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Abstract

In analogy to the various characterizations of Gromov hyperbolicity, we present a list of six mutually equivalent higher rank conditions for metric spaces satisfying some assumption reminiscent of global non-positive curvature.

Keywords Gromov hyperbolicity · Non-positive curvature · Asymptotic rank · Linear isoperimetric inequality · Quasi-minimizer · Morse lemma · Filling radius

Mathematics Subject Classification 53C23 · 51F30 · 20F67

1 Introduction

The concept of Gromov hyperbolicity manifests itself in many different ways. With only mild assumptions on the underlying metric space, the spectrum of equivalent properties includes various thin triangle conditions, the stability of quasi-geodesics (the Morse lemma), a linear isoperimetric filling inequality for closed curves, and a sub-quadratic isoperimetric inequality [3–7, 9, 16, 20, 33–35, 38]. We present a similar list of six equivalent properties in the context of generalized non-positive curvature and higher asymptotic rank. This complements the results in [40] and in the recent paper [28]. We give a largely self-contained proof, providing some improvements and simplifications for the known part.

For an informal statement of the main result, let us focus on the special case that X is a proper CAT(0) or Busemann convex space. In passing from Gromov hyperbolicity to rank $n \geq 2$, the role of closed curves and quasi-geodesics is transferred to n -cycles and n -chains satisfying a suitable quasi-minimality condition, respectively. For the moment, the reader is invited to think of the chain complex of Lipschitz singular chains with integer coefficients in X . Some of the statements below involve a uniform polynomial mass bound of degree n in large balls, and we shall thus speak of n -chains with *controlled density*. This condition holds

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automatically for Lipschitz quasi-geodesics if $n = 1$ or, more generally, for Lipschitz quasi-isometric embeddings of domains in \mathbb{R}^n into X . We show that the following are equivalent:

- the asymptotic rank of X being at most n (see below for the definition);
- a sub-Euclidean isoperimetric inequality for n -cycles, corresponding to a sub-quadratic inequality in the case $n = 1$;
- a linear isoperimetric inequality for n -cycles with controlled density;
- a version of the Morse lemma implying in particular a bound on the Hausdorff distance between (the supports of) two quasi-minimizing n -chains with controlled density and equal boundary;
- a slim $(n + 1)$ -simplex property analogous to the slimness of quasi-geodesic triangles in geodesic Gromov hyperbolic spaces;
- a bound on the filling radius of n -cycles with controlled density.

We now proceed to the details. The actual setup is as in [28]. Suppose that $X = (X, d)$ is a proper metric space, that is, closed bounded subsets are compact. We use the chain complex $\mathbf{I}_{*,c}(X)$ of metric integral currents with compact support, which comprises the singular Lipschitz chains but is more versatile and has suitable compactness properties. The relevant prerequisites from the theory of metric currents will be discussed in Sect. 2. For $n \geq 1$, we say that X satisfies condition (CI_n) if there is a constant c such that any two points x, y in X can be joined by a curve of length $\leq c d(x, y)$, and for $k = 1, \dots, n$, every k -cycle $R \in \mathbf{I}_{k,c}(X)$ in some r -ball is the boundary of an $S \in \mathbf{I}_{k+1,c}(X)$ with mass

$$\mathbf{M}(S) \leq cr \mathbf{M}(R).$$

The cone inequalities (CI_n) hold in particular, for all n , if X is a CAT(0) space or a space with a conical geodesic bicombing [11, 36]. We remark that every hyperbolic group acts geometrically on a proper polyhedral complex with such a bicombing [30], and further classes of groups with this property are discussed in [8, 22, 24, 32]. Moreover, condition (CI_n) holds if X is an n -connected Riemannian manifold with a geometric action of a (quasi-geodesically) combable group; compare Theorem 10.3.5 in [13].

The asymptotic rank of X is the supremum of all $k \geq 0$ for which there exist a sequence $0 < r_i \rightarrow \infty$ and subsets $Y_i \subset X$ such that the rescaled sets $(Y_i, r_i^{-1}d)$ converge in the Gromov–Hausdorff topology to the unit ball in some k -dimensional normed space. This is a quasi-isometry invariant, and if X is a geodesic metric space satisfying (CI_1) , then the asymptotic rank is at most 1 and only if X is Gromov hyperbolic [40]. If X is a cocompact CAT(0) space or a cocompact space with a conical geodesic bicombing, then the asymptotic rank equals the maximal dimension of an isometrically embedded Euclidean or normed space, respectively [10, 27].

Let $S \in \mathbf{I}_{n,c}(X)$. For constants $C \geq 1$ and $a \geq 0$, we say that S has (C, a) -controlled density if for all $x \in X$ and $r > a$, the piece of S in the closed r -ball at x has mass at most Cr^n , or

$$\Theta_{x,r}(S) := \frac{1}{r^n} \mathbf{M}(S \llcorner B_x(r)) \leq C.$$

For almost every $r > 0$, the boundary $\partial(S \llcorner B_x(r))$ has finite mass and $S \llcorner B_x(r)$ is itself an element of $\mathbf{I}_{n,c}(X)$ (see again Sect. 2). Suppose that $Y \subset X$ is a closed set containing the support $\text{spt}(\partial S)$ of ∂S . For $Q \geq 1$ and $a \geq 0$, we say that S is (Q, a) -quasi-minimizing mod Y if for every point $x \in \text{spt}(S)$ at a distance $b > a$ from Y , the inequality

$$\mathbf{M}(S \llcorner B_x(r)) \leq Q \mathbf{M}(T)$$

holds for almost all $r \in (a, b)$ and all $T \in \mathbf{I}_{n,c}(X)$ with $\partial T = \partial(S \lfloor B_x(r))$. A current $S \in \mathbf{I}_{n,c}(X)$ is called a (Q, a) -quasi-minimizer if S is (Q, a) -quasi-minimizing mod $\text{spt}(\partial S)$. As for the analogy with quasi-geodesics, one can easily check that for a Lipschitz quasi-isometric embedding $\gamma: [0, l] \rightarrow X$ of an interval, the associated current $\gamma_{\#}[[0, l]] \in \mathbf{I}_{1,c}(X)$ is a quasi-minimizer with controlled density (compare the case $n = 1$ of Proposition 6.1). For $n > 1$, quasi-minimizers offer more flexibility than quasiflats.

We can now state the main result of this paper. Detailed comments and references are given below.

Theorem 1.1 *Suppose that X is a proper metric space satisfying condition (CI_n) for some $n \geq 1$. Then the following six properties are equivalent:*

- (AR_n) (asymptotic rank) *the asymptotic rank of X is at most n ;*
- (SII_n) (sub-Euclidean isoperimetric inequality) *for all $\epsilon > 0$ there is a constant $M_0 > 0$ such that every cycle $Z \in \mathbf{I}_{n,c}(X)$ is the boundary of a $V \in \mathbf{I}_{n+1,c}(X)$ with mass $\mathbf{M}(V) < \epsilon \max\{M_0, \mathbf{M}(Z)\}^{(n+1)/n}$;*
- (LII_n) (linear isoperimetric inequality) *there is a constant $\nu > 0$, and for all $C > 0$ there is a $\lambda > 0$, such that every cycle $Z \in \mathbf{I}_{n,c}(X)$ with (C, a) -controlled density bounds a $V \in \mathbf{I}_{n+1,c}(X)$ with $\mathbf{M}(V) \leq \max\{\lambda, \nu a\} \mathbf{M}(Z)$;*
- (ML_n) (Morse lemma) *for all $C > 0$ and $Q \geq 1$ there is a constant $l \geq 0$ such that if $Z \in \mathbf{I}_{n,c}(X)$ is a cycle with (C, a) -controlled density and $Y \subset X$ is a closed set such that Z is (Q, a) -quasi-minimizing mod Y , then $\text{spt}(Z)$ is within distance at most $\max\{l, 4a\}$ from Y ;*
- (SS_n) (slim simplices) *for all $L \geq 1$ there is a constant $D \geq 0$ such that if Δ is a Euclidean $(n + 1)$ -simplex and $f: \partial\Delta \rightarrow X$ is a map whose restriction to each facet of Δ is an (L, a) -quasi-isometric embedding, then the image of every facet is within distance at most $D(1 + a)$ of the union of the images of the remaining ones;*
- (FR_n) (filling radius) *for all $C > 0$ there is a constant $h > 0$ such that every cycle $Z \in \mathbf{I}_{n,c}(X)$ with (C, a) -controlled density bounds a $V \in \mathbf{I}_{n+1,c}(X)$ whose support is within distance at most $\max\{h, a\}$ from $\text{spt}(Z)$.*

Note that for $n = 1$, (SII_n) corresponds to a sub-quadratic inequality. The equivalence of (AR_n) and (SII_n) was established in [40] in a more general setup for complete metric spaces. The proof of the forward implication used an elaborate thick-thin decomposition for integral cycles from [39] and also the non-trivial fact that a weakly convergent sequence of cycles converges with respect to the filling volume [37]. We review the entire argument. Employing an elegant new variational result from [23] and introducing a more quantitative approach for the weak convergence of cycles, we reduce the overall complexity substantially. In fact, we prove the sub-Euclidean isoperimetric inequality first in a somewhat restricted form (Theorem 5.1, compare Theorem 4.4 in [28]) and then deduce (LII_n) and (SII_n).

The implication (AR_n) \Rightarrow (LII_n) pertains to a long-standing open problem. In symmetric spaces of non-compact type or homogeneous Hadamard manifolds of rank $\leq n$, a linear isoperimetric inequality holds for all n -cycles (see [21, p. 105], [31], and [25]). It is still unknown whether this generalizes, for instance, to cocompact Hadamard manifolds or CAT(0) spaces of (asymptotic) rank $\leq n$. In our statement, the isoperimetric constant depends on the density bound, so Theorem 1.1 does not resolve this question. Nevertheless, (LII_n) turns out to be equivalent to the remaining properties. Examples of cycles with controlled density include cycles lying near the union of finitely many quasiflats (see in particular the proof of Theorem 7.2).

The proofs of Theorem 5.1 and Theorem 5.2 in [28] show that (SII_n) \Rightarrow (ML_n) \Rightarrow (SS_n), except for a less explicit distance bound in the slim simplex property. The second step involves

an approximation result for quasiflats by quasi-minimizers. We go through the argument in detail, keeping track of the dependence of constants, and thus showing that the bound is linear in the coarseness parameter a . This fact is used in the proof of the backward implication $(SS_n) \Rightarrow (AR_n)$.

The statement of (ML_n) differs formally from the usual stability assertion for quasi-geodesics, but is versatile. If $S \in \mathbf{I}_{n,c}(X)$ with $\text{spt}(\partial S) \subset Y$ is quasi-minimizing mod Y , then the extra assumption we need in order to conclude that S is confined to a bounded neighborhood of Y is that S can be closed up to a cycle $Z = S - S'$ with controlled density and with $\text{spt}(S') \subset Y$. Note that there is no (quasi-)minimality assumption on S' ; the density bound suffices. However, if $S_1, S_2 \in \mathbf{I}_{n,c}(X)$ are two (Q, a) -quasi-minimizers with $\partial S_1 = \partial S_2$, each with (C, a) -controlled density, then S_1 is (Q, a) -quasi-minimizing mod $\text{spt}(S_2)$ and vice-versa, so (ML_n) implies that the Hausdorff distance between the supports is bounded by $\max\{l, 4a\}$ for $l = l(2C, Q)$.

The last assertion of Theorem 1.1 is yet another way of expressing that n -cycles with controlled density (such as geodesic triangles if $n = 1$) are thin. We use an iterative application of the sub-Euclidean isoperimetric inequality to show that the conclusion of (FR_n) holds for every mass minimizing V with $\partial V = Z$. In [40], the filling radius was used to prove that $(SII_n) \Rightarrow (AR_n)$. Similarly, $(FR_n) \Rightarrow (AR_n)$.

The paper is organized as follows. In Sect. 2 we recall the definition of metric currents and collect some basic results. In Sect. 3 we first review an approximation result for cycles from [23] and then use this to give a short proof of a quantitative version of a result from [37], showing that cycles with bounded mass and sufficiently small uniformly bounded density at some fixed scale have small filling volume. We use this further in Sect. 4 to discuss the convergence of cycles. Section 5 is then devoted to isoperimetric inequalities and shows in particular that

$$(AR_n) \Rightarrow (LII_n) \Leftrightarrow (SII_n).$$

In Sect. 6 we prove more explicit versions of two propositions from [28] relating quasiflats and quasi-minimizers. The concluding Sect. 7 then shows in particular that

$$(SII_n) \Rightarrow (ML_n) \Rightarrow (SS_n) \Rightarrow (AR_n) \quad \text{and} \quad (SII_n) \Rightarrow (FR_n) \Rightarrow (AR_n).$$

In fact, we prove all implications (and hence Theorem 1.1) in a stronger form, for any class of proper metric spaces satisfying the respective assumptions uniformly, and with constants depending only on the data involved and on the class, rather than on individual members (see Sects. 5 and 7).

What is missing from the list in Theorem 1.1 is a rank n analog of Gromov’s quadruple definition of δ -hyperbolicity [20, p. 89]. A $2(n + 1)$ -point condition of this type is investigated in [26].

2 Preliminaries

Currents with finite mass in complete metric spaces were introduced by Ambrosio and Kirchheim in [1]. Here, for consistency with [28], we will use the local theory described in [29]. For the class of integral currents with compact support, the principal objects in this paper, the formal difference between the two approaches is marginal. Assuming the underlying metric space to be proper, we will make frequent use of the existence of area minimizing integral currents filling a given cycle (Theorem 2.3). Without this assumption, as an additional twist,

one could still work with almost minimal currents instead (see, for example, Lemma 3.4 in [36]). In particular, Theorem 1.1 and most results in the paper hold more generally for complete metric spaces and Ambrosio–Kirchheim integral currents.

