# Long range order for random field Ising model

**Student Paper**

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**Publication date:**
2024

**Permanent link:**
https://doi.org/10.3929/ethz-b-000672117

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# Long range order for random field Ising model

Semester project

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Wednesday 8th May, 2024

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**Abstract**

We study existence of long range order in the random field Ising model. We define the classical Ising model and use the so-called Griffiths–Peierls argument to prove existence of long range order for low temperatures, in dimension two and above. Then, we introduce the random field Ising model. We use J. Ding and Z. Zhuang recent work in [6], which extends Peierls argument and shows that long range order also exists in this model at low temperatures with the presence of a weak disorder, in dimension three and above.

# Acknowledgments

I would like to acknowledge my supervisors, Prof. Dr. Vincent Tassion and Dr. Barbara Dembin, for giving me the opportunity to get involved in this exciting topic. In particular, I wish to express my sincerest gratitude to Dr. Dembin for her dedication and support through the process of completing this project, as well as her guidance and valuable insights.

# Contents

Chapter 1

# Introduction

The Ising model is a theoretical model in statistical physics, whose primary incentive is to simplify the complex properties of solids by assuming that they can be represented by a lattice arrangement of molecules interacting with their neighbors. The model was initially introduced in 1920 by the German physicist Wilhelm Lenz and developed later on by his student Ernst Ising in his PhD thesis.

Given a macroscopic system made of a substantial number of molecules, providing an accurate description of such system is a strenuous task if one has to keep track of the positions and speed of all the molecules. As an alternative, the Ising model gives a probabilistic description of the system by assuming that each molecule has a random behaviour that can be described with only few parameters. In particular, the Ising model can be used to describe properties of magnets. Consider a piece of iron, that can typically be pictured as a solid made of atoms of iron arranged in a regular crystalline structure.

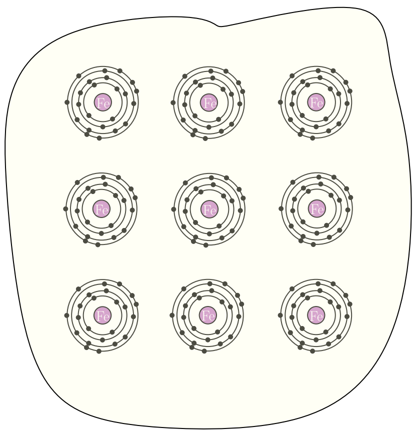

**Figure 1.1:** Piece of iron

One can assume that each atom carries an intrinsic magnetic moment, called its *atomic spin*, that can take one of two possible orientations, represented by $+1$ ("spin up") or $-1$ ("spin down"). Short-range interactions between atoms tend to make neighbouring spins coincide. In particular, when the magnet is immersed in a magnetic field, spins have a tendency to align with this field. When decreasing the intensity of the external field to zero, one can wonder about the behaviour of the spins. A possible scenario is that the global order of the spins will vanish, in which case we say that the piece of iron is *paramagnetic*. The second possible scenario is the one where, even though the influence of the external magnetic field decreases, the interactions among the spins maintain polarization of the spins. In this case, we say that the piece of iron is *ferromagnetic*. This phenomenon depends especially of the temperature of the system, as a higher temperature is associated with more thermal agitation for the atoms, and this in turn interferes with neighbouring spins interactions. As suggested in 1936 by Peierls [13], with a proof that was later made rigorous by Griffiths [9] in 1964, in any dimension $d \geq 2$, when the temperature decreases, the piece of iron in the Ising model undergoes a *phase transition* from a paramagnetic to a ferromagnetic state at a critical temperature $T_C$, called the Curie temperature.

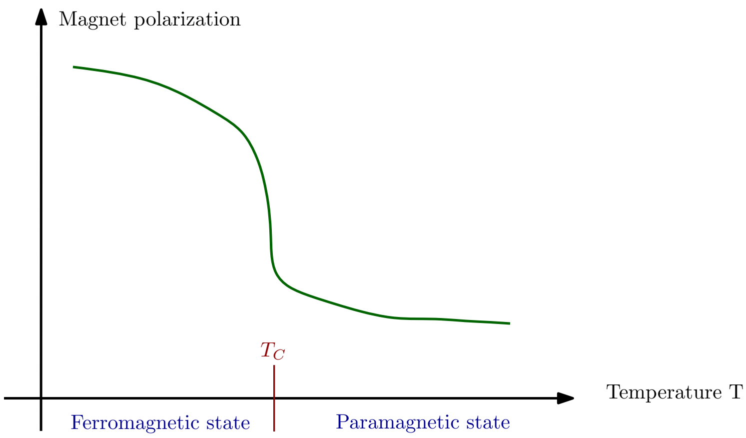


**Figure 1.2:** Ferromagnetic/paramagnetic phase transition

To quantitatively measure whether the piece of iron is in paramagnetic or ferromagnetic state, one can study the average value of the spins. If the spins, on average, have a preferential value after withdrawal of the external magnetic field, i.e. if the solid is in a ferromagnetic state, one talks about the existence of *long range order* in the model.

Until now, we considered the system to be an ideal system. However, when conducting real–life experiments, it is often the case that the system presents some impurities, and that may have significant effects on the outcomes. In particular, a small disorder in the physical system can antagonize the ordering induced by spins interactions. In order to take this into account, one can decide to improve the model by assuming that each atom of the solid is subject to a "noise", that will typically be distributed according to a normal distribution. The global perturbation generated can be identified as a random field, and for that reason the model obtained is called the *random field Ising model* (RFIM).

We are interested, in a first part, in proving existence of long range order at low temperatures in the classical Ising model. Then, in a second part, we are interested in determining under which circumstances does existence of long range order still hold in the random field Ising model. In Chapter 2, we define the classical model and use Griffiths–Peierls argument to prove the existence of long range order in dimension $d \geq 2$ at low temperatures. In Chapter 3, we introduce the RFIM. We demonstrate, as in Ding and Zhuang [6], that in dimension $d \geq 3$, when the disruption caused by the external field is relatively weak, and again at low temperatures, long range order still exists with high probability, the degree of uncertainty being due to the randomness of the perturbation.

Chapter 2

# Classical Ising model

## 2.1 Framework

Let $d \geq 1$ be a dimension. We give the formal definition of the Ising model on a $d$-dimensional square grid graph with nearest neighbors interactions. We consider the $d$-dimensional lattice graph on $\mathbb{Z}^d$, with the set of edges given by

$$E := \{xy = \{x, y\} \subset \mathbb{Z}^d : \|x - y\|_1 = 1\}.$$

We write $x \sim y$ if $xy \in E$. Each vertex $v \in \mathbb{Z}^d$ is characterized by a spin $\sigma_v = \pm 1$. We consider

$$\Omega := \{-1, +1\}^{\mathbb{Z}^d}$$

the set of all *spin configurations*, or *microstates*, of the model.

Consider now a finite-volume $\Lambda \Subset \mathbb{Z}^d$, containing the origin, and representing the physical system. We define the restriction of the spin configurations to the set $\Lambda$ by

$$\Omega_\Lambda := \{-1, +1\}^\Lambda.$$

**Definition 2.1.1** $\forall \sigma \in \Omega_\Lambda$, the energy of the configuration $\sigma$ is given by the *Hamiltonian*

$$H_\Lambda(\sigma) = -\sum_{\substack{xy \subset \Lambda \\ x \sim y}} \beta \sigma_x \sigma_y,$$

where $\beta \geq 0$.

The Hamiltonian is simply obtained by summing the interactions over all neighbouring spins, weighted by the coefficient $\beta$ representing the strength of the interactions. If we choose $T > 0$ to be the temperature of the system, then $\beta$ is related to $T$ by the formula

$$\beta := \frac{k_B}{T},$$

where $k_B$ is the Boltzmann constant.

**Definition 2.1.2** The *Gibbs measure*, also called *Boltzmann distribution*, of the Ising model in $\Lambda$ at parameter $\beta \geq 0$ is the distribution on $\Omega_\Lambda$ given $\forall \sigma \in \Omega_\Lambda$ by

$$\mu_\Lambda[\sigma] = \frac{e^{-H_\Lambda(\sigma)}}{Z_\Lambda},$$

where

$$Z_\Lambda = \sum_{\sigma \in \Omega_\Lambda} e^{-H_\Lambda(\sigma)}$$

is called the *partition function*.

**Remark 2.1** *We denote by* $\langle . \rangle_\Lambda$ *the expectation with respect to the Gibbs measure in* $\Lambda$.

The Gibbs measure is a natural way to define a probability measure on the space of spin configurations. Indeed, one can observe from the Hamiltonian's formula that the more the spins agree for a given configuration, the smaller the associated energy is. Since we consider the system to be an isolated system, a consequence of the second law of thermodynamics is that the system is at equilibrium if and only if its energy reaches a local minimum. Accordingly, the measure defined on $\Omega_\Lambda$ should favor the configurations with smaller energy, which is exactly what the Gibbs measure does.

## 2.2 Boundary conditions

By symmetry of the model, in order to collect information about the behaviour of the spins, it suffices to observe the spin at the origin. Let us consider the value of the expected spin at 0, i.e.

$$\langle \sigma_0 \rangle_\Lambda = \mu_\Lambda[\sigma_0 = 1] - \mu_\Lambda[\sigma_0 = -1].$$

The invariance of the model under global spin flip, also referred as *spin-flip symmetry*, implies that $\mu_\Lambda[\sigma] = \mu_\Lambda[-\sigma]$ for all $\sigma \in \Omega_\Lambda$. It is straightforward to deduce that $\langle \sigma_0 \rangle_\Lambda = 0$, i.e. spins have no preferential value. We would like to polarize the spins and break the spin–flip symmetry. This will be done by immersing the piece of iron in an external magnetic field, whose orientation will influence the value of the spins. In this section, we model mathematically the action of plunging the material into a magnetic field and removing this field afterwards.

From now on, we will assume that $\Lambda \Subset \mathbb{Z}^d$ is a finite hypercube. Specifically, for any $N \geq 1$, let

$$\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$$

be the box of side–length $2N$ centered at the origin 0.

Let $N \geq 1$. Up to now, when defining the model on $\Lambda_N$, we implicitly considered the system to have free boundary conditions, in the sense that we examined spin configurations within $\Lambda_N$, without imposing any constraints on the spins located outside of this volume. Let us now "freeze" the spins outside of $\Lambda_N$.

**Definition 2.2.1** Let $\eta := \{\eta_x\}_{x\in\mathbb{Z}^d\setminus\Lambda_N}$. A configuration of the Ising model in $\Lambda_N$ with boundary condition $\eta$ is an element of the set

$$\Omega^{\eta}_{\Lambda_N} := \{\sigma \in \Omega_{\Lambda_N} : \sigma_x = \eta_x, \forall x \notin \Lambda_N\}.$$

Now, there are interactions between the external magnetic field and the spins that are located on the inner boundary of $\Lambda_N$, since the latters are at distance $l_1$ equal to 1 from the external magnetic field. These interactions contribute to the energy of the system and accordingly, we need to adjust the previous definition of the Hamiltonian.

