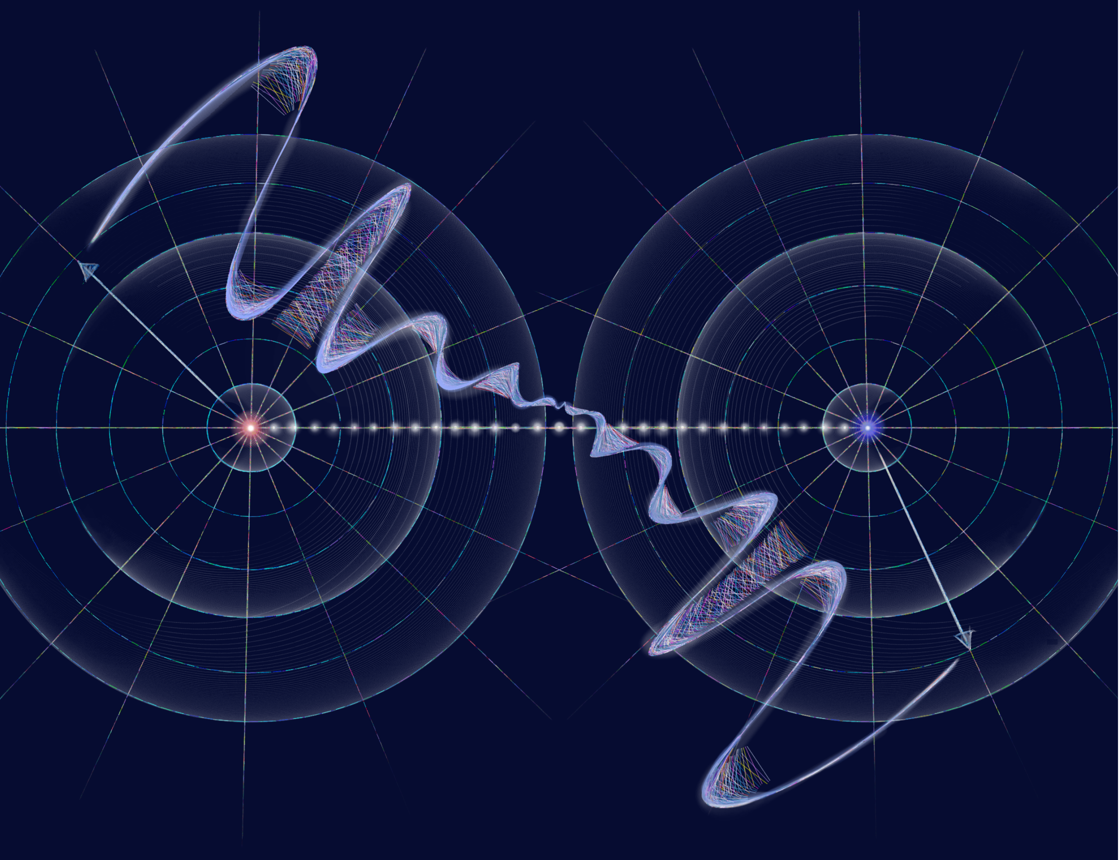
Fermionic Lieb-Schultz-Mattis Theorems and Invertible Phases of Matter

Ömer M. Aksoy



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FERMIONIC LIEB-SCHULTZ-MATTIS THEOREMS AND INVERTIBLE PHASES OF MATTER

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ÖMER M. AKSOY
MSc, ETH Zurich

born on 20.06.1994

accepted on the recommendation of

Prof. Dr. Manfred Sigrist
Prof. Dr. Christopher Mudry
Prof. Dr. Titus Neupert

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To my family

ABSTRACT

This Dissertation studies two closely related topics: fermionic Lieb-Schultz-Mattis (LSM) Theorems and invertible fermionic topological (IFT) phases of matter. The original LSM Theorem was proved in 1961 and applies to quantum antiferromagnetic spin chains. It ties the spectral properties of Hamiltonians to the *presence* of symmetries. It is reminiscent of the Nambu-Goldstone (NG) Theorem (1961), which ties the spectral properties to the *absence*, through spontaneous symmetry breaking, of symmetries. Whereas the NG Theorem had an immediate impact, the power of LSM Theorem was not truly appreciated until the prediction and discovery of crystalline topological phases a decade ago. In the last five years, the original LSM Theorem has been generalized to a set of powerful no-go theorems that provide insights for understanding both gapped phases of matter as well as quantum criticality. In this Dissertation, our **first original** result is the extensions of LSM Theorems to local lattice Hamiltonians built out of fermionic degrees of freedom in any spatial dimensions.

IFT phases of matter with an internal symmetry group G_f realize nondegenerate, gapped, and, G_f-symmetric ground states under periodic boundary conditions, with the caveat that under open boundary conditions nontrivial IFT phases support gapless boundary modes. LSM Theorems explain both features of IFT phases with crystalline symmetries. We apply the insights from fermionic LSM Theorems to identify topological invariants that enumerate all IFT phases of matter with internal symmetry group G_f in one-dimensional space. We derive the so-called fermionic stacking rules that dictate the addition rule for these topological indices. This allows us to compute the Abelian group structure formed by IFT phases with any symmetry group G_f, which is our **second original** result.

Finally, we formulate a correspondence between various (generalized) LSM Theorems and IFT phases with crystalline symmetries. We exemplify this correspondence by the explicit study of a two-dimensional topological superconductor protected by time-reversal, reflection, and, translation symmetries. Our **third original** result is the interpretation of protected gapless boundary modes of this topological superconductor as the consequence of an underlying LSM Theorem.

ABSTRAIT

Cette thèse étudie deux sujets étroitement liés : les théorèmes fermioniques de Lieb-Schultz-Mattis (LSM) et les phases topologiques fermioniques inversibles (IFT) de la matière. Le théorème LSM original a été démontré en 1961 et s'applique aux chaînes de spins antiferromagnétiques quantiques. Il lie les propriétés spectrales des Hamiltoniens à la présence de symétries. Il rappelle le théorème de Nambu-Goldstone (NG) (1961), qui lie les propriétés spectrales à l'absence, par rupture spontanée de symétrie, de symétries. Alors que le théorème NG a eu un impact immédiat, la puissance du théorème LSM n'a été véritablement appréciée qu'avec la prédiction et la découverte des phases topologiques cristallines il y a une décennie. Au cours des cinq dernières années, le théorème LSM original a été généralisé en un ensemble de théorèmes puissants qui fournissent des informations pour comprendre à la fois les phases de matière avec gap et la criticité quantique. Dans cette thèse, notre **premier résultat original** est l'extension des théorèmes LSM aux Hamiltoniens locaux de réseau construits à partir de degrés de liberté fermioniques dans n'importe quelle dimension spatiale.

Les phases IFT de la matière avec un groupe de symétrie interne G_f réalisent des états fondamentaux non dégénérés, à gap et G_f-symétriques sous des conditions aux bords périodiques, avec la réserve que sous des conditions aux bords ouvertes, des phases IFT non triviales soutiennent des modes de bord sans gap. Les théorèmes LSM expliquent les deux caractéristiques des phases IFT avec des symétries cristallines. Nous appliquons les informations tirées des théorèmes LSM fermioniques pour identifier les invariants topologiques qui énumèrent toutes les phases IFT de la matière avec le groupe de symétrie interne G_f dans un espace unidimensionnel. Nous dérivons les règles d'empilement fermioniques qui dictent la règle d'addition pour ces indices topologiques. Cela nous permet de calculer la structure du groupe abélien formée par les phases IFT avec n'importe quel groupe de symétrie G_f, ce qui est notre **deuxième résultat original**.

Enfin, nous formulons une correspondance entre divers théorèmes LSM (généralisés) et les phases IFT avec des symétries cristallines. Nous illustrons cette correspondance par l'étude explicite d'un supraconducteur topologique bidimensionnel protégé par des symétries de renversement du temps, de réflexion et de translation. Notre **troisième résultat original** est l'interprétation des modes de bord sans gap protégés de ce supraconducteur topologique comme conséquence d'un théorème LSM sous-jacent.

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INTRODUCTION

Human brain is hard-wired to recognize patterns. We categorize the objects in our environment according to the similarities they share. Therefore, it is natural for condensed-matter physicists to desire classifying the subject of their study, *matter*, into categories of shared attributes, *phases*. For a given quantum matter, these attributes are determined by the properties related to its ground states such as (i) the presence/absence of a *spectral gap*, (ii) ground-state *degeneracy*, (iii) *orders* (if any) that are supported by the ground states, (iv) and, nature of the low-lying (if any) excitations. Such properties can be probed by experiments such as low-temperature scattering or transport measurements, which can then be put in use to built various devices. For instance, in the band theory of noninteracting electrons subjected to periodic potential of positively charged background of an ionic crystal [1], the existence of a spectral gap defines the insulating phase, while its absence defines the metallic phase, which was an early triumph of solid-state physics that gave us the semiconductor industry.

From a theoretical point of view, to characterize the quantum phase realized by a given Hamiltonian, ideally one seeks to obtain all its eigenvalues and eigenstates. However, often in practice, the best one can hope for is an approximation for the ground state and some low-lying excited states, unless the Hamiltonian is exactly solvable for some particular reason. In fact, even the very first attribute listed above, i.e., the existence of a spectral gap, was shown impossible to numerically determine for generic Hamiltonians [2, 3]. This makes the classification of Hamiltonians into quantum phases of matter a nontrivial problem in physics.

One of the most fruitful ideas in characterizing phases of matter was the so-called Landau-Ginzburg (LG) paradigm [4, 5]. In this scheme, distinct quantum phases of matter are classified by symmetries preserved and broken by the ground states. More specifically, distinct phases are characterized by *local order parameters* which take nonzero expectation value in the ground state and transform nontrivially under the symmetries of the corresponding Hamiltonian. For example, a magnetically ordered phase is characterized by nonzero local magnetization which breaks O(3) spin-rotation and time-reversal symmetries, while in a paramagnetic phase local magnetization vanishes. We say that the O(3) and time-reversal symmetries are spontaneously broken in the ground state.

When studying ordered phases in the LG paradigm, a powerful tool is the Nambu-Goldstone (NG) Theorem [6–8]. It asserts that if a continuous symmetry is spontaneously broken in the ground state, then there exist at least one gapless branch of excitations [1]. These are the so-called Nambu-Goldstone bosons. For example, the ground state of two-dimensional Heisenberg ferromagnet is magnetically ordered. Correspondingly, O(3) symmetry of the Hamiltonian is spontaneously broken to O(2) subgroup by the ground state. There is a single branch of gapless excitations with a quadratic dispersion $\propto p^2$. In contrast, the ground state of two-dimensional Heisenberg antiferromagnet differs from this scenario in that while the symmetry-breaking pattern is the same (from O(3) to its O(2) subgroup), there are two branches of gapless excitations with linear dispersion $\propto |p|$ [10]. Both the nonvanishing local order parameter and the existence of NG bosons can be measured by an experimental probe that couples to the local order parameter. In the case of magnetic order, experimental probes include elastic and inelastic neutron scattering experiments, magnetic susceptibility measurements, muon-spin resonance, and, nuclear magnetic resonance spectroscopy.

From the point of view of LG paradigm and NG Theorem, phases described by non-degenerate, gapped, and, symmetric ground states appears to be rather "boring". The ground state being nondegenerate and symmetric means that there is no nonvanishing local order parameter. At temperatures below the spectral gap, the system does not respond strongly to external probes, i.e., it is insulating. The band insulators of noninteracting electronic systems fall into this category and were thought to be well-understood from a theory point of view. This naive perspective changed dramatically after the discovery of so-called *invertible topological phases*.

Invertible topological phases are described by local Hamiltonians with nondegenerate, gapped, and, symmetric ground states under periodic boundary conditions (PBC) [2]. Remarkably, under open boundary conditions (OBC), certain invertible topological phases support gapless degrees of freedom localized at the spatial boundaries. For this reason, these are called *nontrivial* invertible topological phases. These gapless boundary modes are robust against any symmetric interactions or disorder as long as the interaction/disorder strength is not comparable to the spectral gap when PBC are imposed. The adjective ''topological" here refers to (i) the sensitivity of the ground-state degeneracy or spectral

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1 More precisely, when there is Lorentz-invariance, for each generator of spontaneously-broken symmetry there exist a gapless excitation with linear dispersion. This is not correct when Lorentz-invariance is not present [9].

2 More generally, this condition must be satisfied on any closed space manifold, i.e., a compact topological manifold without boundary.

gap to the boundary conditions [3], (ii) and the robustness of the gapless degrees of freedom under symmetric perturbations. A paradigmatic example is the integer quantum Hall state (IQHS) [11]. Distinct IQHS are described by an integer-valued topological invariant $\nu \in \mathbb{Z}$, namely the Chern number [12]. It corresponds to the total Berry phase [13] accumulated over a closed loop around the first Brillouin zone [14]. When OBC are imposed, a nontrivial IQHS has $|\nu|$ gapless *chiral* boundary modes. The topological invariant ν is associated with the Hall conductance taking the quantized value $\sigma_{xy} = \nu\, e^2/h$ [15, 16].

While the realizations of IQHS require explicit or spontaneous breaking of time-reversal symmetry [17], it is also possible to realize time-reversal symmetric invertible topological phases. These examples include Kitaev's Majorana chains [18, 19], quantum spin Hall states [20–22], and, time-reversal invariant topological insulator [23]. Importantly, these examples imply that not all band insulators are the "same". They can be distinguished by what happens at their boundaries. This idea has become experimentally observable with the application of angle-resolved photoemission spectroscopy (ARPES) [24, 25] techniques which can probe the spectra of gapless boundary modes directly [26–28].

By their definition, invertible topological phases are not described by the LG paradigm as each class of invertible phases share the same set of symmetries. Similarly, NG Theorem does not apply as invertible topological phases are described by gapped and symmetric ground states. Yet, it turns out that symmetry can still be used as an organizing principle for the classification of invertible topological phases [29–36]. As we shall see invertible topological phases are characterized by the *anomalous* realization of symmetries at their boundaries.

In this Dissertation, we take this boundary point of view to study invertible fermionic topological (IFT) phases in one and two dimensions. Just as NG Theorem constraints the spectra of symmetry broken phases in the LG theory, we are going to show that Lieb-Schultz-Mattis (LSM) Theorem [37] and its extensions can be used to understand invertible topological phases in low-dimensional space. In Secs. 1.1 and 1.2, we give historical reviews of LSM theorems and classification of invertible topological phases. Sec. 1.3 concludes with the organization of the Dissertation.

3 One can equivalently say that the ground states of nontrivial invertible phases support nontrivial topology on closed manifolds.

1.1 REVIEW OF LIEB-SCHULTZ-MATTIS THEOREMS

The original Lieb-Schultz-Mattis (LSM) Theorem was proved in the appendix of Ref. [37]. In that form, LSM Theorem establishes that in the spectrum of one-dimensional spin-1/2 Heisenberg antiferromagnet there are excited states with small energies (vanishing in the thermodynamic limit) that are orthogonal to the ground state. Another theorem that is also proved in the same appendix states that the ground state is nondegenerate for any finite chain made out of even number of sites. These two theorems do not have control over the thermodynamic limit. For instance, they do not make any prediction about the absence of long-range order in the thermodynamic limit, which was later established by the Mermin-Wagner Theorem [38]. They also do not put any constraints on the nature of the excitations, as opposed to say NG Theorem which was proved around the same time. The gapless spinon dispersion was computed later [39] using Bethe Ansatz techniques [40].

A surge of interest for deeper understanding emerged after Haldane's conjecture that quantum antiferromagnet has gapped ground state for integer spin values as opposed to the belief that the spectrum is gapless irrespective of the spin quantum number [41, 42]. Experimental evidence confirming this conjecture was later obtained for spin-1 antiferromagnet [43]. Several generalizations then followed. For instance, it has been understood that the SO(3)-spin-rotation symmetry is not an essential requirement for an LSM constraint [44].

LSM type theorems for U(1)-number-conserving Hamiltonians have been established in arbitrary dimensions. These theorems state that systems with noninteger filling fraction, defined as the average number of particles per unit cell, cannot have a translationally invariant, nondegenerate, and short-range entangled ground state [45–52]. Similar constraints have also been worked out for number-conserving Hamiltonians that have non-symmorphic or magnetic space group symmetries [53–56]. A number of LSM type theorems pertaining to discrete internal symmetries combined with crystallographic symmetries have also been worked out [54–64].

A second surge of interest stemmed from the parallels between the boundary modes of invertible topological phases and LSM theorems [58, 60, 65–68]. When open boundary conditions are imposed, the effective low-energy quantum Hamiltonian governing the dynamics of the boundary modes of a nontrivial invertible topological phase supports a ground state that is either (i) gapless, (ii) symmetry-broken, (iii) or topologically ordered if the boundary is no less than two dimensional [31, 57, 69–72], which are reminiscent of the properties of ground states of the bulk Hamiltonians for which LSM type theorems apply. More precisely, the protected *gaplessness* of the boundary modes of weak and crystalline topological phases can be understood as a consequence of an LSM constraint on the boundary.

LSM-type theorems mentioned here can be grouped into two kinds. The first kind is the so-called *filling-constraint* type, which arises due to the presence of translation symmetry and a fractional filling per repeat unit cell which is defined with respect to a global continuous symmetry. For instance, this kind of LSM theorems apply to half-integer spin chains with translation symmetry and $\mathrm{U}(1)$ spin-rotation symmetry along z-axis, or lattice Hamiltonians with translation symmetry, $\mathrm{U}(1)$-charge conservation symmetry, and, a non-integer filling fraction. The second kind of LSM theorems are due to the global symmetries being represented in a *nontrivial projective* manner when restricted to the repeat unit cells of a lattice Hamiltonian. This kind of LSM theorems do not apply to the Hamiltonians with *only* global $\mathrm{U}(1)$ symmetry since the group $\mathrm{U}(1)$ has no nontrivial projective representations. However, both kinds apply to the half-integer spin chains with translation and $\mathrm{SO}(3)$-spin rotation symmetries.

1.2 REVIEW OF INVERTIBLE TOPOLOGICAL PHASES

We define and summarize the properties of invertible topological phases in Sec. 1.2.1. A historical review of various classification schemes of invertible topological phases is presented in Sec. 1.2.2.

1.2.1 *Properties of Invertible Topological Phases*

Invertible topological phases of matter are described by Hamiltonians that are spatially local and support a nondegenerate and gapped ground state on *any closed* spatial manifold [4] in the thermodynamic limit. Here, by locality it is meant that the Hamiltonian is a sum of terms each of which has a finite spatial support. Demanding that ground state to be nondegenerate on any closed manifold rules out the possibility of ground state degeneracies that depend on the topology of underlying space. This is to say that Hamiltonians describing invertible phases do not support nontrivial topological order. Hereby, we consider a tensor product Fock space $\mathfrak{F}$, i.e., $\mathfrak{F}$ is the tensor product of local Fock spaces $\mathfrak{F}_{\boldsymbol{j}}$ on sites $\boldsymbol{j}$ of a lattice Λ. An example ground state on $\mathfrak{F}$ that realize an invertible phase is the product state $|\psi_{\mathrm{triv}}\rangle$, which is obtained by taking the tensor

4 A compact manifold without boundary.

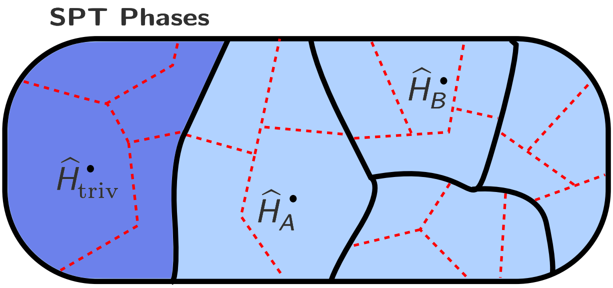

Figure 1.1: Invertible phases are equivalence classes of local Hamiltonians with nondegenerate and gapped ground states on any closed manifold under continuous gap-preserving deformations. The phase boundaries of invertible phases are the thick black lines. The $\widehat{H}_{\text{triv}}$ of which the ground state is a product state is assigned to the trivial invertible phase. Imposing additional symmetries may divide the invertible phases into further equivalence classes. The dashed red lines show the potential division of invertible phases when only the symmetric gap-preserving deformations are allowed. The symmetry protected topological (SPT) phases are the invertible phases that are trivial once the protecting symmetry is broken. The subspace of SPT phases is shown by the shaded area.

product of local states $|\psi_{\boldsymbol{j}}\rangle \in \mathfrak{F}_{\boldsymbol{j}}$. A local Hamiltonian for which $|\psi_{\text{triv}}\rangle$ is the ground state is

$$\widehat{H}_{\text{triv}} = -\sum_{\boldsymbol{j}\in\Lambda} |\psi_{\boldsymbol{j}}\rangle\langle\psi_{\boldsymbol{j}}|. \tag{1.1}$$

By convention, the product state $|\psi_{\text{triv}}\rangle$ is called the trivial state as it is the nondegenerate ground state of Hamiltonian $\widehat{H}_{\text{triv}}$ on any manifold independent of the boundary conditions [5]. Hamiltonians that have the form of $\widehat{H}_{\text{triv}}$ are representatives of the trivial invertible topological phase.

A continuous deformation of a local Hamiltonian is defined to include both the continuous change of short-range couplings between all existing local degrees of freedom or the addition (removal) of decoupled local degrees of freedom that realize a trivial invertible topological phase of their own. Any pair of Hamiltonians with nondegenerate and gapped ground states on any closed manifold are said to be equivalent if they can be continuously deformed into one another without closing the spectral gap. Invertible topological phases are then defined as the equivalence classes of such Hamiltonians under gap-preserving continuous

5 Here, the implicit assumption is that one can define a lattice Λ on the underlying space manifold.

deformations. Fig. 1.1 demonstrate the invertible phases in the space of local Hamiltonians with nondegenerate and gapped ground states on any closed manifold.

While any Hamiltonian realizing an invertible phase have a nondegenerate and gapped ground state on closed manifolds by definition, this is not so true under open boundary conditions. Any Hamiltonian realizing a nontrivial invertible topological phase must support gapless degrees of freedom that are localized at the boundaries. This implies either a ground state degeneracy or gapless states localized at the boundaries.

The classification of invertible topological phases can be enriched by imposing an internal (independent of space) symmetry group G such that two invertible phases are equivalent only if they can be continuously deformed to one another without gap closing and without (neither explicitly nor spontaneously) breaking the G symmetry. Those invertible topological phases that are equivalent to the trivial phase under the continuous deformation that spontaneously or explicitly break the G symmetry are called the *symmetry protected topological* (SPT) phases. When open boundary conditions are imposed, SPT phases support gapless degrees of freedom at the boundaries that are protected by the G symmetry, i.e., the boundary degrees of freedom cannot be gapped unless G symmetry is either explicitly or spontaneously broken.

We can impose an additional algebraic structure on invertible phases under a composition rule called the *stacking rule*. The stacking of any pair of invertible phases consists in creating a new invertible phase by defining the new local degrees of freedom to be the union of the local degrees of freedom from a representative of each invertible phase and by defining the new Hamiltonian acting on the new local degrees of freedom by taking the direct sum of the pair of representative Hamiltonians for each invertible phase. As defined here, stacking is an Abelian operation. The set of invertible phases is closed under the stacking operation. By definitions of invertible phases and gap-preserving continuous deformations, stacking of an invertible phase with the trivial phase is the invertible phase itself, i.e., trivial phase is the identity element of the stacking operation. For each invertible phase there exists an *inverse* phase, hence motivating the name invertible, such that stacking an inverse pair delivers the trivial phase. Therefore, the set of invertible phases together with stacking operation is endowed with an *Abelian group* structure.

1.2.2 *Classification Schemes for IFT Phases*

Even though not identified as such at the time, the first example of an invertible topological phase was polyacetylene, which supports zero-energy bound states localized at the domain walls of bond-density orders [73–75]. The study of topological phases was accelerated after the discovery of IQHS in 1980 [11], where quantized the Hall conductance taking integer values in units of e^2/h was observed.

Soon after the discovery of IQHS, another type of topological phase was discovered, namely, the Abelian fractional quantum hall state (FQHS) [76]. The Abelian FQHS with filling fraction $1/m$ (m odd) [77], supports gapless chiral edge modes and a fractional Hall conductance $\sigma_{xy} = e^2/(m\,h)$. While this is reminiscent of the IQHS, this FQHS has features that are not present in IQHS. For instance, it has has g^m-fold degenerate ground states on a two-dimensional closed manifold of genus g. It supports point point-like excitations with statistical angle $\theta = \pi/m$ [6] that carry fractional charge e/m. FQHS were later identified as examples of topologically ordered phases of matter [78, 79]. While invertible phases can exists in any spatial dimension, topological order is possible only in two or higher dimensional space. In general, the ground-state degeneracy for topological order depends on the topology of underlying manifold. They support excitations with fractional statistics and quantum numbers which are called *anyons* [7]. Anyonic excitations can only be created in inverse pairs by nonlocal operators. A defining feature of the topologically ordered phases is that any local operator acts as the identity when projected on the degenerate ground-state manifold. The adjective “topological” here refers to the sensitivity of the ground-state manifold to the topology of underlying spatial manifold under closed boundary conditions. When OBC are imposed, the boundary degrees of freedom can be either gapped or gapless. Some other examples are Kitaev’s toric code [80, 81] and Levin-Wen string-net models [82]. In this Dissertation, we only focus on invertible phases of matter which are characterized by nondegenerate and gapped ground states on any closed manifold.

The realization of quantum Hall physics relies on explicit or spontaneous breaking of reversal of time. The discovery of time-reversal symmetric quantum spin Hall state (QSHS) by Kane and Mele [20, 21], brought the subject to the domain of band insulators. Various generalizations of Kane and Mele followed both theoretically [22, 23, 83–87] and realized experimentally [26–28, 88–93] in two and three dimensional space.

A systematic classification of topological band insulators and superconductors has been obtained in Refs. [29, 30, 94, 95]. This classification scheme is based on the presence or absence of three discrete symmetries of single-particle Bloch Hamiltonians: time-reversal

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6 This means that the exchange two such excitations produces phase $e^{\mathrm{i}\theta}$ which is neither bosonic ($\theta = 2\pi$) nor fermionic ($\theta = \pi$) for $m > 1$.

7 Strictly speaking, anyons are only supported in two-dimensional space. topologically ordered phases in three or higher dimensions support string-like and brane-like excitations with statistics.

Table 1.1: Tenfold Way classification of topological insulators and superconductors. The leftmost column indicates the Cartan label of the symmetry class. $\mathcal{T} = \pm 1$ and $\mathcal{C} = \pm 1$ means that time-reversal symmetry (TRS) or particle-hole symmetry (PHS) are present in the corresponding symmetry class with $\mathcal{T}^2 = \pm 1$ and $\mathcal{C}^2 = \pm 1$, respectively. When $\mathcal{T} = 0$ or $\mathcal{C} = 0$, TRS or PHS are not present in the corresponding symmetry class. The entry $\mathcal{S} = 1$ indicates the presence of CHS. The entries $\mathbb{Z}$ or $\mathbb{Z}_2$ are the classification at space dimension d while the entry 0 means that only the trivial phase is possible.

Label	$\mathcal{T}$	$\mathcal{C}$	$\mathcal{S}$	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

symmetry (TRS), particle-hole symmetry (PHS), and, their combination that is called sublattice or chiral symmetry (CHS). The consideration of these three symmetries leads to the *Tenfold Way* classification which is summarized in Table 1.1. The left column shows the labels given to each class, which originate from the classification of symmetric spaces by Élie Cartan [96, 97]. Next to each label the presence or absence of each symmetry for that symmetry class is shown. $\mathcal{T}$, $\mathcal{C}$ or $\mathcal{S}$ corresponds to the single-particle representations of TRS, PHS, and, CHS, respectively. The entry 0 indicates that the corresponding transformation is not a symmetry for the class. $\mathcal{T} = \pm 1$ ($\mathcal{C} = \pm 1$) indicates that representation of TRS (PHS) symmetry squares to ± 1 on the single-particle basis. The corresponding many-body operators implementing TRS or PHS squares to identity operator for $+1$ or fermion parity operator for -1. The entry $\mathcal{S} = 0, 1$ means that the chiral symmetry is absent or present,respectively. The entries $\mathbb{Z}$ or $\mathbb{Z}_2$ corresponds to the number of distinct topological phases at space dimension d for each symmetry class. The classification is periodic in space dimension d. For the first two symmetry classes, A and AIII, the periodicity is two whereas for the rest it is eight. All examples of invertible topological phases we have discussed so far are realized by Hamiltonians built out of fermions. For this reason they are examples of invertible fermionic topological (IFT) phases.

The Tenfold Way classification can be extended in different ways. For instance, one can incorporate crystallographic symmetries such as point groups, reflection or inversion

Table 1.2: Exhaustive classification of IFT phases with symmetry classes of Tenfold Way in $d = 0, 1, 2, 3$ spatial dimension. The entries in black denotes the IFT phases for which the classification is the same as noninteracting classification of the Tenfold Way that is given in Table 1.1. The entries in magenta shows the reduction of noninteracting $\mathbb{Z}$ classification in the Tenfold Way. The entries in blue shows the interaction-enabled IFT phases that are only stabilized in the presence of interaction terms. These entries are all trivial in the Tenfold Way classification.

Label	d=0	d=1	d=2	d=3
A	$\mathbb{Z}$	0	$\mathbb{Z} \times \mathbb{Z}$	0
AIII	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_8 \times \mathbb{Z}_2$
AI	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}_8$	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
CII	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
C	0	0	$\mathbb{Z} \times \mathbb{Z}$	0
CI	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_4 \times \mathbb{Z}_2$

to the classification scheme. These lead to the so-called crystalline topological insulators and superconductors where gapless boundary states are protected by a combination of internal and crystallographic symmetries [98–106]. Another extension of the Tenfold Way classification is given by the so-called higher-order topological insulators (HOTI) [107–116]. HOTI in d-dimensional space support protected boundary states that are localized at spatial submanifold with lower than $d-1$ dimensions. For instance, a three-dimensional HOTI supports gapless states not at the two-dimensional boundary but at the one-dimensional hinges or zero-dimensional corners.

The protected boundary states of nontrivial IFT phases are stable against disorder provided that the disorder strength is not larger than the spectral gap in the pristine limit and evade Anderson localization [117]. For this reason, it was initially thought that the protected boundary states would also be stable against interactions. This presumption was shown to be wrong by Fidkowski and Kitaev who established that the noninteracting $\mathbb{Z}$ classification of symmetry class BDI reduces to $\mathbb{Z}_8$ when interactions are added [18, 19]. Following this work, breakdown of Tenfold Way classification and its various extensions have been shown [118–124].

A systematic classification of interacting IFT phases followed the classification of invertible *bosonic* topological (IBT) phases which can only be realized by interacting Hamiltonians. A paradigmatic example in one dimension is the Haldane phase of spin-1

Heisenberg antiferromagnet [41, 125] which was identified as a bosonic SPT (BSPT) phase [8] about three decades after its introduction [126]. After this identification all one dimensional BSPT phases with an arbitrary internal symmetry group were classified [57, 69, 70, 127]. The key idea in this classification scheme was that the protected zero modes at the boundary can be understood from the perspective of boundary projective representations. Therefore, the classification of one-dimensional BSPT phases is equivalent to the classification of projective representations of internal symmetry group which is provided by the second group cohomology. This idea was then generalized in Refs. [31, 128, 129] to a proposal for classification of BSPT phases in any spatial dimension, that is closely related to the classification of certain topological gauge theories in $(d+1)$-dimensional spacetime and $(d+1)$th cohomology group [130]. While this proposal exhaustively classifies all BSPT phases, there are IBT phases that are not SPTs, which are also called "beyond-cohomology phases", in two or higher space dimensions [71, 131–133].

It was proposed by Kapustin in Refs. [33, 34] that an exhaustive classification of IBT phases in any spatial dimension is provided by *cobordism* groups. This approach was then generalized to IFT phases in Ref. [35]. An exhaustive classification for all invertible phases of matter was obtained by Freed and Hopkins in Ref. [36] using stable homotopy theory to classify the invertible topological field theories. This classification delivers Table 1.2 for the symmetry classes in the Tenfold Way for space dimension $d < 3$. There, we colored the entries for which the classification of IFT phases is the same as in Table 1.1. The entries colored in magenta show the breakdown of Tenfold Way classification. Interestingly, there are certain IFT phases that can only be stabilized by interactions, which are colored in blue. For instance, for the class A in $d = 2$, the additional $\mathbb{Z}$ classification is because of quantum thermal hall conductance taking values independent of the quantum charge Hall conductance. This only occurs due to the presence of interactions and can be interpreted as the violation of Wiedemann-Franz law [134]. Other classification and construction schemes for IFT phases with internal and crystalline symmetries are obtained in Refs. [72, 135–150].

1.3 ORGANIZATION OF THE DISSERTATION

This Dissertation is organized in three Parts I, II, and, III, which are based on Refs. [151], [152], and, [153], respectively. Part I focuses on our **first original result**, namely, extension of LSM Theorem to fermionic systems with only discrete symmetries. Part

8 In one-dimensional space all IBT phases are SPTs.

II connects the Fermionic LSM Theorems with the classification of IFT phases in one-dimensional space. Therein, we derive the fermionic stacking rules of one-dimensional IFT phases with any internal symmetry group which is our **second original result**. Part III explores the relation between protected gapless boundary modes of crystalline IFT phases and generalized LSM-type Theorems. Our **third original result** is a concrete application of generalized LSM-type Theorems to a two-dimensional crystalline topological superconductor.

Part I

FERMIONIC LIEB-SCHULTZ-MATTIS THEOREMS

Adapted from:
Ö. M. Aksoy, A. Tiwari, and C. Mudry
"Lieb-Schultz-Mattis type theorems for Majorana models with discrete symmetries",
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Part I is dedicated to the Lieb-Schultz-Mattis (LSM) Theorems that apply to the Hamiltonians that are built out of fermionic degrees of freedom. We start in Chapter 2 by proving LSM Theorem 1, that applies to one-dimensional bosonic or fermionic models with U(1) symmetry. This is a *filling-fraction* type LSM Theorem that can be generalized to any continuous symmetry. It applies to fermionic systems with charge-conservation symmetry.

Chapter 3 presents the main result of this Part. Namely, we prove two LSM Theorems 2 and 3 that apply to one-dimensional Hamiltonians built out of fermions with or without continuous symmetries. As opposed to Theorem 1, these two theorems are due to *projective* nature of the local representations of symmetries. In particular, they do not apply to systems with only U(1) charge-conservation symmetry. Their full power is unleashed when applied to Hamiltonians that are only symmetric under discrete groups such as the mean-field treatment of superconductivity. In Chapter 3, Theorems 2 and 3 are also generalized to higher dimensions in a weaker form.

Chapter 4 introduces the concept of *intrinsically fermionic* LSM Theorems. These are a subset of Theorems 2 and 3 that only applies to Hamiltonians that are built out of fermions. We argue by way of example that intrinsically fermionic LSM Theorems in one-dimension disappear under a Jordan-Wigner (JW) transformation, which is a boson-fermion duality.

In the final Chapter 5, we construct various examples of Hamiltonians for which Theorem 2 applies.

2

LSM THEOREMS IN ONE DIMENSION WITH GLOBAL U(1) SYMMETRY

We will prove a general LSM theorem that applies to Hamiltonians that is supported on a one-dimensional lattice with L sites. The argument we present is adapted from Refs. [47, 49, 52]. We assume that each lattice site hosts p flavors of fermions or bosons. More concretely, we consider the Hamiltonian

$$\widehat{H} = -\sum_{i,j=1}^{L} \hat{\psi}^{\dagger}_{i}\, T_{ij}\, \hat{\psi}_{j} + \sum_{i=1}^{L} \widehat{V}_{i}, \tag{2.1a}$$

where

$$\hat{\psi}^{\dagger}_{i} = \begin{pmatrix} \hat{c}^{\dagger}_{i,1} & \hat{c}^{\dagger}_{i,2} & \cdots & \hat{c}^{\dagger}_{i,p} \end{pmatrix}, \qquad \hat{\psi}_{i} = \begin{pmatrix} \hat{c}_{i,1} & \hat{c}_{i,2} & \cdots & \hat{c}_{i,p} \end{pmatrix}^{\mathsf{T}}, \tag{2.1b}$$

are p-dimensional vectors of creation and annihilation operators, $i, j = 1, \cdots, L$ label the sites of a one-dimensional lattice. The algebra

$$\left\{\hat{c}_{i,\alpha}, \hat{c}^{\dagger}_{j,\beta}\right\} = \delta_{ij}\, \delta_{\alpha,\beta}, \qquad \left\{\hat{c}_{i,\alpha}, \hat{c}_{j,\beta}\right\} = 0, \tag{2.1c}$$

holds if the particles occupying each lattice site are fermionic, and the algebra

$$\left[\hat{c}_{i,\alpha}, \hat{c}^{\dagger}_{j,\beta}\right] = \delta_{ij}\, \delta_{\alpha,\beta}, \qquad \left[\hat{c}_{i,\alpha}, \hat{c}_{j,\beta}\right] = 0, \tag{2.1d}$$

holds if the particles occupying each lattice site are bosonic. The matrix T_{ij} is a $p \times p$ Hermitian matrix of hopping amplitudes that (i) are short-range with range r, and (ii) are bounded in magnitude from above, i.e., for $|i-j| > r$

$$\left[T_{ij}\right]_{\alpha\beta} = 0, \quad \alpha, \beta = 1, \cdots, p, \tag{2.1e}$$

and for $|i-j| \leq r$

$$\left| \left[T_{ij}\right]_{\alpha\beta} \right| \leq \bar{t}, \qquad \alpha, \beta = 1, \cdots, p, \tag{2.1f}$$

where $\bar{t}$ is a finite and positive real number.

The term $\widehat{V}_i$ in (2.1a) is an arbitrary short-range interaction term that is a function of rp number operators acting on the sites i, $i+1$, $\cdots$, $i+r$, i.e.,

$$\widehat{V}_i = \widehat{V}_i \left(\hat{n}_{i,1}, \cdots, \hat{n}_{i,p}, \hat{n}_{i+r,1}, \cdots, \hat{n}_{i+r,p}\right), \qquad \hat{n}_{i,\alpha} := \hat{c}^{\dagger}_{i,\alpha}\, \hat{c}_{i,\alpha}. \tag{2.1g}$$

With these definitions, the Hamiltonian (2.1a) possesses two symmetries. First, there is the global U(1) number conservation symmetry generated by the operator

$$\widehat{U}(\theta) := e^{\mathrm{i}\theta\, \widehat{N}}, \quad \widehat{N} := \sum_{i=1}^{L} \sum_{\alpha=1}^{p} \hat{n}_{i,\alpha} = \sum_{i=1}^{L} \hat{\psi}^{\dagger}_i\, \hat{\psi}_i, \quad \theta \in [0, 2\pi), \tag{2.2a}$$

with the action

$$\widehat{U}(\theta)\, \hat{c}_{i,\alpha}\, \widehat{U}^{\dagger}(\theta) = e^{-\mathrm{i}\theta}\, \hat{c}_{i,\alpha}. \tag{2.2b}$$

Second, there is the lattice translation symmetry generated by the operator $\widehat{T}$ which is defined by its action

$$\widehat{T}\, \hat{c}_{i,\alpha}\, \widehat{T}^{\dagger} = \hat{c}_{i+1,\alpha}, \qquad \widehat{T}^{L} = \mathbb{1}, \tag{2.3}$$

where the second equality imposes periodic boundary conditions. The global symmetry group then is $G_{\mathrm{tot}} = \mathrm{U}(1) \times \mathbb{Z}_L$.

Let $|\psi_{\mathrm{GS}}\rangle$ be a ground state of the Hamiltonian (2.1) that is both U(1) and translation symmetric. If so the identities

$$\widehat{U}(\theta)\, |\psi_{\mathrm{GS}}\rangle = e^{\mathrm{i}\theta\, N}\, |\psi_{\mathrm{GS}}\rangle, \qquad \widehat{T}\, |\psi_{\mathrm{GS}}\rangle = e^{\mathrm{i}\kappa}\, |\psi_{\mathrm{GS}}\rangle, \tag{2.4}$$

where N is the total-number of particles and $\kappa = 2\pi\, n/N$ with $n = 0, 1, \cdots, L-1$, hold. We define the filling fraction by

$$\nu := \frac{N}{L}, \tag{2.5}$$

such that $\nu \in [0, p]$ if the particles are fermionic and $\nu \in [0, \infty)$ if the particles are bosonic.

With these definitions, we will prove the following theorem (originally proved in Refs. [47, 49]).

Theorem 1. Let $\widehat{H}$ be a translationally invariant and U(1)-symmetric Hamiltonian of the form (2.1). Let $|\psi_{\mathrm{GS}}\rangle$ be a ground state of this Hamiltonian and satisfy symmetry conditions (2.4). Then there exists a variational state $|\psi_{\mathrm{var}}\rangle$ such that

(i) the variational energy E_{var} of $|\psi_{\mathrm{var}}\rangle$ compared to the energy E_{GS} of the ground state $|\psi_{\mathrm{GS}}\rangle$ is bounded from above as

$$E_{\mathrm{var}} - E_{\mathrm{GS}} = \langle\psi_{\mathrm{var}}|\widehat{H}\,|\psi_{\mathrm{var}}\rangle - \langle\psi_{\mathrm{GS}}|\widehat{H}\,|\psi_{\mathrm{GS}}\rangle \leq \frac{C}{L}, \tag{2.6a}$$

where C is a constant, and,

(ii) if, in addition, the filling fraction ν defined in (2.5) is not an integer, the state $|\psi_{\mathrm{var}}\rangle$ is orthogonal to the ground-state $|\psi_{\mathrm{GS}}\rangle$, i.e.,

$$\langle\psi_{\mathrm{var}}|\psi_{\mathrm{GS}}\rangle = 0. \tag{2.6b}$$

Proof. We define the unitary operator

$$\widehat{U}_{\mathrm{tw}} := e^{\mathrm{i}\sum_{i=1}^{L} \frac{2\pi\, i}{L} \hat{\psi}_i^{\dagger}\, \hat{\psi}_i}, \tag{2.7a}$$

and the variational state

$$|\psi_{\mathrm{var}}\rangle := \widehat{U}_{\mathrm{tw}}\,|\psi_{\mathrm{GS}}\rangle. \tag{2.7b}$$

The operator $\widehat{U}_{\mathrm{tw}}$ implements a site-dependent U(1) rotation. For the first statement (2.6a), we write the energy difference $\Delta E := E_{\mathrm{var}} - E_{\mathrm{GS}}$ as

$$\begin{aligned}
\Delta E &= \langle\psi_{\mathrm{var}}|\widehat{H}\,|\psi_{\mathrm{var}}\rangle - \langle\psi_{\mathrm{GS}}|\widehat{H}\,|\psi_{\mathrm{GS}}\rangle \\
&= \langle\psi_{\mathrm{GS}}|\left(\widehat{U}^{\dagger}_{\mathrm{tw}}\,\widehat{H}\,\widehat{U}_{\mathrm{tw}} - \widehat{H}\right)|\psi_{\mathrm{GS}}\rangle \\
&= \sum_{i,j=1}^{L}\left(1 - e^{\mathrm{i}\frac{2\pi}{L}(j-i)}\right)\langle\psi_{\mathrm{GS}}|\hat{\psi}_i^{\dagger}\,T_{ij}\,\hat{\psi}_j|\psi_{\mathrm{GS}}\rangle,
\end{aligned} \tag{2.8}$$

where the last line follows since the interaction term $\widehat{V}_i$ being a function of $\hat{n}_{i,\alpha}$ is invariant under local U(1) rotations. Since $|\psi_{\mathrm{GS}}\rangle$ is the ground state we have the bounds

$$0 \leq \langle\psi_{\mathrm{GS}}| \left(\widehat{U}_{\mathrm{tw}}\, \widehat{H}\, \widehat{U}^{\dagger}_{\mathrm{tw}} - \widehat{H}\right) |\psi_{\mathrm{GS}}\rangle, \tag{2.9a}$$

and

$$\begin{aligned}
\Delta E &\leq \langle\psi_{\mathrm{GS}}| \left(\widehat{U}^{\dagger}_{\mathrm{tw}}\, \widehat{H}\, \widehat{U}_{\mathrm{tw}} - \widehat{H}\right) |\psi_{\mathrm{GS}}\rangle + \langle\psi_{\mathrm{GS}}| \left(\widehat{U}_{\mathrm{tw}}\, \widehat{H}\, \widehat{U}^{\dagger}_{\mathrm{tw}} - \widehat{H}\right) |\psi_{\mathrm{GS}}\rangle \\
&= 2 \sum_{i,j=1}^{L} \left(1 - \cos\left(\frac{2\pi}{L}(j-i)\right)\right) \langle\psi_{\mathrm{GS}}|\hat{\psi}^{\dagger}_{i}\, T_{ij}\, \hat{\psi}_{j}|\psi_{\mathrm{GS}}\rangle \\
&\leq 2 \sum_{i,j=1}^{L} \left(1 - \cos\left(\frac{2\pi}{L}(j-i)\right)\right) \sum_{\alpha,\beta=1}^{p} \left| \left[T_{ij}\right]_{\alpha\beta} \right| \left|\langle\psi_{\mathrm{GS}}|\hat{c}^{\dagger}_{i,\alpha}\, \hat{c}_{j,\beta}|\psi_{\mathrm{GS}}\rangle\right|.
\end{aligned} \tag{2.9b}$$

We use Eqs. (2.1e) and (2.1f) to obtain

$$\Delta E \leq 2 \sum_{i=1}^{L} \sum_{j=i-r}^{i+r} \left(1 - \cos\left(\frac{2\pi}{L}(j-i)\right)\right) \bar{t} \sum_{\alpha,\beta=1}^{p} \left|\langle\psi_{\mathrm{GS}}|\hat{c}^{\dagger}_{i,\alpha}\, \hat{c}_{j,\beta}|\psi_{\mathrm{GS}}\rangle\right|, \tag{2.10a}$$

and Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\left|\langle\psi_{\mathrm{GS}}|\hat{c}^{\dagger}_{i,\alpha}\, \hat{c}_{j,\beta}|\psi_{\mathrm{GS}}\rangle\right| &\leq \sqrt{\langle\psi_{\mathrm{GS}}|\hat{n}_{i,\alpha}|\psi_{\mathrm{GS}}\rangle} \sqrt{\langle\psi_{\mathrm{GS}}|\hat{n}_{j,\beta}|\psi_{\mathrm{GS}}\rangle} \\
&\leq \sqrt{\langle\psi_{\mathrm{GS}}| \sum_{\alpha=1}^{p} \hat{n}_{i,\alpha}|\psi_{\mathrm{GS}}\rangle} \sqrt{\langle\psi_{\mathrm{GS}}| \sum_{\beta=1}^{p} \hat{n}_{j,\beta}|\psi_{\mathrm{GS}}\rangle} \\
&= \nu.
\end{aligned} \tag{2.10b}$$

Together, they deliver the bound

$$\Delta E \leq 2 \sum_{i=1}^{L} \sum_{j=i-r}^{i+r} \left(1 - \cos\left(\frac{2\pi}{L}(j-i)\right)\right) \bar{t}\, p^{2}\, \nu. \tag{2.10c}$$

Finally, using the identity

$$\sum_{j=i-r}^{i+r}\left(1-\cos\left(\frac{2\pi}{L}(j-i)\right)\right)\leq\frac{4\pi^2}{L^2}\sum_{j=i-r}^{i+r}(j-i)^2\leq 2r^3\,\frac{4\pi^2}{L^2}, \tag{2.11a}$$

we obtain the bound

$$\Delta E\leq\frac{16\,\pi^2\,r^3\,\bar{t}\,p^2\,\nu}{L}. \tag{2.11b}$$

This proves the first part of Theorem 1.

For the second part, we compute

$$\langle\psi_{\mathrm{var}}|\psi_{\mathrm{GS}}\rangle=\langle\psi_{\mathrm{GS}}|\,\widehat{U}^{\dagger}_{\mathrm{tw}}\,|\psi_{\mathrm{GS}}\rangle=\langle\psi_{\mathrm{GS}}|\,\widehat{T}\,\widehat{U}^{\dagger}_{\mathrm{tw}}\,\widehat{T}^{\dagger}|\psi_{\mathrm{GS}}\rangle, \tag{2.12a}$$

where we have used that the state $|\psi_{\mathrm{GS}}\rangle$ is translationally invariant by assumption. The algebra

$$\begin{aligned}\widehat{T}\,\widehat{U}^{\dagger}_{\mathrm{tw}}\,\widehat{T}^{\dagger}&=e^{-\mathrm{i}\sum_{i=1}^{L}\frac{2\pi\,i}{L}\hat{\psi}^{\dagger}_{i+1}\,\hat{\psi}_{i+1}}\\&=e^{-\mathrm{i}\sum_{i=1}^{L}\frac{2\pi\,i+1}{L}\hat{\psi}^{\dagger}_{i+1}\,\hat{\psi}_{i+1}}\,e^{+\mathrm{i}\sum_{i=1}^{L}\frac{2\pi}{L}\hat{\psi}^{\dagger}_{i}\,\hat{\psi}_{i}}\\&=\widehat{U}^{\dagger}_{\mathrm{tw}}\,e^{+\mathrm{i}\frac{2\pi\,\widehat{N}}{L}},\end{aligned} \tag{2.12b}$$

implies

$$\begin{aligned}\langle\psi_{\mathrm{var}}|\psi_{\mathrm{GS}}\rangle&=\langle\psi_{\mathrm{GS}}|\,\widehat{U}^{\dagger}_{\mathrm{tw}}\,e^{+\mathrm{i}\frac{2\pi\,\widehat{N}}{L}}\,|\psi_{\mathrm{GS}}\rangle\\&=e^{+\mathrm{i}\frac{2\pi\,N}{L}}\,\langle\psi_{\mathrm{GS}}|\,\widehat{U}^{\dagger}_{\mathrm{tw}}\,|\psi_{\mathrm{GS}}\rangle\\&=e^{+\mathrm{i}2\pi\,\nu}\,\langle\psi_{\mathrm{var}}|\psi_{\mathrm{GS}}\rangle.\end{aligned} \tag{2.12c}$$

Unless ν is an integer the exponential does not vanish. For any fractional filling $n\notin\mathbb{Z}$ we have

$$\langle\psi_{\mathrm{var}}|\psi_{\mathrm{GS}}\rangle=0, \tag{2.12d}$$

as claimed in Theorem 1. ☐

The main intuition behind Theorem 1 is the following. Due to the presence of a global continuous symmetry, a local smooth twist by U(1)-rotations is expected to cost little energy. A low-energy variational state then can be constructed by applying the twist operator on a translationally invariant ground state [1]. The corresponding variational energy is vanishing in the $L \to \infty$ limit. However, the variational estimate alone does not guarantee that the states $|\psi_{\text{var}}\rangle$ and $|\psi_{\text{GS}}\rangle$ remain distinct in the thermodynamic limit. The nontrivial observation is that if the filling fraction ν is not an integer, the two states are necessarily orthogonal.

Theorem 1 can be interpreted as the absence of a ground state that is U(1)-symmetric, translationally invariant, nondegenerate, and, separated by a gap from all excited states for any Hamiltonian with the same symmetries at a fractional filling. However, as stated, Theorem 1 does not make any controlled prediction on the ground state in the thermodynamic limit. In Ref. [52], this theorem was generalized by using local twist operators as opposed to a global twist operator (2.7a). It was shown that provided the ground state does not break the translation symmetry, there are infinitely many low-lying states that are orthogonal to the ground state $|\psi_{\text{GS}}\rangle$ and have variational energies vanishing in the limit $L \to \infty$. This means that one of the following three possibilities must occur for the ground state in the thermodynamic limit:

1. The spectrum is gapped and the translation symmetry is spontaneously broken in the ground state manifold.

2. The spectrum is gapless with infinitely many states with arbitrarily small energy above the ground state.

3. The spectrum is gapped and there are infinitely many degenerate ground states [2].

In reaching this conclusion, we proved two separate statements of Theorem 1. First, we showed that the variational energy of the state $|\psi_{\text{var}}\rangle$ is small by using the dynamical data, i.e., our assumptions on the Hamiltonian (2.1) [3]. This argument is independent of the filling fraction since the states $|\psi_{\text{var}}\rangle$ and $|\psi_{\text{GS}}\rangle$ carry same total occupation number.

1 Observe that the total accumulated U(1) phase under the local twist operator (2.7a) is 2π. Indeed, this is nothing but a large gauge transformation that maps the Hamiltonian (2.1) with zero flux through the periodic chain to the same Hamiltonian with one quantum of magnetic flux threading the periodic chain. If the ground state $|\psi_{\text{GS}}\rangle$ is separated by a gap from the excited states, an adiabatic insertion of one quantum of flux should not change the spectral properties. This is to say that $|\psi_{\text{var}}\rangle$ should be either proportional or orthogonal to the ground state $|\psi_{\text{GS}}\rangle$. This argument was used in Refs. [48, 154] to obtain various LSM type theorems in one and higher dimensions.

2 For example, a flat band with noninteger filling falls into this category.

3 This proof can be extended to any Hamiltonian provided that it is local, its matrix elements are bounded and it has U(1) and translation symmetries.

Second, we have showed that when the many-body Fock space $\mathfrak{F}$ is restricted to the Hilbert space $\mathfrak{H}_N$ of N particles, the variation state $|\psi_{\text{var}}\rangle \in \mathfrak{H}_N$ is necessarily orthogonal to $|\psi_{\text{GS}}\rangle$ if $\nu \equiv N/L$ is not an integer. This argument is related to the structure of the N-particle Hilbert space $\mathfrak{H}_N$ and is independent of the energy spectrum. In the remaining of this Chapter, we are going to generalize Theorem 1 to models with discrete symmetry groups. Due to the discreteness of the symmetries involved, there is no notion of a smooth twist operator. In other words, our argument for the first part of Theorem 1 fails immediately. However, as we shall see, the *local* representations of global symmetries constrain the states in a many-body Fock space. A statement analogous to the second part of Theorem 1 can still be made. To establish LSM type theorems, we are going to present arguments alternative to the variational estimate used in proving Theorem 1.

3

LSM THEOREMS WITH GLOBAL DISCRETE SYMMETRIES

In Sec. 2, we proved an LSM type theorem that applies to Hamiltonians with a continuous internal symmetry group. For translationally invariant Hamiltonians with particle number conservation, there is always a global U(1) symmetry and a corresponding LSM constraint. In this section, we will consider Hamiltonians for which there is no necessarily a continuous internal symmetry group e.g., the mean-field Hamiltonian for a superconductor where the global charge conservation symmetry is broken down to fermion parity symmetry. It is particularly useful to represent such Hamiltonians in terms of Majorana degrees of freedom. In what follows, we consider lattice Hamiltonians that are built out of fermionic degrees of freedom. To each site, we attach a number of Majorana degrees of freedom with the constraint that the total number of Majorana degrees of freedom is even. This constraint is required have a well-defined global fermionic Fock space. On such Hamiltonians, we impose (i) the translation symmetry G_{trsl} of the corresponding lattice, and (ii) a global symmetry group G_f [1].

We first consider the case of one-dimensional space, and prove the following two theorems in Sec. 3.3.

Theorem 2. Any one-dimensional lattice Hamiltonian that is local and admits the symmetry group $G_{\mathrm{trsl}} \times G_f$ cannot have a nondegenerate, gapped, and $G_{\mathrm{trsl}} \times G_f$-symmetric ground state that can be described by an even- or odd-parity injective fermionic matrix product state if G_f is realized by a nontrivial projective representation on the local Fock space.

Theorem 3. A local Majorana Hamiltonian (under periodic boundary conditions) with an odd number of Majorana degrees of freedom per repeat unit cell that is invariant under the symmetry group $G_{\mathrm{trsl}} \times G_f$, cannot have a nondegenerate, gapped and translationally invariant ground state.

Several comments are due. Theorem 2 relies on two key concepts: (i) injective fermionic matrix product states (FMPS) and (ii) local projective representations of a group G_f. In one-dimensional space, injective FMPS can be thought as representatives of nondegenerate

1 Here, subscript f stands for a *fermionic* symmetry group, which is explained in detail in Sec. 3.1.1.

and gapped ground states of local Hamiltonians. As opposed to the proof of Theorem 1, where a low-lying variational state is constructed, the proof of Theorem 2 utilizes the properties of injective FMPS to obtain *necessary* conditions to for local Hamiltonians to support nondegenerate, gapped, and, symmetric ground states [2]. Similarly, a *nontrivial* local projective representation plays a role in Theorem 2 that is similar to the role played by *noninteger* filling fraction in Theorem 1. Despite these parallels, Theorem 2 is not a generalization of Theorem 1 since the group U(1) has no projective representations.

Theorem 3 applies to the case where each repeat unit cell contains odd number of Majorana degrees of freedom, where for consistency, we demand the total number of sites to be even. In contrast, Theorem 2 assumes that each repeat cell in the lattice supports a well-defined local fermionic Fock space, i.e., each repeat unit cell contains an even number of Majorana degrees of freedom. In this case, Theorem 3 is inactive.

As stated, Theorem 3 applies to any dimension of space without any restriction on the internal fermionic symmetry group G_f. As for Theorem 2, a weaker form of it holds in any dimension if it is assumed that G_f is Abelian and can be realized locally using unitary operators. Versions of both theorems that apply to any dimension of space are proved using tilted and twisted boundary conditions in Sec. 3.4.

The direct product structure of the symmetry group $G_{\text{trsl}} \times G_f$ is crucial in Theorems 2 and 3, and their generalizations to higher dimensions. Indeed, it has been shown that when the total symmetry group does not have a direct product structure, such as is the case with magnetic translation symmetries, a symmetric, nondegenerate, gapped, and short-range entangled ground state is not ruled out when closed-boundary conditions are imposed [61, 155, 156]. However, such a short-range entangled ground state must then necessarily support gapless symmetry-protected boundary states when open boundary conditions are imposed.

We present an overview of fermionic symmetry groups and their representations in Sec. 3.1. We then introduce the framework of FMPS in Sec. 3.2 which is central to our proofs. The proofs of Theorems 2 and 3 are given in Sec. 3.3 while their generalizations to spatial dimensions larger than one are given in Sec. 3.4.

---

2 The contraposition of Theorem 2 gives a set of necessary conditions for local Hamiltonians to have such ground states.

3.1 REPRESENTATIONS OF FERMIONIC SYMMETRY GROUPS

The fermionic symmetry group G_f and its local representations play a central role in Theorems 2 and 3. In this Section, we will first describe what is meant by a fermionic symmetry group and classification of its representations.

3.1.1 *Fermionic Symmetry Groups*

For quantum systems built out of an even number of local Majorana operators, it is always possible to express all Majorana operators as the real and imaginary parts of local fermionic creation or annihilation operators. We seek to describe the structure of *internal* symmetries with *onsite* action on quantum systems built out of an even number of Majorana operators. In the context of lattice models, internal symmetries are the transformations that preserve locality in the sense that local operators are mapped to nearby local operators. By symmetries with "onsite" action, we mean that the unitary operators implementing the transformations can be written as a composition of unitary operators each of which act on disjoint subsets of adjacent lattice sites [3] [4] [5]. Examples of internal symmetries with an onsite action include, time-reversal or spin-rotation symmetries. We denote the group of all such internal symmetries that are imposed on a fermionic quantum system by G_f.

The parity (evenness or oddness) of the total fermion number is always a constant of the motion. This is to say that any symmetry group G_f contains the fermion parity symmetry at its center, i.e.,

$$G_f \supset \mathbb{Z}_2^{\mathrm{F}} := \{e, p \,|\, e\,p = p\,e = p, \quad e = e\,e = p\,p\}\,, \;\; p\,g = g\,p, \quad \forall g \in G_f, \tag{3.1a}$$

where e is the identity element and we shall interpret the representation of p as the fermion parity operator. It is because of this interpretation of the group element p that we attach the upper index F to the cyclic group $\mathbb{Z}_2$. We denote by G the group that consists any

3 Equivalently, a symmetry acts onsite if the lattice can be divided into disjoint sets of adjacent sites such that the set of local operators supported over each disjoint set is closed under the symmetry transformations.

4 For symmetries that are represented by antiunitary operators, e.g., reversal of time, we call the action onsite if the unitary part of the representation is acting onsite.

5 A symmetry can preserve locality while mapping an operator supported on a single site to an operator acting on several nearby lattice sites. When this happens, the symmetry is said to have a 't Hooft anomaly. See Ref. [128] for an example of anomalous $\mathbb{Z}_2$ symmetry at the one-dimensional boundary of a two-dimensional topological phase.

internal symmetry in G_f other than the fermion parity. The group G is isomorphic to the quotient of G_f by $\mathbb{Z}_2^{\mathrm{F}}$, i.e.,

$$G \cong G_f/\mathbb{Z}_2^{\mathrm{F}}. \tag{3.1b}$$

However, G_f does *not* necessarily has the form of a direct product $G \times \mathbb{Z}_2^{\mathrm{F}}$. For a simple counterexample, consider the groups $G = \mathbb{Z}_2$ and $G_f = \mathbb{Z}_4^{\mathrm{F}}$. Here, the superscript F implies that the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ is contained in the center of $\mathbb{Z}_4$. One observes that

$$G \cong G_f/\mathbb{Z}_2^{\mathrm{F}} = \mathbb{Z}_2, \qquad G_f \ncong \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}, \tag{3.2a}$$

i.e., G_f is not isomorphic to the direct product of G with $\mathbb{Z}_2^{\mathrm{F}}$. In general, given a group G, the fermionic symmetry group G_f is constructed from the central extension of G by $\mathbb{Z}_2^{\mathrm{F}}$. As a set, elements of G_f are given by the pairs $(g,h) \in G \times \mathbb{Z}_2^{\mathrm{F}}$ with the composition rule

$$(g_1,\, h_1) \underset{\gamma}{\circ} (g_2,\, h_2) := \Big(g_1\, g_2, h_1\, h_2\, \gamma(g_1, g_2)\Big), \tag{3.3a}$$

where γ is a map

$$\begin{aligned} \gamma\colon G \times G \ &\to \mathbb{Z}_2^{\mathrm{F}}, \\ (g_1, g_2) \ &\mapsto \gamma(g_1, g_2), \end{aligned} \tag{3.3b}$$

such that

$$\gamma(e,g) = \gamma(g,e) = e, \quad \gamma(g^{-1},g) = \gamma(g,g^{-1}), \tag{3.3c}$$

for all $g \in G$ and

$$\gamma(g_1, g_2)\, \gamma(g_1\, g_2, g_3) = \gamma(g_1, g_2\, g_3)\, \gamma(g_2, g_3), \tag{3.3d}$$

for all $g_1, g_2, g_3 \in G$ [6]. We write any fermionic symmetry group G_f as [7]

$$G_f = G \underset{\gamma}{\times} \mathbb{Z}_2^{\mathrm{F}}. \tag{3.3e}$$

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6 With an abuse of notation, we denote by e the identity element in both G and $\mathbb{Z}_2^{\mathrm{F}}$. From this point on, we will denote the identity element of any group by e.

7 For the example in Eq. (3.2a) we have the map

$$\gamma(e,e) = \gamma(e,g) = \gamma(g,e) = e, \qquad \gamma(g,g) = p,$$

The conditions (3.3c) and (3.3e) ensure that G_f has a group structure, i.e., composition rule (3.3a) is compatible with neutral element, inverse, and, associativity.

Given a group G, two fermionic symmetry groups G_f and G'_f generated by maps γ and γ', respectively, are isomorphic, i.e.,

$$G_f = G \underset{\gamma}{\times} \mathbb{Z}_2^{\mathrm{F}} \cong G \underset{\gamma'}{\times} \mathbb{Z}_2^{\mathrm{F}} = G'_f \tag{3.4a}$$

if there exists a map

$$\begin{aligned} \kappa \colon G &\to \mathbb{Z}_2^{\mathrm{F}}, \\ g &\mapsto \kappa(g), \end{aligned} \tag{3.4b}$$

such that the identity

$$\kappa(g_1\, g_2)\, \gamma(g_1, g_2) = \kappa(g_1)\, \kappa(g_2)\, \gamma'(g_1, g_2) \tag{3.4c}$$

holds for all $g_1, g_2 \in G$, see Appendix A.2. This group isomorphism defines an equivalence relation. We say that the group G_f obtained by extending the group G with the group $\mathbb{Z}_2^{\mathrm{F}}$ through the map γ splits when a map (3.4b) exists such that

$$\kappa(g_1\, g_2)\, \gamma(g_1, g_2) = \kappa(g_1)\, \kappa(g_2) \tag{3.4d}$$

for all $g_1, g_2 \in G$, i.e., G_f splits when it is isomorphic to the direct product $G \times \mathbb{Z}_2^{\mathrm{F}}$.

The task of classifying all the nonequivalent central extensions of G by $\mathbb{Z}_2^{\mathrm{F}}$ through γ is achieved by enumerating all the elements of the second cohomology group $H^2\left(G, \mathbb{Z}_2^{\mathrm{F}}\right)$, see Appendix A.2. We define an index $[\gamma] \in H^2\left(G, \mathbb{Z}_2^{\mathrm{F}}\right)$ to represent such an equivalence class, whereby the index $[\gamma] = 0$ is assigned to the case when G_f splits. We conclude this section by giving some examples of frequently encountered internal symmetry groups G_f when treating fermions in condensed matter physics (see Appendix A.3 for the details):

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where e is the identity element and g is the nontrivial element in $G = \mathbb{Z}_2$. One observes that with respect to the composition rule defined with this γ, the element $(g, e) \in \mathbb{Z}_4^{\mathrm{F}}$ has order 4, i.e.,

$$\begin{aligned} (g, e)^2 &= (g, e) \underset{\gamma}{\circ} (g, e) = (e, p), \\ (g, e)^3 &= (e, p) \underset{\gamma}{\circ} (g, e) = (g, p), \\ (g, e)^4 &= (g, p) \underset{\gamma}{\circ} (g, e) = (e, e). \end{aligned}$$

1. The charge conservation symmetry $G_f = \mathrm{U}(1)^{\mathrm{F}}$ is the nonsplit central extension of $G = \mathrm{U}(1)$ by $\mathbb{Z}_2^{\mathrm{F}}$.

2. The spin rotation symmetry for spinful electrons $G_f = \mathrm{SU}(2)^{\mathrm{F}}$ is the nonsplit central extension of $G = \mathrm{SO}(3)$ by $\mathbb{Z}_2^{\mathrm{F}}$.

3. The time-reversal symmetry for spinful electrons $G_f = \mathbb{Z}_4^{\mathrm{FT}}$ is the nonsplit central extension of $G = \mathbb{Z}_2^{\mathrm{T}}$ by $\mathbb{Z}_2^{\mathrm{F}}$ (We denote by $\mathbb{Z}_2^{\mathrm{T}}$, the $\mathbb{Z}_2$ group where the nontrivial element $t \in \mathbb{Z}_2^{\mathrm{T}}$ is represented by an antiunitary operator).

4. The time-reversal symmetry for spinless electrons $G_f = \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ is the split central extension of $G = \mathbb{Z}_2^{\mathrm{T}}$ by $\mathbb{Z}_2^{\mathrm{F}}$.

3.1.2 *Projective Representations of* G_f

We denote with Λ a d-dimensional lattice with $j \in \mathbb{Z}^d$ labeling the repeat unit cells. We are going to attach to Λ a Fock space on which projective representations of the group G_f constructed in Sec. 3.1.1 are realized. This will be done using four assumptions.

Assumption 1. We attach to each repeat unit cell $j \in \Lambda$ the local Fock space $\mathfrak{F}_j$. This step requires that the number of Majorana degrees of freedom in each repeat unit cell is even. It is then possible to define the local fermion number operator $\hat{f}_j$ and the local fermion-parity operator

$$\hat{p}_j := (-1)^{\hat{f}_j}. \tag{3.5}$$

We assume that all local Fock spaces $\mathfrak{F}_j$ with $j \in \Lambda$ are "identical," in particular they share the same dimensionality $\mathcal{D}$. This assumption is a prerequisite to imposing translation symmetry.

Assumption 2. Each repeat unit cell $j \in \Lambda$ is equipped with a representation $\hat{u}_j(g)$ of G_f through the conjugation

$$\hat{o}_j \;\mapsto\; \hat{u}_j(g)\,\hat{o}_j\,\hat{u}_j^{\dagger}(g), \qquad [\hat{u}_j(g)]^{-1} = \hat{u}_j^{\dagger}(g), \tag{3.6a}$$

of any operator $\hat{o}_j$ acting on the local Fock space $\mathfrak{F}_j$. The representation (3.6a) of $g \in G_f$ can either be unitary or antiunitary. More precisely, let

$$\begin{aligned} \mathfrak{c} : G_f \;&\to\; \{0,1\}\,, \\ g \;&\mapsto\; \mathfrak{c}(g), \end{aligned} \tag{3.6b}$$

be a homomorphism. We then have the decomposition

$$\hat{u}_j(g) := \begin{cases} \hat{v}_j(g), & \text{if } \mathfrak{c}(g) = 0, \\ \\ \hat{v}_j(g)\,\mathsf{K}, & \text{if } \mathfrak{c}(g) = 1, \end{cases} \tag{3.6c}$$

where

$$\hat{v}_j^{-1}(g) = \hat{v}_j^{\dagger}(g), \qquad \hat{p}_j\,\hat{v}_j(g)\,\hat{p}_j = (-1)^{\rho(g)}\,\hat{v}_j(g), \tag{3.6d}$$

is a unitary operator with the fermion parity $\rho(g) \in \{0,1\} \equiv \mathbb{Z}_2$ acting linearly on $\mathfrak{F}_j$ and K denotes complex conjugation on the local Fock space $\mathfrak{F}_j$. Accordingly, the homomorphism $\mathfrak{c}(g)$ dictates if the representation of the element $g \in G_f$ is implemented through a unitary operator $[\mathfrak{c}(g) = 0]$ or an antiunitary operator $[\mathfrak{c}(g) = 1]$. Finally, we always choose to represent locally the fermion parity $p \in \mathbb{Z}_2^{\mathrm{F}}$ by the Hermitian operator $\hat{p}_j$,

$$\hat{u}_j(p) := \hat{p}_j \equiv (-1)^{\hat{f}_j}. \tag{3.6e}$$

Assumption 3. For any two elements $g, h \in G_f$ [to simplify notation, $g \underset{\gamma}{\circ} h \equiv g\,h$ for all $g, h \in G_f$], whereby $e = g\,g^{-1} = g^{-1}\,g$ denotes the neutral element and $g^{-1} \in G_f$ the inverse of $g \in G_f$, we postulate the projective representation

$$\hat{u}_j(e) = \hat{\mathbb{1}}_{\mathcal{D}}, \tag{3.7a}$$

$$\hat{u}_j(g)\,\hat{u}_j(h) = e^{\mathrm{i}\phi(g,h)}\,\hat{u}_j(g\,h), \tag{3.7b}$$

$$\Big[\hat{u}_j(g)\,\hat{u}_j(h)\Big]\,\hat{u}_j(f) = \hat{u}_j(g)\Big[\hat{u}_j(h)\,\hat{u}_j(f)\Big], \tag{3.7c}$$

whereby the identity operator acting on $\mathfrak{F}_j$ is denoted $\hat{\mathbb{1}}_{\mathcal{D}}$ and the function

$$\begin{aligned} \phi : G_f \times G_f &\to [0, 2\pi), \\ (g,h) &\mapsto \phi(g,h), \end{aligned} \tag{3.8a}$$

must be compatible with the associativity in G_f, i.e.,

$$\phi(g,h) + \phi(g\,h, f) = \phi(g, h\,f) + (-1)^{\mathfrak{c}(g)}\,\phi(h,f), \tag{3.8b}$$

for all $g, h, f \in G_f$. The map ϕ taking values in $[0, 2\pi)$ and satisfying (3.8b) is an example of a *2-cocycle* with the group action specified by the $\mathbb{Z}_2$-valued homomorphism $\mathfrak{c}$. In the vicinity of the value 0, ϕ generates the Lie algebra $\mathfrak{u}(1)$. The associated Lie group is denoted U(1). Given the neutral element $e \in G_f$, a *normalized 2-cocycle* obeys the additional constraint

$$\phi(e, g) = \phi(g, e) = 0 \tag{3.8c}$$

for all $g \in G_f$. Two 2-cocycles $\phi(g, h)$ and $\phi'(g, h)$ are said to be equivalent if they can be consistently related through a map

$$\begin{aligned} \xi \colon G_f &\to [0, 2\pi), \\ g &\mapsto \xi(g), \end{aligned} \tag{3.9}$$

as follows. The equivalence relation $\phi \sim \phi'$ holds if the transformation

$$\hat{u}(g) = e^{\mathrm{i}\xi(g)}\, \hat{u}'(g), \tag{3.10a}$$

implies the relation

$$\phi(g, h) - \phi'(g, h) = \xi(g) + (-1)^{\mathfrak{c}(g)}\, \xi(h) - \xi(g\, h), \tag{3.10b}$$

between the 2-cocycle $\phi(g, h)$ associated to the projective representation $\hat{u}(g)$ and the 2-cocycle $\phi'(g, h)$ associated to the projective representation $\hat{u}'(g)$. In particular, $\hat{u}$ is equivalent to an ordinary representation (a trivial projective representation) if $\phi'(g, h) = 0$ for all $g, h \in G_f$. Any $\phi \sim 0$ is called a coboundary. For any coboundary ϕ there must exist a ξ such that

$$\phi(g, h) = \xi(g) + (-1)^{\mathfrak{c}(g)}\, \xi(h) - \xi(g\, h). \tag{3.11}$$

The space of equivalence classes of projective representations is obtained by taking the quotient of 2-cocycles (3.8) by coboundaries (3.11). The resulting set is the second cohomology group $H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$, which has an additive group structure. Appendix A.1 gives more details on group cohomology.

Assumption 4. We attach to Λ the global Fock space $\mathfrak{F}_\Lambda$ by taking the appropriate product over j of the local Fock spaces $\mathfrak{F}_j$. This means that we impose some algebra on all local operators differing by their repeat unit cell labels. We then define the operator

$$\widehat{U}(g) := \begin{cases} \prod_{j\in\Lambda} \hat{v}_j(g)\,[\hat{u}_j(p)]^{\rho(g)}, & \text{if } \mathfrak{c}(g)=0, \\ \left[\prod_{j\in\Lambda} \hat{v}_j(g)\,[\hat{u}_j(p)]^{\rho(g)}\right]\mathsf{K}, & \text{if } \mathfrak{c}(g)=1, \end{cases} \tag{3.12a}$$

that implements globally on the Fock space $\mathfrak{F}_\Lambda$ the operation corresponding to the group element $g \in G_f$. The decomposition (3.12a) ensures that the global representation $\widehat{U}(g)$ and the local representation $\hat{u}_j(g)$ defined in Eq. (3.6c) implement the same transformation rules on the local Majorana degrees of freedom at site j, i.e., the equation

$$\widehat{U}(g)\,\hat{\gamma}_j\,\widehat{U}^\dagger(g) = \hat{u}_j(g)\hat{\gamma}_j\,\hat{u}_j^\dagger(g), \tag{3.12b}$$

holds for any Majorana operator $\hat{\gamma}_j$ at site j. Intuitively, the term $[\hat{u}_j(p)]^{\rho(g)}$ is needed to correct the -1 factors that would arise from anticommutation of $\hat{v}_j(g)$ and $\hat{v}_{\boldsymbol{j'}}(g)$ for $j \neq \boldsymbol{j'}$ when each $\hat{v}_j(g)$ have odd fermion parity. A derivation of the decomposition (3.12a) is presented in Chapter 7.

Theorem 2 presumes the existence of a local projective representation of the symmetry group G_f. This is only possible if the local Fock space $\mathfrak{F}_j$ defined in Sec. 3.1.2 is spanned by an even number of Majorana operators. This hypothesis precludes a situation in which a fermion number operator is well defined globally but not locally, for example when the lattice Λ is made of an even number of repeat unit cells, but a repeat unit cell is assigned an odd number of Majorana operators. (This can happen upon changing the parameters governing the quantum dynamics as is illustrated in Fig. 3.1.) We introduce the index $[\mu] = 0, 1$ to distinguish both possibilities. The case $[\mu] = 0$ applies when the number of local Majorana operators at site $j \in \Lambda$ is even, in which case the number of repeat unit cells in Λ is any positive integer. The case $[\mu] = 1$ applies when the number of local

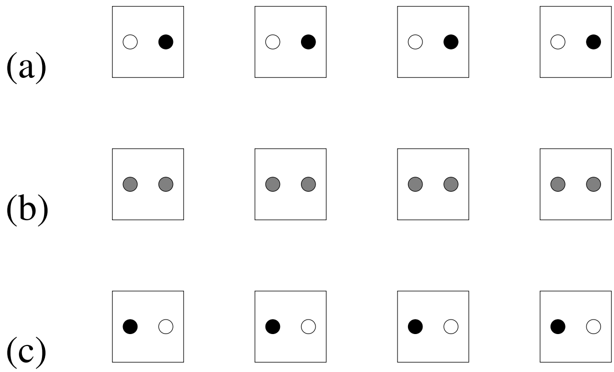

Figure 3.1: The repeat unit cells of a lattice Λ are represented pictorially by squares. The lattice Λ is chosen for simplicity to be one dimensional. (a) The repeat unit cell is decorated with two circles, one empty, the other filled. If periodic boundary conditions are imposed, translations by one repeat unit cell are symmetries. The translation of each circle to the adjacent ones, i.e., translation by half repeat unit cell is not a symmetry. (b) If the filling pattern is smoothly tuned (through an on-site potential whose magnitude is color coded, say) so that both circles in an repeat unit cell have the same filling, then the translation of half a repeat unit cell is a symmetry. One may then choose a smaller repeat unit cell, a square centered about one circle only. (c) Image of (a) under the translation by half repeat unit cell.

Majorana operators at site $j \in \Lambda$ is odd, in which case the number of repeat unit cell in Λ must necessarily be an even positive integer. The doublet

$$([\phi],[\mu]) := \begin{cases} ([\phi],0), & \text{if } [\mu]=0, \\ \\ (0,1), & \text{if } [\mu]=1, \end{cases} \tag{3.13}$$

of indices allows to treat Theorem 3 and 2 together, as we are going to explain.

3.2 FERMIONIC MATRIX PRODUCT STATES

We are going to use the properties of injective FMPS to prove Theorems 3 and 2. In this section, we review the definition and some basic properties of FMPS. Further background can be found in Appendix B and Refs. [135, 136, 139, 157].

Consider a one-dimensional lattice $\Lambda \cong \mathbb{Z}_N$. At the repeat unit cell $j = 1, \cdots, N$, the local fermion number operator is denoted $\hat{f}_j$ and the local Fock space of dimension $\mathcal{D}_j$ is denoted $\mathfrak{F}_j \cong \mathbb{C}^{\mathcal{D}_j}$. We define with

$$|\psi_{\sigma_j}\rangle, \qquad \sigma_j = 1, \cdots, \mathcal{D}_j, \tag{3.14a}$$

an orthonormal basis of $\mathfrak{F}_j$ such that

$$(-1)^{\hat{f}_j}\, |\psi_{\sigma_j}\rangle = (-1)^{|\sigma_j|} |\psi_{\sigma_j}\rangle. \tag{3.14b}$$

The fermion parity eigenvalue of the basis element $|\psi_{\sigma_j}\rangle$ is thus denoted $(-1)^{|\sigma_j|}$ with $|\sigma_j| \equiv 0, 1$. The local Fock space $\mathfrak{F}_j$ admits the direct sum decomposition

$$\mathfrak{F}_j = \mathfrak{F}_j^{(0)} \oplus \mathfrak{F}_j^{(1)} \tag{3.15a}$$

where, given $p = 0, 1$,

$$\mathfrak{F}_j^{(p)} := \mathrm{span}\left\{ |\psi_{\sigma_j}\rangle,\ \sigma_j = 1, \cdots, \mathcal{D}_j \;\middle|\; |\sigma_j| = p \right\}. \tag{3.15b}$$

One verifies that $\dim \mathfrak{F}_j^{(0)} = \dim \mathfrak{F}_j^{(1)} = \mathcal{D}_j/2$. To construct the Fock space $\mathfrak{F}_\Lambda$ for the lattice Λ, we demand that the direct sum (3.15) also holds for $\mathfrak{F}_\Lambda$. This is achieved with the help of the $\mathbb{Z}_2$ tensor product $\otimes_{\mathfrak{g}}$. This tensor product preserves the $\mathbb{Z}_2$-grading structure. We define the reordering rule

$$|\psi_{\sigma_j}\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_{j'}}\rangle \equiv (-1)^{|\sigma_j|\,|\sigma_{j'}|} |\psi_{\sigma_{j'}}\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_j}\rangle \tag{3.16}$$

on any two basis elements $|\psi_{\sigma_j}\rangle$ and $|\psi_{\sigma_{j'}}\rangle$ of $\mathfrak{F}_j$ and $\mathfrak{F}_{j'}$ for any two distinct sites $j \in \Lambda$ and $j' \in \Lambda$, respectively. The rule (3.16) guarantees that states are antisymmetric under the exchange of an odd number of fermions on site j with an odd number of fermions on site j' while symmetric otherwise. We then define the fermionic Fock space $\mathfrak{F}_\Lambda$ for the lattice Λ to be

$$\begin{aligned}\mathfrak{F}_\Lambda := \operatorname{span}\Bigg\{ |\Psi_{\boldsymbol{\sigma}}\rangle \;\Bigg|\; & |\Psi_{\boldsymbol{\sigma}}\rangle \equiv \bigotimes_{j=1}^{N}{}_{\mathfrak{g}}\, |\psi_{\sigma_j}\rangle, \\ & \boldsymbol{\sigma} \equiv (\sigma_1, \cdots, \sigma_N) \in \{1, \cdots, \mathcal{D}_1\} \times \cdots \times \{1, \cdots, \mathcal{D}_N\} \Bigg\}.\end{aligned} \tag{3.17}$$

As the parity $|\sigma_j|$ of the state $|\psi_{\sigma_j}\rangle$ can be generalized to the parity $|\boldsymbol{\sigma}|$ of the state $|\Psi_{\boldsymbol{\sigma}}\rangle$ through the action of the global fermion number operator

$$\widehat{F}_\Lambda := \sum_{j=1}^{N} \hat{f}_j, \qquad |\boldsymbol{\sigma}| \equiv \sum_{j=1}^{N} |\sigma_j| \bmod 2, \tag{3.18}$$

the Fock space (3.17) inherits the direct sum decomposition (3.15a),

$$\mathfrak{F}_\Lambda = \mathfrak{F}_\Lambda^{(0)} \oplus \mathfrak{F}_\Lambda^{(1)}. \tag{3.19}$$

Any state $|\Psi\rangle \in \mathfrak{F}_\Lambda$ has the expansion

$$|\Psi\rangle = \sum_{\boldsymbol{\sigma}} c_{\boldsymbol{\sigma}}\, |\Psi_{\boldsymbol{\sigma}}\rangle \tag{3.20a}$$

with the expansion coefficient $c_{\boldsymbol{\sigma}} \in \mathbb{C}$. Such a state is homogeneous if it belongs to either $\mathfrak{F}_\Lambda^{(0)}$ or $\mathfrak{F}_\Lambda^{(1)}$, in which case it has a definite parity $|\Psi| \equiv 0, 1$. From now on, we assume that all local Fock spaces are pairwise isomorphic, i.e.,

$$\mathcal{D}_j = \mathcal{D}, \qquad \mathfrak{F}_j \cong \mathfrak{F}_{j'} \qquad 1 \leq j < j' \leq N. \tag{3.21}$$

This assumption is needed to impose translation symmetry below. We describe the construction of two families of states that lie in $\mathfrak{F}_\Lambda^{(0)}$ and $\mathfrak{F}_\Lambda^{(1)}$, respectively. To this end, we choose the positive integer M, denote with $\mathbb{1}_M$ the unit $M \times M$ matrix and define the following pair of $2M \times 2M$ matrices

$$P := \begin{pmatrix} \mathbb{1}_M & 0 \\ 0 & -\mathbb{1}_M \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & \mathbb{1}_M \\ -\mathbb{1}_M & 0 \end{pmatrix}. \tag{3.22}$$

The 2×2 grading that is displayed is needed to represent the $\mathbb{Z}_2$ grading in Eq. (3.19) as will soon become apparent. The anticommuting matrices P and Y belong to the set $\mathrm{Mat}(2M, \mathbb{C})$ of all $2M \times 2M$ matrices. This set is a $4M^2$-dimensional vector space over the complex numbers. [8] For any $\sigma_j = 1, \cdots, \mathcal{D}$ with $j \in \Lambda$, we choose the matrices

$$B_{\sigma_j}, C_{\sigma_j}, D_{\sigma_j}, E_{\sigma_j}, G_{\sigma_j} \in \mathrm{Mat}(M, \mathbb{C}) \tag{3.23a}$$

with the help of which we define the matrices

$$A^{(0)}_{\sigma_j} := \begin{cases} \begin{pmatrix} B_{\sigma_j} & 0 \\ 0 & C_{\sigma_j} \end{pmatrix}, & \text{if } |\sigma_j| = 0, \\ \\ \begin{pmatrix} 0 & D_{\sigma_j} \\ E_{\sigma_j} & 0 \end{pmatrix}, & \text{if } |\sigma_j| = 1, \end{cases} \tag{3.23b}$$

and

$$A^{(1)}_{\sigma_j} := \begin{cases} \begin{pmatrix} G_{\sigma_j} & 0 \\ 0 & G_{\sigma_j} \end{pmatrix}, & \text{if } |\sigma_j| = 0, \\ \\ \begin{pmatrix} 0 & G_{\sigma_j} \\ -G_{\sigma_j} & 0 \end{pmatrix}, & \text{if } |\sigma_j| = 1, \end{cases} \tag{3.23c}$$

from $\mathrm{Mat}(2M, \mathbb{C})$. Observe that Eq. (3.23c) is a special case of Eq. (3.23b). For any $\sigma_j = 1, \cdots, \mathcal{D}$ with $j \in \Lambda$, the matrix P commutes (anticommutes) with $A^{(p)}_{\sigma_j}$ when $|\sigma_j| = 0$ ($|\sigma_j| = 1$),

$$P\, A^{(p)}_{\sigma_j} = (-1)^{|\sigma_j|}\, A^{(p)}_{\sigma_j}\, P \tag{3.24}$$

for both $p = 0, 1$. In contrast, the matrix Y commutes with $A^{(1)}_{\sigma_j}$

$$Y\, A^{(1)}_{\sigma_j} = A^{(1)}_{\sigma_j}\, Y \tag{3.25}$$

for all $\sigma_j = 1, \cdots, \mathcal{D}$ with $j \in \Lambda$.

---

8 The set $\mathrm{Mat}(2M, \mathbb{C})$ of all $2M \times 2M$ matrices is a $8M^2$-dimensional vector space over the real numbers.

