

Right-angled Artin groups as finite-index subgroups of their outer automorphism groups

Journal Article**Author(s):**

Wiedmer, Manuel

Publication date:

2024-03

Permanent link:

<https://doi.org/10.3929/ethz-b-000650324>

Rights / license:

[Creative Commons Attribution 4.0 International](#)

Originally published in:

Bulletin of the London Mathematical Society 56(3), <https://doi.org/10.1112/blms.12975>

RESEARCH ARTICLE

Right-angled Artin groups as finite-index subgroups of their outer automorphism groups

Manuel WiedmerDepartment of Mathematics, ETH Zurich,
Zurich, Switzerland**Correspondence**Manuel Wiedmer, Department of
Mathematics, ETH Zurich, Zurich 8092,
Switzerland.Email: manuel.wiedmer@inf.ethz.ch**Present address**Manuel Wiedmer, Institute for
Theoretical Computer Science,
Department of Computer Science, ETH
Zurich, Zurich, Switzerland**Abstract**

We prove that every right-angled Artin group occurs as a finite-index subgroup of the outer automorphism group of another right-angled Artin group. We furthermore show that the latter group can be chosen in such a way that the quotient is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^N$ for some N . For these, we give explicit constructions using the group of pure symmetric outer automorphisms. Moreover, we need two conditions by Day–Wade and Wade–Brück about when this group is a right-angled Artin group and when it has finite index.

MSC 2020

20E36, 20F36 (primary)

1 | INTRODUCTION

Right-angled Artin groups are defined by a presentation using a graph Γ and denoted by A_Γ . They were first introduced by Andreas Baudisch in his paper [2] under the name of ‘semi-free groups’. In recent years, they have been used, among other things, for combinatorial approaches to geometric and topological problems; see, for example, [1, 3]. Right-angled Artin groups can be seen as interpolating between free groups and free abelian groups. Similarly, their outer automorphism groups $\text{Out}(A_\Gamma)$ may be viewed as interpolating between $\text{GL}_n(\mathbb{Z})$ and $\text{Out}(F_n)$. For examples of how this has been used, one can consider [4, 6, 9, 10, 11].

Our aim is to study a topic introduced by Matthew B. Day and Richard D. Wade in [8], which is the main motivation for this paper. We want to understand finite-index subgroups of $\text{Out}(A_\Gamma)$ that are right-angled Artin groups. Day–Wade ask when the group $\text{Out}(A_\Gamma)$ contains a right-angled Artin group as finite-index subgroup; see [8, Question 1.1]. To partially answer this question,

Day–Wade consider the subgroup of so-called pure symmetric outer automorphisms, denoted by $\text{PSO}(A_\Gamma) \leq \text{Out}(A_\Gamma)$. They give a condition on when it is a right-angled Artin group and describe explicitly which one it is in case this condition is satisfied.

We look at the question by Day–Wade from another perspective. Day–Wade fix the graph Γ and ask if $\text{Out}(A_\Gamma)$ has another right-angled Artin group A_Λ as finite-index subgroup. In contrast, we fix the graph Λ and obtain the following main result.

Theorem A. *For any graph Λ , the right-angled Artin group A_Λ is a finite-index subgroup of the outer automorphism group of some other right-angled Artin group A_Γ .*

This theorem is later stated in more detail as Theorem 3.1. The question for which graphs Λ we can find such a graph Γ is already mentioned in [15, Question 3.4]. In that paper, Richard D. Wade and Benjamin Brück study a different topic related to the outer automorphism group of right-angled Artin groups. In particular, they give a condition for when the group of pure symmetric outer automorphisms has finite index in the group of outer automorphisms. We use this condition by Wade–Brück and the one by Day–Wade mentioned above to develop Theorem A.

To prove this theorem, in Section 2, we introduce the background needed for our main result: Among other things, we discuss a set of generators for $\text{Aut}(A_\Gamma)$, the subgroup $\text{PSO}(A_\Gamma) \leq \text{Out}(A_\Gamma)$ and the two conditions by Wade–Brück and Day–Wade mentioned above. We then show the main result in Section 3. For a given graph Λ , we construct a graph Γ such that A_Λ is a finite-index subgroup of $\text{Out}(A_\Gamma)$. More precisely, we show that $A_\Lambda \cong \text{PSO}(A_\Gamma)$ and that $\text{PSO}(A_\Gamma)$ has finite index in $\text{Out}(A_\Gamma)$ using the conditions by Day–Wade and Wade–Brück. Later, in that section, we impose an additional condition on the graph Γ . Namely, it should have no non-trivial graph automorphisms. We show that Theorem A still holds with this additional condition by developing the construction further. This leads to an interesting corollary about the structure of the quotient $\text{Out}(A_\Gamma)/\text{PSO}(A_\Gamma)$ and lets us determine the index of A_Λ in $\text{Out}(A_\Gamma)$. Finally, we conclude this paper by discussing further questions that could be interesting to study in Section 4. In the Appendix, we cover some special cases of small graphs for which the constructions of Section 3 do not work.

This paper is a shortened version of the author’s Master Thesis at ETH Zurich, which can be found as [17]. I would like to express my sincere thanks to the supervisor of this thesis, Prof. Dr. Alessandra Iozzi, for making it possible and to the co-supervisor, Dr. Benjamin Brück, for suggesting this topic and for all his support during this project. He always helped me when I had trouble, answered all questions that came up and was open for many helpful discussions, during the thesis but also while changing it into this paper. Furthermore, I wish to thank the anonymous reviewer for the thorough reading of this text as well as his helpful comments and suggestions for the improvement of it.

2 | PRELIMINARIES

2.1 | Graphs

Let Γ be a graph. We write $V(\Gamma)$ for the set of vertices of Γ and $E(\Gamma)$ for the set of edges of Γ , which is a set of unordered pairs of different vertices. In particular, all the graphs we consider are undirected and do not contain loops or multiple edges. Moreover, we only consider finite graphs, that is, $V(\Gamma)$ is always a finite set. We write $v \sim w$ if v is adjacent to w . For $S \subseteq V(\Gamma)$, we use the

notation $\Gamma - S$ for the induced subgraph with vertex set $V(\Gamma) \setminus S$. When we talk of a component of a graph, we always mean a connected component. Furthermore, we use $\text{lk}(v)$ for the set of neighbours of v and $\text{st}(v)$ for the union of $\text{lk}(v)$ with v itself.