**Definition 2.2.2** $\forall \sigma \in \Omega^{\eta}_{\Lambda_N}$, the Hamiltonian associated with $\sigma$ with boundary conditions $\eta$ is given by

$$H^{\eta}_{\Lambda_N}(\sigma) = -\beta\Bigg(\sum_{\substack{xy\subset\Lambda_N\\ x\sim y}} \sigma_x\sigma_y + \sum_{\substack{x\in\Lambda_N, y\notin\Lambda_N\\ x\sim y}} \sigma_x\eta_y\Bigg).$$

**Definition 2.2.3** The Gibbs measure in $\Lambda_N$ with boundary conditions $\eta$ is defined as

$$\mu^{\eta}_{\Lambda_N}[\sigma] = \frac{e^{-H^{\eta}_{\Lambda_N}(\sigma)}}{Z^{\eta}_{\Lambda_N}},$$

where

$$Z^{\eta}_{\Lambda} = \sum_{\sigma\in\Omega^{\eta}_{\Lambda_N}} e^{-H^{\eta}_{\Lambda_N}(\sigma)}.$$

The presence of the external magnetic field in the Ising model simply corresponds to particular case of boundary conditions, namely by setting $\eta \equiv 1$ or $\eta \equiv -1$, depending whether we want the orientation of the magnetic field to be respectively "+" or "-".

Without loss of generality, we consider the Ising model with "+" boundary conditions. Now, the average value of the spin at 0 when the piece of iron is under the influence of "+" boundary conditions is

$$\langle\sigma_0\rangle^{+}_{\Lambda_N} \geq 0,$$

because the propagation of the influence of "+" boundary conditions through the piece of iron via neighbours interactions can only increase the chances for $\sigma_0$ to be positive. In particular, if $\langle\sigma_0\rangle^{+}_{\Lambda_N}$ is strictly positive, the spin–flip symmetry has been broken.

**Definition 2.2.4** We say that *long range order* exists if the spin–flip symmetry is not restored when taking the thermodynamic limit of the model with "+" boundary conditions, i.e.

$$m_d(\beta) := \lim_{N\to\infty} \langle\sigma_0\rangle^{+}_{\Lambda_N} > 0.$$

The quantity $m_d(\beta)$ is referred to as the *spontaneous magnetization* of the system.

## 2.3 Existence of long range order

Let us prove that in any dimension $d \geq 2$, there exists a temperature $T_C(d) > 0$ under which long range order exists in the Ising model.

**Lemma 2.2** *[14, Proposition 4.3.3] For all $d \leq d'$,*

$$m_d(\beta) \leq m_{d'}(\beta).$$

**Theorem 2.3** *[14, Theorem 4.5.1] $\forall d \geq 2$, there exists $T_C(d) > 0$ such that $\forall 0 \leq T \leq T_C(d)$ and $N \geq 1$,*

$$\mu^+_{\Lambda_N}(\{\sigma_0 = 1\}) \geq \frac{3}{4}.$$

**Proof** By Lemma 2.2, it suffices to prove the result in dimension $d = 2$. In order to do so, we rely on the Griffiths–Peierls argument.

Let $\sigma \in \Omega_{\Lambda_N}$ with $\sigma_0 = -1$. Let $\mathcal{A}_0$ be the maximal simply connected component containing the origin in which all the spins have value "-1". $\mathcal{A}_0$ must be strictly contained in $\Lambda_N$. Indeed, suppose it is not the case. Then, there exists a path starting from the origin, and going all the way to the external boundary of the box, along which all spins have value "-1". However, the spins on the external boundary of the box $\Lambda_N$ have value "+1", hence the contradiction.

We consider the "disagreement loop"

$$\gamma := \{x \in \Lambda_N \backslash \mathcal{A}_0 : \exists y \in \mathcal{A}_0 \text{ s.t. } x \sim y\} \subset V.$$

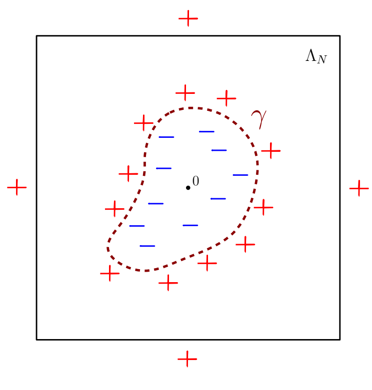


**Figure 2.1:** Disagreement loop $\gamma$. For each vertex $xy$ in the graph such that $\sigma_x = +1$ and $\sigma_y = -1$, one can associate an edge in the dual graph that crosses $xy$ perpendicularly. The union of such edges in the dual graph forms a loop, hence the terminology used here.

Let

$$E_\gamma := \{\sigma \in \Omega_{\Lambda_N} : \gamma \text{ is a disagreement loop}\}$$

and $\forall k \geq 4$, let

$$\Gamma_k := \{\gamma \text{ is a loop surrounding } 0 : |\gamma| = k\}.$$

By union bound,

$$\begin{aligned}\mu^+_{\Lambda_N}\left[\{\sigma_0=-1\}\right] &= \mu^+_{\Lambda_N}\left[\cup_{k\geq 4}\cup_{\gamma\in\Gamma_k} E_\gamma\right] \\ &\leq \sum_{k\geq 4}\sum_{\gamma\in\Gamma_k}\mu^+_{\Lambda_N}[E_\gamma].\end{aligned} \tag{2.1}$$

We prove that the disagreement loops are costly in energy for the system, and that as a consequence, it is not likely to have a configuration with a large disagreement loop. In order to do so, let us consider the auxiliary configuration

$$\tilde{\sigma}_x = \begin{cases} -\sigma_x & x\in \mathrm{Int}(\gamma) \\ \sigma_x & \text{else.}\end{cases}$$

When flipping the spins inside of the disagreement loop, the change in the system's energy is only due to the vertices that are adjacent to the disagreement loop. Specifically, those vertices and their neighbours in the disagreement loop previously had spins of opposite signs and after the flip, they have coinciding spins. On that account, one obtains that

$$H^+_{\Lambda_N}(\tilde{\sigma}) = H^+_{\Lambda_N}(\sigma) - 2\beta|\gamma|.$$

It follows that

$$\begin{aligned}\mu^+_{\Lambda_N}[E_\gamma] &= \frac{\sum_{\sigma\in E_\gamma}\mu^+_{\Lambda_N}[\sigma]}{Z^+_{\Lambda_N}} \\ &\leq e^{-2\beta|\gamma|}\frac{\sum_{\sigma\in E_\gamma}\mu^+_{\Lambda_N}[\tilde{\sigma}]}{Z^+_{\Lambda_N}} \leq e^{-2\beta|\gamma|},\end{aligned}$$

as

$$\frac{\sum_{\sigma\in E_\gamma}\mu^+_{\Lambda_N}[\tilde{\sigma}]}{Z^+_{\Lambda_N}} \leq 1$$

by definition of a probability measure.

Let $k\geq 1$. Let us bound $|\Gamma_k|$. The loop has to go around 0, so we pick any of the four straight lines starting from the origin and going outside of the box, and we know that the loop has to cross that line at some point. It is equivalent to choose any of the four lines as the loop will in any case cross each of these four lines. As we know that the loop has to contain the origin and that it is of length $k$, we necessarily have to start somewhere on the line at a distance less than or equal to $k$ from 0. Thus, we have $k$ choices for the initial step. The loop has to move in a two-dimensional space and has to be self-avoiding, thus we have at most 3 choices for the other steps. In summary, one obtains that

$$|\Gamma_k| \leq k3^k.$$

From (2.1),

$$\begin{aligned}\mu^+_{\Lambda_N}\left[\{\sigma_0=-1\}\right] &\leq \sum_{k\geq 4}|\Gamma_k|e^{-2\beta k} \\ &\leq \sum_{k\geq 4}k(3e^{-\beta})^k \leq \frac{3e^{-\beta}}{(1-e^{-\beta})^2}.\end{aligned}$$

For $\beta$ large enough, we get, $\forall N \geq 1$,

$$\mu^{+}_{\Lambda_N}\left[\{\sigma_0 = -1\}\right] \leq \frac{1}{4}. \qquad \square$$

Chapter 3

# Random field Ising model

In this chapter, we study long range order in the random field Ising model. In case of a strong disorder, it was proved for instance by J. Fröhlich [12] in 1984 that boundary influence decays exponentially in $N$, so there is no long range order. On the other hand, with presence of a weak disorder, a classical result is that long range order exists at low temperatures. This was shown by Imbrie [11] in 1985 and Bricmont and Kupiainen [3] in 1988, using renormalization group theory. Here, we use the proof of Ding and Zhuang [6], which is shorter and simpler. We note that in 2022, Ding et al. [7] extended this result by showing that for any temperature lower than $T_C$ (the critical temperature without disorder), long range order exists as long as the disorder is sufficiently small, depending on the temperature.

## 3.1 Framework

Let $d \geq 2$. As in the previous chapter, we consider the Ising model on the $d$-dimensional lattice graph on $\mathbb{Z}^d$ with nearest neighbours interactions, and a spin configuration in $\Omega$. Let $N \geq 1$. We add some "+/-" boundary conditions on the boundary of the box $\Lambda_N$. To obtain the random field Ising model, one needs to simulate a random perturbation affecting individually each atom of iron. For this, one can choose any probability measure $\mathbb{P}$ on the space $\mathbb{R}^{\mathbb{Z}^d}$, as well as a collection $h := \{h_x\}_{x\in\mathbb{Z}^d}$ of i.i.d. standard Gaussian random variables under $\mathbb{P}$.

**Definition 3.1.1** For $\varepsilon > 0$, we call *external field* the set

$$\varepsilon h := \{\varepsilon h_x\}_{x\in\mathbb{Z}^d}.$$

The external field attributes a magnetic force of random magnitude $\varepsilon h_x$ to each vertex $x \in \mathbb{Z}^d$. The disorder induced by the external field on the system is referred to as a *quenched* disorder, as the environment $h$ is fixed from the beginning and does not evolve in time.

**Definition 3.1.2** $\forall \sigma \in \Omega_{\Lambda_N}$, the *RFIM Hamiltonian* within the box $\Lambda_N$ with +/- boundary conditions and external field $\varepsilon h$ is given by

$$H^{\pm}_{\Lambda_N}(h,\sigma) = -\Big( \sum_{\substack{xy\subset\Lambda_N \\ x\sim y}} \sigma_x\sigma_y \pm \sum_{\substack{x\in\Lambda_N, y\notin\Lambda_N \\ x\sim y}} \sigma_x - \sum_{x\in\Lambda_N} \varepsilon h_x \sigma_x \Big).$$

**Definition 3.1.3** Let $T \geq 0$. The *Gibbs measure* on the space $\Omega_{\Lambda_N}$ at temperature $T$ is given by

$$\mu^{\pm}_{\Lambda_N,h}(\sigma) = \frac{e^{-T^{-1}H^{\pm}_{\Lambda_N,h}(\sigma)}}{Z^{\pm}_{\Lambda_N}(h)},$$

where

$$Z^{\pm}_{\Lambda_N}(h) = \sum_{\sigma\in\Omega_{\Lambda_N}} e^{-T^{-1}H^{\pm}_{\Lambda_N,h}(\sigma)}.$$

## 3.2 Existence of long range order in RFIM

**Definition 3.2.1** We say that *long range order exists for RFIM* if for a typical instance of the disorder $\varepsilon h$,

$$\lim_{N\to\infty} \langle \sigma_0 \rangle^{\pm}_{\Lambda_N,h} = \lim_{N\to\infty} \Big( \mu^{\pm}_{\Lambda_N,h}[\sigma_0 = 1] - \mu^{\pm}_{\Lambda_N,h}[\sigma_0 = -1] \Big) \neq 0.$$

**Theorem 3.1** *[6, Theorem 1.1] Let $d \geq 3$. There exists a constant $C > 0$ such that for all $T \geq 0$, $\varepsilon \leq C$, $N \geq 1$, we have*

$$\mu^{\pm}_{\Lambda_N,h}[\sigma_0 = \pm 1] \geq 1 - e^{-CT^{-1}} - e^{-C\varepsilon^{-2}}$$

*with $\mathbb{P}$-probability at least $1 - e^{-CT^{-1}} - e^{-C\varepsilon^{-2}}$.*

**Remark 3.2** *In the classical Ising model, existence of long range order is proved also in dimension $d = 2$. However, in the RFIM, this result is not true anymore. The random external field is a spatial perturbation of order $\mathcal{O}(N^{d/2})$, and the boundary conditions are of order $\mathcal{O}(N^{d-1})$. In dimension $d = 2$, those two quantities are of the same order and as a consequence, the perturbation cancels the influence of the boundary conditions.*

In the RFIM, the rate of decay of the quantity $\mu^{+}_{\Lambda_N}[\sigma_0 = -1]$ will depend on the environment induced by the perturbation $h$. Hence, we define the quenched probability space

$$\big(\mathbb{R}^{\Lambda_N} \times \Omega^{\pm}_{\Lambda_N}, \mathbb{Q}^{\pm}_{\Lambda_N}\big),$$

with $\mathbb{Q}^{\pm}_{\Lambda_N}$ being the probability measure defined below.

