$. We describe a simplicial subdivision of $\mathrm{Cc}(Y)$ and $\Sigma(gY)$. Let $\mathrm{Bar}(\mathrm{Cox}(Y_2))$ be the barycentric subdivision of $\mathrm{Cox}(Y_2)$. Let $\mathrm{Bar}([0,1])$ be the barycentric subdivision of $[0,1]$. Then, $\mathrm{Bar}(\mathrm{Cox}(Y_2))$, $\mathrm{Bar}[0,1]$, and $\mathrm{Stern}(v)$ for $v \in Y_p$ are piecewise Euclidean simplicial complexes. Moreover, the simplicial complex $\mathrm{Bar}(\mathrm{Cox}(Y_2))$ is isomorphic to $\mathrm{St}_{in}(Y_2, \mathcal{C})$ by Lemma 3.28. For $v \in Y_\infty$, the simplicial complex $\mathrm{Bar}([0,1])$ is isomorphic to $\mathrm{St}_{in}(\{v\}, \mathcal{C})$. For $v \in Y_p$, the simplicial complex $\mathrm{Stern}(v)$ is isomorphic to $\mathrm{St}_{in}(\{v\}, \mathcal{C})$. These isomorphisms induce a scwol structure on the barycentric subdivisions and on $\mathrm{Stern}(v)$. So, the scwol

$$\mathrm{Bar}(\mathrm{Cc}(Y)) = \mathrm{Bar}(\mathrm{Cc}(Y_2)) \times \prod_{v \in Y_\infty} \mathrm{Bar}(\mathrm{Cc}(\{v\})) \times \prod_{v \in Y_p} \mathrm{Cc}(\{v\})$$

is well defined and isomorphic to

$$\mathrm{St}_{in}(Y, \mathcal{C}) = \mathrm{St}_{in}(Y_2, \mathcal{C}) \times \prod_{v \in Y_\infty} \mathrm{St}_{in}(\{v\}, \mathcal{C}) \times \prod_{v \in Y_p} \mathrm{St}_{in}(\{v\}, \mathcal{C}).$$

We endow $\mathrm{Bar}(\mathrm{Cc}(Y))$ with the ℓ_2 metric, so it is a piecewise Euclidean simplicial complex isometric to $\mathrm{Cc}(Y)$. We call $\mathrm{Bar}(\mathrm{Cc}(Y))$ the *nice simplicial subdivision* of $\mathrm{Cc}(Y)$. We call the simplicial subdivision of $\Sigma(gY)$ induced by the isometry j_g the *nice simplicial subdivision* of $\Sigma(gY)$.

The next lemma discusses how $\mathrm{St}_{in}(gY, \mathcal{C})$ can be interpreted as a simplicial subdivision of $\Sigma(gY)$.

Lemma 4.16. *Let $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$. The nice simplicial subdivision of $\Sigma(gY)$ is simplicially isomorphic to the scwol $\mathrm{St}_{in}(gY, \mathcal{C})$. For $Z \in V(\mathcal{X})$ and $kD_Z^f \in D/D_Z^f$ so that $kZ \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$, the isometric embedding $\iota : \Sigma(hZ) \to \Sigma(gY)$ given in Lemma 4.15 preserves the nice simplicial subdivision.*

Proof. The nice simplicial subdivision is isomorphic to $\mathrm{Bar}(\mathrm{Cc}(Y))$ which is isomorphic to $\mathrm{St}_{in}(Y, \mathcal{C})$ which is isomorphic to $\mathrm{St}_{in}(gY, \mathcal{C})$. The second statement follows from the product decomposition of the nice simplicial subdivision, the map ι and the complexes $\Sigma(hZ)$ and $\Sigma(gY)$. ∎

Example 4.17. Let $m, q \in \mathbb{N}_{\geq 2}$. We go back to the example of the Dyer graph $\Gamma_{m,q}$ with associated Dyer group $D_{m,q}$ and scwol $\mathcal{X}_{m,q}$ given in Figure 1, Example 2.6, and Figure 5. Figure 7 shows the subcomplexes $\Sigma(\mathbf{e}\{a,b\})$, $\Sigma(\mathbf{e}\{b,c\})$, $\Sigma(\mathbf{e}\{c,d\})$ and their simplicial subdivision in the case $m = 4$ and $q = 3$.