2.2 | Right-angled Artin groups

Definition 2.1. For a given non-empty graph Γ , the right-angled Artin group A_Γ has the following presentation:

$$A_\Gamma := \langle V(\Gamma) \mid [v, w] = 1 \text{ for } \{v, w\} \in E(\Gamma) \rangle.$$

Recall that $[v, w] := v w v^{-1} w^{-1}$ is a notation for the commutator of v and w . So, the generators of A_Γ correspond to the vertices of Γ and two generators commute if there is an edge between the two corresponding vertices in the graph Γ .

Next, we give a set of generators for the automorphism group of a right-angled Artin group A_Γ . This is based on [14, subsection 2.5]. We refer the reader to this source for more details. For a proof of the statement, one can look at [13] and [12], which are the original sources of this theorem. Servatius conjectured it and proved it for some special cases and Laurence gave a proof for general graphs. Note that the terminology in [13] and [12] is slightly different to the one we use here.

For simplicity, we assume that the graph Γ consists of vertices v_1, \dots, v_n . There are four types of automorphisms that together generate $\text{Aut}(A_\Gamma)$.

- *Γ -legal transvections:* For vertices v_i and v_j that satisfy the condition $\text{lk}(v_i) \subseteq \text{st}(v_j)$, we define the automorphism $T_{i,j}^l$ of A_Γ by mapping the generators as follows: $v_i \mapsto v_j v_i$ and $v_k \mapsto v_k$ for $k \neq i$. These are called Γ -legal (left) transvections. Analogously, again, for v_i and v_j with $\text{lk}(v_i) \subseteq \text{st}(v_j)$, one can define the Γ -legal right transvection $T_{i,j}^r$ by mapping $v_i \mapsto v_i v_j$ and $v_k \mapsto v_k$ for $k \neq i$.
- *Partial conjugations:* Another type of generators are the so-called (Γ -legal) partial conjugations. Here, we often omit the term Γ -legal as we do not introduce other partial conjugations. For a vertex $v_j \in V(\Gamma)$ and a component A of $\Gamma - \text{st}(v_j)$, we define the partial conjugation P_j^A by $v_i \mapsto v_j v_i v_j^{-1}$ for $v_i \in A$ and $v_k \mapsto v_k$ for $v_k \notin A$.
- *Γ -legal permutations:* In order to get another automorphism of A_Γ , we can permute the generators. However, not all permutations give automorphisms, but only those that correspond to automorphisms of the graph Γ . We call these Γ -legal permutations. Due to the fact that these automorphisms of A_Γ are related to graph automorphisms of Γ , Γ -legal permutations are often also called ‘graph automorphisms’.
- *Inversions:* For $j \in \{1, \dots, n\}$, we define the inversion I_j as follows: $v_j \mapsto v_j^{-1}$ and $v_k \mapsto v_k$ for $k \neq j$.

Remark. In a general setting, we write T_{v_i, v_j}^l instead of $T_{i,j}^l$. The same applies to the notation for the other types of generators.

We use the notation from above also for the images of the generators in $\text{Out}(A_\Gamma)$. These elements then generate $\text{Out}(A_\Gamma)$.

2.3 | Pure symmetric outer automorphisms

The following is based on [8, Section 2.2]. An automorphism of A_Γ is called pure symmetric if every generator v_i is mapped to a conjugate of itself. Note that the conjugating element may depend on v_i . The set of such elements forms a subgroup of $\text{Aut}(A_\Gamma)$ as the composition of two pure symmetric automorphisms is again pure symmetric. This subgroup is called the group of pure symmetric automorphisms and denoted by $\text{PSA}(A_\Gamma)$. A generating set for it is the set of partial conjugations; see [8, Theorem 2.5]. We define $\text{PSO}(A_\Gamma)$, the group of pure symmetric outer automorphisms, as the image of $\text{PSA}(A_\Gamma)$ in $\text{Out}(A_\Gamma)$.

Our goal is to first describe conditions for when $\text{PSO}(A_\Gamma)$ has finite-index in $\text{Out}(A_\Gamma)$ and for when it is a right-angled Artin group. Then, we show in Section 3 that every right-angled Artin group A_Λ occurs as $\text{PSO}(A_\Gamma)$ for some graph Γ for which $\text{PSO}(A_\Gamma)$ has finite index in $\text{Out}(A_\Gamma)$.

2.3.1 | Finite-index condition for $\text{PSO}(A_\Gamma)$

The following theorem is based on [15, Appendix A].

Theorem 2.2. *Let Γ be a graph. Then the condition*

$$\forall v, w \in V(\Gamma) : \text{lk}(v) \subseteq \text{st}(w) \implies v = w \quad (1)$$

is equivalent to the group $\text{PSO}(A_\Gamma)$ having finite index in $\text{Out}(A_\Gamma)$.

Note that condition (1) is equivalent to the group A_Γ having no Γ -legal transvections. The idea for the proof that condition (1) implies finite index is to show that

$$\text{Out}(A_\Gamma) / \text{PSO}(A_\Gamma) \cong \text{Inv} \rtimes \text{Per},$$

where Inv and Per denote the subgroups of $\text{Out}(A_\Gamma)$ generated by the inversions, respectively, by the Γ -legal permutations. For the other direction, one can show that the existence of a Γ -legal transvection implies that the quotient has infinitely many elements.

2.3.2 | Condition for when $\text{PSO}(A_\Gamma)$ is a right-angled Artin group

In this subsection, we state a condition about when $\text{PSO}(A_\Gamma)$ is a right-angled Artin group. This condition was developed and proved by Day and Wade in [8]. This is also the source for this subsection, in particular, [8, Chapters 2 and 5]. We first need the following two definitions.

Definition 2.3. Let Γ be a graph and $v \neq w \in V(\Gamma)$. The pair (v, w) is called a separating intersection of links if v is not adjacent to w , that is, $\{v, w\} \notin E(\Gamma)$, and $\Gamma - (\text{lk}(v) \cap \text{lk}(w))$ has a component that contains neither v nor w .

We often use the formulation ‘ (v, w) is an SIL-pair’ for a pair (v, w) that is a separating intersection of links.

Definition 2.4. Let Γ be a graph and $v \in V(\Gamma)$ be a vertex of Γ . We define the support graph S_v^Γ as follows. For every component C of $\Gamma - st(v)$, there is a vertex in S_v^Γ . Two vertices A and B in S_v^Γ are connected by an edge if there is a vertex $b \in B$ such that A is also a component of $\Gamma - st(b)$.

Remark. Note that the definition above is not symmetric. That is, for two components A and B of $\Gamma - st(v)$, the fact that there is a $b \in B$ such that A is also a component of $\Gamma - st(b)$ is not equivalent to the fact that there is an $a \in A$ such that B is also a component of $\Gamma - st(a)$. There is an edge between A and B in the support graph if any of these two conditions holds.