Currents

An integral n -current may roughly be thought of as an oriented n -dimensional Lipschitz surface equipped with a summable integer density function. Formally though, n -currents are defined as functionals; on compactly supported differential n -forms in the classical case (going back to de Rham), and on suitable $(n + 1)$ -tuples of real-valued locally Lipschitz functions for non-smooth ambient spaces. The relating principle (originally proposed by De Giorgi) is that the tuple (f_0, \dots, f_n) , say if the f_i are smooth functions on \mathbb{R}^N , represents the form $f_0 df_1 \wedge \dots \wedge df_n$.

Let $X = (X, d)$ be a proper metric space. For $n \geq 0$, we let $\mathcal{D}^n(X)$ denote the set of all $(n + 1)$ -tuples (f_0, \dots, f_n) of Lipschitz functions $f_i : X \rightarrow \mathbb{R}$ such that f_0 has compact support $\text{spt}(f_0)$ (in [29], f_1, \dots, f_n are merely locally Lipschitz, but the following definition is equivalent). An n -dimensional current S in X is a function $S : \mathcal{D}^n(X) \rightarrow \mathbb{R}$ satisfying the following three conditions:

- (1) S is $(n + 1)$ -linear;
- (2) $S(f_{0,k}, \dots, f_{n,k}) \rightarrow S(f_0, \dots, f_n)$ whenever $f_{i,k} \rightarrow f_i$ pointwise on X with uniformly bounded Lipschitz constants ($i = 0, \dots, n$) and with $\bigcup_k \text{spt}(f_{0,k}) \subset K$ for some compact set $K \subset X$;
- (3) $S(f_0, \dots, f_n) = 0$ whenever one of the functions f_1, \dots, f_n is constant on a neighborhood of $\text{spt}(f_0)$.

It follows from these axioms that S is alternating in the last n arguments. The vector space of all n -dimensional currents in X is denoted $\mathcal{D}_n(X)$. Every function $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ induces a current $\llbracket w \rrbracket \in \mathcal{D}_n(\mathbb{R}^n)$ defined by

$$\llbracket w \rrbracket(f_0, \dots, f_n) := \int w f_0 \det[\partial_j f_i]_{i,j=1}^n dx;$$

note that the partial derivatives $\partial_j f_i$ exist almost everywhere by Rademacher’s theorem. For a Borel set $W \subset \mathbb{R}^n$ we put $\llbracket W \rrbracket := \llbracket \chi_W \rrbracket$, where χ_W denotes the characteristic function. (See Sect. 2 in [29] for details.)

Support, push-forward, and boundary

Let $S \in \mathcal{D}_n(X)$. There exists a smallest closed subset of X , the *support* $\text{spt}(S)$ of S , such that the value $S(f_0, \dots, f_n)$ depends only on the restrictions of f_0, \dots, f_n to this set. Thus, for any closed set $D \subset X$ containing $\text{spt}(S)$, S induces a current in $\mathcal{D}_n(D)$, still denoted by S . For a proper Lipschitz map $\phi : D \rightarrow Y$ into another proper metric space Y , the *push-forward* $\phi_{\#}S \in \mathcal{D}_n(Y)$ is the current with support in $\phi(\text{spt}(S))$ defined by

$$(\phi_{\#}S)(f_0, \dots, f_n) := S(f_0 \circ \phi, \dots, f_n \circ \phi)$$

for all $(f_0, \dots, f_n) \in \mathcal{D}^n(Y)$. In the simplest case, if $\llbracket [a, b] \rrbracket := \llbracket [a, b] \rrbracket$ is the current in $\mathcal{D}_1(\mathbb{R})$ (or $\mathcal{D}_1([a, b])$) associated with an interval, and if $\gamma : [a, b] \rightarrow X$ is a Lipschitz curve,

then

$$\gamma_{\#} \llbracket a, b \rrbracket (f_0, f_1) = \llbracket a, b \rrbracket (f_0 \circ \gamma, f_1 \circ \gamma) = \int_a^b (f_0 \circ \gamma)(f_1 \circ \gamma)' ds$$

for all $(f_0, f_1) \in \mathcal{D}^1(X)$. Similarly, every singular Lipschitz n -chain in X defines an element of $\mathcal{D}_n(X)$; in fact, of $\mathbf{I}_{n,c}(X)$ (see below for the definition, and [2, 17] for some reverse approximation results).

If $S \in \mathcal{D}_n(X)$ and $n \geq 1$, then the *boundary* $\partial S \in \mathcal{D}_{n-1}(X)$ is defined by

$$(\partial S)(f_0, \dots, f_{n-1}) := S(\tau, f_0, \dots, f_{n-1})$$

for all $(f_0, \dots, f_{n-1}) \in \mathcal{D}^{n-1}(X)$ and for any $\tau \in \mathcal{D}^0(X)$ such that $\tau \equiv 1$ in a neighborhood of $\text{spt}(f_0)$. It follows from (1) and (3) that ∂S is well-defined and that $\partial \circ \partial = 0$. Furthermore, $\text{spt}(\partial S) \subset \text{spt}(S)$, and $\phi_{\#}(\partial S) = \partial(\phi_{\#}S)$ for $\phi: D \rightarrow Y$ as above. In the example of a Lipschitz curve, $\partial(\gamma_{\#} \llbracket a, b \rrbracket)(f_0) = f_0(\gamma(b)) - f_0(\gamma(a))$ by the fundamental theorem of calculus. (See Sect. 3 in [29].)

Mass

Let $S \in \mathcal{D}_n(X)$. For an open set $U \subset X$, the *mass* $\|S\|(U) \in [0, \infty]$ of S in U is defined as the supremum of $\sum_k S(f_{0,k}, \dots, f_{n,k})$ over all finite families of tuples $(f_{0,k}, \dots, f_{n,k}) \in \mathcal{D}^n(X)$ such that $\bigcup_k \text{spt}(f_{0,k}) \subset U$, $\sum_k |f_{0,k}| \leq 1$, and $f_{1,k}, \dots, f_{n,k}$ are 1-Lipschitz. This extends to a regular Borel measure $\|S\|$ on X with $\text{spt}(\|S\|) = \text{spt}(S)$, and $\mathbf{M}(S) := \|S\|(X)$ denotes the *total mass*. For Borel sets $W, A \subset \mathbb{R}^n$, $\|\llbracket W \rrbracket\|(A)$ equals the Lebesgue measure of $W \cap A$. For $S, T \in \mathcal{D}_n(X)$,

$$\|S + T\| \leq \|S\| + \|T\|.$$

If the measure $\|S\|$ is locally finite, then

$$|S(f_0, \dots, f_n)| \leq \prod_{i=1}^n \text{Lip}(f_i) \int_X |f_0| d\|S\|$$

for all $(f_0, \dots, f_n) \in \mathcal{D}^n(X)$, where $\text{Lip}(f_i)$ denotes the Lipschitz constant. As a consequence, S extends to tuples whose first entry is merely a bounded Borel function with compact support, and if $u: X \rightarrow \mathbb{R}$ is any bounded Borel function, one can define the *restriction* $S \llcorner u \in \mathcal{D}_n(X)$ by

$$(S \llcorner u)(f_0, \dots, f_n) := S(uf_0, f_1, \dots, f_n)$$

for all $(f_0, \dots, f_n) \in \mathcal{D}^n(X)$. For a Borel set $A \subset X$, $S \llcorner A := S \llcorner \chi_A$. The measure $\|S \llcorner A\|$ agrees with the restriction of $\|S\|$ to A . If $\phi: D \rightarrow Y$ is as above, and $B \subset Y$ is a Borel set, then $(\phi_{\#}S) \llcorner B = \phi_{\#}(S \llcorner \phi^{-1}(B))$ and

$$\|\phi_{\#}S\|(B) \leq \text{Lip}(\phi)^n \|S\|(\phi^{-1}(B)).$$

(See Sect. 4 in [29].)

Integral currents

A current $S \in \mathcal{D}_n(X)$ is *locally integer rectifiable* if $\|S\|$ is locally finite and concentrated on the union of countably many Lipschitz images of compact subsets of \mathbb{R}^n , and for every

Borel set $A \subset X$ with compact closure and every Lipschitz map $\phi: \bar{A} \rightarrow \mathbb{R}^n$, the current $\phi_{\#}(S \llcorner A) \in \mathcal{D}_n(\mathbb{R}^n)$ is of the form $\llbracket w \rrbracket$ for some integer valued $w \in L^1(\mathbb{R}^n)$. Then $\|S\|$ is absolutely continuous with respect to n -dimensional Hausdorff measure. Push-forwards and restrictions to Borel sets of locally integer rectifiable currents are again locally integer rectifiable.

A current $S \in \mathcal{D}_n(X)$ is called a *locally integral current* if S is locally integer rectifiable and, for $n \geq 1$, $\|\partial S\|$ is locally finite; then (by Theorem 8.7 in [29]) ∂S is itself locally integer rectifiable. This gives a chain complex of abelian groups $\mathbf{I}_{n,\text{loc}}(X)$. The subgroups $\mathbf{I}_{n,c}(X)$ of *integral currents* consist of the elements with compact support and, hence, finite total mass. For $X = \mathbb{R}^N$, there is a canonical chain isomorphism from $\mathbf{I}_{*,c}(\mathbb{R}^N)$ to the chain complex of classical (Federer–Fleming) integral currents [15] in \mathbb{R}^N .

For $n \geq 1$, we put $\mathbf{Z}_{n,c}(X) := \{Z \in \mathbf{I}_{n,c}(X) : \partial Z = 0\}$. For $n = 0$, an element of $\mathbf{I}_{0,c}(X)$ is an integral linear combination of currents of the form $\llbracket x \rrbracket$, where $\llbracket x \rrbracket(f_0) = f_0(x)$. We let $\mathbf{Z}_{0,c}(X) \subset \mathbf{I}_{0,c}(X)$ denote the subgroup of linear combinations whose coefficients add up to zero. The boundary of a current in $\mathbf{I}_{1,c}(X)$ belongs to $\mathbf{Z}_{0,c}(X)$. Given $Z \in \mathbf{Z}_{n,c}(X)$, for $n \geq 0$, we will call $V \in \mathbf{I}_{n+1,c}(X)$ a *filling* of Z if $\partial V = Z$.

Slicing

Let $S \in \mathbf{I}_{n,\text{loc}}(X)$, $n \geq 1$, and let $\pi: X \rightarrow \mathbb{R}$ be a Lipschitz function. For $s \in \mathbb{R}$, the *slice* $T_s \in \mathcal{D}_{n-1}(X)$ of S with respect to π is the current

$$T_s := \partial(S \llcorner \{\pi \leq s\}) - (\partial S) \llcorner \{\pi \leq s\}$$

with support in $\{\pi = s\} \cap \text{spt}(S)$. Note that the restrictions are defined since both $\|S\|$ and $\|\partial S\|$ are locally finite. For $a < b$, the coarea inequality

$$\int_a^b \mathbf{M}(T_s) ds \leq \text{Lip}(\pi) \|S\|(\{a < \pi < b\})$$

holds, and if $\pi|_{\text{spt}(S)}$ is proper, then $T_s \in \mathbf{I}_{n-1,c}(X)$ for almost all $s \in \mathbb{R}$. (See Sect. 6 and Theorem 8.5 in [29].)

Convergence and compactness

A sequence (S_i) in $\mathcal{D}_n(X)$ *converges weakly* to a current $S \in \mathcal{D}_n(X)$ if $S_i \rightarrow S$ pointwise as functionals on $\mathcal{D}^n(X)$. Then, for every open set $U \subset X$,

$$\|S\|(U) \leq \liminf_{i \rightarrow \infty} \|S_i\|(U),$$

thus the mass is lower semicontinuous with respect to weak convergence. Furthermore, weak convergence commutes with the boundary operator and with push-forwards. For locally integral currents, the following compactness theorem holds (see Theorem 8.10 in [29]).

Theorem 2.1 *Let X be a proper metric space, and let $n \geq 1$. If (S_i) is a sequence in $\mathbf{I}_{n,\text{loc}}(X)$ such that*

$$\sup_i (\|S_i\| + \|\partial S_i\|)(K) < \infty$$

for every compact set $K \subset X$, then some subsequence (S_{i_k}) converges weakly to a current $S \in \mathbf{I}_{n,\text{loc}}(X)$.

Isoperimetric inequality and Plateau problem

Recall condition (CI_n) from the introduction. Cone inequalities are instrumental for the proof of isoperimetric inequalities of Euclidean type (compare Sect. 3.4 in [19]). For $n \geq 1$, we say that X satisfies (EII_n) if there is a constant $\gamma > 0$ such that every cycle $Z \in \mathbf{Z}_{n,c}(X)$ has a filling $V \in \mathbf{I}_{n+1,c}(X)$ with mass

$$\mathbf{M}(V) \leq \gamma \mathbf{M}(Z)^{(n+1)/n}.$$

To make the constants in (CI_n) and (EII_n) explicit, we will write $(CI_n)[c]$ and $(EII_n)[\gamma]$. The following result was established in a more general form in Theorem 1.2 in [36].

Theorem 2.2 *For all $n \geq 1$ and $c > 0$ there is a constant $\gamma > 0$ such that for every proper metric space X , $(CI_n)[c]$ implies $(EII_n)[\gamma]$.*

(Here the quasi-convexity condition (CI_0) is actually not needed.) By Theorem 2.1 and a well-known application of (EII_n) one gets the following existence result for minimizing integral currents (see the proof of Theorem 2.4 in [28]).

Theorem 2.3 *Let $Z \in \mathbf{Z}_{n,c}(X)$, where X is a proper metric space satisfying (CI_0) if $n = 0$ and $(EII_n)[\gamma]$ if $n \geq 1$. Then there is a filling $V \in \mathbf{I}_{n+1,loc}(X)$ of Z with mass*

$$\mathbf{M}(V) = \inf\{\mathbf{M}(V') : V' \in \mathbf{I}_{n+1,loc}(X), \partial V' = Z\} < \infty.$$

In fact, every such minimizing V has compact support due to the following lower density bound: if $x \in \text{spt}(V)$, $r > 0$, and $B_x(r) \cap \text{spt}(Z) = \emptyset$, then

$$\Theta_{x,r}(V) := \frac{\|V\|(B_x(r))}{r^{n+1}} \geq \delta_0 := \begin{cases} 2 & \text{if } n = 0, \\ (n + 1)^{-(n+1)} \gamma^{-n} & \text{if } n \geq 1; \end{cases}$$

thus $\text{spt}(V)$ is within distance $(\mathbf{M}(V)/\delta_0)^{1/(n+1)}$ from $\text{spt}(Z)$.