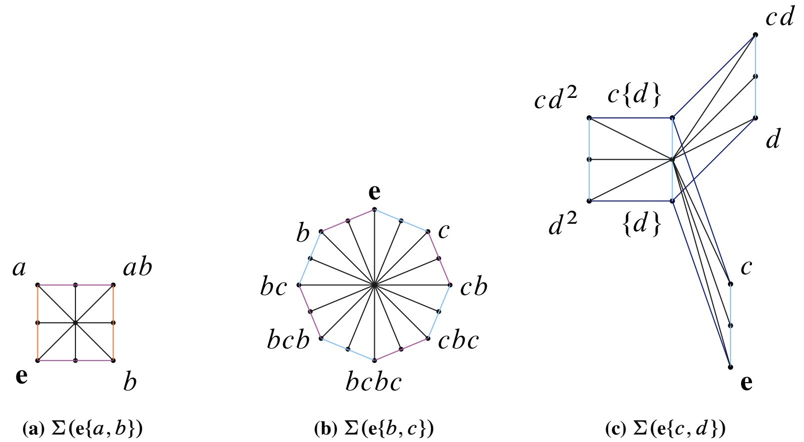


Figure 7. The subcomplexes associated to some vertices of the development of $\mathcal{X}_{m,p}$ and their simplicial subdivision.

The cell complex $\boldsymbol{\Sigma}$. We now have the tools needed to build the cell complex Σ. Consider

$$\Sigma = \bigcup_{gY \in V(\mathcal{C})} \Sigma(gY),$$

where we identify $\Sigma(hZ)$ with $\iota(\Sigma(hZ)) \subset \Sigma(gY)$ whenever $hZ \in \mathrm{St}_{in}(gY, \mathcal{C})$. So, by Lemma 4.15, Σ has a well-defined piecewise Euclidean metric. We endow Σ with the associated length metric. The set of vertices of Σ is

$$V_p(\mathcal{C}) = \{gY \in V(\mathcal{C}) \mid Y \in V(\mathcal{X}),\ Y \subset V_p,\ gD_Y^f \in D/D_Y^f\}.$$

The action of D on $V_p(\mathcal{C})$ induces an action by isometries of D on Σ; in particular, for $d \in D$, we have $d \cdot \Sigma(gY) = \Sigma(dgY)$. By Lemma 4.16, the nice simplicial subdivision of each $\Sigma(gY)$ induces a simplicial subdivision of Σ, which we call the *nice simplicial subdivision* of Σ.

Lemma 4.18. *The scwol $\mathcal{C}$ is isomorphic to the nice simplicial subdivision of Σ. In particular, this implies that Σ is a simply connected metric space.*

Proof. Since $\mathcal{C} = \bigcup_{gY \in V(\mathcal{C})} \mathrm{St}_{in}(gY, \mathcal{C})$ and by Lemma 4.16 every $\mathrm{St}_{in}(gY, \mathcal{C})$ is isomorphic to the nice simplicial subdivision of $\Sigma(gY)$ preserved by ι, the complex $\mathcal{C}$ is isomorphic to the nice simplicial subdivision of Σ. This induces a metric on $\mathcal{C}$ with respect to which the geometric realization $|\mathcal{C}|$ is isometric to Σ. By [2, Theorem III.C.3.14], the scwol $\mathcal{C}$ is simply connected. So, Σ is a well-defined simply connected metric space. ∎

We are finally in a position to show that Σ is CAT(0). Since Σ is simply connected, we only need to understand its local structure, so we are back to studying links of vertices. In order to have a precise description of the links of vertices, we introduce an edge labeling of Σ by $V(\Gamma)$.