Remark. As in Definition 2.4, we often use the same symbol for the vertices in the support graph S_v^Γ and the components of $\Gamma - st(v)$, even though these are not the same.

For $b \in \Gamma - st(v)$, we often write the component of $\Gamma - st(v)$ that contains b as $[b]_v$. So, equivalently, there is an edge between two vertices A and B in the support graph S_v^Γ if there is a $b \in \Gamma - st(v)$ such that $B = [b]_v$ and A is also a component of $\Gamma - st(b)$.

We now define the following graph Θ (depending on Γ). This graph has two types of vertices. We call them vertices of types I and II. More precisely, for every vertex $v \in V(\Gamma)$, we have the following vertices.

- Vertices of type I: For every edge e in S_v^Γ , we have a vertex α_e^v .
- Vertices of type II: We also have a vertex for every component of S_v^Γ except one, that is, we have vertices $\beta_1^v, \dots, \beta_{\min(N(v)-1,0)}^v$, where $N(v)$ is the number of components of S_v^Γ . We take the minimum with 0 to avoid the special case $N(v) = 0$. This happens if $\Gamma - st(v)$ has no vertices and thus also S_v^Γ has no vertices.

Concerning the edges, vertices of type II are connected to all other vertices and vertices α_e^v and α_f^w of type I are connected except when (v, w) is an SIL-pair and the edges are of the form $e = \{[w]_v, L\}$ and $f = \{[v]_w, L\}$, where L is a component of both $\Gamma - st(v)$ and $\Gamma - st(w)$.

We now state the theorem about when $PSO(A_\Gamma)$ is a right-angled Artin group.

Theorem 2.5 [8, Theorem 5.12]. *Let Γ be a graph. Then $PSO(A_\Gamma)$ is a right-angled Artin group if and only if all support graphs of Γ are forests. In this case, $PSO(A_\Gamma) \cong A_\Theta$ for Θ as defined above.*

Remark. Note that in this theorem, in contrast to Definition 2.1, we also treat the group $\{id\} = A_\emptyset$ as a right-angled Artin group, where \emptyset denotes the graph with no vertices. Namely, Θ is the empty graph if all support graphs have at most one vertex. This happens, for example, when Γ is a complete graph. Note that it makes sense that $PSO(A_\Gamma)$ is the trivial group in this case since when Γ is complete, then A_Γ is $\mathbb{Z}^{|V(\Gamma)|}$. Thus, conjugation by any element does nothing, so there are no pure symmetric outer automorphisms except the identity.

In [8, Chapter 5], one can find an explicit construction of the isomorphism between the right-angled Artin group A_Θ and the group $PSO(A_\Gamma)$ in case all support graphs of Γ are forests. They define the vertices of type II more precisely in [8, Definition 5.4], describe the needed generators of $PSO(A_\Gamma)$ in [8, Section 5.1] and give the correspondence between the generators of A_Θ and the generators of $PSO(A_\Gamma)$ in [8, Proposition 5.5].

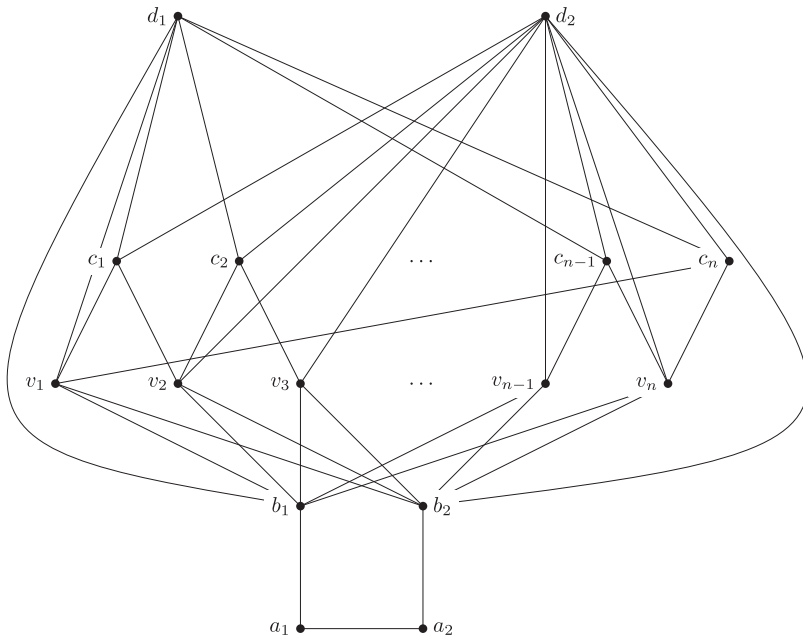


FIGURE 1 Construction of Γ .

3 | RIGHT-ANGLED ARTIN GROUPS AS FINITE-INDEX SUBGROUPS OF $\text{Out}(A_\Gamma)$

In this section, we use Theorems 2.2 and 2.5 to show that every right-angled Artin group is a finite-index subgroup of the outer automorphism group of some other right-angled Artin group. This is Theorem A, which we here restate as Theorem 3.1.

Theorem 3.1. *For any graph Λ , there is a graph $\Gamma = \Gamma(\Lambda)$ such that*

1. $A_\Lambda \cong \text{PSO}(A_\Gamma)$ and
2. $\text{PSO}(A_\Gamma)$ has finite index in $\text{Out}(A_\Gamma)$.

Remark. We first want to comment on how this theorem and also Theorem 3.4 below were developed. We used computer programs that can be found on [16]. Using these, we found out that for any graph with at most four vertices, the corresponding right-angled Artin group occurs as $\text{PSO}(A_\Gamma)$ for some graph Γ . We then generalised these examples step by step until we arrived at the constructions given below. These computer programs are based on and contain parts of programs written by Benjamin Brück, which can be found on [5] and were used for [15].

