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ASPECTS OF QUADRATIC UTILITY: MEAN–VARIANCE HEDGING IN ROUGH VOLATILITY MODELS, AND CAPM-TYPE EQUILIBRIA

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To Maria João, Eugénio and Andreea.

Abstract

The first main topic of this thesis, considered in Chapters I and II, is the study of the mean–variance hedging problem in the rough Heston model. Rough volatility models have become quite popular recently, but the question of hedging in such models is still underexplored. Previous work has focused on perfect hedging in a complete market and on approximate hedging under the risk-neutral measure. We use instead a mean–variance hedging approach under the historical measure, which is more natural for the purposes of risk management. Because the rough volatility process is neither Markovian nor a semimartingale, the rough Heston model poses difficulties to classical techniques in stochastic optimal control. By using the affine structure of the model, we obtain explicit formulas for the optimal mean–variance hedging strategies for a wide class of European-type payoffs, including vanilla call and put options, that can be implemented in practice. We then use those results to find optimal semistatic trading strategies in the underlying asset and a basket of derivatives.

The second part of the thesis, developed in Chapters III and IV, pertains to quadratic market equilibria in continuous time. Many classical results on the existence and uniqueness of Radner equilibria such as the capital asset pricing model (CAPM) require the assumption of a complete market. The study of equilibria in incomplete setups is more challenging due to the absence of Pareto optimality. We obtain an explicit equilibrium in an incomplete semimartingale setup with quadratic utilities by using the linearity properties of mean–variance hedging. We then extend our results to mean–variance preferences and find an explicit solution in the linear case. More generally, we show the stability of the mean–variance hedging problem with respect to the quadratic equilibrium price process by using a novel result on the stability of quadratic backward stochastic differential equations under a BMO condition on the stochastic driver and in a continuous filtration. This yields sufficient conditions for the existence of an equilibrium for general mean–variance utility functions via a fixed-point argument.

Kurzfassung

Das erste Hauptthema dieser Arbeit, das in den Kapiteln I und II behandelt wird, ist das Mean–Variance Hedging (MVH)-Problem im Rough Heston-Modell. Sogenannte Rough Volatility-Modelle sind in letzter Zeit populär geworden, aber das Thema des Hedgings in solchen Modellen ist noch unzureichend erforscht. Frühere Studien konzentrierten sich auf perfektes Hedging in einem vollständigen Markt und auf approximatives Hedging unter einem Martingalmass. Wir versuchen, das MVH-Problem unter einem Semimartingalmass zu lösen, was aus Sicht des Risikomanagements natürlicher ist. Da der Volatilitätsprozess weder Markovsch noch ein Semimartingal ist, können klassische Methoden der stochastischen Kontrolle nicht direkt zur Lösung des Problems angewendet werden. Dank der affinen Struktur des Rough Heston-Modells erhalten wir explizite Formeln für die optimalen Hedging-Strategien einer breiten Klasse europäischer Optionen, einschliesslich Call- und Put-Optionen, die in der Praxis umgesetzt werden können. Anschliessend nutzen wir diese Ergebnisse, um optimale semistatische Handelsstrategien für den Basiswert und einen Korb von Optionen zu finden.

Der zweite Teil der Arbeit, der in den Kapiteln III und IV entwickelt wird, befasst sich mit quadratischen Gleichgewichtsmodellen in stetiger Zeit. Viele klassische Ergebnisse zur Existenz und Eindeutigkeit von Radner-Gleichgewichten wie das Capital Asset Pricing Model (CAPM) erfordern die Annahme eines vollständigen Finanzmarktes. Aufgrund des Fehlens der Pareto-Optimalität ist die Untersuchung von Gleichgewichten im unvollständigen Fall schwieriger. Wir nutzen die Linearität des MVH-Problems, um ein explizites Gleichgewicht in einem unvollständigen Semimartingal-Markt mit quadratischen Nutzenfunktionen zu erhalten. Anschliessend erweitern wir unsere Ergebnisse auf Mean-Variance-Präferenzfunktionen und finden eine explizite Lösung im linearen Fall. Allgemeiner zeigen wir die Stabilität des MVH-Problems in Bezug auf den quadratischen Gleichgewichtspreisprozess. Dafür beweisen wir ein neues Ergebnis zur Stabilität von quadratischen stochastischen Rückwarts-Differentialgleichungen unter einer

BMO-Bedingung für den stochastischen Treiber und in einer stetigen Filtration. Zusammen mit einem Fixpunkt-Argument liefert dies hinreichende Bedingungen für die Existenz eines Gleichgewichts mit allgemeinen Mean-Variance-Präferenzfunktionen.

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Contents

Chapter 0

Introduction

Optimal portfolio selection and the pricing and hedging of contingent claims are two of the central problems in mathematical finance. Simply put, portfolio selection consists of choosing a portfolio with an optimal risk-reward tradeoff, whereas for pricing and hedging a claim H, one looks for a portfolio of underlying assets that replicates or at least approximates H. In mathematical terms, consider a financial market with time horizon $T > 0$ consisting of a riskless asset with constant price 1 and a risky asset with semimartingale price process $S = (S_t)_{0 \leq t \leq T}$. By self-financing trading with initial wealth $x \in \mathbb{R}$ and a strategy $\vartheta = (\vartheta_t)_{0 \leq t \leq T}$ from a set Θ, an agent achieves the final wealth

$$V_T(x, \vartheta) = x + \int_0^T \vartheta_t dS_t.$$

In the *optimal portfolio selection problem*, the agent wants for a given x to maximise profits and minimise risk. Under preferences described by a utility function U on $\mathbb{R}$, the goal is thus to

$$\text{maximise } E\big[U\big(V_T(x, \vartheta)\big)\big] \text{ over all } \vartheta \in \Theta. \tag{0.1}$$

On the other hand, the agent might want to buy or sell a contingent claim with a random payoff H and hedge it by trading in the underlying assets. In a complete market, the *pricing and hedging problem* for H consists of finding $x^H \in \mathbb{R}$ and $\vartheta^H \in \Theta$ such that $H = V_T(x^H, \vartheta^H)$. In that case, x^H is the unique arbitrage-free price for H and ϑ^H the corresponding replicating strategy. But if the financial market is incomplete, such a representation does not exist in general. One can consider instead an approximate version of the problem where the agent looks to

$$\text{minimise } E\big[\ell\big(H - V_T(x,\vartheta)\big)\big] \text{ over all } x \in \mathbb{R},\ \vartheta \in \Theta, \tag{0.2}$$

where ℓ is a loss function on $\mathbb{R}$. Depending on the application, x may be part of the control as in (0.2), or fixed a priori so that the minimisation is only over $\vartheta \in \Theta$.

In much of the mathematical finance literature, the problems (0.1) and (0.2) are studied separately, often with different techniques and different sets of assumptions. The present thesis focuses instead on a setting where both U and ℓ are quadratic. One of the most appealing features of the resulting quadratic problems is that they are closely related to each other and can be solved within the same framework. To see why, consider an agent with quadratic utility $U(x) = x - \frac{1}{2\gamma}x^2$ for some $\gamma > 0$. The portfolio selection problem is then to

$$\text{maximise } E\left[V_T(x,\vartheta) - \frac{1}{2\gamma}V_T^2(x,\vartheta)\right] \text{ over all } \vartheta \in \Theta,$$

and completing the square yields the equivalent problem to

$$\text{minimise } E\big[\big(\gamma - V_T(x,\vartheta)\big)^2\big] \text{ over all } \vartheta \in \Theta.$$

The latter is a hedging problem of the form (0.2) with a constant payoff $H \equiv \gamma$ and the loss function $\ell(y) = y^2$. Thus we have converted the original portfolio selection problem into a *mean–variance hedging* (MVH) problem, that is, a problem of the form (0.2) with a quadratic ℓ. Conversely, a MVH problem can be rewritten as a portfolio selection problem with quadratic utility, where the agent receives the random endowment $\gamma - H$ at time T in addition to the initial wealth x.

The quadratic utility and mean–variance hedging or portfolio selection approaches have some well-known shortcomings; these include the existence of a "bliss point" where maximal utility is attained, and the inability to take into account higher-order moments and heavy-tailed distributions. Nevertheless, these approaches are widely used in practice because of their tractability, which sometimes yields simple and explicit solutions where other approaches fail to do so.

This doctoral thesis comprises two main topics that showcase applications of the quadratic utility and MVH approaches. The first is presented in Chapters I and II, which are based on joint work with Christoph Czichowsky. They are devoted to the study of the mean–variance hedging problem in the so-called rough

Heston model. We obtain for a wide class of European-type payoffs explicit solutions to the MVH problem that can be implemented in practice. The second topic is studied in Chapters III and IV, which are based on joint work with Christoph Czichowsky and Martin Herdegen. We first find an explicit formula for the Radner market equilibrium of agents with quadratic utility in an incomplete market in continuous time. This analysis is then extended to show the existence of an equilibrium under generalised mean–variance preferences. Finally, Appendix A provides some technical results on the existence of solutions to nonlinear Volterra equations that are used in Chapters I and II.

1 Quadratic hedging in the rough Heston model

In the first part of the thesis, we study mean–variance hedging in a specific model with a rough stochastic volatility process. Rough volatility models have become quite popular recently, as they capture both the fractional scaling of the time series of historic volatility (Gatheral et al. [55]) and the asymptotics of the implied volatility surface (Bayer et al. [15], Fukasawa [54]) remarkably well. In these models, S has a volatility process Y that satisfies a stochastic Volterra equation with a fractional kernel that is singular at 0. This leads to a process Y that has rougher paths (i.e., paths of lower Hölder regularity) than Brownian motion, and is no longer a Markov process nor a semimartingale. Hence these models fall outside of the scope of standard techniques in stochastic analysis and present new mathematical challenges.

The problem of pricing under rough volatility has been considered in Bayer et al. [15, 16], and the hedging problem for a complete market with continuous trading in forward variance swaps has been studied in El Euch/Rosenbaum [45]. However, as pointed out in Bayer et al. [15], forward variance swaps are often illiquid and costly to trade. If only the underlying asset S can be traded, the market is incomplete, and the problem of hedging in that setting has so far only been studied in Horvath et al. [66, 67] under a risk-neutral measure. From the point of view of risk management, it is more natural to work under the historical measure P, especially since the risk-neutral measure is not unique. We thus use an approach based on mean–variance hedging (see Schweizer [111] for a recent overview), where we study (0.1) for $\ell(y) = y^2$ and hence we minimise the expected squared difference under P between the claim H and the final wealth $V_T(x, \vartheta)$ of the hedging strategy.

We focus on the rough Heston model of El Euch/Rosenbaum [45] which generalises the classical Heston model by including a fractional kernel in the dynamics of the volatility process Y. Like the classical Heston model, this rough model is affine in the sense that the characteristic function of the log-price admits an exponentially affine representation. However, since Y is not Markovian, it is replaced as a state variable by the infinite-dimensional so-called forward variance curve, and the associated coefficient in the above representation no longer solves a Riccati ODE, but rather a Riccati–Volterra equation. Thus the volatility in the rough Heston model is a so-called affine Volterra process (Abi Jaber et al. [1]). This affine structure preserves the tractability of the classical Heston model and allows the pricing of European options by Fourier transform techniques as shown in [45] for the case of a complete market. We show how to combine these techniques with the mean–variance hedging approach to obtain similar results in the case of an incomplete market.

Our main results provide semi-explicit formulas for the mean–variance hedging strategies for a wide range of European-type payoffs, including vanilla call and put options. The formulas are given in terms of the underlying S and the forward variance curve, with deterministic coefficients that solve one-dimensional Riccati–Volterra equations. These equations do not admit closed-form solutions but can be integrated numerically, so that our formulas are implementable in practice. In order to solve the mean–variance hedging problem and obtain these formulas, we follow the approach in Černý/Kallsen [25] and proceed in two steps.

The first step is the subject of Chapter I and consists of solving the so-called pure investment problem, i.e., (0.2) for $\ell(y) = y^2$, $x = 0$ and $H \equiv 1$. This is equivalent to Markowitz portfolio selection (see Fontana/Schweizer [48]), which has also been recently studied for the rough Heston model in Abi Jaber et al. [2] and Han/Wong [61]. We show an alternative way to solve this via a martingale distortion technique as in Fouque/Hu [49, 50] and give a formula for a generalised moment-generating function in the rough Heston model. This yields formulas for the solution to the pure investment problem and the so-called variance-optimal martingale measure (VOMM) $Q^\star$.

The second step, which is done in Chapter II, is to find for a claim H the mean value process defined by $V_t^H = E_{Q^\star}[H \,|\, \mathcal{F}_t]$ and the pure hedging coefficient Ξ^H which appears as the integrand in the Galtchouk–Kunita–Watanabe decomposition of V^H under $Q^\star$. Because the rough Heston model retains its affine structure under $Q^\star$ (even though its dynamics are now time-inhomogeneous), we are able to obtain explicit formulas for the moment-generating function of the log-price under $Q^\star$. By linearity, this yields formulas for V^H and Ξ^H for payoffs H that can be representated as Mellin transforms, and the mean–variance hedging strategies are then determined via a feedback equation as in Černý/Kallsen [25].

2 Mean–variance equilibrium in continuous time

The second part of the thesis studies equilibria for mean–variance preferences in continuous time and for general semimartingale models in an incomplete setting. The study of market equilibria is a key area of research in economic and mathematical finance theory. The main idea is to model market prices indirectly as outcomes of the interaction between market participants and other exogenous factors. Since every transaction of an asset involves a buyer and a seller, the total number of units bought by some market participants is always equal to the total number of units sold by other participants. This is the so-called *market clearing condition*. In turn, the trading decisions of agents are determined by their own preferences, the market price, and their views about other participants and exogenous economic factors. Thus in an equilibrium, there is a feedback loop between prices, which influence the individual strategies, and agents, which set prices via the market clearing condition. The study of equilibria yields insights into *why* markets display certain behaviours, and can be used to extrapolate the effects of regulatory, fiscal or other large-scale structural changes on financial markets.

Here we work with the well-known concept of a Radner [103] equilibrium. We consider a financial market consisting of one riskless asset with constant price 1 and d risky assets with a d-dimensional semimartingale price process $S = (S_t)_{0\le t\le T}$. Each agent $k = 1, \dots, K$ has initial wealth $x_k \in \mathbb{R}$ and receives a random endowment Ξ^k at time T. Via self-financing trading in $(1, S)$ with a portfolio $\vartheta^k = (\vartheta^k_t)_{0\le t\le T}$, agent k achieves the final wealth

$$V^k_T(\vartheta^k) = x_k + \int_0^T \vartheta^k_t dS_t + \Xi^k.$$

Each agent k has preferences $\mathcal{U}_k$ on $L^0(\mathcal{F}_T)$ and seeks to

$$\text{maximise } \mathcal{U}_k\big(V^k_T(\vartheta^k)\big) \text{ over all } \vartheta^k.$$

Then $(1, S)$ is an equilibrium market if each agent has a unique optimal strategy $\hat\vartheta^k$ and the market clears, i.e., $\sum_{k=1}^K \hat\vartheta^k \equiv 0$. In addition to the market clearing condition, we impose exogenous constraints on the assets, which we divide into *productive* and *financial assets* as in Karatzas et al. [76]. Our goal is then to look for equilibrium markets $(1, S)$ satisfying those constraints.

In Chapter III, we consider preferences given by expected quadratic utility,

$$\mathcal{U}_k(V) = E\left[V - \frac{1}{2\gamma_k}V^2\right], \tag{2.1}$$

for some risk tolerance parameter $\gamma_k > 0$. In Chapter IV, we consider more general mean–variance preferences of the form

$$\mathcal{U}_k(V) = U_k\big(E[V], \sqrt{\mathrm{Var}[V]}\big) \tag{2.2}$$

for a mean–variance utility function U_k on $\mathbb{R} \times \mathbb{R}_+$. If the market is complete, it is well known that the capital asset pricing model (CAPM) gives the equilibrium prices in both cases. That classical result is in line with much of the research on Radner equilibria (see Karatzas/Shreve [78, notes to Chapter IV] for an overview), where it is assumed that the market is complete, or at least that the endowments of the agents can be perfectly hedged.

However, as argued in Kardaras et al. [79], the completeness assumption is not always justified, and it is relevant to study the behaviour of equilibrium markets in which the endowments cannot be perfectly hedged. The main challenge is that equilibria need not to satisfy Pareto optimality in an incomplete market, and so the well-known method of using a representative agent does not work. Despite this difficulty, some positive results on incomplete market equilibria have been obtained in Basak/Cuoco [14], Cheridito et al. [28], Guasoni/Weber [60], Kardaras et al. [79], Koch-Medina/Wenzelburger [85] and Žitković [122], among others. Most relevant for us is [85], where a CAPM equilibrium is found for a discrete-time incomplete market in one period with general mean–variance preferences.

Our results in Chapters III and IV extend to continuous time the existence results of [85] for incomplete market equilibria. So far, results on incomplete market equilibria in continuous time have only been obtained under specific filtrations, such as Brownian ones. In contrast, our results in Chapter III allow general semimartingale price processes with jumps, while the main results in Chapter IV assume only a continuous filtration. To the best of our knowledge, we thus give the first results on equilibria in incomplete markets with general semimartingale dynamics in continuous time.

In Chapter III, we find an explicit formula for the equilibrium under quadratic utilities (2.1). This is done by exploiting the fact that (as seen in the introduction) the individual optimisation problems of the agents are equivalent to certain mean–variance hedging (MVH) problems. The linearity of MVH allows us to deduce that the aggregate demand for the risky assets is equal to the optimal strategy for a nonstandard *representative agent*, i.e., a fictional agent that aggregates the endowments and preferences of the individual agents. We use this insight to obtain a pricing measure for the equilibrium prices, which then yields an explicit formula for the latter.

The next step, which is the subject of Chapter IV, is to extend those results to the case of mean–variance preferences (2.2) by relating them to quadratic utilities (2.1). We show that any equilibrium for mean–variance preferences is also an equilibrium for quadratic utility. At first glance, it may appear that this directly gives the existence of an equilibrium; but it is merely the starting point because the relationship between the quadratic utility and mean–variance preference problems (2.1) and (2.2) depends implicitly on the price process S, which is not known a priori. Surprisingly, in the case of linear mean–variance preferences, we are nevertheless able to obtain an explicit formula for the equilibrium. In the general case, we prove that for some choice of risk tolerance parameters $\gamma_1, \dots, \gamma_K$, the equilibrium market with respect to the quadratic utilities (2.1) is also an equilibrium market for the original mean–variance preferences. We then obtain the existence of an equilibrium via a fixed-point argument on $(\gamma_k)_{k=1}^K$. Because we work in continuous time, this is technically challenging and involves studying the dependence on their parameters of a class of BSDEs.

Chapter I

The pure investment problem for the rough Heston model

1 Introduction

Our goal in Chapters I and II is to solve the mean–variance hedging (MVH) problem for the rough Heston model, as explained in the global introduction from Chapter 0. The present chapter thus plays a supporting role to Chapter II by introducing and defining the rough Heston model, as well as proving several auxiliary results that are needed to tackle the MVH problem. The main subject of this chapter is the so-called pure investment problem of Černý/Kallsen [25], defined as the MVH problem for $x = 0$ and the constant payoff $H \equiv 1$. Writing $\vartheta \bullet S = \int \vartheta dS$ for brevity, one thus looks to

$$\text{minimise } E[(1 - \vartheta \bullet S_T)^2] \text{ over all } \vartheta \in \overline{\Theta}_T(S),$$

where the price process $S = (S_t)_{0 \le t \le T}$ satisfies the rough Heston model and $\overline{\Theta}_T(S)$ is the set of L^2-admissible strategies on $[0, T]$ (see Section 3.1 for the definition). It is well known that the solution $\vartheta^\star$ determines the set of mean–variance efficient portfolios, and hence this problem is equivalent to classical Markowitz portfolio selection; see Fontana/Schweizer [48]. In this chapter, we obtain an explicit solution to the pure investment problem in the rough Heston model. We also find the solution $Q^\star$ to the dual problem, which is to

$$\text{minimise } E[(dQ/dP)^2] \text{ over all } Q \in \mathbb{Q}^2_T(S),$$

where $\mathbb{Q}^2_T(S)$ is the set of equivalent martingale measures Q for S on $[0,T]$ such that $E[(dQ/dP)^2] < \infty$. The measure $Q^\star$ is the so-called variance-optimal martingale measure (VOMM) for S on $[0,T]$, and it plays a key role in solving the MVH problem (Schweizer [110]).

As in Černý/Kallsen [25] and Czichowsky/Schweizer [32], we solve the pure investment problem by determining the opportunity process $L = (L_t)_{0\le t\le T}$, which is the reduced-form value process given by

$$L_t = \operatorname{ess\,inf}_{\vartheta\in\overline{\Theta}_{t,T}(S)} E[(1-\vartheta\bullet S_T)^2 \mid \mathcal{F}_t], \quad 0\le t\le T,$$

where $\overline{\Theta}_{t,T}(S)$ is the set of strategies $\vartheta \in \overline{\Theta}_T(S)$ such that $\vartheta\mathbf{1}_{[\![0,t]\!]} = 0$. It is well known (see e.g. Bobrovnytska/Schweizer [21]) that L satisfies a backward stochastic differential equation (BSDE) with a driver determined by the dynamics of S. Thus our approach is to find a solution $\hat{L}$ to that BSDE via a martingale distortion technique as in Fouque/Hu [49, 50] and then use a verification result of Černý/Kallsen [25] to show that $\hat{L} = L$ is the true opportunity process. The dynamics of L then yield the solution to the pure investment problem as well as a formula for the variance-optimal martingale measure $Q^\star$.

In preparation for the study of the pure investment problem, we also prove a useful result on a generalised moment-generating function in the rough Heston model. This generalisation of El Euch/Rosenbaum [46, Theorem 4.1] allows us to include changes of measure that preserve the affine structure of the model. We use this result throughout this Chapters I and II, where it yields formulas for conditional expectations under $Q^\star$. We note, however, that the moment-generating function can be well defined not for all time horizons, but only up to a positive time. The same therefore applies to our subsequent results on the solution to the pure investment problem. This issue cannot be avoided, since moment explosion is inherent to the rough Heston and other stochastic volatility models (Andersen/Piterbarg [10], Gerhold et al. [57], Keller-Ressel [82]).

This chapter, based on joint work with Christoph Czichowsky, is structured as follows. We introduce in Section 2.1 the rough Heston model and its basic properties. In Section 2.2, we prove the formula for the generalised moment-generating function. Section 3 is then dedicated to solving the pure investment problem for the rough Heston model. After introducing the problem in Section 3.1, we find in Sections 3.2 and 3.3 a martingale distortion representation and an explicit formula, respectively, for the candidate opportunity process $\hat{L}$. In Section 3.4, we prove our main result where we verify that the candidate opportunity process is the true one, i.e., L. We also obtain a formula for the variance-optimal martingale measure. Finally, in Section 3.5, we compare our results to the closely related ones in Černý/Kallsen [27], Abi Jaber et al. [2] and Han/Wong [61].

2 The rough Heston model

2.1 Definition and first properties

We begin by recalling the classical Heston model with drift. We consider a frictionless financial market with a finite time horizon $T \in (0,\infty)$, consisting of one riskless asset with constant price equal to 1 and one risky asset. The price process $(S_t)_{0\leq t\leq T}$ of the risky asset is described by the stochastic differential equations

$$\begin{cases} \dfrac{dS_t}{S_t} = \mu Y_t dt + \sigma\sqrt{Y_t}dW_t, \\ dY_t = \lambda(\theta - Y_t)dt + \zeta\sqrt{Y_t}dB_t \end{cases} \tag{2.1}$$

for $t \in [0,T]$, where W and B are Brownian motions with constant correlation $\varrho \in (-1,1)$, and the parameters $S_0 > 0$, $Y_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, $\lambda > 0$, $\theta > 0$, $\zeta > 0$ and $\varrho \in (-1,1)$ are fixed constants. The parameters μ and σ are related to the rate of return and volatility of S, whereas θ is the long-term mean volatility, λ is the speed of mean reversion of Y, ζ is related to the volatility of Y ("vol of vol"), and ϱ describes the instantaneous correlation between S and Y.

The rough Heston model, introduced in El Euch/Rosenbaum [46], is a generalisation of the classical Heston model where the stochastic volatility process Y is replaced by a process that is rougher than Brownian motion. Rough volatility models have become quite popular recently, as they capture both the fractional scaling of the time series of the historic volatility (Gatheral et al. [55]) and the implied volatility surface (Fukasawa [54], Bayer et al. [15]) remarkably well. One can model a rough volatility process Y by introducing a convolution kernel with a singularity at 0. This is analogous to the construction of the Riemann–Liouville fractional Brownian motion, defined for a Hurst parameter $h \in (0,1)$ as

$$W_t^{(h)} = \int_0^t \frac{(t-s)^{h-\frac{1}{2}}}{\Gamma(h+\frac{1}{2})} dW_s,$$

where W is a Brownian motion and $\Gamma : (0,\infty) \to \mathbb{R}_+$ is the well-known gamma function. The resulting process is then Hölder-continuous of any order strictly smaller than h, and thus rougher than Brownian motion in the case $h < \frac{1}{2}$.

In order to properly define the rough Heston model, we first give some definitions related to the convolution operation.

Notation 2.1. Consider a Borel subset $E \subseteq [0, \infty)$, some constant $p \in [1, \infty)$ and let $F = \mathbb{R}^n$ or $\mathbb{C}^n$ for some $n \in \mathbb{N}$. We say that a Borel-measurable function $f : E \to F$ belongs to $L^p(E; F)$ if $\int_E |f(t)|^p dt < \infty$, and it belongs to $L^p_{\mathrm{loc}}(E; F)$ if $\int_{E \cap [0,T]} |f(t)|^p dt < \infty$ for all $T > 0$. We say that a map $\nu : \bigcup_{T \geq 0} \mathcal{B}([0, T]) \to \mathbb{R}$ is a *σ-finite signed local measure on* $[0, \infty)$ if its restriction to $\mathcal{B}([0, T])$ is a finite signed measure for each $T > 0$. We say that $\nu : \mathcal{B}([0, T]) \to \mathbb{C}$ is a *finite complex measure* on $[0, T]$ if $\nu = \nu_1 + \mathrm{i}\nu_2$ for two finite signed measures ν_1 and ν_2 on $[0, T]$. Similarly, the map $\nu : \bigcup_{T \geq 0} \mathcal{B}([0, T]) \to \mathbb{C}$ is a *σ-finite complex local measure on* $[0, \infty)$ if $\nu = \nu_1 + \mathrm{i}\nu_2$ for two σ-finite signed local measures ν_1 and ν_2.

A σ-finite signed local measure on $\mathbb{R}_+$ can be thought of as the difference of two σ-finite measures on $\mathbb{R}_+$, in the following sense.

Lemma 2.2. *A map $\nu : \bigcup_{T \geq 0} \mathcal{B}([0, T]) \to \mathbb{R}$ is a σ-finite signed local measure if and only if there exist mutually singular σ-finite measures ν_+, ν_- on $([0, \infty), \mathcal{B}([0, \infty)))$ such that*

$$\nu(B) = \nu_+(B) - \nu_-(B) \tag{2.2}$$

for each $B \in \bigcup_{T \geq 0} \mathcal{B}([0, T])$.

Proof. We start with the "only if" statement. As $\nu|_{[0,T]}$ is a finite signed measure for any $T \geq 0$, we have by the Jordan decomposition theorem (Klenke [83, Corollary 7.44]) that there exist unique mutually singular finite measures ν_+^T and ν_-^T on $([0, T], \mathcal{B}([0, T]))$ such that $\nu(B) = \nu_+^T(B) - \nu_-^T(B)$ for each $B \in \mathcal{B}([0, T])$. By the uniqueness of the construction, it is clear that $\nu_\pm^{T'}|_{\mathcal{B}([0,T])} = \nu_\pm^T$ for $T' \geq T \geq 0$, i.e., the measures $(\nu_\pm^T)_{T \geq 0}$ are consistent. We can thus construct functions $\nu_\pm : \bigcup_{T \geq 0} \mathcal{B}([0, T]) \to \mathbb{R}_+$ by $\nu_\pm(B) = \nu_\pm^T(B)$ for each $B \in \mathcal{B}([0, T])$; this is well defined by the consistency of the measures $(\nu_\pm^T)_{T \geq 0}$.

Now let $A = \bigcup_{m \in \mathbb{N}} A_m \in \bigcup_{T \geq 0} \mathcal{B}([0, T])$ for some family of disjoint sets $(A_m)_{m \in \mathbb{N}}$ in $\bigcup_{T \geq 0} \mathcal{B}([0, T])$. There exists some $T \geq 0$ such that $[0, T] \supseteq A \supseteq A_m$ for each $m \in \mathbb{N}$, and hence

$$\nu_\pm(A) = \nu_\pm^T(A) = \sum_{m \in \mathbb{N}} \nu_\pm^T(A_m) = \sum_{m \in \mathbb{N}} \nu_\pm(A_m).$$

Thus we have shown that $\nu_\pm$ is a pre-measure on the ring $\bigcup_{T\geq 0}\mathcal{B}([0,T])$. As $\nu_\pm([0,T]) = \nu_\pm^T([0,T]) < \infty$ for each $T \geq 0$, $\nu_\pm$ is σ-finite. By Carathéodory's extension theorem, $\nu_\pm$ can thus be uniquely extended to a σ-finite measure on $([0,\infty),\mathcal{B}([0,\infty)))$.

As ν_+^T and ν_-^T are mutually singular for each $T = n \in \mathbb{N}$, there exist disjoint sets $A_+^n, A_-^n \in \mathcal{B}([0,n])$ such that $\nu_\pm^n$ is supported on $A_\pm^n$. By the consistency of the measures $(\nu_\pm^n)_{n\in\mathbb{N}}$, we may assume without loss of generality that $A_\pm^{n+1} \supseteq A_\pm^n$ for each $n \in \mathbb{N}$. If we define $A_\pm := \bigcup_{n\in\mathbb{N}} A_\pm^n$, then A_+ and A_- are disjoint, since

$$A_+ \cap A_- = \bigcup_{n\in\mathbb{N}} (A_+ \cap A_- \cap [0,n]) = \bigcup_{n\in\mathbb{N}} (A_+^n \cap A_-^n) = \emptyset.$$

We have $\nu_\pm(B) = \nu_\pm(A_\pm^n \cap B) = \nu_\pm(A_\pm \cap B)$ for any $B \in \mathcal{B}([0,n])$ so that the measures $\nu_\pm$ and $B \mapsto \int_B \mathbf{1}_{A_\pm} d\nu_\pm$ coincide on the ring $\bigcup_{T\geq 0} \mathcal{B}([0,T])$, and hence they are equal by the uniqueness of the Carathéodory extension. This implies that $\nu_\pm$ is supported on $A_\pm$, so that ν_+ and ν_- are mutually singular. Moreover, for each $B \in \bigcup_{T\geq 0} \mathcal{B}([0,T])$, we have

$$\nu(B) = \nu|_{\mathcal{B}([0,T])}(B) = \nu_+^T(B) - \nu_-^T(B) = \nu_+(B) - \nu_-(B),$$

and this shows (2.2).

For the "if" statement, let ν_+ and ν_- be σ-finite measures on $([0,\infty),\mathcal{B}([0,\infty)))$. The equation (2.2) defines a unique map $\nu : \bigcup_{T\geq 0} \mathcal{B}([0,T]) \to \mathbb{R}$, and we have

$$\nu|_{\mathcal{B}([0,T])} = \nu_+|_{\mathcal{B}([0,T])} - \nu_-|_{\mathcal{B}([0,T])}$$

for each $T \geq 0$. As ν_+ and ν_- are σ-finite, this implies that $\nu|_{\mathcal{B}([0,T])}$ is a finite signed measure on $[0,T]$ for each $T \geq 0$, and thus ν is a σ-finite signed local measure. □

As a motivation for this notion of a σ-finite signed local measure, consider the fact that any nonnegative function $f \in L^1_{\text{loc}}([0,\infty);\mathbb{R})$ induces a σ-finite measure $A \mapsto \int_A f(s)ds$ on $([0,\infty),\mathcal{B}([0,\infty)))$, but this need not yield a well-defined signed measure if f is allowed to take both positive and negative values. This is not an issue on each bounded interval $[0,T]$, however, so that $A \mapsto \int_A f(s)ds$ can be seen as a σ-finite signed local measure. One may extend this map to $[0,\infty)$ by $A \mapsto \int_A f^+(s)ds - \int_A f^-(s)ds$ for any set $A \in \mathcal{B}([0,\infty))$ such that this is well defined, i.e., if at least one of the integrals is finite, which is the case for any bounded set A.

We next give the definition of the convolution operation, which plays a key role in rough volatility models such as the rough Heston model. In the following (as well as in Appendix A), we use the textbook by Gripenberg et al. [59] as the main reference on the topic of convolutions and convolution equations.

Definition 2.3. For any pair of functions $f, g \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{C})$, we define the *convolution* $f * g \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{C})$ by

$$(f * g)(t) := \int_0^t f(t-s)g(s)ds = \int_0^t f(s)g(t-s)ds, \quad t \geq 0.$$

The fact that $f * g$ is well defined and belongs to $L^1_{\mathrm{loc}}([0,\infty);\mathbb{C})$ follows from [59, Theorem 2.2.2(i)]. It is clear that the convolution has a *causality property*, i.e., the restriction of $f * g$ to $[0,t]$ only depends on the restrictions of f and g to $[0,t]$ for each $t \geq 0$. Some additional properties of the convolution are listed in [59, Section 2.2]. For instance, if either f or g is a continuous function on $[0,\infty)$, then so is $f * g$, i.e., there exists a continuous representative of $f * g \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{C})$, and it moreover satisfies $(f * g)(0) = 0$. We always choose such a continuous representative for $f * g$ if it exists, although we note that such a representative need not satisfy $(f * g)(0) = 0$ in general. We usually consider functions f and g that are continuous on $(0,\infty)$ but may have an integrable singularity at 0; in that case, there likewise exists a version of $f * g$ that is continuous on $(0,\infty)$ with possibly an integrable singularity at 0. Another notable property is the well-known Young convolution inequality, given in Lemma A.1.5 in the Appendix.

Definition 2.4. We say that $R^k \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{C})$ is the *resolvent of the second kind* (or simply *resolvent*) of a function $k \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{C})$ if it holds that

$$(k * R^k)(t) = k(t) - R^k(t), \quad \text{for Lebesgue-a.a. } t \geq 0.$$

A σ-finite complex local measure L^k on $[0,\infty)$ is the *resolvent of the first kind* of k if it holds that

$$(k * L^k)(t) := \int_{[0,t]} k(t-s)L^k(ds) = 1, \quad \text{for Lebesgue-a.a. } t \geq 0.$$

Any $k \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{C})$ has a resolvent of the second kind, by [59, Theorem 2.3.1]. On the other hand, a resolvent of the first kind does not always exist. Both kinds of resolvents R^k and L^k for k are unique if they exist, by [59, Theorems 2.3.1 and 5.5.2], and they are real-valued if k is. We also note that if k is locally square-integrable, then this property is inherited by R^k, which follows by applying [59, Theorem 2.3.5] in the special case $f = k$ there. Some other properties such as continuity and local boundedness can also be inherited by R^k from k, by the same argument.

Remark 2.5. The resolvents of the first and second kind allow us to solve linear Volterra equations of the first and second kind, respectively. Given functions $f, k \in L^1_{\text{loc}}([0,\infty);\mathbb{C})$, the Volterra equation of the first kind

$$(k * x)(t) = f(t), \quad t \geq 0, \tag{2.3}$$

has the solution $x = \frac{d}{dt}(L^k * f)$, if the resolvent of the first kind L^k exists. Likewise, the Volterra equation of the second kind

$$x(t) + (k * x)(t) = f(t), \quad t \geq 0, \tag{2.4}$$

has the solution $x = f - R^k * f$; see [59, Theorems 2.3.5 and 5.5.3] for references. In such a context, we refer to the function $k \in L^1_{\text{loc}}([0,\infty);\mathbb{C})$ as a *kernel*. This does not impose any additional conditions on k, but rather is a term used in the literature to describe the role of k in convolution equations such as (2.3) and (2.4). Likewise, we refer to the functions κ and $\hat{\kappa}$ (that we define later) as kernels to emphasise their role in the rough Heston model.

Remark 2.6. The connections between the convolution operation, the resolvents and the Laplace transform are also noteworthy; see [59, Theorem 2.2.7 and Sections 2.3 and 5.5] for references. For $f \in L^1_{\text{loc}}([0,\infty);\mathbb{C})$, define the Laplace transform

$$\hat{f}(\rho) := \int_0^\infty e^{-\rho s} f(s) ds$$

for any $\rho \in [0,\infty)$ such that the integral exists. Then for each $f, g \in L^1_{\text{loc}}([0,\infty);\mathbb{C})$ and $\rho \in [0,\infty)$ such that $\hat{f}(\rho)$ and $\hat{g}(\rho)$ exist, we have that $\widehat{f * g}(\rho)$ exists and is given by $\hat{f}(\rho)\hat{g}(\rho)$. Moreover, suppose that $k \in L^1_{\text{loc}}([0,\infty);\mathbb{C})$ admits the resolvents L^k and R^k of the first and second kind, respectively. Then we have

$$\widehat{L^k}(\rho) = \frac{1}{\rho \hat{k}(\rho)} \quad \text{and} \quad \widehat{R^k}(\rho) = \frac{\hat{k}(\rho)}{1 + \hat{k}(\rho)} \tag{2.5}$$

for any ρ such that the Laplace transforms exist, where in the definition of $\widehat{L^k}$, we replace the integrand $f(s)ds$ with $L^k(ds)$. We shall use the formula for $\widehat{R^k}$ below in the proof of Lemma 2.12.

The rough Heston model differs from the classical Heston model due to the introduction of a convolution kernel, which we denote by κ, in the definition of the volatility process Y. We impose some assumptions on the kernel $\kappa : (0,\infty) \to \mathbb{R}_+$, the first three of which ensure the wellposedness of the rough Heston model, while the fourth is helpful for obtaining some additional properties, such as Lemma 2.12. We note that κ may (and often does, as in the examples below) have a singularity at 0, where $\lim_{t\searrow 0} \kappa(t) = \infty$. However, such a singularity must be sufficiently integrable, as specified in the following assumptions.

Assumption 2.7. We assume that the kernel $\kappa \in L^2_{\mathrm{loc}}([0,\infty);\mathbb{R}_+)$ satisfies each of the conditions 1)–4).

1) There exists some $\gamma \in (0,2]$ such that $\int_0^T (\kappa(t+h) - \kappa(t))^2 dt = O(h^\gamma)$ and $\int_0^h \kappa(t)^2 dt = O(h^\gamma)$ for every $T \in (0,\infty)$ and small $h > 0$.

2) The function $\kappa : (0,\infty) \to \mathbb{R}_+$ is not identically zero, nonincreasing and continuous on $(0,\infty)$.

3) κ admits a resolvent of the first kind L^κ that is a nonnegative and nonincreasing measure on $[0,\infty)$, i.e., the map $s \mapsto L^\kappa([s,s+t]) \geq 0$ is nonnegative and nonincreasing in s for all $t \geq 0$.

4) For each $\lambda \geq 0$, the resolvent $R^{\lambda\kappa}$ of the rescaled kernel $\lambda\kappa$ is nonnegative.

We note that although it is natural to impose the condition 4) for all $\lambda \geq 0$, we only use the fact that $R^{\lambda\kappa}$ is nonnegative for the particular choice of $\lambda > 0$ introduced later in the definition (2.8) of the rough Heston model.

It is pointed out in Abi Jaber et al. [1] (which we use as a reference for properties of the rough Heston model) that the conditions 2)–4) of Assumption 2.7 can be replaced with the stronger but simpler condition of complete monotonicity, which may be more convenient for practical applications. We say that a kernel $\kappa : (0,\infty) \to \mathbb{R}_+$ is *completely monotone* if it is infinitely differentiable and it holds that $(-1)^j \kappa^{(j)}(t) \geq 0$ for all $t \in (0,\infty)$ and $j \in \mathbb{N}_0$, where $\kappa^{(j)}$ denotes the j-th derivative. In particular, any completely monotone kernel $\kappa \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{R}_+)$ that is not identically zero satisfies the conditions 2)–4), which follow by the definition and [59, Theorems 5.5.4 and 5.3.1]. In this case, we even obtain that $\lambda\kappa$ and thus $R^{\lambda\kappa}$ are also completely monotone by [59, Theorem 5.3.1] for each $\lambda \geq 0$, and hence $R^{\lambda\kappa}$ also satisfies the conditions 2)–4). We also note that by Lemma A.3.3, $R^{\lambda\kappa}$ always satisfies condition 1) of Assumption 2.7 if κ does. Some notable properties related to complete monotonicity include the fact that sums and products of completely monotone functions are completely monotone, as well as Bernstein's theorem [59, Theorem 5.2.5] which states that a function is completely monotone if and only if it is the Laplace transform of a nonnegative measure on $[0, \infty)$.

Example 2.8. We consider the fractional (Riemann–Liouville) kernel

$$\kappa(t) = \frac{t^{h-1/2}}{\Gamma(h+1/2)}, \quad t \geq 0, \tag{2.6}$$

as our main example, where $\Gamma : (0, \infty) \to \mathbb{R}_+$ is the well-known gamma function and $h \in (0, 1/2]$ a roughness parameter, similar to the Hurst parameter for fractional Brownian motion. As they are completely monotone, the fractional kernel, as well as the more general gamma kernel given by

$$\kappa(t) = \frac{\beta^{h+1/2} t^{h-1/2} e^{-\beta t}}{\Gamma(h+1/2)}, \quad t \geq 0,$$

for some parameter $\beta \geq 0$, satisfy Assumption 2.7 with $\gamma = 2h$. As we will see in (2.8), the fractional kernel with $h = 1/2$, so that $\kappa \equiv 1$, corresponds to the classical Heston model. The resolvents of the first and second kind are given explicitly in [1, Table 1] for fractional and gamma kernels. In the case of the fractional kernel (2.6), we have the resolvent of the second kind

$$R^{\lambda\kappa}(t) = \lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad t \geq 0, \tag{2.7}$$

for each $\lambda \geq 0$, where $\alpha = h + 1/2$ and $E_{\alpha,\beta} : [0, \infty) \to \mathbb{R}_+$ is the generalized Mittag-Leffler function; see Podlubny [101, Chapter 1] or Haubold et al. [63] for the definition and properties of $E_{\alpha,\beta}$. In this case, the resolvent $R^{\lambda\kappa}$ can also be seen as the density of a Mittag-Leffler distribution (see [63, Equation (19.1.2)] in the case $\lambda = 1$), and in particular it is a nonnegative function with $\int_0^\infty R^{\lambda\kappa}(t)dt = 1$.

We are now ready to introduce the rough Heston model. Consider a financial market with finite time horizon $T \in (0, \infty)$, consisting of one riskless asset with constant price equal to 1 and one risky asset. The price process $(S_t)_{0 \leq t \leq T}$ of the risky asset is defined from the unique in law nonnegative weak solution

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P, W, B, S, Y)$ to the Volterra stochastic differential equations

$$\begin{cases} \dfrac{dS_t}{S_t} = \mu Y_t dt + \sigma\sqrt{Y_t} dW_t, \\ Y_t = Y_0 + \displaystyle\int_0^t \kappa(t-s)\big(\lambda(\theta - Y_s)ds + \zeta\sqrt{Y_s}dB_s\big) \end{cases} \tag{2.8}$$

for $t \in [0,T]$, where $(W_t)_{0\le t\le T}$ and $(B_t)_{0\le t\le T}$ are two Brownian motions with constant instantaneous correlation $\varrho \in (-1,1)$. The parameters $S_0 > 0$, $Y_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, $\lambda > 0$, $\theta > 0$, $\zeta > 0$ and $\varrho \in (-1,1)$ are fixed constants, whereas $\kappa : (0,\infty) \to \mathbb{R}_+$ is a fixed kernel satisfying Assumption 2.7. Given a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P, W, B, S, Y)$ to (2.8) such that $S, Y \ge 0$, we say as a shorthand that (S,Y) *satisfies the rough Heston model on* $[0,T]$. We sometimes also consider the Brownian motion $W^\perp := \frac{B-\varrho W}{\sqrt{1-\varrho^2}}$, so that we have the orthogonal decomposition $B = \varrho W + \sqrt{1-\varrho^2}\, W^\perp$. We note that the existence and uniqueness of a nonnegative weak solution (S,Y) to (2.8) follows from Abi Jaber et al. [1, Theorems 3.6 and 7.1] for any kernel κ satisfying the conditions 1)–3) of Assumption 2.7.

The roughness of the volatility process Y is justified by [1, Lemma 2.4 and Theorem 7.1], i.e., we have that Y is Hölder-continuous of any order strictly smaller than $\gamma/2$. This may in general be rougher than Brownian motion: for instance, if κ is the fractional kernel (2.6) with parameter $h \in (0,1/2]$, then Y is only Hölder-continuous of order up to $\gamma/2 = h$, similarly to Riemann–Liouville fractional Brownian motion. The remaining parameters $\mu, \sigma, \lambda, \theta, \zeta$ and ϱ have broadly similar interpretations as in the classical Heston model. Although it is not always true in general that the process Y reverts to a mean θ, we show in Lemma 2.12 that the long-term expectation of Y does converge to θ if we have $\int_0^\infty \kappa(t)dt = \infty$, which is the case for the fractional kernels of the form (2.6). A more general result on the long-term behaviour of the volatility process is shown in Friesen/Jin [53, Theorem 1.3].

We next introduce the forward variance curve. This is a key object that has been used in the literature to deal with the main challenges posed by the rough Heston model: namely, the fact that the rough volatility process Y is in general neither Markov nor a semimartingale. It can be seen as an infinite-dimensional Markovian lift of the volatility process; see Cuchiero/Teichmann [30].

Definition 2.9. The *forward variance curve* on $[0,T]$ is the stochastic process

$(\xi_t(u))_{0\le t\le u\le T}$ defined by

$$\xi_t(u) = E[Y_u \mid \mathcal{F}_t], \quad 0 \le t \le u \le T.$$

Occasionally, it is convenient to extend ξ to $[0,T]\times[0,T]$ by setting

$$\xi_t(u) = E[Y_u \mid \mathcal{F}_t] = \begin{cases} E[Y_u \mid \mathcal{F}_t], & 0 \le t < u \le T, \\ Y_u, & 0 \le u \le t \le T. \end{cases} \tag{2.9}$$

The fact that Y_u is integrable for each $u \ge 0$ is shown in [1, Lemma 4.2]; so the process $(\xi_t(u))_{0\le t\le u}$ is a martingale for any fixed u, by the definition.

Although the properties of the rough Heston model pose challenges for traditional pricing and hedging methods, we can make use of the fact that the process Y inherits some of the linear structure of the classical Heston model, and in particular is an affine Volterra process (in the sense of [1]). This means that the characteristic function of the log-price $X_t = \log S_t$ can be computed in semi-explicit form as an exponentially affine function of the forward variance curve and is given by

$$E[\exp(\mathrm{i}uX_t)] = \exp\left(\mathrm{i}uX_0 + \int_0^t g_{\mathrm{i}u}(t-s)\xi_0(s)ds\right) \tag{2.10}$$

for $t \ge 0$ and $u \in \mathbb{R}$, where $g_{\mathrm{i}u} : [0,T] \to \mathbb{C}$ is a deterministic function satisfying the Riccati–Volterra equation

$$g_{\mathrm{i}u}(t) = \mathrm{i}u\mu + \frac{\sigma^2(-u^2 - \mathrm{i}u)}{2} + \frac{((\hat{\kappa} * g_{\mathrm{i}u})(t))^2}{2} + \mathrm{i}u\varrho\sigma(\hat{\kappa} * g_{\mathrm{i}u})(t), \quad 0 \le t \le T, \tag{2.11}$$

and we set

$$\hat{\kappa}(t) := \frac{\zeta}{\lambda} R^{\lambda\kappa}(t), \quad t \ge 0. \tag{2.12}$$

The kernel $\hat{\kappa}$ appears in many of our results, and its role is shown in the following Lemma 2.10. Equations such as (2.11) are the main topic of Appendix A, and they feature prominently in several results in this chapter as well.

Our main goal in this section is to obtain a considerably more general version of the characteristic function (2.10), which we do in Theorem 2.17. This generalised moment-generating function for the rough Heston model is then used repeatedly in several proofs towards solving the pure investment problem in the next section. In order to prove the theorem, we first need some probabilistic lemmas related to the forward variance curve, as well as some results related to (deterministic) Riccati–Volterra equations, which are given in Appendix A.

In the next lemma, we give a formula for the dynamics of $t \mapsto \xi_t(u)$ in terms of the resolvent of the second kind of $\lambda\kappa$. This is well known in the literature, and also given in [1, Lemma 4.2]. Since the forward variance curve is one of our main tools for working with the rough Heston model, this is a key result, and it also emphasises the role of the kernel $\hat{\kappa}$.

Lemma 2.10. *For any fixed $u \in [0,T]$, the process $(\xi_t(u))_{0\le t\le u}$ is a continuous martingale on $[0,u]$ with the dynamics*

$$d\xi_t(u) = \hat{\kappa}(u-t)\sqrt{Y_t}dB_t, \quad 0 \le t \le u,$$

where $\hat{\kappa} := \frac{\zeta}{\lambda}R^{\lambda\kappa}$ and $R^{\lambda\kappa} : (0,\infty) \to \mathbb{R}$ is the resolvent of the second kind of $\lambda\kappa$. Moreover, we have the initial and terminal values

$$\xi_0(u) = Y_0 + \frac{\lambda(\theta - Y_0)}{\zeta}\int_0^u \hat{\kappa}(s)ds \quad \text{and} \quad \xi_u(u) = Y_u, \quad \text{for each } u \in [0,T]. \tag{2.13}$$

This lemma is a particular case of [1, Lemma 4.2], where in their notation we identify $b_0 := \lambda\theta$, $R_B := R^{\lambda\kappa}$ and

$$E_B := \kappa - R_B * \kappa = \frac{1}{\lambda}\big(\lambda\kappa - R^{\lambda\kappa} * (\lambda\kappa)\big) = \frac{R^{\lambda\kappa}}{\lambda} = \frac{\hat{\kappa}}{\zeta},$$

by the definitions of $R^{\lambda\kappa}$ and $\hat{\kappa}$. We give an outline of the proof (using our notation) in order to illustrate the main idea.

Proof of Lemma 2.10 (sketch). For a fixed $u \in [0,T]$, we have by (2.8) that

$$Y_u = Y_0 + \int_0^u \kappa(u-s)\big(\lambda(\theta - Y_s)ds + \zeta\sqrt{Y_s}dB_s\big).$$

Assuming that the local martingale $M_t := \int_0^t \kappa(u-s)\sqrt{Y_s}dB_s$ is a true martingale on $[0,u]$ and that the finite variation term is integrable, we can formally take conditional expectations with respect to $\mathcal{F}_t$ and use the conditional Fubini theorem to obtain

$$\xi_t(u) = Y_0 + \int_0^u \kappa(u-s)\lambda\big(\theta - \xi_t(s)\big)ds + \zeta\int_0^t \kappa(u-s)\sqrt{Y_s}dB_s, \;\; 0 \le t,u \le T, \tag{2.14}$$

where we consider the extended forward variance curve with $\xi_t(s) = Y_s$ for $t \ge s$.

We can then rearrange (2.14) in the form

$$\xi_t(u) + \int_0^u \lambda\kappa(u-s)\xi_t(s)ds = Y_0 + \int_0^u \lambda\theta\kappa(u-s)ds + \zeta\int_0^t \kappa(u-s)\sqrt{Y_s}dB_s.$$

Using the convolution notation, we can rewrite this as

$$\xi_t(u) + \lambda(\kappa * \xi_t)(u) = Y_0 + \lambda\theta(\kappa * I)(u) + \zeta\int_0^t \kappa(u-s)\sqrt{Y_s}dB_s, \quad 0 \le t, u \le T,$$

where I denotes the constant function $I(s) = 1$. Fixing $t \in [0,T]$, this can be seen as a Volterra equation of the second kind for $(\xi_t(u))_{u\in[0,T]}$ in the sense of (2.4), where $k = \lambda\kappa$. Therefore, this equation can be solved by taking a convolution with $R^{\lambda\kappa}$, which leads to the result after some simplifications. Some care is still needed in order to deal with the stochastic integral; see [1, Lemma 4.2]. □

Note that by Lemma 2.10, the initial value of the forward variance curve $(\xi_0(u))_{0\le u\le T}$ is continuous in u, and the map $t \mapsto \xi_t(u)$ is continuous for each fixed u. It also follows by [1, Theorem 3.4] that the diagonal map $t \mapsto \xi_t(t) = Y_t$ is continuous in t. Thus it is not difficult to show that ξ admits a version that is continuous in each variable t and u separately. Next, we show that the map $(t,u) \mapsto \xi_t(u)$ is jointly continuous and even has the same Hölder regularity as Y, as given in [1, Lemma 2.4]. In particular, this implies that the entire curve $(\xi_t(u))_{0\le t\le u\le T}$ is bounded a.s., which will be useful later. After proving Proposition 2.11, we shall always take a continuous version of $(\xi_t(u))_{0\le t\le u\le T}$.

Proposition 2.11. *For each $\nu \in (0, \gamma/2)$, there exists a version of the forward variance curve $(\xi_t(u))_{0\le t\le u\le T}$ that is a.s. Hölder-continuous of order ν in (t,u).*

Proof. We use the d-dimensional Kolmogorov continuity criterion for the construction; see Revuz/Yor [105, Theorem I.2.1]. Fix $0 \le t_1 \le u_1 \le T$ and $0 \le t_2 \le u_2 \le T$, and assume without loss of generality that $t_2 \ge t_1$. By Lemma 2.10, we have

$$\begin{aligned}\xi_{t_2}(u_2) - \xi_{t_1}(u_1) &= U_1 + \int_0^{t_2} \hat{\kappa}(u_2-s)\sqrt{Y_s}dB_s - \int_0^{t_1} \hat{\kappa}(u_1-s)\sqrt{Y_s}dB_s \\ &= U_1 + U_2 + U_3, \qquad (2.15)\end{aligned}$$

where, using the convention $\int_{u_1}^{u_2} = -\int_{u_2}^{u_1}$ in the case $u_1 > u_2$, we set

$$U_1 := \xi_0(u_2) - \xi_0(u_1) = \frac{\lambda(\theta - Y_0)}{\zeta} \int_{u_1}^{u_2} \hat{\kappa}(s)ds, \tag{2.16}$$

$$U_2 := \int_0^{t_1} \big(\hat{\kappa}(u_2 - s) - \hat{\kappa}(u_1 - s)\big)\sqrt{Y_s}dB_s, \tag{2.17}$$

$$U_3 := \int_{t_1}^{t_2} \hat{\kappa}(u_2 - s)\sqrt{Y_s}dB_s. \tag{2.18}$$

We consider these terms one at a time. Recall that by part 1) of Assumption 2.7, we have $\int_0^h \kappa^2(s)ds = O(h^\gamma)$ and $\int_0^T (\kappa(s+h) - \kappa(s))^2 ds = O(h^\gamma)$. Since κ is nonnegative and nonincreasing by Assumption 2.7.2), we also have

$$\int_t^{t+h} \kappa^2(s)ds \le \int_0^h \kappa^2(s)ds = O(h^\gamma)$$

uniformly in $t \in [0,T]$. Hence by parts 2) and 3) of Lemma A.3.3 and the definition (2.12) of $\hat{\kappa}$, there exist some constants $c_{\hat{\kappa}}, \gamma, \delta > 0$ such that

$$\max\left(\sup_{t\in[0,T]} \int_t^{t+h} \hat{\kappa}^2(s)ds, \int_0^T \big(\hat{\kappa}(s+h) - \hat{\kappa}(s)\big)^2 ds\right) \le c_{\hat{\kappa}} h^\gamma \tag{2.19}$$

for all $h \in [0,\delta]$. Thus for any $p \ge 1$, we have by (2.16) that

$$\begin{aligned} |U_1|^p &= \frac{\lambda^p|\theta - Y_0|^p}{\zeta^p}\left|\int_{u_1}^{u_2} \hat{\kappa}(s)ds\right|^p \\ &\le \frac{\lambda^p|\theta - Y_0|^p}{\zeta^p}|u_2 - u_1|^{p/2}\left|\int_{u_1}^{u_2} |\hat{\kappa}(s)|^2 ds\right|^{p/2} \\ &\le \frac{\lambda^p|\theta - Y_0|^p}{\zeta^p} c_{\hat{\kappa}}^{p/2}|u_2 - u_1|^{p(1+\gamma)/2}. \end{aligned} \tag{2.20}$$

For U_2, (2.17) and the Burkholder–Davis–Gundy inequality yield the bound

$$\begin{aligned} E[|U_2|^p] &\le c_p E\left[\left(\int_0^{t_1} \big(\hat{\kappa}(u_2 - s) - \hat{\kappa}(u_1 - s)\big)^2 Y_s ds\right)^{p/2}\right] \\ &\le c_p E\Big[\sup_{0\le t\le T} Y_t^{p/2}\Big]\left(\int_0^{t_1} \big(\hat{\kappa}(u_2 - s) - \hat{\kappa}(u_1 - s)\big)^2 ds\right)^{p/2} \end{aligned} \tag{2.21}$$

for each $p \ge 1$ and some constant $c_p > 0$. Since the coefficients in the equation (2.8) for Y satisfy a linear growth condition, we have by [1, Lemma 3.1] that $\sup_{0\le t\le T} E[Y_t^q] < \infty$ for every $q \ge 0$. Also by (2.8), the processes $(a_t)_{0\le t\le T}$ and

$(b_t)_{0\le t\le T}$ appearing in the statement of [1, Lemma 2.4] are given by $a_t = \zeta^2 Y_t$ and $b_t = \lambda(\theta - Y_t)$, respectively. Thus [1, Lemma 2.4] with $\alpha = 0$ lets us swap the supremum and the expectation so that $c_Y(q) := E[\sup_{0\le t\le T} Y_t^q] < \infty$ for q large enough (and hence for all $q \ge 1$). Returning to (2.21), by (2.19) we get

$$E[|U_2|^p] \le c_p c_Y(p/2) c_{\hat\kappa}^{p/2} |u_2 - u_1|^{p\gamma/2}. \tag{2.22}$$

In a similar way, (2.18), (2.19) and the Burkholder–Davis–Gundy inequality yield

$$\begin{aligned} E[|U_3|^p] &\le c_p E\bigg[\bigg(\int_{t_1}^{t_2} \hat\kappa(u_2 - s)^2 Y_s ds\bigg)^{p/2}\bigg] \\ &\le c_p E\Big[\sup_{0\le t\le T} Y_t^{p/2}\Big]\bigg(\int_{u_2-t_2}^{u_2-t_1} \hat\kappa(t)^2 dt\bigg)^{p/2} \\ &\le c_p c_Y(p/2) c_{\hat\kappa}^{p/2} |t_2 - t_1|^{p\gamma/2}. \end{aligned} \tag{2.23}$$

Combining (2.20), (2.22) and (2.23) with (2.15) then yields

$$\begin{aligned} E[|\xi_{t_2}(u_2) - \xi_{t_1}(u_1)|^p] &\le 3^{p-1}(|U_1|^p + E[|U_2|^p] + E[|U_3|^p] \\ &\le C_p |(t_2,u_2) - (t_1,u_1)|_\infty^{p\gamma/2} \end{aligned} \tag{2.24}$$

for all $p \ge 1$ and $0 \le t_i \le u_i \le T$ such that $h := |(t_2,u_2) - (t_1,u_1)|_\infty \in [0,\delta]$, where the constant

$$C_p := 3^{p-1} c_{\hat\kappa}^{p/2} \bigg(\frac{\lambda^p |\theta - Y_0|^p}{\zeta^p} + 2 c_p c_Y(p/2)\bigg)$$

does not depend on (t_i, u_i). With (2.24), we are now finally ready to apply the Kolmogorov continuity criterion; note that the proof of [105, Theorem I.2.1] only requires that the inequality hold for pairs (t_1,u_1) and (t_2,u_2) that are close to each other. Then by taking p large enough so that $p\gamma/2 > 2$, [105, Theorem I.2.1] yields that there exists a version of $(\xi_t(u))_{0\le t\le u\le T}$ that is Hölder-continuous of any order $\nu \in (0, \frac{\gamma}{2} - \frac{2}{p})$. By taking $p \to \infty$, it follows that $(\xi_t(u))_{0\le t\le u\le T}$ admits a ν-Hölder-continuous version for any $\nu \in (0, \frac{\gamma}{2})$. □

As mentioned in Example 2.8, we have the explicit formula (2.7) for $R^{\lambda\kappa}$ for a fractional kernel κ given by (2.6), in which case we have $\int_0^\infty R^{\lambda\kappa}(t)dt = 1$, which together with the definition (2.12) of $\hat\kappa$ yields $\int_0^\infty \frac{\lambda}{\zeta}\hat\kappa(t)dt = 1$. Hence by Lemma 2.10 and taking $u \to \infty$ in the forward variance curve $\xi_0(u)$ at time 0, we obtain the limit $\xi_0(u) \to \theta$ as $u \to \infty$. This justifies the interpretation of the parameter

$\theta > 0$ as the long-term volatility in this case. More generally, we show in the following lemma that we have the equality $\int_0^\infty \frac{\lambda}{\zeta}\hat{\kappa}(s)ds = \int_0^\infty R^{\lambda\kappa}(s)ds = 1$ if κ satisfies the additional assumption that $\int_0^\infty \kappa(s)ds = \infty$ (which holds in the fractional case), so that θ can be seen as the long-term volatility for any such choice of κ. We defer the details of the proof to Appendix A; see Lemma A.3.2.

Lemma 2.12. *Let $\kappa \in L^1_{\text{loc}}([0,\infty),\mathbb{R})$ be a kernel satisfying Assumption 2.7 such that $\int_0^\infty \kappa(t)dt = \infty$. Then the resolvent of the second kind $R^{\lambda\kappa}$ is integrable for each $\lambda \geq 0$ with $\int_0^\infty R^{\lambda\kappa}(s)ds = 1$.*

Proof. By Assumption 2.7, κ is nonnegative and nonincreasing, and moreover $R^{\lambda\kappa}$ is nonnegative for each $\lambda \geq 0$. Thus the result follows by Lemma A.3.2 applied to $k := \lambda\kappa$. □

Now that we have established some basic properties of the forward variance curve, we return to the study of the rough Heston model (2.8). To that end, we often consider processes that depend linearly on the forward variance curve, such as the integral term that appears in the characteristic function (2.10). The following two lemmas show how to obtain semimartingale decompositions for two types of processes. In Lemma 2.13, we consider a linear functional of the forward variance curve as well as the past (realised) curve, while in Corollary 2.14, we consider a functional of the forward curve alone.

Lemma 2.13. *Let ν be a finite complex measure on $([0,T],\mathcal{B}([0,T]))$. Then there exists a continuous local martingale $(M_t)_{0\leq t\leq T}$ such that*

$$M_t = \int_{[0,t]} Y_u \nu(du) + \int_{(t,T]} \xi_t(u)\nu(du) \quad \text{for each } 0 \leq t \leq T, \tag{2.25}$$

and it admits the decomposition

$$M_t = M_0 + \int_0^t \left(\int_{[s,T]} \hat{\kappa}(u-s)\nu(du) \right) \sqrt{Y_s} dB_s, \quad 0 \leq t \leq T. \tag{2.26}$$

Proof. The main idea is to apply the stochastic Fubini theorem of Veraar [118, Theorem 2.2] together with the martingale dynamics given in Lemma 2.10. Recall from (2.9) that $\xi_t(u) = Y_u$ for $u \leq t$. We start by taking (2.26) as the definition for M, where we set the initial value as

$$M_0 := \int_{[0,T]} \xi_0(u)\nu(du), \tag{2.27}$$

so that (2.25) is satisfied at $t = 0$. Note that (2.27) is well defined since the map $u \mapsto \xi_0(u)$ is continuous a.s. and the complex measure ν is finite by assumption. We want to show that M is well defined by the stochastic integral and is a continuous local martingale. Note that we have the bound

$$\begin{aligned}\int_{[0,T]} \left(\int_0^{u\wedge T} \hat{\kappa}(u-s)^2 Y_s ds \right)^{\frac{1}{2}} d\nu(u) &\leq \int_{[0,T]} \left(\|\hat{\kappa}\|_{L^2(0,T)} \sup_{s\in[0,T]} \sqrt{|Y_s|} \right) d|\nu|(u) \\ &\leq |\nu|([0,T]) \|\hat{\kappa}\|_{L^2(0,T)} \sup_{s\in[0,T]} \sqrt{|Y_s|} \\ &< \infty \end{aligned} \tag{2.28}$$

almost surely, where $|\nu|([0,T])$ is the total variation of ν on $[0,T]$ (see Rudin [108, Chapter 6]), since Y is a continuous process, hence a.s. bounded on $[0,T]$, ν is finite by assumption and $\hat{\kappa}$ is locally square-integrable. We can then apply [118, Theorem 2.2] with $A = 0$, $X = [0,T]$ and $\psi(u,t,\omega) = \hat{\kappa}(u-s)\sqrt{Y_s(\omega)}\mathbf{1}_{[0,u]}(t)$, since we have shown in (2.28) that the bound in [118, Equation (2.1)] is satisfied. Thus it follows from [118, Theorem 2.2(2)] that the process M defined by (2.26) is a continuous local martingale.

Now we want to show that (2.25) is satisfied. Recall from Lemma 2.10 that

$$\xi_t(u) = \xi_0(u) + \int_0^{u\wedge t} \hat{\kappa}(u-s)\sqrt{Y_s} dB_s, \quad 0 \leq t, u \leq T.$$

We again consider the forward variance curve $(\xi_t(u))_{0\leq t,u\leq T}$ in the extended sense, so that we have $\xi_t(u) = Y_u \mathbf{1}_{[0,t)}(u) + \xi_t(u)\mathbf{1}_{[t,T]}(u)$, and we can rewrite

$$\int_{[0,T]} \xi_t(u)\nu(du) = \int_{[0,t]} Y_u \nu(du) + \int_{(t,T]} \xi_t(u)\nu(du). \tag{2.29}$$

Using the dynamics for the forward variance curve from Lemma 2.10, we have the formal computation

$$\begin{aligned}\int_{[0,T]} \big(\xi_t(u) - \xi_0(u)\big) d\nu(u) &= \int_{[0,T]} \left(\int_0^{u\wedge t} \hat{\kappa}(u-s)\sqrt{Y_s} dB_s \right) d\nu(u) \\ &= \int_0^t \left(\int_{[s,T]} \hat{\kappa}(u-s) d\nu(u) \right) \sqrt{Y_s} dB_s \\ &= M_t - M_0, \quad 0 \leq t \leq T, \end{aligned} \tag{2.30}$$

assuming that the integrals can be interchanged from the first line to the second. Plugging in (2.29) as well as the initial value (2.27) for M_0, this implies (2.25).

The interchanging of the integrals from the first to the second line is justified by [118, Theorem 2.2(3)] since we only need the same bound (2.28) as before. Therefore the equality (2.25) follows from (2.30). □

Corollary 2.14. *For any function $g \in L^1([0,T];\mathbb{C})$, there exists a continuous semimartingale $(Y_t^g)_{0\le t\le T}$ such that*

$$Y_t^g = \int_t^T g(T-u)\xi_t(u)du \quad \text{for each } 0 \le t \le T, \tag{2.31}$$

so that in particular $Y_T^g = 0$, and it has the decomposition

$$Y_t^g = Y_0^g + A_t + M_t, \quad 0 \le t \le T, \tag{2.32}$$

where the continuous finite-variation process $(A_t)_{0\le t\le T}$ and the continuous local martingale $(M_t)_{0\le t\le T}$ are respectively given by

$$A_t = -\int_0^t g(T-s)Y_s ds, \quad M_t = \int_0^t (\hat{\kappa} * g)(T-s)\sqrt{Y_s}dB_s, \quad 0 \le t \le T, \tag{2.33}$$

where we write A, M rather than A^g, M^g for ease of notation.

Proof. Similarly to the proof of Lemma 2.13, we want to take (2.32) and (2.33) as the definition for Y^g, A and M, setting the initial value to be

$$Y_0^g = \int_0^T g(T-u)\xi_0(u)du,$$

which is well defined as $u \mapsto \xi_0(u)$ is continuous and g is integrable by assumption. It is also clear that the process A is continuous and has finite variation, since Y is continuous (hence a.s. bounded on $[0,T]$) and $\int_0^T |g(T-s)|ds < \infty$ as g is integrable. We now check that M is well defined by applying Lemma 2.13. Define the complex measure $\nu(dt) := g(T-t)dt$ on $[0,T]$ and note that ν is finite as g is integrable. Thus we have the identity

$$\begin{aligned}(\hat{\kappa} * g)(T-s) &= \int_0^{T-s} g(T-s-u)\hat{\kappa}(u)du \\ &= \int_s^T g(T-u)\hat{\kappa}(u-s)du = \int_s^T \hat{\kappa}(u-s)\nu(du)\end{aligned}$$

for $s \in [0,T]$, and hence we have

$$M_t = \int_0^t (\hat{\kappa} * g)(T-s)\sqrt{Y_s}dB_s = \int_0^t \left(\int_s^T \hat{\kappa}(u-s)\nu(du) \right) \sqrt{Y_s}dB_s.$$

We can apply Lemma 2.13 to the latter integral, keeping in mind that in this case $M_0 = 0$. By that lemma, M is well defined as a continuous local martingale, and we obtain moreover from (2.30) and the definition of ν that

$$M_t = \int_0^T g(T-u)\xi_t(u)du - \int_0^T g(T-u)\xi_0(u)du, \quad 0 \le t \le T. \tag{2.34}$$

So far, we have shown that the processes A and M are well defined, so that Y^g is a continuous semimartingale satisfying (2.32) by construction. It remains to show the equality (2.31) and that $Y^g_T = 0$. By adding

$$A_t = -\int_0^t g(T-u)Y_u du = -\int_0^t g(T-u)\xi_t(u)du$$

and $Y^g_0 = \int_0^T g(T-u)\xi_0(u)du$ to both sides of (2.34), we obtain that

$$Y^g_t = Y^g_0 + A_t + M_t = \int_t^T g(T-u)\xi_t(u)du, \quad 0 \le t \le T,$$

which shows (2.31). Taking $t \nearrow T$ and changing variables $u \mapsto T-u$, we get

$$\left| \int_t^T g(T-u)\xi_t(u)du \right| \le \int_0^{T-t} |g(u)|du \sup_{0 \le s \le u \le T} \xi_s(u) \longrightarrow 0 \quad \text{as } t \nearrow T,$$

since the forward variance curve is continuous (hence a.s. bounded) due to Proposition 2.11 and by the dominated convergence theorem, as g is an integrable majorant for the family $(\mathbf{1}_{[0,T-t]}g)_{0 \le t \le T}$ of integrable functions on $[0,T]$. Since the semimartingale Y^g is continuous, we then obtain that $Y^g_T = 0$ by taking the limit in (2.31) as $t \nearrow T$. □

2.2 Generalised moment-generating function for the rough Heston model

In order to show in the next section some of our results regarding the pure investment problem, we need an explicit formula for several conditional expectations related to the rough Heston model. This formula is given in Theorem 2.17, and it can be seen as a generalised conditional moment-generating function of the form $E[\exp(z^\top \tilde{X}_T) \mid \mathcal{F}_t]$, where $z \in \mathbb{C}^n$, $t \in [0,T]$ and $\tilde{X}$ is an n-dimensional complex-valued semimartingale whose dynamics depend on the volatility process Y in a linear way. As an immediate application, we can take $n = 1$ and $\tilde{X} = X = \log S$ to be the log-price. We give in Corollary 2.19 some other natural applications, and we use the general statement in the next section, as it also allows us to consider certain changes of measure.

In preparation for Theorem 2.17, we start by giving two Novikov-type criteria for the martingale property of a stochastic exponential. The second version, given in Lemma 2.16, can be applied for stochastic exponentials of complex-valued martingales. This is part of what we need in order to show that the local martingales appearing in Theorem 2.17 are actually true martingales.

Lemma 2.15. *Let $T \in (0,\infty)$ and $(M_t)_{0\le t\le T}$ be a real-valued continuous local martingale. Suppose that there exists some $\delta \in (0,T)$ such that*

$$E\left[\exp\left(\frac{[M]_{t+\delta} - [M]_t}{2}\right)\right] < \infty$$

for all $t \in [0, T-\delta]$. Then the stochastic exponential $\mathcal{E}(M)$ is a martingale on $[0,T]$.

Proof. This follows easily from the Novikov criterion, by subdividing $[0,T]$ into smaller intervals of length at most δ and inductively showing that $\mathcal{E}(M)$ is a martingale on each interval; see Karatzas/Shreve [77, Corollary 3.5.14] for a proof. (While that corollary is only given for an Itô martingale, the proof here is the same.) □

Next, we give a version of the classic Novikov criterion for the stochastic exponential of a complex-valued continuous local martingale $\tilde{M}$, as well as a localised version in which we assume that the terminal value $\mathcal{E}(\tilde{M})_T$ is integrable. This assumption is needed in the complex case, unlike in Lemma 2.15, since the absolute value $|\mathcal{E}(\tilde{M})|$ of the stochastic exponential is in general neither a local martingale nor a supermartingale. This is in contrast to the real case, where we have these properties and hence integrability a priori.

Lemma 2.16. *Let $T \in (0,\infty)$ and consider the complex-valued process $(\tilde{M}_t)_{0\le t\le T}$ given by $\tilde{M}_t = M_t + \mathrm{i}N_t$, where $(M_t)_{0\le t\le T}$ and $(N_t)_{0\le t\le T}$ are continuous real-valued* *local martingales null at* 0. *Define the stochastic exponential* $(Z_t)_{0\le t\le T}$ *of* $\tilde{M}$ *by*

$$Z_t = \mathcal{E}(\tilde{M})_t = \exp\left(M_t + \mathrm{i}N_t - \frac{[M]_t}{2} - \mathrm{i}[M,N]_t + \frac{[N]_t}{2}\right), \quad 0 \le t \le T,$$

and let $p, q \in (1,\infty)$ *be Hölder conjugates, i.e., such that* $1/p + 1/q = 1$.

1) *If* $E[\exp(\frac{p[M]_T}{2} + \frac{q[N]_T}{2})] < \infty$, *then* Z *is a martingale on* $[0,T]$.

2) *Suppose that* $E[|Z_T|] = E[\exp(M_T - \frac{[M]_T}{2} + \frac{[N]_T}{2})] < \infty$ *and there exists* $\delta \in (0,T)$ *such that*

$$E\left[\exp\left(\frac{p\big([M]_{t+\delta} - [M]_t\big)}{2} + \frac{q\big([N]_{t+\delta} - [N]_t\big)}{2}\right)\right] < \infty, \quad 0 \le t \le T-\delta.$$

Then Z *is a martingale on* $[0,T]$.

Proof. **1)** It follows by Itô's formula that

$$dZ_t = Z_t(dM_t + \mathrm{i}dN_t), \quad 0 \le t \le T, \tag{2.35}$$

using the fact that $[M+\mathrm{i}N] = [M] + 2\mathrm{i}[M,N] - [N]$, so that Z is a complex-valued local martingale; see also [71, Theorem I.4.61]. To show that Z is a uniformly integrable martingale, it is thus enough to check that the absolute value process

$$|Z_t| = \exp\left(M_t - \frac{1}{2}[M]_t + \frac{1}{2}[N]_t\right), \quad 0 \le t \le T,$$

is of class (D), i.e., the set $C := \{|Z_\tau| : \tau \le T \text{ a stopping time}\}$ is uniformly integrable. Fix $A \in \mathcal{F}$ and a stopping time $\tau \le T$. Noting that $q(p-1) = p$, it follows by the Hölder inequality that

$$\begin{aligned}
E[|Z_\tau|\,\mathbf{1}_A] &= E\left[\exp\left(M_\tau - \frac{1}{2}[M]_\tau + \frac{1}{2}[N]_\tau\right)\mathbf{1}_A\right] \\
&= E\left[\exp\left(M_\tau - \frac{p}{2}[M]_\tau\right)\exp\left(\frac{p-1}{2}[M]_\tau + \frac{1}{2}[N]_\tau\right)\mathbf{1}_A\right] \\
&\le \left(E\left[\exp\left(pM_\tau - \frac{p^2[M]_\tau}{2}\right)\right]\right)^{\frac{1}{p}}\left(E\left[\exp\left(\frac{p[M]_\tau}{2} + \frac{q[N]_\tau}{2}\right)\mathbf{1}_A\right]\right)^{\frac{1}{q}} \\
&\le \left(E\big[\mathcal{E}(pM)_\tau\big]\right)^{\frac{1}{p}}\left(E\left[\exp\left(\frac{p[M]_T}{2} + \frac{q[N]_T}{2}\right)\mathbf{1}_A\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\mathcal{E}(pM)$ is a nonnegative local martingale and hence a supermartingale on

$[0,T]$, we get that $E[\mathcal{E}(pM)_\tau] \leq 1$ and thus

$$E[|Z_\tau|\,\mathbf{1}_A] \leq \left(E\left[\exp\left(\frac{p[M]_T}{2}+\frac{q[N]_T}{2}\right)\mathbf{1}_A\right]\right)^{\frac{1}{q}}.$$

Taking $A=\Omega$ and using the assumption that $\exp(\frac{p[M]_T}{2}+\frac{q[N]_T}{2})$ is integrable, we obtain that C is bounded in L^1 as the bound is uniform in τ. Moreover, for any $\epsilon>0$, there exists some $\delta>0$ such that

$$E[|Z_\tau|\,\mathbf{1}_A] \leq \left(E\left[\exp\left(\frac{p[M]_T}{2}+\frac{q[N]_T}{2}\right)\mathbf{1}_A\right]\right)^{\frac{1}{q}} < \epsilon$$

for all sets $A\in\mathcal{F}$ with $P[A]<\delta$, since the singleton $\{\frac{p[M]_T}{2}+\frac{q[N]_T}{2}\}$ is uniformly integrable. This implies that C is uniformly integrable by the ϵ-δ-criterion for uniform integrability; see Klenke [83, Theorem 6.24]. Thus we have shown that the process $|Z|$ is of class (D), and so we conclude that Z is a true martingale on $[0,T]$.

2) We want to show that Z is a martingale on $[0,T]$ by induction, going backwards starting from time T. By the assumption, we have that Z is a martingale on the trivial interval $[T,T]=\{T\}$, as Z_T is integrable. For each $t\geq\delta$, we show that if Z is a martingale on $[t,T]$, then it is a martingale on $[t-\delta,T]$. For each $s\in[t-\delta,t]$, observe that we have

$$\begin{aligned}&E\left[\exp\left(\frac{p[\mathbf{1}_{(s,t]}\bullet M]_T}{2}+\frac{q[\mathbf{1}_{(s,t]}\bullet N]_T}{2}\right)\right]\\&=E\left[\exp\left(\frac{p\big([M]_t-[M]_s\big)}{2}+\frac{q\big([N]_t-[N]_s\big)}{2}\right)\right]<\infty\end{aligned}$$

by the assumption in 2), so that $\mathcal{E}(\mathbf{1}_{(s,t]}\bullet\tilde{M})$ is a uniformly integrable martingale on $[0,T]$ by part 1). By the properties of the stochastic exponential, we can write

$$Z_t=\mathcal{E}\big(\mathbf{1}_{(0,t]}\bullet\tilde{M}\big)_t=\mathcal{E}\big(\mathbf{1}_{(0,s]}\bullet\tilde{M}\big)_t\,\mathcal{E}\big(\mathbf{1}_{(s,t]}\bullet\tilde{M}\big)_t=Z_s\,\mathcal{E}\big(\mathbf{1}_{(s,t]}\bullet\tilde{M}\big)_t.$$

Since Z_t is integrable by the inductive hypothesis, we have that

$$\begin{aligned}E[|Z_t|]&=E\Big[E\big[\,|Z_s\,\mathcal{E}(\mathbf{1}_{(s,t]}\bullet\tilde{M})_t|\;\big|\;\mathcal{F}_s\big]\Big]\\&=E\Big[|Z_s|E\big[\,|\mathcal{E}(\mathbf{1}_{(s,t]}\bullet\tilde{M})_t|\;\big|\;\mathcal{F}_s\big]\Big]\geq E[|Z_s|],\end{aligned}$$

since $E[|\mathcal{E}(\mathbf{1}_{(s,t]}\bullet\tilde{M})_t|\mid\mathcal{F}_s]\geq|E[\mathcal{E}(\mathbf{1}_{(s,t]}\bullet\tilde{M})_t\mid\mathcal{F}_s]|=1$ as $\mathcal{E}(\mathbf{1}_{(s,t]}\bullet\tilde{M})$ is a martingale on $[0, T]$. Thus Z_s is integrable, and likewise we obtain

$$E[Z_t \mid \mathcal{F}_s] = E[Z_s \mathcal{E}(\mathbf{1}_{(s,t]} \bullet \tilde{M})_t \mid \mathcal{F}_s] = Z_s.$$

It follows that if Z is a martingale on $[t, T]$ for $t \in [0, T]$, then it is a martingale on $[t - \delta, T]$, as claimed. Therefore the result follows by induction, as we can extend the martingale property to the entire interval $[0, T]$. □

We now proceed to show the first main result in this section, where we give a formula for the conditional moment-generating function $E[\exp(z^\top \tilde{X}_T) \mid \mathcal{F}_t]$ for the terminal values of certain semimartingales $\tilde{X}$ related to the rough Heston model. Namely, we want to show that

$$E[\exp(z^\top \tilde{X}_T) \mid \mathcal{F}_t] = \exp\left(z^\top \tilde{X}_t + \int_t^T g_{z,T}(T-u)\xi_t(u)du\right), \quad 0 \le t \le T, \tag{2.36}$$

for some deterministic function $g_{z,T}$ that depends on the semimartingale $\tilde{X}$, and recalling from Definition 2.9 the forward variance curve $(\xi_t(u))_{0 \le t \le u \le T}$ associated with the rough Heston model. We then provide some applications as a corollary.

Theorem 2.17 improves in some ways on Abi Jaber et al. [1, Theorems 4.3 and 7.1(b)] in the case of the rough Heston model, since we do not assume but rather show that the formula (2.36) produces a true martingale, nor do we impose restrictions on the coefficients to ensure integrability. However, this comes at the cost of restricting to a smaller time interval, since in general such conditional expectations need not be finite for a given time horizon. This is true even in the classical Heston and other popular stochastic volatility models; see e.g. Keller-Ressel [82]. Probabilistically, this has to do with moment explosions in both the classical and rough Heston models, which also correspond to finite-time explosions in the solutions to Riccati ODEs and Riccati–Volterra equations, respectively.

Since we consider a range of possible time horizons $T > 0$, we fix in the following some upper bound $\bar{T} \in (0, \infty)$ and let (S, Y) satisfy the rough Heston model on $[0, \bar{T}]$. Recall from Lemma 2.10 the kernel $\hat{\kappa}$ and forward variance curve $(\xi_t(u))_{0 \le t \le u \le \bar{T}}$ associated with Y, as well as the orthogonal decomposition $B = \varrho W + \sqrt{1 - \varrho^2}\, W^\perp$ given with the definition (2.8) of the rough Heston model.

Theorem 2.17. *Fix $\bar{T} > 0$. Let $\tilde{X} = (\tilde{X}_t^{(1)}, \ldots, \tilde{X}_t^{(n)})_{0 \le t \le \bar{T}}$ be a $\mathbb{C}^n$-valued semimartingale that satisfies the decomposition*

$$\tilde{X}_t^{(k)} = \tilde{X}_0^{(k)} + \int_0^t \left(\mu^{(k)}(s) Y_s ds + \sigma^{(k)}(s)\sqrt{Y_s} dW_s + \tilde{\sigma}^{(k)}(s)\sqrt{Y_s} dW_s^\perp\right) \tag{2.37}$$

for all $t \in [0, \bar{T}]$ *and* $k = 1, \ldots, n$, *some constants* $\tilde{X}_0^{(1)}, \ldots, \tilde{X}_0^{(n)} \in \mathbb{C}$ *and deterministic functions* $\mu \in L^1([0, \bar{T}]; \mathbb{C}^n)$ *and* $\sigma, \tilde{\sigma} \in L^2([0, \bar{T}]; \mathbb{C}^n)$.

1) *Fix* $C > 0$ *and let* $\bar{B}_C(0) \subseteq \mathbb{C}^n$ *be the closed ball of radius* C. *Then there exists some positive time* $\tilde{T} = \tilde{T}(\mu, \sigma, \tilde{\sigma}, C) \in (0, \bar{T}]$ *such that for every* $T \in (0, \tilde{T}]$ *and* $z \in \bar{B}_C(0)$, *there is a unique solution* $g_{z,T} \in L^1([0, T]; \mathbb{C})$ *to the equation*

$$g_{z,T}(t) = z^\top \mu(T - t) + f\big((\hat{\kappa} * g_{z,T})(t); z^\top \sigma(T - t), z^\top \tilde{\sigma}(T - t)\big), \quad 0 \le t \le T, \tag{2.38}$$

where the function $f : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *is defined by*

$$f(x; h, \tilde{h}) = \frac{h^2 + \tilde{h}^2 + x^2}{2} + (\varrho h + \sqrt{1 - \varrho^2}\tilde{h})x. \tag{2.39}$$

Moreover, it holds that $E[\,|\exp(z^\top \tilde{X}_T)|\,] < \infty$, *and we have for* $0 \le t \le T$ *that*

$$E[\exp(z^\top \tilde{X}_T) \mid \mathcal{F}_t] = \exp\left(z^\top \tilde{X}_t + \int_t^T g_{z,T}(T - u)\xi_t(u)du\right). \tag{2.40}$$

2) *Conversely, fix* $z \in \mathbb{C}$ *and* $T \in (0, \bar{T}]$. *If* $E[\,|\exp(z^\top \tilde{X}_T)|\,] < \infty$ *and if there exists a solution* $g_{z,T} \in L^1([0, T]; \mathbb{C})$ *to* (2.38), *then* (2.40) *holds for* $0 \le t \le T$.

As a matter of fact, we directly prove a more general version of Theorem 2.17; that generalisation will be useful in the next chapter. Indeed, we may view $z^\top \tilde{X}_T$ as the terminal value of the process $(z^\top X_t)_{0 \le t \le T}$ which is parametrised by $z \in \bar{B}_C(0)$. By taking the product of z with (2.37), we obtain the dynamics of $z^\top X$ in terms of $z^\top \mu, z^\top \sigma$ and $z^\top \tilde{\sigma}$. We may then ask whether Theorem 2.17 can be extended to a general family of processes $(\tilde{X}^\varphi)_{\varphi \in \Phi}$ that is parametrised by an indexing set Φ instead of $\bar{B}_C(0)$, where the dynamics of $\tilde{X}^\varphi$ are given in terms of some families of functions $(\mu_\varphi)_{\varphi \in \Phi}$, $(\sigma_\varphi)_{\varphi \in \Phi}$ and $(\tilde{\sigma}_\varphi)_{\varphi \in \Phi}$ on $[0, \bar{T}]$. The following result gives a positive answer to that question. We first state the result and show how it implies Theorem 2.17, and then move on to the main task of proving Theorem 2.18.

Theorem 2.18. *Let* $\bar{T} > 0$, Φ *be an indexing set,* $(\mu_\varphi)_{\varphi \in \Phi}$ *a family of functions in* $L^1([0, \bar{T}]; \mathbb{C})$ *and* $(\sigma_\varphi)_{\varphi \in \Phi}, (\tilde{\sigma}_\varphi)_{\varphi \in \Phi}$ *two families of functions in* $L^2([0, \bar{T}]; \mathbb{C})$. *For each* $\varphi \in \Phi$, *let* $\tilde{x}^\varphi \in \mathbb{C}$ *be a constant and define* $(\tilde{X}_t^\varphi)_{0 \le t \le \bar{T}}$ *by*

$$\tilde{X}_t^\varphi = \tilde{x}^\varphi + \int_0^t \big(\mu_\varphi(s) Y_s ds + \sigma_\varphi(s)\sqrt{Y_s} dW_s + \tilde{\sigma}_\varphi(s)\sqrt{Y_s} dW_s^\perp\big), \quad 0 \le t \le \bar{T}. \tag{2.41}$$

1) *Suppose that the families* $(\mu_\varphi)_{\varphi\in\Phi}$*,* $(|\sigma_\varphi|^2)_{\varphi\in\Phi}$ *and* $(|\tilde{\sigma}_\varphi|^2)_{\varphi\in\Phi}$ *are uniformly integrable. Then there exists some positive time* $\tilde{T} \in (0, \bar{T}]$ *(that depends on* (μ_φ)*,* (σ_φ) *and* $(\tilde{\sigma}_\varphi)$*) such that for every* $T \in (0, \tilde{T}]$ *and* $\varphi \in \Phi$*, there is a unique solution* $g_{\varphi,T} \in L^1([0,T]; \mathbb{C})$ *to the equation*

$$g_{\varphi,T}(t) = \mu_\varphi(T-t) + f\big((\hat{\kappa} * g_{\varphi,T})(t); \sigma_\varphi(T-t), \tilde{\sigma}_\varphi(T-t)\big), \quad 0 \le t \le T, \tag{2.42}$$

where the function f *is defined by* (2.39)*. Moreover,* $E[\,|\exp(\tilde{X}_T^\varphi)|\,] < \infty$ *and we have for* $0 \le t \le T$ *that*

$$E[\exp(\tilde{X}_T^\varphi) \mid \mathcal{F}_t] = \exp\left(\tilde{X}_t^\varphi + \int_t^T g_{\varphi,T}(T-u)\xi_t(u)du\right). \tag{2.43}$$

2) *Conversely, fix* $\varphi \in \Phi$ *and* $T \in (0, \bar{T}]$*. If* $E[\,|\exp(\tilde{X}_T^\varphi)|\,] < \infty$ *and if there exists a solution* $g_{\varphi,T} \in L^1([0,T]; \mathbb{C})$ *to* (2.42)*, then* (2.43) *holds for* $0 \le t \le T$*.*

Proof of Theorem 2.17. Consider the families of functions $(\mu_\varphi)_{\varphi\in\mathbb{C}^n}$, $(\sigma_\varphi)_{\varphi\in\mathbb{C}^n}$ and $(\tilde{\sigma}_\varphi)_{\varphi\in\mathbb{C}^n}$ defined by $a_\varphi(t) = \varphi^\top a(t)$ for $a \in \{\mu, \sigma, \tilde{\sigma}\}$ and $t \in [0, \bar{T}]$. Then for each $\varphi \in \mathbb{C}^n$, we obtain by taking the product of both sides of (2.37) with $z = \varphi$ that the process $\tilde{X}^\varphi := \varphi^\top \tilde{X} = z^\top \tilde{X}$ satisfies the dynamics (2.41). We thus obtain part 2) of Theorem 2.17 directly from part 2) of Theorem 2.18, since the equation (2.42) for $g_{\varphi,T}$ reduces to (2.38) after plugging in $z = \varphi$ and $a_\varphi = \varphi^\top a$ for $a \in \{\mu, \sigma, \tilde{\sigma}\}$. If we now consider the indexing set $\Phi = \bar{B}_C(0)$, it is clear that the families $(\mu_\varphi)_{\varphi\in\bar{B}_C(0)}$, $(|\sigma_\varphi|^2)_{\varphi\in\bar{B}_C(0)}$ and $(|\tilde{\sigma}_\varphi|^2)_{\varphi\in\bar{B}_C(0)}$ on $[0, \bar{T}]$ are uniformly integrable by the ϵ-δ-criterion for uniform integrability (see Klenke [83, Theorem 6.24]), since Φ is bounded, $\mu \in L^1([0, \bar{T}]; \mathbb{C}^n)$ and $\sigma, \tilde{\sigma} \in L^2([0, \bar{T}]; \mathbb{C}^n)$. Thus part 1) of Theorem 2.17 likewise follows from part 1) of Theorem 2.18. □

It now remains to show Theorem 2.18. Since the proof is quite technical, we first give an overview of the main ideas. Indeed, the proof is conceptually simple: using Corollary 2.14 and Itô's formula, it is straightforward to check that the expression on the right-hand side of (2.43) matches the terminal value $\exp(\tilde{X}_T^\varphi)$ for any integrable function $g_{\varphi,T}$, and that it is a local martingale if and only if $g_{\varphi,T}$ satisfies the Riccati–Volterra equation (2.42). We then have to show that there exists a solution to that equation and that the resulting process is a true martingale, which is technically more challenging. We use Lemma 2.16 for this task, as well as some of the results in Appendix A on Riccati–Volterra equations.

The main challenge stems from the fact that our results on the existence of solutions to Riccati–Volterra equations ensure this existence only locally, i.e., up to some positive time. Thus it may happen that a solution $g_{\varphi,T}$ only exists up to some smaller time $T' < T$, so that the expression (2.43) is not well defined on all of $[0, T]$. We might then hope to show that (2.43) holds with T' in place of T. However, this could still fail, since the formula for that conditional expectation is given in terms of the solution $g_{\varphi,T'}$ to a new Riccati–Volterra equation, for which a solution may in turn only exist up to an even smaller time $T'' < T'$, and so on. Instead, we take the approach of looking for solutions $g_{\varphi,T}$ to (2.42) for all (parameter) values of T simultaneously, with the goal of showing that there exists some small enough $T' > 0$ such that for each $T \in [0, T']$, there is a solution $g_{\varphi,T}$ on $[0, T]$. For this, we make use of results from Appendix A on the existence of solutions to a family of Riccati–Volterra equations on a common time interval, subject to some uniform bounds on the inputs to the equations. Those results also allow us to show the result simultaneously for all $\varphi \in \Phi$. In this way, we address the issue of showing the existence of solutions $g_{\varphi,T}$ on $[0, T]$ for small enough T.

Once we have obtained the existence of a solution $g_{\varphi,T}$ to (2.42) on the interval $[0, T]$, we still have to show that the expression on the right-hand side of (2.43) gives a true martingale, and this need not be true in general. Instead, we show that there exists some smaller time $\tilde{T} \in (0, T']$ such that for $T \in [0, \tilde{T}]$, there is a solution $\tilde{g}_{\varphi,T}$ on $[0, T]$ to an auxiliary Riccati–Volterra equation (see (2.60) below). We can then use Lemma 2.15 to show that the right-hand side of (2.43) is a true martingale if $T \in [0, \tilde{T}]$. As it turns out, the main issue here is to check that the terminal value $\exp(\tilde{X}_T^{\varphi})$ is actually integrable, and the second restriction on the time horizon ensures that this is the case. If we know a priori that $\exp(\tilde{X}_T^{\varphi})$ is integrable, this second step is not needed, and we show in the proof of part 2) that the existence of $g_{\varphi,T}$ is sufficient. For instance, this can be applied in the setup of Theorem 2.17 if we have $\tilde{X}^{\varphi} = z^{\top}\tilde{X}$, where $\varphi = z \in (\mathrm{i}\mathbb{R})^n$ and $\tilde{X}$ is real-valued, i.e., for calculating the characteristic function of $\tilde{X}_T$.

Proof of Theorem 2.18. **1)** This proof is rather lengthy and goes over several steps.

1a) We first show the existence of some $T' > 0$ such that there are solutions $g_{\varphi,T}$ to the equations (2.42) for all $\varphi \in \Phi$ and $T \in (0, T']$ (later, we further restrict to a smaller time $\tilde{T} \leq T'$). Note that (2.42) depends on T both via the index in $g_{\varphi,T}$ and the time horizon in $0 \leq t \leq T$. To eliminate the second dependence, define for $T \in (0, \bar{T}]$ the functions $\mu_{\varphi,T}, \sigma_{\varphi,T}, \tilde{\sigma}_{\varphi,T} : [0, \bar{T}] \to \mathbb{C}$ by

$$a_{\varphi,T}(t) = a_\varphi(T - t)\mathbf{1}_{[0,T]}(t) \quad \text{for } a \in \{\mu, \sigma, \tilde{\sigma}\}. \tag{2.44}$$

By assumption, the family $(a_\varphi)_{\varphi\in\Phi}$ is uniformly integrable for $a \in \{\mu, |\sigma^2|, |\tilde{\sigma}^2|\}$. Since the indicator functions in (2.44) are bounded, the ϵ-δ-criterion for uniform integrability (see Klenke [83, Theorem 6.24]) yields that for $a \in \{\mu, |\sigma^2|, |\tilde{\sigma}^2|\}$, the family $\{a_{\varphi,T} : \varphi \in \Phi, T \in (0, \bar{T}]\}$ is also uniformly integrable.

We now consider for $T \in (0, \bar{T}]$ and $\varphi \in \Phi$ the extended equation

$$g_{\varphi,T}(t) = \mu_{\varphi,T}(t) + f\big((\hat{\kappa} * g_{\varphi,T})(t); \sigma_{\varphi,T}(t), \tilde{\sigma}_{\varphi,T}(t)\big) \tag{2.45}$$

for $0 \leq t \leq \bar{T}$. These equations now all have the same time horizon $\bar{T}$, and depend on T only through the T-indexed coefficients $\mu_{\varphi,T}, \sigma_{\varphi,T}$ and $\tilde{\sigma}_{\varphi,T}$. By construction, if $g_{\varphi,T}$ is a solution to (2.45), then its restriction to $[0, T]$ is also a solution to (2.42). Indeed, once we restrict to $[0, T]$, the indicator $\mathbf{1}_{[0,T]}$ in (2.44) can be omitted, and the restriction of the convolution $\hat{\kappa} * g_{\varphi,T}$ to $[0, T]$ only depends on the restriction of $g_{\varphi,T}$ to $[0, T]$, due to the causality property of the convolution, as pointed out after Definition 2.3. So plugging (2.44) into (2.45) shows that we have

$$\begin{aligned} g_{\varphi,T}(t) &= \mu_{\varphi,T}(t) + f\big((\hat{\kappa} * g_{\varphi,T})(t); \sigma_{\varphi,T}(t), \tilde{\sigma}_{\varphi,T}(t)\big) \\ &= \mu_\varphi(T - t) + f\big((\hat{\kappa} * g_{\varphi,T})(t); \sigma_\varphi(T - t), \tilde{\sigma}_\varphi(T - t)\big), \quad 0 \leq t \leq T, \end{aligned}$$

which is exactly (2.42).

The extended equations (2.45) need not admit solutions on the whole interval $[0, \bar{T}]$ in general. Nevertheless, we show in the next step that there exists some $T' \in (0, \bar{T}]$ such that (2.45) admits a solution $g_{\varphi,T} \in L^1([0, T']; \mathbb{C})$ on $[0, T']$ for all $\varphi \in \Phi$ and $T \in (0, \bar{T}]$. Restricting as above from $[0, T']$ to $[0, T]$ then yields solutions to (2.42), for $T \leq T'$.

1b) Consider the extended indexing set $\mathcal{J} := \Phi \times (0, \bar{T}]$ and index $j := (\varphi, T)$. Write $y_{\varphi,T} = \mu_{\varphi,T}$, $h_{\varphi,T} = (\sigma_{\varphi,T}, \tilde{\sigma}_{\varphi,T})$ and $k_{\varphi,T} = \hat{\kappa}$, the latter of which does not depend on φ or T, and set $p = q = a = 2$, $m = 2$, $n = 1$ and f as given by (2.39). We now want to apply part 2) of Corollary A.2.7 with respect to these functions and parameters, and so we check its conditions. Define the increasing functions

$\bar{k}, \bar{y}, \bar{h} : [0, \bar{T}] \to [0, \infty)$ by $\bar{k}(t) := \|\hat{\kappa}\|_{L^2(0,t)}$ and

$$\bar{y}(t) := \sup_{\substack{\varphi \in \Phi \\ T \in (0,\bar{T}]}} \|\mu_{\varphi,T}\|_{L^1(0,t)}, \quad \bar{h}(t) := \sup_{\substack{\varphi \in \Phi \\ T \in (0,\bar{T}]}} \big(\|\sigma_{\varphi,T}\|_{L^2(0,t)} + \|\tilde{\sigma}_{\varphi,T}\|_{L^2(0,t)}\big). \quad (2.46)$$

Note that $\hat{\kappa}$ inherits the local square-integrability from κ as pointed out before Remark 2.5, and so $\hat{\kappa} \in L^2([0, \bar{T}]; \mathbb{R}_+)$, i.e., the singleton $\{\mathbf{1}_{[0,\bar{T}]}|\hat{\kappa}^2|\}$ is uniformly integrable. Likewise, we have shown in 1a) the uniform integrability of $(\mu_{\varphi,T})$, $(|\sigma_{\varphi,T}|^2)$ and $(|\tilde{\sigma}_{\varphi,T}|^2)$. It then follows by the uniform integrability of these families and the ϵ-δ-criterion for uniform integrability that $\bar{a}(\bar{T}) < \infty$ and $\bar{a}(t) \searrow 0$ as $t \searrow 0$ for $\bar{a} \in \{\bar{k}, \bar{y}, \bar{h}\}$. Moreover, f satisfies the quadratic bound

$$\begin{aligned} |f(x; h, \tilde{h})| &\le \frac{|h|^2 + |\tilde{h}|^2 + |x|^2}{2} + \big|\varrho h + \sqrt{1-\varrho^2}\tilde{h}\big||x| \\ &\le \frac{(1+\varrho^2)|h|^2 + (2-\varrho^2)|\tilde{h}|^2 + 3x^2}{2}, \end{aligned} \quad (2.47)$$

thanks to the quadratic inequality $|\varrho h_1||x| \le \frac{\varrho^2|h|^2}{2} + \frac{|x|^2}{2}$, and likewise for $\tilde{h}$. Therefore, by part 2) of Corollary A.2.7, there exists some $T' \in (0, \bar{T}]$ such that there are solutions $g_{\varphi,T} \in L^1([0, T']; \mathbb{C})$ to (2.45) on a common interval $[0, T']$ for all $\varphi \in \Phi$ and $T \in (0, \bar{T}]$. For later use, we note here that part 2) of Corollary A.2.7 also yields

$$\sup_{\substack{\varphi \in \Phi \\ T \in (0,T']}} \|g_{\varphi,T}\|_{L^1(0,T')} < \infty \quad \text{and} \quad \lim_{t \searrow 0} \sup_{\substack{\varphi \in \Phi \\ T \in (0,T']}} \|g_{\varphi,T}\|_{L^1(0,t)} = 0. \quad (2.48)$$

As argued at the end of step 1a), we deduce by restricting to $[0, T]$ that $g_{\varphi,T}$ is also a solution to (2.42) for each $\varphi \in \Phi$ and $T \in (0, T']$. The uniqueness of the solution $g_{\varphi,T}$ to (2.42) follows by part 4) of Corollary A.2.7, since f satisfies the Lipschitz-type bound

$$\begin{aligned} \big|f(x; h, \tilde{h}) - f(x'; h, \tilde{h})\big| &\le \frac{|x^2 - (x')^2|}{2} + \big(\varrho h + \sqrt{1-\varrho^2}\tilde{h}\big)|x - x'| \\ &\le |x - x'|\left(\frac{1}{2}(|x| + |x'|) + |\varrho h| + \sqrt{1-\varrho^2}|\tilde{h}|\right). \end{aligned}$$

1c) Now fix $\varphi \in \Phi$, $T \in (0, \bar{T}]$, take the corresponding solution $g_{\varphi,T}$ of (2.42) and define the process $Z = (Z_t)_{0 \le t \le T}$ by

$$Z_t := \exp\left(\tilde{X}_t^\varphi + \int_t^T g_{\varphi,T}(T-u)\xi_t(u)du\right), \quad 0 \le t \le T. \quad (2.49)$$

Because $g_{\varphi,T} \in L^1([0,T];\mathbb{C})$, the integral term vanishes for $t = T$ by Corollary 2.14 so that

$$Z_T = \exp(\tilde{X}_T^{\varphi}). \tag{2.50}$$

Moreover, Corollary 2.14 also gives the semimartingale decomposition

$$d\biggl(\int_t^T g_{\varphi,T}(T-u)\xi_t(u)du\biggr) = -g_{\varphi,T}(T-t)Y_t dt + (\hat{\kappa} * g_{\varphi,T})(T-t)\sqrt{Y_t}dB_t.$$

Recalling the dynamics (2.41) for $\tilde{X}$, we apply Itô's formula to (2.49) and collect the finite variation terms to obtain the decomposition

$$\begin{aligned} dZ_t = Z_t\biggl(&\sqrt{Y_t}\bigl(\sigma_\varphi(t)dW_t + \tilde{\sigma}_\varphi(t)dW_t^\perp\bigr) + (\hat{\kappa} * g_{\varphi,T})(T-t)\sqrt{Y_t}dB_t \\ &+ \Bigl(-g_{\varphi,T}(T-t) + \mu_\varphi(t) + f\bigl((\hat{\kappa} * g_{\varphi,T})(T-t); \sigma_\varphi(t), \tilde{\sigma}_\varphi(t)\bigr)\Bigr) Y_t dt\biggr). \end{aligned} \tag{2.51}$$

Because $g_{\varphi,T}$ satisfies (2.42), the dt-integral in (2.51) vanishes, and so Z is a local martingale. If we can show that it is a true martingale, we obtain from (2.50) and (2.49) that

$$E[\exp(\tilde{X}_T^\varphi) \mid \mathcal{F}_t] = Z_t = \exp\biggl(\tilde{X}_t^\varphi + \int_t^T g_{\varphi,T}(T-u)\xi_t(u)du\biggr), \tag{2.52}$$

for $0 \le t \le T$, which is precisely (2.43). Moreover, Z_T is then also integrable, which means by (2.50) that $E[|\exp(\tilde{X}_T^\varphi)|] < \infty$.

1d) To complete the proof of 1), it remains to show that the local martingale Z from (2.49) is a true martingale. More precisely, we claim that there exists some $\tilde{T} \in (0, T']$ such that for each $\varphi \in \Phi$ and $T \in (0, \tilde{T}]$, the corresponding process Z (or rather $Z = Z^{\varphi,T}$, but we omit the parameters for readability) is a martingale on $[0,T]$. To that end, fix for now φ and T and let M and N be the real-valued local martingales in the decomposition $dZ = Z(dM + \mathrm{i}dN)$. From (2.51), we can identify M and N as

$$\begin{aligned} dM_t =& \sqrt{Y_t}\Bigl(\mathrm{Re}\bigl(\sigma_\varphi(t)\bigr)dW_t + \mathrm{Re}\bigl(\tilde{\sigma}_\varphi(t)\bigr)dW_t^\perp\Bigr) \\ &+ \bigl(\hat{\kappa} * \mathrm{Re}(g_{\varphi,T})\bigr)(T-t)\sqrt{Y_t}dB_t, \quad 0 \le t \le T, \\ dN_t =& \sqrt{Y_t}\Bigl(\mathrm{Im}\bigl(\sigma_\varphi(t)\bigr)dW_t + \mathrm{Im}\bigl(\tilde{\sigma}_\varphi(t)\bigr)dW_t^\perp\Bigr) \\ &+ \bigl(\hat{\kappa} * \mathrm{Im}(g_{\varphi,T})\bigr)(T-t)\sqrt{Y_t}dB_t, \quad 0 \le t \le T. \end{aligned} \tag{2.53}$$

With the identity $(\operatorname{Re} x)^2 + (\operatorname{Im} x)^2 = |x|^2$ for $x \in \mathbb{C}$, this yields

$$[M]_T + [N]_T = \int_0^T \psi_{\varphi,T}(T-u) Y_u du, \tag{2.54}$$

where the coefficient $\psi_{\varphi,T} : [0,T] \to \mathbb{R}$ is given by

$$\begin{aligned} \psi_{\varphi,T}(t) := &\big|\sigma_\varphi(T-t) + \varrho(\hat{\kappa} * g_{\varphi,T})(t)\big|^2 \\ &+ \big|\tilde{\sigma}_\varphi(T-t) + \sqrt{1-\varrho^2}(\hat{\kappa} * g_{\varphi,T})(t)\big|^2, \quad 0 \le t \le T. \end{aligned} \tag{2.55}$$

By part 1) of Lemma 2.16 with $p = q = 2$, the process Z from (2.49) is a true martingale if we can show that

$$U := \exp([M]_T + [N]_T) \text{ is integrable.} \tag{2.56}$$

We show (2.56) in steps 1e) and 1f) below. The main idea is to construct a local martingale $\hat{Z} \ge 0$ of a similar form as Z with the property that $\hat{Z}_T = U$. Because $\hat{Z}$ is a supermartingale, hence integrable, (2.56) will then follow.

1e) To prepare for the construction of $\hat{Z}$, we start by obtaining some bounds on $\psi_{\varphi,T}$ that are uniform in $\varphi \in \Phi$ and $T \in (0, T']$. First, (2.55) and the Cauchy–Schwarz inequality give

$$\begin{aligned} |\psi_{\varphi,T}(s)| &\le 2\big(|\sigma_\varphi(T-s)|^2 + |\tilde{\sigma}_\varphi(T-s)|^2\big) + 2(\varrho^2 + 1 - \varrho^2)|(\hat{\kappa} * g_{\varphi,T})(s)|^2 \\ &= 2\big(|\sigma_\varphi(T-s)|^2 + |\tilde{\sigma}_\varphi(T-s)|^2\big) + 2|(\hat{\kappa} * g_{\varphi,T})(s)|^2, \quad 0 \le s \le T. \end{aligned}$$

Taking the L^1-norm on $(0,t)$ and using Lemma A.1.5 thus gives

$$\begin{aligned} \|\psi_{\varphi,T}\|_{L^1(0,t)} &\le 2\big(\|\sigma_{\varphi,T}\|^2_{L^2(0,t)} + \|\tilde{\sigma}_{\varphi,T}\|^2_{L^2(0,t)}\big) + 2\|\hat{\kappa}\|^2_{L^2(0,t)}\|g_{\varphi,T}\|^2_{L^1(0,t)} \\ &\le 2\big(\bar{h}^2(t) + \bar{k}^2(t)\|g_{\varphi,T}\|^2_{L^1(0,t)}\big), \end{aligned} \tag{2.57}$$

for each $t \in (0,T]$, where we use the coefficients $\sigma_{\varphi,T}, \tilde{\sigma}_{\varphi,T}$ from (2.44) and recall the increasing functions $\bar{h}$ and $\bar{k}$ from (2.46). We have shown in step 1b) that $\bar{h}$ and $\bar{k}$ are finite on $[0,\bar{T}]$ with $\bar{h}(t) \searrow 0$ and $\bar{k}(t) \searrow 0$ as $t \searrow 0$. Together with the first part of (2.48), we obtain by setting $t = T \le T'$ and taking the supremum over φ and T that

$$\sup_{\substack{\varphi\in\Phi \\ T\in(0,T']}} \|\psi_{\varphi,T}\|_{L^1(0,T)} \le 2\Bigg(\bar{h}^2(T') + \bar{k}^2(T') \sup_{\substack{\varphi\in\Phi \\ T\in(0,T']}} \|g_{\varphi,T}\|^2_{L^1(0,T')}\Bigg) < \infty. \tag{2.58}$$

From (2.57) and the second part of (2.48), we also obtain

$$\lim_{t\searrow 0} \sup_{\substack{\varphi\in\Phi \\ T\in(0,T']}} \|\psi_{\varphi,T}\|_{L^1(0,t\wedge T)} \le \lim_{t\searrow 0} 2\left(\bar{h}^2(t) + \bar{k}^2(t) \sup_{\substack{\varphi\in\Phi \\ T\in(0,T']}} \|g_{\varphi,T}\|^2_{L^1(0,t\wedge T)}\right) = 0. \tag{2.59}$$

Now consider the auxiliary Riccati–Volterra equation

$$\tilde{g}_{\varphi,T}(t) = \mathbf{1}_{[0,T]}(t)\psi_{\varphi,T}(t) + \frac{1}{2}\big((\hat{\kappa} * \tilde{g}_{\varphi,T})(t)\big)^2, \tag{2.60}$$

for $0 \le t \le T'$. As in step 2b), we once again check the conditions in order to apply part 2) of Corollary A.2.7 to (2.60) with the indexing set $\mathcal{J} = \Phi \times [0,T']$ and index $j = (\varphi, T)$. Set $m = 0$, $n = 1$, $p = q = a = 2$ and

$$k_{\varphi,T} := k := \hat{\kappa}, \quad \tilde{y}_{\varphi,T} := \mathbf{1}_{[0,T]}\psi_{\varphi,T}, \quad \tilde{f}(x) := \frac{x^2}{2}$$

(note that h and $\bar{h}$ in Corollary A.2.7 are not needed here, since $\tilde{f}$ depends only on x). The quadratic bound on $\tilde{f}(x) = \frac{x^2}{2}$ is trivial, and we showed in step 1b) that $\bar{k}(t) := \|\hat{\kappa}\|_{L^2(0,t)} < \infty$ and $\bar{k}(t) \searrow 0$ as $t \searrow 0$. Moreover, for $\bar{y}_{\text{aux}} : [0,T'] \to [0,\infty)$ defined by

$$\bar{y}_{\text{aux}}(t) := \sup_{\substack{\varphi\in\Phi \\ T\in(0,T']}} \|\tilde{y}_{\varphi,T}\|_{L^1(0,t)}, \quad 0 \le t \le T',$$

we obtain by plugging $\tilde{y}_{\varphi,T} = \mathbf{1}_{[0,T]}\psi_{\varphi,T}$ into the corresponding bounds (2.58) and (2.59) for $\psi_{\varphi,T}$ that $\bar{y}_{\text{aux}}(T') < \infty$ and $\bar{y}_{\text{aux}}(t) \searrow 0$ as $t \searrow 0$. Hence by part 2) of Corollary A.2.7, there exists some $\tilde{T} \in (0,T']$ such that (2.60) has a solution $\tilde{g}_{\varphi,T} \in L^1([0,\tilde{T}];\mathbb{C})$ for all $T \in (0,T']$ and $\varphi \in \Phi$. In particular, for $T \in (0,\tilde{T}]$, the restriction of $g_{\varphi,T}$ to $[0,T]$ satisfies the equation

$$\tilde{g}_{\varphi,T}(t) = \psi_{\varphi,T}(t) + \frac{1}{2}\big((\hat{\kappa} * \tilde{g}_{\varphi,T})(t)\big)^2, \quad 0 \le t \le T, \tag{2.61}$$

as the indicator $\mathbf{1}_{[0,T]}$ can be removed from (2.60). Moreover, $\psi_{\varphi,T}$ is real-valued for all $\varphi \in \Phi$ and $T \in (0,T']$ by its definition (2.55), and $\hat{\kappa} = \frac{\zeta}{\lambda}R^{\lambda\kappa}$ is real-valued because κ is; see before Remark 2.5. As pointed out in Remark A.2.1, the solution $\tilde{g}_{\varphi,T}$ to (2.60) is therefore also real-valued.

1f) We are now ready to construct a local martingale $\hat{Z} \ge 0$ with $\hat{Z}_T = U$ from (2.56). This will complete the proof of 1) as seen at the end of steps 1d) and 1c). Define $\hat{g}_{\varphi,T}(t) := \tilde{g}_{\varphi,T}(t) - \psi_{\varphi,T}(t)$ for $t \in [0,T]$, where $\tilde{g}_{\varphi,T}$ is the solution to (2.60) from step 1e). Then like $\tilde{g}_{\varphi,T}$ and $\psi_{\varphi,T}$, the function $\hat{g}_{\varphi,T}$ is real-valued and in $L^1([0,T];\mathbb{R})$, and plugging into (2.61) shows that it solves the equation

$$\hat{g}_{\varphi,T}(t) = \frac{1}{2}\Big(\big(\hat{\kappa} * (\psi_{\varphi,T} + \hat{g}_{\varphi,T})\big)(t)\Big)^2, \quad 0 \leq t \leq T. \tag{2.62}$$

Define $\hat{X} = (\hat{X}_t)_{0\leq t\leq T}$ by

$$\begin{aligned}\hat{X}_t &:= \int_0^T \psi_{\varphi,T}(T-u)\xi_t(u)du \\ &= \int_0^t \psi_{\varphi,T}(T-u)Y_u du + \int_t^T \psi_{\varphi,T}(T-u)\xi_t(u)du,\end{aligned} \tag{2.63}$$

where we recall from Definition 2.9 that $\xi_t(u) = Y_u$ for $t \geq u$. In particular,

$$\hat{X}_T = \int_0^T \psi_{\varphi,T}(T-u)Y_u du = [M]_T + [N]_T = \log U$$

by (2.54) and (2.56). Then define $\hat{Z} = (\hat{Z}_t)_{0\leq t\leq T}$ by

$$\hat{Z}_t := \exp\left(\hat{X}_t + \int_t^T \hat{g}_{\varphi,T}(T-u)\xi_t(u)du\right). \tag{2.64}$$

Note that because $\hat{g}_{\varphi,T}$ is integrable, Corollary 2.14 implies that

$$\hat{Z}_T = \exp(\hat{X}_T) = U,$$

as required. Moreover, both $\psi_{\varphi,T}$ and $\hat{g}_{\varphi,T}$ are integrable and $u \mapsto \xi_0(u)$ is bounded because ξ is continuous by Proposition 2.11, and hence $\hat{X}_0$ and $\hat{Z}_0$ are finite. We also have that $\psi_{\varphi,T}$, $\hat{g}_{\varphi,T}$ and $u \mapsto \xi_0(u)$ are nonrandom (the latter due to (2.13)), so that $\hat{X}_0$ and $\hat{Z}_0$ are nonrandom as well. It only remains to argue that $\hat{Z}$ is a local martingale, and this is similar to step 1c). By (2.63) and Lemma 2.13 with the measure $\nu(du) = \psi(T-u)du$, we have the decomposition

$$\begin{aligned}\hat{X}_t &= \int_0^T \psi_{\varphi,T}(T-u)\xi_t(u)du = \hat{X}_0 + \int_0^t \left(\int_s^T \psi_{\varphi,T}(T-u)\hat{\kappa}(u-s)du\right)\sqrt{Y_s}dB_s \\ &= \int_0^T \psi_{\varphi,T}(T-u)\xi_t(u)du = \hat{X}_0 + \int_0^t (\hat{\kappa} * \psi_{\varphi,T})(T-s)\sqrt{Y_s}dB_s,\end{aligned}$$

for $t \in [0,T]$. By Corollary 2.14, we also have the dynamics

$$d\bigg(\int_t^T \hat{g}_{\varphi,T}(T-u)\xi_t(u)du\bigg) = -\hat{g}_{\varphi,T}(T-t)Y_t dt + (\hat{\kappa} * \hat{g}_{\varphi,T})(T-t)\sqrt{Y_t}dB_t.$$

Thus by Itô's formula, we obtain from (2.64) the decomposition

$$\begin{aligned}\frac{d\hat{Z}_t}{\hat{Z}_t} &= \big(\hat{\kappa} * (\psi_{\varphi,T} + \hat{g}_{\varphi,T})\big)(T-t)\sqrt{Y_t}dB_t \\ &\quad - \bigg(\hat{g}_{\varphi,T}(T-t) - \frac{1}{2}\Big(\big(\hat{\kappa} * (\psi_{\varphi,T} + \hat{g}_{\varphi,T})\big)(T-t)\Big)^2\bigg)Y_t dt\end{aligned}$$

for $0 \leq t \leq T$. But the dt-integral vanishes because $\hat{g}_{\varphi,T}$ satisfies (2.62); so $\hat{Z}$ is indeed a local martingale, and the proof of 1) is complete.

2) The proof of the second part is likewise divided into steps 2a)–2d), which are similar to the steps 1c)–1f). We do not need to replicate the steps 1a) and 1b), as the existence of a solution $g_{\varphi,T} \in L^1([0,T];\mathbb{C})$ to (2.42) is assumed.

2a) As in step 1c), consider the process $(Z_t)_{0\leq t\leq T}$ defined by

$$Z_t := \exp\bigg(\tilde{X}_t^{\varphi} + \int_t^T g_{\varphi,T}(T-u)\xi_t(u)du\bigg), \quad 0 \leq t \leq T.$$

By the same calculation as in (2.51) using Corollary 2.14 and Itô's formula, Z is a local martingale as the finite variation part vanishes. As $g_{\varphi,T} \in L^1([0,T];\mathbb{C})$, we also get from Corollary 2.14 that the integral term vanishes at $t = T$ and hence $Z_T = \exp(\tilde{X}_T^{\varphi})$. Thus if we show that Z is a true martingale on $[0,T]$, the statement likewise follows by (2.52).

2b) As in step 1d), the idea is to use Lemma 2.16 to show that Z is a true martingale on $[0,T]$. The difference in this case is that we cannot restrict the time interval as T is given, and thus we apply part 2) of the lemma instead of part 1).

We start by once again decomposing $Z = Z_0\mathcal{E}(M + \mathrm{i}N)$ for real-valued continuous local martingales M and N, which are given by the same formula (2.53) as before. Replacing T with t and $t+\delta$ in (2.54) and taking differences, we obtain the equation

$$[M]_{t+\delta} - [M]_t + [N]_{t+\delta} - [N]_t = \int_t^{t+\delta} \psi_{\varphi,T}(T-u)Y_u du \tag{2.65}$$

for each $\delta \in [0,T]$ and $t \in [0, T-\delta]$, where $\psi_{\varphi,T}$ is defined in (2.55), which we recall here; it reads

$$\psi_{\varphi,T}(t) = \left|\sigma_\varphi(T-t) + \varrho(\hat{\kappa} * g_{\varphi,T})(t)\right|^2 + \left|\tilde{\sigma}_\varphi(T-t) + \sqrt{1-\varrho^2}(\hat{\kappa} * g_{\varphi,T})(t)\right|^2, \quad 0 \le t \le T.$$

By part 2) of Lemma 2.16 with $p = q = 2$, Z is a true martingale on $[0,T]$ if there exists some $\delta > 0$ such that

$$U^{t,\delta} := \exp\big([M]_{t+\delta} - [M]_t + [N]_{t+\delta} - [N]_t\big) \in L^1 \text{ for each } 0 \le t \le T-\delta, \tag{2.66}$$

since the terminal value $Z_T = \exp(\tilde{X}_T^\varphi)$ is integrable by assumption. Similarly to the proof of 1), we want to show (2.66) by constructing local martingales $\hat{Z}^{t,\delta} \ge 0$ with the property that $\hat{Z}_T^{t,\delta} = U^{t,\delta}$ for each $0 \le t \le T-\delta$. Then $\hat{Z}^{t,\delta}$ is a supermartingale, hence integrable, and (2.66) will follow.

2c) For $\delta > 0$ and $t \in [0, T-\delta]$, consider the Riccati–Volterra equation

$$\tilde{g}^{t,\delta}_{\varphi,T}(s) = \mathbf{1}_{[T-t-\delta,T-t]}(s)\psi_{\varphi,T}(s) + \frac{1}{2}\big((\hat{\kappa} * \tilde{g}^{t,\delta}_{\varphi,T})(s)\big)^2, \quad 0 \le s \le T. \tag{2.67}$$

This equation is similar to (2.60), but we need a different approach since T is fixed a priori, and hence it is not sufficient to find a solution to (2.67) on a smaller time interval. Instead, we show that $\delta > 0$ can be chosen small enough so that (2.67) admits a solution on $[0,T]$ for each $t \in [0, T-\delta]$.

We use Proposition A.2.8 and its Corollary A.2.9 to show the existence of a solution to (2.67). The equation (2.67) is of the form (A.2.30), with coefficients $a = \mathbf{1}_{[T-t-\delta,T-t]}\psi_{\varphi,T}$, $b = 0$, $c = \frac{1}{2}$ and $k = \hat{\kappa}$; note that only a depends on δ. We also fix the constants $\hat{T} = T$, $B = 0$, $C = \frac{1}{2}$ and some arbitrary $\epsilon > 0$. By Corollary A.2.9, there exist large enough $\gamma > 0$ and small enough $A = A(\gamma) > 0$ such that if

$$\|\mathbf{1}_{[T-t-\delta,T-t]}\psi_{\varphi,T}\|_{L^1(0,T)} \le A, \tag{2.68}$$

then (2.67) admits a solution $g^{t,\delta}_{\varphi,T} \in L^1([0,T];\mathbb{R})$; the fact that $g^{t,\delta}_{\varphi,T}$ is real-valued like a, b, c and κ follows as in Remark A.2.1. Since $\psi_{\varphi,T} \in L^1([0,T];\mathbb{R})$ as shown in (2.57), we have

$$\lim_{\delta \searrow 0} \sup_{t\in[0,T-\delta]} \|\mathbf{1}_{[T-t-\delta,T-t]}\psi_{\varphi,T}\|_{L^1(0,T)} = 0$$

by the ϵ-δ-criterion for uniform integrability applied to the singleton $\{\psi_{\varphi,T}\}$. Thus there exists some $\delta > 0$ such that (2.68) holds for each $0 \le t \le T-\delta$. For that choice of δ, we get by Corollary A.2.9 a solution $g^{t,\delta}_{\varphi,T} \in L^1([0,T];\mathbb{R})$ to (2.67) for each $0 \le t \le T-\delta$.

2d) The last step is now very similar to 1f). Fix δ as in 2c) and some $t \in [0, T-\delta]$, and define

$$\hat{g}^{t,\delta}_{\varphi,T}(s) = \tilde{g}^{t,\delta}_{\varphi,T}(s) - \mathbf{1}_{[T-t-\delta,T-t]}(s)\psi_{\varphi,T}(s), \tag{2.69}$$

where $\tilde{g}^{t,\delta}_{\varphi,T}$ is the solution to (2.67). Then like $\tilde{g}^{t,\delta}_{\varphi,T}$ and $\psi_{\varphi,T}$, the function $\hat{g}^{t,\delta}_{\varphi,T}$ is real-valued and in $L^1([0,T];\mathbb{R})$. Plugging into (2.60), we see that it solves the equation

$$\hat{g}^{t,\delta}_{\varphi,T}(s) = \frac{1}{2}\Big(\big(\hat{\kappa} * (\mathbf{1}_{[T-t-\delta,T-t]}\psi_{\varphi,T} + \hat{g}^{t,\delta}_{\varphi,T})\big)(s)\Big)^2, \quad 0 \le s \le T. \tag{2.70}$$

Similarly to (2.63), we define $\hat{X}^{t,\delta} = (\hat{X}^{t,\delta}_s)_{0\le s\le T}$ by

$$\hat{X}^{t,\delta}_s := \int_t^{t+\delta} \psi_{\varphi,T}(T-u)\xi_s(u)du, \tag{2.71}$$

where we recall from Definition 2.9 that $\xi_t(u) = Y_u$ for $t \ge u$. In particular,

$$\begin{aligned}\hat{X}^{t,\delta}_T &= \int_t^{t+\delta} \psi_{\varphi,T}(T-u)Y_u du \\ &= [M]_{t+\delta} - [M]_t + [N]_{t+\delta} - [N]_t = \log U^{t,\delta}\end{aligned} \tag{2.72}$$

by (2.65) and (2.66). Then define $\hat{Z}^{t,\delta} = (\hat{Z}^{t,\delta}_t)_{0\le t\le T}$ by

$$\hat{Z}^{t,\delta}_t := \exp\left(\hat{X}^{t,\delta}_t + \int_t^T \hat{g}^{t,\delta}_{\varphi,T}(T-u)\xi_t(u)du\right). \tag{2.73}$$

Since $\hat{g}^{t,\delta}_{\varphi,T}$ is integrable, Corollary 2.14 implies that

$$\hat{Z}^{t,\delta}_T = \exp(\hat{X}^{t,\delta}_T) = U^{t,\delta},$$

as required. Moreover, $\psi^{t,\delta}_{\varphi,T}$ and $\hat{g}^{t,\delta}_{\varphi,T}$ are integrable and $u \mapsto \xi_0(u)$ is bounded because it is continuous, and hence $\hat{X}^{t,\delta}_0$ and $\hat{Z}^{t,\delta}_0$ are finite. We also have that $\psi^{t,\delta}_{\varphi,T}$, $\hat{g}^{t,\delta}_{\varphi,T}$ and $u \mapsto \xi_0(u)$ are nonrandom (the latter due to (2.13)), so that $\hat{X}^{t,\delta}_0$ and $\hat{Z}^{t,\delta}_0$ are nonrandom as well. It remains to show that $\hat{Z}^{t,\delta}$ is a local martingale.

By (2.71) and Lemma 2.13 with $\nu(du) := \mathbf{1}_{[t,t+\delta]}(u)\psi_{\varphi,T}(T-u)du$, we have

$$\begin{aligned}\hat{X}_s^{t,\delta} &= \hat{X}_0^{t,\delta} + \int_0^t \left(\int_t^{t+\delta} \psi_{\varphi,T}(T-u)\hat{\kappa}(u-s)du \right) \sqrt{Y_s}dB_s \\ &= \hat{X}_0^{t,\delta} + \int_0^t \big(\hat{\kappa} * (\mathbf{1}_{[T-t-\delta,T-t]}\psi_{\varphi,T})\big)(T-s)\sqrt{Y_s}dB_s,\end{aligned}$$

for $s \in [0,T]$. By Corollary 2.14, we also have the dynamics

$$d\bigg(\int_t^T \hat{g}_{\varphi,T}^{t,\delta}(T-u)\xi_t(u)du \bigg) = -\hat{g}_{\varphi,T}^{t,\delta}(T-t)Y_t dt + (\hat{\kappa} * \hat{g}_{\varphi,T}^{t,\delta})(T-t)\sqrt{Y_t}dB_t.$$

Thus by Itô's formula, we obtain from (2.73) the decomposition

$$\begin{aligned}\frac{d\hat{Z}_t^{t,\delta}}{\hat{Z}_t^{t,\delta}} &= \big(\hat{\kappa} * (\mathbf{1}_{[T-t-\delta,T-t]}\psi_{\varphi,T} + \hat{g}_{\varphi,T}^{t,\delta})\big)(T-t)\sqrt{Y_t}dB_t \\ &\quad - \bigg(\hat{g}_{\varphi,T}^{t,\delta}(T-t) - \frac{1}{2}\Big(\big(\hat{\kappa} * (\mathbf{1}_{[T-t-\delta,T-t]}\psi_{\varphi,T} + \hat{g}_{\varphi,T}^{t,\delta})\big)(T-t)\Big)^2\bigg) Y_t dt\end{aligned}$$

for $0 \le t \le T$. But the dt-integral vanishes because $\hat{g}_{\varphi,T}^{t,\delta}$ satisfies (2.70); so $\hat{Z}^{t,\delta}$ is indeed a local martingale for each $t \in [0, T-\delta]$. Thus the integrability (2.66) follows due to (2.72), as argued at the end of 2b). As in 2a) and 2b), we can then apply part 2) of Lemma 2.16 to show that Z is a true martingale on $[0,T]$, which concludes the proof of 2). □

We collect in the next corollary some examples of conditional moment-generating functions that can be calculated using Theorem 2.17. In the following, we fix the complex parameter $z \in \mathbb{C}$ for simplicity, but one could also use Theorem 2.17 to obtain the result for all $z \in \bar{B}_C(0)$ simultaneously, for some $C > 0$. We also note that the coefficients that we obtain here do not depend on T; so we omit it from the notation. This simplification is possible because the coefficients $\mu, \sigma, \tilde{\sigma}$ below depend on t only indirectly, via the time to maturity $T-t$.

Corollary 2.19. *Fix $z \in \mathbb{C}$ and let $X = \log S$, where (S, Y) satisfies the rough Heston model (2.8). There exists some $\hat{T} > 0$ such that for all $T \in (0, \hat{T}]$, we have for $0 \le t \le T$ that*

$$\begin{aligned}E[e^{zX_T} \mid \mathcal{F}_t] &= \exp\left(zX_t + \int_t^T g_{1,z}(T-u)\xi_t(u)du \right), \\ E[e^{zY_T} \mid \mathcal{F}_t] &= \exp\left(z\xi_t(T) + \int_t^T g_{2,z}(T-u)\xi_t(u)du \right),\end{aligned}$$

$$E\left[e^{z\int_0^T X_s ds} \mid \mathcal{F}_t\right] = \exp\left(z\Big(\int_0^t X_s ds + (T-t)X_t\Big) + \int_t^T g_{3,z}(T-u)\xi_t(u)du\right),$$
$$E\left[e^{z\int_0^T Y_s ds} \mid \mathcal{F}_t\right] = \exp\left(z\Big(\int_0^t Y_s ds + \int_t^T \xi_t(u)du\Big) + \int_t^T g_{4,z}(T-u)\xi_t(u)du\right), \tag{2.74}$$

where $g_{1,z}, g_{2,z}, g_{3,z}, g_{4,z} \in L^1([0,T];\mathbb{C})$ are the unique solutions to the equations

$$g_{1,z}(t) = z\mu + \frac{\sigma^2}{2}(z^2 - z) + \varrho z\sigma(\hat{\kappa} * g_{1,z})(t) + \frac{1}{2}\big((\hat{\kappa} * g_1, z)(t)\big)^2,$$
$$g_{2,z}(t) = \frac{1}{2}\left(z\hat{\kappa}(t) + (\hat{\kappa} * g_{2,z})(t)\right)^2,$$
$$g_{3,z}(t) = z\mu t + \frac{\sigma^2}{2}(z^2t^2 - zt) + \varrho z t\sigma(\hat{\kappa} * g_{3,z})(t) + \frac{1}{2}\big((\hat{\kappa} * g_{3,z})(t)\big)^2,$$
$$g_{4,z}(t) = \frac{1}{2}\left(z\int_0^t \hat{\kappa}(s)ds + (\hat{\kappa} * g_{4,z})(t)\right)^2 \tag{2.75}$$

for $0 \le t \le T$. Moreover, $g_{1,z}$, $g_{3,z}$ and $g_{4,z}$ are continuous and hence bounded on $[0,T]$.

Proof. Fix some time horizon $\bar{T} > 0$ and consider the semimartingales $(\tilde{X}_t^{(i)})_{0\le t\le \bar{T}}$ for $i = 1,2,3,4$ defined by

$$\tilde{X}_t^{(1)} = X_t = \log S_t, \qquad \tilde{X}_t^{(2)} = \xi_t(T),$$
$$\tilde{X}_t^{(3)} = \int_0^t X_s ds + (T-t)X_t, \qquad \tilde{X}_t^{(4)} = \int_0^t Y_s ds + \int_t^T \xi_t(u)du.$$

Recall the orthogonal decomposition $B = \varrho W + \sqrt{1-\varrho^2}\, W^\perp$ introduced after (2.8). By Lemmas 2.10 and 2.13 together with the dynamics (2.8) for the rough Heston model, we have the semimartingale decompositions

$$d\tilde{X}_t^{(1)} = \left(\mu - \frac{\sigma^2}{2}\right)Y_t dt + \sigma\sqrt{Y_t}dW_t,$$
$$d\tilde{X}_t^{(2)} = \hat{\kappa}(T-t)\sqrt{Y_t}(\varrho dW_t + \sqrt{1-\varrho^2}dW_t^\perp),$$
$$d\tilde{X}_t^{(3)} = (T-t)\left(\mu - \frac{\sigma^2}{2}\right)Y_t dt + (T-t)\sigma\sqrt{Y_t}dW_t,$$
$$d\tilde{X}_t^{(4)} = \left(\int_t^T \hat{\kappa}(u-t)du\right)\sqrt{Y_t}(\varrho dW_t + \sqrt{1-\varrho^2}dW_t^\perp),$$

for $0 \le t \le \bar{T}$. Each of these dynamics has the form (2.37), where we identify the coefficients $\mu^{(1)} \equiv \mu - \frac{\sigma^2}{2}, \sigma^{(1)} \equiv \sigma$ and $\tilde{\sigma}^{(1)} \equiv 0$ for $X^{(1)}$, and likewise for $X^{(2)}$,

$X^{(3)}$ and $X^{(4)}$. Note that $\mu^{(i)}, \sigma^{(i)}$ and $\tilde{\sigma}^{(i)}$ are continuous for $i = 1, 3, 4$. Indeed, this is trivial for $i = 1, 3$, and the continuity of $\sigma^{(4)}$ and $\tilde{\sigma}^{(4)}$ follows since

$$\int_t^T \hat{\kappa}(u-t)du = \int_0^{T-t} \hat{\kappa}(s)ds$$

is continuous in t by the dominated convergence theorem; recall that $\hat{\kappa}$ is square-integrable as it inherits the local square-integrability from κ, as noted before Remark 2.5. In the case $i = 2$, we have that $\mu^{(2)} \equiv 0$ and $\sigma^{(2)}, \tilde{\sigma}^{(2)}$ are square-integrable on $[0, \bar{T}]$ as $\hat{\kappa}$ is. Thus the coefficients have the required integrability to apply part 1) of Theorem 2.17, which gives some $\tilde{T} \in (0, \bar{T}]$ such that for $T \in (0, \tilde{T}]$, there exist solutions $g_{i,z}$ to (2.38) on $[0, T]$ for each $i = 1, 2, 3, 4$; the equation (2.38) yields (2.75) by plugging in the respective coefficients for $i = 1, 2, 3, 4$. Moreover, Theorem 2.17 also gives (2.74) for all $0 \leq t \leq T \leq \tilde{T}$.

By Corollary A.2.6, there exists some (possibly smaller) $\hat{T} \in (0, \tilde{T}]$ such that the Riccati–Volterra equations (2.75) for $g_{1,z}, g_{3,z}$ and $g_{4,z}$ admit continuous solutions on $[0, \hat{T}]$, since the respective coefficients are continuous as well. Since Theorem 2.17 gives that the solutions to those equations are unique, we conclude that $g_{1,z}, g_{3,z}$ and $g_{4,z}$ are continuous on $[0, \hat{T}]$. □

Finally, we generalise part 1) of Theorem 2.18 by replacing the exponential term on the left-hand side of (2.40) with a product between an exponential and a linear term; this will be useful later in the proof of Lemma 3.3 and in Chapter II. In the proof, we use the identity $\frac{d}{d\delta} \exp((a+\delta)x)|_{\delta=0} = x \exp(ax)$ to approximate the linear term by an exponential one for which (2.40) holds. It then remains to show that both sides of (2.40) converge for the resulting approximation, which requires some care. In principle, the result could be further extended to polynomial terms rather than linear ones, but such a generalisation is not necessary for our purposes. We show the result directly for the generalised setup of Theorem 2.18, and then apply it to the setup of Theorem 2.17 as a corollary.

