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On notions of braid positivity and knot concordance

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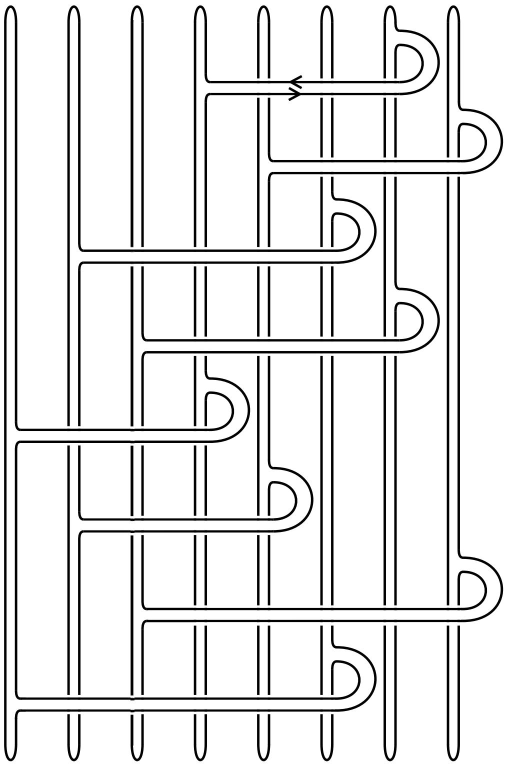
ETH Zurich



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On notions of braid positivity and knot concordance

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Dedicated to Peter Truöl

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Abstract

This work in low-dimensional topology investigates questions on the concordance of knots and links. Links are one-dimensional submanifolds of S^3, the three-dimensional sphere, which are usually studied up to isotopy; knots are connected links. Concordance is an equivalence relation that generalizes the concept of isotopy in dimension 4. In our work, we always consider knots and links as closures of braids, which is possible for every link by a result of Alexander in 1923. We often study knots that are closures of braids on three strands. In addition, various notions of positivity for braids play a special role.

Much of this work has been motivated by Baker's conjecture, which states that strongly quasipositive, fibered knots are isotopic if they are concordant. A counterexample to this conjecture would also provide a counterexample to the long-standing slice-ribbon conjecture of Fox. We show that every non-trivial strongly quasipositive link is concordant to infinitely many pairwise non-isotopic strongly quasipositive links. In particular, we show that Baker's conjecture is false in a strong sense if we drop the requirement that the knots are fibered.

A special case of Baker's conjecture is the following question: are closures of positive braids isotopic if they are concordant? On the way to a possible answer to this question, we study a particular concordance invariant. The upsilon invariant at 1 arises from the broad field of Heegaard Floer theory and was defined in 2017 by Ozsváth, Stipsicz and Szabó. We compute this invariant for all knots that are closures of braids on three strands by constructing cobordisms between these knots and torus knots. As an application, we determine various alternating distances of knots when restricting to positive such braids. For example, it follows from our calculations that the alternation number, the dealternating number and the Turaev genus are given by the sum of the ordinary 3-genus and the upsilon invariant of the knots under consideration.

In the last chapter of this thesis we deal with a related question in the topological category. Again, for knots which are closures of braids on three strands, we would like to determine their topological four-ball genus, i. e. the minimal genus of a topologically locally flatly embedded surface in the four-dimensional ball bounded by S^3. We can classify when, for such knots, the topological four-ball genus coincides with their 3-genus. We use McCoy's twisting method and the Xu normal form of braids on three strands. In addition, we give upper bounds for the topological four-ball genus of closures of positive and strongly quasipositive braids on three strands.

Zusammenfassung

Diese Arbeit im Bereich der niedrigdimensionalen Topologie untersucht Fragen im Zusammenhang mit der Konkordanz von Knoten und Verschlingungen. Verschlingungen sind eindimensionale Untermannigfaltigkeiten der dreidimensionalen Sphäre S^3, die üblicherweise bis auf Isotopie untersucht werden; Knoten sind zusammenhängende Verschlingungen. Konkordanz ist eine Äquivalenzrelation, welche das Konzept der Isotopie in Dimension 4 verallgemeinert. In unserer Arbeit betrachten wir Knoten und Verschlingungen immer als Abschlüsse von Zöpfen, was nach einem Resultat von Alexander aus dem Jahre 1923 für jede Verschlingung möglich ist. Dabei untersuchen wir häufig Knoten, die Abschlüsse von Zöpfen auf drei Strängen sind. Außerdem spielen verschiedene Positivitätsbegriffe für Zöpfe eine besondere Rolle.

Ein großer Teil dieser Arbeit wurde durch eine Vermutung von Baker motiviert, die besagt, dass streng quasipositive, gefaserte Knoten isotop sind, wenn sie konkordant sind. Ein Gegenbeispiel zu dieser Vermutung würde auch ein Gegenbeispiel zu der seit langem bestehenden Slice-Ribbon-Vermutung von Fox liefern. Wir zeigen, dass jede nicht-triviale streng quasipositive Verschlingung zu unendlich vielen, paarweise nicht-isotopen streng quasipositiven Verschlingungen konkordant ist. Insbesondere zeigen wir, dass die Vermutung von Baker in einem starken Sinne falsch ist, wenn wir die Bedingung der Gefasertheit der Knoten weglassen.

Ein Spezialfall von Bakers Vermutung ist die folgende Frage: Sind Abschlüsse von positiven Zöpfen isotop, wenn sie konkordant sind? Auf dem Weg zu einer möglichen Antwort auf diese Frage untersuchen wir eine bestimmte Konkordanzinvariante. Die Upsilon-Invariante an der Stelle 1 stammt aus dem weiten Feld der Heegaard-Floer-Theorie und wurde 2017 von Ozsváth, Stipsicz und Szabó definiert. Wir berechnen diese Invariante für alle Knoten, die Abschlüsse von dreisträngigen Zöpfen sind, indem wir Kobordismen zwischen diesen Knoten und Torusknoten konstruieren. Als Anwendung bestimmen wir verschiedene Alternierungsdistanzen von Knoten, wenn wir uns auf positive solche Zöpfe beschränken. Zum Beispiel ergibt sich aus unseren Berechnungen, dass die Alternierungszahl, die Dealternierungszahl und das Turaev-Geschlecht jeweils durch die Summe des gewöhnlichen 3-Geschlechts und der Upsilon-Invariante des betrachteten Knotens gegeben sind.

Im letzten Kapitel dieser Arbeit behandeln wir eine verwandte Frage in der topologischen Kategorie. Wiederum für Knoten, die als Abschlüsse von von dreisträngigen Zöpfen gegeben sind, wollen wir ihr topologisches Vierball-Geschlecht bestimmen, das minimale Geschlecht einer topologisch lokal flach eingebetteten Fläche im vierdimensionalen Ball, welcher von S^3 berandet wird. Wir können klassifizieren, wann für solche Knoten das topologische Vierball-Geschlecht mit ihrem dreidimensionalen Geschlecht übereinstimmt. Dazu verwenden wir eine Methode zum Entdrehen von Knoten von McCoy und die Xu-Normalform von dreisträngigen Zöpfen. Außerdem geben wir obere Schranken für das topologische Vierball-Geschlecht für Abschlüsse von positiven und streng quasipositiven Zöpfen auf drei Strängen.

1 Introduction

1.1 Overview and outline

This work is in the area of low-dimensional topology, which studies manifolds of dimension 4 and lower. Examples include (knotted) surfaces in 4-dimensional space and classical knots in dimension 3. Knots are non-empty, oriented, connected, closed, smooth, 1-dimensional submanifolds of the 3-dimensional sphere S^3, which are usually studied up to isotopy. A natural generalization in dimension 4 of the question whether certain knots are isotopic to the trivial knot, called unknot, is the concept of concordance. A knot in S^3 is called concordant to the unknot (or slice) if it bounds a slice disk, a "nicely embedded" 2-dimensional disk in B^4, the 4-dimensional ball bounded by S^3. A more general notion is that of the four-ball genus, or 4-genus for short. The 4-genus of a knot in S^3 is the minimal genus of an oriented, connected, compact surface "nicely embedded" in B^4 with boundary the given knot. In the smooth category "nicely embedded" means smoothly embedded, in the topological category we ask the surface to be topologically locally flat. It is one of the curiosities of low-dimensional topology that constructions such as finding slice disks can sometimes be done in the topological category, but fail in the smooth category. We will mainly work in the smooth category, but in Chapter 4 we will address a question in the topological category.

Every knot can be represented as the closure of a braid [Ale23]. Informally, for $n \geqslant 1$, an n-braid is a collection of n non-intersecting, unknotted and never-returning paths in 3-dimensional space connecting n distinguished points to another set of n distinguished points. Isotopy classes of n-braids form a group, the braid group on n strands. Its classical presentation, first introduced by Artin [Art25], provides an algebraic tool for examining knots. In this thesis, we are particularly interested in knots that arise as closures of braids on three strands, so-called 3-braid knots. On the other hand, we study different notions of positivity for braids. We are particularly concerned with positive and strongly quasipositive braids, which behave specially in the context of smooth concordance; more on this below.

Briefly and roughly, this thesis focuses on the study of knot concordance and related concepts like the 4-genus, using the braid group as a tool to represent knots algebraically. Structurally, it consists of three main chapters after this introductory chapter. The contents of these chapters are available on the arXiv preprint server [Tru22, Tru21, BLMT23].

We provide a brief overview of these three chapters before proceeding with a more detailed introduction, including definitions and more background on our results.

Chapter 2, corresponding to [Tru22]**:** This dissertation project was motivated by a conjecture of Baker [Bak16] from knot concordance theory on strongly quasipositive, fibered knots, which is implied by the slice-ribbon conjecture due to Fox [Fox62]. We show that every non-trivial strongly quasipositive link is concordant to infinitely many pairwise non-isotopic strongly quasipositive links. In particular, we show that Baker's conjecture is false in a strong sense when the fiberedness assumption is dropped.

Chapter 3, corresponding to [Tru21]**:** The question of whether concordant braid positive knots are isotopic naturally arises as a special case of Baker's conjecture. Focusing on the smaller subset of positive 3-braid knots, we work towards understanding the concordance classes of these. As our main result of Chapter 3, we provide explicit formulas for the integer-valued smooth concordance invariant $\upsilon(K) = \Upsilon_K(1)$ for every 3-braid knot K. One of the applications is the calculation of several alternating distances for positive 3-braid knots. The contents of Chapter 3 will be published as an article in *Algebraic & Geometric Topology* [Tru21].

Chapter 4, corresponding to [BLMT23]**:** Based on [BLMT23], together with Sebastian Baader, Lukas Lewark and Filip Misev, we classify 3-braid knots whose topological 4-genus coincides with their 3-genus. The tools we use are a technique called twisting [McC21] and a special representation for 3-braids called Xu normal form [Xu92]. In addition, we give upper bounds on the topological 4-genus of positive and strongly quasipositive 3-braid knots.

1.2 Knots and links as closures of braids

A *link* (in S^3) is a non-empty, oriented, closed, smooth, 1-dimensional submanifold of the (oriented) 3-dimensional sphere S^3. We consider links up to *(ambient) isotopy*, i. e. orientation-preserving diffeomorphisms of S^3. *Knots* are links with one connected component. Links can be visualized by viewing them as subsets of the Euclidean 3-space $\mathbb{R}^3$, which we identify with a subset of S^3. Up to an isotopy, we can assume a link L to be in general position with respect to the standard projection $p\colon \mathbb{R}^3 \to \mathbb{R}^2$. The image of L under p together with the additional data of which strand of L is over and which is under at every double point (*crossing*) of the projection is called a *diagram* for L. The *unknot* is the trivial knot that arises as the boundary of the 2-dimensional disk D^2 in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}$. Figure 1.1 shows two examples of knot diagrams.

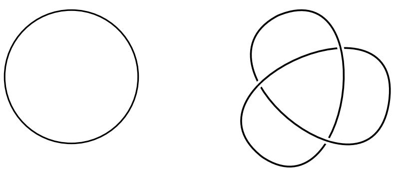


Figure 1.1: Diagrams for the unknot and the trefoil knot.

By a fundamental theorem of Alexander [Ale23], every link can be represented as the closure of an n-braid for some positive integer n. Fix n distinct points $p_1, \ldots, p_n$ in $D^2 \subset \mathbb{R}^2$. A *(geometric) n-braid* or *braid on n strands* is a collection of (oriented) smooth paths $f_i\colon [0,1] \to D^2 \times [0,1]$, $i \in \{1, \ldots, n\}$, called *strands*, with pairwise disjoint images $f_i([0,1])$ and such that for all $i \in \{1, \ldots, n\}$, we have $f_i(t) \in D^2 \times \{t\}$, $f_i(0) = (p_i, 0)$ and $f_i(1) = (p_{\pi(i)}, 1)$ for some permutation π of $\{1, \ldots, n\}$. We study n-braids up to *isotopy*, i. e. two n-braids are considered the same if there is an ambient isotopy of $D^2 \times [0,1]$ fixing the set $D^2 \times \{0,1\}$ pointwise and taking one of the braids to the other.

The *braid group on n strands*, which we denote by B_n, is the group of isotopy classes of n-braids, where the group operation is given by stacking braids on top of each other and rescaling (see Figure 1.3(a) on the next page). The classical presentation for B_n with $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| \geqslant 2 \quad \text{and} \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

was introduced by Artin [Art25]. The generator σ_i corresponds to the n-braid which exchanges p_i and p_{i+1} by a positive half-twist parameterized by $[0,1]$, i. e. the i-th and $(i+1)$-th strand of σ_i cross once positively. We illustrate the generator σ_1 of B_2 with one positive crossing on the left in Figure 1.2. In our figures, braid diagrams are always oriented from bottom to top. We call a word in the generators of B_n and their inverses a *braid word*. Every braid word defines a diagram for a geometric n-braid; see the right-hand side of Figure 1.2 for an example. In the following, we will usually identify braid words with the corresponding geometric braids, and we suppress n if it is clear from the context.

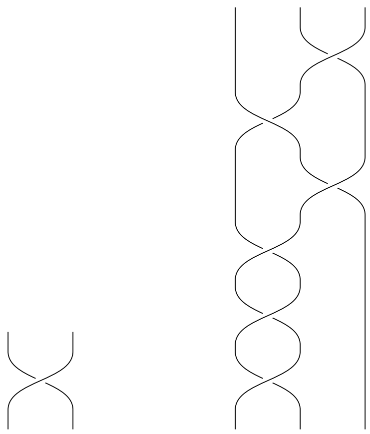
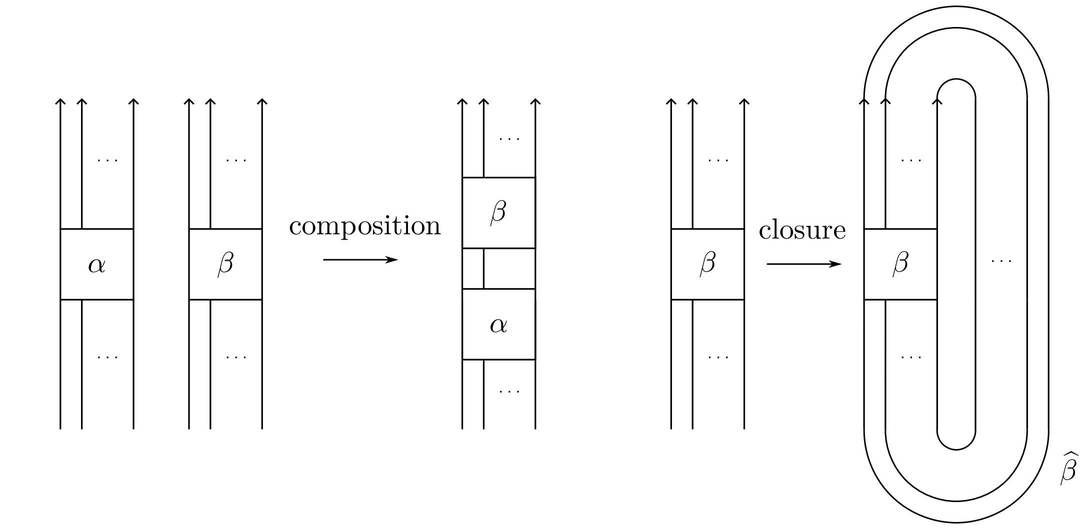


Figure 1.2: The 2-braid that corresponds to the generator σ_1 of B_2 on the left; the geometric 3-braid defined by the braid word $\sigma_1^3\sigma_2\sigma_1^{-1}\sigma_2$ in B_3 on the right.

In Chapters 3 and 4, we will focus on the braid group on three strands B_3. For readability, we will often denote the two generators of B_3 by $a \coloneqq \sigma_1$ and $b \coloneqq \sigma_2$. They are subject to the relation $aba = bab$ in B_3, called *braid relation*.

The *closure* $\widehat{\beta}$ *of a braid* β is the closed 1-dimensional submanifold in $\mathbb{R}^2 \times S^1$ obtained by gluing the ends $(p_i, 0) \in \mathbb{R}^2 \times \{0\}$ of the strands of β to the corresponding ends in $\mathbb{R}^2 \times \{1\}$. The closure $\widehat{\beta}$ of a braid β yields a link in S^3 via a fixed standard embedding of $\mathbb{R}^2 \times S^1$ in S^3; see Figure 1.3(b). Note that conjugate braids $\beta_0, \beta_1 \in B_n$, denoted by $\beta_0 \sim \beta_1$, have isotopic closures $\widehat{\beta_0} = \widehat{\beta_1}$.



(a) The composition $\alpha\beta$ of braids α and β. (b) The closure $\widehat{\beta}$ of a braid β.

Figure 1.3: The composition of and the closure operation on braids.

An *n-braid link* is a link that arises as the closure of an n-braid. If $\beta \in B_n$ induces a permutation with only one cycle on the ends of its n strands, then its closure $\widehat{\beta}$ is a knot and we call it an *n-braid knot*. For example, a 3*-braid knot* is a knot that arises as the closure $\widehat{\beta}$ of a 3-braid β. The *braid index* of a link L is the smallest positive integer n such that L arises as the closure of a braid in B_n.

For a more detailed account on braids and their closures, we refer the reader to [BB05].

1.3 Notions of positivity for braids

Let us define various notions of positivity for braids and their closures, which will be used throughout this thesis. Let $n \geqslant 1$. An n-braid is called *positive* if it can be represented by a braid word $\sigma_{s_1}\sigma_{s_2}\cdots\sigma_{s_l}$ with $s_j \in \{1, \ldots, n-1\}$, i. e. a braid word that is a product of the positive Artin generators σ_i of B_n (no inverses σ_i^{-1}). Moreover, an n-braid is called *quasipositive* if it is a (finite) product of conjugates of the (positive) Artin generators σ_i, and it is called *strongly quasipositive* if the product consists only of certain conjugates of the σ_i, namely the *positive band words* $\sigma_{i,j}$ (see Figure 1.4), where

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\, \sigma_{j-1}\, (\sigma_i \cdots \sigma_{j-2})^{-1} \qquad \text{for} \quad 1 \leqslant i < j \leqslant n. \tag{1.1}$$

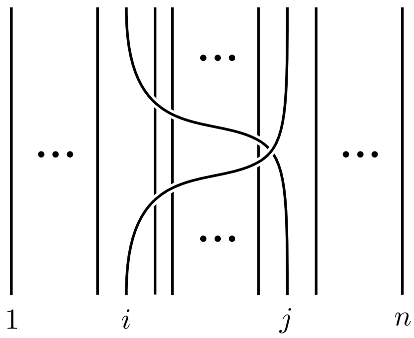


Figure 1.4: The positive band word $\sigma_{i,j}$.

We call a braid word β given as such a product a *(strongly) quasipositive braid word*, and—despite a minor abuse of notation—also denote it by β. Note that $\sigma_{i,i+1} = \sigma_i$. A knot or link is called *braid positive* or *(strongly) quasipositive* if it arises as the closure of a positive or (strongly) quasipositive n-braid for some $n \geqslant 1$, respectively. (Strongly) quasipositive braids and links first appeared in the work of Rudolph [Rud83b, Rud90]. The positive band words $\sigma_{i,j}$ were also used by Birman–Ko–Lee [BKL98] to provide a new solution to the word problem in B_n.

Important examples for Chapter 3 of braid positive knots are the *(positive) torus knots* $T_{p,q}$ for coprime positive integers p, q. The torus knot $T_{p,q}$ is the knot obtained as the closure of the p-braid $(\sigma_1\sigma_2 \ldots \sigma_{p-1})^q$. Positive torus knots are *algebraic*, i. e. they arise as so-called links of isolated singularities of complex algebraic plane curves [Mil68]. Indeed, for small $\varepsilon > 0$ the torus knot $T_{p,q}$ is isotopic to $V(f) \cap S^3_\varepsilon \subset S^3_\varepsilon \subset \mathbb{C}^2$, where $V(f)$ denotes the zero-set of $f\colon \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto x^p - y^q$, and S^3_ε a 3-sphere of radius ε centered at the origin in $\mathbb{C}^2$. Not only positive torus knots, but more generally all algebraic knots are braid positive; see e. g. [BK86, Theorem 12 in Section 8.3].

1.4 Concordances of knots and links

Two ordered links $L_0 = L_0^1 \cup \cdots \cup L_0^b$ and $L_1 = L_1^1 \cup \cdots \cup L_1^b$ of b components are called *(smoothly) concordant* if there exists a smoothly and properly embedded oriented submanifold $A = A_1 \cup \cdots \cup A_b$ of $S^3 \times [0, 1]$, called a *concordance*, such that A is diffeomorphic to a disjoint union of r annuli $S^1 \times [0, 1]$,

$$\partial A_i = L_0^i \times \{0\} \cup L_1^i \times \{1\}, \quad i \in \{1, \ldots, b\},$$

and the induced orientation on ∂A agrees with the orientation of L_0, but is the opposite one on L_1. Concordance defines an equivalence relation on the set of ordered links in S^3.

In large parts of this work, we restrict our study of concordances to knots. Moreover, in Chapter 2, the choice of the order of the considered links does not play an important role. Thus we usually suppress the order from the notation.

Isotopic links are concordant, but the converse is generally not true. For example, for any *non-trivial* knot K (i. e. K is not the unknot), the knot $K\# -K$ is concordant, but not isotopic to the unknot. Here, for a knot K, its *inverse* $-K$ is the image of K under an orientation-reversing diffeomorphism of S^3 with the opposite orientation and $\#$ denotes the connected sum of knots. Note also for later that the image of K under an orientation-reversing diffeomorphism of S^3 is called the *mirror* of K, and the knot K with the opposite orientation is the *reverse* of K. The name inverse is justified by the fact that knots up to concordance form an abelian group, the *concordance group* $\mathcal{C}$. The group operation is induced by connected sum, the unknot represents the identity element, and the additive inverse of a concordance class $[K]$ is $[-K]$. To properly define the connected sum operation we should really think of a knot as a pair of oriented manifolds (S^3, S^1); then the connected sum of two knots is defined in the standard way for oriented pairs.

1.5 The slice-ribbon conjecture and a conjecture by Baker

A knot K is called *slice* if it is concordant to the unknot or, equivalently, if it arises as the boundary of a smoothly embedded 2-dimensional disk D^2 in B^4, the 4-ball bounded by S^3. Furthermore, K is called *ribbon* if it bounds a smoothly embedded disk D in B^4 for which the radial height function on B^4 restricts to a smooth Morse function with no local maxima in the interior of D. Equivalently, we could ask K to bound an immersed disk in S^3 with only so-called ribbon singularities. Ribbon knots are slice and in the 1960s Fox asked whether the converse is true [Fox62].

Conjecture 1.1 (Slice-ribbon conjecture)**.** *Every slice knot is ribbon.*

Informally, the slice-ribbon conjecture states that for slice knots the sliceness can be demonstrated with a disk that can be well visualized in three dimensions. The conjecture is solved for some families of knots, namely 2-bridge knots and certain families of pretzel knots [Lis07, GJ11]. However, it remains open in full generality. In [Bak16], Baker showed that for any two strongly quasipositive, fibered knots K_0 and K_1, if $K_0\# -K_1$ is ribbon (which in particular implies that K_0 and K_1 are concordant), then K_0 is isotopic to K_1. He conjectured the following.

Conjecture 1.2 (Baker's conjecture)**.** *If two strongly quasipositive, fibered knots are concordant, then they are isotopic.*

We will define fibered knots in Section 1.7. Baker's result described above shows that Conjecture 1.1 implies Conjecture 1.2. In other words, either concordance implies isotopy for the set of strongly quasipositive, fibered knots or the slice-ribbon conjecture is false.

1.6 Strongly quasipositive links are concordant to infinitely many such links

In Chapter 2, which is mathematically equivalent to [Tru22], we show the following.

Theorem A. *Every strongly quasipositive link other than an unlink is smoothly concordant to infinitely many pairwise non-isotopic strongly quasipositive links.*

To discuss the context of our result, we focus on knots. In particular, Theorem A shows in a strong way that Baker's conjecture (see Conjecture 1.2) does not hold without the assumption of fiberedness. We can reformulate Theorem A as follows: each equivalence class in the concordance group of a non-trivial strongly quasipositive knot contains infinitely many such knots. Similar statements replacing strongly quasipositivity by other stronger notions would not be true. Consider the following inclusions:

$$\{\text{algebraic knots}\} \subset \{\text{positive knots}\} \subset \{\text{strongly quasipositive knots}\}. \tag{1.2}$$

Positive knots are the knots that admit a diagram in which all crossings are positive. The first inclusion in (1.2) follows from the fact that algebraic knots are braid positive and hence positive, the latter is due to Rudolph and Nakamura [Rud99, Nak00]. In contrast to our result and generalizing earlier results of Stoimenow [Sto08, Sto15], Baader–Dehornoy–Liechti [BDL17] showed that every (topological and thus also smooth) concordance class contains at most finitely many (pairwise non-isotopic) positive knots. Furthermore, considering an even smaller subset, it was shown by Litherland [Lit79] that algebraic knots are isotopic if they are concordant.

We would like to point out that Hedden (see [Bak16, paragraph after Remark 6]) observed the existence of concordance classes that contain at least two strongly quasipositive knots. In fact, his observation was the starting point of the project that led to Theorem A, which shows that every concordance class of a non-trivial strongly quasipositive knot (link) contains infinitely many strongly quasipositive knots (links), respectively; see also Remark 2.2. Given the results on the concordance classes of algebraic and positive knots, which stand in contrast to Theorem A, the following question naturally arises. As far as we know, this question, which is a weaker version of Conjecture 1.2, is open.

Question 1.3. *Are there only finitely many strongly quasipositive, fibered knots in every smooth concordance class?*

1.7 The slice-Bennequin equalities

Throughout this thesis—especially in Chapter 3, but also to justify the non-triviality assumption in Theorem A—we will repeatedly refer to what we call the slice-Bennequin equalities, which are a consequence of the (slice-)Bennequin inequalities. We will recall them here in full generality, introducing some important notions along the way.

Let L be a link. A *Seifert surface* for L is an oriented, compact, not necessarily connected surface in S^3 with oriented boundary L and no closed components. Let $\chi(L)$ be the largest Euler characteristic $\chi(F)$ of any Seifert surface F for L, and let $\chi_4(L)$ be its 4-dimensional analog, i. e. the largest Euler characteristic $\chi(F)$ of any oriented, compact surface F smoothly embedded in the 4-ball B^4 with oriented boundary the link L in $S^3 = \partial B^4$ and no closed components. Moreover, for a knot K, denote by $g(K)$ its 3-*genus*, the minimal genus of a Seifert surface for K, and by $g_4(K)$ its *(smooth)* 4-*genus*, the minimal genus of an oriented, connected, compact surface smoothly embedded in B^4 with oriented boundary K in $S^3 = \partial B^4$. We will define the analog of $g_4(K)$ in the topological category in Section 1.10. Note that we have $\chi(K) = 1 - 2g(K)$ and $\chi_4(K) = 1 - 2g_4(K)$. Finally, let $\mathrm{wr}(\beta)$ denote the *writhe* of a braid $\beta \in B_n$, i. e. the exponent sum of the braid word β with respect to the Artin generators σ_i, which is the image of β under the abelianization $\mathrm{wr}\colon B_n \to \mathbb{Z}$. Building on Kronheimer and Mrowka's proof of the local Thom conjecture [KM93], Rudolph [Rud93] showed that $\chi_4(\widehat{\beta}) \leqslant n - \mathrm{wr}(\beta)$ for every $\beta \in B_n$, $n \geqslant 1$. The analogous statement in three dimensions, $\chi(\widehat{\beta}) \leqslant n - \mathrm{wr}(\beta)$ for every $\beta \in B_n$, $n \geqslant 1$, is due to Bennequin [Ben83]. Bennequin's and Rudolph's results are known as the Bennequin and slice-Bennequin inequalities, respectively. We will see in Section 2.2 that every strongly quasipositive braid word β comes equipped with a canonical Seifert surface F for $L = \widehat{\beta}$ which realizes the equality $\chi(\widehat{\beta}) = n - \mathrm{wr}(\beta)$. Furthermore, every quasipositive braid word β gives rise to an immersed surface with boundary $L = \widehat{\beta}$ with only ribbon singularities which realizes the equality for $\chi_4(L)$ when pushed into the 4-ball. In particular, for every strongly quasipositive braid $\beta \in B_n$, $n \geqslant 1$, such that $K = \widehat{\beta}$ is a knot, we have

$$\text{(slice-Bennequin equalities)} \qquad g_4(K) = g(K) = \frac{\mathrm{wr}(\beta) - n + 1}{2}. \tag{1.3}$$

Stallings [Sta78] showed that (as closures of so-called homogeneous braids), braid positive knots are *fibered*, i. e. their complement in S^3 is the total space of a locally trivial fiber bundle over S^1 whose fibers are the interiors of Seifert surfaces for the knot. Note that fibered knots have a unique (up to isotopy) Seifert surface of minimal genus and the fibers are realized by minimal genus Seifert surfaces [Rud05, Proposition 2.19]. Using Stalling's result, the special case $g(\widehat{\beta}) = \frac{\mathrm{wr}(\beta) - n + 1}{2}$ of (1.3) for positive braids β was also shown in [BW83, Theorem 5.2].

Remark 1.4. The non-triviality assumption in Theorem A is necessary because there exists only one strongly quasipositive, strongly slice link of b components: the unlink of b components. This follows from the slice-Bennequin inequalities for χ and χ_4, using that there is a unique surface in S^3 of Euler characteristic b, with b boundary components and without closed components: the disjoint union of b disks; and a unique link, the unlink of b components, bounded by this surface. Here, a link is called strongly slice if it arises as the boundary of a disjoint union of smoothly embedded disks in B^4.

1.8 On the concordance of positive 3-braid knots

In addition to the inclusions in (1.2), by Stalling's result mentioned above, we have

$$\{\text{braid positive knots}\} \subset \{\text{strongly quasipositive knots}\} \cap \{\text{fibered knots}\}. \tag{1.4}$$

The set on the right hand side of (1.4) consists of the knots from Conjecture 1.2; see [Hed10, Proposition 2.1] for equivalent characterizations of strongly quasipositive, fibered knots. The following question thus arises naturally as a special case of that conjecture.

Question 1.5. *Are concordant braid positive knots isotopic?*

We illustrate in Figure 1.5 how the various notions of positivity for knots are related. Note that all inclusions are strict. For example, the knot 5_2 in Rolfsen's knot table [Rol03, LM23] is a positive knot that is not fibered and therefore not braid positive. Moreover, the knot 4_1 is fibered but not quasipositive, while the knot 10_{145} is strongly quasipositive and fibered, but not positive (see also [Baa05]).

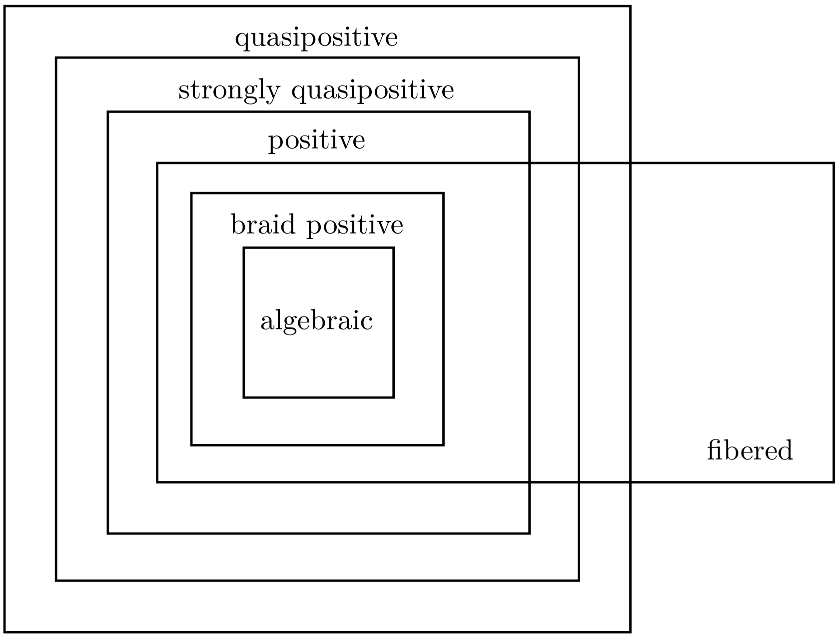


Figure 1.5: Notions of positivity and related notions for knots.

It follows from the slice-Bennequin equalities (1.3) that every concordance class in the knot concordance group contains at most finitely many braid positive knots. To see this, we first observe that there are only finitely many braid positive knots of a given fixed 3-genus, which can be shown, for example, as follows. The writhe of a positive braid β equals the number of generators in the corresponding braid word and is linearly bounded from below by twice the *positive braid index* of $\widehat{\beta}$, the minimal number of strands among the positive braid representatives of $\widehat{\beta}$. For a fixed 3-genus g, there are thus only finitely many possible positive braid indices n and positive braids β with $g = g(\widehat{\beta}) = \frac{\mathrm{wr}(\beta)-n+1}{2}$. Now, since $g_4 = g$ for braid positive knots by (1.3) and g_4 is a concordance invariant, there can only be finitely many braid positive knots in every (smooth) concordance class. This is also true in the topological category: as a corollary of a result by Stoimenow [Sto08] on the signature growth of braid positive knots, there are only finitely many braid positive knots in every concordance class of the topologically locally flat concordance group.

Question 1.5 asks whether there is in fact at most one braid positive knot in each concordance class. One possible approach to answering this question is to explicitly study knots that are the closures of positive braids on a fixed number of strands. In this thesis, we call a knot a *positive 3-braid knot* if it is the closure of a positive braid on three strands. Note that this terminology is somewhat ambiguous, because we do not mean a 3-braid knot that is also positive. The knot 5_2 provides a simple example to show that the two terms are not the same: it is a 3-braid knot (see Figure 1.2 for the 3-braid $\sigma_1^3\sigma_2\sigma_1^{-1}\sigma_2$ with closure 5_2) that is positive but not braid positive. However, it is true that a braid positive knot of braid index 3 is also the closure of a positive 3-braid [Sto17]; see Theorem 4.11.

Focusing on positive 3-braid knots, Question 1.5 seems to be particularly accessible due to classification results on the conjugacy classes of 3-braids [Gar69, Mur74]; see also Proposition 3.8 in Chapter 3. As a corollary of our main theorem in Chapter 3, which corresponds to [Tru21], we provide the following step towards understanding the concordance classes of positive 3-braid knots. Here, $\upsilon(K) = \Upsilon_K(1)$ denotes a (smooth) concordance invariant from knot Floer homology defined by Ozsváth–Stipsicz–Szabó [OSS17a]. Note that a pair of concordant, but non-isotopic positive 3-braid knots would provide a counterexample to the slice-ribbon conjecture (Conjecture 1.1).

Corollary B. *Let K be a positive 3-braid knot, i. e. K is the closure of an element of the braid group $B_3 = \langle a, b \mid aba = bab\rangle$ on three strands that can be written as a word in the generators a and b only (no inverses). Then the minimal r such that K is the closure of $a^{p_1}b^{q_1}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r}$ for integers $p_i, q_i \geqslant 1$, $i \in \{1,\ldots,r\}$, is $r = g(K) + \upsilon(K) + 1$. Moreover, if K and J are concordant positive 3-braid knots, then this minimal r is the same for both K and J.*

To explain the main result of Chapter 3, which implies Corollary B, we begin a little further back.

1.9 The upsilon invariant at 1 of 3-braid knots

Heegaard Floer homology is a very effective tool for understanding 3-manifolds that was developed by Ozsváth and Szabó [OS04b]. There are many generalizations and refinements of this homology theory, which in its simplest form associates to any closed

3-manifold Y a graded chain complex whose chain homotopy type is an invariant of Y. A knot in S^3 induces a filtration on the chain complex associated to S^3. The homology of the associated graded object is known as knot Floer homology, which was defined by Ozsváth–Szabó [OS04a] and independently by Rasmussen [Ras03]. The invariants τ and Υ [OS03, OSS17a] that we consider in this thesis are only two of many concordance invariants coming from knot Floer homology [Hom17].