We assume without loss of generality that $V(\Lambda) = \{v_1, \dots, v_n\}$. We also assume $n \geq 3$. The other cases are covered in the Appendix. For a given graph Λ , we define the graph $\Gamma = \Gamma(\Lambda)$ as follows. We obtain Γ from Λ by adding vertices $a_1, a_2, b_1, b_2, c_1, \dots, c_n, d_1$ and d_2 and the edges depicted in Figure 1: The vertex d_1 is connected to b_1 , to all vertices c_i and to the vertex v_1 , but not to the vertices v_j for $j > 1$. The vertex d_2 is connected to b_2 , all c_i and all v_j except v_1 . The vertices c_i are connected to d_1, d_2 and the vertices v_i and v_{i+1} , where $v_{n+1} := v_1$. In addition to the already

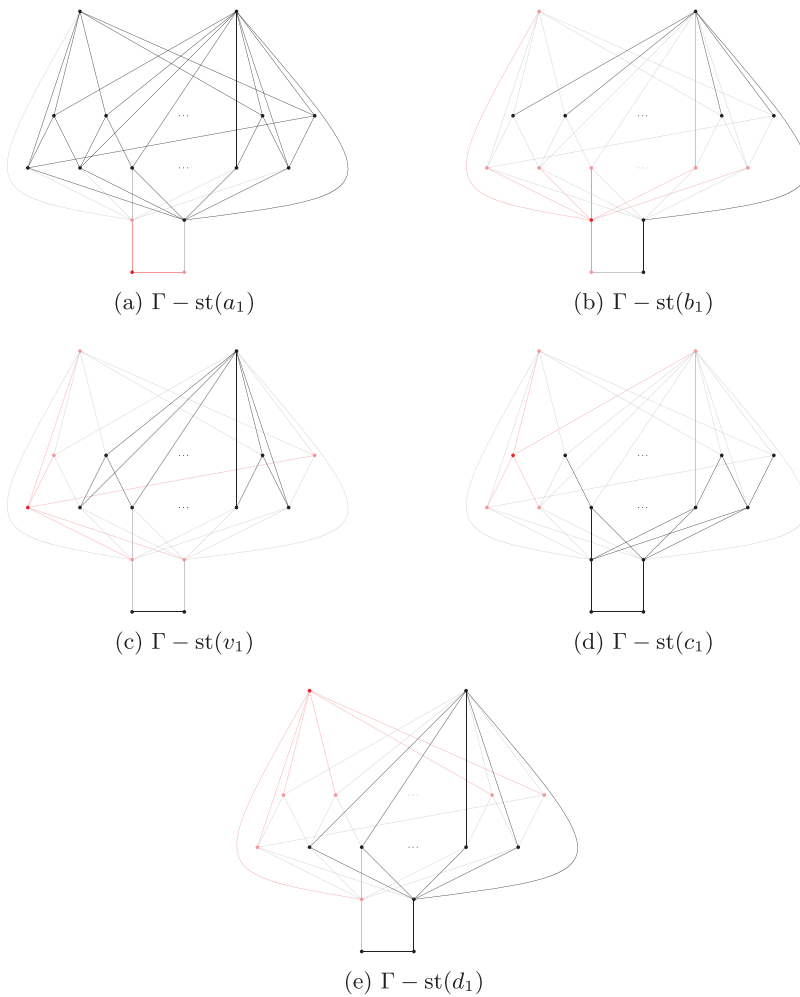


FIGURE 2 $\Gamma - st(w)$ for some important vertices $w \in V(\Gamma)$.

defined edges, the vertices v_j are connected to b_1 and b_2 . Furthermore, we have the edges $\{b_1, a_1\}$, $\{b_2, a_2\}$ and $\{a_1, a_2\}$. Finally, if there are edges in Λ , these are also present in Γ . But as we work with an arbitrary graph Λ , we did not draw them in Figure 1.

We want to prove the following two lemmas, which together imply Theorem 3.1.

Lemma 3.2. *For this graph $\Gamma = \Gamma(\Lambda)$, we have that $PSO(A_\Gamma)$ is isomorphic to A_Λ .*

Lemma 3.3. *For this graph $\Gamma = \Gamma(\Lambda)$, $PSO(A_\Gamma)$ has finite index in $Out(A_\Gamma)$.*

Proof of Lemma 3.2. We want to use Theorem 2.5. So, we first need to show that every support graph is a forest. To compute the support graphs, we need to consider $\Gamma - st(w)$ for all vertices w of Γ . Some of these are shown in Figure 2. The vertex w is coloured in red, the rest of $st(w)$ is coloured in light red and all other deleted edges are coloured in grey. Note that for Figure 2c, it might be that also some of the vertices $\{v_2, \dots, v_n\}$ should be coloured light red, namely all vertices

in $st_\Lambda(v_1)$. For the vertices that we did not consider in Figure 2, $\Gamma - st(w)$ is analogous to one of the vertices we considered there.

We can see that for all vertices w that are not in Λ , $\Gamma - st(w)$ is connected, and thus, the support graph S_w^Γ consists of one vertex. Here, we used that $n \geq 3$, otherwise $\Gamma - st(c_1)$, respectively, $\Gamma - st(c_2)$ have two components as c_2 , respectively, c_1 become isolated. For the vertices v_i , the graph $\Gamma - st(v_i)$ has two components, one of which is $\{a_1, a_2\}$. The second component contains all other vertices that are not in the star of v_i . Using the definition of the support graph, this means that $S_{v_i}^\Gamma$ has two vertices. Whether there is an edge between them depends on whether $V(\Lambda) \setminus st_\Lambda(v_i)$ is empty or not. If $V(\Lambda) \setminus st_\Lambda(v_i) \neq \emptyset$, then there is a $v_j \in V(\Lambda) \setminus st_\Lambda(v_i)$ and $\{a_1, a_2\}$ is also a component of $\Gamma - st(v_j)$. Thus, by Definition 2.4, in this case, there is an edge in the support graph. However, if $V(\Lambda) \setminus st_\Lambda(v_i) = \emptyset$, we cannot satisfy the condition of Definition 2.4, so the support graph has no edge.

In summary, we see that all support graphs are forests, and thus, $PSO(A_\Gamma)$ is a right-angled Artin group by Theorem 2.5. As in this theorem, we call the underlying graph for this right-angled Artin group Θ . Furthermore, we know from the structure of the support graphs that there is exactly one vertex u_i in the graph Θ for every vertex v_i in the graph Γ . Depending on whether $V(\Lambda) \setminus st_\Lambda(v_i) = \emptyset$ or not, it is a vertex of type II or I. If $V(\Lambda) \setminus st_\Lambda(v_i) = \emptyset$, then

$$u_i := \beta_1^{v_i}.$$

If $V(\Lambda) \setminus st_\Lambda(v_i) \neq \emptyset$, then

$$u_i := \alpha_{e_i}^{v_i},$$

where e_i is the edge in the support graph $S_{v_i}^\Gamma$.

We claim that we have $v_i \sim v_j$ if and only if $u_i \sim u_j$, which shows that Λ is isomorphic to Θ . In order to show this claim, we distinguish whether v_i is adjacent to v_j or not.