Edge labeling. Let $Y \in V(\mathcal{X})$ and $hD_Y^f \in D/D_Y^f$ so that $hY \in V(\mathcal{C})$. We start by labeling the edges of $\Sigma(hY)$ by elements of Y. To define this edge labeling, we study when two vertices of $\Sigma(hY)$ are adjacent and then give the corresponding label. Let $X, Z \in V(\mathcal{X})$ and $kD_X^f \in D/D_X^f$, $lD_Z^f \in D/D_Z^f$ so that $kX, lZ \in V_p(hY)$; i.e., they are vertices of $\Sigma(hY)$. Then, kX and lZ are adjacent in $\Sigma(hY)$ if and only if their preimages $j_h^{-1}(kX), j_h^{-1}(lZ) \in V(\mathrm{Cc}(Y))$ are adjacent in $\mathrm{Cc}(Y)$. Let $j_h^{-1}(kX) = (k_2, \lambda_k, k_p X)$, $j_h^{-1}(lZ) = (l_2, \lambda_l, l_p Z) \in V(\mathrm{Cc}(Y))$; hence, $hk_2\phi(\lambda_k)k_p X = kX$ and $hl_2\phi(\lambda_l)l_p Z = lZ$ in $V_p(hY)$. Remember that

$$\mathbb{C}\mathrm{c}(Y) = \mathrm{Cc}(Y_2) \times \mathrm{Cc}(Y_p) \times \mathrm{Cc}(Y_\infty).$$

Then, the vertices $j_h^{-1}(kX), j_h^{-1}(lZ) \in V(\mathrm{Cc}(Y))$ are adjacent in $\mathrm{Cc}(Y)$ if and only if one of the following holds:

(1) k_2, l_2 are adjacent in $V(\mathrm{Cc}(Y_2))$ and $\lambda_k = \lambda_l$ and $k_p X = l_p Z$. Equivalently, $k_2^{-1} l_2 = x_v$ for some $v \in Y_2$ and $\lambda_k = \lambda_l$ and $X = Z$ and $k_p l_p \in D_X^f$.

(2) $k_2 = l_2$ and λ_k, λ_l are adjacent in $V(\mathrm{Cc}(Y_\infty))$ and $k_p X = l_p Z$. This is equivalent to one of the following:
   (a) $k_2 = l_2$ and $\lambda_k \subset \lambda_l$ and $\lambda_l \setminus \lambda_k = \{v\} \subset Y_\infty$ and $X = Z$ and $k_p^{-1} l_p \in D_Z^f$.
   (b) $k_2 = l_2$ and $\lambda_l \subset \lambda_k$ and $\lambda_k \setminus \lambda_l = \{v\} \subset Y_\infty$ and $X = Z$ and $k_p^{-1} l_p \in D_Z^f$.

(3) $k_2 = l_2$ and $\lambda_k = \lambda_l$ and $k_p X, l_p Z$ are adjacent in $V(\mathrm{Cc}(Y_p))$. This is equivalent to one of the following:
   (a) $k_2 = l_2$ and $\lambda_k = \lambda_l$ and $X \subset Z$ and $Z \setminus X = \{x_v\}$ for some $v \in Y_p$ and $k_p^{-1} l_p \in D_Z^f$.
   (b) $k_2 = l_2$ and $\lambda_k = \lambda_l$ and $Z \subset X$ and $X \setminus Z = \{x_v\}$ for some $v \in Y_p$ and $k_p^{-1} l_p \in D_X^f$.

Using the fact that $Y \in V(\mathcal{X})$ so that D_Y is a spherical Dyer group, this leads to the following characterization and labeling of edges by $Y \subset V(\Gamma)$. The vertices $kX, lZ \in V_p(hY)$ are adjacent in $\Sigma(hY)$ if and only if one of the following holds:

(1) $X = Z$ and $k^{-1} l \in x_v D_X^f$ for some $v \in Y_2$. In this case, we label the edge by $v \in Y_2 \subset V(\Gamma)$.

(2) $X = Z$ and $k^{-1} l = x_v^{\pm 1} D_X^f$ for some $v \in Y_\infty$. In this case, we label the edge by $v \in Y_\infty \subset V(\Gamma)$.

(3) (a) $X \subset Z$ and $Z \setminus X = \{x_v\}$ for some $v \in Y_p$ and $k^{-1}l = k_p^{-1}l_p \in D_Z^f$. In this case, we label the edge by $v \in Y_p \subset V(\Gamma)$.

(b) $Z \subset X$ and $X \setminus Z = \{x_v\}$ for some $v \in Y_p$ and $k^{-1}l = k_p^{-1}l_p \in D_X^f$. In this case, we label the edge by $v \in Y_p \subset V(\Gamma)$.