Ozsváth, Stipsicz and Szabó [OSS17a] defined the invariant Υ_K of a knot K using a variant of the chain complex $CFK^-(K)$. For every knot K, the invariant Υ_K takes the form of a real-valued piecewise linear function on the interval $[0, 2]$. In fact, it induces a homomorphism from the concordance group to the group of real-valued piecewise linear functions on $[0, 2]$; see also [Liv17]. The function Υ_K evaluated at $t = 1$, $\upsilon(K) := \Upsilon_K(1)$, induces a homomorphism $\mathcal{C} \to \mathbb{Z}$. In this work, we call $\upsilon(K)$ *upsilon of* K. In Chapter 3, we determine upsilon for all 3-braid knots. More precisely, we show the following.

Theorem C. *Let $\beta = \Delta^{2\ell} a^{-p_1} b^{q_1} a^{-p_2} b^{q_2} \cdots a^{-p_r} b^{q_r}$ be a braid word in the generators a and b of B_3 for some $\ell \in \mathbb{Z}$, and integers $r \geqslant 1$ and $p_i, q_i \geqslant 1$ for $i \in \{1, \ldots, r\}$, where $\Delta^2 = (ab)^3$. Suppose that $K = \widehat{\beta}$ is a knot. Then its upsilon invariant is*

$$\upsilon(K) = \frac{\sum_{i=1}^{r} (p_i - q_i)}{2} - 2\ell.$$

As an application of Theorem C, we show that the following invariants coincide for positive 3-braid knots.

Corollary D. *Let K be a positive 3-braid knot. Then*

$$\mathrm{alt}(K) = \mathrm{dalt}(K) = g_T(K) = \mathcal{A}_s(K) = g(K) + \upsilon(K).$$

Here, the *alternation number* $\mathrm{alt}(K)$, *dealternating number* $\mathrm{dalt}(K)$ and *Turaev genus* $g_T(K)$ are different ways of measuring how far a knot K is from being alternating. The best known among them is certainly the first one: the alternation number $\mathrm{alt}(K)$ of a knot K was first defined by Kawauchi [Kaw10] as the minimal Gordian distance of K to the set of alternating knots. In Section 3.5, we review the precise definitions of the above invariants and prove Corollary D. The invariant $\mathcal{A}_s(K)$ introduced by Friedl, Livingston and Zentner [FLZ17] is defined as the minimal number of double point singularities in a generically immersed concordance from a knot K to an alternating knot.

To prove Theorem C, we will first determine υ for all positive 3-braid knots and then generalize our result to all 3-braid knots. Our main tool is the construction of cobordisms between 3-braid knots and (connected sums of) torus knots. This extension to all 3-braid knots from the positive ones was somewhat unexpected for the author since, in contrast, the same method would not work to determine slice-torus invariants [Liv04, Lew14] such as τ from Heegaard Floer homology or Rasmussen's invariant s from Khovanov homology [Ras10] for all 3-braid knots (see Section 3.4.4.2).

1.10 3-braid knots with maximal topological 4-genus

This section is about a result in the topological category. The *topological 4-genus* $g_4^{\mathrm{top}}(K)$ of a knot K is the minimal genus of an oriented, connected, compact surface with a topologically locally flat embedding in B^4 and oriented boundary the knot K. By definition,

$$g_4^{\mathrm{top}}(K) \leqslant g_4(K) \leqslant g(K)$$

for every knot K. Four decades after Freedman's celebrated work on 4-manifolds [Fre82], the topological 4-genus of knots remains difficult to determine. The first challenge is posed by the figure-eight knot 4_1. From the perspective of invariants that take the form of group homomorphisms on the topologically locally flat concordance group, the figure-eight knot satisfies $g_4^{\mathrm{top}}(4_1) = 1$ for no obvious reason. Its second power $4_1 \# 4_1$ bounds a (smoothly, hence also topologically locally flatly) embedded disk in B^4, causing all its additive lower bounds on the topological 4-genus to be trivial. In particular, we have $\sigma(4_1) = 0$, where $\sigma(K)$ denotes the classical signature of the knot K [Tro62][1]. Due to this example, there is little reason to believe that the inequality

$$|\sigma(K)| \leqslant 2g_4^{\mathrm{top}}(K)$$

has much to tell us about the topological 4-genus of knots in general. In Chapter 4, which corresponds to [BLMT23], we show that 3-braids knots are exceptional in this respect: we will see that the figure-eight knot is exceptional among closures of 3-braids in that it is the only 3-braid knot K that satisfies $|\sigma(K)| < 2g(K)$, yet $g_4^{\mathrm{top}}(K) = g(K)$.

Theorem E. *Let K be a 3-braid knot other than the figure-eight knot. Then*

$$|\sigma(K)| = 2g(K) \quad \Longleftrightarrow \quad g_4^{top}(K) = g(K).$$

These equalities hold precisely if K or its mirror is one of the following knots:

- $T_{2,2m+1} \# T_{2,2n+1}$, *with* $m, n \geqslant 0$,
- $P(2p, 2q+1, 2r+1, 1)$, *with* $p \geqslant 1$, $q, r \geqslant 0$,
- $T_{3,4}$ *or* $T_{3,5}$.

To understand the list in Theorem E, recall from Section 1.3 that $T_{p,q}$ denotes a torus knot. Furthermore, by $P(2p, 2q+1, 2r+1, 1)$ we denote a 4-stranded pretzel knot; see Figure 4.1 in Section 4.3. In Chapter 4, we also give an upper bound for the topological 4-genus of strongly quasipositive 3-braid knots and a more precise bound for certain positive 3-braid knots.

[1] We use the standard signature convention that positive torus knots have negative signatures, e. g. $\sigma(T_{2,3}) = -2$.

2 Strongly quasipositive links are concordant to infinitely many strongly quasipositive links

2.1 Introduction

The main result we prove in this chapter, based on [Tru22], is the following.

Theorem 2.1 (Theorem A)**.** *Every strongly quasipositive link that is not an unlink is smoothly concordant to infinitely many pairwise non-isotopic strongly quasipositive links.*

It follows directly from the definitions that strongly quasipositive links are quasipositive; see Section 1.3 and Figure 1.5 in the introduction. Quasipositive links occur in algebraic geometry as transverse intersections of algebraic curves in $\mathbb{C}^2$ with the 3-sphere $S^3 \subset \mathbb{C}^2$, which provides a geometric characterization of these links [Rud83a, BO01]. In the context of smooth concordance, from now on concordance, (strongly) quasipositive links are special. For example, it follows from the slice-Bennequin inequalities (see Section 1.7) that not every link is concordant to a quasipositive link, and this is contrary to the behavior in the topological category [BF19].

Recall from Sections 1.5 and 1.6 that in contrast to Theorem 2.1, every concordance class in the knot concordance group contains at most finitely many positive knots [BDL17], and at most one algebraic knot [Lit79]. Moreover, it was conjectured in [Bak16] that every concordance class contains at most one strongly quasipositive, fibered knot, and Baker's conjecture is implied by the slice-ribbon conjecture due to Fox [Fox62]. The links constructed in the proof of Theorem 2.1 are however not fibered; see Remark 2.5.

Remark 2.2. In [Bak16], Baker explains a strategy personally communicated to him by Hedden which directly shows that, contrary to the conjectured result for strongly quasipositive, fibered knots, there are (infinitely many) pairs of (ribbon) concordant, strongly quasipositive knots that are not isotopic. Indeed, the positive k-twisted Whitehead doubles of two concordant, non-isotopic knots provide examples of such pairs for negative, sufficiently small k. In particular, there are concordance classes of knots that contain more than one strongly quasipositive knot. Note that the statement of Theorem 2.1 is stronger, since it shows that *every* strongly quasipositive knot other than the unknot is concordant to infinitely many strongly quasipositive knots.

This project began with the observation that, using the above idea of Hedden but being careful about the choice of k, it is not difficult to construct an infinite family of concordant, pairwise non-isotopic, strongly quasipositive knots. Indeed, we can take the positive (-1)-twisted Whitehead doubles of an infinite family of concordant, pairwise non-isotopic knots, all of which have maximal Thurston–Bennequin number TB equal to -1. For instance, for a slice knot R with $\mathrm{TB}(R) = -1$ (see Section 2.2 for an example of such a knot), the connected sums of m copies of R for $m \geqslant 1$ can serve as the latter infinite family. In Remark 2.9, we explain this in more detail.

Organization of this chapter. To prove Theorem 2.1, we will first establish some notations and definitions regarding quasipositive Seifert surfaces and study examples of such surfaces in Section 2.2. In Section 2.3, we will construct from two quasipositive Seifert surfaces F_1 and F_2 for links ∂F_1 and ∂F_2 a third one which has as boundary a link which is concordant, but not isotopic to ∂F_2. The surface F_1 will be one of the quasipositive annuli from Section 2.2. We will finally prove Theorem 2.1 in Section 2.4, leaving the proof of the technical Lemma 2.3 for Section 2.5.

2.2 Quasipositive Seifert surfaces and particular quasipositive annuli

We first define quasipositive Seifert surfaces. Let L be a link that arises as the closure $\widehat{\beta}$ of a strongly quasipositive braid $\beta \in B_n$, $n \geqslant 1$, which is a product of $m \geqslant 0$ positive band words $\sigma_{i,j}$ (see (1.1) in Section 1.3). Recall that we refer to such a product as a strongly quasipositive braid word, and, despite a minor abuse of notation, also denote it by β. There is a canonical Seifert surface of Euler characteristic $n - m$ for L associated to the braid word β. It consists of n copies of disjoint parallel disks and m half-twisted bands connecting these disks [Rud83b, Rud92a]; see Figure 2.1 for an example. Recall that a Seifert surface (for L) is an oriented, compact surface in S^3 (with oriented boundary L) without closed components. We will denote the canonical Seifert surface associated to $L = \widehat{\beta}$ by $F(\beta)$. We call any Seifert surface F for a link $L = \partial F$ *quasipositive* if, for some strongly quasipositive braid word $\beta \in B_n$, $n \geqslant 1$, it is ambient isotopic to the canonical Seifert surface $F(\beta)$. We will be particularly interested in certain quasipositive annuli.

Let R be a non-trivial slice knot that has maximal Thurston–Bennequin number $\mathrm{TB}(R) = -1$, e. g. the mirror of the knot 9_{46} from Rolfsen's knot table [Ng01, Rol03, LM23], which we denote by $\mathrm{m}\,(9_{46})$. For our purposes, we could use any such knot R, but for the sake of concreteness of our illustrations we will fix $R = \mathrm{m}\,(9_{46})$ in the entire chapter. Recall that every knot K has a Legendrian representative (which is at every point in S^3 tangent to the 2-planes of the standard contact structure on S^3) and its maximal Thurston–Bennequin number $\mathrm{TB}\,(K)$ is defined as

$$\mathrm{TB}\,(K) = \max\{\mathrm{tb}(\mathcal{L}) \mid \mathcal{L} \text{ is a Legendrian representative of } K\}.$$

Here, for a Legendrian knot $\mathcal{L}$, $\mathrm{tb}(\mathcal{L})$ denotes its Thurston–Bennequin number; see e. g. [Etn05] for a definition.

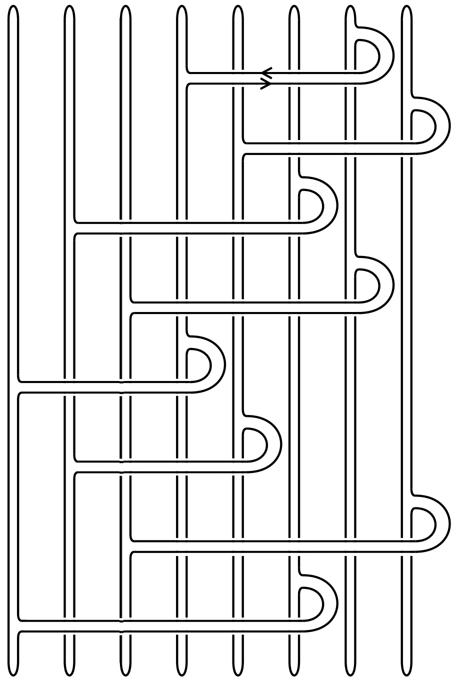


Figure 2.1: The annulus $A(R,-1)$ for $R = \mathrm{m}\,(9_{46})$ is ambient isotopic to $F(\alpha)$ for $\alpha = \sigma_{1,6}\sigma_{3,8}\sigma_{2,5}\sigma_{1,4}\sigma_{3,7}\sigma_{2,6}\sigma_{5,8}\sigma_{4,7} \in B_8$.

Figure 2.2 shows the front projection of a Legendrian representative $\mathcal{L}$ of $\mathrm{m}\,(9_{46})$ with $\mathrm{tb}(\mathcal{L}) = -1$. There is a Lagrangian concordance between the Legendrian unknot $\mathcal{U}$ with $\mathrm{tb}(\mathcal{U}) = -1$ and $\mathcal{L}$; see [Cha15, Figure 4]. In particular, the knot $\mathrm{m}\,(9_{46})$ is slice, and since $\mathrm{TB}(K) \leqslant -1$ for every slice knot K [Rud95], this implies $\mathrm{TB}\,(\mathrm{m}\,(9_{46})) = -1$.

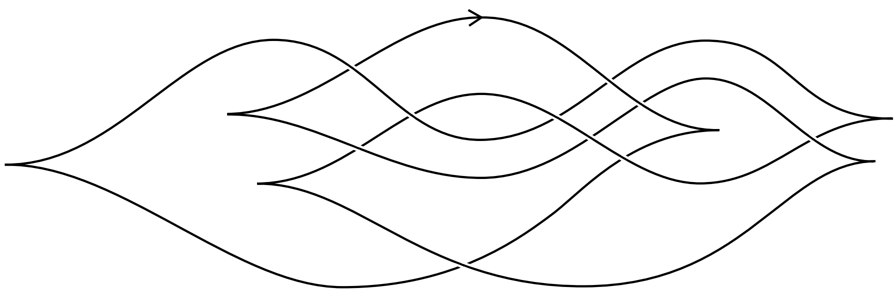


Figure 2.2: Front projection of a Legendrian representative $\mathcal{L}$ of $\mathrm{m}\,(9_{46})$ with $\mathrm{tb}(\mathcal{L}) = -1$; cf. [Cha15, Figure 1].

For a knot K and an integer k, following Rudolph's notation [Rud92b], let $A\,(K,k)$ denote an annulus of type K with k full twists, i. e. $A\,(K,k)$ is an annulus in S^3 with

$K \subset \partial A(K,k)$ and such that the linking number $\operatorname{lk}(K, \partial A(K,k) \setminus K) = -k$. We have

$$\sup\{k \mid A(K,k) \text{ is quasipositive}\} = \operatorname{TB}(K) \qquad \text{[Rud95, Proposition 1]}.$$

Hence, for every knot K with $\operatorname{TB}(K) = -1$, the annulus $A(K,-1)$ is quasipositive, in particular for $K = R = \operatorname{m}(9_{46})$. This is the key observation of this section. It implies the existence of a strongly quasipositive braid word $\alpha \in B_m$ for some $m \geqslant 1$ such that $A(R,-1)$ is ambient isotopic to $F(\alpha)$. For example, we can choose

$$\alpha = \sigma_{1,6}\sigma_{3,8}\sigma_{2,5}\sigma_{1,4}\sigma_{3,7}\sigma_{2,6}\sigma_{5,8}\sigma_{4,7} \in B_8; \tag{2.1}$$

see Figure 2.1. Note that for any knot K with $\operatorname{TB}(K) = k$, a quasipositive diagram for the annulus $A(K,k)$ can be found by taking the Legendrian ribbon of a Legendrian representative $\mathcal{L}$ of K with $\operatorname{tb}(\mathcal{L}) = k$ [Rud84, Rud92b].

2.3 Tying knots into bands of quasipositive Seifert surfaces preserving quasipositivity

Let F be a quasipositive Seifert surface for a link L that is not an unlink. Moreover, let $R = \operatorname{m}(9_{46})$ such that the annulus $A(R,-1)$ is quasipositive (see Section 2.2). As mentioned in Section 2.2, for R we could also use every other non-trivial slice knot with maximal Thurston–Bennequin number $\operatorname{TB}(R) = -1$.

In this section, starting from the quasipositive Seifert surfaces $A(R,-1)$ and F, we will define a new quasipositive Seifert surface F' that will have as boundary a link which is concordant, but not isotopic to $L = \partial F$. To that end, for both $A(R,-1)$ and F, choose strongly quasipositive braid words $\alpha \in B_m$ and $\beta \in B_n$ for $m, n \geqslant 1$, respectively, such that $A(R,-1)$ is ambient isotopic to $F(\alpha)$ and F is ambient isotopic to $F(\beta)$. For example, we can and will choose α as in (2.1) from Section 2.2. Let $\beta = \prod_{k=1}^{\ell} \sigma_{i_k,j_k}$. We can put the surfaces $F(\alpha)$ and $F(\beta)$ in split position in S^3 as sketched in Figure 2.5(a). Concretely, we can take $F(\alpha)$ to lie in the lower hemisphere and $F(\beta)$ to lie in the upper hemisphere of S^3, respectively. Then we can choose a cylinder $Z \subset S^3$ such that the bands B_α of $F(\alpha)$ and B_β of $F(\beta)$ corresponding to the positive band words $\sigma_{4,7}$ and σ_{i_1,j_1}, respectively, intersect Z as indicated in the upper part of Figure 2.3 (ignoring the red curve for now). More precisely, we can choose a cylinder $Z \subset S^3$ and an orientation-preserving diffeomorphism $\varphi\colon Z \to D^2 \times [0,1]$ such that $Z \cap F(\alpha) = Z \cap B_\alpha$, $Z \cap F(\beta) = Z \cap B_\beta$ and φ maps

$$\begin{aligned} Z \cap B_\alpha &\xrightarrow[\varphi]{\cong} \left[-\frac{2}{3}, -\frac{1}{3}\right] \times [0,1], \qquad & Z \cap \partial B_\alpha &\xrightarrow[\varphi]{\cong} \left\{-\frac{2}{3}, -\frac{1}{3}\right\} \times [0,1], \\ Z \cap B_\beta &\xrightarrow[\varphi]{\cong} \left[\frac{1}{3}, \frac{2}{3}\right] \times [0,1], \qquad & Z \cap \partial B_\beta &\xrightarrow[\varphi]{\cong} \left\{\frac{1}{3}, \frac{2}{3}\right\} \times [0,1], \end{aligned} \tag{2.2}$$

where $X \xrightarrow{\cong} Y$ indicates an orientation-preserving diffeomorphism. Here, $[a,b]$ denotes the straight line segment connecting a and b in the closed unit disk D^2 in $\mathbb{C}$. We choose the orientations on $D^2 \times [0,1]$ and $[a,b] \times [0,1]$ induced by the standard orientations on $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3 \subset S^3$ and $\mathbb{R}^2$, respectively; the orientation on $[a,b] \times [0,1]$ also induces an orientation on $\{a,b\} \times [0,1]$.

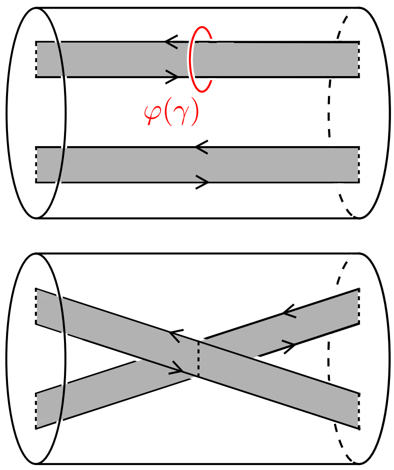


Figure 2.3: Top: The triple $\big(D^2 \times [0,1], [-\frac{2}{3}, -\frac{1}{3}] \times [0,1] \cup [\frac{1}{3}, \frac{2}{3}] \times [0,1], \{\pm\frac{1}{3}, \pm\frac{2}{3}\} \times [0,1]\big)$, which is mapped to $(Z, Z \cap (B_\alpha \cup B_\beta), Z \cap \partial(B_\alpha \cup B_\beta))$ via φ. The red curve depicts $\varphi(\gamma) \subset D^2 \times [0,1]$; see (2.3). Bottom: $B' \subset D^2 \times [0,1]$ for B' as defined in (2.4).

We claim that we can choose Z and $\varphi \colon Z \xrightarrow{\cong} D^2 \times [0,1]$ such that φ satisfies (2.2) and such that there exists a simple closed curve γ in $S^3 \setminus F(\beta)$ that goes once around the band B_β of $F(\beta)$ corresponding to σ_{i_1,j_1} and that is not null-homotopic in $S^3 \setminus \partial F(\beta)$. More precisely, we claim that we can choose Z and φ satisfying (2.2) such that

$$\gamma = \varphi^{-1}\left(C_{\frac{1}{3}}\left(\frac{1}{2}\right) \times \left\{\frac{1}{2}\right\}\right) \subset S^3 \setminus F(\beta) \tag{2.3}$$

is a simple closed curve that is not null-homotopic in $S^3 \setminus \partial F(\beta)$, where $C_{\frac{1}{3}}\left(\frac{1}{2}\right) \subseteq D^2$ denotes the circle with center $\frac{1}{2}$ and radius $\frac{1}{3}$. The situation is shown in the upper part of Figure 2.3 with $\varphi(\gamma) \subset D^2 \times S^1$ in red. The above claim follows from the following lemma.[1]

Lemma 2.3. *Let $F(\beta)$ denote the canonical Seifert surface associated to a strongly quasipositive braid word β such that $\partial F(\beta)$ is not an unlink. Then we can choose a cylinder $Z' \subset S^3$ and an orientation-preserving diffeomorphism φ' of triples of manifolds*

[1]Note that up to conjugation of β or ambient isotopy of $F(\beta)$, that is, up to a different choice of Z and φ, we can choose any positive band word of β to be the first one σ_{i_1,j_1}.

with corners

$$\begin{aligned}\varphi' \colon\ &(Z', Z' \cap F(\beta), Z' \cap \partial F(\beta))\\ &\xrightarrow{\cong} \left(D^2 \times [0,1], \left[-\frac{1}{2}, \frac{1}{2}\right] \times [0,1], \left\{-\frac{1}{2}, \frac{1}{2}\right\} \times [0,1]\right)\end{aligned}$$

such that $B_\beta = Z' \cap F(\beta)$ *is a band of* $F(\beta)$ *corresponding to a positive band word of* β *and such that* $\gamma = (\varphi')^{-1}((\partial D^2) \times \{\frac{1}{2}\})$ *is a simple closed curve in* $S^3 \setminus F(\beta)$ *which is not null-homotopic in* $S^3 \setminus \partial F(\beta)$*. Moreover, either the two components of* $\partial B_\beta \cap \partial F(\beta)$ *belong to two different components of the link* $\partial F(\beta)$ *or we can assume that there exists some quasipositive Seifert surface* G *that is a connected component of* $F(\beta)$ *with boundary a knot* $J = \partial G$ *(which is one of the components of* $\partial F(\beta)$*) such that* $B_\beta \subset G$ *and* γ *is not null-homotopic in* $S^3 \setminus J$*.*

For the proof of Lemma 2.3, we refer the reader to Section 2.5. The necessity of these assumptions will become clear later. Now, let $B = Z \cap (B_\alpha \cup B_\beta)$ for bands B_α and B_β of $F(\beta)$ and a cylinder Z as in (2.2) and define $F' = (F(\alpha) \cup F(\beta)) \setminus B \cup \varphi^{-1}(B')$, where B' is as in the lower part of Figure 2.3. More precisely, we let

$$\begin{aligned}B' = &\left\{(a+t, t) \mid t \in [0,1],\ a \in \left[-\frac{2}{3}, -\frac{1}{3}\right]\right\}\\ &\cup \left\{(a - t + ti, t) \mid t \in \left[0, \frac{1}{2}\right],\ a \in \left[\frac{1}{3}, \frac{2}{3}\right]\right\}\\ &\cup \left\{(a - t + (1-t)i, t) \mid t \in \left[\frac{1}{2}, 1\right],\ a \in \left[\frac{1}{3}, \frac{2}{3}\right]\right\} \subseteq D^2 \times [0,1].\end{aligned} \tag{2.4}$$

In the definition of B' in (2.4) (and only there in this chapter), $i \in \mathbb{C}$ denotes the imaginary unit. We smooth the corners of $\varphi^{-1}(B')$ to obtain a smooth surface F' and claim the following.

Lemma 2.4. *Let* Z *and* φ *be defined as above such that* (2.2) *is satisfied and such that* γ *as in* (2.3) *is a simple closed curve in* $S^3 \setminus F(\beta)$ *that goes once around the band* B_β *of* $F(\beta)$ *and is not null-homotopic in* $S^3 \setminus \partial F(\beta)$*. Moreover, assume that either the two components of* $\partial B_\beta \cap \partial F(\beta)$ *belong to two different components of the link* $\partial F(\beta)$ *or there exists a connected component* G *of* $F(\beta)$ *with boundary a knot* $J = \partial G$ *such that* $B_\beta \subset G$ *and* γ *is not null-homotopic in* $S^3 \setminus J$*. Then the surface* $F' = (F(\alpha) \cup F(\beta)) \setminus B \cup \varphi^{-1}(B')$ *with smoothed corners, where* $B = Z \cap (B_\alpha \cup B_\beta)$ *and* B' *is defined as in* (2.4)*, is a quasipositive Seifert surface for a link* $\partial F'$ *that is concordant, but not isotopic to* $\partial F(\beta)$*.*

Proof of Lemma 2.4. We will show the following two claims separately.

Claim 1. *The surface* F' *is quasipositive.*

Claim 2. *The boundary of* F' *is concordant, but not isotopic to* $\partial F(\beta)$*.*

Proof of Claim 1: The quasipositivity of F' can be shown using an isotopy as depicted in Figures 2.5(b) to 2.5(d) on the next page. A strongly quasipositive braid word δ such that $F(\delta)$ is ambient isotopic to F' can then be read off in Figure 2.5(d). ■

Proof of Claim 2: Observe that the surface F' is obtained from $F(\beta)$ by tying the knot R with framing 0 into the band B_β of $F(\beta)$; see Figure 2.4.

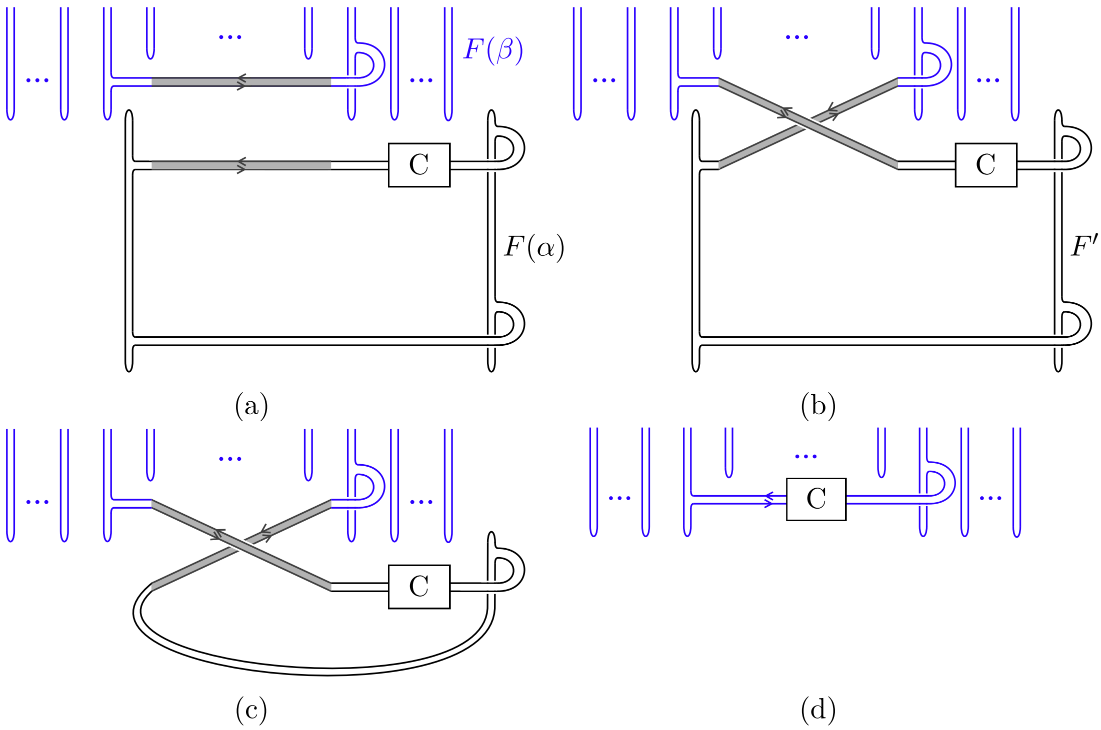


Figure 2.4: The surface F' is obtained from $F(\beta)$ by tying the knot R into the band B_β corresponding to the positive band word σ_{i_1,j_1} of β. Subfigure 2.4(a) shows a schematic representation of the surfaces $F(\alpha)$ and $F(\beta)$, and subfigure 2.4(b) one of F'. Subfigures 2.4(c) and 2.4(d) indicate an ambient isotopy between F' and the surface $F(\beta)$ with the knot R tied into B_β with framing 0.

This amounts to realizing the boundary of F' as a satellite with pattern $\partial F(\beta)$ and companion R. We explain this in detail. The link $\partial F(\beta)$ can be viewed as a link in the solid torus $S^3 \setminus \nu(\gamma)$ given by the complement of an open tubular neighborhood $\nu(\gamma)$ of γ in S^3. We can identify this solid torus with $V = D^2 \times S^1 \subset S^3$ by an orientation-preserving diffeomorphism that takes the preferred longitude of $S^3 \setminus \nu(\gamma)$ to $\{1\} \times S^1 \subset V$. Then $\partial F'$ arising as a satellite link with pattern $\partial F(\beta)$ and companion R means that it is the image of $\partial F(\beta) \subset S^3 \setminus \nu(\gamma) \cong V$ under an orientation-preserving embedding $h\colon V = D^2 \times S^1 \to S^3$ that maps $\{0\} \times S^1$ to R and $\{1\} \times S^1$ to a curve that has linking number 0 with $h(\{0\} \times S^1)$. For more details on satellite constructions and the terms used here, see [Rol03, Sections 2E and 4D]. Our choices ensure that h is faithful in Rolfsen's terminology and that the companion is really R and not the reverse of R.

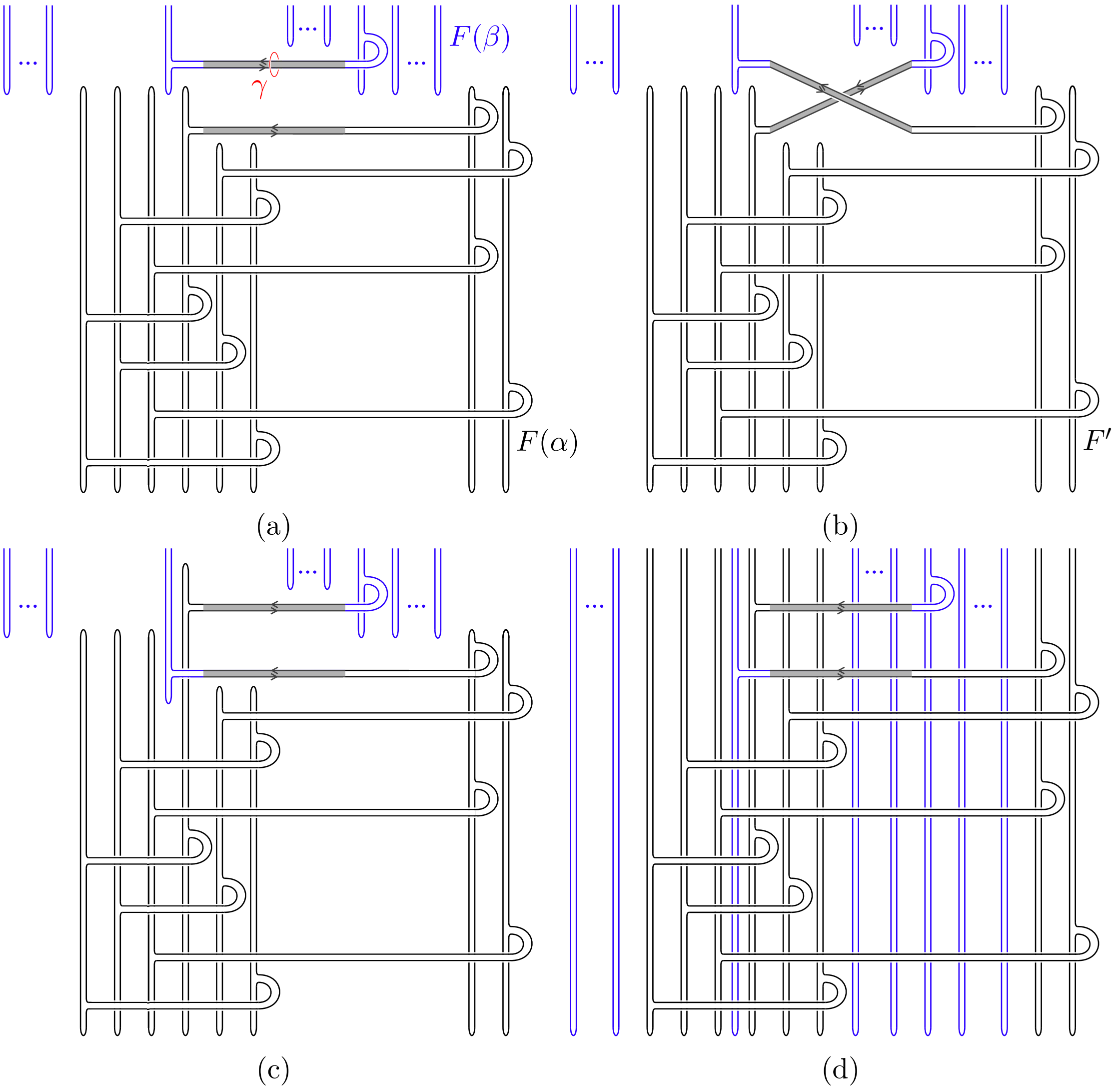


Figure 2.5: Quasipositivity of the surface F' defined from $F(\alpha)$ and $F(\beta)$ (see Lemma 2.4). The surface $F(\alpha)$ is shown in black, the surface $F(\beta)$ in blue, and $Z \cap (F(\alpha) \cup F(\beta))$ in grey. Subfigure 2.5(a) shows $F(\alpha)$ and $F(\beta)$ together with γ as in Lemma 2.4. Subfigure 2.5(b) shows the surface F' which is ambient isotopic to the canonical quasipositive Seifert surface in 2.5(d); an intermediate stage of such an isotopy is shown in 2.5(c).

Now, it is a standard fact from concordance theory that, since R is slice, so concordant to the unknot U, there is a concordance between $\partial F'$, the satellite with pattern $\partial F(\beta)$ and companion R, and $\partial F(\beta)$, the satellite with same pattern but companion U. Indeed, if R and U are concordant via an annulus $A \cong S^1 \times [0,1] \subset S^3 \times [0,1]$, then we can identify $(S^3 \setminus \nu(\gamma)) \times [0,1]$ with a tubular neighborhood of A in $S^3 \times [0,1]$ and the image of $(\partial F(\beta)) \times [0,1]$ in $S^3 \times [0,1]$ under this identification provides us with a concordance between the two satellite links.

On the other hand, we claim that since R is not isotopic to U, the satellite links $\partial F'$ and $\partial F(\beta)$ are not isotopic. To prove this claim, we distinguish two cases. Note that the two components of $\partial B_\beta \cap \partial F(\beta)$ do not necessarily belong to the same component of the link $\partial F(\beta)$.

We first assume that they do, which is the case, for example, if $\partial F(\beta)$ is a knot; and we can further assume that γ is not null-homotopic in the complement of this component J of $\partial F(\beta)$ in S^3 (see the assumptions in the lemma). The claim then follows from work of Kouno and Motegi [KM94, Theorem 1.1] since in this case our satellite operation modifies up to ambient isotopy only the component J of $\partial F(\beta)$ by applying a satellite operation with companion R and pattern J; that $\partial F'$ and $\partial F(\beta)$ are not isotopic follows from the fact that this satellite operation on the non-isotopic knots R and U produces non-isotopic components of $\partial F'$ and $\partial F(\beta)$. Here we need the assumptions on γ, which imply that the pattern J we use in the satellite construction has wrapping number strictly greater than 1. The *wrapping number* $\omega_V(P)$ of a pattern P in the solid torus $V = D^2 \times S^1$ is the minimal geometric intersection number of P and a generic meridional disk of V. Recall that we can consider $\partial F(\beta)$ and hence also its component J as a link in the solid torus $S^3 \setminus \nu(\gamma)$, which we identify with V by an orientation-preserving diffeomorphism that takes the preferred longitude of $S^3 \setminus \nu(\gamma)$ to $\{1\} \times S^1$. Then γ not being null-homotopic in $S^3 \setminus J$ implies that J geometrically intersects non-trivially every meridional disk in $S^3 \setminus \nu(\gamma) \cong V$, so $\omega_V(J) \neq 0$. Since the algebraic winding number of J in $S^3 \setminus \nu(\gamma) \cong V$ is zero (thus even), we get $\omega_V(J) > 1$.