$v_i \sim v_j$: If $V(\Lambda) \setminus st_\Lambda(v_i) = \emptyset$ or $V(\Lambda) \setminus st_\Lambda(v_j) = \emptyset$, then $u_i \sim u_j$ as u_i , respectively, u_j is of type II and thus connected to all other vertices. Otherwise, both are of type I, but (v_i, v_j) is not an SIL-pair as they are connected. Thus, we have $u_i \sim u_j$ as well.

$v_i \not\sim v_j$: Here, both $V(\Lambda) \setminus st_\Lambda(v_i)$ and $V(\Lambda) \setminus st_\Lambda(v_j)$ are non-empty. Thus, both u_i and u_j are of type I. More precisely,

$$u_i = \alpha_{e_i}^{v_i} \text{ and } u_j = \alpha_{e_j}^{v_j},$$

where e_i and e_j are again the corresponding edges in the support graphs $S_{v_i}^\Gamma$ and $S_{v_j}^\Gamma$, respectively.

Also, $\Gamma - (\text{lk}(v_i) \cap \text{lk}(v_j))$ has $\{a_1, a_2\}$ as a component because both b_1 and b_2 are in $\text{lk}(v_i) \cap \text{lk}(v_j)$. As $v_i \not\sim v_j$, we get that (v_i, v_j) is an SIL-pair. Furthermore, the edges in the support graphs of v_i and v_j are of the form $([v_j]_{v_i}, \{a_1, a_2\})$ and $([v_i]_{v_j}, \{a_1, a_2\})$. Hence, there is no edge between u_i and u_j .

Thus, we indeed have that $\Lambda \cong \Theta$. We can conclude that

$$A_\Lambda \cong A_\Theta \cong PSO(A_\Gamma),$$

which is what we wanted to show. □

TABLE 1 Determining when $\text{lk}(u)$ is a subset of $\text{st}(w)$.

	$\text{st}(a_1)$	$\text{st}(a_2)$	$\text{st}(b_1)$	$\text{st}(b_2)$	$\text{st}(v_1)$	$\text{st}(v_l)$	$\text{st}(c_j)$	$\text{st}(d_1)$	$\text{st}(d_2)$
$\text{lk}(a_1)$	✓	b_1	a_2	b_1	a_2	a_2	a_2	a_2	a_2
$\text{lk}(a_2)$	b_2	✓	b_2	a_1	a_1	a_1	a_1	a_1	a_1
$\text{lk}(b_1)$	d_1	d_1	✓	d_1	a_1	a_1	a_1	a_1	a_1
$\text{lk}(b_2)$	d_2	d_2	d_2	✓	a_2	a_2	a_2	a_2	a_2
$\text{lk}(v_1)$	b_2	b_1	b_2	b_1	✓	d_1	b_1	b_2	b_1
$\text{lk}(v_k)$	b_2	b_1	b_2	b_1	d_2	*	b_1	b_2	b_1
$\text{lk}(c_i)$	d_1	d_1	d_2	d_1	d_2	d_1	*	d_2	d_1
$\text{lk}(d_1)$	c_1	c_1	c_1	c_1	c_2	*	b_1	✓	v_1
$\text{lk}(d_2)$	c_1	c_1	c_1	c_1	c_2	*	b_2	v_2	✓

We now prove the second lemma, which states that for this $\Gamma = \Gamma(\Lambda)$, we have that $\text{PSO}(A_\Gamma)$ has finite index in $\text{Out}(A_\Gamma)$.

Proof of Lemma 3.3. In order to prove this, we want to use Theorem 2.2, so we need to show that $\text{lk}(u) \subseteq \text{st}(w)$ implies $u = w$.

We do this with Table 1, which lists all pairs $(\text{lk}(u), \text{st}(w))$. If $\text{lk}(u) \subseteq \text{st}(w)$, we write a ‘✓’; otherwise, we provide an element in $\text{lk}(u) \setminus \text{st}(w)$. Regarding the indices, i and j go from 1 to n and k and l go from 2 to n . For the parts denoted with ‘*’, we need to distinguish what the indices are, which we do in the following. All these special cases need the assumption $n \geq 3$. The problem with $n = 2$ is that we only have two vertices c_i and both of them are connected to all vertices of Λ . We also need this assumption in the table, namely that $c_2 \neq c_n$ and thus, $c_2 \notin \text{st}(v_1)$.

- $\text{lk}(v_k) \subseteq \text{st}(v_l)$ is only true if $k = l$, otherwise there is a vertex of $\{c_1, \dots, c_n\}$ that is in $\text{lk}(v_k) \setminus \text{st}(v_l)$.
- $\text{lk}(c_i) \subseteq \text{st}(c_j)$ holds only if $i = j$, otherwise a vertex of $\{v_1, \dots, v_n\}$ is in $\text{lk}(c_i) \setminus \text{st}(c_j)$.
- $\text{lk}(d_1) \not\subseteq \text{st}(v_l)$ as c_1 or c_n is not in $\text{st}(v_l)$ since $l > 1$.
- $\text{lk}(d_2) \not\subseteq \text{st}(v_l)$ by the same reasoning as for $\text{lk}(d_1) \not\subseteq \text{st}(v_l)$.

With the table and the special cases, we conclude that $\text{lk}(u) \subseteq \text{st}(w)$ implies $u = w$. As discussed in the beginning, this concludes the proof. □

Next, we provide a better construction to strengthen Theorem 3.1.

Theorem 3.4. *For any graph Λ , there is a graph $\Gamma' = \Gamma'(\Lambda)$ such that*

1. $A_\Lambda \cong \text{PSO}(A_{\Gamma'})$,
2. $\text{PSO}(A_{\Gamma'})$ has finite index in $\text{Out}(A_{\Gamma'})$ and
3. Γ' has no non-trivial graph automorphisms.