Note that, for $h'Y' \in \mathrm{St}_{in}(gY, \mathcal{C})$, the labeling of an edge in $\Sigma(h'Y')$ is invariant under the inclusion $\iota : \Sigma(h'Y') \to \Sigma(gY)$. Moreover, the labeling of edges in $\Sigma(\mathbf{e}Y)$ is invariant under the action of D_Y^f. So, this defines a labeling by $V(\Gamma)$ of the edges of Σ. Note that this edge labeling is invariant under the action of D.

4.19. Links of vertices As our goal is to apply Moussong's lemma to Σ, we need to understand links of vertices in Σ. We start with links of vertices in $\Sigma(gY)$. This is crucial to prove later on that Σ is CAT(0). Let $Y \in V(\mathcal{X})$ and $hD_Y^f \in D/D_Y^f$ so that $hY \in V(\mathcal{C})$. Let $X \in V(\mathcal{X})$ and $kD_X^f \in D/D_X^f$ so that $kX \in V_p(hY)$. The edge labeling on Σ and $\Sigma(gY)$ induces a vertex labeling $l : V(\mathrm{Lk}(kX, \Sigma)) \to V$, which restricts to $l : V(\mathrm{Lk}(kX, \Sigma(hY))) \to Y$. Using the map j_h in Lemma 4.14, the link $\mathrm{Lk}(kX, \Sigma(hY))$ is isometric to the link $\mathrm{Lk}(j_h^{-1}(kX), \mathrm{Cc}(Y))$. With Lemma 4.12, this implies that $\mathrm{Lk}(kX, \Sigma(hY))$ can be identified with the spherical flag complex $\Gamma_{Y_2} \star \Gamma_{Y_\infty} \star \Gamma_{Y_p \setminus X} \star (\star_{v \in X}\{v^i \mid 1 \le i \le f(v)\})$. The vertex labeling is given by $l(v) = v$ for every $v \in Y_2 \cup Y_\infty \cup Y_p \setminus X$ and $l(v^i) = v$ for every $v^i \in \{v^i \mid v \in X, 1 \le i \le f(v)\}$. By Lemma 4.12, the edge length in $\mathrm{Lk}(kX, \Sigma(hY))$ is given by

$$d(v, w) = \pi - \pi/m(l(v), l(w)).$$

As $Y \in V(\mathcal{X})$, the matrix $(\cos(d(v, w)))_{v,w \in Y}$ is positive definite by Lemma 4.1. So, $\mathrm{Lk}(kX, \Sigma(hY))$ is a metric flag complex. Additionally, we have that v, w are adjacent vertices in $\mathrm{Lk}(kX, \Sigma(hY))$ if and only if $l(v) \neq l(w)$. As this holds for every $gY \in V(\mathcal{C})$, it implies that if v, w are adjacent vertices in $\mathrm{Lk}(kX, \Sigma)$, we have $l(v) \neq l(w)$. So, for pairwise adjacent vertices $v_1, \dots, v_n \in \mathrm{Lk}(kX, \Sigma)$, we have $l(v_i) \neq l(v_j)$ for every $i \neq j$. To simplify the notation, we will write $\widehat{v_i} = l(v_i) \in V$ when considering pairwise adjacent vertices $v_1, \dots, v_n \in \mathrm{Lk}(kX, \Sigma)$.

Lemma 4.20. *Let $Y \in V(\mathcal{X})$ with $Y \subseteq V_p$ so that $Y \in V_p(\mathcal{C})$. Let the vertices*

$$v_1, \dots, v_k \in V(\mathrm{Lk}(Y, \Sigma))$$

be pairwise adjacent. There exist $Z \in V(\mathcal{X})$ and $g \in D$ such that $Y \in V(\Sigma(gZ))$ and $v_1, \dots, v_k \in V(\mathrm{Lk}(Y, \Sigma(gZ)))$ if and only if $Y \cup \{\widehat{v_1}, \dots, \widehat{v_k}\} \in V(\mathcal{X})$.