3 A variational argument

We start with a slight modification and extension of an effective recent approximation result, Proposition 4.2 in [23]. The main conclusion is that for a cycle Z and any $\eta > 0$ there is a cycle Z' with mass $\leq \mathbf{M}(Z)$ such that $Z - Z'$ has a minimizing filling with mass $\leq \eta \mathbf{M}(Z)$ and Z' satisfies a uniform lower density bound at scales $\lesssim \eta$. This will be used in the proofs of Theorems 3.2 and 5.1. We show in addition that if Z satisfies a uniform upper density bound above some threshold radius, then the same holds for Z' ; see assertion (5) below. This will be employed in Theorem 5.3 (linear isoperimetric inequality).

Proposition 3.1 *Let $n \geq 1$ and $\gamma > 0$. Suppose that X is a proper metric space satisfying $(EII_n)[\gamma]$ and, if $n \geq 2$, also $(EII_{n-1})[\gamma]$. Then for every $Z \in \mathbf{Z}_{n,c}(X)$ and $\eta > 0$ there exists a minimizing $V \in \mathbf{I}_{n+1,c}(X)$ such that the following holds for*

$$Z' := Z - \partial V, \quad \mu := \eta^{-1} \|V\| + \|Z'\|,$$

and some constants $\alpha, \theta > 0$ depending only on n and γ :

- (1) $\mu(X) \leq \mathbf{M}(Z)$, in particular $\mathbf{M}(Z') \leq \mathbf{M}(Z)$ and $\mathbf{M}(V) \leq \eta \mathbf{M}(Z)$;
- (2) $\Theta_{x,r}(Z') \geq \theta$ for all $x \in \text{spt}(Z')$ and $r \in (0, \alpha\eta]$;

- (3) if $B \subset X$ is a closed set and $T := \partial(V \llcorner B) - (\partial V) \llcorner B$ is in $\mathbf{I}_{n,c}(X)$, then $\mu(B) \leq \|Z\|(B) + \mathbf{M}(T)$;
- (4) if $\mathbf{M}(Z) < m := \theta(\alpha\eta)^n$, then $Z' = 0$, and if $\mathbf{M}(Z) \geq m$, then $\text{spt}(Z')$ is within distance at most $\eta(\alpha + \ln(\mathbf{M}(Z)/m))$ from $\text{spt}(Z)$;
- (5) if there exist $C > 0$, $a \geq 0$, and $p \in X$ such that $\Theta_{p,r}(Z) \leq C$ for all $r > a$, then $\mu(B_p(r)) \leq 2^{n+1}Cr^n$ for all $r > \max\{a, 2^{n+1}\eta\}$.

Proof Given $Z \in \mathbf{Z}_{n,c}(X)$ and $\eta > 0$, consider the functional

$$F : \mathbf{I}_{n+1,\text{loc}}(X) \rightarrow [0, \infty], \quad F(V') = \eta^{-1}\mathbf{M}(V') + \mathbf{M}(Z - \partial V').$$

Notice that F is lower semicontinuous with respect to weak convergence, like \mathbf{M} . Moreover, $\mathbf{M}(V') \leq \eta F(V')$ and $\mathbf{M}(\partial V') \leq F(V') + \mathbf{M}(Z)$ for all V' , and $F(0) = \mathbf{M}(Z) < \infty$. We can thus pick a minimizing sequence for F and use Theorem 2.1 to find a $V \in \mathbf{I}_{n+1,\text{loc}}(X)$ that minimizes F . Now if $Z' := Z - \partial V$ and $\mu := \eta^{-1}\|V\| + \|Z'\|$, then

$$\mu(X) = \eta^{-1}\mathbf{M}(V) + \mathbf{M}(Z') = F(V) \leq F(0) = \mathbf{M}(Z),$$

so (1) holds. However, we still have to show that in fact $V \in \mathbf{I}_{n+1,c}(X)$.

We proceed with (2). Let $x \in \text{spt}(Z')$. Put $f(s) := \|Z'\|(B_x(s)) > 0$ for all $s > 0$. For almost every s , the slice $R_s := \partial(Z' \llcorner B_x(s))$ is in $\mathbf{Z}_{n-1,c}(X)$ and satisfies $\mathbf{M}(R_s) \leq f'(s)$. Suppose first that $n \geq 2$. Then by the isoperimetric inequality there exists a filling $T_s \in \mathbf{I}_{n,c}(X)$ of R_s such that

$$\mathbf{M}(T_s) \leq \gamma \mathbf{M}(R_s)^{n/(n-1)} \leq \gamma f'(s)^{n/(n-1)}.$$

Furthermore, the cycle $Z' \llcorner B_x(s) - T_s$ has a filling $W_s \in \mathbf{I}_{n+1,c}(X)$ with

$$\mathbf{M}(W_s) \leq \gamma(f(s) + \mathbf{M}(T_s))^{(n+1)/n}.$$

Since $Z - \partial(V + W_s) = Z' - \partial W_s = Z' \llcorner (X \setminus B_x(s)) + T_s$, we have

$$\begin{aligned} F(V + W_s) &= \eta^{-1}\mathbf{M}(V + W_s) + \mathbf{M}(Z' \llcorner (X \setminus B_x(s)) + T_s) \\ &\leq \eta^{-1}(\mathbf{M}(V) + \mathbf{M}(W_s)) + \|Z'\|(X \setminus B_x(s)) + \mathbf{M}(T_s). \end{aligned}$$

It follows that $0 \leq F(V + W_s) - F(V) \leq \eta^{-1}\mathbf{M}(W_s) - f(s) + \mathbf{M}(T_s)$ and

$$\eta(f(s) - \mathbf{M}(T_s)) \leq \mathbf{M}(W_s) \leq \gamma(f(s) + \mathbf{M}(T_s))^{(n+1)/n}.$$

Hence, if $\mathbf{M}(T_s) \leq f(s)/2$, then $\eta f(s)/2 \leq \gamma(3f(s)/2)^{(n+1)/n}$ and thus

$$f(s) \geq \theta' \eta^n$$

for $\theta' := 2/(3^{n+1}\gamma^n)$. Now if $f(r) < \theta' \eta^n$ for some r , then $f(s) < \theta' \eta^n$ for all $s \in (0, r)$, thus $f(s)/2 < \mathbf{M}(T_s) \leq \gamma f'(s)^{n/(n-1)}$ and

$$f'(s) f(s)^{(1-n)/n} \geq (2\gamma)^{(1-n)/n}$$

for almost every such s , and integration from 0 to r yields

$$f(r) \geq \theta r^n$$

where $\theta := n^{-n}(2\gamma)^{1-n}$. This shows that $f(r) \geq \min\{\theta' \eta^n, \theta r^n\}$ for all $r > 0$. Hence (2) holds with $\alpha := (\theta'/\theta)^{1/n}$ in case $n \geq 2$. Suppose now that $n = 1$. Then every non-zero slice $R_s \in \mathbf{Z}_{0,c}(X)$ has mass at least 2. If $R_s = 0$ for some s , then $Z' \llcorner B_x(s)$ is a cycle, and repeating the above argument with $T_s = 0$ we get that $\eta f(s) \leq \mathbf{M}(W_s) \leq \gamma f(s)^2$, thus

$f(s) \geq \eta/\gamma$. Hence, if $f(r) < \eta/\gamma$ for some r , then $f'(s) \geq \mathbf{M}(R_s) \geq 2$ for almost every $s \in (0, r)$ and so $f(r) \geq 2r$. We conclude that in case $n = 1$, (2) holds with $\theta := 2$ and $\alpha := 1/(2\gamma)$.

Since $\mathbf{M}(Z') < \infty$, it now follows from (2) that Z' has compact support. Hence $\text{spt}(\partial V)$ is compact, and since V has finite mass and is evidently minimizing, by Theorem 2.3 $\text{spt}(V)$ is compact as well.

We prove (3). Let $W := V \llcorner B$. If $T = \partial W - (Z - Z') \llcorner B \in \mathbf{I}_{n,c}(X)$, then also $(Z - Z') \llcorner B \in \mathbf{I}_{n,c}(X)$ and $W \in \mathbf{I}_{n+1,c}(X)$. Since $Z - \partial(V - W) = Z' + \partial W = Z' \llcorner (X \setminus B) + Z \llcorner B + T$, we have

$$\begin{aligned} F(V - W) &= \eta^{-1} \mathbf{M}(V - W) + \mathbf{M}(Z' \llcorner (X \setminus B) + Z \llcorner B + T) \\ &\leq \mu(X \setminus B) + \|Z\|(B) + \mathbf{M}(T). \end{aligned}$$

Since $\mu(X) = F(V) \leq F(V - W)$, it follows that $\mu(B) = \mu(X) - \mu(X \setminus B) \leq \|Z\|(B) + \mathbf{M}(T)$ as claimed.

The first assertion of (4) is clear from (1) and (2). Suppose now that $\mathbf{M}(Z) \geq m = \theta(\alpha\eta)^n$ and $x \in \text{spt}(Z')$ is a point at distance $D > \alpha\eta$ from $\text{spt}(Z)$. Set $g(s) := \mu(B_x(s))$ for all $s \in (0, D)$. For almost every such s , the slice $T_s := \partial(V \llcorner B_x(s)) + Z' \llcorner B_x(s)$ is in $\mathbf{I}_{n,c}(X)$ and satisfies $\mathbf{M}(T_s) \leq \frac{d}{ds} \|V\|(B_x(s))$ as well as $\mathbf{M}(\partial T_s) \leq \frac{d}{ds} \|Z'\|(B_x(s))$. Hence, by (3),

$$g(s) \leq \mathbf{M}(T_s) \leq \mathbf{M}(T_s) + \eta \mathbf{M}(\partial T_s) \leq \eta g'(s).$$

Integrating the inequality $1 \leq \eta g'(s)/g(s)$ from $\alpha\eta$ to $t < D$ we get that

$$t \leq \eta(\alpha + \ln(g(t)) - \ln(g(\alpha\eta))).$$

By (1) and (2), $g(t) \leq \mu(X) \leq \mathbf{M}(Z)$ and $g(\alpha\eta) \geq \|Z'\|(B_x(\alpha\eta)) \geq m$. As this holds for all $t < D$, (4) follows.

It remains to prove (5). We will write B_r for $B_p(r)$. First we choose a sufficiently large $r_0 > 0$ so that

$$\|V\|(B_{r_0}) \leq 2^{n+1} \eta C r_0^n,$$

then we put $r_i := 2^{-i} r_0$ for every integer $i \geq 1$. There exists an $s \in (r_1, r_0)$ such that the slice $T_s := \partial(V \llcorner B_s) - (\partial V) \llcorner B_s \in \mathbf{I}_{n,c}(X)$ satisfies

$$\mu(B_s) \leq \|Z\|(B_s) + \mathbf{M}(T_s)$$

by (3), as well as $\mathbf{M}(T_s) \leq \|V\|(B_{r_0})/(r_0 - r_1) \leq 2^{n+1} \eta C r_0^n / r_1$. Now if $r_1 > \max\{a, 2^{n+1} \eta\}$, then $\|Z\|(B_s) \leq C s^n$ and $\mathbf{M}(T_s) \leq C r_0^n$, hence

$$\mu(B_{r_1}) \leq \mu(B_s) \leq 2C r_0^n = 2^{n+1} C r_1^n.$$

This also yields the above inequality for the next smaller scale,

$$\|V\|(B_{r_1}) \leq 2^{n+1} \eta C r_1^n.$$

Finally, given any $r > \max\{a, 2^{n+1} \eta\}$, we can choose r_0 initially such that $r = r_k = 2^{-k} r_0$ for some $k \geq 1$. When $k \geq 2$, we repeat the above slicing argument successively for $i = 2, \dots, k$, with (r_i, r_{i-1}) in place of (r_1, r_0) . This eventually shows that

$$\mu(B_r) \leq 2^{n+1} C r^n$$

for any $r > \max\{a, 2^{n+1} \eta\}$. □

As a first application of parts (1)–(3) of Proposition 3.1 we give a short proof of a variant of Proposition 5.8 in [37] regarding fillings of thin cycles. This result will play a key role in the next section (see Theorem 4.4). The assumptions on X are as above. The conclusion is that for a cycle $Z \in \mathbf{Z}_{n,c}(X)$, the *filling volume*

$$\text{FillVol}_X(Z) := \inf\{\mathbf{M}(V) : V \in \mathbf{I}_{n+1,c}(X), \partial V = Z\}$$

can be forced to be arbitrarily small by imposing a sufficiently small bound on the supremal mass

$$m_\varrho(Z) := \sup_{x \in X} \|Z\|(B_x(\varrho))$$

in all closed balls of some fixed radius $\varrho > 0$. The proof gives an explicit constant involving a mass bound for Z .