Now, suppose that the two components of $\partial B_\beta \cap \partial F(\beta)$ belong to two different components L_1 and L_2 of the link $\partial F(\beta)$. The satellite operation then has the effect of tying R into both of these components (up to orientation), i. e. the resulting link has components $L_1 \# R$ and $L_2 \# R^r$, where R^r denotes R with the reversed orientation, and all other components unchanged. Note that for our particular choice $R = \mathrm{m}\,(9_{46})$ we have $R = R^r$ [LM23]. This clearly produces a link $\partial F'$ that is not isotopic to $\partial F(\beta)$. ■

This conclues the proof of Lemma 2.4. □

Remark 2.5. The link $\partial F'$ from Lemma 2.4 is not fibered. Indeed, we can use the Seifert–van Kampen theorem to show that the fundamental group of the complement of R embeds into the fundamental group of the complement of F' in S^3, which is thus not free. Hence, the constructed Seifert surface F' is not a fiber surface for $\partial F'$, and since it is a surface with maximal Euler characteristic for $\partial F'$, the link $\partial F'$ cannot be fibered.

2.4 Proof of Theorem 2.1

Recall the statement of Theorem 2.1: Every strongly quasipositive link that is not an unlink is smoothly concordant to infinitely many pairwise non-isotopic such links.

Proof of Theorem 2.1. Let L be a link other than an unlink, let F be a quasipositive Seifert surface for L and let $R = \mathrm{m}\,(9_{46})$ such that the annulus $A\,(R,-1)$ is quasipositive (see Section 2.2). Let α be as in (2.1) from Section 2.2 such that $A\,(R,-1)$ is ambient isotopic to $F(\alpha)$, and, as in Section 2.3, choose a strongly quasipositive braid word $\beta \in B_n$, $n \geqslant 1$, such that F is ambient isotopic to $F(\beta)$.

The statement of Theorem 2.1 will follow from an iterative application of the operation defined in Section 2.3: Given two quasipositive Seifert surfaces $F(\alpha)$ and $F(\beta)$ for links $\partial F(\alpha)$ and $\partial F(\beta)$, respectively, using Lemma 2.3 and Lemma 2.4 we can construct a quasipositive Seifert surface F' with boundary that is concordant, but not isotopic to $\partial F(\beta)$. We will denote this surface by $F(\alpha) \oplus F(\beta) \coloneqq F'$. We define $F_0 = F(\beta)$, $F_1 = F' = F(\alpha) \oplus F(\beta)$ and, inductively, $F_{i+1} = F(\alpha) \oplus F_i$ for all $i \geqslant 1$. The links $\{\partial F_i\}_{i \geqslant 0}$ are then all in the same concordance class (the class of $L = \partial F_0 = \partial F(\beta)$), but pairwise non-isotopic. Let us make this more precise. Recall that we constructed the surface $F_1 = F'$ by tying the knot R into a specific band B_β of $F(\beta)$ (see Section 2.3) which implied that we obtained ∂F_1 as a satellite with pattern $\partial F(\beta)$ and companion R (see the proof of Lemma 2.4). The surfaces $F(\beta)$ and F_1 are both quasipositive Seifert surfaces (see Lemma 2.4) that can again be put in a position where we can choose a cylinder $Z_1 \subset S^3$ and an orientation-preserving diffeomorphism $\varphi_1 \colon Z_1 \to D^2 \times [0,1]$ that satisfies a condition equivalent to the one in (2.2) from Section 2.3 for Z and φ. By Lemma 2.3 and Lemma 2.4, we can choose Z_1 and φ_1 such that the surface F_2 obtained from F_1 by tying the knot R into a specific band of F_1 is quasipositive and has as boundary ∂F_2 a link that is concordant, but not isotopic to ∂F_1. Inductively, F_{i+1} is obtained from F_i by tying R into a band of F_i such that ∂F_{i+1} is concordant, but not isotopic to ∂F_i. However, up to an ambient isotopy of F_1 and F_2, we can assume that for F_2 we tie R into the "same band" of F_1 as B_β of $F(\beta)$.[2] Note that the additional assumptions in Lemma 2.4 about this band will still be satisfied.

As in the proof of Claim 2 in the proof of Lemma 2.4, we now distinguish two cases. If the two components of $\partial B_\beta \cap \partial F(\beta)$ belong to the same component J of the link $\partial F(\beta)$, then we actually obtain ∂F_2 as $P_J(R\#R)$, the satellite with pattern $P_J = J$, but companion $R\#R$. Inductively, we get

$$\partial F_i = P_J(\underbrace{R\#R\ldots\#R}_{i \text{ times}}).$$

Since R is not isotopic to the unknot, the connected sums of i and k copies of R,

[2]To make the term "same band" more precise, we could fix an abstract embedding of the surface $F(\beta)$ throughout.

respectively, are not isotopic for $i \neq k$ (e. g. by arguing with the additivity of the nonzero genus of R). It thus follows from [KM94, Theorem 1.1] that

$$\partial F_i = P_J(\underbrace{R\#R\ldots\#R}_{i \text{ times}}) \quad \text{and} \quad \partial F_k = P_J(\underbrace{R\#R\ldots\#R}_{k \text{ times}})$$

are not isotopic if $i \neq k$. Again, it is important that the pattern J has wrapping number strictly greater than 1 under the assumptions of Lemma 2.4 for $F(\beta)$.

If the two components of $\partial B_\beta \cap \partial F(\beta)$ belong to different components L_1 and L_2 of the link $\partial F(\beta)$, then, by induction, the link ∂F_i has components

$$L_1\#\underbrace{R\#R\ldots\#R}_{i \text{ times}} \quad \text{and} \quad L_2\#\underbrace{R^r\#R^r\ldots\#R^r}_{i \text{ times}}$$

and so again ∂F_i and ∂F_k are not isotopic if $i \neq k$. □

Remark 2.6. A careful generalization of our proof of Theorem 2.1 and in particular Lemma 2.4 shows the following slightly stronger statement. Let F be a quasipositive Seifert surface for a link L other than an unlink. Then there exists an infinite family $\{\Sigma_i \times [0,1]\}_{i\geqslant 1}$ of smoothly and properly embedded 3-manifolds $\Sigma_i \times [0,1]$ in $S^3 \times [0,1]$ where every Σ_i is a surface such that $\partial(\Sigma_i \times [0,1]) = F \times \{0\} \cup \Sigma_i' \times \{1\}$ for some quasipositive Seifert surface Σ_i' with boundary $\partial\Sigma_i'$ non-isotopic to $\partial F = L$ and such that the boundaries $\partial\Sigma_i'$ and $\partial\Sigma_j'$ are non-isotopic for $i \neq j$.

2.5 Proof of Lemma 2.3

In this section, we prove Lemma 2.3, which we recall here (with a slightly different notation) for the reader's convenience.

Lemma 2.7 (Lemma 2.3)**.** *Let $F(\beta)$ denote the canonical Seifert surface associated to a strongly quasipositive braid word β such that $\partial F(\beta)$ is not an unlink. Then we can choose a cylinder $Z \subset S^3$ and an orientation-preserving diffeomorphism φ of triples of manifolds with corners*

$$\begin{aligned} \varphi\colon\ & (Z, Z\cap F(\beta), Z \cap \partial F(\beta)) \\ & \xrightarrow{\cong} \left(D^2 \times [0,1], \left[-\frac{1}{2}, \frac{1}{2}\right] \times [0,1], \left\{-\frac{1}{2}, \frac{1}{2}\right\} \times [0,1]\right) \end{aligned} \tag{2.5}$$

such that $B_\beta = Z \cap F(\beta)$ is a band of $F(\beta)$ corresponding to a positive band word of β and such that $\gamma = \varphi^{-1}((\partial D^2) \times \{\frac{1}{2}\})$ is a simple closed curve in $S^3 \setminus F(\beta)$ which is not null-homotopic in $S^3 \setminus \partial F(\beta)$. Moreover, either the two components of $\partial B_\beta \cap \partial F(\beta)$ belong to two different components of the link $\partial F(\beta)$ or we can assume that there exists some quasipositive Seifert surface G that is a connected component of $F(\beta)$ with boundary a *knot $J = \partial G$ (which is one of the components of $F(\beta)$) such that $B_\beta \subset G$ and γ is not null-homotopic in $S^3 \setminus J$.*

Proof of Lemma 2.7. Let $\beta = \prod_{k=1}^{\ell} \sigma_{i_k,j_k} \in B_n$ for some $n \geqslant 1$. We claim that one of the following is true.

Case 1: There exists a half-twisted band in $F(\beta)$ corresponding to one of the positive band words σ_{i_k,j_k}, $k \in \{1, \dots, \ell\}$, of β such that the boundary of this band intersected with $\partial F(\beta)$ has two components that belong to two different components of the link $\partial F(\beta)$.

Case 2: $F(\beta)$ is a disjoint union of quasipositive Seifert surfaces each of which has only one boundary component.

Here is the argument why: If the half-twisted bands in $F(\beta)$ are such that for each of them the boundary of the band intersected with $\partial F(\beta)$ has two components that belong to the same component of the link $\partial F(\beta)$, then for each of the disks in $F(\beta)$, there is a component of $\partial F(\beta)$ such that the entire boundary of the disk belongs to that component. All the bands emanating from a disk must belong to the same component of $\partial F(\beta)$ as the boundary of that disk, and so every connected component of $F(\beta)$ must have only one component of $F(\beta)$ (a knot) as its boundary.

Let us first assume that we are in case 2, so that $F(\beta)$ is a disjoint union of quasipositive Seifert surfaces each of which a knot as its boundary. By assumption, $F(\beta)$ is not a union of disks. Let G be one of the connected components of $F(\beta)$ which is not a disk. We claim that we can choose Z and φ as in (2.5) such that $B_\beta = Z \cap F(\beta)$ is a band of $G \subset F(\beta)$ corresponding to a positive band word of β and such that $\gamma = \varphi^{-1}((\partial D^2) \times \left\{\frac{1}{2}\right\})$ is a simple closed curve in $S^3 \setminus F(\beta)$ which is not null-homotopic in $S^3 \setminus G$. The claim of Lemma 2.7 will then follow from the more general statement in Lemma 2.8, which can be shown using a standard innermost circle argument. Note that quasipositive Seifert surfaces are of minimal genus [Rud93, KM93] and thus incompressible. For the reader's convenience, we will prove Lemma 2.8 below.

Lemma 2.8. *Let F be an incompressible Seifert surface for a link L and let $\gamma \subset S^3 \setminus F$ be a simple closed curve. If there exists a disk in $S^3 \setminus L$ with boundary γ, then there also exists a disk in $S^3 \setminus F$ with boundary γ.*

So if we find Z and φ as in (2.5) such that $\gamma = \varphi^{-1}((\partial D^2) \times \{\frac{1}{2}\})$ is not null-homotopic in $S^3 \setminus G$, it is also not null-homotopic in $S^3 \setminus \partial G$. To conclude the proof of Lemma 2.7 in case 2, it remains to show that this is always possible.

To that end, we claim that there exists a positive band word σ_{i_ℓ,j_ℓ} in β which fulfills the following condition: the core of the half-twisted band B_β of $F(\beta)$ associated to σ_{i_ℓ,j_ℓ} together with an arc in G the interior of which misses B_β unite to a simple closed curve η in G so that η and a meridian of B_β have linking number ± 1. Under a diffeomorphism $\varphi \colon Z \to D^2 \times S^1$ as in (2.5), we can identify any of the half-twisted bands in G with

$[-\frac{1}{2}, \frac{1}{2}] \subset D^2 \times [0,1]$ for an appropriately chosen cylinder $Z \subset S^3$. For us, a *meridian* of a band B_β of G is then a simple closed curve in $S^3 \setminus G$ which is isotopic to $\varphi^{-1}((\partial D^2) \times \{\frac{1}{2}\})$ under this identification, and the *core* of B_β is $\varphi^{-1}(\{0\} \times [0,1])$. If we find a band B_β with the above requirements, the condition on the linking number of η and the meridian $\gamma = \varphi^{-1}((\partial D^2) \times \{\frac{1}{2}\})$ of B_β will imply that γ cannot be null-homotopic in $S^3 \setminus G$.

The quasipositive Seifert surface G deformation retracts onto a graph Γ in S^3 where vertices of Γ correspond to the disks of G and edges of Γ correspond to the bands of G, respectively. Since G is not disk, Γ is not a tree, hence there must exist an edge e of Γ such that $\Gamma \setminus e$ is not disconnected. This edge e together with a path in Γ connecting the vertices of e, but missing the interior of e, forms a simple closed curve in Γ which has linking number ± 1 with its meridian in $S^3 \setminus \Gamma$. For the desired positive band word σ_{i_ℓ, j_ℓ}, we can take the one corresponding to the edge e.

In summary, we have shown that in case 2 we can choose Z and φ as in (2.5) such that $B_\beta = Z \cap F(\beta)$ is a band of $F(\beta)$ corresponding to a positive band word of β that is contained in one of these Seifert surfaces G and such that $\gamma = \varphi^{-1}((\partial D^2) \times \{\frac{1}{2}\})$ is a simple closed curve in $S^3 \setminus F(\beta)$ that is not null-homotopic in $S^3 \setminus G$ and therefore by Lemma 2.8 not null-homotopic in $S^3 \setminus \partial G$.

Now suppose we are in case 1, so we can choose a cylinder $Z \subset S^3$ and an orientation-preserving diffeomorphism φ as in (2.5) such that $B_\beta = Z \cap F(\beta)$ is a band of $F(\beta)$ corresponding to a positive band word of β where the two components of $\partial B_\beta \cap \partial F(\beta)$ belong to two different components of the link $\partial F(\beta)$. We claim that in this case $\gamma = \varphi^{-1}((\partial D^2) \times \{\frac{1}{2}\})$ is a simple closed curve in $S^3 \setminus F(\beta)$ which is not null-homotopic in $S^3 \setminus F(\beta)$ and thus, by Lemma 2.8, not null-homotopic in $S^3 \setminus \partial F(\beta)$.

Similar as in the argument in case 2 above, the quasipositive Seifert surface $F(\beta)$ deformation retracts onto a graph Γ in S^3 where vertices of Γ correspond to the disks of $F(\beta)$ and edges of Γ correspond to the bands of $F(\beta)$, respectively. Consider the edge e of Γ that corresponds to the band B_β. Since the two components of $\partial B_\beta \cap \partial F(\beta)$ belong to two different components of $\partial F(\beta)$, this edge must be part of a cycle in Γ. This cycle is a simple closed curve in Γ which has linking number ± 1 with its meridian in $S^3 \setminus \Gamma$, so the core of the band B_β together with a certain arc in $F(\beta)$ unite to form a simple closed curve in $F(\beta)$ which has linking number ± 1 with the meridian of B_β and the claim follows. □

Proof of Lemma 2.8. Let $D \subset S^3 \setminus L$ be a disk with $\partial D = \gamma \subset S^3 \setminus F$ and suppose that D intersects F non-trivially. Up to an ambient isotopy, we can assume that D and F intersect transversally in S^3 [GP10]. Then $D \cap F$ is a one-dimensional compact manifold, so a finite collection of simple closed curves. Using the 2-dimensional Schoenflies theorem [Rol03, Section 2A], each of these simple closed curves bounds a disk in D. Let R be one of the simple closed curves in $D \cap F$ which is innermost in the sense that the interior of the disk D' bounded by R in D misses F. Since F is incompressible, R must also bound a disk D'' in F. The union of D' and D'' forms a 2-sphere which, by the 3-dimensional Schoenflies theorem [Rol03, Section 2F], bounds a ball in S^3. We can push F along this ball to obtain a Seifert surface F' for L which is ambient isotopic to F and intersects D in less simple closed curves than F. We repeat this process until we obtain a Seifert surface F'' for L which is ambient isotopic to F and disjoint from D. In summary, up to an ambient isotopy we found the desired disk in $S^3 \setminus F$. □

We conclude with the promised details on the construction in Remark 2.2.

Remark 2.9. We elaborate on how to construct an infinite family of concordant, pairwise non-isotopic, strongly quasipositive knots using Whitehead doubles as sketched in Remark 2.2. Let R be a non-trivial slice knot with maximal Thurston–Bennequin number $\mathrm{TB}(R) = -1$, e. g. $R = \mathrm{m}\,(9_{46})$ (see Section 2.2). For $m \geqslant 1$, let K_m be the connected sum of m copies of R. Then for every $m \geqslant 1$, the knot K_m is slice (since R is) and by inductively using the formula $\mathrm{TB}(L_1 \# L_2) = \mathrm{TB}(L_1) + \mathrm{TB}(L_2) + 1$ for any knots L_1, L_2 [EH03, Tor03], we have $\mathrm{TB}(K_m) = -1$. Note that K_m and K_n are not isotopic for $m \neq n$, since R is non-trivial. Using the notation from [Hed07], we now define $J_m \coloneqq D_+(K_m, -1)$ as the positive (-1)-twisted Whitehead double of K_m. Then $\{J_m\}_{m \geqslant 1}$ is the desired infinite family. Indeed, using $\mathrm{TB}(K_m) \geqslant -1$, by work of Rudolph (see e. g. [Rud05, 102.4]) each J_m is strongly quasipositive. Moreover, as K_m and K_n are not isotopic for $m \neq n$, the knots J_m and J_n are not isotopic either for such m and n [KM94]. On the other hand, J_m and J_n are concordant for every $m \neq n$ as K_m and K_n are. Indeed, as noted in the proof of Lemma 2.4, the satellite operation induces a well-defined map on the concordance group of which taking the positive twisted Whitehead double is a special case.

3 The upsilon invariant at 1 of 3-braid knots

3.1 Introduction

In this chapter, we study knots in S^3, i. e. links of one connected component, as usual up to ambient isotopy. Recall from Section 1.4 that two knots K and J are concordant if there exists an annulus $A \cong S^1 \times [0,1]$ smoothly and properly embedded in $S^3 \times [0,1]$ such that $\partial A = K \times \{0\} \cup J \times \{1\}$ and such that the induced orientation on the boundary of the annulus agrees with the orientation of K, but is the opposite one on J. In [OSS17a], Ozsváth, Stipsicz and Szabó used a variant of the Heegaard Floer chain complex $CFK^-(K)$ to define the invariant Υ_K of a knot K, which induces a homomorphism from the knot concordance group $\mathcal{C}$ to the group of real-valued piecewise linear functions on the interval $[0,2]$. Recall that we call the concordance invariant $\upsilon(K) = \Upsilon_K(1) \in \mathbb{Z}$, the value of the function Υ_K at 1, *upsilon of* K.

Recall further (see Section 1.2) that a 3-braid is an element of the braid group B_3 on three strands with generators $a = \sigma_1$ and $b = \sigma_2$ and relation $aba = bab$, the braid relation. A braid word β—a word in the generators of B_3 and their inverses—defines a diagram for a (geometric) 3-braid; the generators a and b correspond to the geometric 3-braids given by braid diagrams as in Figure 3.1(a). In our figures, braid diagrams will always be oriented from bottom to top. We denote by Δ the braid $aba = bab$ (see Figure 3.1(b)), and note that its square $\Delta^2 = (ab)^3$ (the positive full twist on three strands) generates the center of B_3 [Cho48, Theorem 3]. A (positive) 3-braid knot is a knot that arises as the closure $\widehat{\beta}$ of a (positive) 3-braid β.

(a) The two generators a and b of B_3.   (b) The braid relation $aba = bab$ in B_3.

Figure 3.1: Generators and relation in the braid group B_3.

As our main result in this chapter, based on [Tru21], we determine the upsilon invariant for all 3-braid knots. More precisely, we show the following.

Theorem 3.1 (Theorem C)**.** *Let $\beta = \Delta^{2\ell} a^{-p_1} b^{q_1} a^{-p_2} b^{q_2} \cdots a^{-p_r} b^{q_r}$ be a braid word in the generators a and b of B_3 for some $\ell \in \mathbb{Z}$, and integers $r \geqslant 1$ and $p_i, q_i \geqslant 1$ for $i \in \{1, \ldots, r\}$. Suppose that the closure $K = \widehat{\beta}$ of β is a knot. Then its upsilon invariant is*

$$\upsilon(K) = \frac{\sum_{i=1}^{r} (p_i - q_i)}{2} - 2\ell.$$

Note that it follows from Murasugi's classification of the conjugacy classes of 3-braids [Mur74, Proposition 2.1] that indeed all 3-braid knots, except for the torus knots that are closures of 3-braids, are covered by Theorem 3.1. For torus knots, however, the invariant υ can be computed explicitly by a combinatorial inductive formula in terms of its Alexander polynomial [OSS17a, Theorem 1.15]; see (3.8) below. Hence, we have indeed determined $\upsilon(K)$ for all 3-braid knots K.

As an application of Theorem 3.1, we show that for positive 3-braid knots K, several alternating distances all equal the sum $g(K) + \upsilon(K)$; see Corollary 3.2 below. Recall from Section 1.7 that $g(K)$ denotes the 3-genus of K, the minimal genus of an oriented, connected, compact smooth surface in S^3 with oriented boundary the knot K. In particular, we compute the alternation number, the dealternating number and the Turaev genus for all positive 3-braid knots. We will review the definitions of these invariants and prove Corollary 3.2 in Section 3.5. We will also give upper and lower bounds on the alternation number and dealternating number of each 3-braid knot that differ by 1.

Corollary 3.2 (Corollary D)**.** *Let K be a positive 3-braid knot, i. e. a knot that is the closure of an element of B_3 that can be written as a word in the generators a and b only (no inverses). Then*

$$\mathrm{alt}(K) = \mathrm{dalt}(K) = g_T(K) = \mathcal{A}_s(K) = g(K) + \upsilon(K).$$

Another corollary of Theorem 3.1 for positive 3-braid knots is the following.

Corollary 3.3 (Corollary B)**.** *Let K be a positive 3-braid knot. Then the minimal r such that K is the closure of $a^{p_1} b^{q_1} a^{p_2} b^{q_2} \cdots a^{p_r} b^{q_r}$ for integers $p_i, q_i \geqslant 1$, $i \in \{1, \ldots, r\}$, is $r = g(K) + \upsilon(K) + 1$. Moreover, if K and J are concordant positive 3-braid knots, then this minimal r is the same for both K and J.*

Proposition 3.8 in Section 3.3 provides a normal form for 3-braids, the *Garside normal form*, which is different from the Murasugi normal form mentioned above (see Definition 3.26). The Garside normal form allows us to read off from a braid word whether it is conjugate to a positive braid word. In Corollary 3.43 in Section 3.6, we provide formulas for the fractional Dehn twist coefficient for all 3-braids in Garside normal form.

Proof strategy for Theorem 3.1. A crucial property of the invariant υ is that it provides a lower bound on the 4-genus $g_4(K)$ of a knot K: we have

$$|\upsilon(K)| \leqslant g_4(K) \tag{3.1}$$

for any knot K [OSS17a, Theorem 1.11]. Recall that the (smooth) 4-genus of a knot K is the minimal genus of an oriented, connected, compact surface smoothly embedded in B^4 with oriented boundary K in $S^3 = \partial B^4$ (see Section 1.7). Our general strategy to find $\upsilon(K)$ for every 3-braid knot K will be to construct a cobordism between K and another knot J for which the value of υ is known. A *cobordism* between K and J is an oriented, connected, compact surface C smoothly and properly embedded in $S^3 \times [0,1]$ with boundary $K \times \{0\} \cup J \times \{1\}$ such that the induced orientation on the boundary of C agrees with the orientation of K and disagrees with the orientation of J. Note that a concordance is a cobordism of genus 0. We have

$$|\upsilon(K) - \upsilon(J)| \leqslant g(C) \tag{3.2}$$

for any cobordism C between K and J, where $g(C)$ denotes the genus of the cobordism; see (3.11) in Section 3.4.1. This provides bounds on $\upsilon(K)$ in terms of $\upsilon(J)$ and $g(C)$.

We will find such cobordisms for example by algebraic modifications of a braid word representing K and by saddle moves corresponding to the addition or deletion of generators from such braid words. We will also use repeatedly the trick described in Example 3.13 in Section 3.4.1 of looking at cobordisms of genus 1 between $\widehat{\beta} \# T_{2,2n+1}$ and $\widehat{\beta b^{2n}}$ for 3-braid words β and $n \geqslant 1$.

To prove Theorem 3.1, we will first determine υ for all positive 3-braid knots and then generalize our computations to all 3-braid knots. In Section 3.4.4.2, we will explain why this extension of our technique was somewhat surprising to the author.

Remark 3.4. As we will only use properties of the upsilon invariant (see Section 3.2) and not its definition, we can similarly determine any concordance homomorphism $\mathcal{C} \to \mathbb{Z}$ whose absolute value bounds the 4-genus from below and which takes the same value as υ on torus knots of braid index 2 and 3. An example is $-\frac{t}{2}$ for the concordance invariant t constructed by Ballinger [Bal20] from the $E(-1)$ spectral sequence on Khovanov homology. The invariant t defines a concordance homomorphism valued in the even integers which satisfies $\left|\frac{t(K)}{2}\right| \leqslant g_4(K)$ for any knot K [Bal20, Theorem 1.1]. Moreover, it satisfies $t(T_{p,q}) = -2\upsilon(T_{p,q})$ for the torus knots $T_{p,q}$ for any coprime positive integers p and q [Bal20, p. 22]. The same method we use for the proof of Theorem 3.1 shows that $t(K) = -2\upsilon(K)$ holds for any 3-braid knot K with explicit formulas in terms of braid representatives of K as given by Theorem 3.1.

Remark 3.5. Theorem 3.1 and a result of Erle [Erl99] imply that $\sigma(K) = 2\upsilon(K)$ for all 3-braid knots K except when $K = \pm T_{3,3\ell+k}$ for odd $\ell > 0$ and $k \in \{1, 2\}$. Recall that $\sigma(K)$ denotes the classical signature of the knot K [Tro62]. In the exceptional cases we have $\sigma(K) = 2\upsilon(K) - 2$. This observation improves a result by Feller and Krcatovich who showed that $\left|\upsilon(K) - \frac{\sigma(K)}{2}\right| \leqslant 2$ for all 3-braid knots K [FK17, Proposition 4.4]; see also Section 3.4.4.1.

Organization of this chapter. In Section 3.2, we will review the necessary properties of the upsilon invariant and the knot invariant τ from Heegaard Floer homology before providing a normal form for 3-braids (Proposition 3.8) that we call Garside normal form in Section 3.3. Then in Section 3.4, after a more detailed outline of our proof strategy (Section 3.4.1), we will prove Theorem 3.1 first for positive 3-braid knots (Section 3.4.2) and afterwards in the general 3-braid case (Section 3.4.3). We will prove Corollary 3.3 in Section 3.4.2. Section 3.4.4 will provide further context on our results. Section 3.5 is concerned with the proof of Corollary 3.2 (Section 3.5.1) and the application of our result about the upsilon invariant to alternating distances of general 3-braid knots (Section 3.5.2). Finally, in Section 3.6, we determine the so-called fractional Dehn twist coefficient for all 3-braids in Garside normal form.

3.2 Preliminaries on the concordance invariants τ and Υ

In [OS03], Ozsváth and Szabó constructed the knot invariant τ via the knot filtration on the Heegaard Floer chain complex of S^3; the latter was also defined independently by Rasmussen [Ras03]. The invariant τ induces a group homomorphism $\mathcal{C} \to \mathbb{Z}$ and gives a lower bound on the 4-ball genus $g_4(K)$: we have $|\tau(K)| \leqslant g_4(K)$ for any knot K. For the torus knots $T_{p,q}$, where p and q are coprime positive integers, the invariant τ recovers the 3-genus [OS03, Corollary 1.7], namely we have

$$\tau\left(T_{p,q}\right) = g\left(T_{p,q}\right) = \frac{(p-1)(q-1)}{2}. \tag{3.3}$$

Moreover, it follows from [Liv04, Theorem 4 and Corollary 7] together with the slice-Bennequin equalities (see (1.3) in Section 1.7) that, for any knot K that is the closure of a positive n-braid β, we have

$$\tau(K) = \frac{\mathrm{wr}(\beta) - n + 1}{2} = g_4(K) = g(K). \tag{3.4}$$

Note that (3.4) is true in greater generality for any *slice-torus invariant* μ replacing τ, that is, any homomorphism $\mu\colon \mathcal{C} \to \mathbb{R}$ satisfying $|\mu(K)| \leqslant g_4(K)$ and $\mu\left(T_{p,q}\right) = g_4\left(T_{p,q}\right)$ for all coprime positive integers p and q; see [Liv04] and [Lew14, Proposition 5.6].

The invariant Υ was defined by Ozsváth–Stipsicz–Szabó [OSS17a]. We will not recall the definition of Υ via the knot Floer complex $CFK^\infty(K)$ since the properties of Υ

mentioned below will be enough for our later computations and we will not explicitly use the Heegaard Floer theory behind it. For an overview on the properties of Υ, see the original article [OSS17a] or Livingston's notes on Υ [Liv17]. For every knot K, the knot invariant $\Upsilon_K\colon [0,1] \to \mathbb{R}$ is a continuous, piecewise linear function with the following properties [OSS17a]:

- $\Upsilon_K(0) = 0$,
- the slope of $\Upsilon_K(t)$ at $t = 0$ is given by $-\tau(K)$,
- $\Upsilon_{K_1 \# K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t)$ for all $0 \leqslant t \leqslant 1$ and all knots K_1 and K_2, (3.5)
- $\Upsilon_{-K}(t) = -\Upsilon_K(t)$ for all $0 \leqslant t \leqslant 1$, (3.6)
- $|\Upsilon_K(t)| \leqslant g_4(K)t$ for all $0 \leqslant t \leqslant 1$. (3.7)

Recall that $-K$ is the knot obtained by mirroring K and reversing its orientation, whose concordance class is the inverse of the class of K in $\mathcal{C}$. It follows from (3.5)-(3.7) that Υ induces a homomorphism from $\mathcal{C}$ to the group of real-valued piecewise linear functions on the interval $[0, 1]$.

For some classes of knots, the invariant Υ can be explicitly computed in terms of classical knot invariants like the signature σ and the Alexander polynomial. For a definition of these invariants, see e. g. Rolfsen's book on knots and links [Rol03]. A knot is *alternating* if it admits a diagram such that the crossings alternate between under- and overpasses as one travels along the diagram.

Proposition 3.6 ([OSS17a, Theorem 1.14])**.** *We have $\Upsilon_K(t) = \frac{\sigma(K)}{2}t$ for all alternating (or quasi-alternating) knots K and all $0 \leqslant t \leqslant 1$.*

For torus knots, $\Upsilon_K(t)$ is completely determined by a combinatorial formula in terms of their Alexander polynomial [OSS17a, Theorem 1.15]. For torus knots of braid index 2 or 3, the following holds; see e. g. [Fel16]. For $\ell \geqslant 0$ and $k \in \{1, 2\}$, we have

$$
\begin{aligned}
&\Upsilon_{T_{2,2\ell+1}}(t) = -\tau\left(T_{2,2\ell+1}\right) \cdot t = -\ell \cdot t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1,\\
&\Upsilon_{T_{3,3\ell+1}}(1) = \Upsilon_{T_{3,3\ell+2}}(1) + 1 = -2\ell,\\
&\Upsilon_{T_{3,3\ell+k}}(t) = -\tau(T_{3,3\ell+k})t = -(3\ell + k - 1)t \qquad \text{for all} \quad 0 \leqslant t \leqslant \frac{2}{3} \qquad \text{and}\\
&\Upsilon_{T_{3,3\ell+k}}(t) \text{ is linear on } \left[\frac{2}{3}, 1\right].
\end{aligned} \tag{3.8}
$$

3.3 The Garside normal form for 3-braids

In this section, we provide a classification result on the conjugacy classes of 3-braids; see Proposition 3.8. This result is basically due to work of Garside [Gar69] who gave the first solution to the conjugacy problem for all braid groups B_n, $n \geqslant 3$, in 1965. Proposition 3.8 might be known to the experts, but since the explicit formulas appear to be missing from the literature, we will provide them here.

Throughout, we denote the two generators of the braid group B_3 by $a = \sigma_1$ and $b = \sigma_2$. They are subject to the braid relation $aba = bab$. Recall that the braid $\Delta^2 = (aba)^2 = (ab)^3$ generates the center of B_3.

Remark 3.7. Any 3-braid is conjugate to the same braid with generators a and b interchanged. More precisely, let $\beta = a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r}$ for some $r \geqslant 1$ and integers p_i, q_i, $i \in \{1,\ldots,r\}$, be a 3-braid. Then, using $\Delta a = b\Delta$ and $\Delta b = a\Delta$, we have

$$\beta = \Delta^{-1}\Delta a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r} = \Delta^{-1}b^{p_1}a^{q_1}\cdots b^{p_r}a^{q_r}\Delta \sim b^{p_1}a^{q_1}\cdots b^{p_r}a^{q_r}.$$

In Proposition 3.8, we provide a certain standard form for the conjugacy classes of 3-braids.

Proposition 3.8. *Let β be a 3-braid. Then β is conjugate to one of the 3-braids*

(A) $\Delta^{2\ell}a^p$ *for* $\ell \in \mathbb{Z}$, $p \geqslant 0$,

(B) $\Delta^{2\ell}a^p b$ *for* $\ell \in \mathbb{Z}$, $p \in \{1,2,3\}$,

(C) $\Delta^{2\ell}a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r}$ *for* $\ell \in \mathbb{Z}$, $r \geqslant 1$, $p_i, q_i \geqslant 2$, $i \in \{1,\ldots,r\}$,

(D) $\Delta^{2\ell+1}a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r}$ *for* $\ell \in \mathbb{Z}$, $r \geqslant 1$, $p_r \geqslant 2$, $p_i, q_i \geqslant 2$, $i \in \{1,\ldots,r-1\}$.

If β is a positive 3-braid, then $\ell \geqslant 0$. If $\widehat{\beta}$ is a knot, then only the cases (B)–(D) can occur and p must be odd in case (B), at least one of the p_i and one of the q_i must be odd in case (C), and at least one of the p_i or q_i must be odd in case (D).

We note the following uniqueness result related to Proposition 3.8.

Remark 3.9. Up to cyclic permutation of the exponents $p_1, q_1, \ldots, p_r, q_r$ in (C) and $p_1, q_1, \ldots, p_{r-1}, q_{r-1}, p_r$ in (D), respectively, every 3-braid is conjugate to exactly one of the 3-braids listed in Proposition 3.8. For 3-braids conjugate to braids in (C), we could for example choose as a unique representative the braid in Garside normal form where the tuple $(-2\ell, p_1, q_1, \ldots, p_r, q_r)$ is lexicographically minimal among all words of the form (C) representing the same conjugacy class. This follows from Garside's work [Gar69]. In his notation, each of the 3-braids listed in (A)–(D) in Proposition 3.8 is the standard form of a certain element in the (so-called) summit set of β. For 3-braids of the form (C) or (D), the summit set consists of those 3-braids obtained by cyclic permutation of the powers $p_1, q_1, \ldots, p_r, q_r$ in (C) and $p_1, q_1, \ldots, p_{r-1}, q_{r-1}, p_r$ in (D), respectively.

Definition 3.10. We call a braid word of the form in (A)–(D) a 3-*braid in Garside normal form.*

Remark 3.11. The advantage of the Garside normal form over Murasugi's normal form for 3-braids used later in Section 3.4.3 (see Definition 3.26) is that positive 3-braids are easier to detect in this normal form: if β is a positive 3-braid, then β is conjugate to one of the braids in (A)–(D) with $\ell \geqslant 0$. Since Garside's solution to the conjugacy problem works for any n-braid with $n \geqslant 3$, one might hope to generalize an explicit standard form as in Proposition 3.8 to n-braids for any $n \geqslant 3$.

Remark 3.12. For odd p, case (B) of Proposition 3.8 covers the torus knots of braid index 3. More precisely, if $\beta \sim \Delta^{2\ell}ab = (ab)^{3\ell+1}$, then its closure is $\widehat{\beta} = T_{3,3\ell+1}$ for $\ell \geqslant 0$ and $\widehat{\beta} = -T_{3,3(-\ell-1)+2}$ for $\ell < 0$, and if $\beta \sim \Delta^{2\ell}a^3b \sim (ab)^{3\ell+2}$, then $\widehat{\beta} = T_{3,3\ell+2}$ for $\ell \geqslant 0$ and $\widehat{\beta} = -T_{3,3(-\ell-1)+1}$ for $\ell < 0$.