We are ready to define the FMPS. We define states with either periodic boundary conditions (PBC) or antiperiodic boundary conditions (APBC) denoted by the parameter $b = 0$ or 1, respectively. They are

$$|\{A^{(0)}_{\sigma_j}\}; b\rangle := \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left[P^{b+1} A^{(0)}_{\sigma_1} \cdots A^{(0)}_{\sigma_N}\right] |\Psi_{\boldsymbol{\sigma}}\rangle \tag{3.26a}$$

and

$$|\{A^{(1)}_{\sigma_j}\}; b\rangle := \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left[P^{b}\, Y\, A^{(1)}_{\sigma_1} \cdots A^{(1)}_{\sigma_N}\right] |\Psi_{\boldsymbol{\sigma}}\rangle \tag{3.26b}$$

for any choice of the matrices (3.23b) and (3.23c), respectively, and with the basis (3.17) of the Fock space $\mathfrak{F}_\Lambda$. The following properties follow from the cyclicity of the trace and from the fact that Y is traceless.

Property 1. The FMPS $|\{A^{(p)}_{\sigma_j}\}; b\rangle$ is homogeneous and belongs to $\mathfrak{F}^{(p)}_\Lambda$ for $p = 0, 1$. This claim is a consequence of the identities

$$\sum_{j=1}^{N} |\sigma_j| = 1 \bmod 2 \implies \mathrm{tr}\left(P^b\, P\, A^{(0)}_{\sigma_1} \cdots A^{(0)}_{\sigma_N}\right) = 0, \tag{3.27a}$$

$$\sum_{j=1}^{N} |\sigma_j| = 0 \bmod 2 \implies \mathrm{tr}\left(P^b\, Y\, A^{(1)}_{\sigma_1} \cdots A^{(1)}_{\sigma_N}\right) = 0. \tag{3.27b}$$

Property 2. The FMPS $|\{A^{(p)}_{\sigma_j}\}; b\rangle$ changes by a multiplicative phase under a translation by one repeat unit cell. Indeed, one verifies that

$$\widehat{T}_b\, |\{A^{(p)}_{\sigma_j}\}; b\rangle = |\{A^{(p)}_{\sigma_j}\}; b\rangle, \tag{3.28}$$

where $\widehat{T}_b$ is the generator of translation by one repeat unit cell with boundary conditions $b = 0, 1$ and $k \in \mathbb{Z}$.

Property 3. The FMPS (3.26a) and (3.26b) are not uniquely specified by the choices $\{A^{(p)}_{\sigma_j}\}$ for $p = 0, 1$, respectively. For example, the similarity transformation

$$A^{(0)}_{\sigma_j} \mapsto U\, A^{(0)}_{\sigma_j}\, U^{-1}, \quad \sigma_j = 1, \cdots, D, \quad j = 1, \cdots, N, \tag{3.29}$$

with U any matrix that commutes with P leaves the trace unchanged. Another example occurs if there exists a nonvanishing matrix $Q = Q^2 \in \mathrm{Mat}(2M, \mathbb{C})$ such that

$$Q\, A^{(0)}_{\sigma_j} = Q\, A^{(0)}_{\sigma_j}\, Q, \qquad \sigma_j = 1, \cdots, \mathcal{D}. \tag{3.30}$$

Indeed, one verifies that Eq. (3.30) implies the identity

$$\mathrm{tr}\left[P^{b+1}\, A^{(0)}_{\sigma_1} \cdots A^{(0)}_{\sigma_N}\right] = \mathrm{tr}\left[P^{b+1}\, \tilde{A}^{(0)}_{\sigma_1} \cdots \tilde{A}^{(0)}_{\sigma_N}\right] \tag{3.31a}$$

with $\tilde{A}^{(0)}_{\sigma_j}$ the matrix

$$\tilde{A}^{(0)}_{\sigma_j} := Q\, A^{(0)}_{\sigma_j}\, Q + (\mathbb{1}_M - Q)\, A^{(0)}_{\sigma_j}\, (\mathbb{1}_M - Q). \tag{3.31b}$$

While conditions (3.30) imply that all matrices $A^{(0)}_1, \cdots, A^{(0)}_{\mathcal{D}}$ are reducible, conditions (3.31b) imply that all matrices $\tilde{A}^{(0)}_1, \cdots, \tilde{A}^{(0)}_{\mathcal{D}}$ are decomposable into the same block diagonal form. A necessary and sufficient condition on the $\mathcal{D}$ matrices $A^{(0)}_1, \cdots, A^{(0)}_{\mathcal{D}}$ to prevent that Eq. (3.30) holds for some $Q \in \mathrm{Mat}(2M, \mathbb{C})$ is to demand that there exists an integer $1 \leq \ell^\star \leq N$ such that the vector space spanned by the $\mathcal{D}^{\ell^\star}$ matrix products

$$A^{(0)}_{\sigma_1} \cdots A^{(0)}_{\sigma_{\ell^\star}}, \qquad \sigma_1, \cdots, \sigma_{\ell^\star} = 1, \cdots, \mathcal{D}, \tag{3.32a}$$

is $\mathrm{Mat}(2M, \mathbb{C})$. More precisely, for any $A \in \mathrm{Mat}(2M, \mathbb{C})$, it is possible to find $\mathcal{D}^{\ell^\star}$ coefficients $a^{(0)}_{\sigma_1, \cdots, \sigma_{\ell^\star}} \in \mathbb{C}$ such that [9]

$$A = \sum_{\sigma_1, \cdots, \sigma_{\ell^\star} = 1}^{\mathcal{D}} a^{(0)}_{\sigma_1, \cdots, \sigma_{\ell^\star}}\, A^{(0)}_{\sigma_1} \cdots A^{(0)}_{\sigma_{\ell^\star}}. \tag{3.32b}$$

In order to restrict the redundancy in the choice of the matrices (3.23) that enter the FMPS (3.26), we make the following definitions.

Definition 1. The even-parity FMPS (3.26a) is *injective* if there exists an integer $\ell^\star \geq 1$ such that the $\mathcal{D}^{\ell^\star}$ products $A^{(0)}_{\sigma_1} \cdots A^{(0)}_{\sigma_{\ell^\star}}$ of $2M \times 2M$ matrices span $\mathrm{Mat}(2M, \mathbb{C})$.

9 The basis (3.32a) is in general overcomplete owing to the condition $\mathcal{D}^{\ell^\star} \geq 4M^2$.

Definition 2. The odd-parity FMPS (3.26b) is *injective* if there exists an integer $\ell^\star \geq 1$ such that the $(\mathcal{D}/2)^{\ell^\star}$ products $G_{\sigma_1} \cdots G_{\sigma_{\ell^\star}}$ of $M \times M$ matrices with $|\sigma_1| = \cdots = |\sigma_{\ell^*}| = 0$ span $\mathrm{Mat}(M, \mathbb{C})$.

The need to distinguish the definitions of injectivity for even- and odd-parity FMPS stems from the fact that for an odd-parity FMPS the matrix Y commutes with $A^{(1)}_1, \cdots, A^{(1)}_{\mathcal{D}}$. In other words, Y is in the center of the algebra closed by products of $A^{(1)}_1, \cdots, A^{(1)}_{\mathcal{D}}$. Injectivity requires this center to be generated by $\mathbb{1}_{2M}$ and Y, i.e., the algebra closed by products of matrices $A^{(1)}_1, \cdots, A^{(1)}_{\mathcal{D}}$ is a $\mathbb{Z}_2$-graded simple algebra. For the center to be generated by no more than $\mathbb{1}_{2M}$ and Y, the products of $\mathcal{D}/2$ matrices

$$\left\{ G_{\sigma_1}, \cdots, G_{\sigma_{\mathcal{D}/2}} \,\middle|\, |\sigma_1| = \cdots = |\sigma_{\mathcal{D}/2}| = 0 \right\} \tag{3.33}$$

must close a simple algebra of $M \times M$ matrices, which is precisely the Definition 2. The following properties of FMPS are essential to the proofs of Theorems 3 and 2.

Property 4. Let $\ell \geq \ell^\star$. The $\mathcal{D}^\ell$ products $A^{(0)}_{\sigma_1} \cdots A^{(0)}_{\sigma_\ell}$ of $2M \times 2M$ matrices span $\mathrm{Mat}(2M, \mathbb{C})$ for any injective even-parity FMPS. The $\mathcal{D}/\in^\ell$ products $G_{\sigma_1} \cdots G_{\sigma_\ell}$ with

$$|\sigma_1| = \cdots = |\sigma_\ell| = 0 \tag{3.34}$$

of $M \times M$ matrices span $\mathrm{Mat}(M, \mathbb{C})$ for any $\ell \geq \ell^\star$ injective odd-parity FMPS.

Property 5. If two sets of matrices $\{A^{(p)}_{\sigma_j}\}$ and $\{\widetilde{A}^{(p)}_{\sigma_j}\}$ generate the same injective FMPS, there then exists an invertible matrix V and a phase $\varphi_V \in [0, 2\pi)$ such that [135]

$$\widetilde{A}^{(p)}_{\sigma_j} = e^{\mathrm{i}\varphi_V}\, V\, A^{(p)}_{\sigma_j}\, V^{-1}, \tag{3.35a}$$

for any $\sigma_j = 1, \cdots, \mathcal{D}$, and

$$P = \pm V\, P\, V^{-1}, \tag{3.35b}$$

for $p = 0$, while

$$P = V\, P\, V^{-1}, \qquad Y = \pm V\, Y\, V^{-1}, \tag{3.35c}$$

for $p = 1$. Here, the phase φ_V is needed to compensate for the possibility that the matrix V anticommutes with P or Y. We also observe that the index σ_j that labels the local fermion number is preserved under the conjugation by V. The transformation (3.35) that leaves an injective FMPS invariant is called a gauge transformation.

Property 6. Definitions 1 and 2 ensure that the two-point correlation function of any pair of local operators taken in an injective FMPS decays exponentially fast with their separation. This provides an additional motivation to study them as they can be used to describe nondegenerate and gapped ground states [135].

3.3 PROOFS IN ONE DIMENSION

In this section, we sketch the proofs for Theorems 2 and 3 in 1D space using the machinery of fermionic matrix product states (FMPS). We relegate some intermediate steps and technical details to Appendix C.1. For convenience, we will first prove Theorem 2 and then use it to prove Theorem 3. A separate proof of the latter is also presented in Sec. 3.4.

3.3.1 *Proof of Theorem 2*

Our strategy is inspired by the study of injective bosonic MPS assumed to be $G_{\mathrm{trsl}} \times G$-invariant made by Tasaki in Ref. [158]. For the fermionic case, we shall distinguish the cases of even- and odd-parity FMPS, as each case demands distinct conditions for injectivity. For the case of even-parity injective FMPS, we shall establish the following identity between *any* matrix $A \in \mathrm{Mat}(2M, \mathbb{C})$ and a *given* norm preserving $W \in \mathrm{Mat}(2M, \mathbb{C})$ that is induced by a projective representation of the symmetry group G_f. There exists a phase $\delta \in [0, 2\pi)$ and a nonvanishing positive integer $\ell^\star$ such that

$$A = e^{\mathrm{i}\ell\,\delta}\, W^{-1}\, A\, W, \tag{3.36a}$$

holds for all $\ell{=}\ell^\star, \ell^\star + 1, \ell^\star + 2, \cdots$ and all $A{\in}\mathrm{Mat}(2M, \mathbb{C})$. This is only possible if

$$\delta = 0, \tag{3.36b}$$

which obviously holds when A is the identity matrix $\mathbb{1}_{2M}$. For the case of odd FMPS, we shall establish the same identity as (3.36) for any matrix $A \in \mathrm{Mat}(2M, \mathbb{C})$ that commutes with matrix Y, i.e., Y is in the center of the algebra spanned by such matrices A. Theorem 2 will follow from the interpretation of the condition $\delta = 0$ as the projective representation of G_f defined in Sec. 3.1 to have trivial second group cohomology class.

We start from the even-parity injective FMPS

$$|\{A^{(0)}_{\sigma_j}\}; b\rangle := \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left[P^{b+1}\, A^{(0)}_{\sigma_1} \cdots A^{(0)}_{\sigma_N}\right] |\Psi_{\boldsymbol{\sigma}}\rangle. \tag{3.37}$$

Let g be an element from G_f be represented by the operator $\widehat{U}(g)$ as defined in Sec. 3.1.2.

On the one hand, we have the identity

$$\begin{aligned}\widehat{U}(g)\,|\{A^{(0)}_{\sigma_j}\};b\rangle &= \sum_{\boldsymbol{\sigma}} \operatorname{tr}\left[P^{b+1}A^{(0)}_{\sigma_1}\cdots A^{(0)}_{\sigma_N}\right]\widehat{U}(g)\,|\Psi_{\boldsymbol{\sigma}}\rangle \\ &\equiv \sum_{\boldsymbol{\sigma}} \operatorname{tr}\left[P^{b+1}A^{(0)}_{\sigma_1}(g)\cdots A^{(0)}_{\sigma_N}(g)\right]|\Psi_{\boldsymbol{\sigma}}\rangle,\end{aligned} \tag{3.38a}$$

where

$$A^{(0)}_{\sigma_j}(g) := \sum_{\sigma'_j=1}^{\mathcal{D}} [\mathcal{U}(g)]_{\sigma_j\sigma'_j}\,\mathsf{K}_g\left[A^{(0)}_{\sigma'_j}\right], \tag{3.38b}$$

$$[\mathcal{U}(g)]_{\sigma_j\sigma'_j} := \langle\psi_{\sigma_j}|\left(\hat{u}_j(g)|\psi_{\sigma'_j}\rangle\right), \tag{3.38c}$$

$$\mathsf{K}_g\left[A^{(0)}_{\sigma_j}\right] := \begin{cases} A^{(0)}_{\sigma_j}, & \text{if } \mathfrak{c}(g)=0, \\ \\ \mathsf{K}\,A^{(0)}_{\sigma_j}\,\mathsf{K}, & \text{if } \mathfrak{c}(g)=1. \end{cases} \tag{3.38d}$$

(Complex conjugation is denoted with K.) On the other hand, we have the identity

$$\begin{aligned}\widehat{U}(g)\,|\{A^{(0)}_{\sigma_j}\};b\rangle &= e^{\mathrm{i}\eta(g;b)}\,|\{A^{(0)}_{\sigma_j}\};b\rangle \\ &= |\{e^{\mathrm{i}\eta(g;b)/N}\,A^{(0)}_{\sigma_j}\};b\rangle\end{aligned} \tag{3.39}$$

for some phase $\eta(g;b)\in[0,2\pi)$ if we assume that $\widehat{U}(g)|\{A^{(0)}_{\sigma_j}\};b\rangle$ is an eigenstate of the norm-preserving operator $\widehat{U}(g)$, as it should be if G_f is a symmetry. By the assumption of injectivity, the matrices $A^{(0)}_{\sigma_j}(g)$ and $e^{\mathrm{i}\eta(g;b)/N}\,A^{(0)}_{\sigma_j}$ are related by the similarity transformation (3.35), i.e., there exists an invertible matrix $V(g)$ and a phase $\varphi^{(b)}_{V(g)}\in[0,2\pi)$ such that

$$e^{\mathrm{i}\eta(g;b)/N}\,A^{(0)}_{\sigma_j} = e^{\mathrm{i}\varphi^{(b)}_{V(g)}}\,V(g)\,A^{(0)}_{\sigma_j}(g)\,V^{-1}(g) \tag{3.40}$$

for any σ_j. We massage Eq. (3.40) into

$$e^{\mathrm{i}\theta(g;b)}\, V^{\dagger}(g)\, A^{(0)}_{\sigma_j}\, V(g) = \sum_{\sigma'_j=1}^{\mathcal{D}} \left[\mathcal{U}(g)\right]_{\sigma_j \sigma'_j} \mathsf{K}_g \left[A^{(0)}_{\sigma'_j}\right], \tag{3.41a}$$

where we have introduced the phase

$$\theta(g;b) := \frac{\eta(g;b)}{N} - \varphi^{(b)}_{V(g)}. \tag{3.41b}$$

Consider a second element $h \in G_f$ asides from $g \in G_f$. We can use the relation (3.41a) with g replaced by the composition $g\,h$. We can also iterate the relation (3.41a) by evaluating the composition $\widehat{U}(g)\,\widehat{U}(h)\,|\{A^{(0)}_{\sigma_j}\};b\rangle$. After some algebra (Appendix C.1.1), one finds that (i) the phase

$$\delta(g,h;b) := (-1)^{\mathfrak{c}(g)}\,\theta(h;b) + \theta(g;b) - \phi(g,h) - \theta(g\,h;b) \tag{3.42a}$$

that relates the normalized 2-cocycle defined in Eqs. (3.7) and (3.8) to the phase (3.41a), (ii) the map represented by

$$U(g) := \begin{cases} V(g), & \text{if } \mathfrak{c}(g) = 0, \\ \\ V(g)\,\mathsf{K}, & \text{if } \mathfrak{c}(g) = 1, \end{cases} \tag{3.42b}$$

and the $\mathcal{D}$ matrices $A^{(0)}_{\sigma_j}$, are related by

$$e^{\mathrm{i}\delta(g,h;b)}\, A^{(0)}_{\sigma_j}\, W(g,h) = W(g,h)\, A^{(0)}_{\sigma_j} \tag{3.42c}$$

for any $\sigma_j = 1, \cdots, \mathcal{D}$, where

$$W(g,h) := U(g)\, U(h)\, U^{-1}(g\,h). \tag{3.42d}$$

We are going to make use of the injectivity of the FMPS a second time after massaging Eq. (3.42c) into

$$A^{(0)}_{\sigma_j} = e^{\mathrm{i}\delta(g,h;b)}\, W^{-1}(g,h)\, A^{(0)}_{\sigma_j}\, W(g,h) \tag{3.43}$$

for any $\sigma_j = 1, \cdots, \mathcal{D}$. For any integer $\ell = 1, 2, \cdots$, iteration of Eq. (3.43) gives

$$\prod_{j=1}^{\ell} A^{(0)}_{\sigma_j} = e^{\mathrm{i}\ell\,\delta(g,h;b)}\, W^{-1}(g,h) \left[\prod_{j=1}^{\ell} A^{(0)}_{\sigma_j}\right] W(g,h). \tag{3.44}$$

When $\ell \geq \ell^{*}$, injectivity of the FMPS implies that any matrix $A \in \mathrm{Mat}(2M, \mathbb{C})$ can be written as a linear superposition of all the possible monomials $\prod_{j=1}^{\ell} A^{(0)}_{\sigma_j}$ of order ℓ, each of which obeys Eq. (3.44) [recall Eq. (3.32b)]. Hence, we arrive at the identity

$$A = e^{\mathrm{i}\ell\,\delta(g,h;b)}\, W^{-1}(g,h)\, A\, W(g,h), \qquad \forall \ell \geq \ell^{\star}, \tag{3.45}$$

for any $A \in \mathrm{Mat}(2M, \mathbb{C})$, which implies, in turn, that $W(g,h)$ belongs to the center of the algebra spanned by monomials $\prod_{j=1}^{\ell} A^{(0)}_{\sigma_j}$. For even-parity FMPS, this center is one-dimensional as it is generated by the unit matrix $\mathbb{1}_{2M}$. In particular, we can choose $A = \mathbb{1}_{2M}$ for which

$$\mathbb{1}_{2M} = e^{\mathrm{i}\ell\,\delta(g,h;b)}\, \mathbb{1}_{2M}, \tag{3.46a}$$

which implies that

$$\delta(g,h;b) = 0, \tag{3.46b}$$

and, therefore, $[\phi] = 0$ [recall Eq. (3.11)].

The odd-parity FMPS differs from the even-parity FMPS in that the $\mathcal{D}^{\ell}$ products $A^{(1)}_{\sigma_1} \cdots A^{(1)}_{\sigma_\ell}$ for any $\ell \geq \ell^{\star}$ span a subalgebra of $\mathrm{Mat}(2M, \mathbb{C})$ with the center spanned by $\mathbb{1}_{2M}$ and Y. This difference is of no consequence until reaching the odd-parity counterpart to Eq. (3.40). However, for the odd-parity counterpart to Eq. (3.40) multiplication of $U(g)$ from the left by any element from the center generated by $\mathbb{1}_{2M}$ and Y,

$$\left[a(g)\,\mathbb{1}_{2M} + b(g)\,Y\right] V(g), \qquad |a(g)|^2 + |b(g)|^2 = 1, \tag{3.47}$$

leaves Eq. (3.40) unchanged. To fix this subtlety, we replace $V(g)$ in Eq. (3.40) by $V^{(0)}(g)$ which is defined by

$$V(g) := \left[a(g)\,\mathbb{1}_{2M} + b(g)\,Y\right] V^{(0)}(g), \tag{3.48a}$$

$$P\,V^{(0)}(g)\,P = V^{(0)}(g). \tag{3.48b}$$

With this change in mind, all the steps leading to Eq. (3.42) for the even-parity case can be repeated for the odd-parity case. The analog to the even-parity coboundary condition (3.46b) then follows, thereby completing the proof of Theorem 2.

3.3.2 *Proof of Theorem 3*

Theorem 2 presumes the existence of a local fermionic Fock space, i.e., of an even number of Majorana degrees of freedom per repeat unit cell. This hypothesis precludes translation invariant lattice Hamiltonians with odd number of Majorana operators per repeat unit cell such as

$$\widehat{H}_{\mathrm{K}} := \sum_{j=1}^{2M} \mathrm{i}\hat{\gamma}_j\, \hat{\gamma}_{j+1}. \tag{3.49a}$$

Here, the Hermitian operators $\left(\hat{\gamma}_j = \hat{\gamma}_j^{\dagger}\right)$ obey the Majorana algebra

$$\left\{\hat{\gamma}_j, \hat{\gamma}_{j'}\right\} = 2\delta_{jj'}, \qquad j, j' = 1, \cdots, 2M, \tag{3.49b}$$

and the total number $2M$ of repeat unit cell is an even integer. Hamiltonian $\widehat{H}_{\mathrm{K}}$ realizes the critical point between the two topologically distinct phases of the Kitaev chain. In the continuum limit, it describes a helical pair of Majorana fields and has a gapless spectrum.

Motivated by this example, we now prove a separate LSM constraint on Majorana lattice models with an odd Majorana flavors per repeat unit site. We use Theorem 2 for the proof.

Let $n \geq 0$ be an integer and

$$\hat{\gamma}_j := \left(\hat{\gamma}_{j,1}, \hat{\gamma}_{j,2}, \cdots, \hat{\gamma}_{j,2m+1}\right)^{\mathsf{T}} \tag{3.50}$$

be the spinor made of $2m+1$ Majorana operators. Let the Hamiltonian $\widehat{H}$ be local and translationally invariant. We write

$$\widehat{H} \equiv \sum_{j=1}^{2M} \hat{h}\left(\hat{\gamma}_{j-q}, \ldots, \hat{\gamma}_j, \ldots, \hat{\gamma}_{j+q}\right), \tag{3.51}$$

where $\hat{h}$ is a Hermitian polynomial of $2q$ Majorana spinors $\left\{\hat{\gamma}_{j-q},\dots,\hat{\gamma}_{j+q}\right\}$ with q a positive integer. The finiteness of q renders $\widehat{H}$ local. Hamiltonian (3.51) is defined over $2M$ sites, since an even number of Majorana operators are needed to have a well-defined Fock space. We assume that $\widehat{H}$ has a nondegenerate gapped ground state $|\Psi_0\rangle$. We are going to deliver a contradiction by making use of Theorem 2, thereby proving Theorem 3.

Define the Hamiltonian,

$$\widehat{H}' := \sum_{j=1}^{2M}\sum_{\alpha=1}^{2}\hat{h}\left(\hat{\gamma}^{(\alpha)}_{j-q},\cdots,\hat{\gamma}^{(\alpha)}_{j+q}\right), \tag{3.52a}$$

which is the sum of two copies of Hamiltonian (3.51). The repeat unit cell labeled by $j=1,\cdots,2M$ now contains two Majorana spinors labeled by $\alpha=1,2$. Hamiltonian (3.52) thus acts on a Fock space which is locally spanned by an even number of Majorana flavors. At each site $j=1,\cdots,2M$ one can define a local fermionic Fock space. Since there is no coupling between the two copies $\alpha=1,2$ of Majorana spinors, $\widehat{H}'$ inherits from $\widehat{H}$ the nondegenerate gapped ground state

$$|\Psi'_0\rangle := |\Psi_0\rangle\otimes_{\mathfrak{g}}|\Psi_0\rangle. \tag{3.52b}$$

Since at each site j, there is no term coupling the two copies $\hat{\gamma}^{(1)}_j$ and $\hat{\gamma}^{(2)}_j$, $\widehat{H}'$ is invariant under any local permutation

$$\begin{pmatrix}\hat{\gamma}^{(1)}_j\\ \hat{\gamma}^{(2)}_j\end{pmatrix}\mapsto\begin{pmatrix}\hat{\gamma}^{(2)}_j\\ \hat{\gamma}^{(1)}_j\end{pmatrix}. \tag{3.53a}$$

The local representation of the fermion parity operator is

$$\widehat{P}_j := \prod_{l=1}^{2n+1}\left[\mathrm{i}\,\hat{\gamma}^{(1)}_{j,l}\,\hat{\gamma}^{(2)}_{j,l}\right]. \tag{3.53b}$$

Under the transformation (3.53a), the local fermion parity operator $\widehat{P}_j$ acquires the phase $(-1)^{2n+1}=-1$. Therefore, the symmetry transformation (3.53a) anticommutes with $\widehat{P}_j$. This anticommutation relation implies a nontrivial second group cohomology class $[\phi]\neq 0$ of G_f, independent of the group of onsite symmetries of Hamiltonian (3.51). Therefore by Theorem 2 Hamiltonian $\widehat{H}'$ cannot have a nondegenerate gapped ground state. This is in contradiction with the initial assumption that Hamiltonian (3.51) has the nondegenerate gapped ground state $|\Psi_0\rangle$.

One can interpret Theorem 3 as the inability to write down an injective FMPS for the ground state of translationally invariant Hamiltonians with an odd number of Majorana flavors per repeat unit cell. This is because one cannot define the matrices A_{σ_j} as there is no well-defined Fock space at site j to begin with.

3.4 PROOFS IN HIGHER DIMENSIONS

In this section, we extend Theorem 2 to any dimension d of space when the symmetry group G_f is Abelian and all elements $g \in G_f$ are represented by unitary operators. Our method is inspired by the one used recently in Ref. [159] for quantum spin Hamiltonians.

Consider a d-dimensional lattice Λ with periodic boundary conditions in each linearly independent direction $\alpha = 1, \cdots, d$ such that Λ realizes a d-torus. Let each repeat unit cell be labeled as j and host a local fermionic Fock space $\mathfrak{F}_j$ that is generated by a Majorana spinor $\hat{\gamma}_j$ with $2n$ components $\hat{\gamma}_{j,l}$, $l = 1, \cdots, 2n$. The fermionic Fock space attached to the lattice Λ is denoted by $\mathfrak{F}_\Lambda$. We impose the global symmetry corresponding to the central extension G_f of G by $\mathbb{Z}_2^{\mathrm{F}}$ as defined in Sec. 3.1.1, whereby G_f is assumed to be Abelian. We also impose translation symmetry. If the d-dimensional lattice Λ has N_α repeat unit cell in the α-direction and thus the cardinality

$$|\Lambda| \equiv \prod_{\alpha=1}^{d} N_\alpha, \tag{3.54}$$

the translation group is

$$G_{\mathrm{trsl}} \equiv \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_d}. \tag{3.55}$$

By assumption, the combined symmetry group is the Cartesian product group

$$G_{\mathrm{total}} \equiv G_{\mathrm{trsl}} \times G_f. \tag{3.56}$$

The representation of the translation group (3.55) is generated by the unitary operator $\widehat{T}_\alpha$ whose action on the Majorana spinors is

$$\widehat{T}_\alpha\, \hat{\gamma}_j\, \widehat{T}_\alpha^{-1} = \hat{\gamma}_{j+\boldsymbol{e}_\alpha}, \qquad \widehat{T}_\alpha^{-1} = \widehat{T}_\alpha^{\dagger}, \tag{3.57a}$$

along the α-direction ($\boldsymbol{e}_\alpha$ is a basis-vector along the α-direction). Imposing periodic boundary conditions implies

$$\left(\widehat{T}_\alpha\right)^{N_\alpha+1} = \widehat{T}_\alpha. \tag{3.57b}$$

The representation $\widehat{U}(g)$ of $g \in G_f$ is defined in Sec. 3.1.2. Any translationally and G_f-invariant local Hamiltonian acting on $\mathfrak{F}_\Lambda$ can be written in the form

$$\widehat{H}_{\mathrm{pbc}} := \sum_{\alpha=1}^{d} \sum_{n_\alpha=1}^{N_\alpha} \left(\widehat{T}_\alpha\right)^{n_\alpha} \hat{h}_j \left(\widehat{T}_\alpha^\dagger\right)^{n_\alpha}, \tag{3.58a}$$

where $\hat{h}_j$ is a local Hermitian operator centered at an arbitrary repeat unit cell j. More precisely, it is a finite-order polynomial in the Majorana operators centered at j that is also invariant under all the nonspatial symmetries, i.e.,

$$\hat{h}_j = \widehat{U}(g)\, \hat{h}_j\, \widehat{U}^{-1}(g) = \left(\hat{h}_j\right)^\dagger \tag{3.58b}$$

for any $g \in G_f$. Instead of extracting spectral properties of Hamiltonian $\widehat{H}_{\mathrm{pbc}}$ directly, we shall do so with the family of Hamiltonians indexed by $g \in G_f$ and given by

$$\widehat{H}_{\mathrm{tw}}^{\mathrm{tlt}}(g) := \sum_{a=1}^{|\Lambda|} \left(\widehat{T}_1(g)\right)^a \hat{h}_1^{\mathrm{tlt}} \left(\widehat{T}_1^{-1}(g)\right)^a, \tag{3.59}$$

where $\hat{h}_1^{\mathrm{tlt}}$ is a G_f-symmetric and local Hermitian operator and $\widehat{T}_1(g)$ is the "g-twisted translation operator" to be defined shortly. We shall derive LSM-like constraints for $\widehat{H}_{\mathrm{tw}}^{\mathrm{tlt}}(g)$ and then explain why those LSM-like constraints also apply to $\widehat{H}_{\mathrm{pbc}}$. To this end, we will explain what is meant by the upper index "tlt" for tilted and the lower index "tw" for twisted and how $\widehat{H}_{\mathrm{tw}}^{\mathrm{tlt}}(g)$ and $\widehat{H}_{\mathrm{pbc}}$ differ.

3.4.1 *Case of a $d = 1$-Dimensional Lattice*

As a warm up, we first consider the one-dimensional case, i.e., $\Lambda \cong \mathbb{Z}_N$. We impose two assumptions in addition to those previously assumed. These are that every element in G_f is unitarily represented (**Assumption 5**) and that G_f is an Abelian group (**Assumption 6**). These two assumptions were superfluous when proving Theorem 2 using injective

FMPS in Sec. 3.3. This drawback is compensated by the possibility to extend the proof that follows to any dimension d of space.

Twisted boundary conditions are implemented by defining the symmetry twisted translation operator

$$\widehat{T}_1(g) := \hat{v}_1(g)\,\widehat{T}_1 \tag{3.60a}$$

through its action

$$\widehat{T}_1(g)\,\hat{\gamma}_j\,\widehat{T}_1^{-1}(g) = \begin{cases} (-1)^{\rho(g)}\,\hat{\gamma}_{j+1}, & \text{if } j \neq N, \\ \\ \hat{v}_1(g)\,\hat{\gamma}_1\,\hat{v}_1^{-1}(g), & \text{if } j = N, \end{cases} \tag{3.60b}$$

for $j = 1, \cdots, N$, where $\rho(g) \in \{0,1\} \equiv \mathbb{Z}_2$ is defined in Eq. (3.6d) [see also Eq. (4.10)]. We then consider any Hamiltonian of the form (3.59) where the operator $\hat{h}_1^{\mathrm{tlt}}$ in Eq. (3.59) is nothing but the operator $\hat{h}_j$ in Eq. (3.58a) with Λ restricted to a one-dimensional lattice. Such a twisted boundary condition is equivalent to coupling the Majorana operators to a background Abelian gauge field with a holonomy $g \in G_f$ around the spatial cycle. The effect of turning on such a background field is that it delivers the operator algebra (see Appendix C.2)

$$\left[\widehat{T}_1(g)\right]^N = \widehat{U}(g), \qquad g \in G_f \tag{3.61a}$$

and

$$\widehat{U}(h)^{-1}\,\widehat{T}_1(g)\,\widehat{U}(h) = e^{\mathrm{i}\chi(g,h)}\,\widehat{T}_1(g), \quad h \in G_f, \tag{3.61b}$$

where

$$\chi(g,h) := \phi(h,g) - \phi(g,h) + (N-1)\pi\,\rho(h)[\rho(g)+1]. \tag{3.61c}$$

The same algebra with $\rho(g) \equiv 0$ for all $g \in G_f$ was obtained by Yao and Oshikawa in Refs. [154, 159]. The phase $\chi(g,h)$ is vanishing if and only if the second cohomology class $[\phi]$ is trivial [see Appendix C.2].

If $\chi(g,h)$ mod 2π is nonvanishing, one-dimensional representations of (3.61) are not allowed. The ground state of any Hamiltonian of the form (3.59) is either degenerate or spontaneously breaks the symmetry in the thermodynamic limit. We have rederived

Theorem 2 for the Abelian group G_f that is represented unitarily when twisted boundary conditions apply.

If we assume that the choice of boundary conditions cannot change the ground-state degeneracy when all excited states are separated from the ground states by an energy gap, then Theorem 2 applies to all boundary conditions compatible with translation symmetry that are imposed on the one-dimensional chain Λ and, in particular, to Hamiltonians of the form (3.58) with Λ restricted to a one-dimensional lattice that obey periodic boundary conditions. A necessary condition for this assumption to hold is that all correlation functions between local operators decay sufficiently fast, a condition known to be an attribute of any Hamiltonian with gapped ground states [160].

We emphasize that, in rederiving Theorem 2, we have taken (i) the group G_f to be Abelian and (ii) representation $\hat{u}_j(g)$ to be unitary for all $g \in G_f$. There exist several challenges in relaxing both of these assumptions. When the group is taken to be nonAbelian, one cannot consistently define a twisted Hamiltonian (3.59) that is invariant under both global symmetry transformations $\widehat{U}(h)$ and symmetry twisted translation operators $\widehat{T}_1(g)$ without imposing stricter constraints on local operators $\hat{h}_1^{\mathrm{tlt}}$ than Eq. (3.59). The challenges with imposing antiunitary twisted boundary conditions with the group element $g \in G_f$ are the following. First, complex conjugation is applied on all the states in the Fock space $\mathfrak{F}_\Lambda$. This means that Hamiltonian (3.58) can differ from Hamiltonian (3.59) through an extensive number of terms when $\mathfrak{c}(g) = 1$, in which case it is not obvious to us how to safely tie some spectral properties of Hamiltonians (3.59) and (3.58). Second, not all representations of the group G_f are either even or odd under complex conjugation, in which case conjugation of $\widehat{T}_1(g)$ by $\widehat{U}(h)^{-1}$ need not result anymore in a mere phase factor multiplying $\widehat{T}_1(g)$ when $\mathfrak{c}(g) = 1$. In view of this difficulty with interpreting antiunitary twisted boundary conditions, we observe that the FMPS construction of LSM type constraints is more general than the one using twisted boundary conditions.

3.4.2 *Case of a $d > 1$-Dimensional Lattice*

We now assume that Λ is a $d > 1$-dimensional lattice. We would like to generalize the twisted boundary conditions (3.60) obeyed by the Majorana operators to arbitrary spatial dimensions. There is no unique way for doing so. In what follows, we construct a group of translations $G_{\mathrm{trsl}}^{\mathrm{tlt}}$ that is cyclic. This is achieved by imposing *tilted* or *sheared* boundary conditions. After constructing $G_{\mathrm{trsl}}^{\mathrm{tlt}}$, we twist the boundary conditions in a

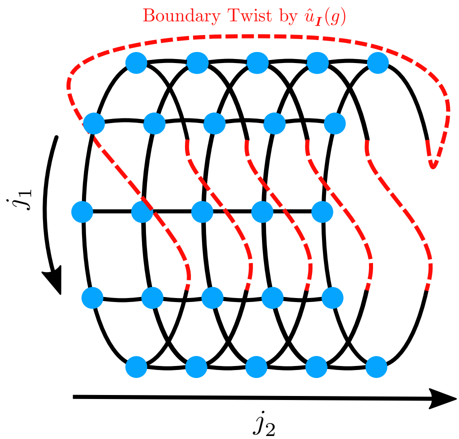

Figure 3.2: Example of a path that visits all the sites of a two-dimensional lattice that decorates the surface of a torus.

particular way using the local representations of elements of the on-site (internal) symmetry group G_f. The operators representing translations on the lattice with tilted and twisted boundary conditions may not commute with the operators representing elements of G_f, even though all elements of $G^{\text{tlt}}_{\text{trsl}}$ commute with all elements of G_f by assumption (3.56). When this is so, the representation of $G^{\text{tlt}}_{\text{total}} = G^{\text{tlt}}_{\text{trsl}} \times G_f$ is necessarily larger than one dimensional, in which case the ground states are either degenerate or the symmetry group $G^{\text{tlt}}_{\text{total}} = G^{\text{tlt}}_{\text{trsl}} \times G_f$ is spontaneously broken.

Our strategy is to construct the counterpart of Eqs. (3.60) and (3.61). To this end, we are going to trade the translation symmetry group (3.55), which is a polycyclic group when $d > 1$, for the cyclic group

$$G^{\text{tlt}}_{\text{trsl}} \equiv \mathbb{Z}_{N_1 \cdots N_d} \tag{3.62}$$

and define the combined symmetry group

$$G^{\text{tlt}}_{\text{total}} \equiv G^{\text{tlt}}_{\text{trsl}} \times G_f. \tag{3.63}$$

The intuition underlying the construction of the tilted translation symmetry group $G^{\mathrm{tlt}}_{\mathrm{trsl}}$ is provided by Fig. 3.2. As a set, the elements of $G^{\mathrm{tlt}}_{\mathrm{trsl}}$ can be labeled by the elements of G_{trsl}, namely

$$G^{\mathrm{tlt}}_{\mathrm{trsl}} := \left\{ \left((t_1)^{n_1}, \cdots, (t_d)^{n_d} \right) \;\middle|\; n_\alpha = 1, \cdots, N_\alpha, \quad \alpha = 1, \cdots, d \right\}. \tag{3.64}$$

However, as a group we would like to label the elements of $G^{\mathrm{tlt}}_{\mathrm{trsl}}$ as those of the cyclic group with $|\Lambda|$ elements, i.e.,

$$G^{\mathrm{tlt}}_{\mathrm{trsl}} := \left\{ t^n \;\middle|\; n = 1, \cdots, |\Lambda| \right\}. \tag{3.65}$$

This is achieved by carefully choosing the group composition for the elements (3.64), i.e., by iterating $d-1$ central extensions.

Step 1. We consider $\mathbb{Z}_{N_1}$ generated by t_1 and extend it by $\mathbb{Z}_{N_{\hat{2}}}$ generated by $t_{\hat{2}}$ through the map

$$\begin{aligned} &\Theta_1 : \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_1} \to \mathbb{Z}_{N_{\hat{2}}}, \\ &\Theta_1\left((t_1)^a, (t_1)^b \right) := (t_{\hat{2}})^{\frac{1}{N_1}\left(a+b-[a+b]_{N_1} \right)}, \end{aligned} \tag{3.66a}$$

for any $a, b = 1, \cdots, N_1$, to obtain $\mathbb{Z}_{N_1 N_{\hat{2}}}$, the group of translations on the tilted lattice restricted to $\mathbb{R}^2$. Here, the notation $[a+b]_n$ is used to denote addition modulo n. This group extension can be summarized by the short exact sequence

$$1 \to \mathbb{Z}_{N_{\hat{2}}} \to \mathbb{Z}_{N_1 N_{\hat{2}}} \to \mathbb{Z}_{N_1} \to 1 \tag{3.66b}$$

and is labeled by the extension classes

$$[\Theta_1] \in H^2\left(\mathbb{Z}_{N_1}, \mathbb{Z}_{N_{\hat{2}}} \right). \tag{3.66c}$$

Using this extension class and the standard expression for group composition in an extended group, we may identify t_1 as the generator of $\mathbb{Z}_{N_1 N_{\hat{2}}}$.

Step 2. We consider $\mathbb{Z}_{N_1\,N_{\hat{2}}}$ generated by t_1 and extend it by $\mathbb{Z}_{N_{\hat{3}}}$ generated by $t_{\hat{3}}$ through the map

$$\begin{aligned}
&\Theta_{\hat{2}} : \mathbb{Z}_{N_1\,N_{\hat{2}}} \times \mathbb{Z}_{N_1\,N_{\hat{2}}} \to \mathbb{Z}_{N_{\hat{3}}}, \\
&\Theta_{\hat{2}}\Big((t_1)^a, (t_1)^b\Big) := (t_{\hat{3}})^{\frac{1}{N_1\,N_{\hat{2}}}\left(a+b-[a+b]_{N_1\,N_{\hat{2}}}\right)},
\end{aligned} \tag{3.67a}$$

for any $a, b = 1, \cdots, N_1\,N_{\hat{2}}$, to obtain $\mathbb{Z}_{N_1\,N_{\hat{2}}\,N_{\hat{3}}}$ the group of translations on the tilted lattice restricted to $\mathbb{R}^3$. This group extension can be summarized by the short exact sequence

$$1 \to \mathbb{Z}_{N_{\hat{3}}} \to \mathbb{Z}_{N_1\,N_{\hat{2}}\,N_{\hat{3}}} \to \mathbb{Z}_{N_1\,N_{\hat{2}}} \to 1 \tag{3.67b}$$

and is labeled by the extension classes

$$[\Theta_{\hat{2}}] \in H^2\big(\mathbb{Z}_{N_1\,N_{\hat{2}}}, \mathbb{Z}_{N_{\hat{3}}}\big). \tag{3.67c}$$

Using this extension class and the standard expression for group composition in an extended group, we may identify t_1 as the generator of $\mathbb{Z}_{N_1\,N_{\hat{2}}\,N_{\hat{3}}}$.

Step $\mathbf{d-1}$. We consider $\mathbb{Z}_{N_1\cdots N_{d-1}}$ generated by t_1 and extend it by $\mathbb{Z}_{N_d}$ generated by t_d through the map

$$\begin{aligned}
&\Theta_{d-1} : \mathbb{Z}_{N_1\cdots N_{d-1}} \times \mathbb{Z}_{N_1\cdots N_{d-1}} \to \mathbb{Z}_{N_d}, \\
&\Theta_{d-1}\Big((t_1)^a, (t_1)^b\Big) := (t_d)^{\frac{1}{N_1\cdots N_{d-1}}\left(a+b-[a+b]_{N_1\cdots N_{d-1}}\right)},
\end{aligned} \tag{3.68a}$$

for any $a, b = 1, \cdots, N_1 \cdots N_{d-1}$, to obtain $\mathbb{Z}_{N_1\cdots\,N_d}$ the group of translations on the tilted lattice Λ. This group extension can be summarized by the short exact sequence

$$1 \to \mathbb{Z}_{N_d} \to \mathbb{Z}_{N_1\cdots N_d} \to \mathbb{Z}_{N_1\cdots N_{d-1}} \to 1 \tag{3.68b}$$

and is labeled by the extension classes

$$[\Theta_{d-1}] \in H^2\big(\mathbb{Z}_{N_1\cdots N_{d-1}}, \mathbb{Z}_{N_d}\big). \tag{3.68c}$$

Using this extension class and the standard expression for group composition in an extended group, we may identify t_1 as the generator of $\mathbb{Z}_{N_1\cdots N_d}$.

At the quantum level, we represent the cyclic group (3.62) by replacing Eq. (3.57) with

$$\widehat{T}_\alpha\, \hat{\gamma}_j\, \widehat{T}_\alpha^{-1} = \hat{\gamma}_{t_\alpha(j)}, \tag{3.69a}$$

where $t_\alpha(j)$ is the action of the cyclic group $G_{\mathrm{trsl}}^{\mathrm{tlt}}$ on the repeat unit cell $j \in \Lambda$. The cyclicity of $G_{\mathrm{trsl}}^{\mathrm{tlt}} \equiv \mathbb{Z}_{N_1\cdots N_d} \equiv \mathbb{Z}_{|\Lambda|}$ is enforced by

$$\left(\widehat{T}_\alpha\right)^{N_\alpha} = \begin{cases} \widehat{T}_{\alpha+1}, & \text{if } \alpha = 1,\cdots,d-1, \\ \widehat{\mathbb{1}}_\Lambda, & \text{if } \alpha = d. \end{cases} \tag{3.69b}$$

With this convention, $(\widehat{T}_1)^a$ with $a = 1,\cdots,|\Lambda|$ represents all the elements $(t_1)^a$ with $a = 1,\cdots,|\Lambda|$ of $G_{\mathrm{trsl}}^{\mathrm{tlt}}$. Equation (3.58) is replaced by

$$\widehat{H}^{\mathrm{tlt}} := \sum_{a=1}^{|\Lambda|} \left(\widehat{T}_1\right)^a\, \hat{h}_1^{\mathrm{tlt}}\, \left(\widehat{T}_1^{-1}\right)^a, \tag{3.70a}$$

whereby

$$\hat{h}_1^{\mathrm{tlt}} = \widehat{U}(g)\, \hat{h}_1^{\mathrm{tlt}}\, \widehat{U}^{-1}(g) = \left(\hat{h}_1^{\mathrm{tlt}}\right)^\dagger \tag{3.70b}$$

holds for any $g \in G_f$. The locality of the polynomial $\hat{h}_1^{\mathrm{tlt}}$ is no longer manifest when comparing the integers that now label the local Majorana operators in $\hat{h}_1^{\mathrm{tlt}}$. The locality of $\hat{h}_1^{\mathrm{tlt}}$ is inherited from the fact that $\hat{h}_j$ is local while $\hat{h}_1^{\mathrm{tlt}}$ is nothing but a mere rewriting of $\hat{h}_j$ in the cyclic representation of $j \in \Lambda$. Hamiltonian (3.70) differs from Hamiltonian (3.58) by a sub-extensive number of terms of order $|\Lambda|/N_1$. The same number of terms would distinguish Hamiltonian (3.58) from the Hamiltonian $\widehat{H}_{\mathrm{tw}}(g)$ obtained by replacing the periodic boundary conditions (3.57b) by twisted one, i.e., by multiplying the right-hand side of Eq. (3.57b) with $\widehat{U}(g)$ for $g \neq e$ and $\alpha = 1$.

Because of the cyclicity of $G^{\text{tlt}}_{\text{trsl}} \equiv \mathbb{Z}_{N_1 \cdots N_d} \equiv \mathbb{Z}_{|\Lambda|}$ and of its quantum representation, we can adapt the definition (3.60) for the twisted translation operator when $d = 1$ to that when $d > 1$. We define for any $g \in G_f$ with $\mathfrak{c}(g) = 0$ the generator of twisted translation

$$\widehat{T}_1(g) := \hat{u}_{\boldsymbol{I}}(g)\, \widehat{T}_1, \qquad \hat{u}^{-1}_{\boldsymbol{I}}(g) = \hat{u}^{\dagger}_{\boldsymbol{I}}(g), \tag{3.71a}$$

through its action

$$\widehat{T}_1(g)\, \hat{\gamma}_j\, \widehat{T}^{-1}_1(g) = \begin{cases} (-1)^{\rho(g)}\, \hat{\gamma}_{t_1(j)}, & \text{if } j \neq \boldsymbol{N}, \\ \\ \hat{u}_{\boldsymbol{I}}(g)\, \hat{\gamma}_{\boldsymbol{I}}\, \hat{u}^{-1}_{\boldsymbol{I}}(g), & \text{if } j = \boldsymbol{N}, \end{cases} \tag{3.71b}$$

on any Majorana operator labeled by $j \in \Lambda$. Here, $\boldsymbol{I} \equiv (1, \cdots, 1) \in \Lambda$, $\boldsymbol{N} = (N_1, \cdots, N_d) \in \Lambda$, and $j = (n_1, \cdots, n_d)$ with $n_\alpha = 1, \cdots, N_\alpha$. One verifies that these twisted translation operators satisfy the twisted operator algebra

$$\widehat{U}(h)^{-1}\, \widehat{T}_1(g)\, \widehat{U}(h) = e^{\mathrm{i}\chi(g,h)}\, \widehat{T}_1(g), \tag{3.72a}$$

where

$$\chi(g,h) := \phi(g,h) - \phi(h,g) + (|\Lambda| - 1)\pi\, \rho(h)[\rho(g) + 1], \tag{3.72b}$$

which is nothing but the algebra (3.61) with the identification $N \to |\Lambda| \equiv N_1 \cdots N_d$. Finally, we define the family of Hamiltonians (3.59) that obey twisted boundary conditions. The proof of Theorem 2 when $d > 1$ for Hamiltonians of the form (3.59) is the same as that when $d = 1$. Because the family of Hamiltonians (3.59) only differ from the family of Hamiltonians (3.58) obeying periodic boundary conditions by a sub-extensive number of terms, the LSM-like conditions characterizing the existence of nondegenerate gapped ground states valid for Hamiltonians of the form (3.59) are conjectured to be also valid for Hamiltonians of the form (3.58).

3.4.3 *Theorem 3 in Any Dimension d*

We have extended Theorem 2 to any spatial dimension d. As discussed at the end of Sec. 3.3.2, if Theorem 2 holds for any d, then so does Theorem 3. It is nevertheless instructive to provide an alternative proof of Theorem 3 for any spatial dimension d without relying on Theorem 2.

We consider a d-dimensional lattice Λ such that at each repeat unit cell labeled by $j \in \Lambda$, there exists a Majorana spinor $\hat{\gamma}_j$ with $2n+1$ components $\hat{\gamma}_{j,l}$, $l = 1, \cdots, 2n+1$. To have a well-defined total Fock space on lattice Λ, we set the total number of sites $|\Lambda|$ in the lattice to be even. On lattice Λ, we impose the tilted translation symmetry group $G^{\mathrm{tlt}}_{\mathrm{trsl}}$ defined in Eq. (3.65). Let $\widehat{T}_1$ be the representation of the generator of the cyclic group $G^{\mathrm{tlt}}_{\mathrm{trsl}}$ with the action (3.69) on the Majorana spinors $\hat{\gamma}_j$.

In terms of the Majorana spinors $\hat{\gamma}_j$, the total fermion parity operator $\widehat{P}$ has the representation

$$\widehat{P} := \mathrm{i}^{|\Lambda|/2} \prod_{j\in\Lambda} \prod_{l=1}^{2n+1} \hat{\gamma}_{j,l}. \tag{3.73}$$

Conjugation of the fermion parity operator $\widehat{P}$ by the tilted translation operator $\widehat{T}_1$ delivers

$$\widehat{T}_1\, \widehat{P}\, \widehat{T}_1^{-1} = (-1)^{|\Lambda|-1} \widehat{P} = -\widehat{P}, \tag{3.74}$$

where we arrived at the last equality by noting that $|\Lambda|$ is an even integer. The factor $(-1)^{|\Lambda|-1}$ arises since each spinor $\hat{\gamma}_j$ consists of an odd number of Majorana operators. The nontrivial algebra (3.74) implies that the ground state of any Hamiltonian that commutes with $\widehat{P}$, the generators of the tilted translation group, and the generators of G_f is either degenerate or spontaneously breaks translation or G_f symmetry. If one assumes that the degeneracy of the ground states when gapped is independent of the choice made for the boundary conditions, we reproduce Theorem 3. We note that the algebra (3.74) was shown in Ref. [161] for a one dimensional Majorana chain and interpreted as the existence of Witten's quantum-mechanical supersymmetry (SUSY) [162].

4

INTRINSICALLY FERMIONIC LSM THEOREMS

Theorem 1 that is proved in Sec. 2 equally applies to Hamiltonians that are built out of both bosonic or fermionic degrees of freedom. In contrasts, Theorem 3 and certain instances of Theorem 2 can only be realized in Hamiltonians that are built out of fermionic degrees of freedom. The distinction between the bosonic and fermionic LSM theorems is due to the presence of the fermion parity symmetry $\mathbb{Z}_2^{\mathrm{F}}$ in the latter. Any Hamiltonian or observable that is built out of fermionic degrees of freedom must be even under the fermion parity. In other words, fermion parity symmetry cannot be broken neither explicitly nor spontaneously. In bosonic system there is no such symmetry with a special status.

In this Section, we motivate the notion of *intrinsically* fermionic LSM theorems, i.e., LSM theorems that apply strictly fermionic models. We focus only on the case of one-dimensional space. A tool that can be used to detect LSM constraints is dualities induced by gauging discrete subgroups of the full internal symmetry group. We define intrinsically fermionic LSM constraints as those which disappear under Jordan-Wigner (JW) transformation [163] which is induced by gauging fermion parity symmetry [164, 165]. For convenience, first, we are going to argue that Theorem 3 ($[\mu] = 1$) is intrinsically fermionic. We will then introduce a decomposition of the index $[\phi]$ that characterizes local projective representations into two indices $[(\nu, \rho)]$. As we shall see, the index ρ will be associated with the instances of Theorem 2 that are intrinsically fermionic. For both cases, we will show that the associated LSM constraint disappears after JW transformation.

4.1 INTRINSICALLY FERMIONIC LSM THEOREM 3

Theorem 3 applies to the system for which there are odd number of Majorana degrees of freedom per repeat unit cell. We specify this possibility by the index $[\mu] = 1$. To demonstrate why the case of $[\mu] = 1$ is intrinsically fermionic, we consider a concrete model. Let $\Lambda = \{1, \cdots, N\}$ be a one-dimensional lattice with $N = 2M$ an even integer and the global fermionic Fock space $\mathfrak{F}_\Lambda$ is of dimension $2^{(2m+1)\,M}$ with $(2m+1)$ the number of local Majorana flavors. By choosing the cardinality $|\Lambda| = 2M$ to be even, we make sure that the lattice is bipartite. We do not impose any internal symmetry, i.e., $G_f = \mathbb{Z}_2^{\mathrm{F}}$.

We define the $2^{(2m+1)M}$-dimensional global Fock space $\mathfrak{F}_\Lambda$ using the $2(2m+1)M$ Majorana operators obeying the algebra

$$\hat{\gamma}^\dagger_{j,a} = \chi_{j,a}, \quad \hat{\gamma}^2_{j,a} = 1, \quad \{\hat{\gamma}_{j,a}, \hat{\gamma}_{j,a'}\} = 2\delta_{j,j'}\,\delta_{a,a'}, \tag{4.1}$$

for $j, j' = 1, \cdots, 2M$ and $a, a' = 1, \cdots, 2m+1$. The most general translation- and G_f-invariant quadratic Hamiltonian with nearest neighbor couplings is

$$\widehat{H}_{\mathrm{pbc}} := \sum_{j=1}^{2M} \mathrm{i}\, \hat{\gamma}^{\mathsf{T}}_j\, \mathsf{M}\, \hat{\gamma}_{j+1} \tag{4.2}$$

where the $(2m+1) \times (2m+1)$-dimensional matrix M is real-valued and antisymmetric. As M has necessarily a zero eigenvalue, the spectrum of $\widehat{H}_{\mathrm{pbc}}$ is gapless. This is consistent with Theorem 3. As discussed in Sec. 3.4.3, in addition to having a gapless spectrum Hamiltonian (4.2) is also supersymmetric [161]. This is because the operator $\widehat{T}$ implementing the lattice translations anticommute with the global fermion parity operator $\widehat{U}(p)$. Furthermore, the Witten index is 0 implying that all states including the ground state is twofold degenerate.

The underlying SUSY of Hamiltonian (4.2) is the signature of its intrinsically fermionic nature. To further illustrate this point, let us focus on the case of $m = 0$, i.e.,

$$\widehat{H}_{\mathrm{K}} := \sum_{j=1}^{2M} \mathrm{i}\hat{\gamma}_j\, \hat{\gamma}_{j+1}, \tag{4.3}$$

which Hamiltonian (3.49) that describe the critical point between the trivial and nontrivial topological phases of the Kitaev chain. In one-dimensional space, there is a duality between the fermionic and bosonic models. It is instructive to map Hamiltonian (4.3) to its dual via JW transformation. We express the Majorana operators in terms of spin-1/2 degrees of freedom

$$\hat{\gamma}_{2j-1} = \left(\prod_{j'<j} \hat{\sigma}^z_{j'}\right) \hat{\sigma}^x_j, \qquad \hat{\gamma}_{2j} = \left(\prod_{j'<j} \hat{\sigma}^z_{j'}\right) \hat{\sigma}^y_j, \tag{4.4a}$$

$$\left[\hat{\sigma}^\alpha_j, \hat{\sigma}^\beta_j\right] = \mathrm{i}2\epsilon_{\alpha\beta\gamma}\hat{\sigma}^\gamma_j, \qquad \hat{\sigma}^\alpha_j\, \hat{\sigma}^\alpha_j = \hat{\mathbb{1}}, \qquad \alpha, \beta, \gamma = x, y, z, \tag{4.4b}$$

which delivers

$$\widehat{H}_{\mathrm{K}} := \sum_{j=1}^{M} \left\{ \mathrm{i}\hat{\gamma}_{2j-1}\,\hat{\gamma}_{2j} + \mathrm{i}\hat{\gamma}_{2j}\,\hat{\gamma}_{2j+1} \right\}$$
$$= -\sum_{j=1}^{M} \hat{\sigma}^{z}_{j} - \sum_{j=1}^{M-1} \hat{\sigma}^{x}_{j}\,\hat{\sigma}^{x}_{j+1} + \left(\prod_{j'=1}^{M} \hat{\sigma}^{z}_{j} \right) \hat{\sigma}^{x}_{M}\,\hat{\sigma}^{x}_{1}. \tag{4.4c}$$

If we impose the constraint

$$\left(\prod_{j=1}^{M} \hat{\sigma}^{z}_{j} \right) \equiv \left(\prod_{j=1}^{M} \mathrm{i}\hat{\gamma}_{2j}\,\hat{\gamma}_{2-1} \right) = \widehat{U}(p) = -\hat{\mathbb{1}}, \tag{4.4d}$$

Hamiltonian becomes

$$\widehat{H}_{\mathrm{I,crit}} := -\sum_{j=1}^{M} \hat{\sigma}^{z}_{j} - \sum_{j=1}^{M} \hat{\sigma}^{x}_{j}\,\hat{\sigma}^{x}_{j+1}, \tag{4.4e}$$

which describes the critical point of the transverse field Ising model with periodic boundary conditions. Hamiltonian (4.4e) also has a gapless spectrum. However, there are two distinctions between Hamiltonians (4.3) and (4.4e).

First, the twofold degeneracy due to the SUSY no longer exists for Hamiltonian (4.4e). This is because the constraint (4.4d) means that under JW transformation only the *odd*-parity sector of Hamiltonian (4.3) with periodic boundary conditions is mapped to Hamiltonian (4.4e) with periodic boundary conditions.

Second, the fermion parity symmetry of Hamiltonian (4.3) is mapped to the $\mathbb{Z}_2$ Ising symmetry generated by the operator

$$\widehat{U}_{I} := \prod_{j=1}^{M} \hat{\sigma}^{z}_{j}. \tag{4.5}$$

However, unlike the fermion parity symmetry, the Ising symmetry can be explicitly or spontaneously broken. The former happens when terms proportional to $\hat{\sigma}^x_j$ or $\hat{\sigma}^y_j$ are added. The latter happens when the coefficient of $\hat{\sigma}^z_j$ term is set to zero, i.e.,

$$\widehat{H}_{\mathrm{I}} := -\sum_{j=1}^{M} \hat{\sigma}^x_j\, \hat{\sigma}^x_{j+1}. \tag{4.6}$$

The Ising symmetry is spontaneously broken while the constraint (4.4d) is not satisfied on the symmetry breaking ground states. In the Majorana language Hamiltonian (4.6) maps to the Majorana chain

$$\widehat{H}_{\mathrm{D}} := \sum_{j=1}^{M} \mathrm{i}\hat{\gamma}_{2j}\, \hat{\gamma}_{2j+1}, \tag{4.7}$$

for which the translation symmetry is explicitly broken. This Hamiltonian realizes the nontrivial topological phase in symmetry class D in the Tenfold way [18, 19, 95]. When open boundary conditions are imposed the Hamiltonian (4.7) has twofold degenerate ground states that are symmetric under fermion parity. This twofold degenerate ground states correspond to the linear combinations of two symmetry breaking ground states of the Hamiltonian (4.6).

It is important to note that there is no LSM type theorem applies to the Hamiltonian (4.4e). This can be seen as the Hamiltonian

$$\widehat{H}_{\mathrm{PM}} := -\sum_{j=1}^{M} \hat{\sigma}^z_j, \tag{4.8}$$

does not break $\mathbb{Z}_2$ Ising symmetry and translation symmetry while its paramagnetic ground state is nondegenerate, gapped, and, symmetric. In light of this discussion, we observe that Hamiltonians for which Theorem 3 applies, does not necessarily map to Hamiltonians built out of bosonic degrees of freedom with a corresponding LSM type constraint. We thus call Theorem 3 intrinsically fermionic.

4.2 INDICES (ν, ρ)

Theorem 2 is only predictive once it is established that a projective representation of G_f is non trivial. We recall that (i) the group G_f is a central extension of the group G by $\mathbb{Z}_2^{\mathrm{F}}$ through the map γ defined in Eq. (3.3b) and (ii) only the equivalence classes $[\gamma] \in H^2\left(G, \mathbb{Z}_2^{\mathrm{F}}\right)$ deliver non-isomorphic groups. Choosing an element $[\gamma] \in H^2\left(G, \mathbb{Z}_2^{\mathrm{F}}\right)$ specifies G_f. In turn, a projective representation of G_f is specified by choosing an element ϕ from the equivalence class $[\phi] \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ where the 2-cocycle ϕ was defined in Eq. (3.8). To describe the cases when Theorem 2 is intrinsically fermionic, we shall distinguish the projective representations of the group G_f that cannot be realized in bosonic models. To this end, we must identify the contribution to the second group cohomology class $[\phi]$ that is purely due to the "bosonic" symmetries $G \cong G_f/\mathbb{Z}_2^{\mathrm{F}}$. Our intuition is that LSM theorems due to the purely bosonic part apply to bosonic Hamiltonians with symmetry group G.

When the group G_f splits, i.e., $[\gamma] = 0$ and $G_f \cong G \times \mathbb{Z}_2^{\mathrm{F}}$, this is achieved by using the Künneth formula

$$
\begin{aligned}
H^2\left(G_f, \mathrm{U}(1)_{\mathfrak{c}}\right) &= H^2\left(G \times \mathbb{Z}_2^{\mathrm{F}}, \mathrm{U}(1)_{\mathfrak{c}}\right) \\
&= H^2\left(G, \mathrm{U}(1)_{\mathfrak{c}}\right) \times H^1\left(G, \mathbb{Z}_2\right). \qquad (4.9)
\end{aligned}
$$

The second cohomology group $H^2(G, \mathrm{U}(1)_{\mathfrak{c}})$ classifies the projective representations of the group G while the first cohomology group $H^1\left(G, \mathbb{Z}_2\right)$ classifies the projective representations of G_f due to the nontrivial algebra of the elements of G with fermion parity $p \in \mathbb{Z}_2^{\mathrm{F}}$. For the second cohomology group $H^2(G, \mathrm{U}(1)_{\mathfrak{c}})$, we assign a 2-cochain $\nu \in C^2(G, U(1))$ which is defined by restricting the domain of ϕ to G from G_f. For the first cohomology $H^1\left(G, \mathbb{Z}_2\right)$, we assign the 1-cochain $\rho \in C^1(G, \mathbb{Z}_2)$ which is defined by

$$
\begin{aligned}
\rho \colon G &\to \mathbb{Z}_2, \\
g &\mapsto \rho(g) := \frac{\phi(g,p) - \phi(p,g)}{\pi} \bmod 2.
\end{aligned} \qquad (4.10)
$$

We recognize that the map (4.10) is the fermion parity $\rho(g) \in \{0,1\} \equiv \mathbb{Z}_2$ defined in Eq. (3.6d). All told, when the central extension G_f of the group G by $\mathbb{Z}_2^{\mathrm{F}}$ splits, the Künneth formula (4.9) predicts that

$$
[\phi] \equiv ([\nu], [\rho]) \qquad (4.11)
$$

When the central extension G_f of the group G by $\mathbb{Z}_2^{\mathrm{F}}$ does not split, i.e., $[\gamma] \neq 0$, then the identification (4.11) is no longer correct. In this case, the equivalence classes $[(\nu, \rho)]$ of the pair $(\nu, \rho) \in C^2(G, \mathrm{U}(1)) \times C^1(G, \mathbb{Z}_2)$ that satisfy the conditions [139]

$$(\delta\nu - \pi\,\rho \smile \gamma, \delta\rho) = (0, 0), \tag{4.12a}$$

are in one to one correspondence with the second cohomology classes $[\phi] \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$, see Appendix A.4. Here, the operation δ is defined in Eq. (A.3), while $\smile$ denotes the cup product defined in Eq. (A.9). Two pairs (ν, ρ) and (ν', ρ') are equivalent to each other if there exists another pair (α, β), whereby

$$\begin{aligned} \alpha\colon G &\to [0, 2\pi), \\ g &\mapsto \alpha(g), \end{aligned} \tag{4.12b}$$

while $\beta \in \mathbb{Z}_2$ such that

$$(\nu, \rho) = (\nu', \rho') + (\delta\alpha + \pi\beta \smile \gamma, \delta\beta)\,. \tag{4.12c}$$

Hence, the second cohomology class is identified with

$$[\phi] \equiv [(\nu, \rho)] \tag{4.12d}$$

It can be seen from Eqs. (4.12a) and (4.12c) that if $[\gamma] = 0$ then $\delta\nu = 0$, which is the defining condition for ν to be a 2-cocycle. We may then identify the gauge equivalent classes $[\nu]$ with the elements of the Abelian group $H^2\left(G, \mathrm{U}(1)_{\mathfrak{c}}\right)$ and the identification (4.12d) reduces to the one in Eq. (4.11). However, when $[\gamma] \neq 0$, the function $\nu\colon G \times G \to \mathrm{U}(1)$ is called a 2-cochain and belongs to the set $C^2\left(G, \mathrm{U}(1)\right)$. In practice, Eq. (4.12) ties the 2-cochain ν to the 2- and 1-cochains γ and ρ that belong to the sets $C^2\left(G, \mathbb{Z}_2^{\mathrm{F}}\right)$ and $C^1\left(G, \mathbb{Z}_2\right)$, respectively. From now on, we shall use the notation (4.12d) to trade in general the second cohomology class $[\phi]$ with the equivalence class $[(\nu, \rho)]$ and reserve the notation $([\nu], [\rho])$ of Eq. (4.11) for the case when the underlying fermionic symmetry group G_f splits (i.e., $[\gamma] = 0$).

4.3 INTRINSICALLY FERMIONIC LSM THEOREM 2 WHEN $\rho \neq 0$

Motivated by Eqs. (4.11) and (4.12), we call LSM constraints following from Theorem 2 intrinsically fermionic if the 1-cochain $\rho \in C^1(G, \mathbb{Z}_2)$ is nonvanishing, i.e., $\rho(g) = 1$ for some $g \in G$. In physical terms, this means that the local projective representation $\hat{u}_j(g)$ of at least one element in g carries odd fermion parity.

Assume that g is one such element with $\rho(g) = 1$. Using (3.12a), the global representation is

$$\widehat{U}(g) = \begin{cases} \prod_{j=1}^{N} \hat{v}_j(g)\, \hat{u}_j(p), & \text{if } \mathfrak{c}(g) = 0, \\ \\ \left[\prod_{j=1}^{N} \hat{v}_j(g)\, \hat{u}_j(p)\right] \mathsf{K}, & \text{if } \mathfrak{c}(g) = 1. \end{cases} \tag{4.13a}$$

When the number of sites N is odd $\widehat{U}(g)$ anticommutes with the global fermion parity operator $\widehat{U}(g)$. As it was the case with global translation operator in Sec. 4.1, any Hamiltonian that commutes with $\widehat{U}(g)$ has quantum mechanical SUSY and all states in its spectrum are at least doubly degenerate. When the number of sites N is even, $\widehat{U}(g)$ carries even fermion parity. We observe that

$$\begin{aligned} \widehat{T}_1\, \widehat{U}(g)\, \widehat{T}_1^{-1} &= \begin{cases} \prod_{j=1}^{N} \hat{v}_{j+1}(g)\, \hat{u}_{j+1}(p), & \text{if } \mathfrak{c}(g) = 0, \\ \\ \left[\prod_{j=1}^{N} \hat{v}_{j+1}(g)\, \hat{u}_{j+1}(p)\right] \mathsf{K}, & \text{if } \mathfrak{c}(g) = 1 \end{cases} \\ &= - \begin{cases} \prod_{j=1}^{N} \hat{v}_j(g)\, \hat{u}_j(p), & \text{if } \mathfrak{c}(g) = 0, \\ \\ \left[\prod_{j=1}^{N} \hat{v}_j(g)\, \hat{u}_j(p)\right] \mathsf{K}, & \text{if } \mathfrak{c}(g) = 1 \end{cases} \\ &= -\widehat{U}(g) \end{aligned} \tag{4.13b}$$

where the minus sign follows from reordering the local representations $\hat{v}_j(g)$ each of which carries odd fermion parity by assumption. This is to say that $\widehat{U}(g)$ and $\widehat{T}_1$ anticommutes and the spectrum of any Hamiltonian that is invariant under these is at least twofold degenerate. We emphasize that this algebra does not occur in Hamiltonians built out of bosonic degrees of freedom where local operators $\hat{v}_j(g)$ and $\hat{v}_{j'}(g)$ commute for any

$j, j' = 1, \cdots, N$. More precisely, consider the situation for bosonic degrees of freedom where we replaced the fermion parity symmetry $\mathbb{Z}_2^{\mathrm{F}}$ with an ordinary $\mathbb{Z}_2$ symmetry. We can also introduce a 1-cochain $\rho(g)$ which encodes whether the local representation $\hat{u}_j(g)$ commute or anticommute with the local representation of the generator of $\mathbb{Z}_2$, say $\hat{u}_j(\tilde{g})$. However, the operators $\hat{v}_j(g)$ and $\hat{v}_{j'}(g)$ commute irrespective of the value of $\rho(g)$. In the fermionic case, the 1-cochain ρ is tied to the statistics of underlying degrees of freedom.

The minus sign in Eq. (4.13b) can be removed by imposing antiperiodic boundary conditions. This is done by twisting the translation operator by fermion parity at one site, say $j = 1$. If so one has

$$\begin{aligned}\widehat{T}_1(p)\,\widehat{U}(g)\,\widehat{T}_1^{-1}(p) &= \hat{u}_1(p)\,\widehat{T}_1\,\widehat{U}(g)\,\widehat{T}_1^{-1}(p)\hat{u}_1(p)\\ &= -\hat{u}_1(p)\,\widehat{U}(g)\,\hat{u}_1(p)\\ &= \widehat{U}(g),\end{aligned} \tag{4.14a}$$

where the last equality follows since conjugating $\hat{v}_1(g)$ by $\hat{u}_1(p)$ gives a minus sign. With antiperiodic boundary conditions, global representation of g and the twisted translation operator commute. Theorem 2 still applies. A Hamiltonian that is invariant under both $\widehat{T}(p)$ and $\widehat{U}(g)$ cannot have a gapped, symmetric, and nondegenerate ground state since under the twisted boundary conditions again (from antiperiodic to periodic boundary conditions) the ground state of the Hamiltonian is at least twofold degenerate (recall the argument presented in Sec. 3.4.1). Similarly, if g is a unitarily represented element, i.e., $\mathfrak{c}(g) = 0$, the minus sign in Eq. (4.13b) can also be removed by imposing boundary conditions twisted by element g as well. In this case, the twisted translation operator $\widehat{T}_1(g)$ anticommutes with the global fermion parity operator. The Hamiltonians symmetric under such boundary conditions are then endowed with SUSY.

To better understand why Theorem 2 when $\rho \neq 0$ is intrinsically fermionic, let us consider the following concrete representation of the group $G_f = \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$

$$\hat{u}_j(g) = \hat{\xi}_j, \qquad \hat{u}_j(p) = \mathrm{i}\hat{\eta}_j\,\hat{\xi}_j, \tag{4.15a}$$

$$\hat{\eta}_j^2 = \hat{\xi}_j^2 = \hat{\mathbb{1}}, \qquad \left\{\hat{\eta}_j,\, \hat{\xi}_j\right\} = 0, \tag{4.15b}$$

$$\widehat{U}(g) = \prod_{j=1}^{2N} \hat{u}_j(g)\hat{u}_j(p) = \mathrm{i}^{2N} \prod_{j=1}^{2N} \hat{\eta}_j, \tag{4.15c}$$

$$\widehat{U}(p) = \prod_{j=1}^{2N} \hat{u}_j(p) = \mathrm{i}^{2N} \prod_{j=1}^{2N} \hat{\eta}_j \hat{\xi}_j. \tag{4.15d}$$

Each repeat unit cell, labeled by $j = 1, \cdots, 2N$, contains two Majorana degrees of freedom $\hat{\xi}_j$ and $\hat{\eta}_j$. Therefore, $[\mu] = 0$ and Theorem 3 is inactive. The local representation $\hat{u}_j(g)$ carries odd fermion parity and therefore $\rho(g) = 1$ [1].

Let $\widehat{H}$ be a local fermionic Hamiltonian that is symmetric under $\widehat{U}(g)$ and translation generated by $\widehat{T}_1$. By the argument we presented, $\widehat{H}$ has at least twofold degenerate ground states, because $\widehat{U}(g)$ anticommutes with the translation operator $\widehat{T}_1$. We use the JW transformation

$$\hat{\xi}_j = \left(\prod_{j'<j} \hat{\sigma}^z_{j'}\right) \hat{\sigma}^x_j, \qquad \hat{\eta}_j = \left(\prod_{j'<j} \hat{\sigma}^z_{j'}\right) \hat{\sigma}^y_j, \tag{4.16a}$$

to map the operators $\widehat{U}(g)$ and $\widehat{U}(p)$ to their bosonic forms

$$\widehat{U}(g) = \prod_{j=1}^{2N} \left(\prod_{j'<j} \hat{\sigma}^z_{j'}\right) \hat{\sigma}^y_j = \prod_{j=1}^{N} \mathrm{i}\hat{\sigma}^x_{2j-1}\, \hat{\sigma}^y_{2j}, \tag{4.16b}$$

$$\widehat{U}(p) = \prod_{j=1}^{2N} \hat{\sigma}^z_j. \tag{4.16c}$$

We observe that after the JW transformation the symmetry operator $\widehat{U}(g)$ while being locality-preserving and onsite, does not commute with translation symmetry as it implements different transformation rules on even and odd sites. Instead, the algebra

$$\widehat{T}_1\, \widehat{U}(g)\, \widehat{T}_1^\dagger = \widehat{U}(g)\widehat{U}(p). \tag{4.16d}$$

This is to say that the total symmetry group $G_f \times \mathbb{Z}_{2N} = \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}} \times \mathbb{Z}_{2N}$ in the fermionic language maps to the group $G_f \rtimes \mathbb{Z}_{2N} = \left(\mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}\right) \rtimes \mathbb{Z}_{2N}$ in the bosonic language under the JW transformation, i.e., the direct product of the internal and spatial symmetries becomes a semidirect product. Because of the semidirect product in the

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1 This is the only nontrivial local projective representation of the group $\mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$.

bosonic language, there is no LSM constraint analogous to Theorem 2 [2]. Intuitively, the representation (4.16b) breaks the translation symmetry as follows. JW transformation maps Majorana operator on a single site to strings of spin-1/2 operators which are nonlocal. The local representation $\hat{u}_j(g)$ defined in Eq. (4.15a) contains odd number of Majorana operators and necessarily mapped to a nonlocal string operator under JW transformation. In particular, $\hat{u}_{2N}(g)$ acts on the entire chain in the bosonic language. Therefore, an even-odd effect appears in the global representation $\widehat{U}(g)$. The fact that $\widehat{U}(g)$ commutes with translation by two sites ($\widehat{T}_1^2$) is related to the fact that when each pair of adjacent sites $2j-1$ and $2j$ are combined into a single site the resulting local representations (acting on the single site comprised of the two sites) carry even fermion parity. One can interpret this as a "trivialization" of the index ρ at the expense of introducing the semidirect product algebra in Eq. (4.16d).

To recapitulate, we have shown that the LSM constraints due to Theorem 3 and Theorem 2 when $\rho \neq 0$ only applies to Hamiltonians that are built out of fermionic degrees of freedom. Specifically, under the JW transformation, the corresponding LSM constraints in the fermionic language disappear. This means that for the resulting bosonic Hamiltonians, there may be symmetric perturbations that stabilize nondegenerate, gapped and symmetric ground state. We, therefore, call the LSM theorems for which either $\rho \neq 0$ or $[\mu] \neq 0$ of the triplet $([\phi],[\mu]) \equiv ([(\nu,\rho)],[\mu])$, *intrinsically fermionic*. All LSM constraints that are due to the presence of nontrivial local projective representations and translation symmetries are contained within the index ν.

2 Recall that the direct product structure of the internal and spatial symmetries is crucial for Theorems 2 and 3 as discussed in Sec. 3.

5

EXAMPLES OF HAMILTONIANS

We supplement the discussion in previous Sections by example Hamiltonians for which Theorem 2 applies [1]. We focus on three fermionic symmetry groups: (i) $G_f = \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$, (ii) $G_f = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$, and, (iii) $G_f = \mathbb{Z}_4^{\mathrm{FT}}$. The first two are split groups ($[\gamma] = 0$) while the last one is a nontrivial central extension of time-reversal symmetry $\mathbb{Z}_2^{\mathrm{T}}$. For simplicity, we consider the case of one-dimensional space and set the index $[\mu] = 0$, i.e., even number of Majorana degrees of freedom at each repeat unit cell. For each group, we then give examples of local representations for various values of the indices $[(\nu, \rho)]$ defined in Sec. 4.2. The details on the corresponding second cohomology groups are derived in Appendix A.1.

The lattice is $\Lambda = \{1, \cdots, 2N\}$, i.e., one-dimensional bipartite lattice of cardinality $|\Lambda| = 2N$. Given any one of these groups, we shall define a global fermionic Fock space $\mathfrak{F}_\Lambda = \mathfrak{F}_0 \oplus \mathfrak{F}_1$ and construct a projective representation that realizes the indices $[(\nu, \rho)]$ labeling $H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$. The global fermionic Fock space $\mathfrak{F}_\Lambda = \mathfrak{F}_0 \oplus \mathfrak{F}_1$ is here always constructed from $2n\,|\Lambda|$ Hermitian operators

$$\hat{\xi}_{j,\alpha} = \hat{\xi}^{\dagger}_{j,\alpha}, \qquad \hat{\eta}_{j,\alpha} = \hat{\eta}^{\dagger}_{j,\alpha}, \qquad j \in \Lambda, \qquad \alpha = 1, \cdots, n, \tag{5.1a}$$

obeying the Majorana (Clifford) algebra

$$\left\{\hat{\eta}_{j,\alpha}, \hat{\eta}_{j',\alpha'}\right\} = \left\{\hat{\xi}_{j,\alpha}, \hat{\xi}_{j',\alpha'}\right\} = 2\delta_{j,j'}\,\delta_{\alpha,\alpha'}, \qquad \left\{\hat{\eta}_{j,\alpha}, \hat{\xi}_{j',\alpha'}\right\} = 0. \tag{5.1b}$$

Since the index $[\mu] = 0$, n is an even integer and we define the fermionic creation and annihilation operators

$$\hat{c}^{\dagger}_{j,\alpha} := \frac{1}{2}\left(\hat{\eta}_{j,\alpha} - \mathrm{i}\hat{\xi}_{j,\alpha}\right), \qquad \hat{c}_{j,\alpha} := \frac{1}{2}\left(\hat{\eta}_{j,\alpha} + \mathrm{i}\hat{\xi}_{j,\alpha}\right), \tag{5.2a}$$

1 Hamiltonians (4.2) and (4.3) provide two examples to which Theorem 3 applies.

with $j \in \Lambda$ and $\alpha = 1, \cdots, n$. The local and global fermionic Fock space $\mathfrak{F}_j$ and $\mathfrak{F}_\Lambda$ are then

$$\mathfrak{F}_j := \operatorname{span}\left\{ \prod_{\alpha=1}^{n} \left(\hat{c}^\dagger_{j,\alpha}\right)^{n_{j,\alpha}} |0\rangle \,\middle|\, n_{j,\alpha} = 0, 1, \quad \hat{c}_{j,\alpha}\,|0\rangle = 0 \right\} \tag{5.2b}$$

and

$$\mathfrak{F}_\Lambda := \operatorname{span}\left\{ \prod_{j\in\Lambda} \prod_{\alpha=1}^{n} \left(\hat{c}^\dagger_{j,\alpha}\right)^{n_{j,\alpha}} |0\rangle \,\middle|\, n_{j,\alpha} = 0, 1, \quad \hat{c}_{j,\alpha}\,|0\rangle = 0 \right\}, \tag{5.2c}$$

respectively. We define the operation of complex conjugation K by its action

$$\mathsf{K} \left(z\, \hat{c}^\dagger_{j,\alpha} + w\, \hat{c}_{j',\alpha'} \right) \mathsf{K} := z^*\, \hat{c}^\dagger_{j,\alpha} + w^*\, \hat{c}_{j',\alpha'} \tag{5.3a}$$

for any pair of complex number $z, w \in \mathbb{C}$. This implies the transformation law

$$\mathsf{K}\, \hat{\eta}_{j,\alpha}\, \mathsf{K} = +\hat{\eta}_{j,\alpha}, \qquad \mathsf{K}\, \hat{\xi}_{j,\alpha}\, \mathsf{K} = -\hat{\xi}_{j,\alpha}, \tag{5.3b}$$

for any $j \in \Lambda$ and $\alpha = 1, \cdots, n$.

Given two example local representations $\hat{u}_{1,j}$ and $\hat{u}_{2,j}$ with indices (ν_1, ρ_1) and (ν_2, ρ_2), we construct a third representation $\hat{u}_{\wedge,j}$ by

$$\hat{u}_{\wedge,j}(g) = \hat{v}_1(g)\, \hat{v}_2(g)\; [\hat{u}_1(p)]^{\rho_2(g)}\; [\hat{u}_2(p)]^{\rho_1(g)}\; \mathsf{K}^{\mathfrak{c}(g)}, \tag{5.4}$$

for any element $g \in G_f$, where $\hat{v}_{1,j}$ and $\hat{v}_{2,j}$ are the unitary parts of the representations $\hat{u}_{1,j}$ and $\hat{u}_{2,j}$ as defined in Eq. (3.6c). Note that Eq. (5.4) resembles Eq. (3.12a) when the number of sites is just two and translation invariance is not imposed (representations on the two sites can carry different indices). This is not a coincidence as Eq. (5.4) is an example of fermionic stacking rules of projective representations. We will devote the entire Chapter 7 to the derivation of general stacking rules of indices $([(\nu, \rho)], [\mu])$ and to their physical significance in classifying invertible phases. For the purposes of this Section, we shall only use Eq. (5.4) to systematically construct local projective representations of the groups $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$, and, $\mathbb{Z}_4^{\mathrm{FT}}$ when $[\mu] = 0$. For each group, this exercise will reveal a group structure of the indices $[(\nu, \rho)]$ under the stacking rule (5.4). Group structures of indices $[(\nu, \rho)]$ for each group are derived in Appendix A.5 using the results from Chapter 7.

5.1 SYMMETRY GROUP $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$

The symmetry group $G_f := \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ is a split group. The group $G := \mathbb{Z}_2^{\mathrm{T}} = \{e, t\}$ corresponds to reversal of time. The local antiunitary representation $\hat{u}_j(t)$ of reversal of time generates a projective representation of the group $\mathbb{Z}_2^{\mathrm{T}}$. The local unitary representation $\hat{u}_j(p)$ of the fermion parity p generates a projective representation of the group $\mathbb{Z}_2^{\mathrm{F}}$. According to Appendix A.5.1, all cohomologically distinct projective representations of $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ are determined by the independent indices $[\nu] = 0, 1$ and $[\rho] = 0, 1$ through the relations [2]

$$\hat{u}_j(t)\,\hat{u}_j(t) = (-1)^{[\nu]}\,\hat{u}_j(e), \tag{5.5a}$$

$$\hat{u}_j(t)\,\hat{u}_j(p) = (-1)^{[\rho]}\,\hat{u}_j(p)\,\hat{u}_j(t), \tag{5.5b}$$

This gives the four distinct group cohomology classes

$$([\nu], [\rho]) \in \Big\{(0,0), (0,1), (1,0), (1,1)\Big\}. \tag{5.5c}$$

Under stacking rule (5.4), these indices form the group $\mathbb{Z}_4$, see Appendix A.5.1. All but the group cohomology class $([\nu], [\rho]) = (1, 0)$ can be realized using $n = 2$ local Majorana flavors. The group cohomology class $([\nu], [\rho]) = (1, 0)$ requires at least $n = 4$ local Majorana flavors for it to be realized. For each possibility in Eq. (5.5c), we will present an example local representation and a Hamiltonian that is invariant under the corresponding symmetry transformation. As we shall see, the local projective representations of the group $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ renders any invariant Hamiltonian to be *intrinsically interacting*, i.e., any symmetric Hamiltonian must consist of terms that are at least quartic in Majorana operators.

5.1.1 *Group Cohomology Class* $(0, 1)$

The local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D} = 2$ is generated by the doublet of Majorana operators

$$\hat{\chi}_j^{\dagger} \equiv \begin{pmatrix} \hat{\eta}_j & \hat{\xi}_j \end{pmatrix}, \qquad j = 1, \cdots, 2N. \tag{5.6}$$

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2 We have chosen the convention of always representing the generator p of $\mathbb{Z}_2^{\mathrm{F}}$ by a Hermitian operator according to Eq. (3.6e)

One verifies that

$$\hat{u}_j(t) := \hat{\eta}_j\,\mathsf{K}, \qquad \hat{u}_j(p) := \mathrm{i}\hat{\xi}_j\,\hat{\eta}_j, \tag{5.7a}$$

realizes the projective algebra (5.5) with $([\nu],[\rho]) = (0,1)$, and implements the transformations

$$\hat{u}_j(t)\,\hat{\eta}_j\,\hat{u}_j^{\dagger}(t) = +\hat{\eta}_j, \qquad \hat{u}_j(t)\,\hat{\xi}_j\,\hat{u}_j^{\dagger}(t) = +\hat{\xi}_j, \tag{5.7b}$$

$$\hat{u}_j(p)\,\hat{\eta}_j\,\hat{u}_j^{\dagger}(p) = -\hat{\eta}_j, \qquad \hat{u}_j(p)\,\hat{\xi}_j\,\hat{u}_j^{\dagger}(p) = -\hat{\xi}_j. \tag{5.7c}$$

One verifies that the Majorana doublet (5.6) is even under conjugation by $\hat{u}_j(t)$ and odd under conjugation by $\hat{u}_j(p)$. Time-reversal symmetry forbids any Hermitian quadratic form for the doublet (5.6). We consider the Hamiltonian

$$\widehat{H}_{\mathrm{pbc}} := \lambda \sum_{j=1}^{2N} \hat{\eta}_j\,\hat{\xi}_j\,\hat{\eta}_{j+1}\,\hat{\xi}_{j+1} = \lambda \sum_{j=1}^{2N} \hat{u}_j(p)\,\hat{u}_{j+1}(p) \qquad \lambda \in \mathbb{R}. \tag{5.8a}$$

This Hamiltonian is nothing but the sum of commuting terms with eigenvalues ± 1. Therefore, it can be equivalently represented by the Ising Hamiltonian

$$\widehat{H}_{\mathrm{pbc}} \equiv \lambda \sum_{j=1}^{2M} \hat{\sigma}_j\,\hat{\sigma}_{j+1}, \qquad \hat{\sigma}_j \equiv \hat{u}_j(p) \tag{5.8b}$$

When $\lambda < 0$, Hamiltonian (5.8b) has twofold degenerate ground states $|+\rangle_{\mathrm{F}}$ and $|-\rangle_{\mathrm{F}}$ that are specified by

$$\hat{u}_j(p)|\pm\rangle_{\mathrm{F}} = \pm|\pm\rangle_{\mathrm{F}}, \qquad j = 1, \cdots, 2N. \tag{5.9a}$$

Since $\hat{u}_j(t)$ and $\hat{u}_j(p)$ anticommute, states $|+\rangle_{\mathrm{F}}$ and $|-\rangle_{\mathrm{F}}$ break spontaneously time-reversal symmetry at zero temperature and thermodynamic limit, while translation and fermion parity symmetries are preserved.