Theorem 3.2 *For all $n \geq 1$ and $\gamma, M, \varrho, \nu > 0$ there exists a constant $\delta = \delta(n, \gamma, M, \varrho, \nu) > 0$ such that if X is a proper metric space satisfying $(\text{EII}_n)[\gamma]$ and, in case $n \geq 2$, also $(\text{EII}_{n-1})[\gamma]$, then every $Z \in \mathbf{Z}_{n,c}(X)$ with $\mathbf{M}(Z) \leq M$ and $m_\varrho(Z) \leq \delta$ has $\text{FillVol}_X(Z) < \nu$.*

Proof Let α and θ be the constants from Proposition 3.1, depending on n and γ . Given $M, \varrho, \nu > 0$, fix $\eta > 0$ such that both ηM and $\gamma(4\eta M/\varrho)^{(n+1)/n}$ are less than $\nu/2$, then put

$$r := \min\left\{\alpha\eta, \frac{\varrho}{4}\right\}, \quad \delta := \frac{1}{2}\theta r^n.$$

Suppose now that $Z \in \mathbf{Z}_{n,c}(X)$ satisfies $\mathbf{M}(Z) \leq M$ and $m_\varrho(Z) \leq \delta$. By Proposition 3.1 there exists $V \in \mathbf{I}_{n+1,c}(X)$ such that, for $Z' := Z - \partial V$,

- (1) $\mathbf{M}(V) \leq \eta \mathbf{M}(Z) \leq \eta M < \nu/2$;
- (2) $\|Z'\|(B_x(r)) \geq \theta r^n = 2\delta$ for all $x \in \text{spt}(Z')$;
- (3) $\|Z'\|(B) \leq \|Z\|(B) + \mathbf{M}(T)$ whenever $B \subset X$ is a closed set and $T := \partial(V \llcorner B) - (\partial V) \llcorner B$ is in $\mathbf{I}_{n,c}(X)$.

If $Z' = 0$, then $\partial V = Z$, and (1) yields the result. Now let $Z' \neq 0$. It remains to show that $\text{FillVol}_X(Z') < \nu/2$. Pick a maximal set $N \subset \text{spt}(Z')$ of distinct points at mutual distance $> 2r$, and put $B_s := \bigcup_{x \in N} B_x(s)$ for all $s > 0$. By (2), since the balls $B_x(r)$ with $x \in N$ are pairwise disjoint, we have $2\delta |N| \leq \|Z'\|(B_r) \leq \mathbf{M}(Z')$, and so

$$\|Z\|(B_\varrho) \leq |N| m_\varrho(Z) \leq \delta |N| \leq \frac{1}{2} \mathbf{M}(Z').$$

Furthermore, since N is maximal, $\text{spt}(Z') \subset B_{2r} \subset B_{\varrho/2}$. Hence, for almost every $s \in (\varrho/2, \varrho)$, the slice $T_s := \partial(V \llcorner B_s) - (\partial V) \llcorner B_s \in \mathbf{I}_{n,c}(X)$ satisfies

$$\mathbf{M}(Z') = \|Z'\|(B_s) \leq \|Z\|(B_s) + \mathbf{M}(T_s) \leq \frac{1}{2} \mathbf{M}(Z') + \mathbf{M}(T_s)$$

by (3). We conclude that $\mathbf{M}(T_s) \geq \mathbf{M}(Z')/2$ and

$$\mathbf{M}(V) \geq \|V\|(B_\varrho) \geq \int_{\varrho/2}^\varrho \mathbf{M}(T_s) ds \geq \frac{\varrho}{4} \mathbf{M}(Z').$$

It now follows from (1) that $\mathbf{M}(Z') \leq 4\eta M/\varrho$, and by the isoperimetric inequality and the choice of η we get that $\text{FillVol}_X(Z') < \nu/2$. □

4 Convergence of cycles

A central result in geometric measure theory says that a weakly convergent sequence $S_i \rightarrow S$ of integral n -currents with supports in a fixed compact set and with $\sup_i (\mathbf{M}(S_i) + \mathbf{M}(\partial S_i)) < \infty$ converges in the *flat metric topology*. This means that there exist integral n -currents T_i and integral $(n + 1)$ -currents V_i such that $S_i - S = T_i + \partial V_i$ and $\mathbf{M}(T_i) + \mathbf{M}(V_i) \rightarrow 0$. In \mathbb{R}^N , this property can be deduced from the deformation theorem (see Theorem 5.5 and Theorem 7.1 in [15]). The result was generalized in [37] to Ambrosio–Kirchheim currents in complete metric spaces satisfying condition (CI_n) locally. It essentially suffices to show that $\text{FillVol}_X(Z_i) \rightarrow 0$ for any bounded sequence of cycles Z_i converging weakly to 0. In this section we prove a somewhat amplified version of this fact for proper metric spaces satisfying (CI_n) globally, so as to facilitate the proof of the sub-Euclidean isoperimetric inequality in the next section. The argument relies on Theorem 3.2 and proceeds along the same lines as [37], but is simplified by the use of a uniform notion of weak convergence.

Let $S \in \mathcal{D}_n(X)$, $n \geq 0$, and suppose that $\text{spt}(S)$ is compact. Note that every such S extends canonically to tuples of Lipschitz functions whose first entry is no longer required to have compact support. We define

$$\mathbf{W}(S) := \sup\{S(f_0, \dots, f_n) : f_0, \dots, f_n \text{ are 1-Lipschitz, } |f_0| \leq 1\}.$$

Evidently $\mathbf{W}(S) \leq \mathbf{M}(S)$, and if $n \geq 1$, then $\mathbf{W}(\partial S) \leq \mathbf{W}(S)$.

The following auxiliary result is an adaptation of Proposition 6.6 in [29] to sequences of cycles in possibly distinct proper metric spaces.

Lemma 4.1 *Suppose that $n \geq 1$, $Z_i \in \mathbf{Z}_{n,c}(X_i)$, $\sup_i \mathbf{M}(Z_i) < \infty$, $\mathbf{W}(Z_i) \rightarrow 0$, and $\pi_i : X_i \rightarrow \mathbb{R}$ is 1-Lipschitz. Then for almost every $s \in \mathbb{R}$ there is a sequence (i_k) such that $Z_{i_k,s} := Z_{i_k} \llcorner \{\pi_{i_k} \leq s\} \in \mathbf{I}_{n,c}(X_{i_k})$ and*

$$\sup_k \mathbf{M}(\partial Z_{i_k,s}) < \infty, \quad \mathbf{W}(\partial Z_{i_k,s}) \leq \mathbf{W}(Z_{i_k,s}) \rightarrow 0 \quad (k \rightarrow \infty).$$

Proof Consider the Borel measures $\mu_i := \pi_{i\#} \|Z_i\|$. Since $\sup_i \mu_i(\mathbb{R}) < \infty$, some subsequence (μ_{i_k}) converges weakly to a finite Borel measure μ on \mathbb{R} . Furthermore, for the slices $\partial Z_{i_k,s}$,

$$\int_{\mathbb{R}} \liminf_{k \rightarrow \infty} \mathbf{M}(\partial Z_{i_k,s}) ds \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}} \mathbf{M}(\partial Z_{i_k,s}) ds \leq \sup_i \mathbf{M}(Z_i) < \infty.$$

We now take s so that $\mu(\{s\}) = 0$ and $\liminf_{k \rightarrow \infty} \mathbf{M}(\partial Z_{i_k,s}) < \infty$, then we adjust the sequence (i_k) , if necessary, to arrange that $\sup_k \mathbf{M}(\partial Z_{i_k,s}) < \infty$. Note that $Z_{i_k,s} \in \mathbf{I}_{n,c}(X_{i_k})$ for all k . Let $\epsilon > 0$, and choose $\delta > 0$ such that $\mu([s, s + \delta]) < \epsilon$. Let $\gamma_{s,\delta} : \mathbb{R} \rightarrow \mathbb{R}$ denote the piecewise affine δ^{-1} -Lipschitz function that is 1 on $(-\infty, s]$ and 0 on $[s + \delta, \infty)$, and put $u_k := \gamma_{s,\delta} \circ \pi_{i_k}$ and $v_k := \chi_{(-\infty,s]} \circ \pi_{i_k}$. If k is sufficiently large, then

$$\mathbf{W}(Z_{i_k} \llcorner (u_k - v_k)) \leq \mu_{i_k}([s, s + \delta]) < \epsilon,$$

$$\mathbf{W}(Z_{i_k} \llcorner u_k) \leq (1 + \delta^{-1})\mathbf{W}(Z_{i_k}) < \epsilon$$

(note that if $f_0 : X_{i_k} \rightarrow \mathbb{R}$ is a 1-Lipschitz function with $|f_0| \leq 1$, then $u_k f_0$ is $(1 + \delta^{-1})$ -Lipschitz), thus $\mathbf{W}(Z_{i_k,s}) = \mathbf{W}(Z_{i_k} \llcorner v_k) < 2\epsilon$. This gives the result. \square

Next, recall that a 0-cycle $Z \in \mathbf{Z}_{0,c}(X)$ is of the form $Z(f) = \sum_i a_i f(x_i)$ for finitely many points $x_i \in X$ and weights $a_i \in \mathbb{Z}$ with $\sum_i a_i = 0$. Note that $Z(f + c) = Z(f)$ for all $c \in \mathbb{R}$. We define

$$\mathcal{W}(Z) := \sup\{Z(f) : f \text{ is 1-Lipschitz}\};$$

in general $\mathscr{W}(Z) \geq \mathbf{W}(Z)$. We have the following optimal result for $n = 0$ (compare Theorem 1.2 in [37]).

Proposition 4.2 *Let X be a proper metric space satisfying $(CI_0)[c]$, that is, any two points $x, y \in X$ can be connected by a curve of length $\leq c d(x, y)$. Then every cycle $Z \in \mathbf{Z}_{0,c}(X)$ has $\text{FillVol}_X(Z) \leq c \mathscr{W}(Z)$, and if $\mathbf{W}(Z) < 2$, then $\mathscr{W}(Z) = \mathbf{W}(Z)$.*

Proof We can write $Z \neq 0$ in the form

$$Z(f) = \sum_{i=1}^k (f(x_i) - f(y_i)) = \int_X f \, d\mu - \int_X f \, d\nu$$

for some (not necessarily distinct) points x_1, \dots, x_k and y_1, \dots, y_k in X and the corresponding measures $\mu := \sum_i \delta_{x_i}$ and $\nu := \sum_i \delta_{y_i}$. Hence, by the Kantorovich–Rubinstein theorem (see [12]), $\mathscr{W}(Z)$ equals the Wasserstein distance $W_1(\mu, \nu)$. It is well-known that for such measures the latter agrees with the minimum of $\sum_i d(x_i, y_{\pi(i)})$ over all permutations π of $\{1, \dots, k\}$. Thus, some such sum is equal to $\mathscr{W}(Z)$. We will give an alternative direct proof of this identity in Lemma 4.3 below. It now follows from condition $(CI_0)[c]$ that Z has a filling with mass less than or equal to $c \mathscr{W}(Z)$.

For the second assertion of the proposition, consider the metric $\delta := \min\{d, 2\}$ on X and let \mathscr{W}_δ denote the corresponding functional. Note that $\mathscr{W}_\delta(Z) = \mathbf{W}(Z)$. If Z is as above, then for some 1-Lipschitz function $f : (X, \delta) \rightarrow \mathbb{R}$ and some permutation π , we have

$$\mathscr{W}_\delta(Z) = Z(f) = \sum_i (f(x_i) - f(y_{\pi(i)})) = \sum_i \delta(x_i, y_{\pi(i)}),$$

in particular $f(x_i) - f(y_{\pi(i)}) = \delta(x_i, y_{\pi(i)}) \geq 0$ for all i . We can assume that the set $\bigcup_i [f(y_{\pi(i)}), f(x_i)]$ is connected; otherwise f can easily be modified so that this holds. Hence, if $\mathscr{W}_\delta(Z) = \mathbf{W}(Z) < 2$, then we can further arrange that $|f| < 1$. It follows that there is no 1-Lipschitz function $g : X \rightarrow \mathbb{R}$ with $Z(g) > \mathbf{W}(Z) = Z(f)$, for otherwise a suitable convex combination $h = (1 - \epsilon)f + \epsilon g$ would satisfy $Z(h) > \mathbf{W}(Z)$ and $|h| \leq 1$. \square

We now provide the alternative argument mentioned above. It is convenient to consider pairwise distinct points but to allow distances to be zero.

Lemma 4.3 *Let (V, d) be a finite pseudo-metric space with a partition $V = V_+ \cup V_-$, where $|V_+| = |V_-|$. If $f : V \rightarrow \mathbb{R}$ is a 1-Lipschitz function that maximizes the quantity $\sum_{x \in V_+} f(x) - \sum_{y \in V_-} f(y)$, then there exists a bijection $\pi : V_+ \rightarrow V_-$ such that $f(x) - f(\pi(x)) = d(x, \pi(x))$ for all $x \in V_+$.*

Proof Given f , define a relation \leq on V such that $x \leq y$ if and only if $f(x) - f(y) = d(x, y)$. Note that if $x \leq y \leq z$, then

$$f(x) - f(z) = d(x, y) + d(y, z) \geq d(x, z),$$

and since f is 1-Lipschitz, $x \leq z$. Thus the relation is transitive. For a set $A \subset V_+$, let $\Gamma(A)$ denote the set of all $y \in V_-$ for which there is an $x \in A$ with $x \leq y$. Suppose first that A is maximal in V_+ in the sense that there is no pair $(x, y) \in A \times (V_+ \setminus A)$ with $x \leq y$. Note that $f(x) - f(y) < d(x, y)$ whenever $x \in A$ and $y \in C := (V_+ \setminus A) \cup (V_- \setminus \Gamma(A))$. By transitivity, the same strict inequality holds whenever $x \in \Gamma(A)$ and $y \in C$. Hence, for some

$\epsilon > 0$, the function f_ϵ obtained from f by increasing the values on $A \cup \Gamma(A)$ by ϵ is still 1-Lipschitz. It follows from the maximality property of f that

$$\epsilon |A| = \sum_{x \in V_+} (f_\epsilon(x) - f(x)) \leq \sum_{y \in V_-} (f_\epsilon(y) - f(y)) = \epsilon |\Gamma(A)|,$$

thus $|A| \leq |\Gamma(A)|$. Let now $A \subset V_+$ be arbitrary. Again by transitivity, the set A' of all points in V_+ with a precursor in A is maximal, and $\Gamma(A') = \Gamma(A)$, so that $|A| \leq |A'| \leq |\Gamma(A')| = |\Gamma(A)|$. This shows that the bipartite graph with edge set $\{(x, y) \in V_+ \times V_- : x \leq y\}$ satisfies the assumption of Hall's marriage theorem. Hence, there is a matching (bijection) π as claimed. \square

In general, for $n \geq 0$, an analog of Proposition 4.2 holds as follows.