Proof. As $v_1, \dots, v_k \in V(\mathrm{Lk}(Y, \Sigma))$ are pairwise adjacent, we have $\widehat{v_i} \neq \widehat{v_j}$. Assume that there exist $Z \in V(\mathcal{X})$ and $g \in D$ such that $Y \in V(\Sigma(gZ))$ and the vertices $v_1, \dots, v_k \in V(\mathrm{Lk}(Y, \Sigma(gZ)))$. Then, $Y \in V_p(gZ)$, so $Y \subset Z$ and $\{\widehat{v_1}, \dots, \widehat{v_k}\} \subset Z$. Hence, $Y \cup \{\widehat{v_1}, \dots, \widehat{v_k}\} \subset Z$ which implies that $Y \cup \{\widehat{v_1}, \dots, \widehat{v_k}\} \in V(\mathcal{X})$.

Now, assume that $Y \cup \{\widehat{v_1}, \dots, \widehat{v_k}\} \in V(\mathcal{X})$. Each vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge in Σ between $\mathbf{e}Y$ and some vertex $h_v Z_v \in \Sigma$. Let us define an element $g_v \in D$.

(i) If $\hat{v} \in V_2$, the vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge between Y and $x_{\hat{v}} Y$. In this case, let $g_v = \mathbf{e}$.

(ii) If $\hat{v} \in V_\infty$, the vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge between Y and $\phi(\hat{v})Y$ or between Y and $\phi(\hat{v})^{-1}Y$. In the first case, let $g_v = \mathbf{e}$. In the second case, let $g_v = \phi(\hat{v})^{-1} = x_{\hat{v}}^{-1}$. Note that only one of these cases can occur as $v_1, \dots, v_k$ are pairwise adjacent.

(iii) For $\hat{v} \in V_p \setminus Y$, the vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge between Y and $Y \cup \{\hat{v}\}$. In this case, we fix $g_v = \mathbf{e}$.

(iv) For $\hat{v} \in Y$, the vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge between Y and $x_{\hat{v}}^t(Y \setminus \{\hat{v}\})$ for some $1 \le t \le f(\hat{v})$. In this case, fix $g_v = \mathbf{e}$.

We claim that for $Z = Y \cup \{\widehat{v_1}, \dots, \widehat{v_k}\}$ and $g = \prod_{i=1}^k g_{v_i}$ we have $Y \in V(\Sigma(gZ))$ and $v_1, \dots, v_k \in V(\mathrm{Lk}(Y, \Sigma(gZ)))$. Since $Y \cup \{\widehat{v_1}, \dots, \widehat{v_k}\} \in V(\mathcal{X})$, we have $g_v g_w = g_w g_v$ for all $v, w \in \{v_1, \dots, v_k\}$. In fact, $g = \phi(\omega)^{-1}$ for $\omega = \{\hat{v} \in Z \mid g_v = x_{\hat{v}}^{-1}\} \subset (Z \setminus Y)_\infty$. Hence, $Y \in V_p(gZ)$. Let $v \in \{v_1, \dots, v_k\}$. Now, we need to show that the element $h_v Z_v \in V_p(gZ)$. We use the case-by-case analysis above to fix the following notation:

(i) If $\hat{v} \in V_2$, we have $h_v Z_v = x_{\hat{v}} Y$, and we set $\lambda_v = \omega \subset (Z \setminus Z_v)_\infty$.

(ii) If $\hat{v} \in V_\infty$ and $h_v Z_v = \phi(\hat{v})Y$, let $\lambda_v = \omega \cup \{v\} \subset (Z \setminus Z_v)_\infty$. If $\hat{v} \in V_\infty$ and $h_v Z_v = \phi(\hat{v})^{-1}Y$, let $\lambda_v = \omega \setminus \{\hat{v}\} \subset (Z \setminus Z_v)_\infty$.

(iii) If $\hat{v} \in V_p \setminus Y$, we have $h_v Z_v = Y \cup \{\hat{v}\}$, and we set $\lambda_v = \omega \subset (Z \setminus Z_v)_\infty$.

(iv) If $\hat{v} \in Y$, we have $h_v Z_v = x_{\hat{v}}^t(Y \setminus \{\hat{v}\})$ for some $1 \le t \le f(\hat{v})$, and we set $\lambda_v = \omega \subset (Z \setminus Z_v)_\infty$.