When $\lambda > 0$, Hamiltonian (5.8b) has twofold degenerate ground states $|+\rangle_{\mathrm{AF}}$ and $|-\rangle_{\mathrm{AF}}$ that are specified by

$$\hat{u}_j(p)|\pm\rangle_{\mathrm{AF}} = \mp(-1)^j|\pm\rangle_{\mathrm{AF}}, \qquad j = 1, \cdots, 2N. \tag{5.9b}$$

Since $\hat{u}_j(t)$ and $\hat{u}_j(p)$ anticommute and translation maps one state to the other, states $|+\rangle_{\mathrm{AF}}$ and $|-\rangle_{\mathrm{AF}}$ break spontaneously time-reversal and translation symmetries at zero temperature and thermodynamic limit, while fermion parity symmetry is preserved.

5.1.2 *Group Cohomology Class* $(1,0)$

The local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D} = 4$ is generated by the quartet of Majorana operators [3]

$$\hat{\chi}_j^{\dagger} \equiv \begin{pmatrix} \hat{\eta}_{j,1} & \hat{\xi}_{j,1} & \hat{\eta}_{j,2} & \hat{\xi}_{j,2} \end{pmatrix}, \qquad j = 1, \cdots, 2N. \tag{5.10}$$

One verifies that stacking two copies of the representation (5.7) according to the rule (5.4) delivers

$$\hat{u}_j(t) := \hat{\xi}_{j,1}\, \hat{\xi}_{j,2}\, \mathsf{K}, \qquad \hat{u}_j(p) := -\hat{\xi}_{j,1}\, \hat{\eta}_{j,1}\, \hat{\xi}_{j,2}\, \hat{\eta}_{j,2}. \tag{5.11a}$$

This representation realizes the projective algebra (5.5) with $([\nu],[\rho]) = (1,0)$ and implements the transformations

$$\hat{u}_j(t)\, \hat{\eta}_{j,\alpha}\, \hat{u}_j^{\dagger}(t) = +\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(t)\, \hat{\xi}_{j,\alpha}\, \hat{u}_j^{\dagger}(t) = +\hat{\xi}_{j,\alpha}, \tag{5.11b}$$

$$\hat{u}_j(p)\, \hat{\eta}_{j,\alpha}\, \hat{u}_j^{\dagger}(p) = -\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(p)\, \hat{\xi}_{j,\alpha}\, \hat{u}_j^{\dagger}(p) = -\hat{\xi}_{j,\alpha}, \tag{5.11c}$$

for $\alpha = 1, 2$. One verifies that the Majorana quartet (5.10) is even under conjugation by $\hat{u}_j(t)$ and odd under conjugation by $\hat{u}_j(p)$. These transformation rules are identical to those in Eq. (5.7c) by construction of the stacking rule (5.4). Hence, two copies of the Hamiltonian (5.8a) is symmetric under the representation (5.7a). Time-reversal symmetry forbids any Hermitian quadratic form for the quartet (5.10).

Another symmetric Hamiltonian is obtained by coupling the two flavors $\alpha = 1$ and $\alpha = 2$. We define

$$\widehat{H}_{\mathrm{pbc}} := \lambda \sum_{j=1}^{2N} \hat{\eta}_{j,1}\, \hat{\xi}_{j,1}\, \hat{\eta}_{j,2}\, \hat{\xi}_{j,2} = -\lambda \sum_{j=1}^{2N} \hat{u}_j(p) \qquad \lambda \in \mathbb{R}. \tag{5.12}$$

---

3 It is not possible to represent the group cohomology class $([\nu],[\rho]) = (1,0)$ with a doublet of Majorana operators.

All terms in this Hamiltonian are mutually commuting and supported on only a single site. Furthermore, one has the identity

$$\left(\hat{\eta}_{j,1}\,\hat{\xi}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,2}\right)^2 = \hat{u}_j^2(p) = \hat{\mathbb{1}}, \tag{5.13}$$

i.e., on the four dimensional local Fock space $\mathfrak{F}_j$ operator $\hat{\eta}_{j,1}\,\hat{\xi}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,2}$ has eigenvalues of ± 1 each of which is twofold degenerate. Therefore, the ground states of Hamiltonian (5.12) are gapped with an excitation gap 2λ and extensively degenerate with total degeneracy 2^{2N} (which diverges exponentially in the thermodynamic limit $N \to \infty$). Since the total number of sites is even, $2N$, all 2^{2N}-fold degenerate ground states of Hamiltonian (5.12) carry even total fermion parity while all 2^{2N}-fold degenerate excited states carry odd total fermion parity. As the representation $\hat{u}_j(t)$ squares to minus identity, the twofold degenerate eigenstates of $\hat{u}_j(p)$ in the local Fock space $\mathfrak{F}_j$ form Kramer's doublets. Hence, 2^{2N}-fold degenerate ground states of Hamiltonian (5.12) form 2^N doublets that map to one another under reversal of time. This means that each one of the 2^{2N}-fold degenerate ground states spontaneously breaks reversal of time.

5.1.3 *Group Cohomology Class* $(1,1)$

The local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D} = 8$ is generated by the sextet of Majorana operators

$$\hat{\chi}_j^{\dagger} \equiv \begin{pmatrix} \hat{\eta}_{j,1} & \hat{\xi}_{j,1} & \hat{\eta}_{j,2} & \hat{\xi}_{j,2} & \hat{\eta}_{j,3} & \hat{\xi}_{j,3} \end{pmatrix}, \qquad j = 1, \cdots, 2N. \tag{5.14}$$

One verifies that stacking representation (5.7) with representation (5.11) according to the rule (5.4) delivers

$$\hat{u}_j(t) := \hat{\xi}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\xi}_{j,3}\,\mathsf{K}, \qquad \hat{u}_j(p) := -\mathrm{i}\hat{\xi}_{j,1}\,\hat{\eta}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,3}. \tag{5.15a}$$

This representation realizes the projective algebra (5.5) with $([\nu],[\rho]) = (1,1)$ and implements the transformations

$$\hat{u}_j(t)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^{\dagger}(t) = +\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(t)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^{\dagger}(t) = +\hat{\xi}_{j,\alpha}, \tag{5.15b}$$

$$\hat{u}_j(p)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^{\dagger}(p) = -\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(p)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^{\dagger}(p) = -\hat{\xi}_{j,\alpha}, \tag{5.15c}$$

for $\alpha = 1, 2, 3$. One verifies that the Majorana sextet (5.14) is even under conjugation by $\hat{u}_j(t)$ and odd under conjugation by $\hat{u}_j(p)$. These transformation rules are identical to those in Eqs. (5.7c) and (5.11c) by construction of the stacking rule (5.4). Hence, three copies of the Hamiltonian (5.8a) is symmetric under the representation (5.7a). Time-reversal symmetry forbids any Hermitian quadratic form for the sextet (5.14).

Another symmetric Hamiltonian is obtained by coupling the three flavors $\alpha = 1, 2, 3$. We define

$$\begin{aligned}\widehat{H}_{\mathrm{pbc}} &:= \lambda \sum_{j=1}^{2N} \left\{ \hat{\eta}_{j,1}\, \hat{\xi}_{j,1}\, \hat{\eta}_{j,2}\, \hat{\xi}_{j,2} + \hat{\eta}_{j,2}\, \hat{\xi}_{j,2}\, \hat{\eta}_{j,3}\, \hat{\xi}_{j,3} + \hat{\eta}_{j,3}\, \hat{\xi}_{j,3}\, \hat{\eta}_{j,1}\, \hat{\xi}_{j,1} \right\} \\ &= -\lambda \sum_{j=1}^{2N} \left\{ \hat{\sigma}_{j,1}\, \hat{\sigma}_{j,2} + \hat{\sigma}_{j,2}\, \hat{\sigma}_{j,3} + \hat{\sigma}_{j,3}\, \hat{\sigma}_{j,1} \right\}, \\ \hat{\sigma}_{j,\alpha} &\equiv \mathrm{i}\hat{\xi}_{j,\alpha}\, \hat{\eta}_{j,\alpha}, \qquad \alpha = 1, \cdots, 3, \qquad \lambda \in \mathbb{R}, \end{aligned} \tag{5.16}$$

All terms in this Hamiltonian are mutually commuting and supported on only a single site. The representation in terms of Ising degrees of freedom $\hat{\sigma}_{j,\alpha}$ implies that Hamiltonian (5.16) is nothing but $2N$ decoupled Ising triangles.

When $\lambda > 0$, the ground states are specified by all local Ising degrees of freedom at site j being aligned. In the eight dimensional local Fock space $\mathfrak{F}_j$ there are two such states, say $|j, \pm\rangle_{\mathrm{F}}$. The states $|j, -\rangle_{\mathrm{F}}$ and $|j, +\rangle_{\mathrm{F}}$ carry opposite fermion parities and mapped to each other under the reversal of time The ground states of Hamiltonian (5.16) has extensive degeneracy of 2^{2N} (which diverges exponentially in the thermodynamic limit $N \to \infty$). Each one of the 2^{2N}-fold degenerate ground states carries even total fermion parity and spontaneously breaks reversal of time.

When $\lambda < 0$, the ground states are specified by all local Ising degrees of freedom at site j being anti-aligned. This is to say that each Ising triangle is frustrated which leads to a twofold local degeneracy In the eight dimensional local Fock space $\mathfrak{F}_j$ there are six such states, say $|j, \pm, \alpha\rangle_{\mathrm{AF}}$ with $\alpha = 1, 2, 3$. For fixed α, the pair of states $|j, -, \alpha\rangle_{\mathrm{AF}}$ and $|j, +, \alpha\rangle_{\mathrm{AF}}$ carry opposite fermion parities and mapped to each other under the reversal of time. Therefore, the ground states of Hamiltonian (5.16) has extensive degeneracy of 6^{2N} (which diverges exponentially in the thermodynamic limit $N \to \infty$). Each one of the 6^{2N}-fold degenerate ground states carries even total fermion parity and spontaneously breaks reversal of time.

5.1.4 *Group Cohomology Class* $(0,0)$

The local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=8$ is generated by the octet of Majorana operators

$$\hat{\chi}_j^\dagger \equiv \begin{pmatrix}\hat{\eta}_{j,1} & \hat{\xi}_{j,1} & \hat{\eta}_{j,2} & \hat{\xi}_{j,2} & \hat{\eta}_{j,3} & \hat{\xi}_{j,3} & \hat{\eta}_{j,4} & \hat{\xi}_{j,4}\end{pmatrix}, \qquad j=1,\cdots,2N. \tag{5.17}$$

One verifies that stacking representation (5.7) with representation (5.15) according to the rule (5.4) delivers

$$\hat{u}_j(t) := \hat{\xi}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\xi}_{j,4}\,\mathsf{K}, \qquad \hat{u}_j(p) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,3}\,\hat{\xi}_{j,4}\,\hat{\eta}_{j,4}. \tag{5.18a}$$

This representation realizes the projective algebra (5.5) with $([\nu],[\rho])=(0,0)$ and implements the transformations

$$\hat{u}_j(t)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^\dagger(t) = +\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(t)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^\dagger(t) = +\hat{\xi}_{j,\alpha}, \tag{5.18b}$$

$$\hat{u}_j(p)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^\dagger(p) = -\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(p)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^\dagger(p) = -\hat{\xi}_{j,\alpha}, \tag{5.18c}$$

for $\alpha=1,\cdots,4$. One verifies that the Majorana octet (5.17) is even under conjugation by $\hat{u}_j(t)$ and odd under conjugation by $\hat{u}_j(p)$. These transformation rules are identical to those in Eqs. (5.7c) (5.11c), and, (5.15c) by construction of the stacking rule (5.4).

Since the indices take the values $([\nu],[\rho])=(0,0)$, Theorem 2 is inoperative. It is possible to find examples of both nondegenerate and degenerate gapped Hamiltonians that are translationally invariant and G_f-symmetric. To prove this claim, consider the Hamiltonian

$$\begin{aligned}\widehat{H}_{\text{pbc}} &:= \sum_{j=1}^{2N}\Big\{\hat{\eta}_{j,1}\,\hat{\xi}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,2} + \hat{\eta}_{j,2}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,3}\,\hat{\xi}_{j,3} \\ &\qquad\qquad + \hat{\eta}_{j,3}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,4}\,\hat{\xi}_{j,4} + \hat{\xi}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\xi}_{j,4}\Big\} \\ &= -\sum_{j=1}^{2N}\left\{\hat{\sigma}_{j,1}\,\hat{\sigma}_{j,2} + \hat{\sigma}_{j,2}\,\hat{\sigma}_{j,3} + \hat{\sigma}_{j,3}\,\hat{\sigma}_{j,4} - \hat{\xi}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\xi}_{j,4}\right\},\end{aligned} \tag{5.19}$$

$$\hat{\sigma}_{j,\alpha} \equiv \mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\eta}_{j,\alpha}, \qquad \alpha=1,\cdots,4, \qquad \lambda\in\mathbb{R}.$$

Hamiltonian (5.19) is sum of pairwise commuting terms. Each term supported on site j has a nondegenerate ground state. To see this, we note that the first three terms in Hamiltonian (5.19) requires the four Ising degrees of freedom $\hat{\sigma}_{j,\alpha}$ to either all align or all anti-align in the ground state manifold. Therefore, the first three terms alone have twofold degenerate ground states, say $|j,\pm\rangle$. The states $|j,\pm\rangle$ both carry even fermion parity and mapped to each other under reversal of time. The last term in Hamiltonian (5.19) then selects the time-reversal symmetric linear combination $|j,+\rangle - |j,-\rangle$ as the nondegenerate and symmetric ground state of each local term in Hamiltonian (5.19). The tensor product

$$\bigotimes_{j=1}^{2N} \Big(|j,+\rangle - |j,-\rangle \Big) \tag{5.20}$$

is then the nondegenerate, translationally invariant, and, G_f-symmetric ground state of Hamiltonian (5.19).

5.2 SYMMETRY GROUP $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$

The symmetry group $G_f := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$ is a split group. As usual, $\mathbb{Z}_2^{\mathrm{F}}$ is generated by p. The Abelian group $G := \mathbb{Z}_2 \times \mathbb{Z}_2$. has two generators g_1 and g_2 that commute pairwise, while each of them squares to the identity. We shall only consider the case when the local number of Majorana flavors $n = 2m$ is an even positive integer. The index μ then takes the value $\mu = 0$.

According to Appendix A.5.2, local projective representations of G_f can be labeled by the pair of indices $[\nu] \in H^2(G, \mathrm{U}(1)_{\mathfrak{c}})$ and $[\rho] = ([\rho]_1, [\rho]_2)$ with $\big([\rho]_1, [\rho]_2\big) \in H^1\big(G, \mathbb{Z}_2\big)$ through the relations [4]

$$\hat{u}(g_1)\,\hat{u}(g_2) = (-1)^{[\nu]}\,\hat{u}(g_2)\,\hat{u}(g_1), \quad [\nu] = 0, 1, \tag{5.21a}$$

$$\hat{u}(g_i)\,\hat{u}(p) = (-1)^{[\rho]_i}\,\hat{u}(p)\,\hat{u}(g_i), \quad [\rho]_i = 0, 1. \tag{5.21b}$$

4 We have chosen the convention of always representing the generator p of $\mathbb{Z}_2^{\mathrm{F}}$ by a Hermitian operator according to Eq. (3.6e)

This gives the eight distinct group cohomology classes

$$([\nu],[\rho]) = \Big\{ \big(0,(0,0)\big), \big(0,(0,1)\big), \big(0,(1,0)\big), \big(0,(1,1)\big), \big(1,(0,0)\big), \big(1,(0,1)\big), \big(1,(1,0)\big), \big(1,(1,1)\big) \Big\}. \tag{5.22}$$

Here, the group cohomology class $\big(0,(0,0)\big)$ is interpreted as the trivial representation. Theorem 2 is predictive for any of the remaining seven group cohomology classes. It is shown in Appendix A.5.2 that these eight distinct group cohomology classes form the (stacking) group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, whereby the group composition is defined by the stacking rule (5.4). We choose the generators of this (stacking) group to be the three group cohomology classes $\big(1,(1,0)\big)$, $\big(1,(0,1)\big)$, and $\big(1,(0,0)\big)$.

5.2.1 *Group Cohomology Class* $\big(1,(1,0)\big)$

The local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=4$ is generated by the quartet of Majorana operators

$$\hat{\chi}^{\dagger}_{j} \equiv \begin{pmatrix} \hat{\eta}_{j,1} & \hat{\xi}_{j,1} & \hat{\eta}_{j,2} & \hat{\xi}_{j,2} \end{pmatrix}, \qquad j=1,\cdots,2N. \tag{5.23}$$

One verifies that

$$\hat{u}_j(g_1) := \hat{\eta}_{j,1}, \qquad \hat{u}_j(g_2) := \hat{\eta}_{j,1}\,\hat{\eta}_{j,2}, \qquad \hat{u}_j(p) := -\hat{\xi}_{j,1}\,\hat{\eta}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,2}, \tag{5.24a}$$

realizes the projective representation (5.21) with $([\nu],[\rho]) = \big(1,(1,0)\big)$ and implements the transformations

$$\hat{u}_j(g_1)\,\hat{\eta}_{j,\alpha}\,\hat{u}^{\dagger}_j(g_1) = (-1)^{\alpha+1}\,\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(g_1)\,\hat{\xi}_{j,\alpha}\,\hat{u}^{\dagger}_j(g_1) = -\hat{\xi}_{j,\alpha}, \tag{5.24b}$$

$$\hat{u}_j(g_2)\,\hat{\eta}_{j,\alpha}\,\hat{u}^{\dagger}_j(g_2) = -\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(g_2)\,\hat{\xi}_{j,\alpha}\,\hat{u}^{\dagger}_j(g_2) = +\hat{\xi}_{j,\alpha}, \tag{5.24c}$$

$$\hat{u}_j(p)\,\hat{\eta}_{j,\alpha}\,\hat{u}^{\dagger}_j(p) = -\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(p)\,\hat{\xi}_{j,\alpha}\,\hat{u}^{\dagger}_j(p) = -\hat{\xi}_{j,\alpha}, \tag{5.24d}$$

for $\alpha = 1, 2$.

An example of a translation- and G_f-invariant Hamiltonian of quadratic order is

$$\widehat{H}_{\mathrm{pbc}} := \sum_{j=1}^{2N} \sum_{\alpha=1}^{2} \left\{ \lambda_\alpha \mathrm{i}\hat{\eta}_{j,\alpha}\, \hat{\eta}_{j+1,\alpha} + \lambda'_\alpha \mathrm{i}\hat{\xi}_{j,\alpha}\, \hat{\xi}_{j+1,\alpha} \right\} + \lambda \sum_{j=1}^{2N} \left\{ \mathrm{i}\hat{\xi}_{j,1}\, \hat{\xi}_{j,2} \right\}, \tag{5.25}$$

with λ_α, λ'_α, $\lambda \in \mathbb{R}$. This Hamiltonian does not conserve the fermion-number in the fermion-number basis (5.2). It can be thought of as four Kitaev chains each of which has an effective index $\mu = 1$. When $\lambda = 0$, all Kitaev chains decouple and are fine-tuned to their quantum critical point (3.49) between their symmetry-protected and topologically inequivalent gapped phases. Two of the Kitaev chains are coupled by the on-site term $\mathrm{i}\hat{\xi}_{j,1}\, \hat{\xi}_{j,2}$, which gaps their spectrum. The low-energy sector of the theory is that of two decoupled quantum critical Kitaev chains containing the $\hat{\eta}_{j,\alpha}$ degrees of freedom. The quadratic term $\mathrm{i}\hat{\eta}_{j,1}\, \hat{\eta}_{j,2}$ that would gap remaining gapless degrees of freedom, thereby delivering a nondegenerate gapped ground state, is odd under conjugation by $\hat{u}_j(g_1)$ and thus forbidden by symmetry. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 2, which also predicts that any G_f-symmetric interaction that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $\big(1,(1,0)\big)$ representation are stacked according to Eq. (5.4), the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D} = 16$ is generated by the octuplet of Majorana operators

$$\hat{\chi}^\dagger_j \equiv \begin{pmatrix} \hat{\eta}_{j,1} & \hat{\xi}_{j,1} & \hat{\eta}_{j,2} & \hat{\xi}_{j,2} & \hat{\eta}_{j,3} & \hat{\xi}_{j,3} & \hat{\eta}_{j,4} & \hat{\xi}_{j,4} \end{pmatrix}, \qquad j = 1, \cdots, 2N. \tag{5.26}$$

One verifies that

$$\hat{u}_j(g_1) := \hat{\xi}_{j,1}\, \hat{\xi}_{j,2}\, \hat{\eta}_{j,2}\, \hat{\xi}_{j,3}\, \hat{\xi}_{j,4}\, \hat{\eta}_{j,4}, \tag{5.27a}$$

$$\hat{u}_j(g_2) := \hat{\eta}_{j,1}\, \hat{\eta}_{j,2}\, \hat{\eta}_{j,3}\, \hat{\eta}_{j,4}, \tag{5.27b}$$

$$\hat{u}_j(p) := \hat{\xi}_{j,1}\, \hat{\eta}_{j,1}\, \hat{\xi}_{j,2}\, \hat{\eta}_{j,2}\, \hat{\xi}_{j,3}\, \hat{\eta}_{j,3}\, \hat{\xi}_{j,4}\, \hat{\eta}_{j,4}, \tag{5.27c}$$

realizes the projective representation (5.21) with $([\nu],[\rho]) = \big(0,(0,0)\big)$, i.e., the trivial projective representation. In this trivial representation, for any Majorana operator there exists another one such that they transform identically under G_f. For instance, one can choose the pairs

$$\left\{\hat{\xi}_{j,1},\,\hat{\xi}_{j,2}\right\},\quad \left\{\hat{\xi}_{j,3},\,\hat{\xi}_{j,4}\right\},\quad \left\{\hat{\eta}_{j,2},\,\hat{\eta}_{j,4}\right\},\quad \left\{\hat{\eta}_{j,1},\,\hat{\eta}_{j,3}\right\}. \tag{5.28}$$

This means that all onsite terms that are coupling these pairs are then G_f symmetric. The ground-state degeneracy of any translation- and G_f-invariant Hamiltonian can be lifted by including the 4 on-site terms, i.e., Theorem 2 is not predictive.

5.2.2 *Group Cohomology Class* $\big(1,(0,1)\big)$

The local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=4$ is generated by the quartet of Majorana operators (5.23). One verifies that

$$\hat{u}_j(g_1) := \hat{\eta}_{j,1}\,\hat{\eta}_{j,2},\qquad \hat{u}_j(g_2) := \hat{\eta}_{j,1},\qquad \hat{u}_j(p) := -\hat{\xi}_{j,1}\,\hat{\eta}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,2}, \tag{5.29a}$$

realizes the projective representation (5.21) with $([\nu],[\rho]) = \big(1,(0,1)\big)$ and implements the transformations

$$\hat{u}_j(g_1)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^{\dagger}(g_1) = -\hat{\eta}_{j,\alpha},\qquad \hat{u}_j(g_1)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^{\dagger}(g_1) = +\hat{\xi}_{j,\alpha}, \tag{5.29b}$$

$$\hat{u}_j(g_2)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^{\dagger}(g_2) = (-1)^{\alpha+1}\,\hat{\eta}_{j,\alpha},\qquad \hat{u}_j(g_2)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^{\dagger}(g_2) = -\hat{\xi}_{j,\alpha}, \tag{5.29c}$$

$$\hat{u}_j(p)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^{\dagger}(p) = -\hat{\eta}_{j,\alpha},\qquad \hat{u}_j(p)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^{\dagger}(p) = -\hat{\xi}_{j,\alpha}, \tag{5.29d}$$

for $\alpha = 1, 2$.

Equation (5.29) differs from Eq. (5.24) by interchanging g_1 and g_2. This difference does not affect the reasoning leading to to the conclusion that the gapless Hamiltonian (5.25) cannot be gapped by onsite quadratic terms without breaking G_f symmetry. This difference also implies that stacking two copies of the $\big(1,(0,1)\big)$ representation (5.29) delivers the trivial projective representation $\big(0,(0,0)\big)$ encoded by Eqs. (5.26) and (5.27), for which Theorem 2 is not predictive anymore.

5.2.3 *Group Cohomology Class* $\big(1,(0,0)\big)$

The local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=4$ is generated by the quartet of Majorana operators (5.23). One verifies that

$$\hat{u}_j(g_1) := \hat{\eta}_{j,1}\,\hat{\eta}_{j,2}, \quad \hat{u}_j(g_2) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,2}, \quad \hat{u}_j(p) := -\hat{\xi}_{j,1}\,\hat{\eta}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,2}, \tag{5.30a}$$

realizes the projective representation (5.21) with $([\nu],[\rho]) = \big(1,(0,0)\big)$ and implements the transformations

$$\hat{u}_j(g_1)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^{\dagger}(g_1) = -\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(g_1)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^{\dagger}(g_1) = +\hat{\xi}_{j,\alpha}, \tag{5.30b}$$

$$\hat{u}_j(g_2)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^{\dagger}(g_2) = (-1)^{\alpha+1}\,\hat{\eta}_{j,\alpha}, \quad \hat{u}_j(g_2)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^{\dagger}(g_2) = (-1)^{\alpha}\,\hat{\xi}_{j,\alpha}, \tag{5.30c}$$

$$\hat{u}_j(p)\,\hat{\eta}_{j,\alpha}\,\hat{u}_j^{\dagger}(p) = -\hat{\eta}_{j,\alpha}, \qquad \hat{u}_j(p)\,\hat{\xi}_{j,\alpha}\,\hat{u}_j^{\dagger}(p) = -\hat{\xi}_{j,\alpha}, \tag{5.30d}$$

An example of a translation- and G_f-invariant Hamiltonian of quadratic order is

$$\widehat{H}_{\mathrm{pbc}} := \sum_{j=1}^{2N}\sum_{\alpha=1}^{2}\left\{\lambda_\alpha \mathrm{i}\hat{\eta}_{j,\alpha}\,\hat{\eta}_{j+1,\alpha} + \lambda'_\alpha \mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\xi}_{j+1,\alpha}\right\}, \tag{5.31}$$

with λ_α, $\lambda'_\alpha \in \mathbb{R}$, i.e., four decoupled Kitaev chains that are fine-tuned to their quantum critical point (3.49) between their symmetry-protected and topologically inequivalent gapped phases. No on-site quadratic term is allowed by the symmetries. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 2. Theorem 2 also predicts that any G_f-symmetric interaction that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $\big(1,(0,0)\big)$ representation are stacked according to Eq. (5.4), the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=16$ is generated by the octuplet of Majorana operators (5.26) with the projective representation

$$\hat{u}_j(g_1) := \hat{\eta}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\eta}_{j,3}\,\hat{\eta}_{j,4}, \tag{5.32a}$$

$$\hat{u}_j(g_2) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,4}, \tag{5.32b}$$

$$\hat{u}_j(p) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,3}\,\hat{\xi}_{j,4}\,\hat{\eta}_{j,4}, \tag{5.32c}$$

that realizes the group cohomology class $([\nu],[\rho],\mu) = \big(0,(0,0),0\big)$, i.e., the trivial group cohomology class. In this trivial representation, it is possible to choose four pairs of Majorana operators such that onsite terms coupling them is G_f-symmetric. The ground-state degeneracy of any translation- and G_f-invariant Hamiltonian can be lifted by including these four on-site terms. Theorem 2 is not predictive.

5.2.4 *Group Cohomology Class* $\big(1,(1,1)\big)$

When representations (5.24) and (5.29) are stacked according to Eq. (5.4), the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=16$ is generated by the octuplet of Majorana operators (5.26). One verifies that

$$\hat{u}_j(g_1) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,3}\,\hat{\eta}_{j,4}, \tag{5.33a}$$

$$\hat{u}_j(g_2) := \hat{\eta}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,4}\,\hat{\xi}_{j,4}, \tag{5.33b}$$

$$\hat{u}_j(p) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,3}\,\hat{\xi}_{j,4}\,\hat{\eta}_{j,4}, \tag{5.33c}$$

realizes the projective representation (5.21) with $([\nu],[\rho]) = \big(1,(1,1)\big)$ and implements the transformations

$$\hat{u}_j(g_1)\,\hat{\chi}^{\dagger}_j\hat{u}^{\dagger}_j(g_1) = \begin{pmatrix} -\hat{\eta}_{j,1} & +\hat{\xi}_{j,1} & +\hat{\eta}_{j,2} & +\hat{\xi}_{j,2} \\ +\hat{\eta}_{j,3} & -\hat{\xi}_{j,3} & +\hat{\eta}_{j,4} & -\hat{\xi}_{j,4} \end{pmatrix}, \tag{5.33d}$$

$$\hat{u}_j(g_2)\,\hat{\chi}^{\dagger}_j\hat{u}^{\dagger}_j(g_2) = \begin{pmatrix} +\hat{\eta}_{j,1} & -\hat{\xi}_{j,1} & +\hat{\eta}_{j,2} & -\hat{\xi}_{j,2} \\ -\hat{\eta}_{j,3} & +\hat{\xi}_{j,3} & +\hat{\eta}_{j,4} & +\hat{\xi}_{j,4} \end{pmatrix}, \tag{5.33e}$$

$$\hat{u}_j(p)\,\hat{\chi}^{\dagger}_j\hat{u}^{\dagger}_j(p) = \begin{pmatrix} -\hat{\eta}_{j,1} & -\hat{\xi}_{j,1} & -\hat{\eta}_{j,2} & -\hat{\xi}_{j,2} \\ -\hat{\eta}_{j,3} & -\hat{\xi}_{j,3} & -\hat{\eta}_{j,4} & -\hat{\xi}_{j,4} \end{pmatrix}. \tag{5.33f}$$

Given the transformation rules (5.33), one cannot construct four disjoint pairs of Majorana operators that transform in the same manner under both symmetries. One can at most obtain three pairs such as

$$\left\{\hat{\xi}_{j,1},\,\hat{\xi}_{j,2}\right\}, \quad \left\{\hat{\eta}_{j,1},\,\hat{\xi}_{j,4}\right\}, \quad \left\{\hat{\eta}_{j,2},\,\hat{\eta}_{j,4}\right\}, \tag{5.34}$$

which leaves the Majorana operators $\hat{\xi}_{j,3}$ and $\hat{\eta}_{j,3}$ unpaired. Therefore, the G_f- and translation-symmetric Hamiltonian

$$\widehat{H}_{\mathrm{pbc}} := \sum_{j=1}^{2N} \sum_{\alpha=1}^{4} \left\{ \lambda_\alpha \mathrm{i}\hat{\eta}_{j,\alpha}\, \hat{\eta}_{j+1,\alpha} + \lambda'_\alpha \mathrm{i}\hat{\xi}_{j,\alpha}\, \hat{\xi}_{j+1,\alpha} \right\}, \tag{5.35}$$

with λ_α, $\lambda'_\alpha \in \mathbb{R}$, that describe eight decoupled Kitaev chains can be partially gapped by three onsite terms. Because of the projective representation (5.33), the low-energy sector described by two decoupled Kitaev chains cannot be gapped. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 2, which also predicts that any G_f-symmetric interaction that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $\big(1,(1,1)\big)$ representation are stacked, the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D} = 256$ is generated by 16 Majorana operators. One verifies that

$$\hat{u}_j(g_1) := \hat{\eta}_{j,1}\, \hat{\xi}_{j,3}\, \hat{\xi}_{j,4}\, \hat{\eta}_{j,5}\, \hat{\xi}_{j,7}\, \hat{\xi}_{j,8}, \tag{5.36a}$$

$$\hat{u}_j(g_2) := \hat{\xi}_{j,1}\, \hat{\xi}_{j,2}\, \hat{\eta}_{j,3}\, \hat{\xi}_{j,5}\, \hat{\xi}_{j,6}\, \hat{\eta}_{j,7}, \tag{5.36b}$$

$$\hat{u}_j(p) := \prod_{\alpha=1}^{4} \mathrm{i}\hat{\xi}_{j,\alpha}\, \hat{\eta}_{j,\alpha}, \tag{5.36c}$$

realizes the group cohomology class $([\nu],[\rho]) = \big(0,(0,0)\big)$, i.e., the trivial group cohomology class. There exists a bijective map $\alpha \mapsto \alpha' := (\alpha+4) \bmod 8$ such that all on-site terms $\mathrm{i}\hat{\xi}_{j,\alpha}\, \hat{\xi}_{j,\alpha'}$ and $\mathrm{i}\hat{\eta}_{j,\alpha}\, \hat{\eta}_{j,\alpha'}$ with $\alpha = 1,\cdots,4$ can be shown to be G_f-symmetric. The ground-state degeneracy of any translation- and G_f-invariant Hamiltonian can be lifted by including the 8 on-site terms $\mathrm{i}\hat{\xi}_{j,\alpha}\, \hat{\xi}_{j,\alpha+4}$ and $\mathrm{i}\hat{\eta}_{j,\alpha}\, \hat{\eta}_{j,\alpha+4}$ with $\alpha = 1,\cdots,4$, i.e., Theorem 2 is not predictive.

5.2.5 *Group Cohomology Class* $\big(0,(1,0)\big)$

When representations (5.24) and (5.30) are stacked according to Eq. (5.4), the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=16$ is generated by the octuplet of Majorana operators (5.26). One verifies that

$$\hat{u}_j(g_1) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,3}\,\hat{\eta}_{j,4}, \tag{5.37a}$$

$$\hat{u}_j(g_2) := \hat{\eta}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,4}, \tag{5.37b}$$

$$\hat{u}_j(p) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,3}\,\hat{\xi}_{j,4}\,\hat{\eta}_{j,4}, \tag{5.37c}$$

realizes the projective representation (5.21) with $([\nu],[\rho])=\big(0,(1,0)\big)$ and implements the transformations

$$\hat{u}_j(g_1)\,\hat{\chi}^{\dagger}_j\,\hat{u}^{\dagger}_j(g_1) = \begin{pmatrix} -\hat{\eta}_{j,1} & +\hat{\xi}_{j,1} & +\hat{\eta}_{j,2} & +\hat{\xi}_{j,2} \\ +\hat{\eta}_{j,3} & -\hat{\xi}_{j,3} & +\hat{\eta}_{j,4} & -\hat{\xi}_{j,4} \end{pmatrix}, \tag{5.37d}$$

$$\hat{u}_j(g_2)\,\hat{\chi}^{\dagger}_j\,\hat{u}^{\dagger}_j(g_2) = \begin{pmatrix} -\hat{\eta}_{j,1} & +\hat{\xi}_{j,1} & -\hat{\eta}_{j,2} & +\hat{\xi}_{j,2} \\ +\hat{\eta}_{j,3} & -\hat{\xi}_{j,3} & -\hat{\eta}_{j,4} & +\hat{\xi}_{j,4} \end{pmatrix}, \tag{5.37e}$$

$$\hat{u}_j(p)\,\hat{\chi}^{\dagger}_j\,\hat{u}^{\dagger}_j(p) = \begin{pmatrix} -\hat{\eta}_{j,1} & -\hat{\xi}_{j,1} & -\hat{\eta}_{j,2} & -\hat{\xi}_{j,2} \\ -\hat{\eta}_{j,3} & -\hat{\xi}_{j,3} & -\hat{\eta}_{j,4} & -\hat{\xi}_{j,4} \end{pmatrix}. \tag{5.37f}$$

Given the transformation rules (5.37), one cannot construct four disjoint pairs of Majorana operators that transform in the same manner under both symmetries. One can at most obtain three pairs such as

$$\left\{\hat{\xi}_{j,1},\,\hat{\xi}_{j,2}\right\}, \quad \left\{\hat{\eta}_{j,1},\,\hat{\xi}_{j,3}\right\}, \quad \left\{\hat{\eta}_{j,2},\,\hat{\eta}_{j,4}\right\}, \tag{5.38}$$

which leaves the Majorana operators $\hat{\xi}_{j,4}$ and $\hat{\eta}_{j,3}$ unpaired. Therefore, the G_f- and translation-symmetric Hamiltonian (5.35) that describe eight decoupled Kitaev chains can be partially gapped by three onsite terms. Because of the projective representation (5.37), the low-energy sector described by two decoupled Kitaev chains cannot be gapped. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 2, which also predicts that any G_f-symmetric interaction that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $\big(0,(1,0)\big)$ representation are stacked, the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=256$ is generated by 16 Majorana operators. One verifies that

$$\hat{u}_j(g_1) := \hat{\eta}_{j,1}\,\hat{\xi}_{j,3}\,\hat{\xi}_{j,4}\,\hat{\eta}_{j,5}\,\hat{\xi}_{j,7}\,\hat{\xi}_{j,8}, \tag{5.39a}$$

$$\hat{u}_j(g_2) := \hat{\eta}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,4}\,\hat{\eta}_{j,5}\,\hat{\eta}_{j,6}\,\hat{\xi}_{j,7}\,\hat{\eta}_{j,8}, \tag{5.39b}$$

$$\hat{u}_j(p) := \prod_{\alpha=1}^{4} \mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\eta}_{j,\alpha}, \tag{5.39c}$$

realizes the group cohomology class $([\nu],[\rho]) = \big(0,(0,0)\big)$, i.e., the trivial group cohomology class. There exists a bijective map $\alpha \mapsto \alpha' := (\alpha+4) \bmod 8$ such that all on-site terms $\mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\xi}_{j,\alpha'}$ and $\mathrm{i}\hat{\eta}_{j,\alpha}\,\hat{\eta}_{j,\alpha'}$ with $\alpha=1,\cdots,4$ can be shown to be G_f-symmetric. The ground-state degeneracy of any translation- and G_f-invariant Hamiltonian can be lifted by including the 8 on-site terms $\mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\xi}_{j,\alpha+4}$ and $\mathrm{i}\hat{\eta}_{j,\alpha}\,\hat{\eta}_{j,\alpha+4}$ with $\alpha=1,\cdots,4$, i.e., Theorem 2 is not predictive.

5.2.6 *Group Cohomology Class* $\big(0,(0,1)\big)$

When representations (5.29) and (5.30) are stacked, the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=16$ is generated by the octuplet of Majorana operators (5.26). One verifies that

$$\hat{u}_j(g_1) := \hat{\eta}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\eta}_{j,3}\,\hat{\eta}_{j,4}, \tag{5.40a}$$

$$\hat{u}_j(g_2) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,4}, \tag{5.40b}$$

$$\hat{u}_j(p) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,1}\,\hat{\xi}_{j,2}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\eta}_{j,3}\,\hat{\xi}_{j,4}\,\hat{\eta}_{j,4}, \tag{5.40c}$$

realizes the projective representation (5.21) with $([\nu],[\rho]) = \big(0,(0,1)\big)$ and implements the transformations

$$\hat{u}_j(g_1)\,\hat{\chi}_j^{\dagger}\hat{u}_j^{\dagger}(g_1) = \begin{pmatrix} -\hat{\eta}_{j,1} & +\hat{\xi}_{j,1} & -\hat{\eta}_{j,2} & +\hat{\xi}_{j,2} \\ -\hat{\eta}_{j,3} & +\hat{\xi}_{j,3} & -\hat{\eta}_{j,4} & +\hat{\xi}_{j,4} \end{pmatrix}, \tag{5.40d}$$

$$\hat{u}_j(g_2)\,\hat{\chi}_j^{\dagger}\hat{u}_j^{\dagger}(g_2) = \begin{pmatrix} -\hat{\eta}_{j,1} & +\hat{\xi}_{j,1} & +\hat{\eta}_{j,2} & +\hat{\xi}_{j,2} \\ -\hat{\eta}_{j,3} & +\hat{\xi}_{j,3} & +\hat{\eta}_{j,4} & -\hat{\xi}_{j,4} \end{pmatrix}, \tag{5.40e}$$

$$\hat{u}_j(p)\,\hat{\chi}_j^{\dagger}\hat{u}_j^{\dagger}(p) = \begin{pmatrix} -\hat{\eta}_{j,1} & -\hat{\xi}_{j,1} & -\hat{\eta}_{j,2} & -\hat{\xi}_{j,2} \\ -\hat{\eta}_{j,3} & -\hat{\xi}_{j,3} & -\hat{\eta}_{j,4} & -\hat{\xi}_{j,4} \end{pmatrix}. \tag{5.40f}$$

Equation (5.40) differs from Eq. (5.37) by interchanging g_1 and g_2 and Majorana operators $\hat{\eta}_{j,3}$ and $\hat{\xi}_{j,3}$. This difference does not affect the reasoning leading to to the conclusion that the gapless Hamiltonian (5.35) cannot be gapped by onsite terms without breaking G_f symmetry. This difference also implies that stacking two copies of the $\big(0,(0,1),0\big)$ representation (5.40) delivers the trivial projective representation $\big(0,(0,0),0\big)$ encoded by Eqs. (5.39), for which Theorem 2 is not predictive anymore.

5.2.7 *Group Cohomology Class* $\big(0,(1,1)\big)$

When representations (5.30) and (5.33) are stacked, the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=64$ is generated by 12 Majorana operators. One verifies that

$$\hat{u}_j(g_1) := \hat{\eta}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\eta}_{j,3}\,\hat{\xi}_{j,5}\,\hat{\xi}_{j,6}, \tag{5.41a}$$

$$\hat{u}_j(g_2) := \hat{\xi}_{j,1}\,\hat{\eta}_{j,2}\,\hat{\xi}_{j,3}\,\hat{\xi}_{j,4}\,\hat{\eta}_{j,5}, \tag{5.41b}$$

$$\hat{u}_j(p) := \prod_{\alpha=1}^{6} \mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\eta}_{j,\alpha}, \tag{5.41c}$$

realizes the projective representation (5.21) with $([\nu],[\rho])=\big(0,(1,1)\big)$ and implements the transformations

$$\hat{u}_j(g_1)\,\hat{\chi}^{\dagger}_j\hat{u}^{\dagger}_j(g_1) = \begin{pmatrix} +\hat{\eta}_{j,1} & -\hat{\xi}_{j,1} & +\hat{\eta}_{j,2} & -\hat{\xi}_{j,2} & +\hat{\eta}_{j,3} & -\hat{\xi}_{j,3} \\ -\hat{\eta}_{j,4} & -\hat{\xi}_{j,4} & -\hat{\eta}_{j,5} & +\hat{\xi}_{j,5} & -\hat{\eta}_{j,6} & +\hat{\xi}_{j,6} \end{pmatrix}, \tag{5.41d}$$

$$\hat{u}_j(g_2)\,\hat{\chi}^{\dagger}_j\hat{u}^{\dagger}_j(g_2) = \begin{pmatrix} -\hat{\eta}_{j,1} & +\hat{\xi}_{j,1} & +\hat{\eta}_{j,2} & -\hat{\xi}_{j,2} & -\hat{\eta}_{j,3} & +\hat{\xi}_{j,3} \\ -\hat{\eta}_{j,4} & +\hat{\xi}_{j,4} & +\hat{\eta}_{j,5} & -\hat{\xi}_{j,5} & -\hat{\eta}_{j,6} & -\hat{\xi}_{j,6} \end{pmatrix}, \tag{5.41e}$$

$$\hat{u}_j(p)\,\hat{\chi}^{\dagger}_j\hat{u}^{\dagger}_j(p) = \begin{pmatrix} -\hat{\eta}_{j,1} & -\hat{\xi}_{j,1} & -\hat{\eta}_{j,2} & -\hat{\xi}_{j,2} & -\hat{\eta}_{j,3} & -\hat{\xi}_{j,3} \\ -\hat{\eta}_{j,4} & -\hat{\xi}_{j,4} & -\hat{\eta}_{j,5} & -\hat{\xi}_{j,5} & -\hat{\eta}_{j,6} & -\hat{\xi}_{j,6} \end{pmatrix}. \tag{5.41f}$$

Given the transformation rules (5.41), one cannot construct six disjoint pairs of Majorana operators that transform in the same manner under both symmetries. One can at most obtain five pairs such as

$$\big\{\hat{\eta}_{j,1},\,\hat{\eta}_{j,3}\big\},\quad \big\{\hat{\xi}_{j,1},\,\hat{\xi}_{j,3}\big\},\quad \big\{\hat{\xi}_{j,2},\,\hat{\eta}_{j,4}\big\},\quad \big\{\hat{\xi}_{j,5},\,\hat{\xi}_{j,6}\big\},\quad \big\{\hat{\xi}_{j,4},\,\hat{\eta}_{j,5}\big\}, \tag{5.42}$$

which leaves the Majorana operators $\hat{\eta}_{j,2}$ and $\hat{\eta}_{j,6}$ unpaired. Therefore, the G_f- and translation-symmetric Hamiltonian

$$\widehat{H}_{\mathrm{pbc}} := \sum_{j=1}^{2N} \sum_{\alpha=1}^{6} \left\{ \lambda_\alpha \mathrm{i}\hat{\eta}_{j,\alpha}\, \hat{\eta}_{j+1,\alpha} + \lambda'_\alpha \mathrm{i}\hat{\xi}_{j,\alpha}\, \hat{\xi}_{j+1,\alpha} \right\}, \tag{5.43}$$

with λ_α, $\lambda'_\alpha \in \mathbb{R}$, that describe twelve decoupled Kitaev chains can be partially gapped by five onsite terms. Because of the projective representation (5.41), the low-energy sector described by two decoupled Kitaev chains cannot be gapped. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 2, which also predicts that any G_f-symmetric interaction that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $\big(0,(1,1)\big)$ representation are stacked, the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D} = 2^{12}$ is generated by 24 Majorana operators. The $\mathbb{Z}_2$-graded tensor product of the projective representation (5.41) with itself realizes the group cohomology class $([\nu],[\rho]) = \big(0,(0,0)\big)$, i.e., the trivial group cohomology class. There exists a bijective map $\alpha \mapsto \alpha' := (\alpha+6) \bmod 12$ such that all on-site terms $\mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\xi}_{j,\alpha'}$ and $\mathrm{i}\hat{\eta}_{j,\alpha}\,\hat{\eta}_{j,\alpha'}$ with $\alpha = 1, \cdots, 6$ can be shown to be G_f symmetric. The ground-state degeneracy of any translation- and G_f-invariant Hamiltonian can be lifted by including the 12 on-site terms $\mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\xi}_{j,\alpha+6}$ and $\mathrm{i}\hat{\eta}_{j,\alpha}\,\hat{\eta}_{j,\alpha+6}$ with $\alpha = 1, \cdots, 6$, i.e., Theorem 2 is not predictive.

5.3 SYMMETRY GROUP $\mathbb{Z}_4^{\mathrm{FT}}$

The symmetry group $G_f := \mathbb{Z}_4^{\mathrm{FT}} := \{t, t^2, t^3, t^4\}$ is the nontrivial central extension of $G \equiv \mathbb{Z}_2^{\mathrm{T}} = \{t, t^2\}$ by $\mathbb{Z}_2^{\mathrm{F}} \equiv \{p, p^2\}$, where the identification $t^2 = p$ is made. The upper index T for the cyclic group $G \equiv \mathbb{Z}_2^{\mathrm{T}} \equiv \{e, t\}$ refers to the interpretation of t as reversal of time (see Appendix A.5.3). As usual, p denotes fermion parity. The symmetry group G_f is thus generated by reversal of time t, whereby reversal of time squares to the fermion parity p.

The local antiunitary representation $\hat{u}_j(t)$ of reversal of time generates a projective representation of the group $\mathbb{Z}_4^{\mathrm{FT}}$. According to Appendix A.5.3, all cohomologically distinct projective representations of $\mathbb{Z}_4^{\mathrm{FT}}$ are determined by the indices $[(\nu,\rho)]$, with $(\nu,\rho)\in C^2(G,\mathrm{U}(1))\times C^1(G,\mathbb{Z}_2)$, through the relations [5]

$$\begin{aligned}([\nu],[\rho]) &= (\rho(t),\rho(t)),\\ \hat{u}_j(t)\,\hat{u}_j(p) &= (-1)^{\rho(t)}\,\hat{u}_j(p)\,\hat{u}_j(t),\end{aligned} \tag{5.44a}$$

where

$$\hat{u}_j^2(t) = e^{\mathrm{i}\phi(t,t)}\,\hat{u}_j(p) \tag{5.44b}$$

and ϕ is the 2-cocycle defined in Eq. (3.7). This gives two distinct group cohomology classes

$$([\nu],[\rho]) \in \{(0,0),(1,1)\}\,. \tag{5.44c}$$

We will start with the nontrivial projective representation in the cohomology class $([\nu],[\rho])=(1,1)$ that we shall represent using two local Majorana flavors. We will then construct a projective representation in the group cohomology classes $([\nu],[\rho])=(0,0)$ by using the graded tensor product, i.e., by considering 4 local Majorana flavors. The indices (5.44c) form group $\mathbb{Z}_2$ under the stacking rule (5.4).

5.3.1 *Group Cohomology Classes* $(1,1)$ *and* $(0,0)$

The local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D}=2$ is generated by the doublet of Majorana operators

$$\hat{\chi}_j^{\dagger} \equiv \begin{pmatrix}\hat{\eta}_j & \hat{\xi}_j\end{pmatrix}, \qquad j=1,\cdots,2N. \tag{5.45}$$

One verifies that

$$\hat{u}_j(t) := \frac{1}{\sqrt{2}}\left(\hat{\eta}_j-\hat{\xi}_j\right)\mathsf{K}, \tag{5.46a}$$

$$\hat{u}_j(p) := \mathrm{i}\hat{\xi}_j\,\hat{\eta}_j, \tag{5.46b}$$

realizes the projective representation (5.44) with $([\nu],[\rho])=(1,1)$. With the help of

$$\left[\hat{u}_j(t)\right]^{-1} = \frac{1}{\sqrt{2}}\left(\hat{\eta}_j+\hat{\xi}_j\right)\mathsf{K}, \tag{5.47}$$

5 We have chosen the convention of always representing the generator p of $\mathbb{Z}_2^{\mathrm{F}}$ by a Hermitian operator according to Eq. (3.6e)

one also verifies that

$$\hat{u}_j(t) \begin{pmatrix} \hat{\eta}_j & \hat{\xi}_j \end{pmatrix} \left[\hat{u}_j(t)\right]^{-1} = \begin{pmatrix} -\hat{\xi}_j \\ +\hat{\eta}_j \end{pmatrix}. \tag{5.48}$$

It follows from Eq. (5.48) that the only on-site Hermitian quadratic form $\mathrm{i}\hat{\eta}_j\,\hat{\xi}_j$ is odd under reversal of time. Consequently,

$$\widehat{H}_{\mathrm{pbc}} := \sum_{j=1}^{2M} \lambda\left(\mathrm{i}\,\hat{\eta}_j\,\hat{\eta}_{j+1} - \mathrm{i}\,\hat{\xi}_j\,\hat{\xi}_{j+1}\right) \tag{5.49}$$

with $\lambda \in \mathbb{R}$ is an example translation- and G_f-invariant Hamiltonian of quadratic order. This Hamiltonian describes two Kitaev chains that have been fine-tuned to their quantum critical point (3.49) between their symmetry-protected and topologically inequivalent gapped phases. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 2. Theorem 2 also predicts that any G_f-symmetric interaction that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of the projective representation (5.46) are stacked according to Eq. (5.4), the local fermionic Fock space $\mathfrak{F}_j$ of dimension $\mathcal{D} = 4$ is generated by 4 Majorana operators. The $\mathbb{Z}_2$-graded tensor product of the projective representation (5.46) with itself realizes the group cohomology class $([\nu],[\rho]) = \big(0,0\big)$, i.e., the trivial group cohomology class. There exists a bijective map $\alpha \mapsto \alpha' := (\alpha+1) \bmod 2$ such that all on-site terms $\mathrm{i}\hat{\eta}_{j,\alpha}\,\hat{\eta}_{j,\alpha'} - \mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\xi}_{j,\alpha'}$ with $\alpha = 1,2$ can be shown to be G_f symmetric. The ground-state degeneracy of any translation- and G_f-invariant Hamiltonian can be lifted by increasing the strength of $\mathrm{i}\hat{\eta}_{j,\alpha}\,\hat{\eta}_{j,\alpha'} - \mathrm{i}\hat{\xi}_{j,\alpha}\,\hat{\xi}_{j,\alpha'}$ with $\alpha = 1,2$, i.e., Theorem 2 is not predictive.

Part II

ONE-DIMENSIONAL INVERTIBLE FERMIONIC TOPOLOGICAL PHASES

Adapted from:
Ö. M. Aksoy and C. Mudry
"Elementary derivation of the stacking rules of invertible fermionic topological phases in one dimension",
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Part II focuses on the study of one-dimensional invertible fermionic topological (IFT) phases. We use the LSM Theorems 3 and 2 to enumerate all one-dimensional IFT phases in Chapter 6. This is achieved by assuming that for each IFT phase one can find a translationally invariant representative Hamiltonian.

The enumeration of IFT phases do not immediately deliver the group structure under stacking operation. Chapter 7 presents a derivation of fermionic stacking rules for any symmetry group G_f by only using elementary means of quantum mechanics and linear algebra. We distinguish the fermionic stacking rules from bosonic ones and show that the latter is a special case of the former. This is one of the main results of this Part.

In Chapter 8, we study the protected ground-state degeneracies associated with non-trivial IFT phases on general grounds. Therein, we also demonstrate that *intrinsically* fermionic invertible phases support robust quantum mechanical supersymmetry (SUSY) at their boundaries.

Chapter 9 deals with construction of bulk representations of symmetries out of zero dimensional projective representations. We also show that for each IFT phase, we can construct a translationally invariant commuting projector Hamiltonian, which closes the loop in our classification scheme.

We close Part II with Chapter 10, where we present an application of the rather abstract framework introduced in the earlier chapters. We consider two closely related families of Hamiltonians, namely, the time-reversal symmetric Majorana chains and spin-1/2 cluster models. The former are representatives of IFT phases with $G_f = \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ symmetry while the latter are representatives of bosonic symmetry protected topological (BSPT) phases with $G = \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2$. For each representative Hamiltonian, we compute the corresponding indices and protected ground-state degeneracies. We also verify the stacking rules derived in Chapter 7 and clarify how the classification of topological phases described by these Hamiltonians are related.

6

FROM LSM THEOREMS TO 1D IFT PHASES

In this Chapter, we explore how the contrapositions of Theorems 2 and 3 lead to the enumeration of IFT phases in one-dimensional space. In Sec. 6.1, we present a strategy that allows the classification of 1D IFT phases. Sec. 6.2 connects the strategy presented in Sec. 6.1 to the projective representations of a fermionic symmetry group G_f that are realized at the boundaries of 1D IFT phases when open boundary conditions are imposed. Therein, an exhaustive classification of 1D IFT phases for any symmetry group G_f is achieved. The derivations of fermionic stacking rules, which impose an additional group structure on the set of IFT phases, are left to Chapter 7.

6.1 STRATEGY FOR CLASSIFYING 1D IFT PHASES

We will present a classification scheme for *invertible fermionic topological* (IFT) phases in one-dimensional space. IFT phases are invertible phases realized by Hamiltonians built out of fermionic degrees of freedom with a fermionic symmetry group G_f. The classification of the IFT phases in one-dimensional space is intimately related to the classification of the projective representations of the fermionic symmetry group G_f, an internal symmetry acting globally on the fermionic Fock space. To illustrate this, we will first consider the representations of G_f on a closed one-dimensional chain and then investigate the consequences of imposing open boundary conditions.

We denote by Λ the set of points on a one-dimensional lattice. Given are the fermionic symmetry group G_f (Appendices A.1 and A.2) and a global fermionic Fock space $\mathfrak{F}_\Lambda$ defined over Λ. We assume that there exists a faithful trivial representation $\widehat{U}_{\mathrm{bulk}}$ of the group G_f on the lattice Λ. In other words, there exists an injective map $\widehat{U}_{\mathrm{bulk}} : G_f \to \mathrm{Aut}\left(\mathfrak{F}_\Lambda\right)$ where $\mathrm{Aut}\left(\mathfrak{F}_\Lambda\right)$ is the set of automorphisms on the fermionic Fock space such that for any $g, h \in G_f$,

$$\widehat{U}_{\mathrm{bulk}}(g)\,\widehat{U}_{\mathrm{bulk}}(h) = \widehat{U}_{\mathrm{bulk}}(g\,h), \tag{6.1a}$$

where $g\,h$ denotes the composition of the elements $g, h \in G_f$. For any $g \in G_f$, its representation $\widehat{U}_{\text{bulk}}(g)$ can be written as [1]

$$\widehat{U}_{\text{bulk}}(g) = \widehat{V}_{\text{bulk}}(g)\,\mathsf{K}^{\mathfrak{c}(g)}, \tag{6.1b}$$

where $\widehat{V}_{\text{bulk}}(g)$ is a unitary operator acting on $\mathfrak{F}_\Lambda$ and K is the complex conjugation map. For each point $j \in \Lambda$, we associate a set of Hermitian Majorana operators

$$\mathfrak{O}_j := \left\{\hat{\gamma}^{(j)}_1, \hat{\gamma}^{(j)}_2, \cdots, \hat{\gamma}^{(j)}_{n_j}\right\}, \tag{6.2a}$$

that realizes the Clifford algebra

$$\mathrm{C}\ell_{n_j} := \mathrm{span}\left\{ \prod_{\iota=1}^{n_j} \left(\hat{\gamma}^{(j)}_\iota\right)^{m_\iota} \;\middle|\; \left\{\hat{\gamma}^{(j)}_\iota, \hat{\gamma}^{(j')}_{\iota'}\right\} = 2\delta_{\iota\iota'},\ m_i = 0,1, \quad \iota,\iota' = 1,\cdots,n_j \right\}. \tag{6.2b}$$

The n_j Majorana operators (6.2a) span a local fermionic Fock space $\mathfrak{F}_j$ if n_j is an even integer. If n_j is odd, the Clifford algebra $\mathrm{C}\ell_{n_j}$ contains a two-dimensional center, reason for which the n_j Majorana operators (6.2a) span a Hilbert space that cannot be interpreted as a fermionic Fock space [2]. The consistent definition of a global fermionic Fock space thus requires the total number of Majorana degrees of freedom to be even, i.e.,

$$\sum_j n_j = 0 \bmod 2. \tag{6.3}$$

1 Recall that $\mathfrak{c}$ is a group homomorphism that specifies if an element $g \in G_f$ is to be represented by a unitary operator, in which case $[\mathfrak{c}(g) = 0]$, or by an antiunitary operator, in which case $[\mathfrak{c}(g) = 1]$.

2 We define the fermionic Fock space of dimension 2^n as follows. We choose n pairs of Majorana operators out of which one can define a conjugate pair of fermionic creation and annihilation operators. The vacuum state that is annihilated by all annihilation operators is the highest weight state from which $2^n - 1$ orthonormal states with the fermion number $n_{\mathrm{f}} = 1, 2, \cdots, n-1, n$ descend by acting on the vacuum state with the product of n_{f} distinct fermionic creation operators. By construction, $\mathrm{C}\ell_{2n}$ acts on a 2^n-dimensional fermionic Fpck space for which each basis element t has a well-defined fermion parity. Because the center of the Clifford algebra $\mathrm{C}\ell_{2n}$ is trivial, multiplication with an element of the center of $\mathrm{C}\ell_{2n}$ thus leaves the fermion parity of each element of the fermionic Fock basis unchanged. This is not so any more for a representation of the Clifford algebra $\mathrm{C}\ell_{2n+1}$ spanned by $2n+1$ Majorana operators. Even though it is still possible to define a Hilbert space of dimension 2^{n+1} on which the Clifford algebra $\mathrm{C}\ell_{2n+1}$ has a nontrivial irreducible representation [166], the center of $\mathrm{C}\ell_{2n+1}$ is a two-dimensional subalgebra. It follows that there is no element in $\mathrm{C}\ell_{2n+1}$, which anticommutes with all the Majorana generators of $\mathrm{C}\ell_{2n+1}$, i.e., it is not possible to distinguish an element in $\mathrm{C}\ell_{2n+1}$, which assigns odd fermion parity to all $2n+1$ Majorana generators. The best one can do is to construct a 2^n-dimensional fermionic Fock space using the generators of a $\mathrm{C}\ell_{2n}$ subalgebra of $\mathrm{C}\ell_{2n+1}$ and a two-dimensional Hilbert space in which states do not have any assigned fermion parity or fermion number.

We define a *local* [3] representation $\hat{u}_j$ of the symmetry group G_f by demanding that on the degrees of freedom localized at site $j \in \Lambda$, $\hat{u}_j$ acts in the same way as the global bulk representation $\widehat{U}_{\mathrm{bulk}}$ does, i.e., the consistency condition

$$\hat{u}_j(g)\,\hat{\gamma}_\iota^{(j)}\,\hat{u}_j^\dagger(g) = \widehat{U}_{\mathrm{bulk}}(g)\,\hat{\gamma}_\iota^{(j)}\,\widehat{U}_{\mathrm{bulk}}^\dagger(g), \tag{6.4}$$

for any $g \in G_f$ and $\iota = 1, \cdots, n_j$ must hold. Hereby, we assume that the bulk representation $\widehat{U}_{\mathrm{bulk}}$ is onsite in the sense that there are no obstructions that prevent decomposing $\widehat{U}_{\mathrm{bulk}}$ into the product of local representations $\hat{u}_j$ (see Refs. [57, 129] for examples when this is not possible). The definition (6.4) implies that the representation $\hat{u}_j$ satisfies for any $g, h \in G_f$

$$\hat{u}_j(g)\,\hat{u}_j(h) = e^{\mathrm{i}\phi_j(g,h)}\,\hat{u}_j(g\,h). \tag{6.5}$$

This is a projective representation as defined in Eq. (3.7c). The phase factor $\phi_j(g,h) \in C^2(G_f, \mathrm{U}(1))$ defines a 2-cochain. Its equivalence classes $[\phi_j]$ takes values in the second cohomology group $H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$, see Appendix A.1.

By definition, local Hamiltonians with the symmetry group G_f that realize IFT phases of matter must necessarily have nondegenerate and gapped ground states that transform as singlets under the symmetry group G_f with any closed boundary conditions. We restrict our attention to IFT phases of matter with translation symmetry G_{trsl} in addition to the internal fermionic symmetry group G_f. In other words, the total symmetry group G_{tot} is by hypothesis the direct product

$$G_{\mathrm{tot}} \equiv G_{\mathrm{trsl}} \times G_f. \tag{6.6}$$

This restriction is justified if we assume that for each invertible phase of matter in one dimension, there exists a translationally invariant representative Hamiltonian [4].

---

3 When n_j is an odd integer, it is not always possible to construct a local representation $\hat{u}_j(g)$ for any $g \in G_f$ only out of the Majorana degrees of freedom in the set $\mathfrak{O}_j$ defined in (6.2a). However, we still call $\hat{u}_j(g)$ a local representation in the sense that it can always be constructed by supplementing the Clifford algebra (6.2b) by an additional Majorana degree of freedom $\hat{\gamma}_{j,\infty}$ that is localized at some other site j' with the number of Majorana operators $n_{j'}$ being an odd integer.

4 Another justification that will be found *a posteriori* is that the classification scheme introduced here accounts for all possible ground state degeneracies of one-dimensional nontrivial IFT phases with open boundary conditions. This ground state degeneracy as we will show is a property of zero-dimensional boundary and independent of whether the one-dimensional bulk is translationally invariant or not. In turn, for each IFT phase, we will show how to construct a translationally invariant representative Hamiltonian.

Imposing translation symmetry G_{trsl} requires the number n_j of Majorana degrees of freedom at each site to be independent of j with the same local representation $\hat{u}_j(g)$ for any element $g \in G_f$. If so Theorems 2 and 3 apply. A nondegenerate and gapped ground state that transforms as a singlet under the symmetry group G_{tot} is permissible if and only if:

1. The number n_j of Majorana degrees of freedom at each site $j \in \Lambda$ is even, i.e., $n_j \equiv 2n$
2. The local representation $\hat{u}_j(g)$ realizes a trivial projective representation, i.e., $[\phi_j] \equiv [\phi] = 0$.

The first condition requires that there exist a *local* fermionic Fock space $\mathfrak{F}_j$ spanned by the even number of local Majorana degrees of freedom (6.2a). Therefore, the global fermionic Fock space $\mathfrak{F}_\Lambda$ decomposes as a $\mathbb{Z}_2$-graded tensor product $\otimes_{\mathfrak{g}}$ of the local Fock spaces $\mathfrak{F}_j$, i.e.,

$$\mathfrak{F}_\Lambda = \bigotimes_{j\in\Lambda}{}_{\mathfrak{g}}\, \mathfrak{F}_j. \tag{6.7}$$

The second condition requires that the local representation $\hat{u}_j \in \mathrm{Aut}\left(\mathfrak{F}_j\right)$ is a representation in the trivial equivalence class $[\phi] = 0$. This implies that the global bulk representation $\widehat{U}_{\mathrm{bulk}}$ decomposes as the product of local representations $\hat{u}_j$, i.e., for any $g \in G_f$

$$\widehat{U}_{\mathrm{bulk}}(g) = \left[\prod_{j\in\Lambda} \hat{v}_j(g)\right] \mathsf{K}^{\mathfrak{c}(g)}. \tag{6.8}$$

Open boundary conditions break the hypothesis of translation symmetry in Theorems 2 and 3. When a closed chain is opened up, the degrees of freedom localized in one or multiple repeat unit cells may be split into two disconnected components Λ_{L} and Λ_{R} of the boundary $\Lambda_{\mathrm{bd}} := \partial\Lambda \equiv \Lambda_{\mathrm{L}} \cup \Lambda_{\mathrm{R}}$, as is illustrated in Fig. 6.1. If so, the two requirements of the LSM constraints need no longer hold at each disconnected component.

Any one-dimensional IFT phase of matter is thus characterized by the following data:

1. There is a $\mathbb{Z}_2$-valued index $[\mu_{\mathrm{B}}] = \{0, 1\}$ (B = L, R) that measures the parity of the number of Majorana degrees of freedom that are localized on either one of the left

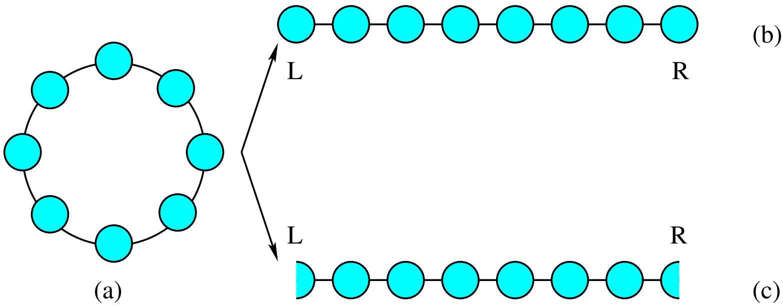

Figure 6.1: The repeat unit cells of a one-dimensional lattices are pictured by colored discs. Each repeat unit cell hosts an even number $2n$ of Majorana degrees of freedom. Without loss of generality, the range of the couplings between Majorana degrees of freedom is one lattice spacing (the thick line between the repeat unit cells). Translation symmetry is imposed by choosing periodic boundary conditions, in which case the one-dimensional lattice is the discretization of a ring. Open boundary conditions break the translation symmetry. This can be achieved by cutting a thick line connecting two repeat unit cell or by cutting open a repeat unit cell. In the former case, the number of Majorana degrees of freedom on any one of the upmost left or right cells is the same even number $2n$ of Majorana degrees of freedom as that in a single repeat unit cell. In the latter case, the number of Majorana degrees of freedom on the upmost left cell is any integer $1 < n_{\mathrm{L}} < 2n$ while that on the upmost cell is $n_{\mathrm{R}} = 2n - n_{\mathrm{L}}$.

(L) or right (R) boundaries (B) of the open chain $\Lambda_{\mathrm{bd}} = \Lambda_{\mathrm{L}} \cup \Lambda_{\mathrm{R}}$. The index $[\mu_{\mathrm{B}}]$ can be viewed as an element of the zero-th cohomology group $H^0\left(G_f, \mathbb{Z}_2\right) = \mathbb{Z}_2$.

2. There is an equivalence class $[\phi_{\mathrm{B}}] \in H^2\left(G_f, \mathrm{U}(1)_{\mathfrak{c}}\right)$ of the second cohomology group (Appendix A.1) that characterizes the projective representation of the internal symmetry group G_f at either one of the left or right boudaries of an open chain $\Lambda_{\mathrm{bd}} = \Lambda_{\mathrm{L}} \cup \Lambda_{\mathrm{R}}$.

Given a disconnected component Λ_{B} of the boundary Λ_{bd}, we assume the existence of a set of boundary Majorana degrees of freedom

$$\mathfrak{O}_{\mathrm{B}} := \left\{\hat{\gamma}^{(\mathrm{B})}_1, \hat{\gamma}^{(\mathrm{B})}_2, \cdots, \hat{\gamma}^{(\mathrm{B})}_{n_{\mathrm{B}}}\right\} \tag{6.9a}$$

that are associated with states exponentially localized in space at the boundary B. The pair of data $([\phi_{\mathrm{B}}], [\mu_{\mathrm{B}}]) \in H^2\left(G_f, \mathrm{U}(1)_{\mathfrak{c}}\right) \times H^0\left(G_f, \mathbb{Z}_2\right)$ are assigned as follows. The index $[\mu_{\mathrm{B}}]$ is nothing but the parity of the number of Majorana degrees of freedom at

Λ_{B}, i.e., $[\mu_{\mathrm{B}}]=n_{\mathrm{B}}$ mod 2. The equivalence class $[\phi_{\mathrm{B}}]$ of the projective phase $\phi_{\mathrm{B}}(g,h)$ is computed by constructing a boundary representation $\widehat{U}_{\mathrm{B}}$ [5]. This is done by demanding the consistency condition

$$\widehat{U}_{\mathrm{B}}(g)\,\hat{\gamma}^{(\mathrm{B})}_{\iota}\,\widehat{U}^{\dagger}_{\mathrm{B}}(g)=\widehat{U}_{\mathrm{bulk}}(g)\,\hat{\gamma}^{(\mathrm{B})}_{\iota}\,\widehat{U}^{\dagger}_{\mathrm{bulk}}(g), \tag{6.9b}$$

for any $g\in G_f$ and $\iota=1,2,\cdots,n_{\mathrm{B}}$.

The index $[\phi_{\mathrm{B}}]\in H^2\big(G_f,\mathrm{U}(1)_{\mathfrak{c}}\big)$ depends both on $[\mu_{\mathrm{B}}]=0,1$ and the fermionic symmetry group G_f. This is so because G_f is the central extension of the internal symmetry group G by the fermion-parity symmetry group $\mathbb{Z}_2^{\mathrm{F}}$ with extension class $[\gamma]\in H^2(G,\mathbb{Z}_2^{\mathrm{F}})$ (Appendix A.2), i.e., G is isomorphic to the group $G_f/\mathbb{Z}_2^{\mathrm{F}}$ [6] As the center of the fermionic symmetry group G_f is the fermion-parity subgroup $\mathbb{Z}_2^{\mathrm{F}}$, its projective representations are sensitive to the values of $[\mu_{\mathrm{B}}]$. This sensitivity can be made explicit if one trades the equivalence classes $[\phi_{\mathrm{B}}]\in H^2\big(G_f,\mathrm{U}(1)_{\mathfrak{c}}\big)$ for the equivalence classes $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]\in\ker\mathcal{D}^2_{\gamma,\mathfrak{c}}\big/\mathrm{im}\,\mathcal{D}^1_{\gamma,\mathfrak{c}}$ where $\mathcal{D}^2_{\gamma,\mathfrak{c}}$ and $\mathcal{D}^1_{\gamma,\mathfrak{c}}$ are modified coboundary operators (Appendix A.4). The construction of the triplet $\big([(\nu_{\mathrm{B}},\rho_{\mathrm{B}})],[\mu_{\mathrm{B}}]\big)$ and their physical meaning will be reviewed in more depth in the next section (Sec. 6.2).

There are two possible scenarios for the fate of the set (6.9a) of boundary degrees of freedom on the boundary Λ_{B} that realize the triplet of boundary data $([(\nu_{\mathrm{B}},\rho_{\mathrm{B}})],[\mu_{\mathrm{B}}])$ when the bulk is perturbed by local and continuous interactions that break neither explicitly nor spontaneously the G_f symmetry. In scenario I, the set (6.9a) is unchanged by the bulk perturbation. If so, the triplet of boundary data $([(\nu_{\mathrm{B}},\rho_{\mathrm{B}})],[\mu_{\mathrm{B}}])$ does not change. In scenario II, the bulk perturbation changes the set (6.9a) by either the addition or removal of boundary degrees of freedom. If the degrees of freedom added to or removed from the boundary Λ_{B} realize the trivial triplet of data, then the resulting triplet of boundary data is unchanged according to the fermionic stacking rules. If the degrees of freedom added to or removed from the boundary Λ_{B} realize a nontrivial triplet of data, then the triplet of boundary data is changed to $([(\nu'_{\mathrm{B}},\rho'_{\mathrm{B}})],[\mu'_{\mathrm{B}}])\neq([(\nu_{\mathrm{B}},\rho_{\mathrm{B}})],[\mu_{\mathrm{B}}])$ according to the fermionic stacking rules. If the bulk-boundary correspondence were to hold, then a gap-closing transition in the bulk that is induced by the bulk perturbations is required to change the boundary triplet of data. This hypothesis is plausible because Bourne and Ogata have shown rigorously in Ref. [143] the existence of triplets of bulk data that take values in the same cohomology groups as the triplets of boundary data

5 We use the capital U here, since $\widehat{U}_{\mathrm{B}}$ is to be treated as a global representation on the $(0+1)$-dimensional boundary.

6 The coset $G_f/\mathbb{Z}_2^{\mathrm{F}}$ is a group since $\mathbb{Z}_2^{\mathrm{F}}$ is in the center of G_f and therefore a normal subgroup.

$([(\nu_{\mathrm{B}},\rho_{\mathrm{B}})],[\mu_{\mathrm{B}}])$, obey the same stacking rules, and offer a bulk classification of IFT phases of matter. We assume without proof this bulk-boundary correspondence.

There is no need to specify the triplets associated with the disconnected components Λ_{L} and Λ_{R} independently. The triplet of data on the left boundary Λ_{L} fixes their counterparts on the right boundary Λ_{R}, owing to the condition that the ground state of a Hamiltonian realizing an IFT phase of matter must be nondegenerate and G_f-symmetric when periodic boundary conditions are selected. Thus, we drop the subscripts when denoting the triplet of data $([(\nu,\rho)],[\mu])$ that characterize the IFT phases.

For any G_f that splits, i.e., G_f is isomorphic to the product $G\times\mathbb{Z}_2^{\mathrm{F}}$, the index $[\mu]$ can take the values 0 or 1. If the group G_f is a nonsplit group, then $[\mu]=0$ is the only possibility. When $[\mu]=1$, the minimal degeneracy of the eigenspace for the ground states when open boundary conditions are selected is two for any split fermionic symmetry group G_f, including the smallest possible fermionic symmetry group $G_f=\mathbb{Z}_2^{\mathrm{F}}$. Hence, one-dimensional Hamiltonians realizing IFT phases of matter with $[\mu]=1$ cannot be deformed adiabatically to a Hamiltonian realizing the trivial IFT phase of matter at the expense of breaking explicitly any of the protecting symmetries in G_f other than $\mathbb{Z}_2^{\mathrm{F}}$. *A forteriori*, these phases of matter are distinct from the fermionic SPT (FSPT) phases of matter in one-dimensional space. In one-dimensional space, FSPT phases of matter are only possible when $[\mu]=0$.

Once the IFT phases in one-dimension are characterized by the triplet $([(\nu,\rho)],[\mu])$, it is imperative to derive the stacking rules, i.e., the group composition rules of the triplets $([(\nu,\rho)],[\mu])$ that are compatible with the $\mathbb{Z}_2$-graded tensor product between fermionic Fock spaces (in physics terminology, antisymmetrization). Stacking rules can be derived by considering the topological indices $([(\nu_\wedge,\rho_\wedge)],[\mu_\wedge])$ of an IFT phase of matter that is constructed by combining the boundary degrees of freedoms of any representatives of two other IFT phases with topological indices $\big([(\nu_1,\rho_1)],[\mu_1]\big)$ and $\big([(\nu_2,\rho_2)],[\mu_2]\big)$, respectively. The stacking rules are essential properties of IFT phases of matter. They enforce a group composition law between IFT phases of matter sharing the same fermionic symmetry group G_f. This group composition law can be interpreted as the physical operation by which two blocks of matter, each realizing IFT phases of matter sharing the same fermionic symmetry group G_f, are brought into contact so as to form a single larger block of matter sharing the same fermionic symmetry group G_f. This group composition law is also needed to implement a consistency condition corresponding to changing from open to closed boundary conditions. Topological data associated with the left and the right disconnected components of the one-dimensional boundary must be the inverse of each other with respect to the stacking rules, i.e., one should obtain the trivial data $([(0,0)],0)$ if the change from open to periodic boundary conditions is interpreted as the stacking of opposite boundaries. One of the main results of Part II is the derivation of fermionic stacking rules from the boundary perspective, which is achieved in Chapter 7.

6.2 BOUNDARY PROJECTIVE REPRESENTATIONS

The internal symmetry group G_f is specified by two pieces of data. The first piece is the central extension class $[\gamma] \in H^2(G, \mathbb{Z}_2^{\mathrm{F}})$ that characterize how the group G and the fermion parity symmetry group $\mathbb{Z}_2^{\mathrm{F}}$ are glued together to produce the group G_f. This is to say that, G_f is not restricted to be the direct product $G_f = G \times \mathbb{Z}_2^{\mathrm{F}}$. The group G_f is such that (i) $\mathbb{Z}_2^{\mathrm{F}}$ is a subgroup of the center of G_f (ii) and G is isomorphic to $G_f/\mathbb{Z}_2^{\mathrm{F}}$. We assign the equivalence class $[\gamma]=0$ to the case of G_f being isomorphic to the direct product $G \times \mathbb{Z}_2^{\mathrm{F}}$ and say that G_f splits and A.2). The second piece is the group homomorphism $\mathfrak{c} : G_f \to \{0,1\}$ that specifies if an element $g \in G_f$ is to be represented by a unitary $[\mathfrak{c}(g)=0]$ operator or by an antiunitary $[\mathfrak{c}(g)=1]$ operator [by definition, $\mathfrak{c}(p)=0$].

In this Section, we summarize the properties of boundary representations $\widehat{U}_{\mathrm{B}}$ of a fermionic symmetry groups G_f. The rather lengthy details are left to Appendix A. Therein, Appendix A.1 introduces group cohomology, while Appendices A.2 and A.4 deals with the construction of symmetry group G_f and the exhaustive characterization of its projective representations, respectively. Various examples are provided in Appendices A.3 and A.5.

6.2.1 *Boundary Fock Spaces*

We denote by Λ the set of points on a one-dimensional lattice that we shall call the bulk. We assume that there exists a nonvanishing boundary

$$\Lambda_{\mathrm{bd}} \equiv \partial\Lambda \tag{6.10a}$$

of the bulk Λ. The boundary Λ_{bd} is the union of two disconnected components Λ_{L} or Λ_{R} of the one-dimensional universe Λ,

$$\Lambda_{\mathrm{bd}} = \Lambda_{\mathrm{L}} \cup \Lambda_{\mathrm{R}}, \qquad \Lambda_{\mathrm{L}} \cap \Lambda_{\mathrm{R}} = \emptyset. \tag{6.10b}$$

The hypothesis that states bound to Λ_{L} or Λ_{R} do not overlap in space only holds for all fermionic invertible topological phases after the thermodynamic limit has been taken. Without loss of generality, we consider any one of Λ_{L} and Λ_{R}, which we denote Λ_{B}. We are going to construct a projective representation of the symmetry group G_f on this component Λ_{B} of the boundary Λ_{bd}, while the opposite component of the boundary must then always be represented by the "inverse" projective representation.

On the boundary Λ_{B}, we assume the existence of a set of n Hermitian Majorana operators

$$\mathfrak{O}_n := \{\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_n\} \tag{6.11a}$$

that realizes the Clifford algebra

$$\mathrm{C}\ell_n := \mathrm{span}\left\{ \prod_{i=1}^{n} (\hat{\gamma}_i)^{m_i} \;\middle|\; \left\{\hat{\gamma}_i, \hat{\gamma}_j\right\} = 2\delta_{ij}, \quad m_i = 0, 1, \quad i, j = 1, \cdots, n \right\}. \tag{6.11b}$$

We assign the index $[\mu] \in \{0, 1\}$ to the parity of n, i.e.,

$$[\mu] = n \bmod 2. \tag{6.11c}$$

We consider the cases of even and odd n separately.

When $[\mu] = 0$, the even number n of Majorana operators from the set (6.11a) span the fermionic Fock space

$$\mathfrak{F}_{\Lambda_{\mathrm{B}},0} := \mathrm{span}\left\{ \prod_{\alpha=1}^{\frac{n}{2}} \left(\frac{\hat{\gamma}_{2\alpha-1} - \mathrm{i}\hat{\gamma}_{2\alpha}}{2} \right)^{n_\alpha} |0\rangle \;\middle|\; \left(\frac{\hat{\gamma}_{2\alpha-1} + \mathrm{i}\hat{\gamma}_{2\alpha}}{2} \right) |0\rangle = 0,\, n_\alpha = 0, 1 \right\} \tag{6.12a}$$

of dimension [7]

$$\dim \mathfrak{F}_{\Lambda_{\mathrm{B}},0} = 2^{n/2}. \tag{6.12b}$$

When $[\mu] = 1$, the odd number n of Majorana operators from the set (6.11a) span a vector space that is not a fermionic Fock space. In order to recover a fermionic Fock

7 The partition of a set of n labels into two pairs of $n/2$ labels is here arbitrary.

space, we add to the set (6.11a) made of an odd number n of Majorana operators the Majorana operator $\hat{\gamma}_{\infty}$ [19],

$$\mathfrak{O}_{n,\infty} := \left\{\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_{2\lfloor n/2\rfloor}, \hat{\gamma}_n, \hat{\gamma}_{\infty}\right\}, \tag{6.13}$$

thereby defining the Clifford algebra $\mathrm{C}\ell_{n+1}$. Here, the lower floor function $\lfloor\cdot\rfloor$ returns the largest integer $\lfloor x\rfloor$ smaller than the positive real number x. We may then define the fermionic Fock space

$$\mathfrak{F}_{\Lambda_{\mathrm{B}},1} := \mathrm{span}\left\{ \prod_{\alpha=1}^{\frac{n+1}{2}} \left(\frac{\hat{\gamma}_{2\alpha-1} - \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right)^{n_\alpha} |0\rangle \;\middle|\; \left(\frac{\hat{\gamma}_{2\alpha-1} + \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right) |0\rangle = 0,\, n_\alpha = 0, 1 \right\} \tag{6.14a}$$

of dimension

$$\dim \mathfrak{F}_{\Lambda_{\mathrm{B}},1} = 2^{(n+1)/2}, \tag{6.14b}$$

where it is understood that $\hat{\gamma}_{n+1} \equiv \hat{\gamma}_{\infty}$. In this fermionic Fock space, all creation and annihilation fermion operators are local, except for one pair. The pair of creation and annihilation operator built out of the pair $\hat{\gamma}_n$ and $\hat{\gamma}_{\infty}$ of Majorana operators is nonlocal as $\hat{\gamma}_{\infty}$ originates from the opposite component of the boundary of one-dimensional space owing to the open boundary conditions, a distance infinitely far away after the thermodynamic limit has been taken. The same is true of the two-dimensional fermionic Fock space

$$\mathfrak{F}_{\mathrm{LR}} := \mathrm{span}\left\{ \left(\frac{\hat{\gamma}_n - \mathrm{i}\hat{\gamma}_{\infty}}{2}\right)^{m_\alpha} |0\rangle \;\middle|\; \left(\frac{\hat{\gamma}_n + \mathrm{i}\hat{\gamma}_{\infty}}{2}\right) |0\rangle = 0 \right\} \tag{6.15}$$

spanned by the pair $\hat{\gamma}_n$ and $\hat{\gamma}_{\infty}$.

Finally, it is assumed that the component Λ_{B} of the boundary Λ_{bd} defined in Eq. (6.10b) is symmetric under the action of G_f in the sense that

$$\widehat{U}_{\mathrm{bulk}}(g)\, \mathrm{C}\ell_n\, \widehat{U}^{\dagger}_{\mathrm{bulk}}(g) \subset \mathrm{C}\ell_n, \qquad \forall g \in G_f. \tag{6.16}$$

We assume that, for any $g \in G_f$, there exists a norm-preserving operator $\widehat{U}_{\mathrm{B}}(g)$ acting on the Fock space $\mathfrak{F}_{\Lambda_{\mathrm{B}},[\mu]}$ as domain of definition such that

$$\widehat{U}_{\mathrm{B}}(g)\,\hat{\gamma}_i\,\widehat{U}^{\dagger}_{\mathrm{B}}(g) = \widehat{U}_{\mathrm{bulk}}(g)\,\hat{\gamma}_i\,\widehat{U}^{\dagger}_{\mathrm{bulk}}(g), \tag{6.17}$$

for $i = 1, 2, \cdots, n$. The boundary representation $\widehat{U}_{\mathrm{B}}(g)$ of any element $g \neq e$, p is not unique since Eq. (6.17) is left invariant by the multiplication from the right of $\widehat{U}_{\mathrm{B}}(g)$ with any norm-preserving element from the center of the Clifford algebra $\mathrm{C}\ell_n$. When n is even this center is trivial and one-dimensional. When n is odd ($[\mu] = 1$) this center is nontrivial and two-dimensional. In contrast, irrespective of $[\mu]$ the representations $\widehat{U}_{\mathrm{B}}(e)$ and $\widehat{U}_{\mathrm{B}}(p)$ of the identity and fermion parity acting on the fermionic Fock space $\mathfrak{F}_{\Lambda_{\mathrm{B}},[\mu]}$ are uniquely determined up to a multiplicative phase factor.