Theorem 4.4 *If X_i satisfies $(CI_n)[c]$ for $i \in \mathbb{N}$, and if the cycles $Z_i \in \mathbf{Z}_{n,c}(X_i)$ satisfy $\sup_i \mathbf{M}(Z_i) < \infty$ and $\mathbf{W}(Z_i) \rightarrow 0$, then $\text{FillVol}_{X_i}(Z_i) \rightarrow 0$.*

Proof The proof is by induction on n . For $n = 0$, the result holds by Proposition 4.2. Assume now that $n \geq 1$ and the assertion holds in dimension $n - 1$. It suffices to show that for every sequence $(Z_i)_{i \in \mathbb{N}}$ as in the statement and for every $\nu > 0$ there is an index j with $\text{FillVol}_{X_j}(Z_j) < \nu$. By Theorem 2.2 there is a constant $\gamma = \gamma(n, c)$ such that every X_i satisfies $(EI_n)[\gamma]$ and, if $n \geq 2$, also $(EI_{n-1})[\gamma]$. Put $M := \sup_i \mathbf{M}(Z_i)$ and choose $\varrho > 0$ such that

$$18c\varrho M < \nu.$$

Let $\delta := \delta(n, \gamma, M, \varrho, \nu/2)$ be the constant from Theorem 3.2. If there is an index j with $m_\varrho(Z_j) \leq \delta$, then $\text{FillVol}_{X_j}(Z_j) < \nu/2$ and we are done.

Suppose now that $m_\varrho(Z_i) > \delta$ for all i . Choose points $x_i \in X_i$ such that

$$\|Z_i\|(B_{x_i}(\varrho)) \geq \delta.$$

By Lemma 4.1 there is an $s \in (\varrho, 2\varrho)$ and an infinite set $I_1 \subset \mathbb{N}$ such that $Z_{i,s} := Z_i \lfloor B_{x_i}(s) \in \mathbf{I}_{n,c}(X_i)$ for all $i \in I_1$ and

$$\sup_{i \in I_1} \mathbf{M}(\partial Z_{i,s}) < \infty, \quad \mathbf{W}(\partial Z_{i,s}) \leq \mathbf{W}(Z_{i,s}) \rightarrow 0 \quad (I_1 \ni i \rightarrow \infty).$$

By the induction assumption there exist fillings $T_i \in \mathbf{I}_{n,c}(X_i)$ of $\partial Z_{i,s}$ such that $\mathbf{M}(T_i) \rightarrow 0$. By reducing the index set further, if necessary, we arrange that $\mathbf{M}(T_i) \leq \delta/2$ and $\text{spt}(T_i) \subset B_{x_i}(3\varrho)$ for all $i \in I_1$ (Theorem 2.3). Then $S_i^1 := Z_{i,s} - T_i$ is a cycle with support in $B_{x_i}(3\varrho)$, and

$$\text{FillVol}_{X_i}(S_i^1) \leq 3c\varrho \mathbf{M}(S_i^1)$$

by the cone inequality. Let $Z_i^1 := Z_i - Z_{i,s} + T_i$ and consider the splitting

$$Z_i = S_i^1 + Z_i^1.$$

Notice that $\mathbf{M}(S_i^1) \leq \mathbf{M}(Z_{i,s}) + \delta/2$ and $\mathbf{M}(Z_i^1) \leq \mathbf{M}(Z_i) - \mathbf{M}(Z_{i,s}) + \delta/2$, moreover $\mathbf{M}(Z_{i,s}) \geq \|Z_i\|(B_{x_i}(\varrho)) \geq \delta$. Hence, for all $i \in I_1$, we have

$$\mathbf{M}(S_i^1) \leq \mathbf{M}(Z_i) - \mathbf{M}(Z_i^1) + \delta,$$

$$\mathbf{M}(Z_i^1) \leq \mathbf{M}(Z_i) - \frac{\delta}{2} \leq M.$$

Note further that $\mathbf{W}(Z_i^1) \leq \mathbf{W}(Z_i) + \mathbf{W}(Z_{i,s}) + \mathbf{M}(T_i) \rightarrow 0$ as $i \rightarrow \infty$.

If $m_\varrho(Z_i^1) > \delta$ for all $i \in I_1$, then we repeat the above argument (with the same constants M, ϱ, δ) and produce similar splittings $Z_i^1 = S_i^2 + Z_i^2$ for all i in an infinite set $I_2 \subset I_1$. If $m_\varrho(Z_i^2) > \delta$ for all $i \in I_2$, we iterate this step, and continue in this manner. This eventually yields an infinite set $I_k \subset \mathbb{N}$, for some $k \geq 1$, and a decomposition

$$Z_i = S_i^1 + \dots + S_i^k + Z_i^k$$

for every $i \in I_k$, such that $m_\varrho(Z_j^k) \leq \delta$ for some $j \in I_k$. In fact, $k \leq 2M/\delta$, because $\mathbf{M}(Z_i^k) \leq \mathbf{M}(Z_i) - k\delta/2$ for all $i \in I_k$. It follows that

$$\begin{aligned} \mathbf{M}(S_i^1) + \dots + \mathbf{M}(S_i^k) &\leq \mathbf{M}(Z_i) - \mathbf{M}(Z_i^k) + k\delta \leq 3M, \\ \text{FillVol}_{X_i}(S_i^1) + \dots + \text{FillVol}_{X_i}(S_i^k) &\leq 9c\varrho M < \frac{\nu}{2}, \end{aligned}$$

and $\text{FillVol}_{X_j}(Z_j^k) < \nu/2$ by Theorem 3.2. Hence, $\text{FillVol}_{X_j}(Z_j) < \nu$. □

The desired result for sequences of cycles with supports in a fixed compact set now follows easily.

Theorem 4.5 *Let X be a proper metric space satisfying condition (CI_n) for some $n \geq 0$. If an \mathbf{M} -bounded sequence of cycles $Z_i \in \mathbf{Z}_{n,c}(X)$ with supports in a fixed compact set $K \subset X$ converges weakly to 0, then $\mathbf{W}(Z_i) \rightarrow 0$ and $\text{FillVol}_X(Z_i) \rightarrow 0$.*

Proof Suppose that $\mathbf{M}(Z_i) \leq M$ for all i . Let \mathcal{F} denote the collection of all 1-Lipschitz functions $f: X \rightarrow \mathbb{R}$ with $|f| \leq \text{diam}(K)/2$. Let $\epsilon > 0$. There is a finite subcollection $\mathcal{G} \subset \mathcal{F}$ such that for all $f_0, \dots, f_n \in \mathcal{F}$ there exist $g_0, \dots, g_n \in \mathcal{G}$ with $\sup_{x \in K} |f_k(x) - g_k(x)| \leq \epsilon/M$ for $k = 0, \dots, n$; then

$$|Z_i(f_0, \dots, f_n) - Z_i(g_0, \dots, g_n)| \leq (n + 1)\epsilon$$

for all i by Lemma 5.2 in [29]. As $Z_i \rightarrow 0$ weakly, if i is sufficiently large, then $Z_i(g_0, \dots, g_n) \leq \epsilon$ for all tuples $(g_0, \dots, g_n) \in \mathcal{G}^{n+1}$, thus $Z_i(f_0, \dots, f_n) \leq (n + 2)\epsilon$ whenever $f_0, \dots, f_n \in \mathcal{F}$. Hence $\mathbf{W}(Z_i) \rightarrow 0$, and $\text{FillVol}_X(Z_i) \rightarrow 0$ by Theorem 4.4. □

5 Isoperimetric inequalities

This section is devoted to isoperimetric inequalities and shows in particular the implications $(\text{AR}_n) \Rightarrow (\text{LII}_n) \Leftrightarrow (\text{SII}_n)$ in Theorem 1.1. In fact we prove some stronger uniform statements. To this end we first extend the notion of asymptotic rank to sequences of metric spaces $X_i = (X_i, d_i)$. A compact metric space Ω is called an *asymptotic subset* of the sequence $(X_i)_{i \in \mathbb{N}}$ if there exist positive numbers $r_i \rightarrow \infty$ and subsets $Y_i \subset X_i$ such that the rescaled sets $(Y_i, r_i^{-1}d_i)$ converge to Ω in the Gromov–Hausdorff topology. The *asymptotic rank* of the sequence (X_i) is the supremum of all $k \geq 0$ such that there exists an asymptotic subset bi-Lipschitz homeomorphic to a compact subset of \mathbb{R}^k with positive Lebesgue measure. The asymptotic rank of a single metric space X , as defined in the introduction, equals the asymptotic rank of the constant sequence $X_i = X$ (see Definition 1.1 and Proposition 3.1 in [40]).

From now on, throughout this section, we assume that \mathcal{X} is a class of proper metric spaces such that for some $n \geq 1$ and $c > 0$, all members of \mathcal{X} satisfy $(\text{CI}_n)[c]$, and every sequence

$(X_i)_{i \in \mathbb{N}}$ in \mathcal{X} has asymptotic rank $\leq n$. The sub-Euclidean isoperimetric inequality for n -cycles in spaces of asymptotic rank at most n was established in greater generality in [40] (Theorem 1.2), and a slightly restricted version was used as a key tool in [28] (Theorem 4.4). First we give a proof of a uniform version of the latter statement.

Theorem 5.1 *For all $C, \epsilon > 0$ there is a constant $\varrho_0 = \varrho_0(\mathcal{X}, n, c, C, \epsilon) > 0$ such that if X belongs to \mathcal{X} , and $Z \in \mathbf{Z}_{n,c}(X)$ satisfies $\mathbf{M}(Z) \leq Cr^n$ and $\text{spt}(Z) \subset B_p(r)$ for some $p \in X$ and $r \geq \varrho_0$, then $\text{FillVol}_X(Z) < \epsilon r^{n+1}$.*

Proof Suppose to the contrary that there exist $C, \epsilon > 0$, a sequence of positive radii $(r_i)_{i \in \mathbb{N}}$ tending to infinity, and cycles $Z_i \in \mathbf{Z}_{n,c}(X_i)$, where $X_i = (X_i, d_i)$ belongs to \mathcal{X} , each with mass $\mathbf{M}(Z_i) \leq Cr_i^n$ and support in some ball $B_{p_i}(r_i)$, such that

$$\text{FillVol}_{X_i}(Z_i) \geq \epsilon r_i^{n+1}.$$

By Theorem 2.2 there is a constant $\gamma = \gamma(n, c)$ such that every X_i satisfies $(\text{EII}_n)[\gamma]$ and, if $n \geq 2$, also $(\text{EII}_{n-1})[\gamma]$. Put $\eta_i := \epsilon r_i / (2C)$ and apply Proposition 3.1 to Z_i to get $V_i \in \mathbf{I}_{n+1,c}(X_i)$ and $Z'_i := Z_i - \partial V_i$ such that

- (1) $\mathbf{M}(Z'_i) \leq Cr_i^n$ and $\mathbf{M}(V_i) \leq \eta_i Cr_i^n = \epsilon r_i^{n+1} / 2$;
- (2) $\Theta_{x,r}(Z'_i) \geq \theta$ whenever $x \in \text{spt}(Z'_i)$ and $0 < r \leq \alpha \eta_i = \alpha \epsilon r_i / (2C)$, where α, θ depend only on n, γ ;
- (3) $\text{spt}(Z'_i) \subset B_{p_i}(\lambda r_i)$ for some constant $\lambda > 1$ depending only on n, γ, C, ϵ (this uses part (4) of Proposition 3.1).

By (1), (3) and the coning inequality $(\text{CI}_n)[c]$ there exists a filling $V'_i \in \mathbf{I}_{n+1,c}(X_i)$ of Z'_i with mass

$$\mathbf{M}(V'_i) \leq c\lambda r_i \mathbf{M}(Z'_i) \leq c\lambda Cr_i^{n+1}.$$

By Theorem 2.3 we can assume that V'_i is minimizing and has support in $B_{p_i}(\lambda' r_i)$ for some $\lambda' > \lambda$ independent of i . Let Y_i denote the set $\text{spt}(V'_i)$ equipped with the metric induced by $r_i^{-1} d_i$. Note that $\mathbf{M}(Z'_i) \leq C$ and $\mathbf{M}(V'_i) \leq c\lambda C$ with respect to this metric, and Y_i has diameter at most $2\lambda'$. It follows from (2) and the lower density bound for V'_i that the family of all Y_i is uniformly precompact. By Gromov’s compactness theorem [18], after passage to subsequences and relabelling, there exist a compact metric space Y and isometric embeddings $\phi_i : Y_i \rightarrow Y$ such that the images $\phi_i(Y_i)$ converge to some compact set $\Omega \subset Y$ in the Hausdorff distance. By Theorem 2.1 we can further assume that the push-forwards $\phi_{i\#} V'_i \in \mathbf{I}_{n+1,c}(Y)$ converge weakly to a current $V \in \mathbf{I}_{n+1,c}(Y)$. Evidently $\text{spt}(V) \subset \Omega$. Since the (sub)sequence $(X_i)_{i \in \mathbb{N}}$ has asymptotic rank $\leq n$, and Ω is an asymptotic subset, it follows that there is no bi-Lipschitz embedding of a compact subset of \mathbb{R}^{n+1} with positive Lebesgue measure into Ω , and therefore V must be zero (compare Theorem 8.3 in [29] and the comments thereafter). Hence, the cycles $\phi_{i\#} Z'_i \in \mathbf{Z}_{n,c}(Y)$ converge weakly to $\partial V = 0$. We can assume that Y satisfies condition (CI_n) ; for example, the injective hull of Y admits a conical geodesic bicombing [30] and is still compact. It then follows from Theorem 4.5 that $\mathbf{W}(\phi_{i\#} Z'_i) \rightarrow 0$. As this is an intrinsic property of the currents, we conclude that $\mathbf{W}(Z'_i) \rightarrow 0$ with respect to the metrics $r_i^{-1} d_i$. However, it follows from the inequality $\text{FillVol}_{X_i}(Z_i) \geq \epsilon r_i^{n+1}$ and (1) that $\text{FillVol}_{X_i}(Z'_i) \geq \epsilon/2$ with respect to $r_i^{-1} d_i$. This contradicts Theorem 4.4 (note that $(X_i, r_i^{-1} d_i)$ still satisfies $(\text{CI}_n)[c]$). \square

As a first application of Theorem 5.1 we derive a density bound for minimizing fillings. This is similar to assertion (5) of Proposition 3.1 and to Proposition 4.5 in [28].