As $Z = Y \cup \{\widehat{v_1}, \dots, \widehat{v_k}\} \in V(\mathcal{X})$, we have $gZ = h_v \phi(\lambda_v)^{-1} Z$ and $Z_v \subset Z$, so $h_v Z_v \in V(\mathrm{St}_{in}(gZ, \mathcal{C}))$. As additionally, $Z_v \subset Z \cap V_p$, we have $h_v Z_v \in V_p(gZ)$. ■

We now have the necessary tools to show the following statement.

Theorem 4.21. *The cell complex Σ is* CAT(0).

Proof. By [2, Theorem II.5.4], Σ is CAT(0) if and only if it is simply connected and the link of every vertex is CAT(1). By Lemma 4.18, the cell complex Σ is simply connected. Let us now prove that the link of every vertex is CAT(1) by using Moussong's Lemma 3.33. Let $Y \in V(\mathcal{X})$ with $Y \subset V_p$, and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\Sigma)$. Assume that $gD_Y^f = D_Y^f$, so

$$gY = \mathbf{e}Y = Y \in V(\Sigma).$$

Claim 1. Every edge in the link $\mathrm{Lk}(Y,\Sigma)$ of Y in Σ has length $\geq \pi/2$.

Proof. Since the vertex $Y \in V(\Sigma)$ is contained in $\Sigma(gZ)$ if and only if the vertex $gZ \in \mathrm{St}_{\mathrm{out}}(Y,\mathcal{C})$, we can describe $\mathrm{Lk}(Y,\Sigma)$ as the union $\bigcup_{gZ\in\mathrm{St}_{\mathrm{out}}(Y,\mathcal{C})}\mathrm{Lk}(Y,\Sigma(gZ))$, where $\mathrm{Lk}(Y,\Sigma(gZ))$ is the link of Y in the subcomplex $\Sigma(gZ)$. By 4.19, for two adjacent vertices $u,v \in V(\mathrm{Lk}(Y,\Sigma(gZ)))$, the length of the corresponding edge is $d(u,v) = \pi - \pi/m(\hat{u},\hat{v}) \geq \pi/2$ as $m(\hat{u},\hat{v}) \geq 2$. So, each edge in $\mathrm{Lk}(Y,\Sigma)$ has length $\geq \pi/2$. ■

Claim 2. The link $\mathrm{Lk}(Y,\Sigma)$ of the vertex Y in the cell complex Σ is metrically flag.

Proof. Consider a set of pairwise adjacent vertices $v_1,\dots,v_k \in \mathrm{Lk}(Y,\Sigma)$. As $\mathrm{Lk}(Y,\Sigma)$ is a piecewise spherical simplicial complex, the vertices $v_1,\dots,v_k$ are pairwise distinct. As mentioned in 4.19, then $\widehat{v_1},\dots,\widehat{v_k}$ are pairwise distinct. So, the map

$$\{v_1,\dots,v_k\} \to \{\widehat{v_1},\dots,\widehat{v_k}\}, \quad (v \mapsto \hat{v})$$

is a bijection. In particular, $Y \cup \{\widehat{v_1},\dots,\widehat{v_k}\}$ spans a complete subgraph of Γ. So, $v_1,\dots,v_k$ span a simplex in $\mathrm{Lk}(Y,\Sigma)$ if and only if $v_1,\dots,v_k$ span a simplex in $\mathrm{Lk}(Y,\Sigma(gZ))$ for some $gZ \in V(\mathcal{C})$. By 4.19, the link $\mathrm{Lk}(Y,\Sigma(gZ))$ is a piecewise spherical flag complex. So, the vertices $v_1,\dots,v_k \in V(\mathrm{Lk}(Y,\Sigma))$ span a simplex in $\mathrm{Lk}(Y,\Sigma)$ if and only if there exists some $gZ \in V(\mathcal{C})$ with $Y \in V(\Sigma(gZ))$ and the vertices $v_1,\dots,v_k \in V(\mathrm{Lk}(Y,\Sigma(gZ)))$. By Lemma 4.20, this is the case if and only if