Proof of Proposition 3.8. The proof will follow from the following claim.

Claim 1. *Let β be a positive 3-braid. Then β is conjugate to one of the 3-braids in (A)–(D) with $\ell \geqslant 0$.*

We first deduce Proposition 3.8 from this claim. To that end, let β be any 3-braid. If β is a positive braid, we are done by Claim 1. If not, then β can be written in the form $\beta = \Delta^m\alpha$ where m is a negative integer and α a positive 3-braid [Gar69, Theorem 5]. In fact, inserting $\Delta^{-1}\Delta$ if m is odd, we can assume β to be of the form $\Delta^{-2n}\alpha$ for some $n \geqslant 1$ and a positive 3-braid α. The proposition then easily follows using the claim for α. It remains to prove Claim 1.

Proof of Claim 1: A positive 3-braid β has the form $\beta = a^{P_1}b^{Q_1}\cdots a^{P_R}b^{Q_R}$ for integers $R \geqslant 1$, $P_i, Q_i \geqslant 0$, $i \in \{1,\dots,R\}$. If all the P_i or all the Q_i are 0, then (possibly using Remark 3.7) β is conjugate to a^p for some $p \geqslant 0$ and we are in case (A) for $\ell = 0$. Possibly after conjugation and reduction of R, we can thus assume that all of the integers P_i, Q_i are non-zero. If $P_1, Q_1 \geqslant 2$ applies for all $i \in \{1,\dots,R\}$, then β is of the form in (C) for $\ell = 0$. If $R = 1$, i. e. $\beta = a^{P_1}b^{Q_1}$ for integers $P_1, Q_1 \geqslant 1$, and $P_1 = 1$ or $Q_1 = 1$, then (possibly using Remark 3.7) β is conjugate to a braid of the form in (B).

It remains to consider the case where $R \geqslant 2$ and at least one of the P_i or Q_i is 1. In that case—if necessary after conjugation—β contains $\Delta = aba = bab$ as a subword and is thus conjugate to $\Delta\alpha$ for some positive 3-braid α. Let $m \geqslant 1$ be maximal with the property that β is conjugate to $\Delta^m\alpha$ for some positive 3-braid α. Then, possibly after conjugation of β, the braid word α must be one of the following:

$$\begin{aligned}
&a^p \quad \text{for } p \geqslant 0,\\
&a^pb \quad \text{for } p \geqslant 1,\\
&a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r} \quad \text{for } r \geqslant 1,\ p_i, q_i \geqslant 2,\ i \in \{1,\dots,r\},\\
&a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r} \quad \text{for } r \geqslant 1,\ p_r \geqslant 2,\ p_i, q_i \geqslant 2,\ i \in \{1,\dots,r-1\}.
\end{aligned} \tag{3.9}$$

Indeed, using Remark 3.7, up to conjugation these are the only possible words such that $\Delta^m\alpha$ does not contain any additional Δ as a subword. Note that α can be the empty word, which is covered by the first case in (3.9) for $p = 0$. Further, note that

$$\begin{gathered}\Delta^{2\ell}a^pb \sim \Delta^{2\ell+1}a^{p-2}, \qquad \Delta^{2\ell+1}a \sim \Delta^{2\ell}a^3b, \qquad \Delta^{2\ell+1}a^pb \sim \Delta^{2\ell+1}a^{p+1}, \\ \Delta^{2\ell+1}a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r} \sim \Delta^{2\ell+1}a^{p_1+q_r}b^{q_1}a^{p_2}\cdots b^{q_{r-1}}a^{p_r} \quad \text{and} \\ \Delta^{2\ell}a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r} \sim \Delta^{2\ell}a^{p_1+p_r}b^{q_1}a^{p_2}\cdots a^{p_{r-1}}b^{p_{r-1}}\end{gathered} \tag{3.10}$$

for any $\ell \geqslant 0$, $p \geqslant 1$, $p_i, q_i \geqslant 2$, $i \in \{1, \ldots, r\}$. It follows from a case by case analysis of the cases in (3.9), using (3.10) and taking the parity of m into account, that any positive 3-braid is conjugate to one of the 3-braids in (A)–(D) with $\ell \geqslant 0$. ■

This concludes the proof of Proposition 3.8. □

3.4 The upsilon invariant of 3-braid knots

In this section, we prove Theorem 3.1. Along the way, we compute the invariant υ for positive 3-braid knots in Garside normal form (Proposition 3.14) and prove Corollary 3.3.

3.4.1 Methodology

First, we recall the inequality (3.2) from Section 3.1—which will be used repeatedly in Section 3.4—in more generality. The *cobordism distance* $d(K, J)$ between two knots K and J is defined as the 4-genus $g_4(K\# -J)$ of the connected sum of K and the inverse of J. Equivalently, the cobordism distance $d(K, J)$ is the minimal genus of a cobordism between K and J (see Section 3.1 for the definition of a cobordism). Suppose the genus of a cobordism C between two knots K and J is $g(C)$. We then have $d(K, J) \leqslant g(C)$, so by the properties (3.5)-(3.7) of Υ from Section 3.2, we get

$$|\Upsilon_K(t) - \Upsilon_J(t)| = |\Upsilon_{K\#-J}(t)| \leqslant g_4(K\# -T)t = d(K, T)t \leqslant g(C)t \tag{3.11}$$

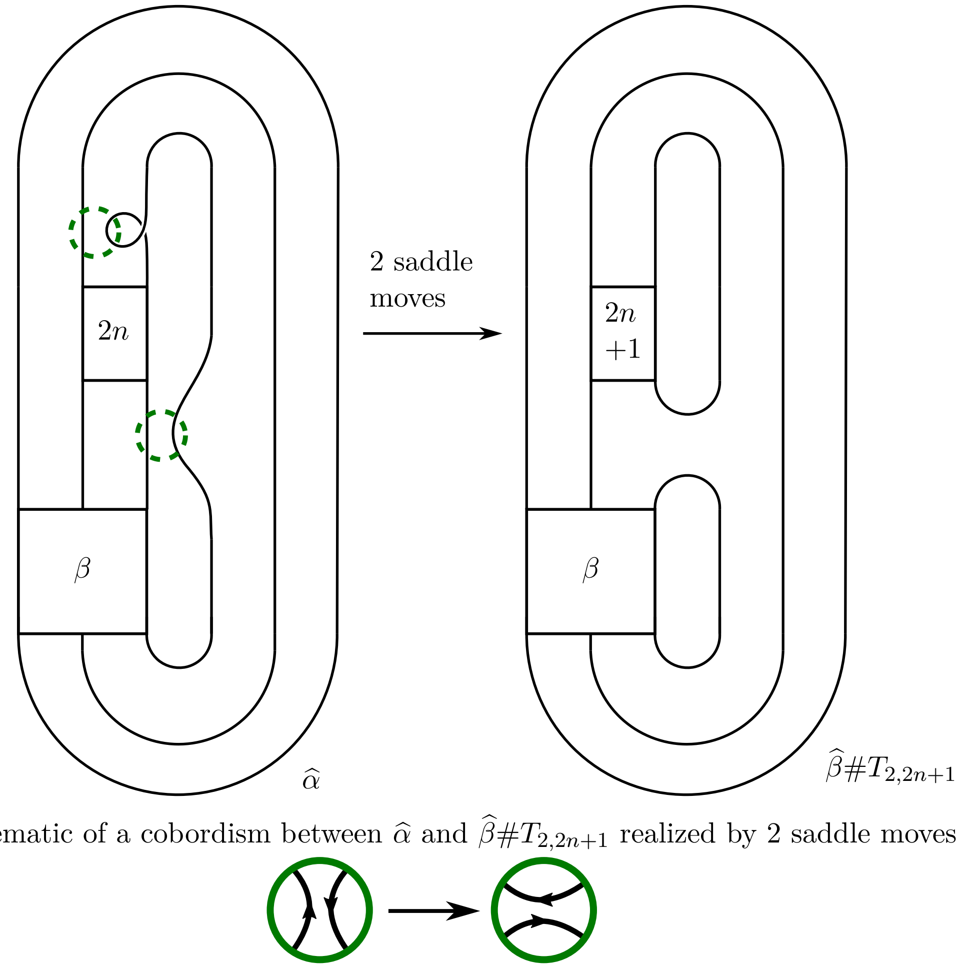
for all $0 \leqslant t \leqslant 1$. This provides bounds on $\Upsilon_K(t)$ in terms of $\Upsilon_J(t)$ and $g(C)$.

We now give an example for the cobordisms we will use later on.

Example 3.13. Among other things, we will frequently use the following trick the author first saw in [FK17, Example 4.5]. Let β be a 3-braid such that $K = \widehat{\beta}$ is a knot. Consider the 3-braid $\alpha := \beta b^{2n}$ for some $n \geqslant 1$. Then $\widehat{\alpha}$ is also a knot and there is a cobordism between $\widehat{\alpha}$ and the connected sum $K\#T_{2,2n+1}$ of genus 1. This cobordism can be realized by two saddle moves (1-handle attachments) of the form shown in Figure 3.2(b) on the next page, performed in the two circled regions of Figure 3.2(a). One of them is used to add a generator b to the braid α to obtain the braid word βb^{2n+1} and the other is used to transform the closure of this new braid word into a connected sum of K and $T_{2,2n+1}$. Recall that our braid diagrams are oriented from bottom to top. Using $\upsilon(T_{2,2n+1}) = -n$ by (3.8) and that the genus of the cobordism is 1, by (3.11) for $t = 1$, we have

$$|\upsilon(\widehat{\alpha}) - \upsilon(K \# T_{2,2n+1})| \leqslant 1 \quad \Longleftrightarrow \quad |\upsilon(\widehat{\alpha}) - \upsilon(K) + n| \leqslant 1, \tag{3.12}$$

which provides the lower bound $\upsilon(K) \geqslant \upsilon(\widehat{\alpha}) + n - 1$ on $\upsilon(K)$.



(a) Schematic of a cobordism between $\widehat{\alpha}$ and $\widehat{\beta} \# T_{2,2n+1}$ realized by 2 saddle moves.

(b) A saddle move.

Figure 3.2: An example illustrating our proof strategy.

3.4.2 The upsilon invariant of positive 3-braid knots

In this subsection, we determine the invariant υ for all positive 3-braid knots.

By Proposition 3.8 and Remark 3.12, positive 3-braid knots are either the torus knots $T_{3,3\ell+k}$ for $\ell \geqslant 0$ and $k \in \{1, 2\}$ which have braid representatives of Garside normal form (B), or closures of positive 3-braids of Garside normal form (C) or (D) (cf. Definition 3.10). The following proposition thus proves Theorem 3.1 for all positive 3-braid knots.

Proposition 3.14. *Let β be a positive 3-braid such that $K = \widehat{\beta}$ is a knot. Then*

$$\upsilon(K) = \begin{cases} -2\ell - \dfrac{p-1}{2} & \text{if } \beta \text{ is conjugate to a braid in (B)}, \\ -\dfrac{\sum_{i=1}^{r}(p_i+q_i)}{2} + r - 2\ell & \text{if } \beta \text{ is conjugate to a braid in (C)}, \\ -\dfrac{\sum_{i=1}^{r-1}(p_i+q_i)+p_r}{2} + r - 2\ell - \dfrac{3}{2} & \text{if } \beta \text{ is conjugate to a braid in (D)}. \end{cases}$$

Remark 3.15. In fact, the formulas from Proposition 3.14 also give the correct upsilon invariant in terms of the Garside normal form of a 3-braid representative of a knot K if K is the closure of any 3-braid in Garside normal form (C) or (D), not necessarily a positive one. This follows from Theorem 3.1 (proved in the next subsection) and the observations of Section 3.4.4.3.

Recall that for the torus knots of braid index 3, we know the invariant υ by (3.8). In the following, we will determine the invariant υ for all knots that are closures of positive 3-braids of Garside normal form (C) or (D).

We first provide an upper bound on $\Upsilon_K(t)$ for all positive 3-braid knots K and $0 \leqslant t \leqslant 1$. The following inequality (3.13) in Lemma 3.16 could also be shown using the dealternating number and a result of Abe–Kishimoto [AK10, Lemma 2.2], whereas the main work for the upper bound on υ for the knots in the second and third case in Proposition 3.14 will be to rewrite the braid words representing these knots. We use the approach below since it will also give bounds on the minimal cobordism distance between any positive 3-braid knot and an alternating knot; see Remark 3.25.

Lemma 3.16. *Let $\beta = a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r}$ be a positive 3-braid, where $r \geqslant 1$ and $p_i, q_i \geqslant 1$, $i \in \{1, \ldots, r\}$, are integers such that $K = \widehat{\beta}$ is a knot. Then*

$$\Upsilon_K(t) \leqslant (-g(K) + r - 1)\, t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1. \tag{3.13}$$

Proof. We claim that there is a cobordism C of genus

$$g(C) = \frac{r - 1 + \varepsilon}{2} \tag{3.14}$$

between K and the connected sum

$$J_\varepsilon = T_{2,\sum_{i=1}^{r} p_i + \varepsilon_p} \# T_{2,q_1+\varepsilon_1} \# T_{2,q_2+\varepsilon_2} \# \ldots \# T_{2,q_r+\varepsilon_r}, \tag{3.15}$$

where $\varepsilon_1, \ldots, \varepsilon_r, \varepsilon_p \in \{0, 1\}$ are chosen such that J_ε is a connected sum of torus knots (rather than links), i. e. such that $\sum_{i=1}^{r} p_i + \varepsilon_p$, $q_1 + \varepsilon_1$, $q_2 + \varepsilon_2, \ldots, q_r + \varepsilon_r$ are all odd, and $\varepsilon := \varepsilon_p + \sum_{i=1}^{r} \varepsilon_i$. This cobordism C can be realized by $r - 1 + \varepsilon$ saddle moves as follows.

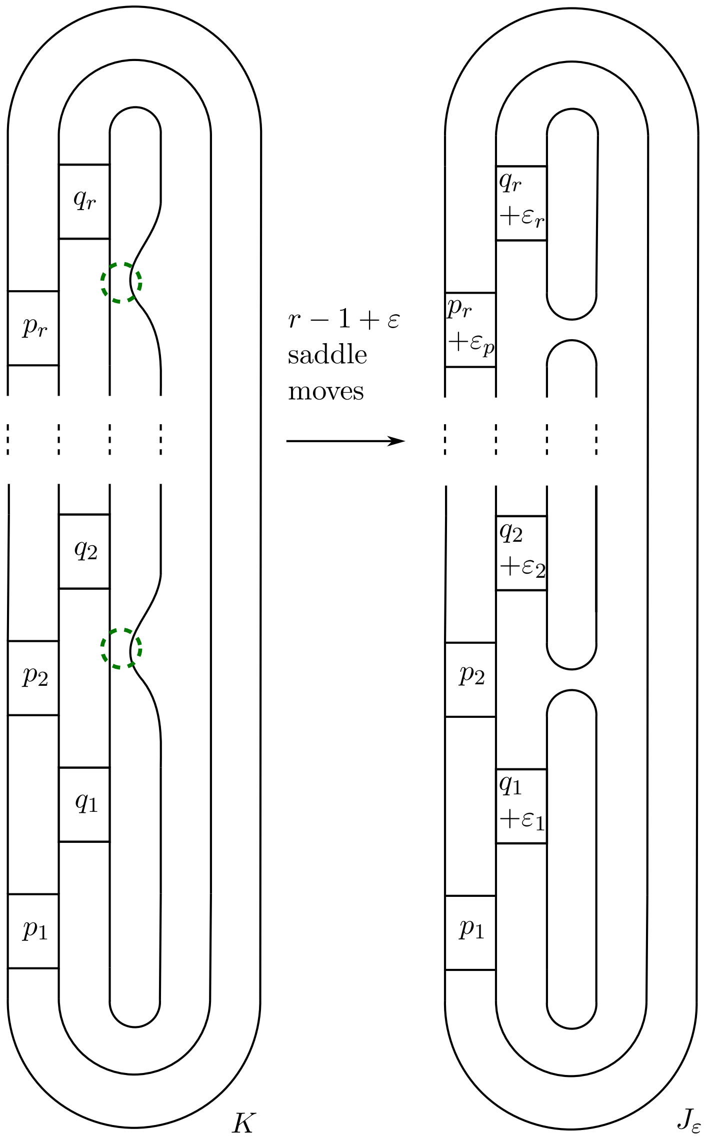


Figure 3.3: A schematic of a cobordism between the knots $K = \widehat{\beta}$ and J_ε as in (3.15) realized by $r - 1 + \varepsilon$ saddle moves.

Following the schematic in Figure 3.3, we add ε generators b by ε saddle moves and additionally perform $r - 1$ saddle moves of the form shown in Figure 3.2(b) in the green circled regions of Figure 3.3. In Figure 3.3, a box on the left labeled p_i or q_i stands for the positive braid a^{p_i} or b^{q_i}, respectively. The Euler characteristic of the cobordism C is $\chi(C) = -r + 1 - \varepsilon$. Since C is connected and—as J_ε and K are knots—has two boundary components, the genus of C is $g(C) = \frac{-\chi(C)}{2} = \frac{r-1+\varepsilon}{2}$ as claimed. By (3.11), we get $|\Upsilon_K(t) - \Upsilon_{J_\varepsilon}(t)| \leqslant g(C)t$ for all $0 \leqslant t \leqslant 1$, hence

$$\Upsilon_K(t) \leqslant \Upsilon_{J_\varepsilon}(t) + g(C)t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1. \tag{3.16}$$

By (3.5) and (3.8) from Section 3.2, we have

$$\begin{aligned}\Upsilon_{J_\varepsilon}(t) &= \left(-\frac{\sum_{i=1}^r p_i + \varepsilon_p - 1}{2} - \frac{q_1 + \varepsilon_1 - 1}{2} - \frac{q_2 + \varepsilon_2 - 1}{2} \cdots - \frac{q_r + \varepsilon_r - 1}{2}\right) t \\ &= -\frac{1}{2}\left(\sum_{i=1}^r (p_i + q_i) - (r+1) + \varepsilon\right) t,\end{aligned}$$

so (3.14) and (3.16) imply

$$\Upsilon_K(t) \leqslant \left(-\frac{\sum\limits_{i=1}^r (p_i + q_i)}{2} + r\right) t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1.$$

The claim follows, since by the slice-Bennequin equalities (1.3), we have

$$g(K) = \frac{\mathrm{wr}(\beta) - 2}{2} = \frac{\sum\limits_{i=1}^r (p_i + q_i) - 2}{2}. \qquad \square$$

The following Lemmas 3.17 and 3.18 improve the upper bound on $\Upsilon_K(t)$ from the last Lemma 3.16 for knots K that are closures of positive 3-braids of Garside normal form (C) or (D), respectively.

Lemma 3.17. *Let $\beta = \Delta^{2\ell+1} a^{p_1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}$ for some $\ell \geqslant 0$, $r \geqslant 1$, $p_r \geqslant 1$ and $p_i, q_i \geqslant 1$ for $i \in \{1, \ldots, r-1\}$ such that $K = \widehat{\beta}$ is a knot. Then*

$$\Upsilon_K(t) \leqslant \left(-\frac{\sum\limits_{i=1}^{r-1} (p_i + q_i) + p_r}{2} + r - 2\ell - \frac{3}{2}\right) t \qquad \textit{for all} \quad 0 \leqslant t \leqslant 1.$$

In the proof of Lemma 3.17, we will use that in B_3, we have

$$(ab)^{3n+1} = ab\Delta^{2n} = a^2ba^3(aba^3)^{n-1}ba^n \qquad \text{for all} \quad n \geqslant 1, \tag{3.17}$$

where $\Delta^2 = (aba)^2 = (ab)^3 = (ba)^3$; see [Fel16, Proof of Prop. 22].

Proof of Lemma 3.17. Let $\Sigma_\beta = \sum\limits_{i=1}^{r-1} (p_i + q_i) + p_r$ and note that using (1.3), we have

$$g(K) = \frac{3(2\ell + 1) + \Sigma_\beta - 2}{2} = \frac{\Sigma_\beta}{2} + 3\ell + \frac{1}{2}. \tag{3.18}$$

If $\ell = 0$, then $\beta = \Delta a^{p_1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}$ is conjugate to

$$\beta_1 = a^{p_1+1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} a^{p_r+1} b$$

and $\widehat{\beta_1} = \widehat{\beta} = K$, so $g\left(\widehat{\beta_1}\right) = \frac{\Sigma_\beta}{2} + \frac{1}{2}$. By Lemma 3.16, we get

$$\Upsilon_K(t) \leqslant \left(-g\left(\widehat{\beta_1}\right) + r - 1\right) t = \left(-\frac{\Sigma_\beta}{2} + r - \frac{3}{2}\right) t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1.$$

For $\ell \geqslant 1$, using $\Delta^{2\ell+1} = (ab)^{3\ell} aba = (ab)^{3\ell+1} a$, we have

$$\begin{aligned}
\beta &= \Delta^{2\ell+1} a^{p_1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} a^{p_r} = (ab)^{3\ell+1} a^{p_1+1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} a^{p_r} \\
&\overset{(3.17)}{=} a^2 b a^3 (aba^3)^{\ell-1} b a^{p_1+\ell+1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} a^{p_r} \\
&\sim a^{p_r+2} b a^3 (aba^3)^{\ell-1} b a^{p_1+\ell+1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} =: \beta_1.
\end{aligned}$$

We have $\widehat{\beta_1} = \widehat{\beta} = K$ and $g\left(\widehat{\beta_1}\right) = \frac{\Sigma_\beta}{2} + 3\ell + \frac{1}{2}$ by (3.18). Again, Lemma 3.16 implies

$$\Upsilon_K(t) \leqslant \left(-g\left(\widehat{\beta_1}\right) + r + \ell - 1\right) t = \left(-\frac{\Sigma_\beta}{2} + r - 2\ell - \frac{3}{2}\right) t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1,$$

which proves the claim of the lemma. ☐

Lemma 3.18. *Let $\beta = \Delta^{2\ell} a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r}$ for some $\ell \geqslant 0$, $r \geqslant 1$ and $p_i, q_i \geqslant 1$ for $i \in \{1, \ldots, r\}$ such that $K = \widehat{\beta}$ is a knot. Then*

$$\Upsilon_K(t) \leqslant \left(-\frac{\sum_{i=1}^{r} (p_i + q_i)}{2} + r - 2\ell\right) t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1.$$

In the proof of Lemma 3.18, we will need the following statement about positive 3-braids.

Lemma 3.19. *In B_3, we have*

$$(ab)^{3n-1} = a^{2n} b (a^2 b^2)^{n-1} a \qquad \text{for all} \quad n \geqslant 1. \tag{3.19}$$

Proof. Starting with the left-hand side we have

$$(ab)^{3n-1} = a(ba)^{3(n-1)} bab = a(ab)^{3(n-1)} aba,$$

which proves (3.19) for $n = 1$. We now show by induction that

$$(ab)^{3(n-1)} a = a^{2n-1} b (a^2 b^2)^{n-2} a^2 b \qquad \text{for all} \quad n \geqslant 2, \tag{3.20}$$

which implies the lemma for all $n \geqslant 1$. For $n = 2$, we have

$$(ab)^3 a = a(ba)^3 = a(ab)^3 = a^2babab = a^3ba^2b.$$

Assuming that (3.20) is true for some $n - 1 \geqslant 2$, we get

$$\begin{aligned}(ab)^{3(n-1)}a = a(ba)^{3(n-1)} = a(ab)^{3(n-1)} &= a^2(ba)^{3(n-2)}babab = a^2(ab)^{3(n-2)}aba^2b \\ &= a^2\left(a^{2n-3}b(a^2b^2)^{n-3}a^2b\right)ba^2b = a^{2n-1}b(a^2b^2)^{n-2}a^2b,\end{aligned}$$

using the induction hypothesis in the second to last equality. □

Proof of Lemma 3.18. Let $\Sigma_\beta = \sum_{i=1}^{r}(p_i + q_i)$. If $\ell = 0$, then by (1.3) and Lemma 3.16 we have

$$\Upsilon_K(t) \leqslant (-g(K) + r - 1)\,t = \left(-\frac{\Sigma_\beta}{2} + r\right)t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1.$$

For $\ell \geqslant 1$, using $\Delta^2 = (ba)^3$ and Lemma 3.19, we have

$$\begin{aligned}\beta = (ba)^{3\ell}a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r} &\sim (ab)^{3\ell-1}a^{p_1+1}b^{q_1}\cdots a^{p_r}b^{q_r+1} \\ &\sim a^{2\ell}b(a^2b^2)^{\ell-1}a^{p_1+2}b^{q_1}\cdots a^{p_r}b^{q_r+1} =: \beta_1.\end{aligned}$$

Note that $\widehat{\beta_1} = \widehat{\beta} = K$ and by (1.3), we have

$$g\left(\widehat{\beta_1}\right) = g(K) = \frac{6\ell + \Sigma_\beta - 2}{2} = \frac{\Sigma_\beta}{2} + 3\ell - 1.$$

Again by Lemma 3.16, we get

$$\Upsilon_K(t) \leqslant \left(-g\left(\widehat{\beta_1}\right) + r + \ell - 1\right)t = \left(-\frac{\Sigma_\beta}{2} + r - 2\ell\right)t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1. \quad \square$$

We will now focus on $\upsilon(K) = \Upsilon_K(1)$ and prove Proposition 3.14 by showing that the upper bounds on $\Upsilon_K(t)$ from Lemma 3.17 and Lemma 3.18 for $t = 1$ are also lower bounds. We will need the following observation used in [FK17, Example 4.5] about 3-braids, which we prove here for completeness.

Lemma 3.20. *In B_3, we have*

$$a^{2n+1}b\left(a^2b^2\right)^n = (ab)^{3n+1} \text{ and } b^{2n+1}a\left(b^2a^2\right)^n = (ba)^{3n+1} \qquad \text{for all} \quad n \geqslant 0. \qquad (3.21)$$

Proof. We prove the first statement by induction. For $n = 0$, the equality is clearly true. For $n = 1$, using $\Delta a = b\Delta$ and $\Delta b = a\Delta$, we have

$$a^3ba^2b^2 = a^2\Delta ab^2 = a^2ba\Delta b = a\Delta^2 b = \Delta^2 ab = (ab)^4.$$

We now assume that (3.21) is true for some $n-1 \geqslant 0$. Using the induction hypothesis and the equality for $n=1$, we get

$$\begin{aligned} a^{2n+1}b\left(a^2b^2\right)^n &= a^2\,(ab)^{3(n-1)+1}\,a^2b^2 = a^3b\Delta^{2(n-1)}a^2b^2 \\ &= \Delta^{2(n-1)}a^3ba^2b^2 = (ab)^{3(n-1)}(ab)^4 = (ab)^{3n+1}. \end{aligned}$$

□

Lemma 3.21. *Let $\beta = \Delta^{2\ell+1}a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r}$ for some $\ell \geqslant 0$, $r \geqslant 1$, $p_r \geqslant 3$ and $p_i, q_i \geqslant 2$ for $i \in \{1,\ldots,r-1\}$ such that $K = \widehat{\beta}$ is a knot. Then*

$$\upsilon(K) = -\frac{\sum_{i=1}^{r-1}(p_i+q_i)+p_r}{2} + r - 2\ell - \frac{3}{2}.$$

Proof of Lemma 3.21. Let $\Sigma_\beta = \sum_{i=1}^{r-1}(p_i+q_i)+p_r$. From Lemma 3.17, it follows directly that $\upsilon(K) = \Upsilon_K(1) \leqslant -\frac{1}{2}\Sigma_\beta + r - 2\ell - \frac{3}{2}$, so we are left to show that

$$\upsilon(K) \geqslant -\frac{1}{2}\Sigma_\beta + r - 2\ell - \frac{3}{2}.$$

To that end, consider

$$\begin{aligned} \beta &= \Delta^{2\ell+1}a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r} \sim \Delta^{2\ell}a\Delta a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r-1} \\ &= \Delta^{2\ell}bab^2a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r-1} =: \beta_1, \end{aligned}$$

where we used $a\Delta = abab = bab^2$. Note that $\widehat{\beta_1} = \widehat{\beta} = K$. Now, define

$$\alpha := b^{2r}\beta_1 = \Delta^{2\ell}b^{2r+1}ab^2a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r-1}$$

and note that $\widehat{\alpha}$ is a knot. By assumption, we have $p_r - 1 \geqslant 2$. There is a cobordism between $\widehat{\alpha}$ and the connected sum $T_{2,2r+1}\#\widehat{\beta_1} = T_{2,2r+1}\#K$ of genus 1 by using two saddle moves similar to the two saddle moves illustrated in Figure 3.2 from Example 3.13. Similarly as in (3.12) from Example 3.13, we have $\upsilon(K) \geqslant \upsilon(\widehat{\alpha}) + r - 1$.

In order to find a lower bound for $\upsilon(\widehat{\alpha})$, note that there is a cobordism C between $\widehat{\alpha}$ and the torus knot $T = T_{3,3(\ell+r)+1}$ of genus $g(C) = \frac{\Sigma_\beta}{2} - 2r + \frac{1}{2}$. Here we think of T as the closure of the braid $\gamma = \Delta^{2\ell}b^{2r+1}a(b^2a^2)^r$, which is equal to $\Delta^{2\ell}(ba)^{3r+1} = (ba)^{3(\ell+r)+1}$ as 3-braid words by Lemma 3.20. The cobordism C between $\widehat{\alpha}$ and $T = \widehat{\gamma}$ can thus be realized by

$$p_1 - 2 + q_1 - 2 + \cdots + p_{r-1} - 2 + q_{r-1} - 2 + p_r - 3 = \Sigma_\beta - 4r + 1$$

saddle moves corresponding to the deletion of the same number of generators a and b from the braid word α to obtain γ. Hence the Euler characteristic of the cobordism C is $\chi(C) = -\Sigma_\beta + 4r - 1$. Since C is connected and has two boundary components (as

$\widehat{\alpha}$ and $T = \widehat{\gamma}$ are knots), the genus of C is indeed $g(C) = \frac{\Sigma_\beta}{2} - 2r + \frac{1}{2}$. By (3.11) and (3.8), we have

$$\upsilon\left(\widehat{\alpha}\right) \geqslant \upsilon\left(T\right) - g(C) = -2\left(\ell + r\right) - \left(\frac{\Sigma_\beta}{2} - 2r + \frac{1}{2}\right) = -\frac{\Sigma_\beta}{2} - 2\ell - \frac{1}{2}.$$

It follows that

$$\upsilon(K) \geqslant \upsilon\left(\widehat{\alpha}\right) + r - 1 \geqslant -\frac{\Sigma_\beta}{2} + r - 2\ell - \frac{3}{2}. \qquad \square$$

This finishes the computation of $\upsilon(K)$ for K the closure of a positive 3-braid in Garside normal form (C).

Lemma 3.22. *Let $\beta = \Delta^{2\ell} a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r}$ for some $\ell \geqslant 0$, $r \geqslant 1$, $p_r, q_r \geqslant 3$ and $p_i, q_i \geqslant 2$ for $i \in \{1, \ldots, r-1\}$ such that $K = \widehat{\beta}$ is a knot. Then*

$$\upsilon(K) = -\frac{\sum_{i=1}^{r}(p_i + q_i)}{2} + r - 2\ell.$$

Proof of Lemma 3.22. The proof uses similar ideas as the proof of Lemma 3.21. Let $\Sigma_\beta = \sum_{i=1}^{r}(p_i + q_i)$. By Lemma 3.18, we have $\upsilon(K) \leqslant -\frac{\Sigma_\beta}{2} + r - 2\ell$, so it remains to show that $\upsilon(K) \geqslant -\frac{\Sigma_\beta}{2} + r - 2\ell$. To that end, we consider

$$\beta = \Delta^{2\ell} a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r} \sim \Delta^{2\ell} b a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r - 1} =: \beta_1.$$

Note that $\widehat{\beta_1} = \widehat{\beta} = K$. We define

$$\alpha := a^{2r}\beta_1 = a^{2r}\Delta^{2\ell} b a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r - 1} \sim \Delta^{2\ell} b a^{2r} b a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r - 2} =: \alpha_1.$$

Then $\widehat{\alpha_1} = \widehat{\alpha}$ is a knot and by assumption we have $q_r - 2 \geqslant 1$. There is a cobordism between $\widehat{\alpha}$ and $T_{2,2r+1} \# \widehat{\beta_1} = T_{2,2r+1} \# K$ of genus 1 by using two saddle moves similar to the cobordism considered in Example 3.13 and in the proof of Lemma 3.21, hence $\upsilon(K) \geqslant \upsilon(\widehat{\alpha_1}) + r - 1$. To find a lower bound for $\upsilon(\widehat{\alpha_1})$, we observe that there is a cobordism C between the knot $\widehat{\alpha_1}$ and the knot $\widehat{\gamma}$, where

$$\gamma = \Delta^{2\ell} b a^{2r} b (a^2 b^2)^{r-1} a^3 b.$$

Using (3.21) from Lemma 3.20 for $n-1$, in B_3, we have

$$b a^{2n} b \left(a^2 b^2\right)^{n-1} a^2 = b a (ab)^{3(n-1)+1} a^2 = b a \Delta^{2(n-1)} a b a^2 = \Delta^{2n} \qquad \text{for all} \quad n \geqslant 1.$$

We thus have $\gamma = \Delta^{2\ell}\Delta^{2r} ab = (ab)^{3(\ell+r)+1}$. Hence the closure of γ is the torus knot $T = T_{3,3(\ell+r)+1}$ with $\upsilon(T) = -2\left(\ell + r\right)$ by (3.8). The cobordism C between $\widehat{\alpha_1}$ and

$T = \widehat{\gamma}$ can be realized by

$$p_1 - 2 + q_1 - 2 + \cdots + p_{r-1} - 2 + q_{r-1} - 2 + p_r - 3 + q_r - 3 = \Sigma_\beta - 4r - 2$$

saddle moves corresponding to the deletion of the same number of generators a and b from the braid word α_1 to obtain γ. By a similar Euler characteristic argument as in the proofs of Lemma 3.16 and Lemma 3.21, the genus of this cobordism is $g(C) = \frac{\Sigma_\beta}{2} - 2r - 1$. Note that here we used $p_r \geqslant 3$ and $q_r \geqslant 3$. Now, by (3.11), we have

$$\upsilon(\widehat{\alpha_1}) \geqslant \upsilon(T) - g(C) = -\frac{\Sigma_\beta}{2} - 2\ell + 1, \qquad \text{hence}$$
$$\upsilon(K) \geqslant \upsilon(\widehat{\alpha_1}) + r - 1 \geqslant -\frac{\Sigma_\beta}{2} + r - 2\ell. \qquad \square$$

Lemma 3.23. *Let $\beta = \Delta^{2\ell} a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r}$ for some $\ell \geqslant 0$, $r \geqslant 2$, $p_i, q_i \geqslant 2$ for $i \in \{1, \ldots, r\}$. Suppose that $q_r \geqslant 3$ and $p_k \geqslant 3$ for some $1 \leqslant k < r$ and that $K = \widehat{\beta}$ is a knot. Then*

$$\upsilon(K) = -\frac{\sum_{i=1}^{r}(p_i + q_i)}{2} + r - 2\ell.$$

Proof. We proceed similar as in the proof of Lemma 3.22, but here we will look at a different cobordism to obtain a lower bound for $\upsilon(\widehat{\alpha_1})$. The steps of the proof are exactly the same until then, so we consider

$$\beta = \Delta^{2\ell} a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r} \sim \Delta^{2\ell} b a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r - 1} =: \beta_1$$

and define

$$\alpha := a^{2r} \beta_1 \sim \Delta^{2\ell} b a^{2r} b a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r - 2} =: \alpha_1.$$

Again, we have $\upsilon(K) \geqslant \upsilon(\widehat{\alpha_1}) + r - 1$. Now, in order to find a lower bound for $\upsilon(\widehat{\alpha_1})$, we observe that there is a cobordism C between $\widehat{\alpha_1}$ and the knot $\widehat{\gamma}$, where

$$\gamma = \Delta^{2\ell} b a^{2r} b (a^2 b^2)^{k-1} a^3 b^2 (a^2 b^2)^{r-k-1} a^2 b.$$

We find the cobordism C by the deletion of generators from the braid word γ to obtain α_1, where we use the assumptions $q_r \geqslant 3$ and $p_k \geqslant 3$. In fact, the cobordism can be realized by

$$\begin{aligned} &p_1 - 2 + q_1 - 2 + \cdots + p_{k-1} - 2 + q_{k-1} - 2 + p_k - 3 + q_k - 2 \\ &\quad + p_{k+1} - 2 + q_{k+1} - 2 + \cdots + p_{r-1} - 2 + q_{r-1} - 2 + p_r - 2 + q_r - 3 \\ &= \Sigma_\beta - 4r - 2 \end{aligned}$$

saddle moves, so its genus is $g(C) = \frac{\Sigma_\beta}{2} - 2r - 1$. Using $a^{2k-1}b(a^2b^2)^{k-1} = (ab)^{3k-2}$ by Lemma 3.20, we have

$$\begin{aligned}\gamma &= \Delta^{2\ell} b a^{2r-2k+1} (ab)^{3k-2} a^3 b^2 (a^2b^2)^{r-k-1} a^2 b \\ &= \Delta^{2\ell} b a^{2r-2k+1} \Delta^{2(k-1)} aba^3 b^2 (a^2b^2)^{r-k-1} a^2 b \\ &\sim \Delta^{2(\ell+k-1)} \Delta a^2 b^2 (a^2b^2)^{r-k-1} a^2 b^2 a^{2r-2k+1} \\ &= \Delta^{2(\ell+k-1)+1} (a^2b^2)^{r-k+1} a^{2r-2k+1} =: \gamma_1.\end{aligned}$$

Note that by our assumptions on ℓ, r and k, we have $\ell + k - 1 \geqslant 0$, $r - k + 1 \geqslant 2$ and $2r - 2k + 1 \geqslant 3$, so γ_1 has the form of the braid words considered in Lemma 3.21. We thus have

$$\begin{aligned}\upsilon(\widehat{\gamma}) = \upsilon(\widehat{\gamma_1}) &= -\frac{4(r-k+1) + 2r - 2k + 1}{2} + (r - k + 2) - 2(\ell + k - 1) - \frac{3}{2} \\ &= -2(\ell + r).\end{aligned}$$

By (3.11), we have

$$\begin{aligned}\upsilon(\widehat{\alpha_1}) &\geqslant \upsilon(\widehat{\gamma}) - g(C) = -\frac{\Sigma_\beta}{2} - 2\ell + 1, \qquad \text{hence} \\ \upsilon(K) &\geqslant \upsilon(\widehat{\alpha_1}) + r - 1 \geqslant -\frac{\Sigma_\beta}{2} + r - 2\ell.\end{aligned}$$ □

Proof of Proposition 3.14. The first case of Proposition 3.14 follows from Remark 3.12 and (3.8). Lemma 3.22 and 3.23 together prove the second case, Lemma 3.21 proves the third case. Note that up to conjugation, by Remark 3.7 and Proposition 3.8, it is no restriction to assume that $p_r \geqslant 3$ in Lemma 3.21 and that $q_r \geqslant 3$ and either $p_r \geqslant 3$ or $p_k \geqslant 3$ for some $1 \leqslant k < r$ in Lemma 3.22 and 3.23, respectively. □

Before we proceed with the general case where the knot K is given as the closure of any 3-braid, let us prove the following corollary of our results in this subsection.