**Definition 3.2.2** Let $A \subset \mathbb{R}^{\Lambda_N}$ and $B \subset \{-1,+1\}^{\Lambda_N}$. We define the mixed joint measure for $(h,\sigma)$ by

$$\mathbb{Q}^{\pm}_{\Lambda_N}[h \in A, \sigma \in B] := \int_A \sum_{\sigma\in B} \nu^{\pm}_{\Lambda_N}(h,\sigma)dh,$$

where $\nu^{\pm}(h,\sigma)$ is the joint density function for $(h,\sigma)$.

Using the fact that the collection $\{h_x\}_{x\in\mathbb{Z}^d}$ is a collection of independent and identically distributed standard Gaussian variables, one has that

$$\nu^{\pm}_{\Lambda_N}(h,\sigma) = \mu^{\pm}_{\Lambda_N,h}[\sigma]\Big(\prod_{x\in\Lambda_N}\Phi[h_x]\Big),$$

where $\Phi$ is the density of a standard Gaussian random variable. It follows that

$$\mathbb{Q}^{\pm}_{\Lambda_N}[h\in A,\sigma\in B] = \int_A \mu^{\pm}_{\Lambda_N,h}[B]d\mathbb{P}(h).$$

One can notice that

$$\begin{aligned}\mathbb{Q}^{\pm}_{\Lambda_N}[\sigma_0=-1] &= \int_{\mathbb{R}^{\Lambda_N}} \mu^{\pm}_{\Lambda_N,h}[\sigma_0=-1]d\mathbb{P}(h)\\ &= \mathbb{E}^{\mathbb{P}}\left[\mu^{\pm}_{\Lambda_N,h}[\sigma_0=-1]\right].\end{aligned} \tag{3.1}$$

For that reason, in order to prove Theorem 3.1, it suffices to control the quantity $\mathbb{Q}^{\pm}_{\Lambda_N}[\sigma_0=-1]$ and apply Markov's inequality.

## 3.3 Proof of Theorem 3.1

### 3.3.1 Preliminaries

**Lemma 3.3 (Gaussian concentration inequality)** *[15, Theorem 3.25] Let $X_1,...,X_n$ be i.i.d. Gaussian random variables, $X_i\sim\mathcal{N}(0,1)$. Then for any $f\in\mathcal{C}^1(\mathbb{R}^n)$, $t\geq 0$, one has that*

$$\mathbb{P}[|f(X_1,...,X_n)-\mathbb{E}(f(X_1,..,X_n)|\geq t]\leq e^{-\frac{t^2}{2\sigma^2}},$$

*where $\sigma^2=\|\|\nabla f\|^2\|_\infty$.*

**Definition 3.3.1** The *free energy* associated to the external field $h$ is defined as

$$\mathcal{F}^{\pm}_{T,\Lambda_N}(h) := -T\log\left(Z^{\pm}_{\Lambda_N}(h)\right)$$

Let $A\subset\mathbb{Z}^d$. For any $(h,\sigma)\in\mathbb{R}^{\mathbb{Z}^d}\times\Omega$, let

$$h^A_x := \begin{cases}-h_x & x\in A\\ h_x & x\notin A\end{cases} \text{ and } \sigma^A_x := \begin{cases}-\sigma_x & x\in A\\ \sigma_x & x\notin A\end{cases}. \tag{3.2}$$

**Definition 3.3.2** The *maximum amount of work* the system can perform while undergoing a flipping operation of the sign of the external field within $A$ is given by

$$\Delta_A(h) := \mathcal{F}^{\pm}_{T,\Lambda_N}(h) - \mathcal{F}^{\pm}_{T,\Lambda_N}(h^A) = -T\log\left(\frac{Z^{\pm}_{\Lambda_N}(h)}{Z^{\pm}_{\Lambda_N}(h^A)}\right).$$

**Lemma 3.4** *[6, Lemma 3.1] Let $A, A'\subset\Lambda_N$ and $\lambda>0$. Then* *(i)* $\Delta_A(h)$ *is a* $\varepsilon^2|A|$*-sub–Gaussian random variable, i.e.*

$$\mathbb{P}[|\Delta_A(h)| > \lambda] \le 2e^{-\frac{\lambda^2}{8\varepsilon^2|A|}}$$

*(ii)* $\Delta_A(h) - \Delta_{A'}(h)$ *is a* $\varepsilon^2|A \oplus A'|$*-sub–Gaussian random variable, i.e.*

$$\mathbb{P}[|\Delta_A(h) - \Delta_{A'}(h)| > \lambda] \le 2e^{-\frac{\lambda^2}{8\varepsilon^2|A\oplus A'|}},$$

*where* $A \oplus A' := (A \cup A') \backslash (A \cap A')$ *denotes the symmetric difference between* $A$ *and* $A'$.

**Proof** Let us first prove (i). The collection $\{h_x\}_{x\in A}$ is distributed symmetrically around 0, hence

$$\mathbb{E}\big[\Delta_A(h)\big|\{h_x : x \in A^c\}\big] = 0.$$

Notice that the map $h \mapsto \Delta_A(h)$ is continuously differentiable. Let $x \in A$.

$$\begin{aligned}
\Big|\frac{\partial}{\partial h_x}\Delta_A(h)\Big| &= \left|\frac{\partial}{\partial h_x} - T\log\left(\frac{Z^{\pm}_{\Lambda_N}(h)}{Z^{\pm}_{\Lambda_N}(h^A)}\right)\right| \\
&= \left| - T\frac{\sum_\sigma e^{-\frac{1}{T}H^{\pm}_{\Lambda_N}(\sigma)}}{Z^{\pm}_{\Lambda_N}(h)}\Big(-\frac{\varepsilon\sigma_x}{T}\Big) + T\frac{\sum_\sigma e^{-\frac{1}{T}H^{\pm}_{\Lambda_N}(\sigma^A)}}{Z^{\pm}_{\Lambda_N}(h^A)}\Big(-\frac{\varepsilon\sigma_x}{T}\Big)\right| \\
&= \varepsilon\Big|\sum_\sigma \mu_{\Lambda_N,h}[\sigma]\sigma_x + \sum_\sigma \mu_{\Lambda_N,h^A}[\sigma]\sigma_x\Big| \\
&= \varepsilon\Big|\langle\sigma_x\rangle_{\Lambda_N,h} + \langle\sigma_x\rangle_{\Lambda_N,h^A}\Big| \le 2\varepsilon.
\end{aligned}$$

Therefore,

$$\left\|\Big|\frac{\partial}{\partial h_x}\Delta_A(h)\Big|^2\right\|_\infty \le \sum_{x\in A}\Big|\frac{\partial}{\partial h_x}\Delta_A(h)\Big|^2 \le |A|4\varepsilon^2.$$

We deduce using Lemma 3.3 that

$$\mathbb{P}\big[|\Delta_A(h)| > \lambda\big|\{h_x : x \in A^c\}\big] \le 2e^{-\frac{1}{2}\frac{\lambda^2}{4\varepsilon^2|A|}}.$$

Thenceforth,

$$\begin{aligned}
\mathbb{P}\big[|\Delta_A(h)| > \lambda\big] &= \sum_{h\in\mathbb{R}^{\Lambda_N}} \mathbb{P}\big[|\Delta_A(h)\big| \ge \lambda|\{h_x : x \in A^c\}\big]\mathbb{P}\big[\{h_x : x \in A^c\}\big] \\
&\le 2e^{-\frac{1}{2}\frac{\lambda^2}{4\varepsilon^2|A|}} \sum_{h\in\mathbb{R}^{\Lambda_N}} \mathbb{P}[\{h_x : x \in A^c\}] \\
&\le 2e^{-\frac{1}{2}\frac{\lambda^2}{4\varepsilon^2|A|}}, \qquad (3.3)
\end{aligned}$$

which yields (i). Let us prove (ii). First, we notice that

$$\Delta_A(h) - \Delta_{A'}(h) = \mathcal{F}^{\pm}(h^{A'}) - \mathcal{F}^{\pm}(h^A)$$

and that the distribution of $(h^A, h^{A'})$ conditioned on $\{h_x : x \in (A \cup A')^c\}$ is the same as the distribution of $(h^{A\oplus A'}, h)$ conditioned on the same set. Thus,

$$\mathcal{F}^{\pm}(h^A) - \mathcal{F}^{\pm}(h^{A'}) = \Delta_A(h) - \Delta_{A'}(h)$$

has the same conditional distribution as

$$\mathcal{F}^{\pm}(h) - \mathcal{F}^{\pm}(h^{A\oplus A'}) = \Delta_{A\oplus A'}(h).$$

From (i), we get

$$\begin{aligned}
&\mathbb{P}\big[|\Delta_{A\oplus A'}(h)| > \lambda \big| \{h_x : x \in (A \cup A')^c\}\big] \\
&\qquad = \mathbb{P}\big[|\Delta_A(h) - \Delta_{A'}(h)| > \lambda \big| \{h_x : x \in (A \oplus A')^c\}\big] \\
&\qquad \le 2e^{-\frac{\lambda^2}{8\varepsilon^2 |A\oplus A'|}},
\end{aligned}$$

and by the same argument as used in (3.3), we conclude that

$$\mathbb{P}\big[|\Delta_{A\oplus A'}(h)| > \lambda\big] \le 2e^{-\frac{\lambda^2}{8\varepsilon^2 |A\oplus A'|}}. \qquad \square$$

### 3.3.2 Extension of the Griffiths–Peierls argument

**Definition 3.3.3** Let $\mathfrak{U}$ be the collection of all simply connected subsets $A \subset \Lambda_N$ containing the origin.

**Definition 3.3.4** Let $A \subset \mathbb{Z}^d$. We define

$$\partial A := \{xy : x \sim y, x \in A, y \notin A\}$$

the *edge boundary* of $A$,

$$\partial_{in} A := \{x \in A : \exists y \notin A, x \sim y\},$$

the *inner vertex boundary* of $A$, and

$$\partial_{ex} A := \{x \notin A : \exists y \in A, x \sim y\},$$

the *external vertex boundary* of $A$.

**Remark 3.5** *The three notions of contours defined above are related by the equation*

$$|\partial_{ex} A| = |\partial A| \le 2d|\partial_{in} A|. \tag{3.4}$$

Let $\mathcal{A}_0 \in \mathfrak{U}$ be the *sign component* of the origin, namely the maximal simply connected component containing the origin in which all the spins have the same sign as the origin. We note that $\mathcal{A}_0$ is the component enclosed by the disagreement loop introduced in the proof of Theorem 2.3 and corresponding to $\partial_{ex}\mathcal{A}_0$ here.