We again assume $n \geq 3$; the other cases are covered in the Appendix. We get $\Gamma' = \Gamma'(\Lambda)$ from Λ by adding the vertices $a_1, a_2, a_3, b_1, b_2, b_3, c_1, \dots, c_n, d_1, d_2$ and d_3 and the edges that are shown in Figure 3: The vertices d_i are connected to all vertices c_j . Furthermore, d_1 is connected to v_1 and b_1 , d_2 to v_2 , d_3 to v_k for $k > 2$ and to b_3 and we have the edge $\{d_2, d_3\}$. Note that d_2 is not connected to b_2 . As for the construction of Γ for Theorem 3.1, c_j is connected to v_j and v_{j+1} ($v_{n+1} := v_1$), the vertices v_k are connected to b_1, b_2 and b_3 and we have

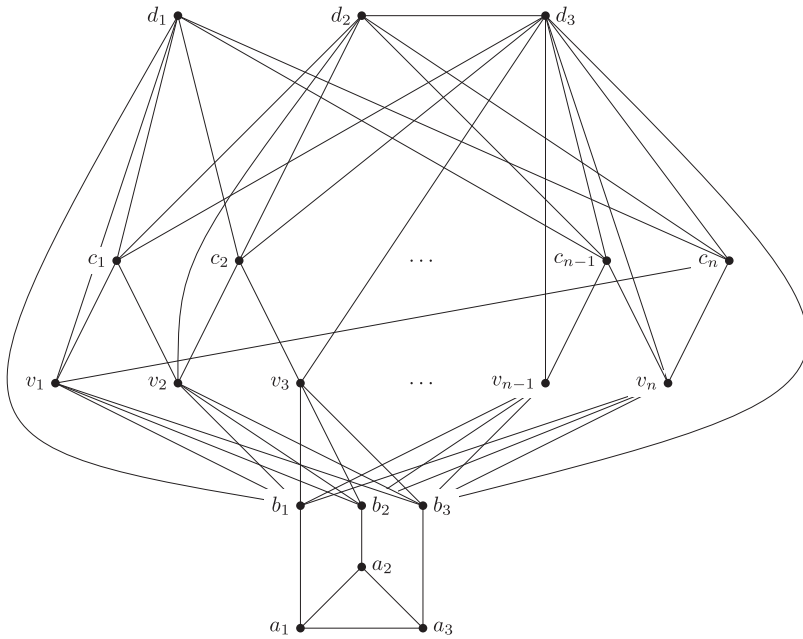


FIGURE 3 Construction of Γ' .

all edges that are present in Λ also in Γ' . Finally, we have the edges $\{b_1, a_1\}, \{b_2, a_2\}, \{b_3, a_3\}, \{a_1, a_2\}, \{a_1, a_3\}$ and $\{a_2, a_3\}$. The proof of the theorem now follows from the following three lemmas.

Lemma 3.5. *For this graph $\Gamma' = \Gamma'(\Lambda)$, we have that $\text{PSO}(A_{\Gamma'})$ is isomorphic to A_Λ .*

Lemma 3.6. *For this graph $\Gamma' = \Gamma'(\Lambda)$, $\text{PSO}(A_{\Gamma'})$ has finite index in $\text{Out}(A_{\Gamma'})$.*

Lemma 3.7. *For this graph $\Gamma' = \Gamma'(\Lambda)$, Γ' has no non-trivial graph automorphisms.*

Lemmas 3.5 and 3.6 can be shown as Lemmas 3.2 and 3.3 using Theorems 2.5 and 2.2. Thus, we only comment on the proof of Lemma 3.7.

Proof sketch of Lemma 3.7. To show this, one uses the fact that a graph automorphism sends adjacent vertices to adjacent vertices and, in particular, preserves the degree of every vertex. One can first show that the levels of Γ' are fixed, that is, the vertices a_i are mapped to vertices a_j by any automorphism, and so on. This is done from bottom to top, that is, starting with the a_i , then the b_j , and so on. Next, one can show that this implies that the vertices also need be fixed pointwise by any automorphism, which concludes the proof. \square

We finish this section with a corollary about the structure of the quotient $\text{Out}(A_{\Gamma'})/\text{PSO}(A_{\Gamma'})$ and compute the index of $\text{PSO}(A_{\Gamma'})$ in $\text{Out}(A_{\Gamma'})$. We still assume $n \geq 3$. One can prove a similar version of this corollary for $n < 3$ using the constructions in the Appendix.

Corollary 3.8. *For the graph $\Gamma' = \Gamma'(\Lambda)$ as described before,*

$$\text{Out}(A_{\Gamma'}) / \text{PSO}(A_{\Gamma'}) \cong (\mathbb{Z}/2\mathbb{Z})^N,$$

and thus, the index of $\text{PSO}(A_{\Gamma'})$ in $\text{Out}(A_{\Gamma'})$ is 2^N , where $N = 2n + 9$ is the number of vertices of Γ' .

Proof. Using the generators of $\text{Aut}(A_{\Gamma'})$ from Section 2.2, one can show that if $\text{PSO}(A_{\Gamma'})$ has finite index in $\text{Out}(A_{\Gamma'})$, one has

$$\text{Out}(A_{\Gamma'}) / \text{PSO}(A_{\Gamma'}) \cong \text{Inv} \rtimes \text{Per}.$$

Here, Inv is the subgroup of $\text{Out}(A_{\Gamma'})$ generated by inversions and Per is the subgroup generated by Γ' -legal permutations. Since Γ' has no non-trivial graph automorphisms, the group Per of Γ' -legal permutations contains only the identity. Using that $\text{Inv} \cong (\mathbb{Z}/2\mathbb{Z})^N$ for N the number of vertices of Γ' , we get that

$$\text{Out}(A_{\Gamma'}) / \text{PSO}(A_{\Gamma'}) \cong \text{Inv} \cong (\mathbb{Z}/2\mathbb{Z})^N.$$

As $|(\mathbb{Z}/2\mathbb{Z})^N| = 2^N$, we also get that the index of $\text{PSO}(A_{\Gamma'})$ in $\text{Out}(A_{\Gamma'})$ is 2^N . To complete the proof, it remains to argue why N satisfies the claimed equality. The graph Γ' as constructed above has $2n + 9$ vertices because Λ has n vertices and we add $n + 9$ vertices. □

4 | CONCLUSION AND OUTLOOK

We conclude this paper by summarising what we achieved and pointing out some possible directions for further research. In the paper [8], Day and Wade gave a condition for when $\text{PSO}(A_\Gamma)$ is a right-angled Artin group and which one it is. In this paper, we asked the question which right-angled Artin groups occur as (finite-index) subgroups of $\text{Out}(A_\Gamma)$ when we vary Γ . Using the condition by Day–Wade and another condition by Wade–Brück, we answered this question in Theorem 3.1 by showing that every right-angled Artin group occurs as $\text{PSO}(A_\Gamma)$ for some graph Γ for which $\text{PSO}(A_\Gamma)$ has finite index in $\text{Out}(A_\Gamma)$.

A possible direction for further research is to simplify the construction or to impose further conditions as we did in Theorem 3.4, where we additionally wanted that the graph Γ has no non-trivial graph automorphisms. The advantage of this compared to the theorem above is that it simplifies the structure of the quotient $\text{Out}(A_\Gamma) / \text{PSO}(A_\Gamma)$. For other simplifications, one could try to reduce the number of vertices or to reduce the index of $\text{PSO}(A_\Gamma)$ in $\text{Out}(A_\Gamma)$.