Finally, we observe two consequences of Eq. (6.17). First, the boundary representation $\widehat{U}_{\mathrm{B}}$ inherits the injectivity of the bulk representation $\widehat{U}_{\mathrm{bulk}}$ of the fermionic symmetry group G_f. Second, for any element $g \in G_f$, the boundary representation $\widehat{U}_{\mathrm{B}}(g)$ has a definite fermion parity. However, unlike the representation $\widehat{U}_{\mathrm{bulk}}$, the representation $\widehat{U}_{\mathrm{B}}$ can be projective.

6.2.2 *Explicit Boundary Representations*

We treat the cases of $[\mu] = 0, 1$ separately. For each case, we list the explicit boundary representation of the fermion parity $\widehat{U}_{\mathrm{B}}(p)$, the general form of $\widehat{U}_{\mathrm{B}}(g)$ for any $g \in G_f$, and, the pair of indices (ν, ρ) that characterize the boundary projective representation.

6.2.2.1 *The Case of* $[\mu] = 0$

When the number n of Majorana operators on the boundary Λ_{B} is even, $[\mu] = 0$. The boundary representation of element $p \in G_f$ that generates the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ is chosen to be

$$\widehat{U}_{\mathrm{B}}(p) := \prod_{\alpha=1}^{n/2} \widehat{P}_{\alpha}, \qquad \widehat{P}_{\alpha} := \mathrm{i}\hat{\gamma}_{2\alpha-1}\,\hat{\gamma}_{2\alpha}. \tag{6.18a}$$

The parity operators $\widehat{P}_1, \cdots, \widehat{P}_{n/2}$ are Hermitian, square to the identity, and are pairwise commuting. Hence, $\widehat{U}_{\mathrm{B}}(p)$ is Hermitian and squares to the identity. We choose a basis in which all parity operators are simultaneously diagonalized and even under complex conjugation K,

$$\mathsf{K}\,\widehat{P}_{\alpha}\,\mathsf{K}=\widehat{P}_{\alpha}, \tag{6.18b}$$

for $\alpha=1,\cdots,n/2$.

The most general form of a representation of element $g\in G_{f}$ is

$$\widehat{U}_{\mathrm{B}}(g):=\widehat{V}_{\mathrm{B}}(g)\,\mathsf{K}^{\mathfrak{c}(g)}, \tag{6.19}$$

where $\widehat{V}_{\mathrm{B}}(g)$ is a unitary operator that belongs to $\mathrm{C}\ell_{n}$ defined in Eq. (6.11).

The projective representation $\widehat{U}_{\mathrm{B}}(g)$ defined in Eq. (6.19) is characterized by a pair of indices (ν,ρ). Here, $\nu\in C^{2}(G,\mathrm{U}(1))$ is a that corresponds to the projective representation of the symmetry group $G\cong G_{f}/\mathbb{Z}_{2}^{\mathrm{F}}$. The index $\rho\in C^{1}(G,\mathbb{Z}_{2})$ is 1-cochain that corresponds to the fermion parity of the representations of elements in G. When defining these indices, it is convenient to label the element in G_{f} by elements (g,h) of the group $G\underset{\gamma}{\times}\mathbb{Z}_{2}^{\mathrm{F}}$ which is isomorphic to G_{f}. The group composition rule in $G\underset{\gamma}{\times}\mathbb{Z}_{2}^{\mathrm{F}}$ is

$$(g_{1},h)\underset{\gamma}{\times}(g',h')=(g_{1}\,g_{2},\,\gamma(g_{1},\,g_{2})\,h_{1}\,h_{2})\,. \tag{6.20}$$

The 2-cochain $\nu\in C^{2}\Big(G,\mathrm{U}(1)\Big)$ is defined by restricting the domain of definition of the 2-cochain ϕ from G_{f} to G,

$$\nu(g_{1},\,g_{2}):=\phi\Big((g_{1},e),(g_{2},e)\Big), \tag{6.21}$$

for any $g_{1},\,g_{2}\in G$, and by e we denote the identity element in both G and $\mathbb{Z}_{2}^{\mathrm{F}}$.

The 1-cochain $\rho\in C^{1}\Big(G_{f},\mathbb{Z}_{2}\Big)$ is defined by the relation

$$\begin{aligned}e^{\mathrm{i}\pi\rho\left(g,h\right)}&\equiv(-1)^{\rho\left(g,h\right)}\\&:=\begin{cases}\widehat{U}_{\mathrm{B}}\Big(g,h\Big)\,\widehat{U}_{\mathrm{B}}\Big(e,p\Big)\,\widehat{U}^{\dagger}_{\mathrm{B}}\Big(g,h\Big)\,\widehat{U}^{\dagger}_{\mathrm{B}}\Big(e,p\Big), & \text{if } \mathfrak{c}\Big(g\Big)=0,\\ \widehat{U}_{\mathrm{B}}\Big(g,h\Big)\,\widehat{U}_{\mathrm{B}}\Big(e,p\Big)\,\widehat{U}^{\dagger}_{\mathrm{B}}\Big(g,h\Big)\,\widehat{U}_{\mathrm{B}}\Big(e,p\Big), & \text{if } \mathfrak{c}\Big(g\Big)=1,\end{cases}\end{aligned} \tag{6.22}$$

for any $(g,h)\in G\underset{\gamma}{\times}\mathbb{Z}_{2}^{\mathrm{F}}$. The 1-cochain $\rho\in C^{1}\Big(G_{f},\mathbb{Z}_{2}\Big)$ takes the values 0 or 1. It is a group homomorphism from G_{f} to $\mathbb{Z}_{2}=\{0,1\}$, since it has a vanishing coboundary and, hence, is a 1-cocycle. It measures the fermion parity of the operator $\widehat{U}_{\mathrm{B}}\big(g,h\big)$. As expected we have $\rho\big(e,p\big)=0$. When a gauge choice is made by choosing the representation $\widehat{U}\big((\mathrm{id},p)\big)$ to be Hermitian, the two cases in the definition (6.22) are equivalent.

In Eq. (6.22), 1-cochain is defined over G_f. With an abuse of notation, we denote its restriction to G by $\rho\in C^1(G,\mathbb{Z}_2)$. It is defined by setting the $h=e$ in Eq. (6.22). The appropriate definition for 1-cocycle ρ should be understood from the context. Index ρ is related to the 2-cochain ϕ by the relation In terms of the 2-cocycle ϕ, $\rho\in C^1\big(G,\mathbb{Z}_2\big)$ is, for any $g\in G$, given by

$$\rho(g)=\frac{1}{\pi}\left[\phi\Big((g,e),(\mathrm{id},p)\Big)-\phi\Big((\mathrm{id},p),(g,e)\Big)+\mathfrak{c}(g,e)\,\phi\Big((\mathrm{id},p),(\mathrm{id},p)\Big)\right]. \tag{6.23}$$

6.2.2.2 *The Case of* $[\mu]=1$

When the number n of Majorana operators on the boundary Λ_{B} is odd, $[\mu]=1$. The boundary representation of element $p\in G_f$ that generates the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ is chosen to be

$$\widehat{U}_{\mathrm{B}}(p):=\widehat{P}\,\widehat{P}_{\mathrm{nonloc}}, \tag{6.24a}$$

$$\widehat{P}:=\prod_{\alpha=1}^{(n-1)/2}\widehat{P}_{\alpha},\quad \widehat{P}_{\alpha}:=\mathrm{i}\hat{\gamma}_{2\alpha-1}\,\hat{\gamma}_{2\alpha}, \tag{6.24b}$$

$$\widehat{P}_{\mathrm{nonloc}}:=\mathrm{i}\hat{\gamma}_{n}\,\hat{\gamma}_{\infty}, \tag{6.24c}$$

for $\widehat{U}_{\mathrm{B}}(p)$ is proportional to the product $\hat{\gamma}_1\,\cdots\,\hat{\gamma}_n\,\hat{\gamma}_\infty$ of all the generators in $C\ell_{n+1}$. As such, $\widehat{U}_{\mathrm{B}}(p)$ anticommutes with all the Majorana operators that span the nonlocal fermionic Fock space (6.14a). The parity operators $\widehat{P}_1,\cdots,\widehat{P}_{(n-1)/2},\widehat{P}_{\mathrm{nonloc}}$ are Hermitian, square to the identity, and are pairwise commuting. We choose a basis in which all parity operators are simultaneously diagonalized and even under complex conjugation K,

$$\mathsf{K}\,\widehat{P}_{\alpha}\,\mathsf{K}=\widehat{P}_{\alpha},\qquad \mathsf{K}\,\widehat{P}_{\mathrm{nonloc}}\,\mathsf{K}=\widehat{P}_{\mathrm{nonloc}}, \tag{6.24d}$$

for $\alpha,\alpha'=1,\cdots,(n-1)/2$.

In addition to defining a representation of the fermion parity p, we need to account for the fact that the center of the Clifford algebra $\mathrm{C}\ell_n$ is two-dimensional when n is odd. We choose to represent the nontrivial element of this center by

$$\widehat{Y}_{\mathrm{B}} := \widehat{P}\,\hat{\gamma}_n, \qquad \widehat{Y}^{\dagger}_{\mathrm{B}} = \widehat{Y}_{\mathrm{B}}, \qquad \widehat{Y}^{2}_{\mathrm{B}} = \hat{\mathbb{1}}_{\mathrm{B},1}. \tag{6.25}$$

By construction, $\widehat{Y}_{\mathrm{B}}$ is proportional to the product $\hat{\gamma}_1 \cdots \hat{\gamma}_n \neq \hat{\mathbb{1}}_{\mathrm{B},1}$. It commutes with the Majorana operators $\hat{\gamma}_1, \cdots, \hat{\gamma}_n$, while it anticommutes with the Majorana operator $\hat{\gamma}_\infty$. The operator $\widehat{Y}_{\mathrm{B}}$ is of odd fermion parity for it anticommutes with the fermion parity operator (6.24). Because of the nontrivial central element $\widehat{Y}_{\mathrm{B}}$ that carry odd fermion parity, the boundary representation $\widehat{U}_{\mathrm{B}}(g)$ satisfying Eq. (6.17) does not have a fixed parity. In other words, $\widehat{U}_{\mathrm{B}}(g)$ and $\widehat{U}_{\mathrm{B}}(g)\,\widehat{Y}_{\mathrm{B}}$ both satisfy Eq.(6.17) while they carry opposite fermion parities.

Since the Clifford algebra $\mathrm{C}\ell_n$ is closed under the action of the boundary representation $\widehat{U}_{\mathrm{bulk}}(g)$, the same must be true for the boundary representation $\widehat{U}_{\mathrm{B}}(g)$ [recall Eqs. (6.16) and (6.17)]. In other words, $\widehat{U}_{\mathrm{B}}(g)$ preserves locality in that its action on those operators whose non-trivial actions are limited to Λ_{B} is merely to mix them. This locality is guaranteed only $\widehat{U}_{\mathrm{B}}(g)$ either commutes or anticommutes with the center $\widehat{Y}_{\mathrm{B}}$ of $\mathrm{C}\ell_n$, i.e.,

$$\widehat{Y}_{\mathrm{B}}\,\widehat{U}_{\mathrm{B}}(g) = \pm\widehat{U}_{\mathrm{B}}(g)\,\widehat{Y}_{\mathrm{B}}. \tag{6.26}$$

Furthermore, this is true only if the decomposition

$$\widehat{U}_{\mathrm{B}}(g) := \widehat{V}_{\mathrm{B}}(g)\,\widehat{Q}_{\mathrm{B}}(g)\,\mathsf{K}^{\mathfrak{c}(g)}, \quad \widehat{Q}_{\mathrm{B}}(g) = [\hat{\gamma}_\infty]^{q(g)}\,, \tag{6.27}$$

holds. Here, $\widehat{V}_{\mathrm{B}}(g) \in \mathrm{C}\ell_n \subset \mathrm{C}\ell_{n+1}$ is a unitary operator with well-defined fermion parity. Here, we used the presence of nontrivial central element $\widehat{Y}_{\mathrm{B}}$ to fix to be even for all $g \in G_f$. In this "gauge", $q(g) = 0, 1$ denotes the fermion parity of the unitary operator $\widehat{V}_{\mathrm{B}}(g)$. Equation (6.27) together with Eqs. (6.24) and (6.25) define the realization of the symmetry group G_f on the boundary Λ_{B} when $[\mu] = 1$.

The projective representation $\widehat{U}_{\mathrm{B}}(g)$ defined in Eq. (6.27) is characterized by a pair of indices (ν, ρ). Here, $\nu \in C^2(G, \mathrm{U}(1))$ is a that corresponds to the projective representation of the symmetry group $G \cong G_f/\mathbb{Z}^{\mathrm{F}}_2$. The index $\rho \in C^1(G, \mathbb{Z}_2)$ is 1-cochain that labels if the representation $\widehat{U}_{\mathrm{B}}(g)$ of an element $g \in G_f$ commutes or anticommutes with $\widehat{Y}_{\mathrm{B}}$. When $[\mu] = 1$, the group G_f necessarily splits, i.e., $[\gamma] = 0$ and G_f must be isomorphic to the direct product $G \times \mathbb{Z}_2^F$ [8]. Therefore, we label the elements in G_f by the pair $(g, h) \in G \times \mathbb{Z}_2^{\mathrm{F}}$.

The 2-cochain $\nu \in C^2\big(G, \mathrm{U}(1)\big)$ is defined by restricting the domain of definition of the 2-cochain ϕ from G_f to G,

$$\nu(g_1, g_2) := \phi\big((g_1, e), (g_2, e)\big), \tag{6.28}$$

for any $g_1, g_2 \in G$, and by e we denote the identity element in both G and $\mathbb{Z}_2^{\mathrm{F}}$.

When $[\mu] = 1$, the Clifford algebra $\mathrm{C}\ell_n$ spanned by the Majorana operators (6.11) has a two-dimensional center, in which case the fermion parity of the boundary representation $\widehat{U}_{\mathrm{B}}\big(g, h\big)$ for any element $(g, h) \in G_f$ can be reversed by multiplying it with the generator $\widehat{Y}_{\mathrm{B}}$ of the two-dimensional center of the Clifford algebra $\mathrm{C}\ell_n$. Moreover, any $\widehat{U}_{\mathrm{B}}\big((g, h)\big)$ must either commute or anticommute with $\widehat{Y}_{\mathrm{B}}$ according to Eq. (6.26).

For this reason, we define the 1-cochain $\rho \in C^1\big(G_f, \mathbb{Z}_2\big)$ through

$$e^{\mathrm{i}\pi\rho\left(g,h\right)} \equiv (-1)^{\rho\left(g,h\right)} := \begin{cases} \widehat{U}_{\mathrm{B}}\big(g,h\big)\,\widehat{Y}_{\mathrm{B}}\,\widehat{U}^{\dagger}_{\mathrm{B}}\big(g,h\big)\,\widehat{Y}^{\dagger}_{\mathrm{B}}, & \text{if } \mathfrak{c}\big(g\big) = 0, \\ \\ \widehat{U}_{\mathrm{B}}\big(g,h\big)\,\widehat{Y}_{\mathrm{B}}\,\widehat{U}^{\dagger}_{\mathrm{B}}\big(g,h\big)\,\widehat{Y}_{\mathrm{B}}, & \text{if } \mathfrak{c}\big(g\big) = 1, \end{cases} \tag{6.29}$$

for any $(g, h) \in G_f$. The 1-cochain $\rho \in C^1\big(G_f, \mathbb{Z}_2\big)$ takes the value 0 and 1. The 1-cochain $\rho \in C^1\big(G_f, \mathbb{Z}_2\big)$ is a group homomorphism from G_f to $\mathbb{Z}_2 = \{0, 1\}$ since it has a vanishing coboundary and, hence, is a 1-cocycle. Since $\widehat{Y}_{\mathrm{B}}$ is of odd fermion parity by definition (6.25), it anticommutes with the representation $\widehat{U}_{\mathrm{B}}\big(e, p\big)$. This implies that $\rho(e, p) = 1$. When a gauge choice is made by choosing the representation $\widehat{Y}_{\mathrm{B}}$ to be Hermitian, the two cases in the definitions (6.29) and (6.29) are equivalent. As was the case for $[\mu] = 0$, with an abuse of notation, we denote the restriction of $\rho \in C^1(G_f, \mathbb{Z}_2)$ to G by $\rho \in C^1(G, \mathbb{Z}_2)$. The latter is defined by setting the $h = e$ in Eq. (6.29). The appropriate definition for 1-cocycle ρ should be understood from the context.

We close Sec. 6.2.2.2 by spelling out two identities that will be convenient when deriving the stacking rules in Sec. 7. We note that definition (6.29) involves conjugation of the

8 This is because the index ρ when $[\mu] = 1$ induces group isomorphisms from $\mathbb{Z}_2^{\mathrm{F}}$ to $\mathbb{Z}_2$ and from G_f to $G \times \mathbb{Z}_2^{\mathrm{F}}$,which implies that γ is a 1-coboundary, see Appendix A.4.3.2 for a proof.

central element $\widehat{Y}_{\mathrm{B}}$ by the boundary representation $\widehat{U}_{\mathrm{B}}(g)$ of some element $g \in G_f$. By definitions (6.24) and (6.25), $\widehat{Y}_{\mathrm{B}}$ can be written as

$$\widehat{Y}_{\mathrm{B}} = -\mathrm{i}\, \widehat{U}_{\mathrm{B}}(p)\, \hat{\gamma}_{\infty}. \tag{6.30a}$$

Using this identity in definition (6.29) allows one to express the complex conjugation of $\hat{\gamma}_{\infty}$ in terms of group homomorphisms $\mathfrak{c}$, q, and ρ. Since Eq. (6.24d) implies that the Majorana operators $\hat{\gamma}_{\infty}$ and $\hat{\gamma}_n$ transform oppositely under complex conjugation, one finds the pair of identities

$$\mathsf{K}^{\mathfrak{c}(g)}\, \hat{\gamma}_{\infty}\, \mathsf{K}^{\mathfrak{c}(g)} = (-1)^{\mathfrak{c}(g)+q(g)+\rho(g)} \hat{\gamma}_{\infty}, \tag{6.30b}$$

$$\mathsf{K}^{\mathfrak{c}(g)}\, \hat{\gamma}_{n}\, \mathsf{K}^{\mathfrak{c}(g)} = (-1)^{q(g)+\rho(g)} \hat{\gamma}_{n}, \tag{6.30c}$$

for any $g \in G_f$

6.3 ENUMERATION OF ONE-DIMENSIONAL IFT PHASES

In Sec. 6.1, we started with a set of representative Hamiltonians with internal G_f-symmetry and translation symmetry for any one-dimensional IFT phase. In turn, using LSM theorems 3 and 2, we argued that the boundary projective representations enumerates distinct IFT phases in one dimension. These correspond to the distinct ways in which the conditions of Theorems 2 and 3 are violated when open boundary conditions are imposed.

In Sec. 6.2, we enumerated boundary projective representations when $[\mu] = 0$ and $[\mu] = 1$ in terms of pair of indices (ν, ρ). This exercise tells us that one-dimensional IFT phases are labeled by the values of the triplet

$$\big((\nu,\rho),[\mu]\big) \in C^2(G,\mathrm{U}(1)) \times C^1(G,\mathbb{Z}_2) \times H^0(G,\mathbb{Z}_2), \tag{6.31}$$

subjected to certain consistency conditions. First, associativity of the group composition rule requires the pair (ν, ρ) to obey the (modified) cocycle conditions

$$\mathcal{D}^2_{\gamma,[\mu]}\,(\nu,\rho) = (0,0), \tag{6.32a}$$

where

$$\mathcal{D}^2_{\gamma,[\mu]}(\nu,\rho) = \begin{cases} \left(\delta^2_{\mathfrak{c}}\nu - \pi\,\rho \smile \gamma,\ \delta^1_{\mathfrak{c}}\rho\right), & \text{if } [\mu]=0, \\ \left(\delta^2_{\mathfrak{c}}\nu,\ \delta^1_{\mathfrak{c}}\rho\right), & \text{if } [\mu]=1. \end{cases} \tag{6.32b}$$

Second, the projective representation $\widehat{U}(g)$ in Eqs. (6.19) and (6.27) are defined up to a multiplicative phase. This means that two pairs (ν,ρ) and (ν',ρ') are equivalent if they are related by a (modified) coboundary

$$(\nu,\rho) = (\nu',\rho') + \mathcal{D}^1_{\gamma,[\mu]}(\alpha,\beta), \tag{6.33a}$$

where

$$\mathcal{D}^1_{\gamma,[\mu]}(\alpha,\beta) = \begin{cases} \left(\delta^1_{\mathfrak{c}}\alpha + \pi\,\beta \smile \gamma,\ \delta^0_{\mathfrak{c}}\beta\right), & \text{if } [\mu]=0, \\ \left(\delta^1_{\mathfrak{c}}\alpha,\ \delta^0_{\mathfrak{c}}\beta\right), & \text{if } [\mu]=1, \end{cases} \tag{6.33b}$$

where $\alpha \in C^1(G,\mathrm{U}(1))$ is a 1-cochain and $\beta \in C^0(G,\mathbb{Z}_2)$ is a 0-cochain. Therefore, the inequivalent boundary projective representations are enumerated by the equivalence classes $\big([(\nu,\rho)],[\mu]\big)$ such that

$$\big([(\nu,\rho)],[\mu]\big) = \frac{\ker\left(\mathcal{D}^2_{\gamma,[\mu]}\right)}{\mathrm{im}\left(\mathcal{D}^1_{\gamma,[\mu]}\right)} \times H^0(G,\mathbb{Z}_2). \tag{6.34}$$

This is to say that the set of equivalence classes $\big([(\nu,\rho)],[\mu]\big)$ is in one-to-one correspondence with the one-dimensional IFT phases.

We emphasize that what is achieved so far is only *enumeration* of IFT phases. Finding all indices $\big([(\nu,\rho)],[\mu]\big)$ does not provide the group structure formed by IFT phases under the operation of stacking. In particular, given the two IFT phases with indices $\big([(\nu_1,\rho_1)],[\mu_1]\big)$ and $\big([(\nu_2,\rho_2)],[\mu_2]\big)$, we want to find the indices $\big((\nu_\wedge,\rho_\wedge),[\mu_{2\wedge}]\big)$ associated with IFT phase obtained from stacking the two. As we shall see, the naive guess of adding the respective indices is not correct, i.e.,

$$\big([(\nu_\wedge,\rho_\wedge)],[\mu_\wedge]\big) \neq \big([(\nu_1+\nu_2,\rho_1+\rho_2)],[\mu_1+\mu_2]\big). \tag{6.35}$$

We will show that this is because of the underlying fermionic nature of the degrees of freedom. In particular, the guess (in a certain form) (6.35) holds for bosonic invertible phases. In other words, the violations of Eq. (6.35) is associated with the *intrinsically fermionic* nature of the corresponding IFT phases.

7

DERIVATION OF THE FERMIONIC STACKING RULES

Given the two triplets $\big((\nu_1,\rho_1),[\mu_1]\big)$ and $\big((\nu_2,\rho_2),[\mu_2]\big)$ associated to the pair $\widehat{U}_1$ and $\widehat{U}_2$ of boundary representations, respectively, we shall construct the triplet $((\nu_\wedge,\rho_\wedge),[\mu_\wedge])$ that is associated with the representation $\widehat{U}_\wedge$, whereby $\widehat{U}_\wedge$ must be compatible with the symmetry group G_f and is obtained from taking the tensor product of the two set of boundary degrees of freedom. We call this operation stacking.

Since the number of boundary Majorana degrees of freedom on which $\widehat{U}_\wedge$ acts is obtained by adding the boundary Majorana degrees of freedom

$$\mathfrak{O}_1 := \left\{\hat{\gamma}^{(1)}_1,\,\hat{\gamma}^{(1)}_2,\,\cdots,\,\hat{\gamma}^{(1)}_{n_1}\right\} \tag{7.1}$$

on which $\widehat{U}_1$ acts to the boundary Majorana degrees of freedom

$$\mathfrak{O}_2 := \left\{\hat{\gamma}^{(2)}_1,\,\hat{\gamma}^{(2)}_2,\,\cdots,\,\hat{\gamma}^{(2)}_{n_2}\right\} \tag{7.2}$$

on which $\widehat{U}_2$ acts, we define the index $[\mu_\wedge]$ of the stacked representation to be

$$[\mu_\wedge] := [\mu_1] + [\mu_2] \bmod 2. \tag{7.3}$$

For any $g \in G_f$, we define the stacked representation $\widehat{U}_\wedge(g)$ to be a norm preserving operator that satisfies the identities

$$\widehat{U}_\wedge(g)\,\hat{\gamma}^{(1)}_i\,\widehat{U}^\dagger_\wedge(g) := \widehat{U}_1(g)\,\hat{\gamma}^{(1)}_i\,\widehat{U}^\dagger_1(g), \tag{7.4a}$$

$$\widehat{U}_\wedge(g)\,\hat{\gamma}^{(2)}_j\,\widehat{U}^\dagger_\wedge(g) := \widehat{U}_2(g)\,\hat{\gamma}^{(2)}_j\,\widehat{U}^\dagger_2(g), \tag{7.4b}$$

for $i = 1,\cdots,n_1$ and $j = 1,\cdots,n_2$. This definition is the natural generalization of Eq. (6.17). Because $\widehat{U}_1(g)$ and $\widehat{U}_2(g)$ act on single Majorana operators in the same way as the bulk representation of the element $g \in G_f$ does, the same is true for the stacked representation $\widehat{U}_\wedge(g)$. The stacked representation $\widehat{U}_\wedge(g)$ is not unique since Eqs. (7.4a) and (7.4b) are left invariant by the multiplications from the right of $\widehat{U}_{\wedge}(g)$ with any norm-preserving element from the center of the Clifford algebra $C\ell_{n_1+n_2}$.

When constructing an explicit representation of $\widehat{U}_{\wedge}(g)$ for any $g \in G_f$, we shall consider the three cases: (i) even-even stacking, $[\mu_1] = [\mu_2] = 0$, (ii) even-odd stacking, $[\mu_1] = 0$, $[\mu_2] = 1$, (iii) and odd-odd stacking, $[\mu_1] = [\mu_2] = 1$. The case of odd-even stacking is to be treated analogously to the case of even-odd stacking.

As is done in Sec. 6.2, we begin with the construction of a representation of the fermion parity $p \in G_f$. When $[\mu_{\wedge}] = 0$, the stacked representation of $\widehat{U}_{\wedge}(p)$ follows from combining Eq. (7.4) with the counterpart to Eq. (6.18). When $[\mu_{\wedge}] = 1$, the stacked representation of $\widehat{U}_{\wedge}(p)$ follows from combining Eq. (7.4) with the counterparts to Eqs. (6.24). More precisely, the stacked representation $\widehat{U}_{\wedge}(p)$ of the fermion parity p is defined to be

$$\widehat{U}_{\wedge}(p) := \begin{cases} \widehat{U}_1(p)\,\widehat{U}_2(p), & \text{if } [\mu_1] = [\mu_2] = 0, \\ \widehat{U}_1(p)\,\widehat{U}_2(p), & \text{if } [\mu_1] = 0,\ [\mu_2] = 1, \\ \widehat{P}_1\,\widehat{P}_2\,\mathrm{i}\hat{\gamma}^{(1)}_{n_1}\,\hat{\gamma}^{(2)}_{n_2}, & \text{if } [\mu_1] = [\mu_2] = 1. \end{cases} \tag{7.5}$$

By construction, we have chosen a Hermitian representation $\widehat{U}_{\wedge}(p)$ of the fermion parity p.

Next, we fix the action of the stacked complex conjugation $\mathsf{K}_{\wedge}$ on the single Majorana operators spanning the fermionic Fock space of the stacked boundary by demanding that some set of mutually commuting fermion parity operators are left invariant under complex conjugation [recall Eqs. (6.18) and (6.24)]. For the cases of even-even ($[\mu_1] = [\mu_2] = 0$) and even-odd stacking ($[\mu_1] = 0$, $[\mu_2] = 1$), we define $\mathsf{K}_{\wedge}$ by

$$\mathsf{K}_{\wedge}\,\hat{\gamma}^{(1)}_i\,\mathsf{K}_{\wedge} := \mathsf{K}_1\,\hat{\gamma}^{(1)}_i\,\mathsf{K}_1, \tag{7.6a}$$

$$\mathsf{K}_{\wedge}\,\hat{\gamma}^{(2)}_j\,\mathsf{K}_{\wedge} := \mathsf{K}_2\,\hat{\gamma}^{(2)}_j\,\mathsf{K}_2, \tag{7.6b}$$

for $i = 1, \cdots, n_1$ and $j = 1, \cdots, n_2$. For the case of odd-odd stacking, we define $\mathsf{K}_{\wedge}$ by

$$\mathsf{K}_{\wedge}\,\hat{\gamma}^{(1)}_i\,\mathsf{K}_{\wedge} := \mathsf{K}_1\,\hat{\gamma}^{(1)}_i\,\mathsf{K}_1, \tag{7.7a}$$

$$\mathsf{K}_{\wedge}\,\hat{\gamma}^{(2)}_j\,\mathsf{K}_{\wedge} := \mathsf{K}_2\,\hat{\gamma}^{(2)}_j\,\mathsf{K}_2, \tag{7.7b}$$

$$\mathsf{K}_\wedge\,\hat{\gamma}^{(1)}_{n_1}\,\mathsf{K}_\wedge := +\hat{\gamma}^{(1)}_{n_1}, \tag{7.7c}$$

$$\mathsf{K}_\wedge\,\hat{\gamma}^{(2)}_{n_2}\,\mathsf{K}_\wedge := -\hat{\gamma}^{(2)}_{n_2}, \tag{7.7d}$$

for $i = 1, \cdots, n_1 - 1$ and $j = 1, \cdots, n_2 - 1$. One verifies that, by construction, the fermion parity operator $\widehat{U}_\wedge(p)$ is invariant under conjugation by $\mathsf{K}_\wedge$. For any operator $\widehat{O}$, we introduce the notations

$$\overline{\widehat{O}}^{1,g} := \mathsf{K}_1^{\mathfrak{c}(g)}\,\widehat{O}\,\mathsf{K}_1^{\mathfrak{c}(g)}, \tag{7.8a}$$

$$\overline{\widehat{O}}^{2,g} := \mathsf{K}_2^{\mathfrak{c}(g)}\,\widehat{O}\,\mathsf{K}_2^{\mathfrak{c}(g)}, \tag{7.8b}$$

$$\overline{\widehat{O}}^{\wedge,g} := \mathsf{K}_\wedge^{\mathfrak{c}(g)}\,\widehat{O}\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}, \tag{7.8c}$$

to denote its complex conjugation by K_1, K_2, and, $\mathsf{K}_\wedge$.

For the stacked representation with $[\mu_\wedge] = 1$ that is achieved with an even-odd stacking, we define the central element $\widehat{Y}_\wedge$ by

$$\widehat{Y}_\wedge := \widehat{U}_1(p)\,\widehat{Y}_2, \tag{7.9}$$

where $\widehat{Y}_2$ is the central element inherited from the representation $\widehat{U}_2$ which by assumption has $[\mu_2] = 1$ for the case of even-odd stacking.

In what follows, we give explicit representations of $\widehat{U}_\wedge(g)$ in terms of the pair $\widehat{U}_1(g)$ and $\widehat{U}_2(g)$ and of $([(\nu_\wedge, \rho_\wedge)], [\mu_\wedge])$ in terms of the pairs $([(\nu_1, \rho_1)], [\mu_1])$ and $([(\nu_2, \rho_2)], [\mu_2])$. The computational details are left to the Appendix D.

7.1 EVEN-EVEN STACKING

For even-even stacking, we have $[\mu_1] = [\mu_2] = 0$. We define

$$[\mu_\wedge] := [\mu_1] + [\mu_2] = 0. \tag{7.10}$$

The representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group G_f are of the form (6.19), i.e., for any $g \in G_f$,

$$\widehat{U}_1(g) = \widehat{V}_1(g)\,\mathsf{K}_1^{\mathfrak{c}(g)}, \qquad \widehat{U}_2(g) = \widehat{V}_2(g)\,\mathsf{K}_2^{\mathfrak{c}(g)}, \tag{7.11}$$

with the pair of unitary operators $\widehat{V}_1(g)$ and $\widehat{V}_2(g)$. The naive guess of $\widehat{V}_1(g)\,\widehat{V}_2(g)\,\mathsf{K}_{\wedge}^{\mathfrak{c}(g)}$ is not a satisfactory definition of $\widehat{U}_{\wedge}(g)$, for one verifies that it fails to satisfy Eq. (7.4). Instead, for any $g \in G_f$, we define

$$\widehat{U}_{\wedge}(g) := \widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)}\,\mathsf{K}_{\wedge}^{\mathfrak{c}(g)}. \tag{7.12}$$

One verifies that this definition satisfies Eq. (7.4) and, *a forteriori*, Eq. (6.17). The parity operators $\widehat{U}_1(p)$ and $\widehat{U}_2(p)$ in definition (7.12) ensure that no additional minus signs are introduced when Majorana operators $\hat{\gamma}_i^{(1)}$ and $\hat{\gamma}_i^{(2)}$ are conjugated by $\widehat{U}_{\wedge}(g)$. This is because, by definition (6.22), the values $\rho_1(g)$ and $\rho_2(g)$ encode the fermion parity of the unitary operators $\widehat{V}_1(g)$ and $\widehat{V}_2(g)$, respectively, and the parity operators $\widehat{U}_1(p)$ and $\widehat{U}_2(p)$ correct for any additional minus signs arising from fermionic algebra between operators from representation $\widehat{U}_1$ and $\widehat{U}_2$ in compliance with Eq. (7.4).

As a sanity check, one verifies that when restricted to the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$, the definition (7.12) of the stacked representation together with definition (6.18) deliver the Hermitian representations

$$\widehat{U}_{\wedge}(e) = \hat{\mathbb{1}}_{\wedge,0}, \quad \widehat{U}_{\wedge}(p) = \widehat{U}_1(p)\,\widehat{U}_2(p), \tag{7.13}$$

that are consistent with the definition (7.5).

The stacking rules are retrieved by composing the representations $\widehat{U}_{\wedge}(g)$ and $\widehat{U}_{\wedge}(h)$ of two elements g and h of G_f. Definition (7.12) delivers (Appendix D.1)

$$\begin{aligned}\widehat{U}_{\wedge}(g)\,\widehat{U}_{\wedge}(h) &= (-1)^{\rho_1(g)\rho_2(h)}\,\widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\,\widehat{V}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g} \\ &\qquad \times \left[\widehat{U}_1(p)\right]^{\rho_2(g\,h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g\,h)}\,\mathsf{K}_{\wedge}^{\mathfrak{c}(g\,h)} \\ &= e^{\mathrm{i}\phi_{\wedge}(g,\,h)}\widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\,\left[\widehat{U}_1(p)\right]^{\rho_2(g\,h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g\,h)}\,\mathsf{K}_{\wedge}^{\mathfrak{c}(g\,h)} \\ &= e^{\mathrm{i}\phi_{\wedge}(g,h)}\,\widehat{U}_{\wedge}(g\,h),\end{aligned} \tag{7.14a}$$

where we have defined

$$\phi_{\wedge}(g,h) := \phi_1(g,h) + \phi_2(g,h) + \pi\,\rho_1(g)\,\rho_2(h). \tag{7.14b}$$

The construction of the indices $([(\nu_\wedge,\rho_\wedge)],[\mu_\wedge])$ in terms of the indices $([(\nu_1,\rho_1)],[\mu_1])$ and $([(\nu_2,\rho_2)],[\mu_2])$ is achieved as follows. According to definition (6.21), the 2-cochain $\nu_\wedge$ is simply obtained by restricting $\phi_\wedge$ to the elements of G, i.e.,

$$\nu_\wedge(g,h)=\nu_1(g,h)+\nu_2(g,h)+\pi\,(\rho_1\smile\rho_2)\,(g,h). \tag{7.15a}$$

In the last step we have used the cup product $\smile$ to construct a 2-cochain $\rho_1\smile\rho_2$ out of the pair of one cochains ρ_1 and ρ_2. For the 1-cochain $\rho_\wedge$, definition (6.22) delivers

$$\rho_\wedge(g)=\rho_1(g)+\rho_2(g), \tag{7.15b}$$

which is nothing but the total fermion parity of the stacked representation $\widehat{U}_\wedge(g)$ of element $g\in G$.

7.2 EVEN-ODD STACKING

For even-odd stacking, we have $[\mu_1]=0$, $[\mu_2]=1$. Hence, we define

$$[\mu_\wedge]:=[\mu_1]+[\mu_2]=1. \tag{7.16}$$

The representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group G_f are of the form (6.19) and (6.27), respectively, i.e., for any $g\in G_f$,

$$\widehat{U}_1(g)=\widehat{V}_1(g)\,\mathsf{K}_1^{\mathfrak{c}(g)},\quad \widehat{U}_2(g)=\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\mathsf{K}_2^{\mathfrak{c}(g)},\quad \widehat{Q}_2(g)=\left[\hat{\gamma}^{(2)}_\infty\right]^{q_2(g)}. \tag{7.17}$$

The naive guess $\widehat{V}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}$ is not a satisfactory definition of $\widehat{U}_\wedge(g)$, for one verifies that it fails to satisfy Eq. (7.4) and to be of even fermion parity. Instead, we define the stacked representation to be

$$\begin{aligned}\widehat{U}_\wedge(g)&:=\widehat{V}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\left[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_\infty\right]^{\rho_1(g)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}\\ &\equiv\widehat{V}_\wedge(g)\,\widehat{Q}_\wedge(g)\,\mathsf{K}_\wedge^{\mathfrak{c}(g)},\end{aligned} \tag{7.18a}$$

$$\widehat{V}_\wedge(g):=\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_1(p)\right]^{\rho_1(g)}, \tag{7.18b}$$

$$\widehat{Q}_\wedge(g):=\widehat{Q}_2(g)\,\left[\hat{\gamma}^{(2)}_\infty\right]^{\rho_1(g)}=\left[\hat{\gamma}^{(2)}_\infty\right]^{q_2(g)+\rho_1(g)}. \tag{7.18c}$$

One verifies that this definition satisfies Eq. (7.4) and, *a forteriori*, Eq. (6.17). For any $g \in G_f$, the definition (7.18) guarantees that $\widehat{U}_\wedge(g)$ is of even fermion parity. This property is inherited from the fact that $\widehat{U}_2(g)$ is of even fermion parity according to Eq. (6.27) and the factor $\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_\infty$ compensates for the fermion parity of the operator $\widehat{V}_1(g)$. The product $\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_\infty$ also compensates for additional minus signs arising from fermionic algebra between the operators from representations $\widehat{U}_1$ and $\widehat{U}_2$ in compliance with Eq. (7.4).

As a sanity check, one verifies that, when restricted to the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$, the definition (7.18) of the stacked representation together with definitions (6.18) and (6.24) deliver the Hermitian representations

$$\widehat{U}_\wedge(e) = \hat{\mathbb{1}}_{\wedge,1}, \quad \widehat{U}_\wedge(p) = \widehat{U}_1(p)\,\widehat{U}_2(p), \tag{7.19}$$

that are consistent with the definition (7.5).

When representations $\widehat{U}_\wedge(g)$ and $\widehat{U}_\wedge(h)$ of two elements $g,h \in G_f$ are composed, we obtain from definition (7.18) (Appendix D.2)

$$\begin{aligned}
\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = (-1)^{\rho_1(g)\,q_2(h)}\,\widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g} \\
\times \left[\hat{\gamma}^{(2)}_\infty\right]^{\rho_1(g)} \left[\overline{\hat{\gamma}^{(2)}_\infty}^{2,g}\right]^{\rho_1(h)} \left[\widehat{U}_1(p)\right]^{\rho_1(g\,h)} \mathsf{K}_\wedge^{\mathfrak{c}(g\,h)} \\
= e^{\mathrm{i}\phi_\wedge(g,h)}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\,\widehat{Q}_1(g\,h)\,\left[\hat{\gamma}^{(2)}_\infty\right]^{\rho_1(h)} \\
= e^{\mathrm{i}\phi_\wedge(g,h)}\,\widehat{U}_\wedge(g\,h),
\end{aligned} \tag{7.20a}$$

where we have defined

$$\phi_\wedge(g,h) := \phi_1(g,h) + \phi_2(g,h) + \pi\rho_1(g)\,q_2(h) + \pi\rho_1(h)\,[\mathfrak{c}(g) + q_2(g) + \rho_2(g)]\,. \tag{7.20b}$$

The projective phase (7.20b) can be simplified by noting that terms that contain the 1-cochain q_2. Therefore, the 2-cochain $\phi_\wedge(g,h)$ defined in (7.20b) is gauge equivalent (Appendix D.2) to

$$\phi'_\wedge(g,h) := \phi_1(g,h) + \phi_2(g,h) + \pi\rho_1(g)\,\rho_2(h) + \pi\rho_1(g)\,\mathfrak{c}(h). \tag{7.21}$$

The construction of the indices $([(\nu_\wedge, \rho_\wedge)], [\mu_\wedge])$ in terms of the indices $([(\nu_1, \rho_1)], [\mu_1])$ and $([(\nu_2, \rho_2)], [\mu_2])$ is achieved as follows.

According to definition (6.28), the 2-cochain $\nu_\wedge$ is simply obtained by restricting $\phi_\wedge$ to the elements of G, i.e.,

$$\nu_\wedge(g,h) := \nu_1(g,h) + \nu_2(g,h) + \pi\left(\rho_1 \smile \rho_2\right)(g,h) + \pi\left(\rho_1 \smile \mathfrak{c}\right)(g,h), \tag{7.22a}$$

where we introduced the cup product $\smile$ to construct a 2-cochain out of 1-cochains.

Since the stacked representation has index $[\mu_\wedge] = 1$, the 1-cochain $\rho_\wedge$ can be either determined by the definition (6.29) or by the identity (6.30b). The definition (7.6) implies

$$\overline{\hat{\gamma}^{(2)}_\infty}^{\wedge,g}\, \hat{\gamma}^{(2)}_\infty = \overline{\hat{\gamma}^{(2)}_\infty}^{2,g}\, \hat{\gamma}^{(2)}_\infty. \tag{7.22b}$$

Using identity (6.30b) for the left and right hand sides separately, and comparing the two we find

$$\rho_\wedge(g) = q_2(g) + q_\wedge(g) + \rho_2(g) = \rho_1(g) + \rho_2(g) \bmod 2, \tag{7.22c}$$

where the value of the 1-cochain $q_\wedge(g) = \rho_1(g) + q_2(g)$ is read off from the fermion parity of the unitary operator $\widehat{V}_\wedge(g)$ defined in Eq. (7.18).

7.3 ODD-ODD STACKING

For odd-odd stacking, we have $[\mu_1] = [\mu_2] = 1$. Hence, we define

$$[\mu_\wedge] := [\mu_1] + [\mu_2] = 0. \tag{7.23}$$

The representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group G_f are of the form (6.27), i.e., for any $g \in G_f$,

$$\widehat{U}_1(g) = \widehat{V}_1(g)\, \widehat{Q}_1(g)\, \mathsf{K}_1^{\mathfrak{c}(g)}, \qquad \widehat{Q}_1(g) = \left[\hat{\gamma}^{(1)}_\infty\right]^{q_1(g)}, \tag{7.24a}$$

$$\widehat{U}_2(g) = \widehat{V}_2(g)\, \widehat{Q}_2(g)\, \mathsf{K}_2^{\mathfrak{c}(g)}, \qquad \widehat{Q}_2(g) = \left[\hat{\gamma}^{(2)}_\infty\right]^{q_2(g)}. \tag{7.24b}$$

The naive guess $\widehat{V}_1(g)\,\widehat{Q}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\mathsf{K}_{\wedge}^{\mathfrak{c}(g)}$ is not a satisfactory definition of $\widehat{U}_{\wedge}(g)$, for one verifies that it fails to satisfy Eq. (7.4). Instead, we define the stacked representation to be

$$\begin{aligned}\widehat{U}_{\wedge}(g) &:= (-\mathrm{i})^{\delta_{g,p}}\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)} \\ &\quad\times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\,\mathsf{K}_{\wedge}^{\mathfrak{c}(g)}, \qquad (7.25a)\\ \widehat{U}_{\wedge}(p) &:= \widehat{P}_1\,\widehat{P}_2\,\mathrm{i}\hat{\gamma}^{(1)}_{n_1}\,\hat{\gamma}^{(2)}_{n_2}, \qquad (7.25b)\end{aligned}$$

where $\widehat{P}_1$ and $\widehat{P}_2$ are the fermion parity operators constructed out of the Majorana operators $\hat{\gamma}^{(1)}_1,\cdots,\hat{\gamma}^{(1)}_{n_1-1}$ and $\hat{\gamma}^{(2)}_1,\cdots,\hat{\gamma}^{(2)}_{n_2-1}$, respectively [recall definitions (6.24) and (7.5)]. The exponent $\delta_{g,p}$ of the multiplicative phase factor $(-\mathrm{i})^{\delta_{g,p}}$ is the Kronecker delta defined over the group G_f.

As a sanity check, one verifies that, when restricted to the center $\mathbb{Z}_2^{\mathrm{F}}\subset G_f$, the definition (7.25) of the stacked representation together with the definition (6.24) deliver the Hermitian representations

$$\widehat{U}_{\wedge}(e)=\hat{\mathbb{1}}_{\wedge,1},\quad \widehat{U}_{\wedge}(p)=\widehat{P}_1\,\widehat{P}_2\,\mathrm{i}\hat{\gamma}^{(1)}_{n_1}\,\hat{\gamma}^{(2)}_{n_2}, \qquad (7.26)$$

that are consistent with the definition (7.5). The choice of the multiplicative phase factor $(-\mathrm{i})^{\delta_{g,p}}$ in Eq. (7.25) is not unique since representation $\widehat{U}(g)$ of any element $g\in G_f$ is defined up to a multiplicative U(1) phase. We observe that the multiplicative factor $(-\mathrm{i})^{\delta_{g,p}}$ in Eq. (7.25) ensures that the stacked representation $\widehat{U}_{\wedge}(p)$ is Hermitian in compliance with the "gauge" choice made in definition (6.24).

Several comments are due. First, one verifies that the definition (7.25) satisfies Eq. (7.4) and, *a forteriori*, Eq. (6.17). Second, the Majorana operators $\hat{\gamma}^{(1)}_{\infty}$ and $\hat{\gamma}^{(2)}_{\infty}$ do not enter the definition (7.25) of the stacked representation $\widehat{U}_{\wedge}$. This is expected as the stacked representation $\widehat{U}_{\wedge}$ has $[\mu_{\wedge}]=0$. Accordingly, $\widehat{U}_{\wedge}$ is constructed solely out of the even number n_1+n_2 of Majorana operators spanning the fermionic Fock space of the stacked boundary [recall definition (6.19)]. Third, the definition (7.25) is not symmetric under exchange of the labels 1 and 2, as is to be expected by inspection of Eq. (7.7).

When representations $\widehat{U}_{\wedge}(g)$ and $\widehat{U}_{\wedge}(h)$ of two elements $g,h\in G_f$ are composed, we obtain from definition (7.25) (Appendix D.3)

$$\widehat{U}_{\wedge}(g)\,\widehat{U}_{\wedge}(h)=\,(-\mathrm{i})^{\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\delta_{h,p}}(-1)^{\mathfrak{c}(g)\left[\mathfrak{c}(h)+q_2(h)+\rho_2(h)\right]}$$

$$
\begin{aligned}
&\times \widehat{V}_1(g)\,\widehat{V}_2(g)\ \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)} \\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\ \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \\
&\times \overline{\widehat{V}_1(h)}^{\wedge,g}\,\overline{\widehat{V}_2(h)}^{\wedge,g}\ \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(h)+\rho_1(h)+\rho_2(h)} \\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)}\ \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}\ \mathsf{K}_\wedge^{\mathfrak{c}(g\,h)} \\
&= e^{\mathrm{i}\phi_{\mathrm{comp}}(g,h)+\mathrm{i}\pi\chi_1(g,h)+\mathrm{i}\pi\chi_{\mathrm{conj}}(g,h)+\mathrm{i}\pi\chi_{\mathrm{ord}}(g,h)+\mathrm{i}\chi_{\mathrm{gag}}(g,h)} \\
&\times (-\mathrm{i})^{\delta_{gh,p}}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\ \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)} \\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g\,h)+\rho_1(g\,h)}\ \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g\,h)+q_2(g\,h)+\rho_2(g\,h)}\ \mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}, \\
&\equiv e^{\mathrm{i}\phi_\wedge(g,h)}\,\widehat{U}_\wedge(g\,h),
\end{aligned}
\tag{7.27a}
$$

where we have defined the phase factors

$$\phi_{\mathrm{comp}}(g,h) := \phi_1(g,h)+\phi_2(g,h)+\pi\,q_1(h)[\mathfrak{c}(g)+\rho_1(g)]+\pi\,q_2(h)[\mathfrak{c}(g)+\rho_2(g)]. \tag{7.27b}$$

$$
\begin{aligned}
\chi_1(g,h) :=\ &\mathfrak{c}(g)\Big[1+\mathfrak{c}(h)+q_1(g)+q_2(g)+\rho_2(h)\Big]+\rho_1(g)\Big[1+q_1(g)+q_1(h)+q_2(h)\Big] \\
&+\rho_2(g)\Big[1+q_1(g)+q_2(g)+q_2(h)\Big]+q_1(g)q_2(g),
\end{aligned}
\tag{7.27c}
$$

$$\chi_{\mathrm{conj}}(g,h) := q_1(h)[\rho_1(g)+q_1(g)]+q_2(h)[\mathfrak{c}(g)+\rho_2(g)+q_2(g)]. \tag{7.27d}$$

$$
\begin{aligned}
\chi_{\mathrm{ord}}(g,h) :=\ &[\mathfrak{c}(g\,h)+\rho_1(g)+\rho_2(g\,h)+q_1(h)][\mathfrak{c}(g)+q_2(g)+\rho_2(g)] \\
&+[\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)+q_2(g\,h)][q_1(g)+\rho_1(g)],
\end{aligned}
\tag{7.27e}
$$

$$\chi_{\mathrm{gag}}(g,h) := \frac{3\pi}{2}\left(\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\delta_{h,p}-\delta_{gh,p}\right). \tag{7.27f}$$

$$\phi_\wedge(g,h) :=\ \phi_{\mathrm{comp}}(g,h)+\pi\,\chi_1(g,h)+\pi\,\chi_{\mathrm{conj}}(g,h)+\pi\,\chi_{\mathrm{ord}}(g,h)+\chi_{\mathrm{gag}}(g,h). \tag{7.27g}$$

After some algebra, one can show that (Appendix D.3) the stacked 2-cochain $\phi_\wedge(g,h)$ in Eq. (7.27g) is gauge equivalent to

$$\phi_\wedge(g,h) := \phi_1(g,h)+\phi_2(g,h)+\pi\,\rho_1(g)\,\rho_2(h). \tag{7.28}$$

The construction of the indices $([(\nu_\wedge,\rho_\wedge)],[\mu_\wedge])$ in terms of the indices $([(\nu_1,\rho_1)],[\mu_1])$ and $([(\nu_2,\rho_2)],[\mu_2])$ is achieved as follows.

The 2-cochain $\nu_\wedge$ is obtained by restricting $\phi_\wedge$ to the elements of G, i.e.,

$$\nu_\wedge(g,h) := \nu_1(g,h)+\nu_2(g,h)+\pi\,(\rho_1\smile\rho_2)\,(g,h), \tag{7.29a}$$

where we introduced the cup product $\smile$ to construct a 2-cochain out of 1-cochains.

Since $[\mu_\wedge] = 0$, we identify the 1-cochain $\rho_\wedge(g)$ as the total fermion parity of the representation of element $g \in G_f$ [recall definition (6.22)]. From the definition (7.25), we thus find

$$\rho_\wedge(g) = \rho_1(g) + \rho_2(g) + \mathfrak{c}(g), \tag{7.29b}$$

where the first two terms originate from $\widehat{V}_1(g)$ and $\widehat{V}_2(g)$, the next two terms originate from $\hat{\gamma}^{(1)}_{n_1}$, and the last three terms originate from $\hat{\gamma}^{(2)}_{n_2}$.

7.4 SUMMARY OF FERMIONIC STACKING RULES

In Secs. 7.1, 7.2, and 7.3, we have explicitly constructed the stacked representation $\widehat{U}_\wedge$ given two representations $\widehat{U}_1$ and $\widehat{U}_2$ in Eqs. (7.12), (7.18), and (7.25). This was achieved by defining for any $g \in G_f$

$$\widehat{U}_\wedge(g) := \widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_1(p)\right]^{\rho_2(g)}\left[\widehat{U}_2(p)\right]^{\rho_1(g)}\mathsf{K}_\wedge^{\mathfrak{c}(g)}, \tag{7.30a}$$

if $[\mu_1] = [\mu_2] = 0$,

$$\widehat{U}_\wedge(g) := \widehat{V}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\left[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_\infty\right]^{\rho_1(g)}\mathsf{K}_\wedge^{\mathfrak{c}(g)}, \tag{7.30b}$$

if $[\mu_1] = 0$, $[\mu_2] = 1$,

$$\begin{aligned}\widehat{U}_\wedge(g) :=& (-\mathrm{i})^{\delta_{g,p}}\widehat{V}_1(g)\widehat{V}_2(g)\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\\ &\times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\mathsf{K}_\wedge^{\mathfrak{c}(g)},\end{aligned} \tag{7.30c}$$

if $[\mu_1] = [\mu_2] = 1$, and deriving Eqs. (7.15), (7.22), and (7.29) by comparing $\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h)$ to $\widehat{U}_\wedge(g\,h)$ for any pair $g, h \in G_f$.

We collect these equations into the fermionic stacking rules of one-dimensional IFT phases

$$([(\nu_1,\,\rho_1)],\,0) \wedge ([(\nu_2,\,\rho_2)],\,0) = ([(\nu_1 + \nu_2 + \pi\,(\rho_1 \smile \rho_2)\,,\,\rho_1 + \rho_2)],\,0)\,, \tag{7.31a}$$

$$([(\nu_1,\,\rho_1)],\,0) \wedge ([(\nu_2,\,\rho_2)],\,1) = ([(\nu_1 + \nu_2 + \pi\,(\rho_1 \smile \rho_2 + \rho_1 \smile \mathfrak{c})\,,\,\rho_1 + \rho_2)],\,1)\,, \tag{7.31b}$$

$$([(\nu_1,\,\rho_1)],\,1) \wedge ([(\nu_2,\,\rho_2)],\,0) = ([(\nu_1 + \nu_2 + \pi\,(\rho_1 \smile \rho_2 + \rho_2 \smile \mathfrak{c})\,,\,\rho_1 + \rho_2)],\,1)\,, \tag{7.31c}$$

$$([(\nu_1,\,\rho_1)],\,1)\wedge([(\nu_2,\,\rho_2)],\,1)=([(\nu_1+\nu_2+\pi\,\rho_1\smile\rho_2,\rho_1+\rho_2+\mathfrak{c})],\,0)\,. \tag{7.31d}$$

They correspond to the even-even, even-odd, odd-even, and odd-odd stacking, respectively. The stacking rules (7.31) agree with the ones derived in Refs. [139] and [143]. We note that the even-odd stacking rule derived in Ref. [139] contains the term $\rho_1 \smile \rho_1$ instead of the term $\rho_1 \smile \mathfrak{c}$. These two terms are gauge equivalent to each other, i.e., they differ by a 2-coboundary $\delta^1_{\mathfrak{c}}\,\xi$ with $\xi = \pi\,\rho_1 \smile \mathfrak{c} - \frac{\pi}{2}\,\rho_1 \smile \rho_1$. The presentation in Eq. (7.31) makes the role of antiunitary symmetries in the stacking rules explicit. If the group G_f consist of only unitary symmetries, i.e., $\mathfrak{c}(g) = 0$ for any $g \in G_f$, the stacking rules (7.31) reduce to

$$([(\nu_1,\,\rho_1)],\,[\mu_1])\wedge([(\nu_2,\,\rho_2)],\,[\mu_2])=([(\nu_1+\nu_2+\pi\,(\rho_1\smile\rho_2)\,,\,\rho_1+\rho_2)],\,[\mu_1]+[\mu_2])\,. \tag{7.32}$$

The stacking rules (7.31) dictate the group structure of IFT phases that are symmetric under the group G_f. This group structure encodes the physical operation by which two open chains realizing IFT phases that are symmetric under group G_f are brought adiabatically into contact so as to realize an IFT phases that is symmetric under group G_f. The stacking rules (7.31a) and (7.31d) each encodes how the left and right boundaries of an open chain realizing an IFT phase that is symmetric under the group G_f are glued back together in such a way that the resulting chain obeying periodic boundary conditions supports a nondegenerate gapped ground state.

We note that the stacking rules (7.31b), (7.31c), and, (7.31d) are only defined when the group G_f splits, since $[\mu] = 1$ is only then possible. Furthermore, these stacking rules are all apply to the case of odd number of Majorana degrees of freedom at the boundaries, which is only possible if the underlying degrees of freedom are fermionic. In this sense, the stacking rules (7.31b), (7.31c), and, (7.31d) are intrinsically fermionic. Similarly, the term $\rho_1 \smile \rho_2$ in stacking rule (7.31a) is present due to the fermionic nature of the underlying degrees of freedom. Had we consider only bosonic degrees of freedom, all operators from the set $\mathfrak{O}_1$ of boundary degrees of freedom would commute with those from the set $\mathfrak{O}_2$. For strictly bosonic systems the stacking rule (7.31a) reduces to

$$[(\nu_1,\,\rho_1)]\wedge[(\nu_2,\,\rho_2)]=[(\nu_1+\nu_2,\,\rho_1+\rho_2)], \tag{7.33}$$

which is the bosonic stacking rule. Hereby, the symmetry imposed on the bosonic system is $\mathcal{G} \equiv G_f$ that is obtained from replacing the femrion parity symmetry $\mathbb{Z}_2^{\mathrm{F}}$ by an ordinary $\mathbb{Z}_2$ symmetry. For a bosonic system, this $\mathbb{Z}_2$ symmetry is not special as it can be explicitly or spontaneously broken. In terms of the 2-cochains ϕ_1 and ϕ_2, the bosonic stacking rule (7.33) is

$$[\phi_\wedge] = [\phi_1 + \phi_2], \tag{7.34}$$

i.e., bosonic invertible phases follows the group structure of $H^2\left(\mathcal{G}, \mathrm{U}(1)_\mathfrak{c}\right)$.

8

PROTECTED GROUND-STATE DEGENERACIES OF 1D IFT PHASES

In Sec. 6.2, we have shown that the distinct IFT phases are characterized by the projective character of the boundary representation $\widehat{U}_{\mathrm{B}}$. In turn this projective character is captured by the triplet of indices $([(\nu,\rho)],[\mu])$. Let us now consider the implications of this triplet being nontrivial, i.e., $([(\nu,\rho)],[\mu]) \neq ([(0,0)],[0])$, for the spectral degeneracy of the boundary states.

The foremost consequence of the nontrivial indices $([(\nu,\rho)],[\mu])$ is the robustness of the boundary degeneracy that is protected by a combination of the symmetry group G_f being represented projectively and the existence of a nonlocal boundary Fock space, denoted $\mathfrak{F}_{\mathrm{LR}}$ in Eq. (6.15), whenever opposite boundaries host odd numbers of Majorana degrees of freedom.

A robust quantum mechanical supersymmetry [162] was shown in Refs. [167–169] to be generically present in nontrivial IFT phases. We are going to recast these results by showing how the quantum mechanical supersymmetry present at the boundaries can be deduced from the indices $([(\nu,\rho)],[\mu])$.

In what follows, we consider the two cases $[\mu]=0$ and $[\mu]=1$ separately. For each case, we first discuss the degeneracies associated with nontrivial pair $[(\nu,\rho)]$ on general grounds.

8.1 THE CASE OF $[\mu]=0$

When $[\mu]=0$, there always are even numbers of Majorana degrees of freedom localized on each disconnected component Λ_{L} and Λ_{R} of the boundary Λ_{bd} [recall definition (6.10b)]. In this case, the boundary Fock space $\mathfrak{F}_{\Lambda_{\mathrm{bd}}}$ spanned by the Majorana degrees of freedom supported on Λ_{bd} decomposes as

$$\mathfrak{F}_{\Lambda_{\mathrm{bd}}} = \mathfrak{F}_{\Lambda_{\mathrm{L}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\mathrm{R}}}, \tag{8.1}$$

where $\otimes_{\mathfrak{g}}$ denotes a $\mathbb{Z}_2$ graded tensor product, while $\mathfrak{F}_{\Lambda_{\mathrm{L}}}$ and $\mathfrak{F}_{\Lambda_{\mathrm{R}}}$ are the Fock spaces spanned by the Majorana degrees of freedom localized at the disconnected components

Λ_{L} and Λ_{R}, respectively. The Fock spaces $\mathfrak{F}_{\Lambda_{\mathrm{L}}}$ and $\mathfrak{F}_{\Lambda_{\mathrm{R}}}$ are defined by Eq. (6.12a). We denote with $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$ the Hamiltonians that act on Fock spaces $\mathfrak{F}_{\Lambda_{\mathrm{L}}}$ and $\mathfrak{F}_{\Lambda_{\mathrm{R}}}$ and govern the dynamics of the local Majorana degrees of freedom localized at boundaries Λ_{L} and Λ_{R}, respectively. By assumption, the Hamiltonians $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$ are invariant under the representations (possibly projective) $\widehat{U}_{\mathrm{L}}$ and $\widehat{U}_{\mathrm{R}}$ of the given symmetry group G_f, respectively.

Since $[\mu]=0$, the only nontrivial IFT phases are those with nontrivial equivalence classes $[(\nu,\rho)]\neq[(0,0)]$, i.e., the FSPT phases. By definition, the indices $([\nu_{\mathrm{L}},\rho_{\mathrm{L}}],0)$ and $([\nu_{\mathrm{R}},\rho_{\mathrm{R}}],0)$ associated with the representations (possibly projective) $\widehat{U}_{\mathrm{L}}$ and $\widehat{U}_{\mathrm{R}}$, respectively, satisfy

$$([(\nu_{\mathrm{L}},\rho_{\mathrm{L}})],0)\wedge([(\nu_{\mathrm{R}},\rho_{\mathrm{R}})],0)=([(0,0)],0) \tag{8.2}$$

under the stacking rule (7.31a).

If we focus on a single boundary (denoted by B), the equivalence class $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]$ characterizes the nontrivial projective nature of the boundary representation $\widehat{U}_{\mathrm{B}}$. Whenever $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]\neq[(0,0)]$, it is guaranteed that there is no state that is invariant under the action of $\widehat{U}_{\mathrm{B}}(g)$ for all $g\in G_f$. In other words, there is no state in the Fock space $\mathfrak{F}_{\Lambda_{\mathrm{B}}}$ that transforms as a singlet under the representation $\widehat{U}_{\mathrm{B}}$. Any eigenenergy of a G_f-symmetric boundary Hamiltonian $\widehat{H}_{\mathrm{B}}$ must be degenerate. The degeneracy is protected by the particular representation $\widehat{U}_{\mathrm{B}}$ of the symmetry group G_f and cannot be lifted without breaking the G_f symmetry. The minimal degeneracy that is protected by the G_f symmetry depends on the explicit structure of the group G_f and the equivalence class $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]$ of the boundary representation $\widehat{U}_{\mathrm{B}}$.

Since for $[\mu]=0$, the boundary representations $\widehat{U}_{\mathrm{L}}$ and $\widehat{U}_{\mathrm{R}}$ act on two independent Fock spaces $\mathfrak{F}_{\Lambda_{\mathrm{L}}}$ and $\mathfrak{F}_{\Lambda_{\mathrm{R}}}$, the total protected ground-state degeneracy $\mathrm{GSD}^{[\mu]=0}_{\mathrm{bd}}$ when open boundary conditions are imposed is nothing but the product of the protected ground-state degeneracies $\mathrm{GSD}^{[\mu]=0}_{\mathrm{L}}$ and $\mathrm{GSD}^{[\mu]=0}_{\mathrm{R}}$ of the Hamiltonians $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$, respectively, i.e.,

$$\mathrm{GSD}^{[\mu]=0}_{\mathrm{bd}}=\mathrm{GSD}^{[\mu]=0}_{\mathrm{L}}\times\mathrm{GSD}^{[\mu]=0}_{\mathrm{R}}. \tag{8.3}$$

When $[\mu]=0$, the 1-cochain $\rho_{\mathrm{B}}(g)=0,1$ encodes the commutation relation between the representations $\widehat{U}_{\mathrm{B}}(g)$ of group element $g\in G_f$ and $\widehat{U}_{\mathrm{B}}(p)$ of fermion parity $p\in G_f$. A nonzero second entry in the equivalence class $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]$ implies that there exists at least one group element $g\in G_f$ with $\rho_{\mathrm{B}}(g)=1$, i.e., the operator $\widehat{U}_{\mathrm{B}}(g)$ is of odd fermion parity.

If this is so, the boundary Hamiltonian $\widehat{H}_{\mathrm{B}}$ must possess an emergent quantum mechanical supersymmetry. The supercharges associated with the boundary supersymmetry are constructed following Ref. [169]. Assume without loss of generality that all energy eigenvalues ε_α of a boundary Hamiltonian $\widehat{H}_{\mathrm{B}}$ are shifted to the positive energies, i.e., $\varepsilon_\alpha > 0$. Also assume that there exists a group element $g \in G_f$ with $\rho_{\mathrm{B}}(g)=1$. For any orthonormal eigenstate $|\psi_\alpha\rangle$ of $\widehat{H}_{\mathrm{B}}$ with energy ε_α, the state

$$|\psi^{\prime}_\alpha\rangle := \widehat{U}_{\mathrm{B}}(g)\,|\psi_\alpha\rangle, \tag{8.4a}$$

is also an orthonormal eigenstate of $\widehat{H}_{\mathrm{B}}$ with the same energy but opposite fermion parity. Since the fermion parities of $|\psi^{\prime}_\alpha\rangle$ and $|\psi_\alpha\rangle$ are different, they are orthogonal. Two supercharges can then be defined as

$$\widehat{Q}_1 := \sum_{\alpha_+} \sqrt{\varepsilon_{\alpha_+}}\left[\left(\widehat{U}_{\mathrm{B}}(g)\,|\psi_{\alpha_+}\rangle\right)\langle\psi_{\alpha_+}| + |\psi_{\alpha_+}\rangle\left(\langle\psi_{\alpha_+}|\,\widehat{U}^{\dagger}_{\mathrm{B}}(g)\right)\right], \tag{8.4b}$$

$$\widehat{Q}_2 := \sum_{\alpha_+} \mathrm{i}\sqrt{\varepsilon_{\alpha_+}}\left[\left(\widehat{U}_{\mathrm{B}}(g)\,|\psi_{\alpha_+}\rangle\right)\langle\psi_{\alpha_+}| - |\psi_{\alpha_+}\rangle\left(\langle\psi_{\alpha_+}|\,\widehat{U}^{\dagger}_{\mathrm{B}}(g)\right)\right], \tag{8.4c}$$

where the summation index α_+ runs over the even fermion parity sector, i.e.,

$$\widehat{U}_{\mathrm{B}}(p)|\psi_{\alpha_+}\rangle = +|\psi_{\alpha_+}\rangle \tag{8.4d}$$

for any α_+. Operators $\widehat{Q}_1$ and $\widehat{Q}_2$ are Hermitian, carry odd fermion parity, and satisfy the defining properties

$$\left\{\widehat{Q}_i, \widehat{Q}_j\right\} = 2\widehat{H}_{\mathrm{B}}\,\delta_{i,j}, \qquad \left[\widehat{Q}_i, \widehat{H}_{\mathrm{B}}\right] = 0, \qquad i,j=1,2, \tag{8.4e}$$

of fermionic supercharges. The precise number of supercharges on the boundary Λ_{B} depends on the pair $[(\nu_{\mathrm{B}}, \rho_{\mathrm{B}})]$ that characterizes the number of symmetry operators $\widehat{U}_{\mathrm{B}}(g)$ that carry odd fermion parity and their mutual algebra.

8.2 THE CASE OF $[\mu]=1$

When $[\mu]=1$, there are odd number of Majorana degrees of freedom localized on each disconnected component Λ_{L} and Λ_{R} of the boundary Λ_{bd} [recall definition (6.10b)].

In this case, the boundary Fock space $\mathfrak{F}_{\Lambda_{\mathrm{bd}}}$ spanned by Majorana degrees of freedom supported on Λ_{bd} decomposes as

$$\mathfrak{F}_{\Lambda_{\mathrm{bd}}} = \mathfrak{F}_{\Lambda_{\mathrm{L}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\mathrm{LR}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\mathrm{R}}}, \tag{8.5}$$

where $\otimes_{\mathfrak{g}}$ denotes a $\mathbb{Z}_2$ graded tensor product. The Fock spaces $\mathfrak{F}_{\Lambda_{\mathrm{B}}}$ with $\mathrm{B} = \mathrm{L}, \mathrm{R}$ is spanned by all the Majorana operators localized at the disconnected components Λ_{B} except one. The two-dimensional Fock space $\mathfrak{F}_{\Lambda_{\mathrm{LR}}}$ is spanned by the two remaining Majorana operators with one localized on the left boundary Λ_{L} and the other localized on the right boundary Λ_{R} of the open chain. Correspondingly, the pair of fermionic creation and annihilation operators that span $\mathfrak{F}_{\Lambda_{\mathrm{LR}}}$ are nonlocal in the sense that they are formed by Majorana operators supported on opposite boundaries. One can define Hamiltonians $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$ that are constructed out of Majorana operators localized at the boundaries Λ_{L} and Λ_{R}. If so, the Hamiltonians $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$ act on Fock spaces

$$\mathfrak{F}_{\Lambda_{\mathrm{L}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\mathrm{LR}}}, \tag{8.6a}$$

and

$$\mathfrak{F}_{\Lambda_{\mathrm{R}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\mathrm{LR}}}, \tag{8.6b}$$

respectively. By assumption, the Hamiltonians $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$ are invariant under the representations $\widehat{U}_{\mathrm{L}}$ and $\widehat{U}_{\mathrm{R}}$ of a given symmetry group G_f, respectively.

On each boundary Λ_{B}, there exists a local Hermitean and unitary operator $\widehat{Y}_{\mathrm{B}}$ that commutes with any other local operator supported on Λ_{B}. The operator $\widehat{Y}_{\mathrm{B}}$ is defined by Eq. (6.25) and is the representation of the nontrivial central element of a Clifford algebra $C\ell_n$ with n an odd number of generators. It therefore carries an odd fermion parity and anticommutes with the representation $\widehat{U}_{\mathrm{B}}(p)$ of fermion parity. It follows that $\widehat{Y}_{\mathrm{B}}$ must commute with $\widehat{H}_{\mathrm{B}}$. We label the simultaneous eigenstates of $\widehat{H}_{\mathrm{B}}$ and $\widehat{Y}_{\mathrm{B}}$ by $|\psi_{\mathrm{B},\alpha,\pm}\rangle$, i.e.,

$$\widehat{Y}_{\mathrm{B}}\,|\psi_{\mathrm{B},\alpha,\pm}\rangle = \pm|\psi_{\mathrm{B},\alpha,\pm}\rangle, \quad \widehat{H}_{\mathrm{B}}\,|\psi_{\mathrm{B},\alpha,\pm}\rangle = \varepsilon_\alpha\,|\psi_{\mathrm{B},\alpha,\pm}\rangle, \tag{8.7}$$

where ε_α is the corresponding energy eigenvalue which we assume without loss of generality to be strictly positive. Hence, all eigenstates of $\widehat{H}_{\mathrm{B}}$ are at least twofold degenerate. Since $\widehat{Y}_{\mathrm{B}}$ carries odd fermion parity, the eigenstates $|\psi_{\mathrm{B},\alpha,\pm}\rangle$ do not have definite fermion parities. The simultaneous eigenstates of $\widehat{H}_{\mathrm{B}}$ and $\widehat{U}_{\mathrm{B}}(p)$ must be the bonding and anti-bonding linear combinations of $|\psi_{\mathrm{B},\alpha,+}\rangle$ and $|\psi_{\mathrm{B},\alpha,-}\rangle$ that are exchanged under the action of $\widehat{Y}_{\mathrm{B}}$. The twofold degeneracy of $\widehat{H}_{\mathrm{B}}$ when $[\mu] = 1$ is due to the presence of the two-dimensional Fock space $\mathfrak{F}_{\mathrm{LR}}$. This twofold degeneracy is of supersymmetric nature and the associated supercharges are

$$\widehat{Q}_1 := \sum_{\alpha} \sqrt{\varepsilon_\alpha}\, \Big(|\psi_{\alpha,+}\rangle\langle\psi_{\alpha,+}| - |\psi_{\alpha,-}\rangle\langle\psi_{\alpha,-}| \Big)\,, \tag{8.8a}$$

$$\widehat{Q}_2 := \sum_{\alpha} \mathrm{i}\sqrt{\varepsilon_\alpha}\, \Big(|\psi_{\alpha,+}\rangle\langle\psi_{\alpha,-}| - |\psi_{\alpha,-}\rangle\langle\psi_{\alpha,+}| \Big)\,. \tag{8.8b}$$

Operators $\widehat{Q}_1$ and $\widehat{Q}_2$ are Hermitian. They carry odd fermion parity since the operator $\widehat{U}_{\mathrm{B}}(p)$ exchanges the states $|\psi_{\alpha,\pm}\rangle$ with $|\psi_{\alpha,\mp}\rangle$. They satisfy the defining properties

$$\Big\{\widehat{Q}_i, \widehat{Q}_j\Big\} = 2\widehat{H}_{\mathrm{B}}\,\delta_{i,j}, \qquad \Big[\widehat{Q}_i, \widehat{H}_{\mathrm{B}}\Big] = 0, \qquad i,j = 1,2, \tag{8.8c}$$

of fermionic supercharges.

There may be other supercharges in addition to the ones defined in Eq. (8.8) due to the representation $\widehat{U}_{\mathrm{B}}$ of the group G_f. The precise number of these additional supercharges on the boundary Λ_{B} depends on the pair $[(\nu_{\mathrm{B}}, \rho_{\mathrm{B}})]$ that characterizes the number of symmetry operators $\widehat{U}_{\mathrm{B}}(g)$ that carry odd fermion parity and their mutual algebra. They can be constructed in the same fashion as in Eq. (8.4).

By definition, the indices $([(\nu_{\mathrm{L}}, \rho_{\mathrm{L}})], 1)$ and $([(\nu_{\mathrm{R}}, \rho_{\mathrm{R}})], 1)$ associated to the representations $\widehat{U}_{\mathrm{L}}$ and $\widehat{U}_{\mathrm{R}}$, respectively, satisfy

$$([(\nu_{\mathrm{L}}, \rho_{\mathrm{L}})], 1) \wedge ([(\nu_{\mathrm{R}}, \rho_{\mathrm{R}})], 1) = ([(0,0)], 0) \tag{8.9}$$

under the stacking rule (7.31d). If we focus on a single boundary (denoted by B), the equivalence class $[(\nu_{\mathrm{B}}, \rho_{\mathrm{B}})]$ characterizes the nontrivial projective nature of the boundary representation $\widehat{U}_{\mathrm{B}}$. Whenever $[(\nu_{\mathrm{B}}, \rho_{\mathrm{B}})] \neq [(0,0)]$, it is guaranteed that there is no state that is invariant under the action of $\widehat{U}_{\mathrm{B}}(g)$ for all $g \in G_f$. In other words, there is no state in the Fock space $\mathfrak{F}_{\Lambda_{\mathrm{B}}}$ that transforms as a singlet under the representation $\widehat{U}_{\mathrm{B}}$. Each eigenstate of a symmetric boundary Hamiltonian $\widehat{H}_{\mathrm{B}}$ must carry degeneracies in addition to the twofold degeneracy due to $[\mu] = 1$. The degeneracy is protected by the particular representation $\widehat{U}_{\mathrm{B}}$ of the symmetry group G_f and cannot be lifted without breaking the G_f symmetry. The exact degeneracy protected by the representation depends on the explicit form of the group G_f, and the boundary representation $\widehat{U}_{\mathrm{B}}$ with the equivalence class $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]$.

Since for $[\mu]=1$, the boundary representations $\widehat{U}_{\mathrm{L}}$ and $\widehat{U}_{\mathrm{R}}$ do not act on two decoupled Fock spaces. The total protected ground-state degeneracy $\mathrm{GSD}^{[\mu]=1}_{\mathrm{bd}}$ when open boundary conditions are imposed cannot be computed by taking the products of degeneracies associated with the Hamiltonians $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$ separately. However, $\mathrm{GSD}^{[\mu]=1}_{\mathrm{bd}}$ can be computed by multiplying the "naive" protected ground state degeneracies of the Hamiltonians at the two boundaries and modding out the twofold degeneracy due to $\mathfrak{F}_{\mathrm{LR}}$ shared by the two Hamiltonians, i.e.,

$$\mathrm{GSD}^{[\mu]=1}_{\mathrm{bd}}=\frac{1}{2}\times\mathrm{GSD}^{[\mu]=1}_{\mathrm{L}}\times\mathrm{GSD}^{[\mu]=1}_{\mathrm{R}}, \tag{8.10}$$

where $\mathrm{GSD}^{[\mu]=1}_{\mathrm{L}}$ and $\mathrm{GSD}^{[\mu]=1}_{\mathrm{R}}$ are the protected ground state degeneracies of $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$, respectively.

9

CONSTRUCTION OF REPRESENTATIVE HAMILTONIANS

The argument presented in Sec. 6.1 for the enumeration of one-dimensional IFT phases is based on the assumption that one can find a translationally invariant representative Hamiltonian with internal G_f symmetry. In this Chapter, we will demonstrate how this can be achieved using the stacked representations (7.30). We will construct a translationally invariant local Hamiltonian that is sum of commuting projectors. This idea is closely related to the so-called *parent Hamiltonians* in matrix product states (MPS) formalism. This idea has been explored extensively in the literature. See Refs. [31, 57, 69, 127, 128] for early examples of parent Hamiltonians of bosonic SPT phases and Refs. [135, 136] for those of IFT phases.

This Chapter mainly serves as a complementary to the earlier chapters and completes our treatment of IFT phases in one dimension. On our way to the construction of representative Hamiltonians, we will also clarify how global representations of symmetry transformations can be built out of local projective representations.

9.1 FROM LOCAL TO GLOBAL REPRESENTATIONS

Invertible fermionic topological phases of matter in one-dimensional space have an internal symmetry group G_f that is represented in the bulk by the faithful representation $\widehat{U}_{\text{bulk}}$ given in Eq. (6.1). Because these symmetries are internal, they induce for any site j of any one-dimensional lattice Λ a faithful representation $\widehat{U}_j$. However, representatives of IFT phases can also accomodate projective representations of the internal symmetry group G_f on the left and right boundaries of Λ provided the stacking of these two boundary representations is gauge-equivalent to a faithful representation of G_f, as is captured by Fig. 6.1.

When treating the translationally invariant Hamiltonians, it is convenient to first list the transformation rules on each repeat unit cell instead of a global bulk representation $\widehat{U}_{\text{bulk}}$. Naively, one then take the composition of local representations to obtain the global representation. However, when each repeat unit cell realizes a projective representation, a mere composition does not work as exemplified by the stacked representations (7.30).

We consider the more general case where there is no translation symmetry, i.e., the case where to each lattice site we can attach a different fermionic Fock space. We are going to construct a bulk representation $\widehat{U}_{\text{bulk}}$ of the symmetry group G_f out of a given set of projective representations $\hat{u}_{\boldsymbol{j}}$ acting on the Clifford algebra

$$\mathrm{C}\ell_{n_{\boldsymbol{j}}} := \mathrm{span}\left\{\hat{\gamma}^{(\boldsymbol{j})}_1, \hat{\gamma}^{(\boldsymbol{j})}_2, \cdots, \hat{\gamma}^{(\boldsymbol{j})}_{n_{\boldsymbol{j}}}\right\} \tag{9.1}$$

spanned by $n_{\boldsymbol{j}}$ Majorana degrees of freedom for any site $\boldsymbol{j}$ from a d-dimensional lattice Λ provided

$$\sum_{\boldsymbol{j}\in\Lambda} n_{\boldsymbol{j}} = 0 \bmod 2. \tag{9.2}$$

We note that stacking three representations $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$ using the definition (7.30) is associative, i.e., it is independent of which two of the three representations are first stacked. This associativity follows from the consistency condition (7.4). This is because, for a Clifford algebra $\mathrm{C}\ell_{2n}$ with an even number of generators, specifying the transformation rules on its generators together with the action of the complex conjugation uniquely (up to a phase factor) determines the representation $\widehat{U}(g)$ of any element $g \in G_f$. For a Clifford algebra $\mathrm{C}\ell_{2n+1}$ with an odd number of generators, this is no longer true since $\mathrm{C}\ell_{2n+1}$ has a two-dimensional center spanned by $\hat{\mathbb{1}}$ and $\widehat{Y}$. We removed the ambiguity consisting in multiplying $\widehat{U}(g)$ by the central element $\widehat{Y}$ by demanding that $\widehat{U}(g)$ is of even fermion parity. Hence, for any $g \in G_f$ and any labeling $\boldsymbol{j}_1, \boldsymbol{j}_2, \cdots, \boldsymbol{j}_{|\Lambda|}$ with $|\Lambda|$ the cardinality of Λ, we can define $\widehat{U}_{\text{bulk}}(g)$ by stacking $\widehat{U}_{\boldsymbol{j}_1}(g)$ with $\widehat{U}_{\boldsymbol{j}_2}(g)$, which we then stack with $\widehat{U}_{\boldsymbol{j}_3}(g)$, and so on. By construction, it follows that

$$\widehat{U}_{\boldsymbol{j}}(g)\,\hat{\gamma}^{(\boldsymbol{j})}_{\iota}\,\widehat{U}^{\dagger}_{\boldsymbol{j}}(g) = \widehat{U}_{\text{bulk}}(g)\,\hat{\gamma}^{(\boldsymbol{j})}_{\iota}\,\widehat{U}^{\dagger}_{\text{bulk}}(g), \tag{9.3}$$

for any $\iota = 1, \cdots, n_{\boldsymbol{j}}$, $\boldsymbol{j} \in \Lambda$, and $g \in G_f$. Equation (9.3) is the counterpart to the consistency condition (6.17) that we used to construct boundary representations. It also follows that the representation

$$\widehat{U}_{\text{bulk}}(g) = \left[\prod_{\boldsymbol{j}\in\Lambda} \hat{v}_{\boldsymbol{j}}(g)\right] \mathsf{K}^{\mathfrak{c}(g)}, \tag{9.4}$$

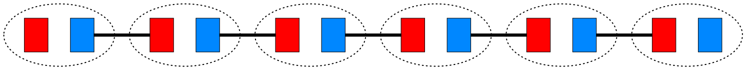

Figure 9.1: Construction of an IFT phase out of inverse local projective representations in one-dimension. Red and blue squares denote degrees of freedom on which inverse representations $\hat{u}_j$ and $\hat{\bar{u}}_j$ are realized. Dashed ellipses denote repeat unit cells, on each of which the trivial projective representation $\hat{u}_{\wedge,j}$ acts. Each solid line represents a projector $-|\psi_{j,j+1}\rangle\langle\psi_{j,j+1}|$ that couples neighboring sites, where $|\psi_{j,j+1}\rangle$ is a state that transforms as a singlet under the stacked representation $\hat{w}_{j,j+1}$ that is obtained by stacking $\hat{\bar{u}}_j$ and $\hat{u}_{j+1}$. When open boundary conditions are imposed, the left and right boundaries support projective representations of the group G_f with indices $([(\nu,\rho)],[\mu])$ and $([(\bar{\nu},\bar{\rho})],[\bar{\mu}])$, respectively.

for any $g \in G_f$ holds if and only if the local representation $\widehat{U}_{\boldsymbol{j}}(g)$ has the indices $\rho_{\boldsymbol{j}}(g) = 0$ and $[\mu_{\boldsymbol{j}}] = 0$ for any $\boldsymbol{j} \in \Lambda$. This implies that the local representation $\widehat{U}_{\boldsymbol{j}}(g)$ of any element $g \in G_f$ is of even fermion parity and the number of Majorana degrees of freedom $n_{\boldsymbol{j}}$ is an even integer for any site $\boldsymbol{j} \in \Lambda$. It is then appropriate to call a local representation $\widehat{U}_{\boldsymbol{j}}$ that has nontrivial indices $\rho_{\boldsymbol{j}}$ and $[\mu_{\boldsymbol{j}}]$ an *intrinsically fermionic* representation. In other words, the decomposition (9.4) is possible if and only if the local representation $\widehat{U}_{\boldsymbol{j}}$ for any site $\boldsymbol{j} \in \Lambda$ is not intrinsically fermionic. In particular, if all local degrees of freedom are bosonic, then the decomposition (9.4) is always valid. However, instead of the decomposition (9.4), $\widehat{U}_{\text{bulk}}(g)$ is obtained in all generality by iterating Eq. (7.30) for any $g \in G_f$.

9.2 COMMUTING PROJECTOR HAMILTONIANS

Given a fermionic symmetry group G_f, we will construct a commuting projector Hamiltonian that realizes an IFT phase with indices $([(\nu,\rho)],[\mu])$. The key observation is the following. Let $\hat{u}_j$ be a nontrivial projective representation acting on a local Fock space $\mathfrak{F}_j$[1], with indices $([(\nu,\rho)],[\mu]) \neq ([(0,0)],0)$. By definition, there is no state in the local Fock space $\mathfrak{F}_j$ that transforms as a singlet under $\hat{u}_j$. Since indices $([(\nu,\rho)],[\mu])$ form an

1 In general, the Fock space $\mathfrak{F}_j$ may be only a subspace of some fermionic Fock space. For instance, the symmetry group $G_f = \mathrm{U}(1)^{\mathrm{F}} \rtimes \mathbb{Z}_2^{\mathrm{T}}$ does not admit a nontrivial projective representation on a four-dimensional fermionic Fock space while it does when restricted to its two-dimensional odd fermion-parity subspace.