Corollary 3.24 (Corollary 3.3)**.** *Let K be a positive 3-braid knot. Then*

$$r = g(K) + \upsilon(K) + 1$$

is minimal among all integers $r \geqslant 1$ such that K is the closure of a positive 3-braid $a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r}$ for integers $p_i, q_i \geqslant 1$, $i \in \{1, \dots, r\}$. If K and J are concordant positive 3-braid knots, then this minimal r is the same for both K and J.

Proof. By Lemma 3.16 we have

$$\upsilon(K) \leqslant -g(K) + r - 1 \quad \Longleftrightarrow \quad g(K) + \upsilon(K) + 1 \leqslant r$$

whenever K is the closure of a positive 3-braid $a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r}$ for integers $r \geqslant 1$ and

$p_i, q_i \geqslant 1$, $i \in \{1, \ldots, r\}$. To prove the first statement of the corollary, it remains to show that we can always find a positive braid representative for K of the form $a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r}$ with $r = g(K) + \upsilon(K) + 1$. We will use Proposition 3.8. In fact, if K is the closure of a positive braid β of the form in (C) with $\ell \geqslant 0$, then $g(K) + \upsilon(K) + 1 = r + \ell$ by (1.3) applied to β, Lemma 3.22 and Lemma 3.23. Moreover, we have

$$\beta = a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r} \qquad \text{if } \ell = 0 \qquad \text{and}$$
$$\beta \sim a^{2\ell} b (a^2 b^2)^{\ell - 1} a^{p_1 + 2} b^{q_1} \cdots a^{p_r} b^{q_r + 1} \qquad \text{if } \ell \geqslant 1$$

by the proof of Lemma 3.18; these give the desired braid representatives for K. Furthermore, if K is represented by a positive braid β of the form in (D) with $\ell \geqslant 0$, then $g(K) + \upsilon(K) + 1 = r + \ell$ by (1.3) and Lemma 3.21, and we have

$$\beta \sim a^{p_1+1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} a^{p_r + 1} b \qquad \text{if } \ell = 0 \qquad \text{and}$$
$$\beta \sim a^{p_r+2} b a^3 (aba^3)^{\ell-1} b a^{p_1 + \ell + 1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} \qquad \text{if } \ell \geqslant 1$$

by the proof of Lemma 3.17. Finally, if $K = T_{3,3\ell+k}$ for $\ell \geqslant 0$ and $k \in \{1, 2\}$, then by (3.3) and (3.8), we have $g(K) + \upsilon(K) + 1 = \ell + 1$ and $T_{3,3\ell+1}$ and $T_{3,3\ell+2}$ are represented by the positive 3-braids $(ab)^{3\ell+1} = a^{2\ell+1} b \left(a^2 b^2\right)^{\ell}$ and $(ab)^{3\ell+2} \sim a^{2\ell+3} b (a^2 b^2)^{\ell}$, respectively, by Lemma 3.20 and Lemma 3.19.

If K and J are concordant, then their 4-genus and their upsilon invariants are equal. So by (1.3) and the above proved first part of Corollary 3.24, positive 3-braids with closures K and J, respectively, will have the same minimal r. □

Remark 3.25. Let $\mathcal{A}_g(K)$ denote the minimal genus of a cobordism between a knot K and an alternating knot, that is the cobordism distance $d(K, \{\text{alternating knots}\})$. By [FLZ17, Theorem 8], we have $\frac{|\tau(K) + \upsilon(K)|}{2} \leqslant \mathcal{A}_g(K)$ for any knot K. It thus follows from our results in this subsection that

$$\frac{r + \ell - 1}{2} \leqslant \mathcal{A}_g(K) \leqslant \frac{r + \ell - 1 + \varepsilon}{2}$$

for any knot K that is the closure of a positive 3-braid in Garside normal form (C) or (D), where $\varepsilon \geqslant 0$ is an integer depending on K. The lower bound uses Proposition 3.14 and (3.4) from Section 3.2; see also the proof of Corollary 3.24. The upper bound follows from the proofs of Lemma 3.17 and Lemma 3.18; see also the proof of Lemma 3.16. Note that for most positive 3-braid knots, we have $\varepsilon > 0$, so we do not get an equality.

A shorter proof of Lemma 3.16 without cobordisms follows from a result of Abe and Kishimoto on the dealternating number of positive 3-braid knots. Indeed, we have

$$|\Upsilon_K(t) + g(K)t| \overset{(3.4)}{=} |\Upsilon_K(t) + \tau(K)t| \leqslant \operatorname{alt}(K)t \overset{(3.23)}{\leqslant} \operatorname{dalt}(K)t$$
$$\overset{(3.26)}{\leqslant} (r-1)\,t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1.$$

The definitions of the dealternating number $\mathrm{dalt}(K)$ and the alternation number $\mathrm{alt}(K)$ of a knot K and more details on the inequalities used here will be provided in Section 3.5.

3.4.3 Proof of Theorem 3.1

It remains to show Theorem 3.1 when K is the closure of a 3-braid that is not necessarily positive. We first recall a result of Murasugi, which implies that indeed all 3-braid knots except for the torus knots of braid index 3 are covered by Theorem 3.1.

Let β be a 3-braid. Then, by [Mur74, Proposition 2.1], β is conjugate to one and only one of the 3-braids

$$\Delta^{2\ell}a^p \quad \text{or} \quad \Delta^{2\ell+1} \qquad \text{for} \quad \ell \in \mathbb{Z},\, p \in \mathbb{Z}, \tag{a}$$

$$\Delta^{2\ell}ab \quad \text{or} \quad \Delta^{2\ell}(ab)^2 \qquad \text{for} \quad \ell \in \mathbb{Z}, \tag{b}$$

$$\Delta^{2\ell}a^{-p_1}b^{q_1}\cdots a^{-p_r}b^{q_r} \qquad \text{for} \quad \ell \in \mathbb{Z},\, r \geqslant 1,\, p_i, q_i \geqslant 1,\, i \in \{1, \dots, r\}. \tag{c}$$

Definition 3.26. We call a braid word of the form in (a)–(c) a 3-*braid in Murasugi normal form.*

Remark 3.27. The closures of the 3-braids in Murasugi normal form (a) are links of two (if p is odd) or three components and the closures of the 3-braids in Murasugi normal form (b) are the torus knots of braid index 3 (cf. Remark 3.12).

If $\ell = 0$ in case (c), the braid word $\beta = a^{-p_1}b^{q_1}\cdots a^{-p_r}b^{q_r}$ for integers $r \geqslant 1$ and $p_i, q_i \geqslant 1$, $i \in \{1, \dots, r\}$, gives rise to an alternating braid diagram. If $K = \widehat{\beta}$ is a knot, by Proposition 3.6 we thus have $\upsilon(K) = \frac{\sigma(K)}{2}$ in that case and the statement of Theorem 3.1 follows directly from a result by Erle on the signature of 3-braid knots.

Proposition 3.28 ([Erl99, Theorem 2.6])**.** *Let* $\beta = \Delta^{2\ell}a^{-p_1}b^{q_1}\cdots a^{-p_r}b^{q_r}$ *for integers* $\ell \in \mathbb{Z}$*,* $r \geqslant 1$ *and* $p_i, q_i \geqslant 1$ *for* $i \in \{1, \dots, r\}$ *such that* $K = \widehat{\beta}$ *is a knot. Then*

$$\sigma(K) = \sum_{i=1}^{r}(p_i - q_i) - 4\ell.$$

We still need to show Theorem 3.1 when K is the closure of a 3-braid in Murasugi normal form (c) with $\ell \neq 0$. The proof will follow from the following two lemmas.

Lemma 3.29. *Let* $\beta = \Delta^{2\ell}a^{-p_1}b^{q_1}\cdots a^{-p_r}b^{q_r}$ *for some* $\ell \geqslant 1, r \geqslant 1$ *and* $p_i, q_i \geqslant 1$ *for* $i \in \{1, \dots, r\}$ *such that* $K = \widehat{\beta}$ *is a knot. Then*

$$\Upsilon_K(t) \leqslant \left(\frac{\sum_{i=1}^{r}(p_i - q_i)}{2} - 2\ell \right) t \qquad \textit{for all} \quad 0 \leqslant t \leqslant 1.$$

Lemma 3.30. *Let $\beta = \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r}$ for some $\ell \geqslant 0$, $r \geqslant 1$ and $p_i, q_i \geqslant 1$ for $i \in \{1, \ldots, r\}$ such that $K = \widehat{\beta}$ is a knot. Then*

$$\upsilon(K) \geqslant \frac{\sum_{i=1}^{r}(p_i - q_i)}{2} - 2\ell.$$

Proof of Theorem 3.1. For $\ell \geqslant 1$, the statement of the theorem follows directly from Lemma 3.29 and Lemma 3.30. If $\ell < 0$, the knot $-K$ is represented by the braid word $\Delta^{-2\ell} a^{-q_r} b^{p_r} \cdots a^{-q_1} b^{p_1}$ with $-\ell \geqslant 1$ and accordingly we have

$$\upsilon(-K) = \frac{\sum_{i=1}^{r}(q_i - p_i)}{2} + 2\ell.$$

Using that $\upsilon(-K) = -\upsilon(K)$ by (3.6) from Section 3.2, this implies the claim. □

The remainder of this subsection is devoted to the proofs of the above Lemmas 3.29 and 3.30.

Proof of Lemma 3.29. We first consider the case where $p_1 \geqslant 2$ and $\ell \geqslant 2$. Using the equality $\Delta a^{-1} = ab$ and

$$(ab)^{3n+2} = b^{n+1} a (b^3 ab)^{n-1} b^3 ab^3 \qquad \text{for all} \quad n \geqslant 1 \quad \text{[Fel16, Proof of Prop. 22]},$$

we have

$$\begin{aligned}\beta &= \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r} = \Delta^{2(\ell-1)+1} aba^{-p_1+1} b^{q_1} \cdots a^{-p_r} b^{q_r} \\ &= (ba)^{3(\ell-1)+2} ba^{-p_1+1} b^{q_1} \cdots a^{-p_r} b^{q_r} \sim (ab)^{3(\ell-1)+2} a^{-p_1+1} b^{q_1} \cdots a^{-p_r} b^{q_r+1} \\ &\sim a(b^3ab)^{\ell-2} b^3 ab^3 a^{-p_1+1} b^{q_1} \cdots a^{-p_r} b^{q_r+\ell+1} =: \beta_1.\end{aligned}$$

Now, we claim that there is a cobordism C of genus $g(C) = \frac{\ell+r-1+\varepsilon}{2}$ between the closure K of β_1 and the connected sum

$$J_\varepsilon = -T_{2,p_1-1-\varepsilon_1} \# -T_{2,p_2-\varepsilon_2} \# \ldots \# -T_{2,p_r-\varepsilon_r} \# T_{2,\sum_{i=1}^{r} q_i + 5\ell - 1 + \varepsilon_q},$$

where we choose $\varepsilon_1, \ldots, \varepsilon_r, \varepsilon_q \in \{0, 1\}$ such that J_ε is a connected sum of torus knots, i. e. such that $\sum_{i=1}^{r} q_i + 5\ell - 1 + \varepsilon_q$, $p_1 - 1 - \varepsilon_1$, $p_2 - \varepsilon_2, \ldots, p_r - \varepsilon_r$ are all odd; and $\varepsilon = \varepsilon_q + \sum_{i=1}^{r} \varepsilon_i$. This cobordism C can be realized using $\ell + r - 1 + \varepsilon$ saddle moves as follows. On the one hand, we add $\sum_{i=1}^{r} \varepsilon_i$ generators a and ε_q generators b to the braid word β_1, on the other hand, we perform $\ell + r - 1$ saddle moves of the form as the $r - 1$ saddle moves used in the proof of Lemma 3.16 to get a connected sum of torus knots. The Euler characteristic of C is $\chi(C) = -\ell - r + 1 - \varepsilon$. Since C is connected and has two boundary components (as K and J_ε are knots), the genus of C

is $g(C) = \frac{-\chi(C)}{2} = \frac{\ell+r-1+\varepsilon}{2}$ as claimed. By (3.5) and (3.8), we have

$$\Upsilon_{J_\varepsilon}(t) = \left(\frac{\sum_{i=1}^{r}(p_i - q_i) - \varepsilon - r - 5\ell + 1}{2} \right) t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1$$

and by (3.11), we get

$$\Upsilon_K(t) \leqslant \Upsilon_{J_\varepsilon}(t) + g(C)t = \left(\frac{\sum_{i=1}^{r}(p_i - q_i)}{2} - 2\ell \right) t \qquad \text{for all} \quad 0 \leqslant t \leqslant 1.$$

If $p_1 \geqslant 2$ and $\ell = 1$, then

$$\beta \sim (ab)^2 a^{-p_1+1} b^{q_1} \cdots a^{-p_r} b^{q_r+1} \sim ab^2 a^{-p_1+1} b^{q_1} \cdots a^{-p_r} b^{q_r+2} =: \beta_1,$$

and similarly as above, there is a cobordism C of genus $g(C) = \frac{r+\varepsilon}{2}$ between the closure K of β_1 and the connected sum

$$J_\varepsilon = -T_{2,p_1-1-\varepsilon_1} \# -T_{2,p_2-\varepsilon_2} \# \ldots \# -T_{2,p_r-\varepsilon_r} \# T_{2,\sum_{i=1}^{r} q_i+4+\varepsilon_q},$$

where we choose $\varepsilon_1, \ldots, \varepsilon_r, \varepsilon_q \in \{0, 1\}$ such that J_ε is a connected sum of torus knots and $\varepsilon = \varepsilon_q + \sum_{i=1}^{r} \varepsilon_i$. The claim follows also in this case from (3.5) and (3.8), and the inequality in (3.11).

It remains to show the claim when $p_1 = 1$. In that case, using $\Delta a^{-1} = ab$, we have

$$\beta = \Delta^{2\ell} a^{-1} b^{q_1} \cdots a^{-p_r} b^{q_r} = \Delta^{2\ell-1} ab^{q_1+1} \cdots a^{-p_r} b^{q_r} \sim \Delta^{2\ell-1} b^{q_1+1} \cdots a^{-p_r} b^{q_r+1}.$$

If $\ell = 1$, then β is conjugate to $\beta_1 = ab^{q_1+2} a^{-p_2} b^{q_2} \cdots a^{-p_r} b^{q_r+2}$ and if $\ell \geqslant 2$, then using (3.17) from Section 3.4.2, we have

$$\begin{aligned} \beta &\sim \Delta^{2(\ell-1)+1} b^{q_1+1} a^{-p_2} b^{q_2} \cdots a^{-p_r} b^{q_r+1} = (ba)^{3(\ell-1)+1} b^{q_1+2} a^{-p_2} b^{q_2} \cdots a^{-p_r} b^{q_r+1} \\ &\sim ab^3 (bab^3)^{\ell-2} ab^{q_1+\ell+1} a^{-p_2} b^{q_2} \cdots a^{-p_r} b^{q_r+3} =: \beta_1. \end{aligned}$$

In both cases, there is a cobordism C of genus $g(C) = \frac{\ell+r-2+\varepsilon}{2}$ between the closure K of β_1 and the connected sum

$$J_\varepsilon = -T_{2,p_2-\varepsilon_2} \# \ldots \# -T_{2,p_r-\varepsilon_r} \# T_{2,\sum_{i=1}^{r} q_i+5\ell-1+\varepsilon_q},$$

where we choose $\varepsilon_1, \ldots, \varepsilon_r, \varepsilon_q \in \{0, 1\}$ such that J_ε is a connected sum of torus knots and $\varepsilon = \varepsilon_q + \sum_{i=1}^{r} \varepsilon_i$. Using (3.5), (3.8) and (3.11) again, the claim follows. □

We will need the following two technical lemmas for the proof of Lemma 3.30.

Lemma 3.31. *Let $\beta = \Delta^{2\ell} a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r}$ for some $\ell \geqslant 0$, $r \geqslant 1$ and integers p_i, q_i such that $p_i < 0$ or $p_i \geqslant 2$, and $q_i < 0$ or $q_i \geqslant 2$, for every $i \in \{1, \ldots, r\}$. Moreover, assume that $K = \widehat{\beta}$ is a knot. Then*

$$\upsilon(K) \geqslant -\frac{\sum_{i=1}^{r}(p_i + q_i)}{2} + r - 2\ell - \#\{i \mid p_i < 0\} - \#\{i \mid q_i < 0\},$$

where $\#A$ denotes the cardinality of the set A.

Lemma 3.32. *Let $\beta = \Delta^{2\ell+1} a^{p_1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}$ for some $\ell \geqslant 0$, $r \geqslant 1$ and integers p_i, q_i such that $p_i < 0$ or $p_i \geqslant 2$ for every $i \in \{1, \ldots, r\}$ and $q_i < 0$ or $q_i \geqslant 2$ for every $i \in \{1, \ldots, r-1\}$. Moreover, assume that $K = \widehat{\beta}$ is a knot. Then*

$$\upsilon(K) \geqslant -\frac{\sum_{i=1}^{r-1}(p_i + q_i) + p_r}{2} + r - 2\ell - \frac{3}{2} - \#\{i \mid p_i < 0\} - \#\{i \mid q_i < 0\}.$$

For the proof of Lemmas 3.31 and 3.32, we refer the reader to the very end of this subsection; we will first prove Lemma 3.30 using these lemmas.

Proof of Lemma 3.30. Let k be the number of exponents q_j of β with $q_j = 1$ and let $\mathcal{J} = \{j_1, \ldots, j_k\}$ for $0 \leqslant k \leqslant r$ be the set of indices such that $q_j = 1$ if and only if $j \in \mathcal{J}$. For all $j \in \mathcal{J}$, we rewrite the subword $a^{-p_j} b^{q_j}$ of β using $\Delta^{-1} ab = a^{-1}$ as

$$a^{-p_j} b^{q_j} = a^{-p_j} b = a^{-p_j} a^{-1} \Delta \Delta^{-1} ab = a^{-p_j - 1} \Delta a^{-1} = \Delta b^{-p_j - 1} a^{-1}.$$

Note that if $j, j+1 \in \mathcal{J}$, then $a^{-p_j} b^{q_j} a^{-p_{j+1}} b^{q_{j+1}} = \Delta^2 a^{-p_j - 1} b^{-p_{j+1} - 2} a^{-1}$. After rewriting $a^{-p_j} b^{q_j}$ for all $j \in \mathcal{J}$, we have $\beta \sim \beta_1 = \Delta^{2\ell+k} \alpha$ for some 3-braid α of the form

$$\alpha = \begin{cases} a^{\widetilde{p_1}} b^{\widetilde{q_1}} \cdots a^{\widetilde{p_n}} b^{\widetilde{q_n}} \text{ for } n = r - \frac{k}{2} & \text{if } k \text{ is even,} \\ b^{\widetilde{p_1}} a^{\widetilde{q_1}} \cdots b^{\widetilde{p_{n-1}}} a^{\widetilde{q_{n-1}}} b^{\widetilde{p_n}} \text{ for } n = r - \frac{k-1}{2} & \text{if } k \text{ is odd,} \end{cases}$$

where $\sum_{i=1}^{n}(\widetilde{p_i} + \widetilde{q_i}) = \sum_{i=1}^{r}(-p_i + q_i) - 3k$ and where the $\widetilde{p_i}$ and $\widetilde{q_i}$ fulfill the assumptions of Lemma 3.31 and Lemma 3.32, respectively, i. e. where $\widetilde{p_i} < 0$ or $\geqslant 2$ and $\widetilde{q_i} < 0$ or $\geqslant 2$ for any i. The number of negative exponents in α equals the number of negative exponents $-p_i$ in β, so $\#\{i \mid \widetilde{p_i} < 0\} + \#\{i \mid \widetilde{q_i} < 0\} = r$. If k is even, by Lemma 3.31, we get

$$\begin{aligned} \upsilon\left(\widehat{\beta}\right) &\geqslant -\frac{\sum_{i=1}^{n}(\widetilde{p_i} + \widetilde{q_i})}{2} + n - (2\ell + k) - \#\{i \mid \widetilde{p_i} < 0\} + \#\{i \mid \widetilde{q_i} < 0\} \\ &= -\frac{\sum_{i=1}^{r}(-p_i + q_i) - 3k}{2} + r - \frac{k}{2} - (2\ell + k) - r = \frac{\sum_{i=1}^{r}(p_i - q_i)}{2} - 2\ell. \end{aligned}$$

Similarly, if k is odd, the claim follows from Lemma 3.32. □

It remains to prove Lemma 3.31 and Lemma 3.32.

Proof of Lemma 3.31. We will modify the braid word β in $2r$ steps, where each step corresponds to one of the $2r$ exponents p_i, q_i, $i \in \{1, \dots, r\}$, of β. In every step, we will either just conjugate β (if the corresponding exponent is positive) or perform a cobordism of genus 1 between the closure of $a^{2n}\beta$ or $b^{2n}\beta$ and the connected sum $T_{2,2n+1}\#\widehat{\beta}$ for some $n \geqslant 0$—similarly as the cobordism described in Example 3.13 and used in the proofs of Lemma 3.21, Lemma 3.22 and Lemma 3.23. We now describe these steps in more detail. First, let $\beta'_{0,q} = \beta$ and define

$$\begin{aligned} a^{-p_1+2+\varepsilon_{1,p}}\beta'_{0,q} &= \Delta^{2\ell}a^{2+\varepsilon_{1,p}}b^{q_1}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r} \\ &\sim \Delta^{2\ell}b^{q_1}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r}a^{2+\varepsilon_{1,p}} =: \beta'_{1,p} && \text{if } p_1 < 0 \text{ and} \\ \beta'_{0,q} &\sim \Delta^{2\ell}b^{q_1}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r}a^{p_1} =: \beta'_{1,p} && \text{if } p_1 > 0 \end{aligned}$$

such that $\beta'_{1,p} = \Delta^{2\ell}b^{q_1}a^{p_2}\cdots a^{p_r}b^{q_r}a^{\widetilde{p_1}}$ for some $\widetilde{p_1} \geqslant 2$ (note that we assumed $p_1 < 0$ or $p_1 \geqslant 2$). Here, if $p_1 < 0$, we choose $\varepsilon_{1,p} \in \{0,1\}$ such that $-p_1+2+\varepsilon_{1,p}$ is even and $\widehat{\beta'_{1,p}}$ is a knot. Second, let $\varepsilon_{1,q} \in \{0,1\}$ such that $-q_1+2+\varepsilon_{1,q}$ is even if $q_1 < 0$, and define

$$\begin{aligned} \beta_{1,q} &= b^{-q_1+2+\varepsilon_{1,q}}\beta'_{1,p} = \Delta^{2\ell}b^{2+\varepsilon_{1,q}}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r}a^{\widetilde{p_1}} \\ &\sim \Delta^{2\ell}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r}a^{\widetilde{p_1}}b^{2+\varepsilon_{1,q}} =: \beta'_{1,q} && \text{if } q_1 < 0 \text{ and} \\ \beta_{1,q} &= \beta'_{1,p} \sim \Delta^{2\ell}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r}a^{\widetilde{p_1}}b^{q_1} =: \beta'_{1,q} && \text{if } q_1 > 0 \end{aligned}$$

such that $\beta'_{1,q} = \Delta^{2\ell}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r}a^{\widetilde{p_1}}b^{\widetilde{q_1}}$ for some $\widetilde{p_1}, \widetilde{q_1} \geqslant 2$. Inductively, for any $1 \leqslant i \leqslant r$, we let

$$\begin{aligned} a^{-p_i+2+\varepsilon_{i,p}}\beta'_{i-1,q} &= \Delta^{2\ell}a^{2+\varepsilon_{i,p}}b^{q_i}a^{p_{i+1}}\cdots a^{p_r}b^{q_r}a^{\widetilde{p_1}}b^{\widetilde{q_1}}\cdots a^{\widetilde{p_{i-1}}}b^{\widetilde{q_{i-1}}} \\ &\sim \Delta^{2\ell}b^{q_i}a^{p_{i+1}}\cdots a^{p_r}b^{q_r}a^{\widetilde{p_1}}b^{\widetilde{q_1}}\cdots a^{\widetilde{p_{i-1}}}b^{\widetilde{q_{i-1}}}a^{2+\varepsilon_{i,p}} =: \beta'_{i,p} && \text{if } p_i < 0 \text{ and} \\ \beta'_{i-1,q} &\sim \Delta^{2\ell}b^{q_i}a^{p_{i+1}}\cdots a^{p_r}b^{q_r}a^{\widetilde{p_1}}b^{\widetilde{q_1}}\cdots a^{\widetilde{p_{i-1}}}b^{\widetilde{q_{i-1}}}a^{p_i} =: \beta'_{i,p} && \text{if } p_i > 0 \end{aligned}$$

such that

$$\beta'_{i,p} = \Delta^{2\ell}b^{q_i}a^{p_{i+1}}\cdots a^{p_r}b^{q_r}a^{\widetilde{p_1}}b^{\widetilde{q_1}}\cdots a^{\widetilde{p_{i-1}}}b^{\widetilde{q_{i-1}}}a^{\widetilde{p_i}}$$

for some integers $\widetilde{p_1}, \widetilde{q_1}, \dots, \widetilde{p_{i-1}}, \widetilde{q_{i-1}}, \widetilde{p_i} \geqslant 2$. Here, we choose $\varepsilon_{i,p} \in \{0,1\}$ such that $-p_i+2+\varepsilon_{i,p}$ is even if $p_i < 0$. Moreover, for any $1 \leqslant i \leqslant r$, we let $\varepsilon_{i,q} \in \{0,1\}$ such that $-q_i+2+\varepsilon_{i,q}$ is even, and define

$$\begin{aligned} \beta_{i,q} &= b^{-q_i+2+\varepsilon_{i,q}}\beta'_{i,p} && \text{if } q_i < 0 \text{ and} \\ \beta_{i,q} &= \beta'_{i,p} && \text{if } q_i > 0; \end{aligned}$$

and we define $\beta'_{i,q}$ similarly as $\beta'_{1,q}$. Inductively, after $2r$ steps, we get the positive 3-braid

$$\beta'_{r,q} = \Delta^{2\ell} a^{\widetilde{p_1}} b^{\widetilde{q_1}} \cdots a^{\widetilde{p_r}} b^{\widetilde{q_r}} \qquad \text{with}$$

$$\widetilde{p_i} = \begin{cases} 2+\varepsilon_{i,p} & \text{if } p_i < 0, \\ p_i & \text{if } p_i > 0, \end{cases} \qquad \text{and} \qquad \widetilde{q_i} = \begin{cases} 2+\varepsilon_{i,q} & \text{if } q_i < 0, \\ q_i & \text{if } q_i > 0, \end{cases}$$

for all $1 \leqslant i \leqslant r$; so that $\widetilde{p_1}, \widetilde{q_1}, \ldots \widetilde{p_r}, \widetilde{q_r} \geqslant 2$. By Proposition 3.14, we have

$$\upsilon\left(\widehat{\beta'_{r,q}}\right) = -\frac{\sum\limits_{\substack{i=1\\p_i>0}}^{r} p_i + \sum\limits_{\substack{i=1\\q_i>0}}^{r} q_i + \sum\limits_{\substack{i=1\\p_i<0}}^{r} (2+\varepsilon_{i,p}) + \sum\limits_{\substack{i=1\\q_i<0}}^{r} (2+\varepsilon_{i,q})}{2} + r - 2\ell.$$

Now, note that if $p_i < 0$ for some $1 \leqslant i \leqslant r$, then there is a cobordism of genus 1 between $\widehat{\beta'_{i,p}}$ and $T_{2,2m+1} \# \widehat{\beta'_{i-1,q}}$ by using two saddle moves, where $m = \frac{-p_i+2+\varepsilon_{i,p}}{2}$, so similarly as in (3.12) from Example 3.13, we have

$$\upsilon\left(\widehat{\beta'_{i-1,q}}\right) \geqslant \upsilon\left(\widehat{\beta'_{i,p}}\right) + m - 1 = \upsilon\left(\widehat{\beta'_{i,p}}\right) + \frac{-p_i + \varepsilon_{i,p}}{2}.$$

Similarly, if $q_i < 0$ for some $1 \leqslant i \leqslant r$, then $\upsilon(\widehat{\beta'_{i,p}}) \geqslant \upsilon(\widehat{\beta'_{i,q}}) + \frac{-q_i+\varepsilon_{i,q}}{2}$. In addition, if $p_i > 0$, we have $\upsilon(\widehat{\beta'_{i,p}}) = \upsilon(\widehat{\beta'_{i-1,q}})$, and if $q_i > 0$, then $\upsilon(\widehat{\beta'_{i,q}}) = \upsilon(\widehat{\beta'_{i,p}})$. We conclude

$$\begin{aligned}
\upsilon\left(\widehat{\beta}\right) = \upsilon\left(\widehat{\beta'_{0,q}}\right) &\geqslant \upsilon\left(\widehat{\beta'_{r,q}}\right) + \sum_{\substack{i=1\\p_i<0}}^{r} \frac{-p_i + \varepsilon_{i,p}}{2} + \sum_{\substack{i=1\\q_i<0}}^{r} \frac{-q_i + \varepsilon_{i,q}}{2} \\
&= -\frac{\sum\limits_{\substack{i=1\\p_i>0}}^{r} p_i + \sum\limits_{\substack{i=1\\q_i>0}}^{r} q_i + \sum\limits_{\substack{i=1\\p_i<0}}^{r} (p_i+2) + \sum\limits_{\substack{i=1\\q_i<0}}^{r} (q_i+2)}{2} + r - 2\ell \\
&= -\frac{\sum\limits_{i=1}^{r} (p_i + q_i)}{2} + r - 2\ell - \#\{i \mid p_i < 0\} - \#\{i \mid q_i < 0\}.
\end{aligned}$$

□

Proof of Lemma 3.32. The strategy of the proof is the same as in the proof of Lemma 3.31. Here, we need $2r-1$ steps corresponding to the $2r-1$ exponents $p_1, q_1, \ldots, p_{r-1}, q_{r-1}, p_r$ of β. The steps are similar as in the proof of Lemma 3.31, the only change is that we multiply $\beta'_{i-1,q}$ by a power of b if $p_i < 0$, and $\beta'_{i,p}$ by a power of a if $q_i < 0$ (since $a\Delta^{2\ell+1} = \Delta^{2\ell+1} b$ and $b\Delta^{2\ell+1} = \Delta^{2\ell+1} a$). Thus, starting with $\beta'_{0,q} = \beta$, after $2r-1$ steps we obtain the positive 3-braid

$$\beta'_{r,p} = \Delta^{2\ell+1} a^{\widetilde{p_1}} b^{\widetilde{q_1}} \cdots a^{\widetilde{p_{r-1}}} b^{\widetilde{q_{r-1}}} a^{\widetilde{p_r}} \qquad \text{with}$$

$$\widetilde{p_i} = \begin{cases} 2+\varepsilon_{i,p} & \text{if } p_i < 0, \\ p_i & \text{if } p_i > 0, \end{cases} \qquad \text{and} \qquad \widetilde{q_i} = \begin{cases} 2+\varepsilon_{i,q} & \text{if } q_i < 0, \\ q_i & \text{if } q_i > 0. \end{cases}$$

By Lemma 3.21, we have

$$\upsilon\left(\beta'_{r,p}\right) = -\frac{\sum\limits_{\substack{i=1\\p_i>0}}^{r} p_i + \sum\limits_{\substack{i=1\\q_i>0}}^{r-1} q_i + \sum\limits_{\substack{i=1\\p_i<0}}^{r} (2+\varepsilon_{i,p}) + \sum\limits_{\substack{i=1\\q_i<0}}^{r-1} (2+\varepsilon_{i,q})}{2} + r - 2\ell - \frac{3}{2}.$$

Since the steps we performed have similar effects on $\upsilon\left(\widehat{\beta}\right)$ as the ones in the proof of Lemma 3.31, we get

$$\begin{aligned}\upsilon\left(\widehat{\beta}\right) = \upsilon\left(\widehat{\beta'_{0,q}}\right) &\geqslant \upsilon\left(\widehat{\beta'_{r,p}}\right) + \sum_{\substack{i=1\\p_i<0}}^{r} \frac{-p_i+\varepsilon_{i,p}}{2} + \sum_{\substack{i=1\\q_i<0}}^{r-1} \frac{-q_i+\varepsilon_{i,q}}{2}\\ &= -\frac{\sum\limits_{\substack{i=1\\p_i>0}}^{r} p_i + \sum\limits_{\substack{i=1\\q_i>0}}^{r-1} q_i + \sum\limits_{\substack{i=1\\p_i<0}}^{r} (p_i+2) + \sum\limits_{\substack{i=1\\q_i<0}}^{r-1} (q_i+2)}{2} + r - 2\ell - \frac{3}{2}\\ &= -\frac{\sum\limits_{i=1}^{r-1}(p_i+q_i)+p_r}{2} + r - 2\ell - \frac{3}{2} - \#\{i \mid p_i<0\} - \#\{i \mid q_i<0\}.\end{aligned}$$ □

3.4.4 Further discussion of Theorem 3.1

In this subsection, we provide some further context on our main result. In particular, in Section 3.4.4.2 we will discuss why it might be surprising that our proof strategy works for all 3-braid knots.

3.4.4.1 Comparison of upsilon and the classical signature

By Theorem 3.1 and Proposition 4.10, we have

$$\sigma(K) = 2\upsilon(K) \tag{3.22}$$

for every knot K which is the closure of a 3-braid $\beta = \Delta^{2\ell}a^{-p_1}b^{q_1}\cdots a^{-p_r}b^{q_r}$ for integers $\ell \in \mathbb{Z}$, $r \geqslant 1$ and $p_i, q_i \geqslant 1$ for $i \in \{1,\dots,r\}$. Computations of the signature for torus knots (and links) of braid index 3, first done by Hirzebruch, Murasugi and Shinora [Mur74, Proposition 9.1, pp. 34-35], together with (3.8) from Section 3.2 imply that the equality in (3.22) is in fact true for all 3-braid knots K except for the cases that $K = \pm T_{3,3\ell+1}$ for odd $\ell > 0$ or $K = \pm T_{3,3\ell+2}$ for odd $\ell > 0$. In the exceptional cases, we have $\sigma(K) = 2\upsilon(K) - 2$. As mentioned in the introduction, this improves the inequality $|\upsilon(K) - \frac{\sigma(K)}{2}| \leqslant 2$ for all 3-braid knots K in [FK17, Proposition 4.4].