Proposition 2.20. *Let Φ be an indexing set, $(\mu_\varphi)_{\varphi\in\Phi}$ a family of functions in $L^1([0, \bar{T}]; \mathbb{C})$ and $(\sigma_\varphi)_{\varphi\in\Phi}, (\tilde{\sigma}_\varphi)_{\varphi\in\Phi}$ two families of functions in $L^2([0, \bar{T}]; \mathbb{C})$ such that $(\mu_\varphi)_{\varphi\in\Phi}$, $(|\sigma_\varphi|^2)_{\varphi\in\Phi}$ and $(|\tilde{\sigma}_\varphi|^2)_{\varphi\in\Phi}$ are uniformly integrable. For each $\varphi \in \Phi$, let $\tilde{x}^\varphi \in \mathbb{C}$ be a constant and define $(\tilde{X}_t^\varphi)_{0\leq t\leq\bar{T}}$ by (2.41). Then there exists some $\tilde{T} \in (0, \bar{T}]$ (which depends on (μ_φ), (σ_φ) and $(\tilde{\sigma}_\varphi)$) such that for all $\varphi_1, \varphi_2 \in \Phi$ and $T \in (0, \tilde{T}]$, there is a unique solution $g_{\varphi_1,T} \in L^1([0, T]; \mathbb{C})$ to (2.42) (with*

$\varphi = \varphi_1$) as well as a unique solution $\tilde{g}_{\varphi_1,\varphi_2,T} \in L^1([0,T];\mathbb{C})$ to the equation

$$\begin{aligned}\tilde{g}_{\varphi_1,\varphi_2,T}(t) = \mu_{\varphi_2}(T-t) + \tilde{f}\big((\hat{\kappa} * \tilde{g}_{\varphi_1,\varphi_2,T})(t); (\hat{\kappa} * g_{\varphi_1,T})(t), \sigma_{\varphi_1}(T-t),\\ \tilde{\sigma}_{\varphi_1}(T-t), \sigma_{\varphi_2}(T-t), \tilde{\sigma}_{\varphi_2}(T-t)\big)\end{aligned} \tag{2.76}$$

for a.a. $t \in [0,T]$, where $\tilde{f} : \mathbb{C}^6 \to \mathbb{C}$ is defined by

$$\begin{aligned}\tilde{f}(x; y, h_1, \tilde{h}_1, h_2, \tilde{h}_2) = (h_1 + \varrho y)(h_2 + \varrho x)\\ + (\tilde{h}_1 + \sqrt{1-\varrho^2}y)(\tilde{h}_2 + \sqrt{1-\varrho^2}x).\end{aligned} \tag{2.77}$$

Moreover, it holds for $0 \le t \le T$ that

$$\begin{aligned}E[\exp(\tilde{X}_T^{\varphi_1})\tilde{X}_T^{\varphi_2} \mid \mathcal{F}_t] = \exp\left(\tilde{X}_t^{\varphi_1} + \int_t^T g_{\varphi_1,T}(T-u)\xi_t(u)du\right)\\ \times \left(\tilde{X}_t^{\varphi_2} + \int_t^T \tilde{g}_{\varphi_1,\varphi_2,T}(T-u)\xi_t(u)du\right).\end{aligned} \tag{2.78}$$

Proof. We divide the proof into three steps.

a) Since $(\mu_\varphi)_{\varphi\in\Phi}$, $(|\sigma_\varphi|^2)_{\varphi\in\Phi}$ and $(|\tilde{\sigma}_\varphi|^2)_{\varphi\in\Phi}$ are uniformly integrable, it follows by the ϵ-δ-criterion for uniform integrability (see Klenke [83, Theorem 6.24]) that each of the families

$$\begin{aligned}&\{\mu_{\varphi_1} + \delta\mu_{\varphi_2} : \varphi_1, \varphi_2 \in \Phi, \delta \in \bar{B}_1(0)\},\\ &\{|\sigma_{\varphi_1} + \delta\sigma_{\varphi_2}|^2 : \varphi_1, \varphi_2 \in \Phi, \delta \in \bar{B}_1(0)\},\\ &\{|\tilde{\sigma}_{\varphi_1} + \delta\tilde{\sigma}_{\varphi_2}|^2 : \varphi_1, \varphi_2 \in \Phi, \delta \in \bar{B}_1(0)\}\end{aligned}$$

is uniformly integrable as well, where $\bar{B}_1(0)$ is the closed unit ball in $\mathbb{C}$. For $\varphi_1, \varphi_2 \in \Phi$ and $\delta \in \bar{B}_1(0)$, define $\tilde{X}^{\varphi_1,\varphi_2,\delta}$ by (2.41), where we substitute a_φ by $a_{\varphi_1} + \delta a_{\varphi_2}$ for $a \in \{\mu, \sigma, \tilde{\sigma}\}$. Then by applying part 1) of Theorem 2.18 to the family of processes $(\tilde{X}^{\varphi_1,\varphi_2,\delta})_{(\varphi_1,\varphi_2,\delta)\in\Phi^2\times\bar{B}_1(0)}$, we obtain that there exists some $\tilde{T} > 0$ such that for all $\varphi_1, \varphi_2 \in \Phi$, $\delta \in \bar{B}_1(0)$ and $T \in (0, \tilde{T}]$, there is a unique solution $g_{\varphi_1,\varphi_2,\delta,T} \in L^1([0,T];\mathbb{C})$ to (2.42) on $[0,T]$ (where we substitute a_{φ_1} by $a_{\varphi_1} + \delta a_{\varphi_2}$ for $a \in \{\mu, \sigma, \tilde{\sigma}\}$), and it holds that $\exp(\tilde{X}_T^{\varphi_1,\varphi_2,\delta})$ is integrable with

$$E[\exp(\tilde{X}_T^{\varphi_1,\varphi_2,\delta}) \mid \mathcal{F}_t] = \exp\left(\tilde{X}_t^{\varphi_1,\varphi_2,\delta} + \int_t^T g_{\varphi_1,\varphi_2,\delta,T}(T-u)\xi_t(u)du\right), \quad 0 \le t \le T. \tag{2.79}$$

By setting $\delta = 0$, this already ensures the existence of $g_{\varphi_1,T} = g_{\varphi_1,\varphi_2,0,T}$ for any $\varphi_1 \in \Phi$ and $T \in (0, \tilde{T}]$. We now fix $T \in (0, \tilde{T}]$, $\varphi_1, \varphi_2 \in \Phi$ and set $a_\delta := a_{\varphi_1} + \delta a_{\varphi_2}$

for $a \in \{\mu, \sigma, \tilde{\sigma}\}$ and $\delta \in [0,1]$. We also write $\tilde{X}^{\delta} := \tilde{X}^{\varphi_1,\varphi_2,\delta} = \tilde{X}^{\varphi_1} + \delta\tilde{X}^{\varphi_2}$ and $g_\delta := g_{\varphi_1,\varphi_2,\delta,T}$ so that $g_0 = g_{\varphi_1,T}$. Plugging into (2.79) yields

$$E_P[Z_T^{(\delta)} \mid \mathcal{F}_t] = \tilde{Z}_t^{(\delta)}, \quad 0 \le t \le T, \tag{2.80}$$

where we define $(Z_t^{(\delta)})_{0\le t\le T}$ and $(\hat{Z}_t^{(\delta)})_{0\le t\le T}$ by

$$\begin{aligned} Z_t^{(\delta)} &:= \exp(\tilde{X}_t^{\delta}) = \exp(\tilde{X}_t^{\varphi_1} + \delta\tilde{X}_t^{\varphi_2}), \\ \hat{Z}_t^{(\delta)} &:= Z_t^{(\delta)} \exp\left(\int_t^T g_\delta(T-u)\xi_t(u)du\right), \quad 0 \le t \le T. \end{aligned} \tag{2.81}$$

Our goal is to differentiate (2.81) and then (2.80) with respect to δ at $\delta = 0$, which requires us to differentiate g_δ in an appropriate sense. To that end, consider the function $y_\delta := (g_\delta - g_0)/\delta$ for $\delta > 0$. After some simplifications, plugging $g_\delta = g_0 + \delta y_\delta$ into (2.42) and collecting the terms that are linear and quadratic in δ yields

$$\begin{aligned} g_\delta(t) &= \mu_\delta(T-t) + f\big((\hat{\kappa} * g_\delta)(t); \sigma_\delta(T-t), \tilde{\sigma}_\delta(T-t)\big) \\ &= g_0(t) + \delta\mu_{\varphi_2}(T-t) \\ &\quad + \delta\tilde{f}\big((\hat{\kappa} * y_\delta)(t); (\hat{\kappa} * g_0)(t), \sigma_{\varphi_1}(T-t), \tilde{\sigma}_{\varphi_1}(T-t), \sigma_{\varphi_2}(T-t), \tilde{\sigma}_{\varphi_2}(T-t)\big) \\ &\quad + \delta^2 f\big((\hat{\kappa} * y_\delta)(t); \sigma_{\varphi_2}(T-t), \tilde{\sigma}_{\varphi_2}(T-t)\big), \quad 0 \le t \le T, \end{aligned}$$

where we recall the definitions (2.39) and (2.77) of f and $\tilde{f}$, respectively. Subtracting g_0 from both sides and dividing by δ yields for $0 \le t \le T$ that

$$\begin{aligned} y_\delta(t) = \tilde{f}\big(&(\hat{\kappa} * y_\delta)(t); (\hat{\kappa} * g_0)(t), \sigma_{\varphi_1}(T-t), \tilde{\sigma}_{\varphi_1}(T-t), \sigma_{\varphi_2}(T-t), \tilde{\sigma}_{\varphi_2}(T-t)\big) \\ &+ \mu_{\varphi_2}(T-t) + \delta f\big((\hat{\kappa} * y_\delta)(t); z_2^\top \sigma(T-t), z_2^\top \tilde{\sigma}(T-t)\big). \end{aligned} \tag{2.82}$$

Although we have defined y_δ only for $\delta \in (0,1]$, note that (2.82) with $\delta = 0$ coincides with (2.76), where we have $\tilde{g}_{\varphi_1,\varphi_2,T}$ in place of y_0 and $g_{\varphi_1,T} = g_0$.

b) Next, we want to show that there exists a solution y_0 to (2.82) with $\delta = 0$, and moreover that $y_\delta \to y_0$ in $L^1([0,T];\mathbb{C})$ as $\delta \searrow 0$. By plugging in the definitions (2.39) and (2.77) of f and $\tilde{f}$ and collecting powers of $\hat{\kappa} * y_\delta$, we re-express (2.82) in the form

$$y_\delta(t) = a_\delta(t) + b_\delta(t)(\hat{\kappa} * y_\delta)(t) + c_\delta(t)\big((\hat{\kappa} * y_\delta)(t)\big)^2, \tag{2.83}$$

where $a_\delta, b_\delta, c_\delta : [0,T] \to \mathbb{C}$ are defined by $c_\delta \equiv \frac{\delta}{2}$ and

$$\begin{aligned} a_\delta(t) &= \mu_{\varphi_2}(T-t) + \sigma_{\varphi_1}(T-t)\sigma_{\varphi_2}(T-t) + \tilde{\sigma}_{\varphi_1}(T-t)\tilde{\sigma}_{\varphi_2}(T-t) \\ &\quad + \big(\varrho\sigma_{\varphi_2}(T-t) + \sqrt{1-\varrho^2}\tilde{\sigma}_{\varphi_2}(T-t)\big)(\hat{\kappa} * g_0)(t) \\ &\quad + \frac{\delta}{2}\big(\sigma^2_{\varphi_2}(T-t) + \tilde{\sigma}^2_{\varphi_2}(T-t)\big), \qquad (2.84) \\ b_\delta(t) &= \varrho\big(\sigma_{\varphi_1}(T-t) + \delta\sigma_{\varphi_2}(T-t)\big) + \sqrt{1-\varrho^2}\big(\tilde{\sigma}_{\varphi_1}(T-t) + \tilde{\sigma}_{\varphi_2}(T-t)\big) \\ &\quad + (\hat{\kappa} * g_0)(t). \qquad (2.85) \end{aligned}$$

Due to the uniform integrability of (μ_φ), $(|\sigma_\varphi|^2)$ and $(|\tilde{\sigma}_\varphi|^2)$, we have

$$C := \sup_{\varphi\in\Phi} \max\{\|\mu_\varphi\|_{L^1}, \|\sigma_\varphi\|_{L^2}, \|\tilde{\sigma}_\varphi\|_{L^2}\} < \infty,$$

where we write $\|\cdot\|_{L^p}$ as a shorthand for $\|\cdot\|_{L^p(0,T)}$. Moreover, $\hat{\kappa}$ inherits the local square-integrability from κ (see before Remark 2.5) so that $\hat{\kappa} \in L^2([0,T];\mathbb{R})$, and we also have $g_0 = g_{\varphi_1,T} \in L^1([0,T];\mathbb{C})$ by construction. By Young's convolution inequality (A.1.2), we thus have

$$\|\hat{\kappa} * g_0\|_{L^2} \le \|\hat{\kappa}\|_{L^2}\|g_0\|_{L^1} < \infty.$$

Hence the Cauchy–Schwarz inequality, (2.84) and (2.85) yield the bounds

$$\begin{aligned} \|a_\delta\|_{L^1} &\le C + 3C^2 + \|\hat{\kappa}\|^2_{L^2}\|g_0\|^2_{L^1} < \infty, \\ \|b_\delta\|_{L^2} &\le 4C + \|\hat{\kappa}\|_{L^2}\|g_0\|_{L^1} < \infty, \\ \|c_\delta\|_{L^\infty} &= \delta/2 < \infty \end{aligned}$$

for $0 \le \delta \le 1$ so that $a_\delta \in L^1([0,T];\mathbb{C})$, $b_\delta \in L^2([0,T];\mathbb{C})$ and $c_\delta \in L^\infty([0,T];\mathbb{C})$. Note in particular that $c_0 \equiv 0$ for $\delta = 0$, i.e., (2.83) is linear in the case $\delta = 0$ and thus of the form (A.2.39) with y_0 in place of x. Hence by Corollary A.2.10, there exists a unique solution $y_0 \in L^1([0,T];\mathbb{C})$ to (2.83) with $\delta = 0$. As pointed out at the end of step a), it follows that $\tilde{g}_{\varphi_1,\varphi_2,T} := y_0$ is also the unique solution to (2.76).

We now want to show that $y_\delta \to y_0$ in L^1. Note that (2.84) and (2.85) yield

$$\begin{aligned} a_\delta(t) - a_0(t) &= \frac{\delta}{2}\big(\sigma^2_{\varphi_2}(T-t) + \tilde{\sigma}^2_{\varphi_2}(T-t)\big), \\ b_\delta(t) - b_0(t) &= \delta\big(\varrho\sigma_{\varphi_2}(T-t) + \sqrt{1-\varrho^2}\tilde{\sigma}_{\varphi_2}(T-t)\big) \end{aligned}$$

for $0 \le t \le T$, and hence

$$\|a_\delta - a_0\|_{L^1} \le C^2\delta \longrightarrow 0 \quad \text{and} \quad \|b_\delta - b_0\|_{L^2} \le 2C\delta \longrightarrow 0$$

as $\delta \searrow 0$, and likewise $\|c_\delta\|_{L^\infty} = \frac{\delta}{2} \to 0$. Hence we have $a_\delta \overset{L^1}{\to} a_0$, $b_\delta \overset{L^2}{\to} b_0$ and $c_\delta \overset{L^\infty}{\to} 0$ as $\delta \searrow 0$. It then follows by applying Proposition A.2.11 to (2.83) that $y_\delta \overset{L^1}{\to} y_0$ as $\delta \searrow 0$.

c) We return to (2.80) and (2.81), which read $E_P[Z_T^{(\delta)} \mid \mathcal{F}_t] = \hat{Z}_t^{(\delta)}$ and

$$\begin{aligned} Z_t^{(\delta)} &:= \exp(\tilde{X}_t^{\varphi_1} + \delta \tilde{X}_t^{\varphi_2}), \\ \hat{Z}_t^{(\delta)} &:= Z_t^{(\delta)} \exp\left(\int_t^T g_\delta(T-u)\xi_t(u)du \right) \end{aligned}$$

for $0 \le t \le T$. We start by differentiating $Z_T^{(\delta)}$ at $\delta = 0$. It is clear that we have

$$U^{(\delta)} := \frac{Z_T^{(\delta)} - Z_T^{(0)}}{\delta} \longrightarrow \exp(\tilde{X}_T^{\varphi_1})\tilde{X}_T^{\varphi_2} \quad P\text{-a.s. as } \delta \searrow 0. \tag{2.86}$$

Using the elementary inequalities $e^x - 1 \le xe^x$ and $x \le e^x$ for $x \ge 0$, we get for $0 < \delta \le \frac{1}{4}$ that

$$\begin{aligned} |U^{(\delta)}| &\le |\exp(\tilde{X}_T^{\varphi_1})| \frac{\exp(\delta|\tilde{X}_t^{\varphi_2}|) - 1}{\delta} \\ &\le |\exp(\tilde{X}_T^{\varphi_1})| \, |\tilde{X}_t^{\varphi_2}| \exp(\delta|\tilde{X}_t^{\varphi_2}|) \\ &= |\exp(\tilde{X}_T^{\varphi_1})| \frac{(1/2-\delta)|\tilde{X}_t^{\varphi_2}|}{1/2-\delta} \exp(\delta|\tilde{X}_t^{\varphi_2}|) \\ &\le |\exp(\tilde{X}_T^{\varphi_1})| \frac{\exp((1/2-\delta)|\tilde{X}_t^{\varphi_2}|)}{1/2-\delta} \exp(\delta|\tilde{X}_t^{\varphi_2}|) \\ &\le 4|\exp(\tilde{X}_T^{\varphi_1})| \exp(|\tilde{X}_t^{\varphi_2}|/2). \end{aligned} \tag{2.87}$$

This gives a bound on $U^{(\delta)}$ that is uniform in $\delta \in (0, 1/4]$. In order to show the integrability of the last term on the right-hand side, we should like to take the absolute value outside the exponential. Note that for every $x \in \mathbb{C}$, we have

$$\begin{aligned} \exp(|x|) &\le \exp\big(|\mathrm{Re}(x)|\big) \exp\big(|\mathrm{Im}(x)|\big) \\ &\le \big(|\exp(x)| + |\exp(-x)|\big)\big(|\exp(\mathrm{i}x)| + |\exp(-\mathrm{i}x)|\big) \\ &= \sum_{\ell=0}^{3} \big|\exp\big(\mathrm{i}^\ell(1+\mathrm{i})x\big)\big|. \end{aligned}$$

Plugging into (2.87) yields

$$|U^{(\delta)}| \le 4\sum_{\ell=0}^{3} \big| \exp\big(\tilde{X}_T^{\varphi_1} + \mathrm{i}^\ell (1+\mathrm{i})\tilde{X}_T^{\varphi_2}/2\big)\big| =: \bar{U}.$$

Since we have $|\mathrm{i}^\ell(1+\mathrm{i})/2| = 1$ for $\ell = 0, \dots, 3$, it follows from the integrability of $\exp(\tilde{X}_T^{\varphi_1} + \delta\tilde{X}_T^{\varphi_2})$ for all $\varphi_1, \varphi_2 \in \Phi$ and $\delta \in \bar{B}_1(0)$, see before (2.79), that $\bar{U}$ is integrable. Hence by the dominated convergence theorem for conditional expectations with majorant $\bar{U}$, we obtain from (2.86) that as $\delta \searrow 0$,

$$\frac{1}{\delta} E[Z_T^{(\delta)} - Z_T^{(0)} \mid \mathcal{F}_t] = E[U^{(\delta)} \mid \mathcal{F}_t] \longrightarrow E[\exp(\tilde{X}_T^{\varphi_1})\tilde{X}_T^{\varphi_2} \mid \mathcal{F}_t] \quad P\text{-a.s.} \tag{2.88}$$

and thus also in probability.

We now consider $\hat{Z}_t^{(\delta)}$. By (2.81), we have

$$\frac{\hat{Z}_t^{(\delta)}}{\hat{Z}_t^{(0)}} = \exp\left(\delta\tilde{X}_t^{\varphi_2} + \delta\int_t^T y_\delta(T-u)\xi_t(u)du\right), \quad 0 \le t \le T, \tag{2.89}$$

where we recall that $g_\delta = g_0 + \delta y_\delta$ for $\delta > 0$ by the definition of y_δ. As shown in step b), we have $y_\delta \to y_0 = \tilde{g}_{z_1,z_2,T}$ in L^1 as $\delta \searrow 0$ so that

$$\left|\int_t^T \big(y_\delta(T-u) - y_0(T-u)\big)\xi_t(u)du\right| \le \|y_\delta - y_0\|_{L^1} \sup_{u\in[t,T]} \xi_t(u) \longrightarrow 0 \quad P\text{-a.s.}$$

as $\delta \searrow 0$, since $u \mapsto \xi_t(u)$ is by Proposition 2.11 continuous, hence bounded, a.s. Since $y_0 = \tilde{g}_{\varphi_1,\varphi_2,T}$, we deduce that

$$\frac{d}{d\delta}\left(\delta\int_t^T y_\delta(T-u)\xi_t(u)du\right)\bigg|_{\delta=0} = \int_t^T \tilde{g}_{\varphi_1,\varphi_2,T}(T-u)\xi_t(u)du \quad P\text{-a.s.}$$

Combining with (2.89) yields

$$\begin{aligned} V^{(\delta)} := \frac{\hat{Z}_t^{(\delta)}/\hat{Z}_t^{(0)} - 1}{\delta} &= \frac{\exp(\delta\tilde{X}_t^{\varphi_2} + \delta\int_t^T y_\delta(T-u)\xi_t(u)du) - 1}{\delta} \\ &\longrightarrow \tilde{X}_t^{\varphi_2} + \int_t^T \tilde{g}_{\varphi_1,\varphi_2,T}(T-u)\xi_t(u)du \quad P\text{-a.s.} \end{aligned} \tag{2.90}$$

as $\delta \searrow 0$. Finally, by using (2.80) in the definition (2.86) of $U^{(\delta)}$ and then using the first equality in (2.90), we have

$$E_P[U^{(\delta)} \mid \mathcal{F}_t] = \hat{Z}_t^{(0)} V^{(\delta)}$$

for each $\delta \in (0, 1/4)$. In the limit $\delta \searrow 0$, we get by (2.88) and (2.90) that

$$E_P[\exp(\tilde{X}_T^{\varphi_1})\tilde{X}_T^{\varphi_2} \mid \mathcal{F}_t] = \tilde{Z}_t^{(0)}\left(\tilde{X}_t^{\varphi_2} + \int_t^T \tilde{g}_{\varphi_1,\varphi_2,T}(T-u)\xi_t(u)du\right),$$

which is precisely (2.78). This concludes the proof. □

The following result follows from Proposition 2.20 as in the proof of Theorem 2.17 by setting $\Phi = \bar{B}_C(0)$, $\varphi = z$, $\mu_\varphi = z^\top \mu$, $\sigma_\varphi = z^\top \sigma$ and $\tilde{\sigma}_\varphi = z^\top \tilde{\sigma}$.

Corollary 2.21. *Let the process $(\tilde{X}_t)_{0 \le t \le \bar{T}}$ be given by (2.37) for some functions $\mu \in L^1([0,\bar{T}];\mathbb{C}^n)$ and $\sigma, \tilde{\sigma} \in L^2([0,\bar{T}];\mathbb{C}^n)$. Fix $C > 0$ and let $\bar{B}_C(0) \subseteq \mathbb{C}^n$ be the closed ball of radius C. Then there exists some $\tilde{T} = \tilde{T}(\mu,\sigma,\tilde{\sigma},C) \in (0,\bar{T}]$ such that for every $T \in (0,\tilde{T}]$ and $z_1, z_2 \in \bar{B}_C(0)$, there is a unique solution $g_{z_1,T} \in L^1([0,T];\mathbb{C})$ to (2.38) (with $z = z_1$) on $[0,T]$ as well as a unique solution $\tilde{g}_{z_1,z_2,T} \in L^1([0,T];\mathbb{C})$ to the equation*

$$\tilde{g}_{z_1,z_2,T}(t) = \tilde{f}_{z_1,z_2}\big((\hat{\kappa} * \tilde{g}_{z_1,z_2,T})(t); (\hat{\kappa} * g_{z_1,T})(t), \mu(T-t), \sigma(T-t), \tilde{\sigma}(T-t)\big) \tag{2.91}$$

for a.a. $t \in [0,T]$, where $\tilde{f}_{z_1,z_2} : \mathbb{C}^5 \to \mathbb{C}$ is defined by

$$\begin{aligned}\tilde{f}_{z_1,z_2}(x;y,b,h_1,h_2) = {} & z_2^\top b + (z_1^\top h_1 + \varrho y)(z_2^\top h_1 + \varrho x) \\ & + (z_1^\top h_2 + \sqrt{1-\varrho^2} y)(z_2^\top h_2 + \sqrt{1-\varrho^2} x).\end{aligned} \tag{2.92}$$

Moreover, it holds for $0 \le t \le T$ that

$$\begin{aligned}E[\exp(z_1^\top \tilde{X}_T) z_2^\top \tilde{X}_T \mid \mathcal{F}_t] = {} & \exp\left(z_1^\top \tilde{X}_t + \int_t^T g_{z_1,T}(T-u)\xi_t(u)du\right) \\ & \times \left(z_2^\top \tilde{X}_t + \int_t^T \tilde{g}_{z_1,z_2,T}(T-u)\xi_t(u)du\right).\end{aligned} \tag{2.93}$$

3 The pure investment problem

3.1 Setup and auxiliary results

Our goal in this section is to study the pure investment problem for the rough Heston model (2.8). The pure investment problem (defined below in (3.1)) is a portfolio selection problem closely related to the well-known Markowitz mean–variance portfolio selection problem, and thus interesting in its own right. For our purposes, the study of the pure investment problem serves mainly as a stepping stone for tackling the mean–variance hedging problem, which we introduce and study in Chapter II for the rough Heston model. This is also the approach taken in Černý/Kallsen [25, 27] for mean–variance hedging in a general semimartingale setup and for obtaining explicit results for the classical Heston model.

Although the Markowitz problem has been recently solved for the rough Heston model in Han/Wong [62] and Abi Jaber et al. [2], our contribution here is twofold. By using a martingale distortion formula as in Fouque/Hu [49] to solve the pure investment problem, we get some additional insight into the structure of the solution to the latter. In particular, this allows us to explain why a relatively simple solution can be obtained for the rough Heston model. Secondly, we connect our results with the general theory developed in [25], and this allows us to obtain results that can later be used for solving the mean–variance hedging problem for the rough Heston model.

Notation 3.1. For a semimartingale $X = (X_t)_{0 \le t \le T}$, we denote by $L_T(X)$ or $L(X)$ the set of predictable X-integrable processes on $[0, T]$; see Jacod/Shiryaev [71, III.6.17]. We say that a predictable process $(A_t)_{0 \le t \le T}$ is *P-integrable on* $[0, T]$ if $\int_0^T |A_t| dt < \infty$ P-a.s. We generally omit P and T if they are unambiguous.

Let (S, Y) satisfy the rough Heston model (2.8) with time horizon $T > 0$. We consider the classical setup of frictionless trading in the asset S, and assume that there is a risk-free asset with constant price 1, i.e., that the interest rate is 0. This means that an agent with initial capital $x \in \mathbb{R}$ may trade in a self-financing way with a *trading strategy* $\vartheta \in L(S)$ to generate the *wealth process* $(V_t(x, \vartheta))_{0 \le t \le T}$ given by

$$V_t(x, \vartheta) := x + \int_0^t \vartheta_s \, dS_s =: x + \vartheta \bullet S_t, \quad 0 \le t \le T.$$

We do not impose any constraints on the agent's positions at a given point in time, so that leverage and short-selling are allowed, but we do specify an admissibility condition on the strategy ϑ to prevent doubling-type behaviour. We thus focus on a subset $\overline{\Theta}_T(S) \subseteq L(S)$ of *admissible trading strategies*, which we take to be the set of L^2-admissible strategies introduced in Černý/Kallsen [25], defined as follows. We say that $(\vartheta_t)_{0 \le t \le T}$ is a *simple integrand* if $\vartheta = \sum_{i=1}^{m-1} \xi_i \mathbf{1}_{]\!]\sigma_i, \sigma_{i+1}]\!]}$ for some $m \in \mathbb{N}$, an increasing sequence of stopping times $0 \le \sigma_1 \le \cdots \le \sigma_m \le T$ such that the stopped process S^{σ_m} is bounded, and bounded $\mathcal{F}_{\sigma_i}$-measurable random variables ξ_i. Then we define $\overline{\Theta}_T(S)$ as the set of integrands $\vartheta \in L(S)$ for which there exists a sequence $(\vartheta^k)_{k\in\mathbb{N}}$ of simple integrands such that we have

$$\textbf{1)}\ \vartheta^k \bullet S_T \xrightarrow{L^2_P} \vartheta \bullet S_T \quad \text{and} \quad \textbf{2)}\ \vartheta^k \bullet S_t \xrightarrow{P} \vartheta \bullet S_t \quad \text{for all } t \in [0,T].$$

An explanation of why this class of strategies is economically reasonable and mathematically useful can be found in [25].

We can now define the *pure investment problem* as the minimisation problem given by

$$E_P[|1 - V_T(0,\vartheta)|^2] = E_P\left[\left|1 - \int_0^T \vartheta_t dS_t\right|^2\right] \longrightarrow \min_{\vartheta \in \overline{\Theta}_T(S)} ! \tag{3.1}$$

In words, an agent starting with initial capital 0 seeks to trade with an admissible strategy ϑ in order to attain the target wealth 1. This may be interpreted as an attempt to find an arbitrage opportunity in the market, which would be achieved if the mean squared error is equal to 0, since in that case $\int_0^T \vartheta_t dS_t = 1$ P-a.s. If that is not possible, then the agent looks instead for a strategy that is closest to achieving that goal, in the sense of a low mean squared error.

We start by introducing some notation. One of our goals is to find an *optimal pure investment strategy*, i.e., a solution $\vartheta^\star$ to the optimisation problem (3.1) with time horizon T, if it exists. In order to find such a strategy, the main step is to compute the *opportunity process* $(L_t)_{0\le t\le T}$, which is defined as the reduced-form value process for the dynamic problem

$$L_t = \operatorname*{ess\,inf}_{\vartheta \in \overline{\Theta}_T(S)} E_P\left[\left(1 - \int_t^T \vartheta_u dS_u\right)^2 \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t \le T. \tag{3.2}$$

More precisely, we say that L is an *opportunity process with time horizon* T if L is a strictly positive càdlàg submartingale, bounded above by 1, and for each $t \in [0,T]$ satisfies (3.2) P-a.s. If such a process L exists, then by (3.2) and the càdlàg property, it is unique up to indistinguishability. We omit the time horizon T when it is clear from the context.

It is also useful to consider the dual problem to (3.1). As is well known in the literature, this leads to the problem of finding a *variance-optimal martingale measure on* $[0,T]$, or VOMM for short, which is defined as a solution $Q^\star = Q^\star(T)$ to the minimisation problem

$$E_P\left[\left(\frac{dQ}{dP}\right)^2\right] \longrightarrow \min_{Q \in \bar{\mathbb{Q}}^2_T(S)} ! \tag{3.3}$$

where $\bar{\mathbb{Q}}^2_T(S)$ is the set of signed measures $Q \approx P$ such that the density dQ/dP is square-integrable under P and Q is a local martingale measure for $(S_t)_{0\leq t\leq T}$, meaning that $Z^Q_0 = 1$ and $Z^Q S$ is a local P-martingale for the density process Z^Q of Q, defined as a càdlàg version of the P-martingale given by $Z^Q_t = E_P[\frac{dQ}{dP} \mid \mathcal{F}_t]$ for $0 \leq t \leq T$. We also define the subset

$$\mathbb{Q}^2_T(S) := \{Q \in \bar{\mathbb{Q}}^2_T(S) : Q \text{ is a probability measure and } Q \approx P\}.$$

If a VOMM exists, then it is unique; this follows from the strict convexity of the problem (3.3) and the fact that $\bar{\mathbb{Q}}^2_T(S)$ is a convex set. The existence of the VOMM is shown in [25, Proposition 3.13] under the assumption that $\mathbb{Q}^2_T(S) \neq \emptyset$ (see [25, Assumption 2.1]), which can be seen as a no-arbitrage condition on S. Under this condition, there exists by [25, Lemma 3.2 and Corollary 3.4] an opportunity process L with time horizon T. We also have by Delbaen/Schachermayer [36, Theorem 1.3] that $Q^\star \in \mathbb{Q}^2_T(S)$ because S is continuous, so that we may replace $\bar{\mathbb{Q}}^2_T(S)$ in (3.3) by $\mathbb{Q}^2_T(S)$ for the rough Heston model. We refer to $Q^\star$ simply as the VOMM (without reference to T) when it is clear from the context.

Given an optimal pure investment strategy $\vartheta^\star \in \overline{\Theta}_T(S)$, we note that the terminal value $\vartheta^\star \bullet S_T$ of the wealth process generated by $\vartheta^\star$ with initial wealth 0 is also the L^2-projection of the random variable 1 onto the set of *attainable gains*, given by

$$\mathcal{G}_T(S) := \{\vartheta \bullet S_T : \vartheta \in \overline{\Theta}_T(S)\} \subseteq L^2_P(\mathcal{F}_T). \tag{3.4}$$

Since $\overline{\Theta}_T(S)$ and thus $\mathcal{G}_T(S)$ are vector spaces, the projection is unique, i.e., for any two optimal pure investment strategies ϑ^1 and ϑ^2, we have $\vartheta^1 \bullet S_T = \vartheta^2 \bullet S_T$ P-a.s. Under the assumption that $\mathbb{Q}^2_T(S) \neq \emptyset$, we have by [25, Lemma 2.11] that the processes $\vartheta^1 \bullet S$ and $\vartheta^2 \bullet S$ are indistinguishable, and so we say that ϑ^1 and ϑ^2 are S-*equivalent* or $\vartheta^1 =_S \vartheta^2$. Thus we have under this assumption that the optimal pure investment strategy $\vartheta^\star$ is unique up to S-equivalence.

The main step in our strategy is to find the opportunity process, since this allows us to solve the pure investment problem in a relatively straightforward manner by using results from [25]. As is well known in the literature, from e.g. [25], Hu et al. [68], Jeanblanc et al. [72] and Mania/Tevzadze [91], the opportunity process L can be characterised as the solution to a backward stochastic differential equation (BSDE). Equivalently, L may also be characterised by an equation (given below in (3.5)) in terms of its differential characteristics, which are defined below for the case of an Itô process. Our approach is to find an explicit solution to (3.5), and then to check the additional conditions that are needed in order to show that this solution is indeed the opportunity process.

In preparation for equation (3.5), we briefly define differential characteristics in the simple case of an Itô process with values in $\mathbb{R}^d$; see [71, Proposition II.2.9] for the general case, and note that $\mathbb{R}$ can be replaced by $\mathbb{C}$ everywhere in the following definition. If $(Z_t)_{0\leq t\leq T}$ is an Itô process taking values in $\mathbb{R}^d$, then there exist predictable integrable processes $(b_t^Z)_{0\leq t\leq T}$ and $(c_t^Z)_{0\leq t\leq T}$ with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$ respectively such that the process $Z - \int b^Z dt$ is a local martingale and we have that $[Z] = \int c^Z dt$. In that case, we say that Z has *differential characteristics* (b^Z, c^Z). Given two Itô processes Z and Z' with values in $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively, we also define the differential characteristic $c^{Z,Z'}$ as the predictable integrable process with values in $\mathbb{R}^{d\times d'}$ such that $[Z, Z'] = \int c^{Z,Z'} dt$. Note that b^Z, c^Z and $c^{Z,Z'}$ are unique up to $(P\otimes dt)$-nullsets.

As previously mentioned, the main step in order to solve the pure investment problem is to find the opportunity process L. Suppose that $\mathbb{Q}_T^2(S)\neq\emptyset$ so that the opportunity process L exists. We make the ansatz that L is an Itô process, which is justified later in Lemma 3.3 and Theorem 3.8; this allows us to use the simplified definition above for the differential characteristics of L. If this ansatz holds, we have by [25, Lemma 3.19] that the differential characteristics of L satisfy the equation

$$
\begin{aligned}
b_t^L &= \frac{L_t}{c_t^R}\left(b_t^R + \frac{c_t^{RL}}{L_t}\right)^2, \quad 0\leq t\leq T, \\
L_T &= 1,
\end{aligned}
\tag{3.5}
$$

where the returns process $(R_t)_{0\leq t\leq T}$ is defined by $dR_t = \frac{dS_t}{S_t} = \mu Y_t dt + \sigma\sqrt{Y_t}dW_t$ and $R_0 = 0$. In particular, we have that $b_t^R = \mu Y_t$ and $c_t^R = \sigma^2 Y_t$.

Our goal is now to solve the equation (3.5), and then to show that the resulting solution is indeed the opportunity process. To that end, we say that $(\hat{L}_t)_{0\leq t\leq T}$ is a *candidate opportunity process* (for S with time horizon T) if it is a strictly positive Itô process bounded above by 1 such that $\hat{L}_T = 1$ and its differential characteristics satisfy (3.5), where we replace L with $\hat{L}$. Naturally, the (true) opportunity process is also a candidate opportunity process, but there may be multiple candidate opportunity processes. Thus we need some additional conditions in order to conclude that a candidate opportunity process is the true one. These conditions are given explicitly in the proof of Theorem 3.8 below.

Given the differential characteristics of the opportunity process L, one can compute the *adjustment process* a, which is a predictable process given by

$$a_t = \frac{1}{S_t c_t^R}\left(b_t^R + \frac{c_t^{RL}}{L_t}\right), \quad 0 \le t \le T. \tag{3.6}$$

Note that a is predictable because S and, by our ansatz, L are both continuous. For convenience, we take (3.6) as the definition for the adjustment process, whereas it is defined in [25, Definition 3.8] via the optimal strategy $\vartheta^\star$ for the pure investment problem. This is justified since a given by (3.6) is an adjustment process in the sense of [25, Definition 3.8] by [25, Theorem 3.25], and conversely, any other adjustment process in the sense of [25, Definition 3.8] is S-equivalent to a by [25, Lemma 3.7]. By replacing L with $\hat{L}$ in (3.6), we can also define the *candidate adjustment process* $\hat{a}$ associated with $\hat{L}$.

Finding a candidate opportunity process $\hat{L}$ is done in two parts. First, we give a martingale distortion formula that gives a process $\hat{L}$ in terms of a conditional expectation. This formula is designed in such a way that its output $\hat{L}$ can be shown to be a candidate opportunity process; we show below how one can reverse-engineer (3.5) to arrive at this formula, and then give a rigorous proof in Lemma 3.2. The next step is to obtain an explicit formula for $\hat{L}$ as an exponentially affine function of the forward variance curve.

Given a candidate opportunity process $\hat{L}$, we then need a verification result to show that it is the true opportunity process, for which we need some additional integrability conditions, given explicitly in [25, Theorem 3.25] and recalled in the proof of Theorem 3.8. These essentially serve to ensure that the candidate adjustment and opportunity processes lead to a candidate optimal strategy that is admissible, and also that the corresponding candidate density process for the variance-optimal martingale measure is a true martingale.

3.2 Martingale distortion

We now start to work towards finding a candidate opportunity process $\hat{L}$. The approach we use was first developed in Zariphopoulou [120] and Tehranchi [116], and more recently applied in Frei/Schweizer [52] to an indifference valuation problem under exponential utility, as well as in Fouque/Hu [49, 50] for portfolio optimisation under a rough stochastic environment. It relies on a martingale distortion formula to express the value function to a stochastic control problem (in our case, the pure investment problem) as a power of an expectation under an equivalent measure. The result we obtain is similar to [50, Proposition 2.2], but we cannot assume that the market price of risk (i.e., the ratio $\frac{\mu Y_t}{\sigma\sqrt{Y_t}} = \frac{\mu\sqrt{Y_t}}{\sigma}$ between the drift and volatility of S) is bounded, as that is not the case in the rough Heston model. The change of measure defined in (3.12) is also used in Han/Wong [61, Equation (2.1)] in order to solve the Markowitz problem for the rough Heston model by a completion-of-squares technique.

We start by giving an intuitive explanation of how the martingale distortion technique can be applied to our problem, and then fill in the details in Lemma 3.2. Recall the definition (2.8) of the rough Heston model, as well as the orthogonal decomposition

$$B = \varrho W + \sqrt{1-\varrho^2}W^{\perp} \tag{3.7}$$

and the returns process $(R_t)_{0\le t\le T}$ given by

$$R_t = \int_0^t \frac{dS_s}{S_s} = \int_0^t (\mu Y_s ds + \sigma\sqrt{Y_s}dW_s), \quad 0 \le t \le T. \tag{3.8}$$

We make the ansatz that there exists a candidate opportunity process $\hat{L}$ that is an Itô process with a local martingale part driven by the Brownian motion B, i.e.,

$$d\hat{L}_t = b_t^{\hat{L}}dt + \eta_t dB_t, \quad 0 \le t \le T, \tag{3.9}$$

for some unknown predictable integrable process $(b_t^{\hat{L}})_{0\le t\le T}$ and predictable integrand $\eta \in L(B)$. This assumption allows us to find a solution $\hat{L}$, but it is ad hoc at this point. We check (3.9) a posteriori in Lemma 3.3, where we find a candidate opportunity process $\hat{L}$ explicitly and show that it satisfies (3.9). The assumption is motivated by the fact that the market price of risk $\frac{\mu}{\sigma}\sqrt{Y_t}$ depends only on the process Y, which is driven by the Brownian motion B and does not depend on W or S; we might then expect that the same is true of the opportunity process, and so it makes sense to look for a candidate opportunity process of this type. If $\hat{L}$ satisfies (3.9), then $b^{\hat{L}}$ and $c^{\hat{L}} = \eta^2$ are the differential characteristics of $\hat{L}$. As the returns process satisfies (3.8), we have that $b_t^R = \mu Y_t$ and $c_t^R = \sigma^2 Y_t$. Noting that $d[R,\hat{L}]_t = \varrho\sigma\sqrt{Y_t}\eta_t dt$ by (3.7), we also obtain the differential characteristic

$c_t^{R\hat{L}} = \varrho\sigma\sqrt{Y_t}\eta_t$. Plugging into (3.9) and using (3.5) for $b^{\hat{L}}$ then yields the BSDE

$$\begin{aligned} d\hat{L}_t &= b_t^{\hat{L}}dt + \eta_t dB_t \\ &= \frac{\hat{L}_t}{c_t^R}\left(b_t^R + \frac{c_t^{R\hat{L}}}{\hat{L}_t}\right)^2 dt + \eta_t dB_t \\ &= \frac{\hat{L}_t}{\sigma^2 Y_t}\left(\mu Y_t + \frac{\varrho\sigma\eta_t\sqrt{Y_t}}{\hat{L}_t}\right)^2 dt + \eta_t dB_t \\ &= \frac{\hat{L}_t}{\sigma^2}\left(\mu^2 Y_t + 2\frac{\varrho\mu\sigma\eta_t\sqrt{Y_t}}{\hat{L}_t} + \frac{\varrho^2\sigma^2\eta_t^2}{\hat{L}_t^2}\right)dt + \eta_t dB_t, \quad 0 \le t \le T, \end{aligned}$$

with terminal condition $\hat{L}_T = 1$ and where $\hat{L}$ and the integrand $\eta \in L(B)$ are unknown. The technique then consists of finding an explicit solution to this BSDE by a change of measure and applying a power function. This takes advantage of the fact that the driver is a quadratic polynomial in η_t. We first want to remove the linear middle term of the driver. As shown later in Lemma 3.2, we can define a new measure $\bar{P} \approx P$ such that $d\bar{B}_t = dB_t + \frac{2\varrho\mu\sqrt{Y_t}}{\sigma}dt$ is a Brownian motion under $\bar{P}$. Hence we can remove the cross term to obtain under $\bar{P}$ the BSDE

$$d\hat{L}_t = \hat{L}_t\left(\frac{\mu^2 Y_t}{\sigma^2} + \frac{\varrho^2\eta_t^2}{\hat{L}_t^2}\right)dt + \eta_t d\bar{B}_t, \quad 0 \le t \le T.$$

In order to also remove the quadratic term in η_t, we apply a power function to $\hat{L}$ with the exponent $\beta = 1 - 2\varrho^2$, which is chosen in order to remove this term. By Itô's formula and using $\frac{\beta-1}{2} = -\varrho^2$, we then obtain

$$\begin{aligned} d(\hat{L}_t^\beta) &= \beta\hat{L}_t^\beta\left(\frac{\mu^2 Y_t}{\sigma^2} + \frac{\varrho^2\eta_t^2}{\hat{L}_t^2}\right)dt + \frac{\beta(\beta-1)}{2}\hat{L}^{\beta-2}\eta_t^2 dt + \beta\eta_t d\bar{B}_t \\ &= \frac{\beta\mu^2}{\sigma^2}\hat{L}_t^\beta Y_t dt + \beta\eta_t d\bar{B}_t, \quad 0 \le t \le T. \end{aligned} \tag{3.10}$$

This is a linear BSDE for the process $\hat{L}^\beta$, and it is well known how to solve such equations explicitly; see e.g. Zhang [121, Proposition 4.1.2]. Together with the terminal condition $\hat{L}_T^\beta = 1$ and under suitable integrability conditions, we can find the explicit solution $\hat{L}^\beta$ as the conditional expectation

$$\hat{L}_t^\beta = E_{\bar{P}}\left[\exp\left(-\frac{\beta\mu^2}{\sigma^2}\int_t^T Y_u du\right)\,\middle|\,\mathcal{F}_t\right], \quad 0 \le t \le T. \tag{3.11}$$

This gives a formula for $\hat{L}$ by taking the power $1/\beta$ on both sides. We note that (3.10) does not give any information in the critical case $\beta = 1 - 2\varrho^2 = 0$, and so the procedure above does not yield a solution in that case. However, by taking a logarithm instead of a power function, we can obtain an explicit formula for the critical case as well, as we see later in Lemma 3.2. That case can also be formally obtained by using the well-known limit $E[X^\beta|\mathcal{H}]^{1/\beta} \to \exp(E[\log X|\mathcal{H}])$ as $\beta \to 0$ (i.e., the power mean converges to the geometric mean as $\beta \to 0$), for a σ-algebra $\mathcal{H}$ and a strictly positive random variable X such that $E[X^\beta + X^{-\beta}] < \infty$ for some $\beta > 0$.

We now want to prove that the ansatz (3.11) indeed yields a candidate opportunity process $\hat{L}$. We first need to show that there exists such an equivalent measure $\bar{P}$ and that the formula (3.11) is well defined, which is nontrivial in the case $\beta < 0$. It then remains to check that $\hat{L}$ satisfies the assumption (3.9), and finally that $\hat{L}$ is a candidate opportunity process, effectively reversing the previous calculations to show that the BSDE is satisfied. Except for checking (3.9), these steps are done in Lemma 3.2, using Theorem 2.17 as our main tool for showing integrability properties. The fact that the resulting process $\hat{L}$ has the form (3.9) is shown later in Lemma 3.3.

In the proof of Lemma 3.2 as well as in some subsequent proofs, we apply Theorem 2.17 to show that certain statements hold for any choice of time horizon $T \in (0, T')$ up to some positive upper bound T', and this T' will generally be different for each of the statements. The overall bound, such as $T_1^\star$ in the case of Lemma 3.2, should then be interpreted as the minimum of these (finitely many) times, so that all considered statements hold simultaneously up to $T_1^\star$. This is made explicit in the proof. When applying Theorem 2.17 to a semimartingale with dynamics of the form (2.37), we do not always identify the respective coefficients μ, σ and $\tilde{\sigma}$ explicitly unless that is needed. We also note that Lemma 3.2 gives the candidate opportunity process $\hat{L}$ for a fixed value of T (if T is small enough), but $\hat{L}$ depends on the choice of T, and so does $\bar{P}$ defined in part 1); one could make the dependence explicit by writing $\hat{L} = \hat{L}(T)$.

Lemma 3.2. *There exists some $T_1^\star > 0$ such that the following statements hold for each choice of time horizon $T \in (0, T_1^\star]$:*

1) *The process $(Z_t^{\bar{P}})_{0 \le t \le T}$ defined by*

$$Z_t^{\bar{P}} = \mathcal{E}\bigg(-\frac{2\mu}{\sigma}\sqrt{Y} \bullet W\bigg)_t, \quad 0 \le t \le T, \tag{3.12}$$

is a strictly positive P-martingale on $[0, T]$, and it is the density process for an equivalent measure $\bar{P} \approx P$ defined by $\frac{d\bar{P}}{dP} := Z_T^{\bar{P}}$.

2) *Suppose that* $\beta := 1 - 2\varrho^2 \neq 0$. *Then there exists a semimartingale* $(\hat{L}_t)_{0 \le t \le T}$ *such that*

$$\hat{L}_t = \left(E_{\bar{P}} \left[\exp\left(- \frac{\beta\mu^2}{\sigma^2} \int_t^T Y_u du \right) \,\middle|\, \mathcal{F}_t \right] \right)^{1/\beta} \quad \text{for each } t \in [0,T]. \tag{3.13}$$

If $\hat{L}$ *is an Itô process of the form*

$$\hat{L}_t = \hat{L}_0 + \int_0^t (b_s^{\hat{L}} ds + \eta_s dB_s), \quad 0 \le t \le T, \tag{3.14}$$

for some integrable predictable process $b^{\hat{L}}$ *and integrand* $\eta \in L(B)$, *then* $\hat{L}$ *is a candidate opportunity process for* S *with time horizon* T.

3) *If* $1 - 2\varrho^2 = 0$, *there exists a semimartingale* $(\hat{L}_t)_{0 \le t \le T}$ *such that*

$$\hat{L}_t = \exp\left(-E_{\bar{P}} \left[\frac{\mu^2}{\sigma^2} \int_t^T Y_u du \,\middle|\, \mathcal{F}_t \right] \right) \quad \text{for each } t \in [0,T]. \tag{3.15}$$

If $\hat{L}$ *is an Itô process of the form* (3.14) *for some integrable predictable process* $b^{\hat{L}}$ *and integrand* $\eta \in L(B)$, *then* $\hat{L}$ *is a candidate opportunity process for* S *with time horizon* T.

Proof. In the following, we write $\psi := \frac{\mu^2}{\sigma^2}$ for readability.

1) The process $Z^{\bar{P}}$ defined in (3.12) is the density process of a probability measure $\bar{P} \approx P$ if it is a strictly positive martingale on $[0,T]$. Since Y is continuous, hence a.s. bounded on $[0,T]$, we have that $\bar{N} := -\frac{2\mu}{\sigma}\sqrt{Y} \bullet W$ is a continuous local martingale on $[0,T]$ and $[\bar{N}]_T = \int_0^T 4\psi Y_t dt < \infty$ a.s. Thus $\bar{N}$ and $[\bar{N}]$ are finite almost surely on $[0,T]$, and so the stochastic exponential $Z^{\bar{P}}$ is strictly positive. To apply Novikov's criterion to $Z^{\bar{P}}$, we need to show that $\exp(\frac{1}{2}[\bar{N}]_T)$ is P-integrable. We can write

$$E_P\left[\exp\left(\frac{[\bar{N}]_T}{2} \right) \right] = E_P\left[\exp\left(\int_0^T 2\psi Y_s ds \right) \right] = E_P[\exp(\tilde{X}_T^{(1)})], \tag{3.16}$$

where we let $\tilde{X}_t^{(1)} := \int_0^t 2\psi Y_s ds$. Then the process $\tilde{X}^{(1)}$ is a semimartingale of the type considered in Theorem 2.17, taking the functions $\mu(t) = 2\psi$ and $\sigma(t) = \tilde{\sigma}(t) = 0$, where μ is integrable and $\sigma, \tilde{\sigma}$ are square-integrable as they are all constant. It then follows from part 1) of Theorem 2.17 that there exists some

$T_1 > 0$ such that for all $T \in (0, T_1]$, we have

$$E_P\left[\exp\left(\frac{[\bar{N}]_T}{2}\right)\right] = E_P[\exp(\tilde{X}_T^{(1)})] < \infty.$$

So by Novikov's criterion, it follows that $Z^{\bar{P}}$ is a strictly positive P-martingale on $[0, T]$. Hence $\bar{P} = \bar{P}(T)$ is an equivalent measure for any choice of $T \in (0, T_1]$.

2) Suppose that $\beta = 1 - 2\varrho^2 \neq 0$. We first want to show that the process $\hat{L}$ is well defined if T is small enough, and we start by showing that the random variable $\exp(-\beta\psi \int_0^T Y_s ds)$ is $\bar{P}$-integrable for small values of $T > 0$. This is clearly the case if $\beta > 0$, as the rest of the exponent is nonpositive, but we also consider the case $\beta < 0$. Plugging in the density for $\bar{P}$, we obtain that

$$\begin{aligned} E_{\bar{P}}\left[\exp\left(-\beta\psi \int_0^T Y_s ds\right)\right] &= E_P\left[\frac{d\bar{P}}{dP}\exp\left(-\beta\psi \int_0^T Y_s ds\right)\right] \\ &= E_P[\exp(\tilde{X}_T^{(2)} + \beta\tilde{X}_T^{(3)})], \end{aligned}$$

where we set $\tilde{X}_t^{(2)} = \int_0^t (-\frac{2\mu}{\sigma}\sqrt{Y_s}dW_s - 2\psi Y_s ds)$ and $\tilde{X}_t^{(3)} = -\int_0^t \psi Y_s ds$. As in 1), we apply part 1) of Theorem 2.17 with $C = \sqrt{2}$ to the process $(\tilde{X}^{(2)}, \tilde{X}^{(3)})$, and since all coefficients of $\tilde{X}^{(2)}$ and $\tilde{X}^{(3)}$ are constant, we get that there exists some $T_1^\star \in (0, T_1]$ such that for each $T \in (0, T_1^\star]$ and $\beta \in [-1, 1]$, we have

$$E_{\bar{P}}\left[\exp\left(- \beta\psi \int_0^T Y_s ds\right)\right] = E_P[\exp(\tilde{X}_T^{(2)} + \beta\tilde{X}_T^{(3)})] < \infty. \tag{3.17}$$

Noting that $\beta = 1 - 2\varrho^2 \in [-1, 1]$ for any $\varrho \in [-1, 1]$, this shows the integrability of $\exp(-\beta\psi \int_0^T Y_s ds)$ for each $T \in (0, T_1^\star]$.

Now fix $T \in (0, T_1^\star]$. By (3.17), we can define the $\bar{P}$-martingale $(M_t)_{0 \le t \le T}$ by

$$M_t = E_{\bar{P}}\left[\exp\left(-\beta\psi \int_0^T Y_s ds\right) \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t \le T.$$

The exponential is strictly positive a.s. as Y is a.s. bounded, and so $M_t > 0$ for each $t \in [0, T]$. Thus we may also define the process $(\hat{L}_t)_{0 \le t \le T}$ by

$$\hat{L}_t = (M_t)^{1/\beta} \exp\left(\psi \int_0^t Y_s ds\right), \quad 0 \le t \le T. \tag{3.18}$$

It is then clear that $\hat{L}$ satisfies (3.13), since we have

$$\begin{aligned}\hat{L}_t^\beta &= E_{\bar{P}}\left[\exp\left(-\beta\psi\int_0^T Y_s ds\right)\,\middle|\,\mathcal{F}_t\right]\exp\left(\beta\psi\int_0^t Y_s ds\right)\\ &= E_{\bar{P}}\left[\exp\left(-\beta\psi\int_t^T Y_s ds\right)\,\middle|\,\mathcal{F}_t\right],\quad 0\le t\le T. \qquad (3.19)\end{aligned}$$

Moreover, it follows by (3.18) and Itô's formula that $\hat{L}$ is a strictly positive semimartingale.

So far we have shown that $\hat{L}$ from (3.18) is well defined and a semimartingale, although it is not clear at this point whether $\hat{L}$ or M are continuous. We now want to show that $\hat{L}$ is a candidate opportunity process under the assumption (3.14), by using the definition (3.18) to find a more explicit semimartingale decomposition. Note that both $\hat{L}$ and $\exp(\psi\int_0^\cdot Y_s ds)$ are strictly positive and continuous, by the assumption (3.14) on $\hat{L}$. Since the exponent $1/\beta$ is nonzero, we get from (3.18) that the $\bar{P}$-martingale M is continuous. Applying Itô's formula to (3.18) and using (3.14), we obtain

$$b_t^{\hat{L}}dt+\eta_t dB_t = d\hat{L}_t = \frac{\hat{L}_t}{\beta M_t}dM_t+\frac{(\frac{1}{\beta}-1)\hat{L}_t}{2\beta M_t^2}d[M]_t+\psi\hat{L}_tY_tdt,\quad 0\le t\le T. \qquad (3.20)$$

Recall that $[B,W]_t=\varrho t$ by (3.7). Thus by Girsanov's theorem and the definition of $\bar{P}$, we obtain a $\bar{P}$-Brownian motion $\bar{B}$ by setting $d\bar{B}_t = dB_t + \frac{2\varrho\mu}{\sigma}\sqrt{Y_t}dt$. Plugging this into (3.20) gives

$$b_t^{\hat{L}}dt+\eta_t d\bar{B}_t-\frac{2\varrho\mu\eta_t\sqrt{Y_t}}{\sigma}dt = \frac{\hat{L}_t}{\beta M_t}dM_t+\frac{(\frac{1}{\beta}-1)\hat{L}_t}{2\beta M_t^2}d[M]_t+\psi\hat{L}_tY_tdt,\quad 0\le t\le T.$$

Since $\bar{B}$ and M are continuous $\bar{P}$-martingales and the remaining terms are continuous and have finite variation, it follows from this equality that $dM_t=\eta_t^M d\bar{B}_t$, where the predictable integrand η^M is given by $\eta_t^M = \frac{\beta M_t\eta_t}{\hat{L}_t}$ and is hence in $L(B)$ like η. Note that $L(\bar{B})$ is the same as $L(B)$, since $\chi\in L(B)$ or $L(\bar{B})$ if $\int_0^T\chi_t^2dt<\infty$ P-a.s. or $\bar{P}$-a.s., respectively, and $\bar{P}\approx P$. Thus we can rewrite (3.20) as

$$\begin{aligned}d\hat{L}_t &= \hat{L}_t\left(\frac{\eta_t^M}{\beta M_t}d\bar{B}_t+\psi Y_tdt+\frac{(\frac{1}{\beta}-1)(\eta_t^M)^2}{2\beta M_t^2}dt\right)\\ &= \hat{L}_t\left(\frac{\eta_t^M}{\beta M_t}dB_t+\frac{2\varrho\mu\sqrt{Y_t}\eta_t^M}{\beta\sigma M_t}dt+\psi Y_tdt+\frac{(\frac{1}{\beta}-1)(\eta_t^M)^2}{2\beta M_t^2}dt\right), \qquad (3.21)\end{aligned}$$

for $0 \le t \le T$. We use this semimartingale decomposition for $\hat{L}$ to find its characteristics. Recalling the definition (3.8) of R and (3.7), we can compute the quadratic covariation

$$d[R, \hat{L}]_t = \frac{\hat{L}_t \varrho \sigma \sqrt{Y_t} \eta_t^M}{\beta M_t} dt. \tag{3.22}$$

From (3.21) and (3.22), we then obtain the differential characteristics $b^{\hat{L}}$ and $c^{R\hat{L}}$ under P as

$$b_t^{\hat{L}} = \hat{L}_t \bigg(\frac{2\varrho\mu\sqrt{Y_t}\eta_t^M}{\beta\sigma M_t} + \psi Y_t + \frac{(\frac{1}{\beta} - 1)(\eta_t^M)^2}{2\beta M_t^2} \bigg), \quad c_t^{R\hat{L}} = \frac{\hat{L}_t \varrho\sigma\sqrt{Y_t}\eta_t^M}{\beta M_t},$$

for $0 \le t \le T$.

We are now ready to check that the differential characteristics of $\hat{L}$ satisfy (3.5). Note that $b_t^R = \mu Y_t$ and $\frac{1}{\beta} - 1 = \frac{2\varrho^2}{\beta}$ because $\beta = 1 - 2\varrho^2$. Thus we get by plugging into the equation for $b^{\hat{L}}$ that

$$\begin{aligned}
b_t^{\hat{L}} &= \hat{L}_t \bigg(\psi Y_t + \frac{2\varrho\mu\sqrt{Y_t}\eta_t^M}{\beta\sigma M_t} + \frac{(\frac{1}{\beta} - 1)(\eta_t^M)^2}{2\beta M_t^2} \bigg) \\
&= \frac{\hat{L}_t}{\sigma^2 Y_t} \bigg(\mu^2 Y_t^2 + \frac{2\varrho\sigma\mu Y_t \sqrt{Y_t}\eta_t^M}{\beta M_t} + \frac{\varrho^2\sigma^2 Y_t (\eta_t^M)^2}{\beta^2 M_t^2} \bigg) \\
&= \frac{\hat{L}_t}{\sigma^2 Y_t} \bigg(\mu Y_t + \frac{\varrho\sigma\sqrt{Y_t}\eta_t^M}{\beta M_t} \bigg)^2 \\
&= \frac{\hat{L}_t}{c_t^R} \bigg(b_t^R + \frac{c_t^{R\hat{L}}}{\hat{L}_t} \bigg)^2, \quad 0 \le t \le T,
\end{aligned}$$

which shows that $\hat{L}$ satisfies (3.5). We have already seen that $\hat{L}$ is strictly positive, it is clear from (3.13) that $\hat{L}_T = 1$, and so it only remains to show that $\hat{L}$ is bounded above by 1. Indeed, if $\beta > 0$, then $\exp(-\beta\psi \int_t^T Y_u du) \le 1$ and hence $\hat{L} \le 1$ by (3.13). If instead $\beta < 0$, then $\exp(-\beta\psi \int_t^T Y_u du) \ge 1$ and again $\hat{L} \le 1$ by (3.13), noting that we take a negative power $1/\beta < 0$ of the expectation. This shows that if $T \in (0, T_1^\star]$, then $\hat{L}$ is a candidate opportunity process with time horizon T for $\beta \ne 0$.

3) If $1 - 2\varrho^2 = 0$, the proof is similar as for 2). Using $x \le e^x$ and recalling from the proof of 2) the processes $\tilde{X}^{(2)}$ and $\tilde{X}^{(3)}$, we start by noting the inequality

$$E_{\bar{P}} \bigg[\int_0^T \psi Y_s ds \bigg] \le E_P \bigg[\frac{d\bar{P}}{dP} \exp\bigg(\int_0^T \psi Y_s ds \bigg) \bigg] \le E_P[\exp(\tilde{X}_T^{(2)} + \tilde{X}_T^{(3)})] < \infty$$

for $T \in (0, T_1^\star]$, due to (3.17) (with $\beta = 1$ there). Fixing some $T \in (0, T_1^\star]$, we can therefore define the $\bar{P}$-martingale $(M_t)_{0\le t\le T}$ by

$$M_t = -E_{\bar{P}}\biggl[\int_0^T \psi Y_s ds \biggm| \mathcal{F}_t\biggr], \quad 0 \le t \le T.$$

We also define the process $(\hat{L}_t)_{0\le t\le T}$ as

$$\begin{aligned}\hat{L}_t &= \exp\left(M_t + \int_0^t \psi Y_s ds\right)\\ &= \exp\biggl(E_{\bar{P}}\biggl[-\int_t^T \psi Y_s ds \biggm| \mathcal{F}_t\biggr]\biggr), \quad 0 \le t \le T,\end{aligned} \tag{3.23}$$

which is a semimartingale by Itô's formula.

The rest of the argument is completely analogous to the case $1 - 2\varrho^2 \neq 0$. First, $\hat{L}$ is strictly positive by (3.23) and as Y is pathwise bounded. Under the assumption (3.14), which implies in particular that $\hat{L}$ is continuous, it follows from (3.23) that M is continuous as well. We then apply Itô's formula and (3.14) to (3.23) to obtain

$$b_t^{\hat{L}} dt - \frac{2\varrho\mu\eta_t\sqrt{Y_t}}{\sigma} dt + \eta_t d\bar{B}_t = d\hat{L}_t = \hat{L}_t\biggl(dM_t + \psi Y_t dt + \frac{1}{2}d[M]_t\biggr), \quad 0 \le t \le T,$$

recalling the $\bar{P}$-Brownian motion $d\bar{B}_t = dB_t + \frac{2\varrho\mu}{\sigma}\sqrt{Y_t}dt$. Comparing the local $\bar{P}$-martingale parts gives $dM = \eta^M d\bar{B}$, where the integrand η^M given by $\eta_t^M = \frac{\eta_t}{\hat{L}_t}$ is in $L(B) = L(\bar{B})$. Plugging in, we get the decomposition

$$d\hat{L}_t = \hat{L}_t\biggl(\eta_t^M dB_t + \frac{2\varrho\mu\sqrt{Y_t}\eta_t^M}{\sigma}dt + \psi Y_t dt + \frac{(\eta_t^M)^2}{2}dt\biggr), \quad 0 \le t \le T,$$

under P, and so we obtain the differential characteristics of $\hat{L}$ as

$$b_t^{\hat{L}} = \hat{L}_t\biggl(\frac{2\varrho\mu\sqrt{Y_t}\eta_t^M}{\sigma} + \psi Y_t + \frac{(\eta_t^M)^2}{2}\biggr), \quad c_t^{R\hat{L}} = \hat{L}_t\varrho\sigma\sqrt{Y_t}\eta_t^M, \quad 0 \le t \le T.$$

We can then check that (3.5) is satisfied. Indeed, as $\varrho^2 = \frac{1}{2}$ in this case, we have

$$\begin{aligned}b_t^{\hat{L}} &= \hat{L}_t\biggl(\psi Y_t + \frac{2\varrho\mu\sqrt{Y_t}\eta_t^M}{\sigma} + \frac{(\eta_t^M)^2}{2}\biggr)\\ &= \frac{\hat{L}_t}{\sigma^2 Y_t}\bigl(\mu^2 Y_t^2 + 2\varrho\sigma\mu Y_t\sqrt{Y_t}\eta_t^M + \varrho^2\sigma^2 Y_t(\eta_t^M)^2\bigr) = \frac{\hat{L}_t}{c_t^R}\biggl(b_t^R + \frac{c_t^{R\hat{L}}}{\hat{L}_t}\biggr)^2\end{aligned}$$

for $0 \le t \le T$ so that (3.5) holds. Finally, we obtain the bound $\hat{L} \le 1$ and the terminal condition $\hat{L}_T = 1$ directly from (3.15), and so $\hat{L}$ is a candidate opportunity process with time horizon T for $\beta = 0$. □

3.3 Explicit formula for the opportunity process

In Lemma 3.2, we have found a candidate opportunity process $\hat{L}$, given by a martingale distortion formula in terms of a conditional expectation under $\bar{P}$ involving the volatility process Y. Thanks to this formula, we shall obtain an explicit expression for $\hat{L}$ in Lemma 3.3. This is possible because the volatility process retains the structure of an affine Volterra process under $\bar{P}$ (see Abi Jaber et al. [1]), so that the conditional expectation (3.13) can be expressed in an exponentially affine form in terms of the forward variance curve $(\xi_t(u))$ as well as the solution $g^\star$ to a Riccati–Volterra equation (3.24) below. A similar representation can be found in Han/Wong [61, Equations (4.13) and (4.14)], where (3.13) is computed in terms of a modified forward variance curve $\xi_t^{\bar{P}}(u) := E_{\bar{P}}[Y_u \mid \mathcal{F}_t]$ calculated under $\bar{P}$.

In Lemma 3.3, we compute the conditional expectations from (3.13) if we have $1 - 2\varrho^2 \neq 0$, and from (3.15) for $1 - \varrho^2 = 0$. In the first case, this can be done by a relatively straightforward application of Theorem 2.17. This does not quite work the critical case $1 - 2\varrho^2 = 0$ since we need to calculate a conditional expectation involving an exponential and a linear term, for which we use Corollary 2.21 instead. As it turns out, both cases yield the same final formula for $\hat{L}$.

We recall the forward variance curve $(\xi_t(u))_{0\le t\le u\le T}$ as well as the kernel $\hat{\kappa}$ from Definition 2.9 and (2.12).

Lemma 3.3. *There exists some $T_2^\star \in (0, T_1^\star]$ such that for each time horizon $T \in (0, T_2^\star]$, there is a unique continuous solution $g^\star : [0,T] \to \mathbb{R}$ to the equation*

$$g^\star(t) = -\frac{\mu^2}{\sigma^2} - \frac{2\varrho\mu}{\sigma}(\hat{\kappa} * g^\star)(t) + \frac{1}{2}(1 - 2\varrho^2)\big((\hat{\kappa} * g^\star)(t)\big)^2, \quad 0 \le t \le T, \tag{3.24}$$

and the process $(\hat{L}_t)_{0\le t\le T}$ given in (3.19) resp. (3.23) satisfies

$$\hat{L}_t = \exp\left(\int_t^T g^\star(T-u)\xi_t(u)du\right), \quad 0 \le t \le T. \tag{3.25}$$

The process $\hat{L}$ is an Itô process with the dynamics

$$d\hat{L}_t = \hat{L}_t\bigg((\hat{\kappa}*g^\star)(T-t)\sqrt{Y_t}dB_t - g^\star(T-t)Y_tdt + \frac{1}{2}\big((\hat{\kappa}*g^\star)(T-t)\big)^2 Y_tdt\bigg) \tag{3.26}$$

for $0 \le t \le T$. *In consequence,* $\hat{L}$ *is a candidate opportunity process.*

Remark 3.4. We note that although $\hat{L}$ depends on the time horizon T, the solution $g^\star$ to (3.24) does not. More precisely, if $T' > T$ and there exists a continuous solution $g' : [0, T'] \to \mathbb{R}$ to (3.24) with time horizon T', then we have $g^\star(t) = g'(t)$ for all $t \in [0, T]$, by uniqueness of the solution $g^\star$ on $[0, T]$.

Proof of Lemma 3.3. We want to apply Theorem 2.17 to compute the conditional expectations in (3.19) and (3.23) in the cases $1 - 2\varrho^2 \neq 0$ and $1 - 2\varrho^2 = 0$, respectively; this is done in steps 1) and 2) below. We then show the dynamics (3.26) at the end, in step 3). The last assertion then follows from Lemma 3.2, since (3.26) ensures that $\hat{L}$ is an Itô process satisfying (3.14).

1) Recall the constant $\psi = \frac{\mu^2}{\sigma^2}$ introduced in the proof of Lemma 3.2. Consider the process $(\tilde{X}_t)_{0 \le t \le T} = (\tilde{Y}_t, \bar{N}_t, [\bar{N}]_t)_{0 \le t \le T}$, where $\tilde{Y}_t := \int_0^t Y_s ds$ and $\bar{N}$ is the stochastic logarithm of the density process $Z^{\bar{P}}$ defined in Lemma 3.2; so

$$\tilde{Y}_t = \int_0^t Y_s ds, \quad \bar{N}_t = \int_0^t -\frac{2\mu}{\sigma}\sqrt{Y_s} dW_s, \quad [\bar{N}]_t = \int_0^t 4\psi Y_s ds, \quad 0 \le t \le T. \tag{3.27}$$

Note that the coefficients $\mu^{\tilde{Y}} \equiv 1$, $\sigma^{\bar{N}} \equiv -\frac{2\mu}{\sigma}$ and $\mu^{[\bar{N}]} \equiv 4\psi$ are constant, hence bounded.

In the case $\beta = 1 - 2\varrho^2 \neq 0$, we have by the Bayes rule that

$$\begin{aligned} E_{\bar{P}}\left[\exp\left(-\beta\psi \int_t^T Y_u du\right) \middle| \mathcal{F}_t\right] &= E_{\bar{P}}\left[\exp\left(-\beta\psi(\tilde{Y}_T - \tilde{Y}_t)\right) \middle| \mathcal{F}_t\right] \\ &= \mathcal{E}(\bar{N})_t^{-1} \exp(\beta\psi\tilde{Y}_t) E_P[\mathcal{E}(\bar{N})_T \exp(-\beta\psi\tilde{Y}_T) \mid \mathcal{F}_t]. \end{aligned} \tag{3.28}$$

In terms of the components of $\tilde{X}$, the term in the conditional expectation can be written as

$$\mathcal{E}(\bar{N})_T \exp(-\beta\psi\tilde{Y}_T) = \exp\left(\bar{N}_T - \frac{[\bar{N}]_T}{2} - \beta\psi\tilde{Y}_T\right).$$

By part 1) of Theorem 2.17 applied to $\tilde{X}$, there exists some $\tilde{T}_2^\star \in (0, T_1^\star]$ such that for $T \in (0, \tilde{T}_2^\star]$, there is a unique solution $x_\beta \in L^1([0, T]; \mathbb{C})$ to the equation

$$x_\beta(t) = -\beta\psi - \frac{2\mu\varrho}{\sigma}(\hat{\kappa} * x_\beta)(t) + \frac{1}{2}\big((\hat{\kappa} * x_\beta)(t)\big)^2, \quad 0 \le t \le T, \tag{3.29}$$

and it holds that

$$E_P[\mathcal{E}(\bar{N})_T \exp(-\beta\psi\tilde{Y}_T) \mid \mathcal{F}_t] = \mathcal{E}(\bar{N})_t \exp\bigg(-\beta\psi\tilde{Y}_t + \int_t^T x_\beta(T-u)\xi_t(u)du\bigg). \tag{3.30}$$

In particular, the conditional expectation on the left-hand side of (3.30) is well defined. Plugging this into (3.28) and using the formula (3.19) for $\hat{L}$ yields

$$\hat{L}_t = \exp\left(\frac{1}{\beta}\int_t^T x_\beta(T-u)\xi_t(u)du\right), \quad 0 \le t \le T. \tag{3.31}$$

To bring this into the form (3.25), we note that for $T \in (0, \tilde{T}_2^\star]$, the function $g^\star := x_\beta/\beta \in L^1([0,T];\mathbb{C})$ is a solution to (3.24) since for $0 \le t \le T$,

$$\begin{aligned} g^\star(t) = \frac{x_\beta(t)}{\beta} &= -\psi - \frac{2\mu\varrho}{\beta\sigma}(\hat{\kappa} * x_\beta)(t) + \frac{1}{2\beta}\big((\hat{\kappa} * x_\beta)(t)\big)^2 \\ &= -\psi - \frac{2\mu\varrho}{\sigma}(\hat{\kappa} * g^\star)(t) + \frac{\beta}{2}\big((\hat{\kappa} * g^\star)(t)\big)^2 \\ &= -\psi - \frac{2\mu\varrho}{\sigma}(\hat{\kappa} * g^\star)(t) + \frac{1}{2}(1-2\varrho^2)\big((\hat{\kappa} * g^\star)(t)\big)^2. \end{aligned} \tag{3.32}$$

Conversely, if $g^\star$ is a solution to (3.24) on $[0,T]$, we similarly get that $x_\beta := \beta g^\star$ solves (3.29). Since the solution x_β to (3.29) is unique due to Theorem 2.17, it follows that the solution $g^\star$ to (3.24) is unique as well. Plugging into (3.31), we therefore obtain (3.25).

It remains to argue that $g^\star$ is continuous and real-valued. To see the latter, note that $\hat{\kappa}$ and all of the constants in (3.24) are real-valued, and hence the complex conjugate $\overline{g^\star}$ is also a solution to (3.24). Thus $\overline{g^\star} = g^\star$ by the uniqueness so that $g^\star$ is real-valued. To show the continuity, note that (3.24) is of the form (A.2.17), where we replace x with $g^\star$ and set $y \equiv 0$ and

$$f(z,s) := -\frac{\mu^2}{\sigma^2} - \frac{2\varrho\mu}{\sigma}z + \frac{1}{2}(1-2\varrho^2)z^2.$$

As f and y are continuous, it follows by Corollary A.2.6 that there exists some (possibly smaller) $T_2^\star \in (0, \tilde{T}_2^\star]$ such that there is a unique continuous solution $\tilde{g}^\star$ to (3.24) on the interval $[0, T_2^\star]$. Note that for any choice of time horizon $T \in (0, T_2^\star]$, the restriction of $\tilde{g}^\star$ to $[0,T]$ is also a solution to (3.24) on the interval $[0,T]$; this follows by the causality property of the convolution mentioned after Definition 2.3. Since $g^\star$ is the unique solution in $L^1([0,T];\mathbb{R})$ to (3.24), we have by uniqueness that $g^\star = \tilde{g}^\star$ on $[0, T]$ and thus it is continuous.

2) The case $1 - 2\varrho^2 = 0$ can be checked similarly to step 1). Recall the process $(\tilde{X}_t)_{0 \le t \le T} = (\tilde{Y}_t, \bar{N}_t, [\bar{N}]_t)_{0 \le t \le T}$ with dynamics given by (3.27). In this case, we need to compute the conditional expectation of $\psi \tilde{Y}_T$ under $\bar{P}$; we use Corollary 2.21 to deal with the linear term. Consider the equation (2.38) with $z = z_1 := (0, 1, -1/2)$. Since the terms that are constant in $g_{z_1,T}$ vanish, it follows that $g_{z_1,T} = 0$ is the unique solution. Thus by Corollary 2.21, there exists some $\tilde{T}_2^\star \in (0, T_1^\star]$ such that for $T \in (0, \tilde{T}_2^\star]$, there exists a unique solution $\tilde{g} \in L^1([0, T]; \mathbb{R})$ to the equation

$$\tilde{g}(t) = \psi - \frac{2\varrho\mu}{\sigma}(\hat{\kappa} * \tilde{g})(t), \quad 0 \le t \le T, \tag{3.33}$$

which corresponds to (2.91) with $z_1 = (0, 1, -1/2)$, $z_2 = (1, 0, 0)$ and $g_{z_1,T} = 0$; the fact that $\tilde{g}$ is real-valued follows by the same argument as at the end of step 1) for $g^\star$. Moreover, Corollary 2.21 also gives that

$$E_P[\mathcal{E}(\bar{N})_T \psi \tilde{Y}_T \mid \mathcal{F}_t] = \mathcal{E}(\bar{N})_t \bigg(\psi \tilde{Y}_t + \int_t^T \tilde{g}(T-u)\xi_t(u)du \bigg), \quad 0 \le t \le T. \tag{3.34}$$

The equation (3.33) does not exactly match (3.24) with $1 - 2\varrho^2 = 0$, since it lacks a minus sign next to the constant term $\psi = \mu^2/\sigma^2$. However, it is clear by plugging $g^\star := -\tilde{g}$ into (3.33) that $g^\star$ is a solution to (3.24) in this case. Conversely, any solution to (3.24) yields a solution to (3.33) by changing the sign, and hence the solution $g^\star$ to (3.24) is unique. By the same argument as in step 1), there exists some $T_2^\star \in (0, \tilde{T}_2^\star]$ such that for any $T \in (0, T_2^\star]$, $g^\star$ is the unique continuous (rather than L^1) solution to (3.24) on $[0, T]$. Plugging $g^\star$ into (3.34) yields

$$E_P[\mathcal{E}(\bar{N})_T \psi \tilde{Y}_T \mid \mathcal{F}_t] = \mathcal{E}(\bar{N})_t \bigg(\psi \tilde{Y}_t - \int_t^T g^\star(T-u)\xi_t(u)du \bigg), \quad 0 \le t \le T,$$

and hence

$$E_{\bar{P}}\bigg[\psi \int_t^T Y_u du \,\bigg|\, \mathcal{F}_t \bigg] = E_{\bar{P}}[\psi(\tilde{Y}_T - \tilde{Y}_t) \mid \mathcal{F}_t] = -\int_t^T g^\star(T-u)\xi_t(u)du, \quad 0 \le t \le T.$$

Plugging into (3.15) directly yields (3.25).

3) It remains to show the dynamics (3.26) for $\hat{L}$. Note that Corollary 2.14

gives the semimartingale decomposition

$$d\Big(\int_t^T g^\star(T-u)\xi_t(u)du\Big) = -g^\star(T-t)Y_t dt + (\hat\kappa * g^\star)(T-t)\sqrt{Y_t}dB_t.$$

Applying Itô's formula to $\hat L_t = \exp(\int_t^T g^\star(T-u)\xi_t(u)du)$ from (3.25), we then obtain that

$$d\hat L_t = \hat L_t\Big(-g^\star(T-t)Y_t dt + (\hat\kappa * g^\star)(T-t)\sqrt{Y_t}dB_t + \frac{1}{2}\big((\hat\kappa * g^\star)(T-t)\big)^2 Y_t dt\Big)$$

for $0 \le t \le T$, and this is (3.26). As argued at the beginning of the proof, this ensures that $\hat L$ is an Itô process satisfying (3.14) and hence a candidate opportunity process by Lemma 3.2. □

For a fixed time horizon $T \in (0, T_2^\star]$, we define $h^\star : [0,T] \to \mathbb{R}$ by

$$h^\star(t) = (\hat\kappa * g^\star)(t) = \int_0^t \hat\kappa(t-s)g^\star(s)ds. \tag{3.35}$$

The function $h^\star$ inherits continuity from $g^\star$, as mentioned after Definition 2.3. Like $g^\star$ in Remark 3.4, also $h^\star$ does not depend on the choice of T because the causality property given after Definition 2.3 says that $h^\star(t) = (\hat\kappa * g^\star)(t)$ only depends on the values taken by $g^\star$ on $[0,t]$.

With the notation $h^\star$, we can directly read off from (3.26) and (3.8) the differential characteristics of $\hat L$ as

$$b_t^{\hat L} = \hat L_t\left(-g^\star(T-t)Y_t + \big(h^\star(T-t)\big)^2 Y_t\right), \quad c_t^{R\hat L} = \hat L_t \varrho\sigma h^\star(T-t)Y_t,$$

for $0 \le t \le T$. This makes it straightforward to find the candidate adjustment process $\hat a$.

Corollary 3.5. *For any $T \in (0, T_2^\star]$, the candidate adjustment process $(\hat a_t)_{0\le t\le T}$ corresponding to the candidate opportunity process from Lemma 3.3 is given by*

$$\hat a_t = \frac{1}{\sigma S_t}\Big(\frac{\mu}{\sigma} + \varrho h^\star(T-t)\Big), \quad 0 \le t \le T. \tag{3.36}$$

Proof. Plugging $b^{\hat L}$ and $c^{R\hat L}$ into the definition (3.6) for $\hat a$ gives

$$\hat a_t = \frac{1}{S_t c_t^R}\left(b_t^R + \frac{c_t^{R\hat L}}{\hat L_t}\right) = \frac{1}{\sigma^2 S_t Y_t}\Big(\mu Y_t + \varrho\sigma h^\star(T-t)Y_t\Big) = \frac{1}{\sigma S_t}\Big(\frac{\mu}{\sigma} + \varrho h^\star(T-t)\Big).$$

□

3.4 Verification

Using the martingale distortion technique and by exploiting the affine structure of the model, we have found in Lemma 3.3 a candidate opportunity process $\hat{L}$ as a solution to the BSDE (3.5) on $[0,T]$. It now remains to be shown that $\hat{L}$ is the true opportunity process, which is done in Theorem 3.8. As mentioned after (3.6), this follows from Černý/Kallsen [25, Theorem 3.25] after checking that certain conditions are satisfied, as we explain in more detail below.

Given $\hat{L}$ and the candidate adjustment strategy $\hat{a}$ computed in Corollary 3.5, we obtain a candidate pure investment strategy $\widehat{\vartheta}^\star := \hat{a}\mathcal{E}(-\hat{a} \bullet S)$ and a candidate variance-optimal martingale measure $\widehat{Q}^\star$ with density $d\hat{Q}^\star/dP = \mathcal{E}(-\hat{a} \bullet S)_T/\hat{L}_0$. The density process $Z^{\widehat{Q}^\star}$ for $\widehat{Q}^\star$ is given below in Lemma 3.6, and we later show in (3.42) in the proof of Theorem 3.8 that we indeed have $Z_T^{\widehat{Q}^\star} = \mathcal{E}(-\hat{a} \bullet S)_T/\hat{L}_0$. If $\hat{L}$ and $\hat{a}$ are the true opportunity and adjustment processes, respectively, then $\widehat{\vartheta}^\star$ and $\widehat{Q}^\star$ are the true pure investment strategy and VOMM, respectively, by results of [25] and as we see later in Corollary 3.9.

Conversely, given a candidate opportunity process $\hat{L}$, we can use [25, Theorem 3.25] to show that $\hat{L}$ is the true opportunity process. For this, we need to check some conditions involving the candidate processes $\hat{a}, \widehat{\vartheta}^\star$ and $Z^{\widehat{Q}^\star}$ associated with $\hat{L}$; namely, that the strategy $\widehat{\vartheta}^\star$ is admissible and the measure $\widehat{Q}^\star$ is an equivalent martingale measure. More precisely, we also need to show a dynamic version of these properties on each stochastic interval $]\!]\tau, T]\!]$ for a stopping time $\tau \leq T$; the exact conditions are given in the proof of Theorem 3.8. Before proving that theorem, we first show two technical results in Lemmas 3.6 and 3.7.

In the following, let (S,Y) satisfy the rough Heston model and $T_2^\star$ and $g^\star$ be as given in Lemma 3.3. We also recall the function $h^\star := \hat{\kappa} * g^\star$ defined in (3.35).

Lemma 3.6. *There exists some $T_3^\star \in (0, T_2^\star]$ such that for each $T \in (0, T_3^\star]$, the process $(Z_t^{\widehat{Q}^\star})_{0\leq t\leq T}$ defined as the stochastic exponential*

$$Z^{\widehat{Q}^\star} = \mathcal{E}\Big(\sqrt{1-\varrho^2}\big(h^\star(T-\cdot)\sqrt{Y}\big) \bullet W^\perp - \frac{\mu}{\sigma}\sqrt{Y} \bullet W\Big) \tag{3.37}$$

is a strictly positive square-integrable P-martingale on $[0,T]$. Then the measure $\widehat{Q}^\star$ defined by the density $d\widehat{Q}^\star/dP = Z_T^{\widehat{Q}^\star}$ belongs to $\mathbb{Q}_T^2(S)$, i.e., it is an equivalent local martingale measure for S such that $E_P[(d\widehat{Q}^\star/dP)^2] < \infty$.

Proof. As $h^\star$ and Y are continuous, $Z^{\widehat{Q}^\star}$ is a strictly positive local P-martingale. The stochastic logarithm $N^{\widehat{Q}^\star}$ of $Z^{\widehat{Q}^\star}$ is given by

$$dN_t^{\widehat{Q}^\star} = \sqrt{1-\varrho^2}\big(h^\star(T-t)\sqrt{Y_t}\big)dW_t^\perp - \frac{\mu}{\sigma}\sqrt{Y_t}dW_t, \quad 0 \le t \le T.$$

Since the Brownian motions W and $W^\perp$ are orthogonal and $\psi = \frac{\mu^2}{\sigma^2}$, we have

$$\begin{aligned} E_P\bigg[\exp\bigg(\frac{1}{2}[N^{\widehat{Q}^\star}]_T\bigg)\bigg] &= E_P\bigg[\exp\bigg(\frac{1}{2}\int_0^T Y_s\Big((1-\varrho^2)\big(h^\star(T-s)\big)^2+\psi\Big)ds\bigg)\bigg] \\ &\le E_P\bigg[\exp\bigg(\frac{\tilde{X}_T^{(4)}}{2}\bigg)\bigg], \end{aligned} \tag{3.38}$$

where we define $(\tilde{X}_t^{(4)})_{0\le t\le T}$ by

$$\tilde{X}_t^{(4)} = \big((1-\varrho^2)\bar{h}^2+\psi\big)\int_0^t Y_s ds$$

for $\bar{h} := \sup_{t\in[0,T]}|h^\star(t)|$. As the coefficient $(1-\varrho^2)\bar{h}^2+\psi$ is constant (hence bounded), we can apply Theorem 2.17 to the process $\tilde{X}^{(4)}$ so that there exists some $\tilde{T}_3^\star > 0$ such that $E_P[\exp(\frac{1}{2}\tilde{X}_T^{(4)})] < \infty$ for $T \in (0, \tilde{T}_3^\star]$. By Novikov's criterion and (3.38), $Z^{\widehat{Q}^\star}$ is thus a P-martingale on $[0,T]$ for any such T.

To show that $Z^{\widehat{Q}^\star}$ is square-integrable, we estimate

$$\begin{aligned} E_P[(Z_T^{\widehat{Q}^\star})^2] &= E_P\big[\exp(2N_T^{\widehat{Q}^\star} - [N^{\widehat{Q}^\star}]_T)\big] = E_P\big[\exp(2N_T^{\widehat{Q}^\star} - 4[N^{\widehat{Q}^\star}]_T + 3[N^{\widehat{Q}^\star}]_T)\big] \\ &\le E_P[\mathcal{E}(4N^{\widehat{Q}^\star})_T]^{1/2} E_P\big[\exp(6[N^{\widehat{Q}^\star}]_T)\big]^{1/2} \le E_P\big[\exp(6[N^{\widehat{Q}^\star}]_T)\big]^{1/2} \end{aligned}$$

by using the Cauchy–Schwarz inequality and the fact that $\mathcal{E}(4N^{\widehat{Q}^\star})$ is a positive local martingale, hence a supermartingale. Note that as in (3.38), we have $6[N^{\widehat{Q}^\star}]_T \le 6\tilde{X}_T^{(4)}$. Thus again applying Theorem 2.17 to $\tilde{X}^{(4)}$, there exists some $T_3^\star \in (0, \tilde{T}_3^\star]$ such that for each $T \in (0, T_3^\star]$, we have

$$E_P[(Z_T^{\widehat{Q}})^2] \le E_P\big[\exp(6[N^{\widehat{Q}^\star}]_T)\big]^{1/2} < \infty.$$

Hence, $Z^{\widehat{Q}}$ is a square-integrable martingale on $[0,T]$ for any such T and $\widehat{Q}^\star$ is a probability measure equivalent to P with $E_P[(d\widehat{Q}^\star/dP)^2] < \infty$.

It remains to show that $\widehat{Q}^\star$ is a local martingale measure for S on $[0,T]$ for any $T \in (0, T_3^\star]$. By Girsanov's theorem and (3.37), $(W_t^\star)_{0\le t\le T}$ defined by

$W_t^\star = W_t + \int_0^t \frac{\mu}{\sigma}\sqrt{Y_s}ds$ is a $\widehat{Q}^\star$-Brownian motion on $[0,T]$. Since

$$dS_t = S_t(\mu Y_t dt + \sigma\sqrt{Y_t}dW_t) = S_t\sigma\sqrt{Y_t}dW_t^\star, \quad 0 \le t \le T,$$

and S, Y are both continuous, S is indeed a $\widehat{Q}^\star$-local martingale. □

Lemma 3.7. *Let $T_3^\star$ be as given in Lemma 3.6 and $\hat{a}$ defined by* (3.36)*. There exists some $T^\star \in (0, T_3^\star]$ such that $E_P[\exp([\hat{a} \bullet S]_T)] < \infty$ for each $T \in (0, T^\star]$.*

Proof. We recall the formula (3.36) for $\hat{a}$ as well as the dynamics (2.8) for S. They are

$$\hat{a}_t = \frac{1}{\sigma S_t}\left(\frac{\mu}{\sigma} + \varrho h^\star(T-t)\right) \quad \text{and} \quad \frac{dS_t}{S_t} = \mu Y_t dt + \sigma\sqrt{Y_t}dW_t, \quad 0 \le t \le T.$$

By plugging in, we thus get

$$E_P\big[\exp([\hat{a} \bullet S]_T)\big] = E_P\left[\exp\left(\int_0^T \left(\frac{\mu}{\sigma} + \varrho h^\star(T-s)\right)^2 Y_s ds\right)\right] \le E_P[\exp(\tilde{X}_T^{(5)})],$$

where we define the process $(\tilde{X}_t^{(5)})_{0 \le t \le T}$ by $\tilde{X}_t^{(5)} = \int_0^t (\frac{|\mu|}{\sigma} + \varrho\bar{h})^2 Y_s ds$, and recall $\bar{h} := \sup_{t\in[0,T]} |h^\star(t)|$ as used in the proof of Lemma 3.6. By Theorem 2.17 applied to $\tilde{X}^{(5)}$, there exists some $T^\star \in (0, T_3^\star]$ such that $E[\exp(\tilde{X}_T^{(5)})] < \infty$ for each $T \in (0, T^\star]$. This gives the result. □

We are now ready to move on to the main theorem.

Theorem 3.8. *Let $T^\star$ be as given in Lemma 3.7. For each $T \in (0, T^\star]$, the opportunity process $L = (L_t)_{0 \le t \le T}$ for S with time horizon T is given by*

$$L_t = \hat{L}_t = \exp\left(\int_t^T g^\star(T-u)\xi_t(u)du\right), \quad 0 \le t \le T, \tag{3.39}$$

where $g^\star : [0,T] \to \mathbb{R}$ is the unique continuous solution to the equation (3.24)*.*

Proof. We already showed in Lemma 3.3 that (3.24) has a unique solution up to some $T_2^\star \ge T^\star > 0$. We now show that L is the true opportunity process for any time horizon $T \in (0, T^\star]$. We claim that this is implied by the conditions (a)–(c) below:

(a) There exists an equivalent local martingale measure for S on $[0,T]$ with square-integrable density.

(b) For each stopping time $\tau \le T$, the process $\mathcal{E}((-\hat{a}\mathbf{1}_{]\!]\tau,T]\!]}) \bullet S)\hat{L}$ is of class (D).

(c) For each stopping time $\tau \le T$, the strategy

$$\hat{\vartheta}^{(\tau)} = \hat{a}\mathcal{E}\big((-\hat{a}\mathbf{1}_{\rrbracket\tau,T\rrbracket})\bullet S\big)\mathbf{1}_{\rrbracket\tau,T\rrbracket} \tag{3.40}$$

belongs to $\overline{\Theta}_T(S)$, i.e., is admissible.

Indeed, the implication follows by Černý/Kallsen [25, Theorem 3.25]. The condition (a) is a standing assumption in [25], while (b) and (c) correspond to condition 4. in that theorem. The remaining conditions 1., 2. and 3. in [25, Theorem 3.25] are satisfied since $\hat{L}$ is a candidate opportunity process (as defined after (3.5)), and hence that theorem ensures that $\hat{L}$ is the true opportunity process under conditions (a)–(c). It now remains to show that the conditions hold.

We have shown (a) in Lemma 3.6 for $T \in (0, T^\star] \subseteq (0, T_3^\star]$. We also get (b) as a consequence of Lemma 3.6, by the same proof as in Černý/Kallsen [27, Proposition 3.2, Step 1]. To see this, note that Itô's formula together with the dynamics (3.26) for $\hat{L}$ gives

$$\begin{aligned}\frac{d(\mathcal{E}(-\hat{a}\bullet S)_t\hat{L}_t)}{\mathcal{E}(-\hat{a}\bullet S)_t\hat{L}_t} =& \Big(-\mu\hat{a}_t S_t - g^\star(T-t) + \big(h^\star(T-t)\big)^2 - \hat{a}_t S_t \varrho\sigma h^\star(T-t)\Big) Y_t dt \\ & - \hat{a}_t S_t \sigma\sqrt{Y_t}dW_t + h^\star(T-t)\sqrt{Y_t}dB_t.\end{aligned}$$

Plugging in (3.36) for $\hat{a}_t$ and $h^\star = \hat{\kappa} * g^\star$ by (3.35), we obtain the dt-integrand

$$\Big(-\frac{\mu^2}{\sigma^2} - \frac{2\mu\varrho}{\sigma}h^\star(T-t) + (1-\varrho^2)\big(h^\star(T-t)\big)^2 - g^\star(T-t)\Big) Y_t dt = 0$$

because $g^\star$ satisfies (3.24). By using again (3.36) for $\hat{a}_t$ and the orthogonal decomposition (3.7) for B, we get

$$\begin{aligned}\frac{d(\mathcal{E}(-\hat{a}\bullet S)_t\hat{L}_t)}{\mathcal{E}(-\hat{a}\bullet S)_t\hat{L}_t} &= -\hat{a}_t S_t \sigma\sqrt{Y_t}dW_t + h^\star(T-t)\sqrt{Y_t}dB_t \\ &= \sqrt{1-\varrho^2}\, h^\star(T-t)\sqrt{Y_t}dW_t^\perp - \frac{\mu}{\sigma}\sqrt{Y_t}dW_t \\ &= \frac{dZ_t^{\widehat{Q}^\star}}{Z_t^{\widehat{Q}^\star}} \end{aligned} \tag{3.41}$$

and therefore

$$\frac{\mathcal{E}(-(\hat{a}\mathbf{1}_{\rrbracket\tau,T\rrbracket})\bullet S)\hat{L}}{\hat{L}_\tau} = \frac{Z^{\widehat{Q}^\star}}{Z_\tau^{\widehat{Q}^\star}} \quad \text{on } \llbracket\tau, T\rrbracket; \tag{3.42}$$

note that the numerator on the left-hand side equals $\hat{L}_\tau$ at time τ. Hence the condition (b) is equivalent to the statement that for each stopping time $\tau \leq T$, the process $\hat{L}_\tau Z^{\widehat{Q}^\star}_{\tau\vee\cdot}/Z^{\widehat{Q}^\star}_\tau$ is of class (D). As $\hat{L}_\tau$ is bounded above by 1, it suffices to show that $Z^{\widehat{Q}^\star}_{\tau\vee\cdot}/Z^{\widehat{Q}^\star}_\tau$ is of class (D). But $Z^{\widehat{Q}^\star}_{\tau\vee\cdot}/Z^{\widehat{Q}^\star}_\tau$ is a positive local martingale by the construction of $Z^{\hat{Q}^\star}$ and hence also a supermartingale that takes the value 1 on $[\![0,\tau]\!]$. Since $E[Z^{\widehat{Q}^\star}_T/Z^{\widehat{Q}^\star}_\tau|\mathcal{F}_\tau] = 1$ as $Z^{\widehat{Q}^\star}$ is a martingale, it follows that $Z^{\widehat{Q}^\star}_{\tau\vee\cdot}/Z^{\widehat{Q}^\star}_\tau$ is also a martingale on $[0,T]$ and thus of class (D).

Finally, (c) follows from Lemma 3.7 by the same argument as in [27, Proposition 3.2, Step 2]. To see this, fix $T \in (0,T^\star]$ and recall the set $\mathbb{Q}^2_T(S)$ of all equivalent local martingale measures for S on $[0,T]$ that have a square-integrable density. For any $Q \in \mathbb{Q}^2_T(S)$, the Cauchy–Schwarz inequality gives

$$E_Q\left[\exp\left(\frac{[\hat{a}\bullet S]_T}{2}\right)\right] \leq \sqrt{E_P\big[\exp([\hat{a}\bullet S]_T)\big]E_P\left[\left(\frac{dQ}{dP}\right)^2\right]} < \infty.$$

Since S is a local Q-martingale on $[0,T]$, it follows by the Novikov criterion that $\mathcal{E}(-(\mathbf{1}_{]\!]\tau,T]\!]}\hat{a})\bullet S)$ is a Q-martingale on $[0,T]$ for each stopping time τ with values in $[0,T]$. Note that the wealth process generated by the strategy $\hat{\vartheta}^{(\tau)}$ from (3.40) is given by

$$\begin{aligned}\hat{\vartheta}^{(\tau)}\bullet S &= \Big(\hat{a}\mathbf{1}_{]\!]\tau,T]\!]}\mathcal{E}\big((-\hat{a}\mathbf{1}_{]\!]\tau,T]\!]})\bullet S\big)\Big)\bullet S\\ &= \mathcal{E}\big((-\hat{a}\mathbf{1}_{]\!]\tau,T]\!]})\bullet S\big)\bullet\big((\hat{a}\mathbf{1}_{]\!]\tau,T]\!]})\bullet S\big)\\ &= 1-\mathcal{E}(-\mathbf{1}_{]\!]\tau,T]\!]}\hat{a})\bullet S,\end{aligned}$$

using the property $\mathcal{E}(M)\bullet M = \mathcal{E}(M)-1$. As $\mathcal{E}(-(\mathbf{1}_{]\!]\tau,T]\!]}\hat{a})\bullet S)$ is a Q-martingale on $[0,T]$, so is then $\hat{\vartheta}^{(\tau)}\bullet S$. Since this holds for any $Q \in \mathbb{Q}^2_T(S)$, it follows by [27, Corollary 2.5] that $\hat{\vartheta}^{(\tau)}$ is admissible. This concludes the proof of the conditions (a)–(c) and hence of Theorem 3.8. □

As a corollary, we can now collect several formulas that follow from Theorem 3.8 by results of Černý/Kallsen [25] since we have shown that $\hat{L}$ from (3.39) is the true opportunity process; this includes a formula for the optimal pure investment strategy $\vartheta^\star$. Thus with the formulas in Corollary 3.9, we have solved the pure investment problem for the rough Heston model, which was our original goal. Looking ahead, in the next chapter, we mostly use the variance-optimal martingale measure $Q^\star$ given by 4) below since it is useful for solving the general mean–variance hedging problem.

In the following, recall the orthogonal decomposition (3.7) for B.

Corollary 3.9. *Fix a time horizon $T \in (0, T^*]$, where $T^\star$ is as in Theorem 3.8, and let (S, Y) satisfy the rough Heston model (2.8) on $[0, T]$. Then the following statements hold:*

1) *The adjustment process $a = (a_t)_{0 \le t \le T}$ is given by*

$$a_t = \frac{\mu + \varrho\sigma h^\star(T-t)}{\sigma^2 S_t}. \tag{3.43}$$

2) *The wealth process $(V_t^\star)_{0 \le t \le T}$ corresponding to the optimal strategy $(\vartheta_t^\star)_{0 \le t \le T}$ for the pure investment problem is given by*

$$\begin{aligned} V_t^\star &= (\vartheta^\star \bullet S)_t = 1 - \mathcal{E}(-a \bullet S)_t \\ &= 1 - \mathcal{E}\Big(\int \Big(-\frac{\mu}{\sigma} + \varrho h^\star(T-s)\Big)\Big(\frac{\mu Y_s}{\sigma} ds + \sqrt{Y_s} dW_s\Big)\Big)_t. \end{aligned}$$

3) *The optimal strategy for the pure investment problem is given by*

$$\vartheta_t^\star = (1 - V_t^\star) a_t = a_t \, \mathcal{E}(-a \bullet S)_t.$$

4) *The density process $(Z_t^{Q^\star})_{0 \le t \le T}$ for the variance-optimal martingale measure $Q^\star$ is given by*

$$Z_t^{Q^\star} = \mathcal{E}\Big(\int \sqrt{Y_s}\Big(\sqrt{1-\varrho^2}\, h^\star(T-s) dW_s^\perp - \frac{\mu}{\sigma} dW_s\Big)\Big)_t. \tag{3.44}$$

Proof. Since $\hat{L} = L$ is the true opportunity process by Theorem 3.8, it follows from [25, Theorem 3.25] that the candidate adjustment process $\hat{a}$ defined in (3.6) is the true adjustment process; we then get (3.43) by Corollary 3.5. The strategy $\vartheta^\star$ is given directly in terms of a in [25, Lemma 3.7], which yields 3), and then we also obtain the expression for $V^\star = \vartheta^\star \bullet S$ in 2). The density process $Z^{Q^\star}$ is given in [25, Proposition 3.13] as

$$Z^{Q^\star} = \frac{L\mathcal{E}(-a \bullet S)}{L_0},$$

and so we get (3.44) by the previous calculations in (3.41), which shows 4). □

Remark 3.10. By Girsanov's theorem, we have the $Q^\star$-Brownian motions $W^\star$

and $B^\star$ defined by

$$W_t^\star = W_t + \int_0^t \frac{\mu}{\sigma}\sqrt{Y_s}ds,$$
$$B_t^\star = B_t + \int_0^t \frac{\varrho\mu}{\sigma}\sqrt{Y_s}ds - \int_0^t (1-\varrho^2)h^\star(T-s)\sqrt{Y_s}ds.$$

Thus by plugging into the dynamics (2.8) for the rough Heston model, we get that the dynamics under $Q^\star$ are given by

$$dS_t = \sigma S_t \sqrt{Y_t} dW_t^\star,$$
$$Y_t = Y_0 + \int_0^t \kappa(t-s)\bigg(\Big(\theta\lambda - \Big(\lambda + \frac{\zeta\varrho\mu}{\sigma} - \zeta(1-\varrho^2)h^\star(T-s)\Big)Y_s\Big)ds + \zeta\sqrt{Y_s}dB_s^\star\bigg).$$

This is similar to the original dynamics under P, with the notable difference that the linear term in the drift of Y now has a time-dependent coefficient $h^\star$. Under $Q^\star$, the process Y can be seen as a time-inhomogeneous affine Volterra process in the sense of Ackermann et al. [5].

3.5 Comparison to the literature

In this section, we relate Theorem 3.8 to two results found in the literature. We start by considering Černý/Kallsen [27], where the pure investment problem is solved for the classical Heston model. We want to show that the opportunity process we obtain in Theorem 3.8 for the rough Heston model coincides with the one given in [27, Proposition 3.1] in the classical case (where the kernel $\kappa \equiv 1$ is trivial), as this is not clear at a first glance. In [27], the opportunity process is given as an exponentially affine function of the spot volatility by

$$L_t = \exp\big(\varkappa_0(t) + \varkappa_1(t)Y_t\big), \quad 0 \le t \le T, \tag{3.45}$$

where the functions $\varkappa_0, \varkappa_1 : [0,T] \to \mathbb{R}$ are the solutions to the two ordinary Riccati differential equations (3.47) and (3.48) below. In these equations, we also include our parameter σ, which is set to 1 in [27], as we can see by comparing (2.8) with [27, (1.1) and (1.2)]. The formula (3.45) for L involves only the spot volatility instead of the forward variance curve, which makes sense as unlike the rough Heston, the classical Heston model is Markovian. Nevertheless, (3.45) yields the same opportunity process as (3.39), and we show the equivalence in the following lemma. To that end, we use the fact that the forward variance curve can be given explicitly as a function of the spot volatility Y_t in this case, which leads to a simplification in the formula for the opportunity process.

Lemma 3.11. *Suppose that the kernel $\kappa \equiv 1$ is trivial, so that the process (S, Y) satisfies the classical Heston model (2.8), and there exists a continuous solution $g^\star : [0, T] \to \mathbb{R}$ to (3.24). Then we have*

$$\int_t^T g^\star(T-u)\xi_t(u)du = \varkappa_0(t) + \varkappa_1(t)Y_t, \quad 0 \le t \le T, \tag{3.46}$$

where the functions $\varkappa_0, \varkappa_1 : [0, T] \to \mathbb{R}$ are continuously differentiable and satisfy the Riccati differential equations

$$-\varkappa_0'(t) = \lambda\theta\varkappa_1(t), \quad 0 \le t \le T, \tag{3.47}$$

$$-\varkappa_1'(t) = -\frac{\mu^2}{\sigma^2} - \left(\lambda + \frac{2\varrho\zeta\mu}{\sigma}\right)\varkappa_1(t) + \frac{\zeta^2(1-2\varrho^2)}{2}\varkappa_1^2(t), \quad 0 \le t \le T, \tag{3.48}$$

with terminal conditions $\varkappa_0(T) = \varkappa_1(T) = 0$.

Proof. As mentioned before, the reason for the simpler formula (3.46) in the classical Heston model is that the forward variance curve $(\xi_t(u))_{0 \le t \le u \le T}$ can be given as a function of the spot volatility Y_t. The formula for the forward variance curve in the Heston model is well known, but for the sake of completeness, we calculate it here using Lemma 2.10. Recall the definition (2.12) of the kernel $\hat{\kappa} = \frac{\zeta}{\lambda}R^{\lambda\kappa}$, where $R^{\lambda\kappa}$ is the resolvent of $\lambda\kappa$. Since the kernel $\kappa \equiv 1$ is trivial, we have $R^{\lambda\kappa}(t) = \lambda e^{-\lambda t}$ by Abi Jaber et al. [1, Table 1], so that $\hat{\kappa}(t) = \zeta e^{-\lambda t}$. Thus by Lemma 2.10, the forward variance curve has the initial value

$$\begin{aligned}\xi_0(u) &= Y_0 + \frac{\lambda(\theta - Y_0)}{\zeta}\int_0^u \zeta e^{-\lambda s} ds \\ &= Y_0 + (\theta - Y_0)(1 - e^{-\lambda u}) = \theta + e^{-\lambda u}(Y_0 - \theta), \quad 0 \le u \le T.\end{aligned}$$

Based on this calculation, we make the ansatz that $\xi_t(u) = \theta + e^{-\lambda(u-t)}(Y_t - \theta)$ for $0 \le t \le u \le T$. For a fixed u, the ansatz matches the initial value $\xi_0(u)$ at $t = 0$, and by using Itô's formula and the dynamics (2.8) for Y in the classical Heston model, we obtain for $0 \le t \le u$ that

$$d\big(\theta + e^{-\lambda(u-t)}(Y_t - \theta)\big) = e^{-\lambda(u-t)}\big(dY_t - \lambda(Y_t - \theta)dt\big) = e^{-\lambda(u-t)}\zeta dB_t = \hat{\kappa}(u-t)dB_t.$$

Since this matches the dynamics of the forward variance curve $\xi_t(u)$ given in Lemma 2.10, we deduce that the ansatz holds, i.e., we have

$$\xi_t(u) = \theta + e^{-\lambda(u-t)}(Y_t - \theta), \quad 0 \le t \le u \le T. \tag{3.49}$$

To show (3.46), we can now plug in (3.49) to obtain that

$$\begin{aligned}\int_t^T g^\star(T-u)\xi_t(u)du &= \int_t^T g^\star(T-u)\big(\theta + e^{-\lambda(u-t)}(Y_t-\theta)\big)du \\ &= \varkappa_0(t) + \varkappa_1(t)Y_t, \quad 0 \le t \le T,\end{aligned}$$

where we define $\varkappa_0, \varkappa_1 : [0,T] \to \mathbb{R}$ by

$$\varkappa_0(t) = \int_t^T \theta g^\star(T-u)(1-e^{-\lambda(u-t)})du, \;\; \varkappa_1(t) = \int_t^T g^\star(T-u)e^{-\lambda(u-t)}du. \tag{3.50}$$

Thus we have (3.46), and it remains to show that $\varkappa_0$ and $\varkappa_1$ satisfy the respective differential equations (3.47) and (3.48). As the integrands are continuous, we get from (3.50) the terminal conditions $\varkappa_0(T) = \varkappa_1(T) = 0$. By the Leibniz integral rule, $\varkappa_0$ and $\varkappa_1$ are differentiable in t since the exponentials are, and their derivatives are given by

$$\begin{aligned}\varkappa_0'(t) &= \int_t^T -\lambda\theta g^\star(T-u)e^{-\lambda(u-t)}du - \theta g^\star(T-t)(1-e^{\lambda(t-t)}) \\ &= -\lambda\theta\varkappa_1(t), \quad 0 \le t \le T,\end{aligned} \tag{3.51}$$

which already shows (3.47), and

$$\begin{aligned}\varkappa_1'(t) &= \int_t^T \lambda g^\star(T-u)e^{-\lambda(u-t)}du - g^\star(T-t)e^{\lambda(t-t)} \\ &= \lambda\varkappa_1(t) - g^\star(T-t), \quad 0 \le t \le T.\end{aligned} \tag{3.52}$$

In particular, (3.51) and (3.52) show that $\varkappa_0$ and $\varkappa_1$ are continuously differentiable as $g^\star$ is continuous by Lemma 3.3. We also have $\zeta\varkappa_1(t) = (\hat{\kappa} * g^\star)(T-t)$ as $\hat{\kappa}(t) = \zeta e^{-\lambda t}$ and by the definition of $\varkappa_1$ in (3.50). Plugging the Riccati–Volterra equation (3.24) for $g^\star$ into (3.52) yields

$$\begin{aligned}\varkappa_1'(t) &= \lambda\varkappa_1(t) + \frac{\mu^2}{\sigma^2} + \frac{2\varrho\mu}{\sigma}(\hat{\kappa} * g^\star)(T-t) - \frac{1}{2}(1-2\varrho^2)\big((\hat{\kappa} * g^\star)(T-t)\big)^2 \\ &= \frac{\mu^2}{\sigma^2} + \left(\frac{2\varrho\mu\zeta}{\sigma} + \lambda\right)\varkappa_1(t) - \frac{\zeta^2(1-2\varrho^2)}{2}\varkappa_1(t)^2, \quad 0 \le t \le T,\end{aligned}$$

which shows (3.48) and concludes the proof. □

The second result we consider is given in Abi Jaber et al. [2], where the Markowitz problem is solved for a general multivariate affine Volterra model. That class of models includes the rough Heston model as a particular case. In [2, Lemma 4.2], they give a formula for a process (there denoted by Γ) which coincides with the opportunity process, as we show below, although they do not identify it as such nor relate it to the pure investment problem. Instead, their process Γ is obtained in [2, Lemma 4.2] as part of a linear–quadratic control for the Markowitz problem. In our setup with a zero interest rate, it is given by

$$\Gamma_t = \tilde{L}_t := \exp\left(\int_t^T \tilde{g}^\star(T-u)\tilde{\xi}_t(u)du\right), \quad 0 \le t \le T, \tag{3.53}$$

where the adjusted forward variance curve $(\tilde{\xi}_t(u))_{0\le t\le u\le T}$ is defined by

$$\tilde{\xi}_t(u) = E_P\left[Y_u + \lambda\int_t^u \kappa(u-s)Y_s ds \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t \le u \le T, \tag{3.54}$$

and in our notation, $\tilde{g}^\star$ satisfies the Riccati–Volterra equation, for $0 \le t \le T$,

$$\tilde{g}^\star(t) = -\frac{\mu^2}{\sigma^2} - \left(\frac{2\mu\varrho\zeta}{\sigma} + \lambda\right)(\kappa * \tilde{g}^\star)(t) + \frac{\zeta^2(1-2\varrho^2)}{2}\big((\kappa * \tilde{g}^\star)(t)\big)^2 \tag{3.55}$$

(this follows from [2, Equations (4.6) and (4.7)], where we replace $F_i(\psi)$ with $\tilde{g}^\star$). Our goal is now to show in Lemma 3.12 below that the formulas (3.53) and (3.39) for the opportunity process coincide. We note that (3.53) is written in terms of $(\tilde{\xi}_t(u))$, which differs from the forward variance curve due to the additional integral term. Moreover, (3.55) is given in terms of the original kernel κ for the rough Heston model (2.8), as opposed to $\hat{\kappa}$.

We start by finding a relationship between the two forward variance curves ξ and $\tilde{\xi}$. By the conditional Fubini theorem, we have

$$\tilde{\xi}_t(u) = \xi_t(u) + \lambda\int_t^u \kappa(u-s)\xi_t(s)ds, \quad 0 \le t \le u \le T. \tag{3.56}$$

Interestingly, in the case of the classical Heston model, we have by plugging in (3.49) and the trivial kernel $\kappa \equiv 1$ that

$$\begin{aligned}\tilde{\xi}_t(u) &= \theta + e^{-\lambda(u-t)}(Y_t - \theta) + \lambda \int_t^u \left(\theta + e^{-\lambda(s-t)}(Y_t - \theta)\right) ds \\ &= \lambda\theta(u - t) + Y_t,\end{aligned} \tag{3.57}$$

so that the adjusted forward variance $\tilde{\xi}_t(u)$ differs from the spot volatility Y_t only by a deterministic term $\lambda\theta(u-t)$. Assume for now that $\tilde{L} = L$, as we show below. For the classical Heston model, we have shown in Lemma 3.11 that L is also given by (3.45). Plugging (3.57) into (3.53), we get in this case that

$$\log \tilde{L}_t = Y_t \int_t^T \tilde{g}^\star(T-u)du + \lambda\theta \int_t^T \tilde{g}^\star(T-u)(u-t)du.$$

Comparing the term linear in Y with (3.45), we then deduce that

$$\varkappa_1(t) = \int_t^T \tilde{g}^\star(T-u)du = \int_0^{T-t} \tilde{g}^\star(s)ds = (\kappa * \tilde{g}^\star)(T-t),$$

since $\kappa \equiv 1$ for the classical Heston model, and we also get $\varkappa_1'(t) = -\tilde{g}^\star(T-t)$. These relationships between $\varkappa_1$ and $\tilde{g}^\star$ in the classical case explain the similarity (up to a change of sign) between the equations (3.48) and (3.58).

We now move on to the main result of showing that the two formulas for the opportunity process coincide. In the following, let $T \in (0,\infty)$ and (S, Y) satisfy the rough Heston model (2.8) on $[0,T]$, and recall the forward and adjusted forward variance curves on $[0,T]$ from Definition 2.9 and (3.54), respectively.

Lemma 3.12. *There exists a unique continuous solution $g^\star : [0,T] \to \mathbb{R}$ to* (3.24) *if and only if there exists a unique continuous solution $\tilde{g}^\star : [0,T] \to \mathbb{R}$ to* (3.55) *on $[0,T]$. In that case, we have*

$$\int_t^T g^\star(T-u)\xi_t(u)du = \int_t^T \tilde{g}^\star(T-u)\tilde{\xi}_t(u)du, \quad 0 \le t \le T, \tag{3.58}$$

and thus the formulas (3.39) *and* (3.53) *for the opportunity process coincide.*

Proof. Let $g^\star : [0,T] \to \mathbb{R}$ be a continuous solution to (3.24). We claim that

$$\tilde{g}^\star := g^\star - \frac{\lambda}{\zeta}\hat{\kappa} * g^\star \tag{3.59}$$

is a solution to (3.55) on $[0,T]$. Note that $\tilde{g}^\star$ is continuous like $g^\star$, as continuity is preserved by the convolution; see Gripenberg et al. [59, Theorem 2.2.2]. We use a resolvent of the second kind in order to invert the Volterra equation of the second kind (3.59), as explained after (2.4), so that $g^\star$ is given in terms of $\tilde{g}^\star$. We show this explicitly here. Recall that $\frac{\lambda}{\zeta}\hat{\kappa} = R^{\lambda\kappa}$ by the definition (2.12) of $\hat{\kappa}$, where the resolvent $R^{\lambda\kappa}$ satisfies the equality $\lambda\kappa * R^{\lambda\kappa} = R^{\lambda\kappa} * \lambda\kappa = \lambda\kappa - R^{\lambda\kappa}$ by Definition 2.4. Taking a convolution with $\lambda\kappa$, we obtain from (3.59) that

$$\begin{aligned}\tilde{g}^\star + \lambda\kappa * \tilde{g}^\star &= g^\star - R^{\lambda\kappa} * g^\star + \lambda\kappa * (g^\star - R^{\lambda\kappa} * g^\star)\\ &= g^\star - R^{\lambda\kappa} * g^\star + \lambda\kappa * g^\star - \lambda\kappa * g^\star + R^{\lambda\kappa} * g^\star = g^\star, \end{aligned} \tag{3.60}$$

using the commutativity and associativity of the convolution; see [59, Corollary 2.2.3]. It also follows that

$$\frac{\lambda}{\zeta}\hat{\kappa} * g^\star = g^\star - \tilde{g}^\star = \lambda\kappa * \tilde{g}^\star,$$

by rearranging (3.60) and (3.59). Thus by plugging into (3.24), we obtain for $0 \le t \le T$ that

$$\tilde{g}^\star(t) + \lambda(\kappa * \tilde{g}^\star)(t) = -\frac{\mu^2}{\sigma^2} - \frac{2\varrho\mu\zeta}{\sigma}(\kappa * \tilde{g}^\star)(t) + \frac{\zeta^2(1-2\varrho^2)}{2}\big((\kappa * \tilde{g}^\star)(t)\big)^2, \tag{3.61}$$

so that $\tilde{g}^\star$ is a solution to (3.55).

Conversely, let $\tilde{g}^\star : [0,T] \to \mathbb{R}$ be a continuous solution to (3.55) on $[0,T]$ and set $g^\star := \tilde{g}^\star + \lambda\kappa * \tilde{g}^\star$. Once again it follows that $g^\star$ is continuous on $[0,T]$ as $\tilde{g}^\star$ is, and similarly to (3.60), we can solve for $\tilde{g}^\star$ to obtain

$$\begin{aligned}g^\star - \frac{\lambda}{\zeta}\hat{\kappa} * g^\star &= \tilde{g}^\star + \lambda\kappa * \tilde{g}^\star - R^{\lambda\kappa} * (\tilde{g}^\star + \lambda\kappa * \tilde{g}^\star)\\ &= \tilde{g}^\star + \lambda\kappa * \tilde{g}^\star - R^{\lambda\kappa} * \tilde{g}^\star - \lambda\kappa * \tilde{g}^\star + R^{\lambda\kappa} * \tilde{g}^\star = \tilde{g}^\star. \end{aligned} \tag{3.62}$$

We also get $\frac{\lambda}{\zeta}\hat{\kappa} * g^\star = g^\star - \tilde{g}^\star = \lambda\kappa * \tilde{g}^\star$ once again. Plugging into (3.55), we obtain for $0 \le t \le T$ that

$$g^\star(t) - \frac{\lambda}{\zeta}(\hat{\kappa} * g^\star)(t) = -\frac{\mu^2}{\sigma^2} - \left(\frac{2\varrho\mu}{\sigma} + \frac{\lambda}{\zeta}\right)(\kappa * \tilde{g}^\star)(t) + \frac{1-2\varrho^2}{2}\big((\kappa * \tilde{g}^\star)(t)\big)^2,$$

so that $g^\star$ solves (3.24). Thus a solution to either one of the equations (3.24) or (3.55) yields a solution to the other.

To see that the uniqueness is preserved, consider two solutions $\tilde{g}_1^\star$ and $\tilde{g}_2^\star$ to

(3.55) on $[0,T]$, and suppose that the solution $g^\star$ to (3.24) on $[0,T]$ is unique. By uniqueness, we then must have

$$\tilde{g}_1^\star + \lambda\kappa * \tilde{g}_1^\star = g^\star = \tilde{g}_2^\star + \lambda\kappa * \tilde{g}_2^\star,$$

since each $\tilde{g}_i^\star$ is a solution to (3.24), as argued above in (3.61). These two equalities are Volterra equations of the second kind for $\tilde{g}_1^\star$ and $\tilde{g}_2^\star$ respectively, and hence by repeating the steps in (3.62) we obtain

$$\tilde{g}_1^\star = g^\star - R^{\lambda\kappa} * g^\star = \tilde{g}_2^\star.$$

Therefore, the uniqueness of the solution to (3.24) implies the uniqueness for (3.55). The proof of the converse statement is analogous.

Finally, we want to show the equality (3.58). This can be done by using the same techniques related to convolutions and resolvents, this time applied to the forward variance processes ξ and $\tilde{\xi}$. For each $t \in [0,T]$, we define the shifted curves $(\Delta_t\tilde{\xi}(s))_{0\le s\le T-t}$ and $(\Delta_t\xi(s))_{0\le s\le T-t}$ by $\Delta_t\tilde{\xi}(s) = \tilde{\xi}_t(t+s)$ and $\Delta_t\xi_t(s) = \xi_t(t+s)$. Thus (3.56) can be rewritten as

$$\Delta_t\tilde{\xi} = \Delta_t\xi + \lambda\kappa * \Delta_t\xi, \quad 0 \le t \le T. \tag{3.63}$$

Similarly to (3.62), we obtain by taking a convolution of (3.63) with $R^{\lambda\kappa}$ that

$$\begin{aligned}\Delta_t\tilde{\xi} - R^{\lambda\kappa} * (\Delta_t\tilde{\xi}) &= \Delta_t\xi + \lambda\kappa * (\Delta_t\xi) - R^{\lambda\kappa} * (\Delta_t\xi + \lambda\kappa * \Delta_t\xi) \\ &= \Delta_t\xi. \end{aligned} \tag{3.64}$$

We recall that $R^{\lambda\kappa} = \frac{\lambda}{\zeta}\hat{\kappa}$, so that (3.64) can be written explicitly as

$$\xi_t(u) = \tilde{\xi}_t(u) - \frac{\lambda}{\zeta}\int_t^u \hat{\kappa}(u-s)\tilde{\xi}_t(s)ds, \quad 0 \le t \le u \le T.$$

Plugging into the left-hand side of (3.58) yields for $0 \le t \le T$ that

$$\int_t^T g^\star(T-u)\xi_t(u)du = \int_t^T g^\star(T-u)\left(\tilde{\xi}_t(u) - \frac{\lambda}{\zeta}\int_t^u \hat{\kappa}(u-s)\tilde{\xi}_t(s)ds\right)du. \tag{3.65}$$

By Fubini's theorem and as $s \mapsto \xi_t(s)$ is a.s. continuous, hence bounded, we can rewrite the double integral as

$$\int_t^T g^\star(T-u)\Big(\int_t^u \hat\kappa(u-s)\tilde\xi_t(s)ds\Big)du = \int_t^T\Big(\int_s^T g^\star(T-u)\hat\kappa(u-s)du\Big)\tilde\xi_t(s)ds$$
$$= \int_t^T (\hat\kappa * g^\star)(T-s)\tilde\xi_t(s)ds.$$

Plugging into (3.65) then yields

$$\int_t^T g^\star(T-u)\xi_t(u)du = \int_t^T \left(g^\star(T-u) - \frac{\lambda}{\zeta}(\hat\kappa * g^\star)(T-u)\right)\tilde\xi_t(u)du$$
$$= \int_t^T \tilde g^\star(T-u)\tilde\xi_t(u)du,$$

where $\tilde g^\star := g^\star - \frac{\lambda}{\zeta}\hat\kappa * g^\star$, which shows (3.58). By the same argument given in the first part of the proof, we have that $\tilde g^\star$ is a continuous solution to (3.55). The conclusion follows immediately by plugging (3.58) into (3.39) and (3.53). □

Finally, we also mention Han/Wong [61], where the Markowitz problem is solved for the rough Heston model as well. Once again, the opportunity process shows up as the process M from [61, Equation (4.6)], up to the factor 2. The fact that $\frac{M}{2}$ coincides with the opportunity process L follows from [61, Equation (4.13)] with zero interest rate, since that is equivalent to the martingale distortion formula (3.13) for L in the case $1-2\varrho^2 \neq 0$. We note, however, that the formula [61, Equation (4.6)] for M is given in terms of yet another modified forward variance curve $(\xi_t^{\bar P}(u))_{0\le t\le u\le T}$ defined by $\xi_t^{\bar P}(u) = E_{\bar P}[Y_u \mid \mathcal{F}_t]$ under the measure $\bar P$ given in (3.12), as opposed to $(\xi_t(u))$, which we use here.

Chapter II

Mean–variance hedging in the rough Heston model

1 Introduction

The derivation of the Black–Scholes formula is one of the most celebrated results in mathematical finance. It shows how to perfectly hedge a European call option by trading in the underlying, in the case where the asset price process is given by geometric Brownian motion. However, such a perfect replication of a claim by a self-financing portfolio is only possible in a complete market. In order to hedge in an incomplete market, one must drop either the requirement of a self-financing portfolio or that of a perfect hedge. We use the mean–variance hedging (MVH) criterion, where we look to approximate a claim H in L^2 by a self-financing portfolio. This natural approach has been widely studied in the literature (see Schweizer [111] for a recent overview). Its main advantage is that it is tractable, i.e., it often provides explicit solutions that can be implemented in practice.

This chapter is the continuation of Chapter I. Our goal is to find explicit solutions to the MVH problem for the rough Heston model; the pure investment problem was considered in the previous chapter. As explained in the introduction from Chapter 0, we consider the MVH problem under the historical measure P, which is natural for risk management. The problem is more difficult under P than under a risk-neutral measure Q, since the solution under Q is given directly by the Galtchouk–Kunita–Watanabe decomposition for H. Practical applications of the MVH criterion have often been done under Q for this reason, but explicit solutions have also been obtained for some specific semimartingale models under P; examples include Biagini et al. [20], Černý/Kallsen [27], Hubalek et al. [69], Kallsen et al. [73] and Kallsen/Vierthauer [75].

Our results in this chapter add the rough Heston model to this list, as we obtain explicit solutions to the MVH problem for European vanilla call and put options and other European-type payoffs including spot volatility and geometric Asian options. We also calculate the hedging errors for European vanilla call and put options. We then extend the analysis to a semistatic setup where in addition to the underlying S, one may also trade at time 0 derivatives that must then be held to maturity. Setups of this type have been studied recently in Acciaio/Larsson [3], Acciaio et al. [4], Di Tella et al. [38, 39] and Nutz et al. [99] because they allow for the introduction of derivative or exotic assets that are too illiquid and/or costly to trade on a continuous basis. We show conceptually how to solve the Markowitz portfolio selection and mean–variance hedging problems in the semistatic setup, and also give explicit formulas when the traded derivatives are European call options.

To obtain explicit solutions to the MVH problem for a claim H, we need to calculate the mean value process V^H, i.e., the successive conditional expectations of H under the VOMM $Q^\star$ that was obtained in Chapter I. We do this via the Fourier transform method of Raible [104], which has worked well for solving mean–variance hedging problems under affine stochastic volatility models (Hubalek et al. [69], Kallsen/Pauwels [74]). We recall that in the rough Heston model, the volatility Y is an affine Volterra process (Abi Jaber et al. [1]) under P, and a time-inhomogeneous affine Volterra process (Ackermann et al. [5]) under $Q^\star$ as we show in Chapter I. Thus the Fourier transform approach also works well in this case, where the usual Riccati equations are replaced by Riccati–Volterra equations. Although the latter do not admit closed-form solutions, they are deterministic equations on $\mathbb{R}$ that can be solved numerically (for instance, by using the fractional Adams method of Diethelm et al. [40, 41]).

This chapter, based on joint work with Christoph Czichowsky, is structured as follows. In Section 2.1, we recall the rough Heston model together with the relevant results from Chapter I and the literature on MVH. In Section 2.2, we introduce the class of payoffs that we consider and calculate the characteristic function of the log-price under $Q^\star$. Our main results are given in Section 2.3, where we solve the MVH problem and calculate the mean squared hedging error for the European call and put options. In Section 3, we introduce the semistatic setup and show how to solve the Markowitz and MVH problems. Finally, in Section 4, we prove a result on the global existence of solutions to nonlinear Volterra equations. This result is used to ensure that the formulas for the optimal hedges in Section 2.3 hold on a nontrivial time interval. This is needed due to the issue of moment explosion; see the introduction to Chapter I.

2 Mean–variance hedging

2.1 Preliminaries

In this chapter, we study the mean–variance hedging problem (defined below) for the rough Heston model, as well as some applications to semistatic portfolio optimisation. This concludes the study of the rough Heston model initiated in Chapter I, and provides a natural application of the results obtained therein.

We start with the same setup as in Chapter I. We fix a time horizon $T > 0$ and a nonnegative weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P, W, B, S, Y)$ to the Volterra stochastic differential equations (I.2.8), which we recall here as

$$\begin{cases} \dfrac{dS_t}{S_t} = \mu Y_t dt + \sigma\sqrt{Y_t}dW_t, \\ Y_t = Y_0 + \displaystyle\int_0^t \kappa(t-s)\big(\lambda(\theta - Y_s)ds + \zeta\sqrt{Y_s}dB_s\big) \end{cases} \tag{2.1}$$

for $t \in [0,T]$, where $(W_t)_{0\le t\le T}$ and $(B_t)_{0\le t\le T}$ are two Brownian motions with constant instantaneous correlation $\varrho \in (-1,1)$, the parameters $S_0 > 0$, $Y_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, $\lambda > 0$, $\theta > 0$ and $\zeta > 0$ are fixed constants, and $\kappa : (0,\infty) \to \mathbb{R}_+$ is a fixed kernel satisfying Assumption I.2.7. For simplicity, we also assume that $\mathcal{F}_0$ is P-trivial. We also recall the *forward variance curve* $(\xi_t(u))_{0\le t\le u\le T}$ given by $\xi_t(u) = E[Y_u \mid \mathcal{F}_t]$. By Lemma I.2.10, ξ has the dynamics

$$d\xi_t(u) = \hat{\kappa}(u-t)\sqrt{Y_t}dB_t, \quad 0 \le t \le u, \tag{2.2}$$

where $\hat{\kappa} := \frac{\zeta}{\lambda}R^{\lambda\kappa}$ and $R^{\lambda\kappa} : (0,\infty) \to \mathbb{R}_+$ is the resolvent of the second kind of $\lambda\kappa$ in the sense of Definition I.2.4. The kernel $\hat{\kappa}$ is locally square-integrable and nonnegative due to Assumption I.2.7 and Gripenberg et al. [59, Theorem 2.3.5], as mentioned before Remark I.2.5.

We consider a financial market with time horizon T, a riskless asset with constant value 1 as well as a risky asset with price process S. We assume that an agent with initial wealth $x \in \mathbb{R}$ may trade frictionlessly in a self-financing way with a strategy $\vartheta \in \overline{\Theta}_T(S)$, which generates the wealth process

$$V_t(x,\vartheta) = x + \vartheta \bullet S_t = x + \int_0^t \vartheta_r dS_r, \quad 0 \le t \le T.$$

The set $\overline{\Theta}_T(S)$ of L^2-admissible strategies is defined in Černý/Kallsen [25, Definition 2.2] and also given in the introduction of Section I.3.

Consider a contingent claim or random payoff at time T, which can be described by an $\mathcal{F}_T$-measurable random variable H. A classical question in mathematical finance is whether one can perfectly hedge or replicate H, i.e., express H as the terminal wealth $V_T(x,\vartheta)$ attained by some self-financing strategy ϑ with initial wealth x. This is always possible in so-called complete markets, whereas in incomplete markets, it is generally impossible to perfectly replicate a claim H in this way. Given this limitation, several different approaches have been introduced in the literature in order to obtain a partial or approximate hedge for a claim H. One natural approach is to consider the mean–variance hedging (MVH) problem, which seeks to approximate H in an L^2-sense by the terminal wealth attained by some self-financing strategy. In other words, for a payoff $H \in L^2_P(\mathcal{F}_T)$, one seeks to minimise the mean squared hedging error

$$\begin{aligned}\varepsilon^2(x,H) &:= \inf_{\vartheta \in \overline{\Theta}_T(S)} E_P\big[\big(H - V_T(x,\vartheta)\big)^2\big] \\ &= \inf_{\vartheta \in \overline{\Theta}_T(S)} E_P\bigg[\bigg(H - x - \int_0^T \vartheta_u dS_u\bigg)^2\bigg], \end{aligned} \tag{2.3}$$

so that the terminal wealth $V_T(x,\vartheta)$ is (on average) close to H. This approach has a long history, and has been widely studied in the literature in increasing levels of generality (Duffie/Richardson [43], Schweizer [109, 110], Delbaen et al. [34], Rheinländer/Schweizer [106], Gourieroux et al. [58], Bertsimas et al. [19], Kohlmann/Tang [86], Černý [23], Černý/Kallsen [25], Mania/Tevzadze [91], Jeanblanc et al. [72], Czichowsky/Schweizer [32], Černý/Czichowsky [24], among others; see also Schweizer [111] for a recent overview). Its main advantage is that it is often tractable and leads to explicit formulas in a number of models, such as the exponential Lévy models studied in Hubalek et al. [69] and the classical Heston model in Černý/Kallsen [27].

Our main goal in this chapter is to find explicit formulas for the optimal hedging strategies associated with a large class of payoffs $H \in L^2_P(\mathcal{F}_T)$ in the rough Heston model. The MVH problem (2.3) is particularly simple to solve when

P is a risk-neutral martingale measure for S, since it then reduces to calculating the Galtchouk–Kunita–Watanabe decomposition for H in terms of S. However, it does not necessarily make sense to measure the hedging error under a risk-neutral measure. For that reason, we consider a semimartingale (or historical) measure P, which explains the presence of a drift term in (2.1). In that case, the study of the MVH problem becomes more involved, but we can still tackle this problem by using some of the results from the literature cited above, namely [25], which we use as a reference.

We now briefly recall the introduction of Section I.3. The first step towards tackling the mean–variance hedging problem (2.3) is to solve the pure investment problem (I.3.1), which reads

$$E_P[|1 - V_T(0,\vartheta)|^2] = E_P\left[\left|1 - \int_0^T \vartheta_t dS_t\right|^2\right] \longrightarrow \min_{\vartheta \in \overline{\Theta}_T(S)}! \tag{2.4}$$

and is also the mean–variance hedging problem (2.3) with initial wealth 0 and constant payoff 1. In connection with (2.4), we also studied in Section I.3 the so-called *opportunity process* $(L_t)_{0\leq t\leq T}$, *adjustment process* $(a_t)_{0\leq t\leq T}$, *variance-optimal martingale measure* $Q^\star$ and *optimal pure investment strategy* $(\vartheta_t^\star)_{0\leq t\leq T}$ (which is the solution to (2.4)). Namely, we showed that in the rough Heston model (2.1), there is some $T^\star > 0$ such that for $T \in (0, T^\star]$, all those quantities are well defined on $(0, T]$ with explicit formulas that we recall below in Theorem 2.3.

Thus if $T \in (0, T^\star]$, we have a variance-optimal martingale measure $Q^\star$ for S in the sense of equation (I.3.3); in particular, there exists an equivalent local martingale measure for S on $[0, T]$ with square-integrable density, namely $Q^\star$ itself. As discussed in the introduction to Section I.3, the existence of such a measure is a necessary no-arbitrage-type condition; without it, the MVH problem (2.3) for a given $H \in L^2_P(\mathcal{F}_T)$ may admit multiple or no solutions. Due to the existence of a VOMM on $[0, T]$ (at least for small $T > 0$), we can apply general results on the mean–variance hedging problem, including those from Černý/Kallsen [25] which we use as our main reference on this topic. It is known from [25, Corollary 2.9] that the existence of an ELMM for S on $[0, T]$ with square-integrable density ensures the closedness of the space of terminal gains

$$\mathcal{G}_T(S) := \{\vartheta \bullet S_T : \vartheta \in \overline{\Theta}_T(S)\}$$

in $L^2_P(\mathcal{F}_T)$. Since the MVH problem (2.3) can be seen as an L^2-projection onto $\mathcal{G}_T(S)$, the closedness already ensures the existence of a unique projection and hence a unique solution $\vartheta \in \overline{\Theta}_T(S)$ to (2.3) for every initial wealth/payoff pair $(x, H) \in \mathbb{R} \times L^2_P(\mathcal{F}_T)$. As mentioned above, the existence of an ELMM for S with square-integrable density can be seen as a no-arbitrage condition for S in an L^2-sense. One can then use an argument based on absence of arbitrage to show that the terminal wealth $V_T(x, \vartheta) = x + \vartheta \bullet S_T$ attained by a self-financing portfolio uniquely determines the wealth process $(V_t(x, \vartheta))_{0 \le t \le T}$; see [25, Lemma 2.11]. Recall from Section I.3 that we identify strategies in $\overline{\Theta}_T(S)$ if they are S-equivalent, i.e., $\vartheta^1 =_S \vartheta^2$ if $(\vartheta^1 \bullet S_t)_{0 \le t \le T}$ and $(\vartheta^2 \bullet S_t)_{0 \le t \le T}$ are indistinguishable. Hence it already follows from general results that there exists a unique solution $\vartheta^H(x)$ to (2.3) for every $x \in \mathbb{R}$ and $H \in L^2_P(\mathcal{F}_T)$.

It is also well known in the literature that the optimal mean–variance hedging strategy can be found as the solution to the feedback equation (2.9) given in [25, Lemma 4.9] and repeated below in Proposition 2.1. However, it can be challenging in practice to obtain explicit formulas for the processes that enter into that feedback equation in particular models, and thus to calculate the mean–variance hedging strategy. We seek to derive such explicit formulas for the rough Heston model and a large class of payoffs H.

Before moving on, we give here some general results from [25] that we later apply to the rough Heston model. Note that we can generalise the mean–variance hedging problem (2.3) by starting at the initial time $t \in [0, T]$ with (random) wealth $x_t \in L^2_P(\mathcal{F}_t)$, which leads to the problem

$$\varepsilon_t^2(x_t, H) = \operatorname{ess\,inf}_{\vartheta \in \overline{\Theta}_{t,T}(S)} E_P\left[\left(H - x_t - \int_t^T \vartheta_u dS_u\right)^2 \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t \le T, \tag{2.5}$$

where $\overline{\Theta}_{t,T}(S) := \{\vartheta \in \overline{\Theta}_T(S) : \vartheta \mathbf{1}_{[\![0,t]\!]} = 0\}$. We denote the corresponding optimal strategy by $(\vartheta^H_u(x_t, t))_{u \in [t,T]}$, or more simply $\vartheta^H(x)$ if $t = 0$ and $x \in \mathbb{R}$ as considered previously in (2.3).

Given a claim $H \in L^2_P(\mathcal{F}_T)$, we define the *mean value process* $(V^H_t)_{0 \le t \le T}$ by

$$V^H_t = E_{Q^\star}[H \mid \mathcal{F}_t], \quad 0 \le t \le T, \tag{2.6}$$

where $Q^\star$ is the variance-optimal martingale measure. We also define the *pure hedge coefficient* $(\Xi^H_t)_{0 \le t \le T}$ as the Radon–Nikodým derivative

$$\Xi^H_t = \frac{d[V^H, S]_t}{d[S]_t}, \quad 0 \le t \le T. \tag{2.7}$$

Since S and V^H are local martingales under $Q^\star$ with S continuous, the pure hedge coefficient Ξ^H can be seen as the integrand in the Galtchouk–Kunita–Watanabe decomposition

$$V^H = V_0^H + \Xi^H \bullet S + M^\perp \tag{2.8}$$

of V^H under $Q^\star$, where $M^\perp$ is some local $Q^\star$-martingale that is strongly orthogonal to S under $Q^\star$. Indeed, given such a decomposition, we must have

$$d[V^H, S]_t = \Xi_t^H d[S]_t + d[M^\perp, S]_t = \Xi_t^H d[S]_t, \quad 0 \leq t \leq T,$$

where $[M^\perp, S] = \langle M^\perp, S\rangle^{Q^\star} = 0$ by the orthogonality and the continuity of S. We note that the mean value process V^H corresponds to that of [25, Lemma 4.1], since the process $(N_t)_{0\leq t\leq T}$ defined in [25, Definition 3.12] is the stochastic logarithm of the density process $Z^{Q^\star}$ of $Q^\star$ so that $\mathcal{E}(N) = Z^{Q^\star}$. Likewise, the pure hedge coefficient corresponds to that of [25, Definition 4.6], since the ratio of modified characteristics (see [25, Equation (1.2)]) simplifies to (2.7) in the case where S is continuous.

The mean value process and pure hedge coefficient play a key role in the mean–variance hedging problem, since we can express the optimal mean–variance hedging strategy as the solution to a feedback equation involving V^H and Ξ^H, together with the adjustment process a. This is recalled in the following result, which is [25, Theorem 4.10].

Proposition 2.1. *Let $H \in L_P^2(\mathcal{F}_T)$, $t \in [0,T]$ and $x_t \in L_P^2(\mathcal{F}_t)$. Suppose that there exists an equivalent local martingale measure with square-integrable density for $(S_t)_{0\leq t\leq T}$. Then there exists a unique mean–variance hedging strategy $\vartheta^H(x_t,t) \in \overline{\Theta}_{t,T}(S)$ starting at time t with initial wealth x_t, and $\vartheta^H(x_t,t)$ satisfies the feedback equation*

$$\vartheta_u^H(x_t,t) = \Xi_u^H + a_u\left(V_u^H - x_t - \int_t^u \vartheta_r^H(x_t,t)dS_r\right), \quad t \leq u \leq T, \tag{2.9}$$

where a is the adjustment process and V^H, Ξ^H are the mean value process and pure hedge coefficient for H, respectively.

Intuitively, this equation can be interpreted as follows. First of all, we note that by [25, Theorem 4.10.2] (and as is clear from the following Proposition 2.2), the mean value process represents the optimal initial wealth in the mean–variance hedging problem. In other words, if one modifies (2.5) so that the initial wealth x_t is also part of the control, then the minimal value is achieved for $x_t = V_t^H$. Hence the second term in (2.9) can be seen as a mean-reversion strategy where the agent seeks to reach V_u^H from her current wealth

$$V_u\big(x_t, \vartheta^H(x_t,t)\big) = x_t + \int_t^u \vartheta_r^H(x_t,t)dS_r, \quad t \le u \le T.$$

In other words, if the current wealth $V_u(x_t, \vartheta^H(x_t,t))$ is not at the optimal level V_u^H, the agent seeks to compensate by trading with the adjustment process a (which is closely related to the pure investment strategy $\vartheta^\star$; see [25, Lemma 3.7]) in proportion to the difference $V_u^H - V_u(x_t, \vartheta^H(x_t,t))$. This helps the agent to "catch up" to V^H if $V_u^H - V_u(x_t, \vartheta^H(x_t,t))$ is positive, or "catch down" otherwise. On the other hand, the pure hedge term Ξ_u^H in (2.9) tries to match the forwards dynamics (2.8) of V^H by trading only in S. In the case where P is a martingale measure for S, we have $a = 0$ so that the pure hedge coefficient is the mean–variance hedging strategy for H starting from any $t \in [0,T]$ and $x_t \in L_P^2(\mathcal{F}_t)$.

Although (2.9) gives the optimal strategy only in feedback form, we note that one can also obtain $\vartheta^H(x_t,t)$ in closed form. Indeed, by integrating (2.9) against S, we obtain

$$\vartheta^H(x_t,t)\bullet S = \Big(\mathbf{1}_{]\!]t,T]\!]}\Big(\Xi^H + a\Big(V^H - x_t - \vartheta^H(x_t,t)\bullet S\Big)\Big)\Big)\bullet S$$

on $[t,T]$, which is a linear stochastic differential equation for $\vartheta^H(x_t,t)\bullet S$ with driver $-(\mathbf{1}_{]\!]t,T]\!]}a)\bullet S$. By standard arguments (see e.g. Protter [102, Theorem V.52]), we obtain the explicit solution

$$\vartheta^H(x_t,t)\bullet S = {}^t\Gamma\bigg(\frac{\mathbf{1}_{]\!]t,T]\!]}(\Xi^H + (V^H - x_t)a)}{{}^t\Gamma}\bullet(S + a\bullet[S])\bigg) \tag{2.10}$$

on $[t,T]$, where we define $({}^t\Gamma_u)_{t\le u\le T}$ by ${}^t\Gamma_u := \mathcal{E}(-(\mathbf{1}_{]\!]t,T]\!]}a)\bullet S)_u$. The formula (2.10) is also given in [25, Corollary 4.11]. After plugging into the right-hand side of (2.9), we obtain a closed formula for $\vartheta^H(x_t,t)$. Alternatively, by integrating (2.9) against S and adding x_t to both sides, we get the equation

$$V\big(x_t,\vartheta^H(x_t,t)\big) = x_t + \Big(\mathbf{1}_{]\!]t,T]\!]}\Big(\Xi^H + a\Big(V^H - V\big(x_t,\vartheta^H(x_t,t)\big)\Big)\Big)\Big)\bullet S,$$

which by [102, Theorem V.52] has the explicit solution

$$V\big(x,\vartheta^H(x_t,t)\big) = {}^t\Gamma\bigg(x_t + \frac{\mathbf{1}_{]\!]t,T]\!]}(\Xi^H + V^H a)}{{}^t\Gamma}\bullet(S + a\bullet[S])\bigg) \tag{2.11}$$

on $[t,T]$. Once again, this can be plugged into (2.9) to obtain another equivalent formula for $\vartheta^H(x_t,t)$. One can directly see the equivalence of (2.10) and (2.11) by adding x_t to (2.10) and checking that

$$1 - {}^t\Gamma\left(\frac{\mathbf{1}_{\rrbracket t,T\rrbracket}a}{{}^t\Gamma}\bullet(S + a\bullet[S])\right) = {}^t\Gamma$$

holds on $[t,T]$, which follows from the fact that $({}^t\Gamma)^{-1} = \mathcal{E}((\mathbf{1}_{\rrbracket t,T\rrbracket}a)\bullet(S+a\bullet[S]))$ by the definition of ${}^t\Gamma$.