Abelian group under stacking, there exists another representation $\hat{\bar{u}}_j$ on a local Fock space $\overline{\mathfrak{F}}_j$ with indices $([(\bar{\nu},\bar{\rho})],[\bar{\mu}])$ such that

$$([(\nu,\rho)],[\mu]) \wedge ([(\bar{\nu},\bar{\rho})],[\bar{\mu}]) = ([(0,0)],0). \tag{9.5}$$

This means that on the stacked local Fock space $\mathfrak{F}_j \otimes_{\mathfrak{g}} \overline{\mathfrak{F}}_j$ [2], there exists a state $|\psi_j\rangle$ that transforms as a singlet under the representation $\hat{u}_{\wedge,j}$ obtained by stacking $\hat{u}_j$ and $\hat{\bar{u}}_j$. The identity

$$\hat{u}_{\wedge,j}(g)|\psi_j\rangle = e^{\mathrm{i}\theta(g)}\,|\psi_j\rangle, \qquad |\psi_j\rangle \in \mathfrak{F}_j \otimes_{\mathfrak{g}} \overline{\mathfrak{F}}_j \tag{9.6}$$

where $\theta(g) \in [0,2\pi)$, holds for any $g \in G_g$. We can choose two sets of basis states that span the Fock spaces $\mathfrak{F}_j$ and $\overline{\mathfrak{F}}_j$, i.e.,

$$\mathfrak{F}_j = \operatorname{span}\left\{|\varphi_{j,1}\rangle,\, |\varphi_{j,2}\rangle,\, \cdots,|\varphi_{j,n}\rangle,\,\right\}, \qquad \dim\left(\mathfrak{F}_j\right) = n, \tag{9.7a}$$

$$\overline{\mathfrak{F}}_j = \operatorname{span}\left\{|\bar{\varphi}_{j,1}\rangle,\, |\bar{\varphi}_{j,2}\rangle,\, \cdots,|\bar{\varphi}_{j,\bar{n}}\rangle,\,\right\}, \qquad \dim\left(\overline{\mathfrak{F}}_j\right) = \bar{n}. \tag{9.7b}$$

The state $|\psi_j\rangle$ then has the expansion

$$|\psi_j\rangle := \sum_{\alpha,\bar{\alpha}} c_{\alpha,\bar{\alpha}}\,|\varphi_{j,\alpha}\rangle \otimes_{\mathfrak{g}} |\bar{\varphi}_{j,\bar{\alpha}}\rangle. \tag{9.7c}$$

Now, we consider a one-dimensional lattice Λ such that the stacked Fock space $\mathfrak{F}_j \otimes_{\mathfrak{g}} \overline{\mathfrak{F}}_j$ resides at each site $j \in \Lambda$. The global representation $\widehat{U}_{\mathrm{bulk}}$ of the group G_f is obtained by stacking the trivial representations $\hat{u}_{\wedge,j}$ according to Eq. (9.4). Since stacking of representations is associative, the global representation $\widehat{U}_{\mathrm{bulk}}$ can be equivalently constructed by stacking the local representations $\hat{w}_{j,j+1}$ that are obtained by stacking $\hat{\bar{u}}_j$ and $\hat{u}_{j+1}$. Furthermore, we can define a state that transforms as a singlet under $\hat{w}_{j,j+1}$

$$|\psi_{j,j+1}\rangle := \sum_{\alpha,\bar{\alpha}} c_{\alpha,\bar{\alpha}}\,|\bar{\varphi}_{j,\bar{\alpha}}\rangle \otimes_{\mathfrak{g}} |\varphi_{j+1,\alpha}\rangle, \qquad \hat{w}_{j,j+1}(g)|\psi_{j,j+1}\rangle = e^{\mathrm{i}\theta(g)}\,|\psi_{j,j+1}\rangle, \tag{9.8}$$

2 Here, $\otimes_{\mathfrak{g}}$ denotes a (potential) $\mathbb{Z}_2$-graded tensor product if the corresponding Fock spaces are fermionic, see Appendix B.

for any $g \in G_f$. The Hamiltonian

$$\widehat{H}_{([(\nu,\rho)],[\mu])} := -\sum_{j\in\Lambda} |\psi_{j,j+1}\rangle\langle\psi_{j,j+1}|, \tag{9.9}$$

is a sum of commuting projectors each of which stabilizes the state $|\psi_{j,j+1}\rangle$ on neighboring sites. When periodic boundary conditions are imposed it realizes a nondegenerate, gapped, translationally invariant, and, G_f-symmetric ground state. When open boundary conditions are imposed, the projector $|\psi_{|\Lambda|,1}\rangle\langle\psi_{|\Lambda|,1}|$ is absent in the Hamiltonian. The boundary degrees of freedom that would be otherwise pinned by this projector form the zero-energy modes that are localized to the left and right boundaries. The total ground-state degeneracy with open boundary conditions is

$$\mathrm{GSD} = \dim\left(\mathfrak{F}_1\right) \times \dim\left(\overline{\mathfrak{F}}_{|\Lambda|}\right) = n\,\bar{n}. \tag{9.10}$$

The total ground-state degeneracy is due to the integrablity of the Hamiltonian (9.9) and can be larger than the minimum protected ground-state degeneracy. The left- and right-boundary degrees of freedom realizes projective representations $\hat{u}_1$ and $\hat{\bar{u}}_{|\Lambda|}$ of the group G_f with nontrivial indices $([(\nu,\rho)],[\mu])$ and $([(\bar{\nu},\bar{\rho})],[\bar{\mu}])$, respectively. Fig. 9.1 sketches the construction of Hamiltonian (9.9).

In Chapter 6.1, we enumerated one-dimensional IFT phases by assuming that for each phase there exists a translationally invariant representative Hamiltonian. By constructing the Hamiltonian (9.9), therefore, we ensure this classification scheme is at least self-consistent.

10

APPLICATIONS TO TIME-REVERSAL INVARIANT MAJORANA AND SPIN CLUSTER MODELS

In this Chapter, we present a concrete application of the toolkit developed in Chapters 6.1, 6.2, and, 7. We will consider two well-known and closely related examples: one-dimensional invertible fermionic topological (IFT) phases with symmetry group $G_f = \mathbb{Z}^T \times \mathbb{Z}_2^F$ (class BDI) and one-dimensional bosonic symmetry protected topological (BSPT) phases with symmetry group $G = \mathbb{Z}_2^T \times \mathbb{Z}_2$.

We start with a short review in Sec. 10.1 of the second cohomology group of $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ and the associated fermionic and bosonic stacking rules, see Appendix A.5.1 for the details.

In Sec. 10.2, we consider the time-reversal symmetric Majorana c chains introduced in Ref. [18], that realize the representatives of the IFT phases in the symmetry class BDI. The label $c \in \mathbb{Z}$ counts the zero modes at the boundaries of noninteracting Bogoliubov-de-Gennes one-dimensional superconductors in the symmetry class BDI. We compute the indices associated with the left and the right boundaries of each Majorana chain and demonstrate that these indices form the cyclic group $\mathbb{Z}_8$ under the stacking rules (7.31). We compute the protected ground-state degeneracy of each representative when open boundary conditions are imposed.

In Sec. 10.3, we consider the quantum spin-1/2 cluster c chains introduced in Ref. [170], that realize the representatives of the BSPT phases with symmetry group $G = \mathbb{Z}_2^T \times \mathbb{Z}_2$. The cluster chains are intimately related to the time-reversal symmetric Majorana chains as they are related by a Jordan-Wigner transformation. This relation has been investigated in Ref. [171]. For each quantum spin-1/2 cluster c chains, we compute the indices associated with the left and the right boundaries and show that they form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ under the bosonic stacking rules. We relate the indices of the BSPT phases to that of $\mathbb{Z}_4$ subgroup of the IFT phases in the symmetry class BDI, which is the group of fermionic symmetry protected topological (FSPT) phases in class BDI.

10.1 REVIEW OF SECOND GROUP COHOMOLOGY OF $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$

The split group $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ is generated by the two generators t and p corresponding to the cyclic subgroups $\mathbb{Z}_2^{\mathrm{T}} \equiv \{e, t\}$ and $\mathbb{Z}_2^{\mathrm{F}} \equiv \{e, p\}$, respectively. The second group cohomology of $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ is given by

$$H^2(\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}, \mathrm{U}(1)_{\mathfrak{c}}) = \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{10.1a}$$

We denote the equivalent classes of $H^2(\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}, \mathrm{U}(1)_{\mathfrak{c}})$ by the index $[(\nu, \rho)]$. Since $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ is a split group, i.e., a trivial extension of $\mathbb{Z}_2^{\mathrm{T}}$ by $\mathbb{Z}_2^{\mathrm{F}}$, we have the identity

$$[(\nu, \rho)] \equiv ([\nu], [\rho]) \in H^2(\mathbb{Z}_2^{\mathrm{T}}, \mathrm{U}(1)_{\mathfrak{c}}) \times H^1(\mathbb{Z}_2^{\mathrm{T}}, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{10.1b}$$

Therefore, the two indices $[\nu] = 0, 1$ and $[\rho] = 0, 1$ characterizes the projective representations of the group $\mathbb{Z}^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$, and therefore, the one-dimensional FSPT phases in class BDI. Given a representation $\widehat{U}$, the indices $[\nu]$ and $[\rho]$ are specified through the identities

$$(-1)^{[\nu]} = \widehat{U}(t)\, \widehat{U}(t), \tag{10.2a}$$

$$(-1)^{[\rho]} = \begin{cases} \widehat{U}(t)\, \widehat{U}(p)\, \widehat{U}^{\dagger}(t)\, \widehat{U}(p), & \text{if } [\mu] = 0, \\ \widehat{U}(t)\, \widehat{Y}\, \widehat{U}^{\dagger}(t)\, \widehat{Y} & \text{if } [\mu] = 1, \end{cases} \tag{10.2b}$$

where $\widehat{Y}$ is the nontrivial center of a Clifford algebra with an odd number of generators as must be the case when $[\mu] = 1$. Together with the index $[\mu] = 0, 1$ that specifies the parity of the number of boundary Majorana degrees of freedom, the triplet $([\nu], [\rho], [\mu]) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ characterizes the eight IFT phases with the symmetry group $\mathbb{Z}^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$.

Given two projective representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group $G_f = \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$, using the stacking rules (7.31), we find

$$[\nu_\wedge] = \begin{cases} [\nu_1] + [\nu_2] + [\rho_1]\,[\rho_2], & \text{if } [\mu_\wedge] \equiv [\mu_1] + [\mu_2] = 0, \\ [\nu_1] + [\nu_2] + [\rho_1]\,[\rho_2] + [\rho_1], & \text{if } [\mu_1] = 0,\ [\mu_2] = 1, \\ [\nu_1] + [\nu_2] + [\rho_1]\,[\rho_2] + [\rho_2], & \text{if } [\mu_1] = 1,\ [\mu_2] = 0, \end{cases} \tag{10.3a}$$

Table 10.1: The indices $([\nu], [\rho], [\mu])$ generated by stacking the indices $(0, 0, 1)$ $(c = 1)$ with itself form a $\mathbb{Z}_8$ under the stacking rules (10.3).

c	$([\nu], [\rho], [\mu])$
1	(0,0,1)
2	(0,1,0)
3	(1,1,1)
4	(1,0,0)
5	(1,0,1)
6	(1,1,0)
7	(0,1,1)
8	(0,0,0)

for the value of the 2-cochain $\nu_\wedge(t,t)$, and

$$[\rho_\wedge] = \begin{cases} [\rho_1] + [\rho_2] + 1, & \text{if } [\mu_1] = 1,\ [\mu_2] = 1, \\ [\rho_1] + [\rho_2], & \text{otherwise,} \end{cases} \tag{10.3b}$$

where we have used the fact that $\mathfrak{c}(t) = 1$. One thus finds that the one-dimensional IFT phases with symmetry group $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ form the cyclic group $\mathbb{Z}_8$ under the stacking rule (10.3). Without loss of generality, the generator of the group $\mathbb{Z}_8$ can be chosen as the IFT phase with indices $([\nu], [\rho], [\mu]) = (0, 0, 1)$. In Table 10.1, the triplet $([\nu], [\rho], [\mu])$ for all elements fo $\mathbb{Z}_8$ are computed using the stacking rules (10.3).

The one-dimensional BSPT phases protected by the group $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2$ are also classified by the second cohomology group (10.1) [31]. The two indices $[\nu] = 0, 1$ and $[\rho] = 0, 1$ generating $H^2(\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2, \mathrm{U}(1)_{\mathfrak{c}})$ can be measured using Eq. (10.2) if we set $[\mu] = 0$ and make the identification between the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ and the cyclic group $\mathbb{Z}_2$ protecting the BSPT phases. There are three main differences as opposed to the fermionic case.

1. First, all bosonic invertible phases in one dimension are BSPT phases. In particular, there is no analogue of the fermionic index $[\mu]$ for bosonic invertible phases.

2. Second, the cyclic subgroup $\mathbb{Z}_2$ for the BSPT phases is an ordinary $\mathbb{Z}_2$ as opposed to the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ in the sense that the former can be explicitly or spontaneously broken whereas the latter cannot.

3. Third, the stacking rules of BSPT phases follow the group composition rule of the second cohomology group (10.1), i.e., one-dimensional BSPT phases with $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2$ symmetry form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ under the stacking operation. In terms of the indices $[\nu]$ and $[\rho]$, we write

$$\big([\nu_{\otimes}],\, [\rho_{\otimes}]\big) \equiv ([\nu_1],\, [\rho_1]) \otimes ([\nu_2],\, [\rho_2]) := ([\nu_1] + [\nu_2],\, [\rho_1] + [\rho_2])\,, \tag{10.4}$$

where $\otimes$ denotes the bosonic stacking operation.

10.2 TIME-REVERSAL INVARIANT MAJORANA CHAINS

Let $c \in \mathbb{Z}$. We consider the family of Hamiltonians introduced in Ref. [18]

$$\begin{aligned} \widehat{H}_n &:= -\sum_{j=1}^{N-c} \left(\hat{c}_j^{\dagger}\, \hat{c}_{j+c} + \hat{c}_j^{\dagger}\, \hat{c}_{j+c}^{\dagger} + \text{H.c.}\right) \\ &= \sum_{j=1}^{N-c} \mathrm{i}\, \hat{\xi}_j\, \hat{\eta}_{j+c}, \end{aligned} \tag{10.5a}$$

where $\hat{c}_j$ and $\hat{c}_j^{\dagger}$ denote the annihilation and creation operators of spinless fermions and $\hat{\xi}_j$ and $\hat{\eta}_j$ denote the Majorana operators defined through the equations

$$\hat{c}_j := \frac{1}{2}\left(\hat{\eta}_j + \mathrm{i}\hat{\xi}_j\right), \qquad \hat{c}_j^{\dagger} := \frac{1}{2}\left(\hat{\eta}_j - \mathrm{i}\hat{\xi}_j\right), \qquad \hat{\eta}_j = \hat{\eta}_j^{\dagger}, \qquad \hat{\xi}_j = \hat{\xi}_j^{\dagger}, \tag{10.5b}$$

together with the algebra

$$\{\hat{c}_i,\, \hat{c}_j^{\dagger}\} = \delta_{ij} \qquad \{\hat{\eta}_i,\, \hat{\eta}_j\} = \{\hat{\xi}_i,\, \hat{\xi}_j\} = 2\delta_{ij}, \tag{10.5c}$$

with all other anticommutators vanishing. The bulk representation of the symmetry group $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}} = \{e, t\} \times \{e, p\}$ is given by

$$\widehat{U}_{\text{bulk}}(t) := \hat{\mathbb{1}}\, \mathsf{K}_{\text{bulk}}, \qquad \widehat{U}_{\text{bulk}}(p) := \prod_{j}^{N} \left(1 - 2\hat{c}_j^{\dagger}\, \hat{c}_j\right) = \prod_{j}^{N} \mathrm{i}\hat{\xi}_j \hat{\eta}_j. \tag{10.6a}$$

We demand that complex conjugation K_{bulk} acts on complex fermion operators $\hat{c}_j$ as

$$\mathsf{K}_{\text{bulk}}\,\hat{c}_j\,\mathsf{K}_{\text{bulk}}=\hat{c}_j,\qquad \mathsf{K}_{\text{bulk}}\,\hat{c}_j^{\dagger}\,\mathsf{K}_{\text{bulk}}=\hat{c}_j^{\dagger}, \tag{10.6b}$$

which implies

$$\widehat{U}_{\text{bulk}}(t)\,\hat{\eta}_j\,\widehat{U}^{\dagger}_{\text{bulk}}(t)=\mathsf{K}_{\text{bulk}}\,\hat{\eta}_j\,\mathsf{K}_{\text{bulk}}=\hat{\eta}_j, \tag{10.6c}$$

$$\widehat{U}_{\text{bulk}}(t)\,\hat{\xi}_j\,\widehat{U}^{\dagger}_{\text{bulk}}(t)=\mathsf{K}_{\text{bulk}}\,\hat{\xi}_j\,\mathsf{K}_{\text{bulk}}=-\hat{\xi}_j. \tag{10.6d}$$

In the family of Hamiltonians (10.5a) open boundary conditions are imposed. Consequently, the Majorana operators $\left\{\hat{\xi}_{N-c+1},\hat{\xi}_{N-c+2},\cdots,\hat{\xi}_{N}\right\}$ on the right and the Majorana operators $\left\{\hat{\eta}_1,\hat{\eta}_2,\cdots,\hat{\eta}_c\right\}$ on the left do not enter a Majorana c open chain. These operators define the zero-energy Majorana degrees of freedom at the right- and left-boundaries, respectively.

10.2.1 *Boundary Representations and Computation of Indices*

We shall construct the boundary representations $\widehat{U}_{\text{L}}(t)$ and $\widehat{U}_{\text{R}}(t)$ that satisfy the compatibility conditions

$$\widehat{U}_{\text{bulk}}(t)\,\hat{\eta}_{\alpha}\,\widehat{U}^{\dagger}_{\text{bulk}}(t)=\widehat{U}_{\text{L}}(t)\,\hat{\eta}_{\alpha}\,\widehat{U}^{\dagger}_{\text{L}}(t),\quad \alpha=1,\cdots,c, \tag{10.7}$$

$$\widehat{U}_{\text{bulk}}(t)\,\hat{\xi}_{\beta}\,\widehat{U}^{\dagger}_{\text{bulk}}(t)=\widehat{U}_{\text{R}}(t)\,\hat{\xi}_{\beta}\,\widehat{U}^{\dagger}_{\text{R}}(t),\quad \beta=N-c+1,\cdots,N. \tag{10.8}$$

To this end, we will construct the boundary representation for the case of $c=1$. For all the remaining cases, we will stack the $c=1$ representation with itself using the explicit form of the stacked representation of reversal of time t

$$\widehat{U}_{\wedge}(t):=\widehat{V}_1(t)\,\widehat{V}_2(t)\,\left[\widehat{U}_1(p)\right]^{[\rho_2]}\,\left[\widehat{U}_2(p)\right]^{[\rho_1]}\,\mathsf{K}_{\wedge}, \tag{10.9a}$$

if $[\mu_1]=[\mu_2]=0$,

$$\widehat{U}_{\wedge}(t):=\widehat{V}_1(t)\,\widehat{V}_2(t)\,\left[\widehat{U}_1(p)\right]^{[\rho_2]}\,\left[\widehat{U}_2(p)\right]^{[\rho_1]}\,\mathsf{K}_{\wedge}, \tag{10.9b}$$

if $[\mu_1]=0$, $[\mu_2]=1$,

$$\widehat{U}_{\wedge}(t):=\widehat{V}_1(t)\,\widehat{V}_2(t)\,\left[\widehat{U}_{\wedge}(p)\right]^{1+[\rho_1]+[\rho_2]}\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(t)+[\rho_1]}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{1+q_2(t)+[\rho_2]}\mathsf{K}_{\wedge}, \tag{10.9c}$$

if $[\mu_1]=[\mu_2]=1$, which is defined in Eq. (7.30). We will then compute the triplets $([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}],[\mu_{\mathrm{L}}])$ and $([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}],[\mu_{\mathrm{R}}])$ that are associated with the left and the right boundaries respectively. We will confirm that these indices are inverses of each other for any c and follow the group structure in Table 10.1 dictated by the stacking rules (10.3).

10.2.1.1 *The Case of $c=1$*

When $c=1$, the sets of Majorana degrees of freedom at the left and the right boundaries are

$$\mathfrak{D}_{\mathrm{L}}=\{\hat{\eta}_1\},\qquad \mathfrak{D}_{\mathrm{R}}=\left\{\hat{\xi}_N\right\}, \tag{10.10}$$

respectively. We choose the complex conjugation on the boundaries to act as

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_1\,\mathsf{K}_{\mathrm{L}}=+\hat{\eta}_1,\qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_N\,\mathsf{K}_{\mathrm{R}}=-\hat{\xi}_N. \tag{10.11}$$

The boundary representations that satisfy the conditions (10.8) are given by

$$\widehat{U}_{\mathrm{L}}(t):=\mathsf{K}_{\mathrm{L}},\qquad \widehat{U}_{\mathrm{R}}(t):=\mathsf{K}_{\mathrm{R}}, \tag{10.12a}$$

$$\widehat{Y}_{\mathrm{L}}:=\hat{\eta}_1,\qquad \widehat{Y}_{\mathrm{R}}:=\hat{\xi}_N. \tag{10.12b}$$

Using the definitions (10.2) delivers the indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}],[\mu_{\mathrm{L}}])=(0,0,1),\qquad ([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}],[\mu_{\mathrm{R}}])=(0,1,1), \tag{10.12c}$$

on the left and the right boundaries, respectively.

10.2.1.2 *The Case of $c=2$*

When $c=2$, the sets of Majorana degrees of freedom at the left and the right boundaries are

$$\mathfrak{D}_{\mathrm{L}}=\{\hat{\eta}_1,\,\hat{\eta}_2\},\qquad \mathfrak{D}_{\mathrm{R}}=\left\{\hat{\xi}_{N-1},\,\hat{\xi}_N\right\}, \tag{10.13}$$

respectively. We choose the complex conjugation on the boundaries to act as

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_{\alpha}\,\mathsf{K}_{\mathrm{L}}=(-1)^{\alpha+1}\hat{\eta}_{\alpha},\qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_{N-\beta}\,\mathsf{K}_{\mathrm{R}}=(-1)^{\beta}\hat{\xi}_{N-\beta}, \tag{10.14}$$

with $\alpha=1,2$, and $\beta=0,1$. Choosing $\widehat{U}_1(t)$ and $\widehat{U}_2(t)$ in Eq. (10.9) to be the $c=1$ representations (10.12a), we obtain the $c=2$ representations

$$\widehat{U}_{\mathrm{L}}(t):=\hat{\eta}_1\,\mathsf{K}_{\mathrm{L}},\qquad \widehat{U}_{\mathrm{R}}(t):=\hat{\xi}_{N-1}\,\mathsf{K}_{\mathrm{R}}, \tag{10.15a}$$

$$\widehat{U}_{\mathrm{L}}(p):=\mathrm{i}\hat{\eta}_2\,\hat{\eta}_1,\qquad \widehat{U}_{\mathrm{L}}(p):=\mathrm{i}\hat{\xi}_{N-1}\,\hat{\xi}_{N}. \tag{10.15b}$$

Note that in the definition of the stacked operator (10.9) for odd-odd stacking, the operator $\gamma^{(2)}_{n_2}$ denotes the Majorana degree of freedom that is odd under stacked complex conjugation. In this case, these are the operators $\hat{\eta}_2$ and $\hat{\xi}_{N-1}$ according to Eq. (10.14). Using the definitions (10.2) delivers the indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}],[\mu_{\mathrm{L}}])=(0,1,0),\qquad ([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}],[\mu_{\mathrm{R}}])=(1,1,0), \tag{10.15c}$$

on the left and the right boundaries, respectively.

10.2.1.3 *The Case of $c=3$*

When $c=3$, the sets of Majorana degrees of freedom at the left and the right boundaries are

$$\mathfrak{O}_{\mathrm{L}}=\{\hat{\eta}_1,\,\hat{\eta}_2,\,\hat{\eta}_3\}\,,\qquad \mathfrak{O}_{\mathrm{R}}=\left\{\hat{\xi}_{N-2},\,\hat{\xi}_{N-1},\,\hat{\xi}_{N}\right\}, \tag{10.16}$$

respectively. We choose the complex conjugation on the boundaries to act as

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_{\alpha}\,\mathsf{K}_{\mathrm{L}}=(-1)^{\alpha+1}\hat{\eta}_{\alpha},\qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_{N-\beta}\,\mathsf{K}_{\mathrm{R}}=(-1)^{\beta}\hat{\xi}_{N-\beta}, \tag{10.17a}$$

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_3\,\mathsf{K}_{\mathrm{L}}=+\hat{\eta}_3,\qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_{N-2}\,\mathsf{K}_{\mathrm{R}}=-\hat{\xi}_{N-2}, \tag{10.17b}$$

with $\alpha=1,2$, and $\beta=0,1$.

Choosing $\widehat{U}_1(t)$ and $\widehat{U}_2(t)$ in Eq. (10.9) to be the $c=2$ representation (10.15a) and the $c=1$ representation (10.12a), respectively, we obtain the $c=3$ representations

$$\widehat{U}_{\mathrm{L}}(t):=\hat{\eta}_2\,\hat{\xi}_{N-2}\,\mathsf{K}_{\mathrm{L}},\qquad \widehat{U}_{\mathrm{R}}(t):=\hat{\xi}_{N}\,\hat{\eta}_3\,\mathsf{K}_{\mathrm{R}}, \tag{10.18a}$$

$$\widehat{Y}_{\mathrm{L}}:=\mathrm{i}\hat{\eta}_3\,\hat{\eta}_2\,\hat{\eta}_1,\qquad \widehat{Y}_{\mathrm{R}}:=\mathrm{i}\hat{\xi}_{N-2}\,\hat{\xi}_{N-1}\,\hat{\xi}_{N}. \tag{10.18b}$$

Using the definitions (10.2) delivers the indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}],[\mu_{\mathrm{L}}]) = (1,1,1), \qquad ([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}],[\mu_{\mathrm{R}}]) = (1,0,1), \tag{10.18c}$$

on the left and the right boundaries, respectively.

10.2.1.4 *The Case of $c = 4$*

When $c = 4$, the sets of Majorana degrees of freedom at the left and the right boundaries are

$$\mathfrak{D}_{\mathrm{L}} = \{\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3, \hat{\eta}_4\}, \qquad \mathfrak{D}_{\mathrm{R}} = \left\{\hat{\xi}_{N-3}, \hat{\xi}_{N-2}, \hat{\xi}_{N-1}, \hat{\xi}_{N}\right\}, \tag{10.19}$$

respectively. We choose the complex conjugation on the boundaries to act as

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_\alpha\,\mathsf{K}_{\mathrm{L}} = (-1)^{\alpha+1}\hat{\eta}_\alpha, \qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_{N-\beta}\,\mathsf{K}_{\mathrm{R}} = (-1)^{\beta}\hat{\xi}_{N-\beta}, \tag{10.20a}$$

with $\alpha = 1,\cdots,4$, and $\beta = 0,\cdots,3$. Choosing $\widehat{U}_1(t)$ and $\widehat{U}_2(t)$ in Eq. (10.9) to be the $c = 3$ representation (10.18a) and the $c = 1$ representation (10.12a), respectively, we obtain the $c = 4$ representations

$$\widehat{U}_{\mathrm{L}}(t) := \hat{\eta}_2\,\hat{\eta}_4\,\mathsf{K}_{\mathrm{L}}, \qquad \widehat{U}_{\mathrm{R}}(t) := \hat{\xi}_{N-2}\,\hat{\xi}_{N}\,\mathsf{K}_{\mathrm{R}}, \tag{10.21a}$$

$$\widehat{U}_{\mathrm{L}}(p) := \hat{\eta}_4\,\hat{\eta}_3\,\hat{\eta}_2\,\hat{\eta}_1, \qquad \widehat{U}_{\mathrm{L}}(p) := \hat{\xi}_{N-3}\,\hat{\xi}_{N-2}\,\hat{\xi}_{N-1}\,\hat{\xi}_{N}. \tag{10.21b}$$

Using the definitions (10.2) delivers the indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}],[\mu_{\mathrm{L}}]) = (1,0,0), \qquad ([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}],[\mu_{\mathrm{R}}]) = (1,0,0), \tag{10.21c}$$

on the left and the right boundaries, respectively.

10.2.1.5 *The Case of $c = 5$*

When $c = 5$, the sets of Majorana degrees of freedom at the left and the right boundaries are

$$\mathfrak{D}_{\mathrm{L}} = \{\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3, \hat{\eta}_4, \hat{\eta}_5\}, \qquad \mathfrak{D}_{\mathrm{R}} = \left\{\hat{\xi}_{N-4}, \hat{\xi}_{N-3}, \hat{\xi}_{N-2}, \hat{\xi}_{N-1}, \hat{\xi}_{N}\right\}, \tag{10.22}$$

respectively. We choose the complex conjugation on the boundaries to act as

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_{\alpha}\,\mathsf{K}_{\mathrm{L}} = (-1)^{\alpha+1}\hat{\eta}_{\alpha}, \qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_{N-\beta}\,\mathsf{K}_{\mathrm{R}} = (-1)^{\beta}\hat{\xi}_{N-\beta}, \tag{10.23a}$$

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_{5}\,\mathsf{K}_{\mathrm{L}} = +\hat{\eta}_{5}, \qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_{N-4}\,\mathsf{K}_{\mathrm{R}} = -\hat{\xi}_{N-4}, \tag{10.23b}$$

with $\alpha = 1, \cdots, 4$, and $\beta = 0, \cdots, 3$. Choosing $\widehat{U}_1(t)$ and $\widehat{U}_2(t)$ in Eq. (10.9) to be the $c = 4$ representation (10.21a) and the $c = 1$ representation (10.12a), respectively, we obtain the $c = 5$ representations

$$\widehat{U}_{\mathrm{L}}(t) := \hat{\eta}_2\,\hat{\eta}_4\,\mathsf{K}_{\mathrm{L}}, \qquad \widehat{U}_{\mathrm{R}}(t) := \hat{\xi}_{N}\,\hat{\xi}_{N-2}\,\mathsf{K}_{\mathrm{R}}, \tag{10.24a}$$

$$\widehat{Y}_{\mathrm{L}} := \hat{\eta}_5\,\hat{\eta}_4\,\hat{\eta}_3\,\hat{\eta}_2\,\hat{\eta}_1, \qquad \widehat{Y}_{\mathrm{R}} := \hat{\xi}_{N-4}\,\hat{\xi}_{N-3}\,\hat{\xi}_{N-2}\,\hat{\xi}_{N-1}\,\hat{\xi}_{N}. \tag{10.24b}$$

Using the definitions (10.2) delivers the indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}],[\mu_{\mathrm{L}}]) = (1,0,1), \qquad ([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}],[\mu_{\mathrm{R}}]) = (1,1,1), \tag{10.24c}$$

on the left and the right boundaries, respectively.

10.2.1.6 *The Case of $c = 6$*

When $c = 6$, the sets of Majorana degrees of freedom at the left and the right boundaries are

$$\mathfrak{D}_{\mathrm{L}} = \{\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3, \hat{\eta}_4, \hat{\eta}_5, \hat{\eta}_6\}, \tag{10.25a}$$

$$\mathfrak{D}_{\mathrm{R}} = \left\{\hat{\xi}_{N-5}, \hat{\xi}_{N-4}, \hat{\xi}_{N-3}, \hat{\xi}_{N-2}, \hat{\xi}_{N-1}, \hat{\xi}_{N}\right\}, \tag{10.25b}$$

respectively. We choose the complex conjugation on the boundaries to act as

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_{\alpha}\,\mathsf{K}_{\mathrm{L}} = (-1)^{\alpha+1}\hat{\eta}_{\alpha}, \qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_{N-\beta}\,\mathsf{K}_{\mathrm{R}} = (-1)^{\beta}\hat{\xi}_{N-\beta}, \tag{10.26a}$$

with $\alpha = 1, \cdots, 6$, and $\beta = 0, \cdots, 5$. Choosing $\widehat{U}_1(t)$ and $\widehat{U}_2(t)$ in Eq. (10.9) to be the $c = 5$ representation (10.24a) and the $c = 1$ representation (10.12a), respectively, we obtain the $c = 6$ representations

$$\widehat{U}_{\mathrm{L}}(t) := \hat{\eta}_1\,\hat{\eta}_3\,\hat{\eta}_5\,\mathsf{K}_{\mathrm{L}}, \qquad \widehat{U}_{\mathrm{R}}(t) := \hat{\xi}_{N-5}\,\hat{\xi}_{N-3}\,\hat{\xi}_{N-1}\,\mathsf{K}_{\mathrm{R}}, \tag{10.27a}$$

$$\widehat{U}_{\mathrm{L}}(p) := \mathrm{i}\prod_{j=1}^{6}\hat{\eta}_{6-j+1}, \qquad \widehat{U}_{\mathrm{L}}(p) := \mathrm{i}\prod_{j=1}^{6}\hat{\xi}_{N-6+j}. \tag{10.27b}$$

Using the definitions (10.2) delivers the indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}],[\mu_{\mathrm{L}}]) = (1,1,0), \qquad ([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}],[\mu_{\mathrm{R}}]) = (0,1,0), \tag{10.27c}$$

on the left and the right boundaries, respectively.

10.2.1.7 *The Case of* $c = 7$

When $c = 7$, the sets of Majorana degrees of freedom at the left and the right boundaries are

$$\mathfrak{D}_{\mathrm{L}} = \{\hat{\eta}_1,\, \hat{\eta}_2,\, \hat{\eta}_3,\, \hat{\eta}_4,\, \hat{\eta}_5,\, \hat{\eta}_6,\, \hat{\eta}_7\}, \tag{10.28a}$$

$$\mathfrak{D}_{\mathrm{R}} = \left\{\hat{\xi}_{N-6},\, \hat{\xi}_{N-5},\, \hat{\xi}_{N-4},\, \hat{\xi}_{N-3},\, \hat{\xi}_{N-2},\, \hat{\xi}_{N-1},\, \hat{\xi}_{N}\right\}, \tag{10.28b}$$

respectively. We choose the complex conjugation on the boundaries to act as

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_{\alpha}\,\mathsf{K}_{\mathrm{L}} = (-1)^{\alpha+1}\hat{\eta}_{\alpha}, \qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_{N-\beta}\,\mathsf{K}_{\mathrm{R}} = (-1)^{\beta}\hat{\xi}_{N-\beta}, \tag{10.29a}$$

$$\mathsf{K}_{\mathrm{L}}\,\hat{\eta}_{7}\,\mathsf{K}_{\mathrm{L}} = +\hat{\eta}_{7}, \qquad \mathsf{K}_{\mathrm{R}}\,\hat{\xi}_{N-6}\,\mathsf{K}_{\mathrm{R}} = -\hat{\xi}_{N-6}, \tag{10.29b}$$

with $\alpha = 1,\cdots,6$, and $\beta = 0,\cdots,5$. Choosing $\widehat{U}_1(t)$ and $\widehat{U}_2(t)$ in Eq. (10.9) to be the $c = 6$ representation (10.27a) and the $c = 1$ representation (10.12a), respectively, we obtain the $c = 7$ representations

$$\widehat{U}_{\mathrm{L}}(t) := \hat{\eta}_2\,\hat{\eta}_4\,\hat{\eta}_6\,\hat{\xi}_{N-6}\,\mathsf{K}_{\mathrm{L}}, \qquad \widehat{U}_{\mathrm{R}}(t) := \hat{\xi}_{N-4}\,\hat{\xi}_{N-2}\,\hat{\xi}_{N}\,\hat{\eta}_7\,\mathsf{K}_{\mathrm{R}}, \tag{10.30a}$$

$$\widehat{Y}_{\mathrm{L}} := \mathrm{i}\prod_{j=1}^{7}\hat{\eta}_{7-j+1}, \qquad \widehat{Y}_{\mathrm{R}} := \mathrm{i}\prod_{j=1}^{7}\hat{\xi}_{N-7+j}. \tag{10.30b}$$

Using the definitions (10.2) delivers the indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}],[\mu_{\mathrm{L}}]) = (0,1,1), \qquad ([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}],[\mu_{\mathrm{R}}]) = (0,0,1), \tag{10.30c}$$

on the left and the right boundaries, respectively.

10.2.1.8 *The Case of* $c = 8$

When $c = 8$, the sets of Majorana degrees of freedom at the left and the right boundaries are

$$\mathfrak{O}_{\mathrm{L}} = \{\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3, \hat{\eta}_4, \hat{\eta}_5, \hat{\eta}_6, \hat{\eta}_7, \hat{\eta}_8\}, \tag{10.31a}$$

$$\mathfrak{O}_{\mathrm{R}} = \left\{\hat{\xi}_{N-7}, \hat{\xi}_{N-6}, \hat{\xi}_{N-5}, \hat{\xi}_{N-4}, \hat{\xi}_{N-3}, \hat{\xi}_{N-2}, \hat{\xi}_{N-1}, \hat{\xi}_N\right\}, \tag{10.31b}$$

respectively. We choose the complex conjugation on the boundaries to act as

$$\mathsf{K}_{\mathrm{L}}\, \hat{\eta}_\alpha\, \mathsf{K}_{\mathrm{L}} = (-1)^{\alpha+1}\hat{\eta}_\alpha, \qquad \mathsf{K}_{\mathrm{R}}\, \hat{\xi}_{N-\beta}\, \mathsf{K}_{\mathrm{R}} = (-1)^{\beta}\hat{\xi}_{N-\beta}, \tag{10.32a}$$

with $\alpha = 1, \cdots, 8$, and $\beta = 0, \cdots, 7$. Choosing $\widehat{U}_1(t)$ and $\widehat{U}_2(t)$ in Eq. (10.9) to be the $c = 7$ representation (10.30a) and the $c = 1$ representation (10.12a), respectively, we obtain the $c = 8$ representations

$$\widehat{U}_{\mathrm{L}}(t) := \hat{\eta}_2\, \hat{\eta}_4\, \hat{\eta}_6\, \hat{\eta}_8\, \mathsf{K}_{\mathrm{L}}, \qquad \widehat{U}_{\mathrm{R}}(t) := \hat{\xi}_{N-6}\, \hat{\xi}_{N-4}\, \hat{\xi}_{N-2}\, \hat{\xi}_N\, \mathsf{K}_{\mathrm{R}}, \tag{10.33a}$$

$$\widehat{U}_{\mathrm{L}}(p) := \prod_{j=1}^{8} \hat{\eta}_{8-j+1}, \qquad \widehat{U}_{\mathrm{L}}(p) := \prod_{j=1}^{8} \hat{\xi}_{N-8+j}. \tag{10.33b}$$

Using the definitions (10.2) delivers the indices

$$([\nu_{\mathrm{L}}], [\rho_{\mathrm{L}}], [\mu_{\mathrm{L}}]) = (0, 0, 0), \qquad ([\nu_{\mathrm{R}}], [\rho_{\mathrm{R}}], [\mu_{\mathrm{R}}]) = (0, 0, 0), \tag{10.33c}$$

on the left and the right boundaries, respectively. Note that the indices on both left and right-boundaries trivialize.

10.2.2 *Protected Ground-State Degeneracies*

We calculate the protected ground-state degeneracy for the symmetry class BDI, for each value of the triplet $([\nu], [\rho], [\mu])$. In Table 10.2, we summarize the protected ground state degeneracies at each boundary and the total protected ground-state degeneracy of each one-dimensional IFT phase in symmetry class BDI.

10.2.2.1 *The Case of* $[\mu]=0$ *for the Symmetry Class BDI*

The case $[\mu]=0$ corresponds to the $c=2,4,6,8$ elements in the Table 10.1, which form a $\mathbb{Z}_4$-group under the stacking rule (10.3). The index $[\nu_{\mathrm{B}}]$ dictates if the boundary representation $\widehat{U}_{\mathrm{B}}(t)$ squares to plus or minus the identity, as defined in Eq. (10.2). The index $[\rho_{\mathrm{B}}]$ characterizes the fermion parity of the representation $\widehat{U}_{\mathrm{B}}(t)$ of reversal of time, as defined in Eq. (10.2). Whenever $[\rho_{\mathrm{B}}]=1$, it was shown in Chapter 8 that all states on the boundary Λ_{B} are at least twofold degenerate because of a quantum mechanical supersymmetry. Whenever $[\nu_{\mathrm{B}}]=1$, there exists a twofold Kramer's degeneracy on the boundary Λ_{B}. When $[\nu_{\mathrm{B}}]=[\rho\,\mathrm{B}]=1$, $\widehat{U}_{\mathrm{B}}(t)$ anticommutes with $\widehat{U}_{\mathrm{B}}(p)$, i.e., it is either the pair $\widehat{H}_{\mathrm{B}}$ and $\widehat{U}_{\mathrm{B}}(t)$ that can be simultanously diagonalized with $\widehat{U}_{\mathrm{B}}(p)$ acting as a ladder operator or the pair $\widehat{H}_{\mathrm{B}}$ and $\widehat{U}_{\mathrm{B}}(p)$ that can be simultanously diagonalized with $\widehat{U}_{\mathrm{B}}(t)$ acting as a ladder operator. Therefore, when $[\mu]=0$, whenever the pair $([\nu_{\mathrm{B}}],[\rho_{\mathrm{B}}])$ is nontrivial, there exist a *twofold* protected degeneracy at the boundary Λ_{B}. The ground state degeneracies are

$$\mathrm{GSD}_{\mathrm{L}}^{[\mu]=0}=2^{[\nu_{\mathrm{L}}]+[\rho_{\mathrm{L}}]-[\nu_{\mathrm{L}}]\,[\rho_{\mathrm{L}}]}, \tag{10.34a}$$

$$\mathrm{GSD}_{\mathrm{R}}^{[\mu]=0}=2^{[\nu_{\mathrm{R}}]+[\rho_{\mathrm{R}}]-[\nu_{\mathrm{R}}]\,[\rho_{\mathrm{R}}]}, \tag{10.34b}$$

$$\mathrm{GSD}_{\mathrm{bd}}^{[\mu]=0}=2^{[\nu_{\mathrm{L}}]+[\rho_{\mathrm{L}}]-[\nu_{\mathrm{L}}]\,[\rho_{\mathrm{L}}]}\times 2^{[\nu_{\mathrm{R}}]+[\rho_{\mathrm{R}}]-[\nu_{\mathrm{R}}]\,[\rho_{\mathrm{R}}]}. \tag{10.34c}$$

We conclude that a one-dimensional nontrivial FSPT phase in the symmetry class BDI has *fourfold* protected ground-state degeneracy when open boundary conditions are imposed. Additional degeneracies are accidental.

10.2.2.2 *The Case of* $[\mu]=1$ *for the Symmetry Class BDI*

The case $[\mu]=1$ corresponds to the $c=1,3,5,7$ elements in the Table 10.1. The index $[\nu_{\mathrm{B}}]$ dictates when the boundary representation $\widehat{U}_{\mathrm{B}}(t)$ squares to plus or minus identity as defined in Eq. (10.2). The index $[\rho_{\mathrm{B}}]$ dictates when the representation $\widehat{U}_{\mathrm{B}}(t)$ of reversal of time commutes or anticommutes with the central element $\widehat{Y}_{\mathrm{B}}$, as defined in Eq. (10.2). Whenever $[\nu_{\mathrm{B}}]=1$, there exist a twofold Kramer's degeneracy at the boundary Λ_{B}. Since by definition (4.10) in the main text, $\widehat{U}_{\mathrm{B}}(t)$ is constructed to have even fermion parity all Kramer's degenerate states carry the same fermion parity. When $[\rho_{\mathrm{B}}]=1$, $\widehat{U}_{\mathrm{B}}(t)$ maps the eigenstates of $\widehat{Y}_{\mathrm{B}}$ with eigenvalues ±1 to the eigenstates with eigenvalues ∓1. However, a nonzero $[\rho_{\mathrm{B}}]$ does not imply any additional protected degeneracy. Hence, any

Table 10.2: The triplets $([\nu_{\mathrm{B}}],[\rho_{\mathrm{B}}],[\mu_{\mathrm{B}}])$ on the left $(\mathrm{B}=\mathrm{L})$ and right $(\mathrm{B}=\mathrm{R})$ boundaries of a Majorana c chain with $c=1,\cdots,8$. The last column GSD is the protected ground-state degeneracies defined in Eqs. (10.34) and (10.35) for $[\mu_{\mathrm{B}}]=0$ and $[\mu_{\mathrm{B}}]=1$, respectively.

c	$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}],[\mu_{\mathrm{L}}])$	$([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}],[\mu_{\mathrm{R}}])$	GSD
1	$(0,0,1)$	$(0,1,1)$	2
2	$(0,1,0)$	$(1,1,0)$	4
3	$(1,1,1)$	$(1,0,1)$	8
4	$(1,0,0)$	$(1,0,0)$	4
5	$(1,0,1)$	$(1,1,1)$	8
6	$(1,1,0)$	$(0,1,0)$	4
7	$(0,1,1)$	$(0,0,1)$	2
8	$(0,0,0)$	$(0,0,0)$	1

protected degeneracy in addition to that due to $[\mu]=1$ comes from the nontrivial index $[\nu_{\mathrm{B}}]$. The ground state degeneracies are

$$\mathrm{GSD}_{\mathrm{L}}^{[\mu]=1}=2\times 2^{[\nu_{\mathrm{L}}]}, \tag{10.35a}$$

$$\mathrm{GSD}_{\mathrm{R}}^{[\mu]=1}=2\times 2^{[\nu_{\mathrm{R}}]}, \tag{10.35b}$$

$$\mathrm{GSD}^{[\mu]=1}=\frac{1}{2}\times 2\times 2^{[\nu_{\mathrm{L}}]}\times 2\times 2^{[\nu_{\mathrm{R}}]}=2\times 2^{[\nu_{\mathrm{L}}]}\times 2^{[\nu_{\mathrm{R}}]}. \tag{10.35c}$$

Hence, we find that a one-dimensional nontrivial IFT phases in the symmetry class BDI with $c=1,3,5,7$ have protected ground-state degeneracy of 2, 8, 8, and 2, respectively, when open boundary conditions are imposed.

10.3 SPIN-1/2 CLUSTER CHAINS

In this section, we consider a family of spin Hamiltonians that realize BSPT phases and are closely related to the Hamiltonians (10.5a) of the time-reversal symmetric Majorana chains. As was done in Sec. 10.2, we will identify the set of gapless boundary degrees of freedom when open-boundary conditions are imposed and compute the associated indices that characterize the boundary projective representations.

We define the spin-1/2 cluster c chains that are described by the family of Hamiltonians

$$\widehat{H}^{(b)}_{c} := -\sum_{j=1}^{2N-b|c|} \widehat{C}_{j}, \tag{10.36a}$$

where $b = 0, 1$ specifies the periodic ($b = 0$) or open ($b = 1$) boundary conditions and we have defined

$$\widehat{C}_{j} := \begin{cases} \widehat{Z}_{j}\,\widehat{X}_{j+1}\,\cdots\,\widehat{X}_{j+c-1}\widehat{Z}_{j+c}, & \text{if } c > 0, \\ -\widehat{X}_{j}, & \text{if } c = 0, \\ \widehat{Y}_{j}\,\widehat{X}_{j+1}\,\cdots\,\widehat{X}_{j+|c|-1}\widehat{Y}_{j+|c|}, & \text{if } c < 0, \end{cases} \tag{10.36b}$$

with $\widehat{X}_{j}$, $\widehat{Y}_{j}$, and $\widehat{Z}_{j}$ the Pauli spin operators at site j realizing the spin-1/2 representation of the $\mathfrak{su}(2)$ Lie algebra. The local terms $\widehat{C}_{j}$ are pairwise commuting and each has eigenvalues ± 1. Therefore, the Hamiltonian (10.36) is a sum of pairwise commuting terms. We set the total number of sites to be even, i.e., $2N$.

The Hamiltonian (10.36) of the spin-1/2 cluster c chains is related to the Hamiltonian (10.5a) of the time-reversal symmetric Majorana c chains by the Jordan-Wigner (JW) transformation:

$$\hat{\eta}_{j} = \left(\prod_{i=1}^{j-1} \widehat{X}_{i}\right) \widehat{Z}_{j}, \qquad \hat{\xi}_{i} = \left(\prod_{i=1}^{j-1} \widehat{X}_{i}\right) \widehat{Y}_{j}, \qquad \mathrm{i}\hat{\xi}_{j}\,\hat{\eta}_{j} = -\widehat{X}_{j}, \tag{10.37a}$$

$$\widehat{Z}_{j} = \left(\prod_{i=1}^{j-1} \mathrm{i}\hat{\eta}_{i}\,\hat{\xi}_{i}\right) \hat{\eta}_{j}, \qquad \widehat{Y}_{j} = \left(\prod_{i=1}^{j-1} \mathrm{i}\hat{\eta}_{i}\,\hat{\xi}_{i}\right) \hat{\xi}_{j}, \qquad \widehat{X}_{j} = -\mathrm{i}\hat{\xi}_{j}\,\hat{\eta}_{j}. \tag{10.37b}$$

In what follows, we will first review the spectra and the internal symmetries of the Hamiltonian (10.36). We will then consider cluster chains with open boundary conditions and construct the boundary projective representations of the protecting symmetries. In doing so, we will also relate the boundary representations of the cluster chains to that of the Majorana chains that are obtained from JW transformation.

10.3.1 *Spectra and Internal Symmetries*

We shall distinguish the cases of c odd and c even. For each cases, we shall single out an internal symmetry group of the spin-1/2 cluster c chain.

10.3.1.1 *The Case of c Even*

When c is an even integer, the Hamiltonian (10.36) has a nondegenerate and gapped ground state when periodic boundary conditions are imposed. This ground state is specified by being the eigenstate of each projector $\widehat{C}_j$ with the eigenvalue $+1$.

The Hamiltonian (10.36) is invariant under the global symmetry group $G_e = \mathbb{Z}_2^{\mathrm{T}'} \times \mathbb{Z}_2 = \{e, t'\} \times \{e, g\}$ with the global representation

$$\widehat{U}_{\mathrm{e}}(t') := \hat{\mathbb{1}}\,\mathsf{K}, \qquad \widehat{U}_{\mathrm{e}}(g) := \prod_{j=1}^{2N} \widehat{X}_j, \tag{10.38a}$$

where $\hat{\mathbb{1}}$ is the identity operator and K is the complex conjugation. The representations (10.38a) act on the local spin-1/2 degrees of freedom as

$$\widehat{U}_{\mathrm{e}}(t') \begin{pmatrix} \widehat{X}_j & \widehat{Y}_j & \widehat{Z}_j \end{pmatrix}^{\mathsf{T}} \widehat{U}^{\dagger}_{\mathrm{e}}(t') = \begin{pmatrix} +\widehat{X}_j & -\widehat{Y}_j & +\widehat{Z}_j \end{pmatrix}^{\mathsf{T}}, \tag{10.38b}$$

$$\widehat{U}_{\mathrm{e}}(g) \begin{pmatrix} \widehat{X}_j & \widehat{Y}_j & \widehat{Z}_j \end{pmatrix}^{\mathsf{T}} \widehat{U}^{\dagger}_{\mathrm{e}}(g) = \begin{pmatrix} +\widehat{X}_j & -\widehat{Y}_j & -\widehat{Z}_j \end{pmatrix}^{\mathsf{T}}. \tag{10.38c}$$

When open boundary conditions are imposed, the Hamiltonian (10.36) may support gapless edge degrees of freedom that are protected by a boundary projective representation of the group G_e. Therefore, cluster c chains with even c realize the BSPT phases with G_e-symmetry. When JW transformation is applied on the Hamiltonian (10.36) with open boundary conditions, one obtains the Hamiltonian (10.5a). These are nothing but the FSPT phases (IFT phases with $[\mu] = 0$) in class BDI which has a $\mathbb{Z}_4$ classification. Under the JW transformation the operators $\widehat{U}_e(t)$ and $\widehat{U}_e(g)$ defined in Eq. (10.38a) are mapped to the reversal of time and fermion parity symmetries of the Majorana chains defined in Eq. (10.6), respectively.

10.3.1.2 *The Case of c Odd*

When c is an odd integer, the Hamiltonian (10.36) has a twofold degenerate ground-state manifold when periodic boundary conditions are imposed. This twofold degeneracy arises owing to the fact that product of all projectors $\widehat{C}_j$ is equal to the identity.

The Hamiltonian (10.36) is invariant under the global symmetry group $\mathrm{G}_{\mathrm{o}} = \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{T}'} \times \mathbb{Z}_2 = \{e, t\} \times \{e, t'\} \times \{e, g\}$ with the global representations

$$\widehat{U}_{\mathrm{o}}(t) := \left(\prod_{j=1}^{2N} \widehat{Y}_j\right) \mathsf{K}, \qquad \widehat{U}_{\mathrm{o}}(t') := \hat{\mathbb{1}}\,\mathsf{K}, \qquad \widehat{U}_{\mathrm{o}}(g) := \prod_{j=1}^{2N} \widehat{X}_j. \tag{10.39a}$$

The actions of these generators on the local spin-1/2 degrees of freedom are

$$\widehat{U}_{\mathrm{o}}(t) \begin{pmatrix} \widehat{X}_j & \widehat{Y}_j & \widehat{Z}_j \end{pmatrix}^{\mathsf{T}} \widehat{U}_{\mathrm{o}}^{\dagger}(t) = \begin{pmatrix} -\widehat{X}_j & -\widehat{Y}_j & -\widehat{Z}_j \end{pmatrix}^{\mathsf{T}}, \tag{10.39b}$$

$$\widehat{U}_{\mathrm{o}}(t') \begin{pmatrix} \widehat{X}_j & \widehat{Y}_j & \widehat{Z}_j \end{pmatrix}^{\mathsf{T}} \widehat{U}_{\mathrm{o}}^{\dagger}(t') = \begin{pmatrix} +\widehat{X}_j & -\widehat{Y}_j & +\widehat{Z}_j \end{pmatrix}^{\mathsf{T}}, \tag{10.39c}$$

$$\widehat{U}_{\mathrm{o}}(g) \begin{pmatrix} \widehat{X}_j & \widehat{Y}_j & \widehat{Z}_j \end{pmatrix}^{\mathsf{T}} \widehat{U}_{\mathrm{o}}^{\dagger}(g) = \begin{pmatrix} +\widehat{X}_j & -\widehat{Y}_j & -\widehat{Z}_j \end{pmatrix}^{\mathsf{T}}. \tag{10.39d}$$

In the thermodynamic limit $N \to \infty$ at zero temperature, the symmetry group G_o is spontaneously broken. Without loss of generality, we project on one of the symmetry breaking ground states and define the mean-field Hamiltonian

$$\widehat{H}^{(b)}_{\mathrm{MF},\, c\in 2\mathbb{Z}+1} := -\sum_{j=1}^{2N-b(|c|-1)} \widehat{M}_j, \tag{10.40a}$$

where the monomial $\widehat{M}_j$ of odd order $|c|$ defined by

$$\widehat{M}_j := \begin{cases} \widehat{Z}_j\,\widehat{Y}_{j+1}\,\widehat{Z}_{j+2}\,\widehat{Y}_{j+3}\,\cdots\,\widehat{Z}_{j+c-3}\,\widehat{Y}_{j+c-2}\,\widehat{Z}_{j+c-1}, & \text{if } c > 0, \\ \widehat{Y}_j\,\widehat{Z}_{j+1}\,\widehat{Y}_{j+2}\,\widehat{Z}_{j+3}\,\cdots\,\widehat{Y}_{j+|c|-3}\,\widehat{Z}_{j+|c|-2}\,\widehat{Y}_{j+|c|-1}, & \text{if } c < 0. \end{cases} \tag{10.40b}$$

This Hamiltonian has a nondegenerate and gapped ground state with closed boundary conditions (b=0). Hamiltonian (10.40a) breaks the $\mathbb{Z}_2^{\mathrm{T}}$ symmetry under reversal of time for $\widehat{M}_j$ is odd under simultaneous sign reversal of all spin operators. In fact, we have the transformations laws

$$\widehat{U}_{\mathrm{o}}(t)\,\widehat{M}_j\,\widehat{U}_{\mathrm{o}}^{\dagger}(t) = -\widehat{M}_j, \tag{10.41a}$$

$$\widehat{U}_{\mathrm{o}}(t')\,\widehat{M}_j\,\widehat{U}_{\mathrm{o}}^{\dagger}(t') = (-1)^{\frac{c-1}{2}}\,\widehat{M}_j, \tag{10.41b}$$

$$\widehat{U}_{\mathrm{o}}(g)\,\widehat{M}_j\,\widehat{U}_{\mathrm{o}}^{\dagger}(g) = -\widehat{M}_j. \tag{10.41c}$$

Nevertheless, by composing pairs of these three operations, it is possible to construct the subgroup

$$\mathrm{G}_{\mathrm{MF}} = \widetilde{\mathbb{Z}}_2^{\widetilde{T}'} \times \widetilde{\mathbb{Z}}_2, \qquad \widetilde{\mathbb{Z}}_2^{\widetilde{T}'} := \{e, \tilde{t}'\}, \qquad \widetilde{\mathbb{Z}}_2 := \{e, \tilde{g}\}, \tag{10.42}$$

with the representations

$$\widehat{U}_{\mathrm{MF}}(\tilde{t}') := \begin{cases} \hat{\mathbb{1}}\,\mathsf{K}, & \text{if } \frac{c-1}{2} = 0 \bmod 2, \\ \left(\prod\limits_{j=1}^{2N} \widehat{Z}_j\right)\mathsf{K}, & \text{if } \frac{c-1}{2} = 1 \bmod 2, \end{cases} \tag{10.43a}$$

$$\widehat{U}_{\mathrm{MF}}(\tilde{g}) := \begin{cases} \prod\limits_{j=1}^{2N} \widehat{Z}_j, & \text{if } \frac{c-1}{2} = 0 \bmod 2, \\ \prod\limits_{j=1}^{2N} \widehat{Y}_j, & \text{if } \frac{c-1}{2} = 1 \bmod 2, \end{cases} \tag{10.43b}$$

such that

$$\widehat{U}_{\mathrm{MF}}(\tilde{t}')\begin{pmatrix}\widehat{X}_j & \widehat{Y}_j & \widehat{Z}_j\end{pmatrix}^{\mathsf{T}}\widehat{U}_{\mathrm{MF}}^{\dagger}(\tilde{t}') = \begin{cases} \begin{pmatrix}+\widehat{X}_j & -\widehat{Y}_j & +\widehat{Z}_j\end{pmatrix}^{\mathsf{T}}, & \text{if } \frac{c-1}{2} = 0 \bmod 2, \\ \begin{pmatrix}-\widehat{X}_j & +\widehat{Y}_j & +\widehat{Z}_j\end{pmatrix}^{\mathsf{T}}, & \text{if } \frac{c-1}{2} = 1 \bmod 2, \end{cases} \tag{10.43c}$$

$$\widehat{U}_{\mathrm{MF}}(\tilde{g})\begin{pmatrix}\widehat{X}_j & \widehat{Y}_j & \widehat{Z}_j\end{pmatrix}^{\mathsf{T}}\widehat{U}_{\mathrm{MF}}^{\dagger}(\tilde{g}) = \begin{cases} \begin{pmatrix}-\widehat{X}_j & -\widehat{Y}_j & +\widehat{Z}_j\end{pmatrix}^{\mathsf{T}}, & \text{if } \frac{c-1}{2} = 0 \bmod 2, \\ \begin{pmatrix}-\widehat{X}_j & +\widehat{Y}_j & -\widehat{Z}_j\end{pmatrix}^{\mathsf{T}}, & \text{if } \frac{c-1}{2} = 1 \bmod 2, \end{cases} \tag{10.43d}$$

and

$$\widehat{M}_j = \widehat{U}_{\mathrm{MF}}(\tilde{t}')\,\widehat{M}_j\,\widehat{U}_{\mathrm{MF}}^{\dagger}(\tilde{t}') = \widehat{U}_{\mathrm{MF}}(\tilde{g})\,\widehat{M}_j\,\widehat{U}_{\mathrm{MF}}^{\dagger}(\tilde{g}), \qquad j = 1, \cdots, 2N. \tag{10.43e}$$

It follows that the group (10.42) is a symmetry group of the mean-field Hamiltonian (10.40a). After projecting the spin-1/2 cluster odd-c Hamiltonian onto the mean-field Hamiltonian (10.40a), the spectrum and symmetries of the latter are the same and isomorphic, respectively, to those of the spin-1/2 cluster $[c-\mathrm{sgn}(c)]$ Hamiltonian. The consequences of the group cohomology that we study next are the same for the spin-1/2 cluster odd-c Hamiltonian and the mean-field projection of the spin-1/2 cluster $[c-(-1)^{\lfloor \frac{c}{4} \rfloor}\mathrm{sgn}(c)]$ Hamiltonian.

10.3.2 *Symmetry Fractionalization in Spin-1/2 Cluster c Chains*

We are going to compute the indices classifying spin-1/2 cluster c models defined in Hamiltonian (10.36). Without loss of generality, this is done as follows when c is even [we consider the mean-field Hamiltonian (10.40a) when c is odd] and the cardinality of the chain $\Lambda := \{j = 1, \cdots, |\Lambda|\}$ is $|\Lambda|$.

1. The cluster operators $\widehat{C}_{|\Lambda|-|c|+1}, \cdots, \widehat{C}_{|\Lambda|}$ are not present in Hamiltonian (10.36), when open boundary conditions are imposed. Consequently, either on the left boundary
$$\Lambda_{\mathrm{L}} := \{j = 1, 2, \cdots, |c|\} \tag{10.44}$$
or on the right boundary
$$\Lambda_{\mathrm{R}} := \{j = 2N - |c| + 1, \cdots, 2N - 1, 2N\}, \tag{10.45}$$
the set of all operators commuting with $\widehat{H}_c^{(|c|)}$ is a Clifford algebra $\mathrm{C}\ell_{|c|}$ with $|c|$ generators represented by $2^{|c|}$-dimensional matrices acting on the Hilbert space $\mathfrak{h}_{\Lambda_{\mathrm{B}}} := \mathbb{C}^{2^{|c|}}$ on either the left (B = L) or the right (B = R) boundary, respectively.

2. The Clifford algebra $\mathrm{C}\ell_{|c|}$ contains the Lie subalgebra
$$\underbrace{\mathfrak{su}(2) \oplus \cdots \oplus \mathfrak{su}(2)}_{|c| \text{ times}} \tag{10.46}$$
which is represented reducibly with $2^{|c|}$-dimensional matrices.

3. It is possible to represent the action of the protecting symmetries on either the left or right boundaries using the generators of the Lie subalgebra (10.46).

4. We construct the boundary representations $\widehat{U}_{\mathrm{e,L}}$ on the boundary degrees of freedom by demanding the consistency condition

$$\widehat{U}_{\mathrm{e,L}}(g)\,\widehat{\mathfrak{S}}\,\widehat{U}^{\dagger}_{\mathrm{e,L}}(g)=\widehat{U}_{\mathrm{e}}(g)\,\widehat{\mathfrak{S}}\,\widehat{U}^{\dagger}_{\mathrm{e}}(g) \tag{10.47}$$

for any $g\in G_e$ and $\widehat{\mathfrak{S}}\in\mathfrak{su}(2)\oplus\cdots\oplus\mathfrak{su}(2)$. For the case of odd c, we shall use the global representations $\widehat{U}_{\mathrm{MF}}(g)$ of the group G_{MF}.

We summarize the results of this exercise in Table 10.3. It is instructive to compare the Table 10.3 with the Table 10.2. We observe that the indices for cluster even-c chains match the indices $([\nu_L],[\rho_{\mathrm{L}}],0)$ of the Majorana chains with c even. On the other hand, the indices associated with the right boundary of the Majorana chains are not picked up by the corresponding cluster even-c chains. This asymmetry is due to the fact that the JW transformation (10.37) is asymmetrical with respect to the left and the right boundaries [171]. The odd-c cluster chains have the same indices as the even-$[c-(-1)^{\lfloor\frac{c}{4}\rfloor}\mathrm{sgn}(c)]$ chains. The protected ground state degeneracies of the cluster odd-c chains match that of the Majorana c chains if we also include the twofold degeneracy due to the spontaneous symmetry breaking.

10.3.2.1 *The Case of* $c=0$

When $c=0$, Hamiltonian (10.36) with open boundary conditions becomes

$$\widehat{H}^{(1)}_{0}:=-\sum_{j=1}^{2N}\widehat{X}_{j}=\widehat{H}^{(0)}_{0}, \tag{10.48}$$

which has a nondegenerate and gapped ground state. All $\widehat{X}_j$ with $j=1,\cdots,2N$ are present in $\widehat{H}^{(1)}_0$ and commute pairwise. In other words, the set of gapless boundary degrees of freedom is the empty set

$$\mathfrak{D}_{\mathrm{B},0}\coloneqq\{\,\},\qquad \mathrm{B}=\mathrm{L},\mathrm{R}. \tag{10.49}$$

By convention, we associate the trivial indices

$$([\nu],[\rho])=(0,0) \tag{10.50}$$

to the spin-1/2 cluster $c=0$ chain.

Table 10.3: The doublet $([\nu_{\mathrm{B}}],[\rho_{\mathrm{B}}])$ of a spin-1/2 cluster c chain that defines the projective representation of the symmetry group (10.38) for even c or (10.39) for odd c that is realized on the left (B=L) or right (B=R) boundaries of an open chain. The time-reversal symmetry is broken spontaneously when c is odd. The column $\dim \mathfrak{h}^{(c)}_{\mathrm{B,min}}$ is the dimensionality of the smallest Hilbert space $\mathfrak{h}^{(c)}_{\mathrm{B,min}}$ for which it is possible to realize the projective algebra on either one of the left or right boundary. Squaring this number gives the topologically protected degeneracy $D(c)$ of the ground states of a spin-1/2 cluster c open chain. The column $\dim \mathfrak{h}^{(c)}_{\mathrm{gs}}$ is the degeneracy of the ground state of the spin-1/2 cluster c open chain that is protected by its integrability if spontaneous symmetry breaking is precluded.

c	$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}])$	$([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}])$	$\dim \mathfrak{h}^{(c)}_{\mathrm{B,min}}$	$D(c)$	$\dim \mathfrak{h}^{(c)}_{\mathrm{gs}}$
0	$(0,0)$	$(0,0)$	1	1	1
1	$(0,0)$	$(0,0)$	1	1	$2=2^1$
2	$(0,1)$	$(0,1)$	2	4	$4=2^2$
3	$(0,1)$	$(0,1)$	2	4	$8=2^3$
4	$(1,0)$	$(1,0)$	2	4	$16=2^4$
5	$(1,1)$	$(1,1)$	2	4	$32=2^5$
6	$(1,1)$	$(1,1)$	2	4	$64=2^6$
7	$(0,0)$	$(0,0)$	1	1	$128=2^7$
8	$(0,0)$	$(0,0)$	1	1	$256=2^8$

10.3.2.2 *The Case of $c=+2$*

When $c=+2$, the choice of open boundary conditions implies that Hamiltonian

$$\widehat{H}^{(2)}_{2} = \sum_{j=1}^{2N-2} \widehat{Z}_{j}\,\widehat{X}_{j+1}\,\widehat{Z}_{j+2} \tag{10.51}$$

has a $2^2=4$-fold degenerate and gapped ground states. Since open boundary conditions are selected, the pair of commuting operators $\widehat{Z}_{2N-1}\,\widehat{X}_{2N}\,\widehat{Z}_{1}$, $\widehat{Z}_{2N}\,\widehat{X}_{1}\,\widehat{Z}_{2}$, are present in $\widehat{H}^{(0)}_{2}$ but are absent in $\widehat{H}^{(1)}_{2}$. The set of all operators that commute with the Hamiltonian (10.51) and have support on the left boundary is the Clifford algebra $C\ell_2$ spanned by the generators

$$\left\{\widehat{X}_{1}\,\widehat{Z}_{2}, \quad \widehat{Z}_{1}\right\}. \tag{10.52}$$

We define define the triplet

$$\widehat{S}^x_{\mathrm{L}} := \frac{1}{2}\,\widehat{X}_1\,\widehat{Z}_2, \quad \widehat{S}^y_{\mathrm{L}} := \frac{1}{2}\,\widehat{Y}_1\,\widehat{Z}_2, \quad \widehat{S}^z_{\mathrm{L}} := \frac{1}{2}\,\widehat{Z}_1, \tag{10.53}$$

of operators that obey the $\mathfrak{su}(2)$ Lie algebra. We deduce the transformation laws

$$\widehat{U}_{\mathrm{e}}(t')\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{e}}(t') = \begin{pmatrix}+\widehat{S}^x_{\mathrm{L}} & -\widehat{S}^y_{\mathrm{L}} & +\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}, \tag{10.54a}$$

$$\widehat{U}_{\mathrm{e}}(g)\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{e}}(g) = \begin{pmatrix}-\widehat{S}^x_{\mathrm{L}} & +\widehat{S}^y_{\mathrm{L}} & -\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}. \tag{10.54b}$$

We define the action K_{L} of complex conjugation on the four-dimensional representation (10.53) of the $\mathfrak{su}(2)$ Lie algebra by demanding that

$$\mathsf{K}_{\mathrm{L}}\,\widehat{S}^x_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^x_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^y_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := -\widehat{S}^y_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^z_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^z_{\mathrm{L}}. \tag{10.55a}$$

Demanding the consistency condition (10.47) to hold, we find the boundary representation

$$\widehat{U}_{\mathrm{e,L}}(t') := \mathbb{1}_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}, \qquad \widehat{U}_{\mathrm{e,L}}(g) := \widehat{S}^y_{\mathrm{L}}. \tag{10.55b}$$

Using the definition (10.2), we associate the pair of indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}]) = (0,1) \tag{10.55c}$$

to the spin-1/2 cluster $c=+2$ chain. One verifies that the Clifford algebra spanned by the generators

$$\left\{\widehat{Z}_{2N-1}\,\widehat{X}_{2N}, \quad \widehat{Z}_{2N}\right\} \tag{10.56a}$$

for the right boundary $\Lambda_{\mathrm{R}} := \{j = 2N-1, 2N\}$ delivers the pair of indices

$$([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}]) = (0,1) \tag{10.56b}$$

for the projective representation of t' and g on the right boundary.

10.3.2.3 *The Case of $c=-2$*

The case $c=-2$ is deduced from the case $c=+2$ by interchanging all the $\widehat{Z}_j$ and $\widehat{Y}_j$ operators for $j=1,\cdots,2N$. The set $\mathfrak{O}_{\mathrm{L}}$ of all operators that commute with the

Hamiltonian $\widehat{H}^{(2)}_{-2}$ and have support on the boundary $\Lambda_{\mathrm{L}} := \{j = 1,2\}$ is the Clifford algebra $C\ell_2$ spanned by the generators

$$\left\{\widehat{X}_1\,\widehat{Y}_2,\quad \widehat{Y}_1\right\}. \tag{10.57}$$

The Clifford algebra with the generators (10.57) contains the $\mathfrak{su}(2)$ Lie algebra generated by the operators

$$\widehat{S}^x_{\mathrm{L}} := \frac{1}{2}\,\widehat{X}_1\,\widehat{Y}_2, \qquad \widehat{S}^y_{\mathrm{L}} := -\frac{1}{2}\,\widehat{Z}_1\,\widehat{Y}_2, \qquad \widehat{S}^z_{\mathrm{L}} := \frac{1}{2}\,\widehat{Y}_1. \tag{10.58}$$

We deduce the transformation laws

$$\widehat{U}_{\mathrm{e}}(t')\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^\dagger_{\mathrm{e}}(t') = \begin{pmatrix}-\widehat{S}^x_{\mathrm{L}} & -\widehat{S}^y_{\mathrm{L}} & -\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}, \tag{10.59a}$$

$$\widehat{U}_{\mathrm{e}}(g)\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^\dagger_{\mathrm{e}}(g) = \begin{pmatrix}-\widehat{S}^x_{\mathrm{L}} & +\widehat{S}^y_{\mathrm{L}} & -\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}. \tag{10.59b}$$

We define the action K_{L} of complex conjugation on the four-dimensional representation (10.58) of the $\mathfrak{su}(2)$ Lie algebra by demanding that

$$\mathsf{K}_{\mathrm{L}}\,\widehat{S}^x_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^x_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^y_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := -\widehat{S}^y_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^z_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^z_{\mathrm{L}}. \tag{10.60a}$$

Demanding the consistency condition (10.47) to hold, we find the boundary representation

$$\widehat{U}_{\mathrm{e,L}}(t') := \widehat{S}^y_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} \qquad \widehat{U}_{\mathrm{e,L}}(g) := \widehat{S}^y_{\mathrm{L}}. \tag{10.60b}$$

Using the definition (10.2), we associate the pair of indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}]) = (1,1) \tag{10.60c}$$

to the spin-1/2 cluster $c = -2$ chain. One verifies that the Clifford algebra spanned by the generators

$$\left\{\widehat{Y}_{2N-1}\,\widehat{X}_{2N},\quad \widehat{Y}_{2N}\right\} \tag{10.61a}$$

for the right boundary $\Lambda_{\mathrm{R}} := \{j = 2N-1, 2N\}$ delivers the pair of indices

$$([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}]) = (1,1) \tag{10.61b}$$

for the projective representation of t' and g on the right boundary.

10.3.2.4 *The Case of* $c = +4$

When $c = +4$, the choice of open boundary conditions implies that Hamiltonian

$$\widehat{H}_4^{(1)} = \sum_{j=1}^{2N-4} \widehat{Z}_j\,\widehat{X}_{j+1}\,\widehat{X}_{j+2}\,\widehat{X}_{j+3}\,\widehat{Z}_{j+4} \tag{10.62}$$

has a $2^4 = 16$-fold degenerate and gapped ground states.

Since open boundary conditions are selected, the quadruplet of pairwise commuting operators

$$\begin{aligned} &\widehat{Z}_{2N-3}\,\widehat{X}_{2N-2}\,\widehat{X}_{2N-1}\,\widehat{X}_{2N}\,\widehat{Z}_1, \qquad \widehat{Z}_{2N-2}\,\widehat{X}_{2N-1}\,\widehat{X}_{2N}\,\widehat{X}_1\,\widehat{Z}_2, \\ &\qquad \widehat{Z}_{2N-1}\,\widehat{X}_{2N}\,\widehat{X}_1\,\widehat{X}_2\,\widehat{Z}_3, \qquad \widehat{Z}_{2N}\,\widehat{X}_1\,\widehat{X}_2\,\widehat{X}_3\,\widehat{Z}_4, \end{aligned} \tag{10.63}$$

are present in $\widehat{H}_4^{(0)}$ but are absent in $\widehat{H}_4^{(1)}$. The set $\mathfrak{O}_{\mathrm{L}}$ of all operators that commute with Hamiltonian (10.62) and have support on the boundary $\Lambda_{\mathrm{L}} := \{j = 1, 2, 3, 4\}$ is the Clifford algebra $\mathrm{C}\ell_4$ spanned by the generators

$$\left\{\widehat{X}_1\,\widehat{X}_2\,\widehat{X}_3\,\widehat{Z}_4, \quad \widehat{X}_1\,\widehat{X}_2\,\widehat{Z}_3, \quad \widehat{X}_1\,\widehat{Z}_2, \quad \widehat{Z}_1\right\}. \tag{10.64}$$

We define the pair of triplets

$$\widehat{S}_{\mathrm{L}}^{x} := \frac{1}{2}\,\widehat{X}_1\,\widehat{X}_2\,\widehat{X}_3\,\widehat{Z}_4, \quad \widehat{S}_{\mathrm{L}}^{y} := -\frac{1}{2}\,\widehat{X}_1\,\widehat{X}_2\,\widehat{Z}_3, \quad \widehat{S}_{\mathrm{L}}^{z} := \frac{1}{2}\,\widehat{Y}_3\,\widehat{Z}_4, \tag{10.65a}$$

$$\widehat{J}_{\mathrm{L}}^{x} := \frac{1}{2}\,\widehat{X}_1\,\widehat{Z}_2\,\widehat{Y}_3\,\widehat{Z}_4, \quad \widehat{J}_{\mathrm{L}}^{y} := -\frac{1}{2}\,\widehat{Z}_1\,\widehat{Y}_3\,\widehat{Z}_4, \quad \widehat{J}_{\mathrm{L}}^{z} := \frac{1}{2}\,\widehat{Y}_1\,\widehat{Z}_2 \tag{10.65b}$$

of operators that obey the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ Lie algebra. We deduce the transformation laws

$$\widehat{U}_{\mathrm{e}}(t')\,\begin{pmatrix}\widehat{S}_{\mathrm{L}}^{x} & \widehat{S}_{\mathrm{L}}^{y} & \widehat{S}_{\mathrm{L}}^{z}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}_{\mathrm{e}}^{\dagger}(t') = \begin{pmatrix}+\widehat{S}_{\mathrm{L}}^{x} & +\widehat{S}_{\mathrm{L}}^{y} & -\widehat{S}_{\mathrm{L}}^{z}\end{pmatrix}^{\mathsf{T}}, \tag{10.66a}$$

$$\widehat{U}_{\mathrm{e}}(g)\,\begin{pmatrix}\widehat{S}_{\mathrm{L}}^{x} & \widehat{S}_{\mathrm{L}}^{y} & \widehat{S}_{\mathrm{L}}^{z}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}_{\mathrm{e}}^{\dagger}(g) = \begin{pmatrix}-\widehat{S}_{\mathrm{L}}^{x} & -\widehat{S}_{\mathrm{L}}^{y} & +\widehat{S}_{\mathrm{L}}^{z}\end{pmatrix}^{\mathsf{T}}, \tag{10.66b}$$

and

$$\widehat{U}_{\mathrm{e}}(t')\,\begin{pmatrix}\widehat{J}_{\mathrm{L}}^{x} & \widehat{J}_{\mathrm{L}}^{y} & \widehat{J}_{\mathrm{L}}^{z}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}_{\mathrm{e}}^{\dagger}(t') = \begin{pmatrix}-\widehat{J}_{\mathrm{L}}^{x} & -\widehat{J}_{\mathrm{L}}^{y} & -\widehat{J}_{\mathrm{L}}^{z}\end{pmatrix}^{\mathsf{T}}, \tag{10.66c}$$

$$\widehat{U}_{\mathrm{e}}(g)\,\begin{pmatrix}\widehat{J}^{x}_{\mathrm{L}} & \widehat{J}^{y}_{\mathrm{L}} & \widehat{J}^{z}_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\widehat{U}^{\dagger}_{\mathrm{e}}(g)=\begin{pmatrix}-\widehat{J}^{x}_{\mathrm{L}} & -\widehat{J}^{y}_{\mathrm{L}} & +\widehat{J}^{z}_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}. \tag{10.66d}$$

We define the action K_{L} of complex conjugation on the four-dimensional representation (10.65) of the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ Lie algebra by demanding that

$$\mathsf{K}_{\mathrm{L}}\,\widehat{S}^{x}_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}:=+\widehat{S}^{x}_{\mathrm{L}},\qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^{y}_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}:=-\widehat{S}^{y}_{\mathrm{L}},\qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^{z}_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}:=+\widehat{S}^{z}_{\mathrm{L}}, \tag{10.67a}$$

$$\mathsf{K}_{\mathrm{L}}\,\widehat{J}^{x}_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}:=-\widehat{J}^{x}_{\mathrm{L}},\qquad \mathsf{K}_{\mathrm{L}}\,\widehat{J}^{y}_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}:=+\widehat{J}^{y}_{\mathrm{L}},\qquad \mathsf{K}_{\mathrm{L}}\,\widehat{J}^{z}_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}:=+\widehat{J}^{z}_{\mathrm{L}}. \tag{10.67b}$$

Demanding the consistency condition (10.47) to hold, we find the boundary representation

$$\widehat{U}_{\mathrm{e,L}}(t'):=\widehat{S}^{x}_{\mathrm{L}}\,\widehat{J}^{x}_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}},\qquad \widehat{U}_{\mathrm{e,L}}(g):=\widehat{S}^{z}_{\mathrm{L}}\,\widehat{J}^{z}_{\mathrm{L}}. \tag{10.67c}$$

Using the definition (10.2), we associate the pair of indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}])=(1,0) \tag{10.67d}$$

to the spin-1/2 cluster $c=+4$ chain. One verifies that the Clifford algebra spanned by the generators

$$\left\{\widehat{Z}_{2N-3}\,\widehat{X}_{2N-2}\,\widehat{X}_{2N-1}\,\widehat{X}_{2N},\quad \widehat{Z}_{2N-2}\,\widehat{X}_{2N-1}\,\widehat{X}_{2N},\quad \widehat{Z}_{2N-1}\,\widehat{X}_{2N},\quad \widehat{Z}_{2N}\right\} \tag{10.68a}$$

for the right boundary $\Lambda_{\mathrm{R}}:=\{j=2N-3,2N-2,2N-1,2N\}$ delivers the pair of indices

$$([\nu_{\mathrm{R}}],[\rho_{\mathrm{R}}])=(1,0) \tag{10.68b}$$

for the projective representation of t' and g on the right boundary.

10.3.2.5 *The Case of $c=-4$*

The case $c=-4$ is deduced from the case $c=+4$ by interchanging all the $\widehat{Z}_j$ and $\widehat{Y}_j$ operators for $j=1,\cdots,2N$. The set $\mathfrak{O}_{\mathrm{L}}$ of all operators that commute with the Hamiltonian $\widehat{H}^{(4)}_{-4}$ and have support on the boundary

$$\Lambda_{\mathrm{L}}:=\{j=1,2,3,4\} \tag{10.69a}$$

is the Clifford algebra $C\ell_4$ spanned by the generators

$$\{\widehat{Y}_1, \quad \widehat{X}_1\,\widehat{Y}_2, \quad \widehat{X}_1\,\widehat{X}_2\,\widehat{Y}_3, \quad \widehat{X}_1\,\widehat{X}_2\,\widehat{X}_3\,\widehat{Y}_4\}. \tag{10.69b}$$

The Clifford algebra with the generators (10.69b) contains the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ Lie algebra generated by the operators

$$\widehat{S}^x_{\mathrm{L}} := \frac{1}{2}\,\widehat{X}_1\,\widehat{X}_2\,\widehat{X}_3\,\widehat{Y}_4, \qquad \widehat{S}^y_{\mathrm{L}} := \frac{1}{2}\,\widehat{X}_1\,\widehat{X}_2\,\widehat{Y}_3, \qquad \widehat{S}^z_{\mathrm{L}} := \frac{1}{2}\,\widehat{Z}_3\,\widehat{Y}_4, \tag{10.70}$$

$$\widehat{J}^x_{\mathrm{L}} := \frac{1}{2}\,\widehat{X}_1\,\widehat{Y}_2\,\widehat{Z}_3\,\widehat{Y}_4, \qquad \widehat{J}^y_{\mathrm{L}} := \frac{1}{2}\,\widehat{Y}_1\,\widehat{Z}_3\,\widehat{Y}_4, \qquad \widehat{J}^z_{\mathrm{L}} := \frac{1}{2}\,Z_1\,Y_2. \tag{10.71}$$

We deduce the transformation laws

$$\widehat{U}_{\mathrm{e}}(t')\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{e}}(t') = \begin{pmatrix}-\widehat{S}^x_{\mathrm{L}} & -\widehat{S}^y_{\mathrm{L}} & -\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}, \tag{10.72a}$$

$$\widehat{U}_{\mathrm{e}}(g)\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{e}}(g) = \begin{pmatrix}-\widehat{S}^x_{\mathrm{L}} & -\widehat{S}^y_{\mathrm{L}} & +\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}, \tag{10.72b}$$

and

$$\widehat{U}_{\mathrm{e}}(t')\,\begin{pmatrix}\widehat{J}^x_{\mathrm{L}} & \widehat{J}^y_{\mathrm{L}} & \widehat{J}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{e}}(t') = \begin{pmatrix}+\widehat{J}^x_{\mathrm{L}} & +\widehat{J}^y_{\mathrm{L}} & -\widehat{J}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}, \tag{10.72c}$$

$$\widehat{U}_{\mathrm{e}}(g)\,\begin{pmatrix}\widehat{J}^x_{\mathrm{L}} & \widehat{J}^y_{\mathrm{L}} & \widehat{J}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{e}}(g) = \begin{pmatrix}-\widehat{J}^x_{\mathrm{L}} & -\widehat{J}^y_{\mathrm{L}} & +\widehat{J}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}. \tag{10.72d}$$

We define the action K_{L} of complex conjugation on the sixteen-dimensional representation (10.58) of the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ Lie algebra by demanding that

$$\mathsf{K}_{\mathrm{L}}\,\widehat{S}^x_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := -\widehat{S}^x_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^y_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^y_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^z_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^z_{\mathrm{L}}, \tag{10.73a}$$

and

$$\mathsf{K}_{\mathrm{L}}\,\widehat{J}^x_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{J}^x_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{J}^y_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := -\widehat{J}^y_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{J}^z_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{J}^z_{\mathrm{L}}. \tag{10.73b}$$

Demanding the consistency condition (10.47) to hold, we find the boundary representation

$$\widehat{U}_{\mathrm{e,L}}(t') := \widehat{S}^x_{\mathrm{L}}\,\widehat{J}^x_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}, \qquad \widehat{U}_{\mathrm{e,L}}(g) := \widehat{S}^z_{\mathrm{L}}\,\widehat{J}^z_{\mathrm{L}}. \tag{10.73c}$$

Using the definition (10.2), we associate the pair of indices

$$([\nu_{\mathrm{L}}],[\rho_{\mathrm{L}}]) = (1,0) \tag{10.73d}$$

to the spin-1/2 cluster $c = -4$ chain. This is the same pair of indices as in Eq.$\tilde{(}$10.67d).

One verifies that the Clifford algebra spanned by the generators

$$\left\{\widehat{Y}_{2N-3}\,\widehat{X}_{2N-2}\,\widehat{X}_{2N-1}\,\widehat{X}_{2N},\quad \widehat{Y}_{2N-2}\,\widehat{X}_{2N-1}\,\widehat{X}_{2N},\quad \widehat{Y}_{2N-1}\,\widehat{X}_{2N},\quad \widehat{Y}_{2N}\ \right\} \tag{10.74a}$$

for the right boundary $\Lambda_{\mathrm{R}} := \{j = 2N-3, 2N-2, 2N-1, 2N\}$ delivers the pair of indices

$$([\nu_{\mathrm{R}}], [\rho_{\mathrm{R}}]) = (1, 0) \tag{10.74b}$$

for the projective representation of t' and g on the right boundary. This is the same pair of indices as in Eq. (10.68b).

10.3.2.6 *The Cases of* $c = \pm 1$

When $c = \pm 1$, the mean-field Hamiltonian (10.40a) becomes

$$\widehat{H}^{(1)}_{\mathrm{MF},\pm 1} = \begin{cases} -\sum_{j=1}^{2N} \widehat{Z}_j, & \text{if } c = +1, \\ \\ -\frac{\hbar\omega}{2} \sum_{j=1}^{2N} \widehat{Y}_j, & \text{if } c = -1, \end{cases} \tag{10.75}$$

which is Hamiltonian (10.48) with $\widehat{X}_j$ replaced by $\widehat{Z}_j$ for $c = +1$ or $\widehat{Y}_j$ for $c = -1$.

Consequently, the set of gapless boundary degrees of freedom is empty, i.e.,

$$\mathfrak{D}_{\mathrm{B},\pm 1} \coloneqq \{\,\}, \qquad \mathrm{B} = \mathrm{L}, \mathrm{R}. \tag{10.76}$$

By convention, we associate the trivial indices

$$([\nu], [\rho]) = (0, 0) \tag{10.77}$$

to the spin-1/2 cluster $c = \pm 1$ chains.