Proposition 5.2 *For all $C, \delta > 0$ there is a $\varrho = \varrho(\mathcal{X}, n, c, C, \delta) > 0$ such that if X belongs to \mathcal{X} , and $Z \in \mathbf{Z}_{n,c}(X)$ is a cycle with $\Theta_{p,r}(Z) \leq C$ for some $p \in X$ and for all $r > a \geq 0$, then every minimizing filling $V \in \mathbf{I}_{n+1,c}(X)$ of Z satisfies*

$$\Theta_{p,r}(V) = \frac{\|V\|(B_p(r))}{r^{n+1}} < \delta$$

for all $r > \max\{\varrho, a\}$.

Proof Let $V \in \mathbf{I}_{n+1,c}(X)$ be a minimizing filling of Z , and set $B_r := B_p(r)$ for all $r > 0$. Choose a sufficiently large radius $r_0 > 0$ such that

$$\delta r_0^{n+1} > \mathbf{M}(V) \geq \|V\|(B_{r_0}),$$

and put $r_i := 2^{-i}r_0$ for every integer $i \geq 1$. There exists an $s \in (r_1, r_0)$ such that the slice $T_s := \partial(V \llcorner B_s) - Z \llcorner B_s$ is in $\mathbf{I}_{n,c}(X)$ and has mass $\mathbf{M}(T_s) \leq \|V\|(B_{r_0})/(r_0 - r_1) < 2\delta r_0^n$. Furthermore, if $r_1 > a$, then $\mathbf{M}(Z \llcorner B_s) \leq Cs^n$ by assumption, thus the cycle $Z_s := Z \llcorner B_s + T_s$ satisfies

$$\mathbf{M}(Z_s) \leq Cs^n + 2\delta r_0^n \leq (C + 2\delta)r_0^n,$$

and $\text{spt}(Z_s) \subset B_{r_0}$. Note that $V \llcorner B_s$ is a minimizing filling of Z_s . By Theorem 5.1 there is a constant $\varrho := 2^{-1}\varrho_0(\mathcal{X}, n, c, C + 2\delta, 2^{-(n+1)}\delta) > 0$ such that if $r_1 > \max\{\varrho, a\}$ and, hence, $r_0 \geq 2\varrho$, then

$$\|V\|(B_{r_1}) \leq \mathbf{M}(V \llcorner B_s) < 2^{-(n+1)}\delta r_0^{n+1} = \delta r_1^{n+1}.$$

Finally, given any $r > \max\{\varrho, a\}$, we can choose r_0 initially such that $r = r_k = 2^{-k}r_0$ for some $k \geq 1$. When $k \geq 2$, we repeat the above slicing argument successively for $i = 2, \dots, k$, with (r_i, r_{i-1}) in place of (r_1, r_0) . This eventually shows that

$$\|V\|(B_r) < \delta r^{n+1}$$

for all $r > \max\{\varrho, a\}$. □

Next we prove a linear isoperimetric inequality for cycles with controlled density. This yields the implication $(\text{AR}_n) \Rightarrow (\text{LI}\Pi_n)$ in Theorem 1.1.

Theorem 5.3 (Linear isoperimetric inequality) *There is a constant $\nu = \nu(n, c) > 0$, and for all $C > 0$ there is a $\lambda = \lambda(\mathcal{X}, n, c, C) > 0$, such that if X belongs to \mathcal{X} and $Z \in \mathbf{Z}_{n,c}(X)$ is a cycle with (C, a) -controlled density, $a \geq 0$, then $\text{FillVol}_X(Z) \leq \max\{\lambda, \nu a\} \mathbf{M}(Z)$.*

Proof Note again that for some $\gamma = \gamma(n, c)$, every member of \mathcal{X} satisfies $(\text{EII}_n)[\gamma]$ and, if $n \geq 2$, also $(\text{EII}_{n-1})[\gamma]$. Suppose that $Z \in \mathbf{Z}_{n,c}(X)$ has (C, a) -controlled density. For any $\eta > 0$, to be specified below, Proposition 3.1 provides a $V \in \mathbf{I}_{n+1,c}(X)$ such that, for $Z' := Z - \partial V \in \mathbf{Z}_{n,c}(X)$ and some constants $\alpha, \theta > 0$ depending only on n and γ ,

- (1) $\eta^{-1}\mathbf{M}(V) + \mathbf{M}(Z') \leq \mathbf{M}(Z)$;
- (2) $\Theta_{x,r}(Z') \geq \theta$ for all $x \in \text{spt}(Z')$ and $r \in (0, \alpha\eta]$;
- (3) Z' has $(2^{n+1}C, \max\{a, 2^{n+1}\eta\})$ -controlled density.

By Theorem 2.3 there exists a minimizing filling $V' \in \mathbf{I}_{n+1,c}(X)$ of Z' , and there is a $\delta_0 = \delta_0(n, \gamma) > 0$ such that $\Theta_{x,r}(V') \geq \delta_0$ whenever $x \in \text{spt}(V')$, $r > 0$, and $B_x(r) \cap \text{spt}(Z') = \emptyset$. On the other hand, by (3) and Proposition 5.2, there is a constant $\varrho' := \varrho(\mathcal{X}, n, c, 2^{n+1}C, \delta_0) > 0$ such that $\Theta_{p,r}(V') < \delta_0$ for all $p \in X$ and $r > \max\{\varrho', a, 2^{n+1}\eta\}$. We now fix η so that

$$2^{n+1}\eta = \max\{\varrho', a\}.$$

Then $\text{spt}(V')$ is within distance at most $2^{n+1}\eta$ from $\text{spt}(Z')$. Pick a maximal set $N \subset \text{spt}(Z')$ of distinct points at mutual distance $> 2\alpha\eta$. The collection of all balls $B_x(2\alpha\eta)$ with $x \in N$ covers $\text{spt}(Z')$, and the corresponding balls with radius

$$r := 2(\alpha + 2^n)\eta$$

cover $\text{spt}(V')$. Furthermore, the balls $B_x(\alpha\eta)$ with $x \in N$ are pairwise disjoint, and $\|Z'\|(B_x(\alpha\eta)) \geq \theta(\alpha\eta)^n$ by (2). Hence, $|N| \leq \mathbf{M}(Z')/(\theta\alpha^n\eta^n)$, and since $r > 2^{n+1}\eta$, we have $\|V'\|(B_x(r)) < \delta_0 r^{n+1}$ for all $x \in N$. (Possibly $V' = 0$ and $|N| = 0$.) Thus

$$\mathbf{M}(V') \leq |N| \delta_0 r^{n+1} \leq \frac{\delta_0 r^{n+1}}{\theta\alpha^n\eta^n} \mathbf{M}(Z') \leq v'\eta \mathbf{M}(Z')$$

for some $v' = v'(n, c) \geq 1$. Now $V + V'$ is a filling of Z with mass

$$\mathbf{M}(V + V') \leq v'(\mathbf{M}(V) + \eta \mathbf{M}(Z')) \leq v'\eta \mathbf{M}(Z)$$

by (1). In view of the choice of η , this gives the result. □

We now turn to the sub-Euclidean isoperimetric inequality as stated in Theorem 1.1. The proof below shows that $(\text{LII}_n) \Rightarrow (\text{SII}_n)$.

Theorem 5.4 (Sub-Euclidean isoperimetric inequality) *For all $\epsilon > 0$ there is a constant $M_0 = M_0(\mathcal{X}, n, c, \epsilon) > 0$ such that if X belongs to \mathcal{X} and $Z \in \mathbf{Z}_{n,c}(X)$, then $\text{FillVol}_X(Z) < \epsilon \max\{M_0, \mathbf{M}(Z)\}^{(n+1)/n}$.*

Proof Given $Z \in \mathbf{Z}_{n,c}(X)$, note that if $t > 0$ and $r > t \mathbf{M}(Z)^{1/n}$, then $\mathbf{M}(Z) < t^{-n}r^n$, thus Z has $(t^{-n}, t \mathbf{M}(Z)^{1/n})$ -controlled density. For $\epsilon > 0$, let $v = v(n, c)$ and $\lambda = \lambda(\mathcal{X}, n, c, C)$ be the constants from Theorem 5.3, where now $C := t^{-n}$ for any fixed $t < \epsilon/v$. Let $M_0 > 0$ be such that $\lambda < \epsilon M_0^{1/n}$. Then

$$\max\{\lambda, vt \mathbf{M}(Z)^{1/n}\} \mathbf{M}(Z) < \epsilon \max\{M_0, \mathbf{M}(Z)\}^{(n+1)/n},$$

and the result follows from Theorem 5.3. □

Finally, we show that Theorem 5.1 follows easily from Theorem 5.4. Given $C, \epsilon > 0$, put $\epsilon' := \epsilon/C^{(n+1)/n}$. If $Z \in \mathbf{Z}_{n,c}(X)$ is a cycle with $\mathbf{M}(Z) \leq Cr^n$, and r is sufficiently large, so that $Cr^n \geq M_0 = M_0(\epsilon')$, then

$$\text{FillVol}_X(Z) < \epsilon'(Cr^n)^{(n+1)/n} = \epsilon r^{n+1}$$

by Theorem 5.4. Since the asymptotic rank assumption in Theorem 5.3 is only used through Theorem 5.1, this also shows that $(\text{SII}_n) \Rightarrow (\text{LII}_n)$.

6 Quasiflats and quasi-minimizers

A map $f : W \rightarrow X$ from another metric space W into X is an (L, a) -quasi-isometric embedding, for constants $L \geq 1$ and $a \geq 0$, if

$$L^{-1}d(x, y) - a \leq d(f(x), f(y)) \leq Ld(x, y) + a$$

for all $x, y \in W$. Propositions 3.6 and 3.7 in [28] show that quasi-isometric embeddings of domains $W \subset \mathbb{R}^n$ into X give rise to quasi-minimizing currents with controlled density, as defined in the introduction. An inspection of the proofs reveals that the statements hold

in a stronger form, in particular with a quasi-minimality constant Q independent of the parameter a . We provide the details for convenience, and also because parts of the proof will be used later. The first result refers to the simpler case when the map is actually Lipschitz.

Proposition 6.1 *For all $n \geq 1$ and $L \geq 1$ there exist $C > 0$ and $Q \geq 1$ such that the following holds. Let $W \subset \mathbb{R}^n$ be a compact set with finite perimeter, so that the associated current $E := \llbracket W \rrbracket$ (with $\text{spt}(E) \subset W$ and $\text{spt}(\partial E) \subset \partial W$) is in $\mathbf{I}_{n,c}(\mathbb{R}^n)$. Suppose that $a \geq 0$ and $f: W \rightarrow X$ is an L -Lipschitz, (L, a) -quasi-isometric embedding into a proper metric space X . Then $S := f_{\#}E \in \mathbf{I}_{n,c}(X)$ has (C, a) -controlled density and is (Q, Qa) -quasi-minimizing mod $f(\partial W)$, furthermore $d(f(x), \text{spt}(S)) \leq Qa$ for all $x \in W$ with $d(x, \partial W) > Qa$.*

Proof Let $B := B_p(r)$ for some $p \in X$ and $r > a$. Then

$$\|f_{\#}E\|(B) \leq L^n \|E\|(f^{-1}(B)),$$

and $f^{-1}(B)$ has diameter $\leq L(2r + a) \leq 3Lr$, thus $\|S\|(B) \leq Cr^n$ for some constant $C = C(n, L)$. This shows that S has (C, a) -controlled density.

Next, let $V \subset W$ be a maximal subset of distinct points at mutual distance $> 2La$ ($V = W$ in case $a = 0$). Note that $d(f(x), f(y)) \geq (2L)^{-1}d(x, y)$ for any $x, y \in V$, thus $f|_V$ has a $2L$ -Lipschitz inverse, which we can extend to an \tilde{L} -Lipschitz map $\tilde{f}: X \rightarrow \mathbb{R}^n$ for some $\tilde{L} = \tilde{L}(n, L)$. Put $h := \tilde{f} \circ f: W \rightarrow \mathbb{R}^n$. For every $x \in W$ there is a $y \in V$ with $d(x, y) \leq 2La$; then $h(y) = y$ and

$$d(h(x), x) \leq d(h(x), h(y)) + d(y, x) \leq (\tilde{L}L + 1)d(x, y) \leq Na,$$

where $N := 2(\tilde{L}L + 1)L$.