When attempting to use the Griffiths–Peierls argument, the first thing we notice is that if we flip the spins located inside sign component of the origin, the sign of the term in the Hamiltonian associated to the action of the random magnetic field will also be flipped. Hence, the change in Hamiltonian depends on each instance of the field $h$. As a result, this change cannot be uniformly bounded, which prevents us from going further in the argument. This can be solved by flipping both the spin and the sign of the external field inside $\mathcal{A}_0$. We do so by considering the mapping

$$\begin{aligned} \mathbb{R}^{\mathbb{Z}^d} \times \Omega &\longrightarrow \mathbb{R}^{\mathbb{Z}^d} \times \Omega \\ (h, \sigma) &\longmapsto (h^A, \sigma^A), \end{aligned}$$

where $(h^A, \sigma^A)$ is defined as in (3.2).

Let $d \geq 3$. We consider without loss of generality that the model has "+" boundary conditions. By flipping both the spin and the sign of the magnetic field component inside $\mathcal{A}_0$, analogously as in the classical Griffiths–Peierls argument, the decrease in the system's energy is only due to the vertices that are not in the sign component but that are adjacent to it. Consequently,

$$\begin{aligned} H^+_{\Lambda_N}(h^{\mathcal{A}_0}, \sigma^{\mathcal{A}_0}) &= H^+_{\Lambda_N}(h, \sigma) - 2|\partial \mathcal{A}_0| \\ &= H^+_{\Lambda_N}(h, \sigma) - 2|\partial_{ex} \mathcal{A}_0|. \end{aligned}$$

Following tightly the strategy of the Griffiths–Peierls argument, we have

$$\begin{aligned} \mathbb{Q}^+_{\Lambda_N}[\mathcal{A}_0 = A] &= \int_{\mathbb{R}^{\Lambda_N}} \mu^+_{\Lambda_N, h}[\mathcal{A}_0 = A] d\mathbb{P}(h) \\ &= \int_{\mathbb{R}^{\Lambda_N}} \sum_{\sigma: \mathcal{A}_0 = A} \frac{e^{-T^{-1} H^+_{\Lambda_N}(h, \sigma)}}{Z^+_{\Lambda_N}(h)} d\mathbb{P}(h) \\ &= \int_{\mathbb{R}^{\Lambda_N}} \sum_{\sigma: \mathcal{A}_0 = A} \frac{e^{-T^{-1} H^+_{\Lambda_N}(h^A, \sigma^A)}}{Z^+_{\Lambda_N}(h^A)} e^{-2T^{-1}|\partial_{ex} A|} \frac{Z^+_{\Lambda_N}(h^A)}{Z^+_{\Lambda_N}(h)} d\mathbb{P}(h) \\ &\leq \sup_{A \in \mathfrak{U}} e^{-2T^{-1}|\partial_{ex} A|} \frac{Z^+_{\Lambda_N}(h^A)}{Z^+_{\Lambda_N}(h)}, \end{aligned} \tag{3.5}$$

where the last equation stems from the fact that

$$\int_{\mathbb{R}^{\Lambda_N}} \sum_{\sigma: \mathcal{A}_0 = A} \frac{e^{-T^{-1} H^+_{\Lambda_N}(h^A, \sigma^A)}}{Z^+_{\Lambda_N}(h^A)} d\mathbb{P}(h) = \mathbb{E}^{\mathbb{P}}[\mu^+_{\Lambda_N, h}(\mathcal{A}_0 = A)] \leq 1$$

by definition of a probability measure.

Let

$$\mathcal{E} := \left\{ (h, \sigma) \in (\Lambda_N, \Omega_{\Lambda_N}) : \sup_{A \in \mathfrak{U}} \frac{Z^+_{\Lambda_N}(h^A)}{Z^+_{\Lambda_N}(h)} e^{-|\partial_{ex} A| T^{-1}} \leq 1 \right\}.$$

On the event $\mathcal{E}$, the probability that $\sigma_0 = -1$ decays exponentially in function of $|\partial_{ex} \mathcal{A}_0|$. Hence, on this event, the rest of the strategy is utterly similar as Griffiths–Peierls argument, and the desired result follows seamlessly. The remaining task consists of proving that the bad event $\mathcal{E}^c$ only happens with very small $\mathbb{P}$-probability. More precisely, we want to show that

$$\mathbb{P}(\mathcal{E}^c) \leq e^{-C\varepsilon^{-2}}.$$

### 3.3.3 A naive first attempt

First, notice that

$$\mathcal{E}^c = \left\{ \sup_{A\in\mathfrak{U}} \frac{Z^+_{\Lambda_N}(h^A)}{Z^+_{\Lambda_N}(h)} e^{-|\partial_{ex}A|T^{-1}} > 1 \right\} = \left\{ \sup_{A\in\mathfrak{U}} T\log\left(\frac{Z^+_{\Lambda_N}(h^A)}{Z^+_{\Lambda_N}(h)}\right) - |\partial_{ex}A| > 0 \right\}$$
$$= \left\{ \sup_{A\in\mathfrak{U}} \frac{\Delta_A(h)}{|\partial_{ex}A|} > 1 \right\},$$

Based on Lemma 3.4, $\Delta_A(h)$ has a sub–Gaussian distribution with respect to the probability measure $\mathbb{P}$, and this is a heavy–tailed distribution. We hope to be able to use this property to bound the $\mathbb{P}$-probability of $\mathcal{E}^c$.

We start by attempting a naive approach, relying on the classical union bound. $\forall n \in \mathbb{N}$, let

$$\Gamma_n := \{A \in \mathfrak{U} : |\partial_{ex}A| = n\}$$

be the set of simply connected component in $\Lambda_N$ containing the origin and having an edge boundary of size $n$. Let us compute $|\Gamma_n|$. First, we notice that because $A \in \Gamma_n$ is simply connected, $\forall x, y \in \partial_{ex}A$,

$$\|x_i - y_j\| \le 2 \tag{3.6}$$

for some $i, j \in \{1, 2\}$, where $x_1, x_2$ and $y_1, y_2$ are respectively the endpoints of $x$ and $y$.

In order to compute $\Gamma_n$, we use the depth-first search (DFS) process, which is a greedy algorithm that consists in exploring one by one all possible paths in the graph. We use it to exhibit all possible contours $\partial_{ex}A$ of size $n$. Let us detail the procedure. To start with, based on Equation (3.6), since we want the origin to be contained in the connected component, one has to choose an initial edge that is at $l_1$-distance less than or equal to $3n$ from the edges adjacent to 0. Accordingly, we can choose any edge in the box $\Lambda_{3n}$. In $\Lambda_{3n}$, there are exactly $(2{\cdot}3n)^{d-1}d$ edges. As $d \ge 3$, there are

$$(2{\cdot}3n)^{d-1}d \le (2dn)^d$$

choices for initiating the process. Next, when exploring the graph to find a component of size $n$, the algorithm will make $2n$ steps (each edge that is crossed will be crossed in both directions by the DFS algorithm). We should now count, at each step, how many possible moves can be realized. Suppose we sit on the edge $x_1x_2$. One can choose to go on one of the edges having one endpoint in common with $x_1x_2$. There are $2d$ edges sharing the endpoint $x_1$ and $2d$ edges sharing the endpoint $x_2$, which adds up to $4d$ choices. Those edges are still at $l_1$-distance 0 of $x_1x_2$, hence one can decide either to choose one of these edges, or to go further. In the second case, one can move on one of the $2d-1$ adjacent edges. Those edges lie at $l_1$-distance 1 of $x_1x_2$, so once more, one can decide to choose one of these edges, or to go further. In the latter case, one can once again move on one of the $2d-1$ adjacent edges.

Therefore, for each of the $n$ steps where a new edge is explored, there are

$$4d{\cdot}\big((2d-1)+1\big){\cdot}\big((2d-1)+1\big) = 4d{\cdot}2d{\cdot}2d = 16d^3$$

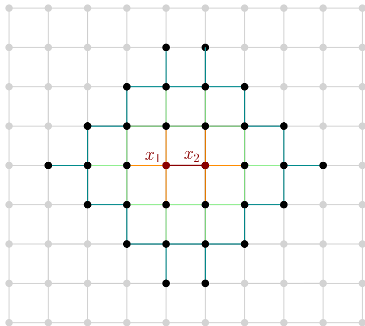


**Figure 3.1:** Edges that can be explored by the DFS at step $k+1$ of the process, knowing that we lie on edge $x_1x_2$ at step $k$. The edges colored in orange are at $l_1$-distance 0 of $x_1x_2$, the edges colored in green are at $l_1$-distance 1 of $x_1x_2$ and the edges colored in blue are at $l_1$-distance 2 of $x_1x_2$. We point out that this picture is in dimension $d=2$, and thus not accurate for representing the situation, as the proof is for dimension $d \geq 3$.

choices in total. The other $n$ steps consist in crossing edges that have already been crossed in the opposite direction, since the DFS algorithm traces back its path. This yields the bound

$$|\Gamma_n| \leq (2nd)^d (16d^3)^{2n}. \tag{3.7}$$

Now, using Lemma 3.4, one has that

$$\begin{aligned}\mathbb{P}\left[\sup_{A\in\mathfrak{U}} \frac{\Delta_A(h)}{|\partial_{ex}A|} > 1\right] &\leq \sum_{n\in\mathbb{N}} \mathbb{P}\left[\sup_{A\in\Gamma_n} \frac{\Delta_A(h)}{n} > 1\right] \\ &\leq \sum_{n\in\mathbb{N}} \sum_{A\in\Gamma_n} \mathbb{P}\Big[\frac{\Delta_A(h)}{n} > 1\Big] \\ &\leq \sum_{n\in\mathbb{N}} (2nd)^d (16d^3)^{2n} 2\exp\left(\frac{-n^2}{8\varepsilon^2 \sup_{A\in\mathfrak{U}}|A|}\right).\end{aligned}$$

Using the isoperimetric inequality

$$|A| \leq C|\partial_{ex}A|^{\frac{d}{d-1}} = Cn^{\frac{d}{d-1}} \tag{3.8}$$

leads to

$$\mathbb{P}\left[\sup_{A\in\mathfrak{U}} \frac{\Delta_A(h)}{|\partial_{ex}A|} > 1\right] \leq 2\exp\Big(C_1\log(n) + C_2 n - C_3\varepsilon^{-2} n^{\frac{d-2}{d-1}}\Big).$$

The dominating term in the above exponential is of order $n$, hence the sum does not converge and this approach does not work.

### 3.3.4 Coarse–graining method

The reason for the previous failure is rather intuitive : if for some $A \in \Gamma_n$, the quantity $\Delta_A(h)$ is greater than $n$, then, with high probability, for a component $A' \in \Gamma_n$ very similar to $A$, this will also be the case. In other words, the events are highly dependent, hence the same phenomenon is counted with big multiplicity in the sum. To address this issue, we want to ”regroup” the components that are almost identical, so that they contribute only once to the sum. The idea, known as the coarse–graining method, is to ”blurry” the elements of $\Gamma_n$, so that the components that appeared previously very similar will then be identical. Instead of summing on all the elements of $\Gamma_n$, we only sum on the possibles images of a contour through this process. We will also have to pay a price for considering coarse–grained components instead of original components. However, because of Lemma 3.4, the difference between $\Delta_A(h)$ and its coarse–grained version is a sub–Gaussian process. To detail the coarse–graining method, we rely on the work of Bovier [2] and Fisher et al. [8].