10.3.2.7 *The Case of* $c = +3$

When $c = +3$, the mean-field Hamiltonian (10.40a) with open boundary conditions becomes

$$\widehat{H}^{(1)}_{\mathrm{MF},+3} = \sum_{j=1}^{2N-1} \widehat{Z}_j\,\widehat{Y}_{j+1}\,\widehat{Z}_{j+2}. \tag{10.78}$$

This mean-field Hamiltonian follows from Hamiltonian (10.51) upon replacing $\widehat{X}_j$ with $\widehat{Y}_j$. One verifies the existence of a Clifford algebra $C\ell_2$ on the left boundary $\Lambda_{\mathrm{L}} = \{j = 1, 2\}$ that (i) commutes with Hamiltonian (10.78) and (ii) contains an $\mathfrak{su}(2)$ Lie algebra whose generators are

$$\widehat{S}^x_{\mathrm{L}} := \frac{1}{2}\,\widehat{Y}_1\,\widehat{Z}_2, \qquad \widehat{S}^y_{\mathrm{L}} := -\frac{1}{2}\,\widehat{X}_1\,\widehat{Z}_2, \qquad \widehat{S}^z_{\mathrm{L}} := \frac{1}{2}\,\widehat{Z}_1. \tag{10.79}$$

We deduce the transformation laws

$$\widehat{U}_{\mathrm{MF}}(\tilde{t}')\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{MF}}(\tilde{t}') = \begin{pmatrix}+\widehat{S}^x_{\mathrm{L}} & -\widehat{S}^y_{\mathrm{L}} & +\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}, \tag{10.80a}$$

$$\widehat{U}_{\mathrm{MF}}(\tilde{g})\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{MF}}(\tilde{g}) = \begin{pmatrix}-\widehat{S}^x_{\mathrm{L}} & +\widehat{S}^y_{\mathrm{L}} & -\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}. \tag{10.80b}$$

We define the action K_{L} of complex conjugation on the four-dimensional representation (10.79) of the $\mathfrak{su}(2)$ Lie algebra by demanding that

$$\mathsf{K}_{\mathrm{L}}\,\widehat{S}^x_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := -\widehat{S}^x_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^y_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^y_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^z_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^z_{\mathrm{L}}. \tag{10.81a}$$

Demanding the consistency condition (10.47) to hold, we find the boundary representation

$$\widehat{U}_{\mathrm{MF,L}}(\tilde{t}') := \widehat{S}^z_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}, \qquad \widehat{U}_{\mathrm{MF,L}}(\tilde{g}) := \widehat{S}^y_{\mathrm{L}}. \tag{10.81b}$$

Using the definition (10.2), we associate the pair of indices

$$([\nu_{\mathrm{L}}], [\rho_{\mathrm{L}}]) = (0, 1) \tag{10.81c}$$

to the spin-1/2 cluster $c = +3$ chain. This is the same pair of indices as in Eq. (10.55c). One verifies that the pair of indices

$$([\nu_{\mathrm{R}}], [\rho_{\mathrm{R}}]) = (0, 1) \tag{10.82}$$

is found for the projective representation of $\tilde{t}'$ and $\tilde{g}$ on the right boundary $\Lambda_{\mathrm{R}} = \{2N-1, 2N\}$.

10.3.2.8 *The Case of $c=-3$*

The case of $c=-3$ is deduced from the case of $c=+3$ by interchanging all the $\widehat{Z}_j$ and $\widehat{Y}_j$ operators for $j=1,\cdots,2N$ in the mean-field Hamiltonian (10.78). One verifies the existence of a Clifford algebra $\mathrm{C}\ell_2$ on the left boundary

$$\Lambda_{\mathrm{L}} = \{j=1,2\} \tag{10.83}$$

that (i) commutes with the mean-field Hamiltonian

$$\widehat{H}^{(1)}_{\mathrm{MF},-3} = \sum_{j=1}^{2N-1} \widehat{Y}_j\,\widehat{Z}_{j+1}\,\widehat{Y}_{j+2} \tag{10.84}$$

and (ii) contains an $\mathfrak{su}(2)$ Lie algebra whose generators are

$$\widehat{S}^x_{\mathrm{L}} := \frac{1}{2}\,\widehat{Z}_1\,\widehat{Y}_2, \qquad \widehat{S}^y_{\mathrm{L}} := \frac{1}{2}\,\widehat{X}_1\,\widehat{Y}_2, \qquad \widehat{S}^z_{\mathrm{L}} := \frac{1}{2}\,\widehat{Y}_1. \tag{10.85}$$

We deduce the transformation laws

$$\widehat{U}_{\mathrm{MF}}(\tilde{t}')\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{MF}}(\tilde{t}') = \begin{pmatrix}-\widehat{S}^x_{\mathrm{L}} & -\widehat{S}^y_{\mathrm{L}} & -\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}, \tag{10.86a}$$

$$\widehat{U}_{\mathrm{MF}}(\tilde{g})\,\begin{pmatrix}\widehat{S}^x_{\mathrm{L}} & \widehat{S}^y_{\mathrm{L}} & \widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}\,\widehat{U}^{\dagger}_{\mathrm{MF}}(\tilde{g}) = \begin{pmatrix}-\widehat{S}^x_{\mathrm{L}} & +\widehat{S}^y_{\mathrm{L}} & -\widehat{S}^z_{\mathrm{L}}\end{pmatrix}^{\mathsf{T}}. \tag{10.86b}$$

Third, we denote by $\mathbb{1}_{\mathrm{L}}$ the unit 4×4 matrix and we define the action K_{L} of complex conjugation on the four-dimensional representation (10.85) of the $\mathfrak{su}(2)$ Lie algebra by demanding that

$$\mathsf{K}_{\mathrm{L}}\,\widehat{S}^x_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := -\widehat{S}^x_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^y_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^y_{\mathrm{L}}, \qquad \mathsf{K}_{\mathrm{L}}\,\widehat{S}^z_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}} := +\widehat{S}^z_{\mathrm{L}}. \tag{10.87a}$$

Demanding the consistency condition (10.47) to hold, we find the boundary representation

$$\widehat{U}_{\mathrm{MF,L}}(\tilde{t}') := \widehat{S}^x_{\mathrm{L}}\,\mathsf{K}_{\mathrm{L}}, \qquad \widehat{U}_{\mathrm{MF,L}}(\tilde{g}) := \widehat{S}^y_{\mathrm{L}}. \tag{10.87b}$$

Using the definition (10.2), we associate the pair of indices

$$([\nu_{\mathrm{L}}], [\rho_{\mathrm{L}}]) = (1, 1) \tag{10.87c}$$

to the spin-1/2 cluster $c = -3$ chain. This is the same pair of indices as in Eq. (10.60c). One verifies that the pair of indices

$$([\nu_{\mathrm{R}}], [\rho_{\mathrm{R}}]) = (1, 1) \tag{10.88}$$

is found for the projective representation of $\tilde{t}'$ and $\tilde{g}$ on the left boundary $\Lambda_{\mathrm{R}} = \{2N - 1, 2N\}$.

Part III

LSM THEOREMS AND CRYSTALLINE TOPOLOGICAL PHASES

Adapted from:
Ö. M. Aksoy, J.-H. Chen, S. Ryu, A. Frusaki and C. Mudry
"Stability against contact interactions of a topological superconductor in two-dimensional space protected by time-reversal and reflection symmetries",
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Part III is devoted to the crystalline invertible topological phases and their relation to the LSM constraints. Chapter 11 relates Theorems 2 and 3 from Part I to the weak topological phases. Therein, generalizations of LSM theorems to crystalline symmetries other than translations are discussed. We provide a view from the literature to propose a classification scheme for certain generalized LSM constraints.

Chapter 12 presents an example of crystalline topological phases, namely, two-dimensional topological superconductor in symmetry class DIIIR. This topological phase has $\mathbb{Z}_8$-classification and which is due to the combination of internal symmetry group $G_f = \mathbb{Z}_4^{\mathrm{FT}}$, reflection symmetry, and, translation symmetry. We study the stability of protected boundary states of this topological superconductor and interpret our results from the perspective of generalized LSM constraints.

11

FROM CRYSTALLINE TOPOLOGICAL PHASES TO LSM THEOREMS

In this Chapter, we are going to present Theorems 2 and 3 from a complementary perspective. This line of thought was explored in Refs. [58, 65, 67, 172] and asserts that there is a one to one correspondence between certain LSM type constraints that involve translation symmetry and the classification of *weak* topological phases. In Sec. 11.1, we illustrate this point of view by recasting Theorems 2 and 3 as ingappability conditions at the boundaries of two-dimensional weak topological phases. In Sec. 11.2, we will argue that this correspondence extends to that between generalized LSM type constraints involving space group symmetries and certain *crystalline* topological phases.

11.1 LSM THEOREMS FROM WEAK TOPOLOGICAL PHASES

Theorems 2 and 3 are statements that rule out the possibility of a symmetric, nondegenerate, and, gapped ground state. Consequently, there are only two ways for a Hamiltonian with an LSM constraint to have a gapped spectrum: (i) either translation or internal symmetries are broken (spontaneously or explicitly), or (ii) for spatial dimensions larger than one the ground state supports topological order. Stated differently, gapless nature of the ground state is protected by a combination of internal and translation symmetries. Such a protected gaplessness is reminiscent of the protected boundary modes of invertible topological phases. While the boundaries of *strong* topological phases are protected merely by internal symmetries, weak and crystalline topological phases have boundary modes that are protected by a combination of internal and spatial symmetries.

To better see these parallels, consider the following setup. Let $\widehat{H}_{1\mathrm{D}}$ be a representative Hamiltonian of a one-dimensional IFT phase with symmetry group G_f and indices $([(\nu,\rho)],[\mu])$. This is a one-dimensional strong topological phase since no spatial symmetry is needed to protect the boundary zero modes. We construct a two-dimensional weak

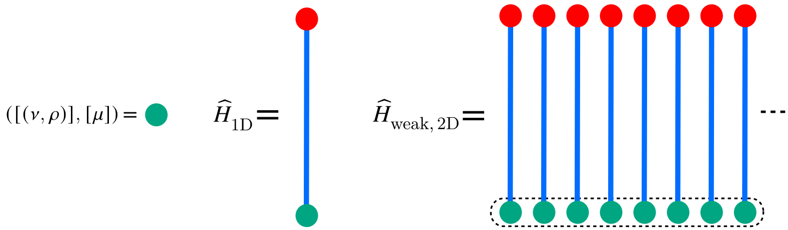

Figure 11.1: Construction of two-dimensional weak topological phase by stacking one-dimensional strong topological phases. Each blue line represents a one-dimensional chain realizing a nontrivial IFT phase with indices $([(\nu,\rho)],[\mu])$. The representative Hamiltonian $\widehat{H}_{\mathrm{weak,\,2D}}$ is obtained by taking an array of one-dimensional Hamiltonians $\widehat{H}_{\mathrm{1D}}$. When open boundary conditions are imposed, LSM Theorems 2 and 3 ensure that the translationally invariant boundaries support protected gapless states.

topological phase by assembling N copies of $\widehat{H}_{\mathrm{1D}}$ in a translationally invariant array. Concretely, we consider the Hamiltonian

$$\widehat{H}_{\mathrm{weak,\,2D}} := \sum_{j=1}^{N} \hat{h}_{\mathrm{1D},j}, \qquad \hat{h}_{\mathrm{1D},j} \equiv \widehat{H}_{\mathrm{1D}}, \tag{11.1}$$

which is nothing but N decoupled copies of one-dimensional chains described by $\widehat{H}_{\mathrm{1D}}$. This construction is shown in Fig. 11.1. When periodic boundary conditions are imposed, Hamiltonian (11.1) has a nondegenerate, gapped, and, symmetric (both translation and G_f) ground state. When open boundary conditions are imposed, each decoupled chain supports zero-energy states at its boundaries, owing to the fact that $\widehat{H}_{\mathrm{1D}}$ itself realizes a nontrivial IFT phase. The one-dimensional boundary (shown by dashed lines in Fig. 11.1) is then a translationally invariant lattice where each repeat unit cell carries a projective representation of G_f that is characterized by indices $([(\nu,\rho)],[\mu])$. If so, LSM Theorems 2 and 3 applies. This means that Hamiltonian (11.1) supports gapless boundary modes when open boundary conditions are imposed. The gaplessness of the boundaries are protected as any gap-opening perturbation must break either translation or G_f symmetry.

So far, we have argued that LSM Theorems 2 and 3 can be used to show that the gapless boundary states of weak topological phases are stable against perturbations (up to symmetry breaking). Conversely, assumption that weak topological phases support protected gapless boundary states, implies certain LSM Theorems. For simplicity, we focus on the case of LSM Theorems that are not intrinsically fermionic, i.e., $\rho = 0$ and $[\mu] = 0$. The only nonzero index is then ν which corresponds to the projective representations of the group $G \cong G_f/\mathbb{Z}_2^{\mathrm{F}}$ (see Appendix A.4.3). This is to say that we are focusing only on LSM Theorems that apply to both bosonic and fermionic models. We are going to use the following two statements without proofs.

Claim 1. Strong bosonic symmetry protected phases (BSPT) in two dimension with G-symmetry are classified by the third cohomology group $H^3(G, \mathrm{U}(1)_{\mathfrak{c}})$ [31].

Claim 2. Given the groups G of internal symmetries, $\mathbb{Z}$ of lattice translations, and, $G_{\mathrm{tot}} = G \times \mathbb{Z}$ of their direct product, all BSPT phases protected by G_{tot} are classified by the third cohomology group $H^3(G_{\mathrm{tot}} = G \times \mathbb{Z}, \mathrm{U}(1)_{\mathfrak{c}})$ [173, 174].

These statements mean that the classification of weak topological phases are contained in the third cohomology group $H^3(G \times \mathbb{Z}, \mathrm{U}(1)_{\mathfrak{c}})$. This becomes transparent upon using the Künneth formula of group cohomology [58, 175–177]

$$H^3(G \times \mathbb{Z}, \mathrm{U}(1)_{\mathfrak{c}}) = H^3(G, \mathrm{U}(1)_{\mathfrak{c}}) \times H^2(G, \mathrm{U}(1)_{\mathfrak{c}}). \tag{11.2}$$

Hereby, $H^3(G, \mathrm{U}(1)_{\mathfrak{c}})$ corresponds to the strong BSPT phases protected by G alone while $H^2(G, \mathrm{U}(1)_{\mathfrak{c}})$ corresponds to the weak BSPT phases protected by translations and internal G-symmetry together. It is not a coincidence that the latter is classified by $H^2(G, \mathrm{U}(1)_{\mathfrak{c}})$ which enumerates the projective representations of the group G. This is nothing but the LSM Theorem 2 at the boundary of weak BSPT phases.

It is not hard to see that construction that lead to Hamiltonian (11.1) can be generalized to obtain higher dimensional weak topological phases. In turn, the classification of these topological phases can be used to obtain various LSM constraints for lower dimensional spaces. From this point of view, generalizations of Theorem 2 to any dimension follows, even though our proof in Sec. 3.4 only applies to the case of Abelian and unitarily represented G_f.

11.2 GENERALIZED LSM THEOREMS AND CRYSTALLINE TOPOLOGICAL PHASES

In our treatment of LSM constraints in Part I, we only considered models with internal and translation symmetries. Recently, LSM constraints have been extended for the Hamiltonians with crystalline symmetries other than translations [59, 60, 62, 63, 172, 178]. For example, it has been rigorously proved in Ref. [63] that for a reflection symmetric spin chain, if the internal symmetry is represented by a nontrivial projective representation at the reflection center, then the ground state cannot be nondegenerate, gapped, and, symmetric. This is reminiscent of what happens at the boundary of a two-dimensional *higher-order* topological phases with reflection symmetry [110]. For such a system boundary is gapped except at the reflection centers where zero-energy modes are localized.

In light of the equivalence between LSM theorems and weak topological phases as described in Sec. 11.1, it is natural to expect a correspondence between generalized LSM constraints and crystalline topological phases. The key step behind this correspondence is the so-called *crystalline equivalence principle* (CEP) [173, 174]. Let us first focus on purely bosonic phases, then CEP asserts the following one-to-one correspondence between crystalline and strong topological phases.

Claim 3 (CEP)**.** Let $G_{\text{tot}} = G \rtimes G_{\text{spc}}$ be a total symmetry group where G and G_{spc} are the groups of internal and crystallographic symmetries. With the semi-direct product $\rtimes$, we allowed for a nontrivial group action of crystallographic symmetries on the internal ones. Then, for any crystalline topological phase with G_{tot}-symmetry that can be realized by a Hamiltonian supported on a contractible spatial manifold, there exists a strong topological phase with internal symmetry group $G_{\text{int}} = G \rtimes \widetilde{G}_{\text{spc}}$. Here, $\widetilde{G}_{\text{spc}}$ is obtained from G_{spc} by replacing all orientation-reversing unitary symmetries by antiunitary ones (e.g., $\mathbb{Z}_2$ inversions in odd-dimensional space are mapped to $\mathbb{Z}_2^T$ reversal of time).

Crystalline equivalence principle 3 is the generalization of the claim 2 and prescribes a classification of generalized LSM constraints. For bosonic systems, enumeration of such theorems in d dimensional space are contained in the cohomology group [31, 177]

$$H^{d+2}\left(G \rtimes \widetilde{G}_{\text{spc}},\, \mathrm{U}(1)_{\mathfrak{c}}\right). \tag{11.3a}$$

This cohomology group contains the classification of strong, crystalline, and, higher-order BSPT phases. The LSM constraints can be obtained after specifying the space group G_{spc}. When the group action of crystalline symmetries is trivial, i.e., $G_{\text{tot}} = G \times G_{\text{spc}}$, an elegant decomposition of Eq. (11.3a) into strong, and higher-order SPTs is possible [177]. For example, the generalized LSM constraints in d dimension with finite space groups (such as point-group symmetries or reflections) are characterized by

$$H^{d}\left(\widetilde{G}_{\text{spc}},\, H^{2}\left(G, \mathrm{U}(1)_{\mathfrak{c}}\right)\right). \tag{11.3b}$$

This group enumerates the projective representations of G per G_{spc}-invariant submanifolds in d-dimensional space, that cannot be trivialized by lattice deformations. Setting $G = \mathbb{Z}_1$, the trivial group, allows one to enumerate LSM constraints that are purely due to spatial symmetries [60, 177], which in d dimension are enumerated by

$$H^{d+2}\left(\widetilde{G}_{\mathrm{spc}},\, \mathrm{U}(1)_{\mathfrak{c}}\right). \tag{11.3c}$$

The CEP correspondence 3 have been generalized to fermionic systems in Refs. [141, 147, 148, 173, 179], i.e., the so-called fermionic cyrstalline equivalence principle (FCEP). The treatment of fermionic systems with crystalline symmetries is subtler since the total symmetry group may contain the central extensions of crystalline symmetries by fermion parity. This is to say that the total symmetry group $G_{\mathrm{tot},f}$ cannot be easily decomposed into spatial and crystalline parts. Apart from this subtlely, FSPT phases with $G_{\mathrm{tot},f}$ symmetry are in one-to-one correspondence with FSPT phases with only internal symmetry $G_f = \widetilde{G}_{\mathrm{tot},f}$ where

1. all crystallographic symmetries are treated as internal symmetries,
2. each orientation reversing crystalline symmetry in $G_{\mathrm{tot},f}$ is replaced by an antiunitary internal symmetry in $\widehat{G}_{\mathrm{tot},f}$, and
3. spinless (spinful) crystallographic symmetries in $G_{\mathrm{tot},f}$ become spinful (spinless) in $\widetilde{G}_{\mathrm{tot},f}$.

The last rule essentially dictates that the central extensions classes associated with $G_{\mathrm{tot},f}$ and $\widetilde{G}_{\mathrm{tot},f}$ must differ. To give an example, in $d=2$ space dimensions, the spinful reflection symmetry is generates the group $\mathbb{Z}_4^{\mathrm{FR}}$ which is a nontrivial central extension of $\mathbb{Z}_2^{\mathrm{R}}$ by fermion parity. When treated as an internal symmetry according to FCEP, the group $\mathbb{Z}_4^{\mathrm{FR}}$ is replaced by internal symmetry group $\mathbb{Z}_2^{\mathrm{R}'}$ of which generator is represented antiunitarily. In the next chapter, we are going to present an application of FCEP.

12

LSM THEOREM AT THE EDGE OF A 2D CRYSTALLINE TOPOLOGICAL SUPERCONDUCTOR

In this Chapter, we explicitly study an example of two-dimensional crystalline topological superconductor, namely, te so-called symmetry class DIIIR [1] [100, 124]. This crystalline topological phase is defined by extending the symmetry class DIII in the Tenfold Way [30, 95] which refers to the strong IFT phase with internal symmetry group $G_f = \mathbb{Z}_4^{\mathrm{FT}}$. The class DIIIR is then obtained by imposing translation and reflection symmetries, which corresponds to the group $\mathbb{Z} \rtimes \mathbb{Z}_4^{\mathrm{FR}}$ of spatial symmetries. The total symmetry group is then

$$G_{\mathrm{tot},f} = \mathbb{Z} \rtimes \frac{\mathbb{Z}_4^{\mathrm{FT}} \times \mathbb{Z}_4^{\mathrm{FR}}}{\mathbb{Z}_2^{\mathrm{F}}}. \tag{12.1}$$

Note that the particular reflection symmetry we consider squares to the fermion parity and, hence, the corresponding group is $\mathbb{Z}_4^{\mathrm{FR}}$.

In two-dimensional space, class DIIIR has the noninteracting classification $\mathbb{Z}$. When open boundary conditions are imposed it features $\nu \in \mathbb{Z}$ pairs of gapless *helical* Majorana modes. When quartic interactions are added it is known that $\nu = 8$ pairs of helical Majorana modes can be gapped out [118]. This is interpreted as the breakdown of noninteracting $\mathbb{Z}$ classification down to interacting $\mathbb{Z}_8$ classification [101, 118, 180].

In what follows, we study the stability of the $\nu < 8$ pair of helical Majorana modes localized at the $(1+1)$-dimensional boundary against quartic contact interactions. For convenience, we focus only on the cases of $\nu = 4, 2, 1$. In Sec. 12.1, we define the boundary Hamiltonian and its single-particle symmetries. We also present a criterion that signals the stability of the boundary theory. Sections 12.2, 12.3, and, 12.4 study the cases of $\nu = 4$, $\nu = 2$, and, $\nu = 1$, respectively. For each case, we explain why the boundary degrees of freedom cannot be gapped without spontaneously or explicitly breaking one of the protecting symmetries.

As explained in Chapter 11.2, the stability of the boundary modes of crystalline topological phases is connected to the presence of an underlying generalized LSM constraint.

1 This class was also labeled as DIIIR_ _ in Refs. [100, 180]

In Sec. 12.5, without proof, we conjecture the underlying LSM constraints for class DIIIR based on the analysis done in the preceding sections.

12.1 DEFINITIONS AND SYMMETRIES

We describe the one-dimensional boundary of a two-dimensional crystalline topological superconductor with the Hamiltonian

$$\widehat{H}_{\mathrm{bd}} := \widehat{H}_0 + \widehat{H}_{\mathrm{int}}, \tag{12.2a}$$

$$\widehat{H}_0 := \int \mathrm{d}x\, \hat{\chi}^{\dagger}\, (\sigma_3 \otimes \mathbb{1}_{\nu}\, \mathrm{i}\partial_x)\, \hat{\chi}, \tag{12.2b}$$

$$\widehat{H}_{\mathrm{int}} := -\int \mathrm{d}x\, \lambda^2 \sum_{l=1}^{\mathrm{N}(\nu)} \left(\hat{\chi}^{\dagger}\, \beta_l\, \hat{\chi}\right)^2 . \tag{12.2c}$$

The Hamiltonian $\widehat{H}_0$ describes ν pairs of left- and right-moving quantum Majorana fields. The components $\hat{\chi}_a$ and $\hat{\chi}^{\dagger}_a$ with $a = 1, \cdots, 2\nu$ of the quantum-fields and their adjoints obey the equal-time algebra

$$\left\{\hat{\chi}_a(x), \hat{\chi}_{a'}(x')\right\} = \delta_{aa'}\, \delta(x - x'), \tag{12.3a}$$

with all other anticommutators vanishing and we impose the Majorana condition

$$\hat{\chi}^{\dagger} = \hat{\chi}^{\mathsf{T}}. \tag{12.3b}$$

The Hamiltonian $\widehat{H}_{\mathrm{int}}$ encodes the quartic contact interactions with coupling constant λ^2 between the ν different flavors. The matrix $\mathbb{1}_{\nu}$ is the identity matrix in flavor space. The label $l = 1, \cdots, \mathrm{N}(\nu)$ enumerates all $2\nu \times 2\nu$ Hermitian matrices such that (i) they square to the identity $\beta_l^2 = \mathbb{1}_{2\nu}$, (ii) any pair $(\beta_l, \beta_{l'})$ anticommutes pairwise as well as with $\sigma_3 \otimes \mathbb{1}_{\nu}$, and (iii) each β_l is odd under complex conjugation. The first two conditions restrict the $\mathrm{N}(\nu)$ interaction channels to the squares of bilinears that are not competing Dirac mass terms. The last condition follows from imposing a Majorana condition on the fermionic quantum fields as we do now. Had we demanded instead of (iii) that each β_l is even under complex conjugation, the bilinear $\hat{\chi}^{\dagger}\, \beta_l\, \hat{\chi}$ would then vanish because of the Majorana condition. We emphasize that $\mathrm{N}(\nu)$ is constant when $2^{n-1} < \nu < 2^n$ for some integer n. This means that the target space corresponding to the normalized dynamical Dirac masses does not change when $2^{n-1} < \nu < 2^n$ for some integer n.

Following Refs. [100, 118], we define the PH, TR, and reflection transformations,

$$\mathcal{C}_{\mathrm{bd},\nu} := \mathbb{1}_2 \otimes \mathbb{1}_\nu \, \mathsf{K}, \tag{12.4a}$$

$$\mathcal{T}_{\mathrm{bd},\nu} := \mathrm{i}\sigma_2 \otimes \mathbb{1}_\nu \, \mathsf{K}, \tag{12.4b}$$

$$\mathcal{R}_{\mathrm{bd},\nu} := \mathrm{i}\sigma_2 \otimes \mathbb{1}_\nu, \tag{12.4c}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and K denotes complex conjugation. They satisfy the defining conditions of the symmetry class DIIIR, i.e.,

$$\mathcal{C}^2_{\mathrm{bd},\nu} = +1, \qquad \mathcal{T}^2_{\mathrm{bd},\nu} = -1, \qquad \mathcal{R}^2_{\mathrm{bd},\nu} = -1, \tag{12.5}$$

with the algebra

$$\left[\mathcal{C}_{\mathrm{bd},\nu}, \mathcal{R}_{\mathrm{bd},\nu}\right] = 0, \qquad \left[\mathcal{T}_{\mathrm{bd},\nu}, \mathcal{R}_{\mathrm{bd},\nu}\right] = 0. \tag{12.6}$$

Had we chosen the Hermitian representation for the reflection transformation (12.4c), i.e., $\mathcal{R}_{\mathrm{bd},\nu} = \sigma_2 \otimes \mathbb{1}_\nu$, it would anticommute with both PH and TR transformations. This is consistent with the definition of DIIIR in Ref. [100]. The anti-Hermitian representation (12.4c) is chosen since the transformation law is then covariant with respect to the Majorana condition (12.3b). Moreover, we demand that transformations (12.4) are (spectral) symmetries of the single-particle Hamiltonian (12.2b)

$$\mathcal{C}_{\mathrm{bd},\nu} \, \mathcal{H}_0(x) \, \mathcal{C}^{-1}_{\mathrm{bd},\nu} = -\mathcal{H}_0(x), \tag{12.7a}$$

$$\mathcal{T}_{\mathrm{bd},\nu} \, \mathcal{H}_0(x) \, \mathcal{T}^{-1}_{\mathrm{bd},\nu} = +\mathcal{H}_0(x), \tag{12.7b}$$

$$\mathcal{R}_{\mathrm{bd},\nu} \, \mathcal{H}_0(-x) \, \mathcal{R}^{-1}_{\mathrm{bd},\nu} = +\mathcal{H}_0(x). \tag{12.7c}$$

When the conditions (12.7) are satisfied and we impose invariance under TS,

$$\widehat{T}(x') \, \widehat{H}_{\mathrm{bd}} \, \widehat{T}^{-1}(x') = \widehat{H}_{\mathrm{bd}}, \qquad \forall\, x' \in \mathbb{R}, \tag{12.8}$$

where $\widehat{T}(x')$ is the operator that implements the translation by x', then Hamiltonian (12.2b) cannot be gapped by adding bilinears of the fermionic fields for any $\nu = 1, 2, 3, \cdots$ [100]. In this case, the noninteracting classification for the class DIIIR is $\mathbb{Z}$. However, bilinears that are odd under reflection are allowed if they are multiplied by a (smooth) function of x, a space-dependent mass, that is odd under $x \to -x$ and must thus vanish at the origin

$x = 0$. Such a mass gaps the single-particle spectrum except for a mid-gap bound state whose envelope decays exponentially fast away from $x = 0$. Any such mass breaks TS and the origin can be thought of as a point defect or a "corner" along the one-dimensional boundary at which the mass term must change sign if it is to respect reflection symmetry. In the presence of such a space-dependent mass, the noninteracting classification reduces to that of the symmetry class DIII, i.e., $\mathbb{Z}_2$ [100, 118, 181].

Alternatively, we can write down the partition function

$$Z_{\mathrm{bd}} := \int \mathcal{D}[\chi]\, e^{-S_{\mathrm{bd}}}, \tag{12.9a}$$

$$S_{\mathrm{bd}} := \int \mathrm{d}\tau \mathrm{d}x\, \mathcal{L}_{\mathrm{bd}}, \tag{12.9b}$$

$$\mathcal{L}_{\mathrm{bd}} := \chi^{\dagger}\left(\partial_{\tau} + \sigma_3 \otimes \mathbb{1}_{\nu}\, \mathrm{i}\partial_x\right)\chi - \lambda^2 \sum_{l=1}^{\mathrm{N}(\nu)} \left(\chi^{\dagger}\, \beta_l\, \chi\right)^2, \tag{12.9c}$$

where the action is defined on (1+1)-dimensional Euclidean space-time. The integration variables are the components of the Grassmann-valued spinor χ, as $\chi^{\dagger}$ is linearly constrained to χ through the Majorana condition (12.3b). The interaction terms can be decoupled via Hubbard-Stratonovich transformation. The partition function (12.9) then takes the form

$$Z_{\mathrm{bd}} = \mathrm{const} \times \int \mathcal{D}[\chi] \int \mathcal{D}[\phi_{\beta_l}]\, e^{-S'_{\mathrm{bd}}}, \tag{12.10a}$$

$$S'_{\mathrm{bd}} = \int \mathrm{d}\tau \mathrm{d}x\, \mathcal{L}'_{\mathrm{bd}}, \tag{12.10b}$$

$$\mathcal{L}'_{\mathrm{bd}} = \chi^{\dagger}\left(\partial_{\tau} + \mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}\right)\chi + \frac{1}{(2\lambda)^2} \sum_{l=1}^{\mathrm{N}(\nu)} \phi_l^2, \tag{12.10c}$$

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}} := +\sigma_3 \otimes \mathbb{1}_{\nu}\, \mathrm{i}\partial_x + \sum_{l=1}^{\mathrm{N}(\nu)} \beta_l\, \phi_l. \tag{12.10d}$$

We have thereby defined the dynamical single-particle boundary Hamiltonian $\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}$. Conditions (12.7) on $\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}$ can be met as follows. PHS imposes that

$$\mathsf{K}\, \beta_l\, \mathsf{K}^{-1} = \beta_l^{*} = -\beta_l, \qquad l = 1, \cdots, \mathrm{N}(\nu), \tag{12.11}$$

for any β_l Hermitian $2\nu \times 2\nu$ matrix. Hence, imposing the Majorana condition trivially satisfies the PHS. Once the maximum number of β_l matrices that are compatible with PHS is found, the symmetry requirements coming from TRS and RS can be satisfied by imposing that ϕ_l is either odd or even under time-reversal and reflection. From now on, we shall use the shorthand notation for the 4^n, $2^n \times 2^n$ Hermitian matrices

$$\begin{aligned} \mathrm{X}_{\mu_1\mu_2\cdots\mu_n} &:= \sigma^{(1)}_{\mu_1} \otimes \sigma^{(2)}_{\mu_2} \otimes \sigma^{(3)}_{\mu_3} \otimes \cdots \otimes \sigma^{(n)}_{\mu_n}, \\ \left(\mathrm{X}_{\mu_1\mu_2\cdots\mu_n}\right)^2 &= \mathbb{1}_{2^n}, \qquad \mu_j = 0,1,2,3, \end{aligned} \tag{12.12}$$

where $\sigma^{(j)}_0$ is $\mathbb{1}_2$, $\boldsymbol{\sigma}^{(j)}$ are the associated Pauli matrices, and $n \in \mathbb{Z}$ is related to ν by the relation $2^{n-1} = \nu$.

The partition function (12.10) is quadratic in Grassmann variables, which therefore can be integrated out to yield an effective action of bosonic fields ϕ_{β_l}, provided the Majorana Pfaffian is nonvanishing. This effective theory is described by the partition function

$$Z = \int \mathcal{D}[\boldsymbol{\phi}]\, \delta\left(\boldsymbol{\phi}^2 - \bar{\phi}^2\right) e^{-\int \mathrm{d}^2x\, \frac{1}{g}(\partial_\mu \boldsymbol{\phi})^2 + \Gamma[\boldsymbol{\phi}]} \tag{12.13}$$

where $\bar{\phi}^2 > 0$ is a real-valued constant, $\boldsymbol{\phi}$ is a $\mathrm{N}(\nu)$-dimensional vector field that is normalized through the nonlinear constraint imposed by the δ function, and the symbol $\Gamma[\boldsymbol{\phi}]$ signifies the existence of a topological obstruction. In other words, the presence of the symbol $\Gamma[\boldsymbol{\phi}]$ implies that the effective action associated to the partition function (12.13) is not merely that of a NLSM. Due to the nonlinear constraint imposed on $\mathrm{N}(\nu)$ bosonic fields, the target space in Eq. (12.13) is the unit sphere $\mathsf{S}^{\mathrm{N}(\nu)-1}$. The symbol $\Gamma[\boldsymbol{\phi}]$ is present in Eq. (12.13) whenever one of the homotopy groups,

$$\begin{aligned} &\pi_0(\mathsf{S}^{\mathrm{N}(\nu)-1}), \\ &\quad \pi_1(\mathsf{S}^{\mathrm{N}(\nu)-1}), \\ &\quad\quad \pi_2(\mathsf{S}^{\mathrm{N}(\nu)-1}), \\ &\quad\quad\quad \cdots \\ &\quad\quad\quad\quad \pi_{d+1}(\mathsf{S}^{\mathrm{N}(\nu)-1}), \end{aligned} \tag{12.14}$$

is nontrivial [182]. (The upper bound $d+1$ is imposed as topological obstructions corresponding to higher homotopy groups modify the equations of motions in a nonlocal way [124].) Such topological obstructions are expected to prevent gapping out the edge modes. If no such topological obstructions exists, the low-energy effective theory is described by no more than a NLSM action. For space-time dimension two, the action then flows to the strong coupling $g \to \infty$ stable fixed point [183]. This is the quantum disordered phase that describes a gapped phase of matter that is symmetric under all protecting symmetries. The original ν gapless edge modes have been gapped by the interactions without any of the preserving symmetries being spontaneously broken. Hence, the noninteracting gapless edge theory is smoothly connected to a strongly interacting gapped edge theory upon switching on local symmetry preserving interactions. The presence of the topological obstruction manifests itself by modifying the renormalization group (RG) flow and preventing the flow to the strong coupling limit $g \to \infty$.

12.2 THE CASE $\nu = 4$

The set (12.12) with $n = 3$ has the 64 elements $\{\mathrm{X}_{\mu\rho\sigma}\}$ with $\mu, \rho, \sigma = 0, \cdots, 3$. This set spans the space of 8×8 Hermitian matrices. For $\nu = 4$, there are at most $\mathrm{N}(4) = 4$ interaction channels allowed by the symmetry conditions (12.7), each of which is labeled by the Hermitian 8×8 matrix β_l. We consider the parametrization

$$\mathcal{H}^{\mathrm{dyn}}_{\mathrm{bd}}(\tau, x) := \beta_0\, \mathrm{i}\partial_x + \sum_{l=1}^{4} \beta_l\, \phi_l(\tau, x) \tag{12.15a}$$

of the dynamical boundary single-particle Hamiltonian, where without loss of generality, we make the choice

$$\beta_0 := \mathrm{X}_{300}, \tag{12.15b}$$

$$\beta_1 := \mathrm{X}_{210}, \tag{12.15c}$$

$$\beta_2 := \mathrm{X}_{230}, \tag{12.15d}$$

$$\beta_3 := \mathrm{X}_{222}, \tag{12.15e}$$

$$\beta_4 := \mathrm{X}_{102} = -\mathrm{X}_{300}\,\mathrm{X}_{210}\,\mathrm{X}_{230}\,\mathrm{X}_{222}. \tag{12.15f}$$

The choice $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ is not unique, but this lack of uniqueness does not affect the subsequent analysis. We define the corresponding partition function

$$Z_{\mathrm{bd}} := \int \mathcal{D}[\chi] \int \mathcal{D}[\Phi]\, e^{-S_{\mathrm{bd}}}, \tag{12.16a}$$

$$S_{\rm bd} := \int \mathrm{d}^2 x \left[\bar{\chi} \left(\mathrm{i}\gamma_\mu \partial_\mu + \Phi \right) \chi + \frac{1}{(2\lambda)^2} \Phi^\dagger \Phi \right], \tag{12.16b}$$

where, following Ref. [184], we have introduced the notations

$$\bar{\chi} := \chi^\dagger (-\mathrm{i}\gamma_0), \tag{12.16c}$$

$$\gamma_0 := \beta_4 = \mathrm{X}_{102}, \tag{12.16d}$$

$$\gamma_1 := \mathrm{i}\beta_4\, \beta_0 = \mathrm{X}_{202}, \tag{12.16e}$$

$$\gamma_5 := \gamma_0\, \gamma_1 = \mathrm{i}\beta_0 = \mathrm{i}\mathrm{X}_{300}, \tag{12.16f}$$

$$\mathrm{Y}_1 := -\,\mathrm{X}_{312}, \tag{12.16g}$$

$$\mathrm{Y}_2 := -\,\mathrm{X}_{332}, \tag{12.16h}$$

$$\mathrm{Y}_3 := -\,\mathrm{X}_{320}, \tag{12.16i}$$

$$\mathrm{Y}_4 := +\mathrm{i}\mathrm{X}_{000}, \tag{12.16j}$$

and have defined the matrix-valued field

$$\Phi(x) := |\boldsymbol{\phi}(x)| \sum_{l=1}^{4} n_l(x)\, \mathrm{Y}_l, \tag{12.16k}$$

$$\boldsymbol{\phi}(x) := |\boldsymbol{\phi}(x)|\, \boldsymbol{n}(x) \in \mathbb{R}^4, \qquad \boldsymbol{n}^2(x) = 1, \tag{12.16l}$$

that parametrizes the dynamical mass profile. We denote the imaginary time and space coordinates by $x = (x_0, x_1) \equiv (\tau, x)$. With the choice of the representation made in Eqs. (12.16), the identities

$$\left\{ \gamma_\mu, \gamma_\nu \right\} = 2\delta_{\mu\nu}, \qquad \{\gamma_\mu, \mathrm{Y}_l\} = 2\delta_{4l}\, \mathrm{Y}_l\, \gamma_\mu, \tag{12.17a}$$

hold for any $\mu, \nu = 0, 1$ and $l = 1, 2, 3, 4$. Performing the Grassmann integration on the partition function (12.16) delivers the bosonic and local effective action

$$Z_{\rm eff} := \int \mathcal{D}\left[\Phi\right] e^{-S_{\rm eff}[\Phi]}, \tag{12.18a}$$

$$S_{\rm eff}[\Phi] := -\frac{1}{2}\, \mathrm{Tr}\, \left[\ln \mathcal{D}_\Phi\right] + \frac{1}{32\lambda^2} \mathrm{Tr}\left[\Phi^\dagger \Phi\right], \tag{12.18b}$$

$$\mathcal{D}_\Phi := \mathrm{i}\gamma_\mu\, \partial_\mu + \Phi. \tag{12.18c}$$

Here, the trace Tr is understood to be over both a plane-wave basis and 8×8 matrices. The local effective action (12.18b) can be written in closed form to any finite order of a gradient expansion [182] as we now sketch.

The solution $\bar{\Phi}$ to the saddle-point equation

$$\frac{\delta S_{\text{eff}}}{\delta \Phi} = 0 \tag{12.19a}$$

is

$$\bar{\Phi} = \bar{\phi} \sum_{\iota=1}^{4} \bar{n}_{\iota} \mathrm{Y}_{\iota}, \tag{12.19b}$$

where

$$\sum_{\iota=1}^{4} \bar{n}_{\iota}^{2} = 1, \qquad \bar{\phi}^{2} := \left(e^{\frac{1}{8\pi\lambda^{2}}} - 1\right)^{-1} \Lambda^{2}. \tag{12.19c}$$

Here, Λ is the UV cutoff introduced to regularize the integration over momenta. The direction of the saddle-point solution $\bar{\boldsymbol{n}}$ is arbitrary.

Next, we first consider the change $\delta S_{\text{eff}}[\Phi]$ of effective action (12.18) when Φ is varied to $\Phi + \delta\Phi$,

$$\delta S_{\text{eff}}[\Phi] = S_{\text{eff}}[\Phi + \delta\Phi] - S_{\text{eff}}[\Phi], \tag{12.20}$$

which is to be expanded around the saddle-point solution (12.19) in powers of $1/\bar{\phi}^{2}$. Taking the limit $\bar{\phi}^{2} \to \infty$ kills all but a finite number of terms on the right-hand side of Eq. (12.20). Integration over $\delta\Phi$ then delivers two terms. The first term is

$$S_{\text{NLSM}} = \int \mathrm{d}^{2}x \, \frac{1}{2g} \left(\partial_{\mu} \boldsymbol{n}\right)^{2}, \qquad g = \pi. \tag{12.21}$$

This is the action of the O(4)-NLSM in two-dimensional Euclidean spacetime with the bare coupling constant $g = \pi$. The second term is

$$\Gamma = \frac{2\mathrm{i}\pi}{3!\mathrm{Area}(\mathsf{S}^{3})} \int \mathrm{d}^{3}\tilde{x} \, \epsilon_{\mu\nu\rho} \, \epsilon_{abcd} \left(\partial_{\mu} \tilde{n}_{a}\right) \left(\partial_{\nu} \tilde{n}_{b}\right) \left(\partial_{\rho} \tilde{n}_{c}\right) \tilde{n}_{d}. \tag{12.22}$$

Indeed, imposing the nonlinear constraint $\boldsymbol{n}^2(x) = 1$ compactifies the target space of the O(4)-NLSM to the three-sphere S^3. This sphere has the nontrivial homotopy group $\pi_3(\mathsf{S}^3) = \mathbb{Z}$. It is then meaningful following Witten [185] to introduce an auxiliary coordinate $u \in [0,1]$ and to extend the domain of definition of the field $\boldsymbol{n}(x)$ from $\mathbb{R}^2$ to $\mathbb{R}^2 \times [0,1]$, $\boldsymbol{n}(x) \to \tilde{\boldsymbol{n}}(x,u)$, $\mathrm{d}^2x \to \mathrm{d}^2x\,\mathrm{d}u \equiv \mathrm{d}^3\tilde{x}$ such that the boundary conditions $\tilde{\boldsymbol{n}}(x,0) = \boldsymbol{n}_0$ for some arbitrary direction $\boldsymbol{n}_0$ and $\tilde{\boldsymbol{n}}(x,1) = \boldsymbol{n}(x)$ are satisfied. This is the WZ [185–188] term for the O(4)-NLSM in two-dimensional Euclidean space-time. This term is not local in the action but its effect on the equations of motion is local. However, this term modifies nonperturbatively the RG flow obeyed by the coupling g. In fact, in the presence of the WZ term, the beta function of g has been conjectured to vanish at the value $g_{\mathrm{c}} = \pi$ that defines a critical point with conformal symmetry [182, 185].

The interaction that we chose has an O(4) symmetry. This symmetry is not sacred. For example, we could have introduced four dimensionless couplings λ_l with $l = 1, \cdots, 4$, one for each dynamical mass β_l in Eqs. (12.15). By treating each dynamical mass β_l as independent Hubbard-Stratonovich fields and integrating over these fields, the interaction is the sum of four quartic contact interactions, each of which is weighted by the multiplicative factor $(2\lambda_l)^{-2}$. This interacting theory can be bosonized with the help of Abelian bosonization rules. The stability analysis then proceeds along the same line as what is done in Sec. 12.3.2 with the same conclusions. The boundary theory is gapped if and only if the protecting symmetries (12.7) or (12.8) are spontaneously broken. One may repeat this exercise with $\nu = 6$ and reach the same conclusion, a gap is necessarily associated with the spontaneous symmetry breaking of the TRS or RS. It is only when ν is an integer multiple of the number 8 that a gap delivers a nondegenerate ground state.

12.3 THE CASE $\nu = 2$

The set (12.12) with $n = 2$ has the 16 elements $\{\mathrm{X}_{\mu\rho}\}$ with $\mu, \rho = 0, \cdots, 3$. This set spans the space of 4×4 Hermitian matrices. For $\nu = 2$, there are at most $\mathrm{N}(2) = 2$ interaction channels allowed by the symmetry conditions (12.7), each of which is labeled by the Hermitian 4×4 matrix β_l. We consider the parametrization

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}(\tau,x) := \beta_0\,\mathrm{i}\partial_x + \sum_{l=1}^{2} \beta_l\,\phi_l(\tau,x) \tag{12.23a}$$

of the dynamical boundary single-particle Hamiltonian. Following the same steps as in Sec. 12.2, we impose the nonlinear constraint:

$$\phi_1^2(\tau,x)+\phi_2^2(\tau,x)=\bar{\phi}^2. \tag{12.24}$$

This condition compactifies the target space of the effective bosonic and local theory to the circle S^1. However, the nontrivial fundamental group $\pi_1(\mathsf{S}^1)=\mathbb{Z}$ implies the existence of a topological obstruction. This topological obstruction takes the form of point defects when the vector $(\phi_1(\tau,x),\phi_2(\tau,x))$ accommodates vortex configurations in $(1+1)$-dimensional space-time. A vortex configuration is singular at the vortex core where its gradient is ill-defined. Direct application of the gradient expansion method employed in Sec. 12.2 is thus invalid. To circumvent this difficulty, we choose the method of Abelian bosonization to derive an effective local bosonic action.

12.3.1 *Abelian bosonization*

We start from

$$\widehat{H}_{\mathrm{bd}}:=\int \mathrm{d}x\left\{\hat{\chi}^{\dagger}\mathrm{X}_{30}\mathrm{i}\partial_x\hat{\chi}-\sum_{l=1}^{2}\lambda_l^2\left(\hat{\chi}^{\dagger}\,\beta_l\,\hat{\chi}\right)^2\right\}, \tag{12.25}$$

i.e., we do not impose the O(2) symmetry resulting from demanding that $\lambda_1^2=\lambda_2^2=\lambda^2$ as is done in Hamiltonian (12.2). Imposing symmetry conditions (12.7a) leads to the identification of two possible sets $\{\beta_l\}$

$$\mathcal{B}_a=\{\mathrm{X}_{12},\mathrm{X}_{20}\},\quad \mathcal{B}_b=\{\mathrm{X}_{21},\mathrm{X}_{23}\}. \tag{12.26}$$

Choosing set $\mathcal{B}_a$ in Eq. (12.25) defines $\widehat{H}_{\mathrm{bd}\,a}$. Choosing set $\mathcal{B}_b$ in Eq. (12.25) defines $\widehat{H}_{\mathrm{bd}\,b}$. We will perform the subsequent analysis for both $\widehat{H}_{\mathrm{bd}\,a}$ and $\widehat{H}_{\mathrm{bd}\,b}$ in parallel. With the convention

$$\hat{\chi}^{\dagger}=\left(\hat{\chi}^1_{\mathrm{L}},\hat{\chi}^2_{\mathrm{L}},\hat{\chi}^1_{\mathrm{R}},\hat{\chi}^2_{\mathrm{R}}\right), \tag{12.27a}$$

Hamiltonians $\widehat{H}_{\mathrm{bd}\,a}$ and $\widehat{H}_{\mathrm{bd}\,b}$ are given by

$$\widehat{H}_{\mathrm{bd}\,a}:=\int \mathrm{d}x\left\{\hat{\chi}^{\dagger}\mathrm{X}_{30}\mathrm{i}\partial_x\hat{\chi}\right.$$

$$- \lambda^{2}_{1,a} \left(\hat{\chi}^{\dagger} \mathrm{X}_{12} \hat{\chi}\right)^{2} - \lambda^{2}_{2,a} \left(\hat{\chi}^{\dagger} \mathrm{X}_{20} \hat{\chi}\right)^{2} \Big\}, \tag{12.27b}$$

and

$$\widehat{H}_{\mathrm{bd}\,b} := \int \mathrm{d}x \, \Big\{ \hat{\chi}^{\dagger}\, \mathrm{X}_{30}\, \mathrm{i}\partial_{x}\, \hat{\chi} - \lambda^{2}_{1,b} \left(\hat{\chi}^{\dagger}\, \mathrm{X}_{21}\, \hat{\chi}\right)^{2} - \lambda^{2}_{2,b} \left(\hat{\chi}^{\dagger}\, \mathrm{X}_{23}\, \hat{\chi}\right)^{2} \Big\}, \tag{12.27c}$$

respectively. This Majorana representation is not well suited for Abelian bosonization. Instead of it, we define the right-moving complex fermion fields

$$\hat{\psi}^{\dagger}_{\mathrm{R}} := \frac{\hat{\chi}^{1}_{\mathrm{R}} - \mathrm{i}\hat{\chi}^{2}_{\mathrm{R}}}{\sqrt{2}}, \qquad \hat{\psi}_{\mathrm{R}} := \frac{\hat{\chi}^{1}_{\mathrm{R}} + \mathrm{i}\hat{\chi}^{2}_{\mathrm{R}}}{\sqrt{2}}, \tag{12.28a}$$

the left-moving complex fermion fields

$$\hat{\psi}^{\dagger}_{\mathrm{L}} := \frac{\hat{\chi}^{1}_{\mathrm{L}} - \mathrm{i}\hat{\chi}^{2}_{\mathrm{L}}}{\sqrt{2}}, \qquad \hat{\psi}_{\mathrm{L}} := \frac{\hat{\chi}^{1}_{\mathrm{L}} + \mathrm{i}\hat{\chi}^{2}_{\mathrm{L}}}{\sqrt{2}}, \tag{12.28b}$$

and the complex fermion basis

$$\hat{\Psi}^{\dagger} := \begin{pmatrix} \hat{\psi}^{\dagger}_{\mathrm{L}} & \hat{\psi}^{\dagger}_{\mathrm{R}} & \hat{\psi}_{\mathrm{L}} & \hat{\psi}_{\mathrm{R}} \end{pmatrix}. \tag{12.28c}$$

In the basis (12.28c), we find the complex fermion representation

$$\widehat{H}_{\mathrm{bd}\,a} := \int \mathrm{d}x \, \Big\{ \hat{\Psi}^{\dagger}\, \mathrm{X}_{03}\, \mathrm{i}\partial_{x}\, \hat{\Psi} - \lambda^{2}_{1,a} \left(\hat{\Psi}^{\dagger}\, \mathrm{X}_{31}\, \hat{\Psi}\right)^{2} - \lambda^{2}_{2,a} \left(\hat{\Psi}^{\dagger}\, \mathrm{X}_{02}\, \hat{\Psi}\right)^{2} \Big\}, \tag{12.29a}$$

and

$$\widehat{H}_{\mathrm{bd}\,b} := \int \mathrm{d}x \, \Big\{ \hat{\Psi}^{\dagger} \mathrm{X}_{03}\, \mathrm{i}\partial_{x}\, \hat{\Psi} - \lambda^{2}_{1,b} \left(\hat{\Psi}^{\dagger}\, \mathrm{X}_{22}\, \hat{\Psi}\right)^{2} - \lambda^{2}_{2,b} \left(\hat{\Psi}^{\dagger}\, \mathrm{X}_{12}\, \hat{\Psi}\right)^{2} \Big\}. \tag{12.29b}$$

The change of basis (12.28) causes a permutation among the matrices $\mathrm{X}_{\mu\rho}$ with $\mu, \rho = 0, 1, 2, 3$. Hamiltonians (12.29a) or (12.29b) are to be normal ordered by using point-splitting and Wick's theorem. These normal-ordered Hamiltonians are then bosonized by using the identities

$$\hat{\psi}^{\dagger}_{\mathrm{R}}(x) =: \hat{\eta}_{\mathrm{R}}\,\frac{e^{-\mathrm{i}\hat{\varphi}_{\mathrm{R}}(x)}}{\sqrt{2\pi\epsilon}}, \qquad \hat{\psi}^{\dagger}_{\mathrm{L}}(x) =: \hat{\eta}_{\mathrm{L}}\,\frac{e^{+\mathrm{i}\hat{\varphi}_{\mathrm{L}}(x)}}{\sqrt{2\pi\epsilon}}, \tag{12.30a}$$

where ϵ is a short-distance cutoff. Hereby, we defined the chiral bosonic fields that obey the algebra

$$[\hat{\varphi}_{\mathrm{R}}(x), \hat{\varphi}_{\mathrm{R}}(x')] = -[\hat{\varphi}_{\mathrm{L}}(x), \hat{\varphi}_{\mathrm{L}}(x')] = \mathrm{i}\pi\,\mathrm{sgn}(x-x'), \qquad [\hat{\varphi}_{\mathrm{R}}(x), \hat{\varphi}_{\mathrm{L}}(x')] = 0, \tag{12.30b}$$

and Klein factors $\hat{\eta}_{\mathrm{R/L}}$ that obey the algebra

$$\{\hat{\eta}_{\mathrm{R}}, \hat{\eta}_{\mathrm{R}}\} = \{\hat{\eta}_{\mathrm{L}}, \hat{\eta}_{\mathrm{L}}\} = 2, \qquad \{\hat{\eta}_{\mathrm{R}}, \hat{\eta}_{\mathrm{L}}\} = 0. \tag{12.30c}$$

Hamiltonian (12.29a) has the bosonic representation

$$\begin{aligned}\widehat{H}_{\mathrm{bd}\,a} = \int \mathrm{d}x \Bigg\{ & \frac{1}{2\pi}\left[(\partial_x\hat{\varphi}_{\mathrm{L}})^2 + (\partial_x\hat{\varphi}_{\mathrm{R}})^2\right] \\ & + \frac{\left(\lambda^2_{1,a}+\lambda^2_{2,a}\right)}{\pi^2}\,(\partial_x\hat{\varphi}_{\mathrm{L}} + \partial_x\hat{\varphi}_{\mathrm{R}})^2 \\ & + \frac{2\left(\lambda^2_{1,a}-\lambda^2_{2,a}\right)}{\pi^2\epsilon^2}\cos\left(2\,\hat{\varphi}_{\mathrm{L}} + 2\,\hat{\varphi}_{\mathrm{R}}\right)\Bigg\}. \end{aligned} \tag{12.31a}$$

Hamiltonian (12.29b) has the bosonic representation

$$\begin{aligned}\widehat{H}_{\mathrm{bd}\,b} = \int \mathrm{d}x \Bigg\{ & \frac{1}{2\pi}\left[(\partial_x\hat{\varphi}_{\mathrm{L}})^2 + (\partial_x\hat{\varphi}_{\mathrm{R}})^2\right] \\ & + \frac{\left(\lambda^2_{1,b}+\lambda^2_{2,b}\right)}{\pi^2}\,(\partial_x\hat{\varphi}_{\mathrm{L}} - \partial_x\hat{\varphi}_{\mathrm{R}})^2 \\ & + \frac{2\left(\lambda^2_{1,b}-\lambda^2_{2,b}\right)}{\pi^2\epsilon^2}\cos\left(2\,\hat{\varphi}_{\mathrm{L}} - 2\,\hat{\varphi}_{\mathrm{R}}\right)\Bigg\}. \end{aligned} \tag{12.31b}$$

In Hamiltonians (12.31), we have removed the Klein factors by diagonalizing the operator $\mathrm{i}\hat{\eta}_{\mathrm{R}}\hat{\eta}_{\mathrm{L}}$ and choosing the eigenvalue $+1$ sector in the Klein Hilbert space. The difference between the two sets $\mathcal{B}_a$ and $\mathcal{B}_b$ in Eq. (12.26) manifests itself as the sign with which

$\hat{\varphi}_{\mathrm{R}}$ enters Hamiltonians (12.31a) and (12.31b), respectively. In Hamiltonian (12.31a), the cosine results from the squares of the backward-scattering term $\propto \hat{\psi}^{\dagger}_{\mathrm{R}}\hat{\psi}_{\mathrm{L}} + \mathrm{H.c.}$ In Hamiltonian (12.31b), the cosine results from the squares of the backward-pairing terms $\propto \hat{\psi}^{\dagger}_{\mathrm{R}}\hat{\psi}^{\dagger}_{\mathrm{L}} + \mathrm{H.c.}$ In the O(2) symmetric case that is defined by the condition

$$\lambda^2_{1,m} = \lambda^2_{2,m}, \qquad m = a, b, \tag{12.32}$$

both cosine interactions vanish and the theory remains gapless. Away from the O(2) symmetric point, the minima of the cosines are two-fold degenerate. If the cosines dominate over the kinetic energy, they open a gap with a two-fold degenerate manifold of ground states. Since the dependence on interaction strengths have the same form in Hamiltonians (12.31a) and (12.31b), the boundaries in the corresponding phase diagrams are identical. However, the phases they separate can be different whenever they break spontaneously distinct symmetries.

The transformation [2]

$$\hat{\varphi}_{\mathrm{L}} \to +\hat{\varphi}_{\mathrm{L}}, \qquad \hat{\varphi}_{\mathrm{R}} \to -\hat{\varphi}_{\mathrm{R}} \tag{12.33}$$

that interchanges Hamiltonians (12.31a) and (12.31b) is nothing but the transformation that interchanges the pair of dual fields

$$\hat{\phi}(x) := \frac{1}{\sqrt{4\pi}}\left[\hat{\varphi}_{\mathrm{L}}(x) + \hat{\varphi}_{\mathrm{R}}(x)\right], \tag{12.34a}$$

$$\hat{\theta}(x) := \frac{1}{\sqrt{4\pi}}\left[\hat{\varphi}_{\mathrm{L}}(x) - \hat{\varphi}_{\mathrm{R}}(x)\right], \tag{12.34b}$$

that satisfy the algebra

$$\left[\hat{\phi}(x), \hat{\theta}(x')\right] = \frac{\mathrm{i}}{2}\mathrm{sgn}(x' - x) \tag{12.34c}$$

with all other commutators vanishing. If one trades the Hamiltonian representation for the Lagrangian representation, one obtains the pair of actions

$$S_a := \int \mathrm{d}^2x \left\{\frac{1}{2g_a}\left(\partial_\mu \phi\right)^2 + \kappa_a \cos\left(\sqrt{16\pi}\phi\right)\right\}, \tag{12.35a}$$

2 Recall that two copies of the helical Majorana fields can be thought of as a low energy description of two copies of the Ising model. Suppose now that a Kramers-Wannier duality transformation is applied to only the second copy of the Ising model via the transformation $\hat{\chi}^2_{\mathrm{L}} \to \hat{\chi}^2_{\mathrm{L}}$ and $\hat{\chi}^2_{\mathrm{R}} \to -\hat{\chi}^2_{\mathrm{R}}$. In the language of complex fermions, the left-handed component $\hat{\psi}_{\mathrm{L}}$ is unchanged, while the right-handed component $\hat{\psi}_{\mathrm{R}}$ is transformed into its dagger, i.e., $\hat{\psi}_{\mathrm{R}} \to \hat{\psi}^{\dagger}_{\mathrm{R}}$. The transformation (12.33) of chiral bosons then follows.

$$S_b := \int \mathrm{d}^2 x \left\{ \frac{1}{2g_b} \left(\partial_\mu \theta\right)^2 + \kappa_b \cos\left(\sqrt{16\pi}\theta\right) \right\}, \tag{12.35b}$$

where ϕ and θ are dual scalar fields satisfying either

$$\partial_\mu \phi = \mathrm{i}\, g_a\, \epsilon_{\mu\nu} \partial_\nu \theta, \tag{12.35c}$$

with $\mu = 0, 1$, $(x_0, x_1) = (v_a \tau, x)$, or

$$\partial_\mu \phi = \mathrm{i}\, g_b\, \epsilon_{\mu\nu} \partial_\nu \theta, \tag{12.35d}$$

with $\mu = 0, 1$, $(x_0, x_1) = (v_b \tau, x)$, respectively. The coupling constants are given by

$$\frac{2}{v_a} = g_a := \frac{1}{\sqrt{1 + 4\frac{\lambda^2_{1,a} + \lambda^2_{2,a}}{\pi}}}, \tag{12.35e}$$

$$\frac{2}{v_b} = g_b := \frac{1}{\sqrt{1 + 4\frac{\lambda^2_{1,b} + \lambda^2_{2,b}}{\pi}}}, \tag{12.35f}$$

whereas the effective interaction strengths are

$$\kappa_a := \frac{4}{\pi^2 \epsilon^2} \sqrt{1 + 4\frac{\lambda^2_{1,a} + \lambda^2_{2,a}}{\pi}} \left(\lambda^2_{1,a} - \lambda^2_{2,a}\right), \tag{12.35g}$$

$$\kappa_b := \frac{4}{\pi^2 \epsilon^2} \sqrt{1 + 4\frac{\lambda^2_{1,b} + \lambda^2_{2,b}}{\pi}} \left(\lambda^2_{1,b} - \lambda^2_{2,b}\right). \tag{12.35h}$$

The two actions (12.35a) and (12.35b) are exchanged if one performs the interchanges $\lambda^2_{i,a} \leftrightarrow \lambda^2_{i,b}$ with $i = 1, 2$ and $\phi \leftrightarrow \theta$. The interaction strengths (12.35g) and (12.35h) change signs depending on whether $\lambda^2_{1,m} > \lambda^2_{2,m}$ or $\lambda^2_{1,m} < \lambda^2_{2,m}$, with $m = a, b$.

Before proceeding, we determine how the symmetries defined in Eqs. (12.4) act on the bosonic fields. The actions of the symmetry transformations on the complex fermionic fields are deduced from their actions on the Majorana fields and given by

$$\widehat{U}^\dagger_{\mathrm{C}} \begin{pmatrix} \hat{\psi}_{\mathrm{L}}(\tau, x) \\ \hat{\psi}_{\mathrm{R}}(\tau, x) \end{pmatrix} \widehat{U}_{\mathrm{C}} = \begin{pmatrix} \hat{\psi}_{\mathrm{L}}(\tau, x) \\ \hat{\psi}_{\mathrm{R}}(\tau, x) \end{pmatrix}, \tag{12.36a}$$

$$\widehat{U}^\dagger_{\mathrm{T}} \begin{pmatrix} \hat{\psi}_{\mathrm{L}}(\tau, x) \\ \hat{\psi}_{\mathrm{R}}(\tau, x) \end{pmatrix} \widehat{U}_{\mathrm{T}} = \begin{pmatrix} +\hat{\psi}^\dagger_{\mathrm{R}}(\tau, x) \\ -\hat{\psi}^\dagger_{\mathrm{L}}(\tau, x) \end{pmatrix}, \tag{12.36b}$$

$$\widehat{U}^{\dagger}_{\mathrm{R}} \begin{pmatrix} \hat{\psi}_{\mathrm{L}}(\tau,x) \\ \hat{\psi}_{\mathrm{R}}(\tau,x) \end{pmatrix} \widehat{U}_{\mathrm{R}} = \begin{pmatrix} +\hat{\psi}_{\mathrm{R}}(\tau,-x) \\ -\hat{\psi}_{\mathrm{L}}(\tau,-x) \end{pmatrix}, \tag{12.36c}$$

where $\widehat{U}_{\mathrm{C}}$, $\widehat{U}_{\mathrm{T}}$, and $\widehat{U}_{\mathrm{R}}$ are PH, reversal of time, and reflection transformations at the many-body level. The operator $\widehat{U}_{\mathrm{T}}$ is defined to be antiunitary, whereas operators $\widehat{U}_{\mathrm{C}}$ and $\widehat{U}_{\mathrm{R}}$ are chosen to be unitary. We note that the PHS is represented by the identity, whereas the TRS involves a PH transformation [3]. These transformation laws together with Eqs. (12.30a) imply the transformation laws

$$\widehat{U}^{\dagger}_{\mathrm{C}} \begin{pmatrix} \hat{\varphi}_{\mathrm{L}}(\tau,x) \\ \hat{\varphi}_{\mathrm{R}}(\tau,x) \end{pmatrix} \widehat{U}_{\mathrm{C}} = \begin{pmatrix} \hat{\varphi}_{\mathrm{L}}(\tau,x) \\ \hat{\varphi}_{\mathrm{R}}(\tau,x) \end{pmatrix}, \tag{12.37a}$$

$$\widehat{U}^{\dagger}_{\mathrm{T}} \begin{pmatrix} \hat{\varphi}_{\mathrm{L}}(\tau,x) \\ \hat{\varphi}_{\mathrm{R}}(\tau,x) \end{pmatrix} \widehat{U}_{\mathrm{T}} = \begin{pmatrix} -\hat{\varphi}_{\mathrm{R}}(\tau,x) \\ -\hat{\varphi}_{\mathrm{L}}(\tau,x)+\pi \end{pmatrix}, \tag{12.37b}$$

$$\widehat{U}^{\dagger}_{\mathrm{R}} \begin{pmatrix} \hat{\varphi}_{\mathrm{L}}(\tau,x) \\ \hat{\varphi}_{\mathrm{R}}(\tau,x) \end{pmatrix} \widehat{U}_{\mathrm{R}} = \begin{pmatrix} -\hat{\varphi}_{\mathrm{R}}(\tau,-x) \\ -\hat{\varphi}_{\mathrm{L}}(\tau,-x) \end{pmatrix}. \tag{12.37c}$$

We note that in deriving transformation rules (12.37), one must take care of the transformation rules on the Klein factors as well. Demanding the invariance of the operator $\mathrm{i}\hat{\eta}_{\mathrm{R}}\hat{\eta}_{\mathrm{L}}$, we find the transformation rules

$$\widehat{U}^{\dagger}_{\mathrm{T}} \begin{pmatrix} \hat{\eta}_{\mathrm{L}} \\ \hat{\eta}_{\mathrm{R}} \end{pmatrix} \widehat{U}_{\mathrm{T}} = \begin{pmatrix} +\hat{\eta}_{\mathrm{R}} \\ +\hat{\eta}_{\mathrm{L}} \end{pmatrix}, \tag{12.38a}$$

$$\widehat{U}^{\dagger}_{\mathrm{R}} \begin{pmatrix} \hat{\eta}_{\mathrm{L}} \\ \hat{\eta}_{\mathrm{R}} \end{pmatrix} \widehat{U}_{\mathrm{R}} = \begin{pmatrix} +\hat{\eta}_{\mathrm{R}} \\ -\hat{\eta}_{\mathrm{L}} \end{pmatrix}. \tag{12.38b}$$

The corresponding transformation rules for the bosonic pair of dual fields are then found to be

$$\widehat{U}^{\dagger}_{\mathrm{C}} \begin{pmatrix} \hat{\phi}(\tau,x) \\ \hat{\theta}(\tau,x) \end{pmatrix} \widehat{U}_{\mathrm{C}} = \begin{pmatrix} \hat{\phi}(\tau,x) \\ \hat{\theta}(\tau,x) \end{pmatrix}, \tag{12.39a}$$

$$\widehat{U}^{\dagger}_{\mathrm{T}} \begin{pmatrix} \hat{\phi}(\tau,x) \\ \hat{\theta}(\tau,x) \end{pmatrix} \widehat{U}_{\mathrm{T}} = \begin{pmatrix} -\hat{\phi}(\tau,x)+\sqrt{\pi}/2 \\ +\hat{\theta}(\tau,x)-\sqrt{\pi}/2 \end{pmatrix}, \tag{12.39b}$$

3 In fact, the unitary particle-hole transformation operator $\widehat{U}_{\mathrm{C}}$ replaces the spinor $\hat{\Psi}$ with its conjugate transpose $\hat{\Psi}^{\dagger}(\tau,x)$ by the transformation rule $\widehat{U}^{\dagger}_{\mathrm{C}}\,\hat{\Psi}(\tau,x)\,\widehat{U}_{\mathrm{C}} = \hat{\Psi}^{\dagger}(\tau,x)\,\mathrm{M}$, where M is a unitary matrix. It follows from the Majorana reality condition (12.3b) and the representation (12.7a) that $\mathrm{M} = \mathrm{X}_{10}$. This implies the transformation rule (12.36a) for the individual components of the spinor $\hat{\Psi}(\tau,x)$.

$$\widehat{U}_{\mathrm{R}}^{\dagger}\begin{pmatrix}\hat{\phi}(\tau,x)\\ \hat{\theta}(\tau,x)\end{pmatrix}\widehat{U}_{\mathrm{R}}=\begin{pmatrix}-\hat{\phi}(\tau,-x)\\ +\hat{\theta}(\tau,-x)\end{pmatrix}. \tag{12.39c}$$

Alternatively, the transformations (12.39) can also be deduced from applying the many-body symmetry transformations on the components of the fermionic two-current.

Equipped with the transformation rules (12.39), we explore the phase diagram corresponding to the actions (12.35). For both actions, the corresponding cosine term has the scaling dimension

$$\Delta_m := \frac{4}{\sqrt{1+4\frac{\lambda_{1,m}^2+\lambda_{2,m}^2}{\pi}}},\qquad m=a,b. \tag{12.40}$$

Therefore, the cosine terms are IR irrelevant when $\lambda_{1,m}^2+\lambda_{2,m}^2<3\pi/4$ and the theory remains critical. Increasing the interaction strengths makes the cosines relevant, in which case the fields θ and ϕ are pinned to the minima of the corresponding cosine terms in the ground state.

Each cosine has four extrema, two of which become minima depending on the difference $\lambda_{1,m}^2-\lambda_{2,m}^2$ being positive or negative. In particular, when this difference is zero, both cosines vanish and the low-energy effective theory is that of a free scalar field, i.e., it also remains critical. This is the O(2)-symmetric line in the parameter space. Away from this line, we observe twofold ground-state degeneracy due to the two minima of the cosine.

For action (12.35a) with $\lambda_{1,a}^2>\lambda_{2,a}^2$, the two ground states are $\phi=\sqrt{\pi}/4$ and $\phi=3\sqrt{\pi}/4$. The transformation rules (12.39) then imply that RS is spontaneously broken. Conversely, when $\lambda_{1,a}^2<\lambda_{2,a}^2$, the ground states correspond to $\phi=0$ and $\phi=\sqrt{\pi}/2$, which implies that TRS is spontaneously broken.

For action (12.35b), the transformation rules (12.39) imply that RS always holds, whereas TRS is broken whenever there are two ground states separated by a shift of $\hat{\theta}$ by $\sqrt{\pi}/2$. This is realized by the cosine interaction in Eq. (12.35b).

In Fig. 12.1, we plot the phase diagrams for both actions (12.35a) ($m=a$) and (12.35b) ($m=b$) as functions of the interaction strengths $\lambda_{1,m}^2$ and $\lambda_{2,m}^2$, respectively. For given $m=a,b$, we define the red line in Fig. 12.1 by

$$\lambda_{1,m}^2+\lambda_{2,m}^2=3\pi/4 \iff \Delta_m=2 \tag{12.41}$$

and the blue line in Fig. 12.1 by

$$\lambda_{1,m}^2=\lambda_{2,m}^2. \tag{12.42}$$

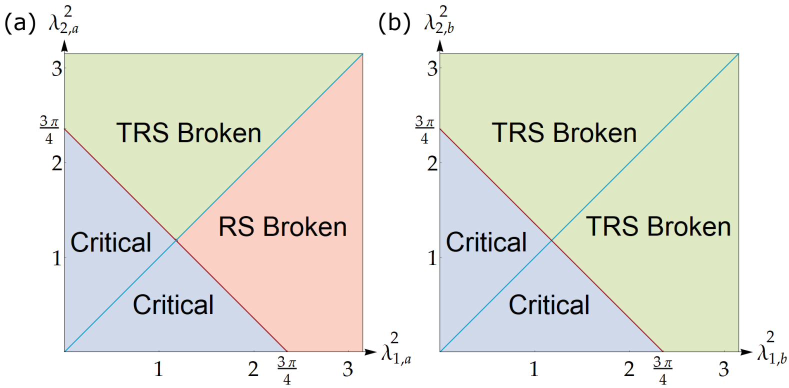

Figure 12.1: Phase diagram for the edge theories defined by the actions (12.35a) ($m = a$) in panel (a) and (12.35b) ($m = b$) in panel (b) as a function of the interaction strengths $\lambda^2_{i,m}$ with $i = 1, 2$ and $m = a, b$. Along the blue line, O(2) symmetry holds and both cosine interactions vanish. Along the red line, both cosine interactions are marginal.

Below the red line (12.41), the cosine interactions are irrelevant as their scaling dimensions are larger than 2. Each point in coupling space is then a critical phase with algebraic correlation functions characterized by scaling exponents that are smooth functions of the couplings $\lambda^2_{1,m}$ and $\lambda^2_{2,m}$. The free Dirac point is defined by the origin $\lambda^2_{1,m} = \lambda^2_{2,m} = 0$ of coupling space. Above the red line (12.41), the cosine interactions are relevant as their scaling dimensions are smaller than 2. Each point in coupling space then belongs to a gapped phase, unless the couplings multiplying the cosine interactions vanish, as they do along the blue line (12.42). Each gapped phase is associated with a pattern of spontaneous symmetry breaking. When $\lambda^2_{1,a} < \lambda^2_{2,a}$ ($\lambda^2_{1,a} > \lambda^2_{2,a}$), TRS (RS) is spontaneously broken as follows from minimizing the cosine interaction. When $\lambda^2_{1,b} \neq \lambda^2_{2,b}$, TRS is spontaneously broken as follows again from minimizing the cosine interaction. Along the blue line (12.42), O(2) symmetry holds and both cosine interactions vanish. Along the red line, both cosine interactions are marginal.

Abelian bosonization reveals that when quartic contact interactions compatible with the DIIIR symmetries are added, gap opening necessarily breaks one of the defining symmetries. Therefore, the $\nu = 2$ edge theory remains stable in the presence of interactions in the sense that it may only be gapped by interactions if any one of the protecting symmetries is either explicitly or spontaneously broken. We will next consider a generic family of symmetry-preserving cosine interactions and demonstrate that any interaction that gaps the edge theory must necessarily break spontaneously one of the protecting symmetries. We also discuss the effect of breaking of TS.

12.3.2 *Haldane criterion*

In Sec. 12.3.1, we bosonized Hamiltonians (12.29). They are of the sine-Gordon type. In light of this result, one may consider a family of bosonic Hamiltonians with generic cosine interactions. These interactions can gap some, most, or all bosonic degrees of freedom. How many bosonic degrees of freedom remain gapless is determined using the so-called Haldane stability criterion [189]. Doing so in a manner compliant with imposing the protecting symmetries, we are going to recover the cosine potentials (12.31).

We consider the Hamiltonian

$$\widehat{H} := \widehat{H}_0 + \widehat{H}_{\mathrm{int}}, \tag{12.43a}$$

which consists of the free Hamiltonian

$$\widehat{H}_0 := \int \mathrm{d}x\, \frac{1}{4\pi} \left(\partial_x \widehat{\Phi}^{\mathsf{T}}\right)(x)\, V \left(\partial_x \widehat{\Phi}\right)(x), \tag{12.43b}$$

that describes free chiral bosonic fields and the interaction

$$\widehat{H}_{\mathrm{int}} := -\int \mathrm{d}x \sum_{T\in\mathbb{H}} h_T(x)\, \mathbf{:}\cos\left(T^{\mathsf{T}}\, K\, \widehat{\Phi}(x) + \alpha_T(x)\right)\mathbf{:} \tag{12.43c}$$

that encodes a countable set of local fermionic interactions describing many-body umklapp processes that we shall call tunneling processes and hence label with the symbol T. The components of the field $\widehat{\Phi}$ obey the commutation relations

$$\left[\widehat{\Phi}_i(x), \widehat{\Phi}_j(x')\right] = -\mathrm{i}\pi \left[K^{-1}_{ij}\mathrm{sgn}(x-x')\right], \tag{12.43d}$$

where K is a 2×2, integer valued, symmetric, and invertible matrix. The static functions

$$h_T(x) \geq 0, \qquad 0 \leq \alpha_T(x) < 2\pi \tag{12.43e}$$

encode the possibility that TS is broken on the edge. The matrix V is a 2×2 symmetric and positive definite matrix. The two-dimensional tunneling vectors T are chosen from a set $\mathbb{H}$, that we will specify later.

Our aim is to compare Hamiltonian (12.43a) with Hamiltonian (12.31a) or Hamiltonian (12.31b) and use the Haldane criterion to identify some "minimal" sets of tunneling vectors $\mathbb{H}$ that would gap the chiral bosonic fields $\widehat{\Phi}$ if the functions h_{T} were "large". By comparing the free Hamiltonian (12.43b) with (12.31), we define the fields

$$\widehat{\Phi}(x) := \begin{pmatrix} \hat{\varphi}_{\mathrm{L}}(x) & \hat{\varphi}_{\mathrm{R}}(x) \end{pmatrix}^{\mathsf{T}}, \tag{12.44a}$$

the universal data

$$Q := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad K := \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{12.44b}$$

and the nonuniversal data

$$V := \begin{pmatrix} v & u \\ u & v \end{pmatrix}, \qquad 0 < v \in \mathbb{R}, \qquad 0 \leq u \in \mathbb{R}. \tag{12.44c}$$

With the universal data (12.44b), the algebra (12.43d) reduces to the algebra (12.30b). The two-dimensional vector Q is the charge vector. The explicit dependence of the positive couplings u and v on the couplings $\lambda^2_{i,m}$, $i = 1, 2$, $m = a, b$ from Hamiltonian (12.31a) will not be needed in the following.

The minimal set of tunneling vectors $\mathbb{H}$ is defined as follows. We first construct the maximal Haldane set

$$\mathbb{L} := \left\{ T \in \mathbb{Z}^2 \,\middle|\, T^{\mathsf{T}} K\, T' = T'^{\mathsf{T}} K\, T = 0, \forall T' \in \mathbb{L} \right\}, \tag{12.45}$$

i.e., the set of elements in $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ such that the bilinear form $T^{\mathsf{T}} K\, T'$ vanishes for any pair T and T' from $\mathbb{L}$. This constraint is the compatibility condition of the Haldane criterion. With it, there is no competition between any pair of cosine interaction entering $\widehat{H}_{\mathrm{int}}$. The vectors $T \in \mathbb{L}$ form a lattice since, for any pair $T, T' \in \mathbb{L}$, the linear combination $n\,T + n'\,T'$ with $n, n' \in \mathbb{Z}$ also satisfies the compatibility condition. We then define the minimal set of tunneling vectors as the subset $\mathbb{H} \subset \mathbb{L}$ such that elements $T \in \mathbb{H}$ constitutes the primitive cell of the lattice $\mathbb{L}$ which is compatible with the symmetry requirements of class DIIIR.

The Haldane criterion then asserts that the Hamiltonian (12.43c) for a given $\mathbb{H}$, removes $2 \times |\mathbb{H}|$-chiral bosonic fields from gapless degrees of freedom by pinning them (the notation $|\mathbb{H}|$ denotes the cardinality of the set $\mathbb{H}$). In our case, Hamiltonian (12.43a) consists of only a single pair of chiral bosonic fields. Therefore, it is enough to find the single tunneling vector making up $\mathbb{H}$ to remove all gapless degrees of freedom.

For a general tunneling vector $T = (m, n)$ of integers $m, n \in \mathbb{Z}$, the Haldane compatibility condition implies that there are two solutions, $n = m$ and $n = -m$. Therefore, there exists two disjoint sets of lattices $\mathbb{L}$ generated by the primitive cells

$$\mathbb{H}_a := \left\{ \begin{pmatrix} +n_a & -n_a \end{pmatrix}^{\mathsf{T}} \,\middle|\, n_a \text{ to be determined} \right\}, \tag{12.46a}$$

$$\mathbb{H}_b := \left\{ \begin{pmatrix} +n_b & +n_b \end{pmatrix}^{\mathsf{T}} \,\middle|\, n_b \text{ to be determined} \right\}. \tag{12.46b}$$

The integers n_a and n_b are not yet determined. To determine how integers n_a and n_b are constrained, we define the pair of interactions

$$\widehat{H}_{\mathrm{int}\,a} := \int \mathrm{d}x\, h_a(x) :\!\cos\left(n_a \left[\hat{\varphi}_{\mathrm{L}} + \hat{\varphi}_{\mathrm{R}}\right](x) + \alpha_a(x)\right)\!:, \tag{12.47a}$$

and

$$\widehat{H}_{\mathrm{int}\,b} := \int \mathrm{d}x\, h_b(x) :\!\cos\left(n_b \left[\hat{\varphi}_{\mathrm{L}} - \hat{\varphi}_{\mathrm{R}}\right](x) + \alpha_b(x)\right)\!:, \tag{12.47b}$$

corresponding to the minimal sets (12.46a) and (12.46b), respectively, on which we shall impose the symmetries under the transformations defined in Eq. (12.39). Observe that, in the strong coupling limit

$$4\pi \sup\{h_a(x)\} \gg \max\{u, v\} \tag{12.48a}$$

[recall that $h_a(x) \geq 0$ for any x and u and v are defined in the velocity matrix (12.44c)], the linear combinations $\hat{\varphi}_{\mathrm{L}}(x) \pm \hat{\varphi}_{\mathrm{R}}(x)$ of the chiral fields are pinned to the minima of the cosine potentials, namely, either

$$n_a \left[\hat{\varphi}_{\mathrm{L}}(x) + \hat{\varphi}_{\mathrm{R}}(x)\right] = 2\pi k + \pi - \alpha_a(x), \tag{12.48b}$$

or

$$n_b \left[\hat{\varphi}_{\mathrm{L}}(x) - \hat{\varphi}_{\mathrm{R}}(x)\right] = 2\pi k + \pi - \alpha_b(x), \tag{12.48c}$$

respectively, for some integer $k \in \mathbb{Z}$.

12.3.2.1 *Symmetry constraints on Hamiltonian* (12.47a)

PHS is trivially satisfied by construction. Imposing TRS by using the transformation rule (12.37b) leads to the constraint

$$\alpha_a(x) = -\alpha_a(x) - n_a \pi \mod 2\pi, \tag{12.49a}$$

which implies

$$\alpha_a(x) = l_a\, \pi - \frac{n_a \pi}{2} \mod 2\pi, \quad l_a = 0, 1, \tag{12.49b}$$

since $\alpha_a(x) \in [0, 2\pi)$. Imposing RS by using the transformation rule (12.37c) leads to the constraint

$$h_a(-x) = h_a(x), \quad \alpha_a(-x) = -\alpha_a(x) \mod 2\pi. \tag{12.50}$$

Combining TRS and RS implies that

$$h_a(x) = h_a(-x) \tag{12.51a}$$

and

$$\alpha_a(x) = \left[f_a(|x|) - \frac{n_a}{2}\right] \pi\, \mathrm{sgn}(x) \mod 2\pi, \tag{12.51b}$$

where $f_a(x)$ is any function such that

$$f_a : [0, \infty) \to \{l_a \ : \ l_a = 0, 1\}\,. \tag{12.51c}$$

We note that for any even n_a, assuming that $f_a(|x|)$ is constant, the discontinuity at $x = 0$ of $\alpha_a(x)$ is an even multiple of 2π so the solution to Eqs. (12.51b) and (12.51c) can be chosen independent of x. This is not the case for odd n_a as $n_a\, \pi\, \mathrm{sgn}(x)/2 \bmod 2\pi$ changes by π mod 2π across $x = 0$. A set of minima for the interaction (12.47a) compatible with TRS and RS that are labeled by the integers l_a and n_a are thus given by

$$n_a \left[\hat{\varphi}_{\mathrm{L}}(x) + \hat{\varphi}_{\mathrm{R}}(x)\right] + \left(l_a - \frac{n_a}{2}\right) \pi \operatorname{sgn}(x) = \pi, \tag{12.51d}$$

where the right-hand side is defined modulo 2π. Here, to minimize the cost in kinetic energy arising from discontinuities, we restrict discontinuities to occur only at $x = 0$ and demand that $h(x)$ vanishes smoothly at $x = 0$ if the argument of the cosine is discontinuous at $x = 0$. From now on, we only consider the cases $n_a = 1$ and $n_a = 2$.

When $n_a = 1$, the minima (12.51d) simplify to

$$\begin{aligned} \hat{\varphi}_{\mathrm{L}}(x) + \hat{\varphi}_{\mathrm{R}}(x) &= \pi + (1/2 - l_a)\, \pi \operatorname{sgn}(x) \mod 2\pi \\ &= \begin{cases} \frac{\pi}{2} \operatorname{sgn}(-x), & \text{if } l_a = 0, \\ \\ \frac{\pi}{2} \operatorname{sgn}(x), & \text{if } l_a = 1. \end{cases} \end{aligned} \tag{12.52}$$

One verifies that

$$\left[\hat{\varphi}_{\mathrm{L}}(x) + \hat{\varphi}_{\mathrm{R}}(x)\right]' = \begin{cases} \frac{\pi}{2} \operatorname{sgn}(-x), & \text{if } l_a = 0, \\ \\ \frac{\pi}{2} \operatorname{sgn}(x), & \text{if } l_a = 1, \end{cases} \tag{12.53}$$

where the prime over the operators on the left-hand side is a short-hand notation for their image under either reversal of time or the reflection as defined by Eq. (12.37). Therefore, for a given phase profile specified by l_a, there exists a unique gapped ground state for the bosonic interaction $\widehat{H}_{\mathrm{int}\,a}$ that is invariant under the action of either TRS or RS. When $n_a = 1$ and the competition between the kinetic energy and the interaction (12.47a) results in the opening of a spectral gap (with a midgap bound state) on the edge, TRS and RS are neither broken explicitly nor spontaneously, while TS is explicitly broken. As announced below Eqs. (12.7) by making use of the bulk-edge correspondence, the noninteracting topological classification $\mathbb{Z}$ of symmetry class DIIIR in (2+1)-dimensional spacetime reduces to the topological classification $\mathbb{Z}_2$ of symmetry class DIII when a RS compliant breaking of TS is allowed [100, 118], since $\widehat{H}_{\mathrm{int}\,a}$ with $n_a = 1$ is nothing but a fermionic mass term in the complex fermion representation. The midgap states bound at the reflection symmetric points are protected by the actions of TRS and RS and cannot be gapped. Such protected “corner" modes are nothing but the signature of a second-order SPT phase induced by the spatially varying mass term. Indeed, it has been shown in Ref. [110] that a two-dimensional superconductor in the symmetry class DIII with RS but no TS along the boundary is an example of a second-order SPT phase [4].