Let $x \in W$. Suppose that $r > 2LN a$ and $B_r := B_{f(x)}(r)$ is disjoint from $f(\partial W)$. For almost every such r , both $S_r := S \llcorner B_r$ and $E_r := E \llcorner f^{-1}(B_r)$ are integral currents, and $f_{\#}E_r = S_r$. Since $f^{-1}(B_r) \cap \text{spt}(\partial E) = \emptyset$, the support of ∂E_r lies in the boundary of $f^{-1}(B_r)$ and is thus at distance at least $L^{-1}r$ from x . The geodesic homotopy from the inclusion map $W \rightarrow \mathbb{R}^n$ to h provides a current $R \in \mathbf{I}_{n,c}(\mathbb{R}^n)$ with $\partial R = h_{\#}(\partial E_r) - \partial E_r$ such that $\text{spt}(R)$ is within distance Na from $\text{spt}(\partial E_r)$. In fact, $R = \tilde{f}_{\#}S_r - E_r$, because $h_{\#}(\partial E_r) = \partial(h_{\#}E_r) = \partial(\tilde{f}_{\#}S_r)$ and $\mathbf{Z}_{n,c}(\mathbb{R}^n) = \{0\}$. By the choice of r we have $L^{-1}r - Na > (2L)^{-1}r$, thus $\text{spt}(R)$ lies outside $B_x((2L)^{-1}r)$. It follows that

$$\mathbf{M}(\tilde{f}_{\#}S_r) = \mathbf{M}(E_r + R) \geq \|E\|(B_x((2L)^{-1}r)) \geq \epsilon r^n$$

for some $\epsilon = \epsilon(n, L) > 0$. Now if $T \in \mathbf{I}_{n,c}(X)$ is such that $\partial T = \partial S_r$, then $\tilde{f}_{\#}T = \tilde{f}_{\#}S_r$, and

$$\mathbf{M}(S_r) \leq Cr^n \leq C\epsilon^{-1}\mathbf{M}(\tilde{f}_{\#}T) \leq Q'\mathbf{M}(T)$$

for $Q' := C\epsilon^{-1}\tilde{L}^n$. This holds for all $x \in W$ and almost all $r > 2LN a$ as long as $B_{f(x)}(r)$ is disjoint from $f(\partial W)$. In particular, since $\text{spt}(S) \subset f(W)$, S is $(Q', 2LN a)$ -quasi-minimizing mod $f(\partial W)$.

Finally, put $Q := \max\{Q', L(2LN + 1)\}$. Let $x \in W$ with $d(x, \partial W) > Qa$. Then $w := d(f(x), f(\partial W)) > L^{-1}Qa - a \geq 2LN a$. For almost every $r \in (2LN a, w)$, the above argument shows that $\mathbf{M}(\tilde{f}_{\#}S_r) > 0$, thus $S_r = S \llcorner B_{f(x)}(r) \neq 0$, and this implies that $d(f(x), \text{spt}(S)) \leq 2LN a \leq Qa$. \square

For the second result, we suppose that the compact set $W \subset \mathbb{R}^n$ is a *triangulated polyhedral set*, that is, W has the structure of a finite simplicial complex all of whose maximal cells are Euclidean n -simplices. We write W^0 and $(\partial W)^0$ for the set of vertices and

boundary vertices of the triangulation, respectively. Furthermore, $[\cdot]_a$ stands for the closed a -neighborhood of a subset of X .

Proposition 6.2 *For all $n \geq 1, c > 0$, and $K, L \geq 1$ there exist $C > 0$ and $Q \geq 1$ such that the following holds. Let X be a proper metric space satisfying condition $(CI_{n-1})[c]$. Suppose that $a > 0$, and $W \subset \mathbb{R}^n$ is a triangulated polyhedral set with simplices of diameter $\leq a$ such that every ball in \mathbb{R}^n of radius $r > a$ intersects at most $Ka^{-n}r^n$ maximal simplices. Let $\mathcal{P}_*(W)$ denote the corresponding chain complex of simplicial integral currents. If $f : W \rightarrow X$ is an (L, a) -quasi-isometric embedding, then there exists a chain map $\iota : \mathcal{P}_*(W) \rightarrow \mathbf{I}_{*,c}(X)$ such that*

- (1) ι maps every vertex $\llbracket x_0 \rrbracket \in \mathcal{P}_0(W)$ to $\llbracket f(x_0) \rrbracket$ and, for $1 \leq k \leq n$, every basic oriented simplex $\llbracket x_0, \dots, x_k \rrbracket \in \mathcal{P}_k(W)$ to a minimizing current with support in $[f(\{x_0, \dots, x_k\})]_{Qa}$;
- (2) $S := \iota \llbracket W \rrbracket \in \mathbf{I}_{n,c}(X)$ has (C, a) -controlled density and is (Q, Qa) -quasi-minimizing mod $[f((\partial W)^0)]_{Qa}$;
- (3) $d(f(x), \text{spt}(S)) \leq Qa$ for all $x \in W$ with $d(x, (\partial W)^0) > Qa$.

Note that by (1), $\text{spt}(S) \subset [f(W^0)]_{Qa}$ and $\text{spt}(\partial S) \subset [f((\partial W)^0)]_{Qa}$.

Proof Put $\mathcal{S}_* := \bigcup_{k=0}^n \mathcal{S}_k$, where \mathcal{S}_k denotes the set of all basic oriented simplices $s = \llbracket x_0, \dots, x_k \rrbracket \in \mathcal{P}_k(W)$ (compare p. 365 in [14] for the notation). We define a map $\iota : \mathcal{S}_* \rightarrow \mathbf{I}_{*,c}(X)$ by induction on k . For $\llbracket x_0 \rrbracket \in \mathcal{S}_0$, we put $\iota \llbracket x_0 \rrbracket := f_{\#} \llbracket x_0 \rrbracket = \llbracket f(x_0) \rrbracket$. Suppose now that $k \in \{1, \dots, n\}$ and ι is defined on \mathcal{S}_{k-1} . For every k -cell of W , we choose an orientation $s = \llbracket x_0, \dots, x_k \rrbracket \in \mathcal{S}_k$, then we let $\iota(s) \in \mathbf{I}_{k,c}(X)$ be a minimizing filling of the cycle

$$\sum_{i=0}^k (-1)^i \llbracket x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k \rrbracket \in \mathbf{Z}_{k-1,c}(X),$$

and we put $\iota(-s) := -\iota(s)$. The resulting map on \mathcal{S}_* readily extends to a chain map $\iota : \mathcal{P}_*(W) \rightarrow \mathbf{I}_{*,c}(X)$. Note that f maps the vertex set of any cell of W to a set of diameter at most $(L + 1)a$. It follows inductively from condition $(CI_{n-1})[c]$ and Theorem 2.3 (if $n \geq 2$, then X satisfies (EII_{n-1}) by Theorem 2.2) that for all $s = \llbracket x_0, \dots, x_k \rrbracket \in \mathcal{S}_k$,

$$\mathbf{M}(\iota(s)) \leq Ma^k$$

and $\text{spt}(\iota(s)) \subset [f(\{x_0, \dots, x_k\})]_{Ma}$ for some constant $M \geq L + 1$ depending only on n, c, L .

Let now $\mathcal{S}_n^+ \subset \mathcal{S}_n$ be the set of all positively oriented n -simplices, whose sum is $\llbracket W \rrbracket$. Put $S := \iota \llbracket W \rrbracket$. To show that S has controlled density, let $p \in X$ and $r > a$, and consider the set of all $s \in \mathcal{S}_n^+$ for which $\text{spt}(\iota(s)) \cap B_p(r) \neq \emptyset$. Every such s has a vertex x^s with $f(x^s) \in B_p(r + Ma)$, thus the set of all x^s has diameter at most $L(2(r + Ma) + a) \leq L(2M + 3)r$. It follows that there are at most $Ka^{-n}(L(2M + 3)r)^n$ such simplices and that

$$\Theta_{p,r}(S) \leq C := KL^n(2M + 3)^n M$$

for $p \in X$ and $r > a$.

Similarly as in the proof of Proposition 6.1, there exists an \bar{L} -Lipschitz map $\tilde{f} : X \rightarrow \mathbb{R}^n$ such that $h := \tilde{f} \circ f : W \rightarrow \mathbb{R}^n$ satisfies

$$d(h(x), x) \leq Na$$

for all $x \in W$, where \bar{L} and N depend only on n, L . Then

$$\bar{\iota} := \bar{f}_{\#} \circ \iota : \mathcal{P}_*(W) \rightarrow \mathbf{I}_{*,c}(\mathbb{R}^n)$$

is a chain map that sends every $\llbracket x_0 \rrbracket \in \mathcal{S}_0$ to $\llbracket h(x_0) \rrbracket$ and every $\llbracket x_0, \dots, x_k \rrbracket \in \mathcal{S}_k$ to a current with support in $\llbracket \{x_0, \dots, x_k\} \rrbracket_{(\bar{L}M+N)a}$. Let $\mathcal{P}_*(\partial W)$ be the complex of simplicial integral currents in ∂W . A similar inductive construction as above, using minimizing fillings of cycles in \mathbb{R}^n , produces a chain homotopy between the inclusion map $\mathcal{P}_*(\partial W) \rightarrow \mathbf{I}_{*,c}(\mathbb{R}^n)$ and the restriction of $\bar{\iota}$ to $\mathcal{P}_*(\partial W)$. This yields an $R \in \mathbf{I}_{n,c}(\mathbb{R}^n)$ with boundary $\partial R = \bar{\iota}(\partial \llbracket W \rrbracket) - \partial \llbracket W \rrbracket$ and support $\text{spt}(R) \subset [(\partial W)^0]_{\bar{M}a}$ for some constant $\bar{M} = \bar{M}(n, c, L) \geq 1$. In fact,

$$R = \bar{f}_{\#}S - \llbracket W \rrbracket,$$

because $\bar{\iota}(\partial \llbracket W \rrbracket) = \partial(\bar{\iota} \llbracket W \rrbracket) = \partial(\bar{f}_{\#}S)$.

Note that $\text{spt}(S) \subset [f(W^0)]_{Ma}$. Let $\bar{x} \in [f(W^0)]_{Ma}$ and $r > 0$ be such that $B_{\bar{x}}(r) \cap f((\partial W)^0) = \emptyset$ and $S_r := S \llcorner B_{\bar{x}}(r) \in \mathbf{I}_{n,c}(X)$. We want to show that if $r > Pa$, for some sufficiently large constant $P = P(n, c, L) \geq 1$, then $\mathbf{M}(\bar{f}_{\#}S_r) \geq \epsilon r^n$ for some $\epsilon = \epsilon(n, L) > 0$. Choose $x \in W^0$ with $d(f(x), \bar{x}) \leq Ma$, and put $B_x := B_x((2L)^{-1}r)$. For all $y \in (\partial W)^0$,

$$r < d(\bar{x}, f(y)) \leq d(f(x), f(y)) + Ma \leq Ld(x, y) + (M + 1)a$$

and thus $d(x, y) > (2L)^{-1}r + \bar{M}a$ for sufficiently large P ; then

$$(\text{spt}(R) \cup \partial W) \cap B_x = \emptyset.$$

Moreover, for every $\bar{y} \in \text{spt}(S - S_r) \subset \text{spt}(S)$ there is a vertex $y \in W^0$ such that $d(f(y), \bar{y}) \leq Ma$,

$$r \leq d(\bar{x}, \bar{y}) \leq d(f(x), f(y)) + 2Ma \leq Ld(x, y) + (2M + 1)a,$$

and $d(x, y) \leq d(x, \bar{f}(\bar{y})) + d(\bar{f}(\bar{y}), h(y)) + Na \leq d(x, \bar{f}(\bar{y})) + (\bar{L}M + N)a$; thus $d(x, \bar{f}(\bar{y})) > (2L)^{-1}r$ for sufficiently large P , implying that

$$\text{spt}(\bar{f}_{\#}(S - S_r)) \cap B_x = \emptyset.$$

Since $\bar{f}_{\#}S_r = \llbracket W \rrbracket + R - \bar{f}_{\#}(S - S_r)$, it then follows that

$$\mathbf{M}(\bar{f}_{\#}S_r) \geq \|\llbracket W \rrbracket\|(B_x) \geq \epsilon r^n$$

for some $\epsilon = \epsilon(n, L) > 0$, as desired. Now if $T \in \mathbf{I}_{n,c}(X)$ is such that $\partial T = \partial S_r$, then $\bar{f}_{\#}T = \bar{f}_{\#}S_r$, and

$$\mathbf{M}(S_r) \leq Cr^n \leq C\epsilon^{-1}\mathbf{M}(\bar{f}_{\#}T) \leq Q'\mathbf{M}(T)$$

for $Q' := C\epsilon^{-1}\bar{L}^n$. Since $\text{spt}(S)$ and $\text{spt}(\partial S)$ are within distance Ma from $f(W^0)$ and $f((\partial W)^0)$, respectively, this shows in particular that S is (Q', Pa) -quasi-minimizing mod $[f((\partial W)^0)]_{Ma}$.

Finally, put $Q := \max\{M, Q', L(P + 1)\}$. Let $x' \in W$ with $d(x', (\partial W)^0) > Qa$. Then $w := d(f(x'), f((\partial W)^0)) > L^{-1}Qa - a \geq Pa$. Note that $f(x') \in [f(W^0)]_{Ma}$, as $M \geq L + 1$. For $\bar{x} = f(x')$ and almost every $r \in (Pa, w)$, the above argument shows that $\mathbf{M}(\bar{f}_{\#}S_r) > 0$, thus $S_r = S \llcorner B_{\bar{x}}(r) \neq 0$, and this implies that $d(f(x'), \text{spt}(S)) \leq Pa \leq Qa$. □

7 Morse lemma, slim simplices, and filling radius

We now turn to the remaining assertions in Theorem 1.1. For the first three results, we assume as in Sect. 5 that \mathcal{X} is a class of proper metric spaces such that for some $n \geq 1$ and $c > 0$, all members of \mathcal{X} satisfy condition $(CI_n)[c]$, and every sequence $(X_i)_{i \in \mathbb{N}}$ in \mathcal{X} has asymptotic rank $\leq n$. We begin with a uniform version of the Morse lemma, analogous to Theorem 5.1 in [28]. The asymptotic rank assumption is only used through Proposition 5.2, or Theorem 5.1, which in turn follows from Theorem 5.4. Hence, $(SII_n) \Rightarrow (ML_n)$.