Another interesting topic to think about is to go beyond the subgroup of pure symmetric outer automorphisms and investigate for which graphs Γ a fixed right-angled Artin group A_Λ is a (finite-index) subgroup of the outer automorphism group of A_Γ . As mentioned above, we showed in this paper, using the subgroup of pure symmetric outer automorphisms, that every right-angled Artin group A_Λ occurs as a finite-index subgroup of $\text{Out}(A_\Gamma)$ for some graph Γ . Also, for fixed Λ , one can, using the condition of Day–Wade, answer the question when $\text{PSO}(A_\Gamma)$ is isomorphic to A_Λ . Namely, $A_\Lambda \cong \text{PSO}(A_\Gamma)$ if and only if the graph Γ satisfies that all its support graphs are forests and the graph described in Theorem 2.5 is isomorphic to Λ . It would be interesting to generalise this to other examples that do not depend on the subgroup of pure symmetric outer automorphisms

and maybe for fixed Λ give conditions that a graph Γ needs to satisfy so that A_Λ is a finite-index subgroup of $\text{Out}(A_\Gamma)$. Such results, even just for some graphs Λ , might give insights about how right-angled Artin groups interact with their outer automorphism groups and help to find more general results. However, a difficulty with this approach is that one cannot look at a fixed subgroup of $\text{Out}(A_\Gamma)$ but needs to consider all subgroups of $\text{Out}(A_\Gamma)$ at the same time.

Another interesting path one could take is to change the point of view and return to the question that Day and Wade asked; see [8, Question 1.1]. This is to not ask for fixed Λ when A_Λ is a (finite-index) subgroup of the outer automorphism group of another right-angled Artin group, but to ask for fixed Γ which right-angled Artin groups occur as a (finite-index) subgroup of $\text{Out}(A_\Gamma)$ and if there are any at all. When restricted to the subgroup of pure symmetric outer automorphisms, this is again answered by the condition of Day–Wade, but it would be interesting to generalise this to arbitrary (finite-index) subgroups.

Finally, one could also try to find examples when the automorphism group $\text{Aut}(A_\Gamma)$ has right-angled Artin groups as (finite-index) subgroups. One could ask the same questions as the ones asked by Day–Wade and in this paper. There is already a result known in this direction. Namely, Charney–Ruane–Stambaugh–Vijayan prove in [7, Theorem 3.6] that the group $\text{PSA}(A_\Gamma)$ is a right-angled Artin group if the graph Γ has no separating intersection of links. For $\text{PSA}(A_\Gamma)$, one can still use the condition for finite index as in Theorem 2.2 because $\text{PSO}(A_\Gamma)$ has finite index in $\text{Out}(A_\Gamma)$ if and only if $\text{PSA}(A_\Gamma)$ has finite index in $\text{Aut}(A_\Gamma)$. To show this, one needs that $\text{Inn}(A_\Gamma)$ is a normal subgroup of $\text{Aut}(A_\Gamma)$ and that $\text{Inn}(A_\Gamma) \subseteq \text{PSA}(A_\Gamma)$. Then, the natural map

$$(\text{Aut}(A_\Gamma)/\text{Inn}(A_\Gamma))/(\text{PSA}(A_\Gamma)/\text{Inn}(A_\Gamma)) \longrightarrow \text{Aut}(A_\Gamma)/\text{PSA}(A_\Gamma)$$

is well defined and bijective. Thus, since

$$\text{Out}(A_\Gamma)/\text{PSO}(A_\Gamma) = (\text{Aut}(A_\Gamma)/\text{Inn}(A_\Gamma))/(\text{PSA}(A_\Gamma)/\text{Inn}(A_\Gamma)),$$

the index of $\text{PSO}(A_\Gamma)$ in $\text{Out}(A_\Gamma)$ is the same as the index of $\text{PSA}(A_\Gamma)$ in $\text{Aut}(A_\Gamma)$. One could try to use [7] and the finite-index condition from Theorem 2.2 to answer the question which right-angled Artin groups occur as a (finite-index) subgroup of $\text{Aut}(A_\Gamma)$ for some Γ . Similarly to this paper, one could use computer programs to find examples and then try to generalise these.

APPENDIX: SMALL GRAPHS

In this appendix, we give examples for the graphs with less than three vertices to complete the proofs of Theorems 3.1 and 3.4. For every such graph Λ , we give a graph Γ such that $\text{PSO}(A_\Gamma) \cong A_\Lambda$, $\text{PSO}(A_\Gamma)$ has finite index in $\text{Out}(A_\Gamma)$ and Γ has no non-trivial graph automorphisms. These graphs were found using computer programs that can be found on [16]. This code can also be used to check that they satisfy the conditions of Theorems 3.1 and 3.4. In this appendix, we only sketch why the graph Θ as defined in Theorem 2.5 is indeed Λ .

Define Λ_1 as the graph with one vertex, Λ_2 as the graph with two vertices and no edge and Λ_3 as the graph with two vertices and an edge. Then, one can show that the graphs Γ_1 , Γ_2 and Γ_3 as in Figures A1a, A1b and A1c satisfy the conditions stated above, that is, $\text{PSO}(A_{\Gamma_i}) \cong A_{\Lambda_i}$ (using Theorem 2.5), $\text{PSO}(A_{\Gamma_i})$ has finite index in $\text{Out}(A_{\Gamma_i})$ (using Theorem 2.2) and Γ_i has no non-trivial graph automorphisms.

As mentioned above, we only sketch $\text{PSO}(A_{\Gamma_i}) \cong A_{\Lambda_i}$; for the complete arguments, the computer programs from [16] can be used. For Γ_1 , v_9 is the only vertex with a separating star. The