It was shown in [OSS17b, Theorem 1.2] that $|\upsilon(K) - \frac{\sigma(K)}{2}|$ gives a lower bound on the *nonorientable smooth 4-genus* of a knot K, denoted $\gamma_4(K)$, the minimal first Betti number of a nonorientable surface in B^4 that meets the boundary S^3 along K. The similarity of the invariant υ and the classical signature σ on 3-braid knots K described above clearly does not lead to a good lower bound on $\gamma_4(K)$. However, the equality $\sigma(K) = 2\upsilon(K)$ for most 3-braid knots is actually no great surprise when noting that in fact $|\upsilon(K) - \frac{\sigma(K)}{2}| \leqslant 1$ must be true for all 3-braid knots K for the following reason. It is not hard to see that for every 3-braid knot K, there is a nonorientable band move to a 2-bridge knot J, which is alternating [Goo72]. This implies that the *nonorientable cobordism distance* $d_\gamma(K, J) = \gamma_4(K\# -J)$ between K and J is bounded from above by 1. On the other hand, using that υ and σ induce homomorphisms $\mathcal{C} \to \mathbb{Z}$ (see Section 3.2 and [Mur65]), the inequality $|\upsilon(K) - \frac{\sigma(K)}{2}| \leqslant \gamma_4(K)$ implies that

$$\left|\upsilon(K) - \frac{\sigma(K)}{2}\right| = \left|\upsilon\left(K\# -J\right) - \frac{\sigma\left(K\# -J\right)}{2}\right| \leqslant d_\gamma(K, J) \leqslant 1,$$

where we used $\upsilon(J) = \frac{\sigma(J)}{2}$ by Proposition 3.6. Note that a similar argument shows that $|\upsilon(K) - \frac{\sigma(K)}{2}| \leqslant 2$ for all 4-braid knots K, using two nonorientable band moves to transform K into a 2-bridge link, which is also alternating.

3.4.4.2 On the proof technique

As mentioned in the introduction, it came as a surprise to the author that our proof strategy works not only for positive 3-braid knots, but for all 3-braid knots. Let us make this more precise.

The proofs in Section 3.4.2 and Section 3.4.3 imply, for any 3-braid knot K, the existence of cobordisms C_1 and C_2 of genus $g(C_1)$ and $g(C_2)$ between K and (connected sums of) torus knots T_1 and T_2, respectively, such that

$$g\left(C_1\right) + g\left(C_2\right) = |\upsilon(T_2) - \upsilon(T_1)| \qquad \text{and}$$
$$\upsilon(K) = \upsilon(T_1) + g(C_1) = \upsilon(T_2) - g(C_2).$$

For example, for knots K that are closures of positive 3-braids of Garside normal form (D), the proof of Lemma 3.17 shows the existence of such a cobordism C_1 for $T_1 = J_\varepsilon$ as in the proof of Lemma 3.16; and the existence of such a cobordism C_2 between K and $T_2 = T_{3,3(\ell+r)+1}\# -T_{2,2r+1}$ follows from the proof of Lemma 3.21.

The same strategy would work to determine the concordance invariants s and τ (or any other slice-torus invariant; see the paragraph after (3.4) for the definition of slice-torus invariants) for all positive 3-braid knots K. Indeed, every positive 3-braid knot can be realized as the slice of a cobordism C between the unknot U and a torus knot T of braid index 3 such that $g(C) = |\tau(U) - \tau(T)| = |s(U) - s(T)|$ [FLL22, Proposition 4.1]. However, in contrast, there are 3-braid knots where this strategy provably fails to determine s and τ. A concrete example is the 3-braid knot 10_{125}, the closure of $a^{-5}ba^3b$ [LM23], which is not squeezed [FLL22, Example 3.1]. This means that every cobordism C between two connected sums of torus knots T_1 and T_2 that has 10_{125} as a slice satisfies $g(C) > |\tau(T_2) - \tau(T_1)| = |s(T_2) - s(T_1)|$.

3.4.4.3 Comparison of the normal forms for 3-braids

An algorithm described in [BM93, Section 7] as Schreier's solution to the conjugacy problem [Sch24] can be used to convert 3-braids in Garside normal form (cf. Definition 3.10) to 3-braids in Murasugi normal form (cf. Definition 3.26): If β is a 3-braid of Garside normal form (C), then

$$\beta \sim \Delta^{2(\ell+r)} a^{-1} b^{p_1-2} a^{-1} b^{q_1-2} \cdots a^{-1} b^{p_r-2} a^{-1} b^{q_r-2},$$

and if β is of Garside normal form (D), then

$$\beta \sim \Delta^{2(\ell+r)} a^{-1} b^{p_1-2} a^{-1} b^{q_1-2} \cdots a^{-1} b^{p_{r-1}-2} a^{-1} b^{q_{r-1}-2} a^{-1} b^{p_r-2}.$$

In addition, it is easy to see how 3-braids of Garside normal form (A) or (B) are conjugate to braids of Murasugi normal form (a) or (b).

3.5 On alternating distances of 3-braid knots

In this section, we prove Corollary 3.2 and provide lower and upper bounds on the alternation number and dealternating number of any 3-braid knot which differ by 1.

3.5.1 Alternating distances of positive 3-braid knots

We will prove the following proposition.

Proposition 3.33. *Let K be a knot that is the closure of a positive 3-braid. Then*

$$\begin{aligned} \operatorname{alt}(K) = \operatorname{dalt}(K) &= \tau(K) + \upsilon(K) \\ &= \begin{cases} \ell & \text{if } K \text{ is the torus knot } T_{3,3\ell+k} \text{ for } \ell \geqslant 0, k \in \{1,2\}, \\ r+\ell-1 & \text{if } K \text{ is the closure of a braid of the form in (C) or (D)}, \end{cases} \end{aligned}$$

where (C) and (D) refer to the Garside normal forms from Proposition 3.8.

Remark 3.34. Some of the cases in Proposition 3.33 have already been proved by other authors. Indeed, Feller, Pohlmann and Zentner used the observation (3.24) below to show that $\operatorname{alt}(T_{3,3\ell+k}) = \ell$ for all $\ell \geqslant 0$, $k \in \{1,2\}$ [FPZ18, Theorem 1.1]. The upper bound they used was provided by [Kan10, Theorem 8]; in fact, the equality had already been shown by Kanenobu in half of the cases, namely when ℓ is even. Moreover, Abe and Kishimoto [AK10, Theorem 3.1] showed that $\operatorname{alt}(K) = \operatorname{dalt}(K) = r+\ell-1$ if K is a knot that is the closure of a positive 3-braid of the form in (C). However, to the best of the author's knowledge, it is new that $\operatorname{alt}(K) = g(K) + \upsilon(K)$ for all positive 3-braid knots K. Recall that $\tau(K) = g(K)$ for all positive 3-braid knots K by (3.4) from Section 3.2.

Before we prove Proposition 3.33, let us provide the necessary definitions and background. The *Gordian distance* $d_G(K, J)$ between two knots K and J is the minimal number of crossing changes needed to transform a diagram of K into a diagram of J, where the minimum is taken over all diagrams of K [Mur85]. The *alternation number* $\mathrm{alt}(K)$ of a knot K is defined as the minimal Gordian distance of the knot K to the set of alternating knots [Kaw10], i. e.

$$\mathrm{alt}(K) = \min\{d_G(K, J) \mid J \text{ is an alternating knot}\}.$$

The *dealternating number* $\mathrm{dalt}(K)$ of a knot K is defined via a more diagrammatic approach [ABB+92]: it is the minimal number m such that K has a diagram that can be turned into an alternating diagram by m crossing changes. It follows from the definitions that

$$\mathrm{alt}(K) \leqslant \mathrm{dalt}(K) \tag{3.23}$$

for any knot K and $\mathrm{alt}(K) = \mathrm{dalt}(K) = 0$ if and only if K is alternating. Note that there are families of knots for which the difference between the alternation number and the dealternating number becomes arbitrarily large [Low15, Theorem 1.1].

In the proof of Proposition 3.33, we will use that

$$|\tau(K) + \upsilon(K)| \leqslant \mathrm{alt}(K) \tag{3.24}$$

for any knot K. In fact, for all alternating knots K, we have

$$\tau(K) = \frac{s(K)}{2} = -\upsilon(K) = -\frac{\Upsilon_K(t)}{t} = -\frac{\sigma(K)}{2} \tag{3.25}$$

for any $t \in (0, 1]$ (see [OS03, Theorem 1.4], [Ras10, Theorem 3] and [OSS17a, Theorem 1.14]), where s denotes Rasmussen's concordance invariant from Khovanov homology [Ras10]. It follows from [Abe09, Theorem 2.1]—which builds on ideas of Livingston [Liv04, Corollary 3]—that the absolute value of the difference of any two of the invariants in (3.25) is a lower bound on $\mathrm{alt}(K)$. It was first observed in [FPZ18] that the upsilon invariant fits very well in this context (see also [FLZ17, Lemma 8]). Another main ingredient of our proof of Proposition 3.33 is the inequality

$$\mathrm{dalt}\left(\widehat{\beta}\right) \leqslant r - 1 \tag{3.26}$$

for any positive 3-braid $\beta = a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r}$ with integers $r \geqslant 1$ and $p_i, q_i \geqslant 1$ for $i \in \{1, \ldots, r\}$ [AK10, Lemma 2.2].

Proof of Proposition 3.33. Let K be a knot that is the closure of a positive 3-braid β of the form in (C) or (D) from Proposition 3.8 with $\ell \geqslant 0$. We claim that then

$$r + \ell - 1 = \tau(K) + \upsilon(K) = |\tau(K) + \upsilon(K)| \leqslant \mathrm{alt}(K) \leqslant \mathrm{dalt}(K) \leqslant r + \ell - 1, \tag{3.27}$$

which implies the statement of the proposition for these knots. The two equalities in (3.27) directly follow from our computations of $\upsilon(K)$ in Proposition 3.14 and (3.4) applied to $K = \widehat{\beta}$. The first two inequalities are direct consequences of the inequalities (3.24) and (3.23). The last inequality follows from (3.26) applied to the particular braid representatives of K considered in the proof of Corollary 3.24.

For torus knots of braid index 3, the statement follows analogously. More precisely, if $K = T_{3,3\ell+k}$ for $\ell \geqslant 0$ and $k \in \{1,2\}$, then $|\tau(K) + \upsilon(K)| = \ell$ by (3.3) and (3.8). In addition, (3.26) applied to the particular braid representatives of K considered in the proof of Corollary 3.24 implies that $\operatorname{dalt}(T_{3,3\ell+k}) \leqslant \ell$. ☐

From Proposition 3.33, it is easy to deduce that the alternating positive 3-braid knots are precisely the unknot and the connected sums $T_{2,2p+1}\#T_{2,2q+1}$ of two torus knots of braid index 2 for $p, q \geqslant 0$. This was already known; in fact, the stronger statement is true that the only prime alternating positive braid knots are the torus knots of braid index 2 [Baa13, Corollary 3]. Note that by [Mor79] (see also [BM93, Corollary 7.2]), the only composite 3-braid knots are the connected sums $T_{2,2p+1}\#T_{2,2q+1}$ for $p, q \in \mathbb{Z}$.

By [Abe09, Theorem 1.1], the only torus knots with alternation number 1 are the torus knots $T_{3,4}$ and $T_{3,5}$. A knot with dealternating number 1 is called *almost alternating*.

Corollary 3.35. *A positive 3-braid knot is almost alternating if and only if it is one of the torus knots $T_{3,4}$ and $T_{3,5}$ or it is represented by a braid of the form*

$$a^{p_1}b^{q_1}a^{p_2}b^{q_2}, \quad \Delta a^{p_1}b^{q_1}a^{p_2}, \quad \Delta^2 a^{p_1}b^{q_1} \quad \text{or} \quad \Delta^3 a^{p_1}$$

for some integers $p_1, p_2, q_1, q_2 \geqslant 2$.

Proof. This follows directly from Proposition 3.33. ☐

Remark 3.36. In particular, the seven positive 3-braid knots with crossing number 12 (cf. [LM23]) are all almost alternating.

Remark 3.37. Our results imply that the Turaev genus equals the alternation number for all positive 3-braid knots. Indeed, let K be a knot that is the closure of a positive braid of the form in (C) or (D) with $\ell \geqslant 0$. Then we have

$$g_T(K) = \operatorname{alt}(K) = \operatorname{dalt}(K) = r + \ell - 1, \tag{3.28}$$

where $g_T(K)$ denotes the Turaev genus of the knot K. The *Turaev genus* $g_T(K)$ of a knot K is another alternating distance [Low15], which was first defined in [DFK+08] as the minimal genus of a Turaev surface $F(D)$, where the minimum is taken over all diagrams D of K. The Turaev surface $F(D)$ is a closed orientable surface embedded in S^3 associated to the diagram D. It is formed by building the natural cobordism between the circles in the two extreme Kauffman states (the *all-A-state* and the *all-B-state*) of the diagram D via adding saddles for every crossing of D, and then capping off the boundary components with disks. More details on the definition can be found e. g. in a survey by Champanerkar and Kofman [CK14].

The equality $g_T(K) = \mathrm{dalt}(K)$ in (3.28) above follows easily from Proposition 3.33, the chain of inequalities $|\tau(K) + \frac{\sigma(K)}{2}| \leqslant g_T(K) \leqslant \mathrm{dalt}(K)$ by [DL11, Theorem 1.1] and [AK10, Cor. 5.4], and the fact that $\sigma(K) = 2\upsilon(K)$ for all knots that are closures of positive braids of Garside normal form (C) or (D) (see Section 3.4.4.1).

It is not known whether the alternation number and the Turaev genus of a knot are comparable in general: it is not known whether $\mathrm{alt}(K) \leqslant g_T(K)$ for all knots K (see [Low15, Question 3]). However, it was shown by Abe and Kishimoto that $g_T(T_{3,3\ell+k}) = \mathrm{dalt}(T_{3,3\ell+k}) = \ell$ for all $\ell \geqslant 0$ and $k \in \{1, 2\}$ [AK10, Theorem 5.9], so $g_T(K) = \mathrm{alt}(K) = \mathrm{dalt}(K)$ is true for all positive 3-braid knots.

Remark 3.38. In [FLZ17], Friedl, Livingston and Zentner introduce the invariant $\mathcal{A}_s(K)$, defined as the minimal number of double point singularities in a generically immersed concordance from a knot K to an alternating knot. In the case that the alternating knot is the unknot, this is the well studied invariant $c_4(K)$ called the 4*-dimensional clasp number* [Shi74]. A sequence of crossing changes in a diagram of a knot K leading to a diagram of an alternating knot J realizes an immersed concordance from K to J where any crossing change gives rise to a double point singularity in the concordance. We thus have $\mathcal{A}_s(K) \leqslant \mathrm{alt}(K)$ for any knot K, which resembles the inequality $c_4(K) \leqslant u(K)$ between the 4-dimensional clasp number and the *unknotting number* $u(K) \coloneqq d_G(K, U)$ of K, where U denotes the unknot. Moreover, we have $|\upsilon(K) + \tau(K)| \leqslant \mathcal{A}_s(K)$ for any knot K [FLZ17, Theorem 18], so Proposition 3.33 implies $\mathcal{A}_s(K) = \mathrm{alt}(K)$ for all positive 3-braid knots K.

We are now ready to prove Corollary 3.2.

Proof of Corollary 3.2. The claim follows directly from Proposition 3.33, Remark 3.37 and Remark 3.38. ☐

3.5.2 Bounds on the alternation number of general 3-braid knots

In the following, we turn our attention to 3-braid knots in general, which are not necessarily the closure of positive 3-braids. We will use that

$$\left|\frac{s(K)}{2} + \upsilon(K)\right| \leqslant \mathrm{alt}(K) \qquad \text{for any knot } K, \tag{3.29}$$

which follows from [Abe09, Theorem 2.1], see also (3.25) from Section 3.5.1. Rasmussen's invariant s was computed for all 3-braid knots in Murasugi normal form (cf. Definition 3.26) by Greene.[1]

[1] These computations were generalized to all links that are closures of 3-braids in [Mar19].

Corollary 3.39. *Let $\beta = \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r}$ for some $\ell \in \mathbb{Z}$, $r \geqslant 1$ and $p_i, q_i \geqslant 1$ for $i \in \{1, \dots, r\}$ such that $K = \widehat{\beta}$ is a knot. Then*

$$|\ell| - 1 \leqslant \operatorname{alt}(K) \leqslant \operatorname{dalt}(K) \leqslant |\ell| \qquad \text{if } \ell \neq 0.$$

Proof of Corollary 3.39. The lower bound on the alternation number follows from (3.29), Theorem 3.1 and the values of the invariant s for $K = \widehat{\beta}$ [Gre14, Proposition 2.4], namely

$$s(K) = \begin{cases} -\sum_{i=1}^{r} (p_i - q_i) + 6\ell - 2 & \text{if } \ell > 0, \\ -\sum_{i=1}^{r} (p_i - q_i) + 6\ell + 2 & \text{if } \ell < 0. \end{cases}$$

Moreover, it follows from [AK10, Theorem 2.5] that $\operatorname{dalt}\left(\widehat{\beta}\right) \leqslant |\ell|$. □

Remark 3.40. An alternative way to prove the upper bound on $\operatorname{dalt}(K)$ in Corollary 3.39 for $\ell \geqslant 1$ follows from our observations in the proof of Lemma 3.29. In fact, the braid diagrams given by the braid representatives β_1 of $K = \widehat{\beta}$ considered in that proof can easily be transformed into alternating diagrams by ℓ crossing changes: it is enough to change the positive crossings corresponding to the single generators a in β_1 to negative crossings; we obtain generators a^{-1} in the corresponding braid words which then correspond to alternating braid diagrams.

Remark 3.41. If K is represented by a 3-braid of Garside normal form (C) or (D) (see Definition 3.10), then using the observations in Section 3.4.4.3, Corollary 3.39 implies

$$\begin{aligned} |r + \ell| - 1 \leqslant \operatorname{alt}(K) \leqslant \operatorname{dalt}(K) \leqslant |r + \ell| \qquad & \text{if } |r + \ell| > 0 \text{ and} \\ \operatorname{alt}(K) = \operatorname{dalt}(K) = 0 \qquad & \text{if } r + \ell = 0. \end{aligned} \tag{3.30}$$

By Proposition 3.33, the lower bound in (3.30) is sharp whenever K is the closure of a positive 3-braid of Garside normal form (C) or (D). However, there are examples where the upper bound in (3.30) is sharp. The two easiest such examples in terms of crossing number are the non-alternating knots 8_{20} and 8_{21}, which are represented by the 3-braids (cf. [LM23])

$$a^3 b^{-1} a^{-3} b^{-1} \sim \Delta^{-3} a^7 \qquad \text{and}$$
$$a^3 b a^{-2} b^2 \sim \Delta^{-2} a^3 b^2 a^2 b^3,$$

respectively. The lower bound on the alternation number from (3.30) is $|r + \ell| - 1 = 0$ in both cases. Indeed, by [Bal08, Theorem 8.6] both knots are quasi-alternating, so all the invariants from (3.25) are equal [Bal08, Proposition 1.4], [MO08], [OSS17a].

Remark 3.42. Similarly to Corollary 3.39, the Turaev genus of all 3-braid knots was determined up to an additive error of at most 1 by Lowrance [Low11, Proposition 4.15] using his computation of the Khovanov width for these knots. More precisely, we have

$$|\ell| - 1 \leqslant g_T(K) \leqslant |\ell| \qquad \text{if } \ell \neq 0$$

for any knot K that is represented by $\beta = \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r}$ for some $\ell \in \mathbb{Z}$, $r \geqslant 1$ and $p_i, q_i \geqslant 1$ for $i \in \{1, \ldots, r\}$.

3.6 The fractional Dehn twist coefficient of 3-braids in Garside normal form

In this section, we compute the fractional Dehn twist coefficient of any 3-braid in Garside normal form (cf. Definition 3.10).

The *fractional Dehn twist coefficient* is a homogeneous quasimorphism on the braid group B_n that assigns to any n-braid β a rational number $\omega(\beta)$. Here, a *quasimorphism* on a group G is any map $\varphi\colon G \to \mathbb{R}$ such that

$$\sup_{(a,b)\in G\times G} |\varphi(ab) - \varphi(a) - \varphi(b)| =: D_\varphi < \infty,$$

where D_φ is called the *defect* of φ. A quasimorphism $\varphi\colon G \to \mathbb{R}$ is called *homogeneous* if $\varphi(a^k) = k\varphi(a)$ for all $k \in \mathbb{Z}$ and $a \in G$. Any homogeneous quasimorphism is invariant under conjugation, so $\omega(\beta)$ is invariant under the conjugacy class of β.

The fractional Dehn twist coefficient first appeared in [GO89] in a different language. It can be defined for mapping classes of general surfaces with boundary, where we here view braids as mapping classes of the n times punctured closed disk. Malyutin defined the fractional Dehn twist coefficient $\omega\colon B_n \to \mathbb{R}$, $n \geqslant 2$, for all braid groups and showed that its defect is 0 if $n = 2$ and 1 if $n \geqslant 3$ [Mal04, Theorem 6.3]. We refer the reader to [Mal04] for a more detailed account.

Corollary 3.43. *Let β be a 3-braid. Then its fractional Dehn twist coefficient is*

$$\omega(\beta) = \begin{cases} \ell & \text{if } \beta \text{ is conjugate to a braid in (A)}, \\ \frac{p+1}{6} + \ell & \text{if } \beta \text{ is conjugate to a braid in (B)}, \\ r + \ell & \text{if } \beta \text{ is conjugate to a braid in (C) or (D)}. \end{cases}$$

where (A)–(D) refer to the Garside normal forms from Proposition 3.8.

Remark 3.44. The fractional Dehn twist coefficient was computed for 3-braids in Murasugi normal form (cf. Definition 3.26) in [HKK+21, Proposition 6.6].

In the proof of Corollary 3.43, we will use that the fractional Dehn twist coefficient of any 3-braid β is completely determined by the writhe $\mathrm{wr}(\beta)$ and the *homogenized* *upsilon invariant* $\widetilde{\upsilon}$ of β: we have

$$\omega(\beta) = \widetilde{\upsilon}(\beta) + \frac{\mathrm{wr}(\beta)}{2} \qquad \text{[FH19, Theorem 1.3]} \tag{3.31}$$

for any 3-braid β. The invariant $\widetilde{\upsilon}$ is another real-valued homogeneous quasimorphism on the braid group B_3 which can be defined as

$$\widetilde{\upsilon}\colon B_3 \to \mathbb{R}, \qquad \beta \mapsto \widetilde{\upsilon}\,(\beta) = \lim_{k\to\infty} \frac{\upsilon\left(\widehat{\beta^{6k}ab}\right)}{6k}.$$

More generally, Brandenbursky [Bra11, Theorem 2.6] showed that a homogeneous quasimorphism $B_n \to \mathbb{R}$ can be assigned to any concordance homomorphism $\mathcal{C} \to \mathbb{R}$ that is bounded above by a constant multiple of the 4-genus. We refer the reader to [Bra11] or [FH19, Appendix A] for more details on homogenized concordance invariants.

Proposition 3.45. *Let β be a 3-braid. Then*

$$\widetilde{\upsilon}(\beta) = \begin{cases} -\frac{p}{2} - 2\ell & \text{if } \beta \text{ is conjugate to a braid in (A)}, \\ -\frac{p+1}{3} - 2\ell & \text{if } \beta \text{ is conjugate to a braid in (B)}, \\ -\frac{\sum_{i=1}^{r}(p_i+q_i)}{2} + r - 2\ell & \text{if } \beta \text{ is conjugate to a braid in (C)}, \\ -\frac{\sum_{i=1}^{r-1}(p_i+q_i)+p_r}{2} + r - 2\ell - \frac{3}{2} & \text{if } \beta \text{ is conjugate to a braid in (D)}. \end{cases}$$

Proof of Proposition 3.45. We will use that $\widetilde{\upsilon}(\alpha\beta) = \widetilde{\upsilon}(\alpha) + \widetilde{\upsilon}(\beta)$ if α and β commute [FH19, Lemma A.1]. In particular, for any 3-braid β and any $\ell \in \mathbb{Z}$, we have

$$\widetilde{\upsilon}\left(\Delta^{2\ell}\beta\right) = \widetilde{\upsilon}\left(\Delta^{2\ell}\right) + \widetilde{\upsilon}(\beta). \tag{3.32}$$

Moreover, by the definition of $\widetilde{\upsilon}$, (3.8) and the homogeneity of $\widetilde{\upsilon}$, we have

$$\widetilde{\upsilon}\left(\Delta^{2\ell}\right) = -2\ell \qquad \text{for all} \quad \ell \in \mathbb{Z}. \tag{3.33}$$

We will now compute $\widetilde{\upsilon}(\beta)$ for the positive 3-braids β of the form (A)–(D), i.e. assuming $\ell \geqslant 0$ in (A)–(D). The statement of Proposition 3.45 then follows from (3.32) and (3.33).

First, let $\beta = \Delta^{2\ell}a^p$ for some $\ell \geqslant 0$, $p \geqslant 0$. If $p = 0$, we have $\widetilde{\upsilon}(\beta) = -2\ell$ by (3.33). If $p \geqslant 1$, we have

$$\beta^{6k}ab = \Delta^{12\ell k}a^{6pk}ab \sim \Delta^{12\ell k+1}a^{6pk-1},$$

so by Lemma 3.21, for $k \geqslant 1$, we get

$$\upsilon\left(\widehat{\beta^{6k}ab}\right) = -\frac{6pk-1}{2} + 1 - 12\ell k - \frac{3}{2} = -3pk - 12\ell k,$$

hence we have

$$\widetilde{\upsilon}(\beta) = \lim_{k\to\infty} \frac{\upsilon\left(\widehat{\beta^{6k}ab}\right)}{6k} = \lim_{k\to\infty} \frac{-3pk - 12\ell k}{6k} = -\frac{p}{2} - 2\ell.$$

Second, let $\beta = \Delta^{2\ell}a^p b$ for some $\ell \geqslant 0$, $p \in \{1,2,3\}$. We have

$$\begin{aligned}
\beta^{6k}ab &= \Delta^{12\ell k}(ab)^{6k}ab = \Delta^{12\ell k+4k}ab && \text{if } p=1,\\
\beta^{6k}ab &= \Delta^{12\ell k}\left(a^2ba^2b\right)^{3k}ab = \Delta^{12\ell k}(ababab)^{3k}ab = \Delta^{12\ell k+6k}ab && \text{if } p=2, \text{ and}\\
\beta^{6k}ab &= \Delta^{12\ell k}\left(a^3ba^3ba^3b\right)^{2k}ab = \Delta^{12\ell k}\left(a^2bababa^2b\right)^{2k}ab\\
&= \Delta^{12\ell k+8k}ab && \text{if } p=3.
\end{aligned}$$

By (3.8), we get

$$\widetilde{\upsilon}(\beta) = \lim_{k\to\infty} \frac{-12\ell k - (2p+2)k}{6k} = -2\ell - \frac{p+1}{3}.$$

Third, let $\beta = \Delta^{2\ell}a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r}$ for some $\ell \geqslant 0$, $r \geqslant 1$, $p_i, q_i \geqslant 2$, $i \in \{1,\dots,r\}$. Then

$$\begin{aligned}
\beta^{6k}ab &= \Delta^{12\ell k}\left(a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r}\right)^{6k}ab\\
&\sim \Delta^{12\ell k+1}a^{p_1-1}b^{q_1}\cdots a^{p_r}b^{q_r}\left(a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r}\right)^{6k-1}\\
&\sim \Delta^{12\ell k+1}\left(b^{q_1}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r}a^{p_1}\right)^{6k-1}b^{q_1}a^{p_2}b^{q_2}\cdots a^{p_r}b^{p_1+q_r-1},
\end{aligned}$$

where $p_1 + q_r - 1 \geqslant 3$. By Lemma 3.21, we have

$$\begin{aligned}
\upsilon\left(\widehat{\beta^{6k}ab}\right) &= -3k\sum_{i=1}^{r}(p_i+q_i) + 6kr - 12\ell k - 1, \qquad \text{hence}\\
\widetilde{\upsilon}(\beta) &= -\frac{1}{2}\sum_{i=1}^{r}(p_i+q_i) + r - 2\ell.
\end{aligned}$$

Finally, let $\beta = \Delta^{2\ell+1}a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r}$ for some $l \geqslant 0$, $r \geqslant 1$, $p_r \geqslant 2$ and $p_i, q_i \geqslant 2, i \in \{1,\dots,r-1\}$. Then

$$\begin{aligned}
\beta^{6k}ab &= \Delta^{12\ell k}\left(\Delta a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r}\right)^{6k}ab\\
&= \Delta^{12\ell k}\left(\Delta^2 b^{p_1}a^{q_1}\cdots b^{p_{r-1}}a^{q_{r-1}}b^{p_r}a^{p_1}b^{q_1}\cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r}\right)^{3k}ab\\
&= \Delta^{12\ell k+6k}\left(b^{p_1}\cdots b^{p_r}a^{p_1}\cdots a^{p_r}\right)^{3k}ab\\
&\sim \Delta^{12\ell k+6k}a^{q_1}b^{p_2}\cdots b^{p_r}a^{p_1}\cdots a^{p_r}\left(b^{p_1}\cdots b^{p_r}a^{p_1}\cdots a^{p_r}\right)^{3k-2}\\
&\quad b^{p_1}\cdots b^{p_r}a^{p_1}\cdots a^{p_r+1}b^{p_1+1},
\end{aligned}$$

where $p_r + 1$, $p_1 + 1 \geqslant 3$. By Lemma 3.22, we have

$$\upsilon\left(\widehat{\beta^{6k}ab}\right) = -3k\left(\sum_{i=1}^{r-1}(p_i + q_i) + p_r\right) + 6kr - 12\ell k - 9k - 1, \qquad \text{hence}$$
$$\widetilde{\upsilon}(\beta) = -\frac{1}{2}\left(\sum_{i=1}^{r-1}(p_i + q_i) + p_r\right) + r - 2\ell - \frac{3}{2}. \qquad \square$$

Proof of Corollary 3.43. Corollary 3.43 follows directly from Proposition 3.45, (3.31), and a straightforward calculation of the writhe of the braids in (A)–(D). □

Remark 3.46. If β is a 3-braid conjugate to a braid of the form in (C) or (D) such that $\widehat{\beta}$ is a knot, then Proposition 3.45 and Theorem 3.1 imply $\widetilde{\upsilon}(\beta) = \upsilon(\widehat{\beta})$. If β additionally is a positive 3-braid, then $\omega(\beta) = r + \ell = g(\widehat{\beta}) + \upsilon(\widehat{\beta}) + 1$ is the minimal number from Corollary 3.3/Corollary 3.24.

Remark 3.47. Our above computation of $\omega(\beta)$ (see Corollary 3.43) together with a result by Feller–Hubbard [FH19, Theorem 1.3] completely determines $\widetilde{\Upsilon(t)}(\beta)$ for all $0 \leqslant t \leqslant 1$ for any 3-braid β, where $\widetilde{\Upsilon(t)}(\beta)$ is the homogenization of the invariant $\Upsilon(t)\colon \mathcal{C} \to \mathbb{R}$, defined similarly as the homogenization $\widetilde{\upsilon}$ of υ.

4 3-braid knots with maximal 4-genus

4.1 Introduction

In this chapter, based on [BLMT23] together with Sebastian Baader, Lukas Lewark and Filip Misev, we prove the following theorem.

Theorem 4.1 (Theorem E)**.** *Let K be a 3-braid knot other than the figure-eight knot. Then*

$$|\sigma(K)| = 2g(K) \iff g_4^{top}(K) = g(K).$$

These equalities hold precisely if K or its mirror is one of the following knots:

- $T_{2,2m+1}\#T_{2,2n+1}$, *with* $m, n \geqslant 0$,
- $P(2p, 2q+1, 2r+1, 1)$, *with* $p \geqslant 1$, $q, r \geqslant 0$,
- $T_{3,4}$ *or* $T_{3,5}$.

The question arises whether the equivalence of $|\sigma(K)| = 2g(K)$ and $g_4^{\text{top}}(K) = g(K)$ holds for other families of knots K. Indeed, it is also true for braid positive knots K [Lie16]. Moreover, we do not know if there exists a knot K of braid index 4 with $g_4^{\text{top}}(K) = g(K)$, but $|\sigma(K)| < 2g(K)$. One may check that such a knot would have to be prime and have crossing number at least 13. For braid index 5 however, there are several knots K in the table with $\sigma(K) = 0 < 2g(K) = 2$ and $g_4^{\text{top}}(K) = g(K) = 1$, such as $K = 8_1$ [LM23].

The proof of Theorem 4.1 is based on a technique called (un)twisting, which was used by McCoy to estimate the topological 4-genus from above [McC21]. We will also make use of a special presentation for 3-braids that goes back to Xu [Xu92], which we call the Xu normal form.

Organization of this chapter. In Section 4.2, we will introduce the Xu normal form of 3-braids and show how it determines the signature invariant (Proposition 4.10), as well as strong quasipositivity and braid positivity (Proposition 4.12) of their closures. Section 4.3 contains the proof of Theorem 4.1, as well as a complementary result (Theorem 4.14): a sharp lower bound on the difference $g(K) - g_4^{\text{top}}(K)$ of strongly quasi-positive 3-braid knots, in terms of two characteristic quantities associated with the Xu normal forms of the corresponding strongly quasipositive 3-braids. In Section 4.4, we determine the topological 4-genus of various families of braid positive 3-braid knots (almost) exactly, and show examples where our technique comes to a limit.

4.2 The Xu normal form of 3-braids

In this chapter, our tool to handle 3-braids is what we call their *Xu normal form*. It was developed by Xu [Xu92] (who called it *representative symbol*), as a variation of the Garside normal form [Gar69]; see also Section 3.3. The reader will note many parallels between the Garside and Xu normal forms. Using the Xu normal form, one may decide whether two given 3-braids are conjugate [Xu92], and whether their closures are equivalent links [BM93, BM08]. Later, the Xu normal form was generalized to braids on arbitrarily many strands by Birman–Ko–Lee [BKL98]. The Garside, Xu and BKL normal forms are all examples of Garside structures on (braid) groups [DDG⁺15].

Recall that a 3-braid β is an element of the braid group $B_3 = \langle a, b \mid aba = bab \rangle$. Let us write $x := a^{-1}ba \in B_3$ and $\delta := ba = ax = xb \in B_3$. In this chapter, by a *Xu word* or simply *word*, we mean a word with letters a, b, x, δ and their inverses. We reserve the equality sign $=$ for equality of braids, and use a dotted equality sign $\doteq$ for equality of words. Moreover, recall that we write $\beta \sim \gamma$ if the two braids β, γ are conjugate. For efficiency, let us also introduce the following notation: for any $i \in \mathbb{Z}$, set $\tau_i \doteq a$ if $i \equiv 1 \pmod 3$, $\tau_i \doteq b$ if $i \equiv 2 \pmod 3$ and $\tau_i \doteq x$ if $i \equiv 0 \pmod 3$.

Definition 4.2. Let w be a word of the form

$$w \doteq \delta^N \tau_1^{u_1} \tau_2^{u_2} \dots \tau_T^{u_T} \qquad \text{for} \quad N \in \mathbb{Z},\, T \geqslant 0,\, u_i \geqslant 1. \tag{4.1}$$

We say that w is in *Xu normal form* if the tuple $(-N, T, u_1, \dots, u_T)$ is lexicographically minimal among all words of the form (4.1) representing the same conjugacy class of 3-braids.

The condition of lexicographic minimality means, in particular, that the Xu normal form maximizes N, and afterwards minimizes T. The term 'normal form' is justified by the following.

Theorem 4.3. *Every 3-braid is conjugate to a unique word in Xu normal form.* □

The following lemma gives a criterion to easily decide whether a word is in Xu normal form.

Lemma 4.4. *A word $w \doteq \delta^N \tau_1^{u_1} \tau_2^{u_2} \dots \tau_T^{u_T}$ for some $N \in \mathbb{Z}$, $T \geqslant 0$, $u_i \geqslant 1$ is in Xu normal form if and only if one of the following conditions holds:*

(a) *$T = 0$. In this case $w \doteq \delta^N$.*

(b) $T = 1$, *and if* $N \equiv 1 \pmod 3$ *then* $u_1 = 1$. *In this case,* $w \doteq \delta^{3k} a^{u_1}$, *or* $w \doteq \delta^{3k+1} a$, *or* $w \doteq \delta^{3k+2} a^{u_1}$.