In [25, Theorem 4.10] the formula (2.12) below is given for the optimal mean squared hedging error $\varepsilon_t^2(x,H)$ attained by $\vartheta^H(x,t)$. The error can be expressed in terms of V^H, Ξ^H as well as the opportunity process L.

Proposition 2.2. *Let $H \in L_P^2(\mathcal{F}_T)$. Suppose that there exists an equivalent local martingale measure with square-integrable density for $(S_t)_{0\le t\le T}$. Then the optimal mean squared hedging error for H starting at time t and initial wealth $x_t \in L_P^2(\mathcal{F}_t)$ is given by*

$$\varepsilon_t^2(x,H) = L_t(x_t - V_t^H)^2 + E_P\left[\int_t^T L_u d[V^H - \Xi^H\bullet S]_u \,\middle|\, \mathcal{F}_t\right]. \tag{2.12}$$

Finally, we restate here the main results from Chapter I that will be useful in our subsequent analysis. The following is a combination of Theorem I.3.8, Corollary I.3.9 and Remark I.3.10; we refer to Section I.3 for the definitions and discussion of L, a, $\vartheta^\star$ and $Q^\star$.

Theorem 2.3. *Let (S,Y) satisfy the rough Heston model (2.1), where κ satisfies Assumption I.2.7. Then there exists some $T^\star > 0$ such that for any $T \in (0,T^\star]$, the following statements hold:*

1) *The opportunity process $(L_t)_{0\le t\le T}$ for S with time horizon T is given by*

$$L_t = \exp\left(\int_t^T g^\star(T-u)\xi_t(u)du\right), \quad 0 \le t \le T, \tag{2.13}$$

where $g^\star : [0,T] \to \mathbb{R}$ is the unique continuous solution to

$$g^\star(t) = -\frac{\mu^2}{\sigma^2} - \frac{2\varrho\mu}{\sigma}(\hat\kappa * g^\star)(t) + \frac{1}{2}(1-2\varrho^2)\big((\hat\kappa * g^\star)(t)\big)^2, \quad 0 \le t \le T. \tag{2.14}$$

2) *The adjustment process* $a = (a_t)_{0\le t\le T}$ *is given by*

$$a_t = \frac{\mu + \varrho\sigma h^\star(T-t)}{\sigma^2 S_t}, \tag{2.15}$$

where $h^\star := \hat{\kappa} * g^\star$.

3) *The wealth process* $(V_t^\star)_{0\le t\le T}$ *corresponding to the optimal strategy* $(\vartheta_t^\star)_{0\le t\le T}$ *for the pure investment problem is given by*

$$\begin{aligned} V_t^\star &= (\vartheta^\star \bullet S)_t = 1 - \mathcal{E}(-a \bullet S)_t \\ &= 1 - \mathcal{E}\Big(\int \Big(-\frac{\mu}{\sigma} + \varrho h^\star(T-s)\Big)\Big(\frac{\mu Y_s}{\sigma}ds + \sqrt{Y_s}dW_s\Big)\Big)_t. \end{aligned}$$

4) *The optimal strategy for the pure investment problem is given by*

$$\vartheta_t^\star = (1 - V_t^\star)a_t = a_t\, \mathcal{E}(-a \bullet S)_t.$$

5) *The density process* $(Z_t^{Q^\star})_{0\le t\le T}$ *for the variance-optimal martingale measure* $Q^\star$ *is given by*

$$Z_t^{Q^\star} = \mathcal{E}\Big(\int \sqrt{Y_s}\Big(\sqrt{1-\varrho^2}\, h^\star(T-s)dW_s^\perp - \frac{\mu}{\sigma}dW_s\Big)\Big)_t. \tag{2.16}$$

6) *The dynamics of* (S, Y) *under* $Q^\star$ *are given by*

$$\begin{aligned} dS_t &= \sigma S_t \sqrt{Y_t}dW_t^\star, \quad 0 \le t \le T, \\ Y_t &= Y_0 + \int_0^t \kappa(t-s)\Big(\Big(\theta\lambda - \Big(\lambda + \frac{\zeta\varrho\mu}{\sigma} - \zeta(1-\varrho^2)h^\star(T-s)\Big)Y_s\Big)ds \\ &\qquad + \zeta\sqrt{Y_s}dB_s^\star\Big), \quad 0 \le t \le T, \end{aligned}$$

where $(W_t^\star)_{0\le t\le T}$ *and* $(B_t^\star)_{0\le t\le T}$ *are* $Q^\star$*-Brownian motions defined by*

$$\begin{aligned} W_t^\star &= W_t + \int_0^t \frac{\mu}{\sigma}\sqrt{Y_s}ds, \quad 0 \le t \le T, \\ B_t^\star &= B_t + \int_0^t \frac{\varrho\mu}{\sigma}\sqrt{Y_s}ds - \int_0^t (1-\varrho^2)(\hat{\kappa} * g^\star)(T-s)\sqrt{Y_s}ds, \quad 0 \le t \le T. \end{aligned}$$

The following two results are Theorem I.2.17 and Proposition I.2.20, which provide formulas for conditional expectations related to the rough Heston model.

Theorem 2.4. *Fix $\bar{T} > 0$ and let $\tilde{X} = (\tilde{X}_t^{(1)}, \dots, \tilde{X}_t^{(n)})_{0 \le t \le \bar{T}}$ be a $\mathbb{C}^n$-valued semimartingale that satisfies the decomposition*

$$\tilde{X}_t^{(j)} = \tilde{X}_0^{(j)} + \int_0^t \big(\mu^{(j)}(s) Y_s ds + \sigma^{(j)}(s)\sqrt{Y_s} dW_s + \tilde{\sigma}^{(j)}(s)\sqrt{Y_s} dW_s^{\perp}\big) \tag{2.17}$$

for all $t \in [0, \bar{T}]$ and $j = 1, \dots, n$, some constants $\tilde{X}_0^{(1)}, \dots, \tilde{X}_0^{(n)} \in \mathbb{C}$ and deterministic functions $\mu \in L^1([0,\bar{T}];\mathbb{C}^n)$ and $\sigma, \tilde{\sigma} \in L^2([0,\bar{T}];\mathbb{C}^n)$. Fix $C > 0$ and let $\bar{B}_C(0) \subseteq \mathbb{C}^n$ be the closed ball of radius C. Then the following statements hold:

1) *There exists some positive time $\tilde{T} = \tilde{T}(\mu, \sigma, \tilde{\sigma}, C) \in (0, \bar{T}]$ such that for every $T \in (0, \tilde{T}]$ and $z \in \bar{B}_C(0)$, there is a unique solution $g_{z,T} \in L^1([0,T];\mathbb{C})$ to the equation*

$$g_{z,T}(t) = z^\top \mu(T-t) + f\big((\hat{\kappa} * g_{z,T})(t); z^\top \sigma(T-t), z^\top \tilde{\sigma}(T-t)\big) \tag{2.18}$$

for $0 \le t \le T$, where the function $f : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is defined by

$$f(x; h_1, h_2) = \frac{h_1^2 + h_2^2 + x^2}{2} + (\varrho h_1 + \sqrt{1-\varrho^2} h_2)x. \tag{2.19}$$

Moreover, it holds that $E[\,|\exp(z^\top \tilde{X}_T)|\,] < \infty$, and we have for $0 \le t \le T$ that

$$E[\exp(z^\top \tilde{X}_T) \mid \mathcal{F}_t] = \exp\left(z^\top \tilde{X}_t + \int_t^T g_{z,T}(T-u)\xi_t(u) du\right). \tag{2.20}$$

2) *Conversely, fix $z \in \mathbb{C}$ and $T \in (0, \bar{T}]$. If $E[\,|\exp(z^\top \tilde{X}_T)|\,] < \infty$ and if there exists a solution $g_{z,T} \in L^1([0,T];\mathbb{C})$ to (2.18), then (2.20) holds for $0 \le t \le T$.*

Proposition 2.5. *Let $\bar{T} > 0$, Φ be an indexing set, $(\mu_\varphi)_{\varphi \in \Phi}$ a family of functions in $L^1([0,\bar{T}];\mathbb{C})$ and $(\sigma_\varphi)_{\varphi \in \Phi}, (\tilde{\sigma}_\varphi)_{\varphi \in \Phi}$ two families of functions in $L^2([0,\bar{T}];\mathbb{C})$ such that (μ_φ), $(|\sigma_\varphi|^2)$ and $(|\tilde{\sigma}_\varphi|^2)$ are uniformly integrable. For each $\varphi \in \Phi$, let $\tilde{x}^\varphi \in \mathbb{C}$ be a constant and define $(\tilde{X}_t^\varphi)_{0 \le t \le \bar{T}}$ by*

$$\tilde{X}_t^\varphi = \tilde{x}^\varphi + \int_0^t \big(\mu_\varphi(s) Y_s ds + \sigma_\varphi(s)\sqrt{Y_s} dW_s + \tilde{\sigma}_\varphi(s)\sqrt{Y_s} dW_s^{\perp}\big), \quad 0 \le t \le \bar{T}. \tag{2.21}$$

Then there exists some $\tilde{T} \in (0, \bar{T}]$ such that for all $\varphi_1, \varphi_2 \in \Phi$ and $T \in (0, \tilde{T}]$, the following statements hold:

1) *There is a unique solution $g_{\varphi_1,T} \in L^1([0,T];\mathbb{C})$ to*

$$g_{\varphi_1,T}(t) = \mu_{\varphi_1}(T-t) + f\big((\hat{\kappa} * g_{\varphi_1,T})(t); \sigma_{\varphi_1}(T-t), \tilde{\sigma}_{\varphi_1}(T-t)\big) \tag{2.22}$$

for a.a. $t \in [0,T]$, where the function $f : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is defined by

$$f(x; h, \tilde{h}) = \frac{h^2 + \tilde{h}^2 + x^2}{2} + (\varrho h + \sqrt{1-\varrho^2}\tilde{h}_2)x. \tag{2.23}$$

2) *There is a unique solution $\tilde{g}_{\varphi_1,\varphi_2,T} \in L^1([0,T];\mathbb{C})$ to the equation*

$$\begin{aligned}\tilde{g}_{\varphi_1,\varphi_2,T}(t) = \mu_{\varphi_2}(T-t) + \tilde{f}\big((\hat{\kappa} * \tilde{g}_{\varphi_1,\varphi_2,T})(t); (\hat{\kappa} * g_{\varphi_1,T})(t), \sigma_{\varphi_1}(T-t),\\ \tilde{\sigma}_{\varphi_1}(T-t), \sigma_{\varphi_2}(T-t), \tilde{\sigma}_{\varphi_2}(T-t)\big)\end{aligned} \tag{2.24}$$

for a.a. $t \in [0,T]$, where $\tilde{f} : \mathbb{C}^6 \to \mathbb{C}$ is defined by

$$\tilde{f}(x; y, h_1, \tilde{h}_1, h_2, \tilde{h}_2) = (h_1 + \varrho y)(h_2 + \varrho x) + (\tilde{h}_1 + \sqrt{1-\varrho^2}y)(\tilde{h}_2 + \sqrt{1-\varrho^2}x). \tag{2.25}$$

3) *For $0 \le t \le T$, we have*

$$\begin{aligned}E[\exp(\tilde{X}_T^{\varphi_1})\tilde{X}_T^{\varphi_2} \mid \mathcal{F}_t] = \exp\left(\tilde{X}_t^{\varphi_1} + \int_t^T g_{\varphi_1,T}(T-u)\xi_t(u)du\right)\\ \times \left(\tilde{X}_t^{\varphi_2} + \int_t^T \tilde{g}_{\varphi_1,\varphi_2,T}(T-u)\xi_t(u)du\right).\end{aligned} \tag{2.26}$$

Finally, we recall Lemma I.2.13 and Corollary I.2.14, which give the dynamics of linear functionals of the forward variance curve.

Lemma 2.6. **1)** *Let ν be a finite complex measure on $([0,T], \mathcal{B}([0,T]))$. Then there exists a continuous local martingale $(\xi_t(\nu))_{0\le t\le T}$ such that*

$$\xi_t(\nu) = \int_{[0,t]} Y_u \nu(du) + \int_{(t,T]} \xi_t(u)\nu(du) \quad \textit{for each } 0 \le t \le T, \tag{2.27}$$

and it admits the decomposition

$$\xi_t(\nu) = \xi_0(\nu) + \int_0^t \left(\int_{[s,T]} \hat{\kappa}(u-s)\nu(du)\right)\sqrt{Y_s}dB_s, \quad 0 \le t \le T. \tag{2.28}$$

2) *For any function $g \in L^1([0,T];\mathbb{C})$, there exists a continuous semimartingale $(Y_t^g)_{0\le t\le T}$ such that*

$$Y_t^g = \int_t^T g(T-u)\xi_t(u)du \quad \textit{for each } 0 \le t \le T, \tag{2.29}$$

so that in particular $Y^g_T = 0$, and Y^g has the decomposition

$$Y^g_t = Y^g_0 + A_t + M_t, \quad 0 \le t \le T, \tag{2.30}$$

where the continuous finite-variation process $(A_t)_{0\le t\le T}$ and the continuous local martingale $(M_t)_{0\le t\le T}$ are respectively given by

$$A_t = -\int_0^t g(T-s)Y_s ds, \quad M_t = \int_0^t (\hat{\kappa} * g)(T-s)\sqrt{Y_s}dB_s, \quad 0 \le t \le T. \tag{2.31}$$

2.2 Setup and moment-generating function under $Q^\star$

Fix a time horizon $T \in (0, T^\star)$ and consider a payoff $H \in L^2_P(\mathcal{F}_T)$. In view of the discussion above, our main task is to find the mean value process V^H, from which we also obtain the pure hedge coefficient Ξ^H and then the optimal MVH strategy for H via Proposition 2.1. Since the mean value process is given by the conditional expectation $V^H_t = E_{Q^\star}[H \mid \mathcal{F}_t]$, we want to find an explicit formula for V^H by applying Theorem 2.4, which gives a generalised moment-generating function for the rough Heston model (more precisely, for the log-price $X = \log S$ and forward variance curve ξ). As we have seen in some of the proofs in Section I.3, Theorem 2.4 can also deal with some changes of measure, and that includes $Q^\star$. As we show below in Corollary 2.16, it is relatively straightforward to find the mean value process for payoffs H of the power type

$$H = S^z_T = \exp(zX_T)$$

for some $z \ge 0$ such that $E[H^2] < \infty$, and indeed, that calculation can be (at least formally) extended to complex-valued payoffs where $z \in \mathbb{C}$. By taking linear combinations of such payoffs, this leads us to consider a class of European payoffs $H = f(S_T)$, where f can be represented as a Mellin transform of a measure on $\mathbb{C}$. An equivalent formulation is $H = g(X_T)$, where g can be represented as a Laplace transform.

Assumption 2.7. We assume that $H = f^H(S_T) \in L^2_P(\mathcal{F}_T)$ for some measurable function $f^H : \mathbb{R} \to \mathbb{R}$ that can be expressed as the Mellin transform

$$f^H(s) = \int_{\mathbb{C}} s^z \pi^H(dz) \quad \text{for } s \in \mathbb{R} \tag{2.32}$$

of a complex-valued bounded variation measure[1] π^H on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Moreover, we suppose that

$$\int_{\mathbb{C}} E_P[S_T^{2\mathrm{Re}(z)}]|\pi^H|(dz) < \infty. \tag{2.33}$$

Sometimes we require additional integrability: We say that H satisfies Assumption 2.7$^+$ if there exists some $\delta > 0$ such that

$$\int_{\mathbb{C}} E_P[S_T^{(2+\delta)\mathrm{Re}(z)}]|\pi^H|(dz) < \infty. \tag{2.34}$$

Remark 2.8. We note that (2.32) and (2.33) already imply that H is square-integrable; indeed, since π^H has bounded variation, we get

$$\begin{aligned} E_P[H^2] &\le E\left[\left(\int_{\mathbb{C}} S_T^{\mathrm{Re}(z)}|\pi^H|(dz)\right)^2\right] \\ &\le E\left[\left(\int_{\mathbb{C}} S_T^{2\mathrm{Re}(z)}|\pi^H|(dz)\right)|\pi^H|(\mathbb{C})\right] \\ &\le |\pi^H|(\mathbb{C}) \int_{\mathbb{C}} E_P[S_T^{2\mathrm{Re}(z)}]|\pi^H|(dz) < \infty \end{aligned}$$

by the Cauchy–Schwarz inequality and Fubini's theorem.

Assumption 2.7 gives the basic setup considered in Hubalek et al. [69], which we use as a reference regarding the Mellin transform approach and where the mean–variance hedging problem is studied for an exponential Lévy price process. As is standard with Fourier pricing and hedging techniques, it is useful to consider a complex-valued integral even though H and f^H are real-valued. Under Assumption 2.7, we can write

$$H = \int_{\mathbb{C}} S_T^z \pi^H(dz),$$

and we want to find the conditional expectation of H under $Q^\star$ by taking the conditional expectation inside the integral. As we shall see later in Proposition 2.15 and Corollary 2.16, we are able to use the structure of the rough Heston model to obtain explicit formulas for the conditional expectation of terms of the form $S_T^z = \exp(zX_T)$. This is due to the fact that although (X, Y) is no longer an affine Volterra process in the sense of Abi Jaber et al. [1] under $Q^\star$, it retains the structure of a so-called time-inhomogeneous affine Volterra process in the sense of Ackermann et al. [5] by the dynamics given in part 6) of Theorem 2.3. This

[1] We say that $\pi^H : \mathcal{B}(\mathbb{C}) \to \mathbb{C}$ is a complex-valued measure if $\pi^H = \pi_1 + \mathrm{i}\pi_2$ for some (real) signed measures π_1 and π_2, and it has bounded variation if $|\pi_1|(\mathbb{C}) + |\pi_2|(\mathbb{C}) < \infty$.

allows us to obtain formulas that are given in terms of the solutions to certain time-inhomogeneous Riccati–Volterra equations such as (2.60) below.

Our main examples of interest are the vanilla European call and put options on the underlying. These can be represented in the form (2.32) since by [69, Lemma 4.1], we have the formulas (2.35) and (2.36) below.

Example 2.9 (European vanilla call and put options)**.** Consider the payoffs $C_{K,T} := (S_T - K)^+$ and $P_{K,T} := (K - S_T)^+$ with strike $K > 0$ and maturity T. We have the Mellin transform representations

$$(s-K)^+ = \frac{1}{2\pi \mathrm{i}} \int_{a_1 - \mathrm{i}\infty}^{a_1 + \mathrm{i}\infty} s^z \frac{K^{1-z}}{z(z-1)} dz = \int_{\mathbb{C}} s^z \pi_{K,a_1}(dz), \tag{2.35}$$

$$(K-s)^+ = \frac{1}{2\pi \mathrm{i}} \int_{a_2 - \mathrm{i}\infty}^{a_2 + \mathrm{i}\infty} s^z \frac{K^{1-z}}{z(z-1)} dz = \int_{\mathbb{C}} s^z \pi_{K,a_2}(dz) \tag{2.36}$$

for all $s > 0$ and any choice of $a_1 \in (1, \infty)$ and $a_2 \in (-\infty, 0)$, where for each $a \in \mathbb{R} \setminus \{0, 1\}$, we define the complex-valued measure $\pi_{K,a}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ by

$$\pi_{K,a}(dz) = \frac{K^{1-z}}{2\pi z(z-1)} \lambda_a(dz), \tag{2.37}$$

and λ_a is the Lebesgue measure on the vertical line $(a - \mathrm{i}\infty, a + \mathrm{i}\infty)$. We note that the dz-integrals in the middle expressions in (2.35) and (2.36) should be interpreted as complex line integrals, whereas $\pi_{K,a}(dz)$ refers to Lebesgue integration. Curiously, the representations of the put and call payoffs have identical formulas but different domains of integration. By the above, these payoffs have the form (2.32), and may satisfy Assumptions 2.7 or 2.7$^+$ depending on the time horizon T and the choice of a_1, a_2 in the representations (2.35) and (2.36); sufficient conditions for Assumption 2.7$^+$ are given below in Corollary 2.19.

The setup from Assumption 2.7 is quite flexible and allows us to consider many types of payoffs that depend only on the terminal value S_T; several examples such as binary options are given in [69, Section 4]. As it turns out, we can use a similar approach to tackle an even wider class of European options, which we now introduce, that may also depend on Y_T and even the paths of S and Y. Once again, the key is to represent H in some sense as a linear combination of simpler payoffs to which we can apply Theorem 2.4. As we shall see, this allows us to derive an explicit formula for the mean value process V^H.

Let $\mathcal{M}([0,T])$ be the set of complex-valued bounded variation measures on $([0,T], \mathcal{B}([0,T]))$, equipped with the topology of weak convergence. For measures

$\nu_1, \nu_2 \in \mathcal{M}([0,T])$, define the processes $(X_t(\nu_1))_{t\in[0,T]}$ and $(\xi_t(\nu_2))_{t\in[0,T]}$ by

$$X_t(\nu_1) = \int_{[0,T]} X_{s\wedge t}\nu_1(ds) = \int_{[0,t)} X_s\nu_1(ds) + X_t\nu_1([t,T]), \tag{2.38}$$

$$\xi_t(\nu_2) = \int_{[0,T]} \xi_t(s)\nu_2(ds) = \int_{[0,t]} Y_s\nu_1(ds) + \int_{(t,T]} \xi_t(u)\nu_2(du). \tag{2.39}$$

We consider simple payoffs of the form $\exp(X_T(\nu_1) + \xi_T(\nu_2))$ for some measures $\nu_1, \nu_2 \in \mathcal{M}([0,T])$. As an example and justification for this choice of notation, note that we have $X_t(\delta_T) = X_t$ and $\xi_t(\delta_T) = \xi_t(T)$ for $0 \le t \le T$, where δ_T is the Dirac mass at $\{T\}$. Hence we can write

$$S_T^z = \exp(zX_T) = \exp\big(zX_T(\delta_T)\big) = \exp\big(X_T(z\delta_T)\big), \tag{2.40}$$

so that the power payoffs S_T^z belong to this class of simple payoffs. We now consider claims H that can be represented in terms of simple payoffs. Concretely, we suppose that H can be given as an integral involving a family of simple payoffs $\exp(\tilde{X}_T(z))$ parametrised by $z := (z_1, \ldots, z_k) \in \mathbb{C}^k$, where we set

$$\tilde{X}_t(z) := X_t(\nu_1^z) + \xi_t(\nu_2^z) \tag{2.41}$$

and $z \mapsto (\nu_1^z, \nu_2^z)$ is a continuous map from $\mathbb{C}^k$ to $\mathcal{M}([0,T])^2$.

Assumption 2.10. We assume that $H = f^H(S_T) \in L_P^2(\mathcal{F}_T)$ is real-valued and has the form

$$H = \int_{\mathbb{C}^k} \exp\big(\tilde{X}_T(z)\big)\pi^H(dz) \tag{2.42}$$

for some complex-valued bounded variation measure π^H on $(\mathbb{C}^k, \mathcal{B}(\mathbb{C}^k))$ and $\tilde{X}$ given by (2.41) for some continuous map $z \mapsto (\nu_1^z, \nu_2^z)$ from $\mathbb{C}^k$ to $\mathcal{M}([0,T])^2$. Moreover, we suppose that

$$\int_{\mathbb{C}^k} E_P\Big[\exp\Big(2\mathrm{Re}\big(\tilde{X}_T(z)\big)\Big)\Big]|\pi^H|(dz) < \infty. \tag{2.43}$$

We say that H satisfies Assumption 2.10$^+$ if there exists some $\delta > 0$ such that

$$\int_{\mathbb{C}^k} E_P\Big[\exp\Big((2+\delta)\mathrm{Re}\big(\tilde{X}_T(z)\big)\Big)\Big]|\pi^H|(dz) < \infty. \tag{2.44}$$

Remark 2.11. In our examples, we often consider $k = 1$ so that $z \in \mathbb{C}$; however, a higher-dimensional parameter is needed in order to represent more exotic options. Indeed, it may be possible to generalise this setup even further by eliminating the parameter z entirely, and instead considering a measure π^H directly on the space $\mathcal{M}([0,T])^2$ together with an appropriate notion of infinite-dimensional integration.

By setting $(\nu_1^z, \nu_2^z) := (z\delta_T, 0)$ for $z \in \mathbb{C}$ (which is continuous in z with respect to the topology of weak convergence), it is clear by (2.40) that Assumption 2.10 is weaker than Assumption 2.7, i.e., it allows for a larger class of payoffs. The integrability condition (2.43) is also the natural generalisation of (2.33). We now give some examples of exotic payoffs that can be represented in the form (2.42).

Example 2.12 (European call and put options on spot volatility)**.** Consider the payoffs $C^Y_{K,T} := (Y_T - K)^+$ and $P^Y_{K,T} := (K - Y_T)^+$ with maturity T and strike $K > 0$. Setting $(\nu_1^z, \nu_2^z) := (0, z\delta_T)$ for $z \in \mathbb{C}$ so that $\tilde{X}_T(z) = zY_T$ by (2.39), we have for any $a_1 > 0$ and $a_2 < 0$ that

$$C^Y_{K,T} = (Y_T - K)^+ = \int_{\mathbb{C}} \exp\left(\tilde{X}_T(z)\right)\pi^Y_{K,a_1}(dz), \tag{2.45}$$

$$P^Y_{K,T} = (K - Y_T)^+ = \int_{\mathbb{C}} \exp\left(\tilde{X}_T(z)\right)\pi^Y_{K,a_2}(dz), \tag{2.46}$$

where $\pi^Y_{K,a}$ is the complex-valued measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ given by

$$\pi^Y_{K,a}(dz) := \frac{e^{-Kz}}{2\pi z^2}\lambda_a(dz), \tag{2.47}$$

and λ_a is the Lebesgue measure on the vertical line $(a - \mathrm{i}\infty, a + \mathrm{i}\infty)$.

Note that these formulas are not analogous to those of Example 2.9 because we require a representation of the function $y \mapsto (y - K)^+$ as a Laplace transform with respect to y, rather than Mellin (which is a Laplace transform in terms of $\log y$). Depending on the particular choice of T, a_1 and a_2, the payoffs $C^Y_{K,T}$ and $P^Y_{K,T}$ may or may not satisfy Assumptions 2.10 or 2.10$^+$. Sufficient conditions for Assumption 2.10$^+$ to be satisfied are given below in Corollary 2.20.

Proof of (2.45) *and* (2.46). Let $g(y) = (y - K)^+$ for $y \in \mathbb{R}$. Integrating by parts, we obtain the two-sided Laplace transform

$$\tilde{g}(z) := \int_{-\infty}^{\infty} g(y)e^{-zy}dy = \int_K^{\infty} (y - K)e^{-zy}dy = \frac{e^{-Kz}}{z^2}$$

for $\mathrm{Re}(z) > 0$. Hence the Bromwich inversion integral (see Hubalek et al. [69, Theorem A.1]) yields the representation

$$g(y) = (y-K)^+ = \frac{1}{2\pi \mathrm{i}} \int_{a_1-\mathrm{i}\infty}^{a_1+\mathrm{i}\infty} e^{zy} \frac{e^{-Kz}}{z^2} dz, \quad y \in \mathbb{R},$$

for any choice of $a_1 \in (0,\infty)$. In an analogous way, we obtain

$$(K-y)^+ = \frac{1}{2\pi \mathrm{i}} \int_{a_2-\mathrm{i}\infty}^{a_2+\mathrm{i}\infty} e^{zy} \frac{e^{-Kz}}{z^2} dz, \quad y \in \mathbb{R},$$

for any $a_2 \in (-\infty, 0)$. Plugging in $y = Y_T$ yields (2.45) and (2.46), using the fact that $\tilde{X}_T(z) = zY_T$ and $dz = \mathrm{i}\lambda_a(dz)$ when integrating over the vertical line $(a - \mathrm{i}\infty, a + \mathrm{i}\infty)$. □

Example 2.13 (Geometric Asian call and put options)**.** Consider the payoffs $C^A_{K,T} := (S^A_T - K)^+$ and $P^A_{K,T} := (K - S^A_T)^+$ with maturity T and strike $K > 0$, where

$$S^A_T := \exp\left(\frac{1}{T}\int_0^T X_t dt\right) = \exp\left(\frac{1}{T}\int_0^T \log S_t dt\right) \tag{2.48}$$

is the geometric mean of the price process $(S_t)_{t\in[0,T]}$. Setting $\nu_1^z(dt) = \frac{z}{T}dt$ and $\nu_2^z = 0$ for $z \in \mathbb{C}$ so that $\tilde{X}_T(z) = \frac{z}{T}\int_0^T X_t dt$ by (2.38), we have

$$C^A_{K,T} = (S^A_T - K)^+ = \int_{\mathbb{C}} \exp\left(\tilde{X}_T(z)\right) d\pi_{K,a_1}(dz), \tag{2.49}$$

$$P^A_{K,T} = (K - S^A_T)^+ = \int_{\mathbb{C}} \exp\left(\tilde{X}_T(z)\right) d\pi_{K,a_2}(dz) \tag{2.50}$$

for any $a_1 > 1$ and $a_2 < 0$, where $\pi_{K,a}$ defined by (2.37) is the same as for the vanilla European call and put options. The payoffs $C^A_{K,T}$ and $P^A_{K,T}$ may or may not satisfy Assumptions 2.10 or 2.10$^+$, depending on the particular choice of T, a_1 and a_2. Sufficient conditions for Assumption 2.10$^+$ to be satisfied are given below in Corollary 2.21.

Proof of (2.49) *and* (2.50). Plugging in $s = S^A_T$ into the Mellin transform representations (2.35) and (2.36) and using the fact that $\exp(\tilde{X}_T(z)) = (S^A_T)^z$ directly yields (2.49) and (2.50). □

As a more general example, we can also consider claims that depend on the values of S and Y at discrete time points and can be represented in the form of a Laplace transform.

Example 2.14. Consider the claim $H = f(X_{t_1}, Y_{t_1}, \dots, X_{t_k}, Y_{t_k}) \in L^2_P(\mathcal{F}_T)$, for some sequence of discrete times $0 \le t_1 < t_2 < \cdots < t_k \le T$ and where the function $f : \mathbb{R}^{2k} \to \mathbb{R}$ can be represented as the Laplace transform

$$f(x_1, y_1, \dots, x_k, y_k) = \int_{\mathbb{C}^k} e^{\sum_{j=1}^k z_j x_j + \sum_{j=1}^k \tilde{z}_j y_j} \pi^H(dz)$$

of some complex-valued bounded variation measure π^H on $(\mathbb{C}^{2k}, \mathcal{B}(\mathbb{C}^{2k}))$. We denote $z = (z_1, \tilde{z}_1, \dots, z_k, \tilde{z}_k)$ and set

$$\nu_1^z = \sum_{j=1}^k z_j \delta_{t_j} \quad \text{and} \quad \nu_2^z = \sum_{j=1}^k \tilde{z}_j \delta_{t_j}, \tag{2.51}$$

so that $\tilde{X}_T(z) = \sum_{j=1}^k z_j X_{t_j} + \sum_{j=1}^k \tilde{z}_j Y_{t_j}$. Then we have

$$H = f(X_{t_1}, Y_{t_1}, \dots, X_{t_k}, Y_{t_k}) = \int_{\mathbb{C}^k} \exp\big(\tilde{X}_T(z)\big) \pi^H(dz).$$

Depending on $t_1, \dots, t_k$ and the choice of measure π^H, the payoff H may or may not satisfy Assumptions 2.10 or 2.10$^+$. For example, both assumptions are satisfied if π^H is finite and supported on $(\mathrm{i}\mathbb{R})^{2k}$.

As previously discussed, by considering claims H that satisfy Assumption 2.10, we are able to find explicit formulas for the mean value process V^H. This approach relies on the following result, where we use Theorem 2.4 to find the conditional expectations of simple payoffs in terms of the solution to a time-dependent Riccati–Volterra equation. For the following, we recall $g^\star$, $h^\star := \hat{\kappa} * g^\star$, $T^\star$ and $Q^\star$ from Theorem 2.3.

Proposition 2.15. *Let ν_1 and ν_2 be complex-valued bounded variation measures on $([0, T^\star], \mathcal{B}([0, T^\star]))$. Then there exists some $T^\star_{\nu_1,\nu_2} \in (0, T^\star]$ such that for all $T \in [0, T^\star_{\nu_1,\nu_2}]$, there is a unique solution $g^\star_{\nu_1,\nu_2,T} \in L^1([0,T]; \mathbb{C})$ to the equation*

$$g^\star_{\nu_1,\nu_2,T}(t) = f^\star\bigg((\hat{\kappa} * g^\star_{\nu_1,\nu_2,T})(t), h^\star(t), \nu_1([T-t, T]), \int_{[T-t,T]} \hat{\kappa}\big(u - (T-t)\big)\nu_2(du)\bigg) \tag{2.52}$$

for $0 \le t \le T$, where the quadratic function $f^\star : \mathbb{C}^4 \to \mathbb{C}$ is defined by

$$\begin{aligned} f^\star(x, h, y_1, y_2) = {} & \frac{\sigma^2(y_1^2 - y_1)}{2} + \bigg(\varrho\sigma y_1 - \frac{\mu\varrho}{\sigma} + (1-\varrho^2)h\bigg)(x + y_2) \\ & + \frac{(x+y_2)^2}{2}. \end{aligned} \tag{2.53}$$

Moreover, for each $T \in (0, T^\star_{\nu_1,\nu_2})$, it holds that $e^{X_T(\nu_1)+\xi_T(\nu_2)}$ is integrable and

$$E_{Q^\star}[e^{X_T(\nu_1)+\xi_T(\nu_2)} \mid \mathcal{F}_t] = \exp\left(X_t(\nu_1) + \xi_t(\nu_2) + \int_t^T g^\star_{\nu_1,\nu_2,T}(T-u)\xi_t(u)du\right) \tag{2.54}$$

for all $t \in [0, T]$.

Proof. By the definition (2.38), we can express $X_t(\nu_1)$ as

$$\begin{aligned} X_t(\nu_1) &= \int_{[0,T]} X_{s\wedge t}\nu_1(ds) \\ &= \int_{[0,T]} \left(X_0 + \int_0^t \mathbf{1}_{[0,s]}(r)dX_r\right)\nu_1(ds) \\ &= \nu_1([0,T])X_0 + \int_{[0,T]} \left(\int_0^t \mathbf{1}_{[0,s]}(r)dX_r\right)\nu_1(ds), \quad 0 \le t \le T. \end{aligned} \tag{2.55}$$

Recall the dynamics $dX_r = d(\log S_r) = (\mu - \frac{\sigma^2}{2})Y_r dr + \sigma\sqrt{Y_r}dB_r$. Note that we have

$$\int_{[0,T]} \left(\int_0^T \sigma^2 Y_r dr\right)|\nu_1|(ds) \le |\nu_1|([0,T])T\sigma^2 \sup_{r\in[0,T]} |Y_r| < \infty$$

and

$$\int_{[0,T]} \left(\int_0^T \left|\mu - \frac{\sigma^2}{2}\right| Y_r dr\right)|\nu_1|(ds) \le |\nu_1|([0,T])T\left|\mu - \frac{\sigma^2}{2}\right| \sup_{r\in[0,T]} |Y_r| < \infty$$

P-a.s., since Y is continuous (hence a.s. bounded) and ν_1 has bounded variation. Hence it follows by the stochastic Fubini theorem (see Veraar [118, Theorem 2.2]) that we can swap the integrals in (2.55); to be more precise, this follows by applying [118, Theorem 2.2] separately to each of the positive and negative parts of the real and imaginary parts of ν_1. Hence we get

$$\begin{aligned} X_t(\nu_1) &= \nu_1([0,T])X_0 + \int_0^t \left(\int_{[0,T]} \mathbf{1}_{[0,s]}(r)\nu_1(ds)\right)dX_r \\ &= \nu_1([0,T])X_0 + \int_0^t \left(\int_{[0,T]} \mathbf{1}_{[r,T]}(s)\nu_1(ds)\right)dX_r \\ &= \nu_1([0,T])X_0 + \int_0^t \nu_1([r,T])dX_r, \quad 0 \le t \le T. \end{aligned}$$

Differentiating this equality and plugging in the dynamics of X yields

$$dX_t(\nu_1) = \nu_1([t,T])\left(\mu - \frac{\sigma^2}{2}\right)Y_t dt + \nu_1([t,T])\sigma\sqrt{Y_t}dW_t, \quad 0 \le t \le T. \tag{2.56}$$

Likewise, the definition (2.39) of $\xi_t(\nu_2)$ and part 1) of Lemma 2.6 yield the dynamics

$$d\xi_t(\nu_2) = \left(\int_{[t,T]} \hat{\kappa}(u-t)\nu_2(du)\right)\sqrt{Y_t}dB_t, \quad 0 \le t \le T. \tag{2.57}$$

Finally, recall from (2.16) that the density process of the VOMM is given by $Z^{Q^\star} = \mathcal{E}(N^{Q^\star})$, where

$$dN_t^{Q^\star} = \sqrt{Y_t}\left(\sqrt{1-\varrho^2}h^\star(T-t)dW_t^\perp - \frac{\mu}{\sigma}dW_t\right), \quad 0 \le t \le T, \tag{2.58}$$

so that

$$d[N^{Q^\star}]_t = Y_t\left((1-\varrho^2)\big(h^\star(T-t)\big)^2 + \frac{\mu^2}{\sigma^2}\right)dt, \quad 0 \le t \le T. \tag{2.59}$$

Setting $(\tilde{X}^{(1)}, \tilde{X}^{(2)}, \tilde{X}^{(3)}, \tilde{X}^{(4)}) := (X(\nu_1), \xi(\nu_2), N^{Q^\star}, [N^{Q^\star}])$, we thus have from (2.56)–(2.59) that the dynamics of each process $\tilde{X}^{(i)}$ has the form (2.17). Next, we check the integrability conditions required in order to apply part 1) of Theorem 2.3. We note that ν_1 and ν_2 have bounded variation, $\hat{\kappa}$ is locally square-integrable (see after (2.2)) and $h^\star$ is bounded. Moreover, we have by the Cauchy–Schwarz inequality and Fubini's theorem that

$$\begin{aligned}\int_0^T \left(\int_{[t,T]} \hat{\kappa}(u-t)\nu_2(du)\right)^2 dt &\le |\nu_2|([0,T]) \int_0^T \int_{[t,T]} \hat{\kappa}^2(u-t)\nu_2(du)dt \\ &= |\nu_2|([0,T]) \int_{[0,T]} \left(\int_0^u \hat{\kappa}^2(u-t)dt\right)\nu_2(du) \\ &\le \big(|\nu_2|([0,T])\big)^2 \|\hat{\kappa}\|^2_{L^2(0,T)} < \infty,\end{aligned}$$

so that the map $t \mapsto \int_{[t,T]} \hat{\kappa}(u-t)\nu_2(du)$ belongs to $L^2([0,T],\mathbb{C})$. It is then clear that the coefficients in the dynamics (2.56)–(2.59) satisfy the integrability conditions required by Theorem 2.3. Thus by part 1) of Theorem 2.3, there exists some $T^\star_{\nu_1,\nu_2} \in (0, T^\star]$ such that for all $T \in [0, T^\star_{\nu_1,\nu_2}]$, there is a unique solution $g^\star_{\nu_1,\nu_2,T} \in L^1([0,T];\mathbb{C})$ to the equation (2.18). Plugging the dynamics of

$(X(\nu_1^z), \xi(\nu_2), N^{Q^\star}, [N^{Q^\star}])$ with $z = (1,1,1,-\frac{1}{2})$ into (2.18) yields

$$\begin{aligned} g^\star_{\nu_1,\nu_2,T}(t) &= \left(\mu - \frac{\sigma^2}{2}\right)\nu_1([t,T]) - \frac{(1-\varrho^2)(h^\star(T-t))^2}{2} - \frac{\mu^2}{2\sigma^2} \\ &+ \frac{((\hat\kappa * g^\star_{\nu_1,\nu_2,T})(t))^2}{2} + \frac{(\sigma\nu_1([t,T]) + \varrho\int_{[t,T]}\hat\kappa(u-t)\nu_2(du) - \frac{\mu}{\sigma})^2}{2} \\ &+ \frac{(1-\varrho^2)(\int_{[t,T]}\hat\kappa(u-t)\nu_2(du) + h^\star(T-t))^2}{2} \\ &+ \varrho\left(\sigma\nu_1([t,T]) + \varrho\int_{[t,T]}\hat\kappa(u-t)\nu_2(du) - \frac{\mu}{\sigma}\right)(\hat\kappa * g^\star_{\nu_1,\nu_2,T})(t) \\ &+ (1-\varrho^2)\left(\int_{[t,T]}\hat\kappa(u-t)\nu_2(du) + h^\star(T-t)\right)(\hat\kappa * g^\star_{\nu_1,\nu_2,T})(t), \end{aligned}$$

which is equivalent to (2.52) after some simplifications. Theorem 2.3 also gives the conditional expectation, for $0 \le t \le T$,

$$\begin{aligned} &E\left[\exp\left(X_T(\nu_1) + \xi_T(\nu_2) + N_T^{Q^\star} - \frac{[N^{Q^\star}]_T}{2}\right) \,\Big|\, \mathcal{F}_t\right] \\ &= \exp\left(X_t(\nu_1) + \xi_t(\nu_2) + N_t^{Q^\star} - \frac{[N^{Q^\star}]_t}{2} + \int_t^T g^\star_{\nu_1,\nu_2}(T-u)\xi_t(u)du\right). \end{aligned}$$

Since $Z^{Q^\star} = \mathcal{E}(N^{Q^\star})$, dividing both sides by $Z_t^{Q^\star}$ gives

$$E_{Q^\star}[e^{X_T(\nu_1)+\xi_T(\nu_2)} \mid \mathcal{F}_t] = \exp\left(X_t(\nu_1) + \xi_t(\nu_2) + \int_t^T g^\star_{\nu_1,\nu_2,T}(T-u)\xi_t(u)du\right),$$

as claimed. □

As a corollary, we obtain a simpler formula for the classical setup from Assumption 2.7.

Corollary 2.16. *For each $z \in \mathbb{C}$, there exists some $T_z^\star \in (0, T^\star]$ such that there is a unique continuous solution $g_z^\star : [0, T_z^\star] \to \mathbb{C}$ to the equation*

$$g_z^\star(t) = f_z^\star\big((\hat\kappa * g_z^\star)(t), h^\star(t)\big), \quad 0 \le t \le T_z^\star, \tag{2.60}$$

where the quadratic function $f_z^\star : \mathbb{C}^2 \to \mathbb{C}$ is defined by

$$f_z^\star(x,h) = \frac{\sigma^2(z^2 - z)}{2} + \left(\varrho\sigma z - \frac{\mu\varrho}{\sigma}\right)x + (1-\varrho^2)xh + \frac{x^2}{2}.$$

Moreover, for any $T \in [0, T_z^\star]$, e^{zX_T} is integrable and we have

$$E_{Q^\star}[e^{zX_T} \mid \mathcal{F}_t] = \exp\left(zX_t + \int_t^T g_z^\star(T-u)\xi_t(u)du\right), \quad 0 \leq t \leq T. \tag{2.61}$$

Proof. By plugging in $\nu_1 = z\delta_{T^\star}$ and $\nu_2 = 0$ into Proposition 2.15 and setting $g_{z,T}^\star := g_{z\delta_{T^\star},0,T}^\star$, we get that there exists some $T_z^\star := T_{z\delta_T,0}^\star \in (0, T^\star]$ such that, for each $T \in [0, T_z^\star]$, there is a unique solution $g_{z,T}^\star \in L^1([0,T];\mathbb{C})$ to the Riccati–Volterra equation (2.52), which simplifies to

$$g_{z,T}^\star(t) = f_z^\star\big((\hat{\kappa} * g_{z,T}^\star)(t), h^\star(t)\big), \quad 0 \leq t \leq T.$$

By the uniqueness of the solution $g_{z,T}^\star$ and since none of the inputs for this equation depends explicitly on T, it follows that $g_{z,T}^\star = g_{z,T_z^\star}^\star$ on $[0,T]$ for each $T \in [0, T_z^\star]$. Setting $g_z^\star := g_{z,T_z^\star}^\star$, we thus have a unique solution $g_z^\star \in L^1([0,T_z^\star];\mathbb{C})$ to (2.60). Moreover, Proposition 2.15 also gives the equality (2.54), which likewise simplifies into (2.61) since in this case we have $X_t(\nu_1) = zX_t$ and $\xi_t(\nu_2) = 0$.

It remains to show the continuity of $g_z^\star$, which is not given by Proposition 2.15. Recall that $g^\star$ is continuous by Theorem 2.3 and hence so is $h^\star := \hat{\kappa} * g^\star$, so that each of the inputs to (2.60) is continuous. Therefore by the uniqueness of $g^\star$ and part 1) of Proposition A.2.2, we can choose some (possibly smaller) $T_z^\star > 0$ such that the solution $g_z^\star$ to (2.60) is continuous on $[0, T_z^\star]$. This concludes the proof. □

2.3 Mean–variance hedging strategies

Thanks to Proposition 2.15, we are now ready to compute the mean value process $(V_t^H)_{0 \leq t \leq T}$ for a payoff H satisfying Assumption 2.10 (or the more restrictive Assumption 2.7).

Proposition 2.17. *Let $T \in (0, T^\star]$ and suppose that $H \in L_P^2(\mathcal{F}_T)$ satisfies Assumption 2.10. Moreover, suppose that*

$$T \leq \inf\{T_{\nu_1^z,\nu_2^z}^\star : z \in \operatorname{supp}(\pi^H)\}, \tag{2.62}$$

where $T_{\nu_1,\nu_2}^\star$ is given by Proposition 2.15. Then the mean value process $(V_t^H)_{t\in[0,T]}$ is given by

$$V_t^H = \int_{\mathbb{C}^k} \tilde{V}_t^z \pi^H(dz), \quad 0 \leq t \leq T, \tag{2.63}$$

where $(\tilde{V}_t^z)_{0\le t\le T}$ *is defined by*

$$\tilde{V}_t^z := \exp\left(\tilde{X}_t(z) + \int_t^T g^\star_{\nu_1^z,\nu_2^z,T}(T-u)\xi_t(u)du\right), \quad 0 \le t \le T, \tag{2.64}$$

for $z \in \operatorname{supp}(\pi^H)$.

Proof. Due to (2.54) and (2.62), we have for each $z \in \operatorname{supp}(\pi^H)$ that the process $\tilde{V}^z$ given by (2.64) is well defined and a $Q^\star$-martingale on $[0,T]$, which yields $\tilde{V}_t^z = E_{Q^\star}[e^{\tilde{X}_T(z)} \mid \mathcal{F}_t]$ for $0 \le t \le T$. By Assumption 2.10 and the Cauchy–Schwarz inequality, we also have the bound

$$\begin{aligned}
&\int_{\mathbb{C}^k} E_{Q^\star}[e^{\operatorname{Re}(\tilde{X}_T(z))}]|\pi^H|(dz) \\
&\le E_P\left[\left(\frac{dQ^\star}{dP}\right)^2\right]^{1/2} \int_{\mathbb{C}^k} E_P[e^{2\operatorname{Re}(\tilde{X}_T(z))}]^{1/2}|\pi^H|(dz) \\
&\le E_P\left[\left(\frac{dQ^\star}{dP}\right)^2\right]^{1/2} \left(\int_{\mathbb{C}^k} E_P[e^{2\operatorname{Re}(\tilde{X}_T(z))}]|\pi^H|(dz)\right)^{1/2} |\pi^H|(\mathbb{C})^{1/2} < \infty. \qquad (2.65)
\end{aligned}$$

Hence the conditional Fubini theorem yields

$$V_t^H = E_{Q^\star}[H \mid \mathcal{F}_t] = \int_{\mathbb{C}^k} E_{Q^\star}[e^{\tilde{X}_T(z)} \mid \mathcal{F}_t]\pi^H(dz) = \int_{\mathbb{C}^k} \tilde{V}_t^z \pi^H(dz)$$

since $\tilde{V}^z$ is a $Q^\star$-martingale on $[0,T]$ for $z \in \operatorname{supp}(\pi^H)$. This shows (2.63). □

Now that we have determined the mean value process V^H, we are able to calculate the pure hedge coefficient $(\Xi_t^H)_{0\le t\le T}$ under the slightly stronger Assumption 2.10^+ (or 2.7^+).

Proposition 2.18. *Let* $T \in (0, T^\star]$ *and suppose that* $H \in L^2_P(\mathcal{F}_T)$ *satisfies Assumption* 2.10^+. *Moreover, suppose that* (2.62) *holds, where* $T^\star_{\nu_1,\nu_2}$ *is given by Proposition 2.15. Then the pure hedge coefficient* $(\Xi_t^H)_{0\le t\le T}$ *for* H *is given by*

$$\Xi_t^H = \frac{1}{\sigma S_t}\int_{\mathbb{C}^k} \tilde{\Xi}_t^z \pi^H(dz), \quad 0 \le t \le T, \tag{2.66}$$

where $(\tilde{\Xi}_t^z)_{0\le t\le T}$ *is defined by*

$$\tilde{\Xi}_t^z := \tilde{V}_t^z\left(\sigma\nu_1^z([t,T]) + \varrho\int_{[t,T]} \hat{\kappa}(u-t)\big(\nu_2^z(du) + g^\star_{\nu_1^z,\nu_2^z,T}(T-u)du\big)\right) \tag{2.67}$$

and $\tilde{V}^z$ *is given by* (2.64) *for* $z \in \operatorname{supp}(\pi^H)$.

Proof. By Proposition 2.17, we have

$$V_t^H = \int_{\mathbb{C}^k} \tilde{V}_t^z \pi^H(dz), \quad 0 \le t \le T. \tag{2.68}$$

In view of the definition (2.7) of Ξ^H, we want to compute the dynamics of $\tilde{V}^z$ and V^H. For the former, we start by differentiating the integral term in (2.64). By part 2) of Lemma 2.6, we have the dynamics

$$\begin{aligned} d&\bigg(\int_t^T g^\star_{\nu_1^z,\nu_2^z,T}(T-u)\xi_t(u)du \bigg) \\ &= -g^\star_{\nu_1^z,\nu_2^z,T}(T-t)Y_t dt + (\hat{\kappa} * g^\star_{\nu_1^z,\nu_2^z,T})(T-t)\sqrt{Y_t}dB_t, \quad 0 \le t \le T, \end{aligned}$$

for $z \in \operatorname{supp}(\pi^H)$. Recall also that $\tilde{X}_t(z) = X_t(\nu_1^z) + \xi_t(\nu_2^z)$, where $X(\nu_1^z)$ and $\xi(\nu_2^z)$ have the dynamics (2.56) and (2.57), respectively. Hence Itô's formula applied to (2.64) yields that the dynamics of $\tilde{V}^z$ is given by

$$d\tilde{V}_t^z = \tilde{V}_t^z \sqrt{Y_t}\big(\nu_1^z([t,T])\sigma dW_t + \varphi_t^z dB_t\big) + dA_t, \quad 0 \le t \le T, \tag{2.69}$$

for some finite variation process $(A_t)_{0\le t\le T}$ that is absolutely continuous with respect to dt and a deterministic coefficient $(\varphi_t^z)_{0\le t\le T}$ given by

$$\varphi_t^z := \int_{[t,T]} \hat{\kappa}(u-t)\big(\nu_2^z(du) + g^\star_{\nu_1^z,\nu_2^z,T}(T-u)du\big), \quad 0 \le t \le T. \tag{2.70}$$

Since $\tilde{V}^z$ is also a $Q^\star$-martingale, it follows from (2.69) that

$$d\tilde{V}_t^z = \tilde{V}_t^z \sqrt{Y_t}\big(\nu_1^z([t,T])\sigma dW_t^\star + \varphi_t^z dB_t^\star\big), \tag{2.71}$$

where $W^\star$ and $B^\star$ are the $Q^\star$-Brownian motions derived from W and B by Girsanov's theorem with respect to $Q^\star$, as given in part 6) of Theorem 2.3.

Next, we want to find the dynamics of V^H. By a similar calculation as in (2.65), Assumption 2.10$^+$ yields

$$\begin{aligned} \int_{\mathbb{C}^k} E_{Q^\star}[|\tilde{V}_T^z|^{1+\delta/2}]|\pi^H|(dz) &= \int_{\mathbb{C}^k} E_{Q^\star}[e^{\mathrm{Re}((1+\delta/2)\tilde{X}_T(z))}]|\pi^H|(dz) \\ &\le C_1 C_2 \left(\int_{\mathbb{C}^k} E_P[e^{(2+\delta)\mathrm{Re}(\tilde{X}_T(z))}]|\pi^H|(dz) \right)^{1/2} \\ &< \infty, \end{aligned} \tag{2.72}$$

where $C_1 := E_P[(dQ^\star/dP)^2]^{1/2}$ and $C_2 := |\pi^H|(\mathbb{C})^{1/2}$ are finite. As $1+\delta/2>1$, there exists by the Burkholder–Davis–Gundy and Doob maximal inequalities some $c_\delta>0$ such that for every real-valued $Q^\star$-martingale M on $[0,T]$,

$$E_{Q^\star}\big[[M]_T^{(2+\delta)/4}\big] \le c_\delta E_{Q^\star}\big[|M|_T^{1+\delta/2}\big].$$

Consider now $M = M^r + \mathrm{i}M^i$ for some real-valued $Q^\star$-martingales $(M_t^r)_{0\le t\le T}$ and $(M_t^i)_{0\le t\le T}$ so that $[M,\overline{M}] = [M^r]+[M^i]$. It follows by considering M^r and M^i separately that

$$E_{Q^\star}\big[[M,\overline{M}]_T^{(2+\delta)/4}\big] \le c_\delta E_{Q^\star}\big[|M|_T^{1+\delta/2}\big]. \tag{2.73}$$

for some (possibly larger) choice of $c_\delta > 0$. Note that by (2.71), we have $d[\tilde{V}^z,\overline{\tilde{V}^z}]_t = \psi_t^z dt$, where $(\psi_t^z)_{0\le t\le T}$ is a nonnegative process given by

$$\psi_t^z = |\tilde{V}_t^z|^2 Y_t\big(|\nu_1^z([t,T])\sigma + \varrho\varphi_t^z|^2 + (1-\varrho^2)|\varphi_t^z|^2\big), \quad 0\le t\le T. \tag{2.74}$$

Hence by Hölder's inequality with $p = 1+\delta/2$ and $q = 1+2/\delta$, Fubini's theorem, (2.73) and (2.72), we obtain

$$\begin{aligned}
E_{Q^\star}\left[\int_{\mathbb{C}^k}\left(\int_0^T \psi_t^z dt\right)^{1/2}|\pi^H|(dz)\right] &\le C_2^{\frac{\delta}{2+\delta}} E_{Q^\star}\left[\int_{\mathbb{C}^k}\left(\int_0^T \psi_t^z dt\right)^{\frac{2+\delta}{4}}|\pi^H|(dz)\right]^{\frac{2}{2+\delta}}\\
&\le C_2^{\frac{\delta}{2+\delta}}\left(c_\delta\int_{\mathbb{C}^k} E_{Q^\star}[|\tilde{V}_T^z|^{1+\delta/2}]|\pi^H|(dz)\right)^{\frac{2}{2+\delta}}\\
&< \infty.
\end{aligned}$$

In particular, since $Q^\star \approx P$,

$$\int_{\mathbb{C}^k}\left(\int_0^T \psi_t^z dt\right)^{1/2}|\pi^H|(dz) < \infty \quad P\text{-a.s.}$$

This corresponds to the inequality in Veraar [118, Equation (2.1)]. Hence the stochastic Fubini theorem [118, Theorem 2.2] together with (2.71) and (2.68) yields that the dynamics of V^H are given by

$$\begin{aligned}
dV_t^H &= \int_{\mathbb{C}^k}\tilde{V}_t^z\sqrt{Y_t}\big(\nu_1^z([t,T])\sigma dW_t^\star + \varphi_t^z dB_t^\star\big)\pi^H(dz)\\
&= \left(\int_{\mathbb{C}^k}\tilde{V}_t^z\nu_1^z([t,T])\pi^H(dz)\right)\sigma\sqrt{Y_t}dW_t^\star\\
&\quad + \left(\int_{\mathbb{C}^k}\tilde{V}_t^z\varphi_t^z\pi^H(dz)\right)\sqrt{Y_t}dB_t^\star, \quad 0\le t\le T.
\end{aligned} \tag{2.75}$$

Note that by (2.67) and (2.70), we have

$$\tilde{V}_t^z \sigma \nu_1^z([t,T]) + \varrho \tilde{V}_t^z \varphi_t^z \pi^H(dz) = \tilde{\Xi}_t^z, \quad 0 \le t \le T.$$

Hence by combining (2.75) with the dynamics $dS_t = S_t \sigma \sqrt{Y_t} dW_t^\star$ from part 6) of Theorem 2.3, we obtain

$$\Xi_t^H = \frac{d[V^H, S]_t}{d[S,S]_t} = \frac{1}{\sigma S_t} \int_{\mathbb{C}^k} \tilde{\Xi}_t^z \pi^H(dz), \quad 0 \le t \le T.$$

This shows (2.66) and concludes the proof. □

Given a time horizon $T \in (0, T^\star)$ and a claim $H \in L_P^2(\mathcal{F}_t)$ satisfying the conditions of Proposition 2.18, we have obtained the formulas (2.63) and (2.66) for the mean value process V^H and Ξ^H, respectively, up to solving the family of Riccati–Volterra equations (2.52), where we set $\nu_1 = \nu_1^z$ and $\nu_2 = \nu_2^z$ for each $z \in \operatorname{supp}(\pi^H)$. While a closed-form solution to (2.52) is not available, one can use numerical methods to solve this (deterministic) equation on $[0,T]$ for each z.

As discussed after Proposition 2.1, by plugging the formulas for V^H and Ξ^H into (2.9) together with the adjustment process a given by (2.15), we thus obtain the mean–variance hedging strategy $\vartheta^H(x,t)$ semi-explicitly as the solution to a feedback equation. In principle, one may even use (2.10) or (2.11) to obtain $\vartheta^H(x,t)$ in a fully explicit form. Therefore, Propositions 2.17 and 2.18 effectively solve the mean–variance hedging problem for claims H that satisfy the assumptions of Proposition 2.18.

While Propositions 2.17 and 2.18 give the solution in a general setup, the associated equations and formulas can often be simplified in practice; we show how to do this for the claims considered in Examples 2.9 and 2.12–2.14.

Corollary 2.19 (European vanilla call and put options)**.** *Suppose that $a_1 > 1$, $a_2 < 0$ and $T \in (0, T^\star]$ are such that*

$$E[S_T^{2a_1+\delta}], E[S_T^{2a_2-\delta}] < \infty \quad \text{and} \quad T \le \inf\left\{T_z^\star : z \in \{a_1, a_2\} + \mathrm{i}\mathbb{R}\right\}$$

for some $\delta > 0$, where $T_z^\star$ is given by Corollary 2.16. Then the European call and put options $C_{K,T}$ and $P_{K,T}$ from Example 2.9 satisfy Assumption 2.7$^+$, and the mean value processes $V^{C_{K,T}}$, $V^{P_{K,T}}$ and pure hedge coefficients $\Xi^{C_{K,T}}$, $\Xi^{P_{K,T}}$ are *given by*

$$V_t^{C_{K,T}} = \int_{\mathbb{C}} \tilde{V}_t^z \pi_{K,a_1}(dz), \qquad \Xi_t^{C_{K,T}} = \frac{1}{\sigma S_t}\int_{\mathbb{C}} \tilde{\Xi}_t^z \pi_{K,a_1}(dz),$$
$$V_t^{P_{K,T}} = \int_{\mathbb{C}} \tilde{V}_t^z \pi_{K,a_2}(dz), \qquad \Xi_t^{P_{K,T}} = \frac{1}{\sigma S_t}\int_{\mathbb{C}} \tilde{\Xi}_t^z \pi_{K,a_2}(dz) \tag{2.76}$$

for $0 \le t \le T$*, where* $\pi_{K,a}$ *is given by* (2.37)*, the processes* $(\tilde{V}_t^z)_{0\le t\le T}$*,* $(\tilde{\Xi}_t^z)_{0\le t\le T}$ *are defined by*

$$\tilde{V}_t^z = \exp\left(zX_t + \int_t^T g_z^\star(T-u)\xi_t(u)du\right), \tag{2.77}$$
$$\tilde{\Xi}_t^z = \big(z\sigma + \varrho(\hat{\kappa} * g_z^\star)(T-t)\big)\tilde{V}_t^z, \tag{2.78}$$

and $g_z^\star : [0,T] \to \mathbb{C}$ *is the continuous solution to* (2.61) *for* $z \in \{a_1 + a_2\} + \mathrm{i}\mathbb{R}$.

Proof. For $\ell \in \{1,2\}$, the measure $\pi_{K,a_\ell}(dz) = \frac{K^{1-z}}{2\pi z(z-1)}\lambda_{a_\ell}(dz)$ (which is supported on $(a_\ell - \mathrm{i}\infty, a_\ell + \mathrm{i}\infty)$) has bounded variation on $\mathbb{C}$ since it decays quadratically as $z \to a_\ell \pm \mathrm{i}\infty$. Thus by assumption, we have for $\ell \in \{1,2\}$ and $\tilde{\delta}_\ell := \delta/|a_\ell|$ that

$$\int_{\mathbb{C}} E_P[S_T^{(2+\tilde{\delta}_\ell)\mathrm{Re}(z)}]|\pi_{K,a_\ell}|(dz)$$
$$= E_P[S_T^{2a_\ell+(-1)^{\ell-1}\delta}]|\pi_{K,a_\ell}|\big((a_\ell - \mathrm{i}\infty, a_\ell + \mathrm{i}\infty)\big) < \infty,$$

which shows (2.34). Hence $C_{K,T}$ and $P_{K,T}$ satisfy Assumption 2.7$^+$. As shown in (2.40), Assumption 2.7$^+$ can be seen as a special case of Assumption 2.10$^+$ with $\nu_1^z = z\delta_T$ and $\nu_2^z = 0$. Since $T_z^\star = T_{z\delta_T,0}^\star$ by definition (see the proof of Corollary 2.16), we also have by the choice of T that (2.62) is satisfied. Thus by applying Propositions 2.17 and 2.18 to $C_{K,T}$ and $P_{K,T}$, we obtain (2.76). Recall that we have $\tilde{X}_t(z) = zX_t$ in the case $\nu_1^z = z\delta_T$ and $\nu_2^z = 0$, so that (2.64) simplifies into (2.77). We also have $g_z^\star = g_{\nu_1^z,\nu_2^z,T}^\star$ for this choice of ν_1^z and ν_2^z, by the construction given in the proof of Corollary 2.16. Plugging this into (2.67) together with $\nu_1^z([t,T]) = z$ for $t \in [0,T]$ and $\nu_2^z = 0$ yields (2.78). □

Corollary 2.20 (European call and put options on spot volatility)**.** *Suppose that* $a_1 > 0$*,* $a_2 < 0$ *and* $T \in (0, T^\star]$ *are such that*

$$E[e^{(2a_1+\delta)Y_T}], E[e^{(2a_2-\delta)Y_T}] < \infty \quad \text{and} \quad T \le \inf\{T_{0,z\delta_T}^\star : z \in \{a_1,a_2\} + \mathrm{i}\mathbb{R}\}$$

for some $\delta > 0$*, where* $T_{\nu_1,\nu_2}^\star$ *is given in Corollary 2.16. Then the spot volatility options* $C_{K,T}^Y$ *and* $P_{K,T}^Y$ *from Example 2.12 satisfy Assumption 2.10*$^+$*, and the* *mean value processes* $V^{C^Y_{K,T}}$, $V^{P^Y_{K,T}}$ *and pure hedge coefficients* $\Xi^{C^Y_{K,T}}$, $\Xi^{P^Y_{K,T}}$ *are given by*

$$\begin{aligned} V_t^{C^Y_{K,T}} &= \int_{\mathbb{C}} \tilde V_t^{z,Y} \pi^Y_{K,a_1}(dz), \qquad \Xi_t^{C^Y_{K,T}} = \frac{1}{\sigma S_t} \int_{\mathbb{C}} \tilde\Xi_t^{z,Y} \pi^Y_{K,a_1}(dz), \\ V_t^{P^Y_{K,T}} &= \int_{\mathbb{C}} \tilde V_t^{z,Y} \pi^Y_{K,a_2}(dz), \qquad \Xi_t^{P^Y_{K,T}} = \frac{1}{\sigma S_t} \int_{\mathbb{C}} \tilde\Xi_t^{z,Y} \pi^Y_{K,a_2}(dz) \end{aligned} \tag{2.79}$$

for $0 \le t \le T$, *where* $\pi^Y_{K,a}$ *is given by* (2.47), *the processes* $(\tilde V_t^{z,Y})_{0\le t\le T}$, $(\tilde\Xi_t^{z,Y})_{0\le t\le T}$ *are defined by*

$$\tilde V_t^{z,Y} = \exp\left(z\xi_t(T) + \int_t^T g^\star_{z,Y}(T-u)\xi_t(u)du \right), \tag{2.80}$$

$$\tilde\Xi_t^{z,Y} = \varrho\big(z\hat\kappa(T-t) + (\hat\kappa * g^\star_{z,Y})(T-t)\big)\tilde V_t^{z,Y}, \tag{2.81}$$

and $g^\star_{z,Y} \in L^1([0,T];\mathbb{C})$ *is the unique solution to the equation*

$$g^\star_{z,Y}(t) = f^\star_{z,Y}\big((\hat\kappa * g^\star_{z,Y})(t), h^\star(t), \hat\kappa(t)\big), \quad 0 \le t \le T, \tag{2.82}$$

for $z \in \{a_1, a_2\} + \mathrm{i}\mathbb{R}$, *where* $f^\star_{z,Y} : \mathbb{C}^3 \to \mathbb{C}$ *is given by*

$$f^\star_{z,Y}(x,h,k) := (x + zk)\left(-\frac{\mu\varrho}{\sigma} + (1-\varrho^2)h + \frac{x+zk}{2} \right).$$

Proof. For $\ell \in \{1,2\}$ the measure $\pi^Y_{K,a_\ell}(dz) := \frac{e^{-Kz}}{2\pi z^2}\lambda_{a_\ell}(dz)$, which is supported on $(a_\ell - \mathrm{i}\infty, a_\ell + \mathrm{i}\infty)$, has bounded variation on $\mathbb{C}$ since it decays quadratically as $z \to a \pm \mathrm{i}\infty$. Setting $\nu_1^z = 0$ and $\nu_2^z = z\delta_T$ for $z \in \mathbb{C}$, we thus have

$$\begin{aligned} &\int_{\mathbb{C}} E_P\Big[\exp\Big((2+\tilde\delta_\ell)\mathrm{Re}\big(\tilde X_T(z)\big)\Big)\Big] |\pi^Y_{K,a_\ell}|(dz) \\ &= E_P\Big[\exp\Big(\big(2a_\ell + (-1)^{\ell-1}\delta\big)Y_T\Big)\Big] |\pi^Y_{K,a_\ell}|\big((a-\mathrm{i}\infty, a+\mathrm{i}\infty)\big) < \infty \end{aligned}$$

by assumption for $\ell \in \{1,2\}$ and $\tilde\delta_\ell := \delta/|a_\ell| > 0$, which shows (2.44). Thus $C^Y_{K,T}$ and $P^Y_{K,T}$ satisfy Assumption 2.10$^+$. Since (2.62) also holds by the choice of T, we may apply Propositions 2.17 and 2.18 to $C^Y_{K,T}$ and $P^Y_{K,T}$, and this yields (2.79).

To show the remaining equations, note that $\tilde X_t(z) = z\xi_t(T)$ in the case $\nu_1^z = 0$ and $\nu_2^z = z\delta_T$, and hence (2.64) simplifies to (2.80). Likewise, (2.52) simplifies into the form (2.82); note that the third argument of $f^\star$ is not needed because $\nu_1^z = 0$. As none of the inputs to (2.82) depends explicitly on T, we may omit the parameter T from $g^\star_{z,Y}$ by the same uniqueness argument used in the proof of

Corollary 2.16. Plugging $g^\star_{z,Y}$ into (2.67) together with $\nu_2^z = z\delta_T$ yields (2.81). $\square$

Corollary 2.21 (Geometric Asian call and put options)**.** *Suppose that $a_1 > 1$, $a_2 < 0$ and $T \in (0, T^\star]$ are such that*

$$E[(S_T^A)^{2a_1+\delta}], E[(S_T^A)^{2a_2-\delta}] < \infty \quad \text{and} \quad T \le \inf\left\{T^\star_{\nu_1^z,0} : z \in \{a_1, a_2\} + \mathrm{i}\mathbb{R}\right\}$$

for some $\delta > 0$, where $\nu_1^z(dt) = \frac{z}{T}dt$, S^A is the geometric mean of S defined in (2.48) and $T^\star_{\nu_1,\nu_2}$ is given in Corollary 2.16. Then the geometric Asian options $C^A_{K,T}$ and $P^A_{K,T}$ from Example 2.13 satisfy Assumption 2.10$^+$, and the mean value processes $V^{C^A_{K,T}}$, $V^{P^A_{K,T}}$ and pure hedge coefficients $\Xi^{C^A_{K,T}}$, $\Xi^{P^A_{K,T}}$ are given by

$$\begin{aligned} V_t^{C^A_{K,T}} &= \int_{\mathbb{C}} \tilde{V}_t^{z,A} \pi_{K,a_1}(dz), \quad & \Xi_t^{C^A_{K,T}} &= \frac{1}{\sigma S_t} \int_{\mathbb{C}} \tilde{\Xi}_t^{z,A} \pi_{K,a_1}(dz), \quad 0 \le t \le T, \\ V_t^{P^A_{K,T}} &= \int_{\mathbb{C}} \tilde{V}_t^{z,A} \pi_{K,a_2}(dz), \quad & \Xi_t^{P^A_{K,T}} &= \frac{1}{\sigma S_t} \int_{\mathbb{C}} \tilde{\Xi}_t^{z,A} \pi_{K,a_2}(dz), \quad 0 \le t \le T, \end{aligned} \tag{2.83}$$

where $\pi_{K,a}$ is given by (2.37), $(\tilde{V}_t^{z,A})_{0\le t\le T}$, $(\tilde{\Xi}_t^{z,A})_{0\le t\le T}$ are defined by

$$\tilde{V}_t^{z,A} = \exp\left(\frac{z}{T}\Big(\int_0^t X_s ds + (T-t)X_t\Big) + \int_t^T g^\star_{z,A}(T-u)\xi_t(u)du\right), \tag{2.84}$$

$$\tilde{\Xi}_t^{z,A} = \left(\frac{\sigma z(T-t)}{T} + \varrho(\hat{\kappa} * g^\star_{z,A})(T-t)\right)\tilde{V}_t^{z,A}, \tag{2.85}$$

and $g^\star_{z,A,T} \in L^1([0,T];\mathbb{C})$ is the unique solution to the equation

$$g^\star_{z,A,T}(t) = f^\star_{z,A,T}\big((\hat{\kappa} * g^\star_{z,A,T})(t), h^\star(t), t\big), \quad 0 \le t \le T, \tag{2.86}$$

for $z \in \{a_1, a_2\} + \mathrm{i}\mathbb{R}$, where $f^\star_{z,A,T} : \mathbb{C}^3 \to \mathbb{C}$ is given by

$$f^\star_{z,A,T}(x,h,t) := \frac{\sigma^2}{2}\left(\frac{z^2t^2}{T^2} - \frac{zt}{T}\right) + \left(\varrho\sigma\frac{zt}{T} - \frac{\mu\varrho}{\sigma}\right)x + (1-\varrho^2)xh + \frac{x^2}{2}.$$

Proof. We already argued in the proof of Corollary 2.19 that $\pi_{K,a}(dz)$ has bounded variation on $\mathbb{C}$. Setting $\nu_1^z(dt) = \frac{z}{T}dt$ and $\nu_2^z = 0$ for $z \in \mathbb{C}$, we have

$$\begin{aligned} &\int_{\mathbb{C}} E_P\big[\big((S_T^A)^{a_\ell}\big)^{2+\tilde{\delta}_\ell}\big] |\pi^Y_{K,a_\ell}|(dz) \\ &= E_P[(S_T^A)^{2a_\ell+(-1)^{\ell-1}\delta}] |\pi^Y_{K,a_\ell}|\big((a_\ell - \mathrm{i}\infty, a_\ell + \mathrm{i}\infty)\big) < \infty \end{aligned}$$

by assumption for $\ell \in \{1,2\}$ and $\tilde{\delta}_\ell := \delta/|a_\ell| > 0$, which shows (2.43). Thus $C^A_{K,T}$ and $P^A_{K,T}$ satisfy Assumption 2.10$^+$ so that by applying Propositions 2.17

and 2.18 to $C^A_{K,T}$ and $P^A_{K,T}$, we have (2.83). Since (2.38) and (2.48) yield

$$\tilde{X}_t(z) = X_t(\nu_1^z) = \frac{z}{T}\int_0^t X_s ds + \frac{z(T-t)X_t}{T}, \quad 0 \le t \le T,$$

it follows that (2.64) simplifies to (2.84). Moreover, the equation (2.52) simplifies into the form (2.90) after plugging in $\nu_2^z = 0$ (so that the fourth argument of $f^\star$ is not needed) and $\nu_1^z([T-t,T]) = zt/T$. Plugging this into (2.67) together with $\nu_1^z([t,T]) = z(T-t)/T$ and $\nu_2^z = 0$ yields (2.85). □

Corollary 2.22 (Options on discrete-time observations)**.** *Suppose that* $T \in (0, T^\star]$ *and the complex-valued bounded variation measure* π^H *on* $(\mathbb{C}^{2k}, \mathcal{B}(\mathbb{C}^{2k}))$ *satisfy the conditions* $T \le \inf\{T^\star_{\nu_1^z,\nu_2^z} : z \in \operatorname{supp}(\pi^H)\}$ *as well as*

$$E\Bigg[\int_{\mathbb{C}^k} \exp\Bigg((2+\delta)\sum_{j=1}^k \operatorname{Re}(z_j)X_{t_j} + (2+\delta)\sum_{j=1}^k \operatorname{Re}(\tilde{z}_j)Y_{t_j}\Bigg)|\pi^H|(dz)\Bigg] < \infty$$

for some $\delta > 0$*, where* ν_1^z *and* ν_2^z *are defined by* (2.51) *and* $T^\star_{\nu_1,\nu_2}$ *is given in Corollary 2.16. Then the mean value process* V^H *and pure hedge coefficient* Ξ^H *for the payoff* H *from Example 2.14 are given by*

$$V_t^H = \int_{\mathbb{C}^k} \tilde{V}_t^{z,H} \pi^H(dz) \quad \text{and} \quad \Xi_t^H = \frac{1}{\sigma S_t}\int_{\mathbb{C}^k} \tilde{\Xi}_t^{z,H}\pi^H(dz) \tag{2.87}$$

for $0 \le t \le T$*, where the processes* $(\tilde{V}_t^{z,H})_{0\le t\le T}$*,* $(\tilde{\Xi}_t^{z,H})_{0\le t\le T}$ *are defined by*

$$\tilde{V}_t^{z,H} = \exp\Bigg(\sum_{j=1}^k z_j X_{t\wedge t_j} + \sum_{j=1}^k \tilde{z}_j \xi_t(t_j) + \int_t^T g^\star_{z,H,T}(T-u)\xi_t(u)du\Bigg), \tag{2.88}$$

$$\tilde{\Xi}_t^{z,H} = \Bigg(\sigma\sum_{j=1}^k z_j \mathbf{1}_{[0,t_j]}(t) + \varrho\sum_{j=1}^k z_j\mathbf{1}_{[0,t_j]}(t)\hat{\kappa}(t_j - t) + \varrho(\hat{\kappa} * g^\star_{z,H,T})(T-t)\Bigg)\tilde{V}_t^{z,H}, \tag{2.89}$$

and $g^\star_{z,H,T} \in L^1([0,T];\mathbb{C})$ *is the unique solution to the equation*

$$g^\star_{z,H,T}(t) = f^\star\Bigg((\hat{\kappa} * g^\star_{z,H,T})(t), h^\star(t), \sum_{j=1}^k z_j\mathbf{1}_{[0,t_j]}(T-t), \sum_{j=1}^k z_j \mathbf{1}_{[0,t_j]}(t)\hat{\kappa}(t_j - t)\Bigg) \tag{2.90}$$

for $0 \le t \le T$ *and* $z \in \operatorname{supp}(\pi^H)$*, where* $f^\star$ *is given by* (2.53).

Proof. Definition (2.51) yields for $0 \leq t \leq T$ that

$$\nu_1^z([t,T]) = \sum_{j=1}^k z_j \mathbf{1}_{[0,t_j]}(t) \quad \text{and} \quad \int_{[t,T]} \hat{\kappa}(u-t)\nu_2^z(du) = \sum_{j=1}^k z_j \mathbf{1}_{[0,t_j]}(t).$$

Thus the result follows directly by plugging into Propositions 2.17 and 2.18. □

Thanks to Propositions 2.17 and 2.18, we have thus been able to obtain explicit formulas for several different payoffs such as the ones considered in Examples 2.9 and 2.12–2.14. However, the results are somewhat unsatisfactory from a theoretical point of view since the assumption (2.62) that

$$T \leq \inf\{T^\star_{\nu_1^z,\nu_2^z} : z \in \operatorname{supp}(\pi^H)\}$$

can in principle be quite restrictive. Indeed, for a general payoff H, measure π^H and associated map $z \mapsto (\nu_1^z, \nu_2^z)$, it is unclear whether this infimum is nonzero. In the case where π^H is supported on a compact subset of $\mathbb{C}^k$, one can strengthen Proposition 2.15 by using Theorem 2.4 (which applies simultaneously for all z belonging to a ball $B \subseteq \mathbb{C}^k$) in order to ensure that (2.54) holds for the measures ν_1^z, ν_2^z on a common interval $[0,T']$ for some T' and all $z \in B$. In this way, we can ensure that the infimum is strictly positive if the support of π^H is compact and hence contained in a ball. However, this approach fails if the support of π^H is unbounded, as is the case for European vanilla call and put options, where π^H is supported on a vertical line of the form $a + \mathrm{i}\mathbb{R} \subseteq \mathbb{C}$.

One partial workaround is to approximate H by truncating the domain of integration. For instance, consider the European call option $C_{K,T} = (S_T - K)^+$. By truncating (2.35), we define the approximate payoff

$$C^m_{K,T} := \frac{1}{2\pi\mathrm{i}} \int_{a_1-\mathrm{i}m}^{a_1+\mathrm{i}m} S_T^z \frac{K^{1-z}}{z(z-1)} dz \tag{2.91}$$

for $m \in \mathbb{N}$ and some fixed $a_1 > 1$. If $T > 0$ is small enough so that $E[S_T^{2a_1}] < \infty$, it follows by Fubini's theorem (in a similar way as in the proof of Corollary 2.19) that $C^m_{K,T} \in L^2_P(\mathcal{F}_T)$ and $C^m_{K,T} \overset{L^2}{\to} C_{K,T}$ as $m \to \infty$. As discussed in the introduction to this chapter, it follows by the general theory of MVH that for each $t \in [0,T]$ and $x_t \in L^2_P(\mathcal{F}_t)$, there exist unique optimal hedging strategies $\vartheta^{C^m_{K,T}}(x_t,t), \vartheta^{C^m_{K,T}}(x_t,t) \in \overline{\Theta}_{t,T}(S)$ for $C^m_{K,T}$ and $C_{K,T}$, respectively. Since the MVH problem can be seen as an L^2-projection problem, which is continuous on $L^2_P(\mathcal{F}_T)$, we also have $(\vartheta^{C^m_{K,T}}(x,t) \bullet S)_T \overset{L^2}{\to} (\vartheta^{C_{K,T}}(x,t) \bullet S)_T$ as $m \to \infty$. Moreover, since the segment $a_1 + \mathrm{i}[-m, m]$ is bounded, there exists by the argument above some $T_m > 0$ such that (2.60) admits for each $z \in a_1 + \mathrm{i}[-m, m]$ a solution $g_z^\star$ on $[0, T_m]$. As in (2.76), this yields

$$V_t^{C_{K,T}^m} = \int_{a_1 - \mathrm{i}m}^{a_1 + \mathrm{i}m} \tilde{V}_t^z \pi_{K,a_1}(dz), \quad \Xi_t^{C_{K,T}^m} = \frac{1}{\sigma S_t} \int_{a_1 - \mathrm{i}m}^{a_1 + \mathrm{i}m} \tilde{\Xi}_t^z \pi_{K,a_1}(dz), \quad 0 \le t \le T,$$

if $T \in (0, T_m]$, and hence we derive a formula for $\vartheta^{C_{K,T}^m}(x_t, t)$ by plugging into the feedback equation (2.9). However, the resulting formula only holds if $T \in (0, T_m]$, and in principle it could be the case that $T_m \to 0$ as $m \to \infty$. Therefore, even though $\vartheta^{C_{K,T}^m}(x_t, t)$ approximates $\vartheta^{C_{K,T}}(x_t, t)$ on $[0, T]$, the explicit formula for $\vartheta^{C_{K,T}^m}(x_t, t)$ may only hold on a small interval that vanishes as $m \to \infty$, so that the limit does not yield an explicit formula for $\vartheta^{C_{K,T}}(x_t, t)$ for any $T > 0$.

Tackling these issues in the general setup of Assumption 2.10 remains a topic for future study. We focus now on the case of European call and put options with the goal of relaxing the assumptions of Corollary 2.19. In the following, we denote by a a fixed $a_1 > 1$ or $a_2 < 0$ corresponding to the call and put options, respectively. Our strategy is to use the particular structure of (2.60) to show that it admits a solution for all $z \in a + \mathrm{i}\mathbb{R}$ on a common interval; this is the main topic of Section 4 and leads to the following theorem.

Theorem 2.23. *Let $a \in \mathbb{R}$, fix $T_a^\star \in (0, T^\star]$ as given by Corollary 2.16 and suppose that the kernel $\kappa : (0, \infty) \to \mathbb{R}_+$ in* (2.1) *is completely monotone. Then there exists a unique continuous solution $g_z^\star : [0, T_a^\star] \to \mathbb{C}$ to* (4.14) *for each $z \in a + \mathrm{i}\mathbb{R}$.*

The proof is deferred to Theorem 4.10. The requirement that κ is completely monotone (in addition to satisfying Assumption I.2.7) is discussed after the statement of Theorem 4.10.

We now use Theorem 2.23 to strengthen Corollaries 2.16 and 2.19. Even though we can solve (2.60) for all $z \in a + \mathrm{i}\mathbb{R}$ on a common time interval, we note that the choice of $T_z^\star$ in Corollary 2.16 not only needs to ensure the existence of a solution $g_z^\star$ to (2.60), but also that (2.61) holds, i.e., that the right-hand side of (2.61) is a true martingale. Thus in principle, even if (2.60) admits a solution $g_z^\star$ on $[0, T]$, (2.61) need not hold. However, by using the second part of Theorem 2.4, we show in the following lemma that (2.61) also holds on a common time interval for all $z \in a + \mathrm{i}\mathbb{R}$.

Lemma 2.24. *Let $a \in \mathbb{R}$ and fix $T_a^\star \in (0, T^\star]$ as given by Corollary 2.16. Then* (2.61) *holds for all $0 \le t \le T \le T_a^\star$ and all $z \in a + \mathrm{i}\mathbb{R}$.*

Proof. Fix $T \in [0, T_a^\star]$ and $z \in a + \mathrm{i}\mathbb{R}$. Note that

$$U := \exp\left(zX_T + N_T^{Q^\star} - \frac{[N^{Q^\star}]_T}{2} \right) = Z_T^{Q^\star} e^{zX_T},$$

where we recall the process $N^{Q^\star}$ as well as the dynamics (2.58) and (2.59) from the proof of Proposition 2.15. In particular, we have

$$E[|U|] = E[|Z_T^{Q^\star} e^{zX_T}|] = E_{Q^\star}[|e^{zX_T}|] = E_{Q^\star}[e^{aX_T}] < \infty$$

since $T \leq T_a^\star$, so that U is integrable. We have already shown in the proofs of Proposition 2.15 and Corollary 2.16 that the equation (2.18) associated with U simplifies in this case to (2.60). By Theorem 2.23, there exists a solution $g_z^\star : [0, T_a^\star] \to \mathbb{C}$ to (2.60) on $[0, T_a^\star]$ for each $z \in a + \mathrm{i}\mathbb{R}$. Thus since U is integrable, part 2) of Theorem 2.4 yields

$$E[U \mid \mathcal{F}_t] = \exp\left(zX_t + \int_t^T g_z^\star(T-u)\xi_t(u)du \right), \quad 0 \leq t \leq T,$$

which is precisely (2.61). □

We are now ready to relax the assumptions of Corollary 2.19.

Proposition 2.25. *Suppose that $T \in (0, T^\star]$, $a_1 > 1$ and $a_2 < 0$ are such that*

$$T \leq \min\{T_{a_1}^\star, T_{a_2}^\star, T_{a_1+\delta}^\star, T_{a_2-\delta}^\star\} \tag{2.92}$$

for some $\delta > 0$, where $T_z^\star$ is given by Corollary 2.16. Then the mean value processes $V^{C_{K,T}}$, $V^{P_{K,T}}$ and pure hedge coefficients $\Xi^{C_{K,T}}$, $\Xi^{P_{K,T}}$ for the European call and put options $C_{K,T}$ and $P_{K,T}$ from Example 2.9 are given by (2.76).

Proof. Since $T \leq \min\{T_{a_1+\delta}^\star, T_{a_2-\delta}^\star\}$, $S_T^{a_1+\delta}$ and $S_T^{a_2-\delta}$ are integrable by Corollary 2.16. Moreover, by Theorem 2.23 and Lemma 2.24, we may set $T_{a+\mathrm{i}b}^\star = T_a^\star$ for $a \in \{a_1, a_2\}$ and $b \in \mathbb{R}$ so that the conclusion of Corollary 2.16 still holds. Therefore $\inf_{b\in\mathbb{R}} T_{a+\mathrm{i}b}^\star = T_a^\star$ for $a \in \{a_1, a_2\}$ and the assumptions of Corollary 2.16 are satisfied, which yields (2.76). □

With Proposition 2.25, we have achieved our goal of proving for the European call and put options that the formulas in (2.76) hold on a nontrivial time interval, since each of the constants on the right-hand side of (2.92) is strictly positive.

We conclude this section by using Proposition 2.2 to obtain an explicit formula for the mean squared hedging error associated with European call and put options.

Calculating this error is of independent interest, but also useful for the application to semistatic portfolio problems that we consider in the next section. We start by showing an auxiliary result on the conditional expectation of terms of the form $L_u \tilde{V}_u^{z_1} \tilde{V}_u^{z_2} Y_u$ for $0 \le u \le T$ and $z_1, z_2 \in \mathbb{C}$.

Proposition 2.26. *For $z_1, z_2 \in \mathbb{C}$, there exists $\tilde{T}_{z_1,z_2} \in (0, T^\star_{z_1} \wedge T^\star_{z_2}]$ such that for all $0 \le u \le T \le \tilde{T}_{z_1,z_2}$, there exist unique solutions $\tilde{g}^{u,T}_{z_1,z_2}, \tilde{h}^{u,T}_{z_1,z_2} \in L^1([0,u];\mathbb{C})$ to the equations*

$$\tilde{g}^{u,T}_{z_1,z_2}(t) = \tilde{f}^g_{z_1,z_2}\big((\hat{\kappa} * \tilde{g}^{u,T}_{z_1,z_2})(t); (\Delta_t \hat{\kappa} * g^\star_{z_1,z_2})(T-u)\big), \tag{2.93}$$

$$\tilde{h}^{u,T}_{z_1,z_2}(t) = \tilde{f}^h_{z_1,z_2}\big((\hat{\kappa} * \tilde{h}^{u,T}_{z_1,z_2})(t); (\hat{\kappa} * \tilde{g}^{u}_{z_1,z_2})(t), (\Delta_t \hat{\kappa} * g^\star_{z_1,z_2})(T-u), \hat{\kappa}(t)\big) \tag{2.94}$$

for $0 \le t \le u$, where $g^\star_{z_1,z_2} = g^\star_{z_1} + g^\star_{z_2} + g^\star$ for $g^\star_z$ and $g^\star$ given by Corollary 2.16 and part 1) of Theorem 2.3, respectively, $\Delta_t \hat{\kappa} : (0,\infty) \to \mathbb{R}_+$ is defined by $\Delta_t \hat{\kappa}(s) = \hat{\kappa}(t+s)$, and $\tilde{f}^g_{z_1,z_2} : \mathbb{C}^2 \to \mathbb{C}$ and $\tilde{f}^h_{z_1,z_2} : \mathbb{C}^4 \to \mathbb{C}$ are defined by

$$\tilde{f}^g_{z_1,z_2}(y;w) = (z_1+z_2)\left(\mu + \frac{(z_1+z_2-1)\sigma^2}{2} + \varrho\sigma(y+w)\right) + \frac{(y+w)^2}{2},$$

$$\tilde{f}^h_{z_1,z_2}(x;y,w,\ell) = \big((z_1+z_2)\varrho\sigma + w + y\big)(\ell + x).$$

Moreover, for any $T \in (0, \tilde{T}_{z_1,z_2}]$ and $u \in [0,T]$, it holds that $L_u \tilde{V}_u^{z_1} \tilde{V}_u^{z_2} Y_u$ is integrable and

$$E_P[L_u \tilde{V}_u^{z_1} \tilde{V}_u^{z_2} Y_u \mid \mathcal{F}_t] = Z_t^{u,T,z_1,z_2} := \tilde{Z}_t^{u,T,z_1,z_2} \xi_t^{u,T,z_1,z_2}, \quad 0 \le t \le u, \tag{2.95}$$

where $(\tilde{Z}_t^{u,T,z_1,z_2})_{0 \le t \le T}$ and $(\xi_t^{u,T,z_1,z_2})_{0 \le t \le T}$ are defined by

$$\tilde{Z}_t^{u,T,z_1,z_2} = S_t^{z_1+z_2} \exp\left(\int_u^T g^\star_{z_1,z_2}(T-r)\xi_t(r)dr + \int_t^u \tilde{g}^{u,T}_{z_1,z_2}(u-r)\xi_t(r)dr\right),$$

$$\xi_t^{u,T,z_1,z_2} = \xi_t(u) + \int_t^u \tilde{h}^{u,T}_{z_1,z_2}(u-r)\xi_t(r)dr, \quad 0 \le t \le T.$$

Remark 2.27. Although it may seem at first glance that the left-hand side of (2.95) does not depend on T, note that the opportunity process L and the process $\tilde{V}^z$ (which may be seen as the mean value process for $H := S_T^z$) implicitly depend on the time horizon T; this becomes apparent from (2.14) and (2.77). On the other hand, the solutions $g^\star$ and $g^\star_z$ to (2.14) and (2.60), respectively, do not depend on T.

Remark 2.28. Since $g^\star_{z_1,z_2}$, $\tilde{f}^g_{z_1,z_2}$ and $\tilde{f}^h_{z_1,z_2}$ are symmetric in (z_1, z_2), it follows by the uniqueness of the solutions to (2.93) and (2.94) that $\tilde{g}^{u,T}_{z_1,z_2}$ and $\tilde{h}^{u,T}_{z_1,z_2}$ are also symmetric in (z_1, z_2), i.e., we have $\tilde{g}^{u,T}_{z_1,z_2} = \tilde{g}^{u,T}_{z_2,z_1}$ and $\tilde{h}^{u,T}_{z_1,z_2} = \tilde{h}^{u,T}_{z_2,z_1}$. Therefore, $\tilde{Z}^{u,T,z_1,z_2}$, ξ^{u,T,z_1,z_2} and Z^{u,T,z_1,z_2} are also symmetric in (z_1, z_2). This observation will be useful later for the proof of Proposition 3.15.

Proof of Proposition 2.26. Fix $z_1, z_2 \in \mathbb{C}$ and write $\bar{T} := T^\star_{z_1} \wedge T^\star_{z_2}$. Consider the indexing set

$$\Phi := \big\{(u, T, j) : 0 \leq u \leq T \leq \bar{T}, j \in \{1, 2\}\big\}$$

with index $\varphi = (u, T, j)$. For now, we only view u and T as parameters. For each $\varphi \in \Phi$, consider the process $(\tilde{X}^\varphi_t)_{0\leq t\leq \bar{T}}$ defined for $0 \leq t \leq \bar{T}$ by

$$\begin{aligned}\tilde{X}^{u,T,1}_t &= \int_u^T g^\star_{z_1,z_2}(T-r)\xi_{t\wedge u}(r)dr + (z_1+z_2)X_{t\wedge u},\\ \tilde{X}^{u,T,2}_t &= \xi_t(u).\end{aligned} \tag{2.96}$$

It is clear that the processes $\tilde{X}^{u,T,1}$ and $\tilde{X}^{u,T,2}$ are constant on $[u, \bar{T}]$. Moreover, due to (2.1), (2.2) and part 1) of Lemma 2.6 with $\nu(dr) = g^\star_{z_1,z_2}(T-r)\mathbf{1}_{[u,T]}(r)dr$, they have on $[0, u]$ the dynamics

$$\begin{aligned}d\tilde{X}^{u,T,1}_t = &\left(\int_u^T g^\star_{z_1,z_2}(T-r)\hat{\kappa}(r-t)dr\right)\sqrt{Y_t}dB_t\\ &+ (z_1+z_2)\left(\left(\mu - \frac{\sigma^2}{2}\right)Y_t dt + \sigma\sqrt{Y_t}dW_t\right), \quad 0 \leq t \leq u,\end{aligned}$$

and

$$d\tilde{X}^{u,T,2}_t = \hat{\kappa}(u-t)\sqrt{Y_t}dB_t, \quad 0 \leq t \leq u.$$

Thus the dynamics of $\tilde{X}^\varphi$ on $[0, \bar{T}]$ can be written in the form

$$d\tilde{X}^\varphi_t = \mu_\varphi(t)Y_t dt + \sigma_\varphi(t)\sqrt{Y_t}dW_t + \tilde{\sigma}_\varphi(t)\sqrt{Y_t}dW^\perp_t, \quad 0 \leq t \leq \bar{T},$$

where we define $\mu_\varphi, \sigma_\varphi, \tilde{\sigma}_\varphi : [0, \bar{T}] \to \mathbb{C}$ by $\mu_{u,T,2} \equiv 0$ and

$$\begin{aligned}\mu_{u,T,1}(t) &= (z_1+z_2)\left(\mu - \frac{\sigma^2}{2}\right)\mathbf{1}_{[0,u]}(t),\\ \sigma_{u,T,1}(t) &= \left((z_1+z_2)\sigma + \varrho\int_u^T g^\star_{z_1,z_2}(T-r)\hat{\kappa}(r-t)dr\right)\mathbf{1}_{[0,u]}(t),\\ \sigma_{u,T,2}(t) &= \varrho\hat{\kappa}(u-t)\mathbf{1}_{[0,u]}(t),\end{aligned}$$

$$\tilde{\sigma}_{u,T,1}(t) = \sqrt{1-\varrho^2}\mathbf{1}_{[0,u]}(t)\int_u^T g^\star_{z_1,z_2}(T-r)\hat{\kappa}(r-t)dr,$$
$$\tilde{\sigma}_{u,T,2}(t) = \sqrt{1-\varrho^2}\hat{\kappa}(u-t)\mathbf{1}_{[0,u]}(t). \tag{2.97}$$

It is clear that the family $(\mu_\varphi)_{\varphi\in\Phi}$ is bounded in L^∞ and hence uniformly integrable. Since $\hat{\kappa}$ is locally square-integrable (see after (2.2)) and hence in $L^2([0,\bar{T}];\mathbb{R}_+)$, we get by the ϵ-δ-criterion for uniformly integrability (see Klenke [83, Theorem 6.24]) that $(|\sigma_{u,T,2}|^2)_{0\le u\le T\le\bar{T}}$ and $(|\tilde{\sigma}_{u,T,2}|^2)_{0\le u\le T\le\bar{T}}$ are uniformly integrable as the indicator functions are bounded. Moreover, since $g^\star_{z_1,z_2}$ is continuous (see Corollary 2.16 and part 1) of Theorem 2.3), hence bounded on $[0,\bar{T}]$, we have

$$\left|\int_u^T g^\star_{z_1,z_2}(T-r)\hat{\kappa}(r-t)dr\right| \le \|g^\star_{z_1,z_2}\|_{L^\infty(0,\bar{T})}\|\hat{\kappa}\|_{L^1(0,\bar{T})} < \infty$$

for all $t\in[0,u]$ and all $0\le u\le T\le\bar{T}$. Thus the families $(|\sigma_{u,T,1}|^2)_{0\le u\le T\le\bar{T}}$ and $(|\tilde{\sigma}_{u,T,1}|^2)_{0\le u\le T\le\bar{T}}$ are uniformly bounded in $L^\infty([0,\bar{T}];\mathbb{C})$, and hence uniformly integrable. Therefore, we may apply Proposition 2.5 with respect to the indexing set Φ and families $(\mu_\varphi),(\sigma_\varphi)$ and $(\tilde{\sigma}_\varphi)$. This yields some $\tilde{T}_{z_1,z_2}\in(0,T^\star_{z_1}\wedge T^\star_{z_2}]$ such that for all $\varphi\in\Phi$ and $T'\in(0,\tilde{T}_{z_1,z_2}]$, statements 1)–3) of Proposition 2.5 hold (with T' in place of T). Now fix $0\le u\le T\le\tilde{T}_{z_1,z_2}$ and set

$$(\varphi_1,\varphi_2,T') := \big((u,T,1),(u,T,2),u\big)\in\Phi\times\Phi\times[0,\tilde{T}_{z_1,z_2}].$$

Since a change of variables yields

$$\begin{aligned}\int_u^T g^\star_{z_1,z_2}(T-r)\hat{\kappa}(r-t)dr &= \int_0^{T-u} g^\star_{z_1,z_2}(r')\hat{\kappa}\big(u-t+(T-u)-r'\big)dr' \\ &= (\Delta_{u-t}\hat{\kappa}*g^\star_{z_1,z_2})(T-u),\quad 0\le t\le u,\end{aligned}$$

we have after plugging in (2.97) and some simplifications that the equations (2.22) and (2.24) for $g_{\varphi_1,u} =: \tilde{g}^{u,T}_{z_1,z_2}$ and $\tilde{g}_{\varphi_1,\varphi_2,u} =: \tilde{h}^{u,T}_{z_1,z_2}$ are equivalent to (2.93) and (2.94), respectively; we note that the indicator functions in the definition (2.97) of $(\mu_\varphi),(\sigma_\varphi)$ and $(\tilde{\sigma}_\varphi)$ do not appear in (2.93) and (2.94) since we have restricted the equations to the interval $[0,u]$. This shows the existence and uniqueness of the solutions to (2.93) and (2.94).

To conclude, we note that setting $t = T' = u$ in (2.96) yields

$$\tilde{X}_u^{\varphi_1} = \int_u^T g^\star_{z_1,z_2}(T-r)\xi_u(r)dr + (z_1+z_2)X_u,$$
$$\tilde{X}_u^{\varphi_2} = Y_u$$

for $\varphi_\ell = (u, T, \ell)$ as before. Thus we have

$$\begin{aligned}\exp(\tilde{X}_u^{\varphi_1})\tilde{X}_u^{\varphi_2} &= \exp\left(\int_u^T g^\star_{z_1,z_2}(T-r)\xi_u(r)dr + (z_1+z_2)X_u\right)Y_u \\ &= L_u \tilde{V}_u^{z_1}\tilde{V}_u^{z_2}Y_u\end{aligned}$$

due to (2.13), (2.77) and the definition of $g^\star_{z_1,z_2} = g^\star_{z_1} + g^\star_{z_2} + g^\star$. Therefore (2.95) follows directly from (2.26) (with time horizon $T' = u$ in place of T). □

We can now apply Proposition 2.26 together with the general result in Proposition 2.2 to obtain an explicit formula for the mean squared hedging error associated with European vanilla call and put options. We note that the same issues discussed after Corollary 2.22 apply to the formulas below. That is, the infimum on the right-hand side of (2.98) could a priori be equal to 0 in some cases, so that the result becomes vacuously true. While that issue was resolved in Proposition 2.25 for the mean value processes and pure hedge coefficients associated with European vanilla call and put options, the question remains open for the hedging error.

Theorem 2.29. *Suppose that $T \in (0, T^\star]$, $a_1 > 1$ and $a_2 < 0$ are such that*

$$T \le \min\{T^\star_{a_1}, T^\star_{a_2}, T^\star_{a_1+\delta}, T^\star_{a_2-\delta}\} \wedge \inf\left\{\tilde{T}_{z_1,z_2} : z_1, z_2 \in \{a_1, a_2\} + \mathrm{i}\mathbb{R}\right\} \tag{2.98}$$

for some $\delta > 0$, where $T^\star_z$ is given by Corollary 2.16 and $\tilde{T}_{z_1,z_2}$ by Proposition 2.26. Then for any $t \in [0, T]$ and initial wealth $x_t \in L^2_P(\mathcal{F}_t)$, the mean squared hedging errors for the call and put options with strike K are given by

$$\varepsilon_t^2(x_t, C_T^K) = L_t(V_t^{C_T^K} - x_t)^2 + \varepsilon_t^2(C_T^K),$$
$$\varepsilon_t^2(x_t, P_T^K) = L_t(V_t^{P_T^K} - x_t)^2 + \varepsilon_t^2(P_T^K),$$

where we define

$$\varepsilon_t^2(C_T^K) := (1-\varrho^2)\int_t^T \bigg(\iint_{(a_1+\mathrm{i}\mathbb{R})^2}\Big(\prod_{m=1,2} h^\star_{z_m}(T-u)\Big) Z_t^{u,z_1,z_2}\pi^{\otimes 2}_{K,a_1}(dz_1,dz_2)\bigg)du,$$

$$\varepsilon_t^2(P_T^K) := (1-\varrho^2)\int_t^T \bigg(\iint_{(a_2+\mathrm{i}\mathbb{R})^2}\Big(\prod_{m=1,2} \tilde{h}^\star_{z_m}(T-u)\Big) Z_t^{u,z_1,z_2}\pi^{\otimes 2}_{K,a_2}(dz_1,dz_2)\bigg)du$$

for $h^\star_{z_m} := \hat{\kappa} * g^\star_{z_m}$, $\pi^{\otimes 2}_{K,a_\ell} := \pi_{K,a_\ell}\otimes\pi_{K,a_\ell}$ *and* Z^{u,z_1,z_2} *given by* (2.95).

Proof. Let $H_1 := C_T^K$ and $H_2 := P_T^K$. Recall that the European call and put options can be represented in terms of the Mellin transforms (2.35) and (2.36) so that due to (2.40), they are of the form (2.42) with $\nu_1^z := z\delta_T$, $\nu_2^z := 0$ and $\pi^{H_\ell} = \pi_{K,a_\ell}$. By Proposition 2.25 and due to the assumption (2.98), Proposition 2.18 applies to H_ℓ for $\ell\in\{1,2\}$. After plugging in these particular choices of ν_1^z, ν_2^z and π^{H_ℓ} together with $g^\star_{\nu_1^z,\nu_2^z} = g^\star_z$ (see the beginning of the proof of Corollary 2.16), (2.70) and (2.75) simplify to $\varphi_t^z = h_z^\star(T-t)$ and

$$dV_t^{H_\ell} = \sqrt{Y_t}\bigg(\sigma\int_{a_\ell+\mathrm{i}\mathbb{R}} z\tilde{V}_t^z\pi_{K,a_\ell}(dz)\,dW_t^\star + \int_{a_\ell+\mathrm{i}\mathbb{R}} h_z^\star(T-t)\tilde{V}_t^z\pi_{K,a_\ell}(dz)\,dB_t^\star\bigg)$$

for $\ell\in\{1,2\}$ and $0\le t\le T$. By (2.78), we also have $\tilde{\Xi}_t^z = (z\sigma+\varrho h_z^\star(T-t))\tilde{V}_t^z$. Recalling that $dS_t = \sigma S_t\sqrt{Y_t}dW_t^\star$ by Theorem 2.3, it follows that

$$d[V^{H_\ell} - \Xi^{H_\ell}\bullet S]_t = (1-\varrho^2)\bigg(\int_{a_\ell+\mathrm{i}\mathbb{R}} h_z^\star(T-t)\tilde{V}_t^z\pi_{K,a_\ell}(dz)\bigg)^2 dt,\quad 0\le t\le T,$$

for $\ell\in\{1,2\}$. By Proposition 2.2 with $t=0$, we have that $\int_0^T L_t d[V^{H_\ell}-\Xi^{H_\ell}\bullet S]_t$ is integrable. Hence by the conditional Fubini theorem, we deduce that

$$\begin{aligned} &E_P\bigg[\int_t^T L_u d[V^{H_\ell}-\Xi^{H_\ell}\bullet S]_u\,\bigg|\,\mathcal{F}_t\bigg] \\ &= (1-\varrho^2)\int_t^T E_P\bigg[L_u\bigg(\int_{a_\ell+\mathrm{i}\mathbb{R}} h_z^\star(T-u)\tilde{V}_u^z\pi_{K,a_\ell}(dz)\bigg)^2\,\bigg|\,\mathcal{F}_t\bigg]du. \end{aligned}$$

After rewriting the inner integral as

$$\bigg(\int_{a_\ell+\mathrm{i}\mathbb{R}} h_z^\star(T-u)\tilde{V}_u^z\pi_{K,a_\ell}(dz)\bigg)^2 = \iint_{(a_\ell+\mathrm{i}\mathbb{R})^2}\Big(\prod_{m=1,2} h^\star_{z_m}(T-u)\tilde{V}_u^{z_m}\Big)\pi^{\otimes 2}_{K,a_\ell}(dz_1,dz_2),$$

the conditional Fubini theorem yields

$$E_P\bigg[\int_t^T L_u d[V^{H_\ell}-\Xi^H\bullet S]_u\,\bigg|\,\mathcal{F}_t\bigg]$$
$$=(1-\varrho^2)\int_t^T\iint_{(a_\ell+\mathrm{i}\mathbb{R})^2}\Big(\prod_{m=1,2}h^\star_{z_m}(T-u)\Big)E_P[L_u\tilde V_u^{z_1}\tilde V_u^{z_2}Y_u\,|\,\mathcal{F}_t]\pi^{\otimes 2}_{K,a_\ell}(dz_1,dz_2)du.$$

By plugging in (2.95), we obtain

$$E\bigg[\int_t^T L_u d[V^{H_\ell}-\Xi^{H_\ell}\bullet S]_u\,\bigg|\,\mathcal{F}_t\bigg]=\varepsilon_t^2(H_\ell),$$

and hence the result follows by Proposition 2.2. □

Since we already have the formulas (2.13) for L and (2.76) for $V^{C_T^K}$ and $V^{P_T^K}$, Theorem 2.29 can therefore be used to calculate the hedging errors associated with european call and put options in a semi-explicit form.

3 An application to semistatic portfolio problems

As an application of our previous results on the optimal mean–variance hedging of claims in the rough Heston model, we now consider the problems of mean–variance portfolio optimisation and mean–variance hedging in a semistatic setup which we define below. We start by studying those problems in a general semimartingale model, and then show how to obtain explicit formulas in the case of the rough Heston model.

Consider a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},P)$ satisfying the usual conditions with time horizon $T>0$; for simplicity, we assume that $\mathcal{F}_0$ is P-trivial. As before, we assume that there exist a riskless asset with constant price 1 as well as a risky asset with price process $(S_t)_{0\le t\le T}$, where S is a semimartingale that we assume to be continuous, which is the case in the rough Heston model. As in Černý/Kallsen [25, Assumption 2.1], we also make the following **standing assumption** for this section.

Assumption 3.1. There exists an equivalent local martingale measure Q for S with square-integrable density dQ/dP.

By Delbaen/Schachermayer [36, Theorem 1.3] and as explained in Section I.3.1, it follows from the continuity of S that there exists a variance-optimal martingale measure $Q^\star\approx P$ for S. Assumption 3.1 holds for the rough Heston model with any time horizon $T \in (0, T^\star]$, where $T^\star > 0$ is given in Theorem 2.3, so that $Q^\star$ given by (2.16) is the variance-optimal martingale measure on $[0, T]$. By results of [25] and as discussed in the introduction of the chapter, Assumption 3.1 ensures that the space

$$\mathcal{G}_T(S) = \{\vartheta \bullet S_T : \vartheta \in \overline{\Theta}_T(S)\} \subseteq L^2_P(\mathcal{F}_T)$$

of attainable gains is closed in $L^2_P(\mathcal{F}_T)$, and also that for every payoff $H \in L^2_P(\mathcal{F}_T)$ and $x \in \mathbb{R}$, there exists a unique mean–variance hedging strategy $\vartheta^H(x)$.

In order to introduce semistatic portfolios, we consider an enlargement of the financial market where in addition to the underlying asset S, we suppose that an agent may trade in a basket $\vec{B} = (B_1, \ldots, B_J)$ of financial derivatives with terminal values[2] $B_j \in L^2_P(\mathcal{F}_T)$ and prices $p_j \in \mathbb{R}$ at time 0 for $j = 1, \ldots, J$; we likewise write $\vec{p} = (p_1, \ldots, p_J)$. Whereas the agent may trade in S with a "dynamic" strategy $\vartheta \in \overline{\Theta}_T(S)$ as before, we assume that she may take only "static" positions in the derivatives B_j, i.e., they can be bought or sold at time 0 and are subsequently held to maturity T. Since we assume that $\mathcal{F}_0$ is P-trivial, the static part of the strategy is represented by a vector $w \in \mathbb{R}^J$, which denotes the number of units of each derivative bought or sold at time 0. By trading in a self-financing way with a *semistatic strategy* $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$, an agent with initial wealth $x \in \mathbb{R}$ attains the *terminal wealth*

$$V_T(x, w, \vartheta) = x + w^\top(\vec{B} - \vec{p}) + \vartheta \bullet S_T. \tag{3.1}$$

We refer to this enlarged market as the *semistatic market* (or "semistatic setup"), and to the original market as the *dynamic market*. While Assumption 3.1 implies the absence of arbitrage in the dynamic market, there may a priori exist arbitrage opportunities using semistatic strategies. We do not need to assume the absence of arbitrage in the enlarged market, but we do require the following **standing assumption**, which rules out the existence of arbitrage opportunities that attain a nonzero terminal wealth with zero variance.

Assumption 3.2. For any pair $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ such that $V_T(0, w, \vartheta) = c$ P-a.s. for some $c \in \mathbb{R}$, we have $c = 0$.

Note that for $w = 0$, Assumption 3.2 implies in particular that $1 \notin \mathcal{G}_T(S)$;

[2] The notation for the basket of claims $\vec{B} = (B_1, \ldots, B_J)$ is unrelated to the Brownian motion B underlying the rough Heston model (2.1); this distinction will always be clear from the context.

this also follows already from Assumption 3.1. We obtain in Lemma 3.10 below an equivalent condition to Assumption 3.2 that can be stated explicitly in terms of quantities related to S and $\vec{B}$. Since the derivatives $\vec{B}$ can only be traded at time 0, Assumptions 3.1 and 3.2 imply that the so-called *law of one price* (see Černý/Czichowsky [24]) holds for the market $(S, \vec{B})$. Indeed, it may be possible to weaken Assumption 3.1 by imposing only the law of one price ([24, Definition 2.5]; see also (i)⇔(v) in [24, Theorem 3.1]) to the dynamic market generated by S, but we do not pursue this further.

The topic of hedging with semistatic strategies has received some attention in recent years; see e.g. Acciaio et al. [4], Acciaio/Larsson [3] and Nutz et al. [99]. Particularly relevant for us are Di Tella et al. [38, 39], where the problem of mean–variance hedging with semistatic strategies has been studied for general stochastic volatility models under a risk-neutral measure. The restriction of taking only static positions is justified in markets characterised by low liquidity or high transaction costs, such as certain over-the-counter markets, where frequent rebalancing may be impossible or too costly. The latter point is further emphasised in [39], where the problem of constructing a sparse portfolio is considered, i.e., one that only takes nonzero positions in a small subset of the derivatives (for instance, in order to reduce trading costs).

3.1 Semistatic portfolio optimisation

Since our study of the rough Heston model is done under a semimartingale measure P, we are interested in the mean–variance portfolio optimisation problem as well as mean–variance hedging. Thus our first goal is to characterise the strategies $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ that are mean–variance efficient in the semistatic setup, in the following sense.

Definition 3.3. Let $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ be a semistatic strategy. We say that ϑ is *mean–variance efficient with respect to* w if there does not exist any other $\tilde{\vartheta} \in \overline{\Theta}_T(S)$ such that we have the inequalities

$$\begin{cases} E_P[V_T(0, w, \tilde{\vartheta})] \geq E_P[V_T(0, w, \vartheta)], \\ \mathrm{Var}_P[V_T(0, w, \tilde{\vartheta})] \leq \mathrm{Var}_P[V_T(0, w, \vartheta)], \end{cases} \tag{3.2}$$

where at least one of the inequalities is strict. Likewise, we say that the pair $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ is *mean–variance efficient* if there does not exist any other pair $(\tilde{w}, \tilde{\vartheta}) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ such that we have the inequalities

$$\begin{cases} E_P[V_T(0, \tilde{w}, \tilde{\vartheta})] \geq E_P[V_T(0, w, \vartheta)], \\ \mathrm{Var}_P[V_T(0, \tilde{w}, \tilde{\vartheta})] \leq \mathrm{Var}_P[V_T(0, w, \vartheta)], \end{cases} \tag{3.3}$$

where at least one of the inequalities is strict.

The concept of mean–variance efficiency is classic, and it is based on the principle that a strategy is "good" (from a mean–variance point of view) if it attains a terminal wealth with high expected value and low variance. In particular, since the strategy $(0, 0)$ attains a terminal wealth with zero variance, it must be mean–variance efficient due to Assumption 3.2. More generally, if $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ is such that $V_T(0, w, \vartheta) = 0$ P-a.s., then (w, ϑ) is mean–variance efficient as well.

There are a number of equivalent definitions for mean–variance efficiency, some of which will be useful later in order to calculate the candidate mean–variance efficient strategies and prove that they are indeed efficient. We refer to Eberlein/Kallsen [44, Rule 10.43] for some of those conditions; in our setup, the first four equivalent conditions given in the rule read as follows.

Lemma 3.4. *Let $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$. Then the following statements are equivalent:*

(∗) *(w, ϑ) is mean–variance efficient.*

(a) *Either $V_T(0, w, \vartheta) = 0$ P-a.s., or $E_P[V_T(0, w, \vartheta)] > 0$ and (w, ϑ) maximises $E_P[V_T(0, \tilde{w}, \tilde{\vartheta})]$ among all pairs $(\tilde{w}, \tilde{\vartheta}) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ such that*

$$\mathrm{Var}_P[V_T(0, \tilde{w}, \tilde{\vartheta})] \leq \mathrm{Var}_P[V_T(0, w, \vartheta)].$$

(b) *(w, ϑ) minimises $\mathrm{Var}_P[V_T(0, \tilde{w}, \tilde{\vartheta})]$ among all pairs $(\tilde{w}, \tilde{\vartheta}) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ such that*

$$E_P[V_T(0, \tilde{w}, \tilde{\vartheta})] \geq E_P[V_T(0, w, \vartheta)].$$

(c) *For some $m \geq 0$, (w, ϑ) minimises $E_P[(m - V_T(0, \tilde{w}, \tilde{\vartheta}))^2]$ among all pairs $(\tilde{w}, \tilde{\vartheta}) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$.*

(d) *Either $V_T(0, w, \vartheta) = 0$ P-a.s., or (w, ϑ) maximises the* Sharpe ratio

$$SR(\tilde{w}, \tilde{\vartheta}) := \frac{E_P[V_T(0, \tilde{w}, \tilde{\vartheta})]}{(\mathrm{Var}_P[V_T(0, \tilde{w}, \tilde{\vartheta})])^{1/2}}$$

among all pairs $(\tilde{w}, \tilde{\vartheta}) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$*, where we set* $0/0 = 0$.

Proof. We can show the equivalence of (a)–(d) by the same proof as in [44, Rule 10.43]; this is possible due to the fact that the terminal wealth $V_T(0, w, \vartheta)$ is linear in (w, ϑ) and the set of admissible strategies $\mathbb{R}^J \times \overline{\Theta}_T(S)$ is a cone (in our case, it is even a vector space). Two more comments are needed in order to make the reasoning fully precise. First, we note that Assumption 3.2 is necessary in the proof of (d) $\Rightarrow$ (a), since otherwise a Sharpe ratio equal to ∞ is attainable and one cannot assume that a competitor $(\tilde{w}, \tilde{\vartheta})$ satisfies $\mathrm{Var}_P[V_T(0, \tilde{w}, \tilde{\vartheta})] > 0$. Second, the converse direction (a) $\Rightarrow$ (d) in [44] only holds with this modified version of (a), where we deal separately with the case $E_P[V_T(0, w, \vartheta)] = 0$. Otherwise, the implication would not hold in the case where $E_P[V_T(0, \tilde{w}, \tilde{\vartheta})] = 0$ for all $(\tilde{w}, \tilde{\vartheta}) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ (i.e., if P is a martingale measure for the extended market). In that case, all strategies satisfy the original statement of [44, Rule 10.43(a)], but the only mean–variance efficient strategies are those that attain the terminal wealth 0. With these additional remarks, the proof of [44, Rule 10.43] goes through and gives the equivalence of (a)–(d).

It remains to show that (a)–(d) are equivalent to $(*)$. If (w, ϑ) is a mean–variance efficient strategy in the sense of Definition 3.3, then it clearly satisfies condition (a), so $(*) \Rightarrow$ (a). Conversely, suppose that (w, ϑ) satisfies both (a) and (b). We consider two cases depending on which of the conditions in (a) is satisfied. If $E_P[V_T(0, w, \vartheta)] > 0$, it is clear from the assumptions (a) and (b) that (w, ϑ) is mean–variance efficient by definition. On the other hand, if $V_T(0, w, \vartheta) = 0$, we have by Assumption 3.2 that there does not exist a competitor $(\tilde{w}, \tilde{\vartheta})$ such that $E_P[V_T(0, \tilde{w}, \tilde{\vartheta})] > 0$ and $\mathrm{Var}_P[V_T(0, \tilde{w}, \tilde{\vartheta})] = 0$; hence (w, ϑ) is mean–variance efficient also in this case. Since we had already shown (a) $\Leftrightarrow$ (b), we obtain $(*) \Rightarrow$ (a) $\Leftrightarrow$ (a)$\wedge$(b) $\Rightarrow (*)$ so that $(*)$ is equivalent to (a)–(d), and this concludes the proof. $\square$

Similarly, we also obtain equivalent conditions for the mean–variance efficiency of $\vartheta \in \overline{\Theta}_T(S)$ with respect to $w = 0$; such a strategy can be seen as mean–variance efficient in the purely dynamic market. By the equivalence of the conditions (a)–(e) in [44, Rule 10.43] and the same argument as for (w, ϑ), we get the following.

Lemma 3.5. *Let* $\vartheta \in \overline{\Theta}_T(S)$*. Then the following statements are equivalent:*

$(*)$ ϑ *is mean–variance efficient with respect to* $w = 0$.

(a) *Either* $V_T(0, 0, \vartheta) = 0$ P*-a.s., or* $E_P[V_T(0, 0, \vartheta)] > 0$ *and* ϑ *maximises*

$E_P[V_T(0,0,\tilde{\vartheta})]$ *among all strategies* $\tilde{\vartheta} \in \overline{\Theta}_T(S)$ *such that*

$$\mathrm{Var}_P[V_T(0,0,\tilde{\vartheta})] \leq \mathrm{Var}_P[V_T(0,0,\vartheta)].$$

(b) ϑ *minimises* $\mathrm{Var}_P[V_T(0,0,\tilde{\vartheta})]$ *among all strategies* $\tilde{\vartheta} \in \overline{\Theta}_T(S)$ *such that*

$$E_P[V_T(0,0,\tilde{\vartheta})] \geq E_P[V_T(0,0,\vartheta)].$$

(c) *For some* $m \geq 0$, ϑ *minimises* $E_P[(m - V_T(0,0,\tilde{\vartheta}))^2]$ *among all strategies* $\tilde{\vartheta} \in \overline{\Theta}_T(S)$.

(d) *Either* $V_T(0,0,\vartheta) = 0$ P*-a.s., or* ϑ *maximises the Sharpe ratio*

$$SR(0,\tilde{\vartheta}) = \frac{E_P[V_T(0,0,\tilde{\vartheta})]}{(\mathrm{Var}_P[V_T(0,0,\tilde{\vartheta})])^{1/2}}$$

among all strategies $\tilde{\vartheta} \in \overline{\Theta}_T(S)$, *where we set* $0/0 = 0$.

(e) $\vartheta =_S m\vartheta^\star$ *for some* $m \geq 0$, *where* $\vartheta^\star$ *denotes the solution to the pure investment problem*

$$E[(1 - \tilde{\vartheta} \bullet S_T)^2] \longrightarrow \min_{\tilde{\vartheta} \in \overline{\Theta}_T(S)} !$$

We note that condition (e) of Lemma 3.5 is equivalent to that of [44, Rule 10.43] in our setup, since by Assumption 3.1 and Černý/Kallsen [25, Corollary 2.5], $\vartheta \bullet S_T = 0$ implies that $\vartheta =_S 0 = 0\vartheta^\star$. Condition (e) of Lemma 3.5 is particularly useful when we consider only dynamic strategies. Indeed, we have already studied the pure investment problem for the rough Heston model in the previous chapter and obtained an explicit formula for $\vartheta^\star$; see Theorem 2.3. Thus condition (e) in Lemma 3.5 parametrises the set of strategies $\vartheta \in \overline{\Theta}_T(S)$ that are mean–variance efficient with respect to $w = 0$.

In order to find explicit formulas for the mean–variance efficient strategies in the semistatic market, our approach is to first determine the mean–variance efficient dynamic strategies $\vartheta \in \overline{\Theta}_T(S)$ with respect to a fixed static strategy $w \in \mathbb{R}^J$, and then to find the static strategies w that correspond to a mean–variance efficient pair (w, ϑ). For the first part, we rely on well-known results on mean–variance hedging, whereas the second part reduces to a finite-dimensional optimisation problem that can be solved explicitly. This approach is also similar to that of Di Tella et al. [38, 39], where the mean–variance hedging problem is decomposed into "inner" and "outer" problems that determine the dynamic and static parts of the optimal strategy, respectively. Our analysis also fits into the general framework of Fontana/Schweizer [48], from which we obtain more explicit results by using the particular structure of the set of terminal gains that are attainable by semistatic strategies.

We start by giving a decomposition of the derivatives B_j in $L^2_P(\mathcal{F}_T)$ that is helpful to our task. This is analogous to the Galtchouk–Kunita–Watanabe (GKW) decomposition for B_j under P considered in [39, Equation (7)], in the case where P is a risk-neutral measure. In general, the decomposition can be obtained by solving a mean–variance hedging problem. We point out for later use that the following result does not require Assumption 3.2, since for now we work only with the dynamic market.

Lemma 3.6. *Every payoff $H \in L^2_P(\mathcal{F}_T)$ admits a unique decomposition*

$$H = c + \vartheta \bullet S_T + H^\perp \tag{3.4}$$

for some $c \in \mathbb{R}$, $\vartheta \in \overline{\Theta}_T(S)$ and $H^\perp \in L^2_P(\mathcal{F}_T)$ such that

$$E_P[H^\perp] = E_P[H^\perp(\tilde{\vartheta} \bullet S_T)] = 0 \tag{3.5}$$

for all $\tilde{\vartheta} \in \overline{\Theta}_T(S)$. Moreover, $c = E_{Q^\star}[H]$ and $\vartheta = \vartheta^H(c)$ is the mean–variance hedging strategy for H starting from time 0 and initial capital c.

Proof. We consider the space of terminal gains attainable by dynamic strategies defined by

$$\mathcal{G}_T(S) = \{\vartheta \bullet S_T : \vartheta \in \overline{\Theta}_T(S)\} \subseteq L^2_P(\mathcal{F}_T).$$

Assumption 3.1 and Černý/Kallsen [25, Corollary 2.5.2] yield $\mathbb{R} \cap \mathcal{G}_T(S) = \{0\}$, which can be seen as a form of absence of arbitrage. Thus the space $L^2_P(\mathcal{F}_T)$ can be decomposed as the direct sum

$$L^2_P(\mathcal{F}_T) = \mathbb{R} \oplus \mathcal{G}_T(S) \oplus \big(\mathbb{R} \oplus \mathcal{G}_T(S)\big)^\perp. \tag{3.6}$$

This yields for any $H \in L^2_P(\mathcal{F}_T)$ a unique decomposition of the form

$$H = c + H_S + H^\perp \tag{3.7}$$

where $c \in \mathbb{R}$, $H_S \in \mathcal{G}_T(S)$ and $H^\perp \in (\mathbb{R} \oplus \mathcal{G}_T(S))^\perp$. Thus we have $H_S = \vartheta \bullet S_T$

for some $\vartheta \in \overline{\Theta}_T(S)$, and because $H^\perp \in (\mathbb{R} \oplus \mathcal{G}_T(S))^\perp$, the condition (3.5) on $H^\perp$ is satisfied. Hence we have obtained a decomposition of the form (3.4) for H.

Note that the choices of c, H_S and $H^\perp$ in (3.7) are unique. It remains to determine c and ϑ, and to show the uniqueness of ϑ. To see the latter, suppose that $\vartheta^i \bullet S_T = H_S$ for $\vartheta^1, \vartheta^2 \in \overline{\Theta}_T(S)$. Since there exists an ELMM $Q^\star$ for S with square-integrable density, we get by the same argument as in the proof of [25, Lemma 2.11] that the processes $\vartheta^1 \bullet S$ and $\vartheta^2 \bullet S$ are indistinguishable as their terminal values coincide. Thus up to S-equivalence, there is a unique choice of $\vartheta \in \overline{\Theta}_T(S)$ such that $\vartheta \bullet S_T = H_S$.

To determine c and ϑ, note that $c + \vartheta \bullet S_T$ is the L^2-projection of H onto $\mathbb{R} \oplus \mathcal{G}_T(S)$ by the decomposition (3.6). This implies that the pair (c, ϑ) attains the infimum

$$\inf_{(c,\vartheta)\in\mathbb{R}\times\overline{\Theta}_T(S)} E[(H - c - \vartheta \bullet S_T)^2] = \inf_{c\in\mathbb{R}} \inf_{\vartheta\in\overline{\Theta}_T(S)} E[(H - c - \vartheta \bullet S_T)^2].$$

The inner infimum on the right-hand side consists of a mean–variance hedging problem for H with initial wealth c, and hence we must have $\vartheta = \vartheta^H(c)$. Moreover, it follows from [25, Theorem 4.10.2] (see also Proposition 2.2) that the minimiser of the outer infimum is $c = V_0^H = E_{Q^\star}[H]$. □

We return to the main problem of finding mean–variance efficient semistatic portfolios. To that end, we introduce some notation. Lemma 3.6 yields for each derivative B_j the decomposition

$$B_j = c^{B_j} + \vartheta^{B_j} \bullet S_T + B_j^\perp, \tag{3.8}$$

and we denote $\vec{c} := (c^{B_1}, \dots, c^{B_J})$, $\vec{\vartheta} = (\vartheta^{B_1}, \dots, \vartheta^{B_J})$ and $\vec{B}^\perp := (B_1^\perp, \dots, B_J^\perp)$. Thus by (3.1), we can decompose the terminal wealth attained by a strategy $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ as

$$\begin{aligned} V_T(0, w, \vartheta) &= \sum_{j=1}^J w_j (c^{B_j} - p_j + \vartheta^{B_j} \bullet S_T + B_j^\perp) + \vartheta \bullet S_T \\ &= w^\top (\vec{c} - \vec{p}) + (w^\top \vec{\vartheta} + \vartheta) \bullet S_T + w^\top \vec{B}^\perp. \end{aligned} \tag{3.9}$$

We also recall the so-called *opportunity process* $(L_t)_{0\le t\le T}$ that was introduced in the previous chapter. The definition of L can be found in (I.3.2), and (2.13) gives an explicit formula for L in the rough Heston model. For this application, we need only the initial value L_0, which takes values in $(0, 1]$ by Černý/Kallsen [25, Lemma 3.10]. Moreover, in the special case $L_0 = 1$, we have by [25, Corollary 3.4] that $\vartheta^\star =_S 0$ and $L \equiv 1$. In that case, the so-called *adjustment process* $(a_t)_{0\le t\le T}$ is also null by [25, Lemma 3.7], and hence S is a local P-martingale due to [25, Proposition 3.13]. It also follows by [25, Lemma 3.2, part 3] that $E_P[\vartheta \bullet S_T] = 0$ for every $\vartheta \in \overline{\Theta}_T(S)$. For this reason, we sometimes need to consider the case $L_0 = 1$ separately.

Since it appears in condition (e) of Lemma 3.5, the pure investment strategy $\vartheta^\star \in \overline{\Theta}_T(S)$ plays a role in determining the mean–variance efficient strategies, and so we now compute the mean and variance of $V_T(0, 0, \vartheta^\star)$ in terms of L_0.

Lemma 3.7. *It holds that*

$$E_P[\vartheta^\star \bullet S_T] = 1 - L_0 \quad \textit{and} \quad \mathrm{Var}_P[\vartheta^\star \bullet S_T] = L_0(1 - L_0).$$

Proof. By [25, Corollary 3.4], we have the two equalities

$$L_0 = E_P[1 - \vartheta^\star \bullet S_T] = E_P[(1 - \vartheta^\star \bullet S_T)^2]. \tag{3.10}$$

This immediately yields $E_P[\vartheta^* \bullet S_T] = 1 - L_0$, and we also get

$$E_P[(\vartheta^\star \bullet S_T)^2] = L_0 + 2E_P[\vartheta^\star \bullet S_T] - 1 = 1 - L_0,$$

so that $\mathrm{Var}_P[\vartheta^\star \bullet S_T] = E_P[(\vartheta^\star \bullet S_T)^2] - E_P[\vartheta^\star \bullet S_T]^2 = L_0(1 - L_0)$. □

In order to calculate the mean and variance of the terminal wealth attained by semistatic portfolios, the last quantity that we need is the covariance matrix Σ^B of the residuals $\vec{B}^\perp$ defined by

$$\Sigma^B = (\Sigma^B_{ij})_{i,j=1}^J := \mathrm{Cov}_P(\vec{B}^\perp) = \big(\mathrm{Cov}_P(B_i^\perp, B_j^\perp)\big)_{i,j=1}^J.$$

The matrix Σ^B encodes the correlation structure of the unhedgeable parts of the derivatives B_j; we return later to the question of how to compute Σ^B explicitly for the rough Heston model. For any pair $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$, we obtain from (3.9) that

$$\begin{aligned} E_P[V_T(0, w, \vartheta)] &= E_P\big[w^\top(\vec{c} - \vec{p}) + (\vartheta + w^\top \vec{\vartheta}) \bullet S_T + w^\top \vec{B}^\perp\big] \\ &= w^\top(\vec{c} - \vec{p}) + E_P[(\vartheta + w^\top \vec{\vartheta}) \bullet S_T], \end{aligned} \tag{3.11}$$

$$\mathrm{Var}_P[V_T(0, w, \vartheta)] = \mathrm{Var}_P[(\vartheta + w^\top \vec{\vartheta}) \bullet S_T] + \mathrm{Var}_P[w^\top \vec{B}^\perp], \tag{3.12}$$

where the last equality follows by (3.5) since

$$\mathrm{Cov}_P(\hat{\vartheta} \bullet S_T, \vec{B}^{\perp}) = E_P[(\hat{\vartheta} \bullet S_T)\vec{B}^{\perp}] = 0$$

for $\hat{\vartheta} := \vartheta + w^\top \vec{\vartheta}$. Given a static portfolio $w \in \mathbb{R}^J$, we can now use (3.11) and (3.12) to find the set of strategies $\vartheta \in \overline{\Theta}_T(S)$ that are mean–variance efficient with respect to w; as it turns out, it consists of linear combinations of the pure investment strategy $\vartheta^\star$ and mean–variance hedging strategies ϑ^{B_j}.

Lemma 3.8. *The following statements hold for each $w \in \mathbb{R}^J$.*

1) *A strategy $\vartheta \in \overline{\Theta}_T(S)$ is mean–variance efficient with respect to w if and only if*

$$\vartheta =_S -w^\top \vec{\vartheta} + \beta \vartheta^\star \tag{3.13}$$

for some $\beta \geq 0$. In that case, the terminal wealth

$$V_T(0, w, \vartheta) = w^\top(\vec{B} - \vec{p}) + \vartheta \bullet S_T$$

attained by (w, ϑ) has expectation and variance given by

$$\begin{cases} E_P[V_T(0, w, \vartheta)] = (1 - L_0)\beta + (\vec{c} - \vec{p})^\top w, \\ \mathrm{Var}_P[V_T(0, w, \vartheta)] = L_0(1 - L_0)\beta^2 + w^\top \Sigma^B w. \end{cases} \tag{3.14}$$

2) *For any $\vartheta \in \overline{\Theta}_T(S)$, there exists some strategy $\tilde{\vartheta} \in \overline{\Theta}_T(S)$ which is mean–variance efficient with respect to w and such that both inequalities in* (3.2) *are satisfied.*

Proof. **1)** For any pair $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$, (3.11) and (3.12) give the expectation and variance of $V_T(0, w, \vartheta)$. Note that the terms $w^\top(\vec{c} - \vec{p})$ and $\mathrm{Var}[w^\top \vec{B}^{\perp}]$ on the right-hand side of those equations do not depend on ϑ. Thus by (3.11) and (3.12), ϑ is mean–variance efficient with respect to w if and only if $\hat{\vartheta} := \vartheta + w^\top \vec{\vartheta}$ is mean–variance efficient with respect to 0, i.e., in the pure investment sense. By Lemma 3.5 $(*) \Leftrightarrow$ (e), this holds if and only if $\hat{\vartheta} = \beta \vartheta^\star$ for some $\beta \geq 0$, which is equivalent to (3.13). Thus ϑ is mean–variance efficient if and only if it is given by (3.13) for some $\beta \geq 0$. By plugging $\hat{\vartheta} =_S \beta \vartheta^\star$ into (3.11) and (3.12) and using Lemma 3.7, we obtain

$$\begin{aligned} E_P[V_T(0, w, \vartheta)] &= w^\top(\vec{c} - \vec{p}) + E_P[\beta \vartheta^\star \bullet S_T] = (1 - L_0)\beta + (\vec{c} - \vec{p})^\top w, \\ \mathrm{Var}_P[V_T(0, w, \vartheta)] &= \mathrm{Var}_P[\beta \vartheta^\star \bullet S_T] + \mathrm{Var}[w^\top \vec{B}^{\perp}] = L_0(1 - L_0)\beta^2 + w^\top \Sigma^B w, \end{aligned}$$

which shows (3.14).

2) Fix some $\vartheta \in \overline{\Theta}_T(S)$. By (3.12), we obtain

$$\mathrm{Var}_P[V_T(0,w,\vartheta)] \geq \mathrm{Var}_P[w^\top \vec{B}^\perp] = w^\top \Sigma^B w.$$

First suppose that $L_0 \in (0,1)$ and set

$$\beta := \sqrt{\frac{\mathrm{Var}_P[V_T(0,w,\vartheta)] - w^\top \Sigma^B w}{L_0(1-L_0)}} \geq 0.$$

It follows by the second part of (3.14) that the strategy $\tilde{\vartheta} := -w^\top \vec{\vartheta} + \beta\vartheta^\star$ satisfies $\mathrm{Var}_P[V_T(0,w,\tilde{\vartheta})] = \mathrm{Var}_P[V_T(0,w,\vartheta)]$. Because $\tilde{\vartheta}$ is mean–variance efficient with respect to w by part 1), we must also have $E_P[V_T(0,w,\tilde{\vartheta})] \geq E_P[V_T(0,w,\vartheta)]$, and this concludes the proof in this case. Suppose now that $L_0 = 1$. As argued before Lemma 3.7, we have $E_P[\hat{\vartheta} \bullet S_T] = 0$ for all $\hat{\vartheta} \in \overline{\Theta}_T(S)$. Then by (3.11), we have $E_P[V_T(0,w,\vartheta)] = w^\top(\vec{c} - \vec{p})$. Setting $\tilde{\vartheta} := -w^\top \vec{\vartheta}$, (3.12) and (3.14) yield

$$\begin{aligned} E_P[V_T(0,w,\vartheta)] &= w^\top(\vec{c} - \vec{p}) = E_P[V_T(0,w,\tilde{\vartheta})], \\ \mathrm{Var}_P[V_T(0,w,\vartheta)] &\geq w^\top \Sigma^B w = \mathrm{Var}_P[V_T(0,w,\tilde{\vartheta})]. \end{aligned}$$

Since $\tilde{\vartheta}$ is mean–variance efficient with respect to w by part 1) with $\beta = 0$, this concludes the proof. □

Lemma 3.8 already gives a good intuition about the structure of the mean–variance portfolio selection problem in the semistatic setup. Indeed, by inspecting (3.13), we conclude that there are two relevant types of investment opportunities. The first is given by the pure investment strategy $\vartheta^\star$, which is an optimal strategy in the purely dynamic setup. By investing in $\beta \geq 0$ units of the strategy $\vartheta^\star$, the agent attains the terminal wealth $\vartheta \bullet S^\star$ and a risk-reward ratio determined by the initial value L_0 of the opportunity process; see Lemma 3.7. The other relevant investment opportunities are the *hedged derivatives*, i.e., where the agent buys a unit of B_j and offsets it with the strategy $-\vartheta^{B_j}$ in the underlying. Due to (3.8), the terminal wealth attained by this strategy is $c^{B_j} - p_j + B_j^\perp$, with expected value equal to the difference between c^{B_j} and the price p_j. Due to Lemma 3.6, $c^{B_j} = E_{Q^\star}[B_j]$ may be interpreted as a "fair value" for B_j under the VOMM $Q^\star$ for S. Moreover, the hedged payoffs are uncorrelated from $\vartheta^\star \bullet S_T$ due to (3.5), and their correlation matrix is given by Σ^B.

Thus the problem effectively reduces to a one-period model with finitely many assets, where an agent may only buy or sell the payoff $\vartheta^\star \bullet S_T$ or the hedged derivatives $c^{B_j} - p_j + B_j^\perp$. Indeed, due to (3.13), it would be suboptimal (from a mean–variance point of view) to use any other type of strategy. In line with Lemma 3.8, we denote the strategies in this reduced market by $(\beta, w) \in \mathbb{R}_+ \times \mathbb{R}_J$. The discussion above suggests that any mean–variance efficient strategy (w, ϑ) in the original market corresponds via (3.13) to a mean–variance efficient strategy (β, w) in the reduced market. We now make a precise statement of this idea.

Lemma 3.9. *Let $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$. The following statements are equivalent:*

(a) *(w, ϑ) is mean–variance efficient.*

(b) *Either $V_T(0, w, \vartheta) = 0$ P-a.s., or there exists some $\beta \geq 0$ such that* (3.13) *holds and the pair (β, w) satisfies $(1 - L_0)\beta + (\vec{c} - \vec{p})^\top w > 0$ and maximises $(1 - L_0)\tilde{\beta} + (\vec{c} - \vec{p})^\top \tilde{w}$ among all pairs $(\tilde{\beta}, \tilde{w}) \in \mathbb{R}_+ \times \mathbb{R}^J$ such that*

$$L_0(1 - L_0)\tilde{\beta}^2 + \tilde{w}^\top \Sigma^B \tilde{w} \leq L_0(1 - L_0)\beta^2 + w^\top \Sigma^B w. \tag{3.15}$$

Proof. "(a) $\Rightarrow$ (b)": Suppose that (w, ϑ) is mean–variance efficient. Then it is clear by Definition 3.3 that ϑ is mean–variance efficient with respect to w, and so (3.13) holds for some $\beta \geq 0$. If $V_T(0, w, \vartheta)$ is not 0, we have by Assumption 3.2 that $\mathrm{Var}_P[V_T(0, w, \vartheta)] > 0$. Because (w, ϑ) is mean–variance efficient, we obtain by comparing with $(0, 0)$ that $E_P[V_T(0, w, \vartheta)] > 0$, and hence (3.14) yields $(1 - L_0)\beta + (\vec{c} - \vec{p})^\top w > 0$. Now suppose for a contradiction that there exists a pair $(\tilde{\beta}, \tilde{w}) \in \mathbb{R}_+ \times \mathbb{R}^J$ such that (3.15) holds and

$$(1 - L_0)\tilde{\beta} + (\vec{c} - \vec{p})^\top \tilde{w} > (1 - L_0)\beta + (\vec{c} - \vec{p})^\top w.$$

Define $\tilde{\vartheta}$ by (3.13) with $\tilde{w}$ and $\tilde{\beta}$ in place of w and β. Then by (3.14) and the assumptions on $(\tilde{\beta}, \tilde{w})$, we have

$$E_P[V_T(0, \tilde{w}, \tilde{\vartheta})] > E_P[V_T(0, w, \vartheta)] \quad \text{and} \quad \mathrm{Var}_P[V_T(0, \tilde{w}, \tilde{\vartheta})] \leq \mathrm{Var}_P[V_T(0, w, \vartheta)],$$

which contradicts the mean–variance efficiency of (w, ϑ). Hence there exists no such pair $(\tilde{\beta}, \tilde{w})$, and (β, w) maximises $(1 - L_0)\tilde{\beta} + (\vec{c} - \vec{p})^\top \tilde{w}$ subject to (3.15). This shows (a) $\Rightarrow$ (b).