When $n_a = 2$, the minima (12.51d) simplify to

$$2\left[\hat{\varphi}_{\mathrm{L}}(x) + \hat{\varphi}_{\mathrm{R}}(x)\right] = \pi + (1 - l_a)\,\pi\,\mathrm{sgn}(x) \mod 2\pi. \tag{12.54a}$$

Because

$$\pi\,\mathrm{sgn}(x) = \pi \mod 2\pi, \qquad -\pi = \pi \mod 2\pi, \tag{12.54b}$$

one may write

$$2\left[\hat{\varphi}_{\mathrm{L}}(x) + \hat{\varphi}_{\mathrm{R}}(x)\right] = \pi\,l_a, \mod 2\pi. \tag{12.54c}$$

We conclude that

$$\hat{\varphi}_{\mathrm{L}}(x) + \hat{\varphi}_{\mathrm{R}}(x) = \begin{cases} 0, & \text{if } l_a = 0, \\ \pi, & \text{if } l_a = 0, \\ \pi/2, & \text{if } l_a = 1, \\ 3\pi/2, & \text{if } l_a = 1. \end{cases} \tag{12.55a}$$

One verifies that

$$\left[\hat{\varphi}_{\mathrm{L}}(x) + \hat{\varphi}_{\mathrm{R}}(x)\right]_{\mathrm{TRS}} = \begin{cases} \pi, & \text{if } l_a = 0, \\ 0, & \text{if } l_a = 0, \\ \pi/2, & \text{if } l_a = 1, \\ 3\pi/2, & \text{if } l_a = 1, \end{cases} \tag{12.55b}$$

4 The topological index belongs to the group $\mathbb{Z}_2$. Hence, the midgap state bound by a dynamical mass supporting a domain wall for the $\nu = 4$ case can be gapped as opposed to the $\nu = 2$ case.

and

$$[\hat{\varphi}_{\mathrm{L}}(x)+\hat{\varphi}_{\mathrm{R}}(x)]_{\mathrm{RS}}=\begin{cases}0, & \text{if } l_a=0,\\ \pi, & \text{if } l_a=0,\\ 3\pi/2, & \text{if } l_a=1,\\ \pi/2, & \text{if } l_a=1,\end{cases} \tag{12.55c}$$

where the subscripts TRS and RS are short-hand notations for the image of the minima under reversal of time and space inversion, respectively. There are two crucial differences between the cases $n_a=1$ and $n_a=2$. The minima (12.55a) transform in a nontrivial way under the actions of TRS and RS. For each choice l_a, two minima are exchanged under the action of either reversal of time or space inversion. Furthermore, the compactness of the chiral fields and the choice $n_a=2$ conspire in such a way that they minimize the interaction $\widehat{H}_{\mathrm{int}\,a}$ without breaking the TS.

The cosine in the interaction $\widehat{H}_{\mathrm{int}\,a}$ with $n_a=2$ is identical to the cosine in Hamiltonian (12.31a). The coupling $h(x)\geq 0$ breaks TS in the interaction $\widehat{H}_{\mathrm{int}\,a}$ when it is not a constant function of x, unlike the coupling that multiplies the cosine in Hamiltonian (12.31a). The two choices for l_a in Eq. (12.55a) correspond to fixing the overall sign of the interaction $\widehat{H}_{\mathrm{int}\,a}$ with $n_a=2$ when evaluated at its translation symmetric minima. In other words, the two choices for l_a in Eq. (12.55a) with $n_a=2$ correspond to choosing which two translation symmetric extrema of the cosine term are the minima. Furthermore, from the transformation rules (12.55b) and (12.55c) we observe that the same patterns for spontaneous symmetry-breaking patterns as with Hamiltonian (12.31a). When $l_a=0$, TRS is spontaneously broken, whereas RS is protected. When $l_a=1$, RS is spontaneously broken, whereas TRS is protected. Hence, even though the interaction (12.47a) breaks TS when $h(x)$ is not a constant function of x, it shares with Hamiltonian (12.31a) the same phase diagram.

Finally, we note that the sign function that interpolates between any two translation symmetric minima of the interaction $\widehat{H}_{\mathrm{int}\,a}$ also minimizes $\widehat{H}_{\mathrm{int}\,a}$. One verifies that this sign function respects TRS and RS but breaks TS. Unlike the translation symmetric minima of the interaction $\widehat{H}_{\mathrm{int}\,a}$, this sign function costs kinetic energy. The competition between the kinetic and interaction terms results in a compromise by which the singularity of the sign function is smoothed. The outcome is a soliton that keeps TRS and RS but breaks TS. This soliton is a gapped excitation that can be interpreted as a pair of helical Majorana modes localized in the region where the soliton energy density is nonvanishing and whose existence is protected by TRS and RS in the Majorana representation of the boundary theory.

12.3.2.2 *Symmetry constraints on Hamiltonian* (12.47b)

PHS is again satisfied trivially by construction. Imposing TRS by using the transformation rule (12.39b) leads to the constraint

$$n_b = 2m, \qquad m \in \mathbb{Z}, \tag{12.56a}$$

i.e., n_b is an even integer. Imposing RS by using the transformation rule (12.39c) leads to the pair of constraints

$$h_b(-x) = h_b(x), \qquad \alpha_b(-x) = \alpha_b(x). \tag{12.56b}$$

A set of minima is given by

$$n_b\,[\hat{\varphi}_{\mathrm{L}}(x) - \hat{\varphi}_{\mathrm{R}}(x)] + \pi\, l_b \;= \pi, \quad \mathrm{mod}\ 2\pi, \tag{12.57}$$

where $l_b = 0, 1$. We only consider the case $n_b = 2$ and conclude that

$$\hat{\varphi}_{\mathrm{L}}(x) - \hat{\varphi}_{\mathrm{R}}(x) = \begin{cases} \pi/2, & \text{if } l_b = 0, \\ 3\pi/2, & \text{if } l_b = 0, \\ 0, & \text{if } l_b = 1, \\ \pi, & \text{if } l_b = 1. \end{cases} \tag{12.58}$$

One verifies that

$$[\hat{\varphi}_{\mathrm{L}}(x) - \hat{\varphi}_{\mathrm{R}}(x)]_{\mathrm{TRS}} = \begin{cases} 3\pi/2, & \text{if } l_b = 0, \\ \pi/2, & \text{if } l_b = 0, \\ \pi, & \text{if } l_b = 1, \\ 0, & \text{if } l_b = 1. \end{cases} \tag{12.59}$$

The four translation symmetric minima (12.58) are invariant under the action of RS. On the other hand, under the action of TRS, two translation symmetric minima corresponding to each l_b are exchanged. Therefore, RS is always protected by the interaction $\widehat{H}_{\mathrm{int}\,b}$ with

$n_b = 2$, whereas TRS is spontaneously broken by its minima. The argument of the cosine in $\widehat{H}_{\mathrm{int}\,b}$ with $n_b = 2$ is identical to that of the cosine in Hamiltonian (12.31b). Hence, both Hamiltonians obey the same pattern of symmetry breaking. Finally, even though the interaction (12.47b) breaks TS when $h(x)$ is not a constant function of x, it shares with Hamiltonian (12.31b) the same phase diagram (Fig. 12.1).

12.4 THE CASE $\nu = 1$

For the $\nu = 1$ case, the boundary theory consists of a single helical pair of Majorana fields. In this case, as we shall explain, it is not possible to employ the gradient expansion method used in Sec. 12.2. Instead, we proceed in two steps. First, we establish that there are two topological sectors in the effective bosonic theory for the boundary. Second, we write down the dominant quartic interaction which we treat within the mean-field approximation.

12.4.1 *Existence of two topological sectors*

The set (12.12) with $n = 1$ has the 4 elements ($\mathrm{X}_\mu \equiv \sigma_\mu$ with $\mu = 0, \cdots 3$). For $\nu = 1$, there is at most $\mathrm{N}(1) = 1$ interaction channel allowed by the symmetry conditions (12.7). Therefore, there is a unique parametrization

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}(\tau, x) := \beta_0\, \mathrm{i}\partial_x + \beta_1\, \phi\,(\tau, x), \tag{12.60}$$

$$\beta_0 := \mathrm{X}_3 \equiv \sigma_3, \quad \beta_1 := \mathrm{X}_2 \equiv \sigma_2, \tag{12.61}$$

of the dynamical boundary single-particle Hamiltonian. If we impose the nonlinear constraint

$$\phi^2(\tau, x) \equiv \bar{\phi}^2 \tag{12.62}$$

for some given real-valued number $\bar{\phi}$, the target manifold is then nothing but two points ± 1 with the only nonvanishing homotopy group $\pi_0(\mathsf{S}^0) = \mathbb{Z}_2$.

When the hard nonlinear constraint (12.62) is strictly imposed, all configurations of $\phi(\tau, x)$ other than the constant field $\phi(\tau, x) = \pm\bar{\phi}$ must be discontinuous at the spacetime points where $\phi(\tau, x)$ switches between $+\bar{\phi}$ and $-\bar{\phi}$. The gradient of $\phi(\tau, x)$ is then ill-defined at singular points and zero everywhere else. If we relax the condition (12.62) by imposing the nonlinear constraint asymptotically,

$$\lim_{\tau\to\pm\infty} \phi^2(\tau, x) \equiv \bar{\phi}^2, \tag{12.63}$$

then smooth deformations of these singular configurations are admissible. However, the continuous function $\phi(\tau, x)$ then necessarily takes the value zero along at least one time slice in $(1 + 1)$-dimensional space-time, which binds zero modes in the spectrum. This prevents employing the gradient expansion approach outlined in Sec. 12.2 since the Pfaffian obtained by integrating out real-valued Grassmann fields,

$$\begin{aligned} Z_{\mathrm{bd}} &\propto \int \mathcal{D}[\phi] \int \mathcal{D}[\chi]\, e^{-\int \mathrm{d}^2x\, \bar{\chi}(\mathrm{i}\gamma_\mu \partial_\mu - \mathrm{i}\phi)\chi} \\ &\propto \int \mathcal{D}[\phi]\, \mathrm{Pf}\left[\mathrm{i}\sigma_2\, D[\phi]\right], \end{aligned} \tag{12.64a}$$

vanishes due to zero eigenvalues of the kernel

$$D := \mathrm{i}\gamma_\mu \partial_\mu - \mathrm{i}\phi, \quad \gamma_0 := -\sigma_2, \quad \gamma_1 := \sigma_1, \tag{12.64b}$$

where $\bar{\chi} = \chi^\dagger(\mathrm{i}\sigma_2)$. Because the kernel $\mathrm{i}\sigma_2 D$ is skew symmetric, the identity

$$\left(\mathrm{Pf}\left[\mathrm{i}\sigma_2\, D[\phi]\right]\right)^2 = \mathrm{Det}\left[\mathrm{i}\sigma_2\, D[\phi]\right] \tag{12.65}$$

holds. Therefore, the Pfaffian of $\mathrm{i}\sigma_2 D$, is nothing but the square root of the functional determinant of $\mathrm{i}\sigma_2 D$.

The idea that we shall develop below is the following. According to Eq. (12.65), computing the Pfaffian of a skew-symmetric operator is akin to taking the square root of a number. Taking the square root of a real-valued number yields two roots differing by their signs. For any pair ϕ and ϕ', it is the relative sign between $\mathrm{Pf}\left[\mathrm{i}\sigma_2\, D[\phi]\right]$ and $\mathrm{Pf}\left[\mathrm{i}\sigma_2\, D[\phi']\right]$ that fixes if ϕ is topologically equivalent to ϕ'. The background ϕ is topologically equivalent to ϕ' if

$$\mathrm{sgn}\left(\frac{\mathrm{Pf}\left[\mathrm{i}\sigma_2\, D[\phi]\right]}{\mathrm{Pf}\left[\mathrm{i}\sigma_2\, D[\phi']\right]}\right) = +1. \tag{12.66}$$

Otherwise, the background ϕ is not topologically equivalent to ϕ'. We are going to show that there are two topological sectors in the effective bosonic theory, i.e., there are two disjoint sets of topologically inequivalent profiles of the field ϕ.

Although the kernel $\mathrm{i}\sigma_2 D[\phi]$ is not Hermitian, the kernel

$$D'[\phi] := \begin{pmatrix} -\phi & +\partial \\ +\bar{\partial} & +\phi \end{pmatrix} = -\mathrm{i}\sigma_1\,\partial_x + \mathrm{i}\sigma_2\,\partial_\tau - \sigma_3\,\phi, \tag{12.67a}$$

$$\partial := \partial_\tau - \mathrm{i}\partial_x, \quad \bar{\partial} := -\partial_\tau - \mathrm{i}\partial_x, \tag{12.67b}$$

(i) shares the same determinant as $\mathrm{i}\sigma_2 D[\phi]$ and (ii) is Hermitian. It follows that the eigenvalues of $D'[\phi]$ are real valued. Moreover, the kernel $D'[\phi]$ obeys the Bogoliubov-de Gennes condition and, hence, the nonvanishing real-valued eigenvalues of $D'[\phi]$ come in pairs of opposite signs. We shall assume that all eigenvalues of $D'[\phi]$ are nonvanishing. The label ι enumerates all pairs of eigenvalues $\pm|\lambda'_\iota| \in \mathbb{R}\setminus\{0\}$ of $D'[\phi]$. We then have the definition

$$\mathrm{Pf}\,[\mathrm{i}\sigma_2\,D[\phi]] := \prod_\iota |\lambda'_\iota| \tag{12.68}$$

that consists of choosing all the positive representatives of the pairs of nonvanishing eigenvalues. The question that immediately arises is if this definition can be done consistently over the entire target space of ϕ. If the answer to this question is positive, then the target space is topologically trivial. Otherwise, it is not.

Our goal is to show that there are two distinct topological sectors as discussed above. To this end, we shall choose an arbitrary profile $\phi(\tau,x)$ that obeys the boundary conditions (12.63).

Claim 4.

$$\mathrm{sgn}\left(\frac{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[\phi]\right]}{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[\bar{\phi}]\right]}\right) = -\mathrm{sgn}\left(\frac{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[\phi]\right]}{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[-\bar{\phi}]\right]}\right), \tag{12.69a}$$

and

$$\mathrm{sgn}\left(\frac{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[\phi]\right]}{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[-\phi]\right]}\right) = -1. \tag{12.69b}$$

Two comments are in order before we prove Eqs. (12.69). Equation (12.69a) implies that profile ϕ is topologically equivalent to either one of the two constant profiles $\pm\bar{\phi}$. In other words, there exist exactly two topological sectors with representative profiles $+\bar{\phi}$ and $-\bar{\phi}$ as measured by Eq. (12.69a). Equation (12.69b) implies that the profiles ϕ and $-\phi$ belong to distinct topological sectors, a fact that originates from a $\mathbb{Z}_2$ global anomaly [188, 190]. Indeed, the transformation

$$\chi = \sigma_3 \chi', \quad \phi = -\phi', \tag{12.70a}$$

leaves the Lagrangian

$$\bar{\chi}\left(\mathrm{i}\gamma_\mu \partial_\mu - \mathrm{i}\phi\right)\chi = \bar{\chi}'\left(\mathrm{i}\gamma_\mu \partial_\mu - \mathrm{i}\phi'\right)\chi' \tag{12.70b}$$

invariant, while the partition function (12.64a) changes according to

$$\begin{aligned} Z_{\mathrm{bd}} &\propto \int \mathcal{D}[\phi']\mathcal{D}[\chi']\mathcal{J}\left[\sigma_3\right] e^{-\int \mathrm{d}^2x\, \bar{\chi}'\left(\mathrm{i}\gamma_\mu\partial_\mu - \mathrm{i}\phi'\right)\chi'} \\ &\propto \int \mathcal{D}[\phi']\mathcal{J}\left[\sigma_3\right] \mathrm{Pf}\left[\mathrm{i}\sigma_2\, D[\phi']\right] \\ &\propto \int \mathcal{D}[\phi]\mathcal{J}\left[\sigma_3\right] \mathrm{Pf}\left[\mathrm{i}\sigma_2\, D[-\phi]\right]. \end{aligned} \tag{12.70c}$$

On the one hand, to reach the right-hand side of the second line, we allowed for a possibly nontrivial Jacobian $\mathcal{J}[\sigma_3]$ associated with the transformation $\chi = \sigma_3\chi'$. On the other hand, to reach the third line, we assumed that the Jacobian associated with the transformation $\phi = -\phi'$ is unity. Equation (12.69b) then implies that $\mathcal{J}\left[\sigma_3\right] = -1$, which is the precise definition of a $\mathbb{Z}_2$ global anomaly, namely the symmetry of the Lagrangian that is not respected by the measure.

Proof. We now prove Eqs. (12.69). To examine whether two profiles $\phi_{\mathrm{i}}(\tau, x)$ and $\phi_{\mathrm{f}}(\tau, x)$ are topologically equivalent, we introduce a parameter $t \in [0, 1]$ and define a continuous function $\phi_t(\tau, x)$ such that

$$\phi_{t=0}(\tau, x) = \phi_{\mathrm{i}}(\tau, x), \quad \phi_{t=1}(\tau, x) = \phi_{\mathrm{f}}(\tau, x). \tag{12.71a}$$

We choose the linear interpolation

$$\phi_t(\tau,x) := (1-t)\,\phi_{\mathrm{i}}(\tau,x) + t\,\phi_{\mathrm{f}}(\tau,x). \tag{12.71b}$$

We impose periodic boundary conditions in both τ and x,

$$\phi(\tau,x+L_x) = \phi(\tau,x), \quad \phi(\tau+L_\tau,x) = \phi(\tau,x). \tag{12.72}$$

Hence, interpolation (12.71) also satisfies these boundary conditions. Boundary conditions (12.72) describe a compact space-time ($\mathsf{S}^1 \times \mathsf{S}^1 = \mathsf{T}^2$). It follows that the spectrum of the kernel $D'[\phi_t]$ defined in Eq. (12.67) is discrete. If one calculates the flow of eigenvalues $\lambda'_{t,\iota}$ of the kernel $D'[\phi_t]$ as a function of t, whenever there is a gap closing, i.e., at least one of the $\lambda'_{t,\iota}$ is 0, there is a π phase change in the Pfaffian. Thus, an odd number of gap closings during the evolution from $t=0$ to $t=1$ means that the initial and final profiles belong to different topological sectors. We will prove Eqs. (12.69) by assuming that the number of gap closings is independent of the choice of the interpolation scheme, without calculating the actual number of gap closings explicitly.

We first examine a special case of Eq. (12.69a) for which $\phi(\tau,x) = +\bar{\phi}$. Consider the linear interpolation

$$\phi^{+,-}_t := (1-t)\bar{\phi} + t(-\bar{\phi}) = (1-2t)\bar{\phi}. \tag{12.73}$$

For any $t \neq 1/2$, $\phi^{+,-}_t$ contributes to the Kernel $D'[\phi^{+,-}_{t\neq 1/2}]$ as a constant nonvanishing mass term. Hence, the spectrum is gapped. This gap closes only at $t=1/2$, in which case the kernel $D'[\phi^{+,-}_{t=1/2}]$ is that of a free Majorana fermion. There exists only a single pair of zero eigenvalues that are labeled by reciprocal vector $(\omega,k)=(0,0)$. Therefore, we find that there is a single crossing between negative and positive eigenvalues of $D'[\phi^{+,-}_t]$ at $t=1/2$. It follows that in the special case $\phi(\tau,x) = +\bar{\phi}$, Eq. (12.69a) holds. For any profile $\phi(\tau,x)$, the manipulation

$$\begin{aligned}\operatorname{sgn}\left(\frac{\operatorname{Pf}\left[\mathrm{i}\sigma_2\,D[\phi]\right]}{\operatorname{Pf}\left[\mathrm{i}\sigma_2\,D[\bar{\phi}]\right]}\right) &= \operatorname{sgn}\left(\frac{\operatorname{Pf}\left[\mathrm{i}\sigma_2\,D[\phi]\right]}{\operatorname{Pf}\left[\mathrm{i}\sigma_2\,D[-\bar{\phi}]\right]}\right)\\ &\quad\times \operatorname{sgn}\left(\frac{\operatorname{Pf}\left[\mathrm{i}\sigma_2\,D[-\bar{\phi}]\right]}{\operatorname{Pf}\left[\mathrm{i}\sigma_2\,D[\bar{\phi}]\right]}\right)\end{aligned} \tag{12.74}$$

then implies Eq. (12.69a). Observe that identity (12.74) is nothing but the interpolation

$$\Phi^{+,-}_t := \begin{cases} (1-2t)\phi(\tau,x) - 2t\,\bar{\phi}, & \text{if } 0 \le t < \frac{1}{2}, \\ (2t-2)\bar{\phi} + (2t-1)\bar{\phi}, & \text{if } \frac{1}{2} \le t \le 1. \end{cases} \tag{12.75}$$

To show Eq. (12.69b), we note that for any $\phi(\tau, x)$,

$$\sigma_2\,\mathsf{K}\,D'[\phi]\,\mathsf{K}\,\sigma_2 = D'[-\phi]. \tag{12.76}$$

Hence, $D'[\phi]$ and $D'[-\phi]$ share the same eigenvalue spectrum. This implies that for the two interpolations

$$\Phi^{+}_t := (1-t)\,\bar{\phi} + t\,\phi(\tau,x), \tag{12.77a}$$

$$\Phi^{-}_t := (1-t)\,(-\bar{\phi}) + t\,(-\phi(\tau,x)) = -\Phi^{+}_t, \tag{12.77b}$$

$D'[\Phi^{+}_t]$ and $D'[\Phi^{-}_t]$ also share the same eigenvalue spectrum. Therefore, one can then show that

$$\operatorname{sgn}\left(\frac{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[\phi]\right]}{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[\bar{\phi}]\right]}\right) = \operatorname{sgn}\left(\frac{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[-\phi]\right]}{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[-\bar{\phi}]\right]}\right), \tag{12.77c}$$

which after rearrangement gives

$$\begin{aligned}\operatorname{sgn}\left(\frac{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[\phi]\right]}{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[-\phi]\right]}\right) &= \operatorname{sgn}\left(\frac{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[\bar{\phi}]\right]}{\mathrm{Pf}\left[\mathrm{i}\sigma_2\,D[-\bar{\phi}]\right]}\right) \\ &= -1. \end{aligned} \tag{12.77d}$$

Any profile $\phi(\tau, x)$ is topologically inequivalent to $-\phi(\tau, x)$, as claimed in Eq. (12.69b). □

12.4.2 *Mean-field treatment of the interaction*

To complement the discussion in the previous subsection, we integrate over the bosonic field ϕ in action (12.64a) and derive the effective action for the Majorana fields $\hat{\chi}_{\mathrm{L}}$ and $\hat{\chi}_{\mathrm{R}}$. The single interaction term has the form $\hat{\chi}_{\mathrm{L}}(x)\,\hat{\chi}_{\mathrm{R}}(x)\,\hat{\chi}_{\mathrm{L}}(x+\epsilon)\,\hat{\chi}_{\mathrm{R}}(x+\epsilon)$ where ϵ is a short-distance cutoff that implements point splitting. For weak coupling strength this interaction term is irrelevant and the boundary remains gapless. In the limit of a strong interaction strength, a gap opens in the spectrum [142, 191]. At the mean-field level, this gap corresponds to the bilinear $\mathrm{i}\hat{\chi}_{\mathrm{L}}\,\hat{\chi}_{\mathrm{R}}$ acquiring a nonvanishing expectation value. This is equivalent to replacing the dynamical field $\phi(\tau,x)$ in action (12.64a) by the constant profiles $\pm\bar{\phi}$. Inserting the mean-field solution for the field $\phi(\tau,x)$ explicitly breaks the TRS since the term $\pm\mathrm{i}\,\bar{\phi}\,\hat{\chi}_{\mathrm{L}}\,\hat{\chi}_{\mathrm{R}}$ is odd under the transformation (12.4b). Gapping the boundary is only possible by spontaneously breaking TRS.

12.5 INTERPRETATION FROM LSM PERSPECTIVE

In the preceding sections, we studied the cases of $\nu = 4$, $\nu = 2$, and, $\nu = 1$ pairs of helical Majorana modes. We now interpret the protected gaplessness of these cases as the presence fo underlying LSM constraints. FCEP described in Sec. 11.2 dictates that the classification of crystalline FSPT phases with total symmetry group

$$G_{\mathrm{tot},f} = \mathbb{Z} \rtimes \frac{\mathbb{Z}_4^{\mathrm{FT}} \times \mathbb{Z}_4^{\mathrm{FR}}}{\mathbb{Z}_2^{\mathrm{F}}}, \tag{12.78a}$$

is equivalent to the classification of FSPT phases with internal symmetry group

$$\widetilde{G}_{\mathrm{tot},f} = \mathbb{Z} \rtimes \mathbb{Z}_2^{\mathrm{R}'} \times \mathbb{Z}_4^{\mathrm{FT}}, \tag{12.78b}$$

where the generator of $\mathbb{Z}_2^{\mathrm{R}'}$ is represented antiunitarily. While obtining the full classification with internal symmetry group $\widetilde{G}_{\mathrm{tot},f}$ is still a nontrivial task, in what follows, we are going to argue that the $\mathbb{Z}_8$ classification corresponds to nontrivial cohomology groups of $\widetilde{G}_{\mathrm{tot},f}$. These then can be interpreted as generalized LSM constraints that apply to the one-dimensional boundary of the two-dimensional topological superconductor. A more in-depth and complete analysis is out of the scope of this dissertation and left as a future work.

The case of $\nu = 1$ is equivalent to the edge theory of the strong FSPT phase in class DIII which has a $\mathbb{Z}_2$-classification. Therefore, for any odd ν pairs of helical Majorana modes, the gapless degrees of freedom must be protected purely by the internal symmetry group $G_f = \mathbb{Z}_4^{\mathrm{FT}}$. In the classification scheme of two-dimensional IFT phases, this is attributed to the nontrivial algebra between the fermion parity symmetry and reversal of time at the boundary [146]. Indeed, intuitively, the local fermion parity operator

$$\widehat{U}(p;x) \sim \mathrm{i}\hat{\chi}_{\mathrm{L}}(x)\,\hat{\chi}_{\mathrm{R}}(x), \tag{12.79}$$

is odd under reversal of time. This is akin to a local projective representation of $\mathbb{Z}_4^{\mathrm{FT}}$ (see Appendix A.5.3) [5].

For any even ν, the protection of gapless Majorana modes due to $\mathbb{Z}_4^{\mathrm{FT}}$ disappears. There is no nontrivial algebra between reversal of time and fermion parity. For any ν, there is also no nontrivial algebra between spatial symmetries and fermion parity. This means that the underlying LSM constraints when ν is even can be captured by only considering the "bosonic" part of the symmetries, i.e.,

$$G_{\mathrm{tot},f}/\mathbb{Z}_2^{\mathrm{F}} = \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z} \rtimes \mathbb{Z}_2^{\mathrm{R}}. \tag{12.80a}$$

We therefore identify the group of internal symmetry

$$G = \mathbb{Z}_2^{\mathrm{T}}, \tag{12.80b}$$

and the group of crystalline symmetries

$$G_{\mathrm{spc}} = \mathbb{Z} \rtimes \mathbb{Z}_2^{\mathrm{R}}. \tag{12.80c}$$

For the case of $\nu = 2$, we have shown in Sec. 12.3.2 that perturbation by a translation-symmetry breaking mass term, the boundaries can be gapped everywhere except at the reflection symmetric points. Such a mass term binds zero-modes, a single Kramer's doublet, at the reflection centers. The same mass term for $\nu = 4$ case binds a pair of Kramer's doublet at each reflection center which can be gapped out. This is to say that imposing only the reflection symmetry $\mathbb{Z}_2^{\mathrm{R}}$ leads to a $\mathbb{Z}_4$ classification of class DIIIR (also see Ref. [110]), where $\nu = 2$ case corresponds to the order two element in $\mathbb{Z}_4$. This classification can be understood by invoking the crystalline equivalence principle 3 and computing the cohomology group (11.3b) with $d = 1$ and $\widetilde{G}_{\mathrm{spc}} = \mathbb{Z}_2^{\mathrm{R}'}$ (which is represented antiunitarily). We find

$$H^1\left(\mathbb{Z}_2^{\mathrm{R}'}, H^2\left(\mathbb{Z}_2^{\mathrm{T}}, \mathrm{U}(1)_{\mathfrak{c}}\right)\right) = H^1\left(\mathbb{Z}_2^{\mathrm{R}'}, \mathbb{Z}_2\right) = \mathbb{Z}_2. \tag{12.81}$$

5 This statement should not be taken at its face value. There is no lattice regularization of a single pair of helical Majorana modes for which $\mathbb{Z}_4^{\mathrm{FT}}$ also act locally.

The $\nu = 2$ case realizes the nontrivial element in the cohomology group (12.81). The underlying LSM constraint is due to the nontrivial projective representation of $\mathbb{Z}_2^{\mathrm{T}}$ at the reflection center.

The case $\nu = 4$ case realizes the trivial element in the cohomology group (12.81). The protected gaplessness of the boundary modes requires all three symmetries reversal of time, translation and reflection to be imposed. This can be understood, by invoking the crystalline equivalence principle 3, as a result of the cohomology group [6]

$$H^3\left(\mathbb{Z} \rtimes \mathbb{Z}_2^{\mathrm{R}}, H^1(\mathbb{Z}_2^{\mathrm{T}}, \mathbb{Z}_c)\right) = H^3\left(\mathbb{Z} \rtimes \mathbb{Z}_2^{\mathrm{R}}, \mathbb{Z}_2\right) \supset \mathbb{Z}_2. \tag{12.82}$$

The $\nu = 4$ case realizes the nontrivial element in the cohomology group (12.82). The $\nu = 8$ case realizes the trivial element in both cohomology groups (12.81) and (12.82).

In summary, together with the $\mathbb{Z}_2$ invariant of strong IFT phase in class DIII, the two LSM constraints (12.81) and (12.82) provide three $\mathbb{Z}_2$-valued indices. Since the noninteracting classification of class DIIIR is the cyclic group $\mathbb{Z}$, we deduce, without derivation, that the three $\mathbb{Z}_2$-valued indices deliver the group $\mathbb{Z}_8$.

6 The physical interpretation of this cohomology group is not clear to the author at the time this dissertation is written. It is left to be understood in the future work.

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CONCLUSION AND OUTLOOK

In this Dissertation, we demonstrated that IFT phases can be organized by using fermionic LSM Theorems that rule out nondegenerate, gapped, and, symmetric ground states. On one hand, contrapositions of these theorems provide a set of necessary condition to realize ground states of IFT phases and can be used to classify them. On the other hand, they also apply to the boundaries of crystalline IFT phases in higher dimensions and explain why the gapless boundary degrees of freedom are protected.

In Part I, we solely focused on fermionic LSM Theorems. We proved Theorems 2 and 3 which apply to local fermionic lattice Hamiltonians with translation symmetry and internal symmetry group G_f. The strength of these two theorems lies in the fact that they can be applied when G_f is only a discrete symmetry group in which case the strategy of constructing low-energy variational states that is used in Chapter 2 is invalid. Interestingly, we have shown in Chapter 4 that certain cases of Theorem 2 and Theorem 3 are intrinsically fermionic, i.e., they only apply to Hamiltonians with underlying fermionic degrees of freedom. Therefore, intrinsically fermionic LSM theorems are nontrivial extension of those that apply to the bosonic Hamiltonians. In particular, despite the exact boson-fermion dualities in one-dimensional space such as Jordan-Wigner transformation, intrinsically fermionic LSM theorems cannot be retrieved from any bosonic LSM theorem by using one of such dualities.

In Part II, starting from the fermionic LSM Theorems, we gave an exhaustive characterization of one-dimensional IFT phases. This was done in three steps. First, given any internal symmetry group G_f, we enumerated in Chapter 6 all one-dimensional IFT phases. Second, we derived in Chapter 7 the corresponding fermionic stacking rules, which prescribes how to add two Hamiltonians describing one-dimensional IFT phases to obtain another such Hamiltonian. This delivers an Abelian group structure of IFT phases for any symmetry group G_f. Third, we showed in Chapter 8, on general grounds, how to compute the protected ground state degeneracy of a one-dimensional IFT phase when open-boundary conditions are imposed. A comprehensive application to time-reversal symmetric Majorana chains and closely-related spin-1/2 cluster models was also presented in Chapter 10.

In the final Part III, we explored the relation between generalized LSM Theorems and IFT phases with crystalline symmetries. We described how LSM theorems with translation symmetry and weak topological phases have a one-to-one correspondence. Starting from this point of view, we argued that there is also a correspondence between generalized LSM Theorems with crystallographic symmetries other than translations and crystalline IFT phases. To demonstrate this, in Chapter 12 we explicitly studied the protected gapless boundary modes of a two-dimensional crystalline topological superconductor. An interpretation in terms of generalized LSM theorems is then given for why the edge states are protected.

APPENDICES

A

PROJECTIVE REPRESENTATIONS OF FERMIONIC SYMMETRY GROUPS

This Appendix provides the mathematical background to construct and classify projective representations of fermionic symmetry groups. A review of group cohomology is presented in Sec. A.1. Sec. A.2 describes the structure of fermionic symmetry groups G_f as central extensions of groups G by fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$. Some examples of frequently encountered fermionic symmetry groups are reviewed in Sec. A.3. We then construct and classify (projective) representations of a given G_f in Sec. A.4. The explicit classification of projective representations of the groups $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$, and $\mathbb{Z}_4^{\mathrm{FT}}$ are computed in Sec. A.5.

A.1 GROUP COHOMOLOGY

Given two groups G and M, an *n-cochain* is the map

$$\begin{aligned} \phi\colon G^n &\to M, \\ (g_1, g_2, \cdots, g_n) &\mapsto \phi(g_1, g_2, \cdots, g_n), \end{aligned} \tag{A.1}$$

that maps an n-tuple $(g_1, g_2, \cdots, g_n)$ to an element $\phi(g_1, g_2, \dots, g_n) \in M$. The set of all n-cochains from G^n to M is denoted by $C^n(G, M)$. We define an M-valued 0-cochain to be an element of the group M itself, i.e., $C^0(G, M) = M$. Henceforth, we will denote the group composition rule in G by $\cdot$ and the group composition rule in M additively by $+$ ($-$ denoting the inverse element).

Given the group homomorphism $\mathfrak{c}\colon G \to \{0, 1\}$, for any $g \in G$, we define the group action

$$\begin{aligned} \mathfrak{C}_g\colon M &\to M, \\ m &\mapsto (-1)^{\mathfrak{c}(g)}\, m. \end{aligned} \tag{A.2}$$

The homomorphism $\mathfrak{c}$ indicates whether and element $g \in G$ is represented unitarily $[\mathfrak{c}(g) = 0]$ or antiunitarily $[\mathfrak{c}(g) = 1]$. We define the map $\delta_{\mathfrak{c}}$

$$\begin{aligned} \delta^{n}_{\mathfrak{c}} \colon C^{n}(G, M) &\to C^{n+1}(G, M), \\ \phi &\mapsto (\delta^{n}_{\mathfrak{c}} \phi)\,, \end{aligned} \tag{A.3a}$$

from n-cochains to $(n+1)$-cochains such that

$$\begin{aligned} (\delta^{n}_{\mathfrak{c}} \phi)\,(g_1, \cdots, g_{n+1}) :=& \mathfrak{C}_{g_1}\left(\phi(g_2, \cdots, g_n, g_{n+1})\right) \\ &+ \sum_{i=1}^{n} (-1)^{i} \phi(g_1, \cdots, g_i \cdot g_{i+1}, \cdots, g_{n+1}) \\ &- (-1)^{n}\, \phi(g_1, \cdots, g_n). \end{aligned} \tag{A.3b}$$

The map $\delta^{n}_{\mathfrak{c}}$ is called a *coboundary operator*. For example, for $n = 1$ and $n = 2$ the corresponding coboundary operators act as

$$\left(\delta^{1}_{\mathfrak{c}} \alpha\right)(g_1, g_2) = (-1)^{\mathfrak{c}(g_1)}\, \alpha(g_2) - \alpha(g_1 \cdot g_2) + \alpha(g_1), \tag{A.4a}$$

$$\left(\delta^{2}_{\mathfrak{c}} \beta\right)(g_1, g_2, g_3) = (-1)^{\mathfrak{c}(g_1)}\, \beta(g_2, g_3) - \beta(g_1 \cdot g_2, g_3) + \beta(g_1, g_2 \cdot g_3) - \beta(g_1, g_2), \tag{A.4b}$$

on 1-cochain $\alpha \in C^1(G, M)$ and 2-cochain $\beta \in C^2(G, M)$, respectively.

Using the coboundary operator, we define two sets

$$Z^{n}(G, M_{\mathfrak{c}}) := \ker(\delta^{n}_{\mathfrak{c}}) = \{\phi \in C^{n}(G, M) \mid \delta^{n}_{\mathfrak{c}} \phi = 0\}\,, \tag{A.5a}$$

and

$$B^{n}(G, M_{\mathfrak{c}}) := \operatorname{im}(\delta^{n-1}_{\mathfrak{c}}) = \left\{\phi \in C^{n}(G, M) \mid \phi = \delta^{n-1}_{\mathfrak{c}} \phi',\ \phi' \in C^{n-1}(G, M)\right\}. \tag{A.5b}$$

The cochains in $Z^{n}(G, M_{\mathfrak{c}})$ are called *n-cocycles*. The cochains in $B^{n}(G, M_{\mathfrak{c}})$ are called *n-coboundaries*. The action of the boundary operator on the elements of the group M is sensitive to the homomorphism $\mathfrak{c}$. For this reason, we label M by $\mathfrak{c}$ in $Z^{n}(G, M_{\mathfrak{c}})$ and $B^{n}(G, M_{\mathfrak{c}})$.

The definition (A.3b) implies for any n-cochain $\phi \in C^{n}(G, M)$ the identity

$$\delta^{n+1}_{\mathfrak{c}}\, \delta^{n}_{\mathfrak{c}} \phi = 0 \tag{A.6}$$

holds, i.e., the coboundary of a coboundary $\Phi = \delta^n_{\mathfrak{c}}\phi$ vanishes. The nth cohomology group is defined as the quotient of the n-cocycles by the n-coboundaries, i.e.

$$H^n(G, M_{\mathfrak{c}}) := \frac{Z^n(G, M_{\mathfrak{c}})}{B^n(G, M_{\mathfrak{c}})} = \frac{\ker\left(\delta^n_{\mathfrak{c}}\right)}{\operatorname{im}\left(\delta^{n-1}_{\mathfrak{c}}\right)}. \tag{A.7}$$

The nth cohomology group $H^n(G, M_{\mathfrak{c}})$ counts the inequivalent cocycles that are not themselves not coboundaries. We denote its elements by $[\phi] \in H^n(G, M_{\mathfrak{c}})$, i.e., the equivalence class of the n-cocycle ϕ. $H^n(G, M_{\mathfrak{c}})$ has a Abelian group structure with the group composition rule

$$[\phi] + [\phi'] = [\phi + \phi']. \tag{A.8}$$

Finally, we define the following operation on the cochains. Given two cochains $\phi \in C^n(G, N)$ and $\theta \in C^m(G, M)$, we produce the cochain $(\phi \cup \theta) \in C^{n+m}(G, N \times M)$ through

$$(\phi \cup \theta)(g_1, \cdots, g_n, g_{n+1}, \cdots, g_m) := \Big(\phi(g_1, \cdots, g_n), \mathfrak{C}_{g_1 \cdot g_2 \cdots g_n}\Big(\theta(g_{n+1}, \cdots, g_{n+m})\Big)\Big). \tag{A.9a}$$

If we compose operation (A.9a) with the pairing map $f : N \times M \to M'$ where M' is an Abelian group, we obtain the cup product

$$(\phi \smile \theta)(g_1, \cdots, g_n, g_{n+1}, \cdots, g_m) := f\left(\phi(g_1, \cdots, g_n), \mathfrak{C}_{g_1 \cdot g_2 \cdots g_n}\Big(\theta(g_{n+1}, \cdots, g_{n+m})\Big)\right). \tag{A.9b}$$

Hence, $(\phi \smile \theta) \in C^{n+m}(G, M')$. For our purposes, both N and M are subsets of the integer numbers, $M' = \mathbb{Z}_2$, while the pairing map f is

$$f(\alpha, \beta) := \alpha(g_1, \cdots, g_n)\, \beta(g_{n+1}, \cdots, g_{n+m}) \bmod 2 \tag{A.10}$$

where multiplication of cochains α and β is treated as multiplication of integers numbers modulo 2. For instance, for the cup product of a 1-cochain $\alpha \in C^1\left(G, \mathbb{Z}_2\right)$ and a 2-cochain $\beta \in C^2\left(G, \mathbb{Z}_2\right)$, we write

$$(\alpha \smile \beta)(g_1, g_2, g_3) = \alpha(g_1)\, \mathfrak{C}_{g_1}(\beta(g_2, g_3)) = \alpha(g_1)\, \beta(g_2, g_3), \tag{A.11}$$

where the cup product takes values in $\mathbb{Z}_2 = \{0, 1\}$ and multiplication of α and β is the multiplication of integers. In reaching the last equality, we have used the fact that the

2-cochain $\beta(g_2, g_3)$ takes values in $\mathbb{Z}_2$ for which $\mathfrak{C}_{g_1}(\beta(g_2, g_3)) = \beta(g_2, g_3)$ for any g_1. The cup product defined in Eq. (A.9b) satisfies

$$\delta_{\mathfrak{c}}^{n+m}(\phi \smile \theta) = (\delta_{\mathfrak{c}}^{n}\phi \smile \theta) + (-1)^n\, (\phi \smile \delta_{\mathfrak{c}}^{m}\theta)\,, \tag{A.12}$$

given two cochains $\phi \in C^n(G, N)$ and $\theta \in C^m(G, M)$. Hence, the cup product of two cocycles is again a cocycle as the right-hand side of Eq. (A.12) vanishes.

A.2 CONSTRUCTION OF FERMIONIC SYMMETRY GROUPS

For quantum systems built out of Majorana degrees of freedom the parity (evenness or oddness) of the total fermion number is always a constant of the motion. We denote the group of two elements e and p

$$\mathbb{Z}_2^{\mathrm{F}} := \{e, p\,|\,e\,p = p\,e = p, \quad e = e\,e = p\,p\}\,, \tag{A.13}$$

whereby e is the identity element and p is the fermion parity operator. It is because of this interpretation of the group element p that we attach the upper index F to the cyclic group $\mathbb{Z}_2$. We denote the group of any symmetries other than the fermion parity by G. Any fermionic symmetry group G_f is then constructed from the group G, via a central extension of G by the fermion parity symmetry $\mathbb{Z}_2^{\mathrm{F}}$. This central extension can be written as the short-exact sequence

$$0 \to \mathbb{Z}_2^{\mathrm{F}} \xrightarrow{i} G_f \xrightarrow{\pi} G \to 0. \tag{A.14a}$$

Hereby, the homomorphisms $i : \mathbb{Z}_2^{\mathrm{F}} \to G_f$ and $b : G_f \to G$ are inclusion and projection maps, respectively. In other words, the map i is an injective homomorphism while π is a surjective homomorphism. The extension (A.14a) is called *central* since the image of i is in the center of G_f, i.e., for any $g_f \in G_f$ and $h \in \mathbb{Z}_2^{\mathrm{F}}$

$$g_f\, i(h) = i(h)\, g_f. \tag{A.14b}$$

The sequence is called *exact* since the identity

$$\mathrm{im}(i) = \ker(\pi) \tag{A.14c}$$

holds, i.e., the kernel of π is equal to the image of i. This means that

$$\begin{aligned} G &\cong G_f \,/\, \ker(\pi) \\ &= G_f \,/\, \mathrm{im}(i) \\ &\cong G_f \,/\, \mathbb{Z}_2^{\mathrm{F}}, \end{aligned} \tag{A.14d}$$

i.e., group G is isomorphic to the coset $G_f/\mathbb{Z}_2^{\mathrm{F}}$ [1]. It is instructive to consider the simple case in which G_f is the direct product of G and $\mathbb{Z}_2^{\mathrm{F}}$, i.e.,

$$G_f = G \times \mathbb{Z}_2^{\mathrm{F}} := \left\{ (g,h) \mid g \in G,\ h \in \mathbb{Z}_2^{\mathrm{F}} \right\}. \tag{A.15a}$$

In this case, the homomorphisms $i : \mathbb{Z}_2^{\mathrm{F}} \to G_f$ and $\pi : G_f \to G$ can be defined as

$$i(e) = (e,e) \in G_f, \qquad i(p) = (e,p) \in G_f, \qquad \pi\Big((g,h)\Big) = g \in G. \tag{A.15b}$$

In general, G_f does not have to be the direct product (A.15a). To see this, we define two more maps

$$s : G \to G_f, \qquad \tau : G_f \to \mathbb{Z}_2^{\mathrm{F}}, \tag{A.16a}$$

such that

$$\pi \circ s = \mathrm{id}_G, \quad \tau \circ i = \mathrm{id}_{\mathbb{Z}_2^{\mathrm{F}}}, \quad \tau(g_f\, i(h)) = \tau(g_f)\, h, \quad g_f \in G_f \quad h \in \mathbb{Z}_2^{\mathrm{F}}, \tag{A.16b}$$

where id_G and $\mathrm{id}_{\mathbb{Z}_2^{\mathrm{F}}}$ are identity maps on the groups G and $\mathbb{Z}_2^{\mathrm{F}}$, respectively. The map s is injective but not necessarily a homomorphism and also called a *section* of the projection map π. The map τ is a surjective map and called a *trivialization* of the central extension. As we shall see, the last condition in Eq. (A.16b) on the map τ guarantees construction of bijections between G_f and $G \times \mathbb{Z}_2^{\mathrm{F}}$.

1 The coset $G \cong G_f/\mathrm{im}(i)$ has a group structure since image of i is in the center of G_f and, therefore, is a normal subgroup.

Using the definition (A.14a) together with the maps (A.16), we are going to define a bijection (and its inverse) between the sets $G \times \mathbb{Z}_2^{\mathrm{F}}$ and G_f. First, we define the map $(s \cdot i)$

$$\begin{aligned}(s \cdot i)\colon G \times \mathbb{Z}_2^{\mathrm{F}} &\to G_f,\\ (g, h) &\mapsto g_f = s(g)\, i(h).\end{aligned} \tag{A.17a}$$

The map $(s \cdot i)$ is injective since the maps s and i are so. Following identities hold:

$$\pi(g_f) = \pi(s(g)\, i(h)) = \pi(s(g))\, \pi(i(h)) = \pi(s(g)) = g, \tag{A.17b}$$

$$\tau(g_f) = \tau\Big(s(g)\, i(h)\Big) = \tau\Big(s(g)\Big)\, h. \tag{A.17c}$$

Hereby, the first equation follows since π is a homomorphism and image of i is in the kernel of π. The second equation follows by condition (A.16b). We define the (left) inverse $(\pi \times \tau)$ of the map $(s \cdot i)$ as

$$\begin{aligned}(\pi \times \tau)\colon G_f &\to G \times \mathbb{Z}_2^{\mathrm{F}},\\ g_f &\mapsto (g, h) = \Big(\pi(g_f), \tau(g_f)\Big).\end{aligned} \tag{A.18}$$

The composition $I := (\pi \times \tau) \circ (s \cdot i)$ is the map

$$\begin{aligned}I\colon G \times \mathbb{Z}_2^{\mathrm{F}} &\to G \times \mathbb{Z}_2^{\mathrm{F}},\\ (g, h) &\mapsto (\tilde{g}, \tilde{h}) = \Big(g, \tau(s(g))\, h\Big).\end{aligned} \tag{A.19a}$$

Observing that each element $(g, h) \in G \times \mathbb{Z}_2^{\mathrm{F}}$ has a unique inverse, i.e.,

$$I(g, h) = I(g', h') \iff g = g', h = h', \tag{A.19b}$$

we conclude that both maps $(s \cdot i)$ and $(\pi \times \tau)$ are bijections between G_f and $G \times \mathbb{Z}_2^{\mathrm{F}}$.

We would like to promote the set bijections between G_f and $G \times \mathbb{Z}_2^{\mathrm{F}}$ to group isomorphisms. Using the bijection (A.17), for any two elements $g_{1,f}, g_{2,f} \in G_f$, one has

$$g_{1,f}\, g_{2,f} = s(g_1)\, i(h_1)\, s(g_2)\, i(h_2) = s(g_1)\, s(g_2)\, i(h_1\, h_2), \tag{A.20a}$$

where we used the fact that image of i is in the center of G_f. We observe that

$$\pi\Big(s(g_1)\,s(g_2)\Big) = \pi\Big(s(g_1)\,\pi\Big(s(g_2)\Big) = g_1\,g_2 = \pi\circ s\Big(g_1\,g_2\Big) = \pi(s(g_1\,g_2)), \qquad \text{(A.20b)}$$

where in the last step we used the fact that $\pi\circ s$ is the identity map by construction. Since the kernel of π is exactly the image of i, Eq. (A.20b) implies that

$$s(g_1)\,s(g_2) = \tilde{\gamma}(g_1,g_2)\,s(g_1\,g_2), \qquad \tilde{\gamma}(g_1,g_2)\in \mathrm{im}(i), \qquad \text{(A.20c)}$$

and the composition rule (A.20a) can be written as

$$s(g_1)\,i(h_1)\,s(g_2)\,i(h_2) = s(g_1\,g_2)i\Big(h_1\,h_2\,\gamma(g_1,g_2)\Big), \qquad \text{(A.20d)}$$

where we defined

$$\gamma(g_1,g_2) := \tau\left(\tilde{\gamma}(g_1,g_2)\right) \in \mathbb{Z}_2^{\mathrm{F}}. \qquad \text{(A.20e)}$$

We want to interpret Eq. (A.20d) as a modified composition rule on the set $G\times\mathbb{Z}_2^{\mathrm{F}}$. In other words, we say that under the isomorphism (A.17), the group

$$G\underset{\gamma}{\times}\mathbb{Z}_2^{\mathrm{F}}, \qquad \text{(A.21a)}$$

with the composition rule

$$(g_1,\,h_1)\underset{\gamma}{\circ}(g_2,\,h_2) = \left(g_1\,g_2,\,h_1\,h_2\,\gamma(g_1,g_2)\right), \qquad \text{(A.21b)}$$

where $\gamma\in C^2(G,\mathbb{Z}_2^{\mathrm{F}})$ is 2-cochain that specifies the central extension, is isomorphic to the group G_f.

The 2-cochain γ can be obtained from the trivialization τ. To see this, we act with the bijection (A.18) the left-hand side of Eq. (A.20d), which becomes

$$(\pi\times\tau)\left(s(g_1)\,i(h_1)\,s(g_2)\,i(h_2)\right) = \left(g_1\,g_2,\,h_1\,h_2\,\tau\left(s(g_1)\,s(g_2)\right)\right). \qquad \text{(A.22a)}$$

Since $(\pi\times\tau)$ is also a group isomorphism from G_f to $G\underset{\gamma}{\times}\mathbb{Z}_2^{\mathrm{F}}$, this must be equal to

$$(\pi\times\tau)\left(s(g_1)\,i(h_1)\right)\underset{\gamma}{\circ}(\pi\times\tau)\left(s(g_2)\,i(h_2)\right) = \left(g_1,\,\tau(s(g_1))\,h_1\right)\underset{\gamma}{\circ}\left(g_2,\,\tau(s(g_2))\,h_2\right)$$

$$= (g_1\, g_2,\, \tau(s(g_1))\, \tau(s(g_2))\, \gamma(g_1, g_2)\, h_1\, h_2)\,, \tag{A.22b}$$

which is true if and only if

$$\gamma(g_1, g_2) = \tau(s(g_1))\, \tau(s(g_2))\, \tau\left(s(g_1)\, s(g_2)\right) = \left(\delta^1_{\mathfrak{c}} \tau\right) \left(s(g_1),\, s(g_2)\right). \tag{A.22c}$$

Therefore, more compactly, 2-cochain γ is given by

$$\gamma = s^*(\delta^1_{\mathfrak{c}} \tau), \tag{A.22d}$$

where $s^*(\delta^1_{\mathfrak{c}} \tau)$ is the pullback of the 2-cocycle $\delta^1_{\mathfrak{c}} \tau \in Z^2\left(G, \mathbb{Z}^{\mathrm{F}}_2\right)$ by the map s. Note that Eq. (A.22c) implies that

$$\gamma\Big(\pi(g_{1,f}), \pi(g_{2,f})\Big) = \tau\Big(g_{1,f}\Big)\, \tau\Big(g_{2,f}\Big)\, \tau\Big(g_{1,f}\, g_{2,f}\Big). \tag{A.23a}$$

This follows from the fact that $s(\pi(g_{1,f}))$ differs from $g_{1,f}$ by an element in the image of the map i. The right-hand side of Eq. (A.22c) is invariant under $s(g_1) \mapsto s(g_1)\, i(h_1)$ for any $h_1 \in \mathbb{Z}^{\mathrm{F}}_2$ by definition (A.16b). Equation (A.23a) can be more compactly written as

$$\pi^* \gamma = \delta^1_{\mathfrak{c}} \tau, \tag{A.23b}$$

where $\pi * \gamma$ is the pullback of the 2-cocycle $\gamma \in Z^2\left(G, \mathbb{Z}^{\mathrm{F}}_2\right)$ by the map π.

One verifies that

$$\gamma(e, e) = \gamma(e, g) = \gamma(g, e) = e \in \mathbb{Z}^{\mathrm{F}}_2, \tag{A.24a}$$

and

$$\delta^2_{\mathfrak{c}} \gamma = 0, \tag{A.24b}$$

i.e., γ itself is a $\mathbb{Z}^{\mathrm{F}}_2$-valued 2-cocycle. The cocycle condition on γ ensures that the composition rule (A.21b) obeys associativity. For any element $(g, h) \in G \underset{\gamma}{\times} \mathbb{Z}^{\mathrm{F}}_2$, its inverse with respect to the composition rule (A.21b) is given by

$$(g^{-1}, [\gamma(g, g^{-1})]^{-1}\, h^{-1}), \tag{A.24c}$$

while the unit element is

$$(e, e). \tag{A.24d}$$

The two groups $G \underset{\gamma}{\times} \mathbb{Z}_2^{\mathrm{F}}$ and $G \underset{\gamma'}{\times} \mathbb{Z}_2^{\mathrm{F}}$ are isomorphic if there exists the bijective map

$$\begin{aligned} \widetilde{\kappa}\colon G \times \mathbb{Z}_2^{\mathrm{F}} &\to G \times \mathbb{Z}_2^{\mathrm{F}}, \\ (g, h) &\mapsto (g, \kappa(g)\, h) \end{aligned} \tag{A.25a}$$

induced by the map

$$\begin{aligned} \kappa\colon G &\to \mathbb{Z}_2^{\mathrm{F}}, \\ g &\mapsto \kappa(g), \end{aligned} \tag{A.25b}$$

such that the condition

$$\widetilde{\kappa}\Big((g_1, h_1) \underset{\gamma}{\circ} (g_2, h_2)\Big) = \widetilde{\kappa}\Big((g_1, h_1)\Big) \underset{\gamma'}{\circ} \widetilde{\kappa}\Big((g_2, h_2)\Big) \tag{A.26}$$

holds for all $(g_1, h_1), (g_2, h_2) \in G \times \mathbb{Z}_2^{\mathrm{F}}$. In other words, γ and γ' generate two isomorphic groups if the identity

$$\kappa(g_1\, g_2)\, \gamma(g_1, g_2) = \kappa(g_1)\, \kappa(g_2)\, \gamma'(g_1, g_2), \tag{A.27}$$

holds for all $g_1, g_2 \in G$, i.e.,

$$\gamma = \gamma'\, \delta_{\mathfrak{c}}^1 \kappa, \tag{A.28}$$

for some 1-cochain $\kappa \in C^1(G, \mathbb{Z}_2^{\mathrm{F}})$. We say that the group G_f obtained by extending the group G with the group $\mathbb{Z}_2^{\mathrm{F}}$ through the map γ splits when a map (A.25b) exists such that

$$\kappa(g_1 \cdot g_2) \cdot \gamma(g_1, g_2) = \kappa(g_1) \cdot \kappa(g_2) \tag{A.29}$$

holds for all $g_1, g_2 \in G$, i.e., G_f splits when it is isomorphic to the direct product (A.15). If the extension splits, then the map s defined in Eq. (A.16a) becomes a group homomorphism.

Two 2-cocycles γ and γ' are equivalent if they are related by a 2-coboundary $\delta_{\mathfrak{c}}^1 \kappa$. This is to say that all non-isomorphic central extensions of G by $\mathbb{Z}_2^{\mathrm{F}}$ through γ are classified by the second group cohomology $H^2\left(G, \mathbb{Z}_2^{\mathrm{F}}\right)$. We define an index $[\gamma] \in H^2\left(G, \mathbb{Z}_2^{\mathrm{F}}\right)$ to represent such an equivalence class, whereby the index $[\gamma]=0$ is assigned to the case when G_f splits.

A.3 EXAMPLES OF FERMIONIC SYMMETRY GROUPS

In this section, we explicitly review the construction of some typical symmetry groups encountered in condensed matter physics. For each case, we will specify the groups G_f and G, their typical (non-projective) representations, and, the maps i, π, s, τ, and, γ as defined in Eqs. (A.14a), (A.16), and, (A.22d).

A.3.1 *Symmetry Group* $\mathrm{U}(1)^{\mathrm{F}}$

The fermionic symmetry group $G_f = \mathrm{U}(1)^{\mathrm{F}}$ implements the charge conservation symmetry. Elements of $\mathrm{U}(1)^{\mathrm{F}}$ are specified by angle $\theta \in [0, 2\pi)$.

For n flavors of fermions, we define the creation and annihilation operators $\hat{c}_i^\dagger$, $\hat{c}_i$ with the algebra

$$\left\{\hat{c}_i,\, \hat{c}_j^\dagger\right\} = \delta_{ij}, \qquad \left\{\hat{c}_i,\, \hat{c}_j\right\} = 0, \qquad i,j = 1,\cdots,n. \tag{A.30a}$$

For element $\theta \in \mathrm{U}(1)^{\mathrm{F}}$, its typical representation is given by the unitary operator

$$\widehat{U}(\theta) := e^{\mathrm{i}\theta \sum_{i=1}^{n} \hat{n}_i}, \qquad \hat{n}_i := \hat{c}_i^\dagger\, \hat{c}_i, \tag{A.30b}$$

which implements the transformation rule

$$\widehat{U}(\theta)\, \hat{c}_i\, \widehat{U}^\dagger(\theta) = e^{-\mathrm{i}\theta}\, \hat{c}_i. \tag{A.30c}$$

One observes that a $\mathrm{U}(1)^{\mathrm{F}}$ rotation by $\theta = \pi$ implements the fermion parity symmetry, i.e.,

$$\widehat{U}(\theta = \pi) = e^{\mathrm{i}\pi \sum_{i=1}^{n} \hat{n}_i} = (-1)^{\sum_{i=1}^{n} \hat{n}_i} \equiv \widehat{U}(p). \tag{A.31}$$

The group $\mathrm{U}(1)^{\mathrm{F}}$ is a nonsplit central extension of group $G = \mathrm{U}(1)$ by fermion parity $\mathbb{Z}_2^{\mathrm{F}}$ that is characterized by the short-exact sequence

$$0 \to \mathbb{Z}_2^{\mathrm{F}} \underset{\tau}{\overset{i}{\rightleftharpoons}} \mathrm{U}(1)^{\mathrm{F}} \underset{s}{\overset{\pi}{\rightleftharpoons}} \mathrm{U}(1) \to 0. \tag{A.32a}$$

Given the elements $\varphi \in \mathrm{U}(1)$ with $\varphi \in [0, 2\pi)$ and $e, p \in \mathbb{Z}_2^{\mathrm{F}}$ the homomorphisms i and π are

$$i(e) = 0, \qquad i(p) = \pi, \qquad \pi(\theta) = 2\,\theta \bmod 2\pi. \tag{A.32b}$$

A section s of the projection π and a trivialization τ are given by

$$s(\varphi) = \frac{1}{2}\varphi \in [0, \pi) \subset \mathrm{U}(1)^{\mathrm{F}}, \qquad \tau(\theta) = p^{\frac{1}{\pi}[\theta - (\theta \bmod \pi)]} \in \mathbb{Z}_2^{\mathrm{F}}. \tag{A.32c}$$

Using definition (A.22d) we find the 2-cocycle γ to be

$$\gamma\left(\varphi, \varphi'\right) = p^{\frac{1}{2\pi}[\varphi + \varphi' - (\varphi + \varphi' \bmod 2\pi)]} \in \mathbb{Z}_2^{\mathrm{F}}. \tag{A.32d}$$

A.3.2 *Symmetry Group* $\mathrm{SU}(2)^{\mathrm{F}}$

The fermionic symmetry group $\mathrm{SU}(2)^{\mathrm{F}}$ implements spin rotation symmetry. Elements of the group $\mathrm{SU}(2)^{\mathrm{F}}$ are labeled by 2×2 unitary matrices with unit determinant, i.e., $M \in \mathrm{Mat}(2, \mathbb{C})$ such that

$$M\,M^{\dagger} = \mathbb{1}_2, \qquad \det M = 1. \tag{A.33a}$$

We can parametrize any such matrix by

$$M_{\theta,\hat{n}} = e^{-\mathrm{i}\frac{\theta}{2}\hat{n}\cdot\boldsymbol{\sigma}}, \tag{A.33b}$$

where $\theta \in [0, 2\pi]$ and $\hat{n} \in S^2 \subset \mathrm{R}^3$ is a unit vector in $\mathbb{R}^3$. The vector $\boldsymbol{\sigma}/2 = \left(\sigma_1, \sigma_2, \sigma_3\right)/2$ is the vector of (normalized) Pauli matrices which span the $\mathfrak{su}(2)$ Lie algebra in its fundamental (two-dimensional) representation. Physically, matrix $M_{\theta,\hat{n}}$ corresponds to rotation of a spinor along $\hat{n}$-direction by θ. Note that in the parametrization (A.33b), we have the identifications

$$M_{0,\hat{n}} = M_{0,\hat{n}'} = \mathbb{1}_2, \qquad M_{2\pi,\hat{n}} = M_{2\pi,\hat{n}'} = -\mathbb{1}_2, \tag{A.33c}$$

for any $\hat{n}, \hat{n'} \in S^2$. This means that the group manifold of $\mathrm{SU}(2)^{\mathrm{F}}$ is the 3-sphere S^3.

We consider n copies of spin-1/2 representations of the group rotation group, i.e., there are $2n$ creation and $2n$ annihilation operators with the algebra

$$\left\{\hat{c}_{\sigma,i}, \hat{c}^{\dagger}_{\sigma',j}\right\} = \delta_{\sigma\,\sigma'}\delta_{ij}, \qquad \left\{\hat{c}_{\sigma,i}, \hat{c}_{\sigma',j}\right\} = 0, \quad \sigma,\sigma' = \uparrow,\downarrow, \quad i,j = 1,\cdots,n. \tag{A.34a}$$

The typical representation of the element $M_{\theta,\hat{n}} \in \mathrm{SU}(2)^{\mathrm{F}}$ is

$$\widehat{U}\left(M_{\theta,\hat{n}}\right) := e^{-\mathrm{i}\frac{\theta}{2}\sum_{i=1}^{n}\hat{n}\cdot\left(\hat{\psi}^{\dagger}_{i}\,\boldsymbol{\sigma}\,\hat{\psi}_{i}\right)}, \qquad \hat{\psi}_i := \begin{pmatrix}\hat{c}_{\uparrow,i}\\ \hat{c}_{\downarrow,i}\end{pmatrix}, \tag{A.34b}$$

which implements the transformation rule

$$\widehat{U}\left(M_{\theta,\hat{n}}\right)\,\hat{\psi}_i\,\widehat{U}^{\dagger}\left(M_{\theta,\hat{n}}\right) = M_{\theta,\hat{n}}\,\hat{\psi}_i. \tag{A.34c}$$

One observes that for $\theta = 2\pi$ and vectors $\hat{n} = (1,0,0)$ $\hat{n} = (0,1,0)$, and, $\hat{n} = (0,0,1)$ the operator $\widehat{U}\left(M_{\theta,\hat{n}}\right)$ is equal to the fermion parity symmetry. For instance, choosing $\theta = 2\pi$ and $\hat{n} = (0,0,1)$ we obtain

$$\widehat{U}\left(M_{2\pi,(0,0,1)}\right) = e^{\mathrm{i}\pi\sum_{i=1}^{n}\left\{\hat{c}^{\dagger}_{\uparrow,i}\,\hat{c}_{\uparrow,i}-\hat{c}^{\dagger}_{\downarrow,i}\,\hat{c}_{\downarrow,i}\right\}} = (-1)^{\sum_{i=1}^{n}\left\{\hat{c}^{\dagger}_{\uparrow,i}\,\hat{c}_{\uparrow,i}+\hat{c}^{\dagger}_{\downarrow,i}\,\hat{c}_{\downarrow,i}\right\}} \equiv \widehat{U}(p), \tag{A.34d}$$

which is interpreted as 2π rotations of spin-1/2 particles being equal to fermion parity.

The group $\mathrm{SU}(2)^{\mathrm{F}}$ is a nonsplit central extension of group $G = \mathrm{SO}(3)$ by fermion parity $\mathbb{Z}_2^{\mathrm{F}}$ that is characterized by the short-exact sequence

$$0 \to \mathbb{Z}_2^{\mathrm{F}} \underset{\tau}{\overset{i}{\rightleftharpoons}} \mathrm{SU}(2)^{\mathrm{F}} \underset{s}{\overset{\pi}{\rightleftharpoons}} \mathrm{SO}(3) \to 0. \tag{A.35}$$

To better understand this extension, it is convenient to introduce the following parametrizations of the groups SO(3) and $\mathrm{SU}(2)^{\mathrm{F}}$. Elements of the group SO(3) are labeled by the 3×3 orthogonal matrices with unit determinant, i.e., $R\in \mathrm{Mat}(3,\mathbb{R})$ such that

$$R\,R^{\mathsf{T}} = \mathbb{1}_3, \qquad \det R = 1. \tag{A.36a}$$

To parametrize such matrices, we first define two disjoint subsets S^2_+ and S^2_- of S^2 as

$$S^2 = S^2_+ \sqcup S^2_-, \tag{A.36b}$$

$$S^2_+ := \left\{ (n_1,\, n_2,\, n_3) \in S^2 \;\middle|\; n_3 > 0,\ \text{or},\ n_3 = 0 \text{ with } 0 \le \arcsin(n_2/n_1) < \pi \right\}, \tag{A.36c}$$

$$S^2_- := \left\{ (n_1,\, n_2,\, n_3) \in S^2 \;\middle|\; n_3 < 0,\ \text{or},\ n_3 = 0 \text{ with } \pi \le \arcsin(n_2/n_1) < 2\pi \right\}. \tag{A.36d}$$

Such matrices are then one-to-one correspondence with

$$R_{\varphi,\hat{m}} = e^{-\mathrm{i}\varphi\,\hat{m}\cdot\boldsymbol{J}}, \tag{A.36e}$$

where $\varphi \in [0,\pi]$ and the unit vector $\hat{m}$ is such that

$$\hat{m} \in \begin{cases} S^2, & \text{if } 0 < \varphi < \pi, \\ S^2_+, & \text{if } \varphi = \pi. \end{cases} \tag{A.37}$$

The vector $\boldsymbol{J} = \left(J_1, J_2, J_3\right)$ is the vector of generators of $\mathfrak{so}(3)$ Lie algebra in its fundamental (3 dimensional) representation. The constraint (A.37) means that the group manifold of SO(3) is obtained by identifying antipodal points on S^3 which is the three dimensional real projective space, i.e., $\mathrm{SO}(3) \cong S^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$.

The homomorphisms i and π are

$$i(e) = M_{0,\hat{n}} = \mathbb{1}_2 \in \mathrm{SU}(2)^{\mathrm{F}}, \qquad i(p) = M_{2\pi,\hat{n}} = -\mathbb{1}_2 \in \mathrm{SU}(2)^{\mathrm{F}}, \tag{A.38a}$$

$$\pi\left(M_{\theta,\hat{n}}\right) = \begin{cases} R_{\theta,\hat{n}}, & \text{if } 0 \le \theta < \pi, \\ R_{\pi,\hat{n}}, & \text{if } \theta = \pi \text{ and } \hat{n} \in S^2_+, \\ R_{\pi,-\hat{n}}, & \text{if } \theta = \pi \text{ and } \hat{n} \in S^2_-, \\ R_{2\pi-\theta,-\hat{n}}, & \text{if } \pi < \theta \le 2\pi, \end{cases} \tag{A.38b}$$

A section s of the projection π and a trivialization τ are then given by

$$s\left(R_{\varphi,\hat{m}}\right) = M_{\varphi,\hat{m}} \in \mathrm{SU}(2)^{\mathrm{F}}, \qquad \tau\left(M_{\theta,\hat{n}}\right) = p^{\alpha} \in \mathbb{Z}_2^{\mathrm{F}}, \tag{A.38c}$$

$$\alpha := \begin{cases} 0, & \text{if } 0 \leq \theta < \pi, \\ 0, & \text{if } \theta = 0, \text{ and, } \hat{n} \in S^2_+ \\ 1, & \text{if } \theta = 0, \text{ and, } \hat{n} \in S^2_- \\ 1, & \text{if } \pi < \theta \leq 2\pi. \end{cases} \tag{A.38d}$$

Using definition (A.22d), we find that the 2-cocycle γ to be

$$\begin{aligned} \gamma\left(R_{\varphi,\hat{m}}, R_{\varphi',\hat{m}'}\right) &= \tau\left(M_{\varphi,\hat{m}}\right)\,\tau\left(M_{\varphi',\hat{m}}\right)\,\tau\left(s\left(R_{\varphi,\hat{m}}\right)\,s\left(R_{\varphi',\hat{m}'}\right)\right) \\ &= \tau\left(s\left(R_{\varphi,\hat{m}}\right)\,s\left(R_{\varphi',\hat{m}'}\right)\right). \end{aligned} \tag{A.38e}$$

In particular, we compute the following values of γ

$$\gamma\left(R_{\pi,\hat{x}},\,R_{\pi,\hat{y}}\right) = \gamma\left(R_{\pi,\hat{y}},\,R_{\pi,\hat{z}}\right) = \gamma\left(R_{\pi,\hat{z}},\,R_{\pi,\hat{x}}\right) e, \tag{A.38f}$$

$$\gamma\left(R_{\pi,\hat{y}},\,R_{\pi,\hat{x}}\right) = \gamma\left(R_{\pi,\hat{z}},\,R_{\pi,\hat{y}}\right) = \gamma\left(R_{\pi,\hat{z}},\,R_{\pi,\hat{x}}\right) = p, \tag{A.38g}$$

$$\gamma\left(R_{\pi,\hat{x}},\,R_{\pi,\hat{x}}\right) = \gamma\left(R_{\pi,\hat{y}},\,R_{\pi,\hat{y}}\right) = \gamma\left(R_{\pi,\hat{z}},\,R_{\pi,\hat{z}}\right) = p, \tag{A.38h}$$

where we used the unit vectors $\hat{x} := (1,0,0)$, $\hat{y} := (0,1,0)$, and, $\hat{z} := (0,0,1)$.

A.3.3 *Symmetry Group* $\mathbb{Z}_4^{\mathrm{FT}}$

The fermionic symmetry group $G_f = \mathbb{Z}_4^{\mathrm{FT}}$ implements the time-reversal symmetry for spin-1/2 particles. The group $\mathbb{Z}_4^{\mathrm{FT}}$ is generated by order four element g, i.e.,

$$\mathbb{Z}_4^{\mathrm{FT}} = \left\{e,\, g,\, g^2 \equiv p, g^3\right\}, \tag{A.39a}$$

where elements g and g^3 are represented antiunitarily and the homomorphism $\mathfrak{c}$

$$\mathfrak{c}(e) = \mathfrak{c}(g^2) = 0, \qquad \mathfrak{c}(g) = \mathfrak{c}(g^3) = 1. \tag{A.39b}$$

We consider n copies of spin-1/2 representations of time-reversal, i.e., there are $2n$ creation and $2n$ annihilation operators with the algebra

$$\left\{\hat{c}_{\sigma,i},\,\hat{c}^{\dagger}_{\sigma',j}\right\}=\delta_{\sigma\,\sigma'}\delta_{ij},\qquad \left\{\hat{c}_{\sigma,i},\,\hat{c}_{\sigma',j}\right\}=0,\quad \sigma,\sigma'=\uparrow,\downarrow,\quad i,j=1,\cdots,n. \tag{A.40a}$$

The generator g has the typical representation

$$\widehat{U}(g):=e^{-\mathrm{i}\frac{\pi}{2}\sum_{i=1}^{n}\left(\hat{\psi}^{\dagger}_{i}\,\sigma_{2}\,\hat{\psi}_{i}\right)}\,\mathsf{K},\qquad \mathsf{K}\,\mathrm{i}\mathsf{K}=-\mathrm{i},\qquad \hat{\psi}_{i}:=\begin{pmatrix}\hat{c}_{\uparrow,i}\\ \hat{c}_{\downarrow,i}\end{pmatrix} \tag{A.40b}$$

which implements the transformation rule

$$\widehat{U}(g)\,\hat{c}_{\sigma,i}\,\widehat{U}^{\dagger}(g)=(-1)^{\sigma}\hat{c}_{-\sigma,i},\qquad \sigma=\uparrow,\downarrow, \tag{A.40c}$$

where $(-1)^{\sigma}=\pm 1$ for $\sigma=\uparrow,\downarrow$, respectively. One observes that $\widehat{U}(g)$ squares to to the fermion parity symmetry, i.e.,

$$\widehat{U}(g^{2})=\widehat{U}(g)\,\widehat{U}(g)=e^{-\mathrm{i}\pi\sum_{i=1}^{n}\left(\hat{\psi}^{\dagger}_{i}\,\sigma_{2}\,\hat{\psi}_{i}\right)}\equiv\widehat{U}(p), \tag{A.41}$$

where the last equality follows from observing that $\widehat{U}(g^{2})$ is the same operator as 2π $\mathrm{SU}(2)^{\mathrm{F}}$ rotation along y-axis. The group $\mathbb{Z}_{4}^{\mathrm{FT}}$ is a nonsplit central extension of group $G=\mathbb{Z}_{2}^{\mathrm{T}}=\{e,t\}$ by fermion parity $\mathbb{Z}_{2}^{\mathrm{F}}$ that is characterized by the short-exact sequence

$$0\to\mathbb{Z}_{2}^{\mathrm{F}}\underset{\tau}{\overset{i}{\rightleftharpoons}}\mathbb{Z}_{4}^{\mathrm{FT}}\underset{s}{\overset{\pi}{\rightleftharpoons}}\mathbb{Z}_{2}^{\mathrm{T}}\to 0. \tag{A.42a}$$

The homomorphisms i and π are

$$i(e)=e,\qquad i(p)=g^{2}, \tag{A.42b}$$

$$\pi(e)=e,\qquad \pi(g)=t,\qquad \pi(g^{2})=e,\qquad \pi(g^{3})=t. \tag{A.42c}$$

A section s of the projection π and a trivialization τ are given by

$$s(e)=e\in\mathbb{Z}_{4}^{\mathrm{FT}},\qquad s(t)=g\in\mathbb{Z}_{4}^{\mathrm{FT}}, \tag{A.42d}$$

$$\tau(e)=e\in\mathbb{Z}_{2}^{\mathrm{F}},\qquad \tau(g)=e\in\mathbb{Z}_{2}^{\mathrm{F}}, \tag{A.42e}$$

$$\tau(g^{2})=p\in\mathbb{Z}_{2}^{\mathrm{F}},\qquad \tau(g^{3})=p\in\mathbb{Z}_{2}^{\mathrm{F}}. \tag{A.42f}$$

Using definition (A.22d) we find the 2-cocycle γ to be

$$\gamma\,(e,e) = \gamma\,(e,t) = \gamma\,(t,e) = e \in \mathbb{Z}_2^{\mathrm{F}}, \qquad \gamma(t,t) = p \in \mathbb{Z}_2^{\mathrm{F}}. \tag{A.42g}$$

A.3.4 *Symmetry Group* $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$

The fermionic symmetry group $G_f = \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ implements the time-reversal symmetry for spinless particles. The group $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ is a split group and has the direct product structure (A.15a). Its elements are

$$\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}} = \{(e,e),\,(t,e),\,(e,p),\,(t,p)\}\,, \tag{A.43a}$$

where elements (t,e) and (t,p) are represented antiunitarily, i.e.,

$$\mathfrak{c}(e,e) = \mathfrak{c}(e,p) = 0, \qquad \mathfrak{c}(t,e) = \mathfrak{c}(t,p) = 1. \tag{A.43b}$$

We consider n creation and n annihilation operators with the algebra

$$\left\{\hat{c}_i,\,\hat{c}_j^{\dagger}\right\} = \delta_{ij}, \qquad \left\{\hat{c}_{\sigma,i},\,\hat{c}_{\sigma',j}\right\} = 0, \quad i,j = 1,\cdots,n. \tag{A.44a}$$

The generators (t,e) and (e,p) of the two subgroups $\mathbb{Z}_2^{\mathrm{T}}$ and $\mathbb{Z}_2^{\mathrm{F}}$ have the typical representations

$$\widehat{U}(t,e) := \hat{\mathbb{1}}\,\mathsf{K}, \qquad \mathsf{K}\,\mathrm{i}\mathsf{K} = -\mathrm{i}, \qquad \widehat{U}(e,p) := (-1)^{\sum_{i=1}^{n}\hat{n}_i} \qquad \hat{n}_i := \hat{c}_i^{\dagger}\,\hat{c}_i \tag{A.44b}$$

which implement the transformation rules

$$\widehat{U}(t,e)\,\hat{c}_i\,\widehat{U}^{\dagger}(t,e) = +\hat{c}_i, \qquad \widehat{U}(e,p)\,\hat{c}_i\,\widehat{U}^{\dagger}(e,p) = -\hat{c}_i. \tag{A.44c}$$

The group $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ is a split central extension of group $G = \mathbb{Z}_2^{\mathrm{T}} = \{e,t\}$ by fermion parity $\mathbb{Z}_2^{\mathrm{F}}$ that is characterized by the short-exact sequence

$$0 \to \mathbb{Z}_2^{\mathrm{F}} \underset{\tau}{\overset{i}{\rightleftharpoons}} \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}} \underset{s}{\overset{\pi}{\rightleftharpoons}} \mathbb{Z}_2^{\mathrm{T}} \to 0. \tag{A.45a}$$

The homomorphisms i and π are

$$i(e) = (e,e), \qquad i(p) = (e,p), \tag{A.45b}$$

$$\pi(e,e)=e, \qquad \pi(t,e)=t, \qquad \pi(e,p)=e, \qquad \pi(t,p)=t. \tag{A.45c}$$

A section s of the projection π and a trivialization τ are given by

$$s(e)=(e,e)\in\mathbb{Z}_2^{\mathrm{T}}\times\mathbb{Z}_2^{\mathrm{F}}, \qquad s(t)=(t,e)\in\mathbb{Z}_2^{\mathrm{T}}\times\mathbb{Z}_2^{\mathrm{F}}, \tag{A.45d}$$

$$\tau(e,e)=e\in\mathbb{Z}_2^{\mathrm{F}}, \qquad \tau(t,e)=e\in\mathbb{Z}_2^{\mathrm{F}}, \tag{A.45e}$$

$$\tau(e,p)=p\in\mathbb{Z}_2^{\mathrm{F}}, \qquad \tau(t,p)=p\in\mathbb{Z}_2^{\mathrm{F}}. \tag{A.45f}$$

Using definition (A.22d) we find the 2-cocycle γ to be

$$\gamma\,(e,e)=\gamma\,(e,t)=\gamma\,(t,e)=\gamma(t,t)=e\in\mathbb{Z}_2^{\mathrm{F}}, \tag{A.45g}$$

which is trivial as $\mathbb{Z}_2^{\mathrm{T}}\times\mathbb{Z}_2^{\mathrm{F}}$ is a split group.

A.4 CLASSIFICATION OF PROJECTIVE REPRESENTATIONS OF G_f

It was described in Appendix A.2, how a global symmetry group G_f for a fermionic quantum system naturally contains the fermion-number parity symmetry group $\mathbb{Z}_2^{\mathrm{F}}$ in its center, i.e., it is a central extension of a group G by $\mathbb{Z}_2^{\mathrm{F}}$. Such group extension are classified by prescribing an element $[\gamma]\in H^2(G,\mathbb{Z}_2^{\mathrm{F}})$. In what follows, we will characterize finite-dimensional quantum mechanical [$(0+1)$-dimensional] projective representations of a given fermionic symmetry group G_f. This is achieved in three steps. First, we specify a set of local degrees of freedom and a fermionic Fock space that the representations act on. Second, given a fermionic Fock space We then construct the general form of the projective representations of G_f. As we shall see, distinct projective representations are classified by the second cohomology class $[\phi]\in H^2(G_f,\mathrm{U}(1)_{\mathfrak{c}})$. Third, we trade $\phi\in C^2(G_f,\mathrm{U}(1))$ for a pair of indices $(\nu,\rho)\in C^2(G,\mathrm{U}(1))\times C^1(G,\mathbb{Z}_2)$. There is a one-to-one correspondence between the second cohomology classes $[\phi]$ and equivalence classes $[(\nu,\rho)]$ of the pair (ν,ρ) under certain equivalence relations.

A.4.1 *Fermionic Fock Spaces*

We assume the existence of n Hermitian Majorana operators

$$\mathfrak{O}_n:=\{\hat{\gamma}_1,\hat{\gamma}_2,\cdots,\hat{\gamma}_n\} \tag{A.46a}$$

that realizes the Clifford algebra

$$\mathrm{C}\ell_n := \mathrm{span}\left\{ \prod_{i=1}^{n} (\hat{\gamma}_i)^{m_i} \;\middle|\; \left\{\hat{\gamma}_i, \hat{\gamma}_j\right\} = 2\delta_{ij}, \quad m_i = 0,1, \quad i,j = 1,\cdots,n \right\}. \tag{A.46b}$$

We assign the index $[\mu] \in \{0,1\}$ to the parity of n, i.e.,

$$[\mu] = n \bmod 2. \tag{A.46c}$$

We consider the cases of even and odd n separately.

When $[\mu] = 0$, the even number n of Majorana operators from the set (A.46) span the fermionic Fock space

$$\mathfrak{F}_0 := \mathrm{span}\left\{ \prod_{\alpha=1}^{n/2} \left(\frac{\hat{\gamma}_{2\alpha-1} - \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right)^{m_\alpha} |0\rangle \;\middle|\; \left(\frac{\hat{\gamma}_{2\alpha-1} + \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right)|0\rangle = 0, \quad m_\alpha = 0,1 \right\} \tag{A.47a}$$

of dimension [2]

$$\dim \mathfrak{F}_0 = 2^{n/2}. \tag{A.47b}$$

When $[\mu] = 1$, the odd number n of Majorana operators from the set (A.46a) span a vector space that is not a fermionic Fock space. In order to recover a fermionic Fock space, we add to the set (A.46a) made of an odd number n of Majorana operators the Majorana operator $\hat{\gamma}_\infty$ [19],

$$\mathfrak{O}_{n,\infty} := \left\{\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_{n-1}, \hat{\gamma}_n, \hat{\gamma}_\infty\right\}, \tag{A.48}$$

thereby defining the Clifford algebra $\mathrm{C}\ell_{n+1}$. Here, the lower floor function $\lfloor\cdot\rfloor$ returns the largest integer $\lfloor x\rfloor$ smaller than the positive real number x. We may then define the fermionic Fock space

$$\mathfrak{F}_1 := \mathrm{span}\left\{ \prod_{\alpha=1}^{(n+1)/2} \left(\frac{\hat{\gamma}_{2\alpha-1} - \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right)^{m_\alpha} |0\rangle \;\middle|\; \left(\frac{\hat{\gamma}_{2\alpha-1} + \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right)|0\rangle = 0,\ m_\alpha = 0,1 \right\} \tag{A.49a}$$

2 The partition of a set of n labels into two pairs of $n/2$ labels is here arbitrary.

of dimension

$$\dim \mathfrak{F}_1 = 2^{(n+1)/2}, \tag{A.49b}$$

where it is understood that $\hat{\gamma}_{n+1} \equiv \hat{\gamma}_{\infty}$.

A.4.2 *Projective Representations of G_f*

Given a set (A.46a) of Majorana degrees of freedom, we want to implement a list of transformation rules on $\mathrm{C}\ell_n$. Each item in the list corresponds to a distinct element $g \in G_f$. This means that there exists a *faithful* representation [3] of the group G_f, i.e., an injective map $\widehat{U} : G_f \to \mathrm{Aut}\,(\mathfrak{F})$ where $\mathrm{Aut}\left(\mathfrak{F}_{[\mu]}\right)$ is the set of automorphisms on the fermionic Fock space $\mathfrak{F}_{[\mu]}$. We assume that the the set (A.46a) of Majorana operators is invariant under the representation $\widehat{U}$, i.e., for all $g \in G_f$

$$\widehat{U}(g)\, \mathrm{C}\ell_n\, \widehat{U}^{\dagger}(g) \subseteq \mathrm{C}\ell_n. \tag{A.50}$$

We construct the general form of the representation $\widehat{U}$ for the cases of $[\mu] = 0, 1$, separately. The distinction between the cases of $[\mu] = 0$ and $[\mu] = 1$ is the following. The representation $\widehat{U}$ implements a list of transformation rules on the set $\mathfrak{O}_n$. When $[\mu] = 0$, the corresponding Clifford algebra $\mathrm{C}\ell_n$ contains all the automorphisms on fermionic Fock space $\mathfrak{F}_0$. In contrasts, when $[\mu] = 1$, the set of automorphisms on fermionic Fock space $\mathfrak{F}_1$ is contained in the the extended Clifford algebra $\mathrm{C}\ell_{n+1}$. Therefore, the Clifford algebra $\mathrm{C}\ell_n$ contains only a subset of automorphisms on $\mathfrak{F}_1$. This leads to an ambiguity when defining $\widehat{U}$ for $[\mu] = 1$ which reflected by the fact that center of the Clifford algebra $\mathrm{C}\ell_n$ is two dimensional when n is odd and one dimensional when n is even.

A.4.2.1 *The Case of $[\mu] = 0$*

When $[\mu] = 0$, the number n of Majorana operators is even. We denote the identity on the local fermionic Fock space (A.47a) by $\hat{\mathbb{1}}_0$. The representation of element $p \in G_f$ that generates the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ is chosen to be

$$\widehat{U}(p) := \prod_{\alpha=1}^{n/2} \widehat{P}_{\alpha}, \quad \widehat{P}_{\alpha} := \mathrm{i}\hat{\gamma}_{2\alpha-1}\, \hat{\gamma}_{2\alpha}. \tag{A.51a}$$

3 The assumption of a faithful representation is crucial when defining the invariants that characterize projective representations.