(c) $T \geqslant 2$, $N + T \equiv 0 \pmod 3$ *and the tuple* $(u_1, \dots, u_T)$ *is lexicographically minimal among its cyclic permutations.*

The proofs of Theorem 4.3 and Lemma 4.4 are essentially contained in Xu's paper [Xu92, Section 4], albeit with slightly other conventions. In our setup, Theorem 4.3 is not actually hard to prove, and makes a good exercise to get acquainted with the calculus of Xu words. The same is true for the 'only if' direction of Lemma 4.4. Let us provide two hints. Firstly, the easily verifiable rules

$$\delta = \tau_i \tau_{i-1}, \qquad \tau_i \delta^N = \delta^N \tau_{i+N}, \qquad \tau_i^{-1} \delta^N = \delta^{N-1} \tau_{i+N+1} \tag{4.2}$$

allow to find Xu words for every 3-braid without the letters a^{-1}, b^{-1}, x^{-1}, and to 'pull all $\delta^{\pm 1}$ to the left' in a Xu word. In this way, one can find a Xu word of the form $\delta^N \tau_{m+1}^{u_1} \dots \tau_{m+T}^{u_T}$ with $m \in \mathbb{Z}$ and $u_i \geqslant 1$ for any 3-braid. Secondly, note that

$$\begin{aligned} \delta^N \tau_1^{u_1} \dots \tau_T^{u_T} \;&=\; \tau_{1-N}^{u_1} \delta^N \tau_2^{u_2} \dots \tau_T^{u_T} \;\sim\; \delta^N \tau_2^{u_2} \dots \tau_T^{u_T} \tau_{1-N}^{u_1} \\ &\sim\; \delta^{N+1} \tau_2^{u_2} \dots \tau_T^{u_T} \tau_{1-N}^{u_1} \delta^{-1} \;=\; \delta^N \tau_1^{u_2} \dots \tau_{T-1}^{u_T} \tau_{-N}^{u_1}. \end{aligned}$$

So cyclically permuting the tuple $(u_1, \dots, u_T)$ results in a conjugate braid if $-N \equiv T \pmod 3$.

Proof of the 'if' direction of Lemma 4.4. Xu proves that condition (c) is sufficient for w to be in Xu normal form (see the definition of the representative symbol and Theorem 5 in [Xu92]), but omits a discussion of conditions (a) and (b). So suppose w satisfies (a) or (b), $w' = \delta^{N'} \tau_1^{u'_1} \dots \tau_{T'}^{u'_{T'}}$ is in Xu normal form, and $w' \sim w$. We need to show that $w \doteq w'$. Let us distinguish the following cases.

- If w satisfies (a), then $w \doteq \delta^N$. The words w and w' must have the same writhe, so $2N = 2N' + u'_1 + \dots + u'_{T'}$. But since w' is in Xu normal form, $N' \geqslant N$. Because $u'_i \geqslant 0$, it follows that $N' = N$ and $T' = 0$, thus $w \doteq w'$ as desired.
- If w satisfies (b) and $w \doteq \delta^N a$, a similar argument applies: the equality of the writhes of w and w' now reads as $2N + 1 = 2N' + u'_1 + \dots + u'_{T'}$. Because of parity, we may again deduce $N' = N$. It follows that $T' = 1, u'_1 = 1$ and $w \doteq w'$ as desired.
- The remaining case is that $w \doteq \delta^N a^{u_1}$ satisfies (b) with $N \not\equiv 1 \pmod 3$ and $u_1 \geqslant 2$. Then the so-called r-index of w defined below Lemma 5 in [Xu92] is 0, and so [Xu92, Theorem 4] implies that N is maximal. It follows that $N' = N$, and hence the minimality of T' implies $T' \leqslant T = 1$. Finally, it follows from the equality of the writhes of w and w' that $T' = 1$ and $u_1 = u'_1$, and thus $w \doteq w'$. ☐

To get a link invariant from the Xu normal form, we need to understand the relationship between conjugacy classes of 3-braids and link equivalence classes of their closures. Birman–Menasco have shown that with a few well-understood exceptions, this relationship is one-to-one.

Theorem 4.5 ([BM93, BM08])**.** *Two 3-braids are conjugate if their closures are equivalent links, except in the following cases:*

(1) The non-conjugate braids $ab, ab^{-1}, a^{-1}b^{-1}$ *have the unknot as closure.*

(2) For $N \in \mathbb{Z} \setminus \{\pm 1\}$*, the non-conjugate braids* $a^N b, a^N b^{-1}$ *have the* $T_{2,N}$ *torus link as closure.*

(3) For pairwise distinct integers $p, q, r \in \mathbb{Z} \setminus \{0, -1, -2\}$*, the two non-conjugate braids* $\beta = a^p b^q x^r$ *and* $\gamma = a^p b^r x^q$ *have the* $P(p, q, r, 1)$ *pretzel as closure (see Figure 4.1); and the two non-conjugate braids* β^{-1} *and* γ^{-1} *have the* $P(-p, -q, -r, -1)$ *pretzel link as closure.*

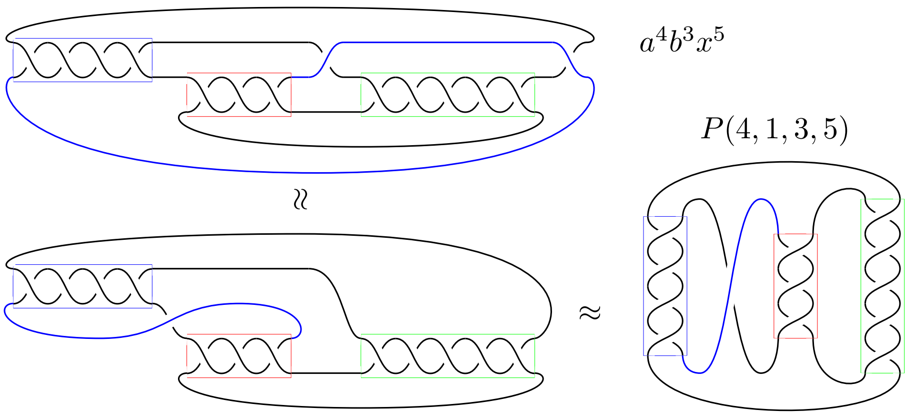


Figure 4.1: Isotopy (denoted $\approx$) from the closure of the braid $a^{u_1}b^{u_2}x^{u_3}$ to the pretzel knot $P(u_1, 1, u_2, u_3) \approx P(u_1, u_2, u_3, 1)$; here $(u_1, u_2, u_3) = (4, 3, 5)$.

The following corollary allows us to sidestep the exceptional cases (1), (2), (3) in Theorem 4.5 by focusing on links of braid index 3 instead of 3-braid links (the latter class of links includes links with braid index 1 and 2, i. e. 2-stranded torus links). Let the *reverse* of a braid $\beta \in B_3$, denoted by $\mathrm{rev}(\beta)$, be the braid given by reading β backwards and switching a with b, and a^{-1} with b^{-1}. Note that $\widehat{\mathrm{rev}(\beta)}$ is obtained from $\widehat{\beta}$ by reversing the link's orientation.

Corollary 4.6. *Let* L *be a link of braid index 3.*

(1) Either there is a unique conjugacy class of 3-braids with closure L*, or there are two of them, such that one consists of the reverses of the braids contained in the other.*

(2) *The numbers N, T, and $U = u_1 + \cdots + u_T$ of the Xu normal form of a braid with closure L do not depend on the choice of braid. Thus N, T and U are link invariants of links with braid index 3.*

Proof. Part (1). This follows quickly from Theorem 4.5, since the unknot and the two-stranded torus links have braid index less than 3, and $\operatorname{rev}(a^p b^q x^r) = x^r a^q b^p \sim a^p b^r x^q$. *Part (2).* Note that $\operatorname{rev}(x) = x$ and so $\operatorname{rev}(\tau_i) = \tau_{-i}$. Also $\operatorname{rev}(\delta) = \delta$. Thus the reverse of $\delta^N \tau_1^{u_1} \tau_2^{u_2} \ldots \tau_T^{u_T}$ is $\tau_{-T}^{u_T} \ldots \tau_{-1}^{u_1} \delta^N$, which has Xu normal form $\delta^N \tau_1^{u_T} \tau_2^{u_{T-1}} \ldots \tau_T^{u_1}$ (up to cyclically permuting the exponents $u_T, \ldots, u_1$). So the numbers N, T and U do not change under braid reversal. Together with (1), this implies (2). □

Xu calls N and T the *power* and *syllable length*, while Birman–Ko–Lee use the terms *infimum* and *canonical length*, respectively.

Since the Xu normal form determines the link type, all link invariants may be read off it. Let us first prove a formula for the signature invariant σ. For that, we will recall from Section 3.3 the Garside normal form [Gar69]. Recall that $\Delta = aba$. A *Garside word* is a word with letters a, b, Δ and their inverses. Again, we use $\doteq$ for equality of words. For the purposes of this chapter, we also use the following notation: for any $i \in \mathbb{Z}$, set $\eta_i \doteq a$ if $i \equiv 1 \pmod 2$ and $\eta_i \doteq b$ if $i \equiv 0 \pmod 2$. We invoke Proposition 3.8 in a slightly modified form.

Proposition 4.7 (Proposition 3.8)**.** *Every 3-braid contains in its conjugacy class a unique Garside word v in* Garside normal form*, i. e. a word*

$$v \doteq \Delta^\ell \eta_1^{p_1} \eta_2^{p_2} \ldots \eta_r^{p_r},$$

with $\ell \in \mathbb{Z}$, $r \geqslant 0$, $p_i \geqslant 1$, satisfying one of the following conditions:

(A) ℓ is even and $r \in \{0, 1\}$, i. e. $v \doteq \Delta^{2k} a^{\geqslant 0}$,

(B) ℓ is even, $r = 2$, $p_1 \in \{1, 2, 3\}$ and $p_2 = 1$, i. e. $v \doteq \Delta^{2k} a^{\{1,2,3\}} b$,

(C)/(D) $r \geqslant 1$, $p_i \geqslant 2$, $\ell \equiv r \pmod 2$, and the tuple $(p_1, \ldots, p_r)$ is lexicographically minimal among its cyclic permutations.

We refer by (C) and (D) to the case that ℓ is even and odd, respectively.

The following lemma tells us how to convert between the Xu and the Garside normal forms.

Lemma 4.8. *Let a word $w \doteq \delta^n \tau_1^{u_1} \tau_2^{u_2} \ldots \tau_T^{u_T}$ in Xu normal form be given. Then the unique word v in Garside normal form representing the same conjugacy class of 3-braids as w is given by the following table.*

Case in Lemma 4.4	*Xu normal form w*	*Garside normal form v*	*Case in Proposition 4.7*
(a)	δ^{3k}	Δ^{2k}	(A)
(a)	δ^{3k+1}	$\Delta^{2k}ab$	(B)
(a)	δ^{3k+2}	$\Delta^{2k}a^3b$	(B)
(b)	$\delta^{3k}a^{u_1}$	$\Delta^{2k}a^{u_1}$	(A)
(b)	$\delta^{3k+1}a$	$\Delta^{2k}a^2b$	(B)
(b)	$\delta^{3k+2}a^{u_1}$	$\Delta^{2k+1}a^{1+u_1}$	(D)
(c)	$\delta^N\tau_1^{u_1}\dots\tau_T^{u_T}$	$\Delta^{(2N-T)/3}\eta_1^{1+u_1}\dots\eta_T^{1+u_T}$	(C)/(D)

Proof. All rows in the table except for the last one may be checked quickly, using $\delta^3 = \Delta^2$. Let us now prove $w \sim v$ for the words w and v in the last row. Let $N = 3k + m$ and $T = 3s + 3 - m$ for $k, s \in \mathbb{Z}$, $s \geqslant 0$, $m \in \{1, 2, 3\}$. In the Xu word w, replace δ^N by $(ba)^m\Delta^{2k}$. Moreover, replace every x^u by $\Delta^{-1}ab^{1+u}a$. These replacements yield a Garside word v_1 with $v_1 = w$ and

$$v_1 \doteq (ba)^m\Delta^{2k}a^{u_1}b^{u_2}(\Delta^{-1}ab^{1+u_3}a)a^{u_4}\dots(\Delta^{-1}ab^{1+u_{3s}}a)a^{u_{3s+1}}b^{u_{3s+2}},$$

setting $u_i = 0$ if $i > T$. Now proceed by 'pulling all the Δ^{-1} to the right', i. e. replacing $\Delta^{-1}a$ by $b\Delta^{-1}$ and $\Delta^{-1}b$ by $a\Delta^{-1}$ as long as possible. These replacements produce a word v_2 with $v_2 = v_1$, where v_2 starts with $(ba)^m\Delta^{2k}a^{u_1}b^{u_2}(ba^{1+u_3}b)b^{u_4}\dots$. Using the η_i-notation and noting that there are precisely s occurrences of Δ^{-1} in v_1, we have

$$\begin{aligned} v_2 &\doteq (ba)^m\Delta^{2k}\eta_1^{u_1}\eta_2^{1+u_2}\eta_3^{1+u_3}\eta_4^{1+u_4}\dots\eta_{3s}^{1+u_{3s}}\eta_{3s+1}^{1+u_{3s+1}}\eta_{3s+2}^{u_{3s+2}}\Delta^{-s} \\ \sim v_3 &\doteq \Delta^{-s}(ba)^m\Delta^{2k}\eta_1^{u_1}\eta_2^{1+u_2}\dots\eta_{3s}^{1+u_{3s}}\eta_{3s+1}^{1+u_{3s+1}}\eta_{3s+2}^{u_{3s+2}}. \end{aligned}$$

Let us now consider the three possibilities for m case by case.

- If $m = 3$, then $u_{3s+2} = u_{3s+1} = 0$ and

$$\begin{aligned} v_3 &\doteq \Delta^{-s}(ba)^3\Delta^{2k}\eta_1^{u_1}\eta_2^{1+u_2}\dots\eta_{3s}^{1+u_{3s}}\eta_{3s+1} \\ &= \Delta^{2k+2-s}\eta_1^{u_1}\eta_2^{u_2+1}\dots\eta_{3s}^{1+u_{3s}}\eta_{3s+1} \\ \sim v_4 &= \Delta^{2k+2-s}\eta_1^{1+u_1}\eta_2^{u_2+1}\dots\eta_{3s}^{1+u_{3s}}. \end{aligned}$$

  We have $v_4 \doteq v$ as desired, since $2k + (m-1) - s = (2N - T)/3$.

- If $m = 2$, then $u_{3s+2} = 0$ and

$$\begin{aligned} v_3 &\doteq \Delta^{-s}(ba)^2\Delta^{2k}\eta_1^{u_1}\eta_2^{1+u_2}\dots\eta_{3s}^{1+u_{3s}}\eta_{3s+1}^{1+u_{3s+1}} \\ &= \Delta^{2k+1-s}\eta_1^{1+u_1}\eta_2^{1+u_2}\dots\eta_{3s}^{1+u_{3s}}\eta_{3s+1}^{1+u_{3s+1}} \doteq v. \end{aligned}$$

– If $m = 1$, then

$$\begin{aligned} v_3 &\doteq \Delta^{-s} ba \Delta^{2k} \eta_1^{u_1} \eta_2^{1+u_2} \dots \eta_{3s}^{1+u_{3s}} \eta_{3s+1}^{1+u_{3s+1}} \eta_{3s+2}^{u_{3s+2}} \\ &= \eta_s \Delta^{2k-s} \eta_1^{1+u_1} \eta_2^{1+u_2} \dots \eta_{3s+1}^{1+u_{3s+1}} \eta_{3s+2}^{u_{3s+2}} \sim v. \end{aligned}$$ □

In [Erl99, Theorem 2.6], Erle provided a formula for the signature of closures of 3-braids in Murasugi normal form (see Proposition 4.10). The following formula for the signature of 3-braid knots using the Garside normal form follows from Remark 3.5, Proposition 3.14 and Remark 3.15.

Proposition 4.9. *Let K be a knot that is the closure of a 3-braid in Garside normal form $\Delta^\ell \eta_1^{p_1} \dots \eta_r^{p_r}$ in case (C)/(D) of Proposition 4.7. Then*

$$\sigma(K) = -\sum_{i=1}^{r} p_i + r - 2\ell.$$

We are now ready to state and prove our signature formula for 3-braids in Xu normal form.

Proposition 4.10. *Let $\delta^N \tau_1^{u_1} \tau_2^{u_2} \dots \tau_T^{u_T}$ be the Xu normal form of a 3-braid whose closure is a knot K of braid index 3. Set $U = u_1 + \cdots + u_T$. If $T > 0$ (equivalently, if K is not a torus knot), then*

$$\sigma(K) = -U - \frac{4}{3}N + \frac{2}{3}T.$$

In the case $T = 0$, i. e. $U = 0$ and $K = T_{3,N}$, the value $\sigma(K) = -\frac{4}{3}N$ given by the above formula is only approximately true, with an error of at most $\frac{4}{3}$. In fact, in that case we have

$$\sigma(K) = -\frac{4}{3}N + \left(2 + 4\left\lfloor \frac{1}{6}N \right\rfloor - \frac{2}{3}N\right) = 2 - 2N + 4\left\lfloor \frac{1}{6}N \right\rfloor.$$

Proof. Our signature formula for torus knots may be seen to agree with the formula given e. g. in [Mur74, Proposition 9.1]. So we are left with the case $T \geqslant 1$, i. e. cases (b) and (c) in Lemma 4.4. Denote the Xu normal form in question by w. Let us first consider case (c). Then w has Garside normal form $\Delta^{(2N-T)/3} \eta_1^{1+u_1} \dots \eta_T^{1+u_T}$, see Lemma 4.8. By Proposition 4.9, we have $\sigma(K) = -4N/3 + 2T/3 + T - (U + T)$, which is equal to the claimed formula. In case (b), since the closure of w is a knot, the only possibility is $w \doteq \delta^{3k+2} a^{u_1}$. Then the Garside normal form of w is $\Delta^{2k+1} a^{1+u_1}$, which has the desired signature, again by Proposition 4.9. □

Next, let us give complete criteria to decide braid positivity and strong quasipositivity for links of braid index 3. Recall from Section 1.3 that a braid positive link is the closure of some *positive word*, i. e. a word in positive powers of the standard generators $\sigma_1, \dots, \sigma_{n-1}$ of the braid group B_n on some number n of strands. Similarly, a link is called strongly quasipositive if it is the closure of a strongly quasipositive word in some B_n. In this chapter, a *strongly quasipositive word* is a word in positive powers of the (positive) band words

$$\sigma_{ij} = \sigma_i^{-1}\sigma_{i+1}^{-1}\dots\sigma_{j-1}^{-1}\sigma_j\sigma_{j-1}\dots\sigma_i$$

with $1 \leqslant i \leqslant j \leqslant n-1$. Note that this convention is different from the one used in Chapters 1 and 2; see (1.1). Thus, in this chapter, for $n = 3$, positive words are words in $a = \sigma_1$, $b = \sigma_2$ and strongly quasipositive words are words in $a = \sigma_{11}$, $b = \sigma_{22}$ and $x = \sigma_{12}$. It is well-known and straightforward to show that an n-braid is the closure of some (strongly quasi-)positive word if and only if the power of Δ (δ) in its Garside (Birman–Ko–Lee) normal form is non-negative, respectively. This makes positivity and strong quasipositivity decidable for braids. For links, the problem is harder because a priori, a braid positive (strongly quasipositive) link with braid index n need not be the closure of a (strongly quasi-)positive word on n strands. For $n = 3$, however, this is the case.

Theorem 4.11 ([Sto17, Theorem 1.1 and 1.3])**.** *The following hold.*

(1) If a strongly quasipositive link is the closure of some 3-braid, then it is the closure of a strongly quasipositive 3-braid.

(2) If a braid positive link is the closure of some 3-braid, then it is the closure of a positive 3-braid.

We are now ready to state and prove our positivity characterizations.

Proposition 4.12. *Let $\delta^N\tau_1^{u_1}\tau_2^{u_2}\dots\tau_T^{u_T}$ be in Xu normal form, with closure a link L of braid index 3. Then the following hold.*

(1) L is a strongly quasipositive link if and only if $N \geqslant 0$.

(2) L is a braid positive link if and only if $N \geqslant T/2$ or $N = 0$ and $T = 1$.

Proof. Part (1). If $N \geqslant 0$, then the Xu normal form yields a strongly quasipositive word for L, so L is strongly quasipositive. For the other direction, assume L is strongly quasipositive and a Xu normal form w with closure L is given. By Theorem 4.11, there is a strongly quasipositive word in B_3 representing L. It may be transformed to its Xu normal form w' just by replacing $\tau_{i+1}\tau_i \to \delta$ and $\tau_i\delta \leftrightarrow \delta\tau_{i+1}$, and by passing from $y\tau_i$ to $\tau_i y$ for y a word in positive powers of a, b, x, δ. None of these transformations create negative powers of δ, and so we find that the Xu normal form w' has $N \geqslant 0$. By Corollary 4.6(2), all Xu normal forms with closure L have the same N, so w has $N \geqslant 0$ as well, which was to be proven.

Part (2). If $N \geqslant T/2$, then one may check using Lemma 4.8 that the Garside normal form of $\delta^N\tau_1^{u_1}\tau_2^{u_2}\dots\tau_T^{u_T}$ starts with a non-negative power of Δ, and thus yields a positive word with closure L. If $N = 0$ and $T = 1$, then the Xu normal form is a^{u_1}, which is already a positive word with closure L. In both cases, it follows that L is braid positive. For the other direction, assume L is braid positive and let a Xu normal form w with closure L be given. By Theorem 4.11, there is a positive word in B_3 representing L. Similarly as in the proof of part (1), one sees that the Garside normal form of this positive word starts with Δ^ℓ with $\ell \geqslant 0$. By going through the rows of the table in Lemma 4.8, one sees that this implies that $N \geqslant T/2$, with the sole exception in the fourth row if $k = 0$ and $u_1 \neq 0$: then, the Xu normal form is a^{u_1}, so $N = 0$ and $T = 1$. Again using Corollary 4.6(2), it follows that w also satisfies $N \geqslant T/2$, or $N = 0$ and $T = 1$. □

If a knot K is the closure of a strongly quasipositive 3-braid in Xu normal form $\delta^N \tau_1^{u_1} \dots \tau_T^{u_T}$, there is also a simple formula for the 3-genus of K:

$$g(K) = \frac{U}{2} + N - 1, \tag{4.3}$$

where $U = u_1 + \cdots + u_T$. This follows from the Bennequin equality; see (1.3).

4.3 Proofs of main theorems

Before beginning with the proof of Theorem 4.1, we describe a technique that we use to detect topological 4-genus defect in a given knot K, that is, to show $g_4^{\mathrm{top}}(K) < g(K)$. The main ingredient is the so-called *generalized crossing change*, also known as *null-homologous twist*, or simply *twist*. A null-homologous twist consists in performing a ± 1 Dehn surgery on the boundary circle of an embedded disk $D \subset S^3$, such that D intersects K transversely in a finite number of interior points, with total algebraic intersection count 0. While ± 1 Dehn surgery on an unknot in S^3 gives back S^3, the effect on K is that a (left- or right-handed) full twist is introduced into the strands of K that cross D (cf. Figure 4.2).

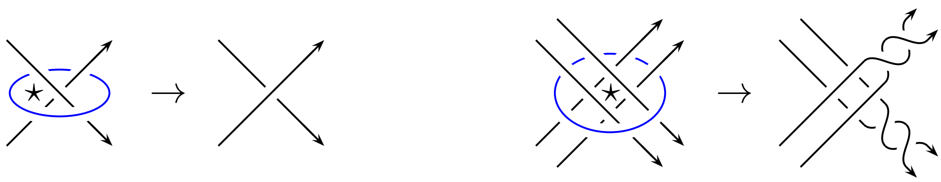


Figure 4.2: Two examples of twists, at the locations marked by $\star$. Here, the boundary of the respective disc D is drawn in blue; in subsequent figures, it will be omitted.

The *untwisting number* $\mathrm{tu}(K)$ of K, introduced by Ince [Inc16], is defined as the minimal number of null-homologous twists needed to turn K into the unknot. Clearly, $\mathrm{tu}(K) \leqslant u(K)$, since crossing changes are special cases of null-homologous twists. Recall that $u(K)$ denotes the unknotting number of K; see Remark 3.38. McCoy [McC21] showed that $g_4^{\text{top}}(K) \leqslant \text{tu}(K)$, that is, the untwisting number of K is an upper bound on the topological 4-genus of K. His result is based on Freedman's theorem, which states that knots of Alexander polynomial 1 are topologically slice [Fre82, FQ90]. This bound can now be used to find topological 4-genus defect: If we find a way to turn a knot K into the unknot with strictly less than $g(K)$ null-homologous twists, this will show $g_4^{\text{top}}(K) < g(K)$. This method was already applied by Baader–Banfield–Lewark [BBL20] to 3-stranded torus knots. For the proof of Theorem 4.1 below, we use a slightly refined version of the method, as follows.

Assume we find a cobordism $C \subset S^3 \times [0,1]$ from a given knot K to some knot K' such that $g(C) = g(K) - g(K')$. If such a C exists, we will write $K \rightsquigarrow K'$. Assume that furthermore $\text{tu}(K') < g(K')$. Then, by McCoy's result, $g_4^{\text{top}}(K') \leqslant \text{tu}(K') < g(K')$. Composing the cobordism C with a topological slice surface for K', we obtain

$$g_4^{\text{top}}(K) \leqslant g_4^{\text{top}}(K') + g(C) < g(K') + g(C) = g(K).$$

In particular, for the topological 4-genus defect we find

$$g(K) - g_4^{\text{top}}(K) \geqslant g(K') - g_4^{\text{top}}(K') \geqslant g(K') - \text{tu}(K'). \tag{4.4}$$

One way to construct cobordisms is to apply saddle moves to knot diagrams, as in Figure 4.3. Note that we have already used this method for finding cobordisms many times in the previous chapter, see in particular Section 3.4.1 and Example 3.13 therein, where we explained the proof strategy for that chapter. Such cobordisms will always be smooth.

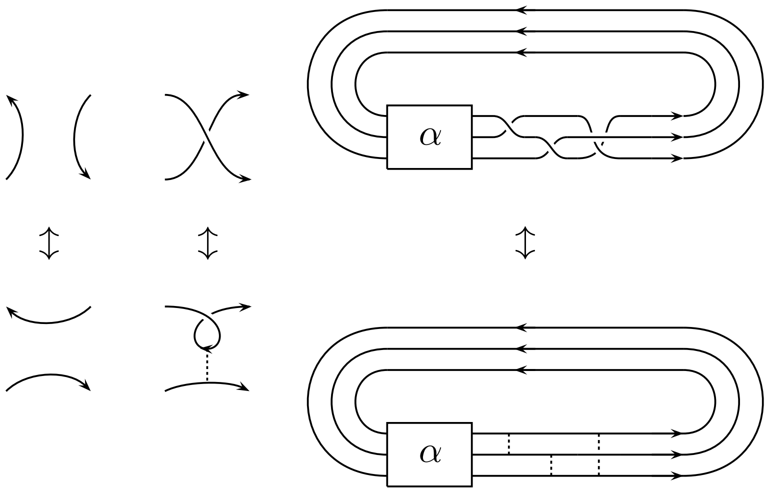


Figure 4.3: Left: Saddle move. Middle: How to use isotopy and a saddle move to add or remove a braid crossing. Right: Example of a cobordism between $\widehat{\alpha}$ and $\widehat{\alpha \cdot abx}$ that consists of three saddle moves, for some 3-braid α.

Suppose the knot K is the closure of a strongly quasipositive 3-braid of the form $\beta = \delta^N \tau_1^{u_1} \cdots \tau_T^{u_T}$ with $N \geqslant 0$ and $u_1, \ldots, u_T \geqslant 1$. Out of saddle moves, one may build a cobordism C that lowers the exponents u_i and N (one saddle move per letter τ_i, two saddle moves per letter δ), or transforms δ into τ_i (one saddle move). Suppose the exponents remain non-negative, and C is a cobordism from K to another knot K'. Then K' is also strongly quasipositive, and it follows from the Bennequin equality (see (4.3) and (1.3)) that $g(C) = g(K) - g(K')$, i. e. $K \rightsquigarrow K'$.

Proof of Theorem 4.1. We organize the proof into two parts, which consist in verifying the following statements.

(1) $|\sigma(K)| = 2g(K)$ for all K in the list of Theorem 4.1 and their mirrors,

(2) $g_4^{\text{top}}(K) < g(K)$ for all other 3-braid knots except the figure-eight.

In light of Kauffman and Taylor's signature bound $|\sigma(J)| \leqslant 2g_4^{\text{top}}(J)$, valid for all knots J, see [KT76, Pow17], these two statements together imply the theorem.

Part (1). The genera and signatures of torus knots are well understood [Lit79]; in particular, we know that the torus knots K with $|\sigma(K)| = 2g(K)$ are precisely the knots $T_{3,4}, T_{3,5}$, $T_{2,2k+1}$, $k \geqslant 0$, and their mirrors. In fact, the signature of $T_{2,2k+1}$, $k \geqslant 0$, is known to be $-2k$. Both 3-genus and signature are additive under connected sum of knots; hence $|\sigma(K)| = 2g(K)$ for all knots K of the first type listed. If K is one of the listed pretzel knots $K = P(2p, 2q+1, 2r+1, 1)$ with $p \geqslant 1, q, r \geqslant 0$, then K has a positive and alternating, hence special alternating diagram; see Figure 4.1. Murasugi shows in this case that $|\sigma(K)| = 2g(K)$; see [Mur65, Corollary 10.3].

Part (2). Let K be a 3-braid knot other than the figure-eight knot such that neither K nor its mirror appears in the list of Theorem 4.1. Our goal now is to show that $g_4^{\text{top}}(K) < g(K)$. We distinguish several cases.

- If K is the closure of a positive 3-braid, this is a special case of the analogous statement about all positive braids, on any number of strands, which is due to Liechti [Lie16, Theorem 1, Corollary 2]. f
- Next, we consider the case in which K is strongly quasipositive without being braid positive; this is the main part of the proof. By Lemma 4.4 and Proposition 4.12, K is the closure of a 3-braid β in Xu normal form $\beta = \delta^N \tau_1^{u_1} \tau_2^{u_2} \cdots \tau_T^{u_T}$ with $u_1, \ldots, u_T \geqslant 1$, $N \geqslant 0$, $T \geqslant 2$, $N + T \equiv 0 \mod 3$, and $2N < T$ (note that the case $N = 0, T = 1$ is excluded since we assume that $\widehat{\beta}$ is a knot). These conditions on N and T leave the following possibilities: $(N, T) = (0, 3)$, or $(N, T) = (1, 5)$, or $T \geqslant 6$. First, if $(N, T) = (0, 3)$, then $K = P(u_1, u_2, u_3, 1)$; see Figure 4.1. If more than one of u_1, u_2, u_3 is even, K is a link of more than one component; the same is true if all three parameters are odd. We may therefore assume that $u_1 = 2p$ is

even and $u_2 = 2q + 1$, $u_3 = 2r + 1$ are odd, with $p \geqslant 1, q, r \geqslant 0$, in which case K is a pretzel knot from the list, which we excluded.

Secondly, if $(N, T) = (1, 5)$, we have $\beta = \delta a^{u_1} b^{u_2} x^{u_3} a^{u_4} b^{u_5}$. If the exponents u_i are all odd, β closes to a two component link, a contradiction to K being a knot. Therefore at least one of the u_i is even. We may assume that u_1 is even, because the exponents $u_1, u_2, \dots, u_T$ may be cyclically permuted without changing the braid closure, as explained in Section 4.2 after Lemma 4.4. In particular, we may assume that $u_1 \geqslant 2$. Then, since $\delta = xb$,

$$\beta \ \sim \ a^{u_1} b^{u_2} x^{u_3} a^{u_4} b^{u_5} xb \ \rightsquigarrow \ a^2 bxabxb \ = \ a(abx)^2 b \ \rightarrowtail \ ab,$$

where '$\sim$' denotes conjugation, '$\rightsquigarrow$' denotes the existence of a cobordism whose genus equals the difference of the 3-genera of the knots it connects and '$\rightarrowtail$' is shown in Figure 4.4. Here, the cobordism '$\rightsquigarrow$' is built from saddles decreasing the exponents u_i. Since the closure of $a(abx)^2 b$ has 3-genus 3 while only 2 twists are used in '$\rightarrowtail$', we obtain $g_4^{\text{top}}(K) \leqslant g(K) - 1 < g(K)$.

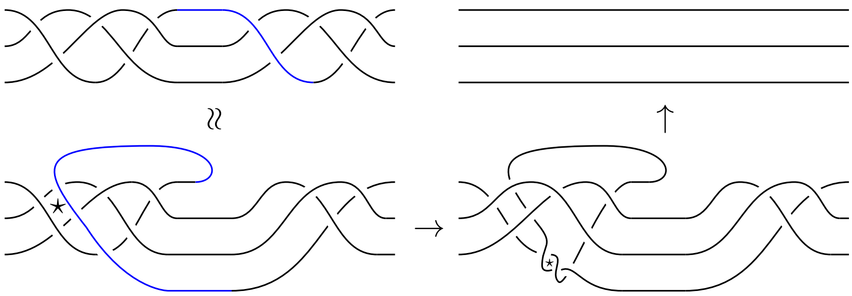


Figure 4.4: $abxabx \rightarrowtail \varnothing$ using one twist on four strands, followed by another twist on two strands, at the locations marked $\star$ (cf. Figure 4.2). The first step is an isotopy, moving the blue strand. The last step is a crossing change near $\star$, followed by an isotopy fixing the endpoints of the braid strands.

Finally, if $T \geqslant 6$, we proceed similarly as in the previous case. First, assume via conjugation that u_1 is even. If $T > 6$ or $N > 0$, then

$$\beta \ \rightsquigarrow \ a^{u_1} b^{u_2} x^{u_3} a^{u_4} b^{u_5} x^{u_6} b \ \rightsquigarrow \ a(abx)^2 b \ \rightarrowtail \ ab.$$

If $T = 6$ and $N = 0$, then $\beta = a^{u_1} b^{u_2} x^{u_3} a^{u_4} b^{u_5} x^{u_6}$. Again, the parity of the exponents u_i determines whether β closes to a knot or a multi-component link. In order to obtain a knot, at least two of them, say u_i and u_j, need to be even. We may assume that $(i, j) = (1, 2)$ or $(i, j) = (1, 4)$. To see this, use cyclic permutation (as in the case $(N, T) = (1, 5)$) and the fact that u_1, u_3 cannot be the only even exponents, again because β would not close to a knot if they were. Therefore, smooth cobordisms bring us to a^2b^2xabx or to a^2bxa^2bx. In the first case,

$$a^2b^2xabx \;=\; a^2b\delta^{-1}\delta bxabx \;=\; a^2b\delta^{-1}b(abx)^2 \;\rightarrowtail\; a^2b\delta^{-1}b \;=\; a^2ba^{-1} \;\sim\; ab.$$

For the second case, Figure 4.5 shows how to turn $a(abxa^2bx)$ into $a\gamma$ using two twists. Here, γ is the tangle shown in the top right corner of the figure. Note that $a\gamma$ describes the unknot when closed like a braid. Since the closure of a^2bxa^2bx has 3-genus 3 (see (4.3)), we obtain $g_4^{\text{top}}(K) \leqslant g(K) - 1 < g(K)$. This concludes the case that K is strongly quasipositive.

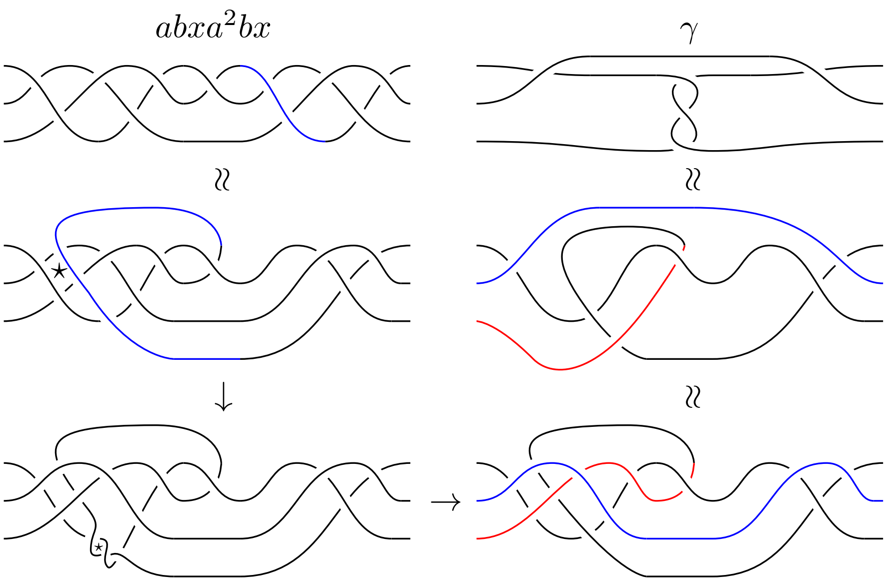


Figure 4.5: How to turn the braid $abxa^2bx$ (top left) into the tangle γ (top right) using one twist on four strands, followed by another twist on two strands, at the locations marked $\star$ (cf. Figure 4.2).