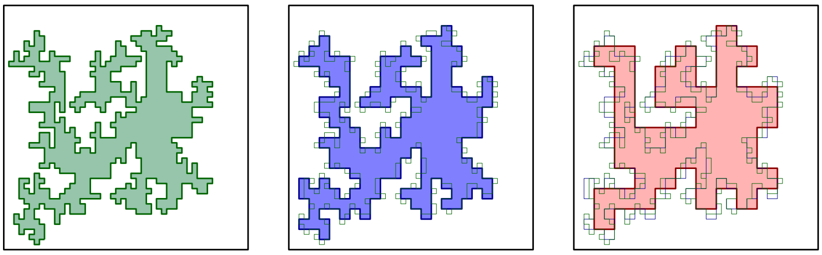

**Figure 3.2:** Successive coarse–graining of a contour

Let $k, n \in \mathbb{N}$. We consider the $2^k\mathbb{Z}^d$-lattice, i.e. the lattice made of boxes of side–length $2^k$, that we call $2^k$-boxes.

**Definition 3.3.5** We define the $2^k$-approximation of $A \in \Gamma_n$ as the set of all $2^k$-boxes being such that at least half of the points in the box also belong to $A$. Namely,

$$\tilde{A}_k := \{C_0 \text{ is a } 2^k\text{-box } : |C_0 \cap A| \geq \frac{1}{2}2^{kd}\}.$$

We call *admissible cubes* the elements of $\tilde{A}_k$. We denote $A_k$ the union of all admissible cubes.

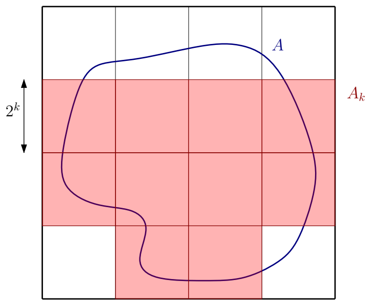


**Figure 3.3:** $2^k$-approximation of $A \in \Gamma_n$

**Definition 3.3.6** Let

$$\mathcal{F}_{n,k} := \{A_k : A \in \Gamma_n\}$$

be the set of possibles images of the elements of $\Gamma_n$ by $2^k$–approximation.

Let $m \geq 1$. Notice that

$$\sup_{A \in \Gamma_n} |\Delta_A(h)| \leq \sup_{A_m \in \mathcal{F}_{n,m}} |\Delta_{A_m}(h)| + \sup_{A_m \in \mathcal{F}_{n,m}} \sup_{A \in \Gamma_n} |\Delta_{A_m}(h) - \Delta_A(h)|.$$

For all $\alpha_1, \alpha_2 \in \mathbb{N}$ such that $\alpha_1 + \alpha_2 \leq n$, it follows that

$$\mathbb{P}\left[\sup_{A \in \Gamma_n} |\Delta_A(h)| > n\right] \leq \mathbb{P}\left[\sup_{A_m \in \mathcal{F}_{n,m}} |\Delta_{A_m}(h)| > \alpha_1\right] + \mathbb{P}\left[\sup_{A_m \in \mathcal{F}_{n,m}} \sup_{A \in \Gamma_n} |\Delta_{A_m}(h) - \Delta_A(h)| > \alpha_2\right].$$

Even though $\Delta_{A_m}(h) - \Delta_A(h)$ is a $\varepsilon^2 |A \oplus A_m|$-sub–Gaussian quantity, for $m$ large, $A$ is too different from its $2^m$-approximation $A_m$ and as a consequence, the symmetric difference $|A \oplus A_m|$ is too large for our purpose. We resolve this by using triangular inequality to only compare the difference between two consecutive box–approximations of $A$. More precisely, for all $(\alpha_i)_{i=1}^m$ such that

$$\sum_{i=1}^m \alpha_i \leq n,$$

it holds that

$$\mathbb{P}\left[\sup_{A \in \Gamma_n} |\Delta_A(h)| > n\right] \leq \mathbb{P}\left[\sup_{A_m \in \mathcal{F}_{n,m}} |\Delta_{A_m}(h)| > \alpha_{m+1}\right] + \sum_{k=1}^m \mathbb{P}\left[\sup_{A_k \in \mathcal{F}_{n,k}} \sup_{A_{k-1} \in \mathcal{F}_{n,k-1}} |\Delta_{A_k}(h) - \Delta_{A_{k-1}}(h)| > \alpha_k\right].$$

Next, by union bound,

$$\mathbb{P}\left[\sup_{A \in \Gamma_n} |\Delta_A(h)| > n\right] \leq \sum_{A_m \in \mathcal{F}_{n,m}} \mathbb{P}\left[|\Delta_{A_m}(h)| > \alpha_{m+1}\right] + \sum_{k=1}^m \sum_{A_k \in \mathcal{F}_{n,k}} \sum_{A_{k-1} \in \mathcal{F}_{n,k-1}} \mathbb{P}\left[|\Delta_{A_k}(h) - \Delta_{A_{k-1}}(h)| > \alpha_k\right].$$

By Lemma 3.4,

$$\mathbb{P}\left[\sup_{A \in \Gamma_n} |\Delta_A(h)| > n\right] \leq 2|\mathcal{F}_{n,m}| \exp\left(\frac{-\alpha_{m+1}^2}{8\varepsilon^2 \sup_{A_m \in \mathcal{F}_{n,m}} |A_m|}\right) + 2\sum_{k=1}^m |\mathcal{F}_{n,k}||\mathcal{F}_{n,k-1}| \exp\left(\frac{-\alpha_k^2}{8\varepsilon^2 \sup_{A_k \in \mathcal{F}_{n,k}, A_{k-1} \in \mathcal{F}_{n,k-1}} |A_k \oplus A_{k-1}|}\right). \tag{3.9}$$

**Upper bounding of the symmetric difference between successive coarse–grained components**

Let us show that $\forall k \in \mathbb{N}$ and $\forall A \in \Gamma_n$,

$$|A_k \oplus A_{k-1}| \leq C2^k n.$$

We rely on the work of Affonso et al. [1]. We start by computing $|A_k \backslash A_{k-1}|$.

**Definition 3.3.7** Let

$$X_k := \{C_k \in A_k : \exists C_{k-1} \in A_{k-1}, C'_{k-1} \notin A_{k-1} \text{ s.t. } C_{k-1} \cap C_k \neq \emptyset, \\ C'_{k-1} \cap C_k \neq \emptyset\}.$$

One readily sees that all contributions to the quantity $|A_k \backslash A_{k-1}|$ will be made by elements of $X_k$, i.e. admissible $2^k$-cubes that contain both admissible $2^{k-1}$-cubes and non–admissible $2^{k-1}$-cubes.

Let $C_k \in X_k$. Let also $C_{k-1} \in A_{k-1}$, $C'_{k-1} \notin A_{k-1}$ such that $C_{k-1} \cap C_k \neq \emptyset, C'_{k-1} \cap C_k \neq \emptyset$. We denote $U := C_{k-1} \cup C'_{k-1}$. Without loss of generality, we assume that $U = [1, 2^{k-1}]^{d-1} \times [1, 2^k]$. By definition of admissibility,

$$\frac{1}{2} 2^{(k-1)d} \leq |A \cap U| \leq \frac{3}{2} 2^{(k-1)d}. \tag{3.10}$$

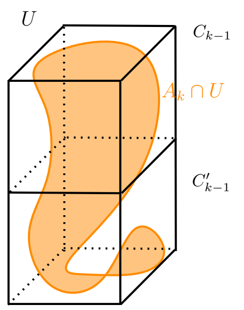


**Figure 3.4:** $U = C_{k-1} \cap C'_{k-1}$.

**Lemma 3.6** *[1, Lemma 3.15, Lemma 3.17]* $\forall k \in \mathbb{N}, C_k \in X_k$, *we have*

$$2^{k(d-1)} \leq C|\partial_{ex} A \cap C_k|.$$

The lemma is intuitively clear as only one cube in $U$ is admissible, so the boundary has to cover a sufficiently large "area" separating points in A and points outside of A. In the most simple case, we can imagine that this area is horizontal and flat, and it cuts the rectangle into two parts. As it is a surface with side length $2^k$ embedded in a space of dimension $d$, it has size $2^{k(d-1)}$. Any boundary that is more sophisticated than that will have a larger size, hence the lower bound.

**Proof** We note that since $|\partial_{ex} A \cap U| \leq |\partial_{ex}(A \cap U)|$, one cannot simply make use of the isoperimetric inequality (3.8). Rather, one can decide to project the points of $\partial_{ex} A \cap U$ onto the faces of $U$ and count the number of projected points on each face. We denote $\{e_i : 1 \leq i \leq d\}$ the standard orthonormal basis of $\mathbb{Z}^d$. For the sake of simplicity, we write $U = \prod_{i=1}^{d}[1, r_i]$, where $r_i \in \{2^{k-1}, 2^k\}$. For all $i \leq d$, we define

$$U_i := \{x \in U : x_i = 1\},$$

the face of $U$ that is perpendicular to the direction $e_i$. The line that connects a point $x \in U_i$ to the opposite face of $U_i$ is

$$l_x^i := \{x + ke_i : 1 \leq k \leq r_i\}.$$

For any set $A \subset \mathbb{Z}^d$, the projection of $A \cap U$ into the face $U_i$ is the set

$$\mathcal{P}_i(A \cap U) := \{x \in U_i : l_x^i \cap A \neq \emptyset\}.$$

If, when traveling via a straight line from $U \cap A$ to a face of the cube, we cross points that are outside of $A$, it is certain that at least once we had to cross the boundary $\partial_{ex} A$. In this case, we say that the projected point is a good point. Otherwise, it is a bad point. The sum of the good points over all the faces of the cubes then gives a lower bound for $\partial_{ex} A \cap U$, and this will be the key for proving the desired statement. We let

$$\mathcal{P}_i^G(A \cap U) := \{x \in \mathcal{P}_i(A \cap U) : l_x^i \cap (U \backslash A) \neq \emptyset\}$$

be the set of good points and

$$\mathcal{P}_i^B(A \cap U) := \mathcal{P}_i(A \cap U) \backslash \mathcal{P}_i^G(A \cap U)$$

be the set of bad points.

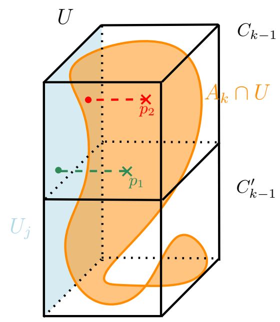


**Figure 3.5:** $p_1$ is a good point and $p_2$ is a bad point for the face $U_j$.

Let $j \leq d$ and $p \in \mathcal{P}_j^B(A \cap U)$ be any bad point in the projection onto the face $j$. Let $r_j := |l_p^j|$. We have

$$\begin{aligned} |\mathcal{P}_j^B(A \cap U)| &= r_j^{-1} \sum_{p \in \mathcal{P}_j^B(A \cap U)} |l_p^j| \\ &\leq r_j^{-1} |A \cap U| \leq \frac{3}{2} 2^{(k-1)d} r_j^{-1}. \end{aligned}$$

We differentiate two cases.