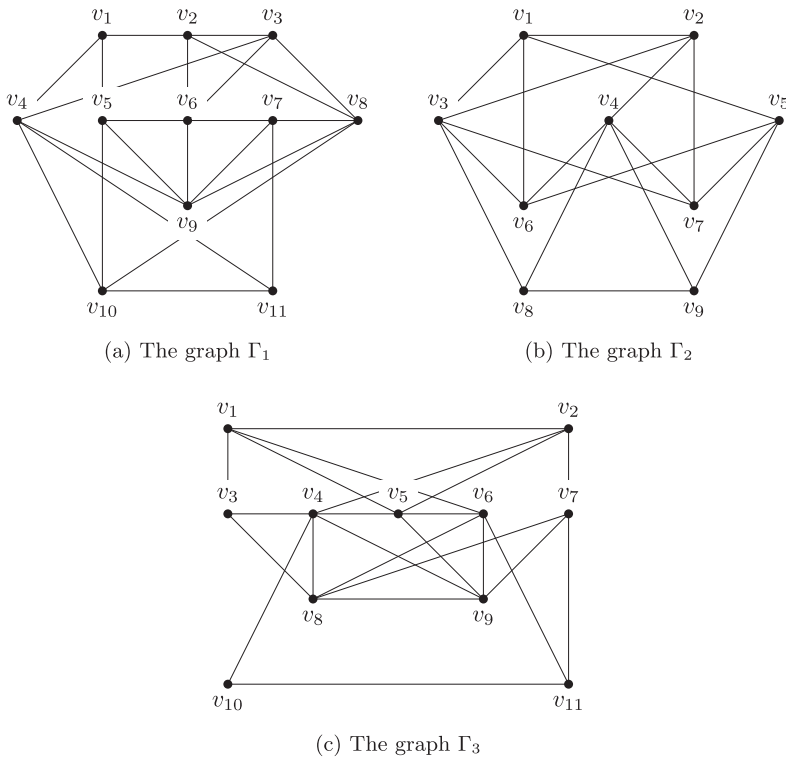


FIGURE A1 Proving Theorems 3.1 and 3.4 for the small graphs.

support graph consists of two vertices without an edge, and thus, we have only one vertex $\beta_1^{v_9}$ in the graph Θ_1 as defined in Theorem 2.5. For all other vertices v_i , $\Gamma_1 - \text{st}(v_i)$ is connected, and hence, they do not define vertices in Θ_1 . Thus, we have $\Theta_1 \cong \Lambda_1$. For Γ_2 , the vertices v_6 and v_7 are the only ones with separating stars. Both their support graphs have two vertices and an edge, so the graph Θ_2 has two vertices $\alpha_{e_6}^{v_6}$ and $\alpha_{e_7}^{v_7}$, where e_6 and e_7 are the edges in the support graphs of v_6 and v_7 , respectively. These two vertices satisfy the condition for when there is no edge between vertices of type I, and thus, the graph Θ_2 has no edge. Thus, we have $\Theta_2 \cong \Lambda_2$. For Γ_3 , the vertices v_8 and v_9 are the only vertices with separating stars and their support graphs have both two vertices and no edge. Thus, in Θ_3 , we have two vertices $\beta_1^{v_8}$ and $\beta_1^{v_9}$ and an edge. Hence, we have $\Theta_3 \cong \Lambda_3$. Using Theorem 2.5, we get that $\text{PSO}(A_{\Gamma_i}) \cong A_{\Lambda_i}$.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. I. Agol, *The virtual Haken conjecture*, Doc. Math. **18** (2013), 1045–1087. With an appendix by Agol, Daniel Groves, and Jason Manning. ISSN: 1431-0635.

2. A. Baudisch, *Kommutationsgleichungen in semifreien Gruppen*, Acta Math. Acad. Sci. Hungar. **29** (1977), no. 3–4, 235–249. ISSN: 0001-5954. <https://doi.org/10.1007/BF01895842>
3. M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), no. 3, 445–470. ISSN: 0020-9910. <https://doi.org/10.1007/s002220050168>
4. C. Bregman, R. Charney, and K. Vogtmann, *Outer space for raags*, 2020. <https://doi.org/10.48550/ARXIV.2007.09725>
5. B. Brück, *Computations for out(raags)*, 2021. Python code. URL: https://github.com/benjaminbrueck/computations_for_raags
6. B. Brück, *Between buildings and free factor complexes: a Cohen-Macaulay complex for Out(RAAGs)*, J. Lond. Math. Soc. (2) **105** (2022), no. 1, 251–307. ISSN: 0024-6107. <https://doi.org/10.1112/jlms.12511>
7. R. Charney, K. Ruane, N. Stambaugh, and A. Vijayan, *The automorphism group of a graph product with no SIL*, Illinois J. Math. **54** (2010), no. 1, 249–262. ISSN: 0019-2082. <https://doi.org/10.1215/ijm/1299679748>
8. M. B. Day and R. D. Wade, *Subspace arrangements, BNS invariants, and pure symmetric outer automorphisms of right-angled Artin groups*, Groups Geom. Dyn. **12** (2018), no. 1, 173–206. ISSN: 1661-7207. <https://doi.org/10.4171/GGD/442>
9. M. B. Day and R. D. Wade, *Relative automorphism groups of right-angled Artin groups*, J. Topol. **12** (2019), no. 3, 759–798. ISSN: 1753-8416. <https://doi.org/10.1112/topo.12101>
10. V. Guirardel and A. Sale, *Vastness properties of automorphism groups of RAAGs*, J. Topol. **11** (2018), no. 1, 30–64. ISSN: 1753-8416. <https://doi.org/10.1112/topo.12047>
11. S. Hensel and D. Kielak, *Nielsen realisation for untwisted automorphisms of right-angled Artin groups*, Proc. Lond. Math. Soc. (3) **117** (2018), no. 5, 901–950. ISSN: 0024-6115. <https://doi.org/10.1112/plms.12150>
12. M. R. Laurence, *A generating set for the automorphism group of a graph group*, J. Lond. Math. Soc. (2) **52** (1995), no. 2, 318–334. ISSN: 0024-6107. <https://doi.org/10.1112/jlms/52.2.318>
13. H. Servatius, *Automorphisms of graph groups*, J. Algebra **126** (1989), no. 1, 34–60. ISSN: 0021-8693. [https://doi.org/10.1016/0021-8693\(89\)90319-0](https://doi.org/10.1016/0021-8693(89)90319-0)
14. K. Vogtmann, *GL(n, \mathbb{Z}), Out(F_n) and everything in between: automorphism groups of RAAGs*, Groups St Andrews 2013, London Math. Soc. Lecture Note Ser., vol. 422, Cambridge University Press, Cambridge, 2015, pp. 105–127. <https://doi.org/10.1017/CBO9781316227343.005>
15. R. D. Wade and B. Brück, *A note on virtual duality and automorphism groups of right-angled Artin groups*, Glasgow Math. J. **65** (2023), no. 3, 573–581. <https://doi.org/10.1017/S0017089523000149>
16. M. Wiedmer, *Raags as subgroups of out(raags)*, 2022. Python code. URL: <https://github.com/manuelwiedmer/RAAGsas-subgroups-of-Out-RAAGs>
17. M. Wiedmer, *Right-angled Artin groups as finite-index subgroups of their outer automorphism groups*, Master's thesis, ETH Zurich, Zurich, 2022. <https://doi.org/10.3929/ethz-b-000538688>

SUPPORTING INFORMATION

Additional supporting information can be found online in the Supporting Information section at the end of this article.