"(b) $\Rightarrow$ (a)": this is immediate if $V_T(0, w, \vartheta) = 0$ due to Assumption 3.2, so we exclude that case without loss of generality. Suppose for a contradiction that (w, ϑ) is not mean–variance efficient. Note that by (3.14) and the assumption, we also have

$$E_P[V_T(0,w,\vartheta)] = (1-L_0)\beta + (\vec{c}-\vec{p})^\top w > 0.$$

Thus by the equivalence (a) $\Leftrightarrow$ (e) in Lemma 3.4, there exists $(\tilde{w},\tilde{\vartheta}) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ such that

$$E_P[V_T(0,\tilde{w},\tilde{\vartheta})] > E_P[V_T(0,w,\vartheta)] \quad \text{and} \quad \mathrm{Var}_P[V_T(0,\tilde{w},\tilde{\vartheta})] \le \mathrm{Var}_P[V_T(0,w,\vartheta)].$$

Due to part 2) of Lemma 3.8, we may assume without loss of generality that $\tilde{\vartheta}$ is mean–variance efficient with respect to $\tilde{w}$, and hence given by (3.13) for some $\tilde{\beta} \ge 0$ and with $\tilde{w}$ in place of w. Then the assumption on $(\tilde{w},\tilde{\vartheta})$ and (3.14) yield that (3.15) holds and

$$(1-L_0)\tilde{\beta} + (\vec{c}-\vec{p})^\top \tilde{w} > (1-L_0)\beta + (\vec{c}-\vec{p})^\top w,$$

which contradicts the optimality of (β, w). Thus the pair (w,ϑ) must be mean–variance efficient. □

Before we proceed to the main result, we take a brief detour and use Lemma 3.8 to characterise Assumption 3.2 explicitly in terms of the matrix Σ^B. As we shall see, this condition arises naturally as a necessary condition when using Lemma 3.9 to find the mean–variance efficient strategies. Of course, in the following result, we temporarily lift Assumption 3.2, and thus we must be careful to avoid using previous results that require that assumption when proving the "if" statement.

Lemma 3.10. *Suppose that Assumption 3.1 holds. Then Assumption 3.2 also holds if and only if* $\vec{c}-\vec{p} \in \mathrm{Ran}\,\Sigma^B$.

Proof. To show the "only if" statement, suppose that $\vec{c}-\vec{p} \notin \mathrm{Ran}\,\Sigma^B$ for a contradiction. As Σ^B is a symmetric positive-semidefinite matrix, it admits an orthogonal basis of eigenvectors so that $\mathrm{Ker}\,\Sigma^B = (\mathrm{Ran}\,\Sigma^B)^\perp$. We can thus decompose $\vec{c}-\vec{p} = w + w^\perp$, where $w \in \mathrm{Ker}\,\Sigma^B$ and $w^\perp \in \mathrm{Ran}\,\Sigma^B$ are orthogonal, and we have $w \neq 0$ as $\vec{c}-\vec{p} \notin \mathrm{Ran}\,\Sigma^B$. Consider now the semistatic portfolio $(bw, b\vartheta)$, where $b > 0$ and $\vartheta = -w^\top \vec{\vartheta}$. Then by Lemma 3.8, ϑ is mean–variance efficient with respect to w and we have

$$\begin{aligned} E_P[V_T(0,bw,b\vartheta)] &= bw^\top(\vec{c}-\vec{p}) = bw^\top(w+w^\perp) = b|w|^2 > 0,\\ \mathrm{Var}_P\big[V_T(0,bw,b\vartheta)\big] &= b^2 w^\top \Sigma^B w = 0, \end{aligned}$$

so that $V_T(0,bw,b\vartheta) = b|w|^2 > 0$ a.s., which contradicts Assumption 3.2.

For the "if" statement, let $(w, \vartheta) \in \overline{\Theta}_T(S)$ be such that $\mathrm{Var}_P[V_T(0, w, \vartheta)] = 0$. We note that Assumption 3.2 is not needed for the proof of Lemma 3.6, and so the calculations in (3.11) and (3.12) still hold. Thus we have

$$0 = \mathrm{Var}_P[(\vartheta + w^\top \vec{\vartheta}) \bullet S_T] + \mathrm{Var}_P[w^\top \vec{B}^\perp]$$

so that each of the terms on the right-hand side is null. By Assumption 3.1, the first term can be null only if $(\vartheta + w^\top \vec{\vartheta}) \bullet S_T = 0$ P-a.s. Moreover, the second term is equal to $w^\top \Sigma^B w$ so that $w \in \operatorname{Ker} \Sigma^B$. Since $\vec{c} - \vec{p} \in \operatorname{Ran} \Sigma^B$ by assumption and $\operatorname{Ker} \Sigma^B = (\operatorname{Ran} \Sigma^B)^\perp$ as Σ^B is symmetric, it follows that $(\vec{c} - \vec{p})^\top w = 0$. But then both terms in the last line of (3.11) are null, and hence $E_P[V_T(0, w, \vartheta)] = 0$, i.e., we must have $V_T(0, w, \vartheta) = 0$ P-a.s. This concludes the proof. □

With Lemma 3.9, we have reduced the problem of finding the mean–variance efficient semistatic strategies $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ to that of solving a quadratic optimisation problem in $\mathbb{R}_+ \times \mathbb{R}^J$. The latter can also be seen as a classical Markowitz mean–variance portfolio selection problem in the reduced market, as discussed after Lemma 3.8. We are now almost ready to find explicit formulas for the mean–variance efficient strategies in the semistatic setup. The last step before giving the main result is to recall how to solve the type of linear–quadratic optimisation problem considered in Lemma 3.9 in an abstract sense. In the following, we denote by A^{-1} the Moore–Penrose inverse (or pseudoinverse) of a square matrix A; see Albert [6, Chapter III] for the definition and basic properties.

Lemma 3.11. *Let $\bar{\Sigma} \in \mathbb{R}^{d \times d}$ be a positive-semidefinite symmetric matrix, $\sigma^2 > 0$ and $D := \{\bar{w} \in \mathbb{R}^d : \bar{w}^\top \Sigma \bar{w} \leq \sigma^2\}$. Then for any $\bar{v} \in \operatorname{Ran} \bar{\Sigma} \setminus \{0\}$, the set of solutions $\bar{w} \in \mathbb{R}^d$ to the problem*

$$\bar{v}^\top \bar{w} \longrightarrow \max_{\bar{w} \in D}!$$

is given by $\hat{w} + \operatorname{Ker} \bar{\Sigma}$, where

$$\hat{w} := \frac{\sigma \bar{\Sigma}^{-1} \bar{v}}{(\bar{v}^\top \bar{\Sigma}^{-1} \bar{v})^{1/2}} \in \operatorname{Ran} \bar{\Sigma}.$$

Proof. First, suppose that $\bar{w}$ is a maximiser. Since $\nabla_{\bar{w}}(\bar{v}^\top \bar{w}) = \bar{v} \neq 0$, $\bar{w}$ cannot belong to the interior of D, and hence $\bar{w}^\top \bar{\Sigma} \bar{w} = \sigma^2$. By the Lagrange multiplier method, we must also have $\bar{v} = \lambda \bar{\Sigma} \bar{w}$ for some $\lambda > 0$; note that $\lambda \neq 0$ as $\bar{v} \neq 0$.

Since $\bar{v} \in \operatorname{Ran} \bar{\Sigma}$, we have $\bar{v} = \bar{\Sigma}\bar{\Sigma}^{-1}\bar{v}$ (see [6, Equation (III.3.9.2)]) and thus

$$0 = \bar{v} - \lambda\bar{\Sigma}\bar{w} = \bar{\Sigma}(\bar{\Sigma}^{-1}\bar{v} - \lambda\bar{w}),$$

i.e., $\bar{\Sigma}^{-1}\bar{v} - \lambda\bar{w} \in \operatorname{Ker}\bar{\Sigma}$. By rearranging, this yields $\bar{w} \in \bar{\Sigma}^{-1}\bar{v}/\lambda + \operatorname{Ker}\bar{\Sigma}$. Since $\bar{w}$ satisfies the boundary constraint and $\lambda > 0$, it follows that $\lambda = (\bar{v}^\top\bar{\Sigma}^{-1}\bar{v})^{1/2}/\sigma$, and hence $\bar{w} \in \hat{w} + \operatorname{Ker}\bar{\Sigma}$. As $\bar{\Sigma}$ is symmetric, we have by [6, Equation (III.3.8.1)] that $\bar{\Sigma}^{-1} = (\bar{\Sigma}^2)^{-1}\bar{\Sigma}$, and hence $\hat{w} \in \operatorname{Ran}\bar{\Sigma}$.

To show that any such $\bar{w}$ is a maximiser, note that $\bar{\Sigma}$ induces a (true) metric on $\operatorname{Ran}\bar{\Sigma}$ so that $D \cap \operatorname{Ran}\bar{\Sigma}$ is compact. Thus there exists at least one maximiser of the continuous function $\bar{w} \mapsto \bar{v}^\top\bar{w}$ in $D \cap \operatorname{Ran}\bar{\Sigma}$, and by the same argument as above, the only possible maximiser is $\hat{w}$. Since $\bar{v} \in \operatorname{Ran}\bar{\Sigma} = (\operatorname{Ker}\bar{\Sigma})^\perp$, we have $\bar{v}^\top\tilde{w} = 0$ and $\tilde{w}^\top\bar{\Sigma}\tilde{w} = 0$ for all $\tilde{w} \in \operatorname{Ker}\bar{\Sigma}$. It then follows trivially that $\hat{w}$ is also a maximiser of $\bar{v}^\top\bar{w}$ on $D = (D \cap \operatorname{Ran}\bar{\Sigma}) \oplus \operatorname{Ker}\bar{\Sigma}$, and so is any element of $\hat{w} + \operatorname{Ker}\bar{\Sigma}$. □

We are now ready to state and prove the main result. In the following, we write $\bar{w} := (\beta, w)$ and $\bar{v} := (1 - L_0, \vec{c} - \vec{p})$ for the strategies and expected values of the payoffs in the reduced market introduced after Lemma 3.8, with payoffs $\vartheta^\star \bullet S_T$ and $c^{B_j} - p_j + B_j^\perp$. Likewise, we consider the covariance matrix of the payoffs and its inverse, given by

$$\Sigma^{\star,B} := \begin{pmatrix} L_0(1-L_0) & \mathbf{0} \\ \mathbf{0} & \Sigma^B \end{pmatrix}, \quad (\Sigma^{\star,B})^{-1} = \begin{pmatrix} \mathbf{1}_{\{L_0 \in (0,1)\}} L_0^{-1}(1-L_0)^{-1} & \mathbf{0} \\ \mathbf{0} & (\Sigma^B)^{-1} \end{pmatrix}$$

due to Lemma 3.7 and (3.5). For notational convenience, we index these vectors and matrices by $\{0, \dots, J\}$ rather than $\{1, \dots, J+1\}$. As in previous results, some care is still needed in order to include both cases $L_0 = 1$ and $L_0 \in (0,1)$.

Proposition 3.12. *The portfolio $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ is mean–variance efficient if and only if*

$$w \in L_0\beta\,(\Sigma^B)^{-1}(\vec{c} - \vec{p}) + \operatorname{Ker}\Sigma^B \quad \text{and} \quad \vartheta =_S -w^\top\vec{\vartheta} + \beta\vartheta^\star \tag{3.16}$$

for some $\beta \geq 0$. In that case, the terminal wealth attained by the portfolio (w, ϑ) has mean and variance

$$\begin{cases} E_P[V_T(0, w, \vartheta)] = \beta\big(1 - L_0 + L_0(\vec{c} - \vec{p})^\top(\Sigma^B)^{-1}(\vec{c} - \vec{p})\big), \\ \operatorname{Var}_P[V_T(0, w, \vartheta)] = L_0\beta^2\big(1 - L_0 + L_0(\vec{c} - \vec{p})^\top(\Sigma^B)^{-1}(\vec{c} - \vec{p})\big). \end{cases} \tag{3.17}$$

Proof. "If": Let (w, ϑ) be given by (3.16) for some $\beta \geq 0$. We note that (3.17) follows directly by plugging the first part of (3.16) into (3.14). We now want to show that (w, ϑ) is mean–variance efficient. Due to (3.17), we have $V_T(0, w, \vartheta) = 0$ P-a.s. if and only if either $\beta = 0$, or if both $L_0 = 1$ and $\vec{c} - \vec{p} = 0$. In those cases, we immediately have by Lemma 3.9 that (w, ϑ) is mean–variance efficient. Otherwise, we can henceforth assume that

$$\beta > 0 \text{ and at least one of } \{1 - L_0, \vec{c} - \vec{p}\} \text{ is nonzero} \tag{3.18}$$

so that $V_T(0, w, \vartheta)$ is not 0. Our goal is to check that (β, ϑ) satisfies the conditions of statement (b) in Lemma 3.9, since that will immediately yield the mean–variance efficiency. First, note that equation (3.13) is the same as the second part of (3.16). By (3.18) and the first part of (3.16), we also obtain

$$(1 - L_0)\beta + (\vec{c} - \vec{p})^\top w = \beta\big((1 - L_0) + (\vec{c} - \vec{p})^\top (\Sigma^B)^{-1}(\vec{c} - \vec{p})\big) > 0.$$

It remains to show the maximality condition on $\bar{w} = (\beta, w)$, i.e., that $\bar{w}$ maximises $\bar{v}^\top \bar{w}'$ over all $\bar{w}' \in \mathbb{R}^{J+1}$ such that $(\bar{w}')^\top \Sigma^{\star,B} \bar{w}' \leq \sigma^2$, where

$$\sigma^2 := \bar{w}^\top \Sigma^{\star,B} \bar{w} = L_0 \beta^2 \big(1 - L_0 + L_0 (\vec{c} - \vec{p})^\top (\Sigma^B)^{-1} (\vec{c} - \vec{p})\big). \tag{3.19}$$

By the first part of (3.16) and (3.18), we have $\sigma^2 > 0$. Next, we note that Assumption 3.2 and Lemma 3.10 yield $\vec{c} - \vec{p} \in \operatorname{Ran} \Sigma^B$. Moreover, we have $\bar{v}_0 = 0$ if and only if $\Sigma^{\star,B}_{00} = 0$, and hence $\bar{v} \in \operatorname{Ran} \Sigma^{\star,B}$. We also have $\bar{v} \neq 0$ by (3.18). Thus by applying Lemma 3.11 to this maximisation problem, we obtain that $\bar{w}$ is a solution if and only if $\bar{w} \in \hat{w} + \operatorname{Ker} \Sigma^{\star,B}$, where $\hat{w} := (\bar{v}^\top (\Sigma^{\star,B})^{-1} \bar{v})^{-1/2} (\sigma (\Sigma^{\star,B})^{-1} \bar{v})$. It remains to check that $\bar{w}$ belongs to this set.

We now distinguish two cases. If $L_0 = 1$ so that $\Sigma^{\star,B}_{00}$ vanishes, then the condition $\bar{w} \in \hat{w} + \operatorname{Ker} \Sigma^{\star,B}$ does not impose any constraint on β. Moreover, the formula in (3.19) for σ^2 and the first part of (3.16) yield

$$w \in \frac{\sigma (\Sigma^B)^{-1} (\vec{c} - \vec{p})}{((\vec{c} - \vec{p})^\top (\Sigma^B)^{-1} (\vec{c} - \vec{p}))^{1/2}} + \operatorname{Ker} \Sigma^B,$$

and hence $\bar{w} \in \hat{w} + \operatorname{Ker} \Sigma^{\star,B}$ in this case. Likewise, if $L_0 \in (0, 1)$, we have by

(3.19) and the first part of (3.16) that

$$\begin{aligned}\bar{w} &\in \frac{\sigma}{\sqrt{\frac{1-L_0}{L_0} + (\vec{c}-\vec{p})^\top (\Sigma^B)^{-1}(\vec{c}-\vec{p})}} \begin{pmatrix} L_0^{-1} \\ (\Sigma^B)^{-1}(\vec{c}-\vec{p}) \end{pmatrix} + \{0\} \times \operatorname{Ker}\Sigma^B \\ &= \hat{w} + \operatorname{Ker}\Sigma^{\star,B}.\end{aligned}$$

Therefore in both cases, it follows by Lemma 3.9 that (w, ϑ) is mean–variance efficient, which proves the "if" statement.

"Only if": Suppose that (w, ϑ) is mean–variance efficient and hence satisfies condition (b) in Lemma 3.9. If $V_T(0, w, \vartheta) = 0$ P-a.s., we have by (3.12) that

$$\operatorname{Var}_P[(\vartheta + w^\top \vec{\vartheta}) \bullet S_T] = \operatorname{Var}_P[w^\top \vec{B}^\perp] = 0.$$

In that case, $w^\top \Sigma^B w = 0$ so that $w \in \operatorname{Ker}\Sigma^B$, and due to Assumption 3.1, $\vartheta + w^\top \vec{\vartheta} =_S 0$. Hence in the case $V_T(0, w, \vartheta) = 0$ P-a.s., (w, ϑ) must satisfy (3.16) with $\beta = 0$. Thus we may assume $V_T(0, w, \vartheta) \not\equiv 0$, and hence Lemma 3.9 yields the second part of (3.16) for some $\beta \geq 0$. We recall $\bar{v}$ and $\Sigma^{\star,B}$ and consider once again $\bar{w} = (\beta, w)$ for this choice of β and w. As before, we have $\bar{v} \in \operatorname{Ran}\Sigma^{\star,B}$. Due to condition (b) of Lemma 3.9, we have $\bar{v}^\top \bar{w} > 0$ so that $\bar{v} \neq 0$. Since $\bar{v} \in \operatorname{Ran}\Sigma^{\star,B}$, the inequality $\bar{v}^\top \bar{w} > 0$ also implies that $\sigma^2 := \bar{w}^\top \Sigma^{\star,B} \bar{w} > 0$. Condition (b) of Lemma 3.9 also gives that $\bar{w}$ maximises $\bar{v}^\top \bar{w}'$ over all $\bar{w}' \in \mathbb{R}^{J+1}$ such that $(\bar{w}')^\top \Sigma^{\star,B} \bar{w}' \leq \sigma^2$. Since we have checked its conditions, we may apply Lemma 3.11 to this problem, which yields that $\bar{w} \in \hat{w} + \operatorname{Ker}\Sigma^{\star,B}$, where $\hat{w} := (\bar{v}^\top (\Sigma^{\star,B})^{-1} \bar{v})^{-1/2} \sigma (\Sigma^{\star,B})^{-1} \bar{v}$.

We once again distinguish the two cases $L_0 = 1$ and $L_0 \neq 1$. If $L_0 = 1$, then the condition $\bar{w} \in \hat{w} + \operatorname{Ker}\Sigma^{\star,B}$ does not give any information on β. On the other hand, it yields for w that

$$w \in \frac{\sigma(\Sigma^B)^{-1}(\vec{c}-\vec{p})}{((\vec{c}-\vec{p})^\top (\Sigma^B)^{-1}(\vec{c}-\vec{p}))^{1/2}} + \operatorname{Ker}\Sigma^B.$$

Then the first part of (3.16) holds if we replace β with

$$\tilde{\beta} := \frac{\sigma}{\sqrt{(\vec{c}-\vec{p})^\top (\Sigma^B)^{-1}(\vec{c}-\vec{p})}}.$$

By the choice of $\beta \geq 0$, we also have that the second part (3.16) holds for the original β. However, since $\vartheta^\star = 0$ in this case as argued after (3.8), the second part of (3.16) also holds if we replace β with $\tilde{\beta}$, and so we conclude that both parts of (3.16) are satisfied for $\tilde{\beta}$.

If $L_0 \in (0,1)$, we have $\Sigma_{00}^{\star,B} \neq 0$, and hence the condition $\bar{w} \in \hat{w} + \operatorname{Ker}\Sigma^{\star,B}$ yields

$$\begin{aligned}\bar{w} &\in \frac{\sigma(\Sigma^{\star,B})^{-1}\bar{v}}{\sqrt{\bar{v}^\top(\Sigma^{\star,B})^{-1}\bar{v}}} + \{0\} \times \operatorname{Ker}\Sigma^B \\ &= \frac{\sigma}{\sqrt{\frac{1-L_0}{L_0} + (\vec{c}-\vec{p})^\top(\Sigma^B)^{-1}(\vec{c}-\vec{p})}} \begin{pmatrix} L_0^{-1} \\ (\Sigma^B)^{-1}(\vec{c}-\vec{p}) \end{pmatrix} + \{0\} \times \operatorname{Ker}\Sigma^B.\end{aligned}$$

By rearranging, we obtain

$$\beta = \frac{\sigma L_0^{-1}}{\sqrt{\frac{1-L_0}{L_0} + (\vec{c}-\vec{p})^\top(\Sigma^B)^{-1}(\vec{c}-\vec{p})}} \quad \text{and} \quad w \in L_0\beta(\Sigma^B)^{-1}(\vec{c}-\vec{p}) + \operatorname{Ker}\Sigma^B \tag{3.20}$$

so that w satisfies the first part of (3.16). The second part of (3.16) also holds by the choice of β, and therefore (w,ϑ) satisfies (3.16). This completes the proof of the equivalence. □

In the process of showing Proposition 3.12, we have already solved the Markowitz problem in this setup, namely the version related to condition (a) of Lemma 3.4, where we seek to maximise the expectation of the terminal wealth under a constraint on the variance. Other versions of the Markowitz problem can be solved in a similar way; see Fontana/Schweizer [48]. It is now also straightforward to find the optimal Sharpe ratio in this setup.

Corollary 3.13. *Suppose that $\vec{c}-\vec{p} \in \operatorname{Ran}\Sigma^B$ with $\vec{c} \neq \vec{p}$ or $L_0 \neq 1$. The optimal Sharpe ratio for the semistatic portfolio optimisation problem, defined by*

$$SR_{S,\vec{B}} := \sup_{\substack{(w,\vartheta)\in\mathbb{R}^J\times\overline{\Theta}_T(S)\\ \operatorname{Var}_P[V_T(0,w,\vartheta)]>0}} SR(w,\vartheta) = \sup_{\substack{(w,\vartheta)\in\mathbb{R}^J\times\overline{\Theta}_T(S)\\ \operatorname{Var}_P[V_T(0,w,\vartheta)]>0}} \frac{E_P[V_T(0,w,\vartheta)]}{(\operatorname{Var}_P[V_T(0,w,\vartheta)])^{1/2}},$$

is given by

$$SR_{S,\vec{B}} = \sqrt{SR_S^2 + (\vec{c}-\vec{p})^\top(\Sigma^B)^{-1}(\vec{c}-\vec{p})}, \tag{3.21}$$

where $SR_S = \sqrt{\frac{1-L_0}{L_0}}$ is the optimal Sharpe ratio attainable by trading only in S.

Proof. By Lemma 3.4 (e) ⇒ (d), it suffices to calculate the Sharpe ratio of any mean–variance efficient strategy (w,ϑ) with $V_T(w,\vartheta) \not\equiv 0$. Due to Proposition 3.12, we may consider the strategy (w,ϑ) given by (3.16) for an arbitrary choice of $\beta > 0$. We note that $\mathrm{Var}_P[V_T(0, w, \vartheta)] > 0$ due to (3.17) and the assumptions. Then the optimal Sharpe ratio is

$$SR_{S,\vec{B}} = SR(w, \vartheta) = \sqrt{\frac{1 - L_0}{L_0} + (\vec{c} - \vec{p})^\top (\Sigma^B)^{-1} (\vec{c} - \vec{p})} \tag{3.22}$$

independently of $\beta > 0$ due to (3.17). Applying this formula in the case $J = 0$ yields that the optimal Sharpe ratio attainable by trading only in S is given by $SR_S = \sqrt{\frac{1-L_0}{L_0}}$; this result for the dynamic case can also be found in Černý/Kallsen [25, Proposition 3.6]. Plugging into (3.22) then yields (3.21). □

We now summarise the main conclusions from the previous results. Under Assumption 3.2, which is equivalent to $\vec{c} - \vec{p} \in \mathrm{Ran}\, \Sigma^B = \mathrm{Ran}\, (\Sigma^B)^{-1}$ by Lemma 3.10, we have $(\vec{c} - \vec{p})^\top (\Sigma^B)^{-1} (\vec{c} - \vec{p}) > 0$ if and only if $\vec{c} - \vec{p} \neq 0$. Thus by (3.21), the additional possibility of taking static positions in the derivatives $B_1, \dots, B_J$ provides a benefit in comparison to being able to trade only S if and only if $\vec{c} \neq \vec{p}$, i.e., if some of the market prices p_j do not coincide with the constants $c^{B_j} = E_{Q^\star}[B_j]$ given by Lemma 3.6. One may interpret the constants c^{B_j} as the "fair values" of B_j under the variance-optimal martingale measure $Q^\star$ for S. Thus if there are discrepancies between the prices and fair values of the derivatives B_j, then (3.16) yields semistatic strategies that achieve a higher Sharpe ratio than what is achievable by trading in S alone. On the other hand, if $\vec{c} = \vec{p}$ and $L_0 < 1$, then by (3.16) it is optimal (from a mean–variance standpoint) to not take any static positions in the derivatives. Finally, consider the case $\vec{c} = \vec{p}$ and $L_0 = 1$ that is excluded in the assumptions of Corollary 3.13. As argued after (3.8), we have that P is a local martingale measure for S and hence also the VOMM for S, i.e., $P = Q^\star$, and we have $E_P[\vartheta \bullet S_T] = 0$ for any $\vartheta \in \overline{\Theta}_T(S)$. Moreover, Lemma 3.6 yields

$$\vec{p} = \vec{c} = E_{Q^\star}[\vec{B}] = E_P[\vec{B}]$$

so that P is also a martingale measure for the static payoffs and a local martingale measure for the enlarged market $(S, \vec{B})$. In that case, it is not possible to achieve a nonzero expected return with any admissible strategy $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$, and hence it is optimal to not trade in either S or $\vec{B}$.

3.2 Semistatic hedging

We now introduce the problem of mean–variance hedging in the semistatic market, which is more closely related to the one considered in Di Tella et al. [38, 39].

We start with the same setup as before, i.e., a risky asset with continuous semimartingale price process $(S_t)_{0\le t\le T}$ and a basket $\vec{B}$ of claims $B_j \in L^2_P(\mathcal{F}_T)$ $(j = 1,\dots,J)$ that may be bought or sold at time 0 for a price $p_j \in \mathbb{R}$. We also make the same Assumptions 3.1 and 3.2. We are interested in solving the mean–variance hedging problem for a claim $H \in L^2_P(\mathcal{F}_T)$ over the set of semistatic portfolios $(w,\vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$, i.e., we consider the problem

$$\varepsilon^2_{\text{semi}}(x,H) = \inf_{(w,\vartheta)\in\mathbb{R}^J\times\overline{\Theta}_T(S)} E_P\left[\left(H - V_T(x,w,\vartheta)\right)^2\right], \tag{3.23}$$

where $x \in \mathbb{R}$ is the (fixed) initial wealth and $V_T(x,w,\vartheta)$ is given by (3.1). In other words, we extend the original MVH problem (2.3) by allowing the use of static strategies $w \in \mathbb{R}^J$. We show below that there exists a unique solution $(w^H(x), \hat{\vartheta}^H(x))$ to (3.23), where we write $\hat{\vartheta}$ instead of ϑ in order to distinguish it from the solution to the MVH problem (2.3) for H in the dynamic market.

As done previously in (3.8) for the portfolio selection problem, we start by using Lemma 3.6 to decompose H_j and $\bar{H}$ as

$$\begin{aligned} B_j &= c^{B_j} + \vartheta^j \bullet S_T + B_j^\perp, \\ H &= c^H + \vartheta^H \bullet S_T + H^\perp, \end{aligned} \tag{3.24}$$

where $c^{B_j}, c^H \in \mathbb{R}$, $\vartheta^{B_j}, \vartheta^H \in \overline{\Theta}_T(S)$ and $B_j^\perp, H^\perp \in L^2_P(\mathcal{F}_T)$ are such that

$$E_P[B_j^\perp] = E_P[B_j^\perp(\tilde{\vartheta}\bullet S_T)] = E_P[H^\perp] = E_P[H^\perp(\tilde{\vartheta}\bullet S_T)] = 0 \tag{3.25}$$

for all $\tilde{\vartheta} \in \overline{\Theta}_T(S)$. As before, let $\Sigma^B = \mathrm{Cov}_P(\vec{B}^\perp)$ be the covariance matrix of the residuals $\vec{B}^\perp = (B_1^\perp,\dots,B_J^\perp)$. We also consider the covariance matrix of the concatenation, given by

$$\Sigma^{H,B} = (\Sigma^{H,B}_{ij})^J_{i,j=0} = \mathrm{Cov}_P\big((H^\perp, \vec{B}^\perp)\big) = \begin{pmatrix} \bar{\Sigma}^{H,B}_{00} & \Sigma^{H,B}_{0\cdot} \\ \Sigma^{H,B}_{\cdot 0} & \Sigma^B \end{pmatrix}, \tag{3.26}$$

where $\Sigma^{H,B}_{00} = \mathrm{Var}_P[H^\perp]$ and

$$\bar{\Sigma}^{H,B}_{0j} = \bar{\Sigma}^{H,B}_{j0} = \mathrm{Cov}_P(B_j^\perp, H^\perp), \quad \text{for } j = 1,\dots,J.$$

Henceforth we assume for simplicity that $\Sigma^{H,B}$ is positive-definite, i.e., invertible. Similarly to Proposition 3.12, one could also consider the case where $\Sigma^{H,B}$ is singular, in which case the optimal hedge would not be unique.

Proposition 3.14. *Suppose that $\Sigma^{H,B}$ is positive-definite. Then there exists a unique solution $(w^H(x), \hat{\vartheta}^H(x)) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ to (3.23). It is given by*

$$w^H(x) = \tilde{\Sigma}^{-1}\tilde{\Sigma}_0(x), \tag{3.27}$$

$$\hat{\vartheta}^H(x) = \vartheta^H - \big(w^H(x)\big)^{\perp}\vec{\vartheta} + \big(c^H - x - (\vec{c}-\vec{p})^{\top} w^H(x)\big)\vartheta^{\star}, \tag{3.28}$$

where $\tilde{\Sigma} := \Sigma^B + L_0(\vec{c}-\vec{p})(\vec{c}-\vec{p})^{\top}$ and $\tilde{\Sigma}_0(x) := \Sigma^{H,B}_{\cdot 0} + L_0(c^H - x)(\vec{c}-\vec{p})$. Moreover, the mean squared error (3.23) attained by $(w^H(x), \hat{\vartheta}^H(x))$ is given by

$$\varepsilon^2_{\mathrm{semi}}(x, H) = \Sigma^{H,B}_{00} + L_0(c^H - x)^2 - \big(\tilde{\Sigma}_0(x)\big)^{\top}\tilde{\Sigma}^{-1}\tilde{\Sigma}_0(x). \tag{3.29}$$

Proof. For any semistatic portfolio $(w, \vartheta) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$, we have by (3.24) that

$$\begin{aligned} &E_P\big[\big(H - V_T(x, w, \vartheta)\big)^2\big] \\ &= E_P\big[\big(c^H - x - w^{\top}(\vec{c}-\vec{p}) + (\vartheta^H - w^{\top}\vec{\vartheta} - \vartheta)\bullet S_T + H^{\perp} - w^{\top}\vec{B}^{\perp}\big)^2\big]. \end{aligned}$$

By the orthogonality properties (3.25) of $H^{\perp}$ and $\vec{H}^{\perp}$, we can decompose this as

$$\mathrm{Var}_P[H^{\perp} - w^{\top}\vec{B}] + E_P\big[\big(c^H - x - w^{\top}(\vec{c}-\vec{p}) + (\vartheta^H - w^{\top}\vec{\vartheta} - \vartheta)\bullet S_T\big)^2\big], \tag{3.30}$$

where the first term does not depend on ϑ, and hence we only need to minimise the second term over $\vartheta \in \overline{\Theta}(S)$. For a fixed $w \in \mathbb{R}^J$, note that the problem

$$E_P\big[\big(c^H - x - w^{\top}(\vec{c}-\vec{p}) - \tilde{\vartheta}\bullet S_T\big)^2\big] \longrightarrow \min_{\tilde{\vartheta}\in\overline{\Theta}_T(S)}! \tag{3.31}$$

is a mean–variance hedging problem for the payoff $c^H - x - w^{\top}(\vec{c}-\vec{p})$ in the dynamic market. By linearity and since the payoff is constant, (3.31) is (up to a scalar factor) equivalent to the pure investment problem (2.4). Thus the optimiser is a multiple of the pure investment strategy, i.e., $\tilde{\vartheta} = (c^H - x - w^{\top}(\vec{c}-\vec{p}))\vartheta^{\star}$, and by (3.10), we have

$$\begin{aligned} E_P\big[\big(c^H - x - w^{\top}(\vec{c}-\vec{p}) - \tilde{\vartheta}\bullet S_T\big)^2\big] &= \big(c^H - x - w^{\top}(\vec{c}-\vec{p})\big)^2 E_P[(1 - \vartheta^{\star}\bullet S_T)^2] \\ &= \big(c^H - x - w^{\top}(\vec{c}-\vec{p})\big)^2 L_0. \end{aligned} \tag{3.32}$$

Hence by linearity, $\vartheta \in \overline{\Theta}_T(S)$ minimises the second term in (3.30) if and only if

$-\vartheta^H + w^\top \vec{\vartheta} + \vartheta = \tilde{\vartheta}$ is a minimiser for (3.31), which yields

$$\vartheta = \vartheta^H - w^\top \vec{\vartheta} + \big(c^H - x - w^\top(\vec{c} - \vec{p})\big)\vartheta^\star.$$

Thus any solution to (3.23) must satisfy (3.28). Plugging (3.32) into (3.30), it remains to solve over w the minimisation problem

$$\mathrm{Var}_P[H^\perp - w^\top \vec{B}^\perp] + L_0\big(c^H - x - w^\top(\vec{c} - \vec{p})\big)^2 \longrightarrow \min_{w\in\mathbb{R}^J}!$$

Recall that we have $L_0 = E_P[(1 - \vartheta^\star \bullet S_T)^2]$ by (3.10). Together with the orthogonality properties from (3.25), we can rewrite the problem as

$$E_P\Big[\Big(H^\perp - w^\top \vec{B}^\perp + \big(c^H - x - w^\top(\vec{c} - \vec{p})\big)(1 - \vartheta^\star \bullet S_T)\Big)^2\Big] \longrightarrow \min_{w\in\mathbb{R}^J}!$$

Collecting the constant and linear terms in w, we obtain

$$E_P\Big[\Big(H^\perp + (c^H - x)(1 - \vartheta^\star \bullet S_T) - w^\top\big(\vec{B}^\perp + (\vec{c} - \vec{p})(1 - \vartheta^\star \bullet S_T)\big)\Big)^2\Big] \longrightarrow \min_{w\in\mathbb{R}^J}! \quad (3.33)$$

This is a linear regression problem in $L^2_P(\mathcal{F}_T)$ of the form

$$E[(\hat{U} - w^\top \tilde{U})^2] \longrightarrow \min_{w\in\mathbb{R}^J}!,$$

where $\tilde{U} := \vec{B}^\perp + (\vec{c} - \vec{p})(1 - \vartheta^\star \bullet S_T)$ and $\hat{U} := H^\perp + (c^H - x)(1 - \vartheta^\star \bullet S_T)$. Note that by (3.10) and (3.25), we have

$$\begin{aligned} E_P[\tilde{U}\tilde{U}^\top] &= E_P\big[\big(\vec{B}^\perp + (\vec{c} - \vec{p})(1 - \vartheta^\star \bullet S_T)\big)\big(\vec{B}^\perp + (\vec{c} - \vec{p})(1 - \vartheta^\star \bullet S_T)\big)^\top\big] \\ &= \Sigma^B + L_0(c - p)(c - p)^\top = \tilde{\Sigma}, \end{aligned} \quad (3.34)$$

and hence $E_P[\tilde{U}\tilde{U}^\top]$ is positive-definite like Σ^B since $L_0(\vec{c} - \vec{p})(\vec{c} - \vec{p})^\top$ is positive-semidefinite. Therefore (3.33) admits the unique minimiser

$$w^H(x) = (E_P[\tilde{U}\tilde{U}^\top])^{-1} E_P[\hat{U}\tilde{U}] \quad (3.35)$$

with minimum error

$$\varepsilon^2_{\mathrm{semi}}(x, H) = E_P[\hat{U}^2] - (E_P[\hat{U}\tilde{U}])^\top (E_P[\tilde{U}\tilde{U}^\top])^{-1} E_P[\hat{U}\tilde{U}]. \quad (3.36)$$

Similarly to (3.34), we also have

$$\begin{aligned}E_P[\hat{U}\tilde{U}] &= E_P\big[\big(H^\perp + (c^H - x)(1 - \vartheta^\star \bullet S_T)\big)\big(\vec{B}^\perp + (\vec{c} - \vec{p})(1 - \vartheta^\star \bullet S_T)\big)\big]\\ &= \Sigma^{H,B}_{\cdot 0} + L_0(c^H - x)(\vec{c} - \vec{p}) = \tilde{\Sigma}_0.\end{aligned}$$

Plugging into (3.35) together with (3.34) yields (3.27). Therefore, the unique solution $(w^H(x), \hat{\vartheta}^H(x)) \in \mathbb{R}^J \times \overline{\Theta}_T(S)$ to (3.23) is given by (3.27) and (3.28). Note that we also obtain from (3.25) that

$$E_P[\hat{U}^2] = E_P\big[\big(H^\perp + (c^H - x)(1 - \vartheta^\star \bullet S_T)\big)^2\big] = \Sigma^{H,B}_{00} + L_0(c^H - x)^2,$$

and plugging into (3.36) yields (3.29). This concludes the proof. □

3.3 European vanilla options in the rough Heston model

Finally, we return to the question of how to obtain explicit formulas in the rough Heston model. As discussed at the beginning of the section, we consider a time horizon $T \in (0, T^\star]$, where $T^\star > 0$ is given by Theorem 2.3 so that the variance-optimal martingale measure $Q^\star$ is well defined. For simplicity, we consider a basket of European call options $B_j = (S_T - K_j)^+$ with strikes $0 < K_1 < \cdots < K_J$ and prices $p_1 > p_2 > \cdots > p_J > 0$. Under the assumption that $\vec{c} - \vec{p} \in \operatorname{Ran} \Sigma^B$, the mean–variance efficient semistatic portfolios (w, ϑ) have by Proposition 3.12 the form

$$w = L_0 \beta\, (\Sigma^B)^{-1}(\vec{c} - \vec{p}) \quad \text{and} \quad \vartheta = -w^\top \vec{\vartheta} + \beta \vartheta^*,$$

for some $\beta \geq 0$. Hence we should like to have explicit formulas for $L_0, \vartheta^\star, c^{B_j}, \vartheta^{B_j}$ and Σ^B. We already have most of those: for instance, we have explicit formulas for $\vartheta^\star$ and L from Theorem 2.3. Likewise, under some assumptions, Corollary 2.19 gives a formula for $c^{B_j} = E_{Q^\star}[B_j] = V_0^{B_j}$. The hedging strategies $\vartheta^{B_j} = \vartheta^{B_j}(c^{B_j})$ can also be determined by solving the feedback equation (2.9) in Proposition 2.1, since we have formulas for each of the inputs. Therefore, the only missing element is the covariance matrix Σ^B. In fact, we already have an explicit formula for the diagonal of Σ^B, since

$$\Sigma^B_{jj} = E_P[(B_j^\perp)^2] = E_P[(B_j - c^{B_j} - \vartheta^{B_j} \bullet S_T)^2] = \varepsilon^2(c^{B_j}, H^{B_j}),$$

where $\varepsilon^2(c^{B_j}, H^{B_j})$ is the mean squared hedging error for B_j in the dynamic market defined in (2.3), which we can calculate with Theorem 2.29. As we show in the following result, the rest of the matrix Σ^B can be obtained by polarisation.

We note that although the discussion above applies to the mean–variance portfolio optimisation problem, the mean–variance hedging problem for another European call option of the form $H = (S_T - K)^+$ can be handled in a similar way. By Lemma 3.14, it suffices to additionally determine c^H, ϑ^H, $\Sigma^{H,B}_{\cdot 0}$ and $\Sigma^{H,B}_{00}$. Once again, we obtain formulas for c^H and ϑ^H from Corollary 2.19 and Proposition 2.1, and the result below can also be used to compute the extended matrix $\Sigma^{H,B}$ from (3.26), which yields $\Sigma^{H,B}_{\cdot 0}$ and $\Sigma^{H,B}_{00}$.

Proposition 3.15. *Suppose that $T \in (0, T^\star]$, $a_1 > 1$ and $a_2 < 0$ are such that*

$$T \le \min\{T^\star_{a_1}, T^\star_{a_2}, T^\star_{a_1+\delta}, T^\star_{a_2-\delta}\} \wedge \inf\left\{\tilde{T}_{z_1,z_2} : z_1, z_2 \in \{a_1, a_2\} + \mathrm{i}\mathbb{R}\right\}$$

for some $\delta > 0$, where $T^\star_z$ is given by Corollary 2.16 and $\tilde{T}_{z_1,z_2}$ by Proposition 2.26. Then for $B_j := (S_T - K_j)^+$ with strikes $0 < K_1 < \cdots < K_J$, the covariance matrix Σ^B of the residuals $B_j^\perp$ is given by

$$\Sigma^B_{jk} = (1-\varrho^2)\int_0^T \left(\iint_{\mathbb{C}^2}\Big(\prod_{m=1,2} h^\star_{z_m}(T-u)\Big) Z_0^{u,T,z_1,z_2}\pi_{K_j,a}(dz_1)\pi_{K_k,a}(dz_2)\right)du \tag{3.37}$$

for $j,k \in \{1,\dots,J\}$, where $\pi_{K_j,a}$ is given by (2.37), Z^{u,T,z_1,z_2} is defined in (2.95) and $h^\star_{z_m} = \hat{\kappa} * g^\star_{z_m}$, where $g^\star_{z_m}$ is the solution to (2.60) with $z = z_m$.

Proof. As noted above, the diagonal term Σ^B_{jj} corresponds to the mean squared hedging error associated with B_j with initial wealth $c_j = V^{B_j}$, and hence by Theorem 2.29 is given by

$$\Sigma^B_{jj} = (1-\varrho^2)\int_0^T \left(\iint_{\mathbb{C}^2}\Big(\prod_{m=1,2} h^\star_{z_m}(T-u)\Big) Z_0^{u,T,z_1,z_2}\pi^{\otimes 2}_{K_j,R}(dz_1,dz_2)du,\right.$$

which immediately yields (3.37) for $j = k$. We can obtain the rest of the covariance matrix by polarisation, since we have

$$\Sigma^B_{jk} = E_P[B_j^\perp B_k^\perp] = \frac{E_P[(B_j^\perp + B_k^\perp)^2] - E_P[(B_j^\perp - B_k^\perp)^2]}{4} \tag{3.38}$$

for $j,k \in \{1,\dots,J\}$. Recall that by (3.4), $B_j^\perp$ is the projection of B_j onto $(\mathbb{R} \oplus \mathcal{G}_T(S))^\perp$. Thus by linearity, $B_j^\perp + B_k^\perp$ is the projection of $B_j + B_k$ onto $(\mathbb{R} \oplus \mathcal{G}_T(S))^\perp$, and hence

$$E_P[(B_j^\perp + B_k^\perp)^2] = \varepsilon_0^2(B_j + B_k),$$

where we recall that $B_j = (S_T - K_j)^+$. By (2.35), we can express

$$(s - K_j)^+ + (s - K_k)^+ = \int_{\mathbb{C}} s^z \pi^+_{j,k}(dz),$$

where we set $\pi^+_{j,k} := \pi_{K_j,a} + \pi_{K_k,a}$ for ease of notation. Hence by replacing π_{K,a_1} with $\pi^+_{j,k}$ in the proof of Theorem 2.29, we obtain that the mean squared hedging error $\varepsilon_0^2((S_T - K_j)^+ + (S_T - K_k)^+)$ is given by

$$(1-\varrho^2)\int_0^T \Big(\iint_{\mathbb{C}^2}\Big(\prod_{m=1,2} h^\star_{z_m}(T-u)\Big) Z_0^{u,T,z_1,z_2}(\pi^+_{j,k})^{\otimes 2}(dz_1,dz_2)\Big)du. \tag{3.39}$$

By an analogous argument, $E[(B_j^\perp - B_k^\perp)^2] = \varepsilon_0^2\big((S_T - K_j)^+ - (S_T - K_k)^+\big)$ equals

$$(1-\varrho^2)\int_0^T \Big(\iint_{\mathbb{C}^2}\Big(\prod_{m=1,2} h^\star_{z_m}(T-u)\Big) Z_0^{u,T,z_1,z_2}(\pi^-_{j,k})^{\otimes 2}(dz_1,dz_2)\Big)du, \tag{3.40}$$

where $\pi^-_{j,k} := \pi_{K_j,R} - \pi_{K_k,R}$. Since we have

$$\begin{aligned}(\pi^+_{j,k})^{\otimes 2} &= \pi^{\otimes 2}_{K_j,a} + \pi_{K_j,a}\otimes\pi_{K_k,a} + \pi_{K_k,a}\otimes\pi_{K_j,a} + \pi^{\otimes 2}_{K_k,a},\\ (\pi^-_{j,k})^{\otimes 2} &= \pi^{\otimes 2}_{K_j,a} - \pi_{K_j,a}\otimes\pi_{K_k,a} - \pi_{K_k,R}\otimes\pi_{K_j,a} + \pi^{\otimes 2}_{K_\ell,a},\end{aligned}$$

it follows that

$$(\pi^+_{j,k})^{\otimes 2} - (\pi^-_{j,k})^{\otimes 2} = 2\pi_{K_j,a}\otimes\pi_{K_k,a} + 2\pi_{K_k,a}\otimes\pi_{K_j,a}.$$

Hence by plugging (3.39) and (3.40) into (3.38) and taking differences, we obtain

$$\begin{aligned}\Sigma^B_{jk} &= \frac{1-\varrho^2}{2}\int_0^T \Big(\iint_{\mathbb{C}^2}\Big(\prod_{m=1,2} h^\star_{z_m}(T-u)\Big) Z_0^{u,T,z_1,z_2}\pi_{K_j,a}(dz_1)\pi_{K_k,a}(dz_2)\Big)du\\ &\quad + \frac{1-\varrho^2}{2}\int_0^T \Big(\iint_{\mathbb{C}^2}\Big(\prod_{m=1,2} h^\star_{z_m}(T-u)\Big) Z_0^{u,T,z_1,z_2}\pi_{K_k,a}(dz_1)\pi_{K_j,a}(dz_2)\Big)du.\end{aligned} \tag{3.41}$$

Finally, we note that the product $\prod_{m=1,2} h^\star_{z_m}(T-u)$ is symmetric in (z_1, z_2), and so is Z_0^{u,T,z_1,z_2} as noted in Remark 2.28. Hence the two integrals in (3.41) are equal, which yields (3.37). □

As discussed above, we have with Proposition 3.15 all of the elements that we need to obtain explicit formulas for the mean–variance portfolio optimisation and hedging problems in the case of European call options via Propositions 3.12 and 3.14, respectively. It is straightforward to obtain an analogous result for European put options. In principle, these results can be extended to other cases such as the ones considered in Examples 2.12–2.14, provided that one can calculate the mean squared hedging errors as in Theorem 2.29, and thus obtain an analogue of Proposition 3.15. As mentioned after Assumption 3.1, it would also be interesting to study whether Propositions 3.12 and 3.14 hold under the weaker assumptions in Černý/Czichowsky [24]. We leave such generalisations to future research.

4 Global existence of solutions to nonlinear Volterra equations

Our goal in this section is twofold. First, we seek to obtain sufficient conditions for the existence of a global solution $x : I \to \mathbb{R}^n$ to a nonlinear Volterra equation of the form

$$x(t) = x_0 + \int_0^t k(t-s) f\big(x(s), s\big) ds, \quad t \in I, \tag{4.1}$$

for given $n \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$, $k \in L^1_{\text{loc}}(I; \mathbb{R}_+)$ (that we refer to as a *kernel*) and $f : \mathbb{R}^n \times I \to \mathbb{R}^n$, where I is an interval of the form $[0, \bar{T}]$ or $[0, \bar{T})$ for some $\bar{T} > 0$ (which may be ∞ if I is right-open). Our second goal is to use this result to show Theorem 2.23, i.e., that if there exists a solution to (2.60) on $[0, \bar{T}]$ for $z = a \in \mathbb{R}$ for some $\bar{T} > 0$, then there is a unique solution to (2.60) on $[0, \bar{T}]$ for all $z \in a + \mathrm{i}\mathbb{R}$. This can be achieved by modifying (2.60) into the form (4.1) and then applying the result from the first part. As argued in Section 2, the existence of solutions to (2.60) on a common time interval for all $z \in a + \mathrm{i}\mathbb{R}$ shows that the Mellin transform approach can be used for solving the mean–variance hedging problem for vanilla European call and put options in the rough Heston model.

4.1 Global existence result

We now outline the strategy for the first part. From Proposition A.2.2 in the Appendix, we know that there exists a continuous solution x to (4.1) at least locally on some small time interval $[0, T]$. As pointed out in Remark A.2.3, such a solution can be extended into a *noncontinuable solution* to (4.1), i.e., a continuous solution $x : [0, \hat{T}) \to \mathbb{R}^n$ to (4.1) on $[0, \hat{T})$ for some $\hat{T} \in (0, \bar{T}]$ such that either $\hat{T} = \bar{T}$ or x blows up at $\hat{T}$ in the sense that $\limsup_{t \nearrow \hat{T}} |x(t)| = \infty$. Thus if we can ensure that such a noncontinuable solution does not blow up, it follows that there exists a global solution to (4.1).

A classic way to prevent such a blow-up is to look for an *invariant set* D that "traps" solutions to (4.1), i.e., such that any solution x stays in D forever if $x_0 \in D$. If D is also compact, hence bounded, it follows that a solution to (4.1) cannot blow up, so that any noncontinuable solution is also a global solution. Thus we look for sufficient conditions that ensure that a given set $D \subseteq \mathbb{R}^n$ traps solutions to (4.1), which we now briefly explain.

As in the case of an ordinary differential equation (where $k \equiv 1$), we look for a set D such that f "pushes" x inwards at all points of the boundary ∂D. This boundary condition is not sufficient to ensure the invariance of D in general, as discussed below in (4.5). Indeed, although f pushes x back towards the interior of D whenever x approaches the boundary, the path-dependent "drift" caused by the convolution with k may outweigh f and push x out of D. However, under some additional assumptions on k, we can show that the path-dependent "drift" is mean-reverting towards a weighted mean of the past trajectory of x. Hence this drift can never push x out if D is convex, so that D is an invariant set.

To prove that the conditions mentioned above imply the invariance of D, we rely on some key observations from the proof of Abi Jaber et al. [1, Theorem 3.6]. There, sufficient conditions are given for the existence of a weak solution, taking values in $\mathbb{R}^n_+$, to a stochastic Volterra equation. Our insight is that those observations can be extended to find an invariant set to (4.1) that need only be convex, rather than an orthant $\mathbb{R}^n_+$ or a convex cone. This generalisation is useful because we need a bounded invariant set D in order to ensure the global existence. More generally, the set D can be replaced with an increasing family of sets $(D_t)_{t\geq 0}$ in order to deal with the time-dependence of f in a more flexible way; this is also given in the main Theorem 4.9 below. After the proof of that theorem, we compare it with a related recent result by Alfonsi [7].

We now start our analysis of the equation (4.1). In this section, the resolvent of the first of kind of k shall play a key role, and so we recall its definition (which is also given in Definition I.2.4).

Definition 4.1. Let $k \in L^1_{\mathrm{loc}}(I;\mathbb{R}_+)$. A measure L^k on $(I, \mathcal{B}(I))$ is a *resolvent of the first kind for* k if it holds that

$$(k * L^k)(t) := \int_{[0,t]} k(t-u) L^k(du) = 1 \quad \text{for Lebesgue-a.a. } t \in I.$$

We say that L^k is *nonincreasing* if the map $s \mapsto L^k([s, s+h])$ is nonincreasing on I_h for each $h \in (0, \bar{T})$, where we write $I_h := [0, \bar{T} - h]$ if I is right-closed and $I_h := [0, \bar{T} - h)$ if I is right-open.

As mentioned after Definition I.2.4, if k admits a resolvent of the first kind L^k, then it is unique by Gripenberg et al. [59, Theorem 5.5.2]. We can now introduce the following standing assumptions on the kernel k, which are the same as the conditions imposed in [1, Theorem 3.6]. We note that the assumptions allow for k to blow up at 0, which is the case for kernels of the fractional type $k(t) \propto t^{\alpha-1}$ for $\alpha \in (0, 1)$.

Assumption 4.2. Throughout this section, we fix a kernel $k \in L^1_{\mathrm{loc}}(I; \mathbb{R}_+)$ satisfying the following conditions:

1) k is not identically zero, nonincreasing and continuous on $I \setminus \{0\}$.

2) k admits a resolvent of the first kind L^k that is a nonnegative and nonincreasing measure on I.

We recall from Chapter I that a kernel $k : (0, \infty) \to \mathbb{R}_+$ is *completely monotone* if it is infinitely differentiable and $(-1)^m \partial^m k \geq 0$ for each $m \in \mathbb{N}$; see after Assumption I.2.7. We argued there that if k is completely monotone, then it satisfies conditions 2)–4) of Assumption I.2.7, and by [59, Theorem 5.5.4], it also satisfies Assumption 4.2. Completely monotone kernels (which include fractional kernels) are our main examples of interest, but the weaker conditions in Assumption 4.2 are sufficient for our results here.

We now start to work towards a proof of the main Theorem 4.9. The key idea is explained below in (4.5) and Lemma 4.5, where we show that for a kernel k satisfying Assumption 4.2, the path-dependent "drift" of a solution to (4.1) caused by the convolution with k is mean-reverting towards the past trajectory of x. To that end, we follow along the lines of the proof of [1, Theorem 3.6], by using Assumption 4.2 to deduce some further properties of k and some related functions. In the following, we define the function $\varphi_h : I_h \to \mathbb{R}_+$ by

$$\varphi_h = \Delta_h k * L^k, \tag{4.2}$$

where $\Delta_h k : I_h \to \mathbb{R}_+$ is the shifted kernel defined by $\Delta_h k(t) = k(t + h)$.

Lemma 4.3. *For any $h \in (0, \bar{T})$, φ_h is right-continuous, nonnegative, nondecreasing and bounded above by* 1.

Proof. Since $\Delta_h k$ and L^k are nonnegative by Assumption 4.2, so is $\varphi_h = \Delta_h k * L^k$. It is shown in the proof of [1, Theorem 3.6], specifically in [1, (3.9) and (3.10)], that φ_h is nondecreasing and bounded above by 1 under Assumption 4.2 on k. To show that φ_h is right-continuous, fix some $t \in [0, \bar{T})$. Since k is continuous on $I \setminus \{0\}$ by Assumption 4.2, we have for each $s \geq 0$ that

$$\mathbf{1}_{[0,t']}(s)k(t' + h - s) \longrightarrow \mathbf{1}_{[0,t]}(s)k(t + h - s)$$

as $t' \searrow t$. Note that both sides above are bounded by $k(h) < \infty$, as k is nonnegative and nonincreasing by Assumption 4.2. It thus follows by the dominated convergence theorem that

$$\begin{aligned}\lim_{t' \searrow t} \varphi_h(t') &= \lim_{t' \searrow t} \int_{[0,t']} k(t' + h - s) L^k(ds) \\ &= \int_{[0,t]} k(t + h - s) L^k(ds) = \varphi_h(t).\end{aligned}$$

□

Next, we show how φ_h can be used to construct for each $t \in I$ a certain probability measure $\mu_{h,t}$ supported on $[0, t]$. The construction involves the Lebesgue–Stieltjes measure associated with φ_h, which is well defined as φ_h is right-continuous and increasing. The observation that $\mu_{h,t}$ is not just a nonnegative measure but a probability measure is a key one, since this is what later allows us to consider invariant sets that are only convex, rather than convex cones as in [1]. The connection between $\mu_{h,t}$ and (4.1) will be shown in Lemma 4.5.

Lemma 4.4. *Fix $h \in (0, \bar{T})$ and $t \in I_h$. There exists a probability measure $\mu_{h,t}$ on $(I, \mathcal{B}(I))$ given by*

$$\mu_{h,t}(A) = \big(1 - \varphi_h(t)\big)\mathbf{1}_A(0) + \varphi_h(0)\mathbf{1}_A(t) + \int_{(0,t]} \mathbf{1}_A(t - s) d\varphi_h(s) \tag{4.3}$$

for each $A \in \mathcal{B}(I)$. Moreover, $\mu_{h,t}$ is supported on $[0, t]$, and for any bounded measurable function $x : I \to \mathbb{R}^n$, it holds that

$$\int_I x(s) d\mu_{h,t}(s) = \big(1 - \varphi_h(t)\big)x(0) + \varphi_h(0)x(t) + \int_{(0,t]} x(t - s) d\varphi_h(s). \tag{4.4}$$

Proof. We have from Lemma 4.3 that φ_h is nonnegative, bounded above by 1 and nondecreasing. From these properties and the definition of $\mu_{h,t}$, we obtain $\mu_{h,t}(A) \geq 0$ for all $A \in \mathcal{B}(I)$ since each of the summands in (4.3) is nonnegative. To show that $\mu_{h,t}$ is a measure, consider a sequence (A_n) of pairwise disjoint sets in $\mathcal{B}(I)$, so that $\mathbf{1}_{\bigcup_{n\in\mathbb{N}} A_n} = \sum_{n\in\mathbb{N}} \mathbf{1}_{A_n}$. As $\mathbf{1}_{\bigcup_{n\in\mathbb{N}} A_n} \le 1$, it follows by the dominated convergence theorem that

$$\begin{aligned}\int_{(0,t]} \mathbf{1}_{\bigcup_{n\in\mathbb{N}} A_n}(t-s)d\varphi_h(s) &= \int_{(0,t]} \sum_{n\in\mathbb{N}} \mathbf{1}_{A_n}(t-s)d\varphi_h(s)\\ &= \sum_{n\in\mathbb{N}} \int_{(0,t]} \mathbf{1}_{A_n}(t-s)d\varphi_h(s).\end{aligned}$$

Thus we have $\mu_{h,t}(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu_{h,t}(A_n)$, so that $\mu_{h,t}$ is a (nonnegative) measure on $[0,\bar{T})$. It is also clear that $\mu_{h,t}$ is supported on $[0,t]$, since each of the terms in (4.3) vanishes if $A \cap [0,t] = \emptyset$. Moreover, we have

$$\mu_{h,t}([0,t]) = \big(1-\varphi_h(t)\big) + \varphi_h(0) + \int_{(0,t]} d\varphi_h(s) = 1,$$

so that $\mu_{h,t}([0,\bar{T})) = \mu_{h,t}([0,t]) = 1$ and $\mu_{h,t}$ is a probability measure. Finally, note that (4.4) follows immediately from the definition (4.3) if $x = \mathbf{1}_A$ for some $A \in \mathcal{B}(I)$. By the dominated convergence theorem, (4.3) then extends to any bounded measurable $x : I \to \mathbb{R}$. We also get (4.3) if x takes values in $\mathbb{R}^n$ by considering each component separately. □

We are now ready to introduce the key idea for the proof of the main result. As motivation, consider a simple integral equation of the form

$$x(t) = x_0 + \int_0^t y(s)ds, \quad t \ge 0,$$

so that $x'(t) = y(t)$. For small $h > 0$, we have the decomposition

$$x(t+h) = x(t) + \int_t^{t+h} y(s)ds.$$

Thus in order to "trap" x within a set D, we need to ensure that whenever $x(t)$ is close to the boundary ∂D, the "drift term" $\int_t^{t+h} y(s)ds$ pushes x away from the boundary. Now consider instead a convolution equation of the form

$$x(t) = x_0 + \int_0^t k(t-s)y(s)ds, \quad t \ge 0.$$

Proceeding in a similar way, we obtain the decomposition

$$\begin{aligned} x(t+h) &= x_0 + \int_0^{t+h} k(t+h-s)y(s)ds \\ &= x_0 + \int_0^t k(t+h-s)y(s)ds + \int_t^{t+h} k(t+h-s)y(s)ds \\ &= \hat{x}_h(t) + \int_0^h k(s)y(t+h-s)ds, \end{aligned} \tag{4.5}$$

where

$$\hat{x}_h(t) := x_0 + (\Delta_h k * y)(t) = x_0 + \int_0^t k(t+h-s)y(s)ds$$

now takes the place of $x(t)$. By analogy, the term $\int_0^h k(s)y(t+h-s)ds$ in (4.5) can be viewed as a "drift" induced by y that pushes x away from ∂D. On the other hand, the "past term" $\hat{x}_h(t)$ now also depends on h, and this can cause issues if $\hat{x}_h(t)$ exits D as h increases, i.e., if the path-dependent "drift" $\hat{x}_h(t) - x(t)$ moves towards ∂D and is able to outweigh the effect of y.

Under Assumption 4.2 on k, we now show that $\hat{x}_h(t)$ belongs to the convex hull $\operatorname{conv}(\{x(s) : s \in [0,t]\})$ for all h. This is the key idea that allows us to find convex invariant sets D for (4.1), since it ensures that $\hat{x}_h(t)$ cannot leave the convex hull of the past trajectory of x and thus stays within D. On the other hand, if D is not convex, then the term $\hat{x}_h(t)$ may pull x outside of D even if the "local term" pushes in the opposite direction; this is unlike the classical case where $x(t) \in D$ by construction.

Lemma 4.5. *Let $x_0 \in \mathbb{R}^n$ and $y : I \to \mathbb{R}^n$ be a continuous function. Define $x : I \to \mathbb{R}^n$ and $\hat{x}_h : I \to \mathbb{R}^n$ by*

$$\begin{aligned} x(t) &= x_0 + \int_0^t k(t-s)y(s)ds, \\ \hat{x}_h(t) &:= x_0 + (\Delta_h k * y)(t), \quad t \in I. \end{aligned}$$

Then for any $h > 0$ and $t \in I$, it holds that

$$\hat{x}_h(t) = \int_{[0,t]} x(s)\mu_{h,t}(ds) \tag{4.6}$$

so that $\hat{x}_h(t)$ belongs to the convex hull $\operatorname{conv}(\{x(s) : s \in [0,t]\})$.

Proof. Fix $h > 0$. We want to apply [1, Lemma 2.6], where we set $K := k$, $L := L^k$, $Z := \int_0^\cdot y(s)ds$, $F := \Delta_h k$ and $X := x$ in their notation. Note that the (deterministic) process

$$K * dZ = k * y = x - x_0 = X - X_0$$

is continuous, where the convolution with respect to $dZ = y\,dt$ is interpreted in the sense of [1, Equation (2.1)]. Recalling the definition (4.2), we also have that

$$F * L = \Delta_h k * L^k = \varphi_h$$

is right-continuous, bounded and nondecreasing by Lemma 4.3. Then [1, Equation (2.15)] gives that (in our notation)

$$(\Delta_h k * y)(t) = \varphi_h(0)x(t) - \varphi_h(t)x(0) + \int_{[0,t]} x(t-s)d\varphi_h(s), \quad t \in I.$$

The equality holds for all $t \in I$ since $\Delta_h k * y$ is continuous like y; see Gripenberg et al. [59, Theorem 2.2.2(i)]. By adding $x_0 = x(0)$ to both sides and plugging in (4.4), we obtain (4.6). By Lemma 4.4, $\mu_{h,t}$ is a probability measure on $[0,t]$ and hence

$$\hat{x}_h(t) = \int_{[0,t]} x(s)\mu_{h,t}(ds) \in \operatorname{conv}(\{x(s) : s \in [0,t]\}),$$

as claimed. (Note that by Dudley [42, Theorem 10.2.6], the integral indeed belongs to the convex hull, not just its closure.) □

We are now almost ready for the main result. Before proceeding, we need to introduce some additional notions from convex geometry on $\mathbb{R}^n$ that help to formalise the intuition that both of the "drift terms" in (4.5) push x away from the boundary of a convex set D.

Definition 4.6. Let $D \subseteq \mathbb{R}^n$ be a compact convex set with nonempty interior. For any $p \in \partial D$, a unit vector n is *outward normal to D at p* if $(x-p)^\top n \leq 0$ for all $x \in D$. We say that $p \in \partial D$ is a *regular boundary point* of D if there exists a unique unit vector n_p that is outward normal to D at p.

The following technical lemma gives a useful property of any regular boundary point p with outward normal n_p. In words, we have for each circular cone with apex p and axis $-n_p$ pointing "inwards" towards D that a small tip of that cone (except for p itself) is contained in the interior D°. We note that c below parametrises the angle between $-n_p$ and the generatrix of the cone, so that the extreme case $c = 1$ corresponds to a ray from p in the direction $-n_p$, whereas

$c = 0$ corresponds to the closed half-space defined by the hyperplane through p with outward normal n_p. This is helpful for dealing with the case where the initial value x_0 of (4.1) belongs to the boundary of D.

Lemma 4.7. *Let $D \subseteq \mathbb{R}^n$ be a closed convex set with nonempty interior. Fix a regular boundary point p with a unique outward normal n_p to D at p. Then for every $c \in (0, 1]$, there exists $\epsilon > 0$ such that*

$$\left\{ y \in B_\epsilon(p) \setminus \{p\} : \frac{(y-p)^\top n_p}{|y-p|} \leq -c \right\} \subseteq D^\circ.$$

Proof. For contradiction, suppose that the statement does not hold for some $c > 0$. Then for each $m \in \mathbb{N}$, there exists some $y_m \in B_{1/m}(p) \setminus \{p\}$ such that $\frac{(y_m-p)^\top n_p}{|y_m-p|} \leq -c$ and $y_m \notin D^\circ$. Define the unit vectors $u_m := \frac{y_m-p}{|y_m-p|}$. By compactness of the unit sphere, we can assume (by taking a subsequence) that $u_m \to u$ as $m \to \infty$, for some unit vector u. Since $u_m^\top n_p \leq -c$ for each m, it also holds that $u^\top n_p \leq -c$.

We claim that the ray $R_{p,u} := \{p + \gamma u : \gamma \geq 0\}$ does not intersect D°. We also show this claim by contradiction, and then return to the main statement. Suppose that $p + \gamma u \in D^\circ$ for some $\gamma \geq 0$. We can thus find an open ball $B_\delta(p+\gamma u) \subseteq D^\circ$ for some $\delta > 0$. Using the convexity property of D with respect to p and all points in $B_\delta(p + \gamma u)$, it follows that

$$\{p + \gamma' u' : \gamma' \in [0, \gamma], u' \in B_\delta(u)\} \subseteq D.$$

By taking the interior of both sets, we deduce that $p+\gamma' u' \in D^\circ$ for all $\gamma' \in (0, \gamma)$ and $u' \in B_\delta(u)$. Note that we have $y_m = p + |y_m - p| u_m$ for each $m \in \mathbb{N}$, where $u_m \to u$ and $|y_m - p| \leq 1/m \to 0$ as $m \to \infty$. Hence we get $u_m \in B_\delta(u)$ and $0 < |y_m - p| < \gamma$ for m large enough, so that $y_m \in D^\circ$. This contradicts the construction of y_m as we assumed that $y_m \notin D^\circ$. Thus we must have $R_{p,u} \cap D^\circ = \emptyset$.

We can now conclude the main argument by contradiction. Since D° and $R_{p,u}$ are convex and we have shown that they are disjoint, by the hyperplane separation theorem (see Rockafellar [107, Theorems 11.1 and 11.3]), there exists a unit vector $\tilde{n}_p$ such that

$$\sup_{q \in D^\circ} \tilde{n}_p^\top q = \sup_{q \in D} \tilde{n}_p^\top q \leq \inf_{q \in R_{p,u}} \tilde{n}_p^\top q \leq \tilde{n}_p^\top p. \tag{4.7}$$

Hence $\tilde{n}_p$ is outward normal at p (as $p \in R_{p,u}$). We have

$$\inf_{q \in R_{p,u}} n_p^\top q = n_p^\top p + \inf_{\gamma \geq 0} \gamma n_p^\top u = -\infty$$

as $n_p^\top u \leq -c < 0$ by the construction of u. Thus $\tilde{n}_p \neq n_p$ since n_p cannot satisfy (4.7). As n_p and $\tilde{n}_p$ are both outward normal at p, this contradicts the fact that p is a regular boundary point. □

We are now ready to move on to the first main theorem, from which we finally obtain as a corollary sufficient conditions for the global existence of a solution to (4.1) Recall that we fix the time interval $I = [0, \bar{T}]$ or $[0, \bar{T})$ as well as a kernel $k \in L^1_{\mathrm{loc}}(I; \mathbb{R}_+)$ satisfying Assumption 4.2.

Theorem 4.8. *Let $x_0 \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times I \to \mathbb{R}^n$ be continuous. Let $\hat{T} \in I$ and suppose that $x : [0, \hat{T}] \to \mathbb{R}^n$ is a continuous solution to the equation*

$$x(t) = x_0 + \int_0^t k(t-s) f\big(x(s), s\big) ds, \quad 0 \leq t \leq \hat{T}. \tag{4.8}$$

Let $D \subseteq \mathbb{R}^n$ be a closed convex set with nonempty interior such that $x_0 \in D$ is either an interior point ($x_0 \in D^\circ$) or a regular boundary point of D. Moreover, suppose that for each boundary point $p \in \partial D$, there exists a unit vector n_p that is outward normal to D at p and such that $f(p,t)^\top n_p < 0$ for all $t \in [0, \hat{T}]$. Then $x(t) \in D^\circ$ for all $t \in (0, \hat{T}]$.

Proof. It suffices to prove that

$$\tau := \inf\{t \in (0, \hat{T}] : x(t) \notin D^\circ\}$$

is equal to ∞. Note that we exclude $t = 0$ as x may start at a regular boundary point. Suppose for a contradiction that $\tau < \infty$; we assume for now that $\tau > 0$. As x is continuous by assumption, we have $x(\tau) \in \partial D$, whereas $x(s) \in D^\circ$ for $0 < s < \tau$. By assumption, there exists an outward normal n_p at $p := x(\tau)$ such that $f(p, \tau)^\top n_p < 0$. Hence by the continuity of x and f, there exists $h \in (0, \tau)$ such that $f(x(s), s)^\top n_p < 0$ for all $s \in [\tau - h, \tau]$. Since x is a solution to (4.8), it follows by (4.5) with $y := f(x(\cdot), \cdot)$ and $t := \tau - h$ that

$$p = x(\tau) = \hat{x}_h(\tau - h) + \int_0^h k(s) f\big(x(\tau - s), \tau - s\big) ds. \tag{4.9}$$

Moreover, Lemma 4.5 with $t = \tau - h$ yields

$$\hat{x}_h(\tau - h) = \int_{[0,\tau-h]} x(s) d\mu_{h,\tau-h}(s) \in \operatorname{conv}(\{x(s) : s \in [0, \tau - h]\})$$

so that $\hat{x}_h(\tau - h) \in D$ since D is convex and $x(s) \in D$ for all $s \in [0, \tau]$ by the definition of τ. Thus we have $(p - \hat{x}_h(\tau - h))^\top n_p \geq 0$ by Definition 4.6 because n_p is outward normal to D at p. By taking the inner product of (4.9) with n_p and rearranging, we obtain

$$\begin{aligned} \int_{\tau-h}^{\tau} k(\tau - s) f\big(x(s), s\big)^\top n_p ds &= \int_0^h k(s) f\big(x(\tau - s), \tau - s\big)^\top n_p ds \\ &= \big(p - \hat{x}_h(\tau - h)\big)^\top n_p \geq 0. \end{aligned}$$

But this leads to a contradiction, since $f(x(s), s)^\top n_p < 0$ for $s \in [\tau - h, \tau]$ by the choice of h, and k is nonnegative, nonincreasing and not identically 0 by Assumption 4.2. Thus we cannot have $\tau \in (0, \hat{T}]$.

We now return to the case $\tau = 0$. By continuity of x, this is only possible if $x_0 \in \partial D$, and by assumption, $p := x_0$ must then be a regular boundary point of D. Let n_p be the unique unit vector that is outward normal to D at p. Since $p \in \partial D$, the boundary condition gives $f(p, 0)^\top n_p < 0$ and in particular $f(p, 0) \neq 0$. By continuity of f and x, we can find $h > 0$, $C > 0$ and $c > 0$ such that

$$C \geq |f(x(s), s)| \geq c \quad \text{and} \quad \frac{f(x(s), s)^\top n_p}{|f(x(s), s)|} \leq -c \quad \text{for all } s \in [0, h]. \tag{4.10}$$

By Lemma 4.7, there exists $\epsilon > 0$ such that

$$\tilde{D} := \left\{ y \in B_\epsilon(p) \setminus \{p\} : \frac{(y - p)^\top n_0}{|y - p|} \leq -c \right\} \subseteq D^\circ. \tag{4.11}$$

We now want to show that there exists some $h' \in (0, h)$ such that $x(t) \in \tilde{D}$ for all $t \in (0, h']$; namely, we need to check that $|x(t) - p| < \epsilon$, $x(t) \neq p$ and that $x(t)$ satisfies the inequality in (4.11) for small enough $t > 0$. To show this claim, note that (4.8) and the first part of (4.10) yield

$$\begin{aligned} |x(t) - p| = \left| \int_0^t k(t - s) f\big(x(s), s\big) ds \right| &\leq \int_0^t k(t - s) \big| f\big(x(s), s\big) \big| ds \\ &\leq C \int_0^t k(s) ds \end{aligned}$$

for $t \in (0,h)$, so that $|x(t)-p| < \epsilon$ if $\int_0^t k(s)ds < \epsilon/C$. Since $\int_0^{h'} k(s)ds \searrow 0$ as $h' \searrow 0$ by the dominated convergence theorem, we can choose h' small enough so that $|x(t)-p| < \epsilon$ for all $t \in (0,h']$. This shows the first condition. By the last inequality in (4.10), we also get

$$\begin{aligned}\big(x(t)-p\big)^\top n_p &= \int_0^t k(t-s) f\big(x(s),s\big)^\top n_p ds \\ &\le -c \int_0^t k(t-s)\big|f\big(x(s),s\big)\big|ds, \quad 0 \le t \le h. \end{aligned} \tag{4.12}$$

Thus for $0 \le t \le h$ we have

$$\big(x(t)-p\big)^\top n_p \le -c \int_0^t k(t-s)\big|f\big(x(s),s\big)\big|ds \le -c^2 \int_0^t k(s)ds.$$

Since k is nonnegative, nonincreasing and not identically 0 on $(0,\infty)$ by Assumption 4.2, we have $\int_0^t k(s)ds > 0$ and hence $x(t) \neq p$ for $t \in (0,h]$. This shows the second condition. Returning to (4.12), plugging in (4.8) yields the bound

$$\begin{aligned}\big(x(t)-p\big)^\top n_p &\le -c \int_0^t k(t-s)\big|f\big(x(s),s\big)\big|ds \\ &\le -c\left|\int_0^t k(t-s) f\big(x(s),s\big)ds\right| = -c|x(t)-p|, \quad 0 \le t \le h.\end{aligned}$$

Rearranging, we thus get

$$\frac{(x(t)-p)^\top n_p}{|x(t)-p|} \le -c, \quad 0 < t \le h.$$

Hence we have $x(t) \in \tilde{D}$ for $t \in (0,h']$, as we have checked each of the conditions in the definition of $\tilde{D}$. Since $\tilde{D} \subseteq D^\circ$ by (4.11), this contradicts the assumption that $\tau = 0$. Thus we have obtained a contradiction in both cases $\tau = 0$ and $\tau \in (0,\hat{T}]$, so that $\tau = \infty$ as claimed, and this concludes the proof. □

As a corollary, we obtain the existence of a global solution to (4.1). As mentioned at the beginning of the section, we generalise slightly by considering an increasing family $(D_t)_{t\in I}$ such that D_t "traps" the solution up to time t; this generalisation can be useful when I is open on the right.

Theorem 4.9. *Let $x_0 \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times I \to \mathbb{R}^n$ be continuous. Suppose that $(D_t)_{t\in I}$ is an increasing family of compact convex sets in $\mathbb{R}^n$ with nonempty interior such that $x_0 \in D_0$ is either an interior point ($x_0 \in D_0^\circ$) or a regular* *boundary point of* D_0. *Suppose moreover that for each* $t \in I$ *and boundary point* $p \in \partial D_t$, *there exists a unit vector* $n_{p,t}$ *that is outward normal to* D_t *at* p *and such that* $f(s,p)^\top n_{p,t} < 0$ *for all* $s \in [0,t]$. *Then there exists a continuous solution* $x : I \to \mathbb{R}^n$ *to the equation*

$$x(t) = x_0 + \int_0^t k(t-s) f\big(x(s), s\big) ds, \quad t \in I. \tag{4.13}$$

Proof. By Gripenberg et al. [59, Theorem 12.1.1] (see also Proposition A.2.2), there is a continuous noncontinuable solution x to (4.13). That is, either there exists a continuous global solution $x : I \to \mathbb{R}^n$ to (4.13), or there exists $\tilde{T} \in (0, \bar{T})$ (or $\tilde{T} \in (0, \bar{T}]$ if I is right-closed) and a continuous solution $x : [0, \tilde{T}) \to \mathbb{R}^n$ to (4.13) on $[0, \tilde{T}) \subsetneq I$ such that $\limsup_{t \nearrow \tilde{T}} |x(t)| = \infty$. Suppose that there exists a solution of the latter type on $[0, \tilde{T})$. If we can show that this leads to a contradiction, then there must exist a global solution on I instead, and that will conclude the proof.

First, we note that if $x(0) = x_0$ is an interior point of D_0, then it is also an interior point of $D_t \supseteq D_0$ for each $t \in I$. On the other hand, if x_0 is a regular boundary point of D_0, we claim that it is also an interior or regular boundary point of D_t for each $t \in I$. Suppose otherwise; since $x_0 \in D_0 \subseteq D_t$, we must have that $x_0 \in D_t \setminus D_t^\circ = \partial D_t$ and x_0 is not a regular boundary point of D_t for some $t \in I$. By Definition 4.6, there exist at least two unit vectors that are outward normal to D_t at x_0. But since $D_0 \subseteq D_t$, those vectors must also be outward normal to D_0 at x_0, which leads to a contradiction since x_0 is a regular boundary point of D_0.

We have shown that $x(0)$ is an interior or regular boundary point of $D_{\hat{T}}$ for each $\hat{T} \in [0, \tilde{T})$, and we now return to the main argument. Since the other conditions in Theorem 4.8 are satisfied by assumption, we apply Theorem 4.8 to (4.13) on $[0, \hat{T}]$ with $D = D_{\hat{T}}$ to obtain that $x(t) \in D_{\hat{T}}$ for all $t \in [0, \hat{T}]$. Taking a union over all $\hat{T} \in [0, \tilde{T})$, it follows that $x(t) \in D_{\tilde{T}}$ for all $t \in [0, \tilde{T})$, and hence x is bounded on $[0, \tilde{T})$ as $D_{\tilde{T}}$ is compact. But this leads to a contradiction with the assumption that $\limsup_{t \nearrow \tilde{T}} |x(t)| = \infty$. Therefore, $\tilde{T} = \bar{T}$ and there exists a continuous global solution to (4.13), as claimed. □

As noted at the beginning of this section, Theorem 4.8 is closely related to the recent result in Alfonsi [7, Theorem 3.2]. In addition to deterministic Volterra equations such as (4.13), [7, Theorem 3.2] also covers stochastic Volterra equations (SVEs) driven by a Brownian motion, and likewise gives sufficient conditions for a closed and convex set D to be invariant. The proof of [7, Theorem 3.2] is based on constructing an approximation to the solution of the SVE and using the fact that completely monotone kernels *preserve monotonicity* in the sense of [7, Definition 2.1] due to [7, Theorem 2.3]. The latter property is used to ensure that the approximate solution is trapped by D, which implies by taking a limit that the true solution stays in D as well.

On the other hand, Theorem 4.8 does have some weaker assumptions: namely, the time interval is not assumed to be $[0,\infty)$, f in (4.8) is allowed to depend on t, and the kernel k need only satisfy Assumption 4.2. The latter condition is weaker than requiring that k be completely monotone (see Abi Jaber et al. [1, Example 3.7]), although completely monotone kernels are our main example of interest. It is an open question whether Assumption 4.2 implies that k preserves monotonicity in the sense of [7, Definition 2.1].

4.2 Application to $g_z^\star$

As mentioned at the beginning of the section, our motivating goal for the first part is to show Theorem 2.23, i.e., the existence of solutions to the Riccati–Volterra equations (2.60) with $z = a + \mathrm{i}b$ for some fixed $a \in \mathbb{R}$ and all $b \in \mathbb{R}$ on a common time interval. We recall here that (2.60) reads

$$g_z^\star(t) = f_z^\star\big((\hat{\kappa} * g_z^\star)(t), h^\star(t)\big) \tag{4.14}$$

for $t \geq 0$, where $f_z^\star : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ is defined by

$$f_z^\star(x,h) = \frac{(z^2 - z)\sigma^2}{2} + \left(z\sigma\varrho - \frac{\mu\varrho}{\sigma}\right)x + (1-\varrho^2)hx + \frac{x^2}{2} \tag{4.15}$$

for $(x,h) \in \mathbb{C}\times\mathbb{R}$, and $h^\star = \hat{\kappa} * g^\star$, where $g^\star$ satisfies the Riccati–Volterra equation

$$g^\star(t) = -\frac{\mu^2}{\sigma^2} - \frac{2\varrho\mu}{\sigma}(\hat{\kappa} * g^\star)(t) + \frac{1}{2}(1-2\varrho^2)\big((\hat{\kappa} * g^\star)(t)\big)^2. \tag{4.16}$$

We now state the main result; the proof is postponed to the end of the section.

Theorem 4.10. *Suppose that $\kappa : (0,\infty) \to \mathbb{R}_+$ is a completely monotone kernel satisfying Assumption I.2.7. Moreover, suppose that for some $\bar{T} > 0$, there exist continuous solutions $g^\star : [0,\bar{T}] \to \mathbb{R}$ and $g_a^\star : [0,\bar{T}] \to \mathbb{R}$ to (4.16) and (4.14) with $z = a \in \mathbb{R}$, respectively. Then there exists a unique continuous solution $g_z^\star : [0,\bar{T}] \to \mathbb{C}$ to (4.14) for each $z \in a + \mathrm{i}\mathbb{R}$.*

We note that the complete monotonicity of κ is inherited by $\hat{\kappa}$ due to Gripenberg et al. [59, Theorem 5.3.1] and the definition (I.2.12) of $\hat{\kappa}$. This is key, since by [59, Theorem 5.5.4], it follows that $\hat{\kappa}$ satisfies Assumption 4.2, which allows us to apply our results from the first part of this section.

We also remark that Theorem 4.10 is not vacuous, i.e., there exists some $\bar{T} > 0$ such that both equations (4.16) and (4.14) with $z = a \in \mathbb{R}$ can be solved simultaneously. Indeed, (4.16) can be solved on $[0, T^\star]$, where $T^\star > 0$ is given in Theorem I.3.8. Then by Corollary 2.16, the equation (4.14) with $z = a \in \mathbb{R}$ admits a solution on a smaller interval $[0, \bar{T}]$, where $\bar{T} := T^\star_a > 0$. As noted in Remark A.2.1 in the Appendix, the solutions $g^\star$ and $g^\star_a$ to (4.16) and (4.14) with $z = a \in \mathbb{R}$ are real-valued, since each of the inputs to the equations are real-valued; this follows by the uniqueness of the solutions to (4.16) and (4.14).

Our strategy to prove Theorem 4.10 is to first transform (4.14) into a suitable form for the application of Theorem 4.9, and then to check that the modified equation satisfies the conditions required by the theorem. The first part is done in the following lemma by a straightforward transformation: we start by taking a convolution with $\hat{\kappa}$ to switch between the two types of Riccati–Volterra equations as explained before Corollary A.2.6, and then we subtract the solution for $z = a$. Indeed, the equation (4.14) can be rewritten in terms of $h^\star_z := \hat{\kappa} * g^\star_z$ as

$$h^\star_z(t) = \int_0^t \hat{\kappa}(t-s) g^\star_z(s) ds = \int_0^t \hat{\kappa}(t-s) f^\star_z\big(h^\star_z(s), h^\star(s)\big) ds, \quad 0 \le t \le \bar{T}. \tag{4.17}$$

For $z = a + \mathrm{i}b$, we make the ansatz $h^\star_z(t) = h^\star_a(t) + \tilde{h}_{a,b}(t)$ for some suitable function $\tilde{h}_{a,b} : [0, \bar{T}] \to \mathbb{C}$ (which is not related to the functions defined in Proposition 2.26). For $h^\star_z$ to satisfy (4.17), $\tilde{h}_{a,b}$ must satisfy the equation

$$\tilde{h}_{a,b}(t) = h^\star_z(t) - h^\star_a(t) = \int_0^t \hat{\kappa}(t-s) \tilde{f}_{a,b}\big(\tilde{h}_{a,b}(s), s\big) ds, \quad 0 \le t \le \bar{T}, \tag{4.18}$$

where

$$\begin{aligned} \tilde{f}_{a,b}\big(\tilde{h}_{a,b}(s), s\big) &= f^\star_z\big(h^\star_z(s), h^\star(s)\big) - f^\star_a\big(h^\star_a(s), h^\star(s)\big) \\ &= f^\star_{a+\mathrm{i}b}\big(h^\star_a(s) + \tilde{h}_{a,b}(s), h^\star(s)\big) - f^\star_a\big(h^\star_a(s), h^\star(s)\big). \end{aligned}$$

We now reverse this argument, i.e., we show that a solution $\tilde{h}_{a,b}$ to (4.18) yields a solution to the original equation (4.14).

Lemma 4.11. *Consider the setup of Theorem 4.10 and fix $b \in \mathbb{R}$. Suppose that* *there exists a continuous solution* $\tilde{h}_{a,b} : [0, \bar{T}] \to \mathbb{C}$ *to* (4.18) *on* $[0, \bar{T}]$, *where* $\tilde{f}_{a,b} : \mathbb{C} \times [0, \bar{T}] \to \mathbb{C}$ *is given by*

$$\begin{aligned} \tilde{f}_{a,b}(x,t) = &\frac{(-b^2 + \mathrm{i}(2ab - b))\sigma^2}{2} + \mathrm{i}b\sigma\varrho h^\star_a(t) \\ &+ \Big((a + \mathrm{i}b)\sigma\varrho - \frac{\mu\varrho}{\sigma} + (1 - \varrho^2)h^\star(t) + h^\star_a(t)\Big)x + \frac{x^2}{2}. \end{aligned} \tag{4.19}$$

Then the function $g^\star_{a+\mathrm{i}b} : [0, \bar{T}] \to \mathbb{C}$ *defined by*

$$g^\star_{a+ib}(t) := f^\star_{a+\mathrm{i}b}\big(h^\star_a(t) + \tilde{h}_{a,b}(t), h^\star(t)\big)$$

is a continuous solution to (4.14) *on* $[0, \bar{T}]$ *with* $z = a + ib$.

Proof. The continuity of $g^\star_{a+ib}$ follows immediately from that of $\tilde{f}_{a,b}$, $h^\star_a$ and $\tilde{h}_{a,b}$. To check that $g^\star_{a+ib}$ satisfies (4.14), note that we obtain

$$\begin{aligned} &f^\star_{a+\mathrm{i}b}\big(h^\star_a(s) + \tilde{h}_{a,b}(s), h^\star(s)\big) \\ &= \frac{((a + \mathrm{i}b)^2 - a - \mathrm{i}b)\sigma^2}{2} + \Big((a + \mathrm{i}b)\sigma\varrho - \frac{\mu\varrho}{\sigma}\Big)\big(h^\star_a(s) + \tilde{h}_{a,b}(s)\big) \\ &\quad + (1 - \varrho^2)h^\star(s)\big(h^\star_a(s) + \tilde{h}_{a,b}(s)\big) + \frac{(h^\star_a(s) + \tilde{h}_{a,b}(s))^2}{2} \\ &= \tilde{f}_{a,b}\big(\tilde{h}_{a,b}(s), s\big) + f^\star_a\big(h^\star_a(s), h^\star(s)\big), \quad 0 \le s \le \bar{T} \end{aligned}$$

after collecting the terms corresponding to each function. Thus by (4.17) and (4.18), we have

$$\begin{aligned} h^\star_a(t) + \tilde{h}_{a,b}(t) &= \int_0^t \hat{\kappa}(t - s)\Big(\tilde{f}_{a,b}\big(\tilde{h}_{a,b}(s), s\big) + f^\star_a\big(h^\star_a(s), h^\star(s)\big)\Big)ds \\ &= \int_0^t \hat{\kappa}(t - s) f^\star_{a+\mathrm{i}b}\big(h^\star_a(s) + \tilde{h}_{a,b}(s), h^\star(s)\big)ds. \end{aligned}$$

Applying $f^\star_{a+\mathrm{i}b}(\cdot, h^\star(t))$ to both sides, we obtain

$$\begin{aligned} g^\star_{a+\mathrm{i}b}(t) &:= f^\star_{a+\mathrm{i}b}\big(h^\star_a(t) + \tilde{h}_{a,b}(t), h^\star(t)\big) \\ &= f^\star_{a+\mathrm{i}b}\Big(\int_0^t \hat{\kappa}(t - s) f^\star_{a+\mathrm{i}b}\big(h^\star_a(s) + \tilde{h}_{a,b}(s), h^\star(s)\big)ds, h^\star(t)\Big) \\ &= f^\star_{a+\mathrm{i}b}\Big(\int_0^t \hat{\kappa}(t - s) g^\star_{a+\mathrm{i}b}(s)ds, h^\star(t)\Big), \quad 0 \le t \le \bar{T}, \end{aligned}$$

so that $g^\star_{a+ib}$ satisfies (4.14) on $[0, \bar{T}]$ with $z = a + \mathrm{i}b$, as claimed. □

The second and more challenging step is to use Theorem 4.9 to deduce the existence of a global solution $\tilde{h}_{a,b}$ to (4.18) on $[0,\bar{T}]$. The main task is to find a suitable "trapping" set D. For an intuitive way to find such a set, note that the leading order term in (4.19) is simply $x^2/2$; so we consider the (ordinary) Riccati equation $z'(t) = z^2(t)$ and the vector field $f(z) = z^2$ on $\mathbb{C}$. For any starting point on the imaginary axis, the solution to the equation starts by moving to the left before curving towards the x-axis and then to the right towards 0; this is also clear from the explicit solution $z(t) = (1/z(0) - t)^{-1}$. Thus the solution to that equation can be trapped between the imaginary axis and and a suitable curve that encloses part of the half-space $\{\mathrm{Re}(z) < 0\}$; the hexagon D defined below is one possible choice. By making the hexagon large enough, we can ensure that only the leading order term in (4.19) plays a role in the behaviour of $\tilde{f}_{a,b}$ along ∂D. The exception to that is the vertical axis, which we cannot scale up away from 0, and hence a finer analysis is needed to make sure that the boundary condition holds there. Luckily, that is the case due to the transformation from Lemma 4.11. Indeed, by subtracting $h^\star_a$ from $h^\star_{a+\mathrm{i}b}$, we have removed the terms that would push the solution to the right starting from the imaginary axis, as can be seen by comparing the real parts of the 0-order terms in (4.15) and (4.19).

Lemma 4.12. *Consider the setup of Theorem 4.10 and fix $b \in \mathbb{R} \setminus \{0\}$. For $C > 0$, define the closed hexagon $D \subseteq \mathbb{C}$ (sketched below) by the six vertices*

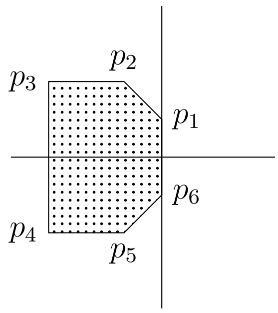
$$\begin{aligned} p_1(t) &:= \mathrm{i}C, & p_2(t) &:= -C + 2\mathrm{i}C, & p_3(t) &:= -3C + 2\mathrm{i}C, \\ p_4(t) &:= -3C - 2\mathrm{i}C, & p_5(t) &:= -C - 2\mathrm{i}C, & p_6(t) &:= -\mathrm{i}C. \end{aligned}$$

Then D is a compact convex set such that $0 \in D$ is a regular boundary point. Moreover, if $C > 0$ is large enough, then for each $p \in \partial D$, there exists an outward normal vector n_p to D at p such that $n_p^\top \tilde{f}_{a,b}(p,t) < 0$ for all $t \in [0,\bar{T}]$, where $\tilde{f}_{a,b}$ is defined by (4.19) and the inner product is taken with respect to the Euclidean metric on $\mathbb{R}^2 \cong \mathbb{C}$.

Proof. For any choice of $C > 0$, the boundary ∂D consists of six edges between the points p_1, p_2, p_3, p_4, p_5, p_6, p_1, which we take in this order (counterclockwise). Denote the respective edges, without the endpoints, by $E_1, E_2, E_3, E_4, E_5, E_6$ (i.e,

$E_1 = (p_1, p_2)$, etc.). We define a family of unit vectors $(n_p)_{p\in\partial D}$ by

$$n_p := \begin{cases} \frac{1}{\sqrt{2}}(1+\mathrm{i}), & p \in E_1 \cup \{p_1\}, \\ \mathrm{i}, & p \in E_2 \cup \{p_2\}, \\ -1, & p \in E_3 \cup \{p_3\}, \\ -\mathrm{i}, & p \in E_4 \cup \{p_4\}, \\ \frac{1}{\sqrt{2}}(1-\mathrm{i}), & p \in E_5 \cup \{p_5\}, \\ 1, & p \in E_6 \cup \{p_6\}, \end{cases}$$



so that n_p is an outward normal vector at p for each $p \in \partial D$. It is also clear that D is compact and convex with $0 \in \partial D$ a regular boundary point.

Note that the inner product $(p_1, p_2) \mapsto p_1^\top p_2$ on $\mathbb{R}^2$ is given by $\mathrm{Re}(p_1\overline{p_2})$ when we identify $\mathbb{R}^2 \cong \mathbb{C}$. Thus we need to check that for $C > 0$ large enough, we have $\mathrm{Re}(\tilde{f}_{a,b}(p,t)\overline{n_p}) < 0$ for all $t \in [0, \bar{T}]$ and $p \in \partial D$. We now consider each edge of D separately, including one vertex in each case. Note that the functions $h^\star$ and $h_a^\star$ in (4.19) are continuous and hence bounded on $[0, \bar{T}]$. Thus by considering the asymptotic behaviour as $C \to \infty$, we show that the inequality holds for each edge if $C > 0$ large enough. As it turns out, for E_1–E_5, we only need to consider the asymptotic behaviour of order C^2, so that only the term $x^2/2$ in (4.19) plays a role in those cases (the remaining terms are linear or constant in C). The analysis is more delicate in the case of E_6.

E_1: We can parametrise $p \in E_1 \cup \{p_1\}$ by $p = (-\gamma + (1+\gamma)\mathrm{i})C$ for some $\gamma \in [0, 1)$, so that

$$\tilde{f}_{a,b}(p,t) = \frac{p^2}{2} + O(C) = \frac{C^2}{2}\big(-1 - 2\gamma - 2(\gamma+\gamma^2)\mathrm{i}\big) + O(C).$$

Thus we have

$$\begin{aligned} \mathrm{Re}\big(\tilde{f}_{a,b}(p,t)\overline{n_p}\big) &= \frac{C^2}{2\sqrt{2}}\mathrm{Re}\Big(\big(-1-2\gamma-2(\gamma+\gamma^2)\mathrm{i}\big)(1-\mathrm{i})\Big) + O(C) \\ &= \frac{(-1-4\gamma-2\gamma^2)C^2}{2\sqrt{2}} + O(C), \end{aligned}$$

where

$$-1 - 4\gamma - 2\gamma^2 \le -1 < 0 \quad \text{for all } \gamma \in [0,1].$$

E_2: We can parametrise $p \in E_2 \cup \{p_2\}$ by $p = (-\gamma + 2\mathrm{i})C$ for some $\gamma \in [1, 3)$, so that

$$\tilde{f}_{a,b}(p,t) = \frac{p^2}{2} + O(C) = \frac{C^2}{2}(\gamma^2 - 4 - 2\gamma \mathrm{i}) + O(C).$$

Thus we have

$$\begin{aligned}\operatorname{Re}\big(\tilde{f}_{a,b}(p,t)\overline{n_p}\big) &= \frac{C^2}{2}\operatorname{Re}\big((\gamma^2 - 4 - 2\gamma\mathrm{i})(-\mathrm{i})\big) + O(C)\\ &= -\gamma C^2 + O(C),\end{aligned}$$

where

$$-\gamma \le -1 < 0 \quad \text{for all } \gamma \in [1,3].$$

E_3: We can parametrise $p \in E_3 \cup \{p_3\}$ by $p = (-3+2\gamma\mathrm{i})C$ for some $\gamma \in (-1,1]$, so that

$$\tilde{f}_{a,b}(p,t) = \frac{p^2}{2} + O(C) = \frac{C^2}{2}(9 - 4\gamma^2 - 6\gamma\mathrm{i}) + O(C).$$

Thus we have

$$\begin{aligned}\operatorname{Re}\big(\tilde{f}_{a,b}(p,t)\overline{n_p}\big) &= \frac{C^2}{2}\operatorname{Re}\big((9 - 4\gamma^2 - 6\gamma\mathrm{i}))(-1)\big) + O(C)\\ &= \frac{(-9+4\gamma^2)C^2}{2} + O(C),\end{aligned}$$

where

$$\frac{-9+4\gamma^2}{2} \le -\frac{5}{2} < 0 \quad \text{for all } \gamma \in [-1,1].$$

E_4: This case is analogous to E_2, up to taking the complex conjugate.

E_5: This case is analogous to E_1, up to taking the complex conjugate.

E_6: This is the most delicate case. As usual, we parametrise $p \in E_6 \cup \{p_6\}$ by $p = \gamma C\mathrm{i}$ for some $\gamma \in [-1,1)$. Since $n_p = 1$, it is enough to calculate the real part $\operatorname{Re}(\tilde{f}_{a,b}(p,t))$. We also recall that $h^\star$ and $h_a^\star$ are real-valued by the assumptions in Theorem 4.10. By plugging $x = p = \gamma C\mathrm{i}$ into (4.19) and gathering the real terms, we have

$$\operatorname{Re}\big(\tilde{f}_{a,b}(p,t)\big) = \frac{-b^2\sigma^2}{2} - b\sigma\varrho\gamma C - \frac{\gamma^2 C^2}{2}.$$

Here, it is not enough to consider the terms of higher order in C, since both vanish at $\gamma = 0$. Nevertheless, we have the inequality

$$\frac{-b^2\sigma^2}{2} - b\sigma\varrho\gamma C - \frac{\gamma^2 C^2}{2} = -\frac{(\gamma C + b\sigma\varrho)^2}{2} - \frac{b^2\sigma^2(1-\varrho^2)}{2} < 0$$

for all $t \in [0, \bar{T}]$, $\gamma \in [-1, 1]$ and $b \in \mathbb{R}\backslash\{0\}$, since $\varrho \in (-1, 1)$ by assumption. Because we have also assumed that $b \neq 0$, this concludes the proof.

□

The proof of Theorem 4.10 is almost complete; it now follows directly from the previous results and concludes this section.

Proof of Theorem 4.10. Since κ is completely monotone by assumption, so is $\hat{\kappa}$ by Gripenberg et al. [59, Theorem 5.3.1] and the definition (I.2.12). Thus by [59, Theorem 5.5.4], $\hat{\kappa}$ satisfies Assumption 4.2. We may then apply Theorem 4.9 to (4.18) with $x_0 = 0$, $f = \tilde{f}_{a,b}$ and $D_t = D$ as given in Lemma 4.12; indeed, the required boundary condition is also checked in Lemma 4.12. Hence we obtain for each $b \in \mathbb{R} \setminus \{0\}$ a continuous solution $\tilde{h}_{a,b} : [0, \bar{T}] \to \mathbb{C} \cong \mathbb{R}^2$ to (4.18). In the case $b = 0$, it is clear that $\tilde{h}_{a,b} \equiv 0$ is a solution to (4.18) by plugging into (4.19). Therefore the existence of a solution $g^\star_{a+\mathrm{i}b}$ to (4.14) follows by Lemma 4.11 for each $b \in \mathbb{R}$. Finally, since $f^\star_z$ is quadratic in x and h and hence satisfies the Lipschitz-type condition (A.2.22), the uniqueness follows by part 4) of Corollary A.2.7 with $y = 0$, $k = \hat{\kappa}$, $h = h^\star$ and $f = f^\star_z$ as well as $p = q =$ "a" $= 2$ (the latter is unrelated to the constant a fixed by the statement of Theorem 4.10). □

Chapter III

Existence and uniqueness of mean–variance equilibria in general semimartingale markets

1 Introduction

The capital asset pricing model (CAPM) of Treynor [117], Sharpe [112], Lintner [89, 90] and Mossin [95] is one of the first general equilibrium models for financial markets. Despite its limitations, it is still one of the cornerstones of modern financial theory and widely used in practice; see [88] for a recent overview. While the early papers focused on the financial implications and shortcomings of the CAPM (see e.g. Banz [13] for an early critique), the existence of an equilibrium was always assumed. The rigorous study of existence and uniqueness of CAPM equilibria was only initiated two decades later by Nielsen [96, 97, 98] and Allingham [8], with more recent important contributions by Berk [18], Dana [33], Hens et al. [64], Wenzelburger [119] and Koch-Medina/Wenzelburger [85], and has so far been mainly considered in one-period models. This strand of literature, with the notable exception of [18], does not study equilibria for preferences described by expected utility but rather by mean-variance functionals, i.e., functionals of the form $U(\mu, \sigma)$, where U is quasiconcave, increasing in the mean μ and decreasing in the volatility σ. This is because without distributional assumptions on the returns, the only utility functions that are compatible with the CAPM (more precisely, the two-fund separation theorem) are quadratic utility functions; see the discussion in Berk [18, after Corollary 3.2]. For expected quadratic utility, existence and uniqueness of CAPM equilibria in one period (and under suitable assumptions) seem to have been regarded as folklore knowledge from early on.

In most of the literature, CAPM equilibria have been studied under the assumption of a complete market. This implies the existence of a so-called representative agent, which simplifies the task of showing existence and uniqueness of an equilibrium. In the complete case, every agent chooses to hedge her idiosyncratic risk and own a fraction of the market portfolio; but this is not possible in an incomplete market. In Koch-Medina/Wenzelburger [85], a similar CAPM-type equilibrium is studied for an incomplete one-period market in discrete time. They find that in the incomplete case, each agent still hedges her individual endowment as best as possible, even though this cannot be done perfectly. Unlike in the complete case, the asset prices are now determined by the so-called extended market portfolio, i.e., the aggregate endowment of all agents, given by the terminal value of the market portfolio together with the unhedgeable parts of the endowments. Our work extends the study of CAPM equilibria to a continuous-time setup with an incomplete market driven by a general semimartingale, where the agents receive unhedgeable endowments at the terminal time T.

We show that analogous results to those of [85] also hold in continuous time. In this chapter, this is done for expected quadratic utility; the more general setup of mean–variance utility functions is studied in Chapter IV. Our proof of existence and uniqueness of an equilibrium is based here on the construction of a nonstandard type of representative agent, i.e., a fictional agent that aggregates the preferences and endowments of the K agents. We show that the market clears if and only if the representative agent does not trade, and this observation yields a pricing measure for the equilibrium market.

A challenge in moving from the one-period setup of [85] to multi-period and continuous time is that for the latter, one needs to impose integrability conditions on the admissible trading strategies. These conditions can preclude the existence of an equilibrium. Indeed, we exhibit an example where the only candidate equilibrium is such that the buy-and-hold strategy for the risky asset is not admissible. If the asset has positive net supply, one of the agents must by linearity use an inadmissible strategy; therefore, this cannot be an equilibrium market. We give sufficient conditions to ensure that the required integrability conditions are satisfied, so that this issue is prevented and an equilibrium exists.

This chapter, which is based on joint work with Christoph Czichowsky and Martin Herdegen, is structured as follows. In Section 2, we introduce the model and give the definition of equilibrium in our setup. We also prove some folklore results on mean–variance hedging (MVH) that are used to show the main results.

In Section 3.1, we demonstrate the connection between the quadratic utility and MVH problems of the individual agents. This is used in Section 3.2 to show that the aggregate demand for the risky assets can obtained via a representative agent. In Section 3.3, we obtain the main result on existence and uniqueness of an equilibrium, as well as an explicit formula. In Section 3.4, we give sufficient conditions for the integrability required by the main result. Finally, in Section 3.5, we provide more general existence results for CAPM equilibria in finite discrete time where the equilibrium markets are not unique and may admit arbitrage opportunities.

2 Model and preliminary results

2.1 Financial market

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}, P)$ with a fixed finite time horizon $T \in (0, \infty)$. We assume that the filtration $\mathbb{F}$ satisfies the usual conditions of right-continuity and completeness, and also that $\mathcal{F}_0$ is P-trivial and $\mathcal{F}_T = \mathcal{F}$.

The financial market consists of $1+d = 1+d_1+d_2$ assets. The first asset, with price process S^0, serves as numéraire and we assume that $(S_t^0)_{0\leq t\leq T} \equiv 1$. In addition, we consider d_1 *financial assets* with price processes $S^{(1)} = (S_t^1, \dots, S_t^{d_1})_{0\leq t\leq T}$ and d_2 *productive assets* (sometimes also referred to as *real assets*) with price processes $S^{(2)} = (S_t^{d_1+1}, \dots, S_t^{d_1+d_2})_{0\leq t\leq T}$. These risky assets are collectively expressed as $S := (S^{(1)}, S^{(2)})$. In the following, we likewise use the notation $x = (x^{(1)}, x^{(2)})$ for each $x \in \mathbb{R}^{d_1+d_2}$ with $x^{(i)} \in \mathbb{R}^{d_i}$. We also write $L^2 = L^2(P)$ where the probability measure is unambiguous.

Our goal is to study a setup where the price processes $S^{(1)}$ and $S^{(2)}$ are not given a priori, but rather determined by a Radner equilibrium between K agents trading in the market according to their individual utility-maximising strategies. In such an equilibrium, that we define precisely later in Definition 2.5, the asset prices should be set in such a way that the total demand for the assets, which is induced by the optimal strategies of the individual agents, equals the total (fixed) supply at all times. The primitives for this problem are the utility functions and endowments for the individual agents, introduced later in Section 2.3, as well as some partial information about the price processes that is given a priori, so that a unique equilibrium price process can be obtained. The financial and productive assetsare distinguished by the type of constraint imposed on their price processes, as detailed below.

We assume that the initial value and volatility structure of the financial assets are predetermined and known by the market participants, i.e., for $j \in \{1, \dots, d_1\}$, we have

$$S^j_t = S^j_0 + M^j_t + A^j_t, \quad 0 \le t \le T, \tag{2.1}$$

where $S^j_0 \in \mathbb{R}$ and the local martingale part $M^j \in \mathcal{M}^2_{0,\mathrm{loc}}$ are given a priori. The predictable finite-variation process $A^j \in \mathrm{FV}_0$ null at time 0 is to be determined in equilibrium. We write $S^{(1)} = S^{(1)}_0 + M^{(1)} + A^{(1)}$. The financial assets may be regarded as securities constructed by the market participants to enable the trading of short-term risks, determined implicitly by the dynamics of $M^{(1)}$, at appropriate prices set by the market, which are reflected in the dynamics of $A^{(1)}$.

We assume that each productive asset $j \in \{d_1 + 1, \dots, d_1 + d_2\}$ with price process S^j entitles the owner to a random terminal dividend $D^j \in L^2$ at time T. In other words, S^j satisfies the terminal condition

$$S^j_T = D^j, \tag{2.2}$$

and the rest of the price process $(S^j)_{0 \le t < T}$ is to be determined by the market in equilibrium. We write $S^{(2)}_T = D^{(2)}$. The random variable $D^{(2)} : \Omega \to \mathbb{R}^{d_2}$ is fixed and known to the agents.

Finally, we also assume that each asset S^j is a *local L^2-semimartingale* for $j \in \{1, \dots, d\}$. This means that there exists a localising sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that each stopped process $S^{j,\tau_n} = (S^j_{\tau_n \wedge t})_{0 \le t \le T}$ is an L^2-semimartingale, in the sense that

$$\sup \big\{ E[(S^{j,\tau_n}_\sigma)^2] : \sigma \text{ stopping time} \big\} < \infty; \tag{2.3}$$

see Delbaen/Schachermayer [35] and Černý/Kallsen [25] for details. We refer to this property by calling $(1, S)$ a *local L^2-market*. Note that by [25, Lemma A.2], a stochastic process is a local L^2-semimartingale if and only if it is a special semimartingale whose local martingale part is locally square-integrable. As a consequence, in view of (2.1), this is only a condition on the productive assets.

Thus we summarise the market setup as follows. Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $T \in (0, \infty)$, $S^{(1)}_0 \in \mathbb{R}^{d_1}$, $M^{(1)} \in \mathcal{M}^2_{0,\mathrm{loc}}$ and $D^{(2)} \in L^2$ are given. Our goal is to study the set of price processes $S = (S^{(1)}, S^{(2)})$ that satisfy (2.1), (2.2) and (2.3), while also leading to an equilibrium between K agents that we introduce below. We formalise the notion of an equilibrium market later in Definition 2.5.

2.2 Admissible strategies

In order to describe trading in the market $(1, S)$, we need to specify which strategies are considered *admissible*. To this end, we follow a two-step approach as in Černý/Kallsen [25], where more details on the rationale can be found.

Definition 2.1. Let $(1, S)$ be a local L^2-market. A *simple integrand for S* is a process of the form $\vartheta = \sum_{i=1}^{m-1} \xi_i \mathbf{1}_{]\!]\sigma_i,\sigma_{i+1}]\!]}$, where $m \in \mathbb{N}$, $0 \leq \sigma_1 \leq \cdots \leq \sigma_m$ are $[0, T]$-valued stopping times, and each ξ_i is a bounded $\mathcal{F}_{\sigma_i}$-measurable random vector, such that each stopped process $S^{j,\sigma_m} = (S^j_{\sigma_m \wedge t})_{0 \leq t \leq T}$ is an L^2-semimartingale for $j = 1, \ldots, d$. We denote by $\Theta_{\mathrm{simple}}(S)$ the linear space of all simple integrands for S. We also let $L(S)$ be the set of predictable S-integrable processes on $[0, T]$; see Jacod/Shiryaev [71, III.6.17].

Definition 2.2. Let $(1, S)$ be a local L^2-market. Then $\vartheta \in L(S)$ is called *L^2-admissible for S* if $\vartheta \bullet S_T \in L^2$ and there exists a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ in $\Theta_{\mathrm{simple}}(S)$ such that

1) $\vartheta^n \bullet S_T \xrightarrow{L^2} \vartheta \bullet S_T$,

2) $\vartheta^n \bullet S_\tau \xrightarrow{P} \vartheta \bullet S_\tau$ for all $[0, T]$-valued stopping times τ,

where $\vartheta \bullet S = (\vartheta \bullet S_t)_{0 \leq t \leq T}$ denotes the stochastic integral $\vartheta \bullet S_t = \int_0^t \vartheta_r dS_r$ for $t \in [0, T]$. The set of all L^2-admissible trading strategies is denoted by $\overline{\Theta}(S)$.

Remark 2.3. (a) Our definition of L^2-admissible strategies slightly differs from the original one given in [25], because we stipulate 2) for all stopping times τ and not only for deterministic times $t \in [0, T]$. However, under [25, Assumption 2.1], i.e., if there exists an equivalent local martingale measure (ELMM) Q for S with $\frac{dQ}{dP} \in L^2(P)$, both definitions coincide. The reason for this change is that it allows us to use dynamic programming arguments even if there does not exist an ELMM Q for S with $\frac{dQ}{dP} \in L^2(P)$, as in Czichowsky/Schweizer [32].

(b) As usual, we assume that market participants choose self-financing portfolios $(\vartheta^0_t, \vartheta_t)_{0 \leq t \leq T}$, where ϑ^0 is a predictable process, $\vartheta \in \overline{\Theta}(S)$ and the self-financing condition $\vartheta^0_t + \vartheta_t^\top S_t = \vartheta^0_0 + \vartheta_0^\top S_0 + \vartheta \bullet S_t$, P-a.s. for all $t \in [0, T]$, is satisfied. Since we shall include the initial wealth of the agents into their endowments, as we explain in the first paragraph of Section 2.3 below, we have $\vartheta^0_0 = 0$ so that a self-financing portfolio can be parametrised in terms of $\vartheta \in \overline{\Theta}(S)$ alone.

We denote by $(e^j_t)_{0 \leq t \leq T} \equiv (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^d$ the buy-and-hold strategy of the j-th risky asset, where 1 is in the j-th position. In general, this strategy will not be L^2-admissible for S because we only assume S^j to be a local L^2-semimartingale. However, if S^j is an L^2-semimartingale, then automatically $e^j \in \Theta_{\mathrm{simple}}(S) \subseteq \overline{\Theta}(S)$. Given a vector $\eta \in \mathbb{R}^d$, we sometimes denote the constant strategy $\sum_{j=1}^d \eta^j e^j$ simply by η.

2.3 Agents and endowments

Consider $K \geq 1$ agents participating in the financial market. We assume that each agent $k \in \{1, \ldots, K\}$ owns a *traded endowment* at time 0, consisting of $\eta^{k,j} \in \mathbb{R}$ units of the asset $j \in \{0, \ldots, d_1 + d_2\}$, and is also entitled to receive a *non-traded endowment* at time T, which consists of a random income $\Xi^{k,n} \in L^2$.

Because we assume zero interest rates and there are no liquidity constraints on the portfolios of the agents, it does not matter whether a fixed amount of cash is received via the traded or non-traded endowment. Thus we may assume that each agent starts with zero cash. We also make the assumption that the financial assets are in *zero net supply*, i.e., that these assets are created and traded internally between the agents so that any long and short positions in the financial assets must net out between the agents, that is, $\sum_{k=1}^K \eta^{k,j} = 0$. Since the initial prices $S_0^0, \ldots, S_0^{d_1}$ are known a priori, each agent is indifferent between receiving an endowment consisting of units of the financial assets or the corresponding cash value via the non-traded endowment. We may thus assume without loss of generality that $\eta^{k,j} = 0$ for $j \in \{0, \ldots, d_1\}$ and $k \in \{1, \ldots, K\}$.

On the other hand, the agents may have a nontrivial endowment consisting of productive assets. We set $\eta^k := (0, \eta^{k,(2)}) \in \mathbb{R}^{d_1+d_2}$ and denote the value of the traded endowment (in the productive assets) of agent k at time T by $\Xi^{k,t} := {\eta^{k,(2)}}^\top D^{(2)} \in L^2$. The *total endowment* of agent k at time T is then given by

$$\Xi^k = \Xi^{k,t} + \Xi^{k,n}. \tag{2.4}$$

Each agent $k \in \{1, \ldots, K\}$ interacts with the market by buying and selling assets according to an L^2-admissible strategy $\vartheta \in \overline{\Theta}(S)$, which includes the original endowment $\eta^{k,(2)}$ of productive assets. Since the agent does not own riskless or financial assets at time 0, her initial wealth is ${\eta^{k,(2)}}^\top S_0^{(2)}$, which is the initial value of her traded endowment. Agent k can then generate the wealth process ${\eta^{k,(2)}}^\top S_0^{(2)} + \vartheta \bullet S$ by trading with the strategy ϑ in a self-financing way. Since she additionally receives the non-traded endowment $\Xi^{k,n}$ at time T, her terminal wealth at time T is given by ${\eta^{k,(2)}}^\top S_0^{(2)} + \vartheta \bullet S_T + \Xi^{k,n}$. Note that the traded endowment has the terminal value

$$\Xi^{k,t} = {\eta^{k,(2)}}^\top D^{(2)} = {\eta^{k,(2)}}^\top S_T^{(2)} = {\eta^{k,(2)}}^\top S_0^{(2)} + \eta^k \bullet S_T,$$

since $\eta^k = (0, \eta^{k,(2)})$ is constant. Thus the terminal wealth of agent k at time T can be equivalently written as

$${\eta^{k,(2)}}^\top S_0^{(2)} + \vartheta \bullet S_T + \Xi^{k,n} = \Xi^{k,t} - \eta^k \bullet S_T + \vartheta \bullet S_T + \Xi^{k,n} = (\vartheta - \eta^k) \bullet S_T + \Xi^k \tag{2.5}$$

in terms of the total endowment defined in (2.4).

From the right-hand side of (2.5), we see that the total wealth at time T consists of the total endowment as well as any gains or losses generated by the strategy $\vartheta - \eta^k$. This difference may be interpreted as a discretionary strategy that is employed by the agent in addition to the fixed endowment η^k. The left-hand side of (2.5) gives an alternative interpretation. Instead of keeping the traded endowment, agent k may immediately sell it for the price of ${\eta^{k,(2)}}^\top S_0^{(2)}$ and then trade with the strategy ϑ; the non-traded endowment $\Xi^{k,n}$ is then added to the wealth at time T. However, we note that the price $S_0^{(2)}$ is not known a priori, but rather determined by the equilibrium. Thus the right-hand side of (2.5) is more useful for solving the equilibrium problem, since the total endowment Ξ^k is fixed by the primitives, so that only the stochastic integral term $(\vartheta - \eta^k) \bullet S_T$ depends on the dynamics of S.

Each agent $k \in \{1, \dots, K\}$ has preferences over terminal wealth at time T described by a functional $\mathcal{U}_k : L^0(P) \to \mathbb{R}$. We consider two types of functionals: the case of a quadratic utility function is the subject of Section 3, and a (generalised) mean–variance functional is considered in Chapter IV. Agent k seeks to maximise utility from terminal wealth at time T, i.e., to solve the problem

$$\mathcal{U}_k\big((\vartheta - \eta^k) \bullet S_T + \Xi^k\big) \longrightarrow \max_{\vartheta \in \overline{\Theta}(S)} ! \tag{2.6}$$

When considering the uniqueness of a solution to the maximisation problem (2.6), we need to view uniqueness on the level of stochastic integrals $\vartheta \bullet S$, rather than on the level of strategies ϑ. To this end, we introduce the following equivalence relation.

Definition 2.4. Let $(1, S)$ be a local L^2-market. Then $\vartheta, \vartheta' \in \overline{\Theta}(S)$ are called *S-equivalent* if $\vartheta \bullet S$ and $\vartheta' \bullet S$ are indistinguishable. In this case, we write

$\vartheta =_S \vartheta'$; see Czichowsky/Schweizer [31] for more details on how to represent different equivalent classes via the so-called projection onto the predictable range.

2.4 Equilibrium

We can now formulate the key notion of an equilibrium market, which we adapt from the classical concept of a Radner equilibrium. We take the primitives $S_0^{(1)}, M^{(1)}, D^{(2)}$, η^k and $\Xi^{k,n}$ defined in Sections 2.1 and 2.3 as given.

Definition 2.5. A local L^2-market $(1, S^{(1)}, S^{(2)})$ is called an *equilibrium market* if it satisfies (2.1) and (2.2) as well as the following conditions:

1) For each agent $k \in \{1, \dots, K\}$, the maximisation problem (2.6) has a solution $\hat{\vartheta}^k \in \overline{\Theta}(S)$ that is unique up to S-equivalence.

2) The market clears, i.e., for $t \in [0, T]$,

$$\sum_{k=1}^{K} \hat{\vartheta}_t^{k,j} =_S \bar{\eta}^j := \begin{cases} 0, & \text{if } j \in \{1, \dots, d_1\}, \\ \sum_{k=1}^{K} \eta^{k,j}, & \text{if } j \in \{d_1 + 1, \dots, d_1 + d_2\}. \end{cases} \tag{2.7}$$

3) $e^j \in \overline{\Theta}(S)$ for $j \in \{d_1 + 1, \dots, d_1 + d_2\}$, i.e., the buy-and-hold strategies of the productive assets are L^2-admissible.

The only slightly non-standard requirement in Definition 2.5 is 3). It ensures that each η^k is L^2-admissible, i.e., just keeping the traded endowment corresponds to an L^2-admissible strategy for each agent, which is a natural condition that the traded endowment should satisfy. Moreover, it is important to ensure that $\hat{\vartheta}^k \in \overline{\Theta}(S)$ if and only if $\hat{\vartheta}^k - \eta^k \in \overline{\Theta}(S)$, since we later show the existence of the optimal strategies $\hat{\vartheta}^k$ by first solving for $\hat{\vartheta}^k - \eta^k$.

Our overall goal is to find an equilibrium market $(1, S^{(1)}, S^{(2)})$ corresponding to the primitives. More precisely, we look for conditions on the primitives that ensure the existence and uniqueness of a corresponding equilibrium market, and we seek to characterise that market. We start by studying the individual optimisation problems for the agents, and then show how the individual decisions of the agents can be aggregated by the concept of a representative agent. To that end, we first consider the mean–variance hedging problem, and then show the relevant connections to the quadratic utility problem defined below in (3.2), which corresponds to (2.6) in the case where the preference functional $\mathcal{U}_k$ is given by expected quadratic utility.

2.5 Preliminaries on mean–variance hedging

The maximisation problem (2.6) for the quadratic utility or (generalised) mean–variance preferences we study later is closely linked to the so-called mean–variance hedging problem; see Schweizer [111] for a recent overview. We introduce here the mean–variance hedging problem, as well as some related results that will be useful later. For a given payoff $H \in L^2$,

- the *mean–variance hedging* (MVH) problem is given by

$$E\big[(\vartheta \bullet S_T - H)^2\big] \longrightarrow \min_{\vartheta \in \overline{\Theta}(S)} ! \tag{2.8}$$

- the *extended mean–variance hedging* (exMVH) problem is given by

$$E\big[(c + \vartheta \bullet S_T - H)^2\big] \longrightarrow \min_{(c,\vartheta) \in \mathbb{R} \times \overline{\Theta}(S)} ! \tag{2.9}$$

Mathematically, the minimisation problems (2.8) and (2.9) are best approximation problems in L^2 for H with respect to the linear subspaces

$$\mathcal{G}_T(S) := \{\vartheta \bullet S_T : \vartheta \in \overline{\Theta}(S)\} \quad \text{and} \quad \mathbb{R} + \mathcal{G}_T(S) := \{c + \vartheta \bullet S_T : c \in \mathbb{R}, \vartheta \in \overline{\Theta}(S)\}$$

of L^2, respectively. Because L^2 is a Hilbert space, the terminal values attained by the solutions to (2.8) and (2.9) are given by the orthogonal projections of H onto $\mathcal{G}_T(S)$ and $\mathbb{R} + \mathcal{G}_T(S)$, which exist provided that $\mathcal{G}_T(S)$ and $\mathbb{R} + \mathcal{G}_T(S)$ are closed in L^2, respectively. If they exist, we denote those terminal values by $\vartheta(H) \bullet S_T$ and $c(H) + \vartheta^{\mathrm{ex}}(H) \bullet S_T$, respectively, for some $c(H) \in \mathbb{R}$ and $\vartheta(H), \vartheta^{\mathrm{ex}}(H) \in \overline{\Theta}(S)$ such that the corresponding terminal values are attained. In that case, one is also interested in whether the choices of $\vartheta(H)$ and $(c(H), \vartheta^{\mathrm{ex}}(H))$ corresponding to these terminal values are unique in the respective spaces. Specifically, we say that (2.8) has a unique solution if $\vartheta^1 =_S \vartheta^2$ for any two solutions $\vartheta^1, \vartheta^2 \in \overline{\Theta}(S)$; we use the same convention for the maximisation problem (2.6). Likewise, we say that (2.9) has a unique solution if $c_1 = c_2$ and $\vartheta^1 =_S \vartheta^2$ for any two solutions $(c_1, \vartheta^1), (c_2, \vartheta^2) \in \mathbb{R} \times \overline{\Theta}(S)$.

From the interpretation of the MVH and exMVH problems as orthogonal projections, we easily deduce that these problems are linear, in the following sense.

Lemma 2.6. *Let $H_1, H_2 \in L^2$ and $\lambda \in \mathbb{R}$.*

1) *Suppose there exist solutions* $\vartheta(H_1), \vartheta(H_2) \in \overline{\Theta}(S)$ *to* (2.8) *for* H_1 *and* H_2, *respectively. Then* $\vartheta(H_1) + \lambda\vartheta(H_2)$ *is a solution to* (2.8) *for the payoff* $H_1 + \lambda H_2$.

2) *Suppose there exist solutions* $(c(H_1), \vartheta^{\mathrm{ex}}(H_1)), (c(H_2), \vartheta^{\mathrm{ex}}(H_2)) \in \mathbb{R} \times \overline{\Theta}(S)$ *to* (2.9) *for* H_1 *and* H_2, *respectively. Then* $(c(H_1) + \lambda c(H_2), \vartheta^{\mathrm{ex}}(H_1) + \lambda\vartheta^{\mathrm{ex}}(H_2))$ *is a solution to* (2.9) *for the payoff* $H_1 + \lambda H_2$.

Proof. **1)** For any $H \in L^2$, $\vartheta \in \overline{\Theta}(S)$ solves (2.8) for H if and only if

$$E[(\vartheta \bullet S_T - H)(\tilde{\vartheta} \bullet S_T)] = 0 \tag{2.10}$$

for all $\tilde{\vartheta} \in \overline{\Theta}(S)$; this follows from the formulation of the MVH problem as an orthogonal projection. Note that we have

$$\begin{aligned} &E\Big[\Big(\big(\vartheta(H_1) + \lambda\vartheta(H_2)\big) \bullet S_T - H_1 - \lambda H_2\Big)(\tilde{\vartheta} \bullet S_T)\Big] \\ &= E\big[\big(\vartheta(H_1) \bullet S_T - H_1\big)(\tilde{\vartheta} \bullet S_T)\big] + \lambda E\big[\big(\vartheta(H_2)\big) \bullet S_T - H_2\big)(\tilde{\vartheta} \bullet S_T)\big] = 0 \end{aligned}$$

for all $\tilde{\vartheta} \in \overline{\Theta}(S)$, since $\vartheta(H_1)$ and $\vartheta(H_2)$ solve (2.8) for H_1 and H_2, respectively. Therefore $\vartheta(H_1) + \lambda\vartheta(H_2)$ solves (2.8) for $H_1 + \lambda H_2$, as claimed.

2) Similarly, for any $H \in L^2$, $(c, \vartheta) \in \mathbb{R} \times \overline{\Theta}(S)$ solves (2.9) if and only if

$$E[(c + \vartheta \bullet S_T - H)(\tilde{c} + \tilde{\vartheta} \bullet S_T)] = 0 \tag{2.11}$$

for all $(\tilde{c}, \tilde{\vartheta}) \in \mathbb{R} \times \overline{\Theta}(S)$. We have

$$\begin{aligned} &E\Big[\Big(c(H_1) + \lambda c(H_2) + \big(\vartheta^{\mathrm{ex}}(H_1) + \lambda\vartheta^{\mathrm{ex}}(H_2)\big) \bullet S_T - H_1 - \lambda H_2\Big)(\tilde{c} + \tilde{\vartheta} \bullet S_T)\Big] \\ &= E\big[\big(c(H_1) + \vartheta^{\mathrm{ex}}(H_1) \bullet S_T - H_1\big)(\tilde{c} + \tilde{\vartheta} \bullet S_T)\big] \\ &\quad + \lambda E\big[\big(c(H_2) + \vartheta^{\mathrm{ex}}(H_2) \bullet S_T - H_2\big)(\tilde{c} + \tilde{\vartheta} \bullet S_T)\big] \\ &= 0 \end{aligned}$$

for all $(\tilde{c}, \tilde{\vartheta}) \in \mathbb{R} \times \overline{\Theta}(S)$, since $(c(H_1), \vartheta^{\mathrm{ex}}(H_1))$ and $(c(H_2), \vartheta^{\mathrm{ex}}(H_2))$ solve (2.9) for H_1 and H_2, respectively. Thus $(c(H_1) + \lambda c(H_2), \vartheta^{\mathrm{ex}}(H_1) + \lambda\vartheta^{\mathrm{ex}}(H_2))$ solves (2.9) for $H_1 + \lambda H_2$, as claimed. □

If S admits an equivalent local martingale measure (ELMM) with square-integrable density, then $\mathcal{G}_T(S)$ and $\mathbb{R} + \mathcal{G}_T(S)$ are closed and the solutions in $\overline{\Theta}(S)$ and $\mathbb{R} \times \overline{\Theta}(S)$ are unique; this follows by Černý/Kallsen [25, Lemma 2.11]. However, without that extra assumption, both closedness of $\mathcal{G}_T(S)$ and $\mathbb{R} + \mathcal{G}_T(S)$ as well as uniqueness of the solutions in $\overline{\Theta}(S)$ and $\mathbb{R} + \overline{\Theta}(S)$ (if they exist) do not hold in general.

In order to deal with the uniqueness issue, it is useful to introduce the notions of *uniqueness of gains processes* and *uniqueness of value processes* associated with a price process S.

Definition 2.7. Let $(1, S)$ be a local L^2-market. It is said to satisfy

- *uniqueness of gains processes* if for any two trading strategies $\vartheta^1, \vartheta^2 \in \overline{\Theta}(S)$, the equality $\vartheta^1 \bullet S_T = \vartheta^2 \bullet S_T$ P-a.s. implies that $\vartheta^1 =_S \vartheta^2$.
- *uniqueness of value processes* if for any two trading strategies $\vartheta^1, \vartheta^2 \in \overline{\Theta}(S)$ and initial values $c_1, c_2 \in \mathbb{R}$, the equality $c_1 + \vartheta^1 \bullet S_T = c_2 + \vartheta^2 \bullet S_T$ P-a.s. implies that $c_1 = c_2$ and $\vartheta^1 =_S \vartheta^2$.

We have the following two equivalent characterisations of uniqueness of gains and value processes. They follow immediately from the linear structure of the (extended) mean–variance hedging problems given in Lemma 2.6, as well as the fact that for $H = 0$, the problems of MVH (2.8) and exMVH (2.9) admit as solutions $\vartheta = 0$ and $(c, \vartheta) = (0, 0)$, respectively.

Proposition 2.8. *Let* $(1, S)$ *be a local* L^2*-market. The following are equivalent:*

(a) $(1, S)$ *satisfies uniqueness of gains processes.*

(b) *For some* $H \in L^2$*, the MVH problem* (2.8) *admits a unique solution.*

(c) *For each* $H \in L^2$ *for which the MVH problem* (2.8) *admits a solution, the solution is unique.*

Proof. (a) $\Rightarrow$ (b): For the particular payoff $H = 0$, we claim that $\vartheta(H) = 0$ is the unique solution to (2.8). Indeed, it is a solution as the hedging error is 0. Moreover, any other solution $\vartheta \in \overline{\Theta}(S)$ must satisfy $\vartheta \bullet S_T = 0 = 0 \bullet S_T$ a.s., and thus $\vartheta =_S 0$ by the uniqueness of gains processes.

(b) $\Rightarrow$ (c): For a contradiction, suppose that the MVH problem for H admits two solutions $\vartheta^1, \vartheta^2 \in \overline{\Theta}(S)$. As we assume (b), there exists some $\tilde{H} \in L^2$ such that the solution $\vartheta(\tilde{H})$ to the MVH problem for $\tilde{H}$ is unique. By Lemma 2.6, $\vartheta(\tilde{H}) + (\vartheta^1 - \vartheta^2)$ is a solution to (2.8) for $\tilde{H} + H - H = \tilde{H}$. Thus we have by uniqueness that $\vartheta(\tilde{H}) + (\vartheta^1 - \vartheta^2) =_S \vartheta(\tilde{H})$, so that $\vartheta^1 =_S \vartheta^2$. This shows that the solution to (2.8) for H is also unique, as claimed.

(c) $\Rightarrow$ (a): Suppose that $\vartheta^1, \vartheta^2 \in \overline{\Theta}(S)$ are such that $\vartheta^1 \bullet S_T = \vartheta^2 \bullet S_T$. By Definition 2.2 and as $\vartheta^1 \in \overline{\Theta}(S)$, we have $H := \vartheta^1 \bullet S_T \in L^2$. Note that both strategies ϑ^1 and ϑ^2 solve the MVH problem for H, as the hedging error generated by either strategy is 0. By (c), we thus get $\vartheta^1 =_S \vartheta^2$, so that $(1, S)$ satisfies uniqueness of gains processes. $\square$

Proposition 2.9. *Let $(1, S)$ be a local L^2-market. The following are equivalent:*

(a) *$(1, S)$ satisfies uniqueness of value processes.*

(b) *For some $H \in L^2$, the exMVH problem (2.9) admits a unique solution.*

(c) *For each $H \in L^2$ for which the exMVH problem (2.9) admits a solution, the solution is unique.*

Proof. The proof is almost identical to that of Proposition 2.8.

(a) $\Rightarrow$ (b): We claim that $(c(0), \vartheta^{\mathrm{ex}}(0)) = (0, 0)$ is the unique solution to (2.9) for the particular payoff $H = 0$. Indeed, it is a solution as the hedging error is 0. Moreover, any other solution $(c, \vartheta) \in \mathbb{R} \times \overline{\Theta}(S)$ must satisfy $c + \vartheta \bullet S_T = 0$ a.s., and thus $c = 0$ and $\vartheta =_S 0$ by the uniqueness of value processes.

(b) $\Rightarrow$ (c): For a contradiction, suppose that the exMVH problem for H admits two solutions $(c_1, \vartheta^1), (c_2, \vartheta^2) \in \mathbb{R} \times \overline{\Theta}(S)$. As we assume (b), there exists some $\tilde{H} \in L^2$ such that the solution $(c(\tilde{H}), \vartheta^{\mathrm{ex}}(\tilde{H}))$ to the exMVH problem for $\tilde{H}$ is unique. By Lemma 2.6, we have that

$$(\tilde{c}, \tilde{\vartheta}) := \big(c(\tilde{H}) + c_1 - c_2, \vartheta(\tilde{H}) + (\vartheta^1 - \vartheta^2)\big)$$

is a solution to (2.8) for $\tilde{H} + H - H = \tilde{H}$, so that $(\tilde{c}, \tilde{\vartheta}) = (c(\tilde{H}), \vartheta^{\mathrm{ex}}(\tilde{H}))$ by the uniqueness of the solution. Thus $c(\tilde{H}) + c_1 - c_2 = c(\tilde{H})$ so that $c_1 = c_2$, and also $\vartheta(\tilde{H}) + (\vartheta^1 - \vartheta^2) =_S \vartheta(\tilde{H})$ so that $\vartheta^1 =_S \vartheta^2$. This shows that the solution to (2.9) for H is unique, as claimed.

(c) $\Rightarrow$ (a): Suppose that $(c_1, \vartheta^1), (c_2, \vartheta^2) \in \mathbb{R} \times \overline{\Theta}(S)$ are such that

$$c_1 + \vartheta^1 \bullet S_T = c_2 + \vartheta^2 \bullet S_T =: H.$$

By Definition 2.2 and as $\vartheta^1 \in \overline{\Theta}(S)$, we have $H \in L^2$. Note that both pairs (c_1, ϑ^1) and (c_2, ϑ^2) solve the exMVH problem for H, as the hedging error generated by either strategy is 0. As we assume that (c) holds, we thus get that $c_1 = c_2$ and $\vartheta^1 =_S \vartheta^2$, and so $(1, S)$ satisfies uniqueness of value processes. $\square$

The following two results show that uniqueness of value processes implies uniqueness of gains processes and link the MVH problem (2.8) and the extended exMVH problem (2.9).

Proposition 2.10. *Let* $(1, S)$ *be a local* L^2*-market satisfying uniqueness of value processes. Then* $(1, S)$ *satisfies uniqueness of gains processes. Moreover, if the MVH problem* (2.8) *for* $H = 1$ *has a solution* $\vartheta(1)$*, then* $E[(\vartheta(1) \bullet S_T - 1)^2] > 0$.

Proof. Since $\vartheta \equiv 0$ is a solution to (2.8) for $H = 0$ with hedging error 0, any solution to (2.8) for 0 is also a solution to (2.9) for 0 with $c = 0$. Thus the first assertion follows from Propositions 2.8 and 2.9, using the fact that if the MVH problem (2.8) for 0 does not have a unique solution, then a fortiori the exMVH problem (2.9) for 0 cannot have a unique solution. The second assertion follows from the fact that if $E[(\vartheta(1) \bullet S_T - 1)^2] = 0$, then the exMVH problem (2.9) for 0 would have two solutions $(-1, \vartheta(1))$ and $(0, 0)$, which would lead to a contradiction due to Proposition 2.9. □

Corollary 2.11. *Let* $(1, S)$ *be a local* L^2*-market satisfying uniqueness of value processes. Suppose that the MVH problems* (2.8) *for* H *and* 1 *have solutions* $\vartheta(H)$ *and* $\vartheta(1)$*, respectively. Then the exMVH problem* (2.9) *for* H *has the solution* $(c(H), \vartheta^{\mathrm{ex}}(H)) := (c(H), \vartheta(H) - c(H)\vartheta(1))$*, where*

$$c(H) := \frac{E[H(1 - \vartheta(1) \bullet S_T)]}{E[(1 - \vartheta(1) \bullet S_T)^2]}. \tag{2.12}$$

Proof. By Lemma 2.6, for fixed $c \in \mathbb{R}$, the MVH problem (2.8) for $H - c$ has the (unique) solution $\vartheta(H) - c\vartheta(1)$. Moreover, as $E[\vartheta(H) \bullet S_T(1 - \vartheta(1) \bullet S_T)] = 0$ by the first-order condition of MVH, it follows that

$$\begin{aligned} &E\Big[\Big(c + \big(\vartheta(H) - c\vartheta(1)\big) \bullet S_T - H\Big)^2\Big] \\ &= c^2 E\big[\big(1 - \vartheta(1) \bullet S_T\big)^2\big] + E\big[\big(H - \vartheta(H) \bullet S_T\big)^2\big] - 2cE\big[H\big(1 - \vartheta(1) \bullet S_T\big)\big]. \end{aligned} \tag{2.13}$$

As a quadratic function of c, the right-hand side of (2.13) has the unique minimiser $c(H)$ given by (2.12). □

The following result gives simple sufficient conditions for uniqueness of gains and value processes and for the existence of solutions to the MVH and exMVH problems (2.8) and (2.9) in terms of a signed local martingale measure for S. The assumption that such a signed measure exists is not necessary for the existence of solutions to the MVH and exMVH problems, but it is a weaker assumption than the existence of an equivalent local martingale measure for S. By Černý/Czichowsky [24, Theorem 2.2], the conditions in part (b) are equivalent to the economic assumption of the so-called law of one price.

Proposition 2.12. *Let $(1, S)$ be a local L^2-market and $Z = (Z_t)_{0\le t\le T}$ a square-integrable martingale such that ZS^j is a local martingale for all $j \in \{1, \dots, d\}$. Then $Z(\vartheta \bullet S)$ is a P-martingale for each $\vartheta \in \overline{\Theta}(S)$. Moreover:*

1) *If $Z_t \neq 0$ P-a.s. for each $t \in [0, T]$, then $(1, S)$ satisfies uniqueness of value processes.*

2) *If $Z_t \neq 0$ and $Z_{t-} \neq 0$ for all $t \in [0, T]$ P-a.s., then the MVH problem (2.8) and exMVH problem (2.9) have unique solutions for each $H \in L^2$.*

Proof. As Z is a martingale and ZS^j is a local martingale for all $j \in \{1, \dots, d\}$, we have that ZS^τ is a local P-martingale on $[0, T]$ for any stopping time τ. Moreover, let σ be another stopping time with values in $[0, T]$ and $A \in \mathcal{F}$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} E\big[\mathbf{1}_A|Z_\tau S^\tau_\sigma|\big] &\le E\big[\mathbf{1}_A|Z_\tau|^2\big]^{1/2} E\big[|S^\tau_\sigma|^2\big]^{1/2} \\ &\le E\Big[\mathbf{1}_A \sup_{t\in[0,T]} |Z_t|^2\Big]^{1/2} \sup\big\{E[|S^\tau_{\sigma'}|^2] : \sigma' \text{ stopping time}\big\}^{1/2}. \end{aligned}$$

Note that the bound on the right-hand side is finite, as S^τ is an L^2-semimartingale and Z is a square-integrable martingale. Since $\sup_{t\in[0,T]} |Z_t|$ is square-integrable, so that the singleton $\{\sup_{t\in[0,T]} |Z_t|^2\}$ is uniformly integrable, it follows by the ε-δ-criterion for uniform integrability that the right-hand side converges to 0 as $P[A] \to 0$. Since the bound on the right-hand side is independent of σ, we have thus shown that the set $\{Z_\sigma S^\tau_\sigma : \sigma \text{ stopping time}\}$ is uniformly integrable, i.e., ZS^τ is of class (D). Therefore, ZS^τ is a true P-martingale on $[0, T]$ for any stopping time τ such that S is an L^2-semimartingale. We deduce that $Z(\vartheta \bullet S)$ is a P-martingale on $[0, T]$ for any strategy of the form $\vartheta = \mathbf{1}_{]\!]0,\tau]\!]}$. By linearity, this martingale property extends to $Z(\vartheta \bullet S)$ for all $\vartheta \in \Theta_{\text{simple}}(S)$.

We now show the same martingale property for $\vartheta \in \overline{\Theta}(S)$. By the definition of $\overline{\Theta}(S)$, there is a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of simple strategies such that $\vartheta^n \bullet S_T \xrightarrow{L^2} \vartheta \bullet S_T$ and $\vartheta^n \bullet S_\tau \xrightarrow{P} \vartheta \bullet S_\tau$ for any $[0, T]$-valued stopping time τ. By the Cauchy–Schwarz inequality, it follows that $Z_T(\vartheta^n \bullet S_T) \xrightarrow{L^1} Z_T(\vartheta \bullet S_T)$. By the L^1-continuity of conditional expectations and the fact that convergence in L^1 implies convergence in probability, we obtain

$$E[Z_T(\vartheta^n \bullet S_T) \mid \mathcal{F}_t] \xrightarrow{P} E[Z_T(\vartheta \bullet S_T) \mid \mathcal{F}_t]$$

for $0 \le t \le T$ as $n \to \infty$. Since ϑ^n is a simple strategy, $Z(\vartheta^n \bullet S)$ is a P-martingale on $[0,T]$ so that $E[Z_T(\vartheta^n \bullet S_T) \mid \mathcal{F}_t] = Z_t(\vartheta^n \bullet S_t)$ and

$$Z_t(\vartheta^n \bullet S_t) \xrightarrow{P} E[Z_T(\vartheta \bullet S_T) \mid \mathcal{F}_t]$$

as $n \to \infty$. But we also have $Z_t(\vartheta^n \bullet S_t) \xrightarrow{P} Z_t(\vartheta \bullet S_t)$ as $n \to \infty$ by the construction of (ϑ^n), which implies that $E[Z_T(\vartheta \bullet S_T) \mid \mathcal{F}_t] = Z_t(\vartheta \bullet S_t)$ for $0 \le t \le T$, i.e., $Z(\vartheta \bullet S)$ is a P-martingale for all $\vartheta \in \overline{\Theta}(S)$. This shows the first statement, and we now proceed to show 1) and 2).