(i) Suppose $|P_j(A \cap U)| > \frac{7}{8}|U_j|$ for some $j \leq d$. Then,

$$\frac{7}{8}|U_j| < |P_j(A \cap U)| \leq |P_j^G(A \cap U)| + |P_j^B(A \cap U)|$$
$$\leq |\partial_{ex} A \cap U| + \frac{3}{2} 2^{d(k-1)} r_j^{-1}$$

Using that $r_j \in \{2^{(k-1)}, 2^k\}$ and $|U_j| \in \{2^{d(k-1)}, 2^{dk}\}$, it holds that

$$\frac{7}{8}|U_j| - \frac{3}{2} 2^{(k-1)d} r_j^{-1} = \frac{1}{8}(7|U_j| - 12.2^{d(k-1)} r_j^{-1})$$
$$\geq \frac{1}{8}(7.2^{(k-1)d} - 6.2^{d(k-1)})$$
$$\geq \frac{1}{4} 2^{d(k-1)}.$$

Finally,

$$2^{d(k-1)} \leq 4|\partial_{ex} A \cap U|. \tag{3.11}$$

(ii) Suppose $|\mathcal{P}_j(A \cap U)| \leq \frac{7}{8}|U_j|$ for all $j \leq d$. We start by proving by induction on the dimension $d$ that

$$\sum_{i=1}^{d} |\mathcal{P}_i(A \cap U)| \leq C|\partial_{ex} A \cap U|.$$

For $d = 2$, $U = [1, r_1] \times [1, r_2]$. If there are no bad points in $\mathcal{P}_1(A \cap U)$, then one readily sees that

$$|\mathcal{P}_1(A \cap U)| = |\mathcal{P}_1^G(A \cap U)| \leq |\partial_{ex} A \cap U|.$$

If there is a bad point $p = (1, p_2) \in \mathcal{P}^B(A \cap U)$, by definition, $l_p^1 \subset A \cap U$. By assumption, $|\mathcal{P}_1(A \cap U)| \leq \frac{7}{8}|U_i| < |U_i|$, so there exists $p' = (1, p_2') \in U_1 \backslash \mathcal{P}_1(A \cap U)$. This implies in turn that $l_{p'}^1 \in A^c \cap U$. Therefore, for any $k \leq r_1$, as $(k, p_2) \in A \cap U$ and $(k, p_2') \notin A \cap U$, we can find a point $p^k = (k, p_2^k) \in \partial_{ex} A \cap U$. We notice that $\forall k_1 \neq k_2$, $p^{k_1} \neq p^{k_2}$, so $r_1 \leq |\partial_{ex} A \cap U|$. Hence

$$|\mathcal{P}_1(A \cap U)| \leq |U_1| = r_2 \leq 2r_1 \leq 2|\partial_{ex} A \cap U|.$$

By the same argument,

$$|\mathcal{P}_2(A \cap U)| \leq 2|\partial_{ex} A \cap U|,$$

and thus,

$$\sum_{i=1}^{2} |\mathcal{P}_i(A \cap U)| \leq 4|\partial_{ex} A \cap U|.$$

Now, if the statement holds in dimension $d - 1$, we prove that it also holds in dimension $d$. We split $U$ into layers by setting

$$L_k := \{x \in \mathbb{Z}^d : x_d = k\}$$

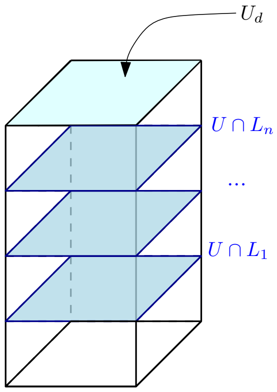


**Figure 3.6:** Splitting $U$ into layers

for every $k \leq r_d$. Then, we have

$$\sum_{i=1}^{d} |\mathcal{P}_i(A \cap U)| = \sum_{i=1}^{d-1} \sum_{k=1}^{r_d} |\mathcal{P}_i(A \cap U) \cap L_k| + |\mathcal{P}_d(A \cap U)|$$
$$= \sum_{k=1}^{r_d} \sum_{i=1}^{d-1} |\mathcal{P}_i(A \cap U) \cap L_k| + |\mathcal{P}_d(A \cap U)|.$$

One can notice that $\mathcal{P}_i(A \cap U) \cap L_k = \mathcal{P}_i(A \cap U \cap L_k)$. Define

$$U^k := U \cap L_k.$$

For all $p \in \mathcal{P}_j^B(A \cap U^k)$, by definition, $l_p^j \subset A \cap U^k$. Let $x \in l_p^j$. We can associate $x$ to a point $x' \in \mathcal{P}_d(A \cap U)$ by setting $x'_m = x_m$ for all $m \leq d-1$, and $x'_d = 1$.

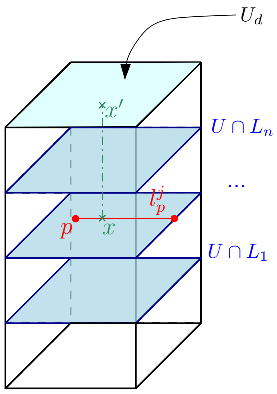


**Figure 3.7:** $x'$ is the projection of $x$ on $U_d$

Hence, it holds that

$$\sum_{p \in \mathcal{P}_j^B(A \cap U^k)} |l_p^j| \leq |\mathcal{P}_d(A \cap U)|.$$

This leads to

$$r_j|\mathcal{P}_j^B(A\cap U^k)| = \sum_{p\in\mathcal{P}_j^B(A\cap U^k)} |l_p^j| \leq |\mathcal{P}_d(A\cap U)|.$$

It follows that

$$\begin{aligned}|\mathcal{P}_j^B(A\cap U^k)| &\leq r_j^{-1}|\mathcal{P}_d(A\cap U)| \\ &\leq \frac{7}{8}|U_d|r_j^{-1} \leq \frac{7}{8}\Big(\prod_{q\neq d} r_q\Big)r_j^{-1} \\ &\leq \frac{7}{8}\Big(\prod_{q\neq d,a\neq r} r_q\Big) = \frac{7}{8}\big|(U_k)_j\big|, \end{aligned} \tag{3.12}$$

where $\big|(U_k)_j\big|$ is simply the projection of $U\cap L_k$ on the face $j$.

We consider again two cases.

- If $|\mathcal{P}_i(A\cap U^k)| \leq \frac{\frac{7}{8}+1}{2}\big|(U_k)_i\big|$ for all $i\leq d-1$, then by induction hypothesis,

$$\sum_{i=1}^{d-1}|\mathcal{P}_i(A\cap U^k)| \leq C|\partial_{ex}A\cap U^k|.$$

- If there exists $j\leq d-1$ such that $|\mathcal{P}_j(A\cap U^k)| > \frac{\frac{7}{8}+1}{2}\big|(U_k)_j\big|$, by Equation (3.12)

$$\begin{aligned}|\mathcal{P}_j^G(A\cap U^k)| &= |\mathcal{P}_j(A\cap U^k)| - |\mathcal{P}_j^B(A\cap U^k)| \\ &\geq |\frac{\frac{7}{8}+1}{2}\big|(U_k)_j\big| - \frac{7}{8}\big|(U_k)_j\big| \geq \frac{\frac{7}{8}-1}{2}\big|(U_k)_j\big|.\end{aligned}$$

Accordingly,

$$\big|(U_k)_j\big| \leq \frac{2}{1-\frac{7}{8}}|\mathcal{P}_j^G(A\cap U^k)| \leq \frac{2}{1-\frac{7}{8}}|\partial_{ex}A\cap U^k|.$$

By noticing that for all $i\leq d$,

$$|(U_k)_i| \leq (2U)^{d-2} \leq 2^{d-2}\big|(U_k)_j\big|,$$

we obtain that

$$\begin{aligned}\sum_{i=1}^{d-1}|\mathcal{P}_i(A\cap U^k)| &\leq \sum_{i=1}^{d-1}|(U^k)_i| \\ &\leq (d-1)2^{d-2}|(U^k)_j| \\ &\leq \frac{(d-1)2^{d-1}}{1-\frac{7}{8}}|\partial_{ex}A\cap U^k| \\ &\leq C|\partial_{ex}A\cap U^k|.\end{aligned}$$

Now, it holds that

$$\begin{aligned}\sum_{i=1}^{d}|\mathcal{P}_i(A\cap U)| &\leq \sum_{i=1}^{r_d}\sum_{i=1}^{d-1}|\mathcal{P}_i(A\cap U^k)| + |\mathcal{P}_d(A\cap U)| \\ &\leq \sum_{i=1}^{r_d} C|\partial_{ex}A\cap U^k| + |\mathcal{P}_d(A\cap U)| \\ &\leq C|\partial_{ex}A\cap U| + |\mathcal{P}_d(A\cap U)|.\end{aligned}$$

One notices that for any $j \leq d$, we can repeat the same strategy by splitting $U$ into layers $\{x \in U : x_j = k\}$, where $k \leq r_j$. This way, we obtain

$$\sum_{i=1}^{d}|\mathcal{P}_i(A\cap U)| \leq C|\partial_{ex}A\cap U| + |\mathcal{P}_j(A\cap U)|$$

for all $j \leq d$. By summing over all possible values of $j$, we conclude

$$\begin{aligned}&\sum_{j=1}^{d}\sum_{i=1}^{d}|\mathcal{P}_i(A\cap U)| \leq \sum_{j=1}^{d} C|\partial_{ex}A\cap U| + \sum_{j=1}^{d}|\mathcal{P}_j(A\cap U)| \\ \iff &\sum_{i=1}^{d}|\mathcal{P}_i(A\cap U)| \leq C|\partial_{ex}A\cap U|, \qquad (3.13)\end{aligned}$$

which ends the induction.

Next, by Remark 3.4 and Equation (3.13),

$$\begin{aligned}\frac{1}{2d}|\partial_{ex}(A\cap U)| &\leq |\partial_{in}(A\cap U)| \\ &\leq |\partial_{in}(A\cap U)\cap\partial_{in}U| + |\partial_{in}(A\cap U)\cap(U\backslash\partial_{in}U)| \\ &\leq |\partial_{in}U| + |\partial_{in}A\cap U| \\ &\leq 2\sum_{i=1}^{d}|\mathcal{P}_i(A\cap U)| + |\partial_{ex}A\cap U| \\ &\leq C|\partial_{ex}A\cap U|. \qquad (3.14)\end{aligned}$$

Furthermore, by isoperimetric inequality (3.8), Equation (3.10) and Equation (3.14),

$$\begin{aligned}2^{k(d-1)} &\leq C|A\cap U|^{\frac{d-1}{d}} \\ &\leq C|\partial_{ex}(A\cap U)| \\ &\leq C|\partial_{ex}A\cap U|. \qquad (3.15)\end{aligned}$$

From Equation (3.11) and Equation (3.15), we obtain

$$2^{d(k-1)} \leq C|\partial_{ex}A\cap U| \leq C|\partial_{ex}A\cap C_k|. \qquad \square$$

$\forall C_k \in A_k$, using Lemma 3.6,

$$\begin{aligned}|C_k| = 2^{kd} &= 2^d 2^{k(d-1)} \\ &\leq 2^k C|\partial_{ex}A\cap C_k|.\end{aligned}$$

Hence,

$$\begin{aligned}|A_k \backslash A_{k-1}| &= \sum_{C_k \in X_k} |A_k \backslash A_{k-1} \cap C_k| \\ &\leq \sum_{C_k \in X_k} |C_k| \leq 2^k \sum_{C_k \in X_k} C|\partial_{ex} A \cap C_k| \\ &\leq 2^k C|\partial_{ex} A| = 2^k Cn.\end{aligned}$$

We apply the same reasoning for the quantity $|A_{k-1} \backslash A_k|$. This yields

$$|A_k \oplus A_{k-1}| = |A_k \backslash A_{k-1}| + |A_{k-1} \backslash A_k| \leq C2^k n.$$

**Upper bounding of the number of possible $2^k$-approximation of a contour in $\Gamma_n$**

Let us prove that

$$|\mathcal{F}_{n,k}| \leq \exp\left(\frac{Ckn}{2^{(d-1)k}}\right). \tag{3.16}$$

In order to do so, we rely on Fisher et al. [8].