$$Y' = Y \cup \{\widehat{v_1},\dots,\widehat{v_k}\} \in V(\mathcal{X}).$$

By Lemma 4.1, $Y' \in V(\mathcal{X})$ if and only if the matrix $(\cos(\pi - \pi/m(u,v)))_{u,v\in Y'}$ is positive definite. As $\pi - \pi/m(u,v) = \pi/2$ for all $u \in Y' \setminus V_2$, $v \in Y' \setminus \{u\}$ and $\pi - \pi/m(u,u) = 0$ for all $u \in Y'$, the matrix $(\cos(\pi - \pi/m(u,v)))_{u,v\in Y'}$ is positive definite if and only if its restriction $(\cos(\pi - \pi/m(u,v)))_{u,v\in Y'\cap V_2}$ is positive definite. As $Y' \cap V_2 = \{\widehat{v_1},\dots,\widehat{v_k}\} \cap V_2$ and $\pi - \pi/m(\hat{u},\hat{v}) = d(u,v)$ for all $\hat{u},\hat{v} \in Y' \cap V_2$, the matrix $(\cos(\pi - \pi/m(\hat{u},\hat{v})))_{\hat{u},\hat{v}\in Y'\cap V_2}$ is positive definite if and only if the matrix $(\cos(d(u,v)))_{\hat{u},\hat{v}\in Y'\cap V_2}$ is positive definite. Finally, $d(u,u) = 0$ for all $\hat{u} \in \{\widehat{v_1},\dots,\widehat{v_k}\}$, and $d(u,v) = \pi/2$ for all $\hat{u} \in \{\widehat{v_1},\dots,\widehat{v_k}\} \setminus V_2$ and $\hat{v} \in \{\widehat{v_1},\dots,\widehat{v_k}\} \setminus \{\hat{u}\}$, so the matrix $(\cos(d(u,v)))_{\hat{u},\hat{v}\in\{\widehat{v_1},\dots,\widehat{v_k}\}\cap V_2}$ is positive definite if and only if

$$(\cos(d(u,v)))_{\hat{u},\hat{v}\in\{\widehat{v_1},\dots,\widehat{v_k}\}}$$

is positive definite. So, we conclude that $v_1,\dots,v_k \in V(\mathrm{Lk}(Y,\Sigma))$ span a simplex if and only if the matrix $(\cos(d(u,v)))_{u,v\in\{v_1,\dots,v_k\}}$ is positive definite. So, $\mathrm{Lk}(Y,\Sigma)$ is metrically flag. ■

■

Remark 4.22. If D is a spherical Dyer group, the scwol $\mathcal{C}$ decomposes as a product $\mathcal{C}_2 \times \mathcal{C}_p \times \mathcal{C}_\infty$. For $i \in \{2, p, \infty\}$, let Σ_i be the cell complex associated to D_i. So, $\Sigma_2 = \mathrm{Cox}(V_2)$, $\Sigma_\infty = \mathbb{R}^{V_\infty}$, and $\Sigma_p = \mathrm{Stern}(V_p)$. Then, $\Sigma = \Sigma_2 \times \Sigma_\infty \times \Sigma_p$, where each factor is known to be CAT(0). So, Σ is CAT(0).

Corollary 4.23. *The Dyer group D is* CAT(0).

Proof. D acts properly discontinuously and cocompactly by isometries on Σ. ■

Remark 4.24. If the Dyer group D is a Coxeter group, Σ is the Davis–Moussong complex described in Theorem 3.30. If the Dyer group D is a right-angled Artin group, Σ is the Salvetti complex described in Section 3.3. The dimension of Σ is $\dim(\Sigma) = \max\{|Y| \mid Y \in V(\mathcal{X})\}$. Consider the Coxeter group W from Theorem 2.8 and its associated Davis–Moussong complex $\Sigma(W)$. The dimension of $\Sigma(W)$ is

$$\dim(\Sigma(W)) = \max\{|S| \mid S \subset V(\Lambda), W_S \text{ is finite}\}.$$

Looking at the construction of the graph Λ, we can see that

$$\dim(\Sigma(W)) = \max\{|Y| + |V_p| + |V_\infty \setminus Y| \mid Y \in V(\mathcal{X})\}.$$

So, we have that $\dim(\Sigma) \leq \dim(\Sigma(W))$.

Remark 4.25. There are many complexes beside Davis–Moussong complexes associated to Coxeter groups. Similarly, there are other complexes, such as Deligne complexes, associated to right-angled Artin groups. In this article, we focused on the Davis–Moussong and the Salvetti complex as the associated actions are geometric, but one could also generalize those other constructions to Dyer groups.

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