1) Fix $c^1, c^2 \in \mathbb{R}$ and $\vartheta^1, \vartheta^2 \in \overline{\Theta}(S)$ such that $c^1 + \vartheta^1 \bullet S_T = c^2 + \vartheta^2 \bullet S_T$ P-a.s. Then $Z(c^1 + \vartheta^1 \bullet S)$ and $Z(c^2 + \vartheta^2 \bullet S)$ are both P-martingales. Since they agree at the terminal time, they are indistinguishable. Since each $Z_t \neq 0$ P-a.s., this implies that $c^1 + \vartheta^1 \bullet S_t = c^2 + \vartheta^2 \bullet S_t$ P-a.s. for each $t \in [0,T]$, whence $c^1 + \vartheta^1 \bullet S$ and $c^2 + \vartheta^2 \bullet S$ are indistinguishable. For $t = 0$, this implies that $c^1 = c^2$, and subtracting the constants yields that $\vartheta^1 \bullet S$ and $\vartheta^2 \bullet S$ are indistinguishable.

2) It suffices to argue existence, as uniqueness follows from part 1) together with Proposition 2.9. By the assumptions on Z, we have $Z = Z_0\,\mathcal{E}(N)$ for some local martingale $N = (N_t)_{0 \le t \le T}$; this is due to Jacod [70, Exercise 6.1] and given in a concise form in Choulli et al. [29, Proposition 2.2]. Hence S is a local $\mathcal{E}$-martingale and $\mathcal{E}(N) = Z/Z_0$ is a square-integrable martingale and so-called regular (since $T_n = T$ for $n \ge 1$; see [29, Definitions 3.4, 3.6 and 3.11]). Existence of a solution to the MVH problem (2.8) then follows from Czichowsky/Schweizer [32, Theorem 2.16]. Finally, the existence of a solution to the exMVH problem (2.9) follows from part 1) and Corollary 2.11. □

We close this section by linking the zero solution of an MVH problem to a local martingale-type condition for S.

Lemma 2.13. *Let $(1,S)$ be a local L^2-market and $H \in L^2$. Define the square-integrable martingale $Z = (Z_t)_{0 \le t \le T}$ by $Z_t := E[H \mid \mathcal{F}_t]$. The following are equivalent:*

(a) *$0 \in \overline{\Theta}(S)$ solves the MVH problem* (2.8).

(b) *ZS^j is a local P-martingale for all $j \in \{1,\dots,d\}$.*

Proof. (a) $\Rightarrow$ (b): Since $0 \in \overline{\Theta}(S)$ is a solution to (2.8), it follows as in (2.10) that H is orthogonal to $\mathcal{G}_T(S)$, i.e., we have $E[(\vartheta \bullet S_T)H] = 0$ for any $\vartheta \in \overline{\Theta}(S)$.

Now fix $j \in \{1,\dots,d\}$ and let σ be a stopping time. Since $(1,S)$ is a local L^2-market, there exists a localising sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, the stopped process S^{j,τ_n} is an L^2-semimartingale. Fix $n \in \mathbb{N}$. Then the strategy $e^{j,\sigma,n} := (0,\dots,0,\mathbf{1}_{]\!]0,\sigma\wedge\tau_n]\!]},0,\dots,0)$, where the indicator process is at the j-th position, belongs to $\Theta_{\mathrm{simple}}(S) \subseteq \overline{\Theta}(S)$. The fact that Z is a P-martingale with $Z_T = H$ and $E[(e^{j,\sigma,n} \bullet S_T)H] = 0$ yield

$$E[S^j_{\sigma\wedge\tau_n} Z_{\sigma\wedge\tau_n} - S^j_0 Z_0] = E[(S^j_{\sigma\wedge\tau_n} - S^j_0)Z_T] = E[(e^{j,\sigma,n} \bullet S_T)H] = 0.$$

Since σ was arbitrary, it follows that $(ZS^j)^{\tau_n}$ is a P-martingale. As (τ_n) is a localising sequence, we conclude that ZS^j is a local P-martingale.

(b) $\Rightarrow$ (a): Fix $\vartheta \in \overline{\Theta}(S)$. By Proposition 2.12, $Z(\vartheta \bullet S)$ is a P-martingale, and hence

$$\begin{aligned} E[(\vartheta \bullet S_T - H)^2] &= E[H^2] - 2E[(\vartheta \bullet S_T)Z_T] + E[(\vartheta \bullet S_T)^2] \\ &= E[H^2] + E[(\vartheta \bullet S_T)^2] \\ &\geq E[(H - 0 \bullet S_T)^2]. \end{aligned}$$

Thus $0 \in \overline{\Theta}(S)$ solves the MVH problem (2.8). □

3 Equilibria for quadratic utilities

In this section, we focus on the case where the preferences of the agents are described by expected utility with a quadratic utility function. More precisely, the preferences of each agent $k \in \{1,\dots,K\}$ are characterised by the expected utility of terminal wealth at time T, with the quadratic utility function

$$U_k(x) = x - \frac{1}{2\gamma_k}x^2, \quad x \in \mathbb{R}, \tag{3.1}$$

where $\gamma_k > 0$ denotes the *risk tolerance* of agent k. Recall from (2.5) the terminal wealth generated by a strategy $\vartheta \in \overline{\Theta}(S)$ together with the traded and non-traded endowments for each agent. Thus the maximisation problem (2.6) of agent k takes the form

$$E\big[U_k\big((\vartheta - \eta^k) \bullet S_T + \Xi^k\big)\big] \to \max_{\vartheta\in\overline{\Theta}(S)}! \tag{3.2}$$

The quadratic utility function U_k is concave and increasing for $x \leq \gamma_k$, so that $\gamma_k > 0$ can also be seen as a *bliss point* for agent k. The fact that U_k is decreasing for $x \geq \gamma_k$ is economically unreasonable, as it implies that the agent would prefer to be less wealthy beyond that point; this is a well-known issue associated with the choice of a quadratic utility function. Nevertheless, this can be a useful model provided that γ_k is large enough, so that it is unlikely that an agent can reach that amount of wealth. A key point in favour of quadratic utility is that it often leads to tractable problems where explicit results can be obtained.

3.1 Individually optimal strategies

We begin by making some observations on the individual maximisation problem (3.2) for a *fixed* local L^2-market $(1, S)$. More precisely, we link (3.2) to an MVH problem for the payoff $H^k := \gamma_k - \Xi^k$. This is the difference between the bliss point and the total endowment for agent k, and may be interpreted as the additional wealth that the agent would like to obtain in order to reach the bliss point.

Lemma 3.1. *Let $(1, S)$ be a local L^2-market and assume that $\eta^k \in \overline{\Theta}(S)$. Then the following are equivalent:*

(a) *The optimisation problem (3.2) of agent k has a unique solution $\hat{\vartheta}^k \in \overline{\Theta}(S)$.*

(b) *The MVH problem*
$$E[(\vartheta \bullet S_T - H^k)^2] \to \min_{\vartheta \in \overline{\Theta}(S)} ! \tag{3.3}$$
has a unique solution $\vartheta(H^k) \in \overline{\Theta}(S)$.

In either case, it holds that $\hat{\vartheta}^k =_S \eta^k + \vartheta(H^k)$, and the market satisfies uniqueness of gains processes.

Proof. Let $\vartheta \in \overline{\Theta}(S)$ and set $\tilde{\vartheta} := \vartheta - \eta^k \in \overline{\Theta}(S)$. Plugging the definition of U_k into (3.2) and rewriting (3.1) as $U_k(x) = -\frac{1}{2\gamma_k}(x - \gamma_k)^2 + \frac{\gamma_k}{2}$ yields

$$\begin{aligned} E\big[U_k\big((\vartheta - \eta^k) \bullet S_T + \Xi^k\big)\big] &= E[U_k(\tilde{\vartheta} \bullet S_T + \Xi^k)] \\ &= -\frac{1}{2\gamma_k} E[(\tilde{\vartheta} \bullet S_T + \Xi^k - \gamma_k)^2] + \frac{\gamma_k}{2} \\ &= -\frac{1}{2\gamma_k} E[(\tilde{\vartheta} \bullet S_T - H^k)^2] + \frac{\gamma_k}{2}. \end{aligned}$$

Because $\overline{\Theta}(S)$ is a vector space, this shows that ϑ is a solution to the maximisation problem (3.2) if and only if $\tilde{\vartheta}$ is a solution to the MVH problem (3.3), and therefore the two problems are equivalent under the assumption that $\eta^k \in \overline{\Theta}(S)$. In particular, (3.2) has a unique solution $\hat{\vartheta}^k$ if and only if (3.3) has a unique solution $\vartheta(H^k)$, in which case we have the relationship $\hat{\vartheta}^k =_S \eta^k + \vartheta(H^k)$ between the solutions. Finally, if (3.3) has a unique solution, then the market satisfies uniqueness of gains processes by Proposition 2.8. □

We proceed to decompose the optimal strategy $\hat{\vartheta}^k$ into a hedging and a pure investment part by considering two problems:

- The *hedging problem of agent k* is the exMVH problem

$$E[(c+\vartheta \bullet S_T - \Xi^k)^2] \to \min_{c\in\mathbb{R},\vartheta\in\overline{\Theta}(S)} ! \tag{3.4}$$

- The *pure investment problem* is the MVH problem

$$E[(\vartheta \bullet S_T - 1)^2] \to \min_{\vartheta\in\overline{\Theta}(S)} ! \tag{3.5}$$

If the market satisfies uniqueness of value processes and there exist (unique) solutions to both (3.2) and the pure investment problem (3.5), then we can decompose the optimal strategy of agent k as follows.

Proposition 3.2. *Let $(1,S)$ be a local L^2-market satisfying uniqueness of value processes. Assume that $\eta^k \in \overline{\Theta}(S)$ and the pure investment problem (3.5) has a solution $\vartheta(1)$. Then the individual optimisation problem (3.2) of agent k has a unique solution $\hat{\vartheta}^k \in \overline{\Theta}(S)$ if and only if the exMVH problem (2.9) for Ξ^k has a unique solution $(c(\Xi^k),\vartheta^{\mathrm{ex}}(\Xi^k)) \in \mathbb{R}\times\overline{\Theta}(S)$. In this case, $\hat{\vartheta}^k$ is given by*

$$\hat{\vartheta}^k =_S \eta^k + \big(\gamma_k - c(\Xi^k)\big)\vartheta(1) - \vartheta^{\mathrm{ex}}(\Xi^k). \tag{3.6}$$

Proof. Suppose that the exMVH problem (2.9) for Ξ^k has a (unique) solution $(c(\Xi^k),\vartheta^{\mathrm{ex}}(\Xi^k)) \in \mathbb{R}\times\overline{\Theta}(S)$. Then the MVH problem (2.8) with payoff $\Xi^k - c(\Xi^k)$ has the unique solution $\vartheta(\Xi^k - c(\Xi^k)) =_S \vartheta^{\mathrm{ex}}(\Xi^k)$. Hence by Lemma 2.6, the MVH problem (2.8) for the payoff

$$H^k = \gamma_k - \Xi^k = \gamma_k - \big(\Xi^k - c(\Xi^k)\big) - c(\Xi^k)$$

has the unique solution

$$\vartheta(H^k) =_S \big(\gamma_k - c(\Xi^k)\big)\vartheta(1) - \vartheta(\Xi^k - c(\Xi^k)) =_S \big(\gamma_k - c(\Xi^k)\big)\vartheta(1) - \vartheta^{\mathrm{ex}}(\Xi^k). \tag{3.7}$$

Thus (3.6) follows by Lemma 3.1.

Conversely, if the individual optimisation problem (3.2) of agent k has a (unique) solution $\hat{\vartheta}^k \in \overline{\Theta}(S)$, then by Lemma 3.1, the MVH problem (2.8) for $H^k = \gamma_k - \Xi^k$ has a unique solution $\vartheta(H^k)$. Hence by Lemma 2.6, the MVH problem (2.8) for Ξ^k has a unique solution $\vartheta(\Xi^k) =_S \gamma_k\vartheta(1) - \vartheta(H^k)$. It now follows from Corollary 2.11 that the exMVH problem (2.9) for Ξ^k has a unique solution $(c(\Xi^k), \vartheta^{\text{ex}}(\Xi^k)) \in \mathbb{R} \times \overline{\Theta}(S)$, where $\vartheta^{\text{ex}}(\Xi^k) = \vartheta(\Xi^k) - c(\Xi^k)\vartheta(1)$. □

3.2 The representative agent

In order to study equilibrium markets in the case of a quadratic utility function, we use the standard idea from financial economics to consider a *representative agent* that holds the aggregate endowment of all agents, i.e., the representative agent owns both $\bar{\eta} = \sum_{k=1}^K \eta^k$ units of the assets, where we recall $\bar{\eta} = (\bar{\eta}^1, \dots, \bar{\eta}^{d_1+d_2}) = (0, \bar{\eta}^{(2)})$ from (2.7), as well as the sum of the non-traded endowments of the agents. Equivalently, the representative agent receives the total endowment $\bar{\Xi} = \sum_{k=1}^K \Xi^k$. By the same argument as in (2.5), the representative agent can attain the terminal wealth $(\vartheta - \bar{\eta}) \bullet S_T + \bar{\Xi}$ by trading with a strategy $\vartheta \in \overline{\Theta}(S)$.

The utility function of the representative agent is defined by

$$\bar{U}_\lambda(x) = \sup\left\{ \sum_{k=1}^K \lambda_k U_k(x_k) : x_1, \dots, x_K \in \mathbb{R}^d, \sum_{k=1}^k x_k = x \right\},$$

where $\lambda = (\lambda_1, \dots, \lambda_K) \in \mathbb{R}_+^K$ is a fixed set of *Negishi weights* summing up to K.[1] We make the ansatz of equal weights $\lambda_1 = \dots = \lambda_K := 1$ and write $\bar{U} := \bar{U}_1$. In this case, denoting by $\bar{\gamma} := \sum_{k=1}^K \gamma_k$ the aggregate risk tolerance, it is not difficult to check that

$$\begin{aligned}\bar{U}(x) &= \sup\left\{ \sum_{k=1}^K \left(x_k - \frac{1}{2\gamma_k} x_k^2 \right) : x_1, \dots, x_K \in \mathbb{R}^d, \sum_{k=1}^K x_k = x \right\} \\ &= \sum_{k=1}^K \left(\frac{x\gamma_k}{\bar{\gamma}} - \frac{1}{2\gamma_k} \left(\frac{x\gamma_k}{\bar{\gamma}} \right)^2 \right) = x - \frac{1}{2\bar{\gamma}} x^2,\end{aligned}$$

so that the utility function of the representative agent is of the same form as the utility function of the individual agents. The representative agent then solves the maximisation problem

$$E\big[\bar{U}\big((\vartheta - \bar{\eta}) \bullet S_T + \bar{\Xi}\big)\big] \to \max_{\vartheta \in \overline{\Theta}(S)} ! \tag{3.8}$$

From a mathematical perspective, (3.8) has exactly the same structure as the

[1] Usually, the convention in the literature is that the Negishi weights sum up to 1, but in our context, the total weight of K leads to neater formulas. Of course, both parametrisations lead to the same set of preferences.

individual maximisation problem (3.2). Thus we get an analogue of Lemma 3.1 for the representative agent. In the following, we set $\bar{H} = \bar{\gamma} - \bar{\Xi}$. Similarly to H^k, the random variable $\bar{H}$ may be interpreted as the aggregate shortfall, i.e., the additional wealth that the agents would (collectively) like to obtain in order to reach the aggregate bliss point $\bar{\gamma}$.

Lemma 3.3. *Let $(1, S)$ be a local L^2-market and assume that $\bar{\eta} \in \overline{\Theta}(S)$. Then the following are equivalent:*

(a) *The optimisation problem* (3.8) *has a unique solution $\bar{\vartheta} \in \overline{\Theta}(S)$.*

(b) *The MVH problem*

$$E[(\vartheta \bullet S_T - \bar{H})^2] \to \min_{\vartheta \in \overline{\Theta}(S)} ! \tag{3.9}$$

has a unique solution $\vartheta(\bar{H}) \in \overline{\Theta}(S)$.

In either case, it holds that $\bar{\vartheta} =_S \bar{\eta} + \vartheta(\bar{H})$, and the market satisfies uniqueness of gains processes.

Analogously to Proposition 3.2, we can also decompose the optimal strategy $\hat{\vartheta}$ of the representative agent into a hedging and a pure investment part, where the *hedging problem of the representative agent* is the (extended) MVH problem

$$E\big[\big(c + \vartheta \bullet S_T - \bar{\Xi}\big)^2\big] \to \min_{c \in \mathbb{R}, \vartheta \in \overline{\Theta}(S)} ! \tag{3.10}$$

and the *pure investment problem* is the MVH problem (3.5).

Proposition 3.4. *Let $(1, S)$ be a local L^2-market satisfying uniqueness of value processes. Assume that $\bar{\eta} \in \overline{\Theta}(S)$ and the pure investment problem* (3.5) *has a unique solution $\vartheta(1)$. Then the optimisation problem* (3.8) *of the representative agent has a unique solution $\bar{\vartheta} \in \overline{\Theta}(S)$ if and only if the exMVH problem* (2.9) *for $\bar{\Xi}$ has a unique solution $(c(\bar{\Xi}), \vartheta^{\mathrm{ex}}(\bar{\Xi})) \in \mathbb{R} \times \overline{\Theta}(S)$, and $\bar{\vartheta}$ is then given by*

$$\bar{\vartheta} =_S \bar{\eta} + \big(\bar{\gamma} - c(\bar{\Xi})\big)\, \vartheta(1) - \vartheta^{\mathrm{ex}}(\bar{\Xi}). \tag{3.11}$$

The following result shows that the optimal strategy for the representative agent is given by the sum of the strategies of the individual agents. This result gives a characterisation of the aggregate demand for the risky assets, which is the key to finding a market equilibrium.

Lemma 3.5. *Let* $(1, S)$ *be a local* L^2*-market. Assume that* $\eta^1, \dots, \eta^K \in \overline{\Theta}(S)$ *and for each agent* $k \in \{1, \dots, K\}$*, the individual optimisation problem* (3.2) *has a unique solution* $\hat{\vartheta}^k$*. Then the optimisation problem* (3.8) *of the representative agent has a unique solution* $\bar{\vartheta}$ *satisfying*

$$\bar{\vartheta} =_S \sum_{k=1}^{K} \hat{\vartheta}^k. \tag{3.12}$$

Proof. By the implication (b) $\Rightarrow$ (c) in Proposition 2.8, the map $H \mapsto \vartheta(H)$ is well defined for all H such that a solution $\vartheta(H)$ to (2.8) exists, since such a solution is unique up to S-equivalence. We also have by Lemma 2.6 that $H \mapsto \vartheta(H)$ is linear where it is defined.

Because $\eta^1, \dots, \eta^K \in \overline{\Theta}(S)$, we get from Lemma 3.1 that the MVH problem (2.8) for H^k has the unique solution $\vartheta(H^k) = \hat{\vartheta}^k - \eta^k$ for each $k \in \{1, \dots, K\}$. Hence there is a unique solution to (2.8) for $\bar{H} = \sum_{k=1}^{K} H^k$, which is given by

$$\vartheta(\bar{H}) =_S \sum_{k=1}^{K} \big(\vartheta(H^k) - \eta^k\big) = \bar{\vartheta} - \bar{\eta}. \tag{3.13}$$

Thus by Lemmas 3.3 and 3.1, we have

$$\bar{\vartheta} =_S \vartheta(\bar{H}) + \bar{\eta} =_S \sum_{k=1}^{K} \big(\vartheta(H^k) + \eta^k\big) =_S \sum_{k=1}^{K} \hat{\vartheta}^k,$$

which shows (3.12) and concludes the proof. □

3.3 Existence and uniqueness of equilibria

So far, we have used the linear structure of the quadratic and MVH problems of the individual agents to characterise the aggregate demand in terms of a representative agent. With this insight, we can now proceed to our main results on the existence and uniqueness of equilibrium markets. We start by giving a characterisation of equilibria from which we will later obtain an explicit formula.

Lemma 3.6. *Suppose that* $(1, S) = (1, S^{(1)}, S^{(2)})$ *is an equilibrium market and let* $(\bar{Z}_t)_{0 \le t \le T}$ *be the (square-integrable)* P*-martingale given by* $\bar{Z}_t = E[\bar{H} \,|\, \mathcal{F}_t]$ *so that* $\bar{Z}_T = \bar{H}$ P*-a.s. Then for each* $j \in \{1, \dots, d_1 + d_2\}$*, the process* $(\bar{Z}_t S^j_t)_{0 \le t \le T}$ *is a local* P*-martingale.*

Proof. Denote by $\hat{\vartheta}^1, \dots, \hat{\vartheta}^K \in \overline{\Theta}(S)$ the unique individually optimal strategies.

Then by Lemma 3.5, $\bar{\vartheta} := \sum_{k=1}^{K} \hat{\vartheta}^k \in \overline{\Theta}(S)$ is the unique solution to the optimisation problem (3.8) of the representative agent. Moreover, the market clearing condition (2.7) yields $\bar{\vartheta} =_S \bar{\eta}$, so that by Lemma 3.3, 0 is the unique solution to the MVH problem (2.8) for $\bar{H}$. Thus Lemma 2.13 yields that $(\bar{Z}_t S_t^j)_{0\le t\le T}$ is a local P-martingale for each $j \in \{1, \dots, d_1 + d_2\}$, as claimed. □

Lemma 3.6 shows that the process $\bar{Z}$ plays a key role, since any equilibrium market S must satisfy the condition that $\bar{Z}S$ is a local P-martingale. We obtain from this insight necessary and sufficient conditions for the existence of a unique equilibrium under the assumption that the process $\bar{Z}$ does not hit 0; for instance, this assumption holds if $\bar{Z}_T = \bar{H} > 0$ so that $\bar{Z}$ is strictly positive. In that case, $\bar{H}/E[\bar{H}]$ is the density of an equivalent local martingale measure for S. It is economically reasonable to assume that the shortfall $\bar{H}$ is strictly positive, since it means that the agents always want to increase their wealth (in aggregate). Nevertheless, for the sake of generality, we allow $\bar{Z}$ to take negative as well as positive values.

Before we proceed to the first main result, it is useful to introduce the Galtchouk–Kunita–Watanabe decomposition of $\bar{Z}$ with respect to $M^{(1)}$ under P, i.e.,

$$\bar{Z}_t = \bar{Z}_0 + \bar{\xi}^{(1)} \bullet M_t^{(1)} + M_t^{\bar{Z}}, \quad 0 \le t \le T, \tag{3.14}$$

where $\bar{\xi}^{(1)} \in L^2(M^{(1)})$ and $M^{\bar{Z}}$ is a square-integrable P-martingale strongly orthogonal to $M^{(1)}$. As we will see, the integrand $\bar{\xi}^{(1)}$ plays an important role in the price dynamics of the financial assets in an equilibrium market.

Remark 3.7. The choice of $\bar{\xi}^{(1)}$ in (3.14) is only unique up to $M^{(1)}$-equivalence. Because the components of $M^{(1)}$ may be linearly dependent, the components $\bar{\xi}^i \bullet M^i$ need not be uniquely (or well) defined, but we can choose a particular integrand $\bar{\xi}^{(1)} = (\bar{\xi}^1, \dots, \bar{\xi}^{d_1})$ as follows. Applying the Gram–Schmidt algorithm to $(M^1, \dots, M^{d_1}, \bar{Z}) \in \mathcal{M}^2_{0,\mathrm{loc}}$ yields a unique decomposition of the form

$$\bar{Z}_t = \bar{Z}_0 + \sum_{i=1}^{d_1} \bar{\xi}^i \bullet M_t^i + M_t^{\bar{Z}}, \quad 0 \le t \le T,$$

where $\sum_{i=1}^{I} \bar{\xi}^i \bullet M^i$ is strongly orthogonal to $\sum_{i=I+1}^{d_1} \bar{\xi}^i \bullet M^i + M^{\bar{Z}}$ for each $I \in \{1, \dots, d_1\}$. This orthogonality property and the square-integrability of $\bar{Z}$ yield that $\sum_{i=1}^{I} \bar{\xi}^i \bullet M^i$ is a square-integrable martingale for each I, and hence so is $\bar{\xi}^i \bullet M^i$. For this choice of $\bar{\xi}^{(1)} := (\bar{\xi}^1, \dots, \bar{\xi}^{d_1})$ and as $M^{(1)}$ is a locally square-integrable martingale by assumption, the predictable quadratic variation

$$\langle \bar{\xi}^i \bullet M^i, M^j \rangle = \bar{\xi}^{i\top} \bullet \langle M^i, M^j \rangle$$

is well defined for each $j \in \{1, \ldots, d_1\}$ and $\bar{\xi}^{i\top} \bullet ([M^i, M^j] - \langle M^i, M^j \rangle)$ is a local P-martingale. In particular, the process

$$\langle \bar{\xi}^{(1)} \bullet M^{(1)}, M^j \rangle = \sum_{i=1}^{d_1} \bar{\xi}^i \bullet \langle M^i, M^j \rangle \tag{3.15}$$

is well defined. Although we choose $\bar{\xi}^{(1)}$ as above, note that the martingale $\bar{\xi}^{(1)} \bullet M^{(1)}$ is independent of that choice due to (3.14). Hence the right-hand side of (3.15) is also independent of any choice of $\bar{\xi}^{(1)}$ such that the individual summands are well defined. Likewise, the finite-variation part in (3.16) below also does not depend on the choice of $\bar{\xi}^{(1)}$.

Theorem 3.8. *Assume that $\bar{Z}_t \neq 0$ and $\bar{Z}_{t-} \neq 0$ for all $t \in [0,T]$ P-a.s. If there exists an equilibrium market $(1, S^{(1)}, S^{(2)})$ that satisfies (2.1) and (2.2), it is unique and explicitly given by*

$$\begin{aligned} S_t^j &= S_0^j + M_t^j - \int_0^t \frac{\mathrm{d}\langle \bar{Z}, M^j \rangle_s}{\bar{Z}_{s-}} \\ &= S_0^j + M_t^j - \sum_{i=1}^{d_1} \int_0^t \frac{\bar{\xi}_s^i}{\bar{Z}_{s-}} \mathrm{d}\langle M^i, M^j \rangle_s, \qquad j \in \{1, \ldots, d_1\}, \qquad (3.16) \\ S_t^j &= \frac{E[\bar{H} D^j \mid \mathcal{F}_t]}{\bar{Z}_t} = \frac{E[\bar{Z}_T D^j \mid \mathcal{F}_t]}{\bar{Z}_t}, \qquad j \in \{d_1 + 1, \ldots, d_1 + d_2\}, \qquad (3.17) \end{aligned}$$

and $(\bar{Z}_t S_t^j)_{0 \le t \le T}$ is a local P-martingale for each $j \in \{1, \ldots, d_1 + d_2\}$. Conversely, $(1, S) = (1, S^{(1)}, S^{(2)})$ defined by (3.16) and (3.17) is an equilibrium market if and only if $S^{(2)}$ is a local L^2-semimartingale such that $e^j \in \overline{\Theta}(S)$ for each $j \in \{d_1 + 1, \ldots, d_1 + d_2\}$.

Proof of Theorem 3.8. **(a)** We start by showing that any equilibrium market $(1, S)$ is given by (3.16) and (3.17). By Lemma 3.6, $\bar{Z} S^j$ is a local P-martingale for each $j \in \{1, \ldots, d_1 + d_2\}$. We first consider $j \in \{1, \ldots, d_1\}$. Recall the decomposition (3.14) for $\bar{Z}$ and the dynamics (2.1) for S^j. Applying the product formula to $\bar{Z}S^j$ and rearranging terms, we obtain

$$\bar{Z}_t S^j_t - \bar{Z}_- \bullet M^j_t - S^j_- \bullet \bar{Z}_t - \sum_{i=1}^{d_1} \bar{\xi}^i \bullet \big([M^i, M^j] - \langle M^i, M^j\rangle\big)_t$$
$$= \bar{Z}_- \bullet A^j_t + \sum_{i=1}^{d_1} \bar{\xi}^i \bullet \langle M^i, M^j\rangle_t \tag{3.18}$$

for $0 \leq t \leq T$. Note that $\bar{Z}S^j$, M^j, $\bar{Z}$ and $\bar{\xi}^i \bullet ([M^i, M^j] - \langle M^i, M^j\rangle)$ are local P-martingales (for the latter, this is shown in Remark 3.7), whereas A^j and $\bar{\xi}^i \bullet \langle M^i, M^j\rangle$ are predictable finite-variation processes. Thus both sides of (3.18) must vanish, as they are null at 0. By assumption, we have $\bar{Z}_t \neq 0$ and $\bar{Z}_{t-} \neq 0$ for all $t \in [0, T]$ P-a.s. Since $\bar{Z}$ is also càdlàg, this implies that $1/\bar{Z}_-$ is finite-valued and càglàd, thus locally bounded. Integrating $1/\bar{Z}_-$ against the right-hand side of (3.18), which vanishes as we have shown, yields (3.16).

Next, consider $j \in \{d_1 + 1, \dots, d_1 + d_2\}$. By (2.2) and as $\bar{Z}_T = \bar{H}$, we have $\bar{Z}_T S^j_T = \bar{H}D^j$. Since $e^j \in \overline{\Theta}(S)$ and $\bar{Z}$ is a square-integrable martingale, it follows from Proposition 2.12 that $\bar{Z}S^j = \bar{Z}S^j_0 + \bar{Z}(e^j \bullet S)$ is a P-martingale, so that $\bar{Z}_t S^j_t = E[\bar{H}D^j \mid \mathcal{F}_t]$. Since $\bar{Z}_t \neq 0$ P-a.s., this yields (3.17). We have thus shown that any equilibrium must satisfy (3.16) and (3.17).

(b) Next, we show the converse statement. Define $(1, S) = (1, S^{(1)}, S^{(2)})$ by (3.16) and (3.17) and assume that $S^{(2)}$ is a local L^2-semimartingale and $e^j \in \overline{\Theta}(S)$ for $j \in \{d_1 + 1, \dots, d_1 + d_2\}$. We claim that $(1, S)$ is an equilibrium market. It is clear from (3.16) and (3.17) that $S^{(1)}$ and $S^{(2)}$ satisfy (2.1) and (2.2), respectively. Note that $S^{(1)}$ is a a special semimartingale and the local martingale part $M^{(1)}$ is locally square-integrable, by assumption. Thus by Černý/Kallsen [25, Lemma A.2], $S^{(1)}$ is also a local L^2-semimartingale so that $(1, S^{(1)}, S^{(2)})$ is a local L^2-market. Next, we want to show that $\bar{Z}S^j$ is a local P-martingale for $j \in \{1, \dots, d_1 + d_2\}$. This is clear for $j \in \{d_1 + 1, \dots, d_1 + d_2\}$ by the construction (3.17). For $j \in \{1, \dots, d_1\}$, we use a result on local $\mathcal{E}$-martingales as in the proof of part 2) of Proposition 2.12. Indeed, the assumptions on $\bar{Z}$ yield $\bar{Z} = \bar{Z}_0\,\mathcal{E}(\bar{N})$ for some local P-martingale $\bar{N} = (\bar{N}_t)_{0 \leq t \leq T}$, namely, $\bar{N} = (1/\bar{Z}_-) \bullet \bar{Z}$. Since for $j \in \{1, \dots, d_1\}$, we have

$$\sum_{i=1}^{d_1} \frac{\bar{\xi}^i_t}{\bar{Z}_{t-}} \mathrm{d}\langle M^i, M^j\rangle_t = \frac{1}{\bar{Z}_{t-}} d\langle \bar{Z}, M^j\rangle_t = d\langle \bar{N}, M^j\rangle_t,$$

we obtain that S^j given by (3.16) is a local $\mathcal{E}$-martingale by Choulli et al. [29, Corollary 3.16] (which generalises Girsanov's theorem to local $\mathcal{E}$-martingales). Thus by [29, Definition 3.11] with $n = 0$, $\bar{Z}S^j$ is a local P-martingale.

Now note that $\bar{Z}_t \neq 0$ and $\bar{Z}_{t-} \neq 0$ for all $t \in [0, T]$ P-a.s. by the assumptions on $\bar{Z}$, and $\bar{Z}S^j$ is a local P-martingale for $j \in \{1, \ldots, d_1 + d_2\}$ as shown above. Hence for each agent $k \in \{1, \ldots, K\}$, the MVH problem (2.8) for H^k has a unique solution $\vartheta(H^k)$ by part 2) of Proposition 2.12. Since moreover $\eta^k \in \overline{\Theta}(S)$ by the assumption on $S^{(2)}$, it follows by Lemma 3.1 that the individual optimisation problem (3.2) for agent k has a unique solution $\hat{\vartheta}^k$. This shows condition 1) in Definition 2.5 of an equilibrium market. Moreover, the strategy 0 solves the MVH problem (2.8) for $\bar{H}$ by Lemma 2.13. Thus Lemmas 3.3 and 3.5 yield $\sum_{k=1}^K \hat{\vartheta}^k = \bar{\vartheta} = \bar{\eta}$, i.e., the market clears and condition 2) is satisfied. Finally, condition 3) is satisfied by assumption, and thus $(1, S)$ is an equilibrium market.

□

3.4 Sufficient conditions for the existence of equilibria

Next, we give sufficient conditions on the primitives to ensure the existence of an equilibrium market; these conditions are generally simpler to check for concrete models than the assumptions of Theorem 3.8. We start by looking at the assumption that $\bar{Z}_t \neq 0$ and $\bar{Z}_{t-} \neq 0$ for all $t \in [0, T]$ P-a.s. Since $\bar{Z}$ is constructed as the martingale $\bar{Z}_t = E[\bar{H} \,|\, \mathcal{F}_t]$, this condition holds automatically if $\bar{H}$ is P-a.s. positive or P-a.s. negative, i.e., if we impose a one-sided boundedness condition on the total endowment $\bar{H}$. As previously mentioned, the most natural case is $\bar{H} = \bar{\gamma} - \bar{\Xi} > 0$ P-a.s., where the aggregate endowment $\bar{\Xi}$ does not exceed the bliss point $\bar{\gamma}$. The case $\bar{H} < 0$ P-a.s. is less significant from an economic point of view, but mathematically, it can be dealt with in the same way.

The remaining assumptions of Theorem 3.8 are that the process $(S_t^{(2)})_{0 \leq t \leq T}$ defined by (3.17) is a local L^2-semimartingale, and also that $e^j \in \overline{\Theta}(S)$ for each $j \in \{d_1 + 1, \ldots, d_1 + d_2\}$. Both of these conditions hold under the stronger assumption that $S^{(2)}$ is an L^2-semimartingale, which yields $e^j \in \Theta_{\mathrm{simple}}(S) \subseteq \overline{\Theta}(S)$ for $j \in \{d_1+1, \ldots, d_1+d_2\}$. We now give sufficient (but not necessary) conditions for $S^{(2)}$ to be an L^2-semimartingale in the case $\bar{H} > 0$ P-a.s.

Lemma 3.9. *Suppose that $\bar{H} > 0$ P-a.s. Then the process $(S_t^{(2)})_{0 \leq t \leq T}$ defined by (3.17) is an L^2-semimartingale if any of the following conditions holds:*

(a) $D^j \in L^\infty(P)$ *for* $j \in \{d_1 + 1, \ldots, d_1 + d_2\}$.

(b) $\bar{H}, \bar{H}^{-1} \in L^\infty(P)$.

(c) $\bar{H} \in L^{1-p_1}(P) \cap L^{p_2}(P)$ *and* $D^j \in L^{2q_1q_2}(P)$ *for all* $j \in \{d_1+1, \dots, d_1+d_2\}$, *where* $p_1, p_2 \in [1, \infty]$ *and* $1/p_i + 1/q_i = 1$ *for* $i \in \{1, 2\}$.[2]

Proof. (a) This is (c) for $p_1 = 1$ and $p_2 = 2$, so that $q_1 = \infty$ and $q_2 = 2$.

(b) This is (c) for $p_1 = p_2 = \infty$.

(c) We only consider the case that $p_1, p_2 \in (1, \infty)$. The arguments for the other cases are very similar and therefore omitted. Fix $j \in \{d_1 + 1, \dots, d_1 + d_2\}$ and define $\bar{Q} \approx P$ by

$$\frac{\mathrm{d}\bar{Q}}{\mathrm{d}P} = \frac{\bar{H}}{E[\bar{H}]} =: Z_T^{\bar{Q}} \in L^{1-p_1}(P) \cap L^{p_2}(P).$$

By (3.17) and the Bayes rule, S^j is a (true) $\bar{Q}$-martingale with $S_T^j = D^j$. Thus the inequalities of Hölder and Doob (with constant C_{q_1}) give

$$\begin{aligned}
E_P\Big[\sup_{t\in[0,T]} |S_t^j|^2\Big] &= E_{\bar{Q}}\bigg[\frac{1}{Z_T^{\bar{Q}}} \sup_{t\in[0,T]} |S_t^j|^2\bigg] \le E_{\bar{Q}}\bigg[\bigg(\frac{1}{Z_T^{\bar{Q}}}\bigg)^{p_1}\bigg]^{1/p_1} E_{\bar{Q}}\Big[\sup_{t\in[0,T]} |S_t^j|^{2q_1}\Big]^{1/q_1} \\
&\le E_P\bigg[\bigg(\frac{1}{Z_T^{\bar{Q}}}\bigg)^{p_1-1}\bigg]^{1/p_1} C_{q_1} E_{\bar{Q}}[(D^j)^{2q_1}]^{1/q_1} \\
&= C_{q_1} E_P\big[(Z_T^{\bar{Q}})^{1-p_1}\big]^{1/p_1} E_P\big[Z_T^{\bar{Q}}(D^j)^{2q_1}\big]^{1/q_1} \\
&\le C_{q_1} E_P\big[(Z_T^{\bar{Q}})^{1-p_1}\big]^{1/p_1} E_P\big[(Z_T^{\bar{Q}})^{p_2}\big]^{1/(q_1p_2)} E_P[(D^j)^{2q_1q_2}]^{1/(q_1q_2)} \\
&< \infty
\end{aligned}$$

by the assumptions. This implies that S^j is an L^2-semimartingale. □

3.5 The case of finite discrete time

Theorem 3.8 provides necessary and sufficient conditions for the existence and uniqueness of an equilibrium under the assumption that $\bar{Z}$ does not hit 0. Although that is an appealing result, the assumption can be relaxed in general. We now study what happens if this assumption is lifted in the case of finite discrete time $t \in \{0, \dots, T\}$ for $T \in \mathbb{N}$. As we shall see, if $\bar{Z}$ is allowed to hit 0 then the equilibrium problem becomes ill-posed, leading to issues of nonexistence or nonuniqueness of equilibria.

In the following, we recall the primitives $S_0^{(1)}$, $(M_t^{(1)})_{t\in\{0,\dots,T\}}$ and $D^{(2)}$ as well as $\bar{H} = \bar{\gamma} - \bar{\Xi}$, where $\bar{\Xi}$ is the aggregate endowment. As in Theorem 3.8, we define the (square-integrable) P-martingale $(\bar{Z}_t)_{t\in\{0,\dots,T\}}$ by $\bar{Z}_t = E[\bar{H} \mid \mathcal{F}_t]$ so that

[2] Note that we can always choose $p_2 \ge 2$ as $\bar{H}$ is square-integrable by assumption.

$\bar{Z}_T = \bar{H}$ P-a.s. We also consider the Galtchouk–Kunita–Watanabe decomposition of $\bar{Z}$ with respect to $M^{(1)}$ under P, i.e.,

$$\bar{Z}_t = \bar{Z}_0 + \sum_{k=1}^{t} \sum_{i=1}^{d_1} \bar{\xi}_k^i \Delta M_k^i + M_t^{\bar{Z}}, \quad t \in [0, T], \tag{3.19}$$

where $\bar{\xi}^{(1)} \in L^2(M^{(1)})$ and $(M_t^{\bar{Z}})_{t \in \{0,\dots,T\}}$ is a square-integrable P-martingale strongly orthogonal to $M^{(1)}$ under P. As usual, $\Delta X_k = X_k - X_{k-1}$ denotes the increment at time k of a stochastic process X in discrete time.

We start by giving necessary conditions for the existence of an equilibrium that are weaker than the assumption that the process $\bar{Z}$ does not hit 0.

Lemma 3.10. *An equilibrium market $(1, S)$ can only exist if both of the conditions*

$$\{\bar{Z}_{t-1} = 0\} \subseteq \{\bar{\xi}_t^i \Delta \langle M^i \rangle_t = 0\} \quad \textit{for } i \in \{1, \dots, d_1\} \textit{ and } t \in \{1, \dots, T\}, \tag{3.20}$$

$$\{\bar{Z}_t = 0\} \subseteq \{E[\bar{H} D^j \mid \mathcal{F}_t] = 0\} \quad \textit{for } j \in \{d_1+1, \dots, d_2\} \textit{ and } t \in \{0, \dots, T-1\} \tag{3.21}$$

hold up to P-null sets.

Proof. Assume that an equilibrium market $(1, S)$ exists. For a contradiction, suppose that (3.20) is not satisfied, i.e., there exist some $i \in \{1, \dots, d_1\}$ and $t \in \{1, \dots, T\}$ such that

$$P[\bar{Z}_{t-1} = 0,\, \bar{\xi}_t^i \Delta \langle M^i \rangle_t \neq 0] > 0. \tag{3.22}$$

By Lemma 3.6, $(\bar{Z}_t S_t)_{t=0,\dots,T}$ is a local P-martingale so that $\mathbf{1}_{\{\bar{Z}_{t-1}=0\}} \Delta(\bar{Z} S^i)_t$ is the increment of a local martingale. We decompose

$$\begin{aligned}
\mathbf{1}_{\{\bar{Z}_{t-1}=0\}} \Delta(\bar{Z} S^i)_t &= \mathbf{1}_{\{\bar{Z}_{t-1}=0\}} (\bar{Z}_{t-1} \Delta S_t^i + S_{t-1}^i \Delta \bar{Z}_t + \Delta \bar{Z}_t \Delta S_t^i) \\
&= \mathbf{1}_{\{\bar{Z}_{t-1}=0\}} S_{t-1}^i \Delta \bar{Z}_t + \mathbf{1}_{\{\bar{Z}_{t-1}=0\}} \Delta \bar{Z}_t \Delta S_t^i \\
&= \mathbf{1}_{\{\bar{Z}_{t-1}=0\}} S_{t-1}^i \Delta \bar{Z}_t + \mathbf{1}_{\{\bar{Z}_{t-1}=0\}} (\Delta [\bar{Z}, S^i]_t - \Delta \langle \bar{Z}, S^i \rangle_t) \\
&\quad + \mathbf{1}_{\{\bar{Z}_{t-1}=0\}} \bar{\xi}_t^i \Delta \langle M^i \rangle_t,
\end{aligned}$$

where the last equality follows since $\Delta \langle \bar{Z}, S^i \rangle_t = \bar{\xi}_t^i \Delta \langle M^i \rangle_t$ by (3.19). Like the left-hand side, the first two terms in the last expression of the right-hand side are increments of local martingales, since $\bar{Z}$ is a martingale and $[\bar{Z}, S^i] - \langle \bar{Z}, S^i \rangle$ a local martingale. It follows that the last term $\mathbf{1}_{\{\bar{Z}_{t-1}=0\}} \bar{\xi}_t^i \Delta \langle M^i \rangle_t$ must also be the increment of a local martingale. However, this term is also $\mathcal{F}_{t-1}$-measurable, and hence null P-a.s. This leads to a contradiction with (3.22) so that (3.20) must hold.

Similarly, suppose that (3.21) does not hold, i.e.,

$$P\big[\bar{Z}_t = 0,\, E[\bar{H}D^j \mid \mathcal{F}_t] \neq 0\big] > 0 \tag{3.23}$$

for some $j \in \{d_1+1,\dots,d_1+d_2\}$ and $t \in \{0,\dots,T-1\}$. Since $\bar{Z}S^j$ is a local P-martingale and $1 \in \bar{\Theta}(S^j)$ by condition 3) of Definition 2.5, Proposition 2.12 yields that $\bar{Z}S^j$ is a true P-martingale. In particular, we have

$$\bar{Z}_t S^j_t = E[\bar{H}D^j \mid \mathcal{F}_t] \quad P\text{-a.s.}$$

This contradicts (3.23), and therefore (3.21) must hold. □

Lemma 3.10 shows what can go wrong when $\bar{Z}$ is allowed to hit 0. To understand (3.20) and (3.21) more clearly, consider the simple setup of a one-period model with $T=1$ where $\mathcal{F}_0$ is P-trivial, and suppose that $d_1 = 0$ and $d_2 = 1$. Thus, there exists a single productive asset S with terminal value $S_1 = D^1 = D$ and unknown initial value $S_0 \in \mathbb{R}$. Suppose that (3.21) is not satisfied, so that $\bar{Z}_0 = 0$ and $E[\bar{H}D] \neq 0$. In this case, there does not exist any value of $S_0 \in \mathbb{R}$ such that $\bar{Z}S$ is a martingale, since $\bar{Z}_0 S_0 = 0$ regardless of that choice. On the other hand, if $\bar{Z}_0 = E[\bar{H}D] = 0$, then $\bar{Z}S$ is a martingale for any choice of $S_0 \in \mathbb{R}$, and one can check that $(1,S)$ defines an equilibrium. This also illustrates the issue of nonuniqueness: namely, if $\bar{Z}_t = E[\bar{H}D \mid \mathcal{F}_t] = 0$ for some $t \in \{0,\dots,T-1\}$, then the price S_t in equilibrium can be set in an arbitrary way.

The issue is similar for the financial assets. Consider now a one-period model with $d_1 = 1$ and $d_2 = 0$ so that there exists a single financial asset S with $S_1 = S_0 + \Delta A_1 + \Delta M_1$, where $\Delta A_1 \in \mathbb{R}$ is unknown and ΔM_1 is the jump of a martingale. If (3.20) does not hold, then $\bar{Z}S$ is not a martingale for any choice of $A \in \mathbb{R}$, since $\bar{Z}$ is a martingale and hence

$$E[Z_1(S_0 + \Delta A_1 + \Delta M_1)] = E[Z_1 \Delta M_1] = \xi_1 \Delta\langle M\rangle_1 \neq 0 = Z_0 S_0.$$

On the other hand, if $\xi_1 \Delta\langle M\rangle_1 = 0$ then $\bar{Z}S$ is a martingale for any value of ΔA_1. Thus if $Z_t = 0$ and (3.20) is satisfied, then we would expect that the value ΔA_{t+1} is arbitrary.

As it turns out, (3.20) and (3.21) are the only significant requirements for the existence of an equilibrium (other than integrability conditions, as we show later in Example 3.12). We now show that if these conditions hold, then there exists an equilibrium which need not be unique. We use the fact that $\bar{Z}S$ is a local martingale, by Lemma 3.6, in order to construct an explicit equilibrium. In comparison to the proof of Theorem 3.8, the construction is here more difficult because $\bar{N}$ is no longer well defined as a stochastic logarithm $(1/\bar{Z}_{-}) \bullet \bar{Z}$. Instead, we define $(\bar{N}_t)_{t\in\{0,\dots,T\}}$ recursively by $\bar{N}_0 := 0$ and

$$\bar{N}_t := \bar{N}_{t-1} + \frac{\bar{Z}_t - \bar{Z}_{t-1}}{\bar{Z}_{t-1}} \mathbf{1}_{\{\bar{Z}_{t-1}\neq 0\}}, \quad t \in \{1,\dots,T\}, \tag{3.24}$$

i.e., we arbitrarily set the increment $\Delta\bar{N}_t$ to 0 whenever $\bar{Z}_{t-1} = 0$. For each $s \in \{0,\dots,T\}$, we also define the local martingale ${}^s\mathcal{E}(\bar{N}) = ({}^s\mathcal{E}(\bar{N})_t)_{t\in\{s,\dots,T\}}$ by

$${}^s\mathcal{E}(\bar{N})_t := \prod_{k=s+1}^{t} (1 + \Delta\bar{N}_k) \tag{3.25}$$

In the case where $\bar{Z}$ does not hit 0, we have ${}^s\mathcal{E}(\bar{N})_t = \bar{Z}_t/\bar{Z}_s$ for each $s \le t$. In other words, we may view ${}^s\mathcal{E}(\bar{N})$ as "restarting" $\bar{Z}$ at time s (in a multiplicative way) with ${}^s\mathcal{E}(\bar{N})_s = 1$. The general case is similar, with the important difference that ${}^s\mathcal{E}(\bar{N})$ is absorbed at 0 whenever $\bar{Z}$ hits 0 from a nonzero value. Thus each process ${}^s\mathcal{E}(\bar{N})$ reproduces the dynamics of $\bar{Z}$ until the latter hits 0. We note once again that the value of ΔN_t may be chosen arbitrarily whenever $\bar{Z}_{t-1} = 0$. Because we set $\Delta N_t = 0$ in that case by (3.24), the equilibrium constructed below defaults to behaving as a local martingale whenever $\bar{Z}$ hits 0.

In the following, we recall the Galtchouk–Kunita–Watanabe decomposition (3.19) for $\bar{Z}$ as well as the conditions (3.20) and (3.21) from Lemma 3.10.

Theorem 3.11. *Assume that ${}^s\mathcal{E}(\bar{N})$ is a square-integrable martingale for each $s \in \{0,\dots,T\}$ and that (3.20) and (3.21) hold up to P-null sets. Define the process $(S_t)_{t\in\{0,\dots,T\}}$ by*

$$S^j_t := S^j_0 + M^j_t + \sum_{k=1}^{t}\sum_{i=1}^{d_1}\left(-\frac{\bar{\xi}^i_k}{\bar{Z}_{k-1}}\mathbf{1}_{\{\bar{Z}_{k-1}\neq 0\}}\Delta\langle M^i, M^j\rangle_k\right), \quad j \in \{1,\dots,d_1\}, \tag{3.26}$$

$$S^j_t := E[{}^t\mathcal{E}(\bar{N})_T D^j \mid \mathcal{F}_t], \quad j \in \{d_1+1,\dots,d_1+d_2\}. \tag{3.27}$$

If $(S^{(2)}_t)_{t\in\{0,\dots,T\}}$ is square-integrable, then $(1,S) = (1,S^{(1)},S^{(2)})$ is an equilibrium market.

Proof. We show in the steps (a)–(d) below that the process $(1, S)$ defined by (3.26) and (3.27) is an equilibrium market. In step (a), we check the conditions required by Definition 2.5 except for 1) and 2). In steps (b) and (c), we show that $({}^s\mathcal{E}(\bar{N})_t S^j_t)_{t\in\{s,\dots,T\}}$ and $(\bar{Z}_t S^j_t)_{t\in\{0,\dots,T\}}$, respectively, are local P-martingales for each $s \in \{0,\dots,T-1\}$ and $j \in \{1,\dots,d_1+d_2\}$. These results are then used in step (d) to check that conditions 1) and 2) are satisfied.

(a) We start by checking (2.1) and (2.2). For $j \in \{1,\dots,d_1\}$, the process $(A^j_t)_{t\in\{0,\dots,T\}}$ of S^j given by

$$A^j_t := \sum_{k=1}^{t}\sum_{i=1}^{d_1}\left(-\frac{\bar{\xi}^i_k}{\bar{Z}_{k-1}}\mathbf{1}_{\{\bar{Z}_{k-1}\neq 0\}}\Delta\langle M^i, M^j\rangle_k\right), \tag{3.28}$$

is predictable, so that by the definition (3.26), $S^{(1)}$ satisfies (2.1). Moreover, plugging $t = T$ into (3.27) yields (2.2) since ${}^T\mathcal{E}(\bar{N})_T = 1$.

We also have to check that $(1, S)$ is a local L^2-market. As argued in the proof of Theorem 3.8, $S^{(1)}$ is a local L^2-semimartingale as $M^{(1)}$ is locally square-integrable and by Černý/Kallsen [25, Lemma A.2]. On the other hand, $S^{(2)}$ is an L^2-semimartingale as it is square-integrable by assumption and the set $\{0,\dots,T\}$ of times is finite. The fact that $S^{(2)}$ is an L^2-semimartingale also implies that condition 3) of Definition 2.5 of an equilibrium market is satisfied.

(b) We first show that $({}^s\mathcal{E}(\bar{N})_t S^j_t)_{t\in\{s,\dots,T\}}$ is a local P-martingale for each $j \in \{1,\dots,d_1\}$ and $s \in \{0,\dots,T-1\}$. We use a similar argument as in the proof of part 2) of Proposition 2.12. Consider the setup of Choulli et al. [29, Section 3] with the family $\mathcal{E} = ({}^s\mathcal{E}(\bar{N}))_{s\in\{0,\dots,T\}}$. Since for $j \in \{1,\dots,d_1\}$, we have

$$\sum_{i=1}^{d_1}\frac{\bar{\xi}^i_t}{\bar{Z}_{t-1}}\mathbf{1}_{\{\bar{Z}_{t-1}\neq 0\}}\Delta\langle M^i, M^j\rangle_t = \frac{\mathbf{1}_{\{\bar{Z}_{t-1}\neq 0\}}}{\bar{Z}_{t-1}}\Delta\langle \bar{Z}, M^j\rangle_t = \Delta\langle \bar{N}, M^j\rangle_t,$$

we obtain that S^j given by (3.26) is a local $\mathcal{E}$-martingale by [29, Corollary 3.16], i.e., ${}^s\mathcal{E}(\bar{N})S^j$ is a local P-martingale for each $s \in \{0,\dots,T\}$.

For $j \in \{d_1+1,\dots,d_1+d_2\}$, we use the definition (3.27) and the square-integrability of ${}^s\mathcal{E}(\bar{N})$ and D^j to obtain for $t \in \{s,\dots,T\}$ that

$${}^s\mathcal{E}(\bar{N})_t S^j_t = {}^s\mathcal{E}(\bar{N})_t E[{}^t\mathcal{E}(\bar{N})_T D^j \mid \mathcal{F}_t] = E[{}^s\mathcal{E}(\bar{N})_T D^j \mid \mathcal{F}_t],$$

and hence ${}^s\mathcal{E}(\bar{N})S^j$ is a true P-martingale for each $s \in \{0,\dots,T\}$.

(c) Next, we show that $\bar{Z}S^j$ is a local P-martingale for $j \in \{1,\dots,d_1+d_2\}$.

For $j \in \{1, \dots, d_1\}$ and $t \in \{1, \dots, T\}$, we have by (3.24) that

$$\begin{aligned}\Delta(\bar{Z}S^j)_t &= \bar{Z}_{t-1}\Delta S_t^j + S_t^j \Delta \bar{Z}_t \\ &= \bar{Z}_{t-1}\Delta S_t^j + S_t^j \big(\mathbf{1}_{\{\bar{Z}_{t-1}\neq 0\}}\bar{Z}_{t-1}\Delta \bar{N}_t + \mathbf{1}_{\{\bar{Z}_{t-1}=0\}}\Delta \bar{Z}_t\big) \\ &= \mathbf{1}_{\{\bar{Z}_{t-1}\neq 0\}}\bar{Z}_{t-1}(\Delta S_t^j + S_t^j \Delta \bar{N}_t) + \mathbf{1}_{\{\bar{Z}_{t-1}=0\}} S_t^j \Delta \bar{Z}_t, \qquad (3.29)\end{aligned}$$

where we use $\bar{Z}_{t-1} = \mathbf{1}_{\{\bar{Z}_{t-1}\neq 0\}}\bar{Z}_{t-1}$ for the last equality. Note that we have ${}^{t-1}\mathcal{E}(\bar{N})_{t-1} = 1$ and

$$\Delta\big({}^{t-1}\mathcal{E}(\bar{N})\big)_t = {}^{t-1}\mathcal{E}(\bar{N})_{t-1}\Delta \bar{N}_t = \Delta \bar{N}_t.$$

By plugging in, this yields

$$\Delta\big({}^{t-1}\mathcal{E}(\bar{N})S^j\big)_t = \Delta S_t^j + S_{t-1}^j \Delta \bar{N}_t + \Delta S_t^j \Delta \bar{N}_t = \Delta S_t^j + S_t^j \Delta \bar{N}_t.$$

Since we have already shown in step (b) that ${}^{s}\mathcal{E}(\bar{N})S^j$ is a local P-martingale for each $s \in \{0, \dots, T\}$, this implies that $\Delta S_t^j + S_t^j \Delta \bar{N}_t$ is the increment of a local P-martingale, and hence so is the first term in the right-hand side of (3.29). We now consider the second term. Since $\Delta \bar{Z}_t = \Delta M_t^{\bar{Z}}$ on $\{\bar{Z}_{t-1} = 0\}$ by (3.19) and the assumption (3.20), we get

$$\mathbf{1}_{\{\bar{Z}_{t-1}=0\}} S_t^j \Delta \bar{Z}_t = \mathbf{1}_{\{\bar{Z}_{t-1}=0\}} (S_{t-1}^j + \Delta A_t^j + \Delta M_t^j)\Delta M_t^{\bar{Z}}.$$

This is the increment of a local P-martingale, as $M^{\bar{Z}}$ and M^j are strongly orthogonal local P-martingales, whereas $S_{t-1}^j + \Delta A_t^j$ is $\mathcal{F}_{t-1}$-measurable. Returning to (3.29), we have thus shown that $\bar{Z}S^j$ is a local P-martingale for $j \in \{1, \dots, d_1\}$.

On the other hand, for $j \in \{d_1 + 1, \dots, d_1 + d_2\}$, we claim that $\bar{Z}S^j$ is even a true P-martingale. We use backward induction to show this statement, starting with $t = T$. Since $\bar{Z}_T = \bar{H}$ and $S_T^j = D^j$ are square-integrable, we get that $\bar{Z}_T S_T^j$ is integrable so that $\bar{Z}S^j$ is a martingale on $\{T\}$. For the inductive step, we claim that if $\bar{Z}S^j$ is a martingale on $\{t+1, \dots, T\}$ for some $t \in \{0, \dots, T-1\}$, then $E[\bar{Z}_{t+1}S_{t+1}^j \mid \mathcal{F}_t] = \bar{Z}_t S_t^j$ P-a.s. so that $\bar{Z}S^j$ is a P-martingale on $\{t, \dots, T\}$. To show this claim, note that the definitions (3.24) and (3.25) yield

$$\begin{aligned}\bar{Z}_{t+1}\mathbf{1}_{\{\bar{Z}_t\neq 0\}} = \bar{Z}_t\left(1 + \frac{\Delta \bar{Z}_{t+1}}{\bar{Z}_t}\right)\mathbf{1}_{\{\bar{Z}_t\neq 0\}} &= \bar{Z}_t(1 + \Delta \bar{N}_{t+1})\mathbf{1}_{\{\bar{Z}_t\neq 0\}} \\ &= \bar{Z}_t\, {}^{t}\mathcal{E}(\bar{N})_{t+1}\mathbf{1}_{\{\bar{Z}_t\neq 0\}}.\end{aligned}$$

By plugging in and recalling that ${}^t\mathcal{E}(\bar{N})S^j$ is a true P-martingale and ${}^t\mathcal{E}(\bar{N})_t = 1$, we get

$$\begin{aligned} E[\bar{Z}_{t+1}S^j_{t+1} \mid \mathcal{F}_t] &= \bar{Z}_t E[{}^t\mathcal{E}(\bar{N})_{t+1}S^j_{t+1} \mid \mathcal{F}_t]\mathbf{1}_{\{\bar{Z}_t\neq 0\}} + E[\bar{Z}_{t+1}S^j_{t+1} \mid \mathcal{F}_t]\mathbf{1}_{\{\bar{Z}_t=0\}} \\ &= \bar{Z}_t S^j_t \mathbf{1}_{\{\bar{Z}_t\neq 0\}} + E[\bar{Z}_{t+1}S^j_{t+1}|\mathcal{F}_t]\mathbf{1}_{\{\bar{Z}_t=0\}}. \end{aligned} \tag{3.30}$$

By the inductive hypothesis and the assumption (3.21), we obtain

$$E[\bar{Z}_{t+1}S^j_{t+1} \mid \mathcal{F}_t]\mathbf{1}_{\{\bar{Z}_t=0\}} = E[\bar{Z}_T S^j_T \mid \mathcal{F}_t]\mathbf{1}_{\{\bar{Z}_t=0\}} = E[\bar{H}D^j \mid \mathcal{F}_t]\mathbf{1}_{\{\bar{Z}_t=0\}} = 0.$$

Plugging into (3.30) yields

$$E[\bar{Z}_{t+1}S^j_{t+1} \mid \mathcal{F}_t] = \bar{Z}_t S^j_t \mathbf{1}_{\{\bar{Z}_t\neq 0\}} = \bar{Z}_t S^j_t.$$

It follows by backward induction that $\bar{Z}S^j$ is a true P-martingale on $\{0, \dots, T\}$ for each $j \in \{d_1 + 1, \dots, d_1 + d_2\}$, as claimed. This also concludes the proof that $\bar{Z}S^j$ is a local P-martingale for all $j \in \{1, \dots, d_1 + d_2\}$.

(d) We are now ready to show that $(1, S)$ satisfies conditions 1) and 2) of Definition 2.5. We first show that $(1, S)$ satisfies uniqueness of value processes. To that end, suppose that $c_1 + \vartheta^1 \bullet S_T = c_2 + \vartheta^2 \bullet S_T$ for some $c_1, c_2 \in \mathbb{R}$ and $\vartheta^1, \vartheta^2 \in \overline{\Theta}(S)$. Recall from (b) that ${}^s\mathcal{E}(\bar{N})S^j$ is a local P-martingale on $\{s, \dots, T\}$ for each $s \in \{0, \dots, T-1\}$. Since ${}^s\mathcal{E}(\bar{N})$ is square-integrable by assumption, it follows by Proposition 2.12 that ${}^s\mathcal{E}(\bar{N})(c_1 + \vartheta^1 \bullet S)$ and ${}^s\mathcal{E}(\bar{N})(c_2 + \vartheta^2 \bullet S)$ are true P-martingales on $\{s, \dots, T\}$. Because ${}^s\mathcal{E}(\bar{N})_s = 1$, this yields

$$\begin{aligned} c_1 + \vartheta^1 \bullet S_s &= E[{}^s\mathcal{E}(\bar{N})_T(c_1 + \vartheta^1 \bullet S_T) \mid \mathcal{F}_s] \\ &= E[{}^s\mathcal{E}(\bar{N})_T(c_2 + \vartheta^2 \bullet S_T) \mid \mathcal{F}_s] = c_2 + \vartheta^2 \bullet S_s. \end{aligned}$$

In particular, taking $s = 0$ gives $c_1 = c_2$. As $s \in \{0, \dots, T\}$ is arbitrary, $\vartheta^1 \bullet S$ and $\vartheta^2 \bullet S$ are indistinguishable, so that $\vartheta^1 =_S \vartheta^2$ and $(1, S)$ satisfies uniqueness of value processes.

Next, we show the existence of solutions to the MVH problem (2.8) for each $H \in L^2$. As in step (b), consider once again the family $\mathcal{E} = ({}^s\mathcal{E}(\bar{N}))_{s\in\{0,\dots,T\}}$, which is square-integrable by assumption. Let τ be a stopping time taking values in $\{0, \dots, T\}$. Since ${}^s\mathcal{E}(\bar{N})$ is a martingale by assumption for any $s \in \{0, \dots, T\}$, so is $1 + \mathbf{1}_{\{\tau=s\}}({}^s\mathcal{E}(\bar{N}) - 1)$ as ${}^s\mathcal{E}(\bar{N}) - 1 = 0$ on $\{0, \dots, s\}$. Thus we obtain that

$$^\tau\mathcal{E}(\bar{N}) = 1 + \sum_{s=0}^{T} \mathbf{1}_{\{\tau=s\}}\big({}^s\mathcal{E}(\bar{N}) - 1\big)$$

is a martingale. As this holds for any stopping time τ, the family $\mathcal{E}$ is so-called regular; see [29, Definitions 3.4 and 3.6]. Thus by Czichowsky/Schweizer [32, Theorem 2.16], the set $\mathcal{G}_T(S)$ is closed in L^2. This implies the existence of a solution to the MVH problem (2.8) for any payoff $H \in L^2$, since it can be seen as a projection problem in L^2. The uniqueness of value processes (and thus of gains processes) together with Proposition 2.8 yields that the solution to (2.8) is unique for each $H \in L^2$. Since $\eta^k \in \overline{\Theta}(S)$ by condition 3) of Definition 2.5, which we already showed in step (a), it follows from Proposition 3.1 that there exists a unique solution $\hat{\vartheta}^k$ to (3.2) for each $k \in \{1, \dots, K\}$, and thus condition 1) is satisfied.

It remains to check that $(1, S)$ satisfies condition 2) of Definition 2.5, for which we use the same argument as in the proof of Theorem 3.8. By Lemma 2.13 and since $\bar{Z}S^j$ is a local P-martingale for each $j \in \{1, \dots, d_1 + d_2\}$, the strategy 0 solves the MVH problem (2.8) for $\bar{H}$. Thus $\sum_{k=1}^K \hat{\vartheta}^k = \bar{\vartheta} = \bar{\eta}$ by Lemmas 3.3 and 3.5, so that the market clears. This concludes the proof that $(1, S)$ is an equilibrium market. □

3.6 Non-existence of equilibria

Finally, we give an example of a setup where an equilibrium market (in the sense of Definition 2.5) fails to exist due to integrability issues, and not because the process $\bar{Z}$ hits 0 as in Section 3.5. We consider the very simplest case of a market with no financial and one productive asset, i.e., $d_1 = 0$ and $d_2 = 1$. The setup is based on the counterexample in Černý/Kallsen [26], which in turn is inspired by the well-known counterexample of Delbaen/Schachermayer [37].

The key point is that the candidate equilibrium price process of the productive asset does not have sufficient integrability for the buy-and-hold strategies to be admissible. In that case, Lemma 3.1 cannot be applied, so that the existence of a solution to the optimisation problem (3.2) is not equivalent to the existence of a solution to the MVH problem (3.3). By part (b) of the proof of Theorem 3.8, there still exists in this case a unique solution to the MVH problem (3.3) for each agent k, but it is unclear whether (3.2) admits a solution.

Example 3.12. After rescaling the time interval $[0,\infty]$ to $[0,T]$, there exists by [26, Lemma 2.2] a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, P)$ supporting two probability measures $\bar{Q}, Q'$ and a continuous process $(X_t)_{0\le t\le T}$ null at 0 with the following properties:

1) The measures $\bar{Q}, Q'$ are equivalent to P, with $\frac{\mathrm{d}\bar{Q}}{\mathrm{d}P}, \frac{\mathrm{d}Q'}{\mathrm{d}P} \in L^2(P)$.

2) The process X is a uniformly integrable martingale under $\bar{Q}$, and a strict local martingale under Q'. Moreover, $X_T \in L^2(P)$.

Fix now some $\bar{\gamma} > 0$, and suppose that $d_1 = 0$, $d_2 = 1$, $D^1 := X_T$ and $\bar{\Xi} := \bar{\gamma} - \frac{\mathrm{d}\bar{Q}}{\mathrm{d}P}$, so that $\bar{H} = \frac{\mathrm{d}\bar{Q}}{\mathrm{d}P} > 0$ P-a.s. Then it follows from Theorem 3.8 that if an equilibrium market exists, it must satisfy $S_t^1 = E_{\bar{Q}}[X_T \mid \mathcal{F}_t] = X_t$. However, since $X = 1 \bullet X$ is not a Q'-martingale, the strategy $e^1 \equiv 1$ is not admissible by Černý/Kallsen [25, Corollary 2.5]. Therefore, an equilibrium market does not exist in this setup.

The filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, P)$ in the example above can be chosen in such a way that the filtration is continuous, that is, every local martingale is continuous. On such a probability space, Example 3.12 is generic in the following sense. Consider the general setup with d_1 financial assets and d_2 productive assets, and suppose that $\mathbb{F}$ is continuous and $\bar{H} > 0$ P-a.s. If an equilibrium market fails to exist, then one can construct a triplet $(\bar{Q}, Q', X)$ with the properties 1) and 2) of Example 3.12. In other words, any such example of a setup where an equilibrium market does not exist corresponds to a counterexample of the type considered in [26, Lemma 2.2].

To see this, consider the processes $S^{(1)}$ and $S^{(2)}$ given in (3.16) and (3.17). Note that we must have $d_2 \ge 1$, as otherwise Theorem 3.8 ensures the existence of an equilibrium. Since $S^{(2)}$ is automatically a local $L^2(P)$-semimartingale by the continuity of the filtration, the last assumption for Theorem 3.8 must fail to hold, i.e., there must exist $j \in \{d_1+1, \dots, d_1+d_2\}$ such that $e^j \notin \overline{\Theta}(S)$. Note that there exists an ELMM for S with square-integrable density, namely $\bar{Q} \approx P$ defined by $\frac{\mathrm{d}\bar{Q}}{\mathrm{d}P} = \frac{\bar{H}}{E[\bar{H}]}$, by the construction of S and as $\bar{H}$ is strictly positive and square-integrable. By [25, Corollary 2.5] and because $e^j \notin \overline{\Theta}(S)$, there must exist an ELMM $Q' \approx P$ for S with square-integrable density such that $S^j = S_0^j + e^j \bullet S$ is a strict local Q'-martingale.

Now we set $X := S^j$. Note that the measures $\bar{Q}, Q' \approx P$ have square-integrable densities, and X is continuous and a strict local martingale under

Q'. Moreover, since $\bar{Z}X$ is a uniformly integrable P-martingale by the definition (3.17), it follows by the Bayes formula that X is a uniformly integrable martingale under $\bar{Q}$, since the density process of $\bar{Q}$ is given by $\bar{Z}/E[\bar{H}]$. Thus we recover $(\bar{Q}, Q', X)$ satisfying the properties 1) and 2) in Example 3.12, as claimed.

Chapter IV

Equilibrium under general mean–variance preferences

1 Introduction

As in the previous chapter, we want to find mean–variance equilibria for an incomplete market in continuous time. In Chapter III, we found an explicit formula for the equilibrium prices under quadratic utility preferences of the form $E[V - V^2/\gamma_k]$, where V denotes the final wealth and $\gamma_k > 0$ is the risk tolerance parameter of agent k. We now look to extend our results to the case of mean–variance preferences of the form $U_k(E[V], \sqrt{\mathrm{Var}[V]})$ for mean–variance utility functions U_k on $\mathbb{R} \times \mathbb{R}_+$. At first, this may seem like a simple task. Indeed, Koch-Medina/Wenzelburger [85] study the same equilibrium problem for the one-period model directly under mean–variance preferences, but the extension becomes considerably more involved in continuous time. In one period, the unknown time-0 prices are constant, and hence the set of random variables that can be replicated by a self-financing portfolio $(x, \vartheta) \in \mathbb{R} \times \Theta$ is known a priori. In the continuous-time case, the price process is both random and unknown, so that the intertemporal dynamics of equilibrium prices play a bigger role.

Our approach is to relate the mean–variance preference and quadratic utility problems in order to apply the results from Chapter III. Both problems are versions of classical Markowitz portfolio selection, and hence any solution to either must be a mean–variance efficient strategy. Thus for a fixed mean–variance preference maximisation problem, there exist risk tolerance parameters $(\gamma_k)_{k=1}^K$ for the K agents such that each quadratic utility problem with risk tolerance γ_k admits the same solution as the original one. This argument yields that any mean–variance equilibrium is also a quadratic equilibrium. However, finding the equilibrium prices is more involved. Since the choice of $(\gamma_k)_{k=1}^K$ depends implicitly on the dynamics of the equilibrium, which is unknown, we cannot directly apply our previous results. The only exception is the case of linear mean–variance preferences, where the structure of the quadratic equilibrium yields an explicit formula for the mean–variance equilibrium.

For the general case, we show the existence of an equilibrium via a fixed-point argument. For a set of parameters $(\gamma_k)_{k=1}^K$, Theorem III.3.8 gives the quadratic equilibrium price process $S(\bar{\gamma})$ which depends only on the aggregate risk tolerance $\bar{\gamma} = \sum_{k=1}^K \gamma_k$. Given $S(\bar{\gamma})$, there exist *implicit parameters* $\tilde{\gamma}_k$ such that for each agent k, the mean–variance and quadratic utility problems with respect to $S(\bar{\gamma})$ have the same solutions. Thus $S(\bar{\gamma})$ is a mean–variance equilibrium if and only if $\gamma_k = \tilde{\gamma}_k$ for all k; this can be written as a fixed-point condition $\bar{\gamma} = \Psi(\bar{\gamma})$ on the aggregate risk tolerance. We prove sufficient conditions for the existence of such a fixed point, and hence of an equilibrium.

We show that a fixed point exists by proving the continuity of Ψ and obtaining bounds on its output. The hardest step is to show that the $\tilde{\gamma}_k$ depend continuously on $S(\bar{\gamma})$, which can be seen as a stability result for the Markowitz and mean–variance hedging problems with respect to the price process. For utility maximisation problems, some recent results of this type are given in Bayraktar et al. [17], Kardaras/Žitković [80], Larsen/Žitković [87] and Mocha/Westray [94]. Most relevant for us are the results of [94], where the stability of utility maximisation is shown via stability results for BSDEs based on Mocha/Westray [93]. As in [94], we assume that the filtration is continuous. The difference to our work is that we cannot assume that all exponential moments of the so-called mean–variance tradeoff process are finite. Instead, we obtain new results on the stability of mean–variance hedging and quadratic BSDEs under a BMO condition, which can be ensured to hold via assumptions on the model primitives.

This chapter, based on joint work with Christoph Czichowsky and Martin Herdegen, is structured as follows. In Section 2, we recall the basic setup and show the first results on the relationship between quadratic and mean–variance equilibria. This provides a characterisation of mean–variance equilibria in terms of a fixed-point problem. In Section 3, we show how to find an explicit solution in the case of linear mean–variance preferences. In Section 4, we state and prove our main results on the existence of a solution of a fixed point and thus of a mean–variance equilibrium. In particular, we prove in Section 4.2 our results on the continuity of the mean–variance hedging problem. The proofs of some results used in Section 4 are deferred to Sections 5 and 6 which are self-contained. In Section 5, we analyse mean–variance preference functions as well as an abstract mean–variance optimisation problem. Finally, in Section 6, we show the stability of quadratic BSDEs under a *BMO* bound on the stochastic driver.

2 Setup and connections to quadratic equilibria

2.1 Setup

We start with the same basic setup as in Chapter III, which we briefly recall here; a more extended discussion can be found in Section III.2 and at the beginning of Section III.3. Fix a time horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, where $(\mathcal{F}_t)_{0\le t\le T}$ satisfies the usual conditions, $\mathcal{F} = \mathcal{F}_T$ and $\mathcal{F}_0$ is P-trivial. We consider a market consisting of a risk-free asset with constant value 1, as well as d_1 *financial assets* and d_2 *productive* (or *real*) *assets*. The prices of the risky assets are determined as equilibrium prices, as we shall explain below. We start by imposing some conditions that the price processes must satisfy irrespective of any equilibrium considerations. First, we assume that the price processes of the financial assets satisfy the semimartingale decomposition

$$S_t^j = S_0^j + A_t^j + M_t^j, \quad 0 \le t \le T, \tag{2.1}$$

for $j \in \{1, \dots, d_1\}$, where $S_0^j \in \mathbb{R}$ and $(M_t^j)_{0\le t\le T} \in \mathcal{M}^2_{0,\mathrm{loc}}$ are fixed, whereas the predictable finite-variation parts $(A_t^j)_{0\le t\le T} \in \mathrm{FV}_0$ are to be determined in equilibrium. On the other hand, the real assets satisfy the terminal condition

$$S_T^j = D^j \tag{2.2}$$

for $j \in \{d_1 + 1, \dots, d_1 + d_2\}$, i.e., the terminal values are given by a random *dividend* $D^j \in L^2$; the rest of the price process $(S_t^j)_{0\le t<T}$ is to be determined in equilibrium. We denote the prices of the financial assets by $S^{(1)} = (S^1, \dots, S^{d_1})$ and those of the productive assets by $S^{(2)} = (S^{d_1+1}, \dots, S^{d_1+d_2})$ so that the prices of all assets are given by $(1, S) = (1, S^{(1)}, S^{(2)})$. In the following, we likewise use the notation $x = (x^{(1)}, x^{(2)})$ for each $x \in \mathbb{R}^{d_1+d_2}$ with $x^{(i)} \in \mathbb{R}^{d_i}$. We say that a semimartingale $(S_t)_{0\le t\le T}$ is an L^2*-semimartingale* if for $j \in \{1, \dots, d_1 + d_2\}$,

$$\sup\{E[|S_\sigma^j|^2] : \sigma \le T \text{ a stopping time}\} < \infty, \tag{2.3}$$

and that $(1, S)$ is a *local L^2-market* if S is a local L^2-semimartingale. We impose the condition that any equilibrium market $(1, S)$ must be a local L^2-market.

We suppose that the market consists of K agents who trade with each other continuously on the time interval $[0, T]$. Each agent $k \in \{1, \dots, K\}$ receives a *traded endowment* $\Xi^{k,t} = (\eta^{k,(2)})^\top S^{(2)}_T = (\eta^{k,(2)})^\top D^{(2)}$ consisting of $\eta^{k,(2)} \in \mathbb{R}^{d_2}$ units of the productive assets and a *non-traded endowment* $\Xi^{k,n} \in L^2$ at time T, so that the *total endowment* is $\Xi^k := \Xi^{k,t} + \Xi^{k,n}$. It is natural from an economic point of view to make the following **standing assumption** about the endowments that will be useful later in this chapter (see Lemma 2.25 below). We note that the assumption depends only on the fixed quantities $\eta^{k,(2)}, D^{(2)}$ and $\Xi^{k,n}$, and not on the (unknown) equilibrium prices.

Assumption 2.1. The total endowment Ξ^k of each agent k is bounded, nonnegative and not identically 0.

In addition to receiving an endowment, each agent trades in a frictionless and self-financing way with an *admissible strategy* $\vartheta \in \overline{\Theta}(S)$, where the set $\overline{\Theta}(S)$ of admissible strategies is the one considered in Černý/Kallsen [25, Definition 2.2] and Definition III.2.2, that we restate below as Definition 2.3.

Definition 2.2. Let $(1, S)$ be a local L^2-market. A *simple integrand for S* is a process of the form $\vartheta = \sum_{i=1}^{m-1} \xi_i \mathbf{1}_{]\!]\sigma_i, \sigma_{i+1}]\!]}$, where $m \in \mathbb{N}$, $0 \le \sigma_1 \le \dots \le \sigma_m$ are $[0, T]$-valued stopping times and each ξ_i is a bounded $\mathcal{F}_{\sigma_i}$-measurable random vector, such that each stopped process $S^{j,\sigma_m} = (S^j_{\sigma_m \wedge t})_{0 \le t \le T}$ is an L^2-semimartingale for $j = 1, \dots, d_1 + d_2$. We denote by $\Theta_{\mathrm{simple}}(S)$ the set of all simple integrands for S.

For a semimartingale X, we denote by $L(X)$ the set of *predictable X-integrable processes* on $[0, T]$; see Jacod/Shiryaev [71, III.6.17]. Note that we identify integrands up to *X-equivalence*, i.e., for $\eta^1, \eta^2 \in L(X)$, we write $\eta^1 =_X \eta^2$ if the processes $\eta^1 \bullet X$ and $\eta^2 \bullet X$ are indistinguishable.

Definition 2.3. Let $(1, S)$ be a local L^2-market. A strategy $\vartheta \in L(S)$ is called *L^2-admissible for S* if $\vartheta \bullet S_T \in L^2$ and there exists a sequence $(\vartheta^n)_{n \in \mathbb{N}}$ in $\Theta_{\mathrm{simple}}(S)$ such that

1) $\vartheta^n \bullet S_T \xrightarrow{L^2} \vartheta \bullet S_T$,

2) $\vartheta^n \bullet S_\tau \xrightarrow{P} \vartheta \bullet S_\tau$ for all $[0, T]$-valued stopping times τ, where $\vartheta \bullet S = (\vartheta \bullet S_t)_{0 \leq t \leq T}$ denotes the stochastic integral $\vartheta \bullet S_t = \int_0^t \vartheta_u dS_u$ for $t \in [0, T]$. The set of all L^2-admissible trading strategies is denoted by $\overline{\Theta}(S)$, and we identify L^2-admissible strategies for S up to S-equivalence.

As shown in (III.2.5), agent k attains the terminal wealth

$$V_T^k(\vartheta) = (\vartheta - \eta^k) \bullet S_T + \Xi^k \tag{2.4}$$

by receiving the total endowment Ξ^k and trading with a strategy $\vartheta \in \overline{\Theta}(S)$, where we write η^k for the constant strategy $(0, \eta^{k,(2)})$. We note that the assumption that S is a local L^2-semimartingale does not ensure that $\eta^k \in \overline{\Theta}(S)$, i.e., holding the traded endowment without any additional trading activity need not be an admissible strategy for agent k. For this reason, we impose the additional assumption that $e^j \in \overline{\Theta}(S)$ for each $j \in \{d_1 + 1, \dots, d_1 + d_2\}$, where e^j is the constant strategy that buys and holds one unit of asset j from time 0. Since the agents do not receive an endowment of financial assets, this assumption is not required for $j \in \{1, \dots, d_1\}$.

We suppose that the agents consume their wealth at the terminal time T, so that each agent k trades with the goal of maximising the utility from her terminal consumption. In general, each agent's preferences may be described by a functional $\mathcal{U}_k : L^0(P) \to \mathbb{R}$ that assigns higher values to more desirable outcomes. The strategy chosen by the agent k is then the solution to the problem

$$\mathcal{U}_k\big((\vartheta - \eta^k) \bullet S_T + \Xi^k\big) \longrightarrow \max_{\vartheta \in \overline{\Theta}(S)} ! \tag{2.5}$$

We view $S_0^{(1)}$, $M^{(1)}$ and $D^{(2)}$ together with the endowments $\Xi^{k,t}$, $\Xi^{k,n}$ and preference functionals $\mathcal{U}_k$ as being exogenously determined and fixed a priori; we refer to these as the *primitives*. We now recall Definition III.2.5 of an equilibrium market with respect to these primitives.

Definition 2.4. A local L^2-market $(1, S^{(1)}, S^{(2)})$ is called an *equilibrium market* if it satisfies (2.1) and (2.2) as well as the following conditions:

1) For each agent $k \in \{1, \dots, K\}$, the maximisation problem (2.5) has a solution $\hat{\vartheta}^k \in \overline{\Theta}(S)$ that is unique up to S-equivalence.

2) The market clears, i.e.,

$$\sum_{k=1}^K \hat{\vartheta}^{k,j} =_S \bar{\eta}^j := \begin{cases} 0, & \text{if } j \in \{1, \dots, d_1\}, \\ \sum_{k=1}^K \eta^{k,j}, & \text{if } j \in \{d_1 + 1, \dots, d_1 + d_2\}. \end{cases} \tag{2.6}$$

3) $e^j \in \overline{\Theta}(S)$ for $j \in \{d_1+1,\dots,d_1+d_2\}$, i.e., the buy-and-hold strategies of the productive assets are L^2-admissible.

In both Chapter III and the present one, our goal is to study the equilibrium markets, i.e., the family of price processes $(1, S_t)_{0\le t\le T}$ that satisfy Definition 2.4 for a given set of primitives. In particular, we should like to know whether an equilibrium market exists and if it is unique. The difference between the previous chapter and the current one lies in the choice of preference functionals $\mathcal{U}_k$. Previously, we assumed that the preferences are described by *expected quadratic utility*, i.e., that

$$\mathcal{U}_k(V) := E[U_k^Q(V)]$$

for $V \in L^2$ and $k \in \{1,\dots,K\}$, where $U_k^Q : \mathbb{R} \to \mathbb{R}$ is the quadratic utility function given by

$$U_k^Q(x) := x - \frac{x^2}{2\gamma_k} \tag{2.7}$$

for $x \in \mathbb{R}$ and some *risk tolerance* $\gamma_k \in (0,\infty)$. Thus each agent k would solve the maximisation problem (III.3.2), which reads

$$E\big[U_k^Q\big((\vartheta-\eta^k)\bullet S_T + \Xi^k\big)\big] \longrightarrow \max_{\vartheta\in\overline{\Theta}(S)} ! \tag{2.8}$$

In contrast to Chapter III, we suppose here that the preferences of the agents are described by so-called mean–variance utility functions, defined as follows.

Definition 2.5. A function $U : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ is called a *mean–variance utility function* if it is strictly quasiconcave, strictly increasing in the first variable μ, strictly decreasing in the second variable σ, twice continuously differentiable and nondegenerate in the sense that $|\nabla U(\mu,\sigma)| > 0$ for all $\mu \in \mathbb{R}$ and $\sigma \ge 0$.

We note that U is actually defined as a function of the standard deviation σ rather than the variance σ^2; we use this convention for consistency with Koch-Medina/Wenzelburger [85]. This choice also leads to a slightly more general definition because the strict quasiconcavity and differentiability conditions with respect to σ are weaker than those with respect to σ^2. From a notational point of view, we nevertheless refer to U as a mean–variance utility function. If U is given by

$$U(\mu,\sigma) = \mu - \frac{\sigma^2}{2\lambda}, \quad (\mu,\sigma) \in \mathbb{R}\times\mathbb{R}_+, \tag{2.9}$$

for some $\lambda > 0$, then we say that U is a *linear mean–variance utility function*. These linear U play a special role in Section 3 below. Other examples of mean–

variance utility functions can be constructed by setting $U(\mu, \sigma) = f(\mu) - g(\sigma)$, where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ are twice continuously differentiable functions with $f'(\mu) > 0$, $f''(\mu) < 0$, $g'(\sigma) > 0$ and $g''(\sigma) > 0$ for all $\mu \in \mathbb{R}$ and $\sigma \geq 0$.

We assume that each agent has an associated mean–variance utility function U_k so that the preference functional $\mathcal{U}_k$ is given by

$$\mathcal{U}_k(V) := U_k(E[V], \sqrt{\operatorname{Var}[V]})$$

for $V \in L^2$ and $k \in \{1, \dots, K\}$. Thus by (2.4), agent k seeks to solve the problem

$$U_k\big(E[(\vartheta - \eta^k) \bullet S_T + \Xi^k], \sqrt{\operatorname{Var}[(\vartheta - \eta^k) \bullet S_T + \Xi^k]}\big) \longrightarrow \max_{\vartheta \in \overline{\Theta}(S)} ! \tag{2.10}$$

A similar setup has been studied in Koch-Medina/Wenzelburger [85] for a one-period market, but the problem becomes substantially more challenging in multiperiod markets. Note that for a one-period equilibrium, one only needs to determine the initial price $S_0^{(2)}$ of the real assets and the predictable increment $\Delta A_1^{(1)}$ for the financial assets, both of which are deterministic as $\mathcal{F}_0$ is P-trivial. Thus even before any equilibrium considerations, the primitives determine the price process S up to a deterministic shift in the price increment ΔS_1 (more precisely, a deterministic shift in $S_1^{(1)}$ for the financial assets and $S_0^{(2)}$ for the real assets). Since any admissible strategy ϑ is in this case uniquely determined by ϑ_1, which is $\mathcal{F}_0$-measurable and hence constant, it follows that the set of attainable payoffs depends only on the primitives. By the same argument, the variance of the terminal gain $\vartheta_1 \Delta S_1$ attained by a given strategy ϑ is also determined uniquely by the primitives, unlike the expected return, which is sensitive to a deterministic shift in the increment ΔS_1.

The latter observation is used in [85] to show that every equilibrium in the sense of Definition 2.4 corresponds to an equilibrium in a so-called market for risk (see [85, Definition 2]) and vice versa, which results in a simplification of the problem. In the multiperiod case, $\operatorname{Var}[\vartheta \bullet S_T]$ depends on equilibrium prices $(S_t)_{0 \leq t \leq T}$ that are not known a priori, and not just on the primitives. Thus a different approach is needed. Our main idea is to exploit the relationship between the mean–variance utility maximisation problem (2.10) and the quadratic utility maximisation problem (2.8).

Definition 2.6. A local L^2-market $(1, S^{(1)}, S^{(2)})$ is called a *mean–variance equilibrium market* (with respect to the mean–variance utility functions $U_1, \dots, U_K$) if it is an equilibrium market in the sense of Definition 2.2, where the problem

(2.5) has the particular form (2.10). Likewise, $(1, S^{(1)}, S^{(2)})$ is called a *quadratic equilibrium* market (with *risk tolerances* $\gamma_1, \dots, \gamma_K > 0$) if it is an equilibrium market in the sense of Definition 2.2, where (2.5) is given by (2.8).

We recall that Theorem III.3.8 in the previous chapter gives sufficient conditions for the existence and uniqueness of a quadratic equilibrium, as well as explicit formulas for the equilibrium market. We occasionally also use results from Section III.2.5 on the mean–variance hedging problem (defined below in (2.14)). **Throughout this chapter**, we consider the mean–variance utility functions U_k to be fixed, whereas the risk tolerances γ_k in the definition (2.7) of U_k^Q may vary. By showing that a mean–variance equilibrium is a quadratic equilibrium and vice versa (under some assumptions), we are able to obtain sufficient conditions for the existence of mean–variance equilibria and to characterise them.

More precisely, the overall strategy is as follows. First, we study the individual optimisation problem (2.10) for agent k with respect to a fixed price process S. As we shall see in Proposition 2.13, the maximiser of (2.10) for a given mean–variance utility function U_k is also optimal for the quadratic utility problem (2.8) for a suitable choice of risk tolerance $\gamma_k > 0$ that depends on U_k as well as S. We can use this observation to show that a mean–variance equilibrium market is also a *generalised* quadratic equilibrium (see Definition 2.21 below) for some risk tolerances $\gamma_1, \dots, \gamma_K$. Our goal is then to reverse this procedure, i.e., to use Theorem III.3.8 to construct a generalised quadratic equilibrium S for some choice of parameters $\gamma_1, \dots, \gamma_K$ such that S is also a mean–variance equilibrium. This is still challenging, however, since the correct choice of parameters $\gamma_1, \dots, \gamma_K$ depends not only on the mean–variance utility functions U_k but also (implicitly) on the equilibrium price process S, which is unknown.

We show in Section 2.3 that the quadratic equilibrium with risk tolerances $\gamma_1, \dots, \gamma_K$ is a mean–variance equilibrium if and only if the *aggregate risk tolerance* $\bar{\gamma} := \sum_{k=1}^K \gamma_k > 0$ solves a fixed point problem on $\mathbb{R}$. We then consider two cases. First, if each U_k is linear of the form (2.9), we show that the fixed point problem can be solved explicitly for $\bar{\gamma}$, so that we can identify the mean–variance equilibrium as the quadratic equilibrium with aggregate risk tolerance $\bar{\gamma}$. We then consider the general case. By studying the fixed point problem, we find sufficient conditions for the existence of a solution $\bar{\gamma}$, although we do not obtain an explicit formula in this case. The existence of a fixed point $\bar{\gamma}$ then ensures the existence of a mean–variance equilibrium.

2.2 Individual optimisation problems

Our first goal is to study the individual optimisation problem (2.10) for each agent k with respect to a fixed price process $(S_t)_{0\leq t\leq T}$, and to relate it to the quadratic optimisation problem (2.8). We note that strictly speaking, the quadratic optimisation problem (2.8) is not a mean–variance optimisation problem. Indeed, for a quadratic utility function U_k^Q of the form (2.7), we have

$$E[U_k^Q(V)] = E\left[V - \frac{V^2}{2\gamma_k}\right] = E[V] - \frac{E[V]^2}{2\gamma_k} - \frac{\mathrm{Var}[V]}{2\gamma_k}$$

for all $V \in L^2$. The problem (2.8) could thus be written in the form (2.10), where

$$\tilde{U}_k(\mu, \sigma) := \mu - \frac{\mu^2}{2\gamma_k} - \frac{\sigma^2}{2\gamma_k}, \quad (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+.$$

However, $\tilde{U}_k$ is not a mean–variance utility function in the sense of Definition 2.5 since it is not increasing in μ due to the middle term. In particular, this means that the quadratic utility problem (2.8) is not equivalent to the problem for linear mean–variance utility (2.9), despite the apparent similarity.

Remark 2.7. The fact that mean–variance utility functions are strictly increasing in μ means that agents always prefer to increase their wealth. Thus mean–variance utility can be considered to be more natural than quadratic utility from an economic point of view. However, mean–variance utility functions do not completely preclude economically irrational behaviour, since an agent may still reject a positive random endowment (a "free lunch") if the increase in volatility outweighs the higher expected return.

We fix a price process $(S_t)_{0\leq t\leq T}$ satisfying the following **standing assumption** for the remainder of this section, which is a necessary condition for S to be an equilibrium in the sense of Definition 2.4.

Assumption 2.8. We suppose that S is a fixed local L^2-semimartingale such that $e^j \in \overline{\Theta}(S)$ for each $j \in \{d_1 + 1, \ldots, d_1 + d_2\}$.

In order to solve (2.10), we start by considering three related problems. The first is the classic *Markowitz portfolio optimisation problem*, where an agent seeks to achieve the highest possible expected return for a certain level of risk (represented by the variance), or conversely, to achieve a certain expected return with the minimum possible variance. One classic way to express the Markowitz problem is in terms of so-called mean–variance efficient strategies that we now define for our setup; we refer to Eberlein/Kallsen [44, Rule 10.43] for some equivalent definitions.

Definition 2.9. A strategy $\vartheta \in \overline{\Theta}(S)$ is *mean–variance efficient with respect to the endowment* $H \in L^2$ if there does not exist any other strategy $\vartheta' \in \overline{\Theta}(S)$ such that both of the following inequalities hold, with one of them strict:

$$E[\vartheta' \bullet S_T + H] \geq E[\vartheta \bullet S_T + H], \tag{2.11}$$

$$\mathrm{Var}[\vartheta' \bullet S_T + H] \leq \mathrm{Var}[\vartheta \bullet S_T + H]. \tag{2.12}$$

We say that ϑ is *mean–variance efficient for agent* k if $\vartheta - \eta^k$ is mean–variance efficient with respect to $H = \Xi^k$. The *mean–variance efficient frontier* for agent k is defined as the set

$$\mathcal{E}^k := \Big\{ \big(E[V_T^k(\vartheta)], \sqrt{\mathrm{Var}[V_T^k(\vartheta)]}\big) : \vartheta \in \overline{\Theta}(S) \text{ is mean-variance efficient for agent } k \Big\} \subseteq \mathbb{R} \times \mathbb{R}_+, \tag{2.13}$$

where we recall the formula (2.4) for $V_T^k(\vartheta)$.

Lemma 2.10. *If* $\hat{\vartheta}^k \in \overline{\Theta}(S)$ *is a solution to* (2.10), *then* $\hat{\vartheta}^k$ *is mean–variance efficient for agent* k.

Proof. Suppose by way of contradiction that $\hat{\vartheta}^k$ is not mean–variance efficient for agent k. Then there exists some $\vartheta' \in \overline{\Theta}(S)$ satisfying

$$E[\vartheta' \bullet S_T + \Xi^k] \geq E[(\hat{\vartheta}^k - \eta^k) \bullet S_T + \Xi^k],$$
$$\mathrm{Var}[\vartheta' \bullet S_T + \Xi^k] \leq \mathrm{Var}[(\hat{\vartheta}^k - \eta^k) \bullet S_T + \Xi^k],$$

where one of the inequalities is strict. Since U_k is strictly increasing in μ and strictly decreasing in σ by Definition 2.5, we have

$$U_k\Big(E[(\hat{\vartheta}^k - \eta^k) \bullet S_T + \Xi^k], \big(\mathrm{Var}[(\hat{\vartheta}^k - \eta^k) \bullet S_T + \Xi^k]\big)^{1/2}\Big)$$
$$< U_k\Big(E[(\tilde{\vartheta} - \eta^k) \bullet S_T + \Xi^k], \big(\mathrm{Var}[(\tilde{\vartheta} - \eta^k) \bullet S_T + \Xi^k]\big)^{1/2}\Big),$$

where $\tilde{\vartheta} := \vartheta' + \eta^k \in \overline{\Theta}(S)$, and this contradicts the optimality of $\hat{\vartheta}^k$ for (2.10). □

In view of Lemma 2.10, it is sufficient to consider mean–variance efficient strategies for agent k when looking for candidate solutions to (2.10). As we shall see, the mean–variance efficient strategies for the agents can be given in terms of the solutions to certain *mean–variance hedging* (MVH) and *extended mean–variance hedging* (exMVH) problems, which are defined as follows (see also Section III.2.5, where a more detailed discussion can be found). For a claim $H \in L^2$, we consider the MVH and exMVH problems

$$\varepsilon^2(H) := \inf_{\vartheta \in \overline{\Theta}(S)} E[(\vartheta \bullet S_T - H)^2], \tag{2.14}$$

$$\varepsilon^2_{\mathrm{ex}}(H) := \inf_{\substack{c \in \mathbb{R} \\ \vartheta \in \overline{\Theta}(S)}} E[(c + \vartheta \bullet S_T - H)^2], \tag{2.15}$$

respectively, i.e., we want to find the minimisers for the right-hand side of (2.14) and (2.15) as well as the *mean squared hedging errors* $\varepsilon^2(H)$ and $\varepsilon^2_{\mathrm{ex}}(H)$. Later, we shall sometimes write $\varepsilon^2(H;S)$ and $\varepsilon^2_{\mathrm{ex}}(H;S)$ to specify the price process S in (2.14) and (2.15), respectively. In order to apply classical results on MVH (for which we use Černý/Kallsen [25] as a reference), we make for the remainder of this section the following **standing assumption** on S. In our existence results for equilibrium prices below, we provide sufficient conditions on the primitives to ensure that Assumption 2.11 is satisfied for the equilibrium price.

Assumption 2.11. We suppose that there exists an equivalent local martingale measure (ELMM) $Q \approx P$ for S with density $dQ/dP \in L^2(P)$.

Assumption 2.11 can be seen as a no-free-lunch condition in an L^2-sense; see Stricker [115, Théorème 2]. It is well known that if Assumption 2.11 is satisfied, there exist unique (up to S-equivalence) minimisers $\vartheta^{\mathrm{MVH}}(H) \in \overline{\Theta}(S)$ for (2.14) and $(c(H), \vartheta^{\mathrm{ex}}(H)) \in \mathbb{R} \times \overline{\Theta}(S)$ for (2.15); see e.g. [25, Lemma 2.11]. An important instance of the MVH problem is the so-called *pure investment problem*, which is (2.14) with $H \equiv 1$. In that case, we say that $\vartheta^{\mathrm{MVH}}(1)$ is the *pure investment strategy* and denote the mean squared hedging error by $\ell := \varepsilon^2(1)$, i.e.,

$$\ell = \inf_{\vartheta \in \overline{\Theta}(S)} E[(1 - \vartheta \bullet S_T)^2] = E\big[\big(1 - \vartheta^{\mathrm{MVH}}(1) \bullet S_T\big)^2\big]. \tag{2.16}$$

We choose this notation since $\ell = L_0$ is also the initial value of the so-called opportunity process $(L_t)_{0 \le t \le T}$ from [25, Definition 3.3] that we introduce in Section 3 below. Under Assumption 2.11, we have $\ell \in (0,1]$ by [25, Lemma 3.10]. Note that by (2.16) and the uniqueness of the solution $\vartheta^{\mathrm{MVH}}(1)$, we have $\ell = 1$ if and only if $\vartheta^{\mathrm{MVH}}(1) =_S 0$. By Lemma III.2.13 with $H \equiv 1$, both properties are equivalent to the statement that S is a local martingale.

The following lemma shows the relationship between the pure investment strategy $\vartheta^{\mathrm{MVH}}(1)$ and the set of mean–variance efficient strategies with respect to $H \equiv 0$. This is a kind of folklore result with several versions appearing in the literature; see e.g. Černý/Kallsen [25, Lemma 3.10] and Fontana/Schweizer [48, Proposition 3.6] with $Y \equiv 1$. We use this later to find the mean–variance efficient strategies for the agents.

Lemma 2.12. *A strategy $\vartheta \in \overline{\Theta}(S)$ is mean–variance efficient with respect to $H \equiv 0$ if and only if $\vartheta =_S y\vartheta^{\mathrm{MVH}}(1)$ for some $y \geq 0$. In that case, we have*

$$E[\vartheta \bullet S_T] = y(1-\ell) \quad \textit{and} \quad \mathrm{Var}[\vartheta \bullet S_T] = y^2\ell(1-\ell). \tag{2.17}$$

Proof. Since a strategy is mean–variance efficient if and only if it satisfies the equivalent conditions (a) and (b) of Eberlein/Kallsen [44, Rule 10.43], the first assertion follows directly from the equivalence with condition (e) in [44, Rule 10.43]. Then (2.17) is given by [44, Rule 10.47]. □

We now return to the task of finding the mean–variance efficient strategies for the agents. For each k, let $(c(\Xi^k), \vartheta^{\mathrm{ex}}(\Xi^k))$ be the unique solution to the exMVH problem (2.15) with $H = \Xi^k$, and denote $c_k := c(\Xi^k)$ and $\varepsilon_k^2 := \varepsilon_{\mathrm{ex}}^2(\Xi^k)$. We start by using the structure of the mean–variance hedging problem to obtain more explicit formulas for the expectation and variance of the terminal wealth attained by an arbitrary strategy $\vartheta \in \overline{\Theta}(S)$. Indeed, since $(c_k, \vartheta^{\mathrm{ex}}(\Xi^k))$ is the unique solution to the exMVH problem (2.15) with $H = \Xi^k$, we have by (III.2.11) that $c_k + \vartheta^{\mathrm{ex}}(\Xi^k) \bullet S_T$ is the orthogonal projection of Ξ^k onto the set

$$\{x + \vartheta \bullet S_T : x \in \mathbb{R}, \vartheta \in \overline{\Theta}(S)\} \subseteq L^2,$$

which is closed in L^2 by Černý/Kallsen [25, Lemma 2.9] and Assumption 2.11. Thus we obtain an orthogonal decomposition of the form

$$\Xi^k = c_k + \vartheta^{\mathrm{ex}}(\Xi^k) \bullet S_T + \tilde{\Xi}^k \tag{2.18}$$

where $\tilde{\Xi}^k \in L^2$ is such that $E[\tilde{\Xi}^k] = E[(\tilde{\vartheta} \bullet S_T)\tilde{\Xi}^k] = 0$ for all $\tilde{\vartheta} \in \overline{\Theta}(S)$. Moreover, we have by (2.15) that

$$\mathrm{Var}[\tilde{\Xi}^k] = \varepsilon_{\mathrm{ex}}^2(\Xi^k) = \varepsilon_k^2.$$

For any $\vartheta \in \overline{\Theta}(S)$, plugging (2.18) into the formula (2.4) for $V_T^k(\vartheta)$ yields

$$E[V_T^k(\vartheta)] = E[(\vartheta - \eta^k) \bullet S_T + c_k + \vartheta^{\mathrm{ex}}(\Xi^k) \bullet S_T + \tilde{\Xi}^k]$$
$$= c_k + E\big[\big((\vartheta - \eta^k + \vartheta^{\mathrm{ex}}(\Xi^k)\big) \bullet S_T\big], \tag{2.19}$$
$$\mathrm{Var}[V_T^k(\vartheta)] = \mathrm{Var}[(\vartheta - \eta^k) \bullet S_T + c_k + \vartheta^{\mathrm{ex}}(\Xi^k) \bullet S_T + \tilde{\Xi}^k]$$
$$= \mathrm{Var}\big[\big(\vartheta - \eta^k + \vartheta^{\mathrm{ex}}(\Xi^k)\big) \bullet S_T\big] + \varepsilon_k^2. \tag{2.20}$$

In the following result, we use these formulas to show that a mean–variance efficient strategy ϑ for agent k is also the solution to the quadratic utility problem (2.8) for some choice of risk tolerance $\gamma_k \geq c_k$, and that ϑ can be represented as a linear combination of $\vartheta^{\mathrm{MVH}}(1)$ and $\vartheta^{\mathrm{ex}}(\Xi^k)$. In the subsequent corollary, we obtain an explicit parametrisation for the mean–variance efficient frontier $\mathcal{E}^k$ (see (2.13)) in terms of the triplet $(\ell, c_k, \varepsilon_k^2) \in (0,1] \times \mathbb{R} \times \mathbb{R}_+$.

Proposition 2.13. *For $\vartheta \in \overline{\Theta}(S)$, the following statements are equivalent:*

(a) *ϑ is mean–variance efficient for agent k.*

(b) *$\vartheta =_S \vartheta^k(y)$ for some $y \geq 0$, where $\vartheta^k(y) := y\vartheta^{\mathrm{MVH}}(1) + \eta^k - \vartheta^{\mathrm{ex}}(\Xi^k)$.*

(c) *$\vartheta - \eta^k$ is the unique solution to the MVH problem* (2.14) *for the payoff $H^k(\gamma_k) := \gamma_k - \Xi^k$ for some $\gamma_k \geq c_k$.*

The constants y and γ_k in (b) *and* (c) *can be chosen so that $y + c_k = \gamma_k$. Moreover, statement (c) holds for some $\gamma_k > 0$ if and only if*

(d) *ϑ is the unique solution to the quadratic utility problem* (2.8) *with risk tolerance γ_k.*

Proof. (a) $\Leftrightarrow$ (b): Since c_k and ε_k^2 do not depend on the choice of ϑ, it follows by (2.19) and (2.20) together with Definition 2.9 that ϑ is mean–variance efficient for agent k if and only if $\vartheta - \eta^k + \vartheta^{\mathrm{ex}}(\Xi^k)$ is mean–variance efficient with respect to 0. By Lemma 2.12, the latter statement is equivalent to

$$\vartheta - \eta^k + \vartheta^{\mathrm{ex}}(\Xi^k) = y\vartheta^{\mathrm{MVH}}(1)$$

for some $y \geq 0$. Thus we have (a) $\Leftrightarrow$ (b).

(b) $\Leftrightarrow$ (c): Since $(c_k, \vartheta^{\mathrm{ex}}(\Xi^k))$ is the unique solution to the exMVH problem (2.15) with payoff Ξ^k, it follows by fixing c_k that $\vartheta^{\mathrm{ex}}(\Xi^k)$ is also the unique solution to the MVH problem (2.14) with payoff $\Xi^k - c_k$ so that $\vartheta^{\mathrm{ex}}(\Xi^k) = \vartheta^{\mathrm{MVH}}(\Xi^k - c_k)$.

Then by the linearity of MVH (see Lemma III.2.6), we have for any $y \geq 0$ that

$$\vartheta^k(y) = y\vartheta^{\mathrm{MVH}}(1) + \eta^k - \vartheta^{\mathrm{ex}}(\Xi^k) = \vartheta^{\mathrm{MVH}}(y + c_k - \Xi^k) + \eta^k,$$

and hence

$$\vartheta^k(y) - \eta^k = \vartheta^{\mathrm{MVH}}(\gamma_k - \Xi^k), \tag{2.21}$$

where $\gamma_k := y + c_k \geq c_k$. Thus by (2.21), we have for a strategy $\vartheta \in \overline{\Theta}(S)$ that $\vartheta = \vartheta^k(y)$ if and only if $\vartheta - \eta^k$ is the unique solution to the MVH problem (2.14) with payoff $H^k(\gamma_k) = \gamma_k - \Xi^k$. This shows (b) $\Leftrightarrow$ (c), where the constants γ_k and y are related by $\gamma_k = y + c_k$.

(c) $\Leftrightarrow$ (d): By Lemma III.3.1, ϑ is a solution to the quadratic utility problem (2.8) with risk tolerance $\gamma_k > 0$ if and only if $\vartheta - \eta^k$ is a solution to the MVH problem (2.14) for $H^k(\gamma_k) = \gamma_k - \Xi^k$; in particular, the solution to (2.8) is unique by the uniqueness of the solution to (2.14). This shows (c) $\Leftrightarrow$ (d) for $\gamma_k > 0$. □

Remark 2.14. Let $Q^\star$ be the so-called variance-optimal martingale measure for S; see the definition above Schweizer [110, Lemma 1] or in Equation (2.38) below. Note that S satisfies Assumption 2.11 and hence [110, Assumption (1.2)] so that $Q^\star$ exists. Then by [110, Proposition 2], $c_k = c(\Xi^k) = E_{Q^\star}[\Xi^k]$. If the asset prices S^j are continuous, we have $Q^\star \approx P$ by Delbaen/Schachermayer [36, Theorem 1.3] so that $c_k > 0$ due to Assumption 2.1. Therefore, the assumption $\gamma_k > 0$ for part (d) of Proposition 2.13 is automatically satisfied in the case of continuous asset prices. However, $Q^\star$ may in general be a signed measure for a right-continuous price process satisfying Assumption 2.11 (see [110, Example 3]), and so Assumption 2.1 does not imply $c_k > 0$ in general.

Due to Proposition 2.13, it is now straightforward to identify the mean–variance efficient frontier for agent k.

Corollary 2.15. *The mean–variance efficient frontier for agent k is given by*

$$\mathcal{E}^k = \big\{\big(\mu_k(y), \sigma_k(y)\big) = \big(c_k + (1-\ell)y, \sqrt{\varepsilon_k^2 + \ell(1-\ell)y^2}\big) : y \geq 0\big\}. \tag{2.22}$$

Proof. By the equivalence (a) $\Leftrightarrow$ (b) in Proposition 2.13, we have

$$\mathcal{E}^k = \Big\{\Big(E\big[V_T^k\big(\vartheta^k(y)\big)\big], \sqrt{\mathrm{Var}\big[V_T^k\big(\vartheta^k(y)\big)\big]}\Big) : y \geq 0\Big\},$$

where $\vartheta^k(y) := y\vartheta^{\mathrm{MVH}}(1) + \eta^k - \vartheta^{\mathrm{ex}}(\Xi^k)$. Then (2.17), (2.19) and (2.20) yield

$$E\big[V_T^k\big(\vartheta^k(y)\big)\big] = c_k + E[y\vartheta^{\mathrm{MVH}}(1)\bullet S_T] = c_k + (1-\ell)y = \mu_k(y), \tag{2.23}$$

$$\mathrm{Var}\big[V_T^k\big(\vartheta^k(y)\big)\big] = \mathrm{Var}[y\vartheta^{\mathrm{MVH}}(1)\bullet S_T] + \varepsilon_k^2 = \varepsilon_k^2 + \ell(1-\ell)y^2 = \sigma_k^2(y), \tag{2.24}$$

which shows (2.22). □

We are finally ready to tackle the individual optimisation problem (2.10). As shown in Lemma 2.10, any solution to (2.10) must be mean–variance efficient for agent k. Hence by Proposition 2.13, an optimal strategy for (2.10) (if it exists) is of the form $\hat{\vartheta}^k = \vartheta^k(\hat{y}_k)$ for some $\hat{y}_k \geq 0$ and is also the solution to the quadratic utility problem (2.8) with risk tolerance $\gamma_k = \hat{y}_k + c_k$. In the following result, we prove the stronger statement that a strategy maximises (2.10) over all strategies if and only if it maximises (2.10) over the set of mean–variance efficient strategies for agent k. In other words, we discard all other strategies not only as candidate solutions to (2.10), but also as competitors to a candidate solution. Thus the infinite-dimensional problem (2.10) reduces to the one-dimensional problem of finding the values of $y \geq 0$ such that $\vartheta^k(y)$ is a solution to (2.10).

Corollary 2.16. *A strategy $\hat{\vartheta}^k \in \overline{\Theta}(S)$ is a solution to the maximisation problem* (2.10) *if and only if $\hat{\vartheta}^k = \vartheta^k(\hat{y}_k)$, where $\hat{y}_k \geq 0$ is a maximiser for the problem*

$$U_k\big(\mu_k(y), \sigma_k(y)\big) \longrightarrow \max_{y \geq 0}! \tag{2.25}$$

and μ_k and σ_k are given in (2.22).

Proof. We first claim that for any strategy $\vartheta \in \overline{\Theta}(S)$, there exists a mean–variance efficient strategy $\vartheta' \in \overline{\Theta}(S)$ for agent k such that

$$E[V_T^k(\vartheta')] \geq E[V_T^k(\vartheta)], \tag{2.26}$$

$$\mathrm{Var}[V_T^k(\vartheta')] \leq \mathrm{Var}[V_T^k(\vartheta)]. \tag{2.27}$$

Indeed, we note that (2.19) and (2.20) hold for any $\vartheta \in \overline{\Theta}(S)$. Thus in the case $\ell \in (0,1)$, we have by (2.20) and (2.24) that the mean–variance efficient strategy $\vartheta' = \vartheta^k(y)$ for agent k with

$$y := \sqrt{\frac{\mathrm{Var}[(\vartheta - \eta^k + \vartheta^{\mathrm{ex}}(\Xi^k))\bullet S_T]}{\ell(1-\ell)}}$$

satisfies $\mathrm{Var}[V_T^k(\vartheta')] = \mathrm{Var}[V_T^k(\vartheta)]$. Thus ϑ' must also satisfy (2.26) because it is mean–variance efficient. In the case $\ell = 1$, i.e., if $0 \in \overline{\Theta}(S)$ is a solution to the pure investment problem (2.16), we have by Lemma III.2.13 that S is a local P-martingale, and hence $(\vartheta - \eta^k + \vartheta^{\mathrm{ex}}(\Xi^k)) \bullet S$ is a P-martingale due to Proposition III.2.12. Then the second term on the right-hand side of (2.19) is null so that by (2.19), (2.20), (2.23) and (2.24), we get (2.26) and (2.27) for the mean–variance efficient strategy $\vartheta' = \vartheta^k(0) = \eta^k - \vartheta^{\mathrm{ex}}(\Xi^k)$. This shows the claim.

By the above and because each mean–variance utility function U_k is strictly increasing in μ and strictly decreasing in σ by Definition 2.5, we deduce that a strategy $\hat{\vartheta}^k \in \overline{\Theta}(S)$ is a solution to the problem (2.10) if and only if it maximises (2.10) among all mean–variance efficient strategies for agent k. Then the result follows by plugging (2.23) and (2.24) into (2.10). □

The next step is to solve (2.25), which is a one-dimensional problem involving the mean–variance utility function U_k and the triplet $(\ell, c_k, \varepsilon_k^2) \in (0,1] \times \mathbb{R} \times \mathbb{R}_+$. We note that $\ell = 1$ leads to a degenerate case since the functions μ_k and σ_k are then constant and every $y \geq 0$ is a solution to (2.25). Since we also have here $\vartheta^{\mathrm{MVH}}(1) = 0$, it follows by Proposition 2.13 that the only mean–variance efficient strategy is $\hat{\vartheta}^k = \eta^k - \vartheta^{\mathrm{ex}}(\Xi^k)$, and hence this is the unique solution to (2.10).

We now consider (2.25) in the nontrivial case $\ell \in (0,1)$. More precisely, we begin by studying an abstract version of (2.25) as follows. Given constants $(\ell, c, \varepsilon^2) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$ and a mean–variance utility function U, we consider the optimisation problem

$$U\big(c + (1-\ell)y, \sqrt{\varepsilon^2 + \ell(1-\ell)y^2}\big) \longrightarrow \max_{y \geq 0}! \tag{2.28}$$

Aside from a generic choice of U, the main difference between (2.28) and (2.25) is that we consider arbitrary constants $(\ell, c, \varepsilon^2) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$, as opposed to the constants $(\ell, c_k, \varepsilon_k^2)$ defined before Proposition 2.13 for the given price process S and payoff Ξ^k. This allows us to study not only the existence and uniqueness of a solution $\hat{y}_U$ to (2.28), but also whether $\hat{y}_U$ depends continuously on (ℓ, c, ε^2). Although it is not necessary for solving the individual optimisation problems with respect to S, the continuity of $\hat{y}_U$ will play a crucial role when we later study the equilibrium problem for the K agents, since S is not known a priori and hence neither are the parameters $(\ell, c_k, \varepsilon_k^2)$.

In order to tackle (2.28), we first introduce some standard notions related to mean–variance utility functions; see Koch-Medina/Wenzelburger [85, Section

3.3]. For a mean–variance utility function U, we denote by

$$\mathcal{I}_U(\mu,\sigma) := \{(\mu',\sigma') \in \mathbb{R}\times\mathbb{R}_+ : U(\mu',\sigma') = U(\mu,\sigma)\}$$

the *indifference curve* through (μ,σ). We also define $\mathcal{S}_U : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ by

$$\mathcal{S}_U(\mu,\sigma) = -\frac{\partial_\sigma U(\mu,\sigma)}{\partial_\mu U(\mu,\sigma)}, \tag{2.29}$$

so that $\mathcal{S}_U(\mu,\sigma)$ denotes the *slope* of $\mathcal{I}_U(\mu,\sigma)$ at (μ,σ). As in [85], we switch the order of the coordinates for the geometric interpretation of the "slope". That is, we plot $\mathcal{I}_U$ on the σ-μ-plane with σ on the horizontal and μ on the vertical axis, as is customary in the literature on mean–variance analysis. We now show the existence and uniqueness of a solution to (2.28) given suitable bounds on $\mathcal{S}_U$.

Proposition 2.17. *Let U be a mean–variance utility function. Suppose there exist constants $\lambda^{\min}, \lambda^{\max} \in (0,\infty)$ such that*

$$\frac{\sigma}{\lambda^{\max}} \le \mathcal{S}_U(\mu,\sigma) \le \frac{\sigma}{\lambda^{\min}} \quad \textit{for all } (\mu,\sigma) \in \mathbb{R}\times\mathbb{R}_+. \tag{2.30}$$

Then there exists a continuous map $\hat{y}_U : (0,1)\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ such that $\hat{y}_U(\ell,c,\varepsilon^2)$ is the unique solution to (2.28) for each $(\ell,c,\varepsilon^2) \in (0,1)\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$. Moreover, the map $\hat{y}_U$ satisfies the bounds

$$\frac{\lambda^{\min}}{\ell} \le \hat{y}_U(\ell,c,\varepsilon^2) \le \frac{\lambda^{\max}}{\ell} \quad \textit{for all } (\ell,c,\varepsilon^2) \in (0,1)\times\mathbb{R}\times\mathbb{R}_+. \tag{2.31}$$

Proof. The proof is deferred to the end of Section 5. □

Remark 2.18. The existence, uniqueness and continuity of the solution $\hat{y}_U$ to (2.25) hold under weaker conditions than (2.28); this is explained in more detail in Section 5 (see Corollary 5.4). However, we need the extra condition (2.30) in order to obtain the bounds (2.31) which are used later.

We now return once again to the individual optimisation problem (2.10). To apply Proposition 2.17, we make the following **standing assumption** on the mean–variance utility functions U_k.

Assumption 2.19. For each $k \in \{1,\dots,K\}$, there exist constants $\lambda_k^{\min}, \lambda_k^{\max} > 0$ such that

$$\frac{\sigma}{\lambda_k^{\max}} \le \mathcal{S}_{U_k}(\mu,\sigma) \le \frac{\sigma}{\lambda_k^{\min}} \quad \text{for all } (\mu,\sigma) \in \mathbb{R}\times\mathbb{R}_+. \tag{2.32}$$

We note that Assumption 2.19 is somewhat restrictive on the choice of U_k, and indeed, it is not necessary for the existence of an equilibrium in the one-period model; see Koch-Medina/Wenzelburger [85, Theorem 3]. It remains an open question whether this assumption can be removed or relaxed in our subsequent results for the multiperiod case. In any case, Assumption 2.19 still allows for some flexibility in the choice of utility functions for the agents. Most notably, (2.32) holds for linear mean–variance utility functions of the form (2.9), which reads

$$U_k(\mu, \sigma) = \mu - \frac{\sigma^2}{2\lambda_k}$$

for some $\lambda_k \geq 0$ with $\lambda_k^{\min} = \lambda_k^{\max} = \lambda_k$. By analogy with the linear case (2.9), the ratio $\frac{\sigma}{\mathcal{S}_U(\mu,\sigma)}$ may in general be interpreted as the local risk tolerance at (μ, σ). Therefore, (2.32) assumes that the local risk tolerance is bounded within the range $[\lambda_k^{\min}, \lambda_k^{\max}]$.

Under Assumption 2.19, we can now combine our previous results to yield the existence and uniqueness of solutions to (2.10) for each agent k.

Theorem 2.20. *Let $(S_t)_{0 \leq t \leq T}$ be a fixed price process satisfying Assumptions 2.8 and 2.11. For each $k \in \{1, \dots, K\}$, define the triplet $(\ell, c_k, \varepsilon_k^2) \in (0,1] \times \mathbb{R} \times \mathbb{R}_+$ by $\ell := \varepsilon^2(1)$, $c_k := c(\Xi^k)$ and $\varepsilon_k^2 := \varepsilon_{\mathrm{ex}}^2(\Xi^k)$.*

1) *Suppose that $\ell = 1$. Then for each $k \in \{1, \dots, K\}$, the unique solution to* (2.10) *is $\hat{\vartheta}^k = \eta^k - \vartheta^{\mathrm{ex}}(\Xi^k)$.*

2) *Suppose that $\ell \in (0,1)$ and Assumption 2.19 holds. Then for $k \in \{1, \dots, K\}$, the unique solution to the individual optimisation problem* (2.10) *is*

$$\hat{\vartheta}^k := \vartheta^k\big(\hat{y}_{U_k}(\ell, c_k, \varepsilon_k^2)\big) = \hat{y}_{U_k}(\ell, c_k, \varepsilon_k^2)\vartheta^{\mathrm{MVH}}(1) + \eta^k - \vartheta^{\mathrm{ex}}(\Xi^k), \tag{2.33}$$

where the map $\hat{y}_{U_k} : (0,1) \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ is given by Proposition 2.17. Moreover, we have the bounds

$$\frac{\lambda_k^{\min}}{\ell} \leq \hat{y}_{U_k}(\ell, c_k, \varepsilon_k^2) \leq \frac{\lambda_k^{\max}}{\ell} \tag{2.34}$$

for each $k \in \{1, \dots, K\}$, and $\hat{\vartheta}^k - \eta^k$ is the unique solution to the MVH problem (2.14) *with payoff $H^k(\gamma_k) = \gamma_k - \Xi^k$, where $\gamma_k := c_k + \hat{y}_{U_k}(\ell, c_k, \varepsilon_k^2)$. If $\gamma_k > 0$, then $\hat{\vartheta}^k$ is the unique solution to the quadratic utility problem* (2.8) *with risk tolerance γ_k.*

Proof. **1)** This follows from the discussion after Corollary 2.16.

2) By Proposition 2.17, $\hat{y}_{U_k}(\ell, c_k, \varepsilon_k^2)$ is the unique solution to the problem (2.25) and satisfies the bounds (2.34). Thus by Corollary 2.16, the strategy

$\hat{\vartheta}^k \in \overline{\Theta}(S)$ defined by (2.33) is the unique solution to (2.10). By combining (2.33) with (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) in Proposition 2.13 (the latter of which holds for $\gamma_k > 0$), we obtain the last two statements. □

2.3 Characterisation of the equilibrium as a fixed point

So far, we have analysed the individual optimisation problem (2.10) and shown that it is related to the quadratic utility problem (2.8). Namely, by Proposition 2.13, a solution to the former is also a solution to the latter for some risk tolerance γ_k, and so it should follow that a mean–variance equilibrium is also a quadratic equilibrium for some parameters $(\gamma_1, \dots, \gamma_K)$.

To make this argument fully precise, we need to take care with the assumption $\gamma_k > 0$ in Proposition 2.13, especially in view of Remark 2.14. A convenient way to circumvent this is to bypass the quadratic utility problem (2.8) by directly working with the MVH problem (2.14) for the payoff $H^k(\gamma_k) := \gamma_k - \Xi^k$. By Lemma III.3.1 and as noted in the proof of Proposition 2.13, that MVH problem is equivalent to (2.8) for any $\gamma_k > 0$, but the MVH problem is well posed even if γ_k is nonpositive. Note that by the linearity of MVH, $\vartheta - \eta^k$ solves (2.14) with payoff $H^k(\gamma_k)$ if and only if ϑ solves (2.14) with payoff $H^k(\gamma_k) + \eta^k \bullet S_T$. This leads to the following extension of Definition 2.6.

Definition 2.21. A local L^2-market $(1, S^{(1)}, S^{(2)})$ is called a *generalised quadratic equilibrium market* (with *risk tolerances* $\gamma_1, \dots, \gamma_K \in \mathbb{R}$ and *aggregate risk tolerance* $\bar{\gamma} = \sum_{k=1}^K \gamma_k$) if it is an equilibrium market in the sense of Definition 2.2, where for each $k \in \{1, \dots, K\}$, (2.5) is replaced by the MVH problem (2.14) for the payoff $H^k(\gamma_k) + \eta^k \bullet S_T$.

From the previous discussion, we directly get the following result.

Corollary 2.22. *For any $\gamma_1, \dots, \gamma_K > 0$, the market $(1, S)$ is a quadratic equilibrium with risk tolerances $\gamma_1, \dots, \gamma_K$ if and only if it is a generalised quadratic equilibrium with risk tolerances $\gamma_1, \dots, \gamma_K$.*

We also note that the only difference in the Definitions 2.6 and 2.21 of mean–variance and generalised quadratic equilibria, respectively, is that the optimal strategies for the agents are required to solve different utility maximisation problems. For later reference, this observation can be formulated as the following simple result.

Lemma 2.23. **1)** *Suppose that* $(1, S)$ *is a mean–variance equilibrium market and for each* k*, there exists some* $\gamma_k \in \mathbb{R}$ *such that the unique solution* $\hat{\vartheta}^k \in \overline{\Theta}(S)$ *to* (2.10) *is also the unique solution to* (2.14) *with payoff* $H^k(\gamma_k) + \eta^k \bullet S_T$*. Then* $(1, S)$ *is a generalised quadratic equilibrium market with risk tolerances* $\gamma_1, \dots, \gamma_K$*.*

2) *Conversely, suppose that* $(1, S)$ *is a generalised quadratic equilibrium market with risk tolerances* $\gamma_1, \dots, \gamma_K$*. If for each* k*, the unique solution to* (2.14) *with payoff* $H^k(\gamma_k) + \eta^k \bullet S_T$ *is also the unique solution* $\hat{\vartheta}^k \in \overline{\Theta}(S)$ *to* (2.10)*, then* $(1, S)$ *is a mean–variance equilibrium market.*

Proof. **1)** Since $(1, S)$ is a mean–variance equilibrium market, we have by Definitions 2.6 and 2.4 that $(1, S)$ is a local L^2-market and $e^j \in \overline{\Theta}(S)$ for each $j \in \{d_1 + 1, \dots, d_1 + d_2\}$. Moreover, by assumption, there exists a unique solution $\hat{\vartheta}^k$ to (2.14) with payoff $H^k(\gamma_k) + \eta^k \bullet S_T$, and since $(1, S)$ is a mean–variance equilibrium market with the same optimal strategies, the market clears, i.e., (2.6) holds. Therefore $(1, S)$ is a generalised quadratic equilibrium because we have checked all conditions in Definition 2.4 as required by Definition 2.6.

2) The proof of the converse statement is completely analogous. □

We are now ready to formalise the argument above to show that a mean–variance equilibrium $(1, S)$ is a *generalised* quadratic equilibrium, even in the case where it is not a quadratic equilibrium due to issues related to Remark 2.14.

Lemma 2.24. *Let* $U_1, \dots, U_K$ *be mean–variance utility functions. Suppose that* $(S_t)_{0 \le t \le T}$ *satisfies Assumption 2.11 and* $(1, S)$ *is a mean–variance equilibrium market with respect to* $U_1, \dots, U_K$*. Then there exist parameters* $\gamma_1, \dots, \gamma_K \in \mathbb{R}$ *such that* $(1, S)$ *is also a generalised quadratic equilibrium market with respect to the risk tolerances* $\gamma_1, \dots, \gamma_K$*.*

Proof. Since $(1, S)$ is a mean–variance equilibrium in the sense of Definition 2.6, there exists for each agent $k \in \{1, \dots, K\}$ a unique solution $\hat{\vartheta}^k$ to (2.10). By Lemma 2.10, $\hat{\vartheta}^k$ is mean–variance efficient for agent k so that by Proposition 2.13 and the linearity of MVH (see Lemma III.2.6), $\hat{\vartheta}^k$ is also the unique solution to the MVH problem (2.14) with payoff $H^k(\gamma_k) + \eta^k \bullet S_T$ for some $\gamma_k \ge c_k$. Then by part 1) of Lemma 2.23, $(1, S)$ is also a generalised quadratic equilibrium. □

Lemma 2.24 already gives a good characterisation of mean–variance equilibria that satisfy Assumption 2.11 since it tells us that they are also generalised quadratic equilibria. We have already studied quadratic equilibria in the previous chapter, and so we now want to apply Theorem III.3.8. This gives explicit formulas for quadratic equilibria that depend only on the aggregate risk tolerance $\bar\gamma = \sum_{k=1}^K \gamma_k$, but not directly on the individual tolerances γ_k. More precisely, note that due to Assumption 2.1, we have

$$0 < \bar\gamma_0 := \operatorname{ess\,sup} \bar\Xi < \infty. \tag{2.35}$$

Thus for $\bar\gamma > \bar\gamma_0$, the process $(\bar Z_t(\bar\gamma))_{0\le t\le T}$ defined by $\bar Z_t(\bar\gamma) = \bar\gamma - E[\bar\Xi \,|\, \mathcal{F}_t]$ is a strictly positive bounded martingale with terminal value $\bar H(\bar\gamma) := \bar\gamma - \bar\Xi$. For the following result, we recall from the previous chapter that Lemma III.3.9 gives sufficient conditions to ensure that the candidate price process for the productive assets is an L^2-semimartingale.

Lemma 2.25. *For $\bar\gamma > \bar\gamma_0$, the process $(1, S_t(\bar\gamma))_{0\le t\le T}$ defined by*

$$S_t^j(\bar\gamma) := S_0^j + M_t^j - \int_0^t \frac{d\langle \bar Z(\bar\gamma), M^j\rangle_s}{\bar Z_{s-}(\bar\gamma)}, \qquad j \in \{1,\dots,d_1\}, \tag{2.36}$$

$$S_t^j(\bar\gamma) := \frac{E[\bar H(\bar\gamma) D^j \mid \mathcal{F}_t]}{\bar Z_t(\bar\gamma)}, \qquad j \in \{d_1+1,\dots,d_1+d_2\}, \tag{2.37}$$

is the unique generalised quadratic equilibrium market with respect to any choice of parameters $\gamma_1,\dots,\gamma_K \in \mathbb{R}$ such that $\sum_{k=1}^K \gamma_k = \bar\gamma$; for short, we say that $(1, S(\bar\gamma))$ is the generalised quadratic equilibrium with aggregate risk tolerance $\bar\gamma$. Moreover, $S(\bar\gamma)$ satisfies Assumptions 2.8 and 2.11, $S^{(2)}(\bar\gamma)$ is an L^2-semimartingale, and the measure $Q(\bar\gamma) \approx P$ with bounded density $dQ(\bar\gamma)/dP(\bar\gamma) = H(\bar\gamma)/Z_0(\bar\gamma)$ is a local martingale measure for $S(\bar\gamma)$.

Proof. First, we note that Theorem III.3.8 still holds for $\gamma_1,\dots,\gamma_K \in \mathbb{R}$ if we replace “quadratic equilibrium” with “generalised quadratic equilibrium”. Indeed, by Lemma III.3.6 (the proof of which relies on Lemma III.3.5), $\bar Z S$ is a local P-martingale for any quadratic equilibrium market $(1, S)$. It is clear from the proof of Lemma III.3.5 that its conclusion still follows if we only assume that $\hat\vartheta^k - \eta^k$ solves the MVH problem with payoff $H^k(\gamma_k)$, and so the claim that $\bar Z S$ is a local P-martingale still holds. The rest of the first part of the proof of Theorem III.3.8 is unchanged if we consider a generalised quadratic equilibrium. Likewise, in the proof of the converse direction that (III.3.16) and (III.3.17) define a quadratic equilibrium market, we already show that the optimal strategies $\hat\vartheta^k$ are such that $\hat\vartheta^k - \eta^k$ solves the MVH problem with payoff $H^k(\gamma_k)$; that argument still holds if the γ_k are allowed to be nonpositive. Therefore Theorem III.3.8 can be extended to generalised quadratic equilibria.

Now fix $\bar\gamma > \bar\gamma_0$. Note that we have the bounds $\bar\gamma - \bar\gamma_0 \le \bar H(\bar\gamma) \le \bar\gamma$, and that (2.36) and (2.37) are the same as (III.3.16) and (III.3.17) with $\bar{Z}(\bar{\gamma})$ in the place of $\bar{Z}$. Because condition (b) in Lemma III.3.9 is satisfied, $S^{(2)}(\bar{\gamma})$ is an L^2-semimartingale. As discussed before Lemma III.3.9, this implies that each buy-and-hold strategy e^j belongs to $\overline{\Theta}(S(\bar{\gamma}))$ for $j \in \{d_1+1, \dots, d_1+d_2\}$, as required by Theorem III.3.8. We also note that the process $\bar{Z}(\bar{\gamma})$ is strictly positive and hence never hits 0. Therefore by Theorem III.3.8 (extended as above to generalised quadratic equilibria), $S(\bar{\gamma})$ is the unique generalised quadratic equilibrium with respect to any choice of parameters $\gamma_1, \dots, \gamma_K$ such that $\sum_{k=1}^K \gamma_k = \bar{\gamma}$. Theorem III.3.8 also gives that $\bar{Z}(\bar{\gamma})S(\bar{\gamma})$ is a local P-martingale. Thus $Q(\bar{\gamma})$ is a local martingale measure for $S(\bar{\gamma})$ such that $Q(\bar{\gamma}) \approx P$, and $dQ(\bar{\gamma})/dP$ is bounded because $\bar{Z}(\bar{\gamma})$ is strictly positive and bounded. This also implies that $S(\bar{\gamma})$ satisfies Assumption 2.11, and Assumption 2.8 follows since $(1, S(\bar{\gamma}))$ is a local L^2-market by the definition of a generalised quadratic equilibrium. □

By Lemma 2.24, we know that a mean–variance equilibrium is also a generalised quadratic equilibrium for some parameters $(\gamma_1, \dots, \gamma_K)$, and if we have $\bar{\gamma} := \sum_{k=1}^K \gamma_k > \bar{\gamma}_0$, then the mean–variance equilibrium is $(1, S(\bar{\gamma}))$ with $S(\bar{\gamma})$ given by (2.36) and (2.37). However, it is still not straightforward to find a mean–variance equilibrium or determine whether it is unique since we cannot determine the parameters γ_k or $\bar{\gamma}$ directly from the primitives. To circumvent this issue, a natural approach is to "try" every possible value of $\bar{\gamma} > \bar{\gamma}_0$ and check whether $S(\bar{\gamma})$ produces the desired equilibrium. Thus our next goal is to characterise the values of $\bar{\gamma} > \bar{\gamma}_0$ that generate a mean–variance equilibrium $(1, S(\bar{\gamma}))$.

Before moving on, we want to deal with the case $\ell = 1$ since it requires special treatment in several results in Section 2.2. In particular, it is useful to determine the values of $\bar{\gamma} > \bar{\gamma}_0$ such that $\ell(\bar{\gamma}) = 1$, where $\ell(\bar{\gamma})$ is the constant $\ell = \varepsilon^2(1)$ taken with respect to $S(\bar{\gamma})$; see (2.14). We start by considering the asymptotic behaviour of ℓ as $\bar{\gamma} \to \infty$.

Lemma 2.26. *We have $\ell(\bar{\gamma}) \to 1$ as $\bar{\gamma} \to \infty$.*

Proof. For each $\bar{\gamma} > \bar{\gamma}_0$, define $\mathcal{Q}(S(\bar{\gamma}))$ as the set of absolutely continuous measures $Q \ll P$ such that the density process $(Z_t^Q)_{0 \le t \le T}$ is a square-integrable martingale with $Z_0^Q = 1$ and $Z^Q S(\bar{\gamma})$ is a local P-martingale. By Lemma 2.24, $Q(\bar{\gamma})$ is an equivalent local martingale measure for $S(\bar{\gamma})$ with bounded density. Thus $Q(\bar{\gamma}) \in \mathcal{Q}(S(\bar{\gamma}))$ so that $\mathcal{Q}(S(\bar{\gamma}))$ is nonempty and Assumption (1.2) of Schweizer [110] is satisfied. Hence there exists a *variance-optimal martingale measure* $Q^\star(\bar{\gamma})$

for $S(\bar{\gamma})$, which is defined as the unique solution to the problem

$$E_P\left[\left(\frac{dQ}{dP}\right)^2\right] \longrightarrow \inf_{Q\in\mathcal{Q}(S(\bar{\gamma}))}! \tag{2.38}$$

By part (b) of [110, Lemma 1], we have

$$\frac{dQ^\star(\bar{\gamma})}{dP} = \frac{1-(\vartheta^{\mathrm{MVH}}(1;S(\bar{\gamma}))\bullet S(\bar{\gamma}))_T}{1-E_P[(\vartheta^{\mathrm{MVH}}(1;S(\bar{\gamma}))\bullet S(\bar{\gamma}))_T]}, \tag{2.39}$$

where we define $\vartheta^{\mathrm{MVH}}(1;S(\bar{\gamma}))$ as the unique solution to the MVH problem (2.14) with $H\equiv 1$ and $S=S(\bar{\gamma})$. Due to (2.17) with $y=1$, we obtain

$$\begin{aligned} E_P\left[\left(\frac{dQ^\star(\bar{\gamma})}{dP}\right)^2\right] &= \frac{E_P[(1-(\vartheta^{\mathrm{MVH}}(1;S(\bar{\gamma}))\bullet S(\bar{\gamma}))_T)^2]}{E_P[1-(\vartheta^{\mathrm{MVH}}(1;S(\bar{\gamma}))\bullet S(\bar{\gamma}))_T]^2} \\ &= \frac{\ell^2(\bar{\gamma})+\ell(\bar{\gamma})(1-\ell(\bar{\gamma}))}{\ell^2(\bar{\gamma})} = \frac{1}{\ell(\bar{\gamma})}. \end{aligned}$$

On the other hand, since $Q(\bar{\gamma})\in\mathcal{Q}(S(\bar{\gamma}))$, we have by the optimality of $Q^\star(\bar{\gamma})$ and the definition of $Q(\bar{\gamma})$ that

$$E_P\left[\left(\frac{dQ^\star(\bar{\gamma})}{dP}\right)^2\right] \le E_P\left[\left(\frac{dQ(\bar{\gamma})}{dP}\right)^2\right] = 1+\frac{\mathrm{Var}_P[\bar{H}(\bar{\gamma})]}{(\bar{\gamma}-E_P[\bar{\Xi}])^2} = 1+\frac{\mathrm{Var}_P[\bar{\Xi}]}{(\bar{\gamma}-E_P[\bar{\Xi}])^2},$$

and hence

$$\ell(\bar{\gamma}) = \frac{1}{E_P[(dQ^\star(\bar{\gamma})/dP)^2]} \ge \left(1+\frac{\mathrm{Var}_P\bar{\Xi}}{(\bar{\gamma}-E_P[\bar{\Xi}])^2}\right)^{-1} \longrightarrow 1$$

as $\bar{\gamma}\to\infty$. Since $\ell(\bar{\gamma})\le 1$ for each $\bar{\gamma}>\bar{\gamma}_0$, the result follows. □

We now return to the question of determining for which (finite) values of $\bar{\gamma}>\bar{\gamma}_0$ we have $\ell(\bar{\gamma})=1$. The following result provides a conclusive answer, namely, that it holds either on the empty set or on the whole set $(\bar{\gamma}_0,\infty)$. To show this, we define the square-integrable martingale $(\bar{Z}^0_t)_{0\le t\le T}$ by

$$\bar{Z}^0_t := \bar{Z}_t(\bar{\gamma}_0) = E[\bar{H}(\bar{\gamma}_0)\mid\mathcal{F}_t] = \bar{\gamma}_0 - E[\bar{\Xi}\mid\mathcal{F}_t],\quad 0\le t\le T,$$

as well as the martingales $(M^{D,j}_t)_{0\le t\le T}$ by

$$M^{D,j}_t := E[D^j\mid\mathcal{F}_t],\quad 0\le t\le T,\ j\in\{d_1,\dots,d_1+d_2\},$$

and set $M^{D,(2)} := (M^{D,d_1+1},\dots,M^{D,d_1+d_2})$. In the following, we write $M\perp N$ if

M, N are two strongly orthogonal local martingales.

Lemma 2.27. *The following statements are equivalent:*

(a) *There exists some* $\bar{\gamma}' > \bar{\gamma}_0$ *such that* $S(\bar{\gamma}')$ *is a local martingale.*

(b) *There exists some* $\bar{\gamma}' > \bar{\gamma}_0$ *such that* $\ell(\bar{\gamma}') = 1$.

(c) *It holds for all* $\bar{\gamma} > \bar{\gamma}_0$ *that* $\ell(\bar{\gamma}) = 1$.

(d) *It holds that* $M^j \perp \bar{Z}^0$ *for each* $j \in \{1, \dots, d_1\}$ *and* $M^{D,j} \perp \bar{Z}^0$ *for each* $j \in \{d_1 + 1, \dots, d_1 + d_2\}$.

(e) *It holds for all* $\bar{\gamma} > \bar{\gamma}_0$ *that* $S(\bar{\gamma}) = (S_0^{(1)} + M^{(1)}, M^{D,(2)})$.

Proof. We show (b) ⇒ (a) ⇒ (d) ⇒ (e) ⇒ (c) ⇒ (b) in this order.

(b) ⇒ (a): Suppose that $\ell(\bar{\gamma}') = 1$. Then $\vartheta^{\mathrm{MVH}}(1; S(\bar{\gamma}')) = 0$ is a minimiser for (2.16) with respect to $S = S(\bar{\gamma}')$. It follows from Lemma III.2.13 with $H \equiv 1$ that $S(\bar{\gamma}')$ is a local martingale.

(a) ⇒ (d): Suppose that $S(\bar{\gamma}')$ is a local martingale. By Lemma 2.25, $S^{(2)}(\bar{\gamma})$ is also an L^2-semimartingale, and hence it is of class (D) and a true martingale on $[0, T]$. Thus (2.1) and (2.2) yield

$$S_t^j(\bar{\gamma}') = S_0^j + M_t^j, \qquad j \in \{1, \dots, d_1\}, \tag{2.40}$$

$$S_t^j(\bar{\gamma}') = E[D^j \mid \mathcal{F}_t] = M_t^{D,j}, \quad j \in \{d_1 + 1, \dots, d_1 + d_2\}. \tag{2.41}$$

Since $\bar{Z}(\bar{\gamma}')$ is a martingale by construction and $\bar{Z}(\bar{\gamma}')S(\bar{\gamma}')$ is a local martingale by Lemma 2.25, $(M^{(1)}, M^{D,(2)})$ and $\bar{Z}(\bar{\gamma}')$ are strongly orthogonal. Thus $(M^{(1)}, M^{D,(2)})$ and $\bar{Z}^0$ are also strongly orthogonal because $\bar{Z}^0 = \bar{\gamma}_0 - \bar{\gamma}' + \bar{Z}(\bar{\gamma}')$, which shows (d).