**Remark 3.7** *By a slight abuse of notation, we denote*

$$\partial_{in} A_k = \{x \in A_k : \exists y \notin A_k, x \sim y\}$$

*i.e. $\partial_{in} A_k$ is the surface delimiting the coarse–grained set $A_k$.*

It is clear that $A_k$ defines uniquely $\partial_{in} A_k$. Conversely, if we are only given the boundary $\partial_{in} A_k$, it is also the case that it uniquely defines $A_k$. Indeed, given $x \in \partial_{in} A_k$, as $A_k$ is connected and as it is a finite union of cubes, we have sufficiently enough information to determine unambiguously which neighbours $y \sim x$ belong to $A_k$ and this way, we can recover the whole set $A_k$. As a consequence, $|\mathcal{F}_{n,k}|$ is also equal to the number of possible coarse–grained contours $\partial_{in} A_k$.

**Lemma 3.8** *[8, Proposition 1] $\forall k \in \mathbb{N}$,*

$$|\partial_{in} A_k| \leq Cn.$$

**Proof** Recalling Definition 3.3.7, and because the surface of a $2^k$-cube is equal to $2d \cdot 2^{k(d-1)}$, one can notice that

$$|\partial_{in} A_k| \leq \sum_{C_k \in X_k} 2d \cdot 2^{(d-1)k}.$$

Using Lemma 3.6, it holds that

$$\begin{aligned}|\partial_{in} A_k| &\leq 2d \sum_{C_k \in X_k} 2^{(d-1)k} \\ &\leq \sum_{C_k \in X_k} C|\partial_{ex} A \cap C_k| \\ &\leq C|\partial_{ex} A| = Cn.\end{aligned}$$

□

Let us prove Equation (3.16). It might be that $A_k$ is not connected, but that it is made of a collection of connected components $\{A_k^\alpha\}_\alpha$. Using Lemma 3.8,

$$\frac{|\partial_{in} A_k|}{2d \cdot 2^{(d-1)k}} \leq \frac{Cn}{2^{(d-1)k}} =: \alpha^*$$

so there are less than $\lfloor \alpha^* \rfloor$ such connected components. Moreover,

$$\sum_\alpha |\partial_{in} A_k^\alpha| = |\partial_{in} A_k| \leq Cn. \tag{3.17}$$

For $\{x_\alpha\}_\alpha$ a set of lattice points in $2^k \mathbb{Z}^d$, we define $d_1 := \|x^1\|_1$ and for any $\alpha$,

$$d_\alpha := \|x_\alpha - x_{\alpha-1}\|_1.$$

Let $\Gamma_k\big(\{x_\alpha, a_\alpha\}\big)$ be the number of contours $\partial_{in} A_k$ such that for all $\alpha$, $x_\alpha \in \partial_{in} A_k^\alpha$ and $|\partial_{in} A_k^\alpha| = a_\alpha$. By Grimmett [10, (4.24)], $\forall a \in \mathbb{R}, x \in \mathbb{Z}^d$,

$$\#\{E : E \text{ connected component of size } a \text{ containing } x\} \leq 7^{da}. \tag{3.18}$$

$\forall \alpha \leq \lfloor \alpha^* \rfloor$, $\partial_{in} A_k^\alpha$ is a connected component. Hence, using Equation (3.17) and Equation (3.18) yields

$$\begin{aligned} \Gamma_k\big(\{x_\alpha, a_\alpha\}\big) &\leq \prod_{\alpha \leq \lfloor \alpha^* \rfloor} 7^{d a_\alpha} \\ &\leq \big(7^{d a_\alpha}\big)^{\lfloor \alpha^* \rfloor} \\ &\leq \exp\left(\frac{C \sum_\alpha a_\alpha}{2^{(d-1)k}}\right) \leq \exp\left(\frac{Cn}{2^{(d-1)k}}\right). \end{aligned} \tag{3.19}$$

Let us bound the number of possible choices of $\{d_\alpha\}_\alpha$. If $A_k$ is connected, we set $D := A_k$. If $A_k$ is made of at least two connected components, we set

$$D := \partial_{in} A \cup \left( \bigcup_{\alpha \leq \lfloor \alpha^* \rfloor} \partial_{in} A_k^\alpha \right)$$

and one can check that it is a connected set. In both cases, Equation (3.17) yields

$$|D| \leq Cn.$$

Since $D$ is a connected set, it is proved for instance in Diestel [5, Proposition 1.5.6.] that we can remove edges from $D$ until we extract a spanning tree from it. One can explore this tree by using depth–first search process. We decide to order the $x_\alpha$ according to the order of exploration of those vertices by the DFS process. We note that $\forall \alpha \leq \lfloor \alpha^* \rfloor$, the $l_1$-distance $d_\alpha$ is less than the distance between $x_\alpha$ and $x_{\alpha-1}$ in the tree. The number of edges in $D$ is less than $\frac{2d|D|}{2} = d|D|$ and each edge is used at most twice during the exploration process. Hence, it holds that

$$\sum_{\alpha \leq \lfloor \alpha^* \rfloor} d_\alpha \leq 2d|D|.$$

By a classical counting argument,

$$\begin{aligned}
\#\Big\{\big(d_1,...,d_{\lfloor\alpha^*\rfloor}\big) \in \mathbb{Z}_+^{\lfloor\alpha^*\rfloor} : \sum_{\alpha\le\lfloor\alpha^*\rfloor} d_\alpha \le 2d|D|\Big\} \\
&= \sum_{N=1}^{2d|D|} \#\Big\{\big(d_1,...,d_{\lfloor\alpha^*\rfloor}\big) \in \mathbb{Z}_+^{\lfloor\alpha^*\rfloor} : \sum_{\alpha\le\lfloor\alpha^*\rfloor} d_\alpha = N\Big\} \\
&= \sum_{N=1}^{2d|D|} \binom{N-1+\lfloor\alpha^*\rfloor}{\lfloor\alpha^*\rfloor} \qquad (3.20)\\
&\le \sum_{N=1}^{2d|D|} \Big(\frac{CN}{\lfloor\alpha^*\rfloor}\Big)^{\lfloor\alpha^*\rfloor} \le \exp\Big(\frac{Ckn}{2^{(d-1)k}}\Big). \qquad (3.21)
\end{aligned}$$

Hence, we get at most

$$\exp\Big(\frac{Ckn}{2^{(d-1)k}}\Big)$$

choices for the collection $\{d_\alpha\}_\alpha$.

Let us bound the number of possible choices of the collection $\{x_\alpha\}_\alpha$. Suppose that the collection $\{d_\alpha\}_\alpha$ is fixed. For $x^1$, one can choose any point among the $a_1$ points of $\partial_{ex}A_k^1$. There are less than $Cn$ points in $\partial_{ex}A_k$, hence $a_1 \le Cn$. Then, to choose $x^2$, one has to pick a point at $l_1$-distance $d_2$ of $x^1$, so a point among the $Cd_2^d$ points in the $\|.\|_1$-ball of radius $d_2$. We repeat the same operation for all the $\alpha$'s. We get

$$Cn \prod_{\alpha\le\lfloor\alpha^*\rfloor} Cd_\alpha^d$$

possible outcomes for this process. Using the method of Lagrange's multiplier yields that the above product is maximal when all the $x_\alpha$ are equidistant and $d_\alpha = \frac{n}{\lfloor\alpha^*\rfloor}$ for all $\alpha$. Hence, for $\{d_\alpha\}_\alpha$ fixed, we have at most

$$\begin{aligned}
Cn\Big(\frac{n}{\alpha^*}\Big)^{d\lfloor\alpha^*\rfloor} &= Cn\big(C2^{(d-1)k}\big)^{d\lfloor\alpha^*\rfloor} \\
&\le \exp\big(C\ln(n) + \ln(2)d^2k\lfloor\alpha^*\rfloor\big) \\
&\le \exp\Big(\frac{Ckn}{2^{(d-1)k}}\Big). \qquad (3.22)
\end{aligned}$$

choices for $\{x_\alpha\}_\alpha$.

Let us now bound the number of possibilities for $\{a_\alpha\}_\alpha$. We have that

$$\sum_{\alpha\le\lfloor\alpha^*\rfloor} a_\alpha \le n,$$

and that $a_\alpha$ is divisible by $2^{(d-1)k}$ for every $\alpha$. Let $\alpha \le \lfloor\alpha^*\rfloor$. There exists

$m_\alpha \in \mathbb{N}$ such that $a_\alpha = m_\alpha 2^{(d-1)k}$. Hence,

$$\begin{aligned}\#\Big\{\big(a_1,...,a_{\lfloor\alpha^*\rfloor}\big) \in \mathbb{Z}_+^{\lfloor\alpha^*\rfloor} : \sum_{\alpha\leq\lfloor\alpha^*\rfloor} a_\alpha \leq n\Big\} \\ &= \#\Big\{\big(a_1,...,a_{\lfloor\alpha^*\rfloor}\big) \in \mathbb{Z}_+^{\lfloor\alpha^*\rfloor} : \sum_{\alpha\leq\lfloor\alpha^*\rfloor} m_\alpha \leq \frac{n}{2^{(d-1)k}}\Big\} \\ &= \sum_{N=1}^{n2^{-(d-1)k}} \#\Big\{\big(a_1,...,a_{\lfloor\alpha^*\rfloor}\big) \in \mathbb{Z}_+^{\lfloor\alpha^*\rfloor} : \sum_{\alpha\leq\lfloor\alpha^*\rfloor} m_\alpha = N\Big\}.\end{aligned}$$

Similarly as in Equation (3.20), we get at most

$$\exp\left(\frac{Ckn}{2^{(d-1)k}}\right) \tag{3.23}$$

possibilities for $\{a_\alpha\}_\alpha$.

In summary, $|\mathcal{F}_{n,k}|$ is given by

$$\Gamma_k\big(\{x_\alpha, a_\alpha\}\big)\cdot\#\big\{\text{ choices for } \{x_\alpha\}_\alpha\big\}\cdot\#\big\{\text{ choices for } \{d_\alpha\}_\alpha\big\}\cdot\#\big\{\text{ choices for } \{a_\alpha\}_\alpha\big\},$$

and by Equations (3.23), (3.19), (3.22) and (3.20), it holds that

$$|\mathcal{F}_{n,k}| \leq \exp\left(\frac{Ckn}{2^{(d-1)k}}\right).$$

as desired.