- If the mirror of K is strongly quasipositive, we apply the above argument to the mirror of K; since both g_4^{top} and g are invariant under taking mirror images, we obtain $g_4^{\text{top}}(K) < g(K)$ again.

- If K or its mirror is the knot $T_{2,2m+1}\#T_{2,-2n-1}$ with $m \geqslant n \geqslant 1$, then it has a Seifert surface S of genus $m+n = g(K)$ that contains a copy of the ribbon knot $R \coloneqq T_{2,2n+1}\#T_{2,-2n-1}$, bounding a subsurface of S of genus $g(R) = 2n$. A surgery that cuts this subsurface off S and replaces it with a slice disk for R gives rise to a smooth surface of genus $m+n-2n = m-n$ embedded in the four-ball, with boundary K. This shows that $g_4(K) \leqslant m-n \leqslant m+n-2 < g(K)$, because $n \geqslant 1$ by assumption. Recall that $g_4(K)$ denotes the smooth 4-genus. In particular, we obtain $g_4^{\text{top}}(K) < g(K)$.

– We are left with the 3-braid knots K such that neither K nor its mirror is among the following: strongly quasipositive, a connected sum of the form $T_{2,2m+1}\#T_{2,-2n-1}$ with $m \geqslant n \geqslant 1$, or the figure-eight knot. For such knots K, Lee–Lee [LL13] prove a bound on their unknotting number $u(K)$, namely $u(K) < g(K)$. Since $g_4^{\text{top}}(J) \leqslant g_4(J) \leqslant u(J)$ holds for all knots J, this implies $g_4^{\text{top}}(K) < g(K)$ and completes the proof. □

For comparison, we note the following analog of Theorem 4.1, in which the smooth 4-genus $g_4(K)$ replaces its analog $g_4^{\text{top}}(K)$ in the topological category, and Rasmussen's invariant $s(K)$ from Khovanov homology [Ras10] plays the role of the signature $\sigma(K)$. Here, we could replace s by any other slice-torus invariant [Liv04], e. g. the Heegaard Floer τ-invariant [OS03]; see the paragraph after (3.4) in Section 3.2 for the definition of slice-torus invariants.

Proposition 4.13. *Let K be a 3-braid knot other than the figure-eight knot. Then*

$$|s(K)| = 2g(K) \quad \Longleftrightarrow \quad g_4(K) = g(K).$$

These equalities hold precisely when K or its mirror is strongly quasipositive.

Proof. First, if K or its mirror is strongly quasipositive, then $|s(K)| = 2g(K)$ follows, see [Shu07, Proposition 1.7]. Moreover, the implication $|s(K)| = 2g(K) \Rightarrow g_4(K) = g(K)$ holds for all knots K, because of the inequalities $|s(K)| \leqslant 2g_4(K) \leqslant 2g(K)$. It remains to prove that if K is a 3-braid knot other than the figure-eight knot with $g_4(K) = g(K)$, then K or its mirror is strongly quasipositive. This follows from Lee–Lee's results [LL13]. More precisely, for a 3-braid knot K with $g_4(K) = g(K)$ Theorem 1.1 in [LL13] implies that $u(K) = g(K)$. By Theorem 1.3 of the same paper, K or its mirror is either strongly quasipositive or a connected sum of two-strand torus knots $T_{2,2m+1}\#T_{2,-2n-1}$ with $m \geqslant n \geqslant 1$. The latter can be excluded as in the proof of Theorem 4.1 by showing $g_4(K) \leqslant m - n \leqslant m + n - 2 < g(K)$, a contradiction to $g_4(K) = g(K)$. □

Theorem 4.14. *Let K be a strongly quasipositive 3-braid knot, written as the closure of a 3-braid β in Xu normal form $\beta = \delta^N \tau_1^{u_1}\tau_2^{u_2}\cdots\tau_T^{u_T}$, with $u_1, \ldots, u_T \geqslant 1$ and $N \geqslant 0$. Then the topological 4-genus defect of K is bounded as follows:*

$$\frac{N}{3} + \frac{T}{3} - 1 \geqslant g(K) - g_4^{top}(K) \geqslant \frac{N}{3} + \frac{T}{6} - 3.$$

The constants $\frac{1}{3}$ and $\frac{1}{6}$ in the second inequality are optimal in the following sense: Whenever $C > \frac{1}{3}$ or $D > \frac{1}{6}$, and $E \in \mathbb{R}$, there exists a 3-braid in Xu normal form as above with $N \geqslant 0$ such that its braid closure K satisfies

$$g(K) - g_4^{top}(K) < CN + DT - E.$$

Proof. We begin with the upper bound $\frac{N}{3} + \frac{T}{3} - 1 \geqslant g(K) - g_4^{\mathrm{top}}(K)$. The case $T = 0$, in which $K = T_{3,N}$, is covered by [BBL20, Theorem 1]:

$$g(K) - g_4^{\mathrm{top}}(K) = N - 1 - \left\lceil \frac{2N}{3} \right\rceil \leqslant \frac{N}{3} - 1.$$

For $T > 0$, we first use Proposition 4.10 to compute the absolute value of the signature $|\sigma(K)| = -\sigma(K) = U + \frac{4}{3}N - \frac{2}{3}T$, where $U = u_1 + \ldots + u_T$, and recall from (4.3) in Section 4.2 that $g(K) = \frac{U}{2} + N - 1$. The bound then follows directly from Kauffman and Taylor's classical bound $|\sigma(K)| \leqslant 2g_4^{\mathrm{top}}(K)$.

To establish the lower bound, we first apply a smooth cobordism from K to a knot K', by suitably lowering the exponents $N, u_1, u_2, \ldots, u_T$ in β, as explained in Figure 4.3 above. First, assume $N > 0$. We set $K' := \widehat{\delta^m (abx)^{\frac{s}{3}}}$, where

$$s = 6 \left\lfloor \frac{T}{6} \right\rfloor \quad \text{and} \quad m = \begin{cases} N-2 & \text{if } N \equiv 0 \mod 3 \\ N-1 & \text{if } N \equiv 2 \mod 3 \\ N & \text{if } N \equiv 1 \mod 3. \end{cases}$$

We have $K \rightsquigarrow K'$, and so this is a smooth cobordism which does not increase the topological 4-genus defect. In other words,

$$g(K) - g_4^{\mathrm{top}}(K) \geqslant g(K') - g_4^{\mathrm{top}}(K'),$$

as in (4.4). Next, we apply the untwisting move $(abx)^2 \rightarrowtail \varnothing$ from Figure 4.4 exactly $\frac{s}{6}$ times to the knot K', resulting in $\widehat{\delta^m} = T_{3,m}$. Since $m \equiv 1 \mod 3$, this is a knot again. For $m \geqslant 4$, [BBL20, Lemma 5 (1)] yields $\mathrm{tu}(T_{3,m}) \leqslant \frac{2}{3}m + \frac{1}{3}$. This inequality still holds for $m = 1$ and adds up to

$$g_4^{\mathrm{top}}(K') \leqslant \mathrm{tu}(K') \leqslant 2 \cdot \frac{s}{6} + \frac{2}{3}m + \frac{1}{3}.$$

Since $g(K') = \frac{s}{2} + m - 1$ by (4.3), and since $\frac{s}{6} \geqslant \frac{t}{6} - \frac{5}{6}$ and $m \geqslant N - 2$, we obtain

$$g(K) - g_4^{\mathrm{top}}(K) \geqslant \frac{s}{6} + \frac{m}{3} - \frac{4}{3} \geqslant \frac{t}{6} + \frac{N}{3} - 3.$$

In the case $N = 0$, the above procedure fails because m, as defined above, is negative and the smooth cobordism to K' might therefore increase the 4-genus defect. However, a simple cosmetic modification allows for a cobordism that only increases the 4-genus defect by at most one. Specifically, we set $m = 1$ (instead of $m = -2$) and $K' = \widehat{\delta (abx)^{\frac{s}{3}}}$. The cobordism from K to K' is then given by lowering the exponents $u_1, u_2, \ldots, u_T$ in β as above while increasing the exponent of δ from 0 to 1. This gives

$$g(K) - g_4^{\mathrm{top}}(K) \geqslant g(K') - g_4^{\mathrm{top}}(K') - 1.$$

Now we apply the $\frac{s}{6}$ untwisting moves $(abx)^2 \rightarrowtail \varnothing$ as above. The result is the unknot $\widehat{\delta}$. Since $N = 0$, we again obtain the claimed bound

$$g(K) - g_4^{\mathrm{top}}(K) \geqslant \frac{s}{6} - 1 \geqslant \frac{t}{6} - \frac{5}{6} - 1 \geqslant \frac{T}{6} + \frac{N}{3} - 3.$$

In order to demonstrate optimality of the constants, we consider the two special families of 3-braids δ^N and $(abx)^N$, which we slightly modify to δ^{3k+1} and $(abx)^{2k}abx^2abx^2$, to make sure that their braid closures are connected.

- For $K = T_{3,3k+1}$ with $k \geqslant 1$, the closure of the braid δ^{3k+1} in Xu normal form with $N = 3k+1$ and $t = 0$, we have $g(K) = 3k$ and $g_4^{\mathrm{top}}(K) = 2k+1$ by [BBL20], hence $g(K) - g_4^{\mathrm{top}}(K) = k - 1$. Whenever $C > \frac{1}{3}$ and D, E are arbitrary constants, we will therefore have

$$g(K) - g_4^{\mathrm{top}}(K) = k - 1 < C \cdot (3k+1) + D \cdot 0 - E$$

  for sufficiently large k.

- If K is the closure of $(abx)^{2k}abx^2abx^2$, which is in Xu normal form with $N = 0$ and $t = 6k+6$, then $g(K) = 3k+3$. By Gambaudo–Ghys [GG16, Corollary 4.4], for any 3-braid β with closure J, the Levine–Tristram signature function of J, $[0,1] \to \mathbb{Z}$, $t \mapsto \sigma_{e^{2\pi i t}}(J)$, grows linearly on $(0, \frac{1}{3})$ with slope -2 times the writhe of β, up to a pointwise error of at most 2 (see e. g. Figure 4.6 at the end of Section 4.4). For strongly quasipositive β with J a knot, that slope is $-4(g(J)+1)$, see (4.3), and hence

$$\widehat{\sigma}(J) \coloneqq \max_{\omega \in S^1 \setminus \Delta_J^{-1}(0)} |\sigma_\omega(J)| \geqslant \frac{4}{3}(g(J)+1) - 2. \tag{4.5}$$

  Since $\frac{1}{2}|\sigma_\omega(J)| \leqslant g_4^{\mathrm{top}}(J)$ whenever $\omega \in S^1$ is not a root of the Alexander polynomial of J (see [KT76, Pow17]), we obtain $\frac{2}{3}(g(K)+1) - 1 = 2k + \frac{5}{3} \leqslant g_4^{\mathrm{top}}(K)$. Hence, if C, E are arbitrary constants and $D > \frac{1}{6}$, we have

$$g(K) - g_4^{\mathrm{top}}(K) \leqslant 3k + 3 - 2k - \frac{5}{3} = k + \frac{4}{3} < C \cdot 0 + D \cdot (6k+6) - E,$$

  for sufficiently large k. In this case, it does not suffice to consider the (classical) signature bound on $g_4^{\mathrm{top}}(K)$. Indeed, $|\sigma(K)| = 2k+4$ (see Proposition 4.10) is roughly half the maximal Levine–Tristram signature $\widehat{\sigma}(K)$. Substituting $|\sigma(K)|$ for $\widehat{\sigma}(K)$ in the above argument would therefore not work. □

4.4 The topological 4-genus of positive 3-braid knots

The methods of Section 4.3 allow us to determine the topological 4-genus up to an error of 1 for knots which are closures of positive 3-braids under an additional assumption on their Xu normal form (see Proposition 4.18). In certain cases, we can determine g_4^{top} exactly (see Examples 4.16 and 4.17 and Remark 4.19). For these results, we use the $\frac{1}{2}|\sigma| \leqslant g_4^{\mathrm{top}}$. We end with a discussion of examples where this bound is not sharp (Remark 4.21) and give examples where instead of untwisting, we determine g_4^{top} via their algebraic genus (Example 4.20). We are led by the following questions.

Question 4.15. *Does the equality $\frac{1}{2}\widehat{\sigma}(K) = g_4^{top}(K)$ hold for all braid positive 3-braid knots? For all strongly quasipositive 3-braid knots? For all braid positive knots?*

Throughout, we will use the Xu normal form of 3-braids from Section 4.2. Recall that we write $\delta = ba = ax = xb$ such that $a\delta = \delta b$, $b\delta = \delta x$ and $x\delta = \delta a$ (see (4.2) in Section 4.2).

Example 4.16. Consider the 3-braid $\beta = \delta^{3\ell+2}a^{u_1}$ for $\ell \geqslant 0, u_1 \geqslant 1$ in Xu normal form with closure a knot K. Note that u_1 must be even for K to be a knot. We claim that

$$g_4^{\mathrm{top}}(K) = \mathrm{tu}(K) = \frac{|\sigma(K)|}{2} = \frac{u_1}{2} + 2\ell + 1. \tag{4.6}$$

The last equality follows directly from Proposition 4.10. To prove (4.6), using the inequalities $\frac{1}{2}|\sigma(K)| \leqslant g_4^{\mathrm{top}}(K) \leqslant \mathrm{tu}(K)$ (as explained in the beginning of Section 4.3) it is enough to show that $\mathrm{tu}(K) \leqslant \frac{1}{2}|\sigma(K)|$. By $\frac{u_1-2}{2}$ crossing changes from positive crossings of β to negative crossings we obtain the braid $\delta^{3\ell+2}a^2$. We will prove by induction that this braid can be untwisted with $2\ell+2$ twists, which implies $\mathrm{tu}(K) \leqslant \frac{u_1}{2} + 2\ell + 1$. For $\ell = 0$, we have $\delta^2 a^2 = baba^3 = aba^4$ which becomes ab (with closure the unknot) by two crossing changes. For $\ell = 1$, we have

$$\begin{aligned} \delta^5 a^2 &= \delta^4 axa^2 = \delta^3 x^2 bxa^2 = \delta^2 b^3 abxa^2 = \delta a^4 xabxa^2 = x^5 bxabxa^2 \\ &\sim b^5 abxabxx \rightarrowtail b^5 x, \end{aligned}$$

which turns into $bx \sim \delta$ using two crossing changes. Recall that the two twists needed to untwist $abxabx \rightarrowtail \varnothing$ are shown in Figure 4.4 of Section 4.3. Now, for $\ell \geqslant 2$, we have

$$\begin{aligned} \delta^{3\ell+2}a^2 &= \delta^{3\ell-3}x^5 bxabxa^2 \sim \delta^{3\ell-3}b^5 abxabxx \rightarrowtail \delta^{3\ell-3}b^5 x \\ &\sim \delta^{3\ell-3}\delta b^4 = \delta^{3\ell-3}a\delta b^3 \sim \delta^{3\ell-1}b^2 \sim \delta^{3(\ell-1)+2}a^2, \end{aligned}$$

where we again used two twists for '$\rightarrowtail$'. Inductively this shows that $\delta^{3\ell+2}a^2$ can be untwisted with $2\ell+2$ twists as claimed.

Example 4.16 combined with the results from [BBL20] for 3-strand torus knots shows that the equalities $g_4^{\mathrm{top}} = \mathrm{tu} = \frac{\widehat{\sigma}}{2}$ hold for all strongly quasipositive 3-braid knots in Xu normal form (a) or (b) from Lemma 4.4, where $\widehat{\sigma} = |\sigma|$ except for certain torus knots of braid index 3; see also Remark 4.21. We next consider a sub-case of case (c).

Example 4.17. Let K be a knot that is the closure of a 3-braid in Xu normal form $\beta = \delta^{3\ell+1}a^{u_1}b^{u_2}$ for $\ell \geqslant 0, u_1, u_2 \geqslant 1$. Note that u_1 and u_2 must both be even for K to be a knot. We claim that

$$g_4^{\text{top}}(K) = \text{tu}(K) = \frac{|\sigma(K)|}{2} = \frac{u_1+u_2}{2} + 2\ell.$$

The proof works as in Example 4.16. After $\frac{u_1+u_2-4}{2}$ positive to negative crossing changes in β we obtain the braid $\delta^{3\ell+1}a^2b^2$, which we can untwist with $2\ell+2$ twists as follows. For $\ell = 0$, the braid δa^2b^2 turns into δ by two crossing changes. For $\ell = 1$, we have

$$\begin{aligned}\delta^4 a^2b^2 &= \delta^3 xaxab^2 = \delta^2 bx^2bxab^2 = \delta ab^3abxab^2 = xa^4xabxab^2\\ &\sim ab^4abxabxx \rightarrowtail ab^4x,\end{aligned}$$

which can be untwisted using two crossing changes. For $\ell \geqslant 2$, we have

$$\begin{aligned}\delta^{3\ell+1}a^2b^2 &= \delta^{3\ell-3}xa^4xabxab^2 \sim \delta^{3\ell-3}ab^4abxabxx \rightarrowtail \delta^{3\ell-3}ab^4x\\ &= \delta^{3\ell-4}xa^3babx = \delta^{3\ell-5}bx^3a^2xabx = \delta^{3\ell-6}ab^3x^3bxabx\\ &\sim \delta^{3\ell-6}a^3b^3abxabx \rightarrowtail \delta^{3\ell-6}a^3b^3 \sim \delta^{3\ell-5}a^2b^2 = \delta^{3(\ell-2)+1}a^2b^2,\end{aligned}$$

which we can untwist inductively using the two base cases above.

The following proposition improves the statement from Theorem 4.14 for braid positive 3-braid knots under the additional assumption $u_i \geqslant 2$ for the exponents in the Xu normal form of their braid representatives. In fact, we can determine $g_4^{\text{top}}(K)$ in this case up to an error of 1, using $\frac{1}{2}|\sigma(K)|$ as a lower bound.

Proposition 4.18. *Let K be a knot that is the closure of a 3-braid in Xu normal form*

$$\delta^N \tau_1^{u_1}\tau_2^{u_2}\dots\tau_T^{u_T} \qquad \textit{for} \quad T \geqslant 1,\, N \geqslant \frac{T}{2},\, u_1,\dots,u_T \geqslant 2.$$

Then K is a braid positive knot and

$$\frac{N+T}{3} - 1 = g(K) - \frac{|\sigma(K)|}{2} \geqslant g(K) - g_4^{top}(K) \geqslant \frac{N+T}{3} - 2. \tag{4.7}$$

Proof. Let

$$\beta = \delta^N \tau_1^{u_1}\tau_2^{u_2}\dots\tau_T^{u_T} \qquad \text{for} \quad T \geqslant 1,\, N \geqslant \frac{T}{2},\, u_1,\dots,u_T \geqslant 2$$

such that its closure is a knot K. Set $U = u_1 + \dots + u_T$. Proposition 4.12 implies that K is braid positive. Moreover, we have $\frac{|\sigma(K)|}{2} = \frac{U}{2} + \frac{2N}{3} - \frac{T}{3}$ by Proposition 4.10 and $g(K) = \frac{U}{2} + N - 1$ by (4.3). Using $\frac{1}{2}|\sigma(K)| \leqslant g_4^{\text{top}}(K)$, it remains to show that $g_4^{\text{top}}(K) \leqslant \frac{|\sigma(K)|}{2} + 1$. We distinguish two cases depending on the parity of T.

First, let $T = 2r$ be even for $r \geqslant 1$. The conditions $2N \geqslant T$ and $N + T \equiv 0 \pmod 3$ imply that we can write $N = 3\ell + r$ for $\ell \geqslant 0$, and we have

$$\beta = \delta^{3\ell+r}\tau_1^{u_1}\tau_2^{u_2}\dots\tau_{2r}^{u_{2r}}.$$

The case $r = 1$ ($T = 2$) is covered by Example 4.17, so we can further assume that $r \geqslant 2$. There is a smooth cobordism of Euler characteristic $4r - U - 4$ from K to the knot that is the closure of

$$\beta' = \tau_{1-r}\delta^{3\ell+r-1}\prod_{i=1}^{r-2}\tau_i^2\tau_{r-1}\tau_r\tau_{r+1}\prod_{i=r+2}^{2r}\tau_i^2.$$

Indeed, we can use $U - 4r + 3$ saddle moves to replace all but three of the exponents u_i by 2 and the other three by 1. We use a last saddle move to replace δ by τ_{1-r}. We will prove by induction on r that β' turns into $\delta^{3\ell+1}$ by $2r - 2$ twists. Since the closure of $\delta^{3\ell+1}$ is the torus knot $T_{3,3\ell+1}$, this will imply

$$\operatorname{tu}\left(\widehat{\beta'}\right) \leqslant \operatorname{tu}(T_{3,3\ell+1}) + 2r - 2 = \begin{cases} 2\ell + 2r - 1 & \text{if } \ell \geqslant 1, \\ 2r - 2 & \text{if } \ell = 0, \end{cases} \tag{4.8}$$

where the equality follows from [BBL20, Lemma 5 and Theorem 1]. Recall that we have $\delta = \tau_{i+1}\tau_i$ and $\tau_i\delta = \delta\tau_{i+1}$ by (4.2), and $\tau_i = \tau_{i+3m}$ for all $m \in \mathbb{Z}, i \in \mathbb{Z}$. For $r = 2$, we thus have $\tau_{1-r} = \tau_2$ and

$$\begin{aligned}\beta' &= \tau_2\delta^{3\ell+1}\tau_1\tau_2\tau_3\tau_4^2 = \delta^{3\ell}\tau_2\tau_1\tau_0\tau_1\tau_2\tau_3\tau_4^2 = \delta^{3\ell}\tau_0\tau_{-1}\tau_0\tau_1\tau_2\tau_3\tau_4^2 \\ &\sim \delta^{3\ell}\tau_2\tau_1\tau_2\tau_3\tau_4\tau_5\tau_6^2 \rightarrowtail \delta^{3\ell}\tau_2\tau_6 \sim \delta^{3\ell+1},\end{aligned}$$

so β' indeed turns into $\delta^{3\ell+1}$ using $2r-2 = 2$ twists in this case. Now, for $r \geqslant 3$, consider

$$\begin{aligned}\beta' &= \tau_{1-r}\delta^{3\ell+r-2}\prod_{i=1}^{r-3}\tau_{i-1}^2\delta\tau_{r-2}^2\tau_{r-1}\tau_r\tau_{r+1}\tau_{r+2}^2\prod_{i=r+3}^{2r}\tau_i^2 \\ &= \tau_{1-r}\delta^{3\ell+r-2}\prod_{i=1}^{r-3}\tau_{i-1}^2\tau_{r-3}\tau_{r-2}\tau_{r-3}\tau_{r-2}\tau_{r-1}\tau_r\tau_{r+1}\tau_{r+2}^2\prod_{i=r+3}^{2r}\tau_i^2 \\ &\sim \tau_{(1-r)-(r-4)}\delta^{3\ell+r-2}\prod_{i=1}^{r-3}\tau_{i-1-(r-4)}^2\tau_1\tau_2\tau_1\tau_2\tau_3\tau_4\tau_5\tau_6^2\prod_{i=r+3}^{2r}\tau_{i-(r-4)}^2 \\ &\rightarrowtail \tau_{(1-r)-r+1}\delta^{3\ell+r-2}\prod_{i=1}^{r-3}\tau_{i-r}^2\tau_1\tau_2\tau_3\prod_{i=r+3}^{2r}\tau_{i-r+1}^2 \\ &\sim \tau_{(1-r)+1}\delta^{3\ell+r-2}\prod_{i=1}^{r-3}\tau_i^2\tau_{r-2}\tau_{r-1}\tau_r\prod_{i=r+1}^{2(r-1)}\tau_i^2.\end{aligned}$$

The braid β' hence turns into

$$\tau_{(1-r)+1}\delta^{3\ell+r-2}\prod_{i=1}^{r-3}\tau_i^2\tau_{r-2}\tau_{r-1}\tau_r\prod_{i=r+1}^{2(r-1)}\tau_i^2$$

by two twists and inductively we get that β' turns into

$$\tau_{(1-r)+r-2}\delta^{3\ell+1}\tau_1\tau_2\tau_3\tau_4^2$$

by $2(r-2)$ twists. Since $\tau_{(1-r)+r-2}=\tau_2$, this braid is the same as the one from the base case $r=2$ and therefore can be untwisted with two twists. We obtain that β' becomes $\delta^{3\ell+1}$ by $2r-2$ twists. Thus (4.8) follows and we get

$$\begin{aligned} g_4^{\text{top}}(K) &\leqslant g_4^{\text{top}}\left(\widehat{\beta'}\right)+\frac{U-4r+4}{2}\leqslant \text{tu}\left(\widehat{\beta'}\right)+\frac{U}{2}-2r+2 \\ &\leqslant \begin{cases} \frac{U}{2}+2\ell+1=\frac{|\sigma(K)|}{2}+1 & \text{if } \ell\geqslant 1, \\ \frac{U}{2}=\frac{|\sigma(K)|}{2} & \text{if } \ell=0. \end{cases} \end{aligned}$$

Next, let $T=2r+1$ be odd for $r\geqslant 0$. The conditions $2N\geqslant T$ and $N+T\equiv 0 \pmod 3$ imply that we can write $N=3\ell+r+2$ for $\ell\geqslant 0$, and we have

$$\beta=\delta^{3\ell+r+2}\tau_1^{u_1}\tau_2^{u_2}\dots\tau_{2r+1}^{u_{2r+1}}.$$

The case $T=1$ is covered by Example 4.16, so we can further assume that $r\geqslant 1$. There is a smooth cobordism of Euler characteristic $4r-U-2$ from K to the knot that is the closure of

$$\beta'=\tau_{1-r}\delta^{3\ell+r+1}\prod_{i=1}^{r-1}\tau_i^2\tau_r\tau_{r+1}\tau_{r+2}\prod_{i=r+3}^{2r+1}\tau_i^2,$$

similar to the cobordism considered in the above case. We prove by induction on r that β' turns into $\delta^{3(\ell+1)+1}$ by $2r-2$ twists, hence

$$\text{tu}\left(\widehat{\beta'}\right)\leqslant \text{tu}(T_{3,3(\ell+1)+1})+2r-2=2\ell+2r+1.$$

For $r=1$, we have

$$\beta'=\tau_0\delta^{3\ell+2}\tau_1\tau_2\tau_3=\delta^{3\ell+3}\tau_2\tau_3\sim\delta^{3(\ell+1)+1}. \tag{4.9}$$

For $r=2$, we have

$$\begin{aligned}\beta'&=\tau_2\delta^{3\ell+3}\tau_1^2\tau_2\tau_3\tau_4\tau_5^2=\tau_2\delta^{3\ell+2}\tau_0\tau_1\tau_0\tau_1\tau_2\tau_3\tau_4\tau_5^2\\ &\sim\tau_0\delta^{3\ell+2}\tau_1\tau_2\tau_1\tau_2\tau_3\tau_4\tau_5\tau_6^2\rightarrowtail\tau_0\delta^{3\ell+2}\tau_1\tau_2\tau_6\sim\delta^{3(\ell+1)+1}\end{aligned}$$

using (4.9) in the last step. Now, for $r \geqslant 3$, consider

$$\begin{aligned}
\beta' &= \tau_{1-r}\delta^{3\ell+r}\prod_{i=1}^{r-2}\tau_{i-1}^2\delta\tau_{r-1}^2\tau_r\tau_{r+1}\tau_{r+2}\tau_{r+3}^2\prod_{i=r+4}^{2r+1}\tau_i^2 \\
&= \tau_{1-r}\delta^{3\ell+r}\prod_{i=1}^{r-2}\tau_{i-1}^2\tau_{r-2}\tau_{r-1}\tau_{r-2}\tau_{r-1}\tau_r\tau_{r+1}\tau_{r+2}\tau_{r+3}^2\prod_{i=r+4}^{2r+1}\tau_i^2 \\
&\sim \tau_{(1-r)-(r-3)}\delta^{3\ell+r}\prod_{i=1}^{r-2}\tau_{i-1-(r-3)}^2\tau_1\tau_2\tau_1\tau_2\tau_3\tau_4\tau_5\tau_6^2\prod_{i=r+4}^{2r+1}\tau_{i-(r-3)}^2 \\
&\rightarrowtail \tau_{(1-r)-r}\delta^{3\ell+r}\prod_{i=1}^{r-2}\tau_{i-r-1}^2\tau_1\tau_2\tau_3\prod_{i=r+4}^{2r+1}\tau_{i-r}^2 \\
&\sim \tau_{(1-r)+1}\delta^{3\ell+r}\prod_{i=1}^{r-2}\tau_i^2\tau_{r-1}\tau_r\tau_{r+1}\prod_{i=r+2}^{2(r-1)+1}\tau_i^2.
\end{aligned}$$

Inductively we get that β' turns into $\tau_{(1-r)+r-2}\delta^{3\ell+3}\tau_1^2\tau_2\tau_3\tau_4\tau_5^2 = \tau_2\delta^{3\ell+3}\tau_1^2\tau_2\tau_3\tau_4\tau_5^2$ by $2(r-2)$ twists, so into $\delta^{3(\ell+1)+1}$ by $2r-2$ twists. We obtain

$$\begin{aligned}
g_4^{\text{top}}(K) &\leqslant g_4^{\text{top}}\left(\widehat{\beta'}\right) + \frac{U-4r+2}{2} \leqslant \operatorname{tu}\left(\widehat{\beta'}\right) + \frac{U}{2} - 2r + 1 \\
&\leqslant \frac{U}{2} + 2\ell + 2 = \frac{|\sigma(K)|}{2} + 1.
\end{aligned}$$ □

Remark 4.19. The proof of Proposition 4.18 (more precisely, the first case with $\ell = 0$) shows that the first inequality in (4.7) is an equality when $2N = T$.

Example 4.20. Let us try to determine the topological 4-genera of the knots arising as closures of the following positive 3-braids:

$$\begin{aligned}
\delta^3a^2b^2xabx &\sim a^3b^3a^2b^2a^2b^2 \\
\delta^4a^2bxab &\sim \Delta a^3b^2a^2b^2a^2 \\
\delta^4a^4bxab &\sim \Delta a^5b^2a^2b^2a^2 \\
\delta^4a^2b^2xa^2b &\sim \Delta a^3b^3a^2b^3a^2 \\
\delta^6a^2bx &\sim \Delta^3a^3b^2a^2.
\end{aligned}$$

First, we note that for all of these knots K, we have the lower bound

$$\frac{1}{2}\widehat{\sigma}(K) = \frac{1}{2}|\sigma(K)| = g(K) - 2 \leqslant g_4^{\text{top}}(K).$$

Second, in the search for upper bounds, we are able to find a knot J such that $K \rightsquigarrow J$ and $\operatorname{tu}(J) = g(J) - 1$, thus proving $g_4^{\text{top}}(K) \leqslant g(K) - 1$, for each of these knots K. However, we are unable to find J with $K \rightsquigarrow J$ and $\operatorname{tu}(J) = g(J) - 2$. Nevertheless, we can prove $g_4^{\text{top}}(K) = g(K) - 2$ for all of these knots K in a different way, which is practical for individual knots with low genus. Namely, a computer search [Lew23] reveals that the *algebraic genus* $g_{\mathrm{alg}}(K)$, which is defined in terms of Seifert matrices of K and provides an upper bound $g_4^{\mathrm{top}}(K) \leqslant g_{\mathrm{alg}}(K)$ [FL18], satisfies $g_{\mathrm{alg}}(K) \leqslant g(K) - 2$ for all those knots K.

Remark 4.21. The maximal Levine–Tristram signature, see (4.5) in Section 4.3, provides a good computable lower bound for g_4^{top}:

$$\widehat{\sigma}(K) = \max_{\omega \in S^1 \setminus \Delta_K^{-1}(0)} |\sigma_\omega(K)| \leqslant g_4^{\mathrm{top}}(K).$$

The function $S^1 \to \mathbb{Z}, \omega \mapsto \sigma_\omega(K)$, is piecewise constant and jumps only at zeroes of the Alexander polynomial Δ_K. A priori, its maximum absolute value may be assumed anywhere on S^1.

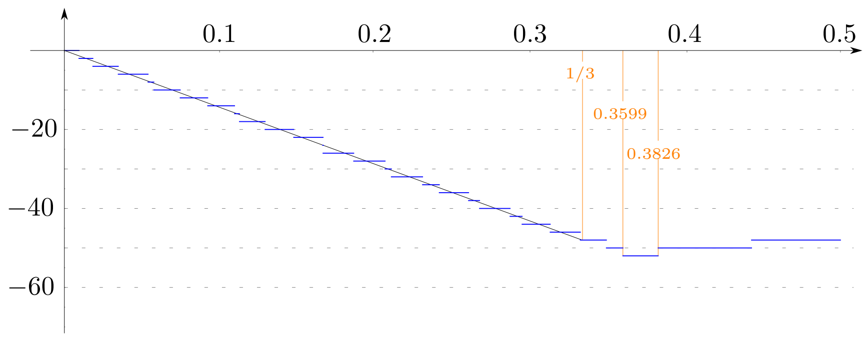


Figure 4.6: In blue, the graph of the Levine–Tristram signature $\sigma_{e^{2\pi it}}(K)$ for $t \in [0, \frac{1}{2}]$ and K the closure of the 3-braid $\left(a^2b^2\right)^8 \left(a^5b^5\right)^4 \sim \delta^{12} \left(a^4b^4x^4\right)^2 a^4b^4(xab)^5x$. In black, the linear approximation by [GG16, Corollary 4.4] for $t \in [0, \frac{1}{3}]$. The maximum absolute value $\widehat{\sigma}(K)$ of $\sigma_{e^{2\pi it}}(K)$, which equals $|\sigma(K)| + 4 = |\sigma_{e^{2\pi i/3}}(K)| + 4$, is assumed between 0.3599 and 0.3826 (rounded).

In the above examples and Proposition 4.18 we have seen that for certain families of 3-braid knots, $\widehat{\sigma} = |\sigma| = 2g_4^{\mathrm{top}}$, where $\sigma = \sigma_{-1} = \sigma_{e^{\pi i}}$ is the classical knot signature. This also holds for the $T_{3,3k+m}$ torus knots with $m \in \{1, 2\}$ and odd $k \geqslant 1$ [BBL20]. For even k on the other hand, e. g. for $T_{3,7}$, one finds $\widehat{\sigma} = |\sigma_\omega| = |\sigma| + 2$ for ω chosen only one jump-point of the Levine–Tristram signature away from $e^{\pi i}$, i. e. $\omega = e^{2\pi it}$ for

$$t \in \left(\frac{1}{2} - \frac{5}{18k + 6m}, \frac{1}{2} - \frac{1}{18k + 6m}\right).$$

This observation relies on the fact that the jumps of the Levine–Tristram signatures of torus knots are well understood [Lit79, Ban22]. Moreover, we have seen examples where $\widehat{\sigma} = \sigma_{e^{2\pi i/3}}$, namely the closure of $(abx)^{2k}abx^2abx^2$ for $k \geqslant 0$; see the proof of Theorem 4.14 in Section 4.3. Overall, whenever we could precisely determine the topological 4-genus of a 3-braid knot K, then the maximum absolute value of $\sigma_{e^{2\pi it}}$ was either assumed at $t = \frac{1}{3}$, or at $t = \frac{1}{2}$, or close to $t = \frac{1}{2}$. However, there are 3-braid knots K for which $\widehat{\sigma}(K) \geqslant |\sigma_{e^{2\pi i/3}}(K)| + 4$ and $\widehat{\sigma}(K) \geqslant |\sigma(K)| + 4$; see Figure 4.6. In fact, we conjecture that the difference $\widehat{\sigma}(K) - \max\left(|\sigma_{e^{2\pi i/3}}(K)|, |\sigma(K)|\right)$ is unbounded for K ranging over closures of 3-braids of the form $\left(a^2b^2\right)^m \left(a^5b^5\right)^n$. Determining $\widehat{\sigma}$ and g_4^{top} for all 3-braid knots, or even just closures of positive 3-braids, thus appears to be a hard problem.

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