(d) ⇒ (e): Let $\bar{\gamma} > \bar{\gamma}_0$. By assumption, we have $(M^{(1)}, M^{D,(2)}) \perp \bar{Z}^0$ and hence $(M^{(1)}, M^{D,(2)}) \perp \bar{Z}(\bar{\gamma})$. Plugging into (2.36) directly yields

$$S_t^j(\bar{\gamma}) = S_0^j + M_t^j, \quad j \in \{1, \dots, d_1\}. \tag{2.42}$$

Moreover, for each $j \in \{d_1 + 1, \dots, d_1 + d_2\}$, $\bar{Z}^0 M^{D,j}$ is a local martingale by the strong orthogonality. Since $\bar{H}(\bar{\gamma}_0)$ is bounded and $D^j \in L^2$, we have that $\bar{Z}^0$ is a bounded martingale and $M^{D,j}$ is an L^2-martingale, so that $\bar{Z}^0 M^{D,j}$ is a true martingale and hence

$$\bar{Z}_t^0 M_t^{D,j} = E[\bar{H}(\bar{\gamma}_0) D^j \mid \mathcal{F}_t], \quad 0 \le t \le T.$$

Thus we have

$$\begin{aligned}E[\bar{H}(\bar{\gamma})D^j \mid \mathcal{F}_t] &= E\big[\big(\bar{\gamma} - \bar{\gamma}_0 + \bar{H}(\bar{\gamma}_0)\big)D^j \mid \mathcal{F}_t\big] \\ &= (\bar{\gamma} - \bar{\gamma}_0)M_t^{D,j} + \bar{Z}_t^0 M_t^{D,j} \\ &= \bar{Z}_t(\bar{\gamma})M_t^{D,j}, \quad 0 \le t \le T.\end{aligned}$$

Plugging this equality into (2.37) yields

$$S_t^j(\bar{\gamma}) = M_t^{D,j}, \quad j \in \{d_1 + 1, \dots, d_1 + d_2\}. \tag{2.43}$$

Together with (2.42), this shows (e).

(e) $\Rightarrow$ (c): Because $S(\bar{\gamma})$ is a local martingale, it follows as explained after (2.16) that $\ell(\bar{\gamma}) = 1$ for each $\bar{\gamma} > \bar{\gamma}_0$.

(c) $\Rightarrow$ (b): This is trivial. □

With the characterisation from Lemma 2.27, we can now show the existence of a solution to the mean–variance equilibrium problem in the case $\ell = 1$.

Corollary 2.28. *Suppose that $(M^{(1)}, M^{D,(2)}) \perp \bar{Z}^0$ and define the price process $(S_t)_{0 \le t \le T}$ by $S := (S_0^{(1)} + M^{(1)}, M^{D,(2)})$. Then $(1, S)$ is a mean–variance equilibrium for any choice of mean–variance utility functions U_k for $k = 1, \dots, K$.*

Proof. Fix some $\gamma_1, \dots, \gamma_K$ such that $\bar{\gamma} := \sum_{k=1}^K \gamma_k > \bar{\gamma}_0$. Then by Lemma 2.27, we have $S = S(\bar{\gamma})$ so that $(1, S)$ is a generalised quadratic equilibrium with respect to the parameters $\gamma_1, \dots, \gamma_K$. Let $\hat{\vartheta}^k \in \overline{\Theta}(S)$ be the unique solution to the MVH problem (2.14) with payoff $H^k(\gamma_k) + \eta^k \bullet S_T$. Since S is a local martingale, we have $\ell = 1$ and $\vartheta^{\mathrm{MVH}}(1) = 0$ with respect to the price process S. Then the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) in Proposition 2.13 yield that $\hat{\vartheta}^k = \eta^k - \vartheta^{\mathrm{ex}}(\Xi^k)$ is the unique mean–variance efficient strategy for agent k. Thus by part 1) of Theorem 2.20, $\hat{\vartheta}^k$ is also the unique solution to the mean–variance utility problem (2.10), and hence by part 2) of Lemma 2.23, $(1, S)$ is a mean–variance equilibrium. □

We note that Corollary 2.28 does not give uniqueness for the mean–variance equilibrium, and it may indeed not be unique in general. One could try to argue the uniqueness as follows. By Lemma 2.24, any mean–variance equilibrium market $(1, S)$ is also a generalised quadratic equilibrium for some $\gamma_1, \dots, \gamma_K \in \mathbb{R}$, and if $\bar{\gamma} := \sum_{k=1}^K \gamma_k > \bar{\gamma}_0$, then Lemma 2.27 yields $S = S(\bar{\gamma}) = (S_0^{(1)} + M^{(1)}, M^{D,(2)})$. However, the case $\bar{\gamma} \le \bar{\gamma}_0$ is also possible, in which case Theorem III.3.8 and Lemma 2.27 need not apply since the process $\bar{Z}(\bar{\gamma})$ can hit 0. In general, there may exist multiple or no (generalised) quadratic equilibria for parameters $(\gamma_1, \dots, \gamma_K)$ such that $\bar{\gamma} \leq \bar{\gamma}_0$, as discussed after Theorem III.3.11 for the discrete-time setup. In particular, such an equilibrium cannot exist if $\bar{\gamma}$ and the primitives violate either of the conditions in Lemma III.3.10.

Due to the previous considerations, we do not fully address the issue of uniqueness for the mean–variance equilibrium. Instead, we focus on the narrower question of whether there exists some $\bar{\gamma} > \bar{\gamma}_0$ such that $S(\bar{\gamma})$ is a mean–variance equilibrium, and if so, whether $\bar{\gamma}$ is unique. Under the assumption $(M^{(1)}, M^{D,(2)}) \perp \bar{Z}^0$, we have that every $\bar{\gamma} > \bar{\gamma}_0$ produces the same mean–variance equilibrium $(M^{(1)}, M^{D,(2)})$ by Lemma 2.27 and Corollary 2.28. Thus in this case, we have existence of a mean–variance equilibrium by Corollary 2.28 and uniqueness in the sense explained above. We can henceforth exclude this trivial case with the following **standing assumption**.

Assumption 2.29. We suppose that the local martingales $(M^{(1)}, M^{D,(2)})$ and $\bar{Z}^0$ are not strongly orthogonal.

We now focus on the nontrivial case where Assumption 2.29 holds; by Lemma 2.27, we have $\ell(\bar{\gamma}) < 1$ for all $\bar{\gamma} > \bar{\gamma}_0$. We also get the following corollary to Lemma 2.27, which shows that the map $\bar{\gamma} \mapsto S(\bar{\gamma})$ is injective.

Corollary 2.30. *Suppose that Assumption 2.29 holds. Then for all $\bar{\gamma} > \bar{\gamma}_0$ and $\bar{\gamma}' \in \mathbb{R}$, $S(\bar{\gamma})$ is a generalised quadratic equilibrium with aggregate risk tolerance $\bar{\gamma}'$ if and only if $\bar{\gamma} = \bar{\gamma}'$. In particular, for $\bar{\gamma}' > \bar{\gamma}_0$, we have $S(\bar{\gamma}) = S(\bar{\gamma}')$ if and only if $\bar{\gamma} = \bar{\gamma}'$.*

Proof. By Lemma 2.25, $S(\bar{\gamma})$ is the unique generalised quadratic equilibrium with aggregate risk tolerance $\bar{\gamma}$ and $\bar{Z}(\bar{\gamma})S(\bar{\gamma})$ is a local martingale. This shows the "if" statement. To prove the converse, suppose for a contradiction that $S(\bar{\gamma})$ is a generalised quadratic equilibrium with respect to some risk tolerances $\gamma_1, \dots, \gamma_K$ such that $\sum_{k=1}^K \gamma_k =: \bar{\gamma}' \neq \bar{\gamma}$. As argued at the beginning of the proof of Lemma 2.25, the conclusion of Lemma III.3.6 still holds for generalised quadratic equilibria, and thus the process $\bar{Z}(\bar{\gamma}')S(\bar{\gamma})$ is also a local martingale. Taking differences yields that

$$\big(\bar{Z}(\bar{\gamma}') - \bar{Z}(\bar{\gamma})\big)S(\bar{\gamma}) = (\bar{\gamma}' - \bar{\gamma})S(\bar{\gamma})$$

is a local martingale as well, and so is $S(\bar{\gamma})$ because $\bar{\gamma}' \neq \bar{\gamma}$. Thus the implication (a) $\Rightarrow$ (d) in Lemma 2.27 contradicts Assumption 2.29, so that $\bar{\gamma}' \neq \bar{\gamma}$ cannot hold. This concludes the proof of the first statement. The second statement follows immediately from the first since $S(\bar\gamma')$ is the unique generalised quadratic equilibrium with aggregate risk tolerance $\bar\gamma'$ in the case $\bar\gamma' > \bar\gamma_0$. $\square$

We now return to the idea of "trying" values of $\bar\gamma$ in order to find an equilibrium. It is useful to express this procedure in more concrete terms. In our previous results, we have implicitly used the four maps ψ_i defined below, where we denote by $\mathcal{S}(\mathbb{R}^{d_1+d_2})$ the set of $\mathbb{R}^{d_1+d_2}$-valued semimartingales on $[0,T]$.

Definition 2.31. We define the maps $\psi_1, \psi_2, \psi_3, \psi_4$ and Ψ as follows:

- $\psi_1 : (\bar\gamma_0, \infty) \to \mathcal{S}(\mathbb{R}^{d_1+d_2})$ is defined by

$$\psi_1(\bar\gamma) := \big(S_t(\bar\gamma)\big)_{0\le t\le T}, \quad \bar\gamma > \bar\gamma_0, \tag{2.44}$$

where $S(\bar\gamma)$ is the generalised quadratic equilibrium price process with aggregate risk tolerance $\bar\gamma$, given by (2.36) and (2.37) which read

$$\begin{aligned} S_t^j(\bar\gamma) &:= S_0^j + M_t^j - \int_0^t \frac{d\langle \bar Z(\bar\gamma), M^j\rangle_s}{\bar Z_{s-}(\bar\gamma)}, \quad 0\le t\le T,\ j\in\{1,\dots,d_1\},\\ S_t^j(\bar\gamma) &:= \frac{E[\bar H(\bar\gamma)D^j \mid \mathcal{F}_t]}{\bar Z_t(\bar\gamma)}, \quad 0\le t\le T,\ j\in\{d_1+1,\dots,d_1+d_2\}. \end{aligned}$$

- $\psi_2 : \operatorname{Ran}\psi_1 \to (0,1)\times(\mathbb{R}\times\mathbb{R}_+)^K$ is defined by

$$\psi_2\big((S_t)_{0\le t\le T}\big) := \big(\ell, (c_k, \varepsilon_k^2)_{k=1}^K\big), \tag{2.45}$$

where we define ℓ, c_k and ε_k^2 as in (2.14) and (2.15) in terms of S by

$$\ell(S) := \varepsilon^2(1;S) = \min_{\vartheta\in\overline{\Theta}(S)} E[(1-\vartheta\bullet S_T)^2], \tag{2.46}$$

$$\varepsilon_k^2(S) := \varepsilon_{\mathrm{ex}}^2(\Xi^k;S) = \min_{\substack{c\in\mathbb{R}\\ \vartheta\in\overline{\Theta}(S)}} E[(\Xi^k - c - \vartheta\bullet S)^2], \tag{2.47}$$

$$c_k(S) = c(\Xi^k;S), \tag{2.48}$$

and $(c(\Xi^k;S), \vartheta^{\mathrm{ex}}(\Xi^k;S))$ is the unique minimiser to (2.47).

- $\psi_3 : (0,1)\times(\mathbb{R}\times\mathbb{R}_+)^K \to \mathbb{R}^K$ is defined by

$$\psi_3\big(\ell, (c_k,\varepsilon_k^2)_{k=1}^K\big) := (\gamma_k)_{k=1}^K := \big(c_k + \hat y_{U_k}(\ell, c_k, \varepsilon_k^2)\big)_{k=1}^K, \tag{2.49}$$

where $\hat y_{U_k} : (0,1)\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ is the function given by Proposition 2.17 so that $\hat y_{U_k}(\ell, c_k, \varepsilon_k^2)$ is the unique maximiser to the problem

$$U_k\big(c_k + (1-\ell)y, \sqrt{\varepsilon_k^2 + \ell(1-\ell)y^2}\big) \longrightarrow \max_{y\ge 0}!$$

- $\psi_4 : \mathbb{R}^K \to \mathbb{R}$ is defined by

$$\psi_4(\gamma_1, \dots, \gamma_K) := \sum_{k=1}^{K} \gamma_k. \tag{2.50}$$

- $\Psi : (\bar{\gamma}_0, \infty) \to \mathbb{R}$ is the composition

$$\Psi := \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1. \tag{2.51}$$

We note that ψ_2 is well defined since $S(\bar{\gamma})$ satisfies Assumptions 2.8 and 2.11 for $\bar{\gamma} > \bar{\gamma}_0$ by Lemma 2.25, and because $\ell(\bar{\gamma}) \neq 1$ under Assumption 2.29 due to Lemma 2.27. We also have that ψ_3 is well defined under Assumption 2.19 due to Proposition 2.17. The maps ψ_1, ψ_2 and ψ_3 are studied more closely in Sections 4.1, 4.2 and 5, respectively. Nevertheless, we can already prove the first main result which states that $\bar{\gamma} > \bar{\gamma}_0$ yields a mean–variance equilibrium if and only if $\bar{\gamma}$ is a fixed point of Ψ. Intuitively, we may interpret $\tilde{\gamma} := \Psi(\bar{\gamma})$ as the aggregate risk tolerance that is implied by the optimal strategies of the agents with respect to $S(\bar{\gamma})$ and U_k, when we instead view those strategies as the solutions to quadratic utility problems of the form (2.8). Thus the implied aggregate risk tolerance $\tilde{\gamma}$ should equal $\bar{\gamma}$ if and only if $S(\bar{\gamma})$ is a mean–variance equilibrium, and this is what we prove now.

Before giving the main result, some bookkeeping is in order regarding the assumptions. We do not need Assumptions 2.8 and 2.11 any more; those assumptions were imposed on a general price process S, but they are automatically satisfied by $S(\bar{\gamma})$ for $\bar{\gamma} > \bar{\gamma}_0$ due to Lemma 2.25. On the other hand, we still require Assumptions 2.1, 2.19 and 2.29; note that these are conditions only on the primitives and not on the equilibrium prices.

Theorem 2.32. *Suppose that the primitives $S_0^{(1)}, M^{(1)}, D^{(2)}$, η^k and $\Xi^{k,n}$ (for $k = 1, \dots, K$) are such that Assumptions 2.1, 2.19 and 2.29 hold, and let $\bar{\gamma} > \bar{\gamma}_0$. Then $(1, S(\bar{\gamma}))$ is a mean–variance equilibrium if and only if $\bar{\gamma} = \Psi(\bar{\gamma})$.*

Proof. For a fixed $\bar{\gamma} > \bar{\gamma}_0$, define $S(\bar{\gamma}) = \psi_1(\bar{\gamma})$ as well as the constants

$$\Big(\ell(\bar{\gamma}), \big(c_k(\bar{\gamma}), \varepsilon_k^2(\bar{\gamma})\big)_{k=1}^{K}\Big) := \psi_2 \circ \psi_1(\bar{\gamma}) \tag{2.52}$$

associated with the MVH problems with respect to $S(\bar{\gamma})$, where we use the shorthand $\ell(\bar{\gamma}) = \ell(S(\bar{\gamma}))$, etc. By Lemma 2.25, $S(\bar{\gamma})$ satisfies Assumptions 2.8 and 2.11, and hence we may apply our results from Section 2.2 with respect to $S(\bar{\gamma})$.

We start by proving "only if". If $(1, S(\bar{\gamma}))$ is a mean–variance equilibrium, there exists a unique solution $\hat{\vartheta}^k \in \overline{\Theta}(S)$ to the mean–variance utility problem

(2.10) for each $k \in \{1,\dots,K\}$. By part 2) of Theorem 2.20, $\hat{\vartheta}^k$ is also the unique solution to the MVH problem (2.14) with payoff $H^k(\gamma_k) = \gamma_k - \Xi^k$, where

$$\gamma_k(\bar{\gamma}) = c_k(\bar{\gamma}) + \hat{y}_{U_k}\big(\ell(\bar{\gamma}), c_k(\bar{\gamma}), \varepsilon_k^2(\bar{\gamma})\big).$$

Thus by part 1) of Lemma 2.23, $S(\bar{\gamma})$ is a generalised quadratic equilibrium with risk tolerances $\gamma_1(\bar{\gamma}),\dots,\gamma_K(\bar{\gamma})$, where (2.52) and the definition of ψ_3 yield

$$\big(\gamma_1(\bar{\gamma}),\dots,\gamma_K(\bar{\gamma})\big) = \psi_3 \circ \psi_2 \circ \psi_1(\bar{\gamma}).$$

However, by Lemma 2.25, the unique generalised quadratic equilibrium with risk tolerances $\gamma_1(\bar{\gamma}),\dots,\gamma_K(\bar{\gamma})$ is $S(\widetilde{\gamma})$, where

$$\widetilde{\gamma} := \sum_{k=1}^{K} \gamma_k(\bar{\gamma}) = \psi_4\big(\gamma_1(\bar{\gamma}),\dots,\gamma_K(\bar{\gamma})\big) = \Psi(\bar{\gamma}).$$

Thus we must have $S(\bar{\gamma}) = S(\Psi(\bar{\gamma}))$. By Corollary 2.30, this implies $\Psi(\bar{\gamma}) = \bar{\gamma}$ which proves the "only if" statement.

To show the "if" statement, let $\bar{\gamma}$ be a fixed point of Ψ and set

$$\big(\gamma_1(\bar{\gamma}),\dots,\gamma_K(\bar{\gamma})\big) = \psi_3 \circ \psi_2 \circ \psi_1(\bar{\gamma}).$$

Then by part 2) of Theorem 2.20, there exists a unique optimal strategy $\hat{\vartheta}^k$ to (2.10) with respect to $S(\bar{\gamma})$ for each $k \in \{1,\dots,K\}$, and $\hat{\vartheta}^k - \eta^k$ is the unique solution to the MVH problem (2.14) with payoff $H^k(\gamma_k(\bar{\gamma}))$. By the linearity of MVH (see Lemma III.2.6), $\hat{\vartheta}^k$ is also the unique solution to (2.14) with payoff $H^k(\gamma_k(\bar{\gamma})) + \eta^k \bullet S_T$. On the other hand, we have by assumption that

$$\sum_{k=1}^{K} \gamma_k(\bar{\gamma}) = \psi_4\big(\gamma_1(\bar{\gamma}),\dots,\gamma_k(\bar{\gamma})\big) = \Psi(\bar{\gamma}) = \bar{\gamma}$$

so that by Lemma 2.25, $(1, S(\bar{\gamma}))$ is the unique generalised quadratic equilibrium market with risk tolerances $\gamma_1(\bar{\gamma}),\dots,\gamma_K(\bar{\gamma})$. We have already shown that the problems (2.10) and (2.14) with payoff $H^k(\gamma_k(\bar{\gamma})) + \eta^k \bullet S_T$ have the same unique solution $\hat{\vartheta}^k$ for each $k \in \{1,\dots,K\}$. Therefore, it follows by part 2) of Lemma 2.23 that $(1, S(\bar{\gamma}))$ is a mean–variance equilibrium market. $\square$

3 The linear case

In the previous section, we have introduced the problem of finding a mean–variance equilibrium for a given set of primitives $S_0^{(1)}, M^{(1)}, D^{(2)}$, η^k, $\Xi^{k,n}$ and mean–variance utility functions U_k as in Definition 2.3. We have shown in Theorem 2.32 that the generalised quadratic equilibrium market $(1, S(\bar{\gamma}))$ is also a mean–variance equilibrium market if and only if $\bar{\gamma} \in (\bar{\gamma}_0, \infty)$ is a fixed point of the map $\Psi = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1$. We now study the existence and uniqueness of such a fixed point in the case of linear mean–variance utility (2.9), which reads

$$U_k(\mu, \sigma) = \mu - \frac{\sigma^2}{2\lambda_k}, \quad (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+, \tag{3.1}$$

for some $\lambda_k > 0$. As it turns out, there exists in this case at most one fixed point $\bar{\gamma}$ for which we even obtain an explicit formula. In order to study the mean–variance equilibria in the linear case, the first step is to use the particular form (3.1) of U_k to solve the optimisation problems (2.25) for each $k \in \{1, \dots, K\}$. By doing so, we obtain a relatively simple formula for the map ψ_3.

Lemma 3.1. *Suppose that each U_k has the form* (3.1)*. Then we have*

$$\psi_3\big(\ell, (c_k, \varepsilon_k^2)_{k=1}^K\big) = \left(c_k + \frac{\lambda_k}{\ell}\right)_{k=1}^K \tag{3.2}$$

for all $(\ell, (c_k, \varepsilon_k^2)_{k=1}^K) \in (0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K$.

Proof. We start by considering the abstract mean–variance problem (2.28), i.e.,

$$U\Big(c + (1-\ell)y, \sqrt{\varepsilon^2 + \ell(1-\ell)y^2}\Big) \longrightarrow \max_{y \geq 0}!$$

for some $(\ell, c, \varepsilon^2) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$. Plugging in $U(\mu, \sigma) := \mu - \frac{\sigma^2}{2\lambda}$ for some $\lambda > 0$ yields the problem

$$c + (1-\ell)y - \frac{\varepsilon^2 + \ell(1-\ell)y^2}{2\lambda} \longrightarrow \max_{y \geq 0}!$$

This is elementary, and the unique minimiser is given by

$$\hat{y}_U(\ell, c, \varepsilon^2) = \frac{\lambda}{\ell} > 0.$$

Hence by plugging into Definition 2.31, we obtain

$$\psi_3\big(\ell,(c_k,\varepsilon_k^2)_{k=1}^K\big) = \big(c_k + \hat{y}_{U_k}(\ell,c_k,\varepsilon_k^2)\big)_{k=1}^K = \left(c_k + \frac{\lambda_k}{\ell}\right)_{k=1}^K,$$

as claimed. □

Lemma 3.1 shows part of the reason why finding an equilibrium is simpler in the linear case: (3.2) yields a simple explicit formula for ψ_3 which is not available in general. In particular, ψ_3 does not depend on the hedging errors ε_k^2 in this case. However, that is not the full story. To see why, fix $\bar{\gamma} > \bar{\gamma}_0$ and write

$$\Big(\ell(\bar{\gamma}),\big(c_k(\bar{\gamma}),\varepsilon_k^2(\bar{\gamma})\big)_{k=1}^K\Big) := (\psi_2 \circ \psi_1)(\bar{\gamma}).$$

Then by (3.2) and Definition 2.31, $\bar{\gamma}$ is a fixed point of Ψ if and only if it satisfies the equation

$$\bar{\gamma} = \sum_{k=1}^K \left(c_k(\bar{\gamma}) + \frac{\lambda_k}{\ell(\bar{\gamma})}\right). \tag{3.3}$$

Although this simplifies the fixed point condition, it is still not obvious how to solve (3.3) for $\bar{\gamma}$. At first glance, it looks as though we need to study the maps $\bar{\gamma} \mapsto \ell(\bar{\gamma})$ and $\bar{\gamma} \mapsto c_k(\bar{\gamma})$, as we do in the next section for the case of general mean–variance utility functions. However, this turns out not to be necessary here, due to a relationship (given below) between $\ell(\bar{\gamma})$ and $\bar{c}(\bar{\gamma}) := \sum_{k=1}^K c_k(\bar{\gamma})$ that arises from the probabilistic structure of the problem. Surprisingly, that relationship even allows us to find an explicit solution to (3.3).

In preparation for the main results, we first recall some notions from the theory of mean–variance hedging; we use Černý/Kallsen [25] as a reference. By Lemma 2.25, $S(\bar{\gamma})$ admits an equivalent local martingale measure $Q(\bar{\gamma})$ with bounded density so that [25, Assumption 2.1] is satisfied. We introduce the *opportunity process* $L(\bar{\gamma})$ given by

$$L_t(\bar{\gamma}) = \operatorname{ess\,inf}_{\vartheta \in \overline{\Theta}_{t,T}(S(\bar{\gamma}))} E\Big[\Big(1 - \big(\vartheta \bullet S(\bar{\gamma})\big)_T\Big)^2 \,\Big|\, \mathcal{F}_t\Big], \quad 0 \le t \le T, \tag{3.4}$$

where $\overline{\Theta}_{t,T}(S(\bar{\gamma})) \subseteq \overline{\Theta}(S(\bar{\gamma}))$ is the set of admissible strategies ϑ such that $\vartheta \mathbf{1}_{[\![0,t]\!]} = 0$. We have by [25, Corollary 3.4 and Lemma 3.10] that $L(\bar{\gamma})$ is an $(0,1]$-valued submartingale with $L_T(\bar{\gamma}) = 1$. Setting $t = 0$ in (3.4) together with (2.46) yields $L_0(\bar{\gamma}) = \ell(\bar{\gamma})$. By [25, Lemma 3.1], there exists for each $t \in [0,T]$ a unique optimal strategy $\vartheta^{(t)}(1; S(\bar{\gamma})) \in \overline{\Theta}_{t,T}(S(\bar{\gamma}))$ to (3.4); we say that it is the *optimal pure investment strategy started at time t.*

Next, we introduce the *mean value process* $(\bar V_t(\bar\gamma))_{0\le t\le T}$ for $\bar H(\bar\gamma)=\bar\gamma-\bar\Xi$ in the sense of [25, Definition 4.2]. By [25, Lemmas 3.7 and 4.1 and Proposition 3.13.1], $\bar V(\bar\gamma)$ is the unique semimartingale such that

$$\bar V_t(\bar\gamma)=\frac{1}{L_t(\bar\gamma)}E\Big[\bar H(\bar\gamma)\Big(1-\big(\vartheta^{(t)}(1;S(\bar\gamma))\bullet S(\bar\gamma)\big)_T\Big)\;\Big|\;\mathcal F_t\Big],\quad 0\le t\le T.$$

In particular, $\bar V_T(\bar\gamma)=\bar H(\bar\gamma)$. Moreover, [25, Lemma 4.1] also gives that the process $(\bar V_s(\bar\gamma)M_s^{(t)}(\bar\gamma))_{t\le s\le T}$ is a P-martingale on $[t,T]$ for any $t\in[0,T]$, where $(M_s^{(t)}(\bar\gamma))_{t\le s\le T}$ is a P-martingale (see [25, Lemma 3.2]) defined by

$$M_s^{(t)}(\bar\gamma):=L_s(\bar\gamma)\Big(1-\big(\vartheta^{(t)}(1;S(\bar\gamma))\bullet S(\bar\gamma)\big)_s\Big),\quad 0\le t\le s\le T. \tag{3.5}$$

The key property from our point of view is that $\bar V_t(\bar\gamma)$ satisfies the inequality

$$\begin{aligned}&\operatorname{ess\,inf}_{\vartheta\in\overline\Theta_{t,T}(S(\bar\gamma))}E\Big[\Big(\bar H(\bar\gamma)-\bar V_t(\bar\gamma)-\big(\vartheta\bullet S(\bar\gamma)\big)_T\Big)^2\;\Big|\;\mathcal F_t\Big]\\&\le\operatorname{ess\,inf}_{\vartheta\in\overline\Theta_{t,T}(S(\bar\gamma))}E\Big[\Big(\bar H(\bar\gamma)-U-\big(\vartheta\bullet S(\bar\gamma)\big)_T\Big)^2\;\Big|\;\mathcal F_t\Big]\end{aligned}$$

for any $\mathcal F_t$-measurable random variable U; this follows by [25, Theorem 4.10.2]. In particular, $\bar V_0(\bar\gamma)$ is the first component of the solution to the exMVH problem (2.15) for $\bar H(\bar\gamma)$, and hence $\bar V_0(\bar\gamma)=c(\bar H(\bar\gamma);S(\bar\gamma))=\bar\gamma-c(\bar\Xi;S(\bar\gamma))$.

Remark 3.2. As pointed out in Remark 2.14, the variance-optimal martingale measure $Q^\star(\bar\gamma)$ for $S(\bar\gamma)$ (see (2.38) for the definition) is equivalent to P if $S(\bar\gamma)$ is continuous. In that case, the process $\bar V(\bar\gamma)$ can be written more simply as $\bar V_t(\bar\gamma)=E_{Q^\star(\bar\gamma)}[\bar H(\bar\gamma)\mid\mathcal F_t]$ by [25, Equation (4.1)].

We are now ready to state and prove the property of $S(\bar\gamma)$ that allows us to solve (3.3). For a more intuitive explanation of why we obtain (3.8) below, we first consider the continuous case so that

$$\bar V_t(\bar\gamma)=E_{Q^\star(\bar\gamma)}[\bar H(\bar\gamma)\mid\mathcal F_t]=\frac{E_P[Z_T^{Q^\star}(\bar\gamma)\bar H(\bar\gamma)\mid\mathcal F_t]}{Z_t^{Q^\star}(\bar\gamma)},\quad 0\le t\le T, \tag{3.6}$$

where $(Z_t^{Q^\star}(\bar\gamma))_{0\le t\le T}$ is the density process of $Q^\star(\bar\gamma)$ with respect to P. For readability, we temporarily omit the parameter $\bar\gamma$ below and write $\vartheta^{(t)}$ as a shorthand for $\vartheta^{(t)}(1; S(\bar{\gamma}))$. By [25, Lemma 3.7 and Proposition 3.13], we have the formula

$$Z_t^{Q^\star} = \frac{M_t^{(0)}}{M_0^{(0)}} = \frac{L_t(1 - \vartheta^{(0)} \mathbin{\bullet} S_t)}{L_0}, \quad 0 \le t \le T,$$

so that $dQ^\star/dP = (1 - \vartheta^{(0)} \mathbin{\bullet} S_T)/L_0$. By Bayes' rule, plugging into (3.6) yields

$$\begin{aligned} \bar{V}_t &= \frac{E_P[(1 - \vartheta^{(0)} \mathbin{\bullet} S_T)\bar{H} \mid \mathcal{F}_t]}{L_t(1 - \vartheta^{(0)} \mathbin{\bullet} S_t)} \\ &= \frac{E_P[\bar{H} \mid \mathcal{F}_t]}{L_t} - \frac{E_P[\bar{H}(\vartheta^{(0)} \mathbin{\bullet} S_T - \vartheta^{(0)} \mathbin{\bullet} S_t) \mid \mathcal{F}_t]}{L_t(1 - \vartheta^{(0)} \mathbin{\bullet} S_t)}, \quad 0 \le t \le T. \end{aligned} \tag{3.7}$$

So far, these calculations hold for the mean value process $\bar{V}$ of any payoff $\bar{H} \in L^2$. However, by the construction of $S = S(\bar{\gamma})$ as a generalised quadratic equilibrium, we also know that $\bar{H} = \bar{H}(\bar{\gamma})$ is bounded and (up to a scalar factor) the density of an equivalent local martingale measure for S. Since $\vartheta^{(0)} \in \overline{\Theta}(S)$, the second term on the right-hand side of (3.7) vanishes by Proposition III.2.12, and hence

$$\bar{V}_t(\bar{\gamma})L_t(\bar{\gamma}) = E_P[\bar{H}(\bar{\gamma}) \mid \mathcal{F}_t] = Z_t(\bar{\gamma}), \quad 0 \le t \le T.$$

Thus in the continuous case, (3.8) below follows from the fact that the payoff $\bar{H}(\bar{\gamma}) = \bar{\gamma} - \bar{\Xi}$ also induces an equivalent local martingale measure for $S(\bar{\gamma})$.

We now give the proof for the general case. This is somewhat more technical because the variance-optimal martingale measure $Q^\star(\bar{\gamma})$ can in general be a signed measure, and so we do not work with it directly. Here we do not obtain an analogue of the decomposition (3.7) which holds for all payoffs, but we nevertheless arrive at the same formula in the end.

Proposition 3.3. *For $\bar{\gamma} > \bar{\gamma}_0$, we have*

$$\bar{Z}_t(\bar{\gamma}) = \bar{V}_t(\bar{\gamma})L_t(\bar{\gamma}), \quad 0 \le t \le T. \tag{3.8}$$

In particular, $t = 0$ yields $\bar{\gamma} - E_P[\bar{\Xi}] = (\bar{\gamma} - \bar{c}(\bar{\gamma}))\ell(\bar{\gamma})$, where $\bar{c}(\bar{\gamma}) = \sum_{k=1}^K c_k(\bar{\gamma})$.

Proof. As above, we fix $\bar{\gamma} > \bar{\gamma}_0$ and drop it for readability, and write $\vartheta^{(t)}$ as a shorthand for $\vartheta^{(t)}(1; S(\bar{\gamma}))$. Recall that $\bar{Z}S$ is a local P-martingale by Lemma 2.25, where $\bar{Z}$ is a bounded and strictly positive P-martingale. Fix $t \in [0, T]$. Since $\vartheta^{(t)} \in \overline{\Theta}(S)$, it follows by Proposition III.2.12 that $\bar{Z}(\vartheta^{(t)} \mathbin{\bullet} S)$ is a true P-martingale, and hence so is $\bar{Z}(1 - \vartheta^{(t)} \mathbin{\bullet} S)$. We also know that $(\bar{V}_s M_s^{(t)})_{t \le s \le T}$

is a P-martingale. By (3.5) and as $L_T = 1$, we have

$$\bar{V}_T M_T^{(t)} = \bar{H} L_T \big(1 - (\vartheta^{(t)} \bullet S)_T\big) = \bar{Z}_T \big(1 - (\vartheta^{(t)} \bullet S)_T\big).$$

Thus by taking conditional expectations $E_P[\,\cdot\,|\,\mathcal{F}_s]$, we obtain

$$\bar{V}_s M_s^{(t)} = \bar{Z}_s \big(1 - (\vartheta^{(t)} \bullet S)_s\big), \quad t \le s \le T.$$

Then (3.8) follows immediately by taking $s = t$ because $M_t^{(t)} = L_t$ by (3.5). Since $L_0(\bar{\gamma}) = \ell(\bar{\gamma})$, setting $t = 0$ in (3.8) yields

$$\bar{\gamma} - E_P[\Xi] = \bar{Z}_0(\bar{\gamma}) = \bar{V}_0(\bar{\gamma}) L_0(\bar{\gamma}) = c\big(\bar{H}(\bar{\gamma}); S(\bar{\gamma})\big)\ell(\bar{\gamma}),$$

where

$$c\big(\bar{H}(\bar{\gamma}); S(\bar{\gamma})\big) = c\big(\bar{\gamma} - \bar{\Xi}; S(\bar{\gamma})\big) = \bar{\gamma} - c\big(\bar{\Xi}; S(\bar{\gamma})\big).$$

This implies that $\bar{\gamma} - E_P[\bar{\Xi}] = (\bar{\gamma} - c(\bar{\Xi}; S(\bar{\gamma})))\ell(\bar{\gamma})$. By the linearity of exMVH (see Lemma III.2.6) and (2.48), we have

$$c\big(\bar{\Xi}; S(\bar{\gamma})\big) = \sum_{k=1}^{K} c\big(\Xi^k; S(\bar{\gamma})\big) = \sum_{k=1}^{K} c_k(\bar{\gamma}) = \bar{c}(\bar{\gamma}),$$

and this concludes the proof. □

Finally, we use Proposition 3.3 to obtain a unique solution to (3.3) and hence a mean–variance equilibrium via Theorem 2.32. In order to apply the latter, we retain Assumptions 2.1 and 2.29 from the previous section. On the other hand, Assumption 2.19 is automatically satisfied for linear mean–variance utility functions given by (3.1) and thus not needed here.

Theorem 3.4. *Suppose that Assumptions 2.1 and 2.29 hold, and that U_k has the form (3.1) for each $k \in \{1, \dots, T\}$ and some $\lambda_1, \dots, \lambda_K > 0$. Define*

$$\bar{\gamma} := \sum_{k=1}^{K} \lambda_k + E_P[\bar{\Xi}]. \tag{3.9}$$

Then Ψ admits a fixed point if and only if $\bar{\gamma} > \bar{\gamma}_0$. In that case, $\bar{\gamma}$ is the unique fixed point and $S(\bar{\gamma})$ is a mean–variance equilibrium.

Proof. Let $\tilde{\gamma} > \bar{\gamma}_0$ and write

$$\Big(\ell(\tilde{\gamma}), \big(c_k(\tilde{\gamma}), \varepsilon_k^2(\tilde{\gamma})\big)_{k=1}^K\Big) := (\psi_2 \circ \psi_1)(\tilde{\gamma}).$$

As explained after Lemma 3.1, $\tilde{\gamma}$ is a fixed point if and only if (3.3) holds with $\tilde{\gamma}$ in place of $\bar{\gamma}$. Note that $\ell(\tilde{\gamma}) < 1$ by Assumption 2.29 and Lemma 2.27. Hence by rearranging (3.3), $\tilde{\gamma}$ is a fixed point of Ψ if and only if

$$\big(\tilde{\gamma} - \bar{c}(\tilde{\gamma})\big)\ell(\tilde{\gamma}) = \sum_{k=1}^K \lambda_k. \tag{3.10}$$

Moreover, Proposition 3.3 gives

$$\tilde{\gamma} - E_P[\bar{\Xi}] = \big(\tilde{\gamma} - \bar{c}(\tilde{\gamma})\big)\ell(\tilde{\gamma}). \tag{3.11}$$

By plugging (3.11) into (3.10), we conclude that $\tilde{\gamma} \in (\bar{\gamma}_0, \infty)$ is a fixed point if and only if $\tilde{\gamma} - E_P[\bar{\Xi}] = \sum_{k=1}^K \lambda_k$, i.e., if and only if $\tilde{\gamma} = \bar{\gamma}$. Therefore $\bar{\gamma}$ is the only possible fixed point of the map Ψ, and it is indeed a fixed point if $\bar{\gamma} > \bar{\gamma}_0$; otherwise, $\bar{\gamma}$ cannot be a fixed point since ψ_1 is only defined on $(\bar{\gamma}_0, \infty)$. If $\bar{\gamma} > \bar{\gamma}_0$, then $S(\bar{\gamma})$ is a mean–variance equilibrium by Theorem 2.32. □

With Theorem 3.4, we conclude that there exists at most one mean–variance equilibrium of the form $S(\bar{\gamma})$ for $\bar{\gamma} > \bar{\gamma}_0$, and $\bar{\gamma}$ is explicitly given by (3.9). Note that the constants λ_k in (3.1) can be seen as risk tolerance parameters for the agents, and the condition $\bar{\gamma} > \bar{\gamma}_0$ in Theorem 3.4 is equivalent to

$$\sum_{k=1}^K \lambda_k > \operatorname{ess\,sup} \bar{\Xi} - E_P[\bar{\Xi}]. \tag{3.12}$$

Thus there exists an equilibrium if and only if the aggregate risk tolerance $\sum_{k=1}^K \lambda_k$ is larger than the uncertainty of the aggregate endowment $\bar{\Xi}$ as measured by $\operatorname{ess\,sup} \bar{\Xi} - E_P[\bar{\Xi}] \geq 0$. We note that (3.12) is not necessary for the one-period model considered in Koch-Medina/Wenzelburger [85]. We include it here because as argued after Corollary 2.28, it is difficult to characterise (generalised) quadratic equilibria with aggregate risk tolerance $\bar{\gamma} \leq \bar{\gamma}_0$ in continuous time. For the same reason, we do not claim in Theorem 3.4 that the mean–variance equilibrium is unique, but rather that it is unique among the set of generalised quadratic equilibria $S(\bar{\gamma})$ for $\bar{\gamma} > \bar{\gamma}_0$. Nevertheless, since the quadratic equilibria with $\bar{\gamma} > \bar{\gamma}_0$ are the most meaningful from an economic point of view (as can be seen from the results in Section III.3.5 for finite discrete time), this is still a satisfactory uniqueness result. This concludes our analysis of the linear case (3.1), and we now turn to the general case.

4 The general case

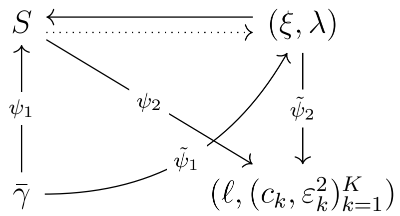
We return to the case of general mean–variance utility functions U_k and seek sufficient conditions for the existence of a mean–variance equilibrium in the sense of Definition 2.6, where each agent $k = 1, \dots, K$ solves the mean–variance utility problem (2.10). Our main tool is Theorem 2.32 which states that for $\bar{\gamma} > \bar{\gamma}_0$, a generalised quadratic equilibrium of the form $S(\bar{\gamma})$ (see (2.36) and (2.37)) is also a mean–variance equilibrium if and only if $\bar{\gamma}$ is a fixed point of the map Ψ introduced in Definition 2.31. Thus we want to show the existence of such a fixed point $\bar{\gamma}$ under suitable assumptions.

The overall strategy is as follows. The first step is to show the continuity of the composition $\Psi = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1$ by considering each map ψ_i individually. The continuity of ψ_4 is trivial, and that of ψ_3 follows from Proposition 2.17. We postpone the proof of the latter to Section 5 because it requires a careful study of the abstract mean–variance optimisation problem (2.28) that is self-contained and may be of independent interest for other mean–variance utility problems.

On the other hand, the question of the continuity of ψ_1 and ψ_2 is more involved; these two maps are studied in more detail in Sections 4.1 and 4.2, respectively. Whereas ψ_1 is indeed continuous, as we shall see in Lemma 4.4 below, we cannot expect ψ_2 to be continuous in general; see Example 4.5. To overcome this issue, we note that the continuity of $\psi_2 \circ \psi_1$ is sufficient for our purposes, even if ψ_2 itself is not continuous. We show the continuity of the composition by refactoring it as $\psi_2 \circ \psi_1 = \tilde{\psi}_2 \circ \tilde{\psi}_1$, where $\tilde{\psi}_1$ can be seen as an "enriched" version of ψ_1 that maps $\bar{\gamma}$ to the coefficients $(\xi(\bar{\gamma}), \lambda(\bar{\gamma}))$ associated with a structure-condition-type (SC) decomposition for $S(\bar{\gamma})$ of the form

$$dS_t(\bar{\gamma}) = \xi_t^\top(\bar{\gamma}) d\langle \bar{M} \rangle_t \xi_t(\bar{\gamma}) \lambda_t(\bar{\gamma}) + \xi_t(\bar{\gamma}) d\bar{M}_t, \quad 0 \le t \le T$$

for a fixed local martingale $\bar{M}$ that does not depend on $\bar{\gamma}$. The existence of such a decomposition is shown below in Lemma 4.11 and precise definitions for $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are given below in Definitions 4.12 and 4.24, respectively. The maps $\psi_1, \psi_2, \tilde{\psi}_1, \tilde{\psi}_2$ are represented by the following diagram:



(4.1)

Although the map $(\xi, \lambda) \mapsto S$ (defined below by (4.16)) is injective and hence admits on its range an inverse map (represented by the dashed arrow in (4.1)), the latter is not continuous in general. Thus the continuity of $\tilde{\psi}_2$ does not imply that of ψ_2; indeed, the discontinuity of $S \mapsto (\xi, \lambda)$ turns out to be the main obstacle to the continuity of ψ_2. We show in Theorem 4.18 and Corollary 4.23 below that $\xi(\bar{\gamma})$ and $\lambda(\bar{\gamma})$ (and hence $\tilde{\psi}_1$) depend continuously on $\bar{\gamma}$ in an appropriate sense, and this allows us to sidestep the dashed arrow to obtain the continuity of $\tilde{\psi}_2 \circ \tilde{\psi}_1$. More specifically, we show that $\tilde{\psi}_1$ is continuous by obtaining an explicit decomposition for $S(\bar{\gamma})$ and using the fact that $S(\bar{\gamma})$ admits an equivalent local martingale measure satisfying the so-called reverse Hölder inequality $R_2(P)$ (see Definition 4.8 below). We then express the outputs ℓ, c_k and ε_k^2 of $\tilde{\psi}_2$ as the initial values of solutions to a certain set of quadratic backward stochastic differential equations (BSDEs) with stochastic coefficients that depend on ξ and λ; see (4.56)–(4.58) below. This allows us to show the continuity of $\tilde{\psi}_2$ in Theorem 4.26 via a stability result (Theorem 6.6) for a certain class of BSDEs. Similarly to the study of the map ψ_3, this BSDE stability result can be shown in an abstract setting and may be of independent interest for other applications, and hence we postpone its statement and proof to Section 6.

So far, we have outlined how to show the continuity of the map Ψ. The second element that we need in order to find a fixed point is a bound on its range. Indeed, if there exist constants $\bar{\gamma}_2 \geq \bar{\gamma}_1 > \bar{\gamma}_0$ such that $\Psi(\bar{\gamma}_1) \geq \bar{\gamma}_1$ and $\Psi(\bar{\gamma}_2) \leq \bar{\gamma}_2$, then the existence of a fixed point follows immediately by the intermediate value theorem. In order to obtain such a condition on Ψ, we first consider the range of $\psi_4 \circ \psi_3$. We already have bounds on the outputs of ψ_3 under Assumption 2.19 by Proposition 2.17. Since ψ_4 is just a sum, those bounds pass trivially to the output of $\psi_4 \circ \psi_3$ and can then be applied to the map Ψ.

We now proceed to the main work of defining the maps $\tilde{\psi}_1$ and $\tilde{\psi}_2$ and showing their continuity. At the end of the section, we then combine these results to show the existence of mean–variance equilibria. We note once again that the subsequent Sections 5 and 6 are self-contained and do not rely on any results from this section, and therefore we may use here some of the results proven therein.

4.1 Continuity of $\tilde{\psi}_1$

Our first main goal is to show that the generalised quadratic equilibrium $S = S(\bar{\gamma})$ given in Theorem III.3.8 depends continuously on the aggregate risk tolerance $\bar{\gamma}$, i.e., that the maps ψ_1 and $\tilde{\psi}_1$ (the latter of which is still to be defined) are continuous in an appropriate sense.

We start by introducing some notation related to multidimensional stochastic integration. In this section, we always work with column vectors and thus identify $\mathbb{R}^n = \mathbb{R}^{n\times 1}$; in particular, we use the conventional definition of a vector stochastic integral $\xi \bullet X$ for $\mathbb{R}^{n\times 1}$-valued processes ξ and X. We then extend this notation to matrix-valued integrands by considering $\mathbb{R}^{m\times 1}$-valued stochastic integrals of the form $\xi \bullet X$ for $\mathbb{R}^{n\times m}$-valued ξ and $\mathbb{R}^{n\times 1}$-valued X.

Notation 4.1. Let $T > 0$, $n \in \mathbb{N}$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a stochastic basis satisfying the usual conditions. We denote by $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{M}^2(\mathbb{R}^n)$, $\mathcal{M}^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{P}(\mathbb{R}^n)$ the sets of $\mathbb{R}^n$-valued semimartingales, L^2-martingales, local L^2-martingales and predictable processes on $[0,T]$, respectively. For $X \in \mathcal{S}(\mathbb{R}^n)$, we denote by $L(X) = L(X;\mathbb{R}^n)$ the set of $\mathbb{R}^{n\times 1}$-valued predictable processes $(\xi_t)_{0\le t\le T}$ that are X-integrable in the sense of Jacod/Shiryaev [71, III.6.17]. We also define for $m \in \mathbb{N}$ the set

$$L(X;\mathbb{R}^{n\times m}) = \{\xi = (\xi^1, \dots, \xi^m) : \xi^1, \dots, \xi^m \in L(X)\},$$

and for $\xi \in L(X;\mathbb{R}^{n\times m})$, we write $\xi \bullet X = (\xi^1 \bullet X, \dots, \xi^m \bullet X)^\top$ with values in $\mathbb{R}^{m\times 1}$. We identify $\xi^{(1)}, \xi^{(2)} \in L(X;\mathbb{R}^{n\times m})$ if they are X-*equivalent*, i.e., if $\xi^{(1)} \bullet X$ and $\xi^{(2)} \bullet X$ are indistinguishable; we write in that case $\xi^{(1)} =_X \xi^{(2)}$.

With this notation, we have for all processes $X^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$, $X^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$, $\xi^{(1)} \in L(X^{(1)};\mathbb{R}^{n_1\times m_1})$ and $\xi^{(2)} \in L(X^{(2)};\mathbb{R}^{n_2\times m_2})$ that the quadratic covariation $[\xi^{(1)} \bullet X^{(1)}, \xi^{(2)} \bullet X^{(2)}]$ takes values in $\mathbb{R}^{m_1\times m_2}$ and satisfies

$$d[\xi^{(1)} \bullet X^{(1)}, \xi^{(2)} \bullet X^{(2)}]_t = (\xi^{(1)}_t)^\top d[X^{(1)}, X^{(2)}]_t \xi^{(2)}_t, \quad 0 \le t \le T. \tag{4.2}$$

Moreover, we have for $\xi^{(3)} \in L(\xi^{(1)} \bullet X^{(1)}, \mathbb{R}^{m_1\times m_3})$ the associative property

$$\xi^{(3)} \bullet (\xi^{(1)} \bullet X^{(1)}) = (\xi^{(1)}\xi^{(3)}) \bullet X^{(1)}. \tag{4.3}$$

We also introduce the *Émery distance* on $\mathcal{S}(\mathbb{R}^n)$. For two semimartingales $X, Y \in \mathcal{S}(\mathbb{R}^n)$, we write

$$d_{\mathcal{S}}(X,Y) = \sup\{E[1 \wedge |\eta \bullet X_T - \eta \bullet Y_T|] : \eta \in \mathcal{P}(\mathbb{R}^n), \|\eta\|_\infty \le 1\}.$$

By Émery [47, Lemme 7], the space $\mathcal{S}(\mathbb{R}^n)$ is a complete topological vector space under $d_{\mathcal{S}}$. The topology induced by $d_{\mathcal{S}}$ is the so-called *semimartingale topology.* We denote convergence in the semimartingale topology by "$\overset{\mathcal{S}}{\to}$".

Recall the definitions and assumptions related to the agents $k \in 1, \dots, K$ as well as the financial and real assets introduced in Section 2.1 (including Assumption 2.1). We define the local martingale $\bar{M} := (M^{(1)}, M^{D,(2)}, \bar{\Pi}, \bar{Z}^0)$, where $(M_t^{(1)})_{0 \le t \le T}$ is the local martingale part of $S^{(1)}$ (see (2.2)) and $(M_t^{D,(2)})_{0 \le t \le T}$, $(\bar{\Pi}_t)_{0 \le t \le T}$ and $(\bar{Z}_t^0)_{0 \le t \le T}$ are square-integrable martingales defined by

$$M_t^{D,(2)} = E[D^{(2)} \mid \mathcal{F}_t], \quad \bar{\Pi}_t = E[(\bar{\gamma}_0 - \bar{\Xi})D^{(2)} \mid \mathcal{F}_t], \quad \bar{Z}_t^0 = E[\bar{\gamma}_0 - \bar{\Xi} \mid \mathcal{F}_t] \tag{4.4}$$

for $0 \le t \le T$, where we recall $\bar{\gamma}_0 := \operatorname{ess\,sup} \bar{\Xi}$ (see (2.35)) so that $\bar{Z}^0 \ge 0$. We note that $M^{(1)}$ takes values in $\mathbb{R}^{d_1}$, $M^{D,(2)}$ and $\bar{\Pi}$ in $\mathbb{R}^{d_2}$ and $\bar{Z}^0$ in $\mathbb{R}$, so that $\bar{M}$ takes values in $\mathbb{R}^{\bar{d}}$ for $\bar{d} := d_1 + 2d_2 + 1$. The local martingale $\bar{M}$ plays an important role in the representation of the martingale part of $S(\bar{\gamma})$ since it is defined directly in terms of the primitives and does not depend on $\bar{\gamma}$. We also make the following **standing assumption**.

Assumption 4.2. The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ is *continuous*, i.e., every $\mathbb{F}$-adapted martingale admits a continuous version.

Assumption 4.2 is somewhat restrictive, and it remains an open question whether one can relax it in our subsequent results. As we shall see, its inclusion eases the task of showing the continuity of $\tilde{\psi}_1$ and $\tilde{\psi}_2$. This assumption automatically holds in setups where the filtration is generated by a Brownian motion since the continuity of $\mathbb{F}$ is ensured there by the Itô representation theorem. Under Assumption 4.2, we may and always do take a continuous version of any martingale or local martingale. In particular, Assumption 4.2 implies that $\bar{M}$ is continuous. Moreover, for local martingales, we can replace the square brackets $[\cdot, \cdot]$ by the sharp brackets $\langle \cdot, \cdot \rangle$.

For some fixed aggregate risk tolerance $\bar{\gamma} > \bar{\gamma}_0$, we recall the generalised quadratic equilibrium $S(\bar{\gamma})$ given in (2.36) and (2.37) by

$$S_t^j(\bar{\gamma}) := S_0^j + M_t^j - \int_0^t \frac{d\langle \bar{Z}(\bar{\gamma}), M^j \rangle_s}{\bar{Z}_s(\bar{\gamma})}, \qquad j \in \{1, \dots, d_1\}, \tag{4.5}$$

$$S_t^j(\bar{\gamma}) := \frac{E[\bar{H}(\bar{\gamma})D^j \mid \mathcal{F}_t]}{\bar{Z}_t(\bar{\gamma})}, \qquad j \in \{d_1 + 1, \dots, d_1 + d_2\} \tag{4.6}$$

for $0 \le t \le T$, where the martingale $(\bar{Z}_t(\bar{\gamma}))_{0\le t\le T}$ defined by

$$\bar{Z}_t(\bar{\gamma}) := E[H(\bar{\gamma}) \mid \mathcal{F}_t] = \bar{\gamma} - \bar{\gamma}_0 + \bar{Z}^0_t, \quad 0 \le t \le T, \tag{4.7}$$

is bounded and strictly positive by Assumption 2.1. Note that $\bar{Z}(\bar{\gamma})$ is continuous by Assumption 4.2, so that we have $\bar{Z}_s(\bar{\gamma}) = \bar{Z}_{s-}(\bar{\gamma})$ in the integrand of (4.5). We now want to study the map $\psi_1 : (\bar{\gamma}_0, \infty) \to \mathcal{S}(\mathbb{R}^{d_1+d_2}), \bar{\gamma} \mapsto S(\bar{\gamma})$, and the first step is to rewrite (4.5) and (4.6) in terms of the components of $\bar{M}$; recall (4.4).

Lemma 4.3. *For each $\bar{\gamma} > \bar{\gamma}_0$, we have the decompositions*

$$S^{(1)}_t(\bar{\gamma}) = S^{(1)}_0 + M^{(1)}_t + A^{(1)}_t(\bar{\gamma}), \quad 0 \le t \le T, \tag{4.8}$$
$$S^{(2)}_t(\bar{\gamma}) = S^{(2)}_0(\bar{\gamma}) + M^{(2)}_t(\bar{\gamma}) + A^{(2)}_t(\bar{\gamma}), \quad 0 \le t \le T, \tag{4.9}$$
$$S^{(2)}_0(\bar{\gamma}) = \frac{(\bar{\gamma} - \bar{\gamma}_0)M^{D,(2)}_0 + \bar{\Pi}_0}{\bar{\gamma} - \bar{\gamma}_0 + \bar{Z}^0_0}, \tag{4.10}$$

where the local martingale $M^{(2)}(\bar{\gamma}) \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{R}^{d_2})$ is given by

$$M^{(2)}_t(\bar{\gamma}) = \int_0^t \left(\frac{(\bar{\gamma} - \bar{\gamma}_0)dM^{D,(2)}_s + d\bar{\Pi}_s}{\bar{\gamma} - \bar{\gamma}_0 + \bar{Z}^0_s} - \frac{((\bar{\gamma} - \bar{\gamma}_0)M^{D,(2)}_s + \bar{\Pi}_s)d\bar{Z}^0_s}{(\bar{\gamma} - \bar{\gamma}_0 + \bar{Z}^0_s)^2} \right) \tag{4.11}$$

for $0 \le t \le T$, and the finite-variation processes $(A^{(1)}_t(\bar{\gamma}))_{0\le t\le T}$ and $(A^{(2)}_t(\bar{\gamma}))_{0\le t\le T}$ which take values in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively, are given by

$$A^{(1)}_t(\bar{\gamma}) = -\int_0^t \frac{d\langle M^{(1)}, \bar{Z}^0\rangle_s}{\bar{\gamma} - \bar{\gamma}_0 + \bar{Z}^0_s} \quad \text{and} \quad A^{(2)}_t(\bar{\gamma}) = -\int_0^t \frac{d\langle M^{(2)}(\bar{\gamma}), \bar{Z}^0\rangle_s}{\bar{\gamma} - \bar{\gamma}_0 + \bar{Z}^0_s} \tag{4.12}$$

for $0 \le t \le T$.

Proof. The decomposition (4.8) for $S^{(1)}$, where $A^{(1)}(\bar{\gamma})$ is given by (4.12), follows directly by plugging (4.7) into (4.5). Likewise, plugging the decomposition $\bar{H}(\bar{\gamma}) = \bar{\gamma} - \bar{\gamma}_0 + \bar{\gamma}_0 - \bar{\Xi}$ together with (4.4) and (4.7) into (4.6) yields

$$S^{(2)}_t(\bar{\gamma}) = \frac{(\bar{\gamma} - \bar{\gamma}_0)M^{D,(2)}_t + \bar{\Pi}_t}{\bar{\gamma} - \bar{\gamma}_0 + \bar{Z}^0_t}, \quad 0 \le t \le T.$$

Setting $t = 0$ yields (4.10). By Itô's formula, we obtain the semimartingale decomposition

$$
\begin{aligned}
dS_t^{(2)}(\bar\gamma) = {} & \frac{(\bar\gamma-\bar\gamma_0)dM_t^{D,(2)} + d\bar\Pi_t}{\bar\gamma-\bar\gamma_0+\bar Z_t^0} - \frac{((\bar\gamma-\bar\gamma_0)M_t^{D,(2)}+\bar\Pi_t)d\bar Z_t^0}{(\bar\gamma-\bar\gamma_0+\bar Z_t^0)^2} \\
& - \frac{d\langle(\bar\gamma-\bar\gamma_0)M^{D,(2)}+\bar\Pi,\bar Z^0\rangle_t}{(\bar\gamma-\bar\gamma_0+\bar Z_t^0)^2} + \frac{((\bar\gamma-\bar\gamma_0)M_t^{D,(2)}+\bar\Pi_t)d\langle\bar Z^0\rangle_t}{(\bar\gamma-\bar\gamma_0+\bar Z_t^0)^3}
\end{aligned}
\tag{4.13}
$$

for $0 \le t \le T$. By separating the local martingale and finite variation parts, we obtain (4.11). Moreover, the finite variation part is given by

$$
\begin{aligned}
dA_t^{(2)}(\bar\gamma) &= -\frac{d\langle(\bar\gamma-\bar\gamma_0)M^{D,(2)}+\bar\Pi,\bar Z^0\rangle_t}{(\bar\gamma-\bar\gamma_0+\bar Z_t^0)^2} + \frac{((\bar\gamma-\bar\gamma_0)M_t^{D,(2)}+\bar\Pi_t)d\langle\bar Z^0\rangle_t}{(\bar\gamma-\bar\gamma_0+\bar Z_t^0)^3} \\
&= -\frac{d\langle M^{(2)}(\bar\gamma),\bar Z^0\rangle_t}{\bar\gamma-\bar\gamma_0+\bar Z_t^0},
\end{aligned}
$$

which yields the second part of (4.12), and then (4.9) follows from (4.13). □

The decompositions (4.8) and (4.9) already let us show the continuity of ψ_1.

Lemma 4.4. *The map $\psi_1 : (\bar\gamma_0,\infty) \to \mathcal{S}(\mathbb{R}^{d_1+d_2})$, $\bar\gamma \mapsto S(\bar\gamma)$ is continuous with respect to the semimartingale topology on $\mathcal{S}(\mathbb{R}^{d_1+d_2})$.*

Proof. Fix a sequence $(\bar\gamma_n)_{n\in\mathbb{N}}$ in $(\bar\gamma_0,\infty)$ with $\lim_{n\to\infty}\bar\gamma_n = \bar\gamma_\infty > \bar\gamma_0$. By Lemma 4.3, it suffices to show that $S_0^{(2)}(\bar\gamma_n) \to S_0^{(2)}(\bar\gamma_\infty)$, $A^{(1)}(\bar\gamma_n) \overset{\mathcal{S}}{\to} A^{(1)}(\bar\gamma_\infty)$, $M^{(2)}(\bar\gamma_n) \overset{\mathcal{S}}{\to} M^{(2)}(\bar\gamma_\infty)$ and $A^{(2)}(\bar\gamma_n) \overset{\mathcal{S}}{\to} A^{(2)}(\bar\gamma_\infty)$ as $n\to\infty$. The first convergence follows from (4.10) because $\bar\gamma_* := \inf_{n\in\mathbb{N}}\bar\gamma_n > \bar\gamma_0$ and $\bar Z^0 \ge 0$, so that

$$
S_0^{(2)}(\bar\gamma_n) = \frac{(\bar\gamma_n-\bar\gamma_0)M_0^{D,(2)}+\bar\Pi_0}{\bar\gamma_n-\bar\gamma_0+\bar Z_0^0} \longrightarrow \frac{(\bar\gamma_\infty-\bar\gamma_0)M_0^{D,(2)}+\bar\Pi_0}{\bar\gamma_\infty-\bar\gamma_0+\bar Z_0^0} = S_0^{(2)}(\bar\gamma_\infty).
$$

Using (4.12) together with the elementary inequality $|\frac{1}{a}-\frac{1}{b}| \le |a-b|/(\bar\gamma_*-\bar\gamma_0)^2$ for $a,b \ge \bar\gamma_*-\bar\gamma_0 > 0$ yields

$$
\begin{aligned}
& \sup_{\substack{\eta\in\mathcal{P}(\mathbb{R}^n)\\ \|\eta\|_\infty\le 1}} \left| \left(\eta\bullet\left(A^{(1)}(\bar\gamma_n)-A^{(1)}(\bar\gamma_\infty)\right)\right)_T \right| \\
& \le \sup_{\substack{\eta\in\mathcal{P}(\mathbb{R}^n)\\ \|\eta\|_\infty\le 1}} \int_0^T \left| \frac{1}{\bar\gamma_n-\bar\gamma_0+\bar Z_s^0} - \frac{1}{\bar\gamma_\infty-\bar\gamma_0+\bar Z_s^0} \right| |\eta_s||d\langle M^{(1)},\bar Z^0\rangle_s| \\
& \le \frac{|\bar\gamma_n-\bar\gamma_\infty|}{(\bar\gamma_*-\bar\gamma_0)^2} \int_0^T |d\langle M^{(1)},\bar Z^0\rangle_s| \longrightarrow 0 \quad P\text{-a.s. as } n\to\infty
\end{aligned}
$$

so that $A^{(1)}(\bar\gamma_n) \overset{\mathcal{S}}{\to} A^{(1)}(\bar\gamma_\infty)$. For the martingale parts, we decompose (4.11) in the form

$$M^{(2)}(\bar\gamma_n) - M^{(2)}(\bar\gamma_\infty) = X^{1,n} + X^{2,n} + X^{3,n}, \tag{4.14}$$

where $X^{1,n}, X^{2,n}, X^{3,n} \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{R}^{d_2})$ are defined by

$$\begin{aligned}
X^{1,n} &:= \left(\frac{\bar\gamma_n - \bar\gamma_0}{\bar\gamma_n - \bar\gamma_0 + \bar Z^0} - \frac{\bar\gamma_\infty - \bar\gamma_0}{\bar\gamma_\infty - \bar\gamma_0 + \bar Z^0}\right) \bullet M^{D,(2)},\\
X^{2,n} &:= \left(\frac{1}{\bar\gamma_n - \bar\gamma_0 + \bar Z^0} - \frac{1}{\bar\gamma_\infty - \bar\gamma_0 + \bar Z^0}\right) \bullet \bar\Pi,\\
X^{3,n} &:= \left(\frac{(\bar\gamma_n - \bar\gamma_0)M^{D,(2)} + \bar\Pi}{(\bar\gamma_n - \bar\gamma_0 + \bar Z^0)^2} - \frac{(\bar\gamma_\infty - \bar\gamma_0)M^{D,(2)} + \bar\Pi}{(\bar\gamma_\infty - \bar\gamma_0 + \bar Z^0)^2}\right) \bullet \bar Z^0.
\end{aligned}$$

Let $\bar\gamma^* := \sup_{n\in\mathbb{N}} \bar\gamma_n < \infty$. For each n, we have the inequality

$$\frac{|(\bar\gamma_n - \bar\gamma_0)M^{D,(2)}_s + \bar\Pi_s|}{(\bar\gamma_n - \bar\gamma_0 + \bar Z^0_s)^2} \le \frac{(\bar\gamma^* - \bar\gamma_0)|M^{D,(2)}_s| + |\bar\Pi_s|}{(\bar\gamma_* - \bar\gamma_0 + \bar Z^0_s)^2} =: \varphi_s, \quad 0 \le s \le T,$$

where $(\varphi_t)_{0\le t\le T}$ is continuous P-a.s., and hence $\int_0^T \varphi_s^2 d\langle \bar Z^0\rangle_s < \infty$ P-a.s. Thus the dominated convergence theorem yields that

$$\langle X^{3,n}\rangle_T = \int_0^T \left(\frac{(\bar\gamma_n - \bar\gamma_0)M^{D,(2)}_s + \bar\Pi_s}{(\bar\gamma_n - \bar\gamma_0 + \bar Z^0_s)^2} - \frac{(\bar\gamma_\infty - \bar\gamma_0)M^{D,(2)}_s + \bar\Pi_s}{(\bar\gamma_\infty - \bar\gamma_0 + \bar Z^0_s)^2}\right)^2 d\langle \bar Z^0\rangle_s \longrightarrow 0$$

P-a.s. as $n \to \infty$. We obtain by similar arguments that $\langle X^{1,n}\rangle_T, \langle X^{2,n}\rangle_T \to 0$ P-a.s. as $n \to \infty$. Hence (4.14) and the Cauchy–Schwarz inequality yield

$$\langle M^{(2)}(\bar\gamma_n) - M^{(2)}(\bar\gamma_\infty)\rangle_T \longrightarrow 0 \quad P\text{-a.s.} \tag{4.15}$$

as $n \to \infty$. By Émery [47, Lemma 6], this implies that $M^{(2)}(\bar\gamma_n) \xrightarrow{\mathcal{S}} M^{(2)}(\bar\gamma_\infty)$.

Finally, note that (4.12), the Kunita–Watanabe inequality and (4.15) yield

$$\begin{aligned}
&\sup_{\substack{\eta\in\mathcal{P}(\mathbb{R}^n)\\ \|\eta\|_\infty\le 1}} \left|\Big(\eta \bullet \big(A^{(2)}(\bar\gamma_n) - A^{(2)}(\bar\gamma_\infty)\big)\Big)_T\right|\\
&\le \int_0^T \left|\frac{1}{\bar\gamma_n - \bar\gamma_0 + \bar Z^0_s} - \frac{1}{\bar\gamma_\infty - \bar\gamma_0 + \bar Z^0_s}\right| |d\langle M^{(2)}(\bar\gamma_\infty), \bar Z^0\rangle_s|\\
&\quad + \int_0^T \left|\frac{1}{\bar\gamma_n - \bar\gamma_0 + \bar Z^0_s}\right| |d\langle M^{(2)}(\bar\gamma_n) - M^{(2)}(\bar\gamma_\infty), \bar Z^0\rangle_s|\\
&\le \frac{|\bar\gamma_n - \bar\gamma_\infty|}{(\bar\gamma_* - \bar\gamma_0)^2}\int_0^T |d\langle M^{(2)}(\bar\gamma_\infty), \bar Z^0\rangle_s| + \frac{\langle M^{(2)}(\bar\gamma_n) - M^{(2)}(\bar\gamma_\infty)\rangle_T^{1/2}\langle \bar Z^0\rangle_T^{1/2}}{\bar\gamma_* - \bar\gamma_0} \longrightarrow 0
\end{aligned}$$

P-a.s. as $n \to \infty$, so that $A^{(2)}(\bar\gamma_n) \xrightarrow{\mathcal{S}} A^{(2)}(\bar\gamma_\infty)$. This concludes the proof. □

Lemma 4.4 gives a continuity result for ψ_1. However, it is insufficient for our original purpose of showing the continuity of $\psi_2 \circ \psi_1$, since ψ_2 is in general not continuous with respect to the semimartingale topology.

Example 4.5. Recall from (2.45) that $\ell(S) = \varepsilon^2(1; S)$ is the first component of $\psi_2(S)$. Consider the sequence of semimartingales $(S^n)_{n\in\mathbb{N}}$ on $[0, T]$ defined by $S^n_t = (t + W_t)/n$ so that $S^n \xrightarrow{\mathcal{S}} S^\infty \equiv 0$ as $n \to \infty$. For $n \in \mathbb{N} \cup \{\infty\}$, let $\ell_n := \varepsilon^2(1; S^n) \in (0, 1]$. Then we have $\ell_\infty = 1$ as argued after (2.16) since S^∞ is a martingale. On the other hand, $\ell_1 = \ell_2 = \cdots \neq 1$ since S^1 is not a local martingale and the set of wealth processes attainable by trading S^n is the same for all $n \in \mathbb{N}$. Hence we cannot have $\psi_2(S^n) \to \psi_2(S^\infty)$ as $n \to \infty$.

In view of the counterexample above, we need further insight into the structure of $S(\bar\gamma)$ and its dependence on $\bar\gamma$ in order to prove the continuity of $\psi_2 \circ \psi_1$. First, we rewrite the martingale decomposition in Lemma 4.3 more explicitly in terms of $\bar M$. This already gives the first output $\xi(\bar\gamma)$ of the map $\tilde\psi_1$ that we define below.

Lemma 4.6. *For each $\bar\gamma > \bar\gamma_0$, there exists a unique (up to $\bar M$-equivalence) process $\xi(\bar\gamma) \in L^2_{\mathrm{loc}}(\bar M; \mathbb{R}^{\bar d\times(d_1+d_2)})$ such that*

$$M(\bar\gamma) := \big(M^{(1)}, M^{(2)}(\bar\gamma)\big) = \xi(\bar\gamma)\bullet \bar M. \tag{4.16}$$

Moreover, the family $\{\xi(\bar\gamma) : \bar\gamma > \bar\gamma_0\}$ can be chosen such that

$$P\Big[\lim_{\bar\gamma'\to\bar\gamma} \xi_t(\bar\gamma') = \xi_t(\bar\gamma) \text{ for all } 0 \le t \le T \text{ and } \bar\gamma > \bar\gamma_0\Big] = 1, \tag{4.17}$$

$$P[\xi_t(\bar\gamma) \text{ has full rank for all } 0 \le t \le T \text{ and } \bar\gamma > \bar\gamma_0] = 1. \tag{4.18}$$

Proof. We construct $(\xi_t(\bar\gamma))_{0\le t\le T}$ explicitly by setting each entry $(\xi_t(\bar\gamma))_{ij}$ to

$$\begin{cases} 1, & 1 \le i = j \le d_1,\\ (\bar\gamma - \bar\gamma_0)/\bar Z_t(\bar\gamma), & d_1 + 1 \le i = j \le d_1 + d_2,\\ 1/\bar Z_t(\bar\gamma), & d_1 + d_2 + 1 \le i \le d_1 + 2d_2,\ j = i - d_2,\\ -\big((\bar\gamma - \bar\gamma_0)(M^{D,(2)}_t)_{i-d_1} + (\bar\Pi_t)_{i-d_1})/(\bar Z_t(\bar\gamma)\big)^2, & i = \bar d,\ d_1 + 1 \le j \le d_1 + d_2,\\ 0, & \text{otherwise.} \end{cases} \tag{4.19}$$

Visually, $\xi_t(\bar\gamma)$ can be represented as a $\bar d \times (d_1 + d_2)$ block matrix of the shape $\begin{pmatrix} A & 0\\ 0 & B\\ 0 & C\\ 0 & v\end{pmatrix}$, where $A \in \mathbb{R}^{d_1\times d_1}$ and $B, C \in \mathbb{R}^{d_2\times d_2}$ are diagonal matrices and $v \in \mathbb{R}^{1\times d_2}$ is a row vector. Then (4.16) follows directly by plugging into (4.11); the fact that

$\xi(\bar\gamma)$ satisfies (4.16) immediately implies the uniqueness up to $\bar M$-equivalence. Because $\bar Z^0$ is nonnegative and continuous, we have $P[E]=1$, where

$$E := \{\bar Z^0_t \geq 0 \text{ for all } 0 \leq t \leq T\}.$$

On the set E, we have that $(\bar Z_t(\bar\gamma))^{-1} = (\bar\gamma - \bar\gamma_0 + \bar Z^0_t)^{-1}$ is well defined for all $t \in [0,T]$ and $\bar\gamma > \bar\gamma_0$. Hence by (4.7) and the construction (4.19), it is clear that $\lim_{\bar\gamma' \to \bar\gamma} Z_t(\bar\gamma') = Z_t(\bar\gamma)$ and $\lim_{\bar\gamma' \to \bar\gamma} \xi_t(\bar\gamma') = \xi_t(\bar\gamma)$ for all $0 \leq t \leq T$ and $\bar\gamma > \bar\gamma_0$ on E; this shows (4.17). We also have on E that the first $d_1 + d_2$ rows of $\xi_t(\bar\gamma)$ are linearly independent for all $t \in [0,T]$ and $\bar\gamma > \bar\gamma_0$, since the submatrix formed by these columns has nonzero entries along the diagonal and null entries elsewhere. Thus (4.18) holds. □

The next step is to study the ELMM $Q(\bar\gamma) \approx P$ for $S(\bar\gamma)$ given by Lemma 2.25. Namely, we show that $Q(\bar\gamma)$ satisfies the so-called reverse Hölder inequality $R_2(P)$. The following definitions are given in Delbaen et al. [34, Definitions 2.8 and 2.11]; note that [34, Definitions 2.8 and 2.9] are equivalent in our setup since all martingales are continuous due to Assumption 4.2.

Definition 4.7. We say that a martingale $M \in \mathcal{M}^2(\mathbb{R})$ belongs to BMO if there exists a constant $C_M \geq 0$ such that

$$E[\langle M\rangle_T - \langle M\rangle_\tau \mid \mathcal{F}_\tau] \leq C_M$$

for every stopping time τ that takes values in $[0,T]$. We write $\|M\|_{BMO} := \sqrt{C^*_M}$, where C^*_M is the infimum of all such constants C_M. The set of BMO martingales null at 0 is a Banach space with norm $\|\cdot\|_{BMO}$ (see Protter [102, Section IV.4]), whereas the latter is only a seminorm on the space of all BMO martingales.

Definition 4.8. We say that a strictly positive martingale $Z \in \mathcal{M}^2(\mathbb{R})$ satisfies the *reverse Hölder inequality* $R_2(P)$ if there exists a constant $C_Z \geq 1$ such that

$$E\left[\left(\frac{Z_T}{Z_t}\right)^2 \,\middle|\, \mathcal{F}_t\right] \leq C_Z, \quad 0 \leq t \leq T. \tag{4.20}$$

Similarly, we say that an equivalent measure $Q \approx P$ satisfies the *reverse Hölder inequality* $R_2(P)$ if the density process $(Z^Q)_{0\leq t\leq T}$ of Q satisfies $R_2(P)$.

Lemma 4.9. *For each $\bar\gamma > \bar\gamma_0$, the process $(\bar Z_t(\bar\gamma))_{t\in[0,T]}$ satisfies $R_2(P)$ and the* *stochastic logarithm* $(\bar{N}_t(\bar{\gamma}))_{0\le t\le T}$ *defined by*

$$\bar{N}(\bar{\gamma}) := \frac{1}{\bar{Z}(\bar{\gamma})} \bullet \bar{Z}(\bar{\gamma}) \tag{4.21}$$

belongs to BMO. *Moreover, for each* $\bar{\gamma}_* > \bar{\gamma}_0$, *the family* $(\bar{N}(\bar{\gamma}))_{\bar{\gamma}\ge\bar{\gamma}_*}$ *is bounded in* BMO *and there exists some constant* $C = C_{\bar{\gamma}_*} > 0$ *such that for every* $\bar{\gamma} \ge \bar{\gamma}_*$, *the process* $\bar{Z}(\bar{\gamma})$ *satisfies* $R_2(P)$ *with constant* C *on the right-hand side of* (4.20).

Proof. Since $0 \le \bar{Z}^0 \le \bar{\gamma}_0$ due to Assumption 2.1 and the definitions (2.35) and (4.4) of $\bar{\gamma}_0$ and $\bar{Z}^0$, we have by (4.7) that $\bar{\gamma} - \bar{\gamma}_0 \le \bar{Z}(\bar{\gamma}) \le \bar{\gamma}$ and hence

$$E\left[\left(\frac{\bar{Z}_T(\bar{\gamma})}{\bar{Z}_t(\bar{\gamma})}\right)^2 \,\middle|\, \mathcal{F}_t\right] \le \frac{\bar{\gamma}^2}{(\bar{\gamma}-\bar{\gamma}_0)^2} < \infty, \quad 0 \le t \le T,$$

so that $\bar{Z}(\bar{\gamma})$ satisfies $R_2(P)$. Since the function $\bar{\gamma} \mapsto \bar{\gamma}^2/(\bar{\gamma}-\bar{\gamma}_0)^2$ is decreasing on $(\bar{\gamma}_0, \infty)$, we can set $C_{\bar{\gamma}_*} = \bar{\gamma}_*^2/(\bar{\gamma}_* - \bar{\gamma}_0)^2$ so that for every $\bar{\gamma} \ge \bar{\gamma}_*$, $\bar{Z}(\bar{\gamma})$ satisfies (4.20) with the constant $C_{\bar{\gamma}_*}$. Thus by Delbaen et al. [34, Lemma 4.2], the stochastic logarithm $\bar{N}(\bar{\gamma})$ belongs to BMO. We can rewrite (4.21) as

$$\bar{N}(\bar{\gamma}) = \frac{1}{\bar{Z}(\bar{\gamma})} \bullet \bar{Z}(\bar{\gamma}) = \frac{1}{\bar{\gamma}-\bar{\gamma}_0+\bar{Z}^0} \bullet (\bar{\gamma}-\bar{\gamma}_0+\bar{Z}^0) = \frac{1}{\bar{\gamma}-\bar{\gamma}_0+\bar{Z}^0} \bullet \bar{Z}^0. \tag{4.22}$$

Thus for $0 \le s \le t \le T$ and $\bar{\gamma}' \le \bar{\gamma}$, we have

$$\begin{aligned}\langle N(\bar{\gamma})\rangle_t - \langle N(\bar{\gamma})\rangle_s &= \int_s^t \frac{1}{(\bar{\gamma}-\bar{\gamma}_0+\bar{Z}^0_u)^2} d\langle \bar{Z}^0\rangle_u \\ &\le \int_s^t \frac{1}{(\bar{\gamma}'-\bar{\gamma}_0+\bar{Z}^0_u)^2} d\langle \bar{Z}^0\rangle_u = \langle N(\bar{\gamma}')\rangle_t - \langle N(\bar{\gamma}')\rangle_s. \end{aligned} \tag{4.23}$$

Thus for any $\bar{\gamma} \ge \bar{\gamma}_* > \bar{\gamma}_0$, the increments of $\langle N(\bar{\gamma})\rangle$ are bounded by those of $\langle N(\bar{\gamma}_*)\rangle$ so that $\|N(\bar{\gamma})\|_{BMO} \le \|N(\bar{\gamma}_*)\|_{BMO}$ and $(\bar{N}(\bar{\gamma}))_{\bar{\gamma}\ge\bar{\gamma}_*}$ is bounded in BMO. □

The following lemma is well known and helps to explain why the reverse Hölder inequality $R_2(P)$ for $Q(\bar{\gamma})$ gives important information about $S(\bar{\gamma})$: it implies that the sets $\Theta(S(\bar{\gamma}))$ and $\overline{\Theta}(S(\bar{\gamma}))$ of admissible strategies of Schweizer [109] and Černý/Kallsen [25], respectively, coincide in this case. Recall that the latter is the one we use throughout this chapter; see Definition 2.3.

Lemma 4.10. *Let* $(S_t)_{0\le t\le T}$ *be a continuous semimartingale with canonical decomposition* $S = S_0 + M + A$, *where* M *is a local martingale and* A *a finite-*

variation process. Define the set $\Theta(S) := L^2(M) \cap L^2(A)$. *If there exists an equivalent local martingale measure* Q *for* S *that satisfies* $R_2(P)$, *then* $\Theta(S) = \overline{\Theta}(S)$ *and the set of attainable gains* $\mathcal{G}_T(\Theta(S)) := \{\vartheta \bullet S_T : \vartheta \in \Theta(S)\}$ *is closed in* L^2.

Proof. Because Q satisfies $R_2(P)$, the implication (2) $\Rightarrow$ (1) in Delbaen et al. [34, Theorem 4.1] gives that $\mathcal{G}_T(\Theta(S))$ is closed in L^2. Thus by Černý/Kallsen [25, Corollary 2.9], $\mathcal{G}_T(\Theta(S)) = \mathcal{G}_T(\overline{\Theta}(S))$. By Proposition III.2.12, the terminal gain $\vartheta \bullet S_T$ of a strategy $\vartheta \in \overline{\Theta}(S)$ uniquely determines ϑ up to S-equivalence. Thus the equality $\mathcal{G}_T(\Theta(S)) = \mathcal{G}_T(\overline{\Theta}(S))$ implies that $\Theta(S) = \overline{\Theta}(S)$. □

We are now ready to define the second output $\lambda(\bar{\gamma})$ of $\tilde{\psi}_1$ via the so-called (SC) decomposition (4.24) of $S(\bar{\gamma})$. We prove the existence of such a decomposition by using Lemma 4.9 together with results of Delbaen et al. [34].

Lemma 4.11. *For each* $\bar{\gamma} > \bar{\gamma}_0$, *there exists a process* $\lambda(\bar{\gamma}) \in L^2(M(\bar{\gamma}); \mathbb{R}^{d_1+d_2})$ *(which is unique up to* $M(\bar{\gamma})$*-equivalence) such that*

$$S_t(\bar{\gamma}) = S_0(\bar{\gamma}) + \int_0^t \big(d\langle M(\bar{\gamma})\rangle_s \lambda_s(\bar{\gamma}) + dM(\bar{\gamma})_s\big) \tag{4.24}$$

$$= S_0(\bar{\gamma}) + \int_0^t \big(\xi_s^\top(\bar{\gamma}) d\langle \bar{M}\rangle_s \xi_s(\bar{\gamma})\lambda_s(\bar{\gamma}) + \xi_s(\bar{\gamma}) d\bar{M}_s\big), \quad 0 \le t \le T. \tag{4.25}$$

Proof. Since $Q(\bar{\gamma})$ has the density process $(Z_t(\bar{\gamma}))_{0\le t\le T}$ and $S(\bar{\gamma})$ is continuous, Lemmas 2.25 and 4.9 yield that $Q(\bar{\gamma})$ is an equivalent local martingale measure for $S(\bar{\gamma})$ and satisfies $R_2(P)$. Hence by Lemma 4.10, $\mathcal{G}_T(\Theta(S(\bar{\gamma})))$ is closed in L^2. Thus because $Q(\bar{\gamma})$ is an equivalent local martingale measure for $S(\bar{\gamma})$ with square-integrable density, we obtain from Delbaen et al. [34, Theorem 3.7] that $S(\bar{\gamma})$ satisfies the so-called inequality $D_2(P)$. Since the martingale part of $S(\bar{\gamma})$ is $M(\bar{\gamma}) = (M^{(1)}, M^{(2)}(\bar{\gamma}))$ by Lemma 4.3, we obtain from [34, Lemma 3.1] a process $\lambda(\bar{\gamma}) \in L^2(M(\bar{\gamma}); \mathbb{R}^{d_1+d_2})$ such that

$$A(\bar{\gamma}) = \int_0^\cdot d\langle M(\bar{\gamma})\rangle_s \lambda_s(\bar{\gamma}), \tag{4.26}$$

and hence the (SC) decomposition (4.24) for $S(\bar{\gamma})$ is satisfied. By plugging $M(\bar{\gamma}) = \xi(\bar{\gamma}) \bullet \bar{M}$ (see (4.16)) and (4.2) into (4.24), we obtain (4.25). To show the uniqueness of $\lambda(\bar{\gamma})$, note that due to (4.24), $\lambda(\bar{\gamma})$ is unique $P \otimes \langle M(\bar{\gamma})\rangle$-a.e. Since $M(\bar{\gamma})$ is a local martingale, this implies the uniqueness up to $M(\bar{\gamma})$-equivalence. □

We are finally ready to formally define the map $\tilde{\psi}_1$.

Definition 4.12. We define $\tilde{\psi}_1 : (\bar{\gamma}_0, \infty) \to L^2_{\mathrm{loc}}(\bar{M}; \mathbb{R}^{\bar{d}\times(d_1+d_2)}) \times \mathcal{P}(\mathbb{R}^{d_1+d_2})$ by

$$\tilde{\psi}_1(\bar{\gamma}) := \big(\xi(\bar{\gamma}), \lambda(\bar{\gamma})\big), \tag{4.27}$$

where $\xi(\bar{\gamma})$ and $\lambda(\bar{\gamma})$ are given in Lemmas 4.6 and 4.11.

We now want to show an integrability condition on $(\xi(\bar{\gamma}), \lambda(\bar{\gamma}))$, namely that $\lambda(\bar{\gamma}) \bullet M(\bar{\gamma}) = \lambda(\bar{\gamma}) \bullet (\xi(\bar{\gamma}) \bullet \bar{M})$ belongs to BMO. This follows by Lemma 4.9 together with the next (folklore) result which we state in a general form for later use, where we bound the increments of $\langle \lambda(\bar{\gamma}) \bullet M(\bar{\gamma})\rangle$ by those of $\langle \bar{N}(\bar{\gamma})\rangle$.

Lemma 4.13. *Let $(S_t)_{0\le t\le T}$ be a continuous semimartingale of the form*

$$S_t = S_0 + \int_0^t d\langle M\rangle_s \lambda_s + M_t, \quad 0 \le t \le T,$$

where M is a local martingale and $\lambda \in L^2_{\mathrm{loc}}(M)$. If there exists an equivalent local martingale measure Q for S with continuous density process $Z = \mathcal{E}(N)$ for some local martingale $(N_t)_{0\le t\le T}$, then the local martingales M and $N + \lambda \bullet M$ are strongly orthogonal, and it holds for all $0 \le s \le t \le T$ that

$$\langle \lambda \bullet M\rangle_t - \langle \lambda \bullet M\rangle_s \le \langle N\rangle_t - \langle N\rangle_s. \tag{4.28}$$

Proof. Because Q is an equivalent local martingale measure for S, we obtain by Girsanov's theorem that

$$d\langle M, N\rangle_t = -d\langle M\rangle_t \lambda_t = -d\langle M, \lambda \bullet M\rangle_t, \quad 0 \le t \le T.$$

Hence we find $\langle M, N + \lambda \bullet M\rangle \equiv 0$, which shows the strong orthogonality. We thus have the orthogonal decomposition $N = -\lambda \bullet M + (N + \lambda \bullet M)$ which yields for all $0 \le s \le t \le T$ that

$$\langle \lambda \bullet M\rangle_t - \langle \lambda \bullet M\rangle_s + \langle N + \lambda \bullet M\rangle_t - \langle N + \lambda \bullet M\rangle_s = \langle N\rangle_t - \langle N\rangle_s,$$

and this implies (4.28). □

The next step is to study the continuity of $\lambda(\bar{\gamma})$ in $\bar{\gamma}$. For that, we consider the *semimartingale characteristics* of $\bar{M}$; see Jacod/Shiryaev [71, Definition II.2.6]. Since $\bar{M}$ is a continuous local martingale, only the characteristic $(C^{\bar{M}}_t)_{0\le t\le T}$ given by $C^{\bar{M}}_t = \langle \bar{M}\rangle_t$ is nonzero. Define the predictable increasing process $(I_t)_{0\le t\le T}$ by $I_t = \mathrm{tr}\langle \bar{M}\rangle_t$ for $0 \le t \le T$. We may and do choose versions of $C^{\bar{M}}$ and I

such that P-a.s., we have for all $0 \leq s \leq t \leq T$ that $C_t^{\bar{M}} - C_s^{\bar{M}}$ is symmetric and positive semidefinite with $\operatorname{tr}(C_t^{\bar{M}} - C_s^{\bar{M}}) = I_t - I_s$. Since $\|C\|_{\mathrm{op}} \leq \operatorname{tr} C$ for any symmetric positive semidefinite matrix C, where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm on $\mathbb{R}^{\bar{d}\times\bar{d}}$, we obtain

$$P[\|C_t^{\bar{M}} - C_s^{\bar{M}}\|_{\mathrm{op}} \leq I_t - I_s \text{ for all } 0 \leq s < t \leq T] = 1. \tag{4.29}$$

Hence there exists a Radon–Nikodým derivative for $C^{\bar{M}}$ with respect to I, i.e., a predictable process $(c_t^{\bar{M}})_{0\leq t\leq T}$ with values in the set of $\bar{d}\times\bar{d}$ symmetric positive semidefinite matrices such that

$$C_t^{\bar{M}} = \langle \bar{M}\rangle_t = \int_0^t c_s^{\bar{M}} dI_s, \quad 0 \leq t \leq T. \tag{4.30}$$

The choice of $c^{\bar{M}}$ is unique $P\otimes I$-a.e. Moreover, we have

$$\int_0^t dI_s = I_t = \operatorname{tr}\langle \bar{M}\rangle_t = \int_0^t (\operatorname{tr} c_s^{\bar{M}}) dI_s, \quad 0 \leq t \leq T,$$

so that $\operatorname{tr} c^{\bar{M}} = 1$ $P\otimes I$-a.e. Thus we may and do choose a version of $c^{\bar{M}}$ such that $P[\operatorname{tr} c_t^{\bar{M}} = 1 \text{ for all } t \in [0,T]] = 1$. The process $c^{\bar{M}}$ can be seen as a *differential characteristic* of $\bar{M}$ with respect to I; see Eberlein/Kallsen [44, Section 4.4]. We now show the continuity of the map $\bar{\gamma} \mapsto \lambda(\bar{\gamma})$ (in the sense of (4.31) below) under the assumption that $c^{\bar{M}}$ is $P\otimes I$-a.e. invertible.

Lemma 4.14. **1)** *Let $\bar{\gamma} > \bar{\gamma}_0$ and suppose that $\xi^\top(\bar{\gamma})c^{\bar{M}}\xi(\bar{\gamma})$ is $P\otimes I$-a.e. invertible. Then $\lambda(\bar{\gamma}') \to \lambda(\bar{\gamma})$ $P\otimes I$-a.e. as $\bar{\gamma}' \to \bar{\gamma}$ and*

$$\lim_{\bar{\gamma}'\to\bar{\gamma}} \big\langle \lambda(\bar{\gamma}')\bullet\big(\xi(\bar{\gamma}')\bullet\bar{M}\big) - \lambda(\bar{\gamma})\bullet\big(\xi(\bar{\gamma})\bullet\bar{M}\big)\big\rangle_T = 0 \quad P\text{-a.s.} \tag{4.31}$$

2) *If $c_t^{\bar{M}}$ is $P\otimes I$-a.e. invertible, then $\xi^\top(\bar{\gamma})c^{\bar{M}}\xi(\bar{\gamma})$ is $P\otimes I$-a.e. invertible for all $\bar{\gamma} > \bar{\gamma}_0$.*

Proof. **1)** Since $\langle \bar{M}, \bar{Z}^0\rangle \ll \langle \bar{M}\rangle \ll I$, there exists an I-integrable predictable process $(c_t^{\bar{M},\bar{Z}^0})_{0\leq t\leq T}$ such that $d\langle \bar{M}, \bar{Z}^0\rangle = c^{\bar{M},\bar{Z}^0} dI$. Thus (4.16) yields

$$d\langle \bar{M}(\bar{\gamma}), \bar{Z}^0\rangle = \xi^\top(\bar{\gamma})c^{\bar{M},\bar{Z}^0} dI$$

as well as $d\langle \bar{M}(\bar{\gamma})\rangle = \xi^\top(\bar{\gamma})c^{\bar{M}}\xi(\bar{\gamma})$. By plugging into (4.12), we get

$$dA_t(\bar{\gamma}) = -\frac{\xi_t^\top(\bar{\gamma})c_t^{\bar{M},\bar{Z}^0}dI_t}{\bar{\gamma}-\bar{\gamma}_0+\bar{Z}_t^0}, \quad 0 \le t \le T.$$

It thus follows from (4.26), (4.16) and the invertibility assumption that

$$\lambda(\bar{\gamma}) = -\big(\xi^\top(\bar{\gamma})c^{\bar{M}}\xi(\bar{\gamma})\big)^{-1}\frac{\xi^\top(\bar{\gamma})c^{\bar{M},\bar{Z}^0}}{\bar{\gamma}-\bar{\gamma}_0+\bar{Z}^0} \quad P\otimes I\text{-a.e.} \tag{4.32}$$

Fix now $\bar{\gamma} > \bar{\gamma}_0$. Note that by (4.24), we have $\xi_t(\bar{\gamma}') \to \xi_t(\bar{\gamma})$ as $\bar{\gamma}' \to \bar{\gamma}$ for all $0 \le t \le T$ P-a.s. For any sequence of square matrices $(C_n)_{n\in\mathbb{N}}$ that converge to an invertible limit C, it holds that $C_n^\dagger \to C^\dagger = C^{-1}$ (see e.g. Stewart [114, Equation (1.5)]), where $C_n^\dagger$ denotes the Moore–Penrose inverse of C_n; see Albert [6, Chapter III] for the definition and basic properties. Hence it follows from (4.32) and the invertibility of $\xi^\top(\bar{\gamma})c^{\bar{M}}\xi(\bar{\gamma})$ that

$$\lambda(\bar{\gamma}') \longrightarrow \lambda(\bar{\gamma}) \quad P\otimes I\text{-a.e. as } \bar{\gamma}' \to \bar{\gamma}. \tag{4.33}$$

Since $\bar{Z}^0$ is a component of $\bar{M}$ so that $\langle \bar{Z}^0\rangle \ll I$, there exists a predictable process $(c_t^{\bar{Z}^0})_{0\le t\le T}$ such that $d\langle \bar{Z}^0\rangle = c^{\bar{Z}^0}dI$. Indeed, $c^{\bar{Z}^0}$ can be taken to be the $(\bar{d},\bar{d})$-entry of $c^{\bar{M}}$. Now fix some $\bar{\gamma}_* \in (\bar{\gamma}_0,\bar{\gamma})$ and recall $\bar{N}(\bar{\gamma}_*) = \frac{1}{\bar{Z}(\bar{\gamma}_*)} \bullet \bar{Z}(\bar{\gamma}_*)$ as in (4.21). Then by applying the inequalities (4.23) and (4.28) (for the latter with $\lambda = \lambda(\bar{\gamma}_*)$, $M = M(\bar{\gamma}_*) = \lambda(\bar{\gamma}_*) \bullet (\xi(\bar{\gamma}_*) \bullet \bar{M})$, and $N = N(\bar{\gamma}_*)$), we obtain for $\bar{\gamma}' \ge \bar{\gamma}_*$ and $0 \le s \le t \le T$ that

$$\big\langle \big(\xi(\bar{\gamma}')\lambda(\bar{\gamma}')\big)\bullet\bar{M}\big\rangle_t - \big\langle \big(\xi(\bar{\gamma}')\lambda(\bar{\gamma}')\big)\bullet\bar{M}\big\rangle_s \le \langle \bar{N}(\bar{\gamma}_*)\rangle_t - \langle \bar{N}(\bar{\gamma}_*)\rangle_s.$$

Differentiating with respect to I and plugging in the dynamics (4.22) yields

$$0 \le \lambda^\top(\bar{\gamma}')\xi^\top(\bar{\gamma}')c^{\bar{M}}\xi(\bar{\gamma}')\lambda(\bar{\gamma}') \le \frac{c^{\bar{Z}^0}}{(\bar{\gamma}_*-\bar{\gamma}_0+\bar{Z}^0)^2} \quad P\otimes I\text{-a.e.} \tag{4.34}$$

for $\bar{\gamma}' \ge \bar{\gamma}_*$. Since we have

$$\int_0^T \frac{c_t^{\bar{Z}^0}}{(\bar{\gamma}_*-\bar{\gamma}_0+\bar{Z}_t^0)^2}dI_t \le \frac{\langle \bar{Z}^0\rangle_T}{(\bar{\gamma}_*-\bar{\gamma}_0)^2} < \infty,$$

it follows by (4.33), (4.34) and the dominated convergence theorem that

$$\int_0^T \big(\lambda_t^\top(\bar\gamma')\xi_t^\top(\bar\gamma') - \lambda_t^\top(\bar\gamma)\xi_t^\top(\bar\gamma)\big)c_t^{\bar M}\big(\xi_t(\bar\gamma')\lambda_t(\bar\gamma') - \xi_t(\bar\gamma)\lambda_t(\bar\gamma)\big) \longrightarrow 0$$

P-a.s. as $\bar\gamma' \to \bar\gamma$; this shows (4.31).

2) Suppose that $c^{\bar M}$ is invertible and hence positive definite, $P \otimes I$-a.e. Thus on a set of full $P \otimes I$-measure, we have for all $x \in \mathbb{R}^{d_1+d_2}$ that

$$\xi^\top(\bar\gamma)c^{\bar M}\xi(\bar\gamma)x = 0 \quad\Longrightarrow\quad x^\top\xi^\top(\bar\gamma)c^{\bar M}\xi(\bar\gamma)x = 0 \quad\Longrightarrow\quad \xi(\bar\gamma)x = 0,$$

and hence $\ker \xi^\top(\bar\gamma)c^{\bar M}\xi(\bar\gamma) = \ker \xi(\bar\gamma)$ $P \otimes I$-a.e. On the other hand, $\xi_t(\bar\gamma)$ has full rank for all $0 \le t \le T$ P-a.s. by Lemma 4.6. Since $\xi(\bar\gamma)$ has dimensions $\bar d \times (d_1 + d_2)$ with $\bar d > d_1 + d_2$, it follows that

$$\ker \xi^\top(\bar\gamma)c^{\bar M}\xi(\bar\gamma) = \ker \xi(\bar\gamma) = \{0\} \quad P \otimes I\text{-a.e.}$$

Since $\xi^\top(\bar\gamma)c^{\bar M}\xi(\bar\gamma)$ is a square matrix, it is thus invertible $P \otimes I$-a.e. □

In order to apply part 2) of Lemma 4.14, we henceforth assume that $c^{\bar M}$ is invertible. This **standing assumption** will also be helpful later for showing the continuity of $\tilde\psi_2$; see Lemma 4.30 below.

Assumption 4.15. We suppose that $\{(\omega, t) : c_t^{\bar M}(\omega) \text{ is singular}\}$ is $P \otimes I$-null.

We can interpret Assumption 4.15 as a requirement that $d\langle \bar M\rangle$ have full rank, so that any integrand $\xi \in L^2_{\mathrm{loc}}(\bar M)$ satisfies $\xi \bullet \bar M = 0$ if and only if $\xi = 0$ $P \otimes I$-a.e. Like Assumption 4.2, this assumption is somewhat restrictive, and it remains an open question whether our main results still hold if it is removed or relaxed. In the case of a Brownian filtration, one can in principle ensure that Assumption 4.15 is satisfied by perturbing each component of $\bar M$ with an independent Brownian motion. We note that due to the definition of $\bar M$ (see (4.4)), this assumption depends only on the primitives. Assumption 4.15 (together with (4.18)) prevents degenerate situations where the components of the martingale parts of the of $S(\bar\gamma)$ become correlated for some value of $\bar\gamma > \bar\gamma_0$, which can lead to a discontinuity as in Example 4.5.

Under Assumptions 4.2 and 4.15, we have shown the continuity of the map $\bar\gamma \mapsto (\xi(\bar\gamma), \lambda(\bar\gamma))$ in the sense of (4.17) and (4.31). We now want to translate our results into more explicit statements on the range of continuity of $\tilde\psi_1$. We first define a set $\mathcal{D}(\bar M)$ that contains the range of $\tilde\psi_1$ (as we show below), and will also serve as the domain for the map $\tilde{\psi}_2$.

Definition 4.16. We define $\mathcal{D}(\bar{M}) \subseteq L^2_{\text{loc}}(\bar{M};\mathbb{R}^{\bar{d}\times(d_1+d_2)}) \times \mathcal{P}(\mathbb{R}^{d_1+d_2})$ as the set of pairs (ξ,λ) such that ξ has full rank $P\otimes I$-a.e., $\lambda \in L^2_{\text{loc}}(\xi\bullet\bar{M})$ and for some $S_0\in\mathbb{R}$, the process $(S_t(\xi,\lambda))_{0\le t\le T}$ defined by the (SC) decomposition

$$S_t(\xi,\lambda) = \int_0^t (\xi_s^\top d\langle\bar{M}\rangle_s\xi_s\lambda_s + \xi_s d\bar{M}_s), \quad 0\le t\le T \tag{4.35}$$

admits an equivalent local martingale measure $Q = Q(\xi,\lambda)$ that satisfies $R_2(P)$. For $C\ge 1$, we also define the set $\mathcal{D}_C(\bar{M})$ of pairs $(\xi,\lambda)\in\mathcal{D}(\bar{M})$ such that there exists at least one such measure $Q(\xi,\lambda)$ that satisfies $R_2(P)$ with constant C on the right-hand side of (4.20).

By comparing (4.25) and (4.35), we obtain

$$S(\bar{\gamma}) = S_0(\bar{\gamma}) + S\big(\xi(\bar{\gamma}),\lambda(\bar{\gamma})\big). \tag{4.36}$$

Thus up to the initial value $S_0(\bar{\gamma})$, (4.35) allows us to reconstruct the price process $\psi_1(\bar{\gamma}) = S(\bar{\gamma})$ in terms of $\tilde{\psi}_1(\bar{\gamma}) = (\xi(\bar{\gamma}),\lambda(\bar{\gamma}))$. We leave out $S_0(\bar{\gamma})$ in order to simplify the notation and because a constant shift in the asset prices does not affect the MVH and exMVH problems (2.14) and (2.15).

For later use, we show that any $(\xi,\lambda)\in\mathcal{D}(\bar{M})$ satisfies a BMO bound that depends only on the constant in the inequality $R_2(P)$ for $Q(\xi,\lambda)$; this is a folklore result on the (SC) decomposition.

Lemma 4.17. *For any $(\xi,\lambda)\in\mathcal{D}(\bar{M})$, we have $\lambda\bullet(\xi\bullet\bar{M}) = (\xi\lambda)\bullet\bar{M}\in BMO$. Moreover, there exists an increasing function $f:(0,\infty)\to(0,\infty)$ such that*

$$\|\lambda\bullet(\xi\bullet\bar{M})\|_{BMO}\le f(C), \tag{4.37}$$

where C is the constant on the right-hand side of the inequality $R_2(P)$ (4.20) for the density process $Z(\xi,\lambda)$ of $Q(\xi,\lambda)$.

Proof. Fix $(\xi,\lambda)\in\mathcal{D}(\bar{M})$ and write $S = S(\xi,\lambda)$; likewise for Q and Z. Let $(N_t)_{0\le t\le T}$ be the stochastic logarithm of Z. Then Delbaen et al. [34, Lemma 4.2] gives that N belongs to BMO, and an inspection of its proof reveals that $\|N\|_{BMO}$ is bounded by $f(C)$ for some increasing function $f:(0,\infty)\to(0,\infty)$. By (4.28) with $M = \xi\bullet\bar{M}$, the increments of $\langle\lambda\bullet(\xi\bullet\bar{M})\rangle$ are bounded by those of N, and this yields (4.37). □

We now collect our main results so far on the range and continuity of $\tilde{\psi}_1$.

Theorem 4.18. *Suppose that Assumptions 2.1, 4.2 and 4.15 hold. Then the map $\tilde{\psi}_1$ satisfies the following properties:*

1) *The range of $\tilde{\psi}_1$ is contained in $\mathcal{D}(\bar{M})$.*

2) *For any sequence $(\bar{\gamma}_n)_{n\in\mathbb{N}}$ in $(\bar{\gamma}_0,\infty)$ such that $\bar{\gamma}_n \to \bar{\gamma}_\infty > \bar{\gamma}_0$, it holds that*

(i) $P[\lim_{n\to\infty} \xi_t(\bar{\gamma}_n) = \xi_t(\bar{\gamma}_\infty)$ *for all* $0 \le t \le T] = 1$.

(ii) $\lim_{n\to\infty}\langle \lambda(\bar{\gamma}_n) \bullet (\xi(\bar{\gamma}_n) \bullet \bar{M}) - \lambda(\bar{\gamma}_\infty) \bullet (\xi(\bar{\gamma}_\infty) \bullet \bar{M})\rangle_T = 0 \quad P$*-a.s.*

3) *For each $\bar{\gamma}_* > \bar{\gamma}_0$, the family $\{\lambda(\bar{\gamma}) \bullet (\xi(\bar{\gamma}) \bullet \bar{M}) : \bar{\gamma} \ge \bar{\gamma}_*\}$ is bounded in BMO. Moreover, there exists a constant $C = C_{\bar{\gamma}_*} > 0$ such that $\tilde{\psi}_1(\bar{\gamma}) \in \mathcal{D}_C(\bar{M})$ for all $\bar{\gamma} \ge \bar{\gamma}_*$.*

Proof. **1)** This follows since $\xi(\bar{\gamma})$ has full rank by (4.18), $Q(\bar{\gamma})$ is an equivalent local martingale measure for $S(\bar{\gamma})$ with density process $\bar{Z}(\bar{\gamma})$ due to Lemma 2.25, and $\bar{Z}(\bar{\gamma})$ satisfies $R_2(P)$ by Lemma 4.9.

2) The assertions (i) and (ii) were shown in Lemmas 4.6 and 4.14, respectively.

3) For all $\bar{\gamma} \ge \bar{\gamma}_*$ and stopping times τ taking values in $[0,T]$, we have by (4.28) with $\lambda = \lambda(\bar{\gamma})$, $M = \xi(\bar{\gamma}) \bullet \bar{M}$ and $N = \bar{N}(\bar{\gamma})$ and Lemma 4.9 that

$$\begin{aligned}
&E[\langle \lambda(\bar{\gamma}) \bullet (\xi(\bar{\gamma}) \bullet \bar{M})\rangle_T - \langle \lambda(\bar{\gamma}) \bullet (\xi(\bar{\gamma}) \bullet \bar{M})\rangle_\tau \mid \mathcal{F}_\tau] \\
&\le E[\langle \bar{N}(\bar{\gamma})\rangle_T - \langle \bar{N}(\bar{\gamma})\rangle_\tau \mid \mathcal{F}_\tau] \\
&\le E[\langle \bar{N}(\bar{\gamma}_*)\rangle_T - \langle \bar{N}(\bar{\gamma}_*)\rangle_\tau \mid \mathcal{F}_\tau] \le \|\bar{N}(\bar{\gamma}_*)\|^2_{BMO} \quad P\text{-a.s.}
\end{aligned}$$

and hence $\|\lambda(\bar{\gamma}) \bullet (\xi(\bar{\gamma}) \bullet \bar{M})\|_{BMO} \le \|\bar{N}(\bar{\gamma}_*)\|_{BMO}$. Lemma 4.9 also gives that $Q(\bar{\gamma})$ satisfies $R_2(P)$ with constant $C_{\bar{\gamma}_*}$ for every $\bar{\gamma} \ge \bar{\gamma}_*$. Together with part 1), this yields $\tilde{\psi}_1(\bar{\gamma}) \in \mathcal{D}_C(\bar{M})$ for all $\bar{\gamma} \ge \bar{\gamma}_*$ and completes the proof of 3). □

We conclude the study of $\tilde{\psi}_1$ by restating part 2) of Theorem 4.18 more precisely as the statement that $\tilde{\psi}_1$ is continuous in an appropriate sense; for that purpose, we first need to define a topology on $\mathcal{D}(\bar{M})$. We do this by inducing a metric on $\mathcal{D}(\bar{M})$ via the function $f_{\mathcal{D}(\bar{M})} : \mathcal{D}(\bar{M}) \to L^2_{\mathrm{loc}}(\bar{M};\mathbb{R}^{\bar{d}\times(d_1+d_2)}) \times \mathcal{M}^2(\mathbb{R}^{d_1+d_2})$ given by

$$f_{\mathcal{D}(\bar{M})}(\xi,\lambda) := \big(\xi, \lambda \bullet (\xi \bullet \bar{M})\big). \tag{4.38}$$

By Lemma 4.17, $\lambda \bullet (\xi \bullet \bar{M}) \in BMO \subseteq \mathcal{M}^2$ for $(\xi,\lambda) \in \mathcal{D}(\bar{M})$ so that $f_{\mathcal{D}(\bar{M})}$ is well defined.

Definition 4.19. We endow the space $L^2_{\text{loc}}(\bar{M};\mathbb{R}^{(d_1+d_2)\times\bar{d}}) \times \mathcal{M}^2(\mathbb{R}^{d_1+d_2})$ with the metric d' defined by

$$d'\big((\xi^1,M^1),(\xi^2,M^2)\big) = E\left[\frac{\int_0^T (1\wedge \|\xi^1_t-\xi^2_t\|)dI_t}{I_T+1}\right] + E[\langle M^1-M^2\rangle]^{1/2} \quad (4.39)$$

for $(\xi^1,M^1),(\xi^2,M^2)\in L^2_{\text{loc}}(\bar{M};\mathbb{R}^{\bar{d}\times(d_1+d_2)})\times\mathcal{M}^2(\mathbb{R}^{d_1+d_2})$, where $\|\cdot\|$ denotes the Frobenius norm on $\mathbb{R}^{\bar{d}\times(d_1+d_2)}$. We endow $\mathcal{D}(\bar{M})$ with the pseudometric[1] $d_{\mathcal{D}(\bar{M})}$ induced by d' and $f_{\mathcal{D}(\bar{M})}$, i.e.,

$$d_{\mathcal{D}(\bar{M})}\big((\xi^1,\lambda^1),(\xi^2,\lambda^2)\big) = d'\big(f_{\mathcal{D}(\bar{M})}(\xi^1,\lambda^1), f_{\mathcal{D}(\bar{M})}(\xi^2,\lambda^2)\big) \quad (4.40)$$

for $(\xi^1,\lambda^1),(\xi^2,\lambda^2)\in\mathcal{D}(\bar{M})$, as well as the *pseudometric topology* on $\mathcal{D}(\bar{M})$ generated by the open balls

$$B_r(\xi^1,\lambda^1) = \big\{(\xi^2,\lambda^2)\in\mathcal{D}(\bar{M}) : d_{\mathcal{D}(\bar{M})}\big((\xi^1,\lambda^1),(\xi^2,\lambda^2)\big) < r\big\}$$

for $r>0$ and $(\xi^1,\lambda^1)\in\mathcal{D}(\bar{M})$.

We note that d' is indeed a metric, since it is a combination of the $\mathcal{M}^2$-norm and the $L^0(P\otimes I)$-metric on $L^2_{\text{loc}}(\bar{M};\mathbb{R}^{\bar{d}\times(d_1+d_2)})$, where we include the factor $(I_T+1)^{-1}$ to ensure finiteness. Thus it is clear by the construction that $d_{\mathcal{D}(\bar{M})}$ is a pseudometric. In general, $d_{\mathcal{D}(\bar{M})}$ is not a true metric because $f_{\mathcal{D}(\bar{M})}$ is not injective on $\mathcal{D}(\bar{M})$.

Remark 4.20. One could convert $d_{\mathcal{D}(\bar{M})}$ into a true metric on the quotient space of $\mathcal{D}(\bar{M})$ with respect to $\xi\bullet\bar{M}$-equivalence, i.e., where $(\xi^1,\lambda^1)\sim(\xi^2,\lambda^2)$ if and only if $\xi^1 =_{\bar{M}} \xi^2$ and $\lambda^1 =_{\xi^1\bullet\bar{M}} \lambda^2$. Indeed, the map induced by $f_{\mathcal{D}(\bar{M})}$ is injective on that space, and we are only interested in λ only up to $\xi\bullet\bar{M}$-equivalence (see Lemma 4.13).

Remark 4.21. Replacing $\mathcal{M}^2$ and its associated norm in Definition 4.19 with BMO and $\|\cdot\|_{BMO}$ would result in a stronger topology on $\mathcal{D}(\bar{M})$. However, in that case, the continuity of $\tilde{\psi}_1$ would not follow from part 2) of Theorem 4.18, even with the uniform BMO bound given by part 3), as illustrated in the following example.

Example 4.22. Let $T=2$ and X be an unbounded (but finite) nonnegative random variable that is $\mathcal{F}_1$-measurable. Consider a Brownian motion $(W_t)_{0\le t\le 2}$

[1]We say that a map $d: A\times A\to\mathbb{R}_+$ on a set A is a *pseudometric* if it satisfies all of the axioms for a metric except positivity, i.e., there may exist $x\neq y$ such that $d(x,y)=0$.

and the family of martingales $(N^n)_{n\in\mathbb{N}}$ on $[0,2]$ defined by

$$N_t^n := \mathbf{1}_{\{X\geq n\}}\mathbf{1}_{[1,2]}(t)(W_t - W_1), \quad 0 \leq t \leq 2.$$

Then $E[\langle N^n\rangle_2] = P[X \geq n] \to 0$ as $n \to \infty$, so that $N^n \to 0$ in $\mathcal{M}^2$ as $n \to \infty$. On the other hand, we have

$$1 \geq \|N^n\|_{BMO} \geq \operatorname{ess\,sup} E[\langle N^n\rangle_2 - \langle N^n\rangle_1 \mid \mathcal{F}_1] = \operatorname{ess\,sup} \mathbf{1}_{\{X\geq n\}} = 1,$$

i.e., $\|N^n\|_{BMO} = 1$ for each $n \in \mathbb{N}$. Thus the sequence $(N^n)_{n\in\mathbb{N}}$ is bounded in BMO, but $N^n \not\to 0$ in BMO as $n \to \infty$.

Now that we have defined a pseudometric and a topology on $\mathcal{D}(\bar{M})$, we can restate parts 1) and 2) of Theorem 4.18 as follows.

Corollary 4.23. *Suppose that Assumptions 2.1, 4.2 and 4.15 hold. Then the map $\tilde{\psi}_1 : (\bar{\gamma}_0, \infty) \to \mathcal{D}(\bar{M})$ is continuous.*

Proof. By part 1) of Theorem 4.18, the range of $\tilde{\psi}_1$ is contained in $\mathcal{D}(\bar{M})$. To show the continuity, consider a sequence $(\bar{\gamma}_n)_{n\in\mathbb{N}}$ in $(\bar{\gamma}_0, \infty)$ such that $\bar{\gamma}_n \to \bar{\gamma}_\infty > \bar{\gamma}_0$. We want to show that $\tilde{\psi}_1(\bar{\gamma}_n)$ converges to $\tilde{\psi}_1(\bar{\gamma}_\infty)$ with respect to $d_{\mathcal{D}(\bar{M})}$. Recall the definitions (4.38)–(4.40) of $f_{\mathcal{D}(\bar{M})}$, d' and $d_{\mathcal{D}(\bar{M})}$, respectively. Thus we need to show that $f_{\mathcal{D}(\bar{M})}(\tilde{\psi}_1(\bar{\gamma}_n)) = (\xi(\bar{\gamma}_n), \lambda_n \bullet (\xi(\bar{\gamma}_n) \bullet \bar{M}))$ converges with respect to d'. We start by considering the first component. Let $\bar{\gamma}_* := \inf_{n\in\mathbb{N}} \bar{\gamma}_n > \bar{\gamma}_0$. Recall that by part 2)(i) of Theorem 4.18, we have

$$P\Big[\lim_{n\to\infty} \xi_t(\bar{\gamma}_n) = \xi_t(\bar{\gamma}_\infty) \text{ for all } 0 \leq t \leq T\Big] = 1.$$

Then by twice applying the dominated convergence theorem with respect to $\int_0^T \cdot\, dI$ and $E[\,\cdot\,]$ with majorant 1 in both cases, we obtain

$$E\left[\frac{1}{I_T + 1}\int_0^T \big(1 \wedge \|\xi_t(\bar{\gamma}_n) - \xi_t(\bar{\gamma}_\infty)\|\big)\, dI_t\right] \longrightarrow 0 \quad \text{as } n \to \infty. \tag{4.41}$$

For the second component, note that the family $\{\lambda(\bar{\gamma}) \bullet (\xi(\bar{\gamma}) \bullet \bar{M}) : \bar{\gamma} \geq \bar{\gamma}_*\}$ is bounded in BMO by part 3) of Theorem 4.18. Hence by well-known results on BMO martingales (see Corollary 6.8 below, which is based on results of Kazamaki [81]), the set $\{\langle \lambda(\bar{\gamma}) \bullet (\xi(\bar{\gamma}) \bullet \bar{M})\rangle_T : \bar{\gamma} \geq \bar{\gamma}_*\}$ is uniformly integrable. Thus the bound $\langle M - N\rangle \leq 2\langle M\rangle + 2\langle N\rangle$ for local martingales M and N yields that

$$\big\{\big\langle \lambda(\bar{\gamma}_n)\bullet\big(\xi(\bar{\gamma}_n)\bullet\bar{M}\big) - \lambda(\bar{\gamma}_\infty)\bullet\big(\xi(\bar{\gamma}_\infty)\bullet\bar{M}\big)\big\rangle_T : n \in \mathbb{N}\big\}$$

is also uniformly integrable. Recall that by part 2)(ii) of Theorem 4.18, we have

$$\lim_{n\to\infty} \big\langle \lambda(\bar\gamma_n)\bullet\big(\xi(\bar\gamma_n)\bullet\bar M\big) - \lambda(\bar\gamma_\infty)\bullet\big(\xi(\bar\gamma_\infty)\bullet\bar M\big)\big\rangle_T = 0 \quad P\text{-a.s.}$$

so that taking expectations yields

$$\lim_{n\to\infty} E\big[\big\langle \lambda(\bar\gamma_n)\bullet\big(\xi(\bar\gamma_n)\bullet\bar M\big) - \lambda(\bar\gamma_\infty)\bullet\big(\xi(\bar\gamma_\infty)\bullet\bar M\big)\big\rangle_T\big] = 0.$$

In other words, we have $\lambda(\bar\gamma_n)\bullet(\xi(\bar\gamma_n)\bullet\bar M) \overset{\mathcal{M}^2}{\to} \lambda(\bar\gamma_\infty)\bullet(\xi(\bar\gamma_\infty)\bullet\bar M)$ as $n\to\infty$. By combining this with (4.41), we obtain $f_{\mathcal{D}(\bar M)}(\tilde\psi_1(\bar\gamma_n)) \overset{d'}{\to} f_{\mathcal{D}(\bar M)}(\tilde\psi_1(\bar\gamma_\infty))$ as $n\to\infty$ so that $\tilde\psi_1(\bar\gamma_n) \overset{d_{\mathcal{D}(\bar M)}}{\to} \tilde\psi_1(\bar\gamma_\infty)$ as $n\to\infty$, and therefore $\tilde\psi_1$ is continuous. □

4.2 Continuity of $\tilde\psi_2$

In the following, we retain the **standing Assumptions** 2.1, 4.2 and 4.15 and recall the processes $\bar M$, I and $c^{\bar M}$ as well as the set $\mathcal{D}(\bar M)$; see (4.4), (4.30) and Definition 4.16. We are now ready to define the map $\tilde\psi_2$ on $\mathcal{D}(\bar M)$ and to study its continuity. Fix $(\xi,\lambda)\in\mathcal{D}(\bar M)$ as well as $S = S(\xi,\lambda)$ and $Q = Q(\xi,\lambda)$ as given by Definition 4.16. Then S satisfies Assumption 2.11 and as discussed in Section 2.2, it follows that the MVH and exMVH problems (2.14) and (2.15) for any $H\in L^2$ with respect to S admit unique solutions. Thus we can define $\ell = \varepsilon^2(1;S)$, $c_k = c(\Xi^k;S)$ and $\varepsilon_k^2 = \varepsilon^2_{\mathrm{ex}}(\Xi^k;S)$ for $k=1,\dots,K$ with respect to $S = S(\xi,\lambda)$ in the same way as in (2.46)–(2.48).

Definition 4.24. We define the map $\tilde\psi_2 : \mathcal{D}(\bar M) \to (0,1]\times(\mathbb{R}\times\mathbb{R}_+)^K$ by

$$\tilde\psi_2(\xi,\lambda) = \Big(\varepsilon^2\big(1;S(\xi,\lambda)\big), \Big(c\big(\Xi^k;S(\xi,\lambda)\big), \varepsilon^2_{\mathrm{ex}}\big(\Xi^k;S(\xi,\lambda)\big)\Big)_{k=1}^K\Big) \tag{4.42}$$

for $(\xi,\lambda)\in\mathcal{D}(\bar M)$.

As explained at the beginning of the section (see Diagram 4.1), we have constructed the maps $\tilde\psi_1$ and $\tilde\psi_2$ with the goal of refactoring the original composition $\psi_2\circ\psi_1$ as $\tilde\psi_2\circ\tilde\psi_1$. Recall that by Definition 2.31, we have for $\bar\gamma > \bar\gamma_0$ that

$$\psi_2\circ\psi_1(\bar\gamma) = \Big(\varepsilon^2\big(1;S(\bar\gamma)\big), \Big(c\big(\Xi^k;S(\bar\gamma)\big), \varepsilon^2_{\mathrm{ex}}\big(\Xi^k;S(\bar\gamma)\big)\Big)_{k=1}^K\Big). \tag{4.43}$$

We now show that the two compositions $\psi_2\circ\psi_1$ and $\tilde\psi_2\circ\tilde\psi_1$ are indeed equal.

Lemma 4.25. *We have $\tilde\psi_2\circ\tilde\psi_1 = \psi_2\circ\psi_1$.*

Proof. Let $\bar{\gamma} \in (\bar{\gamma}_0, \infty)$. Since $\tilde{\psi}_1(\bar{\gamma}) = (\xi(\bar{\gamma}), \lambda(\bar{\gamma}))$ (see (4.27)), $\tilde{\psi}_2 \circ \tilde{\psi}_1(\bar{\gamma})$ is given by the right-hand side of (4.42) with $\xi = \xi(\bar{\gamma})$ and $\lambda = \lambda(\bar{\gamma})$. So it suffices to show that $\varepsilon^2(1; S(\bar{\gamma})) = \varepsilon^2(1; S(\xi(\bar{\gamma}), \lambda(\bar{\gamma})))$, and likewise for the other components in (4.42) and (4.43). By (4.36), the price processes $S(\bar{\gamma})$ and $S(\xi(\bar{\gamma}), \lambda(\bar{\gamma}))$ differ only by a constant shift $S_0(\bar{\gamma})$ which does not affect the MVH problems because they depend only on stochastic integrals with respect to the price process. Therefore, the constants on the right-hand side of (4.43) and (4.42) with $(\xi, \lambda) = \tilde{\psi}_1(\bar{\gamma})$ are equal so that $\tilde{\psi}_2 \circ \tilde{\psi}_1(\bar{\gamma}) = \psi_2 \circ \psi_1(\bar{\gamma})$ for all $\bar{\gamma} > \bar{\gamma}_0$. □

We now proceed to study the continuity of $\tilde{\psi}_2$ with respect to the pseudometric $d_{\bar{M}}$ given in Definition 4.19. More precisely, we claim that for each $C \geq 1$, ψ_2 is continuous on the set $\mathcal{D}_C(\bar{M}) \subseteq \mathcal{D}(\bar{M})$ which we recall from Definition 4.16. This is the main theorem on $\tilde{\psi}_2$ that we prove in this section.

Theorem 4.26. *Suppose that Assumptions 2.1, 4.2 and 4.15 are satisfied. Then* $\tilde{\psi}_2 : \mathcal{D}(\bar{M}) \to (0, 1] \times (\mathbb{R} \times \mathbb{R}_+)^K$ *is continuous on* $\mathcal{D}_C(\bar{M})$ *for each* $C \geq 1$.

The proof of Theorem 4.26 is postponed to the end of the section. Given Theorem 4.26, we can already connect it with our previous results on $\tilde{\psi}_1$ to show the continuity of the composition $\tilde{\psi}_2 \circ \tilde{\psi}_1$.

Corollary 4.27. *Suppose that Assumptions 2.1, 4.2 and 4.15 are satisfied. Then the map* $\tilde{\psi}_2 \circ \tilde{\psi}_1 = \psi_2 \circ \psi_1$ *is continuous on* $(\bar{\gamma}_0, \infty)$.

Proof. We have $\tilde{\psi}_2 \circ \tilde{\psi}_1 = \psi_2 \circ \psi_1$ by Lemma 4.25 and $\tilde{\psi}_1 : (\bar{\gamma}_0, \infty) \to \mathcal{D}(\bar{M})$ is continuous by Corollary 4.23. By part 3) of Theorem 4.18, we have for every $\bar{\gamma}_* > \bar{\gamma}_0$ that $\tilde{\psi}_1((\bar{\gamma}_0, \infty)) \subseteq \mathcal{D}_C(\bar{M})$ for some $C > 0$. Since $\tilde{\psi}_2$ is continuous on $\mathcal{D}_C(\bar{M})$ by Theorem 4.26, it follows that $\tilde{\psi}_2 \circ \tilde{\psi}_1$ is continuous on $[\bar{\gamma}_*, \infty)$, and $\bar{\gamma}_* > \bar{\gamma}_0$ is arbitrary. □

Our goal is now to prove Theorem 4.26 by studying the dependence of the MVH and exMVH problems (2.46) and (2.47) on ξ and λ. Instead of directly considering the continuity of $\tilde{\psi}_2$, it is notationally more convenient to start by studying the continuity in (ξ, λ) (via the asset prices $S = S(\xi, \lambda)$) of the generic MVH problems (2.14) and (2.15) for a given payoff H. We can then apply these findings to $H = \Xi^k$ for each $k \in \{1, \ldots, K\}$.

To that end, let $C > 0$, $(\xi^n, \lambda^n) \to (\xi^\infty, \lambda^\infty)$ in $\mathcal{D}_C(\bar{M})$ as $n \to \infty$ and write $S^n = S(\xi^n, \lambda^n)$ and $Z^n = Z(\xi^n, \lambda^n)$. We also write $\hat{\ell}_n = \varepsilon^2(1; S^n)$, $\hat{c}_n = c(H; S^n)$ and $\hat{\varepsilon}_n^2 = \varepsilon_{\text{ex}}^2(H; S^n)$, where the accents are used to distinguish from the notation for the K agents. We want to show that $(\hat{\ell}_n, \hat{c}_n, \hat{\varepsilon}_n^2) \to (\hat{\ell}_\infty, \hat{c}_\infty, \hat{\varepsilon}_\infty^2)$. To achieve this, the main idea is to relate $\hat{\ell}_n, \hat{c}_n$ and $\hat{\varepsilon}_n^2$ to the initial values of the solutions to a certain family of backward stochastic differential equations (BSDEs) and show the convergence of the solutions by using a BSDE stability result given later in Section 6. We do this via some results from the analysis of mean–variance hedging via dynamic programming, for which we use Jeanblanc et al. [72] as a reference. We define for $t \in [0,T]$ and $x_t \in L^2(\mathcal{F}_t)$ the dynamic version of (2.14) as

$$\varepsilon^2(t, H - x_t; S^n) := \operatorname{ess\,inf}_{\vartheta \in \overline{\Theta}_{t,T}(S^n)} E[(H - x_t - \vartheta \bullet S_T^n)^2 \mid \mathcal{F}_t], \tag{4.44}$$

where $\overline{\Theta}_{t,T}(S^n) := \{\vartheta \in \overline{\Theta}(S^n) : \vartheta \mathbf{1}_{[\![0,t]\!]} = 0\}$. We note that the set of admissible strategies considered in [72] is $\Theta(S^n)$ (and $\Theta_{t,T}(S^n)$, respectively); by Lemma 4.10, this coincides with $\overline{\Theta}(S^n)$ because Definition 4.16 gives an ELMM Q^n for S^n that satisfies $R_2(P)$. Thus the conditions of [72, Theorem 1.4] are satisfied (see also the Remark before [72, Lemma 1.5]), and [72, Equation (1.6)] yields

$$\varepsilon^2(t, H - x_t; S^n) = x_t^2 \hat{Y}_t^{(2),n} - 2x_t \hat{Y}_t^{(1),n} + \hat{Y}_t^{(0),n} \tag{4.45}$$

for all $0 \le t \le T$ and $x_t \in L^2(\mathcal{F}_t)$, where $\hat{Y}^{(2),n}, \hat{Y}^{(1),n}$ and $\hat{Y}^{(0),n}$ are semimartingales that do not depend on x_t; moreover, $\hat{Y}^{(2),n}$ also does not depend on H. To see the relationship between $\hat{Y}^{(i),n}$ and $(\hat{\ell}_n, \hat{c}_n, \hat{\varepsilon}_n^2)$, note that by the linearity of the MVH problem (see Lemma III.2.6), we have

$$\varepsilon^2(0, 0 - x; S^n) = x^2 \varepsilon^2(0, 0 - 1; S^n) = x^2 \varepsilon^2(0, 1 - 0; S^n) = x^2 \hat{\ell}_n$$

for $x \in \mathbb{R}$. Since the $\hat{Y}^{(i),n}$ in (4.45) do not depend on x, we obtain

$$\hat{\ell}_n = \varepsilon^2(0, 1 - 0; S^n) = \hat{Y}_0^{(2),n}. \tag{4.46}$$

Next, recall that $\hat{c}_n = c(H; S^n)$, where $(c(H; S^n), \vartheta^{\mathrm{ex}}(H; S^n))$ is the minimiser of the exMVH problem (2.15) so that

$$\hat{c}_n = \operatorname*{arg\,min}_{c \in \mathbb{R}} \inf_{\vartheta \in \overline{\Theta}(S^n)} E[(H - c - \vartheta \bullet S_T^n)^2].$$

By (4.44), the infimum on the right-hand side is equal to $\varepsilon^2(0, H - c; S^n)$. Thus by plugging in the right-hand side of (4.45) and minimising this quadratic function over $c \in \mathbb{R}$, we obtain

$$\hat{c}_n = \frac{\hat{Y}_0^{(1),n}}{\hat{Y}_0^{(2),n}} = \frac{\hat{Y}_0^{(1),n}}{\hat{\ell}_n}. \tag{4.47}$$

Finally, we have by the definitions of $\hat{\varepsilon}_n^2$ and $\hat{c}_n$ that

$$\hat{\varepsilon}_n^2 = \varepsilon^2_{\mathrm{ex}}(H; S_n) = \inf_{c\in\mathbb{R}} \inf_{\vartheta\in\overline{\Theta}(S^n)} E[(H - c - \vartheta \bullet S^n_T)^2] = \varepsilon^2(0, H - \hat{c}_n; S^n)$$

and hence (4.45) and (4.47) yield

$$\hat{\varepsilon}_n^2 = \hat{Y}_0^{(0),n} - \frac{(\hat{Y}_0^{(1),n})^2}{\hat{Y}_0^{(2),n}} = \hat{Y}_0^{(0),n} - \hat{\ell}_n \hat{c}_n^2. \tag{4.48}$$

Thus if $\hat{Y}_0^{(i),n} \to \hat{Y}_0^{(i),\infty}$ as $n \to \infty$ for $i = 0, 1, 2$, we have by (4.46)–(4.48) that $(\hat{\ell}_n, \hat{c}_n, \hat{\varepsilon}_n^2) \to (\hat{\ell}_\infty, \hat{c}_\infty, \hat{\varepsilon}_\infty^2)$ as $n \to \infty$. Therefore we want to study the processes $\hat{Y}^{(i),n}$, and we prove later in Proposition 4.32 that they converge as $n \to \infty$ if H is bounded as in Assumption 2.1.

In the following, we also assume that H is strictly positive and bounded away from 0; we note that this additional assumption can be made without loss of generality. That is because for any bounded payoff H, there exists some constant $b > 0$ such that $\tilde{H} := H + b$ is bounded away from 0. Then the decomposition (4.45) for the hedging error associated with H can be converted into a decomposition for the hedging error associated with $\tilde{H}$ and vice versa. Indeed, if we write $\hat{Y}^{(i),n}(\tilde{H})$ and $\hat{Y}^{(i),n}(H)$ for the processes given by [72, Equation (1.6)] for $\tilde{H}$ and H, respectively, we have by (4.45) that

$$\hat{Y}_t^{(2),n}(H) = \hat{Y}_t^{(2),n}(\tilde{H}) = \hat{Y}_t^{(2),n}, \qquad 0 \le t \le T, \tag{4.49}$$

$$\hat{Y}_t^{(1),n}(H) = \hat{Y}_t^{(1),n}(\tilde{H}) - b\hat{Y}_t^{(2),n}, \qquad 0 \le t \le T, \tag{4.50}$$

$$\hat{Y}_t^{(0),n}(H) = \hat{Y}_t^{(0),n}(\tilde{H}) - 2b\hat{Y}_t^{(1),n}(\tilde{H}) + b^2\hat{Y}_t^{(2),n}, \quad 0 \le t \le T. \tag{4.51}$$

Therefore, we can work with the payoff $\tilde{H} > 0$, and the convergence of the processes $\hat{Y}^{(i),n}(\tilde{H})$ is equivalent to that of the processes $\hat{Y}^{(i),n}(H)$.

In view of the discussion above, the next (and main) step in proving the continuity of $\tilde{\psi}_2$ is to show the convergence of the processes $\hat{Y}^{(i),n}$ associated with a strictly positive bounded payoff H. We now fix the setup **for the remainder of Section 4.2** until we return to the proof of Theorem 4.26. Namely, we fix $C \ge 1$ and a sequence (ξ^n, λ^n) in $\mathcal{D}_C(\bar{M})$ with $(\xi^n, \lambda^n) \overset{d_{\mathcal{D}(\bar{M})}}{\to} (\xi^\infty, \lambda^\infty) \in \mathcal{D}_C(\bar{M})$ as $n \to \infty$, as well as a payoff $H > 0$ that is bounded above and below away from 0. Our goal is to show that the processes $\hat{Y}^{(i),n}$ $(i = 0, 1, 2)$ given by (4.44) and (4.45) with respect to H and $S^n := S(\xi^n, \lambda^n)$ converge as $n \to \infty$. We start by showing bounds on the sequence (ξ^n, λ^n) and the processes $\hat{Y}^{(i),n}$.

Lemma 4.28. *The sequence* $(\lambda^n \bullet (\xi^n \bullet \bar{M}))_{n\in\mathbb{N}\cup\{\infty\}}$ *is bounded in* BMO.

Proof. By assumption, $(\xi^n, \lambda^n) \in \mathcal{D}_C(\bar{M})$ for each $n \in \mathbb{N}\cup\{\infty\}$ so that $S(\xi^n, \lambda^n)$ admits an ELMM $Q(\xi^n, \lambda^n)$ that satisfies $R_2(P)$ with a constant C which is independent of n. The result then follows by Lemma 4.17. □

Lemma 4.29. *For all* $n \in \mathbb{N}\cup\{\infty\}$ *and* $0 \le t \le T$, *we have the inequalities*

$$0 < C^{-1} \le \hat{Y}_t^{(2),n} \le 1, \tag{4.52}$$

$$0 < C^{-1} \operatorname{ess\,inf} H \le \hat{Y}_t^{(1),n} \le \operatorname{ess\,sup} H, \tag{4.53}$$

$$0 \le \hat{Y}^{(0),n} \le \operatorname{ess\,sup} H^2. \tag{4.54}$$

Proof. Because each Q^n satisfies $R_2(P)$ with constant C, Jeanblanc et al. [72, Lemmas 1.5 and 2.1] gives $\delta \le \hat{Y}^{(2),n} \le 1$, where the explicit lower bound $\delta = C^{-1}$ is given at the end of the proof of [72, Lemma 2.1]. This shows (4.52). Moreover, the remark after [72, Proposition 2.6] yields for each $n \in \mathbb{N}\cup\{\infty\}$ that

$$\hat{Y}_t^{(1),n} = \hat{Y}_t^{(2),n} E_{Q^{\star,n}}[H \mid \mathcal{F}_t], \quad 0 \le t \le T, \tag{4.55}$$

where $Q^{\star,n}$ is the variance-optimal martingale measure (VOMM) for S^n (see (2.38)). Since $S^n = S(\xi^n, \lambda^n)$ given by (4.35) is continuous, the VOMM is equivalent to P (as opposed to being a signed measure) by Delbaen/Schachermayer [36, Theorem 1.3]. Since H is bounded above and below away from 0, (4.49) and (4.55) yield (4.53). Finally, note that (4.45) with $x_t = 0$ gives $\hat{Y}_t^{(0),n} = \varepsilon^2(t, H; S^n)$ for $0 \le t \le T$ which immediately gives the lower bound in (4.54). The upper bound then follows by plugging the (suboptimal) strategy $\vartheta \equiv 0$ into the right-hand side of (4.44) for $\varepsilon^2(t, H; S^n)$. □

In order to show the convergence of the processes $\hat{Y}^{(i),n}$, we characterise them as solutions to BSDEs. Fix some $n \in \mathbb{N}$, which we temporarily omit from the notation for readability so that $S = S^n$, etc. Note that $dS^n = d\langle M^n\rangle\lambda^n + dM^n$ by (4.35), where $M = M^n := \xi^n \bullet \bar{M}$. Thus by [72, Theorem 3.1] (see also Mania/Tevzadze [92, Theorem 4.1]), the processes $\hat{Y}^{(i)}$ satisfy the BSDEs

$$d\hat{Y}_t^{(2)} = \frac{1}{\hat{Y}_t^{(2)}}(\hat{Y}_t^{(2)}\lambda_t + \hat{\psi}_t^{(2)})^\top d\langle M\rangle_t(\hat{Y}_t^{(2)}\lambda_t + \hat{\psi}_t^{(2)}) + \hat{\psi}_t^{(2)} dM_t + d\hat{N}_t^{(2)}, \tag{4.56}$$

$$d\hat{Y}_t^{(1)} = \frac{1}{\hat{Y}_t^{(2)}}(\hat{Y}_t^{(2)}\lambda_t + \hat{\psi}_t^{(2)})^\top d\langle M\rangle_t(\hat{Y}_t^{(1)}\lambda_t + \hat{\psi}_t^{(1)}) + \hat{\psi}_t^{(1)} dM_t + d\hat{N}_t^{(1)}, \tag{4.57}$$

$$d\hat{Y}_t^{(0)} = \frac{1}{\hat{Y}_t^{(2)}}(\hat{Y}_t^{(1)}\lambda_t + \hat{\psi}_t^{(1)})^\top d\langle M\rangle_t(\hat{Y}_t^{(1)}\lambda_t + \hat{\psi}_t^{(1)}) + \hat{\psi}_t^{(0)} dM_t + d\hat{N}_t^{(0)} \tag{4.58}$$

for $0 \leq t \leq T$ with terminal conditions $\hat{Y}^{(2)}_T = 1$, $\hat{Y}^{(1)}_T = H$ and $\hat{Y}^{(0)}_T = H^2$, where $\hat{\psi}^{(i)} \in L^2_{\mathrm{loc}}(M)$ and $\hat{N}^{(i)}$ is a local martingale strongly orthogonal to M for $i = 0, 1, 2$. Note that $\hat{N}^{(i)}$ is continuous for $i \in \{0, 1, 2\}$ due to Assumption 4.2, and hence so is $\hat{Y}^{(i)}$, as pointed out in the remark before [72, Lemma 2.3].

The equations (4.56)–(4.58) are partially coupled BSDEs where the solution to (4.56) appears in (4.57) and (4.58), and the solution to (4.57) appears in (4.58). Thus it makes sense to study the equations in this order. At a glance, we observe that the driver (i.e., the drift term) of (4.56) grows quadratically with $\hat{\psi}^{(2)}$ and depends on the exogenous stochastic parameter λ. Although the term $\frac{1}{\hat{Y}^{(2)}}$ could in principle cause some issues, we know a priori that the process $\hat{Y}^{(2)}$ bounded away from 0 due to (4.52); so this is not a concern. The next BSDE (4.57) is linear in $\hat{Y}^{(1)}$ and $\hat{\psi}^{(1)}$ with stochastic parameters $\lambda, \hat{Y}^{(2)}$ and $\hat{\psi}^{(2)}$, of which the latter two are determined by the solution to (4.56). Finally, (4.58) can be solved explicitly by taking a conditional expectation, since the driver does not depend at all on $\hat{Y}^{(0)}$ or $\hat{\psi}^{(0)}$.

We leave (4.58) aside for the moment. In order to study the stability of (4.56) and (4.57) in $n \in \mathbb{N}$, we transform the equations into a more amenable form. The first step is to take a logarithm in order to remove the dependence of the drivers on $\hat{Y}^{(2)}$ and $\hat{Y}^{(1)}$, respectively. Because $\hat{Y}^{(2)}$ and $\hat{Y}^{(1)}$ are strictly positive by Lemma 4.29, we may define $Y^{(i)} = \log \hat{Y}^{(i)}$ for $i = 1, 2$, and Itô's formula yields

$$dY^{(i)}_t = \frac{d\hat{Y}^{(i)}_t}{\hat{Y}^{(i)}_t} - \frac{d\langle \hat{Y}^{(i)} \rangle_t}{2(\hat{Y}^{(i)}_t)^2}, \quad 0 \leq t \leq T.$$

Plugging into (4.56) and (4.57) yields the BSDEs

$$\begin{aligned} dY^{(2)}_t &= (\lambda_t + \psi^{(2)}_t)^\top d\langle M \rangle_t (\lambda_t + \psi^{(2)}_t) - \frac{(\psi^{(2)}_t)^\top d\langle M \rangle_t \psi^{(2)}_t + d\langle \tilde{N}^{(2)} \rangle_t}{2} \\ &\quad + \psi^{(2)}_t dM_t + d\tilde{N}^{(2)}_t, \quad 0 \leq t \leq T, \qquad (4.59) \\ dY^{(1)}_t &= (\lambda_t + \psi^{(2)}_t)^\top d\langle M \rangle_t (\lambda_t + \psi^{(1)}_t) - \frac{(\psi^{(1)}_t)^\top d\langle M \rangle_t \psi^{(1)}_t + d\langle \tilde{N}^{(1)} \rangle_t}{2} \\ &\quad + \psi^{(1)}_t dM_t + d\tilde{N}^{(1)}_t, \quad 0 \leq t \leq T, \qquad (4.60) \end{aligned}$$

with $Y^{(2)}_T = \log 1 = 0$ and $Y^{(2)}_T = \log H$, where $\psi^{(i)} := \hat{\psi}^{(i)} / \hat{Y}^{(i)} \in L^2_{\mathrm{loc}}(M)$ and $\tilde{N}^{(i)} := \frac{1}{\hat{Y}^{(i)}} \bullet \hat{N}^{(i)}$ is a local martingale orthogonal to M for $i = 1, 2$.