#### Upper bounding of the bad event $\mathcal{E}^c$

Let us come back to Equation (3.9). At least half of the elements in the boxes belonging to the $2^m$-approximation of $A$ must also belong to $A$, thus there must be at most $\frac{|A|}{2^{dm-1}}$ boxes in the $2^m$-approximation of $A$. Using the isoperimetric inequality (3.8) yields

$$|A_m| \leq 2^{dm}\frac{|A|}{2^{dm-1}} = 2|A| \leq Cn^{\frac{d}{d-1}}.$$

Accordingly, Equation (3.9) becomes

$$\begin{aligned}\mathbb{P}\left[\sup_{A\in\Gamma_n}|\Delta_A(h)| \geq n\right] &\leq 2|\mathcal{F}_{n,m}|\exp\left(\frac{-\alpha_{m+1}^2}{8\varepsilon^2 Cn^{\frac{d}{d-1}}}\right) + \sum_{k=1}^m 2|\mathcal{F}_{n,k}||\mathcal{F}_{n,k-1}|\exp\left(\frac{-\alpha_k^2}{8\varepsilon^2 C2^k n}\right) \\ &\leq 2\exp\left(\frac{C_1 mn}{2^{m(d-1)}} - \frac{C_2\alpha_{m+1}^2 n^{\frac{d}{1-d}}}{\varepsilon^2}\right) + \sum_{k=1}^m 2\exp\left(\frac{C_3 n}{2^{k(d-1)}} - \frac{C_4\alpha_k^2}{\varepsilon^2 2^k n}\right)\end{aligned}$$

We choose $m \in \mathbb{N}$ such that $2^m = n^{\frac{1}{3}}$ and for all $k \leq m$, $\alpha_k := nk^{-2}$. The first term in the equation becomes

$$\begin{aligned}2\exp\left(\frac{C_1 mn}{2^{m(d-1)}} - \frac{C_2\alpha_{m+1}^2 n^{\frac{d}{1-d}}}{\varepsilon^2}\right) &\leq 2\exp\Big(C_1\ln(n)n^{\frac{4-d}{3}} - C_2 n^{\frac{d-2}{d-1}}\ln(n)^{-4}\varepsilon^{-2}\Big) \\ &\leq 2\exp\Big(C_1\ln(n)n^{\frac{1}{3}} - C_2 n^{\frac{d-2}{d-1}}\ln(n)^{-4}\varepsilon^{-2}\Big) \\ &\leq 2\exp\Big(-n^{\frac{d-2}{d-1}}\big(C_2\varepsilon^{-2}\ln(n)^{-4} - C_1 n^{\frac{5-2d}{3d-3}}\ln(n)^4\big)\Big)\end{aligned}$$

As $d \geq 3$, we have that $\frac{5-2d}{3d-3} < 0$, so for $n$ large, the term is of order

$$\exp\Big(-Cn^{\frac{d-2}{d-1}}\varepsilon^{-2}\Big) \leq \exp\Big(-Cn^{\frac{1}{2}}\varepsilon^{-2}\Big).$$

As for the second term in the equation, it holds that

$$\begin{aligned}
\sum_{k=1}^{m} 2\exp\left(\frac{C_3 n}{2^{k(d-1)}} - \frac{C_4\alpha_k^2}{\varepsilon^2 2^k n}\right) &\leq 2\sum_{k=1}^{m}\exp\left(\frac{C_3 n}{2^{k(d-1)}} - C_4 n^2 k^{-4}\varepsilon^{-2}2^{-k}\right)\\
&\leq 2m\exp\left(\frac{C_3 n 2^{d+1}}{2^{md}} - C_4 n^2 m^{-4}\varepsilon^{-2}2^{-m}\right)\\
&\leq 2m\exp\Big(C_3 n^{1-\frac{d}{3}}2^d - C_4 n^{\frac{2}{3}}\ln(n)^{-4}\varepsilon^{-2}\Big)
\end{aligned}$$

Once again, we have $1-\frac{d}{3} < 0$, so for $n$ large, the term is of order

$$\exp\big(-Cn^{\frac{2}{3}}\varepsilon^{-2}\big).$$

In summary, we get that

$$\mathbb{P}\left[\sup_{A\in\Gamma_n}|\Delta_A(h)| > n\right] \leq \exp\big(-Cn^{\frac{2}{3}}\varepsilon^{-2}\big).$$

It follows that

$$\begin{aligned}
\mathbb{P}[\mathcal{E}^c] &= \mathbb{P}\left[\bigcup_{n\in\mathbb{N}}\left\{\sup_{A\in\Gamma_n}|\Delta_A(h)| > n\right\}\right]\\
&\leq \sum_{n\in\mathbb{N}}\mathbb{P}\left[\sup_{A\in\Gamma_n}|\Delta_A(h)| > n\right]\\
&\leq \sum_{n\in\mathbb{N}}\exp\big(-Cn^{\frac{2}{3}}\varepsilon^{-2}\big) \leq e^{-C_1\varepsilon^{-2}}. \qquad (3.24)
\end{aligned}$$

### 3.3.5 Conclusion of the proof

Let us bound $\mathbb{Q}^+[\sigma_0 = -1]$. The event $\{\sigma_0 = -1\}$ can be split into two subevents, depending on whether $\mathcal{E}$ is happening or not.

$$\begin{aligned}
\mathbb{Q}^+[\sigma_0 = -1] &= \mathbb{Q}^+[\{\sigma_0 = -1\}\cap\mathcal{E}] + \mathbb{Q}^+[\{\sigma_0 = -1\}\cap\mathcal{E}^c]\\
&\leq \mathbb{Q}^+[\{\sigma_0 = -1\}\cap\mathcal{E}] + \mathbb{P}[\mathcal{E}^c].
\end{aligned}$$

According to Equation (3.5),

$$\begin{aligned}
\mathbb{Q}^+[\{\sigma_0 = -1\}\cap\mathcal{E}] &= \sum_{A\in\mathfrak{U}}\mathbb{Q}^+[\mathcal{E},\{\sigma_0 = -1\}\cap\mathcal{A}_0 = A]\\
&\leq \sum_{A\in\mathfrak{U}}\sup_{\mathcal{A}_0=A,h\in\mathcal{E}} e^{-2T^{-1}|\partial_{ex}A|}\frac{Z^+_{\Lambda_N}(h^A)}{Z^+_{\Lambda_N}(h)}\\
&\leq \sum_{A\in\mathfrak{U},A\in\Gamma_n}\sup_{\mathcal{A}_0=A,h\in\mathcal{E}} e^{-2T^{-1}|\partial_{ex}A|}\\
&\leq \sum_{n\geq 1}\sum_{A\in\mathfrak{U},A\in\Gamma_n} e^{-2nT^{-1}}.
\end{aligned}$$

Using Equation (3.7),

$$\begin{aligned}
\mathbb{Q}^+[\{\sigma_0 = -1\} \cap \mathcal{E}] &\leq \sum_{n\geq 1} |\Gamma_n| e^{-2nT^{-1}} \\
&\leq \sum_{n\geq 1} (2nd)^d (16d^3)^{2n} e^{-nT^{-1}} \\
&\leq \sum_{n\geq 1} \exp\left(\frac{n}{T}\Big(dT\log(2d) + 2T\log(16d^3) - 1\Big)\right).
\end{aligned}$$

Let us choose

$$T < \frac{1}{d\log(2d) + 2\log(16d^3)},$$

so that

$$dT\log(2d) + 2T\log(16d^3) - 1 < 0.$$

One obtains

$$\mathbb{Q}^+[\{\sigma_0 = -1\} \cap \mathcal{E}] \leq \sum_{n\geq 1} e^{-CnT^{-1}} \leq e^{-C_2T^{-1}}. \tag{3.25}$$

Putting together Equation (3.24) and Equation (3.25) and choosing a positive constant $C \leq \min\left\{\frac{C_1}{2}, \frac{C_2}{2}\right\}$, one gets

$$\mathbb{Q}^+[\sigma_0 = -1] \leq e^{-2CT^{-1}} + e^{-2C\varepsilon^{-2}}.$$

Ultimately, by using Markov's inequality and Equation (3.1),

$$\begin{aligned}
\mathbb{P}\Big[\mu^+[\sigma_0 = -1] \geq e^{-CT^{-1}} + e^{-C\varepsilon^{-2}}\Big] &\leq \frac{\mathbb{E}^{\mathbb{P}}\left[\mu^+[\sigma_0 = -1]\right]}{e^{-CT^{-1}} + e^{-C\varepsilon^{-2}}} \\
&\leq \frac{\mathbb{Q}^+[\sigma_0 = -1]}{e^{-CT^{-1}} + e^{-C\varepsilon^{-2}}} \\
&\leq \frac{e^{-2CT^{-1}} + e^{-2C\varepsilon^{-2}}}{e^{-CT^{-1}} + e^{-C\varepsilon^{-2}}} \leq e^{-CT^{-1}} + e^{-C\varepsilon^{-2}},
\end{aligned}$$

which yields exactly the desired result.

# Bibliography


[1] Lucas Affonso, Rodrigo Bissacot, and João Maia. Phase transitions in long-range random field Ising models in higher dimensions. 2023. URL https://www.researchgate.net/publication/372654476_Phase_Transition_for_Long-Range_Random_Field_Ising_Model_in_Higher_Dimensions.

[2] A. Bovier. *Statistical Mechanics of Disordered Systems: A Mathematical Perspective*, chapter 7. Cambridge University Press, 2006. doi: 10.1017/CBO9780511616808.009. URL https://www.cambridge.org/core/books/statistical-mechanics-of-disordered-systems/randomfield-ising-model/07A9340DA27A179CE6582003E860A1E1.

[3] J. Bricmont and A. Kupiainen. Phase transition in the 3d random field Ising model. *Communications in Mathematical Physics*, 116(4):539 – 572, 1988. doi: 10.1007/BF01224901. URL https://doi.org/10.1007/BF01224901.

[4] Stephen G. Brush. History of the Lenz-Ising model. *Rev. Mod. Phys.*, 39:883–893, Oct 1967. doi: 10.1103/RevModPhys.39.883. URL https://link.aps.org/doi/10.1103/RevModPhys.39.883.

[5] Reinhard Diestel. *Graph Theory*. Springer Berlin, Heidelberg, 2017. doi: 10.1007/978-3-662-53622-3. URL https://link.springer.com/book/10.1007/978-3-662-53622-3.

[6] J. Ding and Z. Zhuang. Long range order for random field Ising and Potts models. *Comm. Pure Appl. Math.*, 77:37–51, 2024. doi: 10.1002/cpa.22127. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/cpa.22127.

[7] Jian Ding, Yu Liu, and Aoteng Xia. Long range order for three-dimensional random field Ising model throughout the entire low temperature regime, 2022. URL https://arxiv.org/abs/2209.13998.

[8] D.S. Fisher, J. Fröhlich, and T. Spencer. The Ising model in a random magnetic field. *Journal of Statistical Physics*, 34:863–870, 1984. URL https://link.springer.com/article/10.1007/BF01009445.

[9] Robert B. Griffiths. Peierls proof of spontaneous magnetization in a two-dimensional Ising ferromagnet. *Phys. Rev.*, 136:A437–A439, Oct 1964.

doi: 10.1103/PhysRev.136.A437. URL https://link.aps.org/doi/10.1103/PhysRev.136.A437.

[10] G. Grimmett. *Percolation*. Springer, 1999. URL https://link.springer.com/book/10.1007/978-3-662-03981-6.

[11] John Z. Imbrie. The ground state of the three-dimensional random-field Ising model. *Communications in Mathematical Physics*, 98(2):145 – 176, 1985. doi: 10.1007/BF01220505. URL https://doi.org/10.1007/BF01220505.

[12] J.Z. Imbrie J. Fröhlich. Improved perturbation expansion for disordered systems: Beating Griffiths singularities. *Communications in Mathematical Physics*, (94):145 – 180, 1984. doi: 10.1007/BF01240218. URL https://doi.org/10.1007/BF01240218.

[13] R. Peierls. On Ising's model of ferromagnetism. *Mathematical Proceedings of the Cambridge Philosophical Society*, 32:477–481, 1936. doi: 10.1017/S0305004100019174. URL https://www.cambridge.org/core/journals/mathematical-proceedings-of-the-cambridge-philosophical-society/article/on-isings-model-of-ferromagnetism/C0584C5711BC3D25830B63A4C2F09609.

[14] V. Tassion. Ising model. Lectures notes. URL https://metaphor.ethz.ch/x/2021/hs/401-3822-17L/.

[15] R. van Handel. Probability in high dimension. Lectures notes. URL https://web.math.princeton.edu/~rvan/APC550.pdf.

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