Theorem 7.1 (Morse lemma) *For all $C > 0$ and $Q \geq 1$ there is a constant $l = l(\mathcal{X}, n, c, C, Q) \geq 0$ such that if X belongs to \mathcal{X} , and $Z \in \mathbf{Z}_{n,c}(X)$ has (C, a) -controlled density and is (Q, a) -quasi-minimizing mod Y , where $Y \subset X$ is a closed set and $a \geq 0$, then the support of Z is within distance at most $\max\{l, 4a\}$ from Y .*

Proof Let $x \in \text{spt}(Z) \setminus Y$. Essentially the same argument as for the second part of Theorem 2.3 (using (EII_{n-1}) if $n \geq 2$) shows that there is a constant $\delta'_0 = \delta'_0(n, c) > 0$ such that $\Theta_{x,s}(Z) \geq \delta'_0 Q^{1-n}$ whenever $s > 2a$ and $B_x(s) \cap Y = \emptyset$ (see Lemma 3.3 in [28]). Now let $V \in \mathbf{I}_{n+1,c}(X)$ be a minimizing filling of Z , and suppose that $r > 4a$ and $B_x(r) \cap Y = \emptyset$. For almost every $s \in (2a, r)$, the slice $T_s = \partial(V \llcorner B_x(s)) - Z \llcorner B_x(s) \in \mathbf{I}_{n,c}(X)$ satisfies

$$Q \mathbf{M}(T_s) \geq \mathbf{M}(Z \llcorner B_x(s)) \geq \delta'_0 Q^{1-n} s^n,$$

and integrating the inequality $\mathbf{M}(T_s) \geq \delta'_0 Q^{-n} s^n$ from $r/2 > 2a$ to r we get that $\Theta_{x,r}(V) \geq \delta$ for some $\delta = \delta(n, c, Q) > 0$ (compare Lemma 3.4 in [28]). On the other hand, by Proposition 5.2 there is a constant $l := \varrho(\mathcal{X}, n, c, C, \delta) > 0$ such that $\Theta_{x,r}(V) < \delta$ for all $r > \max\{l, a\}$. Hence, $r \leq \max\{l, 4a\}$. \square

The next statement strengthens Theorem 5.2 in [28]. The proof shows that $(ML_n) \Rightarrow (SS_n)$. A facet of an $(n + 1)$ -simplex is an n -dimensional face.

Theorem 7.2 (Slim simplices) *For all $L \geq 1$ there is a constant $D = D(\mathcal{X}, n, c, L) \geq 0$ such that the following holds. Let Δ be a Euclidean $(n + 1)$ -simplex, X a member of \mathcal{X} , and $a \geq 0$. Suppose that $f : \partial\Delta \rightarrow X$ is a map whose restriction to each facet of Δ is an (L, a) -quasi-isometric embedding. Then the image of every facet is within distance at most $D(1 + a)$ from the union of the images of the remaining ones.*

Proof Let $W_0, \dots, W_{n+1} \subset \partial\Delta$ be an enumeration of the (closed) facets of Δ , and let $E_i := (\partial\llbracket\Delta\rrbracket) \llcorner W_i \in \mathbf{I}_{n,c}(\mathbb{R}^{n+1})$ denote the corresponding currents, whose sum is the boundary cycle $\partial\llbracket\Delta\rrbracket \in \mathbf{Z}_{n,c}(\mathbb{R}^{n+1})$.

Suppose that $a > 0$. Choose a triangulation of $\partial\Delta$ with simplices of diameter $\leq a$ such that, for some constant $K = K(n)$ and for each i , any ball in \mathbb{R}^{n+1} of radius $r > a$ intersects at most $Ka^{-n}r^n$ maximal simplices in W_i . Let $\mathcal{P}_*(\partial\Delta)$ be the corresponding chain complex of simplicial integral currents. A slight adaptation of Proposition 6.2 provides a chain map $\iota : \mathcal{P}_*(\partial\Delta) \rightarrow \mathbf{I}_{*,c}(X)$ such that the following properties hold for each $S_i := \iota(E_i) \in \mathbf{I}_{n,c}(X)$ and for some constants C, Q depending only on n, c, L :

- (1) $\text{spt}(S_i) \subset [f(W_i)]_{Qa}$ and $\text{spt}(\partial S_i) \subset [f(\partial W_i)]_{Qa}$;
- (2) S_i has (C, a) -controlled density and is (Q, Qa) -quasi-minimizing mod $[f(\partial W_i)]_{Qa}$;
- (3) $d(f(x), \text{spt}(S_i)) \leq Qa$ for all $x \in W_i$ with $d(x, \partial W_i) > Qa$.

Here $[\cdot]_{Qa}$ stands again for the closed Qa -neighborhood, and ∂W_i denotes the relative boundary of W_i . Let M_i denote the union of all W_j with $j \neq i$. The cycle

$Z := \iota(\partial[\Delta]) = \sum_{i=0}^{n+1} S_i$ has $((n + 2)C, a)$ -controlled density and is (Q, Qa) -quasi-minimizing mod $[f(M_i)]_{Qa}$ for every i . It then follows from Theorem 7.1 that the set $\text{spt}(S_i) \setminus [f(M_i)]_{Qa} = \text{spt}(Z) \setminus [f(M_i)]_{Qa}$ is within distance at most $\max\{l', 4Qa\}$ from $[f(M_i)]_{Qa}$ for some $l' = l'(\mathcal{X}, n, c, L)$. Hence, for any $x \in W_i$, it follows from (3) that $d(f(x), f(M_i))$ is less than or equal to $2Qa + \max\{l', 4Qa\}$ if $d(x, \partial W_i) > Qa$ and less than or equal to $LQa + a$ otherwise.

Note that if the restriction of f to each facet of Δ is L -Lipschitz in addition, or if $a = 0$, then the proof can be simplified by using Proposition 6.1 instead of Proposition 6.2. \square

The proof of the following result relies again on Proposition 5.2; thus $(\text{SII}_n) \Rightarrow (\text{FR}_n)$.

Theorem 7.3 (Filling radius) *For all $C > 0$ there is a constant $h = h(\mathcal{X}, n, c, C) > 0$ such that if X belongs to \mathcal{X} and $Z \in \mathbf{Z}_{n,c}(X)$ has (C, a) -controlled density for some $a \geq 0$, then the support of every minimizing filling $V \in \mathbf{I}_{n+1,c}(X)$ of Z is within distance at most $\max\{h, a\}$ from $\text{spt}(Z)$.*

Proof Suppose that $x \in \text{spt}(V) \setminus \text{spt}(Z)$. By Theorems 2.2 and 2.3 there are constants $\gamma = \gamma(n, c)$ and $\delta_0 = \delta_0(n, \gamma) > 0$ such that $\Theta_{x,r}(V) \geq \delta_0$ whenever $r > 0$ and $B_x(r) \cap \text{spt}(Z) = \emptyset$. On the other hand, Proposition 5.2 shows that there is a constant $h = h(\mathcal{X}, n, c, C, \delta_0) > 0$ such that $\Theta_{x,r}(V) < \delta_0$ for all $r > \max\{h, a\}$. Thus there is no point $x \in \text{spt}(V)$ at distance bigger than $\max\{h, a\}$ from $\text{spt}(Z)$. \square

We now prove the implication $(\text{SS}_n) \Rightarrow (\text{AR}_n)$, which holds without further assumptions on the metric space X .

Proposition 7.4 *Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of metric spaces $X_i = (X_i, d_i)$, let $n \geq 1$, and suppose that for every $L \geq 1$ there exists $D \geq 0$ such that every X_i satisfies (SS_n) with constant $D = D(L)$. Then the sequence $(X_i)_{i \in \mathbb{N}}$ has asymptotic rank $\leq n$.*

Proof Suppose to the contrary that the sequence $(X_i)_{i \in \mathbb{N}}$ has asymptotic rank $> n$. Then there exist a compact set $K \subset \mathbb{R}^{n+1}$ with positive Lebesgue measure, an L -bi-Lipschitz map $\phi: K \rightarrow \Omega$ onto some metric space Ω , and a sequence of $(1, \delta_i)$ -quasi-isometric embeddings $h_i: \Omega \rightarrow (X_i, r_i^{-1}d_i)$, where $L \geq 1$, $\delta_i \rightarrow 0$, and $r_i \rightarrow \infty$. We can assume that $0 \in \mathbb{R}^{n+1}$ is a Lebesgue density point of K . Let $B := B_0(1) \subset \mathbb{R}^{n+1}$. For all $k \in \mathbb{N}$ there is a $\lambda_k > 0$ such that every point in $\lambda_k B$ is at distance $\leq (2k)^{-1} \lambda_k$ from some point in K , thus there exist $(1, k^{-1} \lambda_k)$ -quasi-isometric embeddings $\psi_k: \lambda_k B \rightarrow K$. Choose $i(k) \in \mathbb{N}$ such that $s_k := \lambda_k r_{i(k)} \rightarrow \infty$ and $\epsilon_k := \lambda_k^{-1} \delta_{i(k)} + k^{-1} L \rightarrow 0$. It is straightforward to check that the map

$$f_k: s_k B \rightarrow (X_{i(k)}, d_{i(k)})$$

defined by $f_k(s_k x) = h_{i(k)} \circ \phi \circ \psi_k(\lambda_k x)$ for all $x \in B$ is an $(L, \epsilon_k s_k)$ -quasi-isometric embedding.

Now let Δ be any $(n + 1)$ -simplex inscribed in B , and pick a point x in a facet of Δ such that x is a distance $\delta > 0$ away from the union M of the remaining facets. For every k , the point $f_k(s_k x)$ is at distance at least $L^{-1} \delta s_k - \epsilon_k s_k$ from $f_k(s_k M)$. On the other hand, by assumption, there is a constant $D = D(L)$ such that the distance is at most $D(1 + \epsilon_k s_k)$. This leads to the inequality $L^{-1} \delta - \epsilon_k \leq D(s_k^{-1} + \epsilon_k)$, which contradicts the fact that $s_k \rightarrow \infty$ and $\epsilon_k \rightarrow 0$. \square

Lastly, we show that $(\text{FR}_n) \Rightarrow (\text{AR}_n)$. This is similar to Theorem 6.1 in [40].

Proposition 7.5 *Let $n \geq 1$ and $c > 0$, and let $(X_i)_{i \in \mathbb{N}}$ be a sequence of proper metric spaces $X_i = (X_i, d_i)$ satisfying $(CI_n)[c]$. Suppose further that for every $C > 0$ there exists $h > 0$ such that every X_i satisfies (FR_n) with constant $h = h(C)$. Then the sequence $(X_i)_{i \in \mathbb{N}}$ has asymptotic rank $\leq n$.*

Proof Suppose to the contrary that $(X_i)_{i \in \mathbb{N}}$ has asymptotic rank $> n$. Let L, s_k, ϵ_k and $f_k : s_k B \rightarrow X_{i(k)}$ be given as in the first part of the proof of Proposition 7.4. Let again Δ be any $(n + 1)$ -simplex inscribed in B . For every k , fix a triangulation of $\partial\Delta$ with simplices of diameter at most ϵ_k such that, for some constant $K = K(n)$, every ball in \mathbb{R}^{n+1} of radius $r > \epsilon_k$ meets at most $K(r/\epsilon_k)^n$ maximal simplices in each facet of Δ . It then follows as in the proof of Proposition 6.2 that for every k , and for some constants C, \bar{L}, \bar{M} depending only on n, c, L , there exist a cycle $Z_k \in \mathbf{Z}_{n,c}(X_{i(k)})$ with $(C, \epsilon_k s_k)$ -controlled density, an \bar{L} -Lipschitz map $\tilde{f}_k : X_{i(k)} \rightarrow \mathbb{R}^{n+1}$, and a current $R_k \in \mathbf{I}_{n+1,c}(\mathbb{R}^{n+1})$ such that $\partial R_k = \tilde{f}_{k\#} Z_k - \partial \llbracket s_k \Delta \rrbracket$ and

$$\text{spt}(R_k) \cup \tilde{f}_k(\text{spt}(Z_k)) \subset [\partial(s_k \Delta)]_{\bar{M}\epsilon_k s_k}.$$

Fix a point $x \in \Delta$ a distance $\delta > 0$ away from $\partial\Delta$. Suppose that k is so large that $\bar{M}\epsilon_k < \delta$, and $V_k \in \mathbf{I}_{n+1,c}(X_{i(k)})$ is any filling of Z_k . Then $\partial \llbracket s_k \Delta \rrbracket = \partial(\tilde{f}_{k\#} V_k) - \partial R_k$, hence $\llbracket s_k \Delta \rrbracket = \tilde{f}_{k\#} V_k - R_k$ and $s_k \Delta \subset \text{spt}(\tilde{f}_{k\#} V_k) \cup \text{spt}(R_k)$. Since $s_k x \notin \text{spt}(R_k)$, there is a point $y_k \in \text{spt}(V_k)$ such that $\tilde{f}_k(y_k) = s_k x$. For every $z_k \in \text{spt}(Z_k)$, we have

$$s_k \delta = d(s_k x, s_k \Delta) \leq d(s_k x, \tilde{f}_k(z_k)) + \bar{M}\epsilon_k s_k \leq \bar{L} d_{i(k)}(y_k, z_k) + \bar{M}\epsilon_k s_k.$$

On the other hand, by assumption, there exists a filling V_k of Z_k whose support is within distance $\max\{h, \epsilon_k s_k\}$ from $\text{spt}(Z_k)$, where $h = h(C)$. This leads to the inequality $\delta \leq \bar{L} \max\{s_k^{-1}h, \epsilon_k\} + \bar{M}\epsilon_k$, which contradicts the fact that $s_k \rightarrow \infty$ and $\epsilon_k \rightarrow 0$. \square

The uniform statements in Sect. 5 and above can be combined to show that the implications in Theorem 1.1 that we proved through (AR_n) hold with constants independent of X . We exemplify this for $(FR_n) \Rightarrow (SII_n)$. If \mathcal{X} denotes the class of all proper metric spaces satisfying $(CI_n)[c]$ and (FR_n) for some fixed $c > 0$ and $h = h(C)$, then Proposition 7.5 shows that every sequence $(X_i)_{i \in \mathbb{N}}$ in \mathcal{X} has asymptotic rank $\leq n$. Hence, by Theorem 5.4, (SII_n) holds for some constant $M_0 = M_0(\mathcal{X}, n, c, \epsilon) > 0$, which depends only on n, c, ϵ and the function $h = h(C)$.

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