This change of variables makes it simpler to show the stability of (4.59) and (4.60) in comparison to (4.56) and (4.57) since the drivers do not depend on $Y^{(i)}$. We now reintroduce the superscript n, i.e., we write $\hat{Y}^{(i),n}$, $Y^{(i),n}$, and so on, for the processes corresponding to (S^n, ξ^n, λ^n). We want to show $Y_0^{(i),n} \to Y_0^{(i),\infty}$ as $n \to \infty$ for $i = 1, 2$. In comparison to classical results on quadratic BSDEs such as those in Kobylanski [84], (4.59) and (4.60) present two main difficulties. The first is that the exogenous coefficients λ^n are not bounded; instead, we have by Lemma 4.28 that $\lambda^n \bullet M^n = \lambda^n \bullet (\xi^n \bullet \bar{M})$ is a BMO-martingale. The issues related to the stochastic coefficient are discussed and dealt with in Section 6, where we show in Theorem 6.6 a stability result for a class of quadratic BSDEs with a stochastic exogenous parameter satisfying a BMO condition. We later use Theorem 6.6 to show the convergence of the processes $\hat{Y}^{(i),n}$.

Before we proceed, we need to deal with the second main difficulty, which is that the local martingales $M^n := \xi^n \bullet \bar{M}$ depend on $n \in \mathbb{N}$; recall that in (4.56)–(4.60), we have written M instead of M^n to alleviate the notation. This causes difficulties because the orthogonality requirement $\tilde{N}^{(i),n} \perp M^n$ also depends on $n \in \mathbb{N}$, which is nonstandard for BSDE stability results. Some results in this direction were obtained recently in Papapantoleon et al. [100], but only under the assumption of a Lipschitz bound on the driver, which does not hold for (4.59) and (4.60). Our next step is thus to reexpress (4.59) and (4.60) in terms of $\bar{M}$ rather than M^n. For $i = 1, 2$, we have the Galtchouk–Kunita–Watanabe decomposition for $\psi^{(i),n} \bullet M^n + \tilde{N}^{(i),n}$ in terms of $\bar{M}$ given by

$$\psi^{(i),n} \bullet M^n + \tilde{N}^{(i),n} = \zeta^{(i),n} \bullet \bar{M} + N^{(i),n} \tag{4.61}$$

for some $\zeta^{(i),n} \in L^2_{\text{loc}}(\bar{M})$ and a local martingale $N^{(i),n}$ which is strongly orthogonal to $\bar{M}$ and thus also to $M^n = \xi^n \bullet \bar{M}$. We recall that $\bar{M}$ and M^n take values in $\mathbb{R}^{\bar{d} \times 1}$ and $\mathbb{R}^{(d_1+d_2)\times 1}$, respectively, while the integrands $\xi^n, \psi^{(i),n}$ and $\zeta^{(i),n}$ take values in $\mathbb{R}^{\bar{d} \times (d_1+d_2)}$, $\mathbb{R}^{(d_1+d_2)\times 1}$ and $\mathbb{R}^{\bar{d}\times 1}$, respectively, so that equation (4.61) is real-valued. In order to eliminate $\psi^{(i),n}$ from the drivers in (4.59) and (4.60), we need to express $\psi^{(i),n}$ in terms of $\zeta^{(i),n}$. Since both $N^{(i),n}$ and $\tilde{N}^{(i),n}$ are strongly orthogonal to $\bar{M}$, $\psi^{(i),n}$ is determined uniquely by $\zeta^{(i),n}$ because (4.61) yields the Galtchouk–Kunita–Watanabe decomposition

$$\zeta^{(i),n} \bullet \bar{M} = \psi^{(i),n} \bullet M^n + \tilde{N}^{(i),n} - N^{(i),n}$$

of $\zeta^{(i),n} \bullet \bar{M}$ with respect to M^n. In the next result, we show how to find $\xi^n \psi^{(i),n}$ explicitly via a predictable $\mathbb{R}^{\bar{d}\times\bar{d}}$-valued process $(\pi^n)_{0\le t\le T}$ that gives the orthogonal projection onto the (random) range of $c^{\bar{M}}\zeta^n$ in $\mathbb{R}^{\bar{d}}$ with respect to the metric induced by $c^{\bar{M}}$. In the following, we denote the Moore–Penrose inverse of a matrix C by $C^\dagger$ (see Albert [6, Chapter III] for the definition and basic properties);

it coincides with the usual inverse if C is square and invertible.

Lemma 4.30. *There exist for $n \in \mathbb{N} \cup \{\infty\}$ predictable processes $(\pi_t^n)_{0\le t\le T}$ with values in $\mathbb{R}^{\bar d\times\bar d}$ such that each of the following statements holds $P\otimes I$-a.e.:*

1) $(\pi^n)^\top c^{\bar M} = c^{\bar M}\pi^n = (\pi^n)^\top c^{\bar M}\pi^n$ *for $n \in \mathbb{N}\cup\{\infty\}$.*

2) $y^\top(\pi^n)^\top c^{\bar M}\pi^n y \le y^\top c^{\bar M} y$ *for all $y\in\mathbb{R}^{\bar d}$ and $n\in\mathbb{N}\cup\{\infty\}$.*

3) $\xi^n\psi^{(i),n} = \pi^n\zeta^{(i),n}$ *for $n\in\mathbb{N}\cup\{\infty\}$ and $i=1,2$.*

4) $\pi^n \to \pi^\infty$ *as $n\to\infty$.*

Proof. By Assumption 4.15, the predictable process $(c_t^{\bar M})_{0\le t\le T}$ takes values in the set of $\bar d\times\bar d$ symmetric positive definite matrices $P\otimes I$-a.e. By Ando/van Hemmen [11, Proposition 3.2], the map $C\mapsto C^{1/2}$ is continuous and hence measurable on the set of symmetric positive definite matrices (equipped with the Borel σ-algebra). Thus the process $((c_t^{\bar M})^{1/2})_{0\le t\le T}$ is predictable and takes values $P\otimes I$-a.e. in the set of symmetric positive definite $\bar d\times\bar d$ matrices. Now fix $n\in\mathbb{N}\cup\{\infty\}$ and recall from Definition 4.16 that ξ^n takes values in $\mathbb{R}^{\bar d\times(d_1+d_2)}$. The map $C\mapsto C^{-1}$ is continuous (and thus measurable) on the set of invertible matrices; see e.g. Stewart [114, Equation (1.5)]. For a (possibly nonsquare) $m\times n$ matrix A, the construction in Albert [6, Theorem 3.4] gives

$$A^\dagger = \lim_{\delta\searrow 0}(A^\top A + \delta^2\,\mathrm{id})^{-1}A^\top,$$

where id is the $n\times n$ identity matrix. Because the inversion map is measurable, it follows that $A\mapsto A^\dagger$ is measurable on the set of all $m\times n$ matrices. Hence we can define the predictable process $(((c_t^{\bar M})^{1/2}\xi_t^n)^\dagger)_{0\le t\le T}$ with values in $\mathbb{R}^{\bar d\times(d_1+d_2)}$ as the Moore–Penrose inverse of $(c^{\bar M})^{1/2}\xi^n$. Moreover, by [6, Corollary 3.5], the orthogonal projection on the range of $(c^{\bar M})^{1/2}\xi^n$ with respect to the Euclidean metric is given $P\otimes I$-a.e. by the predictable process $(\tilde\pi_t^n)_{0\le t\le T}$ defined as

$$\tilde\pi^n := (c^{\bar M})^{1/2}\xi^n\big((c^{\bar M})^{1/2}\xi^n\big)^\dagger. \tag{4.62}$$

Finally, we construct the predictable process $(\pi_t^n)_{0\le t\le T}$ by

$$\pi^n = (c^{\bar M})^{-1/2}\tilde\pi^n(c^{\bar M})^{1/2} = \xi^n\big((c^{\bar M})^{1/2}\xi^n\big)^\dagger(c^{\bar M})^{1/2}. \tag{4.63}$$

It remains to check that $(\pi^n)_{n\in\mathbb{N}\cup\{\infty\}}$ satisfies conditions 1)–4).

1) The first equality follows since $P\otimes I$-a.e.,

$$(\pi^n)^\top c^{\bar M} = (c^{\bar M})^{1/2}\tilde\pi^n (c^{\bar M})^{-1/2} c^{\bar M} = (c^{\bar M})^{1/2}\tilde\pi^n (c^{\bar M})^{1/2} =: \tilde a_n,$$
$$c^{\bar M}\pi^n = c^{\bar M}(c^{\bar M})^{-1/2}\tilde\pi^n (c^{\bar M})^{1/2} = \tilde a_n.$$

As $\tilde\pi^n$ is a projection, $\tilde\pi^n\tilde\pi^n = \tilde\pi^n$ $P\otimes I$-a.e., and hence we also have

$$\begin{aligned}(\pi^n)^\top c^{\bar M}\pi^n &= (c^{\bar M})^{1/2}\tilde\pi^n (c^{\bar M})^{-1/2} c^{\bar M}(c^{\bar M})^{-1/2}\tilde\pi^n (c^{\bar M})^{1/2}\\ &= (c^{\bar M})^{1/2}\tilde\pi^n\tilde\pi^n (c^{\bar M})^{1/2} = \tilde a_n, \quad P\otimes I\text{-a.e.}\end{aligned}$$

2) Since $\tilde\pi^n$ is by construction $P\otimes I$-a.e. an orthogonal projection with respect to the Euclidean metric, we have $x^\top x \ge x^\top\tilde\pi^n x$ for all $x\in\mathbb{R}^{\bar d}$ $P\otimes I$-a.e. Setting $x=(c^{\bar M})^{1/2}y$ then yields that $P\otimes I$-a.e., we have for all $y\in\mathbb{R}^{\bar d}$

$$y^\top(\pi^n)^\top c^{\bar M}\pi^n y = y^\top\tilde a^n y = \big((c^{\bar M})^{1/2}y\big)^\top\tilde\pi^n\big((c^{\bar M})^{1/2}y\big) = x^\top\tilde\pi^n x \le x^\top x = y^\top c^{\bar M} y.$$

3) Since $\tilde N^{(i),n}$ and $N^{(i),n}$ are strongly orthogonal to M^n, taking the quadratic covariation of (4.61) with $M^n = \xi^n \bullet \bar M$ yields

$$\big(\xi^n\psi^{(i),n}\big)^\top c^{\bar M}\xi^n = (\zeta^{(i),n})^\top c^{\bar M}\xi^n \quad P\otimes I\text{-a.e.}$$

By taking differences, it follows that the predictable process $(d^n_t)_{0\le t\le T}$ defined by

$$d^n := (c^{\bar M})^{1/2}(\xi^n\psi^{(i),n} - \zeta^{(i),n})$$

satisfies $(d^n)^\top(c^{\bar M})^{1/2}\xi^n = 0$ $P\otimes I$-a.e. Since $\tilde\pi^n$ is the Euclidean projection on the range of $(c^{\bar M})^{1/2}\xi^n$, we thus have $\tilde\pi^n d^n = 0$, i.e.,

$$\tilde\pi^n(c^{\bar M})^{1/2}\zeta^{(i),n} = \tilde\pi^n(c^{\bar M})^{1/2}\xi^n\psi^{(i),n} \quad P\otimes I\text{-a.e.}$$

As $(c^{\bar M})^{1/2}\xi^n\psi^{(i),n}$ belongs to the range of $(c^{\bar M})^{1/2}\xi^n$, it is by (4.62) invariant under $\tilde\pi^n$, and hence we may omit $\tilde\pi^n$ from the right-hand side. Expressing $\tilde\pi^n$ by (4.63) in terms of π^n on the left-hand side then yields

$$(c^{\bar M})^{1/2}\pi^n\zeta^{(i),n} = (c^{\bar M})^{1/2}\xi^n\psi^{(i),n} \quad P\otimes I\text{-a.e.}$$

The result follows immediately by the invertibility of $(c^{\bar M})^{1/2}$.

4) By assumption, $\xi^n\to\xi^\infty$ $P\otimes I$-a.e. as $n\to\infty$. Moreover, by the definition of $\mathcal{D}(\bar M)$, each ξ^n has $P\otimes I$-a.e. full rank d_1+d_2, and so does $(c^{\bar M})^{1/2}\xi^n$ as $(c^{\bar M})^{1/2}$

is invertible. Hence by Stewart [114, Theorem 4.3], we have

$$\lim_{n\to\infty}\big((c^{\bar M})^{1/2}\xi^n\big)^\dagger = \big((c^{\bar M})^{1/2}\xi^\infty\big)^\dagger \quad P\otimes I\text{-a.e.}$$

Thus (4.62) and (4.63) yield $\tilde\pi^n \to \tilde\pi^\infty$ and $\pi^n\to\pi^\infty$ $P\otimes I$-a.e. as $n\to\infty$. □

Now that we have defined the processes $(\pi^n)_{n\in\mathbb{N}}$, we can return to (4.59), (4.60) and the orthogonal decomposition (4.61). The latter allows us to replace the local martingale parts in (4.59) and (4.60), and we can use the identity $M^n = \xi^n \bullet \bar M$ together with part 3) of Lemma 4.30 to reexpress the first term on the right-hand side of (4.59) and (4.60) in terms of $\zeta^{(i),n}$ and $\bar M$. For the second term in (4.59) and (4.60), taking the quadratic variation of (4.61) yields

$$\begin{aligned}(\psi_t^{(i),n})^\top d\langle M^n\rangle_t\psi_t^{(i),n} + d\langle\tilde N^{(i),n}\rangle_t &= d\langle\psi^{(i),n}\bullet\bar M + \tilde N^{(i),n}\rangle_t\\ &= d\langle\zeta^{(i),n}\bullet\bar M + N^{(i),n}\rangle_t\\ &= (\zeta_t^{(i),n})^\top d\langle\bar M\rangle_t(\zeta_t^{(i),n}) + d\langle N^{(i),n}\rangle_t\end{aligned}$$

for $0\le t\le T$. Inserting these into (4.59) and (4.60) yields the BSDEs

$$\begin{aligned}dY_t^{(2),n} &= \left((\tilde\lambda_t^n + \pi_t^n\zeta_t^{(2),n})^\top c_t^{\bar M}(\tilde\lambda_t^n+\pi_t^n\zeta_t^{(2),n}) - \frac{(\zeta_t^{(2),n})^\top c^{\bar M}\zeta_t^{(2),n}}{2}\right)dI_t\\ &\quad - \frac{d\langle N^{(2),n}\rangle_t}{2} + \zeta_t^{(2),n}d\bar M_t + dN_t^{(2),n},\quad 0\le t\le T, \qquad (4.64)\\ dY_t^{(1),n} &= \left((\tilde\lambda_t^n + \pi_t^n\zeta_t^{(2),n})^\top c_t^{\bar M}(\tilde\lambda_t^n+\pi_t^n\zeta_t^{(1),n}) - \frac{(\zeta_t^{(1),n})^\top c^{\bar M}\zeta_t^{(1),n}}{2}\right)dI_t\\ &\quad - \frac{d\langle N^{(1),n}\rangle_t}{2} + \zeta_t^{(1),n}d\bar M_t + dN_t^{(1),n},\quad 0\le t\le T, \qquad (4.65)\end{aligned}$$

with $Y_T^{(2),n} = \log 1 = 0$ and $Y_T^{(1).n} = \log H$, where $N^{(i),n}\perp\bar M$ and $\tilde\lambda^n := \xi^n\lambda^n$. We write (4.64) and (4.65) in the form

$$dY_t^{(2),n} = f_t^{(2),n}(\zeta_t^{(2),n},\tilde\lambda_t^n)dI_t - \varrho_t d\langle N^{(2),n}\rangle_t + \zeta_t^{(2),n}d\bar M_t + dN_t^{(2),n}, \qquad (4.66)$$
$$dY_t^{(1),n} = f_t^{(1),n}(\zeta_t^{(1),n},\zeta_t^{(2),n},\tilde\lambda_t^n)dI_t - \varrho_t d\langle N^{(1),n}\rangle_t + \zeta_t^{(1),n}d\bar M_t + dN_t^{(1),n} \qquad (4.67)$$

for $0\le t\le T$, where we define $\varrho\equiv\frac12$ and $f^{(2),n}:\Omega\times[0,T]\times\mathbb{R}^{\bar d}\times\mathbb{R}^{\bar d}\to\mathbb{R}$ and $f^{(1),n}:\Omega\times[0,T]\times\mathbb{R}^{\bar d}\times\mathbb{R}^{\bar d}\times\mathbb{R}^{\bar d}\to\mathbb{R}$ as the predictable functions given by

$$f_t^{(2),n}(\zeta_2,\tilde\lambda) = (\tilde\lambda+\pi_t^n\zeta_2)^\top c_t^{\bar M}(\tilde\lambda+\pi_t^n\zeta_2) - \frac{\zeta_2^\top c_t^{\bar M}\zeta_2}{2}, \qquad (4.68)$$

$$f_t^{(1),n}(\zeta_1,\zeta_2,\tilde{\lambda}) = (\tilde{\lambda}+\pi_t^n\zeta_2)^\top c_t^{\bar{M}}(\tilde{\lambda}+\pi_t^n\zeta_1) - \frac{\zeta_1^\top c_t^{\bar{M}}\zeta_1}{2} \tag{4.69}$$

for $0 \le t \le T$ and $\zeta_1, \zeta_2, \tilde{\lambda} \in \mathbb{R}^{\bar{d}}$. Our goal is now to show the stability of the BSDEs (4.66) and (4.67), i.e., to show that $Y^{(i),n} \to Y^{(i),\infty}$ in a suitable sense as $n \to \infty$ for $i = 1, 2$; at the end, we return to $i = 0$.

We start with some observations about (4.66) and (4.67). Thanks to our transformations, we have obtained two relatively standard quadratic BSDEs where the drivers $f^{(i),n}$ grow quadratically in $\zeta^{(i),n}$ and do not depend on $Y^{(i),n}$. However, as mentioned after (4.59) and (4.60), the stochastic coefficients $\tilde{\lambda}^n = \xi^n\lambda^n$ are not bounded. Instead, $\lambda^n \bullet (\xi^n \bullet \bar{M}) = \tilde{\lambda}^n \bullet \bar{M}$ is a BMO martingale by Lemma 4.28. This is the key condition that allows us to show the stability of (4.66) despite the unboundedness of the stochastic coefficient. We show below that $\zeta^{(i),n} \bullet \bar{M}$ also belongs to BMO, as is typically true of solutions to quadratic BSDEs. Thus there is a certain symmetry in that the exogenous coefficient $\tilde{\lambda}^n$ and the endogenous coefficients $\zeta^{(i),n}$ all satisfy the same condition $\chi \bullet \bar{M} \in BMO$ for $\chi \in \{\tilde{\lambda}^n, \zeta^{(2),n}, \zeta^{(1),n}\}$. This is particularly relevant for $\zeta^{(2),n}$ which is both endogenous for (4.66) and exogenous for (4.67).

The BMO properties allow us to establish in Theorem 6.6 a stability result for quadratic BSDEs of this form, as well as an a priori bound in Proposition 6.2. We postpone further discussion of BSDEs of this type to Section 6 and now deduce the stability of (4.66) and (4.67) by checking the conditions of Theorem 6.6, which we state below for the convenience of the reader. In the following, we write $|x|_C := (x^\top Cx)^{1/2}$ and $\|A\|_C := \mathrm{tr}(A^\top CA)^{1/2}$ for $x \in \mathbb{R}^{\bar{d}}$, $A \in \mathbb{R}^{\bar{d}\times m}$ and symmetric positive semidefinite $C \in \mathbb{R}^{\bar{d}\times\bar{d}}$. Denoting the rows of A by $x_1, \dots, x_m \in \mathbb{R}^{\bar{d}\times 1}$, we have by elementary linear algebra that $A^\top CA = (x_i^\top Cx_j)_{i,j=1}^m$ and hence

$$\|A\|_C^2 = \mathrm{tr}(A^\top CA) = \sum_{i=1}^m |x_i|_C^2. \tag{4.70}$$

Theorem 6.6. *Suppose that $(Y^n)_{n\in\mathbb{N}\cup\{\infty\}}$ is a sequence of continuous semimartingales on $[0,T]$ such that Y^n for $n \in \mathbb{N}\cup\{\infty\}$ satisfies the equation*

$$\begin{aligned} dY_t^n &= f_t^n(\zeta_t^n,\chi_t^n)dI_t - \varrho_t d\langle N^n\rangle_t + \zeta_t^n d\bar{M}_t + dN_t^n, \quad 0 \le t \le T, \\ Y_T^n &= G^n, \end{aligned} \tag{6.14}$$

where N^n is a continuous local martingale orthogonal to $\bar{M}$, $\chi^n \in L^2_{\mathrm{loc}}(\bar{M};\mathbb{R}^{\bar{d}\times m})$ and $\zeta^n \in L^2_{\mathrm{loc}}(\bar{M};\mathbb{R}^{\bar{d}})$ are predictable integrands, $(\varrho_t)_{0\le t\le T}$ is a bounded predictable *process,* $G^n \in L^\infty(\mathcal{F}_T)$ *and* $f^n : \Omega \times [0,T] \times \mathbb{R}^{\bar d} \times \mathbb{R}^{\bar d \times m} \to \mathbb{R}$ *is a predictable function. Suppose that the following conditions hold:*

(a) $C_Y := \sup_{n\in\mathbb{N}\cup\{\infty\}} \| \sup_{t\in[0,T]} |Y_t^n| \|_\infty < \infty$.

(b) *The processes* $\chi^n \bullet \bar M$ *are BMO martingales for each* $n \in \mathbb{N}\cup\{\infty\}$ *with*

$$\sup_{n\in\mathbb{N}\cup\{\infty\}} \|\chi^n \bullet \bar M\|_{BMO} < \infty.$$

Moreover, $\operatorname{tr}\langle(\chi^n - \chi^\infty)\bullet \bar M\rangle_T \overset{P}{\to} 0$ *as* $n \to \infty$.

(c) *For some* $C_f, L_f > 0$*, the functions* f^n *satisfy* $P \otimes I$*-a.e. for all* $n \in \mathbb{N}$ *and* $\tilde x_1, \tilde x_2 \in \mathbb{R}^{\bar d} \times \mathbb{R}^{\bar d \times m} = \mathbb{R}^{\bar d \times (m+1)}$ *the bounds*

$$|f^n(\tilde x_1)| \le C_f \|\tilde x_1\|^2_{c^{\bar M}}, \tag{6.7}$$

$$|f^n(\tilde x_1) - f^n(\tilde x_2)| \le L_f(\|\tilde x_1 - \tilde x_2\|_{c^{\bar M}})(\|\tilde x_1\|_{c^{\bar M}} + \|\tilde x_2\|_{c^{\bar M}}). \tag{6.8}$$

(d) *For any* $z \in \mathbb{R}^{\bar d}$ *and* $x \in \mathbb{R}^{\bar d\times m}$*, it holds that*

$$\lim_{n\to\infty} f^n(z,x) = f^\infty(z,x) \quad P \otimes I\text{-a.e.}$$

(e) $G^n \overset{P}{\to} G^\infty$ *as* $n \to \infty$.

Then the families of martingales $(\zeta^n \bullet \bar M)_{n\in\mathbb{N}\cup\{\infty\}}$ *and* $(N^n)_{n\in\mathbb{N}\cup\{\infty\}}$ *are bounded in* BMO*, and it holds as* $n \to \infty$ *that*

$$\sup_{t\in[0,T]} |Y_t^n - Y_t^\infty| \overset{P}{\longrightarrow} 0, \tag{6.15}$$

$$\langle(\zeta^n - \zeta^\infty)\bullet \bar M\rangle_T + \langle N^n - N^\infty\rangle_T \overset{P}{\longrightarrow} 0. \tag{6.16}$$

We now show how to use Theorem 6.6 together with our previous results to deduce the convergence of the processes $Y^{(i),n}$. We also obtain a uniform bound on the BMO-norms of the martingale parts of $Y^{(2),n}$ and $Y^{(1),n}$; this will be useful later to show the convergence of $\hat Y^{(0),n}$.

In the following, we recall the setup that we have introduced so far: as mentioned after (4.51), we fix some $C \ge 1$ and a sequence (ξ^n, λ^n) in $\mathcal{D}_C(\bar M)$ (see Definition 4.19) such that $(\xi^n, \lambda^n) \overset{d_{\mathcal{D}(\bar M)}}{\to} (\xi^\infty, \lambda^\infty) \in \mathcal{D}_C(\bar M)$. We then define the processes $\hat Y^{(i),n}$ by (4.45) and $Y^{(i),n} = \log \hat Y^{(i),n}$, as well as $\tilde\lambda^n := \xi^n \lambda^n$ and the coefficients $\zeta^{(i),n}$ and $N^{(i),n}$ via the BSDEs (4.64) and (4.65).

Theorem 4.31. *We have as $n \to \infty$ that $\langle (\tilde{\lambda}^n - \tilde{\lambda}^\infty) \bullet \bar{M} \rangle_T \xrightarrow{P} 0$ and*

$$\sup_{t \in [0,T]} |Y_t^{(i),n} - Y_t^{(i),\infty}| \xrightarrow{P} 0, \tag{4.71}$$

$$\langle (\zeta^{(i),n} - \zeta^{(i),\infty}) \bullet \bar{M} \rangle_T + \langle N^{(i),n} - N^{(i),\infty} \rangle_T \xrightarrow{P} 0, \tag{4.72}$$

for $i = 2$, and the sequences $(\tilde{\lambda}^n \bullet \bar{M})_{n \in \mathbb{N}}$, $(\zeta^{(2),n} \bullet \bar{M})_{n \in \mathbb{N}}$ and $(N^{(2),n})_{n \in \mathbb{N}}$ are bounded in BMO. If H is a bounded payoff with $\operatorname{ess\,inf} H > 0$, *then* (4.71) *and* (4.72) *also hold as $n \to \infty$ for $i = 1$, and the sequences $(\zeta^{(1),n} \bullet \bar{M})_{n \in \mathbb{N}}$ and $(N^{(1),n})_{n \in \mathbb{N}}$ are bounded in BMO.*

Proof. **1)** We start by considering $Y^{(2),n}$ which does not depend on H as noted after (4.45). The BSDE (4.66) is of the form (6.14) with $\chi^n := \tilde{\lambda}^n$ and $G^n := 0$. By (4.52), we have $-\log C \le Y_t^{(2),n} \le 0$ for all $t \in [0,T]$ and $n \in \mathbb{N} \cup \{\infty\}$ so that condition (a) of Theorem 6.6 is satisfied. By Lemma 4.28, $(\tilde{\lambda}^n \bullet \bar{M})_{n \in \mathbb{N} \cup \{\infty\}}$ is bounded in BMO because $\tilde{\lambda}^n = \xi^n \lambda^n$. Then because $(\xi^n, \lambda^n) \to (\xi^\infty, \lambda^\infty)$ in $\mathcal{D}(\bar{M})$, we have $\xi^n \to \xi^\infty$ $P \otimes I$-a.e. and $(\tilde{\lambda}^n - \tilde{\lambda}^\infty) \bullet \bar{M} \xrightarrow{\mathcal{M}^2} 0$ as $n \to \infty$. The latter convergence yields $\langle (\tilde{\lambda}^n - \tilde{\lambda}^\infty) \bullet \bar{M} \rangle_T \to 0$ P-a.s. so that condition (b) in Theorem 6.6 is satisfied. Condition (e) also holds because $G^n = 0$ for all $n \in \mathbb{N} \cup \{\infty\}$.

Next, we show that the sequence of predictable functions $(f^{(2),n})_{n \in \mathbb{N} \cup \{\infty\}}$ satisfies conditions (c) and (d) in Theorem 6.6. By part 4) of Lemma 4.30, $\pi^n \to \pi^\infty$ P-a.e. so that plugging into (4.68) yields (d). It remains to show (c). By (4.68), the triangle inequality and part 2) of Lemma 4.30, we have the bound

$$|f^{(2),n}(\zeta, \tilde{\lambda})| \le |\tilde{\lambda} + \pi^n \zeta|_{c^{\bar{M}}}^2 + \frac{1}{2}|\zeta|_{c^{\bar{M}}}^2 \le 2|\tilde{\lambda}|_{c^{\bar{M}}}^2 + \frac{5}{2}|\zeta|_{c^{\bar{M}}}^2 \quad P \otimes I\text{-a.e.}$$

for all $\zeta, \tilde{\lambda} \in \mathbb{R}^{\bar{d}}$ and $n \in \mathbb{N} \cup \{\infty\}$, which is of the form (6.7) by (4.70). In the following, we use for two variables x, x' the notation $x_d := x - x'$ and $x_s := x + x'$. The Cauchy–Schwarz inequality $x^\top c^{\bar{M}} x' \le |x|_{c^{\bar{M}}} |x'|_{c^{\bar{M}}}$ and the identity

$$x^\top c^{\bar{M}} x' - (x')^\top c^{\bar{M}} x = (x - x')^\top c^{\bar{M}} (x + x') = x_d^\top c^{\bar{M}} x_s, \quad x, x' \in \mathbb{R}^{\bar{d}},$$

together with part 2) of Lemma 4.30, give $P \otimes I$-a.e. for $\zeta, \zeta', \tilde{\lambda}, \tilde{\lambda}' \in \mathbb{R}^{\bar{d}}$ that

$$\begin{aligned} |f^{(2),n}(\zeta, \tilde{\lambda}) - f^{(2),n}(\zeta', \tilde{\lambda}')| &\le |\tilde{\lambda}_d + \pi^n \zeta_d|_{c^{\bar{M}}} |\tilde{\lambda}_s + \pi^n \zeta_s|_{c^{\bar{M}}} + \frac{1}{2}|\zeta_d|_{c^{\bar{M}}} |\zeta_s|_{c^{\bar{M}}} \\ &\le (|\tilde{\lambda}_d|_{c^{\bar{M}}} + |\zeta_d|_{c^{\bar{M}}})(|\tilde{\lambda}_s|_{c^{\bar{M}}} + |\zeta_s|_{c^{\bar{M}}}) + \frac{1}{2}|\zeta_d|_{c^{\bar{M}}} |\zeta_s|_{c^{\bar{M}}} \\ &\le \frac{3}{2}(|\tilde{\lambda}_d|_{c^{\bar{M}}} + |\zeta_d|_{c^{\bar{M}}})(|\tilde{\lambda}_s|_{c^{\bar{M}}} + |\zeta_s|_{c^{\bar{M}}}). \end{aligned}$$

This is a bound of the form (6.8), and hence condition (c) in Theorem 6.6 is satisfied. So all the conditions of Theorem 6.6 are satisfied by (4.66), and therefore (4.71) holds for $i = 2$, and $(\zeta^{(2),n} \bullet \bar{M})_{n\in\mathbb{N}}$ and $(N^{(2),n})_{n\in\mathbb{N}}$ are bounded in BMO.

2) Suppose now that $H > 0$ is bounded above and away from 0. We likewise check for $Y^{(1),n}$ that the conditions of Theorem 6.6 are satisfied by (4.67) with $\chi^n := (\zeta^{(2),n}, \tilde{\lambda}^n)$ and $G^n := \log H$. Condition (a) is satisfied due to the uniform bound (4.53) on $\hat{Y}^{(1),n}$ which yields a uniform bound on the logarithms $Y^{(1),n}$. We have already checked that condition (b) is satisfied by the component $(\tilde{\lambda}^n \bullet \bar{M})_{n\in\mathbb{N}}$, and it is also satisfied by $(\zeta^{(2),n} \bullet \bar{M})_{n\in\mathbb{N}}$ by step 1). Condition (e) holds because $G^n = \log H$ for all $n \in \mathbb{N}$, and condition (d) once again follows immediately because $\pi^n \to \pi^\infty$ by part 4) of Lemma 4.30. To show condition (c), note that by part 2) of Lemma 4.30 and (4.69), we have the bound

$$\begin{aligned} |f^{(1),n}(\zeta_1, \zeta_2, \tilde{\lambda})| &\le |\tilde{\lambda} + \pi^n \zeta_2|_{c^{\bar{M}}} |\tilde{\lambda} + \pi^n \zeta_1|_{c^{\bar{M}}} + \frac{1}{2}|\zeta_1|^2_{c^{\bar{M}}} \\ &\le \frac{1}{2}\left(|\tilde{\lambda} + \pi^n \zeta_2|^2_{c^{\bar{M}}} + |\tilde{\lambda} + \pi^n \zeta_1|^2_{c^{\bar{M}}} + |\zeta_1|^2_{c^{\bar{M}}} \right) \\ &\le 2|\tilde{\lambda}|^2_{c^{\bar{M}}} + \frac{3}{2}|\zeta_1|^2_{c^{\bar{M}}} + |\zeta_2|^2_{c^{\bar{M}}} \quad P \otimes I\text{-a.e.} \end{aligned}$$

for $\zeta_1, \zeta_2, \tilde{\lambda} \in \mathbb{R}^{\bar{d}}$, which has the form (6.8). We also have for $\zeta_1, \zeta_2, \zeta_1', \zeta_2', \tilde{\lambda}, \tilde{\lambda}' \in \mathbb{R}^{\bar{d}}$ that

$$\begin{aligned} &(\tilde{\lambda} + \pi^n \zeta_2)^\top c^{\bar{M}} (\tilde{\lambda} + \pi^n \zeta_1) - (\tilde{\lambda}' + \pi^n \zeta_2')^\top c^{\bar{M}} (\tilde{\lambda}' + \pi^n \zeta_1') \\ &= (\tilde{\lambda}_d + \pi^n \zeta_{2,d})^\top c^{\bar{M}} (\tilde{\lambda} + \pi^n \zeta_1) + (\tilde{\lambda}' + \pi^n \zeta_2')^\top c^{\bar{M}} (\tilde{\lambda}_d + \pi^n \zeta_{1,d}) \quad P \otimes I\text{-a.e.} \end{aligned}$$

Combining this with part 2) of Lemma 4.30 and (4.69) yields that

$$\begin{aligned} &|f^{(1),n}(\zeta_1, \zeta_2, \tilde{\lambda}) - f^{(1),n}(\zeta_1', \zeta_2', \tilde{\lambda}')| \\ &\le |\tilde{\lambda}_d + \pi^n \zeta_{2,d}|_{c^{\bar{M}}} |\tilde{\lambda} + \pi^n \zeta_1|_{c^{\bar{M}}} + |\tilde{\lambda}_d + \pi^n \zeta_{1,d}|_{c^{\bar{M}}} |\tilde{\lambda}' + \pi^n \zeta_2'|_{c^{\bar{M}}} + \frac{|\zeta_{1,d}|_{c^{\bar{M}}} |\zeta_{1,s}|_{c^{\bar{M}}}}{2} \\ &\le (|\tilde{\lambda}_d|_{c^{\bar{M}}} + |\zeta_{1,d}|_{c^{\bar{M}}} + |\zeta_{2,d}|_{c^{\bar{M}}})(|\tilde{\lambda}|_{c^{\bar{M}}} + |\tilde{\lambda}'|_{c^{\bar{M}}} + |\zeta_1|_{c^{\bar{M}}} + |\zeta_2'|_{c^{\bar{M}}}) + \frac{|\zeta_{1,d}|_{c^{\bar{M}}} |\zeta_{1,s}|_{c^{\bar{M}}}}{2} \end{aligned}$$

holds $P \otimes I$-a.e, and this is a bound of the form (6.8). So all the conditions of Theorem 6.6 are satisfied by (4.66), and therefore (4.71) holds for $i = 1$ as well, and $(\zeta^{(1),n} \bullet \bar{M})_{n\in\mathbb{N}}$ and $(N^{(1),n})_{n\in\mathbb{N}}$ are bounded in BMO. □

We are now ready to show the convergence of the original processes $\hat{Y}^{(i),n}$ for $i = 0, 1, 2$. This follows directly from Theorem 4.31 for $i = 1, 2$ because

$\hat{Y}^{(i),n} = \exp(Y^{(i),n})$. In the case $i = 0$, some work is still required to show the stability of the BSDE for $\hat{Y}^{(0),n}$, where we likewise add the superscript n to each of the processes on the right-hand side of (4.58) to obtain

$$d\hat{Y}_t^{(0),n} = \frac{(\hat{Y}_t^{(1),n}\lambda_t^n + \hat{\psi}_t^{(1),n})^\top d\langle M^n\rangle_t(\hat{Y}_t^{(1),n}\lambda_t^n + \hat{\psi}_t^{(1),n})}{\hat{Y}_t^{(2),n}} + \hat{\psi}_t^{(0),n} dM_t^n + d\hat{N}_t^{(0),n} \tag{4.73}$$

with terminal condition $\hat{Y}_T^{(0),n} = H^2$ for each n. Note that the drift term in (4.73) does not depend on $\hat{Y}^{(0),n}$ or $\hat{\psi}^{(0),n}$, and hence one could solve (4.73) explicitly. Nevertheless, it is convenient to once again use Theorem 6.6 to show the stability of (4.73), since that avoids some of the work otherwise needed to obtain suitable bounds for $\hat{Y}^{(0),n}$ (which would mimic parts of the proof of Theorem 6.6).

Since the driver of the equation (4.73) does not explicitly depend on the solution $\hat{Y}^{(0),n}$ (unlike (4.56) and (4.57)), we do not need to a take a logarithm as for $\hat{Y}^{(2),n}$ and $\hat{Y}^{(1),n}$. On the other hand, we again have the issue that the martingale driver $M^n = \xi^n \bullet \bar{M}$ depends on n. As in (4.61), we have the Galtchouk–Kunita–Watanabe decomposition

$$\psi^{(0),n} \bullet M^n + \hat{N}^{(0),n} = \zeta^{(0),n} \bullet \bar{M} + N^{(0),n} \tag{4.74}$$

for some $\zeta^{(0),n} \in L^2_{\mathrm{loc}}(\bar{M})$ and a local martingale $N^{(0),n}$ strongly orthogonal to $\bar{M}$. Plugging in $\tilde{\lambda}^n = \xi^n \lambda^n$ (see after (4.65)), $\hat{\psi}^{(1),n} = \hat{Y}^{(1),n}\psi^{(1),n}$ (see after (4.60)) and $\xi^n\psi^{(1),n} = \pi^n\zeta^{(1),n}$ due to part 3) of Lemma 4.30, we can thus rewrite (4.73) in the form

$$d\hat{Y}_t^{(0),n} = f_t^{(0),n}(\zeta_t^{(1),n}, \tilde{\lambda}_t^n)dI_t + \zeta_t^{(0),n} d\bar{M}_t + dN_t^{(0),n}, \quad 0 \le t \le T, \tag{4.75}$$

with $\hat{Y}_T^{(0),n} = H^2$, where $f^{(0),n} : \Omega \times [0,T] \times \mathbb{R}^{\bar{d}} \times \mathbb{R}^{\bar{d}} \to \mathbb{R}$ is the predictable function defined by

$$f_t^{(0),n}(\zeta_1, \tilde{\lambda}) = \frac{(\hat{Y}_t^{(1),n})^2(\tilde{\lambda} + \pi_t^n\zeta_1)^\top c_t^{\bar{M}}(\tilde{\lambda} + \pi_t^n\zeta_1)}{\hat{Y}_t^{(2),n}} \tag{4.76}$$

for $0 \le t \le T$ and $\zeta_1, \tilde{\lambda} \in \mathbb{R}^{\bar{d}}$. Since (4.75) is in the form (6.6), we can apply Theorem 6.6 to show its stability. Because we consider the original processes $\hat{Y}^{(i),n}$ and not their logarithms $Y^{(i),n}$, we can now remove the assumption that $H > 0$ as explained in (4.49)–(4.51).

Proposition 4.32. *Let $C \ge 1$. For any sequence (ξ^n, λ^n) in $\mathcal{D}_C(\bar{M})$ such that*

$(\xi^n, \lambda^n) \overset{d_{\mathcal{D}(\bar{M})}}{\to} (\xi^\infty, \lambda^\infty) \in \mathcal{D}_C(\bar{M})$, *it holds that*

$$\sup_{t\in[0,T]} |\hat{Y}_t^{(i),n} - \hat{Y}_t^{(i),\infty}| \xrightarrow{P} 0 \quad as\ n \to \infty \tag{4.77}$$

for $i = 2$ *as* $n \to \infty$. *If* H *is a bounded payoff, then* (4.77) *also holds for* $i = 0, 1$.

Proof. The statement for $i = 2$ follows immediately from Theorem 4.31 since $\hat{Y}^{(2),n} = \exp(Y^{(2),n})$ and the exponential is continuous. Now let H be bounded; by the argument in (4.49)–(4.51), we may assume without loss of generality that $\operatorname{ess\,inf} H > 0$. Then (4.77) for $i = 1$ likewise follows immediately from Theorem 4.31. We now consider $\hat{Y}^{(0),n}$. The BSDE (4.75) is of the form (6.14) (where $f^{(0),n}$ does not depend on $\zeta^{(0),n}$, so we omit that argument) with $\chi^n := (\zeta^{(1),n}, \tilde{\lambda}^n)$, $\varrho \equiv 0$ and $G^n := H^2$ for each n. Thus condition (e) of Theorem 6.6 is trivially satisfied. By the bounds in (4.54), condition (a) of Theorem 6.6 is also satisfied, and so is condition (b) due to Theorem 4.31.

It remains to show that the predictable functions $(f^{(0),n})_{n\in\mathbb{N}\cup\{\infty\}}$ defined by (4.76) satisfy conditions (c) and (d) in Theorem 6.6. Condition (d) follows immediately from part 4) of Lemma 4.30 together with (4.77), which we already showed for $i = 1, 2$. Next, recall that by (4.52) and (4.53), we have $\hat{Y}^{(2),n} \geq C^{-1}$ and $|\hat{Y}^{(1),n}| \leq \operatorname{ess\,sup} H$ for all $n \in \mathbb{N} \cup \{\infty\}$. Thus by (4.76), the triangle inequality and part 2) of Lemma 4.30, we have with $a := C \operatorname{ess\,sup} H^2$ that

$$|f^{(0),n}(\zeta, \tilde{\lambda})| \leq a|\tilde{\lambda} + \pi^n\zeta|^2_{c^{\bar{M}}} \leq 2a|\tilde{\lambda}|^2_{c^{\bar{M}}} + 2a|\zeta|^2_{c^{\bar{M}}} \quad P \otimes I\text{-a.e.}$$

for all $\zeta, \tilde{\lambda} \in \mathbb{R}^{\bar{d}}$ and $n \in \mathbb{N} \cup \{\infty\}$, which is a bound of the form (6.7) by (4.70). We write once again $x_d = x - x'$ and $x_s = x + x'$ for variables x, x'. By the Cauchy–Schwarz inequality for $|\cdot|_{c^{\bar{M}}}$, the identity $x^\top c^{\bar{M}} x - (x')^\top c^{\bar{M}} x' = x_d^\top c^{\bar{M}} x_s$ and part 2) of Lemma 4.30, we have $P \otimes I$-a.e. for $\zeta, \zeta', \tilde{\lambda}, \tilde{\lambda}' \in \mathbb{R}^{\bar{d}}$ that

$$\begin{aligned} |f^{(0),n}(\zeta, \tilde{\lambda}) - f^{(0),n}(\zeta', \tilde{\lambda}')| &\leq a|\tilde{\lambda}_d + \pi^n\zeta_d|_{c^{\bar{M}}}|\tilde{\lambda}_s + \pi_t^n\zeta_s|_{c^{\bar{M}}} \\ &\leq a(|\tilde{\lambda}_d|_{c^{\bar{M}}} + |\zeta_d|_{c^{\bar{M}}})(|\tilde{\lambda}_s|_{c^{\bar{M}}} + |\zeta_s|_{c^{\bar{M}}}). \end{aligned}$$

This is a bound of the form (6.8), and hence condition (c) in Theorem 6.6 is satisfied. So (4.75) satisfies all the conditions of Theorem 6.6, and therefore (4.77) holds for $i = 0$. □

We are finally ready to complete the proof of Theorem 4.26, which states that $\tilde{\psi}_2$ is continuous. This now follows straightforwardly from Proposition 4.32.

Proof of Theorem 4.26. Let $C > 0$. Since the topology of $\mathcal{D}_C(\bar{M})$ is generated by a pseudometric $d_{\mathcal{D}(\bar{M})}$, we have by Arkhangel'skii/Pontryagin [12, Proposition I.2.9 and Definitions I.1.9, I.2.4] that the space $\mathcal{D}_C(\bar{M})$ is sequential, and hence by [12, Proposition I.3.3], it suffices to check that $\tilde{\psi}_2$ is sequentially continuous.

Suppose that $(\xi^n, \lambda^n) \to (\xi^\infty, \lambda^\infty)$ in $\mathcal{D}_C(\bar{M})$ and define $S^n := S(\xi^n, \lambda^n)$ by (4.35) with $S_0^n = 0$. By Assumption 2.1, Ξ^k is bounded for each $k \in \{1, \ldots, K\}$. Denote by $\hat{Y}^{(i),n,k}$ the process $\hat{Y}^{(i),n}$ given by (4.45) for the payoff Ξ^k. Then by Proposition 4.32, $\hat{Y}_0^{(i),n,k} \to \hat{Y}_0^{(i),\infty,k}$ pointwise as $n \to \infty$, since we assumed at the beginning of Section 2 that $\mathcal{F}_0$ is P-trivial. It follows by (4.46)–(4.48) that $\varepsilon^2(1; S^n) \to \varepsilon^2(1; S^\infty)$, $c(\Xi^k; S^n) \to c(\Xi^k; S^\infty)$ and $\varepsilon^2_{\mathrm{ex}}(\Xi^k; S^n) \to \varepsilon^2_{\mathrm{ex}}(\Xi^k; S^\infty)$ as $n \to \infty$ for each $k = 1, \ldots, K$. Thus $\tilde{\psi}_2(\xi^n, \lambda^n) \to \tilde{\psi}_2(\xi^\infty, \lambda^\infty)$ by the definition (4.42) so that $\tilde{\psi}_2$ is sequentially continuous and hence continuous on $\mathcal{D}_C(\bar{M})$. □

4.3 Existence of a mean–variance equilibrium

We are finally ready to state and prove sufficient conditions for the existence of a mean–variance equilibrium in the sense of Definition 2.6 for general mean–variance utility functions U_k. In addition to our previous results on $\tilde{\psi}_1$ and $\tilde{\psi}_2$, we also use the continuity of the map ψ_3. Under Assumption 2.19, the latter is ensured by Proposition 2.17, the proof of which is postponed to Section 5. We also recall Assumption 2.29 which excludes the trivial case $\ell(\bar{\gamma}) = 1$ for all $\bar{\gamma} > \bar{\gamma}_0$ that was already considered in Corollary 2.28.

As outlined at the beginning of the section, we can now combine these results to show the existence of an equilibrium. We note that the bounds in (4.78) below can be seen as the analogue of (3.9) for the linear case since $\lambda_k^{\min} = \lambda_k^{\max} = \lambda_k$ for mean–variance utility functions U_k of the linear form (3.1). In the following, recall Definition 2.31 of the maps $\psi_1, \psi_2, \psi_3, \psi_4$ and Ψ, as well as Definitions 4.12 and 4.24 of $\tilde{\psi}_1$ and $\tilde{\psi}_2$, respectively.

Theorem 4.33. *Suppose that Assumptions 2.1, 2.19, 2.29, 4.2 and 4.15 hold and define the constants $\bar{\gamma}_2 \geq \bar{\gamma}_1$ by*

$$\bar{\gamma}_1 = E_P[\bar{\Xi}] + \sum_{k=1}^K \lambda_k^{\min}, \quad \bar{\gamma}_2 = E_P[\bar{\Xi}] + \sum_{k=1}^K \lambda_k^{\max}. \tag{4.78}$$

Then any fixed point of $\Psi : (\bar{\gamma}_0, \infty) \to \mathbb{R}$ is contained in the interval $[\bar{\gamma}_1, \bar{\gamma}_2]$. Moreover, if $\bar{\gamma}_1 > \bar{\gamma}_0$, then Ψ admits a fixed point $\bar{\gamma} \in [\bar{\gamma}_1, \bar{\gamma}_2]$ and $(1, S(\bar{\gamma}))$ is a mean–variance equilibrium market.

Proof. We want to find a fixed point of $\Psi = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1 : (\bar{\gamma}_0, \infty) \to \mathbb{R}$, where $\bar{\gamma}_0 = \operatorname{ess\,sup} \bar{\Xi}$; see (2.35). By Assumption 2.19 and Proposition 2.17, each map $\hat{y}_{U_k}$ is well defined and continuous on $(0,1) \times \mathbb{R} \times \mathbb{R}_+$, and hence $\psi_3 : (0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K \to \mathbb{R}^K$ given by

$$\psi_3\big(\ell, (c_k, \varepsilon_k^2)_{k=1}^K\big) = \big(c_k + \hat{y}_{U_k}(\ell, c_k, \varepsilon_k^2)\big)_{k=1}^K$$

is also well defined and continuous, and so is $\psi_4 : \mathbb{R}^K \to \mathbb{R}, (\gamma_k)_{k=1}^K \mapsto \sum_{k=1}^K \gamma_k$. Moreover, under Assumptions 2.1, 4.2 and 4.15, we have by Corollary 4.27 that

$$\psi_2 \circ \psi_1 = \tilde{\psi}_2 \circ \tilde{\psi}_1 : (\bar{\gamma}_0, \infty) \to (0,1] \times (\mathbb{R} \times \mathbb{R}_+)^K \to \mathbb{R}^K$$

is continuous as well. Due to Assumption 2.29 and Lemma 2.27, we have $\ell(\bar{\gamma}) \neq 1$ for $\bar{\gamma} > \bar{\gamma}_0$, where $\ell(\bar{\gamma}) = \ell(S(\bar{\gamma}))$ is the first component of $\psi_2 \circ \psi_1(\bar{\gamma})$. Thus the range of $\psi_2 \circ \psi_1$ is contained in $(0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K \to \mathbb{R}^K$ which is the domain of ψ_3. Therefore, the composition $\Psi : (\bar{\gamma}_0, \infty) \to \mathbb{R}$ is well defined and continuous.

We next show some bounds on the range of the composition. Fix some arbitrary $(\ell, (c_k, \varepsilon_k^2)_{k=1}^K) \in (0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K \to \mathbb{R}^K$ and let

$$(\gamma_1, \dots, \gamma_K) := \psi_3\big(\ell, (c_k, \varepsilon_k^2)_{k=1}^K\big).$$

Then by Assumption 2.19 and Proposition 2.17, we have

$$c_k + \frac{\lambda_k^{\min}}{\ell} \leq \gamma_k := c_k + \hat{y}_{U_k}(\ell, c_k, \varepsilon_k^2) \leq c_k + \frac{\lambda_k^{\max}}{\ell}$$

for $k \in \{1, \dots, K\}$. Summing over k yields with $\bar{c} := \sum_{k=1}^K c_k$ that

$$\bar{c} + \sum_{k=1}^K \frac{\lambda_k^{\min}}{\ell} \leq \sum_{k=1}^K \gamma_k \leq \bar{c} + \sum_{k=1}^K \frac{\lambda_k^{\max}}{\ell}.$$

Since $\sum_{k=1}^K \gamma_k = (\psi_4 \circ \psi_3)(\ell, (c_k, \varepsilon_k^2)_{k=1}^K)$, we thus have

$$\bar{c} + \sum_{k=1}^K \frac{\lambda_k^{\min}}{\ell} \leq (\psi_4 \circ \psi_3)\big(\ell, (c_k, \varepsilon_k^2)_{k=1}^K\big) \leq \bar{c} + \sum_{k=1}^K \frac{\lambda_k^{\max}}{\ell} \tag{4.79}$$

for all $(\ell, (c_k, \varepsilon_k^2)_{k=1}^K) \in (0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K \to \mathbb{R}^K$. For $\bar{\gamma} > \bar{\gamma}_0$, we set

$$\Big(\ell(\bar{\gamma}), \big(c_k(\bar{\gamma}), \varepsilon_k^2(\bar{\gamma})\big)_{k=1}^K\Big) := (\psi_2 \circ \psi_1)(\bar{\gamma}).$$

Then by Proposition 3.3, we have

$$\bar{c}(\bar{\gamma}) := \sum_{k=1}^{K} c_k(\bar{\gamma}) = \bar{\gamma} - \frac{\bar{\gamma} - E_P[\bar{\Xi}]}{\ell(\bar{\gamma})}.$$

By plugging into (4.79) and recalling that $\Psi = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1$, we obtain

$$\bar{\gamma} - \frac{\bar{\gamma} - E_P[\bar{\Xi}] - \sum_{k=1}^{K} \lambda_k^{\min}}{\ell(\bar{\gamma})} \leq \Psi(\bar{\gamma}) \leq \bar{\gamma} - \frac{\bar{\gamma} - E_P[\bar{\Xi}] - \sum_{k=1}^{K} \lambda_k^{\max}}{\ell(\bar{\gamma})}. \tag{4.80}$$

Then if $\bar{\gamma}$ is a fixed point of Ψ, we must have

$$\bar{\gamma} - E_P[\bar{\Xi}] - \sum_{k=1}^{K} \lambda_k^{\max} \leq 0 \leq \bar{\gamma} - E_P[\bar{\Xi}] - \sum_{k=1}^{K} \lambda_k^{\min},$$

which implies that $\bar{\gamma} \in [\bar{\gamma}_1, \bar{\gamma}_2]$ by the definition (4.78). To show the existence statement, suppose that $\bar{\gamma}_2 \geq \bar{\gamma}_1 > \bar{\gamma}_0$. We have the inequalities $\Psi(\bar{\gamma}_1) \geq \bar{\gamma}_1$ and $\Psi(\bar{\gamma}_2) \leq \bar{\gamma}_2$ by plugging (4.78) into the left and right bounds in (4.80), respectively. Therefore, the function $\bar{\gamma} \mapsto \Psi(\bar{\gamma}) - \bar{\gamma}$ is continuous and changes sign between $\bar{\gamma}_1$ and $\bar{\gamma}_2$. By the intermediate value theorem, there exists some $\bar{\gamma} \in [\bar{\gamma}_1, \bar{\gamma}_2]$ such that $\Psi(\bar{\gamma}) - \bar{\gamma} = 0$, i.e., $\bar{\gamma}$ is a fixed point of Ψ. Therefore $(1, S(\bar{\gamma}))$ is a mean–variance equilibrium market by Theorem 2.32. □

The following two sections complete the proof of Theorem 4.33 by showing the results so far given without proof. Namely, it still remains to prove Proposition 2.17, which is related to the continuity of ψ_3, as well as the BSDE results in Theorem 6.6 that was used to obtain the continuity of $\tilde{\psi}_2$. Both of these topics are studied in abstract settings, since they have some independent interest beyond their role in proving Theorem 4.33. So that there are no circular arguments, those results are proven without using any of the results from the previous sections.

5 An abstract mean–variance problem

The main goal of this section is to study the abstract mean–variance optimisation problem defined in (2.28), which reads

$$U\big(c + (1-\ell)y, \sqrt{\varepsilon^2 + \ell(1-\ell)y^2}\big) \longrightarrow \max_{y \geq 0}! \tag{5.1}$$

for a mean–variance utility function U and constants $(\ell, c, \varepsilon^2) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$. We refer to Section 2 for a financial interpretation of this problem. In particular, we want to prove Proposition 2.17 which we now recall.

Proposition 2.17. *Let U be a mean–variance utility function. Suppose that there exist constants $\lambda^{\min}, \lambda^{\max} \in (0,\infty)$ such that*

$$\frac{\sigma}{\lambda^{\max}} \leq \mathcal{S}_U(\mu,\sigma) \leq \frac{\sigma}{\lambda^{\min}} \quad \textit{for all } (\mu,\sigma) \in \mathbb{R} \times \mathbb{R}_+. \tag{2.30}$$

Then there exists a continuous map $\hat{y}_U : (0,1)\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ such that $\hat{y}_U(\ell,c,\varepsilon^2)$ is the unique solution to (5.1) *for each $(\ell,c,\varepsilon^2) \in (0,1)\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$. Moreover, the map $\hat{y}_U$ satisfies the bounds*

$$\frac{\lambda^{\min}}{\ell} \leq \hat{y}_U(\ell,c,\varepsilon^2) \leq \frac{\lambda^{\max}}{\ell} \quad \textit{for all } (\ell,c,\varepsilon^2) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+. \tag{2.31}$$

On the way to proving Proposition 2.17, we show some basic properties of general mean–variance utility functions U and study the existence and uniqueness of solutions to (5.1) for a fixed parameter $\theta := (\ell, c, \varepsilon^2)$. The next step is to study the subset of parameters $\theta \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$ such that a unique solution exists, and to show that it depends continuously on θ. We then prove the bounds (2.31) on the solution to (5.1), and this yields Proposition 2.17 (see after Lemma 5.6 below). Finally, we discuss how these results may be applied to the equilibrium problem of Section 2 under weaker assumptions, i.e., in the absence of the bounds (2.30) on the mean–variance utility functions of the agents.

We note that some of our results in this section are well known/folklore in the economics literature. In particular, condition (c) in Theorem 5.3 below, which is equivalent to the existence and uniqueness of solutions to (5.1), is already given in Koch-Medina/Wenzelburger [85, Proposition 1]. We nevertheless give full proofs of these results in order to keep this section self-contained. This also yields auxiliary results that are helpful for proving our results on the continuity of the solution map $\theta \mapsto \hat{y}_U(\theta)$; the latter (starting from Proposition 5.5 below) are new to the best of our knowledge.

We begin by recalling some notions related to mean–variance utility functions, with the same notation as in [85]; note that we always write $\mathbb{R}_+ = [0,\infty)$. Recall that by Definition 2.5, $U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is a *mean–variance utility function* if it is strictly quasiconcave, strictly increasing in the first variable, strictly decreasing in the second variable, twice continuously differentiable and nondegenerate in the sense that $|\nabla U(\mu,\sigma)| > 0$ for all $\mu \in \mathbb{R}$ and $\sigma \geq 0$. As introduced before

Proposition 2.17, the *indifference curve* through $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$ is given by

$$\mathcal{I}_U(\mu, \sigma) := \{(\tilde{\mu}, \tilde{\sigma}) \in \mathbb{R} \times \mathbb{R}_+ : U(\tilde{\mu}, \tilde{\sigma}) = U(\mu, \sigma)\},$$

and the *slope* of $\mathcal{I}_U(\mu, \sigma)$ is given by the function $\mathcal{S}_U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$, where

$$\mathcal{S}_U(\mu, \sigma) = -\frac{\partial_\sigma U(\mu, \sigma)}{\partial_\mu U(\mu, \sigma)}. \tag{5.2}$$

This is the usual notion of the slope of $\mathcal{I}_U(\mu, \sigma)$ when it is viewed on a plane with σ on the horizontal and μ on the vertical axis. We show below that $\partial_\mu U$ is strictly positive so that $\mathcal{S}_U$ is well defined and nonnegative on $\mathbb{R} \times \mathbb{R}_+$. On the other hand, $\partial_\sigma U$ can take the value 0 at $\sigma = 0$; indeed, this must hold for mean–variance utility functions that satisfy the bound on the right-hand side of (2.30) including linear ones (2.9) of the form $U(\mu, \sigma) = \mu - \frac{1}{2\lambda}\sigma^2$. Finally, we also define the *limiting slope* of $\mathcal{I}_U(\mu, \sigma)$ by

$$\rho_U(\mu, \sigma) := \sup \big\{ \mathcal{S}_U(\tilde{\mu}, \tilde{\sigma}) : (\tilde{\mu}, \tilde{\sigma}) \in \mathcal{I}_U(\mu, \sigma) \big\}. \tag{5.3}$$

The following technical result gives a characterisation of the indifference curves associated with a mean–variance utility function.

Lemma 5.1. *Let $U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a mean–variance utility function. The following statements hold:*

1) *$\partial_\mu U(\mu, \sigma) > 0$ for all $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$. Hence $\mathcal{S}_U$ is well defined and nonnegative.*

2) *For each $\mu_0 \in \mathbb{R}$, there exist $\sigma^*(\mu_0) \in (0, \infty]$ and $f_{\mu_0} : [0, \sigma^*(\mu_0)) \to \mathbb{R}$ such that the indifference curve through $(\mu_0, 0)$ is given by*

$$\mathcal{I}_U(\mu_0, 0) = \big\{ \big(f_{\mu_0}(\sigma), \sigma\big) : \sigma \in \big[0, \sigma^*(\mu_0)\big) \big\}. \tag{5.4}$$

Moreover, f_{μ_0} is strictly convex, strictly increasing, twice continuously differentiable and satisfies $f_{\mu_0}(0) = \mu_0$ and $f_{\mu_0}(\sigma) \nearrow \infty$ as $\sigma \nearrow \sigma^*(\mu_0)$.

3) *Every indifference curve for U has the form (5.4) for some $\mu_0 \in \mathbb{R}$.*

4) *$\mathcal{S}_U(f_{\mu_0}(\sigma), \sigma) = f'_{\mu_0}(\sigma)$ for each $\mu_0 \in \mathbb{R}$ and $\sigma \in [0, \sigma^*(\mu_0))$. As a consequence, $\mathcal{S}_U(\mu, \sigma) > 0$ for all $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+ \setminus \{0\}$.*

5) *For $\mu_1 > \mu_2$, it holds that $\sigma^*(\mu_1) \le \sigma^*(\mu_2)$ and $f_{\mu_1}(\sigma) > f_{\mu_2}(\sigma)$ for all $\sigma \in [0, \sigma^*(\mu_1))$.*

6) *The map $\mu \mapsto \rho_U(\mu, 0) = \lim_{\sigma \nearrow \sigma^*(\mu)} f'_\mu(\sigma)$ takes strictly positive values,* *is nondecreasing and left-continuous. If U is concave, then the map is constant, i.e.,* $\rho_U(\mu,\sigma)=\rho_U$ *for some* $\rho_U\in(0,\infty]$ *and all* $(\mu,\sigma)\in\mathbb{R}\times\mathbb{R}_+$.

Proof. **1)** Fix $(\mu,\sigma)\in\mathbb{R}\times\mathbb{R}_+$ and some arbitrary $\mu'>\mu$. Because U is strictly increasing in μ, we have $U(\mu',\sigma)>U(\mu,\sigma)$. By continuity of U, there exists some small $\delta>0$ such that $U(\mu',\sigma+\delta)>U(\mu,\sigma)$. On the other hand, we have $U(\mu,\sigma+\delta)<U(\mu,\sigma)$ since U is strictly decreasing in σ. Thus, we have the bounds

$$U(\mu,\sigma+\delta)<U(\mu,\sigma)<U(\mu',\sigma+\delta).$$

The intermediate value theorem yields $\tilde\mu\in(\mu,\mu')$ such that $U(\tilde\mu,\sigma+\delta)=U(\mu,\sigma)$. By the quasiconcavity of U, we have

$$U\big(\lambda\tilde\mu+(1-\lambda)\mu,\sigma+\lambda\delta\big)\geq U(\mu,\sigma)$$

for all $\lambda\in[0,1]$. Differentiating with respect to λ at $\lambda=0$, we obtain

$$(\tilde\mu-\mu)\partial_\mu U(\mu,\sigma)+\delta\partial_\sigma U(\mu,\sigma)\geq 0. \tag{5.5}$$

Note that $\partial_\mu U\geq 0$ and $\partial_\sigma U\leq 0$ by the monotonicity properties of U. From the nondegeneracy assumption, we know that $\partial_\mu U(\mu,\sigma)$ and $\partial_\sigma U(\mu,\sigma)$ cannot both be zero. Hence if $\partial_\mu U(\mu,\sigma)=0$, then we must have $\partial_\sigma U(\mu,\sigma)<0$, which together with $\delta>0$ contradicts (5.5). Therefore, we must have $\partial_\mu U(\mu,\sigma)>0$ so that $\mathcal{S}_U$ is well defined and nonnegative on $\mathbb{R}\times\mathbb{R}_+$.

2) As U is twice continuously differentiable, it follows that $\mathcal{S}_U$ is continuously differentiable and in particular locally Lipschitz. Thus we can find a unique local solution f_{μ_0} to the ordinary differential equation

$$\begin{cases} f'_{\mu_0}(\sigma)=\mathcal{S}_U\big(f_{\mu_0}(\sigma),\sigma\big), & \sigma\geq 0,\\ f_{\mu_0}(0)=\mu_0. \end{cases} \tag{5.6}$$

This gives a continuous solution f_{μ_0} on a a maximal domain $[0,\sigma^*(\mu_0))$, where $\sigma^*(\mu_0)\in(0,\infty]$. Since $\mathcal{S}_U$ is nonnegative, f_{μ_0} is increasing. Hence if $\sigma^*(\mu_0)<\infty$, we must have $\lim_{\sigma\nearrow\sigma^*(\mu_0)}f_{\mu_0}(\sigma)=\infty$; otherwise, f_{μ_0} could be extended continuously to $[0,\sigma^*(\mu_0)]$ and then to a larger open interval $[0,\tilde\sigma)$ as a solution to (5.6), contradicting the maximality of $\sigma^*(\mu_0)$.

To show (5.4), we first note that (5.6) and the definition of $\mathcal{S}_U$ yield

$$\begin{aligned}\frac{d}{d\sigma}U\big(f_{\mu_0}(\sigma),\sigma\big) &= f'_{\mu_0}(\sigma)\partial_\mu U\big(f_{\mu_0}(\sigma),\sigma\big) + \partial_\sigma U\big(f_{\mu_0}(\sigma),\sigma\big) \\ &= \mathcal{S}\big(f_{\mu_0}(\sigma),\sigma\big)\partial_\mu U\big(f_{\mu_0}(\sigma),\sigma\big) + \partial_\sigma U\big(f_{\mu_0}(\sigma),\sigma\big) = 0 \qquad (5.7)\end{aligned}$$

for $\sigma \in [0,\sigma^*(\mu_0))$, so that U is constant on $\tilde{I}_{\mu_0} := \{(f_{\mu_0}(\sigma),\sigma) : \sigma \in [0,\sigma^*(\mu_0))\}$, and this gives the inclusion "$\supseteq$" in (5.4). To show "$\subseteq$" , suppose for a contradiction that $(\mu,\sigma) \in \mathcal{I}_U(\mu_0,0) \setminus \tilde{I}_{\mu_0}$. If $\sigma < \sigma^*(\mu_0)$, then we have

$$U(\mu,\sigma) \neq U\big(f_{\mu_0}(\sigma),\sigma\big) = U(\mu_0,0)$$

as $\mu \neq f_{\mu_0}(\sigma)$ and U is strictly increasing in μ, so that (μ,σ) cannot belong to $\mathcal{I}_U(\mu_0,0)$. If $\sigma \geq \sigma^*(\mu_0)$, we must have $\sigma^*(\mu_0) < \infty$ and $\lim_{\sigma\nearrow\sigma^*(\mu_0)} f_{\mu_0}(\sigma) = \infty$ so that there exists $\sigma' < \sigma^*(\mu_0)$ such that $\mu' := f_{\mu_0}(\sigma') > \mu$. Since $\mu' > \mu$ and $\sigma' < \sigma^*(\mu_0) \leq \sigma$, we obtain $U(\mu',\sigma') > U(\mu,\sigma)$. Observing that

$$U(\mu',\sigma') = U\big(f_{\mu_0}(\sigma'),\sigma'\big) = U(\mu_0,0) = U(\mu,\sigma)$$

since $(\mu,\sigma) \in \mathcal{I}_U(\mu_0,0)$, this leads to a contradiction. So (5.4) holds.

It remains to prove the properties of f_{μ_0}. By 1) and since U is twice continuously differentiable, $\mathcal{S}_U$ is continuously differentiable. Thus differentiating (5.6) with the chain rule yields that f_{μ_0} is twice continuously differentiable. The initial value $f_{\mu_0}(0) = \mu_0$ is given by (5.6). To show that f_{μ_0} is strictly convex, take $\lambda \in (0,1)$ and $\sigma_1,\sigma_2 \in [0,\sigma^*(\mu_0))$. By (5.4), we have with $\overline{\lambda} := 1-\lambda$ that

$$U\big(f_{\mu_0}(\sigma_1),\sigma_1\big) = U\big(f_{\mu_0}(\sigma_2),\sigma_2\big) = U\big(f_{\mu_0}(\lambda\sigma_1+\overline{\lambda}\sigma_2),\lambda\sigma_1+\overline{\lambda}\sigma_2\big).$$

The strict quasiconcavity of U then yields

$$U\big(\lambda f_{\mu_0}(\sigma_1)+\overline{\lambda}f_{\mu_0}(\sigma_2),\lambda\sigma_1+\overline{\lambda}\sigma_2\big) > U\big(f_{\mu_0}(\lambda\sigma_1+\overline{\lambda}\sigma_2),\lambda\sigma_1+\overline{\lambda}\sigma_2\big) \qquad (5.8)$$

so that $\lambda f_{\mu_0}(\sigma_1)+\overline{\lambda}f_{\mu_0}(\sigma_2) > f_{\mu_0}(\lambda\sigma_1+\overline{\lambda}\sigma_2)$ as U is strictly increasing in μ. Since $\sigma_1,\sigma_2 \in [0,\sigma^*(\mu_0))$ are arbitrary, this shows that f_{μ_0} is strictly convex. Since f_{μ_0} is also increasing as seen above, we deduce that it is strictly increasing with $\lim_{\sigma\to\infty} f_{\mu_0}(\sigma) = \infty$ in the case $\sigma^*(\mu_0) = \infty$. Since we have already shown that $\lim_{\sigma\to\sigma^*(\mu_0)} f_{\mu_0}(\sigma) = \infty$ for $\sigma^*(\mu_0) < \infty$, this concludes the proof of 2).

3) Fix $(\tilde{\mu},\tilde{\sigma}) \in \mathbb{R}\times\mathbb{R}_+$ and consider the indifference curve $\mathcal{I}_U(\tilde{\mu},\tilde{\sigma})$. We consider the backward differential equation

$$\begin{cases} g'(\sigma) = \mathcal{S}_U\big(g(\sigma), \sigma\big), & \sigma \leq \tilde{\sigma}, \\ g(\tilde{\sigma}) = \tilde{\mu}. \end{cases} \tag{5.9}$$

Similarly to (5.6), (5.9) admits a solution on a maximal open interval $(\sigma_*, \tilde{\sigma}]$ for some $\sigma_* \in [0, \tilde{\sigma})$. Moreover, since $g' \geq 0$ so that g is increasing, we must have by maximality that $\lim_{\sigma \searrow \sigma_*} g(\sigma) = -\infty$ if $\sigma_* > 0$. As in the proof of 2), we note that $\tilde{I}_{(\tilde{\mu}, \tilde{\sigma})} := \{(g(\sigma), \sigma) : \sigma \in (\sigma_*, \tilde{\sigma}]\}$ is contained in $\mathcal{I}_U(\tilde{\mu}, \tilde{\sigma})$ since $\frac{d}{d\sigma} U(g(\sigma), \sigma) \equiv 0$ by the same calculation as in (5.7). Likewise as in (5.8), it follows by the quasiconcavity of U that g is convex on $(\sigma_*, \tilde{\sigma}]$. Hence the derivative $g'(\sigma) = \mathcal{S}_U(g(\sigma), \sigma)$ is nonnegative and increasing on $(\sigma_*, \tilde{\sigma}]$ with a maximum at $\tilde{\sigma}$. Thus we have $0 \leq g'(\sigma) \leq g'(\tilde{\sigma})$ for $\sigma \in (\sigma_*, \tilde{\sigma}]$ and $g(\tilde{\sigma}) = \tilde{\mu}$, which yields

$$\tilde{\mu} - g'(\tilde{\sigma})(\tilde{\sigma} - \sigma) \leq g(\sigma) \leq \tilde{\mu}$$

for each $\sigma \in (\sigma_*, \tilde{\sigma}]$. The left-hand side is bounded below by $\tilde{\mu} - g'(\tilde{\sigma})\tilde{\sigma}$ independently of σ so that g is bounded. Since $\lim_{\sigma \searrow \sigma_*} g(\sigma) = -\infty$ cannot hold, we must have $\sigma_* = 0$. Because g is bounded and increasing, it can thus be extended continuously to the closed interval $[0, \tilde{\sigma})$ and we likewise have $(g(0), 0) \in \mathcal{I}_U(\tilde{\mu}, \tilde{\sigma})$. Thus for $\mu_0 := g(0)$, we have $U(\mu_0, 0) = U(\tilde{\mu}, \tilde{\sigma})$ and $\mathcal{I}_U(\mu_0, 0) = \mathcal{I}_U(\tilde{\mu}, \tilde{\sigma})$ by the definition of the indifference curve $\mathcal{I}_U$. Hence part 2) yields

$$\mathcal{I}_U(\tilde{\mu}, \tilde{\sigma}) = \mathcal{I}_U(\mu_0, 0) = \big\{\big(f_{\mu_0}(\sigma), \sigma\big) : \sigma \in \big[0, \sigma^*(\mu_0)\big)\big\},$$

and this proves 3).

4) The fact that $\mathcal{S}_U(f_{\mu_0}(\sigma), \sigma) = f'_{\mu_0}(\sigma)$ for all $\sigma \geq 0$ follows by the construction of f_{μ_0} as a solution to (5.6). Fix now some $\mu \in \mathbb{R}$ and $\sigma > 0$. By 3), there exists some $\mu_0 \in \mathbb{R}$ such that

$$\mathcal{I}_U(\mu, \sigma) = \mathcal{I}_U(\mu_0, 0) = \big\{\big(f_{\mu_0}(\sigma), \sigma\big) : \sigma \in \big[0, \sigma^*(\mu_0)\big)\big\}.$$

Since $U(\mu, \sigma) = U(f_{\mu_0}(\sigma), \sigma)$, the monotonicity of U yields $\mu = f_{\mu_0}(\sigma)$ so that $\mathcal{S}_U(\mu, \sigma) = f'_{\mu_0}(\sigma)$. Since f_{μ_0} is increasing, we have $f'_{\mu_0} \geq 0$ and in particular $f'_{\mu_0}(0) \geq 0$. Since f'_{μ_0} is strictly increasing as f_{μ_0} is strictly convex, we obtain $\mathcal{S}_U(\mu, \sigma) = f'_{\mu_0}(\sigma) > 0$.

5) Fix $\mu_1 > \mu_2$. By 2) and as U is strictly increasing in μ, we have

$$U\big(f_{\mu_1}(\sigma),\sigma\big) = U(\mu_1,0) > U(\mu_2,0) = U\big(f_{\mu_2}(\sigma),\sigma\big), \tag{5.10}$$

so that $f_{\mu_1}(\sigma) > f_{\mu_2}(\sigma)$ for all $\sigma \in [0,\sigma^*(\mu_1)\wedge\sigma^*(\mu_2))$. Assume for a contradiction that $\sigma^*(\mu_1) > \sigma^*(\mu_2)$. Note that (5.10) yields

$$\lim_{\sigma\nearrow\sigma^*(\mu_2)} f_{\mu_1}(\sigma) \geq \lim_{\sigma\nearrow\sigma^*(\mu_2)} f_{\mu_2}(\sigma) = \infty.$$

This leads to a contradiction since f_{μ_1} is continuous and finite on the interval $[0,\sigma^*(\mu_2)] \subseteq [0,\sigma^*(\mu_1))$. Hence we must have $\sigma^*(\mu_1) \leq \sigma^*(\mu_2)$. In that case, we have already shown with (5.10) that $f_{\mu_1}(\sigma) > f_{\mu_2}(\sigma)$ for all $\sigma \in [0,\sigma^*(\mu_1))$, which concludes the proof.

6) Let $\mu \in \mathbb{R}$. By (5.3), (5.4), 4) and since f_μ is convex by 2), we have

$$\rho_U(\mu,0) = \sup_{\sigma\in[0,\sigma^*(\mu))} \mathcal{S}_U\big(f_\mu(\sigma),\sigma\big) = \sup_{\sigma\in[0,\sigma^*(\mu))} f'_\mu(\sigma) = \lim_{\sigma\nearrow\sigma^*(\mu)} f'_\mu(\sigma). \tag{5.11}$$

Choosing an arbitrary $\sigma \in (0,\sigma^*(\mu))$, we have by 4) that $f'_\mu(\sigma) > 0$ so that $\rho_U(\mu,0) \geq f'_\mu(\sigma) > 0$, i.e., ρ_U takes strictly positive values.

To show that $\mu \mapsto \rho_U(\mu,0)$ is increasing, we fix $\mu_1 > \mu_2$ and claim that $\rho_U(\mu_1,0) \geq \rho_U(\mu_2,0)$. If $\sigma^*(\mu_1) < \infty$, then $\lim_{\sigma\nearrow\sigma^*(\mu_1)} f_{\mu_1}(\sigma) = \infty$ so that f'_{μ_1} cannot be bounded above. In that case, we have $\rho_U(\mu_1,0) = \infty \geq \rho_U(\mu_2,0)$ and the claim holds. Thus we may assume $\sigma^*(\mu_1) = \infty$ so that $\sigma^*(\mu_2) = \infty$ by 5). Suppose now for a contradiction that $\rho_U(\mu_2,0) > \rho_U(\mu_1,0)$. By (5.11), we have $f'_{\mu_2}(\sigma) - f'_{\mu_1}(\sigma) \to \rho_U(\mu_2,0) - \rho_U(\mu_1,0)$ as $\sigma \to \infty$. This yields

$$f'_{\mu_2}(\sigma) - f'_{\mu_1}(\sigma) \geq \frac{1}{2}\big(\rho_U(\mu_2,0) - \rho_U(\mu_1,0)\big) > 0$$

for all large enough σ; in words, f_{μ_2} must grow asymptotically faster that f_{μ_1}. Integrating this inequality, we deduce that $f_{\mu_2}(\sigma) > f_{\mu_1}(\sigma)$ for large enough $\sigma > 0$, which leads to a contradiction due to 5). Therefore, the map $\mu \mapsto \rho_U(\mu,0)$ is increasing, as claimed.

To show the left-continuity, consider an increasing sequence $(\mu_n)_{n\in\mathbb{N}}$ such that $\mu_n \nearrow \mu_\infty$. Thus $U(\mu_n,0) \nearrow U(\mu_\infty,0)$ as $n \to \infty$ by the continuity and monotonicity of U. For small $\delta > 0$, we can find by (5.11) some $\sigma' \in [0,\sigma^*(\mu_\infty))$ such that $f'_{\mu_\infty}(\sigma') \geq \rho_U(\mu_\infty, 0) - \delta$. Setting $\mu' := f_{\mu_\infty}(\sigma')$, it follows by (5.4) that

$$U\big(f_{\mu_n}(\sigma'), \sigma'\big) = U(\mu_n, 0) \nearrow U(\mu_\infty, 0) = U(\mu', \sigma')$$

as $n \to \infty$ and hence $f_{\mu_n}(\sigma') \nearrow \mu'$ since U is continuous and strictly increasing in μ. Combining this with the first equality in (5.11), the continuity of $\mathcal{S}_U$ and the choice of $(\mu', \sigma') \in \mathcal{I}_U(\mu_\infty, 0)$ yields

$$\lim_{n\to\infty} \rho_U(\mu_n, 0) \geq \lim_{n\to\infty} \mathcal{S}_U\big(f_{\mu_n}(\sigma'), \sigma'\big) = \mathcal{S}_U(\mu', \sigma') \geq \rho_U(\mu_\infty, 0) - \delta.$$

Therefore $\lim_{n\to\infty} \rho_U(\mu_n, 0) \geq \rho_U(\mu_\infty, 0)$ as $\delta > 0$ is arbitrary. Since we have already shown that $\mu \mapsto \rho_U(\mu, 0)$ is increasing, this proves the left-continuity.

Finally, suppose that U is concave (not just quasiconcave) and fix some $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$. By part 3), there exists some $\mu_0 \in \mathbb{R}$ such that the indifference curve through (μ, σ) is given by (5.4). Since U is strictly increasing in the first variable and strictly decreasing in the second variable, it follows that the closed superlevel set of U at the level $U(\mu, \sigma)$ is the convex set given by

$$\begin{aligned} SL_U(\mu, \sigma) &:= \{(\mu', \sigma') \in \mathbb{R} \times \mathbb{R}_+ : U(\mu', \sigma') \geq U(\mu, \sigma)\} \\ &= \{(\mu', \sigma') \in \mathbb{R} \times \mathbb{R}_+ : \sigma' < \sigma^*(\mu_0) \text{ and } \mu' \geq f_{\mu_0}(\sigma')\}. \end{aligned}$$

We know that f'_{μ_0} is nondecreasing with $\lim_{\sigma \nearrow \sigma^*(\mu_0)} f'_{\mu_0}(\sigma) = \rho_U(\mu_0, 0) = \rho_U(\mu, \sigma)$. Thus a vector of the form $(1, a)$ belongs to the asymptotic cone of $SL_U(\mu, \sigma)$ (see Hiriart-Urruty/Lemaréchal [65, Section III.2.2] for the definition) if and only if $a \geq \rho_U(\mu, \sigma)$ (including in the case $\rho_U(\mu, \sigma) = \infty$). On the other hand, since U is concave (i.e., $-U$ is convex), we have by [65, Proposition IV.3.2.5] that the asymptotic cone of the superlevel set $SL_U(\mu, \sigma)$ does not depend on the choice of (μ, σ). Therefore the map ρ_U is constant, as claimed. □

We now return to the abstract mean–variance optimisation problem (5.1). The following result provides a first-order condition for a solution to (5.1), which later allows us to obtain necessary and sufficient conditions for the existence of a unique solution. For now, we fix a mean–variance utility function U and a triplet $(\ell, c, \varepsilon^2) \in (0, 1) \times \mathbb{R} \times \mathbb{R}_+$. We also define $\mu : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\mu(y) := c + (1 - \ell)y, \quad \sigma^2(y) := \varepsilon^2 + \ell(1 - \ell)y^2, \quad y \geq 0. \tag{5.12}$$

We note the change of notation: henceforth, μ and σ^2 are no longer constants as in Lemma 5.1, but rather the "mean" and "variance" (as implied in the statement of (5.1)) that are achieved by $y \geq 0$ for a given triplet (ℓ, c, ε^2).

In the following, we define the function $s : \mathbb{R}_+ \to [0, \infty]$ by

$$s(y) = \begin{cases} \frac{\sigma(y)}{\ell y}, & y > 0, \\ \infty, & y = 0 \text{ and } \varepsilon^2 > 0, \\ \sqrt{\frac{1-\ell}{\ell}}, & y = 0 \text{ and } \varepsilon^2 = 0. \end{cases} \tag{5.13}$$

It follows straightforwardly from (5.12) that $s(y) \to s(0)$ as $y \searrow 0$. The function $s : \mathbb{R}_+ \to [0, \infty]$ represents the slope of the curve $\{(\sigma(y), \mu(y)) : y \geq 0\}$ in the σ-μ-plane because $s(y) = \mu'(y)/\sigma'(y)$ for $y > 0$. It will also be helpful to reparametrise $\mu(y)$ in terms of $\sigma(y)$ rather than y. By solving the second equation in (5.12) for y and plugging into the first equation, we obtain for $y \geq 0$ that

$$y = \sqrt{\frac{\sigma^2(y) - \varepsilon^2}{\ell(1-\ell)}} \quad \text{and} \quad \mu(y) = c + \sqrt{\frac{(1-\ell)(\sigma^2(y) - \varepsilon^2)}{\ell}} =: g\big(\sigma(y)\big), \tag{5.14}$$

where $g : [\varepsilon, \infty) \to \mathbb{R}$ has the derivative

$$g'(\sigma) = \begin{cases} \sqrt{\frac{1-\ell}{\ell}} \frac{\sigma}{\sqrt{\sigma^2 - \varepsilon^2}}, & \sigma > \varepsilon, \\ \infty, & \sigma = \varepsilon > 0, \\ \sqrt{\frac{1-\ell}{\ell}}, & \sigma = \varepsilon = 0. \end{cases}$$

Thus by plugging in the first part of (5.14), we obtain that the slope of the mean–variance efficient frontier $\mathcal{E} := \{(g(\sigma), \sigma) : \sigma \geq \varepsilon\}$ at $(\mu(y), \sigma(y))$ (in the σ-μ-plane) is given in all cases by

$$g'\big(\sigma(y)\big) = s(y), \quad y \geq 0. \tag{5.15}$$

In order to find a solution to the problem (5.1), we obtain a first-order condition by comparing the slopes $\mathcal{S}_U$ (see (5.2)) and s of the indifference curve and the mean–variance efficient frontier, respectively, at the point $(\mu(y), \sigma(y))$ for $y \geq 0$.

Proposition 5.2. *Fix $y \geq 0$. If $\mathcal{S}_U(\mu(y), \sigma(y)) \geq s(y)$, then*

$$U\big(\mu(y), \sigma(y)\big) > U\big(\mu(\tilde{y}), \sigma(\tilde{y})\big) \tag{5.16}$$

for all $\tilde{y} > y$*. If* $\mathcal{S}_U(\mu(y), \sigma(y)) \le s(y)$*, then*

$$U\big(\mu(y), \sigma(y)\big) > U\big(\mu(\tilde{y}), \sigma(\tilde{y})\big) \tag{5.17}$$

for all $\tilde{y} < y$*. In particular, if* $\mathcal{S}_U(\mu(y), \sigma(y)) = s(y)$*, then* y *is the unique solution to* (5.1).

Proof. Fix $y \ge 0$ and write $\sigma_y := \sigma(y)$ and $\mu_y := \mu(y) = g(\sigma_y)$. By part 3) of Lemma 5.1, there exists some $\mu_0 \in \mathbb{R}$ such that $(\mu_y, \sigma_y) = (f_{\mu_0}(\sigma_y), \sigma_y)$. Then the indifference curve through (μ_y, σ_y) is given by $\{(f_{\mu_0}(\sigma), \sigma) : \sigma \in [0, \sigma^*(\mu_0))\}$. We now compare f_{μ_0} and g in the vicinity of σ_y. Note that $f_{\mu_0}(\sigma_y) = g(\sigma_y) = \mu_y$ and f_{μ_0} is strictly convex by part 2) of Lemma 5.1, whereas g is concave by its definition in (5.14). If $y = 0$ and $s(y) = \infty$, both statements (5.16) and (5.17) are vacuously true since neither $\tilde{y} < y$ nor $\mathcal{S}_U(\mu(y), \sigma(y)) \ge s(y)$ can hold. Thus we need only consider the case where $s(y)$ is finite, and $\mathcal{S}_U$ is always finite by part 1) of Lemma 5.1.

First, $\mathcal{S}_U(\mu(y), \sigma(y)) \ge s(y)$ implies $f'_{\mu_0}(\sigma_y) \ge g'(\sigma_y)$ by (5.15) and part 4) of Lemma 5.1. As f_{μ_0} is strictly convex and g is concave (so that $f''_{\mu_0} > 0 \ge g''$), we have $f'_{\mu_0}(\tilde{\sigma}) > g'(\tilde{\sigma})$ for all $\tilde{\sigma} \in (\sigma, \sigma^*(\mu_0))$. Together with $f_{\mu_0}(\sigma_y) = g(\sigma_y) = \mu_y$, it follows that $f_{\mu_0}(\tilde{\sigma}) > g(\tilde{\sigma})$ for all $\tilde{\sigma} \in (\sigma_y, \sigma^*(\mu_0))$. Since U is strictly increasing in μ, we deduce that

$$U(\mu_y, \sigma_y) = U\big(f_{\mu_0}(\tilde{\sigma}), \tilde{\sigma}\big) > U\big(g(\tilde{\sigma}), \tilde{\sigma}\big), \quad \tilde{\sigma} \in (\sigma_y, \sigma^*(\mu_0)). \tag{5.18}$$

Consider now some $\tilde{\sigma} \ge \sigma^*(\mu_0)$. Since $f_{\mu_0}(\hat{\sigma}) \nearrow \infty$ as $\hat{\sigma} \nearrow \sigma^*(\mu_0)$ by part 2) of Lemma 5.1, there exists some $\hat{\sigma} < \sigma^*(\mu_0)$ such that $f_{\mu_0}(\hat{\sigma}) > g(\tilde{\sigma})$. We then have

$$U(\mu_y, \sigma_y) = U\big(f_{\mu_0}(\hat{\sigma}), \hat{\sigma}\big) > U\big(g(\tilde{\sigma}), \tilde{\sigma}\big)$$

by the monotonicity properties of U, since $f_{\mu_0}(\hat{\sigma}) > g(\tilde{\sigma})$ and $\hat{\sigma} < \sigma^*(\mu_0) \le \tilde{\sigma}$. Together with (5.18), we have thus shown $U(\mu_y, \sigma_y) > U(g(\tilde{\sigma}), \tilde{\sigma})$ for all $\tilde{\sigma} > \sigma_y$. As σ is strictly increasing in y, this is equivalent to (5.16) for all $\tilde{y} > y$.

Similarly, $\mathcal{S}_U(\mu(y), \sigma(y)) \le s(y)$ yields $f'_{\mu_0}(\sigma_y) \le g'(\sigma_y)$, and we also have $\mu_y = f_{\mu_0}(\sigma_y) = g(\sigma_y)$ and $f''_{\mu_0} > 0 \ge g''$. Since f'_{μ_0} is strictly increasing and g' is decreasing, we deduce that $f'_{\mu_0}(\tilde{\sigma}) < g'(\tilde{\sigma})$ and hence $f_{\mu_0}(\tilde{\sigma}) > g(\tilde{\sigma})$ for all $\tilde{\sigma} \in [0, \sigma_y)$. Since U is strictly increasing in μ, we then have

$$U(\mu_y, \sigma_y) = U\big(f_{\mu_0}(\tilde{\sigma}), \tilde{\sigma}\big) > U\big(g(\tilde{\sigma}), \tilde{\sigma}\big)$$

for all $\tilde{\sigma} \in [0, \sigma)$. Rewriting this in terms of y with $\tilde{\sigma} = \sigma(\tilde{y})$ yields (5.17).

Finally, if $\mathcal{S}_U(\mu(y), \sigma(y)) = s(y)$, both (5.16) and (5.17) hold so that we have $U(\mu(y), \sigma(y)) > U(\mu(\tilde{y}), \sigma(\tilde{y}))$ for all $\tilde{y} \neq y$. Therefore y is the unique solution to (5.1). □

Proposition 5.2 provides a first-order condition to identify a solution to (5.1), and the inequalities (5.16) and (5.17) ensure that such a solution must be unique. However, Proposition 5.2 does not say whether there exists a solution to (5.1) (or, equivalently, to the first-order condition). The question of existence is answered by the following result, where the equivalence (a) $\Leftrightarrow$ (c) below is also given in [85, Proposition 1]. We give the proof in full for the sake of completeness. We also show the equivalence to condition (b) below, which will be useful for the subsequent results. As before, we fix the triplet $(\ell, c, \varepsilon^2) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$ as well as the functions μ, σ defined by (5.12).

Theorem 5.3. *The following statements are equivalent:*

(a) *There exists a solution $\hat{y}$ to* (5.1)*.*

(b) *There exists some $y > 0$ such that*

$$\mathcal{S}_U\big(\mu(y), \sigma(y)\big) > \frac{\sigma(y)}{\ell y}.$$

(c) *It holds that*

$$\sup_{y>0} \rho_U\big(\mu(y), \sigma(y)\big) > \sqrt{(1-\ell)/\ell}.$$

If any of the assertions (a)–(c) *holds, then the solution $\hat{y}$ to* (5.1) *is unique.*

Proof. In order to show (a) $\Leftrightarrow$ (b), we distinguish three cases:

1) $\varepsilon^2 = 0$ and $\mathcal{S}_U(c, 0) \geq \sqrt{\frac{1-\ell}{\ell}}$.

2) $\varepsilon^2 = 0$ and $\mathcal{S}_U(c, 0) < \sqrt{\frac{1-\ell}{\ell}}$.

3) $\varepsilon^2 > 0$.

The proof is structured in steps A)–C). In step A), we prove (a) $\Leftrightarrow$ (b) in case 1). In step B), we show (5.19) below for both cases 2) and 3); this allows us to prove (a) $\Leftrightarrow$ (b) for 2) and 3) simultaneously. In step C), we show (b) $\Leftrightarrow$ (c), where we no longer need to distinguish the cases.

A) We start by considering case 1), where by (5.12) and (5.13), we have

$$\mathcal{S}_U\big(\mu(0), \sigma(0)\big) = \mathcal{S}_U(c, 0) \geq \sqrt{\frac{1-\ell}{\ell}} = s(0),$$

and hence (5.16) gives $U(\mu(0),\sigma(0)) > U(\mu(\tilde{y}),\sigma(\tilde{y}))$ for all $\tilde{y} > 0$. Thus $\hat{y} = 0$ is the unique solution to (5.1) so that (a) always holds in this case and the solution is unique. Moreover, we must have $\mathcal{S}_U(\mu(y),\sigma(y)) > \frac{\sigma(y)}{\ell y}$ for all $y > 0$ since otherwise (5.17) would contradict the optimality of $\hat{y} = 0$. Thus (b) always holds in this case as well. So (a) and (b) always hold and hence (a) $\Leftrightarrow$ (b) in case 1).

B) In case 3), we have $\sigma(0) > 0$ so that $s(y) = \frac{\sigma(y)}{\ell y} \to \infty$ as $y \searrow 0$. Since μ, σ and $\mathcal{S}_U$ are continuous (the latter due to part 1) of Lemma 5.1), we obtain

$$\mathcal{S}_U\big(\mu(y),\sigma(y)\big) < s(y) \tag{5.19}$$

for small enough $y \geq 0$. In case 2), we have $s(0) = \sqrt{\frac{1-\ell}{\ell}} < \mathcal{S}_U(\mu(0),\sigma(0))$ so that the assumption and the continuity of μ, σ and $\mathcal{S}_U$ also yield (5.19) for small y. We can now show (a) $\Leftrightarrow$ (b) in both cases 2) and 3) simultaneously since we have (5.19) in both cases.

(b) $\Rightarrow$ (a): Suppose that (b) holds for some $y > 0$. By the continuity of $\mathcal{S}_U$ and the intermediate value theorem together with (5.19), there exists some $\hat{y} \in (0, y)$ such that

$$\mathcal{S}_U\big(\mu(\hat{y}),\sigma(\hat{y})\big) = \frac{\sigma(\hat{y})}{\ell\hat{y}} = s(\hat{y}). \tag{5.20}$$

By Proposition 5.2, $\hat{y}$ is the unique solution to (5.1), and this shows (a) as well as the uniqueness of $\hat{y}$.

(a) $\Rightarrow$ (b): Suppose that $\hat{y} \geq 0$ is a solution to (5.1). We have by (5.19) and (5.17) (for a small $y > 0$) that 0 cannot be a solution to (5.1), and hence $\hat{y} > 0$. We claim that

$$\mathcal{S}_U\big(\mu(y'),\sigma(y')\big) > \frac{\sigma(y')}{\ell y'} = s(y') \tag{5.21}$$

holds for all $y' > \hat{y}$. To show this, suppose for a contradiction that (5.21) does not hold for some $y' > \hat{y}$. Then (5.17) in Proposition 5.2 (with $y := y'$) yields

$$U\big(\mu(y'),\sigma(y')\big) > U\big(\mu(\hat{y}),\sigma(\hat{y})\big),$$

and this contradicts the optimality of $\hat{y}$. Thus (5.21) holds for all $y' > \hat{y}$, and this shows (b).

C) We have now shown (a) $\Leftrightarrow$ (b) in all cases 1)–3), and it remains to show (b) $\Leftrightarrow$ (c) for which we no longer need to distinguish the cases.

(b) $\Rightarrow$ (c): Suppose that (b) holds for some $y > 0$. Then we have

$$\sup_{y' \geq 0} \rho_U\big(\mu(y'), \sigma(y')\big) \geq \mathcal{S}_U\big(\mu(y), \sigma(y)\big) > \frac{\sigma(y)}{\ell y} \geq \sqrt{\frac{1-\ell}{\ell}},$$

since $\sigma(y) \geq \sqrt{\ell(1-\ell)y}$ by the definition (5.12); this shows (c).

(c) $\Rightarrow$ (b): If (c) holds, then for some $y > 0$, we have

$$\rho_U\big(\mu(y), \sigma(y)\big) > \sqrt{(1-\ell)/\ell}.$$

If $\mathcal{S}_U(\mu(y), \sigma(y)) > \frac{\sigma(y)}{\ell y}$, there is nothing to prove. If $\mathcal{S}_U(\mu(y), \sigma(y)) = \frac{\sigma(y)}{\ell y}$, then (a) holds by Proposition 5.2, and we have already shown (a) $\Leftrightarrow$ (b). Thus we may suppose without loss of generality that

$$\mathcal{S}_U\big(\mu(y), \sigma(y)\big) < \frac{\sigma(y)}{\ell y}. \tag{5.22}$$

Recall the function g defined by (5.14) and the function f_{μ_0} that corresponds to $\mathcal{I}_U(\mu(y), \sigma(y))$ by part 2) of Lemma 5.1, so that

$$g\big(\sigma(y)\big) = f_{\mu_0}\big(\sigma(y)\big) = \mu(y).$$

Moreover, note that (5.22) can be written as $f'_{\mu_0}(\sigma(y)) < g'(\sigma(y))$ by part 4) of Lemma 5.1 and (5.15), so that $f_{\mu_0}(\sigma(y+\delta)) < g(\sigma(y+\delta))$ for some small $\delta > 0$. On the other hand, note that by the assumption (c), part 6) of Lemma 5.1 and (5.15), we have

$$\lim_{\sigma' \nearrow \sigma^*(\mu_0)} f'_{\mu_0}(\sigma) = \rho_U\big(\mu(y), \sigma(y)\big) > \frac{\sigma(y)}{\ell y} = \lim_{y' \to \infty} g'_{\mu_0}\big(\sigma(y')\big).$$

Hence if $\sigma^*(\mu_0) = \infty$, then $f_{\mu_0}(\sigma(\cdot))$ grows asymptotically faster than $g(\sigma(\cdot))$. Thus by the intermediate value theorem, there exists some $y' > y + \delta$ with $f_{\mu_0}(\sigma(y')) = g(\sigma(y'))$. The same is true if $\sigma^*(\mu_0) < \infty$ as $\lim_{\sigma' \to \sigma^*(\mu_0)} f_{\mu_0}(\sigma') = \infty$ by part 2) of Lemma 5.1 so that $f_{\mu_0}(\sigma(\cdot))$ must cross $g(\sigma(\cdot))$. In either case, f_{μ_0} is strictly convex and g is concave, and they meet at both points $\sigma(y) < \sigma(y')$. By the convexity and concavity, respectively, it follows that $f'_{\mu_0}(\sigma(y')) > g'(\sigma(y'))$ at the larger crossing point. Using part 4) of Lemma 5.1 and (5.15) once again, this means that $\mathcal{S}_U(\mu(y'), \sigma(y')) > \frac{\sigma(y')}{\ell y'}$, and therefore (b) holds. This concludes the proof of (c) $\Rightarrow$ (b). $\square$

By combining Proposition 5.2 and Theorem 5.3, we now obtain a characterisation of the solution to (5.1). This will be useful later to show that the solution depends continuously on the parameters (ℓ, c, ε^2).

Corollary 5.4. *The following statements hold:*

1) *If* $\varepsilon^2 = 0$ *and* $\mathcal{S}_U(c, 0) \geq \sqrt{\frac{1-\ell}{\ell}}$, *then* $\hat{y} = 0$ *is the unique solution to* (5.1).

2) *Suppose any of the conditions* (a)–(c) *in Theorem 5.3 holds and that either* $\varepsilon^2 > 0$, *or* $\varepsilon^2 = 0$ *and* $\mathcal{S}_U(c, 0) < \sqrt{\frac{1-\ell}{\ell}}$. *Then there exists a unique solution* $\hat{y} > 0$ *to* (5.1), *and* $\hat{y}$ *is also the unique solution to the first-order condition*

$$\mathcal{S}_U\big(\mu(y), \sigma(y)\big) = \frac{\sigma(y)}{\ell y}. \tag{5.23}$$

Moreover, in both cases, it holds that

$$\mathcal{S}_U\big(\mu(y), \sigma(y)\big) \gtrless \frac{\sigma(y)}{\ell y} \tag{5.24}$$

for $0 < y \gtrless \hat{y}$, *respectively.*

Proof. **1)** The assumptions are those of case 1) in the proof of Theorem 5.3, and we already showed in step A) of that proof that $\hat{y} = 0$ is the unique solution and (5.24) holds for $y > 0$.

2) By the assumption, we have one of the cases 2) or 3) of the proof of Theorem 5.3. We showed in step B) for both cases that if a solution exists, then it is unique, strictly positive and satisfies the first-order condition (5.23) due to (5.20). It remains to show (5.24). For a contradiction, suppose that there exists $y > \hat{y}$ with $\mathcal{S}_U(\mu(y), \sigma(y)) \leq \frac{\sigma(y)}{\ell y} = s(y)$. Then (5.17) holds for $\tilde{y} = \hat{y}$, and this contradicts the optimality of $\hat{y}$. Likewise, if $0 < y < \hat{y}$ with $\mathcal{S}_U(\mu(y), \sigma(y)) \geq \frac{\sigma(y)}{\ell y} = s(y)$, then (5.16) holds with $\tilde{y} = \hat{y}$ and contradicts the optimality of $\hat{y}$. Therefore (5.23) holds in this case, and this concludes the proof. ☐

Theorem 5.3 and Corollary 5.4 provide necessary and sufficient conditions for the existence of a unique solution to (5.1), and we can in principle find the solution by solving the first-order condition (5.23) for $\hat{y}$. This concludes our study of the problem (5.1) for a fixed parameter $\theta = (\ell, c, \varepsilon^2) \in (0, 1) \times \mathbb{R} \times \mathbb{R}_+$.

Our next goal is to study the set of parameters θ for which a solution $\hat{y} = \hat{y}_U(\theta)$ to (5.1) exists, and whether the map $\theta \mapsto \hat{y}_U(\theta)$ is continuous. We show that $\hat{y}_U$ is well defined and continuous on a nonempty open subset $V \subseteq (0, 1) \times \mathbb{R} \times \mathbb{R}_+$ which depends on the choice of mean–variance utility function U. In the following, we define $\mu(\cdot) = \mu(\cdot\,;\theta)$ and $\sigma^2(\cdot) = \sigma^2(\cdot\,;\theta)$ as before by (5.12), where the dependence on the underlying parameter $\theta = (\ell, c, \varepsilon^2)$ is now made explicit.

Proposition 5.5. *Let U be a mean–variance utility function. Then the set*

$$V := \{\theta = (\ell, c, \varepsilon^2) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+ : \text{ a solution } \hat{y}_U(\theta) \text{ to (5.1) exists}\}$$

is a nonempty open subset of $(0,1) \times \mathbb{R} \times \mathbb{R}_+$ (with the relative topology), and it is given by

$$V = \left\{\theta \in (0,1) \times \mathbb{R} \times \mathbb{R}_+ : \sup_{y>0} \rho_U\big(\mu(y;\theta), \sigma(y;\theta)\big) > \sqrt{\frac{1-\ell}{\ell}}\right\}. \tag{5.25}$$

Moreover, the map $\theta \mapsto \hat{y}_U(\theta)$ is continuous on V.

Proof. By Theorem 5.3, a solution $\hat{y}_U(\theta)$ to (5.1) exists if and only if there is some $y > 0$ such that

$$\mathcal{S}_U\big(\mu(y;\theta), \sigma(y;\theta)\big) > \frac{\sigma(y;\theta)}{\ell y}.$$

Thus V is the projection on $(0,1) \times \mathbb{R} \times \mathbb{R}_+$ of the set

$$\tilde{V} := \{(y, \ell, c, \varepsilon^2) \in (0,\infty) \times (0,1) \times \mathbb{R} \times \mathbb{R}_+ : d(y; \ell, c, \varepsilon^2) > 0\},$$

where the map $d : (0,\infty) \times (0,1) \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$d(y;\theta) = \mathcal{S}_U\big(\mu(y;\theta), \sigma(y;\theta)\big) - \frac{\sigma(y;\theta)}{\ell y}.$$

Note that d is a composition of continuous functions and hence continuous on $(0,\infty) \times (0,1) \times \mathbb{R} \times \mathbb{R}_+$. Thus $\tilde{V}$ is an open subset of $(0,\infty) \times (0,1) \times \mathbb{R} \times \mathbb{R}_+$ with the relative topology. Since the projection is an open map, it follows that $V \subseteq (0,1) \times \mathbb{R} \times \mathbb{R}_+$ is open in the relative topology.

The alternative description (5.25) follows immediately from the equivalence (a) $\Leftrightarrow$ (c) in Theorem 5.3. To show that V is nonemtpy, fix some arbitrary $(c, \varepsilon^2) \in \mathbb{R} \times \mathbb{R}_+$. By part 6) of Lemma 5.1, we have $\rho_U(c, \sqrt{\varepsilon^2 + 1}) > 0$. Note that the map $\mu \mapsto \rho_U(\mu, \sigma)$ is increasing for any $\sigma \geq 0$, since for $\mu_1 > \mu_2$, we have

$$\rho_U(\mu_1, \sigma) = \rho_U(\tilde{\mu}_1, 0) > \rho_U(\tilde{\mu}_2, 0) = \rho_U(\mu_2, 0)$$

for some $\tilde{\mu}_1 > \tilde{\mu}_2$ by parts 2), 3) and 6) of Lemma 5.1. Hence for any $\ell \in (0,1)$

and setting $y := \frac{1}{\sqrt{\ell(1-\ell)}}$, we have by (5.12) that

$$\begin{aligned}\rho_U\big(\mu(\sqrt{\ell(1-\ell)}^{-1};\theta), \sigma(\sqrt{\ell(1-\ell)}^{-1};\theta)\big) &= \rho_U\big(\mu(\sqrt{\ell(1-\ell)}^{-1};\theta), \sqrt{\varepsilon^2+1}\big)\\ &> \rho_U(c, \sqrt{\varepsilon^2+1}) > 0.\end{aligned}$$

It follows that

$$\sup_{y>0} \rho_U\big(\mu(y;\ell,c,\varepsilon^2), \sigma\big(y;\ell,c,\varepsilon^2\big)\big) > \rho_U(c, \sqrt{\varepsilon^2+1}) > 0$$

for all $\ell \in (0,1)$. On the other hand, we have $\sqrt{(1-\ell)/\ell} \to 0$ as $\ell \nearrow 1$ so that

$$\sup_{y>0} \rho_U\big(\mu(y;\ell,c,\varepsilon^2), \sigma\big(y;\ell,c,\varepsilon^2\big)\big) > \sqrt{(1-\ell)/\ell} \tag{5.26}$$

for ℓ close enough to 1. Thus by (5.25), there exists for any $(c,\varepsilon^2) \in \mathbb{R}\times\mathbb{R}_+$ some $\ell \in (0,1)$ such that $(\ell, c, \varepsilon^2) \in V$. Thus the projection of V on $\mathbb{R}\times\mathbb{R}_+$ is $\mathbb{R}\times\mathbb{R}_+$, which implies in particular that V is nonempty.

It remains to prove the continuity of $\hat{y}$. Fix a sequence $(\theta_n)_{n\in\mathbb{N}}$ in V with $\theta_n \to \theta_\infty \in V$ as $n \to \infty$. Writing $y_n := \hat{y}_U(\theta_n)$ for $n \in \mathbb{N}\cup\{\infty\}$, we want to show that $y_n \to y_\infty$ as $n \to \infty$. Equation (5.24) yields

$$\mathcal{S}_U\big(\mu(y_\infty+\delta;\theta_\infty), \sigma(y_\infty+\delta;\theta_\infty)\big) > \frac{\sigma(y_\infty+\delta;\theta_\infty)}{\ell_\infty(y_\infty+\delta)}$$

for each $\delta > 0$. By the continuity of $\mathcal{S}_U$, μ and σ, it follows that

$$\mathcal{S}_U\big(\mu(y_\infty+\delta;\theta_n), \sigma(y_\infty+\delta;\theta_n)\big) > \frac{\sigma(y_\infty+\delta;\theta_n)}{\ell_n(y_\infty+\delta)} \tag{5.27}$$

for $\delta > 0$ and all large enough $n \in \mathbb{N}$. By applying (5.24) with parameter θ_n, we deduce from (5.27) that $y_n < y_\infty + \delta$ for n large enough. Thus since $\delta > 0$ is arbitrary, we have $\limsup_{n\to\infty} y_n \le y_\infty$. If $y_\infty = 0$, this already shows $y_n \to y_\infty$ as $n \to \infty$. If $y_\infty > 0$, we use the left inequality in (5.24) at $y_\infty - \delta$ to obtain

$$\mathcal{S}_U\big(\mu(y_\infty-\delta;\theta_\infty), \sigma(y_\infty-\delta;\theta_\infty)\big) < \frac{\sigma(y_\infty-\delta;\theta_\infty)}{\ell_\infty(y_\infty-\delta)}$$

so that by the continuity of $\mathcal{S}_U$, μ and σ, we have

$$\mathcal{S}_U\big(\mu(y_\infty-\delta;\theta_n), \sigma(y_\infty-\delta;\theta_n)\big) < \frac{\sigma(y_\infty-\delta;\theta_n)}{\ell_n(y_\infty-\delta)}$$

for n large enough. Hence $y_n > y_\infty - \delta$ for n large enough due to (5.24). Therefore $\liminf_{n\to\infty} y_n \geq y_\infty$ as $\delta > 0$ is arbitrary, and hence $y_n \to y_\infty$ as $n \to \infty$. This proves the continuity of $\hat{y}_U$. □

With Proposition 5.5 we have shown the continuity of $\hat{y}_U$ and characterised its domain V. We are now almost ready to prove Proposition 2.17; it only remains to show how each of the bounds in (2.32) yields information about the map $\hat{y}_k$. The result will then follow by combining both bounds.

Lemma 5.6. *Let U be a mean–variance utility function. Then the following statements hold:*

1) *If there exists $\lambda^{\min} \in (0, \infty)$ such that*

$$\mathcal{S}_U(\mu, \sigma) \leq \frac{\sigma}{\lambda^{\min}} \tag{5.28}$$

for all $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$, then we have

$$\hat{y}_U(\ell, c, \varepsilon^2) \geq \frac{\lambda^{\min}}{\ell} > \lambda^{\min} \tag{5.29}$$

for all $(\ell, c, \varepsilon^2) \in V$, where V is given by (5.25).

2) *If there exists $\lambda^{\max} \in (0, \infty)$ such that*

$$\mathcal{S}_U(\mu, \sigma) \geq \frac{\sigma}{\lambda^{\max}} \tag{5.30}$$

for all $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$, then $V = (0, 1) \times \mathbb{R} \times \mathbb{R}_+$ and

$$\hat{y}_U(\ell, c, \varepsilon^2) \leq \frac{\lambda^{\max}}{\ell} \tag{5.31}$$

for all $(\ell, c, \varepsilon^2) \in (0, 1) \times \mathbb{R} \times \mathbb{R}_+$.

Proof. **1)** Let $\theta := (\ell, c, \varepsilon^2) \in V$ so that $\hat{y}_U(\theta)$ is well defined by the definition of V. Note that part 2) of Corollary 5.4 applies to θ even for $\varepsilon^2 = 0$ since (5.28) yields $\mathcal{S}_U(c, 0) = 0 < \sqrt{\frac{1-\ell}{\ell}}$. Thus we have $\hat{y}_U(\theta) > 0$, and rearranging (5.23) yields

$$\hat{y}_U(\theta) = \frac{\sigma(\hat{y}_U(\theta); \theta)}{\ell \mathcal{S}_U\big(\mu\big(\hat{y}_U(\theta); \theta\big), \sigma\big(\hat{y}_U(\theta); \theta\big)\big)} \geq \frac{\lambda^{\min}}{\ell} > \lambda^{\min}$$

by (5.28) and as $\ell < 1$. This shows (5.29).

2) Fix $\theta := (\ell, c, \varepsilon^2) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$. By (5.30), we have

$$\mathcal{S}_U\big(\mu(y;\theta), \sigma(y;\theta)\big) \geq \frac{\sigma(y;\theta)}{\lambda^{\max}} > \frac{\sigma(y;\theta)}{\ell y} \tag{5.32}$$

for any $y > \frac{\lambda^{\max}}{\ell}$ so that condition (b) in Theorem 5.3 is satisfied. Thus $\hat{y}_U(\theta)$ is well defined, and since θ is arbitrary, it follows that $V = (0,1) \times \mathbb{R} \times \mathbb{R}_+$. Then (5.24) and (5.32) yield $\hat{y}_U(\theta) < y$ for any $y > \frac{\lambda^{\max}}{\ell}$, and this shows (5.31). □

We now collect our results to show Proposition 2.17; the statement is given at the beginning of the section.

Proof of Proposition 2.17. Since (2.30) gives both bounds (5.28) and (5.30), it follows from Lemma 5.6 that $V = (0,1) \times \mathbb{R} \times \mathbb{R}_+$ and we have the bounds (2.31). By Proposition 5.5, $\hat{y}_U$ is well defined and continuous on $(0,1) \times \mathbb{R} \times \mathbb{R}_+$, and by definition, $\hat{y}_U(\ell, c, \varepsilon^2)$ is a solution to (5.1) for each (ℓ, c, ε^2). The uniqueness of the solution follows from Theorem 5.3. □

Our last goal in this section is to show how these results may be applied to the equilibrium problem of Section 2 in a more general setting. Consider K agents with mean–variance utility functions $U_k : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, $k = 1, \dots, K$. In the previous sections, we worked under Assumption 2.19 which we now recall.

Assumption 2.19. We suppose that for each $k \in \{1, \dots, K\}$, there exist constants $\lambda_k^{\min}, \lambda_k^{\max} \in (0, \infty)$ such that

$$\frac{\sigma}{\lambda_k^{\max}} \leq \mathcal{S}_{U_k}(\mu, \sigma) \leq \frac{\sigma}{\lambda_k^{\min}} \quad \text{for all } (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+. \tag{2.32}$$

Assumption 2.19 can be interpreted in terms of the indifference curves for U_k. By parts 3) and 4) of Lemma 5.1, Assumption 2.19 is satisfied if and only if

$$\frac{\sigma}{\lambda_k^{\max}} \leq f'_{k,\mu_0}(\sigma) \leq \frac{\sigma}{\lambda_k^{\min}} \quad \text{for all } \mu_0 \in \mathbb{R}, \sigma \in \big[0, \sigma_k^*(\mu_0)\big),$$

where $\mathcal{I}_{U_k}(\mu_0, 0) = \{\big(f_{k,\mu_0}(\sigma), \sigma\big) : \sigma \in \big[0, \sigma_k^*(\mu_0)\big)\}$ is the indifference curve for U_k through $(\mu_0, 0)$. For $\lambda_k^{\min} = \lambda_k^{\max}$, the indifference curves are thus parabolas; this is the case of linear mean–variance utility (2.9). In general, the curves can be sandwiched between parabolas, but their precise shape may differ. For example, consider for $a_k > b_k > 0$ the function $g_k : \mathbb{R}_+ \to \mathbb{R}$ given by

$$g(x) = \frac{a_k x^2}{2} - b_k e^{-x}(x+1), \quad x \geq 0,$$

so that $g_k'(x) = x(a_k - b_k e^{-x}) > 0$ and $g''(x) = a_k - b_k e^{-x} + b_k x e^{-x} > 0$ for $x > 0$. Thus g_k is strictly increasing and strictly convex, and it is straightforward to check that the function $U_k : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ given by

$$U_k(\mu, \sigma) := \mu - g_k(\sigma), \quad (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ \tag{5.33}$$

is a mean–variance utility function as in Definition 2.5. Moreover, we have

$$\mathcal{S}_{U_k}(\mu, \sigma) = -\frac{\partial_\sigma U_k(\mu, \sigma)}{\partial_\mu U_k(\mu, \sigma)} = \sigma(a_k - b_k e^{-\sigma}), \quad (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+,$$

and hence U_k satisfies (2.32) with $\lambda_k^{\min} = \frac{1}{a_k}$ and $\lambda_k^{\max} = \frac{1}{a_k - b_k}$. Other examples of mean–variance utility functions of the semilinear form (5.33) can be constructed for suitable choices of $g_k : \mathbb{R}_+ \to \mathbb{R}$.

So while Assumption 2.19 is rather restrictive on the choice of functions U_k, it still allows some flexibility. We now discuss the consequences of omitting it. Recall from Definition 2.31 the map ψ_3 given by

$$\psi_3\big(\ell, (c_k, \varepsilon_k^2)_{k=1}^K\big) := (\gamma_k)_{k=1}^K := \big(c_k + \hat{y}_{U_k}(\ell, c_k, \varepsilon_k^2)\big)_{k=1}^K.$$

In Definition 2.31, the map ψ_3 has the domain $(0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K$. This is justified because Assumption 2.19 and Proposition 2.17 yield that each map $\hat{y}_{U_k}$ is well defined on $(0,1) \times \mathbb{R} \times \mathbb{R}_+$. However, without Assumption 2.19, the map ψ_3 may not be defined on the whole set $(0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K$, and this would pose problems for the techniques used in Section 4.

The following result shows that in general, ψ_3 is at least well defined and continuous on a nonempty set. Indeed, by Proposition 5.5 (which holds in general), there exists a unique solution $\hat{y}_k(\ell, c_k, \varepsilon_k^2)$ to (5.1) with U_k in place of U if and only if $(\ell, c_k, \varepsilon_k^2) \in V_k$, where $V_k \subseteq (0,1) \times \mathbb{R} \times \mathbb{R}_+$ is given as in (5.25). Moreover, the solution $\hat{y}_k$ depends continuously on $(\ell, c_k, \varepsilon_k^2)$. In order to define the maps $\hat{y}_k$ on a common domain for all $k = 1, \dots, K$, we consider the set $V^{(K)} \subseteq (0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K$ given by

$$V^{(K)} = \Big\{\big(\ell, (c_k, \varepsilon_k^2)_{k=1}^K\big) \in (0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K : (\ell, c_k, \varepsilon_k^2) \in V_k \text{ for } k = 1, \dots, K\Big\}.$$

We can now define ψ_3 on $V^{(K)}$ without Assumption 2.19.

Corollary 5.7. *The map* $\psi_3 : V^{(K)} \to \mathbb{R}_+^K$ *defined by*

$$\psi_3\big(\ell, (c_k, \varepsilon_k^2)_{k=1}^K\big) := \big(c_k + \hat{y}_k(\ell, c_k, \varepsilon_k^2)\big)_{k=1}^K \tag{5.34}$$

is well defined and continuous on $V^{(K)} \neq \emptyset$. *Moreover, the projection of the set* $V^{(K)} \subseteq (0,1) \times (\mathbb{R} \times \mathbb{R}_+)^K$ *on* $(\mathbb{R} \times \mathbb{R}_+)^K$ *is surjective, so that* $V^{(K)}$ *is nonempty.*

Proof. Fix $(c_k, \varepsilon_k^2)_{k=1}^K \in (\mathbb{R} \times \mathbb{R}_+)^K$. By the argument used in the proof of Proposition 5.5 to show that V is nonempty (see after (5.26)), there exists for each $k = 1, \dots, K$ some $\ell_{*,k} \in (0,1)$ such that $(\ell, c_k, \varepsilon_k^2) \in V_k$ for all $\ell \geq \ell_{*,k}$. Thus we must have $(\ell, (c_k, \varepsilon_k^2)_{k=1}^K) \in V^{(K)}$ for $\ell \geq \max_k \ell_{*,k}$ so that $V^{(K)}$ is nonempty. Since $(c_k, \varepsilon_k^2)_{k=1}^K$ is arbitrary, we also obtain that the projection of $V^{(K)}$ on $(\mathbb{R} \times \mathbb{R}_+)^K$ is surjective. Finally, Proposition 5.5 yields that for each $k = 1, \dots, K$, the map $(\ell, (c_k, \varepsilon_k^2)_{k=1}^K) \mapsto \hat{y}_k(\ell, c_k, \varepsilon_k^2)$ is well defined and continuous on $V^{(K)}$. Thus ψ_3 is continuous by the definition (5.34). □

In order to apply these results in the proof of Theorem 4.33, the main requirement is the continuity of ψ_3 which holds in full generality due to Corollary 5.7. Thus it may indeed be possible to weaken or omit Assumption 2.19; this is supported by the fact that such an assumption is not necessary in the one-period model, as shown in Koch-Medina/Wenzelburger [85]. However, one has to deal with the fact that ψ_3 may be defined on a proper subset $V^{(K)} \subseteq (0,1) \times \mathbb{R} \times \mathbb{R}_+$, and so it is no longer clear whether the map $\Psi = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1$ is well defined. One also lacks the a priori bounds (2.31) on the output of ψ_3. One approach to circumvent these issues would be study the range of the map $\psi_2 \circ \psi_1$, possibly under stronger assumptions on the primitives of the model. If this range is contained in $V^{(K)}$, one may then be able to extend the proof of Theorem 4.33. However, this seems challenging in the general case.

6 A stability result for quadratic BSDEs

6.1 BSDE stability

We now turn to the study of the stability of quadratic BSDEs of the type considered in Section 4.2. The results that we obtain here are stated in a general form and are therefore of independent interest. For our purposes, they are used in the proofs of Theorem 4.31 and Proposition 4.32 to obtain the continuity of $\tilde{\psi}_2$. Although we consider an abstract setting in this section, we retain a similar notation to Section 4.2 for the sake of consistency. We impose the **standing Assumption** 4.2 on the filtration, that is, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, P)$ such that $\mathcal{F}_0$ is P-trivial and $\mathbb{F}$ satisfies the usual assumptions and is continuous. We also fix a (continuous) local martingale $(\bar{M}_t)_{0\le t\le T}$ taking values in $\mathbb{R}^{\bar{d}}$ for some $\bar{d} \in \mathbb{N}$, and define the increasing process $(I_t)_{0\le t\le T}$ by $I_t = \operatorname{tr}\langle \bar{M}\rangle_t$. As after (4.29), we introduce a predictable process $(c_t^{\bar{M}})_{0\le t\le T}$ taking values in the set of $\bar{d}\times\bar{d}$ symmetric positive semidefinite matrices such that $d\langle\bar{M}\rangle_t = c_t^{\bar{M}}dI_t$; moreover, we may and do assume that $\operatorname{tr} c_t^{\bar{M}} = 1$ for all $t \in [0,T]$ P-a.s. Assumption 4.15 is not necessary for the results in this section, i.e., $c^{\bar{M}}$ **need not** be invertible.

For $m \in \mathbb{N}$, we call a function $f : \Omega \times [0,T] \times \mathbb{R}^{\bar{d}\times m} \to \mathbb{R}$ *predictable* if it is measurable with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{\bar{d}\times m})$, where $\mathcal{P}$ is the predictable σ-algebra on $\Omega\times[0,T]$. We denote by $|\cdot|$ and $\|\cdot\|$ the Euclidean and Frobenius norms on $\mathbb{R}^{\bar{d}}$ and $\mathbb{R}^{\bar{d}\times m}$, respectively. For a symmetric positive semidefinite matrix $C \in \mathbb{R}^{\bar{d}\times\bar{d}}$, $x \in \mathbb{R}^{\bar{d}}$ and $A \in \mathbb{R}^{\bar{d}\times m}$, we write

$$|x|_C := (x^\top Cx)^{1/2} \quad \text{and} \quad \|A\|_C := \operatorname{tr}(A^\top CA)^{1/2} \tag{6.1}$$

so that $|\cdot|_C$ and $\|\cdot\|_C$ can be seen as the seminorm and Frobenius seminorm, respectively, under the pseudometric induced by C. Since $\operatorname{tr} c_t^{\bar{M}} = 1$, we have $|\cdot|_{c_t^{\bar{M}}} \le |\cdot|$ and $\|\cdot\|_{c_t^{\bar{M}}} \le \|\cdot\|$ for $0\le t\le T$. In the following, we shall sometimes refer to results given in Section 6.2 below; these are well-known and folklore results on BMO martingales, for which we use Kazamaki [81] as a reference.

We consider the backward stochastic differential equation (BSDE)

$$\begin{aligned} dY_t &= f_t(\zeta_t, \chi_t)dI_t - \varrho_t d\langle N\rangle_t + \zeta_t d\bar{M}_t + dN_t, \quad 0\le t\le T, \\ Y_T &= G \end{aligned} \tag{6.2}$$

for a given $G \in L^\infty$, a predictable bounded process $(\varrho_t)_{0\le t\le T}$, a predictable function $f : \Omega\times[0,T]\times\mathbb{R}^{\bar{d}}\times\mathbb{R}^{\bar{d}\times m} \to \mathbb{R}$ with quadratic growth (see (6.4) below), and an exogenous predictable process $\chi \in L^2(\bar{M};\mathbb{R}^{\bar{d}\times m})$ for some $m\in\mathbb{N}$ such that $\chi\bullet\bar{M} \in BMO$. We say that (Y,ζ,N) is a *solution* to the BSDE (6.2) if Y is a bounded semimartingale, $\zeta \in L^2(\bar{M};\mathbb{R}^{\bar{d}})$ and N is a continuous local martingale strongly orthogonal to $\bar{M}$ such that (6.2) and the terminal condition are satisfied. Our goal is to show sufficient conditions for the stability of solutions to (6.2).

Equation (6.2) is a BSDE where the driver has quadratic growth in ζ and an unbounded stochastic parameter χ. BSDEs of this type have been studied in Frei et al. [51], Briand/Hu [22] and Mocha/Westray [93] for a continuous filtration; in the latter two, the stochastic parameter is replaced by a stochastic bound on the driver. However, each of the Assumptions [51, 2.3], [22, (A.2)(iv)] and [93, 1(i)] implies for (6.2) that the random variable

$$K := \operatorname{tr}\langle \chi \mathbin{\bullet} \bar{M} \rangle_T = \int_0^T \operatorname{tr}(\chi_t^\top c_t^{\bar{M}} \chi_t) dI_t$$

admits exponential moments of all positive orders. Under our assumption that $\chi \mathbin{\bullet} \bar{M} \in BMO$, K admits by Lemma 6.7 below an exponential moment of some positive order, but not necessarily all. It turns out that weakening this condition has significant implications for the study of (6.2). Indeed, whereas existence results are obtained in [51, 22, 93], our assumption does not ensure the existence of a solution to (6.2) as shown in the following example.

Example 6.1. Fix a process $\chi \in L^2(\bar{M})$ such that $\chi \mathbin{\bullet} \bar{M} \in BMO$. Suppose that for some $a > 0$, there exists a solution $(Y_t)_{0 \le t \le T}$ to the BSDE

$$dY_t = -a\chi_t^\top c_t^{\bar{M}} \chi_t dI_t - \zeta_t^\top c_t^{\bar{M}} \zeta_t dI_t + \zeta_t d\bar{M}_t, \quad 0 \le t \le T, \tag{6.3}$$

with $Y_T = 0$ for some $\zeta \in L^2_{\mathrm{loc}}(\bar{M})$, where we drop N for simplicity (i.e., in this example, we assume that $\bar{M}$ has the *martingale representation property* for $\mathbb{F}$ so that any local martingale strongly orthogonal to $\bar{M}$ is constant). Since the first term on the right-hand side of (6.3) does not depend on Y or ζ, we can set $\tilde{Y}_t := Y_t + \int_0^t a\chi_s^\top c_s^{\bar{M}} \chi_s dI_s$ so that $\tilde{Y}$ satisfies the BSDE

$$d\tilde{Y}_t = -\zeta_t^\top c_t^{\bar{M}} \zeta_t dI_t + \zeta_t d\bar{M}_t, \quad 0 \le t \le T,$$

with $\tilde{Y}_T = aK$, where $K := \int_0^T \chi_t^\top c_t^{\bar{M}} \chi_t dI_t$. Itô's formula yields

$$\frac{d(\exp(2\tilde{Y}_t))}{\exp(2\tilde{Y}_t)} = -2\zeta_t^\top c_t^{\bar{M}} \zeta_t dI_t + \zeta_t d\bar{M}_t + 2d\langle \zeta \mathbin{\bullet} \bar{M} \rangle_t = \zeta_t d\bar{M}_t, \quad 0 \le t \le T,$$

and hence $\exp(2\tilde{Y})$ is a nonnegative local martingale with terminal value $\exp(2aK)$ and some finite initial value $\exp(2Y_0)$, which is deterministic as $\mathcal{F}_0$ is P-trivial. Thus the existence of a solution to (6.3) implies that $\exp(2\tilde{Y})$ is a supermartingale and $E[\exp(2aK)] < \infty$. Conversely, a solution to (6.3) cannot exist if $\chi \in L^2(\bar{M})$ is such that $\exp(2aK)$ is not integrable. The condition that $\chi \mathbin{\bullet} \bar{M} \in BMO$ is not sufficient to ensure the finiteness of all exponential moments of K, and thus it does not guarantee the existence of a solution to (6.3) for all $a > 0$.

Example 6.1 shows why the finiteness of a large enough exponential moment of K is typically needed to obtain the well-posedness of (6.2), and that condition does not hold in our setup. As we shall see in Theorem 6.6, we can still show the stability of equations of the form (6.2) without this assumption, but circumventing it comes with some tradeoffs. First, in (6.2) we have assumed that the driver does not explicitly depend on Y; this is also the reason why it was necessary to take the logarithms of the original processes $\hat{Y}^{(i)}$ to obtain (4.59) and (4.60) in Section 4.2. More importantly, we assume rather than show the existence of a solution to (6.2) for some given inputs χ, $\bar{M}$, f, ϱ and G. Finally, we obtain only a partial a priori bound in Proposition 6.2 below, in the following sense. Typically, one would like to obtain bounds on $\|Y\|_\infty$, $\|\zeta \bullet \bar{M}\|_{BMO}$ and $\|N\|_{BMO}$ that depend only on the inputs. Instead, we show in Proposition 6.2 how to bound $\|\zeta \bullet \bar{M}\|_{BMO}$ and $\|N\|_{BMO}$ in terms of the inputs as well as $\|Y\|_\infty$, but we do not obtain a bound for Y. While the latter bounds are less satisfactory, they suffice for our main goal of showing the stability of (6.2).

We now give the partial a priori bound, which in particular ensures that the martingale parts of a solution Y to (6.2) belong to BMO if Y is bounded and f satisfies a quadratic growth condition. The main idea of the proof is standard; we want to use (6.2) to construct a certain nonnegative submartingale so that a moment estimate on its terminal value yields a corresponding estimate on the whole process. We do this along similar lines as in part (ii) of the proof of Zhang [121, Theorem 7.2.1]; we do not require part (i) because we do not give an a priori bound on $\|Y\|_\infty$. We need some additional care since the exogenous coefficient χ is not bounded and we include an orthogonal part N which we only assume to be a continuous local martingale. The first issue turns out not to make a significant difference in the resulting bounds, and the second can be dealt with by constructing a nonnegative local submartingale (rather than a true one).

Proposition 6.2. *Let $G \in L^\infty(\mathcal{F}_T)$, $\chi \in L^2(\bar{M}; \mathbb{R}^{\bar{d}\times m})$, $(\varrho_t)_{0\le t\le T}$ be a bounded predictable process and $f : \Omega \times [0,T] \times \mathbb{R}^{\bar{d}} \times \mathbb{R}^{\bar{d}\times m} \to \mathbb{R}$ a predictable function. Suppose that (Y, ζ, N) is a solution to (6.2), where $(Y_t)_{0\le t\le T}$ is a bounded semimartingale, $\zeta \in L^2_{\mathrm{loc}}(\bar{M}; \mathbb{R}^{\bar{d}})$ and N is a continuous local martingale orthogonal to $\bar{M}$. Moreover, suppose that $\chi \bullet \bar{M} \in BMO$ and that f satisfies the inequality*

$$|f(z, x)| \le C_f(|z|_{c^{\bar{M}}} + \|x\|_{c^{\bar{M}}})^2 \quad P \otimes I\text{-a.e.} \tag{6.4}$$

for some constant $C_f > 0$ and all $z \in \mathbb{R}^{\bar{d}}$ and $x \in \mathbb{R}^{\bar{d}\times m}$. Then $\zeta \bullet \bar{M}$ and N are

BMO martingales with

$$\max(\|\zeta \bullet \bar{M}\|_{BMO}, \|N\|_{BMO}) < C,$$

where C depends only on C_f, $\|\varrho\|_\infty$, $\|Y\|_\infty$ and $\|\chi \bullet \bar{M}\|_{BMO}$.

Proof. Define $\tilde{X}_t := \exp(aY_t)$, $0 \le t \le T$, for some $a > 0$ to be chosen later. By Itô's formula,

$$\begin{aligned}\frac{d\tilde{X}_t}{\tilde{X}_t} &= af_t(\zeta_t, \chi_t)dI_t - a\varrho_t d\langle N\rangle_t + \frac{a^2(\zeta_t^\top c_t^{\bar{M}}\zeta_t dI_t + d\langle N\rangle_t)}{2} \\ &\quad + a\zeta_t d\bar{M}_t + adN_t \end{aligned} \tag{6.5}$$

for $0 \le t \le T$. We claim that the process $(X_t)_{0\le t\le T}$ defined by

$$X_t = \tilde{X}_t + 2aC_f \int_0^t \tilde{X}_s \|\chi_s\|^2_{c_s^{\bar{M}}} dI_s - \delta \int_0^t \tilde{X}_s |\zeta_s|^2_{c_s^{\bar{M}}} dI_s - \delta(\tilde{X} \bullet \langle N\rangle)_t \tag{6.6}$$

is a local submartingale for some $a > 0$ and $\delta > 0$. It suffices to show that the finite-variation part of X is increasing. We first consider the dI-component, which by (6.5) and (6.6) is given by

$$\left(af_t(\zeta_t, \chi_t) + 2aC_f\|\chi_t\|^2_{c_t^{\bar{M}}} + \frac{(a^2 - 2\delta)}{2}|\zeta_t|^2_{c_t^{\bar{M}}}\right)\tilde{X}_t dI_t, \quad 0 \le t \le T.$$

If we choose a and δ such that $a > 4C_f$ and $0 < \delta < \frac{a^2 - 4aC_f}{2}$, then (6.4) yields

$$\begin{aligned} &af_t(\zeta_t, \chi_t) + 2aC_f\|\chi_t\|^2_{c_t^{\bar{M}}} + \frac{(a^2 - 2\delta)}{2}|\zeta_t|^2_{c_t^{\bar{M}}} \\ &\ge -2aC_f(|\zeta_t|^2_{c_t^{\bar{M}}} + \|\chi_t\|^2_{c_t^{\bar{M}}}) + 2aC_f\|\chi_t\|^2_{c_t^{\bar{M}}} + \frac{(a^2 - 2\delta)}{2}|\zeta_t|^2_{c_t^{\bar{M}}} \\ &= \frac{1}{2}(a^2 - 4aC_f - 2\delta)|\zeta_t|^2_{c_t^{\bar{M}}} \ge 0, \quad 0 \le t \le T. \end{aligned}$$

Thus the dI-component of X is increasing, and the conditions on a and δ depend only on C_f. We now consider the $d\langle N\rangle$-component of X which by (6.5) and (6.6) is given by

$$\left(\frac{a^2}{2} - a\varrho_t - \delta\right)\tilde{X}_t d\langle N\rangle_t, \quad 0 \le t \le T.$$

This is nonnegative if $a > 2\|\varrho\|_\infty$ and $\delta < \frac{a^2}{2} - a\|\varrho\|_\infty$. Thus we may fix some large $a > 0$ and small $\delta > 0$ that depend only on C_f and $\|\varrho\|_\infty$ such that X is a local submartingale. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence of stopping times such that X^{τ_n} is a submartingale for each $n \in \mathbb{N}$. Then for any stopping time τ with values in $[0,T]$, we have $E[X_{\tau_n} \mid \mathcal{F}_\tau] \geq X_{\tau\wedge\tau_n}$. By rearranging (6.6), we obtain

$$\delta E\bigg[\int_{\tau\wedge\tau_n}^{\tau_n} \tilde{X}_s|\zeta_s|^2_{c_s^{\bar{M}}} dI_s + \int_{\tau\wedge\tau_n}^{\tau_n} \tilde{X}_s d\langle N\rangle_s \bigg| \mathcal{F}_\tau\bigg] \\ \leq E\bigg[\tilde{X}_{\tau_n} - \tilde{X}_{\tau\wedge\tau_n} + 2aC_f\int_{\tau\wedge\tau_n}^{\tau_n} \tilde{X}_s\|\chi_s\|^2_{c_s^{\bar{M}}} dI_s \bigg| \mathcal{F}_\tau\bigg]$$

for $n \in \mathbb{N}$. Since $\tilde{X} = \exp(aY)$, we have $\|\tilde{X}\|_\infty \leq \exp(a\|Y\|_\infty)$ so that

$$E\bigg[\int_{\tau\wedge\tau_n}^{\tau_n} \tilde{X}_s|\zeta_s|^2_{c_s^{\bar{M}}} dI_s + \int_{\tau\wedge\tau_n}^{\tau_n} \tilde{X}_s d\langle N\rangle_s \bigg| \mathcal{F}_\tau\bigg] \leq \frac{2\exp(a\|Y\|_\infty)(1+aC_f\|\chi \bullet \bar{M}\|^2_{BMO})}{\delta}$$

for $n \in \mathbb{N}$. Taking $n \to \infty$, the conditional monotone convergence theorem and the lower bound $\inf_{0\leq t\leq T} \tilde{X}_t \geq \exp(-a\|Y\|_\infty)$ yield

$$E\bigg[\int_\tau^T |\zeta_s|^2_{c_s^{\bar{M}}} dI_s + \langle N\rangle_T - \langle N\rangle_\tau \bigg| \mathcal{F}_\tau\bigg] \leq \frac{2\exp(2a\|Y\|_\infty)(1+C_f\|\chi \bullet \bar{M}\|^2_{BMO})}{\delta}.$$

As this holds for any stopping time τ, it follows that $\zeta \bullet \bar{M}, N \in BMO$ and the bounds depend only on C_f, $\|\varrho\|_\infty$, $\|Y\|_\infty$ and $\|\chi \bullet \bar{M}\|_{BMO}$. □

Before proceeding to the main result on the stability of (6.2), we first give two technical lemmas.

Lemma 6.3. *Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ be sequences of nonnegative random variables such that $X_nY_n \xrightarrow{P} 0$ as $n \to \infty$ and*

$$C_{\tilde{\kappa}} := \sup_{n\in\mathbb{N}} E[X_n^{-\tilde{\kappa}}] < \infty$$

for some $\tilde{\kappa} > 0$. Then $Y_n \xrightarrow{P} 0$ as $n \to \infty$.

Proof. By the Markov inequality, $P[X_n \leq \epsilon] = P[X_n^{-\tilde{\kappa}} \geq \epsilon^{-\tilde{\kappa}}] \leq C_{\tilde{\kappa}}\epsilon^{\tilde{\kappa}}$ for all $\epsilon > 0$ and $n \in \mathbb{N}$. Since $X_nY_n \xrightarrow{P} 0$ as $n \to \infty$, we obtain for any $\delta, \epsilon > 0$ that

$$\limsup_{n\to\infty} P[Y^n > \delta] \leq \limsup_{n\to\infty}(P[X_nY_n > \epsilon\delta] + P[X_n \leq \epsilon]) \leq C_{\tilde{\kappa}}\epsilon^{\tilde{\kappa}}.$$

Hence because $\epsilon > 0$ is arbitrary, $P[Y^n > \delta] \to 0$ and $Y^n \xrightarrow{P} 0$ as $n \to \infty$. □

We now introduce some notation in order to state the next lemma and the main result in Theorem 6.6.

Definition 6.4. Let $m \in \mathbb{N}$. We say that a family $(f^n)_{n\in\mathbb{N}\cup\{\infty\}}$ of predictable functions $f^n : \Omega \times [0,T] \times \mathbb{R}^{\bar{d}\times m} \to \mathbb{R}$ satisfies a *uniform quadratic growth bound* (with respect to $\bar{M}$) if there exists a constant $C_f > 0$ such that

$$|f^n(x)| \le C_f \|x\|^2_{c^{\bar{M}}} \quad P \otimes I\text{-a.e.} \tag{6.7}$$

for $n \in \mathbb{N}\cup\{\infty\}$ and $x \in \mathbb{R}^{\bar{d}\times m}$. We say that $(f^n)_{n\in\mathbb{N}\cup\{\infty\}}$ satisfies a *uniform local Lipschitz bound* (with respect to $\bar{M}$) if there exists a constant $L_f > 0$ such that

$$|f^n(x_1) - f^n(x_2)| \le L_f(\|x_1 - x_2\|_{c^{\bar{M}}})(\|x_1\|_{c^{\bar{M}}} + \|x_2\|_{c^{\bar{M}}}) \quad P \otimes I\text{-a.e.} \tag{6.8}$$

for $n \in \mathbb{N} \cup \{\infty\}$ and $x_1, x_2 \in \mathbb{R}^{\bar{d}\times m}$.

We now show that the pointwise convergence of predictable functions f^n satisfying the bounds (6.7) and (6.8) implies the convergence of the stochastic processes generated by replacing the spatial coordinate x of f^n with a predictable process $(\chi_t)_{0\le t\le T}$.

Lemma 6.5. *Let $f^n : \Omega \times [0,T] \times \mathbb{R}^{\bar{d}\times m} \to \mathbb{R}$ be a predictable function for each $n \in \mathbb{N} \cup \{\infty\}$ and some $m \in \mathbb{N}$. Suppose that $(f^n)_{n\in\mathbb{N}\cup\{\infty\}}$ satisfies the uniform quadratic growth and local Lipschitz bounds* (6.7) *and* (6.8)*, and that*

$$\lim_{n\to\infty} f^n(x) = f^\infty(x) \quad P \otimes I\text{-a.e.} \tag{6.9}$$

for each $x \in \mathbb{R}^{\bar{d}\times m}$. Then for any predictable process $(\chi_t)_{0\le t\le T} \in L^2_{\mathrm{loc}}(\bar{M}; \mathbb{R}^{\bar{d}\times m})$, it holds that

$$\lim_{n\to\infty} \int_0^T |f^n_t(\chi_t) - f^\infty_t(\chi_t)| dI_t = 0 \quad P\text{-a.s.} \tag{6.10}$$

Proof. We start by approximating the process χ by a simple process $\tilde{\chi}$ with values in $\mathbb{R}^{\bar{d}\times m}$, and for that we partition the latter space into hypercubes. Fix some $\delta > 0$ and consider the partition $(\tilde{D}_z)_{z\in\mathbb{Z}^{\bar{d}\times m}}$ of $\mathbb{R}^{\bar{d}\times m}$, where $\tilde{D}_z$ is the hypercube given by $\prod_{j=1}^{\bar{d}\times m}[z_j, z_j + 1)$. For each $z \in \mathbb{Z}^{\bar{d}\times m}$, we then further partition $\tilde{D}_z$ into $q_z^{\bar{d}\times m}$ smaller hypercubes of side length $1/q_z$, where

$$q_z := \left\lceil \frac{\sup_{x\in\tilde{D}_z} |x|}{(m\bar{d})^{1/2}\delta} \right\rceil.$$

Thus if we enumerate the resulting family of smaller hypercubes by $(D_i)_{i\in\mathbb{N}}$ in some arbitrary order, it follows that the D_i form a partition of $\mathbb{R}^{\bar{d}\times m}$. Moreover, let $i \in \mathbb{N}$ and take $z \in \mathbb{Z}^{\bar{d}\times m}$ such that $D_i \subseteq \tilde{D}_z$. Then we have the inequality

$$\operatorname{diam}(D_i) \sup_{x\in D_i} |x| \leq \frac{(\bar{d}m)^{1/2}}{q_z} \sup_{x\in\tilde{D}_z} |x| \leq \delta \tag{6.11}$$

by the choice of q_z; this holds for all $i \in \mathbb{N}$.

We can now use the partition (D_i) to show (6.10). Without loss of generality, we may assume that $f^\infty = 0$. For each $i \in \mathbb{N}$, pick an arbitrary element $x_i \in D_i$. By (6.7) and since $\|\cdot\|_{c_t^{\bar{M}}} \leq \|\cdot\|$, we have for $i \in \mathbb{N}$ that $|f^n(x_i)| \leq C_f \|x_i\|^2$ $P \otimes I$-a.e. for all $n \in \mathbb{N}$, where

$$\int_0^T C_f \|x_i\|^2 dI_t = C_f \|x_i\|^2 \operatorname{tr} \langle \bar{M} \rangle_T < \infty \quad P\text{-a.s. for } i \in \mathbb{N}.$$

Thus (6.9) with $f^\infty = 0$ and the dominated convergence theorem with majorant $C_f \|\chi_i\|^2$ for each i yield

$$P\left[\lim_{n\to\infty} \int_0^T |f_t^n(x_i)| dI_t = 0, \forall i \in \mathbb{N}\right] = 1. \tag{6.12}$$

Now fix a process $\chi \in L^2_{\mathrm{loc}}(\bar{M}; \mathbb{R}^{\bar{d}\times m})$. Consider the random sets

$$A_i := \{(\omega, t) \in \Omega \times [0, T] : \chi_t(\omega) \in D_i\}, \quad i \in \mathbb{N},$$

and define the process $\tilde{\chi} := \sum_{i\in\mathbb{N}} \mathbf{1}_{A_i} x_i$. By construction, we have on A_i that $\chi \in D_i$ and $\tilde{\chi} = x_i \in D_i$. Then by (6.7), (6.8) and (6.11), we have $P \otimes I$-a.e. that

$$\begin{aligned} |f^n(\chi)| &\leq C_f \|\chi\|^2_{c^{\bar{M}}}, \\ |f^n(\chi) - f^n(\tilde{\chi})| &\leq L_f(\|\chi - \tilde{\chi}\|_{c^{\bar{M}}})(\|\chi\|_{c^{\bar{M}}} + \|\tilde{\chi}\|_{c^{\bar{M}}}) \\ &\leq \sum_{i\in\mathbb{N}} L_f \mathbf{1}_{A_i}(\|\chi - x_i\|)(\|\chi\| + \|x_i\|) \leq 2L_f\delta. \end{aligned} \tag{6.13}$$

In particular, $|f^n(\tilde{\chi})| \leq C_f \|\chi\|^2_{c^{\bar{M}}} + 2L_f\delta$ so that

$$y_{i,n} := \int_0^T \mathbf{1}_{A_i} |f_t^n(\tilde{\chi}_i)| dI_t \leq \int_0^T \mathbf{1}_{A_i}(C_f \|\chi_t\|^2_{c_t^{\bar{M}}} + 2L_f\delta) dI_t =: \bar{y}_i \quad P\text{-a.s.}$$

for each $i \in \mathbb{N}$ and uniformly in $n \in \mathbb{N}$, where the majorant defined by the map

$i \mapsto \bar{y}_i$ is summable since

$$\sum_{i\in\mathbb{N}} \bar{y}_i = \int_0^T (C_f \|\chi_t\|^2_{c_t^{\bar{M}}} + 2L_f\delta) dI_t = C_f \mathrm{tr}\,\langle \chi \bullet \bar{M}\rangle_T + 2L_f\delta \mathrm{tr}\,\langle \bar{M}\rangle_T < \infty \quad P\text{-a.s.}$$

Thus by (6.12) and the dominated convergence theorem with respect to the counting measure on $\mathbb{N}$, we have

$$\begin{aligned}\lim_{n\to\infty} \int_0^T |f_t^n(\tilde{\chi}_t)| dI_t &= \lim_{n\to\infty} \sum_{i\in\mathbb{N}} y_{i,n} \\ &= \sum_{i\in\mathbb{N}} \lim_{n\to\infty} y_{i,n} = \sum_{i\in\mathbb{N}} \lim_{n\to\infty} \int_0^T \mathbf{1}_{A_i} |f_t^n(x_i)| dI_t = 0 \quad P\text{-a.s.}\end{aligned}$$

Hence (6.13) yields

$$\limsup_{n\to\infty} \int_0^T |f_t^n(\chi_t)| dI_t \le \int_0^T 2\delta dI_t + \lim_{n\to\infty} \int_0^T |f_t^n(\tilde{\chi}_t)| dI_t \le 2\delta \mathrm{tr}\,\langle \bar{M}\rangle_T \quad P\text{-a.s.}$$

As we assumed $f^\infty = 0$ and $\delta > 0$ is arbitrary, (6.10) follows by taking $\delta \searrow 0$. □

We are now ready to give a stability result for equations of the form (6.2), where we replace the inputs f, χ and G with sequences $(f^n)_{n\in\mathbb{N}\cup\{\infty\}}, (\chi^n)_{n\in\mathbb{N}\cup\{\infty\}}$ and $(G^n)_{n\in\mathbb{N}\cup\{\infty\}}$ such that $f^n \to f^\infty$, $\chi^n \to \chi^\infty$ and $G^n \to G^\infty$ as $n \to \infty$ in the sense of (d), (b) and (e) below. As discussed after Example 6.1, we assume that solutions (Y^n, ζ^n, N^n) to the corresponding equations are given a priori. Additionally, we suppose that the sequence of solutions $(Y^n)_{n\in\mathbb{N}}$ is uniformly bounded in L^∞; this is needed since Proposition 6.2 does not provide an a priori bound for Y^n. In addition to the previous results, we also use in the proof some well-known facts about continuous BMO martingales that are given afterwards in Section 6.2 for convenience.

We follow the basic structure of the proof of Zhang [121, Theorem 7.3.4]: namely, we obtain a BSDE for $Y^n - Y^\infty$ and simplify it by a change of measure. More precisely, we find the dynamics of $Z^n(Y^n - Y^\infty)$ for a suitable positive martingale Z^n started at 1, which by Girsanov's theorem is equivalent to finding the dynamics of $Y^n - Y^\infty$ under the equivalent measure with density Z^n_T. We can then obtain bounds for $|Y^n - Y^\infty|$ as well as for the differences of the martingale parts. The change of measure is here more delicate than in [121] since the coefficients that define Z^n are not bounded in this case, and hence more care is required to obtain the subsequent bounds.

Theorem 6.6. *Suppose that* $(Y^n)_{n\in\mathbb{N}\cup\{\infty\}}$ *is a sequence of continuous semimartingales on* $[0,T]$ *such that* Y^n *for* $n \in \mathbb{N}\cup\{\infty\}$ *satisfies the equation*

$$dY^n_t = f^n_t(\zeta^n_t, \chi^n_t)dI_t - \varrho_t d\langle N^n\rangle_t + \zeta^n_t d\bar{M}_t + dN^n_t, \quad 0 \le t \le T, \tag{6.14}$$
$$Y^n_T = G^n,$$

where N^n *is a continuous local martingale orthogonal to* $\bar{M}$, $\chi^n \in L^2(\bar{M};\mathbb{R}^{\bar{d}\times m})$ *and* $\zeta^n \in L^2_{\mathrm{loc}}(\bar{M};\mathbb{R}^{\bar{d}})$ *are predictable integrands,* $(\varrho_t)_{0\le t\le T}$ *is a bounded predictable process,* $G^n \in L^\infty(\mathcal{F}_T)$ *and* $f^n : \Omega\times[0,T]\times\mathbb{R}^{\bar{d}}\times\mathbb{R}^{\bar{d}\times m}\to\mathbb{R}$ *is a predictable function. Suppose that the following conditions hold:*

(a) $C_Y := \sup_{n\in\mathbb{N}\cup\{\infty\}} \|\sup_{t\in[0,T]}|Y^n_t|\|_\infty < \infty$.

(b) *The processes* $\chi^n \bullet \bar{M}$ *are BMO martingales for each* $n\in\mathbb{N}\cup\{\infty\}$ *with*

$$\sup_{n\in\mathbb{N}\cup\{\infty\}} \|\chi^n\bullet\bar{M}\|_{BMO} < \infty.$$

Moreover, $\operatorname{tr}\langle(\chi^n-\chi^\infty)\bullet\bar{M}\rangle_T \xrightarrow{P} 0$ *as* $n\to\infty$.

(c) $(f^n)_{n\in\mathbb{N}\cup\{\infty\}}$ *satisfies the uniform quadratic growth and local Lipschitz bounds* (6.7) *and* (6.8) *(with* $(z,x)\in\mathbb{R}^{\bar{d}\times(m+1)}$ *in place of* x*).*

(d) *For any* $z\in\mathbb{R}^{\bar{d}}$ *and* $x\in\mathbb{R}^{\bar{d}\times m}$*, it holds that*

$$\lim_{n\to\infty} f^n(z,x) = f^\infty(z,x) \quad P\otimes I\text{-a.e.}$$

(e) $G^n \xrightarrow{P} G^\infty$ *as* $n\to\infty$.

Then the families of martingales $(\zeta^n\bullet\bar{M})_{n\in\mathbb{N}\cup\{\infty\}}$ *and* $(N^n)_{n\in\mathbb{N}\cup\{\infty\}}$ *are bounded in BMO, and it holds as* $n\to\infty$ *that*

$$\sup_{t\in[0,T]} |Y^n_t - Y^\infty_t| \xrightarrow{P} 0, \tag{6.15}$$
$$\langle(\zeta^n-\zeta^\infty)\bullet\bar{M}\rangle_T + \langle N^n - N^\infty\rangle_T \xrightarrow{P} 0. \tag{6.16}$$

Proof. **(i)** We first derive a BSDE for $\delta^n Y := Y^n - Y^\infty$. In the following, we also write $\delta^n X := X^n - X^\infty$ and $\sigma^n X := X^n + X^\infty$ for $X\in\{\zeta,N,\chi,G,f\}$. Taking differences in (6.14) yields for each $n\in\mathbb{N}$ and $0\le t\le T$ the BSDE

$$\begin{aligned} d\delta^n Y_t =& \big(f^n_t(\zeta^n_t,\chi^n_t) - f^\infty_t(\zeta^\infty_t,\chi^\infty_t)\big)dI_t - \varrho_t d\langle\delta^n N,\sigma^n N\rangle_t \\ &+ \delta^n\zeta_t d\bar{M}_t + d\delta^n N_t \end{aligned} \tag{6.17}$$

with terminal condition $\delta^n Y_T = \delta^n G$. We decompose the drift term as

$$\begin{aligned} f_t^n(\zeta_t^n, \chi_t^n) - f_t^\infty(\zeta_t^\infty, \chi_t^\infty) &= \delta^n f_t(\zeta_t^\infty, \chi_t^\infty) + \big(f_t^n(\zeta_t^\infty, \chi_t^n) - f_t^n(\zeta_t^\infty, \chi_t^\infty)\big) \\ &\quad + \big(f_t^n(\zeta_t^n, \chi_t^n) - f_t^n(\zeta_t^\infty, \chi_t^n)\big), \quad 0 \le t \le T. \end{aligned} \tag{6.18}$$

Define the predictable processes $(\alpha_t^n)_{0\le t\le T}$, $(\beta_t^n)_{0\le t\le T}$ for $0 \le t \le T$ as

$$\alpha_t^n := \mathbf{1}_{\{|\delta^n \zeta_t|_{c_t^{\bar M}} \neq 0\}} \frac{f_t^n(\zeta_t^n, \chi_t^n) - f_t^n(\zeta_t^\infty, \chi_t^n)}{|\delta^n \zeta_t|_{c_t^{\bar M}}}, \tag{6.19}$$

$$\beta_t^n := \mathbf{1}_{\{\|\delta^n \chi_t\|_{c_t^{\bar M}} \neq 0\}} \frac{f_t^n(\zeta_t^\infty, \chi_t^n) - f_t^n\big(\zeta_t^\infty, \chi_t^\infty\big)}{\|\delta^n \chi_t\|_{c_t^{\bar M}}} \tag{6.20}$$

so that by (6.8), we have

$$\max(|\alpha^n|, |\beta^n|) \le 2L_f(|\zeta^n|_{c^{\bar M}} + |\zeta^\infty|_{c^{\bar M}} + \|\chi^n\|_{c^{\bar M}} + \|\chi^\infty\|_{c^{\bar M}}) \quad P \otimes I\text{-a.e.} \tag{6.21}$$

By plugging (6.19) and (6.20) into (6.18), we can rewrite (6.17) as

$$\begin{aligned} d\delta^n Y_t &= \big(\delta^n f_t(\zeta_t^\infty, \chi_t^\infty) + \beta_t^n \|\delta^n \chi_t\|_{c_t^{\bar M}} + \alpha_t^n |\delta^n \zeta_t|_{c_t^{\bar M}}\big) d\bar B_t \\ &\quad - \varrho_t d\langle \delta^n N, \sigma^n N\rangle_t + dM_t^{Y,n}, \quad 0 \le t \le T, \end{aligned} \tag{6.22}$$

where we define $(M_t^{Y,n})_{0\le t\le T}$ by

$$M^{Y,n} := \delta^n \zeta \bullet \bar M + \delta^n N. \tag{6.23}$$

Next, we want to remove the drift terms from (6.22) involving $\delta^n \zeta$ and $\delta^n N$. To that end, consider the stochastic exponential $(Z_t^n)_{0\le t\le T}$ given by $Z^n := \mathcal{E}(M^{Z,n})$, where we define the local martingale $(M_t^{Z,n})_{0\le t\le T}$ by

$$M^{Z,n} := -\left(\frac{\alpha^n}{|\delta^n \zeta|_{c^{\bar M}}} \delta^n \zeta\right) \bullet \bar M + \varrho \bullet \sigma^n N \tag{6.24}$$

for $n \in \mathbb{N}$. For later use, we also define the local martingales $(M_t^{Z,Y,n})_{0\le t\le T}$ and $(\hat M_t^{Z,Y,n})_{0\le t\le T}$ by

$$M^{Z,Y,n} := \delta^n Y \bullet M^{Z,n} + M^{Y,n}, \tag{6.25}$$

$$\hat M^{Z,Y,n} := \delta^n Y \bullet M^{Z,Y,n} + \delta^n Y \bullet M^{Y,n}. \tag{6.26}$$

We now check that $M^{Y,n}, M^{Z,n}, M^{Z,Y,n}$ and $\hat M^{Z,Y,n}$ are martingales that are bounded in BMO over $n \in \mathbb{N} \cup \{\infty\}$. Note that the sequence $(\chi^n \bullet \bar{M})_{n\in\mathbb{N}\cup\{\infty\}}$ is bounded in BMO by assumption (b). Thus by Proposition 6.2 and the uniform bounds in (a), (b) and (c), the sequences $(\zeta^n \bullet \bar{M})_{n\in\mathbb{N}\cup\{\infty\}}$ and $(N^n)_{n\in\mathbb{N}\cup\{\infty\}}$ are also bounded in BMO; this already shows the first statement of the theorem. Hence $(M^{Y,n})_{n\in\mathbb{N}}$ is bounded in BMO since each of the terms in (6.23) is. For $M^{Z,n}$, with $\tilde{\alpha}^n := -\frac{\alpha^n}{|\delta^n\zeta|_{c^{\bar{M}}}}\delta^n\zeta$, the bound (6.21) yields

$$
\begin{aligned}
d\langle \tilde{\alpha}^n \bullet \bar{M}\rangle_t &= \frac{|\alpha_t^n|^2(\delta^n\zeta_t)^\top c_t^{\bar{M}}\delta^n\zeta_t}{|\delta^n\zeta_t|^2_{c_t^{\bar{M}}}}dI_t \\
&= |\alpha_t^n|^2 dI_t \\
&\le 16L_f^2(|\zeta_t^n|^2_{c_t^{\bar{M}}} + |\zeta_t^\infty|^2_{c_t^{\bar{M}}} + \|\chi_t^n\|^2_{c_t^{\bar{M}}} + \|\chi_t^\infty\|^2_{c_t^{\bar{M}}})dI_t \qquad (6.27)
\end{aligned}
$$

for $0 \le t \le T$. Since we have for any $\varphi \in L^2_{\mathrm{loc}}(\bar{M})$ that

$$
\langle \varphi \bullet \bar{M}\rangle_T - \langle \varphi \bullet \bar{M}\rangle_t = \int_t^T (\varphi_u)^\top c_u^{\bar{M}}\varphi_u dI_u = \int_t^T |\varphi_u|^2_{c_u^{\bar{M}}} dI_u, \quad 0 \le t \le T, \quad (6.28)
$$

it follows from (6.27) that the increments of $\langle \tilde{\alpha}^n \bullet \bar{M}\rangle$ are bounded by a linear combination of the increments of $\langle \zeta^n \bullet \bar{M}\rangle$, $\langle \zeta^\infty \bullet \bar{M}\rangle$, $\langle \chi^n \bullet \bar{M}\rangle$ and $\langle \chi^\infty \bullet \bar{M}\rangle$, and so $\tilde{\alpha}^n \in L^2(\bar{M})$ and $\tilde{\alpha}^n \bullet \bar{M}$ is also a BMO martingale. Since the constant $16L_f^2$ in (6.27) is independent of n, also $(\tilde{\alpha}^n \bullet \bar{M})_{n\in\mathbb{N}}$ is bounded in BMO. Likewise, ϱ is bounded and $(\sigma^n N)_{n\in\mathbb{N}}$ is bounded in BMO so that $(\varrho \bullet \sigma^n N)_{n\in\mathbb{N}}$ is bounded in BMO, and hence so is $(M^{Z,n})_{n\in\mathbb{N}}$ by (6.24). Finally, $(M^{Z,Y,n})_{n\in\mathbb{N}}$ and $(\hat{M}^{Z,Y,n})_{n\in\mathbb{N}}$ are bounded in BMO like $(M^{Y,n})_{n\in\mathbb{N}}$ and $(M^{Z,n})_{n\in\mathbb{N}}$ by (6.25) and (6.26) because $(Y^n)_{n\in\mathbb{N}}$ is uniformly bounded by assumption (a).

Returning to (6.22), the orthogonality $N^n, N^\infty \perp \bar{M}$ and (6.24) yield

$$
\begin{aligned}
d\langle M^{Z,n}, \delta^n Y\rangle_t &= -\frac{\alpha_t^n}{|\delta^n\zeta_t|_{c_t^{\bar{M}}}}(\delta^n\zeta_t)^\top c_t^{\bar{M}}\delta^n\zeta_t dI_t + \varrho_t d\langle \delta^n N, \sigma^n N\rangle_t \\
&= -\alpha_t^n|\delta^n\zeta_t|_{c_t^{\bar{M}}} dI_t + \varrho_t d\langle \delta^n N, \sigma^n N\rangle_t, \quad 0 \le t \le T. \qquad (6.29)
\end{aligned}
$$

By the product rule, plugging (6.24) and (6.29) into (6.22) yields

$$
\frac{d(Z_t^n\delta^n Y_t)}{Z_t^n} = \big(\beta_t^n\|\delta^n\chi_t\|_{c_t^{\bar{M}}} + \delta^n f_t(\zeta_t^\infty, \chi_t^\infty)\big)dI_t + dM_t^{Z,Y,n}, \quad 0 \le t \le T, \quad (6.30)
$$

where we recall $M^{Z,Y,n}$ from (6.25). For later use, we also derive a BSDE for

$Z_t^n(\delta^n Y_t)^2$. By (6.23)–(6.25), we have

$$\begin{aligned}
d\langle M^{Y,n}, M^{Z,Y,n}\rangle_t &= \delta^n Y_t d\langle M^{Y,n}, M^{Z,n}\rangle_t + d\langle M^{Y,n}\rangle_t \\
&= -\delta^n Y_t\big(\alpha_t^n|\delta^n\zeta_t|_{c_t^{\bar M}} dI_t - \varrho_t d\langle \delta^n N, \sigma^n N\rangle_t\big) + d\langle M^{Y,n}\rangle_t.
\end{aligned}$$

Thus by applying the product rule to (6.22) and (6.30), we get

$$\begin{aligned}
\frac{d(Z_t^n(\delta^n Y_t)^2)}{Z_t^n} &= \delta^n Y_t\bigg(\frac{d(Z_t^n\delta^n Y_t)}{Z_t^n} + d\delta^n Y_t\bigg) + d\langle M^{Y,n}, M^{Z,Y,n}\rangle_t \\
&= 2\delta^n Y_t\big(\beta_t^n\|\delta^n\chi_t\|_{c_t^{\bar M}} + \delta^n f_t(\zeta_t^\infty, \chi_t^\infty)\big)dI_t \\
&\quad + d\langle M^{Y,n}\rangle_t + d\hat M_t^{Z,Y,n}, \quad 0 \le t \le T,
\end{aligned} \tag{6.31}$$

where we recall $\hat M^{Z,Y,n}$ from (6.26).

(ii) Next, we show that $Z^n \bullet M^{Z,Y,n}$ is a true martingale in order to take conditional expectations in (6.30). By Lemma 6.9 below and since $(M^{Z,n})_{n\in\mathbb N}$ is bounded in BMO, each $Z^n = \mathcal E(M^{Z,n})$ is a strictly positive martingale on $[0,T]$ satisfying the bound

$$\sup_{t\in[0,T]} E\bigg[\sup_{u\in[t,T]}\bigg|\frac{Z_u^n}{Z_t^n}\bigg|^{1+\kappa}\,\bigg|\,\mathcal F_t\bigg] \le C_Z \tag{6.32}$$

for $n \in \mathbb N\cup\{\infty\}$ and some constants $\kappa > 0$, $C_Z > 0$ independent of n. Since $M^{Z,Y,n} \in BMO$, we have by (6.32) and Lemma 6.11 below with $\gamma = Z^n$ that $Z^n \bullet M^{Z,Y,n}$ is a (true) P-martingale on $[0,T]$. Taking conditional expectations in (6.30) and recalling the terminal condition $\delta^n Y_T = \delta^n G$, we thus obtain

$$Z_t^n\delta^n Y_t = E\bigg[Z_T^n\delta^n G - \int_t^T Z_u^n\big(\beta_u^n\|\delta^n\chi_u\|_{c_u^{\bar M}} + \delta^n f_u(\zeta_u^\infty, \chi_u^\infty)\big)dI_u\,\bigg|\,\mathcal F_t\bigg]$$

for $0 \le t \le T$ and $n \in \mathbb N\cup\{\infty\}$. Dividing by Z_t^n yields

$$\delta^n Y_t = E\bigg[\frac{Z_T^n\delta^n G}{Z_t^n} - \int_t^T \frac{Z_u^n}{Z_t^n}\big(\beta_u^n\|\delta^n\chi_u\|_{c_u^{\bar M}} + \delta^n f_u(\zeta_u^\infty, \chi_u^\infty)\big)dI_u\,\bigg|\,\mathcal F_t\bigg] \tag{6.33}$$

for $0 \le t \le T$. Taking absolute values in (6.33), we obtain the bound

$$|\delta^n Y_t| \le E\bigg[Z_{t,T}^{n,*}\bigg(|\delta^n G| + \int_t^T \big(|\beta_u^n|\|\delta^n\chi_u\|_{c_u^{\bar M}} + |\delta^n f_u(\zeta_u^\infty, \chi_u^\infty)|\big)dI_u\bigg)\,\bigg|\,\mathcal F_t\bigg] \tag{6.34}$$

for $0 \le t \le T$, where $Z^{n,*}_{t,T} := \sup_{u\in[t,T]} |Z^n_u/Z^n_t|$.

(iii) We now use (6.34) to show the convergence of the sequence $(\delta^n Y)$ as $n \to \infty$. By (6.32), $(Z^{n,*}_{t,T})_{n\in\mathbb{N}}$ is bounded in $L^{1+\kappa}$. Due to assumptions (a) and (e), $(\delta^n G)_{n\in\mathbb{N}}$ is uniformly bounded with $\delta^n G \overset{P}{\to} 0$ as $n \to \infty$. Next we consider the integral terms in (6.34). By the Cauchy–Schwarz inequality,

$$\int_0^T |\beta^n_u| \|\delta^n \chi_u\|_{c^{\bar{M}}_u} dI_u \le \left(\int_0^T |\beta^n_u|^2 dI_u\right)^{1/2} \left(\int_0^T \|\delta^n \chi_u\|^2_{c^{\bar{M}}_u} dI_u\right)^{1/2}. \tag{6.35}$$

We want to show that the first factor on the right-hand side of (6.35) is bounded in L^p uniformly in $n \in \mathbb{N}$ and the second converges to 0 as $n \to \infty$. Note that by (6.21) and (6.7), we have for each $\varphi^n \in \{\beta^n, \delta^n f(\zeta^\infty, \chi^\infty)\}$ that

$$\int_t^T |\varphi^n_u|^2 dI_u \le C \int_t^T (|\zeta^n_u|^2_{c^{\bar{M}}_u} + |\zeta^\infty_u|^2_{c^{\bar{M}}_u} + \|\chi^n_u\|^2_{c^{\bar{M}}_u} + \|\chi^\infty_u\|^2_{c^{\bar{M}}_u}) dI_u$$

for some constant $C > 0$ that does not depend on n. Combining this with (6.28) and since $(\zeta^n \bullet M)_{n\in\mathbb{N}\cup\{\infty\}}$ and $(\chi^n \bullet M)_{n\in\mathbb{N}\cup\{\infty\}}$ are bounded in BMO, we obtain

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} E\left[\int_t^T |\varphi^n_u|^2 dI_u \,\middle|\, \mathcal{F}_t\right] < \infty.$$

Hence by Corollary 6.8 below and because $(\delta^n \chi \bullet M)_{n\in\mathbb{N}}$ is bounded in BMO with $\operatorname{tr} d\langle \delta^n \chi \bullet M\rangle_u = \|\delta^n \chi_u\|^2_{c^{\bar{M}}_u} dI_u$, the sets

$$\left\{\left|\int_0^T |\varphi^n_u|^2 dI_u\right|^p : n \in \mathbb{N}\right\} \quad \text{and} \quad \left\{\left|\int_0^T \|\delta^n \chi_u\|^2_{c^{\bar{M}}_u} dI_u\right|^p : n \in \mathbb{N}\right\} \tag{6.36}$$

are uniformly integrable for each $p \in [1, \infty)$ and $\varphi^n \in \{\beta^n, \delta^n f(\zeta^\infty, \chi^\infty)\}$. Moreover, by assumption (b), we have

$$\int_0^T \|\delta^n \chi_u\|^2_{c^{\bar{M}}_u} dI_u = \int_0^T \operatorname{tr}\big(\delta^n \chi_u c^{\bar{M}}_u (\delta^n \chi_u)^\top\big) dI_u = \operatorname{tr} \langle \delta^n \chi \bullet \bar{M}\rangle_T \overset{P}{\longrightarrow} 0$$

as $n \to \infty$, and hence we obtain

$$\int_0^T \|\delta^n \chi_u\|^2_{c^{\bar{M}}_u} dI_u \overset{L^p}{\longrightarrow} 0 \quad \text{as } n \to \infty \text{ for each } p \in [1, \infty). \tag{6.37}$$

Returning to (6.35), it follows from (6.36) for $\varphi^n = \beta^n$, (6.37) and Hölder's inequality that

$$\int_0^T |\beta_u^n| \|\delta^n \chi_u\|_{c_u^{\bar{M}}} dI_u \xrightarrow{L^p} 0 \quad \text{as } n \to \infty \text{ for each } p \in [1, \infty). \tag{6.38}$$

For the second term inside the integral in (6.34), assumption (d) and Lemma 6.5 yield

$$\int_0^T |\delta^n f_u(\zeta_u^\infty, \chi_u^\infty)| dI_u \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

Together with (6.36) for $\varphi^n = \delta^n f(\zeta^\infty, \chi^\infty)$, this yields

$$\int_0^T |\delta^n f_u(\zeta_u^\infty, \chi_u^\infty)| dI_u \xrightarrow{L^p} 0 \quad \text{as } n \to \infty \text{ for each } p \in [1, \infty). \tag{6.39}$$

We can now return to (6.34). By Hölder's inequality, we have

$$\sup_{t \in [0,T]} |\delta^n Y_t| \le \sup_{t \in [0,T]} E[(Z_{t,T}^{n,*})^{1+\kappa} \mid \mathcal{F}_t]^{\frac{1}{1+\kappa}} \sup_{t \in [0,T]} E\big[\tilde{G}_n^{\frac{\kappa+1}{\kappa}} \bigm| \mathcal{F}_t\big]^{\frac{\kappa}{\kappa+1}}, \tag{6.40}$$

where we recall $Z_{t,T}^{n,*} := \sup_{u \in [t,T]} |Z_u^n / Z_t^n|$ and $\kappa > 0$ from (6.32) and define

$$\tilde{G}_n := |\delta^n G| + \int_0^T \big(|\beta_u^n| \|\delta^n \chi_u\|_{c_u^{\bar{M}}} + |\delta^n f_u(\zeta_u^\infty, \chi_u^\infty)|\big) dI_u. \tag{6.41}$$

Since $(\delta^n G)_{n \in \mathbb{N}}$ is uniformly bounded by assumption (a) and $\delta^n G \xrightarrow{P} 0$ as $n \to \infty$, we also have by the dominated convergence theorem that $\delta^n G \xrightarrow{L^p} 0$ as $n \to \infty$ for each $p \ge 1$. Combining with (6.38) and (6.39) yields

$$\tilde{G}_n \xrightarrow{L^p} 0 \quad \text{as } n \to \infty \text{ for each } p \in [1, \infty). \tag{6.42}$$

Equivalently, $\tilde{G}_n^{\frac{\kappa+1}{\kappa}} \xrightarrow{L^p} 0$ as $n \to \infty$ for each $p \ge 1$. Doob's L^p-inequality then yields

$$\sup_{0 \le t \le T} E[\tilde{G}_n^{\frac{\kappa+1}{\kappa}} \mid \mathcal{F}_t] \longrightarrow 0 \quad \text{as } n \to \infty$$

in L^p for each $p \ge 1$ as well as in probability. Plugging into (6.40) and using the fact that the first term on the right-hand side there is uniformly bounded by (6.32), we finally obtain (6.15), i.e., $\sup_{t \in [0,T]} |Y_t^n - Y_t^\infty| \xrightarrow{P} 0$ as $n \to \infty$.

(iv) It remains to show (6.16), i.e., that the martingale parts $\zeta^n \bullet \bar{M}$ and N^n converge. To that end, recall the BSDE (6.31) for $Z^n(\delta^n Y)^2$. We want to take conditional expectations in (6.31) similarly to (6.30). Since $\hat{M}^{Z,Y,n} \in BMO$ by (6.26), we have by (6.32) and Lemma 6.11 below with $\gamma = Z^n$ that $Z^n \bullet \hat{M}^{Z,Y,n}$

is a (true) P-martingale on $[0,T]$. We can then take conditional expectations in (6.31) with $(\delta^n Y_T)^2 = (\delta^n G)^2$ to obtain

$$(\delta^n Y_t)^2 = E\bigg[\frac{Z^n_T}{Z^n_t}(\delta^n G)^2 - 2\int_t^T \frac{Z^n_u}{Z^n_t}\delta^n Y_u\big(\beta^n_u\|\delta^n\chi_u\|_{c^{\bar M}_u} + \delta^n f_u(\zeta^\infty_u,\chi^\infty_u)\big)dI_u \\ - \int_t^T \frac{Z^n_u}{Z^n_t} d\langle M^{Y,n}\rangle_u \,\bigg|\, \mathcal{F}_t\bigg], \quad 0 \le t \le T.$$

Taking $t=0$ and rearranging yields

$$E\bigg[\int_0^T Z^n_u d\langle M^{Y,n}\rangle_u\bigg] = -E\bigg[2\int_0^T Z^n_u \delta^n Y_u\big(\beta^n_u\|\delta^n\chi_u\|_{c^{\bar M}_u} + \delta^n f_u(\zeta^\infty_u,\chi^\infty_u)\big)dI_u\bigg] \\ + E[Z^n_T(\delta^n G)^2] - (\delta^n Y_0)^2. \tag{6.43}$$

The last term on the right-hand side converges to 0 as $n\to\infty$ by (6.15). Recalling the definitions (6.41) of $\tilde G^n$ and $Z^{n,*}_{0,T} := \sup_{u\in[0,T]}|Z^n_u|$, we have by assumption (a), Hölder's inequality and (6.32) that

$$\begin{aligned} &E\bigg[2\int_0^T Z^n_u|\delta^n Y_u|\big|\beta^n_u\|\delta^n\chi_u\|_{c^{\bar M}_u} + \delta^n f_u(\zeta^\infty_u,\chi^\infty_u)\big|dI_u\bigg] + E[Z^n_T(\delta^n G)^2] \\ &\le 4C_Y E\bigg[Z^{n,*}_{0,T}\int_0^T \big|\beta^n_u\|\delta^n\chi_u\|_{c^{\bar M}_u} + \delta^n f_u(\zeta^\infty_u,\chi^\infty_u)\big|dI_u\bigg] + 2C_Y E[Z^{n,*}_{0,T}|\delta^n G|] \\ &\le 4C_Y E[Z^{n,*}_{0,T}\tilde G_n] \le 4C_Y C_Z^{\frac{1}{1+\kappa}} E\big[\tilde G_n^{\frac{\kappa+1}{\kappa}}\big]^{\frac{\kappa}{\kappa+1}}. \end{aligned}$$

Plugging into (6.43), we obtain due to (6.42) that

$$E\bigg[\int_0^T Z^n_u d\langle M^{Y,n}\rangle_u\bigg] \longrightarrow 0 \quad \text{as } n\to\infty. \tag{6.44}$$

The final step is to remove the integrand Z^n from (6.44) to show that $\langle M^{Y,n}\rangle_T$ converges to 0. Note that

$$Z^n_T\langle M^{Y,n}\rangle_T = \int_0^T Z^n_u d\langle M^{Y,n}\rangle_u + \int_0^T \langle M^{Y,n}\rangle_u dZ^n_u \tag{6.45}$$

by the product rule. We also recall that $M^{Y,n}, M^{Z,n} \in BMO$ as shown after (6.28). Thus by (6.32), Corollary 6.8 below and Hölder's inequality with $p = \frac{1+\kappa}{1+\kappa/2}$ and $q = \frac{1+\kappa}{\kappa/2}$, we obtain

$$E\Big[\sup_{t\in[0,T]}(|Z^n_t|\langle M^{Y,n}\rangle_t)^{1+\kappa/2}\Big] \le E[(Z^{n,*}_{0,T})^{1+\kappa}]^{1/p}E[\langle M^{Y,n}\rangle_T^{q(1+\kappa/2)}]^{1/q} < \infty.$$

By combining this bound with the definition $Z^n = \mathcal{E}(M^{Z,n})$, Lemma 6.11 with $\gamma = Z^n\langle M^{Y,n}\rangle$ and (6.32), we obtain that $\langle M^{Y,n}\rangle \bullet Z^n = (Z^n\langle M^{Y,n}\rangle) \bullet M^{Z,n}$ is a true martingale. After taking expectations in (6.45), (6.44) gives

$$E[Z^n_T\langle M^{Y,n}\rangle_T] = E\bigg[\int_0^T Z^n_u d\langle M^{Y,n}\rangle_u\bigg] \longrightarrow 0 \quad \text{as } n\to\infty.$$

In particular, we have $Z^n_T\langle M^{Y,n}\rangle_T \overset{P}{\to} 0$ as $n\to\infty$. Moreover, since $(M^{Z,n})_{n\in\mathbb{N}}$ is bounded in BMO, Lemma 6.10 below gives for some constants $\tilde{\kappa}, C_{\tilde{\kappa}} > 0$ that

$$E[(Z^n_T)^{-\tilde{\kappa}}] \leq C_{\tilde{\kappa}} \quad \text{for all } n\in\mathbb{N}.$$

Thus applying Lemma 6.3 to the random variables Z^n_T and $\langle M^{Y,n}\rangle_T$ yields that $\langle M^{Y,n}\rangle_T \overset{P}{\to} 0$ as $n\to\infty$. Finally, by the definition (6.23) of $M^{Y,n}$ and since $N^n, N^\infty \perp \bar{M}$, we have

$$\langle M^{Y,n}\rangle_T = \langle \delta^n\zeta \bullet \bar{M}\rangle_T + \langle \delta^n N\rangle_T \overset{P}{\longrightarrow} 0 \quad \text{as } n\to\infty,$$

which shows (6.16) and completes the proof, since we have shown after (6.26) that $(\zeta^n \bullet \bar{M})_{n\in\mathbb{N}\cup\{\infty\}}$ and $(N^n)_{n\in\mathbb{N}\cup\{\infty\}}$ are bounded in BMO, and (6.15) was proven at the end of step (iii). □

6.2 Lemmas on BMO martingales

We collect here some useful results on BMO martingales that are well known in the literature and needed for the proof of Theorem 6.6; we use Kazamaki [81] as a reference.

Lemma 6.7. *Let $(M_t)_{0\leq t\leq T}$ be a continuous BMO martingale. Then there exist constants $\alpha, C_\alpha > 0$ that depend only on $\|M\|_{BMO}$ such that*

$$\operatorname{ess\,sup}_{t\in[0,T]} E\big[\exp\big(\alpha(\langle M\rangle_T - \langle M\rangle_t)\big) \bigm| \mathcal{F}_t\big] \leq C_\alpha. \tag{6.46}$$

Proof. Choose $\alpha > 0$ such that $\alpha\|M\|_{BMO} = \|\alpha M\|_{BMO} < 1$. Then by applying [81, Theorem 2.2] to αM and all (constant) stopping times $t\in[0,T]$, we obtain (6.46) for the constant $C_\alpha = \frac{1}{1-\alpha^2\|M\|^2_{BMO}} < \infty$. □

Corollary 6.8. *Let $(M^n)_{n\in\mathbb{N}}$ be a sequence of continuous martingales on $[0,T]$ that is bounded in BMO. Then the set $\{\langle M^n\rangle^p_T : n\in\mathbb{N}\}$ is uniformly integrable for each $p\in[1,\infty)$.*

Proof. Fix $p \geq 1$. Note that Lemma 6.7 yields (6.46) simultaneously for all M^n with constants $\alpha, C_\alpha > 0$ that depend only on $\sup_{n\in\mathbb{N}} \|M^n\|_{BMO}$. By the elementary inequality $x \leq \exp(cx)/c$ with $c = \alpha/(p+1) > 0$, we have

$$E[\langle M^n\rangle_T^{p+1}] \leq \left(\frac{p+1}{\alpha}\right)^{p+1} E[\exp(\alpha\langle M^n\rangle_T)] \leq \left(\frac{p+1}{\alpha}\right)^{p+1} C_\alpha.$$

As $\langle M^n\rangle^{p+1} = (\langle M^n\rangle_T^p)^{\frac{p+1}{p}}$, the set $\{\langle M^n\rangle_T^p : n \in \mathbb{N}\}$ is bounded in $L^{\frac{p+1}{p}}$ and hence uniformly integrable. □

Lemma 6.9. *Let $(M_t)_{0\leq t\leq T}$ be a continuous BMO martingale. Then $\mathcal{E}(M)$ is a uniformly integrable martingale on $[0,T]$ and there exist constants $\kappa, C_\kappa > 0$ that depend only on $\|M\|_{BMO}$ such that*

$$\sup_{t\in[0,T]} E\left[\sup_{u\in[t,T]} \frac{\mathcal{E}(M)_u^{1+\kappa}}{\mathcal{E}(M)_t^{1+\kappa}} \,\middle|\, \mathcal{F}_t\right] \leq C_\kappa. \tag{6.47}$$

Proof. By [81, Theorem 2.3], $\mathcal{E}(M)$ is a uniformly integrable martingale. As pointed out in [81, after Equation (3.4)], the function Φ in [81, Theorem 3.1] is continuous and decreasing on $(1,\infty)$ with $\lim_{p\searrow 1}\Phi(p) = \infty$. Thus there exists $\kappa > 0$ that depends only on $\|M\|_{BMO}$ such that $\Phi(1+\kappa) > \|M\|_{BMO}$, and hence [81, Theorem 3.1 and Definition 3.1] yield (6.47) for some $C_\kappa > 0$. By inspecting [81, Equation (3.5)] and the last equation in the proof of [81, Theorem 3.1], we see that C_κ depends only on $\|M\|_{BMO}$ and the choice of $\kappa > 0$. □

Lemma 6.10. *Let $(M_t)_{0\leq t\leq T}$ be a continuous BMO martingale. Then there exist constants $\tilde{\kappa}, C_{\tilde{\kappa}} > 0$ that depend only on $\|M\|_{BMO}$ such that*

$$\sup_{t\in[0,T]} E\left[\frac{\mathcal{E}(M)_T^{-\tilde{\kappa}}}{\mathcal{E}(M)_t^{-\tilde{\kappa}}} \,\middle|\, \mathcal{F}_t\right] \leq C_{\tilde{\kappa}}. \tag{6.48}$$

Proof. For $\alpha, \kappa > 0$ given by Lemmas 6.7 and 6.9, define the constants

$$p := \frac{\alpha+1+\kappa}{\alpha} > 1, \quad q := \frac{\alpha+1+\kappa}{1+\kappa} > 1, \quad \tilde{\kappa} := \frac{\alpha(1+\kappa)}{\alpha+1+\kappa} > 0,$$

so that $\frac{1}{p}+\frac{1}{q} = 1$ with $\tilde{\kappa}p = 1+\kappa$ and $\tilde{\kappa}q = \alpha$. Note that

$$\frac{\mathcal{E}(M)_T^{-1}}{\mathcal{E}(M)_t^{-1}} = \frac{\mathcal{E}(-M)_T}{\mathcal{E}(-M)_t}\exp(\langle M\rangle_T - \langle M\rangle_t).$$

Then by Hölder's inequality, we have

$$
\begin{aligned}
E\bigg[\frac{\mathcal{E}(M)_T^{-\tilde\kappa}}{\mathcal{E}(M)_t^{-\tilde\kappa}}\ \bigg|\ \mathcal{F}_t\bigg] &\le E\bigg[\frac{\mathcal{E}(-M)_T^{\tilde\kappa p}}{\mathcal{E}(-M)_t^{\tilde\kappa p}}\ \bigg|\ \mathcal{F}_t\bigg]^{1/p} E\big[\exp\big(\tilde\kappa q(\langle M\rangle_T-\langle M\rangle_t)\big)\ \big|\ \mathcal{F}_t\big]^{1/q}\\
&= E\bigg[\frac{\mathcal{E}(-M)_T^{1+\kappa}}{\mathcal{E}(-M)_t^{1+\kappa}}\ \bigg|\ \mathcal{F}_t\bigg]^{1/p} E\big[\exp\big(\alpha(\langle M\rangle_T-\langle M\rangle_t)\big)\ \big|\ \mathcal{F}_t\big]^{1/q}\\
&\le C_\kappa^{1/p}C_\alpha^{1/q} =: C_{\tilde\kappa}
\end{aligned}
$$

for $0\le t\le T$, which shows (6.48). □

Lemma 6.11. *Let $(M_t)_{0\le t\le T}$ be a continuous BMO martingale and $(\gamma_t)_{0\le t\le T}$ a predictable process such that $E[\sup_{t\in[0,T]}|\gamma_t|^{1+\delta}]<\infty$ for some $\delta>0$. Then $\gamma\bullet M$ is an $\mathcal{H}^1$-martingale on $[0,T]$.*

Proof. Set $\gamma^*:=\sup_{t\in[0,T]}|\gamma_t|$. By the assumption, Corollary 6.8 and Hölder's inequality, we have

$$
\begin{aligned}
E[\langle\gamma\bullet M\rangle_T^{1/2}] = E\bigg[\bigg(\int_0^T\gamma_t^2 d\langle M\rangle_t\bigg)^{1/2}\bigg] &\le E[\gamma^*\langle M\rangle_T^{1/2}]\\
&\le E[(\gamma^*)^{1+\kappa}]^{\frac{1}{1+\kappa}} E[\langle M\rangle_T^{\frac{\kappa+1}{2\kappa}}]^{\frac{\kappa}{\kappa+1}}<\infty.
\end{aligned}
$$

Hence by the Burkholder–Davis–Gundy inequality, we have

$$
E\Big[\sup_{t\in[0,T]}|\gamma\bullet M_t|\Big]\le CE[\langle\gamma\bullet M\rangle_T^{1/2}]<\infty
$$

for some $C>0$ so that $\gamma\bullet M$ is an $\mathcal{H}^1$-martingale on $[0,T]$. □

Appendix A

Volterra equations

In this appendix, we give some results on the existence and uniqueness of solutions to convolution equations, particularly of Riccati–Volterra type. These results are used in Chapter I for proving Theorem I.2.17, which is needed in several proofs related to the mean–variance hedging problem for the rough Heston model. Throughout this appendix, we use the textbook by Gripenberg et al. [59] as the main reference for results related to this topic.

We start by citing three well-known theorems: the Kolmogorov–Riesz compactness criterion, the Schauder fixed point theorem and Young's convolution inequality. The latter is included in Lemma 1.5, where we show a slightly stronger result. We also give an auxiliary result in Lemma 1.6. The main results in this section are then Propositions 2.2 and 2.4, which give general conditions for the existence and uniqueness of solutions $x : [0, T] \to \mathbb{C}^n$ to a convolution equation of the form $x = k * f(x)$, for a given nonlinear function f and kernel k. These two propositions give the existence of continuous and L^p-integrable solutions, respectively. In Corollaries 2.6 and 2.7, we show the existence and uniqueness of solutions to equations of the alternative form $x = f(k * x)$, which we use most often in Chapters I and II. We also give explicit bounds for the solutions of Riccati–Volterra equations as well as a stability result. Finally, we conclude with the proofs of two results directly related to Chapter I.

1 Preliminaries

We recall the definition of the convolution operation.

Notation 1.1. In the following, we generally work with the spaces $L^q([0, T]; \mathbb{C})$ or $L^q([0, T]; \mathbb{C}^n)$ for $T > 0$, $q \in [1, \infty)$ and $n \in \mathbb{N}$, where the integrability is defined with respect to the Lebesgue measure on $[0,T]$. For ease of notation, we denote the norms of either space by $\|\cdot\|_{L^q(0,T)}$, $\|\cdot\|_{L^q}$ or $\|\cdot\|_q$, when it is clear from the context. Given $T\in(0,\infty]$, we also consider the space $L^q_{\mathrm{loc}}([0,T);\mathbb{C}^n)$ of measurable functions $f:[0,T)\to\mathbb{C}^n$ such that $\|f\|_{L^q(0,t)}<\infty$ for each $t\in(0,T)$.

Definition 1.2. For $T\in(0,\infty)$, we define the *convolution* $k*y$ of two functions $k\in L^1([0,T];\mathbb{C})$ and $y\in L^1([0,T];\mathbb{C}^n)$ by

$$(k*y)(t):=\int_0^t k(t-s)y(s)ds=\int_0^t k(s)y(t-s)ds,\quad 0\le t\le T,\tag{1.1}$$

so that $k*y\in L^1([0,T];\mathbb{C}^n)$ (see below). Alternatively, if $k\in L^1_{\mathrm{loc}}([0,T);\mathbb{C})$ and $y\in L^1_{\mathrm{loc}}([0,T);\mathbb{C}^n)$, then we define $k*y$ by (1.1) for $t\in[0,T)$, and in this case $k*y\in L^1_{\mathrm{loc}}([0,T);\mathbb{C}^n)$.

The fact that $k*y$ is well defined, measurable and belongs to $L^1([0,T];\mathbb{C}^n)$ or $L^1_{\mathrm{loc}}([0,T);\mathbb{C}^n)$, respectively, follows from [59, Theorem 2.2.2(i)]; this can also be shown using Fubini's theorem and the Young convolution inequality that we give later in Theorem 1.5, with $p=q=r=1$. Definition 1.2 differs slightly from Definition I.2.3, as we allow $n>1$ and fix a terminal time T, which is convenient for the type of results that we consider here. Note that the convolution has a *causality property*, i.e., $(k*y)\mathbf{1}_{[0,T]}$ only depends on $k\mathbf{1}_{[0,T]}$ and $y\mathbf{1}_{[0,T]}$ by the definition (1.1). Thus the convolution $k*y$ does not depend on the terminal time, in the sense that $(k*y)(t)$ is the same for any choice of time horizon $T\ge t$.

We start by giving two well-known theorems from functional analysis that are helpful for proving the subsequent results; see Alt/Nürnberg [9, Theorem 4.16] and Gripenberg et al. [59, Theorem 12.1.4], respectively, for references. The first is a compactness criterion for L^p-spaces on $\mathbb{R}^n$, analogous to the Arzelà–Ascoli theorem for spaces of continuous functions, and the second is a fixed point theorem that we use for showing the existence of solutions to Volterra equations.

Theorem 1.3 (Kolmogorov–Riesz compactness criterion)**.** *Let $p\in[1,\infty)$ and $n,m\in\mathbb{N}$. Then a subset $D\subseteq L^p(\mathbb{R}^n;\mathbb{R}^m)$ is relatively compact (i.e., the closure $\overline{D}\subseteq L^p(\mathbb{R}^n;\mathbb{R}^m)$ is compact) with respect to the L^p-norm-topology if and only if the following conditions hold:*

1) *D is bounded, i.e., $\sup_{f\in D}\|f\|_{L^p(\mathbb{R}^n)}<\infty$.*

2) *D is equicontinuous, i.e., $\lim_{h\searrow 0}\sup_{f\in D}\|f(\cdot+h)-f(\cdot)\|_{L^p(\mathbb{R}^n)}=0$.*

3) *D is equitight, i.e., $\lim_{R\to\infty}\sup_{f\in D}\|\mathbf{1}_{\mathbb{R}^n\setminus B_R(0)}f\|_{L^p(\mathbb{R}^n)}=0$.*

Theorem 1.4 (Schauder fixed point theorem)**.** *Let B be a Banach space and $D \subseteq B$ a closed, bounded and convex subset. If $\Phi : D \to D$ is a continuous function such that $\overline{\Phi(D)} \subseteq B$ is compact, then Φ has a fixed point in D.*

We now give the well-known Young convolution inequality, together with an additional statement on the compactness of the convolution map on L^p-spaces. This is useful for the proofs of the subsequent propositions on the existence and uniqueness of solutions to Volterra equations, and generalises [59, Theorem 2.2.5] in the case where $p, q > 1$.

Lemma 1.5. *Fix $n \in \mathbb{N}$, $\bar{T} \in (0, \infty)$ and $p, q, r \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. For all $k \in L^p([0, \bar{T}]; \mathbb{C})$ and $y \in L^q([0, \bar{T}]; \mathbb{C}^n)$, it holds that*

$$\|k * y\|_{L^r(0,\bar{T})} \leq \|k\|_{L^p(0,\bar{T})} \|y\|_{L^q(0,\bar{T})}, \tag{1.2}$$

and hence $k * y \in L^r([0, \bar{T}]; \mathbb{C}^n)$. Moreover, for fixed $k \in L^p([0, \bar{T}]; \mathbb{C})$, the linear map $y \mapsto k * y$ from $L^q([0, \bar{T}]; \mathbb{C}^n)$ to $L^r([0, \bar{T}]; \mathbb{C}^n)$ is compact, i.e., the set

$$\tilde{D} := k * D := \{k * y : y \in D\} \subseteq L^r([0, \bar{T}]; \mathbb{C}^n)$$

is relatively compact for any bounded subset $D \subseteq L^q([0, \bar{T}]; \mathbb{C}^n)$.

Proof. The inequality (1.2) is Young's convolution inequality; see Sogge [113, Theorem 0.3.1]. Since we need a slightly more general version of (1.2) in order to show the compactness, we also give here a full proof. We start by defining the constants

$$b = \frac{p}{r}, \quad c = \frac{q}{r}, \quad a = 1 - b, \quad d = 1 - c. \tag{1.3}$$

Since $p, q, r \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ by assumption, we must have $r \geq \max(p, q)$, and thus we get $a, b, c, d \in [0, 1]$. We also have the equality

$$\frac{a}{p} + \frac{1}{r} + \frac{d}{q} = \left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{r} + \left(\frac{1}{q} - \frac{1}{r}\right) = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1. \tag{1.4}$$

Now recall the generalisation of Hölder's inequality to n functions on a measure space $(\Omega, \mathcal{A}, \mu)$ (see Alt/Nürnberg [9, Theorem 3.18]), which gives that

$$\left\| \prod_{j=1}^{n} f_j \right\|_{L^1(\Omega,\mu)} \leq \prod_{j=1}^{n} \|f_j\|_{L^{p_j}(\Omega,\mu)}$$

for functions $f_j \in L^{p_j}(\Omega, \mu)$ and constants $p_j \in [1, \infty]$ such that $\sum_j \frac{1}{p_j} = 1$. In particular, we can apply Hölder's inequality for three functions on $[0, \bar{T}]$ with powers $\frac{p}{a}$, r and $\frac{q}{d}$, due to (1.4). Recalling also (1.3), we can bound $k * y$ by

$$\begin{aligned}|(k*y)(t)| = \left|\int_0^t k(t-s)y(s)ds\right| &\leq \int_0^t |k(t-s)|^a|k(t-s)|^b|y(s)|^c|y(s)|^d ds \\ &\leq \left(\int_0^t |k(t-s)|^p ds\right)^{\frac{a}{p}}\left(\int_0^t |k(t-s)|^p|y(s)|^q ds\right)^{\frac{1}{r}}\left(\int_0^t |y(t-s)|^q ds\right)^{\frac{d}{q}} \\ &\leq \|k\|^a_{L^p(0,\bar{T})}\|y\|^d_{L^q(0,\bar{T})}\left(\int_0^t |k(t-s)|^p|y(s)|^q ds\right)^{\frac{1}{r}},\end{aligned}$$

for each $t \in [0,\bar{T}]$. Now consider a subset $E \subseteq [0,\bar{T}]$ with Lebesgue measure $\ell := |E| \in [0,\bar{T}]$. Integrating the previous inequality on E, we obtain by Fubini's theorem that

$$\begin{aligned}\|\mathbf{1}_E(k*y)\|_{L^r(0,\bar{T})} &\leq \|k\|^a_{L^p(0,\bar{T})}\|y\|^d_{L^q(0,\bar{T})}\left(\int_E\int_0^t |k(t-s)|^p|y(s)|^q ds dt\right)^{\frac{1}{r}} \\ &= \|k\|^a_{L^p(0,\bar{T})}\|y\|^d_{L^q(0,\bar{T})}\left(\int_0^{\bar{T}} |y(s)|^q\left(\int_{E\cap[s,\bar{T}]} |k(t-s)|^p dt\right)ds\right)^{\frac{1}{r}} \\ &\leq \|k\|^a_{L^p(0,\bar{T})}\|y\|^d_{L^q(0,\bar{T})}\left(\int_0^{\bar{T}} |y(s)|^q ds\right)^{\frac{1}{r}}\big(n_k(\ell)\big)^{\frac{p}{r}} \\ &= \|k\|^a_{L^p(0,\bar{T})}\|y\|_{L^q(0,\bar{T})}\big(n_k(\ell)\big)^{\frac{p}{r}},\end{aligned} \tag{1.5}$$

using that $d + \frac{q}{r} = d + c = 1$ and where we define $n_k : (0,\bar{T}] \to [0,\infty)$ by

$$n_k(\ell) := \sup\left\{\|k\mathbf{1}_{E'}\|_{L^p(0,\bar{T})} : E' \subseteq [0,\bar{T}], |E'| \leq \ell\right\}. \tag{1.6}$$

Indeed as $|E| = \ell$, the inner integral in the second line is taken over a set of measure at most ℓ, so that it can be bounded by $(n_k(\ell))^p$. In the particular case $E = [0,\bar{T}]$, so that $\ell = \bar{T}$ and $n_k(\bar{T}) = \|k\|_{L^p(0,\bar{T})}$, we obtain (1.2) by plugging into (1.5), since $a + \frac{p}{r} = a + b = 1$.

We have thus shown Young's convolution inequality (1.2), and so the map $y \mapsto k * y$ is linear and continuous. It remains to show that it is compact. For a given kernel $k \in L^p([0,\bar{T}];\mathbb{C})$, we need to show that the set

$$\tilde{D} := k * D := \{k * y : y \in D\} \subseteq L^r([0,\bar{T}];\mathbb{C}^n)$$

is relatively compact for any bounded set $D \subseteq L^q([0,\bar{T}];\mathbb{C}^n)$. In the following, we embed $L^r([0,\bar{T}];\mathbb{C}^n)$ into $L^r(\mathbb{R};\mathbb{C}^n)$ by setting functions to 0 on $\mathbb{R} \setminus [0,\bar{T}]$, and denote the norm in $L^r(\mathbb{R};\mathbb{C}^n)$ by $\|\cdot\|_{L^r}$. We also define the shift operator

$\Delta_\delta : L^r(\mathbb{R};\mathbb{C}^n) \to L^r(\mathbb{R};\mathbb{C}^n)$ by $(\Delta_\delta z)(t) = z(t+\delta)$ for $\delta > 0$. Note that each $z \in \tilde{D}$ is supported on $[0,\bar{T}]$, so that

$$\Delta_\delta z(t) - z(t) = \begin{cases} z(t+\delta), & t \in [-\delta, 0), \\ z(t+\delta) - z(t), & t \in [0, \bar{T}-\delta), \\ z(t), & t \in [\bar{T}-\delta, \bar{T}], \\ 0, & t \in \mathbb{R} \setminus [-\delta, \bar{T}]. \end{cases}$$

Hence we get the equality

$$\|\Delta_\delta z - z\|_{L^r} = \big(\|z(\mathbf{1}_{[0,\delta]} + \mathbf{1}_{[\bar{T}-\delta,\bar{T}]})\|_{L^r} + \|(\Delta_\delta z - z)\mathbf{1}_{[0,\bar{T}-\delta]}\|_{L^r}\big) \tag{1.7}$$

for each $z \in \tilde{D}$. Our goal is now to show that $\sup_{z\in\tilde{D}} \|\Delta_\delta z - z\|_{L^r} \to 0$ as $\delta \searrow 0$, for which we use (1.7).

To show the convergence, fix $y \in D$, $z := k * y \in \tilde{D}$ and $\delta > 0$. Considering the last term in (1.7), we have for $0 \le t \le \bar{T} - \delta$ that

$$\begin{aligned} |\Delta_\delta z(t) - z(t)| &= \left| \int_0^{t+\delta} k(t+\delta-s)y(s)ds - \int_0^t k(t-s)y(s)ds \right| \\ &= \left| \int_t^{t+\delta} k(t+\delta-s)y(s)ds + \int_0^t \big(k(t+\delta-s) - k(t-s)\big)y(s)ds \right| \\ &\le \left| \int_t^{t+\delta} k(t+\delta-s)y(s)ds \right| + \big|\big((\Delta_\delta k - k) * y\big)(t)\big|. \end{aligned} \tag{1.8}$$

Taking the absolute value inside the integral, we can bound the first term on the right-hand side of (1.8) by

$$\begin{aligned} \left| \int_t^{t+\delta} k(t+\delta-s)y(s)ds \right| &= \left| \int_0^{\delta} k(s)y(t+\delta-s)ds \right| \\ &\le \int_0^{t+\delta} \big|k(s)y(t+\delta-s)\mathbf{1}_{[0,\delta]}(s)\big|ds \\ &= \big(|k\mathbf{1}_{[0,\delta]}| * |y|\big)(t+\delta). \end{aligned} \tag{1.9}$$

Hence taking the L^r-norm on $[0, \bar{T}-\delta]$, we obtain from (1.8), (1.9) and Young's convolution inequality (1.2) that

$$\begin{aligned}\big\|(\Delta_\delta z - z)\mathbf{1}_{[0,\bar{T}-\delta]}\big\|_{L^r} &\le \big\||k\mathbf{1}_{[0,\delta]}| * |y|\big\|_{L^r} + \|(\Delta_\delta k - k) * y\|_{L^r} \\ &\le \big(\|k\mathbf{1}_{[0,\delta]}\|_{L^p} + \|\Delta_\delta k - k\|_{L^p}\big)\|y\|_{L^q} \\ &\le \big(n_k(\delta) + \|\Delta_\delta k - k\|_{L^p}\big)\|y\|_{L^q}, \end{aligned} \tag{1.10}$$

where we recall the definition (1.6) of n_k. Returning to (1.7) and considering now the first term, the inequality (1.5) yields the bound

$$\|z(\mathbf{1}_{[0,\delta]} + \mathbf{1}_{[\bar{T}-\delta,\bar{T}]})\|_{L^r} \le \|k\|_{L^p}^{1-\frac{p}{r}}\|y\|_{L^q} n_k(2\delta)^{\frac{p}{r}}. \tag{1.11}$$

Then by plugging in the two bounds (1.10) and (1.11) into (1.7), we obtain

$$\|\Delta_\delta z - z\|_{L^r} \le \big(\|k\|_{L^p}^{1-\frac{p}{r}} n_k(2\delta)^{\frac{p}{r}} + n_k(\delta) + \|\Delta_\delta k - k\|_{L^p}\big)\|y\|_{L^q} \tag{1.12}$$

for $z = k * y$. Since the singleton $\{k\} \in L^p$ is compact and hence equicontinuous by the Kolmogorov–Riesz compactness criterion, we have $\lim_{\delta\searrow 0}\|\Delta_\delta k - k\|_{L^p} = 0$. We also have that as $\delta \searrow 0$,

$$n_k(\delta) = \sup\big\{\|k\mathbf{1}_{E'}\|_{L^p(0,\bar{T})} : E' \subseteq [0,\bar{T}], |E'| \le \delta\big\} \searrow 0$$

since the singleton $\{|k|^p\}$ is uniformly integrable and by the ϵ-δ-criterion for uniform integrability; see Klenke [83, Theorem 6.24]. Because $\sup_{y\in D}\|y\|_{L^q} < \infty$ and the remaining terms in (1.12) are independent of y and converge to 0, we obtain from (1.12) that

$$\sup_{y\in D}\|\Delta_\delta(k * y) - k * y\|_{L^r} \longrightarrow 0 \quad \text{as } \delta \searrow 0.$$

This shows the equicontinuity of $\tilde{D}$. Note that $\tilde{D}$ is also equitight as all elements of $\tilde{D}$ vanish outside of $[0,\bar{T}]$ by construction, and it is bounded by Young's convolution inequality (1.2), as $\|y\|_{L^q(0,\bar{T})}$ is uniformly bounded, and hence so is $\|k * y\|_{L^r(0,\bar{T})} \le \|k\|_{L^p(0,\bar{T})}\|y\|_{L^q(0,\bar{T})}$. It then follows by the Kolmogorov–Riesz compactness criterion that $\tilde{D}$ is relatively compact. Since this holds for any choice of bounded set D, we conclude that the map $y \mapsto k*y$ is compact, as claimed. □

The following result is also well known, and it shows that any continuous function $f : \mathbb{C}^m \to \mathbb{C}^n$ satisfying a power growth condition induces by composition a continuous map on L^p-spaces. This is used in the proof of Proposition 2.4.

Although we only need this result for L^p-spaces on intervals of real numbers equipped with Lebesgue measure, we note that this result would also hold on any finite measure space $(\Omega, \mathcal{F}, \mu)$.

Lemma 1.6. *Fix $m, n \in \mathbb{N}$, $\bar{T} \in (0, \infty)$ and $a, q \in [1, \infty)$ such that $a \le q$. Let $f : \mathbb{C}^m \to \mathbb{C}^n$ be a continuous function and suppose that it satisfies the growth condition $|f(\tilde{x})| \le C(1 + |\tilde{x}|^a)$ for some $C > 0$ and all $\tilde{x} \in \mathbb{C}^m$. Then the map $x \mapsto f \circ x$ is continuous from $L^q([0, \bar{T}]; \mathbb{C}^m)$ to $L^{q/a}([0, \bar{T}]; \mathbb{C}^n)$.*

Proof. We note that the map is well defined into $L^{q/a}([0, \bar{T}]; \mathbb{C}^n)$, since

$$\|f \circ x\|_{L^{q/a}(0,\bar{T})} \le \|C(1 + |x|^a)\|_{L^{q/a}(0,\bar{T})} \le C\big(\bar{T}^{a/q} + \|x\|^a_{L^q(0,\bar{T})}\big) < \infty.$$

To show the continuity, let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to some x in $L^q([0, \bar{T}]; \mathbb{C}^m)$ and pick an arbitrary subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Then we claim that there exist a further subsequence $(x_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ and some $\bar{x} \in L^q([0, \bar{T}]; \mathbb{C}^m)$ such that $x_{n_{k_\ell}} \to x$ a.s. as $\ell \to \infty$ and $\max(|x_{n_{k_\ell}}|, |x|) \le \bar{x}$ a.s. for each $\ell \in \mathbb{N}$. To see this, we first use the L^q-convergence to find a subsequence of $(x_{n_k})_{k\in\mathbb{N}}$ that converges to x almost surely and in L^q. Then, we find a rapidly convergent further subsequence $(x_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ so that $\|x_{n_{k_{\ell+1}}} - x_{n_{k_\ell}}\|_{L^q(0,\bar{T})} \le 2^{-\ell}$ for each $\ell \in \mathbb{N}$. By construction, we still have that $x_{n_{k_\ell}} \to x$ a.s. as $\ell \to \infty$, and we also have a majorant $\bar{x} := |x_{n_{k_1}}| + \sum_{\ell=1}^{\infty} |x_{n_{k_{\ell+1}}} - x_{n_{k_\ell}}|$. Note that the triangle inequality yields $|x_{n_{k_\ell}}| \le \bar{x}$ a.s. for each ℓ, and by taking pointwise limits we obtain $|x| \le \bar{x}$ a.s. As we can bound the L^q-norm of $\bar{x}$ by a geometric series, we have $\bar{x} \in L^q([0, \bar{T}]; \mathbb{C}^m)$. Thus by the continuity of f, we have $f \circ x_{n_{k_\ell}} \to f \circ x$ almost surely as $\ell \to \infty$. We can bound

$$\begin{aligned} \big|f\big(x_{n_{k_\ell}}(t)\big) - f\big(x(t)\big)\big|^{q/a} &\le \Big(C\big(2 + \big|x_{n_{k_\ell}}(t)\big|^a + |x(t)|^a\big)\Big)^{q/a} \\ &\le \Big(2C\big(1 + |\bar{x}(t)|^a\big)\Big)^{q/a}, \end{aligned}$$

where the right-hand side belongs to $L^1([0, \bar{T}]; \mathbb{C}^m)$ since $|\bar{x}|^a \in L^{q/a}([0, \bar{T}]; \mathbb{C}^m)$. It follows by the dominated convergence theorem that $f \circ x_{n_{k_\ell}} \to f \circ x$ in $L^{q/a}([0, \bar{T}]; \mathbb{C}^n)$ as $\ell \to \infty$. Therefore, as every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ has a further subsequence $(x_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ such that $f \circ x_{n_{k_\ell}} \to f \circ x$ in $L^{q/a}([0, \bar{T}]; \mathbb{C}^n)$ as $\ell \to \infty$, we must also have $f \circ x_n \to f \circ x$ in $L^{q/a}([0, \bar{T}]; \mathbb{C}^n)$ as $n \to \infty$ for the original sequence. This shows the continuity of the map $x \mapsto f \circ x$. □

2 Existence and uniqueness results

We are now ready to prove the following two results about the existence of solutions to convolution equations. The first version is a particular case of Gripenberg et al. [59, Theorem 12.1.1], and it gives sufficient conditions for the existence of a continuous solution on a small time interval. In Proposition 2.4, we show a version of this result for existence in an L^q-space. In both cases, we also obtain uniqueness under a suitable Lipschitz condition.

Remark 2.1. In the following results, namely Propositions 2.2, 2.4 and 2.8 as well as Corollaries 2.6 and 2.7, we have that the solutions to the corresponding Volterra equations are real-valued if all the inputs are real-valued. This can be seen by replacing $\mathbb{C}$ with $\mathbb{R}$ in all statements and proofs.

Proposition 2.2. *Fix* $n \in \mathbb{N}$, $\bar{T} \in (0, \infty]$ *and* $k \in L^1_{\mathrm{loc}}([0, \bar{T}); \mathbb{C})$, *and suppose that* $y : [0, \bar{T}) \to \mathbb{C}^n$ *and* $f : \mathbb{C}^n \times [0, \bar{T}) \to \mathbb{C}^n$ *are continuous functions. Then the following statements hold:*

1) *There exists a positive time* $\hat{T} = \hat{T}(k, y, f) \in (0, \bar{T})$ *such that there is a continuous solution* $x : [0, \hat{T}] \to \mathbb{C}^n$ *to the equation*

$$x(t) = y(t) + \int_0^t k(t-s) f\big(x(s), s\big) ds, \tag{2.1}$$

for $0 \leq t \leq \hat{T}$.

2) *If* $\hat{T} < \bar{T}$ *and* $x : [0, \hat{T}] \to \mathbb{C}^n$ *is a continuous solution to the equation* (2.1) *on* $[0, \hat{T}]$, *then there exists some* $\tau > 0$ *such that* x *can be extended to a continuous solution to* (2.1) *on the interval* $[0, \hat{T} + \tau]$. *That is, there exists a continuous solution* $\hat{x} : [0, \hat{T} + \tau] \to \mathbb{C}^n$ *to* (2.1) *for* $0 \leq t \leq \hat{T} + \tau$ *such that* $\hat{x}(t) = x(t)$ *for all* $t \in [0, \hat{T}]$.

3) *Suppose that for all* $B \in \mathbb{R}_+$, *there exists some* $L(B) > 0$ *such that*

$$\sup \left\{ \frac{|f(x_1, t) - f(x_2, t)|}{|x_1 - x_2|} : |x_1|, |x_2| \leq B, x_1 \neq x_2, \, t \in [0, \bar{T} - B^{-1}] \right\} \leq L(B), \tag{2.2}$$

i.e., f *is locally Lipschitz-continuous with respect to the first variable. Then for each* $\hat{T} > 0$, *there is at most one bounded solution to* (2.1) *on* $[0, \hat{T}]$.

Remark 2.3. By 1) and a repeated application of 2) (see also Gripenberg et al. [59, Theorem 12.1.1]), one can find a *noncontinuable solution* to (2.1), i.e., a solution x that cannot be extended to any larger interval. Such a noncontinuable solution x takes values on an interval of the form $[0,\hat{T})$ for some $\hat{T} \in (0,\bar{T}]$, and it satisfies (2.1) for $0 \le t < \hat{T}$. Moreover, if $\hat{T} < \bar{T}$, then $\limsup_{t\nearrow\hat{T}} |x(t)| = \infty$, i.e., the solution blows up at the terminal time $\hat{T}$.

Proof. **1)**, **2)** The existence and continuability of solutions to (2.1) follow directly from Gripenberg et al. [59, Theorem 12.1.1], by identifying $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and replacing k with the kernel $\tilde{k}(t,s) := k(t-s)\mathbf{1}_{[0,t]}(s)$, as this does not change the equation (2.1). The fact that $\tilde{k}$ is a Volterra kernel of continuous type in the sense of [59, Definition 9.5.2] follows from the remark after that definition, since $\tilde{k}$ is a convolution kernel. Hence [59, Theorem 12.1.1] gives 1) and 2). If we consider the real case as mentioned in Remark 2.1, we can apply [59, Theorem 12.1.1] directly in $\mathbb{R}^n$ to obtain a real-valued solution.

3) To show the uniqueness, suppose that x and x' are two bounded solutions to (2.1) on $[0,\hat{T}]$. Define the constant

$$B = \sup_{t\in[0,\hat{T}]} (|x(t)| \vee |x'(t)|) \vee (\bar{T}-\hat{T})^{-1} < \infty.$$

The equation (2.1) and the Lipschitz condition (2.2) imply that

$$\begin{aligned} |x(t)-x'(t)| &= \left| \int_0^t k(t-u)\Big(f\big(x(u),u\big) - f\big(x'(u),u\big)\Big)du \right| \\ &\le \int_0^t L(B)|k(t-u)|\,|x(u)-x'(u)|du, \qquad 0 \le t \le \hat{T}, \end{aligned} \tag{2.3}$$

as $|x(u)|, |x'(u)| \le B$ and $u \le \bar{T} - B^{-1}$ for $u \in [0,\hat{T}]$ by the construction of B. Note that for $\beta \in \mathbb{R}$ and $y, z \in L^1_{\text{loc}}([0,\bar{T});\mathbb{C})$, we have the identity

$$\begin{aligned} e^{-\beta t}(y*z)(t) &= \int_0^t e^{-\beta(t-s)}y(t-s)e^{-\beta s}z(s)ds \\ &= \big((e^{-\beta\cdot}y)*(e^{-\beta\cdot}z)\big)(t), \quad 0 \le t < \bar{T}. \end{aligned}$$

Setting $\phi := |x-x'|$ in (2.3) and multiplying with $e^{-\beta t}$, we obtain the bound

$$\begin{aligned} \|e^{-\beta\cdot}\phi\|_{L^1(0,\hat{T})} &\le L(B)\||e^{-\beta\cdot}k| * |e^{-\beta\cdot}\phi|\|_{L^1(0,\hat{T})} \\ &\le L(B)\|e^{-\beta\cdot}k\|_{L^1(0,\hat{T})}\|e^{-\beta\cdot}\phi\|_{L^1(0,\hat{T})}, \end{aligned} \tag{2.4}$$

using Young's convolution inequality (1.2) with $p = q = 1$ for the second inequality. Since $\lim_{\beta\to\infty} \|e^{-\beta\cdot}k\|_{L^1(0,\hat{T})} = 0$ by the dominated convergence theorem, we can choose $\beta > 0$ large enough so that $L(B)\|e^{-\beta\cdot}k\|_{L^1(0,\hat{T})} < 1$. Thus, (2.4) implies that $\|e^{-\beta\cdot}\phi\|_{L^1(0,\hat{T})} = 0$ for large $\beta > 0$, and hence $\phi = 0$ a.e. on $[0,\hat{T}]$. Therefore we get $x = x'$ a.e. on $[0,\hat{T}]$, and so the solution is unique. □

Next, we give a version of Proposition 2.2 for L^q-spaces, which is for us the main setup of interest. In comparison to Proposition 2.2, we give here an additional statement on the simultaneous solvability of a family of equations of this type. That is given in the second part of the following result, which generalises the first part. This additional result is particularly relevant in Chapter I for the proof of Theorem I.2.17, where we solve such a family of equations on a fixed time interval.

Proposition 2.4. *Fix $m \in \mathbb{N}_0, n \in \mathbb{N}$ and $\bar{T} \in (0,\infty)$. Suppose that $p,q \in [1,\infty)$ and $a \in [1,q]$ are such that $\frac{1}{p} + \frac{a-1}{q} = 1$. Let $k \in L^p([0,\bar{T}];\mathbb{C})$, $y \in L^q([0,\bar{T}];\mathbb{C}^n)$, $h \in L^q([0,\bar{T}];\mathbb{C}^m)$ and suppose that $f : \mathbb{C}^{n+m} \to \mathbb{C}^n$ is a continuous function satisfying the growth condition $|f(\tilde{x},\tilde{h})| \le C(1+|\tilde{x}|^a+|\tilde{h}|^a)$ for some $C > 0$ and all $\tilde{x} \in \mathbb{C}^n$ and $\tilde{h} \in \mathbb{C}^m$. Then the following statements hold:*

1) *There exists some positive time $\hat{T} \in (0,\bar{T}]$ such that there is a solution $x \in L^q([0,\hat{T}];\mathbb{C}^n)$ to the equation*

$$x(t) = y(t) + \int_0^t k(t-s) f\big(x(s),h(s)\big)ds \tag{2.5}$$

for a.a. $t \in [0,\hat{T}]$.

2) *Consider an indexing set $\mathcal{J}$ and families of functions $(k_j)_{j\in\mathcal{J}}$ in $L^p([0,\bar{T}];\mathbb{C})$, $(y_j)_{j\in\mathcal{J}}$ in $L^q([0,\bar{T}];\mathbb{C}^n)$ and $(h_j)_{j\in\mathcal{J}}$ in $L^q([0,\bar{T}];\mathbb{C}^m)$. Define the functions $\bar{k},\bar{y},\bar{h} : (0,\bar{T}] \to [0,\infty]$ by*

$$\bar{k}(t) = \sup_{j\in\mathcal{J}} \|k_j\|_{L^p(0,t)}, \quad \bar{y}(t) = \sup_{j\in\mathcal{J}} \|y_j\|_{L^q(0,t)}, \quad \bar{h}(t) = \sup_{j\in\mathcal{J}} \|h_j\|_{L^q(0,t)} \tag{2.6}$$

for $0 \le t \le \bar{T}$. Suppose that we have

$$\bar{k}(\bar{T}), \bar{y}(\bar{T}), \bar{h}(\bar{T}) < \infty \quad \textit{and} \quad \lim_{t\searrow 0} \bar{k}(t) = \lim_{t\searrow 0} \bar{y}(t) = \lim_{t\searrow 0} \bar{h}(t) = 0.$$

Then there exists some time $\hat{T} \in (0,\bar{T}]$ such that for each $j \in \mathcal{J}$, the equation

$$x_j(t) = y_j(t) + \int_0^t k_j(t-s) f\big(x_j(s),h_j(s)\big)ds \quad \textit{for a.a. } t \in [0,\hat{T}] \tag{2.7}$$

admits a solution $x_j \in L^q([0,\hat{T}];\mathbb{C}^n)$. *Moreover, we have*

$$\sup_{j\in\mathcal{J}} \|x_j\|_{L^q(0,\hat{T})} < \infty \quad \text{and} \quad \lim_{t\searrow 0} \sup_{j\in\mathcal{J}} \|x_j\|_{L^q(0,t)} = 0. \tag{2.8}$$

3) *If* $\hat{T} < \bar{T}$ *and* $x \in L^q([0,\hat{T}];\mathbb{C}^n)$ *is a solution to the equation* (2.5) *on* $[0,\hat{T}]$, *then there exists some* $\tau > 0$ *such that* x *can be extended to a solution to* (2.5) *on the interval* $[0,\hat{T}+\tau]$, *i.e., there is a solution* $\hat{x}$ *on* $L^q([0,\hat{T}+\tau];\mathbb{C}^n)$ *to* (2.5) *on* $[0,\hat{T}+\tau]$ *such that* $x = \hat{x}|_{[0,\hat{T}]}$ *a.s.*

4) *If* f *satisfies the local Lipschitz-type condition*

$$|f(\tilde{x}_2,\tilde{h}) - f(\tilde{x}_1,\tilde{h})| \le L\big(1 + |\tilde{x}_1|^{a-1} + |\tilde{x}_2|^{a-1} + |\tilde{h}|^{a-1}\big)|\tilde{x}_1 - \tilde{x}_2| \tag{2.9}$$

for some $L > 0$ *and all* $\tilde{x}_1, \tilde{x}_2 \in \mathbb{C}^n$ *and* $\tilde{h} \in \mathbb{C}^m$, *then there exists at most one solution to* (2.5) *in* $L^q([0,\hat{T}];\mathbb{C}^n)$.

Remark 2.5. Similarly to Remark 2.3, it follows by 3) that any solution to (2.5) can either be extended to the whole interval $[0,\bar{T}]$, or to a maximal solution $x \in L^q_{\text{loc}}([0,\hat{T});\mathbb{C}^n)$ for some $\hat{T} \in (0,\bar{T}]$ such that $\|x\|_{L^q(0,\hat{T})} = \infty$.

Proof of Proposition 2.4. **1)** This follows as a special case of 2) (shown below) by setting $\mathcal{J}$ to be a singleton and $k_j = k$, $y_j = y$ and $h_j = h$. Indeed, note that the functions $\bar{k}(t) := \|k\|_{L^p(0,t)}$, $\bar{y}(t) := \|y\|_{L^q(0,t)}$ and $\bar{h}(t) := \|h\|_{L^q(0,t)}$ are finite on $[0,\bar{T}]$ by the integrability of k, y and h. We also get that $\bar{k}(t), \bar{y}(t), \bar{h}(t) \to 0$ as $t \searrow 0$ by the dominated convergence theorem, where we use k, y and h as majorants for $k\mathbf{1}_{(0,t)}, y\mathbf{1}_{(0,t)}$ and $h\mathbf{1}_{(0,t)}$, respectively. Thus 2) applies and we get the existence of a solution on $[0,\hat{T}]$ for some $\hat{T} \in (0,\bar{T}]$.

2) We construct solutions to the equations (2.7) by a fixed point argument. For $B > 0$ and $\hat{T} \in (0,\bar{T}]$ to be specified later, consider the closed ball

$$D := \big\{x \in L^q([0,\hat{T}];\mathbb{C}^n) : \|x\|_{L^q(0,\hat{T})} \le B\big\} \tag{2.10}$$

and define the maps $\Phi_j : D \to L^q([0,\hat{T}];\mathbb{C}^n)$ by $\Phi_j(x) = y_j + k_j * f(x(\cdot),h_j(\cdot))$. Then each Φ_j is well defined since for $x \in D$, we can bound

$$\begin{aligned}
\|\Phi_j(x)\|_{L^q(0,\hat{T})} &\le \|y_j\|_{L^q(0,\hat{T})} + \big\|k_j * f\big(x(\cdot),h_j(\cdot)\big)\big\|_{L^q(0,\hat{T})} \\
&\le \|y_j\|_{L^q(0,\hat{T})} + \|k_j\|_{L^p(0,\hat{T})}\big\|f\big(x(\cdot),h_j(\cdot)\big)\big\|_{L^{q/a}(0,\hat{T})} \\
&\le \|y_j\|_{L^q(0,\hat{T})} + \|k_j\|_{L^p(0,\hat{T})} C\big(\hat{T}^{a/q} + \|x\|^a_{L^q(0,\hat{T})} + \|h_j\|^a_{L^q(0,\hat{T})}\big) \\
&\le \bar{y}(\hat{T}) + C\bar{k}(\hat{T})\big(\hat{T}^{a/q} + B^a + \bar{h}(\hat{T})^a\big) < \infty
\end{aligned} \tag{2.11}$$

by using Young's convolution inequality (1.2), the growth condition on f and the definition (2.6) of $\bar{k}, \bar{y}$ and $\bar{h}$. Thus $\Phi_j(x) \in L^q([0,\hat{T}];\mathbb{C}^n)$ for each $j \in \mathcal{J}$ and $x \in D$.

Fix some arbitrary $B > 0$. Since $\bar{k}(t), \bar{y}(t), \bar{h}(t) \to 0$ as $t \searrow 0$, we can find $\hat{T} > 0$ small enough (depending on B) so that the bound from (2.11) satisfies

$$\bar{y}(\hat{T}) + C\bar{k}(\hat{T})\big(\hat{T}^{a/q} + \bar{h}(\hat{T})^a + B^a\big) \leq B. \tag{2.12}$$

For this choice of $B > 0$ and $\hat{T} > 0$, we have by (2.11) and the definition (2.10) that $\Phi_j(x) \in D$ for all $x \in D$ and $j \in \mathcal{J}$, and thus $\Phi_j(D) \subseteq D$. Note that D is a closed convex set and each Φ_j is a continuous map, since it is the composition of the maps $x \mapsto f(x(\cdot), h_j(\cdot))$ and $x \mapsto y_j + k_j * x$ which are continuous by Lemmas 1.6 and 1.5, respectively. Moreover, the image of D under the map $x \mapsto f(x(\cdot), h_j(\cdot))$ is bounded in $L^{q/a}([0,\hat{T}];\mathbb{C}^n)$ due to the growth condition on f. Since the map $x \mapsto y_j + k_j * x$ is compact by Lemma 1.5, it follows that $\Phi_j(D)$ is relatively compact. Therefore by Theorem 1.4 applied to $\Phi_j : D \to D$, there exists a fixed point $x_j \in D \subseteq L^q([0,\hat{T}];\mathbb{C}^n)$ of Φ_j. By the definition of Φ_j, the fixed point $x_j = \Phi_j(x_j)$ is a solution to (2.5).

Since x_j belongs to D, we have $\|x_j\|_{L^q(0,\hat{T})} \leq B$; so we get a bound on $\|x_j\|_{L^q(0,\hat{T})}$ that is uniform in $j \in \mathcal{J}$. This gives the first part of (2.8). Repeating the estimate (2.11) with t in place of $\hat{T}$ and using the fact that $\|x_j\|_{L^q(0,\hat{T})} \leq B$, we also obtain the bound

$$\|x_j\|_{L^q(0,t)} = \|\Phi_j(x_j)\|_{L^q(0,t)} \leq \bar{y}(t) + C\bar{k}(t)\big(t^{a/q} + \bar{h}(t)^a + B^a\big) \longrightarrow 0$$

uniformly in $j \in \mathcal{J}$ as $t \searrow 0$. This shows the second part of (2.8).

3) To show that a solution $x \in L^q([0,\hat{T}];\mathbb{C}^n)$ can be extended, consider the new convolution equation

$$x'(s) = y'(s) + \big(k * f(x', h')\big)(s), \tag{2.13}$$

for $0 \leq s \leq \bar{T} - \hat{T}$, where we omit the arguments in $x' = x'(\cdot)$ and $h' = h'(\cdot)$ for readability and for $0 \leq s \leq \bar{T} - \hat{T}$ define

$$h'(s) = h(\hat{T} + s), \quad y'(s) = y(\hat{T} + s) + \Big(k * \big(\mathbf{1}_{[0,\hat{T}]} f(x, h)\big)\Big)(\hat{T} + s). \tag{2.14}$$

Note that $h' \in L^q([0, \bar{T} - \hat{T}];\mathbb{C}^m)$ by the L^q-integrability of h. Likewise, by Young's convolution inequality (1.2), the growth bound on f and as we have

$x \in L^q([0,\hat{T}];\mathbb{C}^n)$ and $y \in L^q([0,\bar{T}];\mathbb{C}^n)$ by assumption, we get

$$\begin{aligned}\|y'\|_{L^q(0,\bar{T}-\hat{T})} &\le \|y\|_{L^q(0,\bar{T})} + \big\|k * \big(\mathbf{1}_{[0,\hat{T}]} f(x,h)\big)\big\|_{L^q(0,\bar{T})} \\ &\le \|y\|_{L^q(0,\bar{T})} + \|k\|_{L^p(0,\bar{T})} \|\mathbf{1}_{[0,\hat{T}]} f(x,h)\|_{L^{q/a}(0,\bar{T})} \\ &\le \|y\|_{L^q(0,\bar{T})} + \|k\|_{L^p(0,\bar{T})} C\big(\hat{T}^{a/q} + \|x\|_{L^q(0,\hat{T})} + \|h\|_{L^q(0,\hat{T})}\big) \\ &< \infty,\end{aligned}$$

so that $y' \in L^q([0,\bar{T}-\hat{T}];\mathbb{C}^n)$. Hence we can apply the existence result 1) to (2.13) and get a solution $x' \in L^q([0,\tau];\mathbb{C}^n)$ for some $\tau > 0$. We can then extend x to $[0,\hat{T}+\tau]$ by setting $x(\hat{T}+s) := x'(s)$ for $s \in [0,\tau]$. It is clear by construction that $x \in L^q([0,\hat{T}+\tau];\mathbb{C}^n)$, and we need to check that x is a solution to (2.5) on $[0,\hat{T}+\tau]$. By the definition (2.12) of h' and the construction of x on $[\hat{T},\hat{T}+\tau]$, we have

$$\begin{aligned}\big(k * f(x',h')\big)(s) &= \int_0^s k(s-u) f\big(x(\hat{T}+u), h(\hat{T}+u)\big)du \\ &= \int_{\hat{T}}^{\hat{T}+s} k(\hat{T}+s-u) f\big(x(u),h(u)\big)du \\ &= \int_0^{\hat{T}+s} \mathbf{1}_{[\hat{T},\hat{T}+\tau]}(u) k(\hat{T}+s-u) f\big(x(u),h(u)\big)du \\ &= \Big(k * \big(\mathbf{1}_{[\hat{T},\hat{T}+\tau]} f(x,h)\big)\Big)(\hat{T}+s), \quad 0 \le s \le \tau.\end{aligned}$$

Hence by the construction of x on $[\hat{T},\hat{T}+\tau]$, the definition (2.14) of y' and h' and the equation (2.13) for x', we get

$$\begin{aligned}x(\hat{T}+s) &= x'(s) = y'(s) + \big(k * f(x',h')\big)(s) \\ &= y(\hat{T}+s) + \Big(k * \big(\mathbf{1}_{[0,\hat{T}]} f(x,h)\big)\Big)(\hat{T}+s) + \Big(k * \big(\mathbf{1}_{[\hat{T},\hat{T}+\tau]} f(x,h)\big)\Big)(\hat{T}+s) \\ &= y(\hat{T}+s) + \big(k * f(x,h)\big)(\hat{T}+s), \quad 0 \le s \le \tau,\end{aligned}$$

using the linearity of the convolution to obtain the last line. Therefore, x satisfies (2.5) on $[\hat{T},\hat{T}+\tau]$. Since x also satisfies (2.5) on $[0,\hat{T}]$ by assumption, this concludes the proof of 3).

4) Suppose that the condition (2.9) holds. Let $x_1, x_2 \in L^q([0,\hat{T}];\mathbb{C}^n)$ be two solutions to (2.5) on $[0,\hat{T}]$ and set

$$\phi := x_1 - x_2 = \big(y + k * f(x_1,h)\big) - \big(y + k * f(x_2,h)\big) = k * \big(f(x_1,h) - f(x_2,h)\big).$$

We note the identity $e^{-\beta\cdot}(y * z) = (e^{-\beta\cdot}y) * (e^{-\beta\cdot}z)$ for any $\beta \in \mathbb{R}$, which was also used in the proof of part 3) in Proposition 2.2. This yields

$$e^{-\beta\cdot}\phi = (e^{-\beta\cdot}k) * \Big(e^{-\beta\cdot}\big(f(x_1,h) - f(x_2,h)\big)\Big). \tag{2.15}$$

Recall that $\frac{1}{p} + \frac{a-1}{q} = 1$ by the assumption on a, so that

$$q = \frac{p(a-1)}{p-1} \quad \text{and} \quad \frac{1}{p} + \frac{a(p-1)}{p(a-1)} = \frac{1}{p} + \frac{a}{q} = 1 + \frac{1}{q}\,. \tag{2.16}$$

Thus by Young's convolution inequality (1.2) with powers p, $\frac{p(a-1)}{a(p-1)}$ and q, we obtain from (2.15) that

$$\begin{aligned} \|e^{-\beta\cdot}\phi\|_q &= \Big\|(e^{-\beta\cdot}k) * \Big(e^{-\beta\cdot}\big(f(x_1,h) - f(x_2,h)\big)\Big)\Big\|_q \\ &\le \|e^{-\beta\cdot}k\|_p \big\|e^{-\beta\cdot}\big(f(x_1,h) - f(x_2,h)\big)\big\|_{\frac{p(a-1)}{a(p-1)}}\,. \end{aligned}$$

By the local Lipschitz condition (2.9) and Hölder's inequality with powers a and $\frac{a}{a-1}$, we get

$$\begin{aligned} \|e^{-\beta\cdot}\phi\|_q &\le \|e^{-\beta\cdot}k\|_p \, \|C(e^{-\beta\cdot}|\phi|)(1 + |x_1|^{a-1} + |x_2|^{a-1} + |h|^{a-1})\|_{\frac{p(a-1)}{a(p-1)}} \\ &\le C\|e^{-\beta\cdot}k\|_p \, \|e^{-\beta\cdot}\phi\|_{\frac{p(a-1)}{p-1}} \, \big\|1 + |x_1|^{a-1} + |x_2|^{a-1} + |h|^{a-1}\big\|_{\frac{p}{p-1}} \\ &\le C\|e^{-\beta\cdot}k\|_p \, \|e^{-\beta\cdot}\phi\|_q\big(1 + \|x_1\|_q^{a-1} + \|x_2\|_q^{a-1} + \|h\|_q^{a-1}\big) < \infty, \end{aligned}$$

once again recalling (2.16) to obtain the last inequality. By the dominated convergence theorem, we can make $\|e^{-\beta\cdot}k\|_p$ arbitrarily small by choosing β large enough. This leads to a contradiction in the inequality above, unless we have $\|e^{-\beta\cdot}\phi\|_q = 0$ for large β, so that $\phi = 0$ a.e. on $[0,\hat{T}]$. This shows that $x_1 = x_2$ a.e., and hence the uniqueness. □

It is well known that equations of the form $x = k * f(x)$ as in Propositions 2.2 and 2.4 can be converted into equations of the form $y = f(k * y)$, and vice versa, by setting $x = k * y$ and $y = f(x)$; see e.g. Gatheral/Keller-Ressel [56, Theorem A.5 and Corollary A.7]. We show this equivalence in the following two corollaries, which give the existence of solutions to equations of the latter type. We start by giving a result in the continuous setting of Proposition 2.2, where we only need the analogue of part 1) in Proposition 2.2, although one could also give corresponding versions for parts 2) and 3). We then give a full analogue of

Proposition 2.4 for this type of equation in the L^q-setting.

Corollary 2.6. *Fix $n \in \mathbb{N}$, $\bar{T} \in (0,\infty]$ and $k \in L^1_{\mathrm{loc}}([0,\bar{T});\mathbb{C})$, and suppose that $y : [0,\bar{T}) \to \mathbb{C}^n$ and $f : \mathbb{C}^n \times [0,\bar{T}) \to \mathbb{C}^n$ are continuous functions. Then there exists a positive time $\hat{T} = \hat{T}(k,y,f) \in (0,\bar{T})$ such that there is a continuous solution $x : [0,\hat{T}] \to \mathbb{C}^n$ to the equation*

$$x(t) = y(t) + f\biggl(\int_0^t k(t-s)x(s)ds, t\biggr), \quad 0 \le t \le \hat{T}. \tag{2.17}$$

Proof. By Proposition 2.2, there exists a positive time $\hat{T} = \hat{T}(k,y,f) \in (0,\bar{T})$ such that there is a continuous solution $\hat{x} : [0,\hat{T}] \to \mathbb{C}^n$ to the equation

$$\hat{x}(t) = \hat{y}(t) + \int_0^t k(t-s)f\bigl(\hat{x}(s),s\bigr)ds, \quad 0 \le t \le \hat{T}, \tag{2.18}$$

where $\hat{y} := k * y$ is continuous like y; see Gripenberg et al. [59, Section 2.2]. Defining $x : [0,\hat{T}] \to \mathbb{C}^n$ by $x(t) := y(t) + f(\hat{x}(t),t)$, note that x is also continuous as y, f and $\hat{x}$ are, and the definition of x yields

$$\begin{aligned} x(t) &= y(t) + f\bigl(\hat{x}(t),t\bigr) \\ &= y(t) + f\biggl(\hat{y}(t) + \int_0^t k(t-s)f\bigl(\hat{x}(s),s\bigr)ds, t\biggr) \\ &= y(t) + f\biggl(\int_0^t k(t-s)\Bigl(y(s) + f\bigl(\hat{x}(s),s\bigr)\Bigr)ds, t\biggr) \\ &= y(t) + f\biggl(\int_0^t k(t-s)x(s)ds, t\biggr), \quad 0 \le t \le \hat{T}, \end{aligned}$$

using the definition of $\hat{y}$. Thus x is a solution to (2.17) on $[0,\hat{T}]$. □

We now consider the L^q-case as in Proposition 2.4 and give analogues to each of the statements 1)–4). Note that the assumptions are largely the same, except that we make here the weaker assumption that y is $L^{q/a}$-integrable, and the solution is likewise only $L^{q/a}$-integrable. The reason for these differences will be apparent from the way in which the two equations are related.

Corollary 2.7. *Fix $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $\bar{T} \in (0,\infty)$. Suppose that $p,q \ge 1$ and $a \in [1,q]$ are such that $\frac{1}{p} + \frac{a-1}{q} = 1$. Let $k \in L^p([0;\bar{T}];\mathbb{C})$, $y \in L^{q/a}([0,\bar{T}];\mathbb{C}^n)$, $h \in L^q([0,\bar{T}];\mathbb{C}^m)$ and $f : \mathbb{C}^{n+m} \to \mathbb{C}^n$ be a continuous function satisfying the growth condition $|f(\tilde{x},\tilde{h})| \le C(1 + |\tilde{x}|^a + |\tilde{h}|^a)$ for some $C > 0$ and all $\tilde{x} \in \mathbb{C}^n$ and $\tilde{h} \in \mathbb{C}^m$. Then the following statements hold:*

1) *There exists some* $\hat{T} \in (0, \bar{T}]$ *such that there is a solution* $x \in L^{q/a}([0, \hat{T}]; \mathbb{C}^n)$ *to the equation*

$$x(t) = y(t) + f\big((k * x)(t), h(t)\big) \tag{2.19}$$

for a.a. $t \in [0, \hat{T}]$.

2) *Consider an indexing set* $\mathcal{J}$ *and families of functions* $(k_j)_{j \in \mathcal{J}}$ *in* $L^p([0, \bar{T}]; \mathbb{C})$, $(y_j)_{j \in \mathcal{J}}$ *in* $L^{q/a}([0, \bar{T}]; \mathbb{C}^n)$ *and* $(h_j)_{j \in \mathcal{J}}$ *in* $L^q([0, \bar{T}]; \mathbb{C}^m)$. *Suppose that the functions* $\bar{k}, \bar{y}, \bar{h} : (0, \bar{T}] \to [0, \infty]$, *defined for* $0 \le t \le \bar{T}$ *by*

$$\bar{k}(t) = \sup_{j \in \mathcal{J}} \|k_j\|_{L^p(0,t)}, \quad \bar{y}(t) = \sup_{j \in \mathcal{J}} \|y_j\|_{L^{q/a}(0,t)}, \quad \bar{h}(t) = \sup_{j \in \mathcal{J}} \|h_j\|_{L^q(0,t)},$$

are finite and satisfy the limits $\bar{k}(t), \bar{y}(t), \bar{h}(t) \to 0$ *as* $t \searrow 0$. *Then there exists some time* $\hat{T} \in (0, \bar{T}]$ *such that for each* $j \in \mathcal{J}$, *the equation*

$$x_j(t) = y_j(t) + f\big((k_j * x_j)(t), h_j(t)\big), \tag{2.20}$$

for a.a. $t \in [0, \hat{T}]$, *admits a solution* $x_j \in L^{q/a}([0, \hat{T}]; \mathbb{C}^n)$. *Moreover, we have*

$$\sup_{j \in \mathcal{J}} \|x_j\|_{L^{q/a}(0,\hat{T})} < \infty \quad \text{and} \quad \limsup_{t \searrow 0} \sup_{j \in \mathcal{J}} \|x_j\|_{L^{q/a}(0,t)} = 0. \tag{2.21}$$

3) *If* $\hat{T} < \bar{T}$ *and* $x \in L^{q/a}([0, \hat{T}]; \mathbb{C}^n)$ *is a solution to* (2.19), *then there exists some* $\tau > 0$ *such that* x *can be extended to a solution to the equation* (2.19) *on the interval* $[0, \hat{T} + \tau]$.

4) *If* f *satisfies the local Lipschitz-type condition*

$$|f(\tilde{x}_1, \tilde{h}) - f(\tilde{x}_2, \tilde{h})| \le L\big(1 + |\tilde{x}_1|^{a-1} + |\tilde{x}_2|^{a-1} + |\tilde{h}|^{a-1}\big)|\tilde{x}_1 - \tilde{x}_2| \tag{2.22}$$

for some $L > 0$ *and all* $\tilde{x}_1, \tilde{x}_2 \in \mathbb{C}^n$ *and* $\tilde{h} \in \mathbb{C}^m$, *then there exists at most one solution to* (2.19) *in* $L^{q/a}([0, \hat{T}]; \mathbb{C}^n)$.

Proof. **1)** This follows from 2) by taking $\mathcal{J}$ to be a singleton, by the same argument as in the proof of statement 1) of Proposition 2.4.

2) For each $j \in \mathcal{J}$, consider the modified equation

$$\hat{x}_j(t) = \hat{y}_j(t) + \big(k_j * f(\hat{x}_j, h_j)\big)(t), \quad 0 \le t \le \hat{T}, \tag{2.23}$$

where $\hat{y}_j := k_j * y_j$ and we omit the argument in $h_j = h_j(\cdot)$ and $\hat{x}_j = \hat{x}_j(\cdot)$ for readability. We want to show the existence of solutions $\hat{x}_j$ to (2.23), and then use these to construct solutions x_j to the original equations (2.20). Note that we have

$$\bar{y}'(t) := \sup_{j\in\mathcal{J}} \|\hat{y}_j\|_{L^q(0,t)} = \sup_{j\in\mathcal{J}} \|k_j * y_j\|_{L^q(0,t)} \le \bar{k}(t)\bar{y}(t)$$

by Young's convolution inequality (1.2), so that $\bar{y}'$ is finite with $\bar{y}'(t) \to 0$ as $t \searrow 0$. Thus we can apply part 2) of Proposition 2.4 to (2.23), since the required assumptions on $\bar{k}$, $\bar{y}'$ and $\bar{h}$ are satisfied. This shows the existence of solutions $\hat{x}_j \in L^q([0,\hat{T}];\mathbb{C}^n)$ to (2.23). We also get that

$$\sup_{j\in\mathcal{J}} \|\hat{x}_j\|_{L^q(0,\hat{T})} < \infty \quad \text{and} \quad \lim_{t\searrow 0} \sup_{j\in\mathcal{J}} \|\hat{x}_j\|_{L^q(0,t)} = 0. \tag{2.24}$$

Now define the functions $x_j : [0,\hat{T}] \to \mathbb{C}^n$ by $x_j := y_j + f(\hat{x}_j(\cdot), h_j(\cdot))$. By the definitions of $\bar{k}, \bar{y}$ and $\bar{h}$ and the growth condition on f, we have the bound

$$\sup_{j\in\mathcal{J}} \|x_j\|_{L^{q/a}(0,t)} \le \bar{y}(t) + C\Big(t^{a/q} + \sup_{j\in\mathcal{J}} \|\hat{x}_j\|^a_{L^q(0,t)} + \bar{h}(t)\Big). \tag{2.25}$$

Setting $t = \hat{T}$, we see that the x_j are uniformly bounded in $L^{q/a}([0,\hat{T}];\mathbb{C}^n)$ by the first part of (2.24). Letting $t \searrow 0$, we get $\sup_{j\in\mathcal{J}} \|x_j\|_{L^{q/a}(0,t)} \to 0$ by the assumptions on $\bar{y}$ and $\bar{h}$ and the second part of (2.24). This shows (2.21).

It remains to check that x_j satisfies the original equation (2.20). Plugging in (2.23) and the definition $\hat{y}_j = k_j * y_j$, we obtain

$$\begin{aligned} x_j(t) &= y_j(t) + f\big(\hat{x}_j(t), h_j(t)\big) \\ &= y_j(t) + f\Big(\hat{y}_j(t) + \big(k_j * f(\hat{x}_j, h_j)\big)(t), h_j(t)\Big) \\ &= y_j(t) + f\Big(\Big(k_j * \big(y_j + f(\hat{x}_j, h_j)\big)\Big)(t), h_j(t)\Big) \\ &= y_j(t) + f\big((k_j * x_j)(t), h_j(t)\big), \qquad 0 \le t \le \hat{T}. \end{aligned} \tag{2.26}$$

Thus each $x_j \in L^{q/a}([0,\hat{T}];\mathbb{C}^n)$ is a solution to (2.19), as claimed.

3) Suppose that $x \in L^{q/a}([0,\hat{T}];\mathbb{C}^n)$ is a solution to (2.19). Defining $\hat{x} := k*x$, we have $\hat{x} \in L^q([0,\hat{T}];\mathbb{C}^n)$ by Young's convolution inequality (1.2). Similarly to (2.26), $\hat{x}$ satisfies the equation

$$\begin{aligned} \hat{x}(t) = (k * x)(t) &= \Big(k * \big(y(t) + f(k * x, h)\big)\Big)(t) \\ &= \hat{y}(t) + \big(k * f(\hat{x}, h)\big)(t), \quad 0 \le t \le \hat{T}, \end{aligned} \tag{2.27}$$

where $\hat{y} := k * y \in L^q([0,\hat{T}];\mathbb{C}^n)$ by Young's convolution inequality (1.2). Thus it follows from part 3) of Proposition 2.4 that the solution $\hat{x}$ to (2.27) can be extended to a larger interval $[0,\hat{T}+\tau]$ for some $\tau > 0$. We can then extend x by setting $x(t) := y(t) + f(\hat{x}(t), h(t))$ for $t \in [0, \hat{T}+\tau]$. Note that this coincides with the original x on $[0,\hat{T}]$ since

$$y(t) + f\big(\hat{x}(t), h(t)\big) = y(t) + f\big((k * x)(t), h(t)\big) = x(t), \quad 0 \le t \le \hat{T}.$$

We also have $x \in L^{q/a}([0,\hat{T}+\tau]; \mathbb{C}^n)$ by the same bound as in (2.25), and x is a solution to (2.19) by repeating the steps in (2.26). Thus we have extended x to a solution to (2.19) on $[0,\hat{T}+\tau]$.

4) Suppose that (2.22) holds and let x_1 and x_2 be two solutions to (2.19) on $[0,\hat{T}]$. Once again by plugging in, we obtain that $\hat{x}_1 := \kappa * x_1$ and $\hat{x}_2 := \kappa * x_2$ are two solutions to (2.27), and $\hat{x}_1, \hat{x}_2 \in L^q([0,\hat{T}]; \mathbb{C}^n)$ by Lemma 1.5. We also have

$$x_i = f(k * x_i, h) = f(\hat{x}_i, h).$$

Since we must have $\hat{x}_1 = \hat{x}_2$ a.e. by the uniqueness in part 4) of Proposition 2.4, it follows that $x_1 = f(\hat{x}_1, h) = f(\hat{x}_2, h) = \hat{x}_2$ a.e. This shows the uniqueness. □

We can obtain a version of Corollary 2.7 with improved bounds for the particular case of a Riccati–Volterra equation, that is, when the nonlinear function f is quadratic. This is done in the following proposition. Later, we shall use these bounds to obtain weaker conditions under which the solution to a Riccati–Volterra equation of the form (2.19) is small in the L^1-norm, as well as a result on the stability of solutions to Riccati–Volterra equations. For simplicity, we consider only the one-dimensional case.

Proposition 2.8. *Let $a, b, c, k : [0,\infty) \to \mathbb{C}$ be measurable functions. Let $\hat{T} > 0$, $\gamma \in \mathbb{R}$ and $A, B, C, K \in [0,\infty)$ be constants such that*

$$\begin{aligned} \|e^{-\gamma \cdot} a\|_{L^1(0,\hat{T})} &\le A, & \|c\|_{L^\infty(0,\hat{T})} &\le C, \\ \|b\|_{L^2(0,\hat{T})} &\le B, & \|e^{-\gamma \cdot} k\|_{L^2(0,\hat{T})} &\le K, \end{aligned} \tag{2.28}$$

and suppose that $\hat{T}, \gamma, A, B, C$ and K satisfy the inequalities

$$BK < 1 \quad \textit{and} \quad (1 - BK)^2 \ge 4 e^{\gamma \hat{T}} A C K^2. \tag{2.29}$$

Then there exists a unique solution $x \in L^1([0,\hat{T}]; \mathbb{C})$ to the equation

$$x(t) = a(t) + b(t)(k * x)(t) + c(t)\big((k * x)(t)\big)^2 \quad \textit{for a.a. } t \in [0,\hat{T}], \tag{2.30}$$

and x satisfies the bound

$$\|e^{-\gamma\cdot}x\|_{L^1(0,\hat{T})} \le \frac{2A}{1-BK+\sqrt{(1-BK)^2-4e^{\gamma\hat{T}}ACK^2}}. \tag{2.31}$$

Proof. To show the existence, we proceed similarly as in the proof of Proposition 2.4, by using a fixed point argument to find a solution to (2.30) (more precisely, we first solve a modified version of the equation). Assume for the moment that $C>0, K>0$. Define the set

$$D := \{x \in L^1([0,\hat{T}];\mathbb{C}) : \|x\|_{L^1(0,\hat{T})} \le r_-\}, \tag{2.32}$$

where we choose $r_- \ge 0$ as the smallest real root of the quadratic function

$$g(r) := Ce^{\gamma\hat{T}}K^2r^2 - (1-BK)r + A, \quad r \in \mathbb{R}, \tag{2.33}$$

which is given explicitly by

$$r_- = \frac{1-BK-\sqrt{(1-BK)^2-4e^{\gamma\hat{T}}ACK^2}}{2Ce^{\gamma\hat{T}}K^2},$$

so that $g(r_-)=0$. It follows from (2.29) that r_- is real-valued and nonnegative as claimed, since

$$1-BK = \sqrt{(1-BK)^2} \ge \sqrt{(1-BK)^2-4e^{\gamma\hat{T}}ACK^2} \ge 0.$$

Now consider the map $\Phi : D \to L^1([0,\hat{T}];\mathbb{C})$ defined by

$$\Phi(\hat{x}) = e^{-\gamma\cdot}a + b\big((e^{-\gamma\cdot}k) * \hat{x}\big) + e^{-\gamma\cdot}c\big((e^{-\gamma\cdot}k) * \hat{x}\big)^2. \tag{2.34}$$

In the following, we abbreviate $L^p(0,\hat{T})$ to L^p for readability, as the time horizon $\hat{T}$ is fixed. By Young's convolution inequality (1.2), the Cauchy–Schwarz inequality and the definition (2.33) of g, we obtain

$$\begin{aligned}\|\Phi(\hat{x})\|_{L^1} &\le \|e^{-\gamma\cdot}a\|_{L^1} + \|b\|_{L^2}\|(e^{-\gamma\cdot}k)*\hat{x}\|_{L^2} + e^{\gamma\hat{T}}\|c\|_\infty\|(e^{-\gamma\cdot}k)*\hat{x}\|_{L^2}^2 \\ &\le \|e^{-\gamma\cdot}a\|_{L^1} + \|b\|_{L^2}\|e^{-\gamma\cdot}k\|_{L^2}\|\hat{x}\|_{L^1} + e^{\gamma\hat{T}}\|c\|_\infty\|e^{-\gamma\cdot}k\|_{L^2}^2\|\hat{x}\|_{L^1}^2 \\ &\le A + BKr_- + Ce^{\gamma\hat{T}}K^2r_-^2 \\ &= g(r_-) + r_- = r_- \end{aligned} \tag{2.35}$$

for each $\hat{x} \in D$, so that $\Phi(\hat{x}) \in D$. Thus the map Φ is well defined and we have

$\Phi(D) \subseteq D$. It is clear that D is a closed and convex set. As in the proof of Proposition 2.4, Φ is continuous as the composition of the map $x \mapsto a+bx+cx^2$, which is continuous from $L^2([0,\hat{T}];\mathbb{C})$ to $L^1([0,\hat{T}];\mathbb{C})$ by Lemma 1.6, with the map $x \mapsto k*x$, which is continuous and compact from $L^1([0,\hat{T}];\mathbb{C})$ to $L^2([0,\hat{T}];\mathbb{C})$ by Lemma 1.5. Hence $\Phi(D)$ is also compact. Therefore by Theorem 1.4, we obtain a fixed point $\hat{x} = \Phi(\hat{x}) \in D$.

The fixed point $\hat{x}$ is not yet a solution to (2.30), but rather to the modified equation

$$\hat{x}(t) = \Phi(\hat{x})(t) = e^{-\gamma t}a(t) + b(t)(e^{-\gamma \cdot}k*\hat{x})(t) + e^{\gamma t}c(t)\big((e^{-\gamma \cdot}k*\hat{x})(t)\big)^2.$$

Setting now $x = e^{\gamma \cdot}\hat{x}$, we have $x \in L^1([0,\hat{T}];\mathbb{C})$ because $\hat{x} \in D \subseteq L^1([0,\hat{T}];\mathbb{C})$ and $e^{\gamma \cdot}$ is bounded on $[0,\hat{T}]$. Moreover, by using that $e^{-\gamma \cdot}(y*z) = (e^{-\gamma \cdot}y * e^{-\gamma \cdot}z)$ and plugging into the equation above, we have

$$\begin{aligned} x = e^{\gamma \cdot}\hat{x} &= a + be^{\gamma \cdot}(e^{-\gamma \cdot}k * e^{-\gamma \cdot}x) + ce^{2\gamma \cdot}(e^{-\gamma \cdot}k * e^{-\gamma \cdot}x)^2 \\ &= a + b(k*x) + c(k*x)^2. \end{aligned} \tag{2.36}$$

Therefore, there exists a solution $x \in L^1([0,\hat{T}];\mathbb{C})$ to (2.30). We can obtain the bound (2.31) for x by noting that

$$\begin{aligned} \|e^{-\gamma \cdot}x\|_{L^1} = \|\hat{x}\|_{L^1} \leq r_- &= \frac{1-BK-\sqrt{(1-BK)^2-4e^{\gamma\hat{T}}ACK^2}}{2Ce^{\gamma\hat{T}}K^2} \\ &= \frac{(1-BK)^2-((1-BK)^2-4e^{\gamma\hat{T}}ACK^2)}{\big(1-BK+\sqrt{(1-BK)^2-4e^{\gamma\hat{T}}ACK^2}\big)2Ce^{\gamma\hat{T}}K^2} \\ &= \frac{2A}{1-BK+\sqrt{(1-BK)^2-4e^{\gamma\hat{T}}ACK^2}}, \end{aligned}$$

as claimed. Thus, we have shown the existence of a solution and the bound (2.31) under the assumption that both C and K are strictly positive.

We now return to the cases $K=0$ and $C=0$. The case $K=0$ is straightforward, since we must have $k=0$ a.e. and thus the only solution to (2.30) is $x=a$ a.e. In that case, we have $\|xe^{-\gamma \cdot}\|_{L^1(0,\hat{T})} = \|ae^{-\gamma \cdot}\|_{L^1(0,\hat{T})} = A$, which precisely corresponds to the bound (2.31) after plugging in $K=0$.

Finally, we suppose that $C=0$ and may now assume that $K>0$. The existence in this case follows by reproducing the proof for the case $C>0$ with minor adjustments. We set $C=0$ in (2.33), so that g is linear, and replace the root with $r_- = \frac{A}{1-BK} \geq 0$ by (2.29). We then define D and Φ in the same way by

(2.32) and (2.34) and obtain the same bound (2.35) with $C = 0$. Hence Theorem 1.4 gives a fixed point $\hat{x}$ to Φ by the same argument as before. It then follows that $x := e^{\gamma\cdot}\hat{x}$ is a solution to (2.30) by (2.36) and satisfies the bound

$$\|e^{-\gamma\cdot}x\|_{L^1} = \|\hat{x}\|_{L^1} \leq r_- = \frac{A}{1-BK}.$$

Once again, this precisely matches the bound (2.31) in the case $C = 0$. Thus we have shown the existence of a solution as well as the bound (2.31) in all cases.

It only remains to show the uniqueness of the solution, which also follows by a similar argument as in the proof of Proposition 2.4. Let $x_1, x_2 \in L^1([0,\hat{T}];\mathbb{C})$ be two solutions to (2.30) and set $\phi := x_1 - x_2$. By taking the difference in (2.30), ϕ satisfies the equation

$$\phi = b(k*\phi) + c(k*\phi)\big((k*(x_1+x_2)\big).$$

Thus by the property $e^{-\beta\cdot}(y*z) = (e^{-\beta\cdot}y * e^{-\beta\cdot}z)$, the Cauchy–Schwarz inequality and Young's convolution inequality (1.2) with $p = 2$ and $q = r = 1$, we obtain

$$\begin{aligned}\|e^{-\beta\cdot}\phi\|_{L^1} &= \Big\|e^{-\beta\cdot}\Big(b(k*\phi) + c(k*\phi)\big((k*(x_1+x_2)\big)\Big)\Big\|_{L^1} \\ &= \Big\|\big((e^{-\beta\cdot}k)*(e^{-\beta\cdot}\phi)\big)b + \big((e^{-\beta\cdot}k)*(e^{-\beta\cdot}\phi)\big)c\big((k*(x_1+x_2)\big)\Big\|_{L^1} \\ &\leq \|(e^{-\beta\cdot}k)*(e^{-\beta\cdot}\phi)\|_{L^2}\big(\|b\|_{L^2} + \|c\|_\infty\|k*(x_1+x_2)\|_{L^2}\big) \\ &\leq \|e^{-\beta\cdot}k\|_{L^2}\|e^{-\beta\cdot}\phi\|_{L^1}\big(\|b\|_{L^2} + \|c\|_\infty\|k\|_{L^2}\|x_1+x_2\|_{L^1}\big)\end{aligned}$$

for any $\beta \in \mathbb{R}$. Note that $\|x_1 + x_2\|_{L^1} < \infty$ by the assumption on the solutions, and we have $\|k\|_{L^2} < e^{\gamma\hat{T}}K < \infty$ by the assumption on k. Since $\|e^{-\beta\cdot}k\|_{L^2} \to 0$ as $\beta \to \infty$ by the dominated convergence theorem, the above inequality leads to a contradiction unless $\phi = 0$ a.s. Therefore, we have $x_1 = x_2$ a.s. and this shows the uniqueness. □

As a corollary, we deduce from Proposition 2.8 that a solution x to (2.29) is small if a is small, in the following sense.

Corollary 2.9. *Fix $\hat{T}, B, C, \epsilon > 0$ as well as a kernel $k \in L^2([0,\hat{T}];\mathbb{C})$. Then there exist a large enough $\gamma > 0$ and a small enough $A > 0$ so that for any functions $a \in L^1([0,\hat{T}],\mathbb{C})$, $b \in L^2([0,\hat{T}],\mathbb{C})$ and $c \in L^\infty([0,\hat{T}],\mathbb{C})$ such that*

$$\|e^{-\gamma\cdot}a\|_{L^1(0,\hat{T})} \leq A, \quad \|b\|_{L^2(0,\hat{T})} \leq B \quad \textit{and} \quad \|c\|_{L^\infty(0,\hat{T})} \leq C, \tag{2.37}$$

there exists a solution $x \in L^1([0,\hat{T}];\mathbb{C})$ *to* (2.30), *and it holds that*

$$\|x\|_{L^1(0,\hat{T})} \le e^{\gamma \hat{T}} \|e^{-\gamma \cdot} x\|_{L^1(0,\hat{T})} \le \epsilon. \tag{2.38}$$

In particular, the first bound in (2.37) *holds if* $\|a\|_{L^1(0,\hat{T})} \le A$, *since we have* $\|e^{-\gamma \cdot} a\|_{L^1(0,\hat{T})} \le \|a\|_{L^1(0,\hat{T})}$.

Proof. We have $\|e^{-\gamma \cdot} k\|_{L^2(0,\hat{T})} \to 0$ as $\gamma \to \infty$ by the dominated convergence theorem. Thus we can choose $\gamma > 0$ large enough so that $K := \|e^{-\gamma \cdot} k\|_{L^2(0,\hat{T})}$ satisfies the inequality $BK < 1$, which is the first inequality in (2.29). If we now fix γ and K in addition to $\hat{T}, B, C$ and ϵ, note that the right-hand side of the second inequality in (2.29) converges to 0 as $A \searrow 0$, and thus we can choose some $A = A(B, C, K, \gamma, \hat{T}, \epsilon) > 0$ small enough so that the second inequality in (2.29) holds as well. By choosing A to be possibly even smaller, we can also ensure that the right-hand side in (2.31) is smaller than $e^{-\gamma \hat{T}} \epsilon$.

With these choices of constants, Proposition 2.8 directly gives the existence of a solution $x \in L^1([0,\hat{T}];\mathbb{C})$ to (2.30) for any functions a, b, c satisfying (2.37), since the inequalities (2.29) hold. The bound (2.38) also follows directly from (2.31) and the choice of A, since we get $\|e^{-\gamma \cdot} x\|_{L^1(0,\hat{T})} \le e^{-\gamma \hat{T}} \epsilon$. □

In the next corollary to Proposition 2.8, we show that if (2.30) is linear in x, i.e., if $c \equiv 0$, then (2.30) admits a solution on $[0, \infty)$.

Corollary 2.10. *For any given functions* $a \in L^1_{\text{loc}}([0,\infty);\mathbb{C})$, $b \in L^2_{\text{loc}}([0,\infty);\mathbb{C})$ *and* $k \in L^2_{\text{loc}}([0,\infty);\mathbb{C})$, *there exists a unique solution* $x \in L^1_{\text{loc}}([0,\infty);\mathbb{C})$ *to the equation*

$$x(t) = a(t) + b(t)(k * x)(t), \quad t \ge 0. \tag{2.39}$$

Proof. Fix some arbitrary $T > 0$ and set $B := \|b\|_{L^2(0,T)}$. Since $k \in L^2_{\text{loc}}([0,\infty);\mathbb{C})$, we have $\|e^{-\gamma \cdot} k\|_{L^2(0,T)} \to 0$ as $\gamma \to \infty$ by the dominated convergence theorem with majorant $|k|$. Thus we may choose some $\gamma > 0$ large enough so that $BK < 1$, where $K := \|e^{-\gamma \cdot} k\|_{L^2(0,T)}$. We fix this choice of γ and set $A := \|e^{-\gamma \cdot} a\|_{L^1(0,\hat{T})}$. Hence the first inequality in (2.29) is satisfied, and the second inequality automatically holds because $C := \|c\|_\infty = 0$. It then follows from Proposition (2.8) that there exists a unique solution $x_T \in L^1_{\text{loc}}([0,\infty);\mathbb{C})$ to (2.39) on $[0, T]$.

By varying the choice of $T > 0$, we obtain a family of solutions $(x_T)_{T \ge 0}$ to (2.39) on each interval $[0, T]$. By the uniqueness of the solutions, we have that x_T is equal to the restriction of $x_{T'}$ to $[0, T]$ for each $0 \le T \le T'$. Thus we can define $x(t) = \sum_{n=1}^\infty \mathbf{1}_{t \in [n-1,n)} x_n$, and it is clear that the restriction of x to $[0, T]$ is equal to x_T for any $T \geq 0$. Hence $x \in L^1_{\text{loc}}([0,\infty);\mathbb{C})$ and x is a solution to (2.39). Conversely, any solution to (2.39) must coincide with x_T on $[0,T]$ for $T \geq 0$ due to the uniqueness of the solutions, and therefore x is the unique solution. □

Finally, as a useful application of Corollary 2.9, we give a stability result for equations of the form (2.30). Note that we only assume the existence of solutions to the perturbed equations (2.40) below. This may be ensured with Proposition 2.8 if the inequalities in (2.29) hold, but the latter are not necessary conditions.

Proposition 2.11. *Fix $T \in (0,\infty)$ and $k \in L^2([0,T];\mathbb{C})$. For each $n \in \mathbb{N}\cup\{\infty\}$, let $a_n \in L^1([0,T];\mathbb{C})$, $b_n \in L^2([0,T];\mathbb{C})$ and $c_n \in L^\infty([0,T];\mathbb{C})$, and suppose that there exists a solution $x_n \in L^1([0,T];\mathbb{C})$ to the equation*

$$x_n(t) = a_n(t) + b_n(t)(k * x_n)(t) + c_n(t)\big((k * x_n)(t)\big)^2 \quad \text{for a.a. } t \in [0,T]. \tag{2.40}$$

Moreover, suppose that $a_n \xrightarrow{L^1} a_\infty$, $b_n \xrightarrow{L^2} b_\infty$ and $c_n \xrightarrow{L^\infty} c_\infty$ as $n \to \infty$. Then $x_n \xrightarrow{L^1} x_\infty$ as $n \to \infty$.

Proof. Consider the differences $\tilde{f}_n := f_n - f_\infty$ for $f = a, b, c, x$. Taking differences in (2.40) and using the linearity of the convolution yields

$$\begin{aligned}\tilde{x}_n(t) = \tilde{a}_n(t) + \tilde{b}_n(t)(k * x_\infty)(t) + b_n(t)(k * \tilde{x}_n)(t) + \tilde{c}_n(t)\big((k * x_\infty)(t)\big)^2 \\ + c_n(t)\big(2(k * x_\infty)(t) + (k * \tilde{x}_n)(t)\big)(k * \tilde{x}_n)(t) \quad \text{for a.a. } t \in [0,T].\end{aligned}$$

Collecting the powers of $k * \tilde{x}_n$, this can be rewritten as

$$\tilde{x}_n(t) = \hat{a}_n(t) + \hat{b}_n(t)(k * \tilde{x}_n)(t) + c_n(t)\big((k * \tilde{x}_n)(t)\big)^2 \quad \text{for a.a. } t \in [0,T], \tag{2.41}$$

where we define the coefficients $\hat{a}_n$ and $\hat{b}_n$ by

$$\begin{aligned}\hat{a}_n(t) &:= \tilde{a}_n(t) + \tilde{b}_n(t)(k * x_\infty)(t) + \tilde{c}_n(t)\big((k * x_\infty)(t)\big)^2, \\ \hat{b}_n(t) &:= b_n(t) + 2c_n(t)(k * x_\infty)(t), \quad 0 \leq t \leq T.\end{aligned}$$

Recall that by Young's convolution inequality (1.2), we have $\|k*\tilde{x}\|_{L^2} \leq \|k\|_{L^2}\|\tilde{x}\|_{L^1}$. Thus the Cauchy–Schwarz inequality yields

$$\|\hat{a}_n\|_{L^1} \leq \|\tilde{a}_n\|_{L^1} + \|\tilde{b}_n\|_{L^2}\|k\|_{L^2}\|x_\infty\|_{L^1} + \|\tilde{c}_n\|_{L^\infty}\|k\|_{L^2}^2\|x_\infty\|_{L^1}^2, \tag{2.42}$$

$$\|\hat{b}_n\|_{L^2} \leq \|b_n\|_{L^2} + 2\|c_n\|_{L^\infty}\|k\|_{L^2}\|x_\infty\|_{L^1}. \tag{2.43}$$

The sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are bounded in L^2 and L^∞, respectively, since $b_n \xrightarrow{L^2} b_\infty$ and $c_n \xrightarrow{L^\infty} c_\infty$ as $n\to\infty$. This implies $C := \sup_{n\in\mathbb{N}} \|c_n\|_{L^\infty} < \infty$, and by (2.43), we also have $B := \sup_{n\in\mathbb{N}} \|\hat{b}_n\|_{L^2} < \infty$. Fix now some arbitrary $\epsilon > 0$, and let $A > 0$ and $\gamma > 0$ be given by Corollary 2.9 with respect to T, B, C, ϵ and k. Recall that $\|\tilde{a}_n\|_{L^1(0,T)} \to 0$, $\|\tilde{b}_n\|_{L^2(0,T)} \to 0$ and $\|\tilde{c}_n\|_{L^\infty(0,T)} \to 0$ as $n\to\infty$ by assumption, and hence $\hat{a}_n \xrightarrow{L^1} 0$ as $n \to \infty$ by (2.42). Thus there exists some $N \in \mathbb{N}$ such that

$$\|e^{-\gamma\cdot}\tilde{a}_n\|_{L^1(0,T)} \leq \|\tilde{a}_n\|_{L^1(0,T)} \leq A$$

for all $n \geq N$. Since the last two inequalities in (2.37) are also satisfied by b_n and c_n for each $n \geq N$ by the choice of B and C, it follows by Corollary 2.9 that the solution $\tilde{x}_n$ to (2.41) satisfies $\|\tilde{x}_n\|_{L^1(0,T)} \leq \epsilon$ for $n \geq N$. Since $\epsilon > 0$ is arbitrary, it follows that $\tilde{x}_n = x_n - x_\infty \xrightarrow{L^1} 0$ as $n \to \infty$, and this concludes the proof. □

3 Auxiliary results for Chapter I

Finally, we give two technical results involving convolutions and Volterra equations that are used directly in Chapter I. First, we show that the solution $g^\star$ to the equation (3.1) below takes nonpositive values. This equation is the same as (I.3.24) from Chapter I, after plugging in the particular kernel $k = \hat{\kappa}$ defined in (I.2.12). Thus the following result yields that the function $g^\star$ from Theorem I.3.8 is nonnegative.

Lemma 3.1. *Let $T > 0$ and $k : [0,T] \to [0,\infty)$ be continuous, nonnegative and nonincreasing. Suppose that $g^\star : [0,T] \to \mathbb{R}$ is a continuous solution to the equation*

$$g^\star(t) = f\big((k * g^\star)(t)\big), \quad 0 \leq t \leq T, \tag{3.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = -\frac{\mu^2}{\sigma^2} - \frac{2\varrho\mu x}{\sigma} + \frac{1}{2}(1-2\varrho^2)x^2 \tag{3.2}$$

for fixed constants $\mu \in \mathbb{R}$, $\sigma > 0$ and $\varrho \in [-1,1]$. If $\mu \neq 0$, then $g^\star$ takes strictly negative values on $[0,T]$. If $\mu = 0$, then $g^\star \equiv 0$.

Proof. The quadratic function f satisfies the Lipschitz-type condition

$$|f(x_1) - f(x_2)| \leq \left(\frac{2|\varrho|\,|\mu|}{\sigma} + \frac{1}{2}(1-2\varrho^2)(|x_1|+|x_2|)\right)|x_1 - x_2|, \quad x_1, x_2 \in \mathbb{R}.$$

Note that (3.1) is of the form (2.19) with $g^\star$ in place of x, where we set the constants $p = q = a = 1$, $y \equiv 0$, $n = 1$ and $m = 0$, so that h is absent. Hence by part 4) of Corollary 2.7, there is at most one integrable solution to (3.1). Since $g^\star$ is continuous by assumption, it is thus the unique continuous solution. In the case $\mu = 0$, we have that the constant function 0 is a solution to (3.1), so that $g^\star \equiv 0$ as claimed. Henceforth we assume without loss of generality that $\mu \neq 0$.

Since k is integrable and $g^\star$ is continuous, we have by Gripenberg et al. [59, Theorem 2.2.2] that $k * g^\star$ is continuous. We also have the limit

$$|(k * g^\star)(t)| \leq \int_0^t |k(t-s)||g(s)|ds \leq \|g\|_\infty \|k\|_{L^1(0,t)} \longrightarrow 0$$

as $t \searrow 0$ by the dominated convergence theorem with majorant k. This implies that $(k * g^\star)(0) = 0$, and hence $g^\star(0) = f(0) = -\frac{\mu^2}{\sigma^2} < 0$ by (3.1) and the assumption that $\mu \neq 0$. Now we consider two cases depending on the shape of the quadratic defined by f: either $f(x) < 0$ for all $x \leq 0$, or there exists a negative root x_- of f, so that $f(x_-) = 0$. If such a negative root is not unique, we take x_- to be the larger one, i.e., the one closer to 0.

In the first case where $f(x) < 0$ for all $x \leq 0$, define the (deterministic) time

$$\tau = \inf\{t \in [0, T] : g^\star(t) \geq 0\}.$$

As $g^\star(0) < 0$, we have $\tau > 0$. If $\tau \leq T$, then $g^\star(\tau) = 0$ by continuity of $g^\star$, and we also have $g^\star < 0$ on $[0, \tau)$. As $k \geq 0$, we obtain $(k * g^\star)(\tau) \leq 0$, and hence

$$0 = g^\star(\tau) = f\big((k * g^\star)(\tau)\big) < 0,$$

which leads to a contradiction. Thus $\tau = \infty$, so that $g^\star$ takes strictly negative values on $[0, T]$, as claimed.

In the second case, let x_- be the negative root. If $1 - 2\varrho^2 \geq 0$, then f is convex by the definition (3.2), and it is clear that x_- is the unique negative root in this case as $f(0) = -\frac{\mu^2}{\sigma^2} < 0$. We thus have checked the conditions in Gatheral/Keller-Ressel [56, Assumption A.1], where we set $a \equiv 0$, $\kappa = k$, $H = f$ and $g = g^\star$ in their notation, and let $w_* = x_-$ and $w_{\max} = 0$. We also have that $a \equiv 0 = w_{\max}$ is nondecreasing, and the kernel k is nonnegative, nondecreasing and continuous on $[0, T]$ by assumption. Thus by applying [56, Corollary A.7(a')] to f and $g^\star$, we get that $g^\star(t) < 0$ for $t \in [0, T]$.

If instead $1 - 2\varrho^2 < 0$ so that f is concave, we define the new functions

$\bar{g}(t) = -g^\star(t)$ and $\bar{f}(x) = -f(-x)$. Plugging in yields that $\bar{g}$ satisfies the equation

$$\bar{g} = -g^\star = -f(\hat{\kappa} * g^\star) = \bar{f}(\hat{\kappa} * \bar{g})$$

on $[0, T]$. In this case, $\bar{f}$ is a convex quadratic function with $\bar{f}(0) = \frac{\mu^2}{\sigma^2} > 0$ and a positive root $-x_-$. By convexity, it follows that any other root of $\bar{f}$ must also be positive, since $\bar{f}$ is decreasing on $(-\infty, 0]$. Therefore, [56, Corollary A.7(c)] applies to $\bar{f}$ and $\bar{g}$ with $a \equiv 0$ and $w_* = -x_- > 0$, so that $\bar{g}(t) > 0$ for $t \in [0, T]$. Returning to the original function, we get $g^\star(t) = -\bar{g}(t) < 0$ for $t \in [0, T]$, as claimed.

□

Next, we show a result that is used in Chapter I to justify the interpretation of the parameter θ as the long-term volatility in the rough Heston model; see Lemma I.2.12. By definition, for $k \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$ we say that $R^k \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$ is the *resolvent of the second kind* of k if

$$(k * R^k)(t) = k(t) - R^k(t) \quad \text{for a.a. } t \geq 0. \tag{3.3}$$

By [59, Theorem 2.3.1], such a resolvent of the second kind R^k exists for each $k \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$.

Lemma 3.2. *Let $k \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$ be a nonnegative and nonincreasing kernel such that $\int_0^\infty k(t)dt = \infty$, and suppose that the resolvent of the second kind R^k is integrable, i.e., $\int_0^\infty |R^k(t)|dt < \infty$. Then it holds that $\int_0^\infty R^k(t)dt = 1$. In particular, R^k is integrable if it is nonnegative.*

Proof. Since k is nonnegative, nonincreasing and $k \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$, we have

$$\tilde{k}(\rho) := \int_0^\infty e^{-\rho t} k(t)dt \leq \int_0^1 k(t)dt + k(1) \int_0^\infty e^{-\rho t}dt < \infty$$

for each $\rho > 0$, i.e., the (one-sided) Laplace transform $\tilde{k}$ of k is well defined on $(0, \infty)$. Moreover, the assumption $\int_0^\infty k(t)dt = \infty$ implies that $\lim_{\rho \searrow 0} \tilde{k}(\rho) = \infty$ by the monotone convergence theorem. If R^k is integrable, then its (one-sided) Laplace transform $\tilde{R}^k$ is well defined and continuous on $\mathbb{R}_+ + \mathrm{i}\mathbb{R}$. As given in a remark after Gripenberg et al. [59, Theorem 2.3.5], the Laplace transform $\tilde{R}^k$ is given in terms of $\tilde{k}$ by

$$\tilde{R}^k(\rho) := \int_0^\infty e^{-\rho t} R^k(t)dt = \frac{\tilde{k}(\rho)}{1 + \tilde{k}(\rho)}, \quad \rho > 0,$$

as both sides are well defined for $\rho > 0$. Since $\tilde{R}^k$ is continuous on $\mathbb{R}_+$, taking the limit as $\rho \searrow 0$ yields

$$\int_0^\infty R^k(t)dt = \tilde{R}^k(0) = \lim_{\rho \searrow 0} \frac{\tilde{k}(\rho)}{1+\tilde{k}(\rho)} = 1,$$

as claimed.

Assume now that R^k is nonnegative, and for a contradiction suppose that $\int_0^\infty R^k(t)dt = \infty$. Thus, there exists some $T_0 > 0$ such that $\int_0^T R^k(t)dt \geq 2$ for all $T \geq T_0$. Recall from (3.3) that we have

$$R^k(t) = k(t) - (k * R^k)(t), \quad t \geq 0.$$

Since R^k is nonnegative by assumption, this yields the inequality

$$(k * R^k)(t) = k(t) - R^k(t) \leq k(t), \quad t \geq 0. \tag{3.4}$$

By the assumption on R^k and as k is nonincreasing, we can bound

$$(k * R^k)(t) = \int_0^t k(t-s)R^k(s)ds \geq \int_0^t k(t)R^k(s)ds \geq 2k(t), \quad t \geq T_0.$$

Since k is locally integrable, nonnegative, nonincreasing and $\int_0^\infty k(t)dt = \infty$, we must have $k(t) > 0$ for all $t \geq 0$. But then we have

$$(k * R^k)(t) \geq 2k(t) > k(t)$$

for $t \geq T_0$, which contradicts (3.4). Thus the assumption that $\int_0^\infty R^k(t)dt = \infty$ cannot hold, which shows that R^k is integrable. □

Finally, we show that the resolvent of the second kind R^k inherits the continuity property in Assumption I.2.7.1) (which we repeat below) from k. This is a simple consequence of Lemma 1.5 together with some auxiliary results shown in the proof of that lemma.

Lemma 3.3. *Let $p \geq 1$, $\gamma > 0$ and $k \in L^p_{\text{loc}}([0,\infty;\mathbb{R}))$.*

1) *If $\int_0^h |k(t)|^p dt = O(h^\gamma)$ for small $h > 0$, then $\int_0^h |R^k(t)|^p dt = O(h^\gamma)$ for small $h > 0$.*

2) *If 1) holds and additionally $\int_0^T |k(t+h) - k(t)|^p dt = O(h^\gamma)$ for each $T > 0$ and small $h > 0$, then $\int_0^T |R^k(t+h) - R^k(t)|^p dt = O(h^\gamma)$ for each $T > 0$*

and small $h > 0$.

3) *If $\int_t^{t+h} |k(t)|^p dt = O(h^\gamma)$ for small $h > 0$ uniformly in $t \in [0,T]$, then $\int_t^{t+h} |R^k(t)|^p dt = O(h^\gamma)$ for small $h > 0$ uniformly in $t \in [0,T]$.*

Proof. **1)** By Gripenberg et al. [59, Theorem 2.3.1], the resolvent of the second kind R^k for k exists and belongs to $L^1_{\mathrm{loc}}([0,\infty);\mathbb{R})$. Fix some $T > 0$ and let $C := \|R^k\|_{L^1(0,T)} < \infty$. By the definition (3.3) of R^k, we have $R^k = k - k * R^k$. Thus by applying Lemma 1.5 with $q = 1$ and $r = p$, we obtain

$$\|R^k\|_{L^p(0,h)} \leq \|k\|_{L^p(0,h)} + \|k\|_{L^p(0,h)}\|R^k\|_{L^1(0,h)} \leq (C+1)\|k\|_{L^p(0,h)}$$

for small $h > 0$ so that the assumption yields

$$\|R^k\|^p_{L^p(0,h)} \leq (C+1)^p\|k\|^p_{L^p(0,h)} = O(h^\gamma).$$

2) Fix some $T > 0$ and define $C := \|R^k\|_{L^1(0,T+1)} < \infty$ and $z := k * R^k$. By (1.8) and (1.9) with $y := R^k$ and $\delta := h$, we obtain the bound

$$|z(t+h) - z(t)| \leq \big(|k\mathbf{1}_{[0,h]}| * |R^k|\big)(t+h) + \big((\Delta_h k - k) * R^k\big)(t) \tag{3.5}$$

for $t \geq 0$ and $h > 0$, where we define $\Delta_h f \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{R})$ by $\Delta_h f(t) = f(t+h)$ for $f \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{R})$. Then as in the first two lines of (1.10), integrating (3.5) yields by Young's convolution inequality (1.2) with $q = 1$ that

$$\begin{aligned}\|\Delta_h z - z\|_{L^p(0,T)} &\leq \||k\mathbf{1}_{[0,h]}| * |R^k|\|_{L^p(0,T+h)} + \|(\Delta_h k - k) * R^k\|_{L^p(0,T)} \\ &\leq \|k\mathbf{1}_{[0,h]}\|_{L^p(0,T+h)}\|R^k\|_{L^1(0,T+h)} + \|\Delta_h k - k\|_{L^p(0,T)}\|R^k\|_{L^1(0,T)} \\ &\leq C(\|k\|_{L^p(0,h)} + \|\Delta_h k - k\|_{L^p(0,T)}) \end{aligned} \tag{3.6}$$

for $h \in (0,1)$. Once again by the definition (3.3), we have

$$\Delta_h R^k(t) - R^k(t) = \Delta_h k(t) - k(t) + \Delta_h z(t) - z(t)$$

for $t > 0$ and $h > 0$. Integrating on $[0,T]$ and plugging in (3.6) yields

$$\begin{aligned}\|\Delta_h R^k - R^k\|^p_{L^p(0,T)} &\leq (\|\Delta_h k - k\|_{L^p(0,T)} + \|\Delta_h z - z\|_{L^p(0,T)})^p \\ &\leq (C+1)^p(\|k\|_{L^p(0,h)} + \|\Delta_h k - k\|_{L^p(0,T)})^p \\ &\leq 2^{p-1}(C+1)^p(\|k\|^p_{L^p(0,h)} + \|\Delta_h k - k\|^p_{L^p(0,T)}) = O(h^\gamma)\end{aligned}$$

for $h > 0$ by the assumption on k; this concludes the proof of 2).

3) Once again, set $C := \|R^k\|_{L^1(0,T+1)} < \infty$. We claim that we have the bound

$$\|k * R^k\|_{L^p(t,t+h)} \leq C\tilde{n}_k(h), \tag{3.7}$$

where $\tilde{n}_k(h) := \sup\{\|k\|_{L^p(t,t+h)} : t \in [0,T]\}$. Indeed, (3.7) essentially follows by (1.5) with $y = R^k$, $q = 1$, $r = p$, $a = 0$ and $E = [t, t+h]$. More precisely, note that $\tilde{n}_k$ is smaller than the n_k defined in (1.6), since we only take the supremum over intervals rather than all sets with Lebesgue measure h. However, (3.7) still follows as in (1.5) because E is an interval, and hence the inner integral in the second line of (1.5) is taken over an interval in this case. This shows (3.7) so that

$$\|k * R^k\|^p_{L^p(t,t+h)} \leq C\tilde{n}^p_k(h) \leq C\tilde{C}h^\gamma$$

for some $\tilde{C} > 0$, small $h > 0$ and all $t \in [0,T]$ by the assumption. Thus by (3.3) and the assumption, we get

$$\|R^k\|^p_{L^p(t,t+h)} \leq (\|k\|_{L^p(t,t+h)} + \|k * R^k\|_{L^p(t,t+h)})^p \leq 2^{p-1}(C+1)^p\tilde{C}^p h^\gamma$$

for small $h > 0$ and all $t \in [0,T]$; this concludes the proof. □

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