The parity operators $\widehat{P}_1, \cdots, \widehat{P}_{n/2}$ are Hermitian, square to the identity, and are pairwise commuting. Hence, $\widehat{U}(p)$ is Hermitian and squares to the identity. Since operators $\widehat{P}_1, \cdots, \widehat{P}_{n/2}$ are pairwise commuting, we can simultaneously diagonalize them and choose any one of them to be even under complex conjugation K,

$$\mathsf{K}\, \widehat{P}_\alpha\, \mathsf{K} = \widehat{P}_\alpha, \tag{A.51b}$$

for $\alpha = 1, \cdots, n/2$. The most general form of a representation of element $g \in G_f$ is

$$\widehat{U}(g) := \widehat{V}(g)\, \mathsf{K}^{\mathfrak{c}(g)}, \tag{A.52}$$

where $\widehat{V}(g)$ is a unitary operator that belongs to $C\ell_n$ defined in Eq. (A.46).

A.4.2.2 *The Case of* $[\mu] = 1$

When $[\mu] = 1$, the number n of Majorana operators is odd. We denote the identity on the nonlocal fermionic Fock space (6.14a) by $\hat{\mathbb{1}}_1$. The representation of element $p \in G_f$ that generates the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ is chosen to be

$$\widehat{U}(p) := \widehat{P}\, \widehat{P}_{\mathrm{nonloc}}, \tag{A.53a}$$

$$\widehat{P} := \prod_{\alpha=1}^{(n-1)/2} \widehat{P}_\alpha, \quad \widehat{P}_\alpha := \mathrm{i}\hat{\gamma}_{2\alpha-1}\, \hat{\gamma}_{2\alpha}, \tag{A.53b}$$

$$\widehat{P}_{\mathrm{nonloc}} := \mathrm{i}\hat{\gamma}_n\, \hat{\gamma}_\infty, \tag{A.53c}$$

for $\widehat{U}(p)$ is proportional to the product $\hat{\gamma}_1 \cdots \hat{\gamma}_n\, \hat{\gamma}_\infty$ of all the generators in $C\ell_{n+1}$. As such, $\widehat{U}(p)$ anticommutes with all the Majorana operators that span the nonlocal fermionic Fock space (A.49a). The parity operators $\widehat{P}_1, \cdots, \widehat{P}_{(n-1)/2}, \widehat{P}_{\mathrm{nonloc}}$ are Hermitian, square to the identity, and are pairwise commuting. We choose to diagonalize them simultaneously and choose each of them to be even under complex conjugation K,

$$\mathsf{K}\, \widehat{P}_\alpha\, \mathsf{K} = \widehat{P}_\alpha, \qquad \mathsf{K}\, \widehat{P}_{\mathrm{nonloc}}\, \mathsf{K} = \widehat{P}_{\mathrm{nonloc}}, \tag{A.53d}$$

for $\alpha, \alpha' = 1, \cdots, (n-1)/2$.

In addition to defining a representation of the fermion parity p, we need to account for the fact that the center of the Clifford algebra $C\ell_n$ is two-dimensional when n is odd. We choose to represent the nontrivial element of this center by

$$\widehat{Y} := \widehat{P}\,\hat{\gamma}_n, \qquad \widehat{Y}^\dagger = \widehat{Y}, \qquad \widehat{Y}^2 = \hat{\mathbb{1}}_1. \tag{A.54}$$

By construction, $\widehat{Y}$ is proportional to the product $\hat{\gamma}_1 \cdots \hat{\gamma}_n \neq \hat{\mathbb{1}}_1$. It commutes with the Majorana operators $\hat{\gamma}_1, \cdots, \hat{\gamma}_n$, while it anticommutes with the Majorana operator $\hat{\gamma}_\infty$. The operator $\widehat{Y}$ is of odd fermion parity for it anticommutes with the fermion parity operator (A.53). Because $\widehat{Y}$ commutes with all the elements of $C\ell_n$, it follows that

$$\widehat{U}(g)\,\hat{\gamma}_i\,\widehat{U}^\dagger(g) = \widehat{U}'(g)\,\hat{\gamma}_i\,\widehat{U}'^\dagger(g), \qquad \widehat{U}'(g) := \widehat{U}(g)\,\widehat{Y}, \tag{A.55}$$

for any $i = 1, \cdots, n$, i.e., the operators $\widehat{U}(g)$ and $\widehat{U}'(g)$ implement the same transformation rule. However, the two operators have opposite fermion parities owing to the fact that $\widehat{Y}$ carries odd fermion parity.

The Clifford algebra $C\ell_n$ is closed under the action of the representation $\widehat{U}(g)$ [recall Eq. (A.50)]. In other words, $\widehat{U}(g)$ preserves locality in that its action on those operators whose non-trivial actions are limited to $\mathfrak{O}_n$ is merely to mix them. This locality is guaranteed only if the condition

$$\left[\widehat{U}(g)\,\hat{\gamma}_i\,\widehat{U}^\dagger(g), \widehat{Y}\,\right] = 0, \tag{A.56}$$

is satisfied for any $g \in G_f$ and $i = 1, \cdots, n$.

Claim 5. The condition (A.56) implies that $\widehat{U}(g)$ either commutes or anticommutes with the center $\widehat{Y}$ of $C\ell_n$, i.e.,

$$\widehat{Y}\,\widehat{U}(g) = \pm \widehat{U}(g)\,\widehat{Y}. \tag{A.57}$$

Furthermore, this is true only if the decomposition

$$\widehat{U}(g) := \widehat{V}(g)\,\widehat{Q}(g)\,\mathsf{K}^{\mathfrak{c}(g)} \tag{A.58}$$

holds. Here, $\widehat{V}(g) \in C\ell_n \subset C\ell_{n+1}$ is a unitary operator with well-defined fermion parity and the operator $\widehat{Q}(g)$ is either proportional to the identity operator in $C\ell_{n+1}$ or to the operator $\hat{\gamma}_\infty$.

Proof. The most general form of a norm-preserving representation of element $g \in G_f$ acting on the nonlocal fermionic Fock space (A.49a) is

$$\widehat{U}(g) := \widehat{W}(g)\,\mathsf{K}^{\mathfrak{c}(g)}, \tag{A.59}$$

Choose any $i = 1, \cdots, n$ and any $g \in G_f$. We rewrite condition (A.56) as

$$\widehat{U}(g)\,\hat{\gamma}_i\,\widehat{U}^{\dagger}(g)\,\widehat{Y} = \widehat{Y}\,\widehat{U}(g)\,\hat{\gamma}_i\,\widehat{U}^{\dagger}(g). \tag{A.60a}$$

After isolating $\hat{\gamma}_i$ on the left-hand side, we find that

$$\begin{aligned}\hat{\gamma}_i &= \left[\widehat{U}^{\dagger}(g)\,\widehat{Y}\,\widehat{U}(g)\right]\,\hat{\gamma}_i\,\left[\widehat{U}^{\dagger}(g)\,\widehat{Y}\,\widehat{U}(g)\right] \\ &= \widehat{Z}(g)\,\hat{\gamma}_i\,\widehat{Z}(g),\end{aligned} \tag{A.60b}$$

where we have defined the Hermitian operator

$$\widehat{Z}(g) := \widehat{U}^{\dagger}(g)\,\widehat{Y}\,\widehat{U}(g) = \widehat{Z}^{\dagger}(g) \tag{A.60c}$$

that squares to the identity $\hat{\mathbb{1}}_1$ in $\mathrm{C}\ell_{n+1}$. We observe that Eq. (A.60b) implies that $\widehat{Z}(g)$ commutes with $\hat{\gamma}_i$. As $i = 1, \cdots, n$ was arbitrarily chosen, $\widehat{Z}(g)$ must belong to the center of the Clifford algebra $\mathrm{C}\ell_n \subset \mathrm{C}\ell_{n+1}$.

Moreover, $\widehat{Z}(g)$ must have odd fermion parity, for $\widehat{U}(g)$ and $\widehat{U}^{\dagger}(g)$ have the same fermion parity and $\widehat{Y}$ has odd fermion parity. Hence,

$$\widehat{Z}(g) = \zeta\,\widehat{Y}, \qquad \zeta \in \mathbb{C}, \quad |\zeta|^2 = 1. \tag{A.61}$$

Because $\widehat{Z}(g)$ squares to the identity $\hat{\mathbb{1}}_1$ according to Eq. (6.25), we find that $\zeta = \pm 1$ and

$$\widehat{Z}(g) = \widehat{U}^{\dagger}(g)\,\widehat{Y}\,\widehat{U}(g) = \pm\widehat{Y} \implies \widehat{U}(g)\,\widehat{Y} = \pm\widehat{Y}\,\widehat{U}(g). \tag{A.62}$$

As g was arbitrarily chosen from G_f, we have completed the proof that $\widehat{U}(g)$ either commutes or anticommutes with $\widehat{Y}$ for any $g \in G_f$.

Next, we note that if the algebra (A.62) between the representation $\widehat{U}(g)$ and the center $\widehat{Y}$ holds, then the same algebra must hold between $\widehat{W}(g)$ defined in Eq. (A.59), i.e.,

$$\widehat{W}(g)\,\widehat{Y} = \pm\widehat{Y}\,\widehat{W}(g), \qquad \forall g \in G_f. \tag{A.63}$$

This is so because complex conjugation K (if present) either commutes or anticommutes with the center $\widehat{Y}$ by the very definition of $\widehat{Y}$.

Finally, owing to the fact that a general element of the Clifford algebra $\mathrm{C}\ell_{n+1}$ is of the form

$$\widehat{A}\,\hat{\mathbb{1}}_1 + \widehat{B}\,\hat{\gamma}_\infty, \tag{A.64}$$

where $\widehat{A}, \widehat{B} \in \mathrm{C}\ell_n \subset \mathrm{C}\ell_{n+1}$, we observe that the operator $\widehat{W}(g)$ with well-defined fermion parity that satisfies Eq. (A.63) must have the form of either

$$\widehat{W}(g) = \widehat{A}(g)\,\hat{\mathbb{1}}_1 \tag{A.65}$$

with $\widehat{A}(g)$ unitary and of well-defined fermion parity or

$$\widehat{W}(g) = \widehat{B}(g)\,\hat{\gamma}_\infty, \tag{A.66}$$

with $\widehat{B}(g)$ unitary and of well-defined fermion parity. But, this is nothing but the decomposition (A.58) whereby $\widehat{V}(g)$ is either $\widehat{A}(g)$ or $\widehat{B}(g)$. □

The invariance of Eq. (A.50) under the G_f-resolved transformation (A.55) allows to fix the fermion parity of $\widehat{U}(g)$ to be even for all $g \in G_f$. In this "gauge",

$$\widehat{U}(g) = \widehat{V}(g)\,\widehat{Q}(g)\,\mathsf{K}^{\mathfrak{c}(g)}, \quad \widehat{Q}(g) = [\hat{\gamma}_\infty]^{q(g)}\,, \tag{A.67}$$

where $q(g) = 0, 1$ denotes the fermion parity of the unitary operator $\widehat{V}(g)$. Equation (A.67) together with Eqs. (A.53) and (6.25) define the realization of the symmetry group G_f when $[\mu] = 1$.

A.4.3 *Indices* (ν, ρ)

We consider a boundary representation $\widehat{U}\colon G_f \to \mathrm{Aut}\big(\mathfrak{F}_{[\mu]}\big)$, where $\mathrm{Aut}\big(\mathfrak{F}_{[\mu]}\big)$ denotes the set of automorphisms on the fermionic Fock space $\mathfrak{F}_{[\mu]}$. We demand that this map satisfies, for any $g, h, f \in G_f$,

$$\widehat{U}(g)\,\widehat{U}(h) = e^{\mathrm{i}\phi(g,h)}\,\widehat{U}(g\,h), \tag{A.68a}$$

where $g\,h$ denotes the composition of the elements $g,h\in G_f$. The map $\phi(\cdot,\cdot)\in C^2\big(G_f,\mathrm{U}(1)\big)$ is a $\mathrm{U}(1)$-valued 2-cochain [4]. Furthermore, to ensure the compatibility with the associativity of the composition law of G_f, we demand that, for any $g,h,f\in G_f$,

$$\phi(g,h)+\phi(gh,f)=(-1)^{\mathfrak{c}(g)}\,\phi(h,f)+\phi(g,hf). \tag{A.68b}$$

The 2-cochains that satisfy this condition are called 2-cocycles. The map (A.68) defines a projective representation of the symmetry group G_f. Under the gauge transformation

$$\widehat{U}(g)\mapsto e^{\mathrm{i}\xi(g)}\,\widehat{U}(g), \tag{A.69a}$$

the phase $\phi(g,h)$ entering any projective representation of the symmetry group G_f changes by

$$\phi'(g,h)-\phi(g,h)=\xi(g\,h)-\xi(g)-(-1)^{\mathfrak{c}(g)}\,\xi(h) \tag{A.69b}$$

for any $g,h\in G_f$. Two 2-cochains ϕ and ϕ' are equivalent if they are related by a gauge transformation. The 2-cochains ϕ that vanish under a gauge transformation, i.e., the identity

$$\phi(g,h)=\xi(g\,h)-\xi(g)-(-1)^{\mathfrak{c}(g)}\,\xi(h) \tag{A.69c}$$

for any $g,h\in G_f$ holds, are called 2-coboundaries. The set of equivalence classes $[\phi]$ of 2-cocycles under the gauge transformations is the second cohomology group $H^2\big(G_f,\mathrm{U}(1)_{\mathfrak{c}}\big)$.

Elements of G_f were referred to, so far, by single letters g, with e reserved for the identity and p reserved from the fermion parity. We will use the group isomorphism $(\pi\times\tau)$ that is defined in Eq. (A.18) to map the group G_f to the group $G\underset{\gamma}{\times}\mathbb{Z}_2^{\mathrm{F}}$. As we shall see, this mapping will allow us to represent the 2-cochain ϕ in terms of the pair $(\nu,\rho)\in C^2\big(G,\mathrm{U}(1)\big)\times C^1\big(G,\mathbb{Z}_2\big)$. Here, the 2-cochain $\nu\in C^2\big(G,\mathrm{U}(1)\big)$ captures the projective representation (A.68) of the elements $\pi(g)\in G$ for any $g\in G_f$. When $[\mu]=0$, the 1-cochain $\rho\in C^1\big(G,\mathbb{Z}_2\big)$ measures if an operator representing an element of G commutes or anticommutes with the operator representing the fermion parity p. When $[\mu]=1$, the 1-cochain $\rho\in C^1\big(G,\mathbb{Z}_2\big)$ measures if an operator representing an element

4 Note that we denote the elements of the set of 2-cochains $C^2(G_f,\mathrm{U}(1))$ by the phase $\phi(g_1,g_2)$ as opposed to its exponential as is the usual convention. This is because we impose an additive composition rule on the group $\mathrm{U}(1)$ as opposed to a multiplicative one.

of G commutes or anticommutes with the central element $\widehat{Y}$ of the Clifford algebra $\mathrm{C}\ell_n$. We will show that it is possible to organize $C^2\big(G,\mathrm{U}(1)\big)\times C^1\big(G,\mathbb{Z}_2\big)$ into a coset of equivalence classes $\{[(\nu,\rho)]\}$ such that there is a one-to-one correspondence between any element $[\phi]\in H^2\big(G_f,\mathrm{U}(1)_{\mathfrak{c}}\big)$ and $[(\nu,\rho)]$ (also see Ref. [139]). When defining the indices (ν,ρ), there is an implicit choice for the projection π and trivialization τ. The projection π is defined up to group automorphisms of group G while the trivialization τ is defined up to certain group isomorphisms on G_f. In Sec. A.4.4, we explain how the pair (ν,ρ) changes under isomorphisms relating different representatives of the central extension class $[\gamma]\in H^2(G,\mathbb{Z}_2^{\mathrm{F}})$.

A.4.3.1 *The Case of* $[\mu]=0$

Recall that under the isomorphism (A.18), elements g and h in G_f are mapped to the pairs $(\pi(g),\tau(g))$ and $(\pi(h),\tau(h))$ in $G\underset{\gamma}{\times}\mathbb{Z}_2^{\mathrm{F}}$ with the composition rule

$$\Big(\pi(g),\tau(g)\Big)\underset{\gamma}{\circ}\Big(\pi(h),\tau(h)\Big)=\Big(\pi(g)\,\pi(h),\tau(g)\,\tau(h)\,\gamma\Big(\pi(g),\pi(h)\Big)\Big). \tag{A.70}$$

When $[\mu]=0$, the number n of Majorana operators is even. The 2-cochain $\nu\in C^2\big(G,\mathrm{U}(1)\big)$ is defined by restricting the domain of definition of the 2-cochain ϕ from G_f to G, i.e.,

$$\nu\left(\pi(g),\pi(h)\right):=\phi\Big(\Big(\pi(g),e\Big),\Big(\pi(h),e\Big)\Big). \tag{A.71}$$

Note that for any element $g_b\in G$, there exists an element $g\in G_f$ such that $\pi(g)=g_b$ and $\tau(g)=e$. Equation (A.71) asserts that the 2-cochain ν is retrieved by inserting such elements g and h in 2-cocycle ϕ. Another equivalent definition of ν is that

$$\nu\left(g_b,h_b\right):=\phi\Big(s(g_b),s(h_b)\Big). \tag{A.72}$$

for any g_b, $h_b\in G$.

The 1-cochain $\rho\in C^1\big(G,\mathbb{Z}_2\big)$ is defined by

$$e^{\mathrm{i}\pi\rho(\pi(g))}\equiv(-1)^{\rho(\pi(g))}:=\begin{cases}\widehat{U}\big(g\big)\,\widehat{U}\big(p\big)\,\widehat{U}^{\dagger}\big(g\big)\,\widehat{U}^{\dagger}\big(p\big), & \text{if }\mathfrak{c}\big(g\big)=0,\\[2ex] \widehat{U}\big(g\big)\,\widehat{U}\big(p\big)\,\widehat{U}^{\dagger}\big(g\big)\,\widehat{U}\big(p\big), & \text{if }\mathfrak{c}\big(g\big)=1,\end{cases} \tag{A.73a}$$

for any $g \in G_f$. In terms of the 2-cocycle ϕ, $\rho \in C^1\big(G, \mathbb{Z}_2\big)$ is, for any $g \in G_f$, given by

$$\rho(\pi(g)) = \frac{1}{\pi}\left[\phi\big(g, p\big) - \phi\big(p, g\big) + \mathfrak{c}(g)\,\phi\big(p, p\big)\right]. \tag{A.73b}$$

The definition (A.73) is made so that the 1-cochain ρ is invariant under the gauge transformation (A.69a). The 1-cochain $\rho \in C^1\big(G, \mathbb{Z}_2\big)$ is a group homomorphism from G to $\mathbb{Z}_2 = \{0, 1\}$, since it has a vanishing coboundary and, hence, is a 1-cocycle [5]. We note that when a gauge choice is made by choosing the representation $\widehat{U}\big(p\big)$ to be Hermitian, the two cases in the definition (A.73) are equivalent.

By definition, we have $\rho\,(\pi(p)) = \rho\,(\pi(e)) = 0$. Since both ρ and π are group homomorphisms, the 1-cocycle $\rho \circ \pi \in C^1(G_f, \mathbb{Z}_2)$ denotes the fermion the fermion parity of the element g for any $g \in G_f$. With an abuse of notation, we use ρ to denote both $\rho : G \to \mathbb{Z}_2$ and $\rho \circ \pi : G_f \to \mathbb{Z}_2$, when the distinction is clear from the context.

It is possible to construct the 2-cocycle ϕ in terms of 2-cochain ν, 1-cochain ρ, and the trivialization map τ as follows.

Claim 6. When $[\mu] = 0$, the 2-cocycle ϕ is gauge equivalent to

$$\phi \sim \nu + \pi\,\rho \smile \tau, \tag{A.74}$$

where the pair $(\nu, \rho) \in C^2\big(G, \mathrm{U}(1)\big) \times C^1\big(G, \mathbb{Z}_2\big)$ and $\tau : G_f \to \mathbb{Z}^{\mathrm{F}}{}_2$ is the trivialization map. Hereby, for the cup product of ρ and τ to be well-defined, we take $\tau(g) = 0, 1$ when $\tau(g) = e, p$ for any $g \in G_f$, respectively.

Proof. Using the group isomorphism $\pi \times \tau$, we write the representation of $g \in G_f$ as

$$\widehat{U}(g) = \widehat{U}(\pi \times \tau(g)) \equiv \widehat{U}\big(\pi(g), \tau(g)\big). \tag{A.75}$$

We can then write the composition rule (A.68a) as

$$\widehat{U}\big(\pi(g), \tau(g)\big)\,\widehat{U}\big(\pi(h), \tau(h)\big) = e^{\mathrm{i}\phi(g,h)}\,\widehat{U}\big(\pi(g\,h), \tau(g\,h)\big). \tag{A.76}$$

5 A $\mathbb{Z}_2$ valued 1-cocycle $\rho \in Z^1(G, \mathbb{Z}_2)$ satisfies by definition $(\delta^1_{\mathfrak{c}}\rho)(g, h) = \rho(g) + \mathfrak{c}(g)\rho(h) - \rho(g\,h) = 0$ for any $g, h \in G$. Since in the group $\mathbb{Z}_2$, $\rho(h) = \pm\rho(h)$ for any $h \in G_f$, the 1-cocycle ρ is a group homomorphism.

Note that this equation implies the identity

$$\widehat{U}\Big(\pi(g),e\Big)\,\widehat{U}\Big(e,\tau(g)\Big)=e^{\mathrm{i}\phi\Big(\big(\pi(g),e\big),\big(e,\tau(g)\big)\Big)}\,\widehat{U}\Big(\pi(g),\tau(g)\Big). \tag{A.77}$$

Without loss of generality let us choose a gauge such that $\phi\Big((\pi(g),e),(e,\tau(g))\Big)=0$. This is achieved by a gauge transformation (A.69a), where we choose the 1-cochain $\xi(g)=-\phi\Big((\pi(g),e\big),\big(e,\tau(g)\big)\Big)$. The left-hand side of Eq. (A.76) then becomes

$$\begin{aligned}\widehat{U}\Big(\pi(g),\tau(g)\Big)\,\widehat{U}\Big(\pi(h),\tau(h)\Big)&=\widehat{U}\Big(\pi(g),e\Big)\,\widehat{U}\Big(e,\tau(g)\Big)\,\widehat{U}\Big(\pi(h),e\Big)\,\widehat{U}\Big(e,\tau(h)\Big)\\&=e^{+\mathrm{i}\pi\,\tau(g)\,\rho\big(\pi(h)\big)}\,\widehat{U}\Big(\pi(g),e\Big)\,\widehat{U}\Big(\pi(h),e\Big)\,\widehat{U}\Big(e,\tau(g)\,\tau(h)\Big),\end{aligned} \tag{A.78}$$

where the phase factor $\mathrm{i}\pi\,\tau(g)\,\rho\big(\pi(h)\big)$ is to take into account the fermionic statistics when representation of h carries odd fermion parity $[\rho(h)=1]$ and $(e,\tau(g))\equiv p\in G_f$ $[\tau(g)\equiv 1]$. Using the composition rule (A.76) twice more delivers

$$\begin{aligned}\widehat{U}\Big(\pi(g),\tau(g)\Big)\,\widehat{U}\Big(\pi(h),\tau(h)\Big)&=e^{\mathrm{i}\phi\Big(\big(\pi(g),e\big),\big(\pi(h),e\big)\Big)+\mathrm{i}\pi\,\tau(g)\,\rho\big(\pi(h)\big)}\\&\quad\times\widehat{U}\Big(\pi(g\,h),\gamma\Big(\pi(g),\pi(h)\Big)\,\tau(g)\,\tau(h)\Big)\\&=e^{\mathrm{i}\nu\big(\pi(g),\pi(h)\big)+\mathrm{i}\pi\,\tau(g)\,\rho\big(\pi(h)\big)}\,\widehat{U}\Big(\pi(g\,h),\tau(g\,h)\Big),\end{aligned} \tag{A.79}$$

where we used the definition (A.71) of ν. Comparison with Eq. (A.76) implies the equivalence

$$\phi\sim\nu+\pi\,\rho\smile\tau, \tag{A.80}$$

where we used the gauge equivalence $\tau\smile\rho\sim\rho\smile\tau$ for the cup product of any $\mathbb{Z}_2$-valued 1-cochains. □

Since ϕ is a 2-cocycle, Eq. (A.74) implies that

$$0=\delta^2_{\mathfrak{c}}\phi=\delta^2_{\mathfrak{c}}\nu+\pi\rho\smile\delta^1_{\mathfrak{c}}\tau=\delta^2_{\mathfrak{c}}\nu+\pi\rho\smile\gamma, \tag{A.81a}$$

i.e.,

$$\delta^2_{\mathfrak{c}}\nu=\pi\rho\smile\gamma. \tag{A.81b}$$

Therefore, it is convenient to define the modified 2-coboundary operator

$$\mathcal{D}^2_\gamma\,(\nu,\,\rho) := \Big(\delta^2_{\mathfrak{c}}\nu - \pi\,\rho \smile \gamma, \delta^1_{\mathfrak{c}}\rho\Big)\,, \tag{A.82}$$

acting on a tuple of cochains $(\nu, \rho) \in C^2\big(G, \mathrm{U}(1)\big) \times C^1\big(G, \mathbb{Z}_2\big)$ together with the modified 1-coboundary operator

$$\mathcal{D}^1_\gamma\,(\alpha, \beta) := \Big(\delta^1_{\mathfrak{c}}\alpha + \pi\beta \smile \gamma, \delta^0_{\mathfrak{c}}\beta\Big) \tag{A.83}$$

acting on a tuple of cochains $(\alpha,\,\beta) \in C^1\big(G, \mathrm{U}(1)\big) \times C^0\big(G, \mathbb{Z}_2\big)$. Being a 0-cochain β does not take any arguments and takes values in $\mathbb{Z}_2$, i.e., $\beta \in \mathbb{Z}_2$. Note that for the 0-cochain β, the coboundary operator (A.3b) acts as

$$(\delta^0_{\mathfrak{c}}\beta)(g) = \mathfrak{C}_g(\beta) - \beta, \tag{A.84}$$

which in fact vanishes for any $g \in G$ since β takes values in $\mathbb{Z}_2$ and $\mathfrak{C}_g(\beta) = \beta$. Using Eq. (A.12) and the fact that γ is a cocycle, i.e., $\delta^2_{\mathfrak{c}}\gamma = 0$, one verifies that

$$\mathcal{D}^2_\gamma\, \mathcal{D}^1_\gamma(\alpha, \beta) = (0, 0) \tag{A.85}$$

for any tuple $(\alpha,\,\beta) \in C^1\big(G, \mathrm{U}(1)\big) \times C^0\big(G, \mathbb{Z}_2\big)$.

As we have shown (also see Ref. [139]), one may assign to any 2-cocycle $[\phi] \in H^2\big(G_f, \mathrm{U}(1)_{\mathfrak{c}}\big)$ an equivalence class $[(\nu, \rho)]$ of those tuples $(\nu, \rho) \in C^2\big(G, \mathrm{U}(1)\big) \times C^1\big(G, \mathbb{Z}_2\big)$ that satisfy the cocycle condition under the modified 2-coboundary operator (A.82) given by

$$\mathcal{D}^2_\gamma\,(\nu, \rho) = \Big(\delta^2_{\mathfrak{c}}\nu - \pi\,\rho \smile \gamma,\, \delta^1_{\mathfrak{c}}\rho\Big) = (0, 0). \tag{A.86}$$

Indeed, two tuples (ν, ρ) and (ν', ρ') that satisfy Eq. (A.86) are said to be equivalent if there exists a tuple $(\alpha,\,\beta) \in C^1\big(G, \mathrm{U}(1)\big) \times C^0\big(G, \mathbb{Z}_2\big)$ such that

$$(\nu,\,\rho) = (\nu',\,\rho') + \mathcal{D}^1_\gamma(\alpha,\,\beta) = (\nu' + \delta^1_{\mathfrak{c}}\alpha + \pi\,\beta \smile \gamma,\, \delta^0_{\mathfrak{c}}\beta). \tag{A.87}$$

In other words, using this equivalence relation we define an equivalence class $[(\nu,\rho)]$ of the tuple (ν,ρ) as an element of the set

$$[(\nu,\rho)] \in \frac{\ker(\mathcal{D}^2_\gamma)}{\mathrm{im}(\mathcal{D}^1_\gamma)}. \tag{A.88}$$

We note that when the $[\gamma]=0$, i.e., the group G_f splits as $G_f = G\times\mathbb{Z}_2^{\mathrm{F}}$, the modified coboundary operators (A.82) and (A.83) reduce to the coboundary operator (A.3b) with $n=2$ and $n=1$, respectively. If so the cochains ν and ρ are both cocycles, i.e., $(\nu,\,\rho)\in Z^2(G,\mathrm{U}(1)_{\mathfrak{c}})\times Z^1\big(G,\mathbb{Z}_2\big)$. The equivalence classes $[(\nu,\rho)]$ of the tuple (ν,ρ) is then equal to the equivalence cohomology classes of each of its components, i.e.,

$$[(\nu,\rho)] = ([\nu],[\rho]) \in H^2(G,\mathrm{U}(1)_{\mathfrak{c}})\times H^1\big(G,\mathbb{Z}_2\big). \tag{A.89}$$

We use the notation $([\nu],[\rho])$ for the two indices whenever the group G_f splits ($[\gamma]=0$). The notation $[(\nu,\rho)]$ applies whenever the group G_f does not split ($[\gamma]\neq 0$).

A.4.3.2 *The Case of* $[\mu]=1$

When $[\mu]=1$, the number n of Majorana operators is odd. The 2-cochain $\nu\in C^2\big(G,\mathrm{U}(1)\big)$ is defined by restricting the domain of definition of the 2-cochain ϕ from G_f to G, i.e.,

$$\nu\left(\pi(g),\pi(h)\right) := \phi\Big(\big(\pi(g),e\big),\big(\pi(h),e\big)\Big). \tag{A.90}$$

Note that for any element $g_b\in G$, there exists an element $g\in G_f$ such that $\pi(g)=g_b$ and $\tau(g)=e$. Equation (A.90) asserts that the 2-cochain ν is retrieved by inserting such elements g and h in 2-cocycle ϕ. Another equivalent definition of ν is that

$$\nu\left(g_b,h_b\right) := \phi\Big(s(g_b),s(h_b)\Big). \tag{A.91}$$

for any g_b, $h_b\in G$.

When $[\mu]=1$, the Clifford algebra $\mathrm{C}\ell_n$ spanned by the Majorana operators (6.11) has a two-dimensional center, in which case the fermion parity of the boundary representation $\widehat{U}(g)$ for any element $g\in G_f$ can be reversed by multiplying it with the generator $\widehat{Y}$ of the two-dimensional center of the Clifford algebra $\mathrm{C}\ell_n$. Moreover, any $\widehat{U}(g)$ must either commute or anticommute with $\widehat{Y}$ according to Eq. (6.26).

For this reason, we define the 1-cochain $\rho\in C^1\left(G_f,\mathbb{Z}_2\right)$ through

$$e^{\mathrm{i}\pi\rho(g)}\equiv(-1)^{\rho(g)}:=\begin{cases}\widehat{U}(g)\,\widehat{Y}\,\widehat{U}^\dagger(g)\,\widehat{Y}^\dagger, & \text{if } \mathfrak{c}(g)=0,\\ \\ \widehat{U}(g)\,\widehat{Y}\,\widehat{U}^\dagger(g)\,\widehat{Y}, & \text{if } \mathfrak{c}(g)=1,\end{cases} \tag{A.92}$$

for any $g\in G_f$. The 1-cochain $\rho\in C^1\left(G_f,\mathbb{Z}_2\right)$ takes the value 0 and 1. The 1-cochain $\rho\in C^1\left(G_f,\mathbb{Z}_2\right)$ is a group homomorphism from G_f to $\mathbb{Z}_2=\{0,1\}$ since it has a vanishing coboundary and, hence, is a 1-cocycle. Since $\widehat{Y}$ is of odd fermion parity by definition (6.25), it anticommutes with the representation $\widehat{U}(p)$. This implies that $\rho(p)=1$. More generally, the 1-cochain $\rho\in C^1\left(G_f,\mathbb{Z}_2\right)$ measures if the representation $\widehat{U}(g,h)$ of $(g,h)\in G_f$ commutes or anticommutes with $\widehat{Y}$.

The 1-cochain $\rho\in C^1\left(G,\mathbb{Z}_2\right)$ is defined by restricting the domain of definition of $\rho\in C^1\left(G_f,\mathbb{Z}_2\right)$ from G_f to G, i.e.,

$$e^{\mathrm{i}\pi\rho\left(\pi(g)\right)}\equiv(-1)^{\rho\left(\pi(g)\right)}:=\begin{cases}\widehat{U}\left(\pi(g),e\right)\,\widehat{Y}\,\widehat{U}^\dagger\left(\pi(g),e\right)\,\widehat{Y}^\dagger, & \text{if } \mathfrak{c}(g)=0,\\ \\ \widehat{U}\left(\pi(g),e\right)\,\widehat{Y}\,\widehat{U}^\dagger\left(\pi(g),e\right)\,\widehat{Y}, & \text{if } \mathfrak{c}(g)\big)=1,\end{cases} \tag{A.93}$$

for any $g\in G_f$ with $\tau(g)=e$. The definitions (A.92) and (A.93) are made so that the 1-cochain ρ is invariant under the gauge transformation (A.69a). We note that when a gauge choice is made by choosing the representation $\widehat{Y}$ to be Hermitian, the two cases in the definitions (A.92) are equivalent.

The fact that the 1-cochain $\rho\in C^1\left(G_f,\mathbb{Z}_2\right)$ defined in Eq. (A.92) is a group homomorphism puts constraints on the structure of the internal symmetry group G_f.

Claim 7. Compatibility between the existence of the group homomorphism $\rho\in C^1\left(G_f,\mathbb{Z}_2\right)$ which is defined in Eq. (A.92) and the group composition rule in G_f requires that the central extension class $[\gamma]\in H^2(G_f,\mathbb{Z}_2^{\mathrm{F}})$ is trivial, i.e., $[\gamma]=0$ and G_f is the split group.

Proof. We then have the identity

$$\begin{aligned}\left(\pi(g)\,\pi(h),e\right)&=\left(\pi(g),\,e\right)\underset{\gamma}{\circ}\left(\pi(h),\gamma\left(\pi(g),\,\pi(h)\right)\right)\\ &=\left(\pi(g),e\right)\underset{\gamma}{\circ}\left(\pi(h),e\right)\underset{\gamma}{\circ}\left(e,\gamma\left(\pi(g),\,\pi(h)\right)\right)\end{aligned} \tag{A.94}$$

for any $g, h \in G_f$. Note that the 1-cochain $\rho \in C^1\left(G_f, \mathbb{Z}_2\right)$ encodes the information on the algebra between the center $\widehat{Y}$ and the representation $\widehat{U}(g)$ of an element $g \in G_f$. As such it is a group homomorphism from G_f to the additive group $\mathbb{Z}_2$. Applying ρ on both sides of Eq. (A.94) gives

$$\rho\Big(\pi(g)\,\pi(h), e\Big) = \rho\Big(\pi(g), e\Big) + \rho\Big(\pi(h), e\Big) + \rho\Big(e, \gamma\Big(\pi(g),\, \pi(h)\Big)\Big) \bmod 2 \tag{A.95}$$

for any $g,\, h \in G_f$. We want to isolate the 2-cochain $\gamma\Big(\pi(g),\, \pi(h)\Big)$ and express it as a coboundary of a 1-cochain. To this end we define the group isomorphism $t : \mathbb{Z}_2 \to \mathbb{Z}_2^{\mathrm{F}}$ as

$$t(h) := \begin{cases} e, & \text{if } h = 0, \\ p, & \text{if } h = 1, \end{cases} \tag{A.96a}$$

and the 1-cochain $\tilde{\kappa}\colon G_f \to \mathbb{Z}_2^{\mathrm{F}}$

$$\tilde{\kappa}(\pi(g), \tau(g)) := (t \circ \rho)\, (\pi(g), \tau(g)), \tag{A.96b}$$

for any $g \in G_f$. The 1-cochain $\tilde{\kappa}$ is a group homomorphism since it is a composition of the group homomorphism ρ and the group isomorphism t. We note that

$$\tilde{\kappa}\Big(e, \gamma\Big(\pi(g),\, \pi(h)\Big)\Big) = \begin{cases} e, & \text{if } \rho\Big(e, \gamma\Big(\pi(g),\, \pi(h)\Big)\Big) = 0, \\ & \\ p, & \text{if } \rho\Big(e, \gamma\Big(\pi(g),\, \pi(h)\Big)\Big) = 1, \end{cases} \tag{A.97a}$$

and

$$\tilde{\kappa}\Big(e, \gamma\Big(\pi(g),\, \pi(h)\Big)\Big) = \gamma\Big(\pi(g),\, \pi(h)\Big) \tag{A.97b}$$

for any $g, h \in G_f$. Because $\tilde{\kappa}$ is a group homomorphism, applying $\tilde{\kappa}$ on both sides of Eq. (A.94) gives

$$\begin{aligned} \tilde{\kappa}\Big(\pi(g)\,\pi(h), e\Big) &= \tilde{\kappa}\Big(\pi(g), e\Big)\, \tilde{\kappa}\Big(\pi(h), e\Big)\, \tilde{\kappa}\Big(e, \gamma\Big(\pi(g),\, \pi(h)\Big)\Big) \\ &= \tilde{\kappa}\Big(\pi(g), e\Big)\, \tilde{\kappa}\Big(\pi(h), e\Big)\, \gamma\Big(\pi(g),\, \pi(h)\Big), \end{aligned} \tag{A.98a}$$

i.e.,

$$\gamma = \delta^{1}_{\mathfrak{c}}\,\kappa, \tag{A.98b}$$

where the 1-cochain $\kappa \in C^1(G, \mathbb{Z}_2^{\mathrm{F}})$ is defined by

$$\begin{aligned}\kappa\colon G &\to \mathbb{Z}_2^{\mathrm{F}},\\ g &\mapsto \kappa(\pi(g)) := \tilde{\kappa}\big(\pi(g), e\big).\end{aligned} \tag{A.98c}$$

In other words, γ is necessarily a coboundary, i.e.,

$$[\gamma] = 0. \tag{A.99}$$

□

Incompatibility, between nonsplit fermionic groups G_f and $[\mu] = 1$ stems from the fact that when restricted to the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$, the homomorphism $\rho \in C^1\big(G_f, \mathbb{Z}_2\big)$ is a group isomorphism This is not true when $[\mu] = 0$. The group homomorphism $\rho \in C^1\big(G_f, \mathbb{Z}_2\big)$ defined in Eq. (A.73) takes the values $\rho\big(e,e\big) = \rho\big(e,p\big) = 0$. Therefore, when restricted to the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$, it is not an isomorphism. In other words, the only internal symmetry groups G_f compatible with an odd number of Majorana degrees of freedom ($[\mu] = 1$) are those that split, i.e, $G_f \cong G \times \mathbb{Z}_2^{\mathrm{F}}$. This means that the second group cohomology $H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ splits via Künneth formula

$$H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}}) = H^2(G \times \mathbb{Z}_2^{\mathrm{F}}, \mathrm{U}(1)_{\mathfrak{c}}) = H^2(G, \mathrm{U}(1)_{\mathfrak{c}}) \times H^1(G, \mathbb{Z}_2). \tag{A.100}$$

If so, equivalence classes $[\phi]$ are in one-to-one correspondence with the equivalence classes of the pair $[(\nu,\rho)] = ([\nu],[\rho])$ when $[\mu] = 1$. Both ν and ρ are cocycles, i.e.,

$$\delta^{2}_{\mathfrak{c}}\nu = \delta^{1}_{\mathfrak{c}}\rho = 0. \tag{A.101}$$

The pair (ν,ρ) is equivalent to the pair (ν',ρ') if they are related by the pair of coboundaries $(\delta^1_{\mathfrak{c}}\alpha, \delta^0_{\mathfrak{c}}\beta)$ where $(\alpha,\beta) \in C^1(G, \mathrm{U}(1)) \times C^0(G, \mathbb{Z}_2)$.

We close this section by relating 2-cocycle ϕ to 2-cocycle ν and 1-cocycle ρ defined in Eqs. (A.90) and (A.93).

Claim 8. When $[\mu] = 1$, the 2-cocycle ϕ is gauge equivalent to

$$\phi \sim \nu, \tag{A.102}$$

where the pair $\nu \in C^2\big(G, \mathrm{U}(1)\big)$.

Proof. Since the group splits $[\gamma] = 0$, without loss of generality we set $\gamma = 0$. This means that any element $g \in G_f$ is represented as

$$\widehat{U}(g) = \widehat{U}\Big(\pi(g), \tau(g)\Big). \tag{A.103}$$

With the composition rule

$$\widehat{U}\Big(\pi(g), \tau(g)\Big)\, \widehat{U}\Big(\pi(h), \tau(h)\Big) = e^{\mathrm{i}\phi(g,h)}\, \widehat{U}\Big(\pi(g)\,\pi(h),\, \tau(g)\,\tau(h)\Big). \tag{A.104}$$

Furthermore, by definition (A.67), all representations $\widehat{U}\Big(\pi(g), \tau(g)\Big)$ carry even fermion parity, i.e., we can set

$$\phi\Big((\pi(g), e), (e, \tau(g))\Big) = \phi\Big((e, \tau(g)), (\pi(g), e)\Big) = 0, \tag{A.105a}$$

$$\phi\Big((\pi(g), \tau(g)), (e, p)\Big) = \phi\Big((e, p), (\pi(g), \tau(g))\Big) = 0. \tag{A.105b}$$

The composition rule (A.104) then reads

$$\begin{aligned}\widehat{U}\Big(\pi(g), \tau(g)\Big)\, \widehat{U}\Big(\pi(h), \tau(h)\Big) &= \widehat{U}\Big(\pi(g), e\Big)\, \widehat{U}\Big(e, \tau(g)\Big)\, \widehat{U}\Big(\pi(h), e\Big)\, \widehat{U}\Big(e, \tau(h)\Big) \\ &= \widehat{U}\Big(\pi(g), e\Big)\, \widehat{U}\Big(\pi(h), e\Big)\, \widehat{U}\Big(e, \tau(g)\,\tau(h)\Big) \\ &= e^{\mathrm{i}\nu\Big(\pi(g), \pi(h)\Big)}\, \widehat{U}\Big(\pi(g)\,\pi(h),\, \tau(g)\,\tau(h)\Big)\end{aligned} \tag{A.106}$$

Comparison with Eq. (A.104) implies the equivalence

$$\phi \sim \nu. \tag{A.107}$$

□

We observe that ϕ in Eq. (A.102) does not contain the term $\rho \smile \tau$ in Eq. (A.74). This is due to the fact that we explicitly assumed that the representation $\widehat{U}(g)$ defined in Eq. (A.67) has even fermion parity. This can be thought as a "gauge fixing" in which the term $\rho \smile \tau$ is effectively set to zero. This gauge choice is inconsequential for the transformation rules of local physical degrees of freedom $\mathfrak{O}_n$ defined in Eq. (A.46a) and enumerating the equivalence classes $([\nu], [\rho])$.

A.4.4 *Change in Indices (ν,ρ) under Group Isomorphisms*

As explained in the Appendix A.2, the fermionic symmetry group G_f can be constructed as the set of pairs $(g,h)\in G\times\mathbb{Z}_2^{\mathrm{F}}$ with the composition rule (A.21b) specified by the 2-cochain $\gamma\in C^2(G,\mathbb{Z}_2^{\mathrm{F}})$. The distinct central extensions G_f of G are then classified by the equivalence classes $[\gamma]\in H^2(G,\mathbb{Z}_2^{\mathrm{F}})$. In other words, the central extension G_f is determined up to the group isomorphisms (A.25a) under which the equivalence class $[\gamma]$ is invariant.

In Sec. A.4.3, we defined the pair of indices $(\nu,\rho)\in C^2(G,\mathrm{U}(1))\times C^1(G,\mathbb{Z}_2)$ for a given index $[\mu]=0,1$. The definitions (A.71) and (A.90) of ν and the definitions (A.73) and (A.93) are not invariant under group isomorphisms. In particular, when restricting the domain of definition of the 2-cochain ϕ from G_f to G, we made an implicit choice of trivialization τ and therefore γ. In this Section, we discuss how the pair $(\nu,\rho)\in C^2(G,\mathrm{U}(1))\times C^1(G,\mathbb{Z}_2)$ is shifted under the group isomorphism (A.25a) for the cases $[\mu]=0,1$.

Let G_f be a fermionic symmetry group obtained by centrally extending the symmetry group G by $\mathbb{Z}_2^{\mathrm{F}}$ through the 2-cochain γ. We denote the elements of G_f by the pairs $(g,h)\in G\times\mathbb{Z}_2^{\mathrm{F}}$. Let G'_f be a fermionic symmetry group isomorphic to G_f through the group isomorphism

$$\begin{aligned}\widetilde{\kappa}\colon G_f &\to G'_f,\\ (g,h) &\mapsto (g',h')=(g,p^{\kappa(g)}\,h)\end{aligned}\tag{A.108a}$$

where $\kappa(g)=0,1$ for any $g\in G$ and we introduced the shorthand notation $p^0=e$ and $p^1=p$ for the elements in $\mathbb{Z}_2^{\mathrm{F}}$. In other words, G'_f is the central extension of G by $\mathbb{Z}_2^{\mathrm{F}}$ through the 2-cochain γ' such that

$$\gamma'(g_1,g_2)=\gamma(g_1,g_2)\,p^{\kappa(g_1)+\kappa(g_2)+\kappa(g_1\,g_2)},\tag{A.108b}$$

for any $g_1,g_2\in G$.

One verifies that the pairs $(g',h')\in G'_f$ are identified with the pairs $(g=g',p^{\kappa(g)}h')\in G_f$ under the group isomorphism $\widetilde{\kappa}$. The identity

$$(g,h)\underset{\gamma}{\circ}(e,p^{\kappa(g)})=(g,h\,p^{\kappa(g)}),\tag{A.109a}$$

which holds for any $g \in G$ and $h \in \mathbb{Z}_2^{\mathrm{F}}$, then suggests that the representation $\widehat{U}'$ of element $(g', h') \in G'_f$ is related to the boundary representation $\widehat{U}$ of element $(g = g', p^{\kappa(g)}\, h') \in G_f$ via the relation

$$\widehat{U}'\Big(g', h'\Big) \propto \widehat{U}\Big(g = g', h'\Big)\ \Big[\widehat{U}\Big(e, p\Big)\Big]^{\kappa(g)}, \tag{A.109b}$$

i.e., the operator $\widehat{U}'\Big((g', h')\Big)$ must act up to a multiplicative phase factor as the operator $\widehat{U}\Big((g, h')\Big)$ composed with the fermion parity operator $\widehat{U}\Big((e, p)\Big)$ if $\kappa(g) = 1$. Hereby, the exponent $\kappa(g)$ ensures that the operators $\widehat{U}'\Big((g', h')\Big)$ and $\widehat{U}\Big((g, h')\Big)$ act identically, if $\kappa(g) = 0$. Without loss of generality, we take the proportionality in (A.109b) to be equality. We shall treat the cases of $[\mu] = 0$ and $[\mu] = 1$ separately.

A.4.4.1 *The Case of* $[\mu] = 0$

On the one hand, invoking the definition (6.21) for the 2-cochain ν' associated with the group G'_f delivers

$$\begin{aligned}\widehat{U}'\Big(g'_1, e\Big)\,\widehat{U}'\Big(g'_2, e\Big) &= e^{\mathrm{i}\nu'(g'_1, g'_2)}\,\widehat{U}'\Big(g'_1\, g'_2, \gamma'(g'_1, g'_2)\Big)\\ &= e^{\mathrm{i}\nu'(g_1, g_2)}\,\widehat{U}\Big(g_1\, g_2, \gamma(g_1, g_2)\, p^{\kappa(g_1)+\kappa(g_2)+\kappa(g_1\, g_2)}\Big)\ \Big[\widehat{U}\Big(e, p\Big)\Big]^{\kappa(g_1\, g_2)},\end{aligned} \tag{A.110a}$$

where in reaching the last line we have used Eqs. (A.108b) and (A.109b). Applying the identity (A.109a), we find

$$\widehat{U}'\Big(g'_1, e\Big)\,\widehat{U}'\Big(g'_2, e\Big) = e^{\mathrm{i}\nu'(g_1, g_2)}\,\widehat{U}\Big(g_1\, g_2, \gamma(g_1, g_2)\ \Big[\widehat{U}\Big(e, p\Big)\Big]^{\kappa(g_1)+\kappa(g_2)}, \tag{A.110b}$$

where the equality holds up to a multiplicative phase factor that can be gauged away, reason for which it is omitted for convenience. On the other hand, inserting Eq. (A.109b) on the left-hand side delivers

$$\begin{aligned}\widehat{U}'\Big(g'_1, e\Big)\,\widehat{U}'\Big(g'_2, e\Big) &= \widehat{U}\Big(g_1, e\Big)\ \Big[\widehat{U}\Big(e, p\Big)\Big]^{\kappa(g_1)}\ \widehat{U}\Big(g_2, e\Big)\ \Big[\widehat{U}\Big(e, p\Big)\Big]^{\kappa(g_2)}\\ &= e^{\mathrm{i}\nu(g_1, g_2)+\mathrm{i}\pi\,\kappa(g_1)\rho(g_2)}\,\widehat{U}\Big(g_1\, g_2, \gamma(g_1, g_2)\Big)\ \Big[\widehat{U}\Big(e, p\Big)\Big]^{\kappa(g_1)+\kappa(g_2)},\end{aligned} \tag{A.110c}$$

where the phase factor $e^{\mathrm{i}\nu(g_1,g_2)}$ arises from the definition (6.21) of 2-cochain ν and the phase factor $e^{\mathrm{i}\kappa(g_1)\rho(g_2)}$ arises when the operators $\widehat{U}\left(g_2,e\right)$ and $\left[\widehat{U}\left(e,p\right)\right]^{\kappa(g_1)}$ are interchanged. Comparing Eqs. (A.110b) and (A.110c), we make the identification

$$\nu'(g_1,g_2)=\nu(g_1,g_2)+\pi\,(\kappa\smile\rho)(g_1,g_2). \tag{A.111}$$

The index ρ by definition (6.22) measures the fermion parity of the representation of the element $(g,h)\in G_f$. One notes that the relation (A.109b) implies that the representations $\widehat{U}$ and $\widehat{U}'$ have the same fermion fermion parity since $\widehat{U}((e,p))$ is fermion parity even. Hence, the indices ρ and ρ' associated with G_f and G'_f, respectively, coincide.

We conclude that under the isomorphism (A.108a) the pair of indices $((\nu',\rho'),0)$ and $((\nu,\rho),0)$ are related as

$$\Big((\nu',\rho'),\,0\Big)=((\nu+\pi\,(\kappa\smile\rho),\,\rho),0)\,. \tag{A.112}$$

A.4.4.2 *The Case of* $[\mu]=1$

When $[\mu]=1$, the definition (6.28) of the index ν is the same as it is when $[\mu]=0$. However, by definition (A.67) all representations $\widehat{U}(g,h)$ have even fermion parity. This is to say that the term $\kappa\smile\rho$ in Eq. (A.111) does not arise. Therefore, ν is unchanged under isomorphism. In contrasts, from Eqs. (A.92) and (A.109b), one observes that under the isomorphism (A.108a) the index ρ gets shifted by κ, i.e.,

$$\rho'(g)=\rho(g)+\kappa(g). \tag{A.113}$$

This is because computation of the index ρ' involves an additional conjugation of $\widehat{Y}$ by fermion parity operator $\widehat{U}\left(e,p\right)$, which brings an additional factor of $(-1)^{\kappa(g)}$. We conclude that under the isomorphism (A.108a) the pair of indices $((\nu',\rho'),1)$ and $((\nu,\rho),1)$ are related as

$$\Big((\nu',\rho'),\,1\Big)=((\nu,\,\rho+\kappa),1)\,. \tag{A.114}$$

Under the group isomorphism (A.108a) the values of the indices (ν,ρ) and their respective equivalence classes may change (according to Eqs. (A.112) and (A.114)). However, the number of equivalence classes $([(\nu,\rho)],[\mu])$ and their stacking rules remain the same, i.e., Eqs. (7.30) commute with the relations (A.112) and (A.114).

A.5 EXAMPLES OF PROJECTIVE REPRESENTATIONS

We compute the values of equivalence classes $[(\nu,\rho)]$ of the groups $\mathbb{Z}_2^{\mathrm{T}}\times\mathbb{Z}_2^{\mathrm{F}}$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2^{\mathrm{F}}$, and $\mathbb{Z}_4^{\mathrm{FT}}$. For convenience, we denote by g both an element in G_f as well as its projection $\pi(g)$ onto G. Similarly, p denotes fermion parity in both G_f and $\mathbb{Z}_2^{\mathrm{F}}$.

A.5.1 *Symmetry Group* $\mathbb{Z}_2^{\mathrm{T}}\times\mathbb{Z}_2^{\mathrm{F}}$

The group $\mathbb{Z}_2^{\mathrm{T}}\times\mathbb{Z}_2^{\mathrm{F}}$, where the upper index T for the cyclic group $\mathbb{Z}_2^{\mathrm{T}}\equiv\{e,t\}$ refers to the interpretation of t as time, is a split group. Since the group splits ($[\gamma]=0$), one finds that $[\phi]\in H^2(G_f,\mathrm{U}(1)_{\mathfrak{c}})$ separates into the pair of independent indices $[\nu]\in H^2(\mathbb{Z}_2^{\mathrm{T}},\mathrm{U}(1)_{\mathfrak{c}})$ and $[\rho]\in H^1(\mathbb{Z}_2^{\mathrm{T}},\mathbb{Z}_2)$. Both $[\mu]=0$ and $[\mu]=1$ are possible. As we shall see

$$H^2(\mathbb{Z}_2^{\mathrm{T}},\mathrm{U}(1)_{\mathfrak{c}})=\mathbb{Z}_2,\qquad H^1(\mathbb{Z}_2^{\mathrm{T}},\mathbb{Z}_2)=\mathbb{Z}_2, \tag{A.115a}$$

i.e.,

$$H^2\left(\mathbb{Z}_2^{\mathrm{T}}\times\mathbb{Z}_2^{\mathrm{F}},\mathrm{U}(1)_{\mathfrak{c}}\right)=\left\{\left([\nu],[\rho]\right)\,\middle|\,[\nu]=0,1,\quad[\rho]=0,1\right\}. \tag{A.115b}$$

Below we compute these cohomology groups and the group structure of the triplet $([\nu],[\rho],[\mu])$ under the stacking rules (7.31).

Claim 9. $[\nu]=0,1.$

Proof. Any cochain ν belonging to the equivalence class $[\nu]$ is defined by the substitutions $G=\mathbb{Z}_2^{\mathrm{T}}$ in Eqs. (A.71) and (A.90). It must satisfy and must satisfy the cocycle and coboundary conditions in (A.68b) and (A.69b), respectively. If one chooses $g=h=f=t$ in Eq. (A.68b),one finds

$$\nu(t,t)+\nu(e,t)=\nu(t,e)-\nu(t,t)\ \mathrm{mod}\ 2\pi\implies\nu(t,t)=0,\pi. \tag{A.116}$$

Equation (A.116) is nothing but the statement that the representation of time reversal should square to either the identity or minus the identity. These two possibilities are not connected by a coboundary. Hence, they correspond to different second cohomology classes. To see this, assume they were connected by a coboundary, i.e., they satisfy the equivalence condition (A.69b). On the one hand, choosing $g = t$ and $h = t$ in Eq. (A.69b) implies that

$$\nu(t,t) - \nu'(t,t) = \xi(t) - \xi(t) - \xi(e) = -\xi(e) \implies \xi(e) = \pi \tag{A.117}$$

if $\nu'(t,t) = \pi$ and $\nu(t,t) = 0$. However, on the other hand, choosing $g = t$ and $h = e$ in Eq. (A.69b) implies that

$$\nu(t,e) - \nu'(t,e) = \xi(t) - \xi(e) - \xi(t) = -\xi(e) \implies \xi(e) = 0, \tag{A.118}$$

since $\nu(g,e) = \nu(e,g) = 0$ for all g. Equations (A.117) and (A.118) contradict each other. This contradiction implies that one cannot consistently define a gauge transformation φ that interpolates between ν such that $\nu(t,t) = \pi$ to ν' such that $\nu(t,t) = 0$. We denote the cases $\nu(t,t) = \pi, 0$ with the equivalence classes $[\nu] = 1, 0$, respectively. □

Claim 10. $[\rho] = 0, 1$.

Proof. For the second index $[\rho] \in H^1(\mathbb{Z}_2^{\mathrm{T}}, \mathbb{Z}_2)$, two 1-cochains ρ and ρ' are equivalent if and only if they are 1-cocycles that differ by a coboundary of a 0-cochain. But, by definition, a $\mathbb{Z}_2$-valued 0-cochain has a vanishing coboundary. Hence, the coset $H^1(\mathbb{Z}_2^{\mathrm{T}}, \mathbb{Z}_2)$ is just the set of all distinct 1-cocycles. By definition, a 1-cocycle ρ must obey [recall Eq. (A.4a)]

$$\rho(g) + \mathfrak{c}(g)\rho(h) - \rho(g\,h) = 0. \tag{A.119a}$$

Choosing $g = t$ and $h = t$ delivers

$$\rho(t) = \rho(t) + \rho(e). \tag{A.119b}$$

Since $\rho(e) = 0$ by definition, the cocycle condition (A.119a) is trivially satisfied. The elements in $H^1(\mathbb{Z}_2^{\mathrm{T}}, \mathbb{Z}_2)$ are labeled by the values $\rho(t) = 0, 1$. Equivalently, this is to say that there are two distinct group homomorphisms between $\mathbb{Z}_2^{\mathrm{T}}$ and $\mathbb{Z}_2$. We assign the indices $[\rho] = 0, 1$ to the values $\rho(t) = 0, 1$, respectively. □

Given two projective representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group $G_f = \mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ acting on the Fock spaces $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively, we now derive the indices associated with the projective representation $\hat{U}_\wedge$ acting on the Fock space $\mathfrak{F}_\wedge$ constructed from the graded tensor product of the Fock spaces $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Using the stacking rules derived in the Sec. 7, [Eq. (7.31)], we find

$$\nu_\wedge(t,t) = \begin{cases} \nu_1(t,t) + \nu_2(t,t) + \pi\,\rho_1(t)\,\rho_2(t), & \text{if } [\mu_1]+[\mu_2]=0, \\ \nu_1(t,t) + \nu_2(t,t) + \pi\,\rho_1(t)\,\rho_2(t) + \pi\,\rho_1(t)\,\mathfrak{c}(t), & \text{if } [\mu_1]=0,\ [\mu_2]=1, \\ \nu_1(t,t) + \nu_2(t,t) + \pi\,\rho_1(t)\,\rho_2(t) + \pi\,\rho_2(t)\,\mathfrak{c}(t), & \text{if } [\mu_1]=1,\ [\mu_2]=0, \end{cases} \tag{A.120a}$$

for the value of the 2-cochain $\nu_\wedge(t,t)$, and

$$\rho_\wedge(t) = \begin{cases} \rho_1(t) + \rho_2(t) + \mathfrak{c}(t), & \text{if } [\mu_1]=1,\ [\mu_2]=1, \\ \rho_1(t) + \rho_2(t), & \text{otherwise,} \end{cases} \tag{A.120b}$$

for the value of the 1-cochain $\rho_\wedge(t)$. Assignments of indices $[\nu_\wedge]$ and $[\rho_\wedge]$ to the projective representations of the group $\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{F}}$ and Eq. (A.120) imply that the indices of the tensor product representation are related to the indices of the constituent representations via

$$[\nu_\wedge] = \begin{cases} [\nu_1] + [\nu_2] + [\rho_1]\,[\rho_2], & \text{if } [\mu_\wedge] \equiv [\mu_1]+[\mu_2]=0, \\ [\nu_1] + [\nu_2] + [\rho_1]\,[\rho_2] + [\rho_1], & \text{if } [\mu_1]=0,\ [\mu_2]=1, \\ [\nu_1] + [\nu_2] + [\rho_1]\,[\rho_2] + [\rho_2], & \text{if } [\mu_1]=1,\ [\mu_2]=0, \end{cases} \tag{A.121a}$$

for the value of the 2-cochain $\nu_\wedge(t,t)$, and

$$[\rho_\wedge] = \begin{cases} [\rho_1] + [\rho_2] + 1, & \text{if } [\mu_1]=1,\ [\mu_2]=1, \\ [\rho_1] + [\rho_2], & \text{otherwise.} \end{cases} \tag{A.121b}$$

One thus finds that the triplets $([\nu],[\rho],[\mu])$ form the cyclic group $\mathbb{Z}_8$ under the stacking rule (A.121). Without loss of generality, the generator of the group $\mathbb{Z}_8$ can be chosen as the triplet $([\nu],[\rho],[\mu]) = (0,0,1)$. This is nothing but the $\mathbb{Z}_8$ classification of Class BDI in the Tenfold Way [18, 19].

A.5.2 *Symmetry Group* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$

As in Sec. A.5.1, the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$ is a split group. We denote the two generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by g_1 and g_2, both of which are represented by unitary operators. Because of the Cartesian products, $[\phi] \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ separates into the pair of independent indices $[\nu] \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1)_{\mathfrak{c}})$ and $[\rho] \in H^1(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$. Both $[\mu] = 0$ and $[\mu] = 1$ are possible. As we shall see

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1)_{\mathfrak{c}}) = \mathbb{Z}_2, \qquad H^1(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{A.122a}$$

i.e.,

$$H^2\left(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}, \mathrm{U}(1)_{\mathfrak{c}}\right) = \left\{ \left([\nu], [\rho]\right) \;\middle|\; [\nu] = 0, 1, \quad [\rho] = \left([\rho]_1, [\rho]_2\right), \quad [\rho]_1, [\rho]_2 = 0, 1 \right\}. \tag{A.122b}$$

Below we compute these cohomology groups and the group structure of the triplet $([\nu], [\rho], [\mu])$ under the stacking rules (7.31).

Claim 11. $[\nu] = 0, 1$.

Proof. Since the group representation is unitary (and hence linear as opposed to antilinear), there is no negative sign that appears on the right-hand side of the equality in Eq. (A.116). It is not possible to constrain the possible values of $\nu(g_1, g_1)$ or $\nu(g_2, g_2)$ as was done in Eq. (A.116). Cocycle conditions that are akin to Eq. (A.116) are trivially satisfied. This is to say that we can choose a gauge for which $\nu(g_1, g_1) = \nu(g_2, g_2) = 0$. If so the only nonvanishing values of ν occurs for $\nu(g_1, g_2)$ and $\nu(g_2, g_1)$ Group structure of $\mathbb{Z}_2 \times \mathbb{Z}_2$ dictates that the projective representations of g_1 and g_2 must either commute or anticommute with each other, i.e.,

$$\nu(g_1, g_2) - \nu(g_2, g_1) = 0, \pi. \tag{A.123}$$

Hence, these two possible values constitute the two inequivalent cohomology classes for the index $[\nu]$. To show that they are not connected by a 2-coboundary, we assume that ν and ν' the two are related by Eq. (A.69b). One finds

$$\nu(g_1, g_2) - \nu'(g_1, g_2) = \xi(g_1) + \xi(g_2) - \xi(g_1\, g_2), \tag{A.124a}$$

$$\nu(g_2, g_1) - \nu'(g_2, g_1) = \xi(g_2) + \xi(g_1) - \xi(g_2\, g_1). \tag{A.124b}$$

Because $G \equiv \mathbb{Z}_2 \times \mathbb{Z}_2$ is Abelian, this pair of equations implies that

$$\nu(g_1, g_2) - \nu(g_2, g_1) = \nu'(g_1, g_2) - \nu'(g_2, g_1). \tag{A.124c}$$

Therefore, the projective representations of $g_1 \in G$ and $g_2 \in G$ that either commute pairwise or anticommute pairwise must belong to distinct second cohomology classes. We assign the values $[\nu] = 0, 1$ to $\nu(g_1, g_2) - \nu(g_2, g_1) = 0, \pi$, respectively. □

Claim 12. $[\rho] = \big([\rho]_1, [\rho]_2\big) = (0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.

Proof. Since a $\mathbb{Z}_2$-valued 0-cochain has a vanishing coboundary, enumerating the elements of $H^1(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ corresponds to enumerating the distinct 1-cochains ρ. Equation (A.4a) implies for 1-cocycle ρ

$$\rho(g_1\, g_2) = \rho(g_1) + \rho(g_2), \tag{A.125}$$

i.e., ρ is a group homomorphism. Therefore, $[\rho]$ retains the $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure. We assign a pair of indices

$$[\rho] = \big([\rho]_1, [\rho]_2\big), \qquad [\rho]_1 = 0, 1, \qquad [\rho]_2 = 0, 1, \tag{A.126}$$

to the values $\rho(g_1) = 0, 1$ and $\rho(g_2) = 0, 1$, respectively. □

Given two projective representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group $G_f = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$ acting on the Fock spaces $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively, we now derive the indices associated with the projective representation $\hat{U}_\wedge$ acting on the Fock space $\mathfrak{F}_\wedge$ constructed from the graded tensor product of the Fock spaces $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Using the stacking rules derived in the Sec. 7, [Eqs. (7.31) and (7.32)], for any $[\mu_1]$ and $[\mu_2]$ we find

$$\nu_\wedge(g_1,\, g_2) = \nu_1(g_1,\, g_2) + \nu_2(g_1,\, g_2) + \pi\, \rho_1(g_1)\, \rho_2(g_2), \tag{A.127a}$$

$$\nu_\wedge(g_2,\, g_1) = \nu_1(g_2,\, g_1) + \nu_2(g_2,\, g_1) + \pi\, \rho_1(g_2)\, \rho_2(g_1), \tag{A.127b}$$

for the values of the 2-cochain $\nu_\wedge(g_1,\, g_2)$ and $\nu_\wedge(g_2,\, g_1)$, and

$$\rho_\wedge(g_1) = \rho_1(g_1) + \rho_2(g_1), \qquad \rho_\wedge(g_2) = \rho_1(g_2) + \rho_2(g_2) \tag{A.127c}$$

for the value of the 1-cochains $\rho_\wedge(g_1)$ and $\rho_\wedge(g_2)$. Assignments of indices $[\nu_\wedge]$ and $[\rho_\wedge]$ to the projective representations of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{F}}$ and Eq. (A.127) imply that the indices of the tensor product representation are related to the indices of the constituent representations via

$$[\nu_\wedge] = [\nu_1] + [\nu_2] + [\rho_1]_1\,[\rho_2]_2 + [\rho_1]_2\,[\rho_2]_1, \tag{A.128a}$$

$$[\rho_\wedge] = \left([\rho_1]_1 + [\rho_2]_1,\ [\rho_1]_2 + [\rho_2]_2\right), \tag{A.128b}$$

for any $[\mu_1]$ and $[\mu_2]$. One thus finds that the triplets $([\nu],[\rho],[\mu])$ form the group $\mathbb{Z}_2^4 = \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ under the stacking rule (A.128). Hence, each triplet $([\nu],[\rho],[\mu])$ is its inverse. Without loss of generality, the four generators a_i, of the group $\mathbb{Z}_2^4$ can be chosen as

$$\Big([\nu],\,([\rho]_1,[\rho]_2)\,,\,[\mu]\Big) = \Big(1,(0,0)\,,0\Big), \tag{A.129}$$

$$\Big([\nu],\,([\rho]_1,[\rho]_2)\,,\,[\mu]\Big) = \Big(0,(1,0)\,,0\Big), \tag{A.130}$$

$$\Big([\nu],\,([\rho]_1,[\rho]_2)\,,\,[\mu]\Big) = \Big(0,(0,1)\,,0\Big), \tag{A.131}$$

$$\Big([\nu],\,([\rho]_1,[\rho]_2)\,,\,[\mu]\Big) = \Big(0,(0,0)\,,1\Big). \tag{A.132}$$

A.5.3 *Symmetry Group* $\mathbb{Z}_4^{\mathrm{FT}}$

The group $\mathbb{Z}_4^{\mathrm{FT}}$ is the nontrivial central extension of $G \equiv \mathbb{Z}_2^{\mathrm{T}}$ by $\mathbb{Z}_2^{\mathrm{F}}$. This central extension of time reversal by fermion parity is specified by the map γ with $\gamma(t,t) = p$, which implies the group composition rule $t\,t = p$ [see Sec. A.3]. Since $[\gamma] \neq 0$, only $[\mu] = 0$ is possible. If so, ν is not a cocycle but a cochain with nonvanishing coboundary according to Eq. (A.86). On the other hand, ρ is a 1-cocycle. As we shall see

$$H^2\Big(\mathbb{Z}_4^{\mathrm{FT}}, \mathrm{U}(1)_{\mathfrak{c}}\Big) = \Big\{\Big[\Big(\nu,\rho\Big)\Big]\ \Big|\ \Big[\Big(\nu,\rho\Big)\Big] = (0,0),(1,1)\Big\}.. \tag{A.133}$$

Claim 13. $\Big[\Big(\nu,\rho\Big)\Big] = (0,0),(1,1)$.

Proof. Two tuples (ν,ρ) and (ν',ρ') are not in the same equivalence class if $\rho(t) \neq \rho'(t)$. Since $\rho(t)$ can take two values, 0 or 1, there exist at least two distinct equivalence classes of the tuple (ν,ρ), labeled by $\rho(t)$. Given this value of $\rho(t)$, we shall construct the distinct equivalence classes of (ν,ρ) corresponding to different values of $\nu \in C^2(G,\mathrm{U}(1))$. Choosing $g = h = f = t$ in Eq. (A.86) delivers

$$\nu(t,e) - \nu(t,t) - \nu(t,t) - \nu(e,t) = \pi\,\rho(t)\,\gamma(t,t) \bmod 2\pi. \tag{A.134}$$

With the choice of the convention $\gamma(t,t) = p \equiv 1$ for the nonsplit group $\mathbb{Z}_4^{\mathrm{FT}}$, we find the pair of solutions to Eq. (A.134) given by

$$\nu(t,t) = -\frac{\pi}{2}\,\rho(t), \qquad \nu(t,t) = -\frac{\pi}{2}\rho(t) + \pi. \tag{A.135}$$

The multiplicative factor π appears on the right-hand sides since ν takes values in U(1) and is thus defined modulo 2π. Now, the two solutions (A.135) are equivalent under the equivalence relation (A.87) as can be seen by choosing $\alpha = 0$ and $\beta = p \equiv 1$ in Eq. (A.87). Indeed, the term $\pi\beta \smile \gamma = \pi$ then cancels the factor π between the two solutions (A.135). Thus, for each value of $\rho(t) = 0, 1$, there exists a single distinct equivalence class $[(\nu, \rho)]$. We assign $[(\nu, \rho)] = (1, 1)$ to the case $(\nu(t,t), \rho(t)) = (-\pi/2, 1) \sim (+\pi/2, 1)$ and $[(\nu, \rho)] = (0, 0)$ to the case $(\nu(t,t), \rho(t)) = (0, 0) \sim (\pi, 0)$. □

Given two projective representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group $G_f = \mathbb{Z}_4^{\mathrm{FT}}$ acting on the Fock spaces $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively, we now derive the indices associated with the projective representation $\hat{U}_\wedge$ acting on the Fock space $\mathfrak{F}_\wedge$ constructed from the graded tensor product of the Fock spaces $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Using the stacking rules derived in the Sec. 7, [Eq. (7.31)], we find, where $[\mu_1] = [\mu_2] = 0$,

$$\nu_\wedge(t,\, t) = \nu_1(t,\, t) + \nu_2(t,\, t) + \pi\, \rho_1(t)\, \rho_2(t), \tag{A.136a}$$

for the values of the 2-cochain $\nu_\wedge(t,\, t)$ and

$$\rho_\wedge(t) = \rho_1(t) + \rho_2(t), \tag{A.136b}$$

for the value of the 1-cochain $\rho_\wedge(t)$. Assignments of indices $\left[\left(\nu_\wedge, \rho_\wedge\right)\right]$ to the projective representations of the group $\mathbb{Z}_4^{\mathrm{FT}}$ and Eq. (A.136) imply that the indices of the tensor product representation are related to the indices of the constituent representations via

$$\left[\left(\nu_\wedge, \rho_\wedge\right)\right] = \left[\left(\rho_1 + \rho_2,\, \rho_1 + \rho_2\right)\right], \tag{A.137a}$$

for $[\mu_1] = [\mu_2] = 0$. One thus finds that the triplets $\left([(\nu, \rho)], 0\right)$ form the group $\mathbb{Z}_2$ under the stacking rule (A.137). This is nothing but the $\mathbb{Z}_2$ classification of Class DIII in the Tenfold Way [19].

B

REVIEW OF FERMIONIC MATRIX PRODUCT STATES (FMPS) FORMALISM

In this Appendix, we review the construction of fermionic matrix product states (FMPS). We refer the reader to Refs. [135, 157] and references therein on the topic of matrix product states (MPS). As in bosonic matrix product states (BMPS), FMPS can be expressed as a contraction of objects belonging to a graded tensor product of vector spaces. The need for *graded tensor product of vector spaces* stems from the underlying fermionic algebra.

B.1 $\mathbb{Z}_2$-GRADED VECTOR SPACES

Any fermionic Fock space $\mathfrak{F}$ can be seen, in the basis that diagonalizes the total fermionic number operator, to be the direct sum over a subspace $\mathfrak{F}_0$ with even total fermionic number and a subspace $\mathfrak{F}_1$ with odd total fermionic number. This property endows fermionic Fock space with a natural $\mathbb{Z}_2$-grading.

A $\mathbb{Z}_2$-graded vector space V admits the direct sum decomposition

$$V = V_0 \oplus V_1. \tag{B.1}$$

We shall identify the subscripts 0 and 1 as the elements of the additive group $\mathbb{Z}_2$. We say that V_0 (V_1) has parity 0 (1). Any vector space is $\mathbb{Z}_2$-graded since the choice $V_0 = V$ and $V_1 = \emptyset$ is always possible. Any subspace of V_0 shares its parity 0. Any subspace of V_1 shares its parity 1. A vector $|v\rangle \in V$ is called *homogeneous* if it entirely resides in either one of the subspaces V_0 and V_1. The parity $|v|$ of the homogeneous state $|v\rangle$ is either 0 if $|v\rangle \in V_0$ or 1 if $|v\rangle \in V_1$. These observation on the $\mathbb{Z}_2$-grading of a vector space V only become useful when one demands that any operation acting on V preserves the $\mathbb{Z}_2$-grading.

For example, certain operations need to be defined carefully between two $\mathbb{Z}_2$-graded vector space V and W that preserve their $\mathbb{Z}_2$ structure. One such operation is the

$\mathbb{Z}_2$-graded tensor product. Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be two $\mathbb{Z}_2$-graded vector spaces. We define their graded tensor product as the map

$$\otimes_{\mathfrak{g}} \colon V \times W \to V \otimes W, \tag{B.2a}$$

such that

$$V_i \otimes_{\mathfrak{g}} W_j \subseteq (V \otimes W)_{(i+j) \bmod 2}, \qquad i, j = 0, 1. \tag{B.2b}$$

By design, the operation $\otimes_{\mathfrak{g}}$ carries the $\mathbb{Z}_2$-grading of V and W to their $\mathbb{Z}_2$-graded tensor product. In particular, for any homogeneous vectors $|v\rangle \in V$ with parity $|v| = 0, 1$ and $|w\rangle \in W$ with parity $|w| = 0, 1$, the graded tensor product $|v\rangle \otimes_{\mathfrak{g}} |w\rangle$ of two homogeneous vectors has the parity

$$\Big| |v\rangle \otimes_{\mathfrak{g}} |w\rangle \Big| := (|v| + |w|) \bmod 2. \tag{B.2c}$$

The connection between the $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ and fermionic Fock spaces $\mathfrak{F} = \mathfrak{F}_0 \oplus \mathfrak{F}_1$, is established through the identifications $\mathfrak{F}_0 \to V_0$ and $\mathfrak{F}_1 \to V_1$. However, a fermionic Fock space has more structure than a mere $\mathbb{Z}_2$-graded vector space. Wave functions in a fermionic Fock space are fully antisymmetric under the permutation of two fermions. This requirement can be implemented as follows on a $\mathbb{Z}_2$-graded vector space. The exchange of two fermions can be represented by the isomorphism

$$R \colon V \otimes_{\mathfrak{g}} W \to W \otimes_{\mathfrak{g}} V, \tag{B.3a}$$

by which the graded tensor product of the homogeneous vectors $|v\rangle \in V$ and $|w\rangle \in W$ obeys

$$|v\rangle \otimes_{\mathfrak{g}} |w\rangle \mapsto (-1)^{|v|\,|w|} \, |w\rangle \otimes_{\mathfrak{g}} |v\rangle. \tag{B.3b}$$

The map R is called the *reordering* operation. It is invertible with itself as inverse since R^2 is the identity map.

For every $\mathbb{Z}_2$-graded vector space V, we define the dual $\mathbb{Z}_2$-graded vector space V^*. We denote an element of the dual $\mathbb{Z}_2$-graded vector space V^* by $\langle v|$, the dual to the vector $|v\rangle \in V$. The dual $\mathbb{Z}_2$-graded vector space V^* inherits a $\mathbb{Z}_2$ grading from assigning the parity $|v|$ to the vector $\langle v| \in V^*$ if $|v\rangle \in V$ is homogeneous with parity $|v|$. The *contraction* $\mathcal{C}$ is the map

$$\begin{aligned} \mathcal{C} : V^* \otimes_{\mathfrak{g}} V &\to \mathbb{C}, \\ \langle\psi| \otimes_{\mathfrak{g}} |\phi\rangle &\mapsto \langle\psi|\phi\rangle, \end{aligned} \tag{B.4a}$$

where $\langle\psi|\phi\rangle$ denotes the scalar product between the pair $|\psi\rangle, |\phi\rangle \in V$. Hence,

$$\mathcal{C}\left(\langle i| \otimes_{\mathfrak{g}} |j\rangle\right) = \delta_{ij} \tag{B.4b}$$

holds for any pair of orthonormal and homogeneous basis vectors $|i\rangle, |j\rangle \in V$. The *contraction* $\mathcal{C}^*$ is the map $\mathcal{C}^* : V \otimes_{\mathfrak{g}} V^* \to \mathbb{C}$ defined by its action

$$\begin{aligned} \mathcal{C}^*\left(|i\rangle \otimes_{\mathfrak{g}} \langle j|\right) &:= \mathcal{C}\Big(R\left(|i\rangle \otimes_{\mathfrak{g}} \langle j|\right)\Big) \\ &= \mathcal{C}\Big((-1)^{|i||j|} \langle j| \otimes_{\mathfrak{g}} |i\rangle\Big) \\ &= (-1)^{|i||j|} \langle j|i\rangle \\ &= (-1)^{|i||j|} \delta_{ij} \end{aligned} \tag{B.4c}$$

for any pair of orthonormal basis vectors $|i\rangle, |j\rangle \in V$. It is common practice to use the same symbol $\mathcal{C}$ for both $\mathcal{C}$ and $\mathcal{C}^*$. Any linear operator

$$M : V \to V \tag{B.5a}$$

can be represented in the orthonormal and homogeneous basis $\{|i\rangle\}$ of V by the matrix

$$M_{ij} = (-1)^{|i||j|}\, M_{ji} \tag{B.5b}$$

through the expansion

$$M := \sum_{i,j} M_{ij}\, |i\rangle \otimes_{\mathfrak{g}} \langle j| \in V \otimes_{\mathfrak{g}} V^*. \tag{B.5c}$$

The linear operator M has a well defined parity if and only if each term $|i\rangle \otimes_{\mathfrak{g}} \langle j|$ in the summation has the same parity, in which case

$$|M| := (|i| + |j|) \bmod 2. \tag{B.5d}$$

More generally, if we define

$$T := \sum_{i_i,\cdots,i_n} T_{i_i,\cdots,i_n}\, |i_1\rangle \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} |i_n\rangle \in V_{i_1} \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} V_{i_n} \tag{B.6a}$$

we can assign the parity

$$|T| := (|i_1| + \cdots + |i_n|) \mod 2 \tag{B.6b}$$

when all $|i_1\rangle \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} |i_n\rangle$ share the same parity.

B.2 DEFINITION OF FMPS

We attach to each integer $j = 1, \cdots, N$ three $\mathbb{Z}_2$-graded vector spaces

$$V_j := \mathrm{span}\{|\alpha) \mid \alpha = 1, \cdots, \mathcal{D}_{\mathrm{v},j}\}, \tag{B.7a}$$

$$\mathfrak{F}_j := \mathrm{span}\{|\psi_\sigma\rangle \mid \sigma = 1, \cdots, \mathcal{D}_j\}, \tag{B.7b}$$

$$V_j^* := \mathrm{span}\{(\beta| \mid \beta = 1, \cdots, \mathcal{D}_{\mathrm{v},j}\}. \tag{B.7c}$$

The basis states $|\alpha)$ and $(\beta|$ of the dual pair V_j and V_j^* of $\mathbb{Z}_2$-graded vector spaces are virtual (auxiliary) states. They are denoted by rounded kets and bras and are introduced for convenience. Each auxiliary basis state has a well defined parity by assumption. The basis states $\{|\psi_\sigma\rangle\}$ span the physical fermionic Fock space $\mathfrak{F}_j$. Each physical basis state $|\psi_\sigma\rangle$ has a well defined parity by assumption, as follows from working in the fermion-number basis of $\mathfrak{F}_j$ say. The auxiliary $\mathbb{Z}_2$-graded vector space V_j has the dimension $\mathcal{D}_{\mathrm{v},j}$. The physical fermionic Fock space $\mathfrak{F}_j$ has dimension $\mathcal{D}_j$. A fermionic matrix product state (FMPS) takes the form

$$|\Psi\rangle := \mathcal{C}_{\mathrm{v}}\Big(Q(b)\, Y\, \mathsf{A}[1] \otimes_{\mathfrak{g}} \mathsf{A}[2] \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} \mathsf{A}[N]\Big) \tag{B.7d}$$

and has a well defined parity provided the objects $Q(b)$, Y, $\mathsf{A}[1]$, $\mathsf{A}[2]$, $\cdots$, $\mathsf{A}[N]$ are defined as follows. For any $j = 1, \cdots, N$, element $\mathsf{A}[j] \in V_j \otimes_{\mathfrak{g}} \mathfrak{F}_j \otimes_{\mathfrak{g}} V_{j+1}^*$ is defined by

$$\mathsf{A}[j] := \sum_{\alpha_j=1}^{\mathcal{D}_{\mathrm{v},j}} \sum_{\sigma_j=1}^{\mathcal{D}_j} \sum_{\beta_j=1}^{\mathcal{D}_{\mathrm{v},j+1}} (A_{\sigma_j})_{\alpha_j \beta_j}\, |\alpha_j) \otimes_{\mathfrak{g}} |\psi_{\sigma_j}\rangle \otimes_{\mathfrak{g}} (\beta_j| \tag{B.7e}$$

once the matrices A_{σ_j}, labeled as they are by the basis elements of the local Fock space $\mathfrak{F}_j$ and with the matrix elements $(A_{\sigma_j})_{\alpha_j\,\beta_j}$, have been chosen. The contraction $\mathcal{C}_{\mathrm{v}}$ labeled by the lower index v is understood to be over all virtual indices belonging to the dual pair (V_j^*, V_j) of auxiliary $\mathbb{Z}_2$-graded vector spaces, thereby producing the tensor proportional to

$$T_{\alpha_1\cdots\alpha_N|\beta_1\cdots\beta_N} := \delta_{\beta_1\,\alpha_2}\,\delta_{\beta_2\,\alpha_3}\cdots\delta_{\beta_{N-1}\,\alpha_N}\,\delta_{\beta_N\,\alpha_1} \tag{B.7f}$$

if $Q(b) \in V_1 \otimes_{\mathfrak{g}} V_1^*$ and $Y \in V_1 \otimes_{\mathfrak{g}} V_1^*$ were chosen to be the identity

$$Q(b) \equiv Y \equiv \sum_{\alpha} |\alpha) \otimes_{\mathfrak{g}} (\alpha|. \tag{B.7g}$$

The integer $b = 0, 1$ labels the boundary conditions selected by $Q(b) \in V_1 \otimes_{\mathfrak{g}} V_1^*$. The element $Y \in V_1 \otimes_{\mathfrak{g}} V_1^*$ is needed to fix the fermion parity of $|\Psi\rangle$. More precisely, we demand that the parity (B.5d) of $Q(b) \in V_1 \otimes_{\mathfrak{g}} V_1^*$ and the parity (B.6b) of $\mathsf{A}[j] \in V_j \otimes_{\mathfrak{g}} \mathfrak{F}_j \otimes_{\mathfrak{g}} V_{j+1}^*$ are *both* even, while the parity (B.5d) of $Y \in V_1 \otimes_{\mathfrak{g}} V_1^*$ is *either* even *or* odd. Consequently, the parity of $|\Psi\rangle$ is determined by the parity of Y since

$$|\Psi| = \left(|Q(b)| + |Y| + \sum_{j=1}^{N} |\mathsf{A}[j]|\right) \bmod 2 = |Y|. \tag{B.7h}$$

A prerequisite to imposing translation symmetry on any FMPS is that all dimensions $\mathcal{D}_{\mathrm{v},j}$ and $\mathcal{D}_j$ are independent of $j = 1, \cdots, N$. Hence, we assume from now on that

$$\mathcal{D}_{\mathrm{v},j} \equiv \mathcal{D}_{\mathrm{v}}, \qquad \mathcal{D}_j \equiv \mathcal{D}, \qquad j = 1, \cdots, N. \tag{B.7i}$$

B.2.1 *Even-Parity FMPS*

The FMPS

$$|\Psi\rangle_0^b := \mathcal{C}_{\mathrm{v}}\Big(Q(b)\,Y\,\mathsf{A}[1] \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} \mathsf{A}[N]\Big) \tag{B.8a}$$

is an even-parity FMPS obeying periodic ($b=0$) or antiperiodic ($b=1$) boundary conditions if, for any $j=1,\cdots,N$,

$$\mathsf{A}[j] := \sum_{\alpha_j=1}^{\mathcal{D}_\mathrm{v}} \sum_{\sigma_j=1}^{\mathcal{D}} \sum_{\beta_j=1}^{\mathcal{D}_\mathrm{v}} \left(A^{(0)}_{\sigma_j}\right)_{\alpha_j\,\beta_j} |\alpha_j) \otimes_{\mathfrak{g}} |\psi_{\sigma_j}\rangle \otimes_{\mathfrak{g}} (\beta_j|, \tag{B.8b}$$

$$|\mathsf{A}[j]| = \Big(|\alpha_j| + |\sigma_j| + |\beta_j|\Big) \mod 2 = 0, \tag{B.8c}$$

$$Y := \sum_{\alpha=1}^{\mathcal{D}_\mathrm{v}} |\alpha) \otimes_{\mathfrak{g}} (\alpha|, \tag{B.8d}$$

$$Q(b=0) := \sum_{\alpha=1}^{\mathcal{D}_\mathrm{v}} |\alpha) \otimes_{\mathfrak{g}} (\alpha|, \tag{B.8e}$$

$$Q(b=1) := \sum_{\alpha=1}^{\mathcal{D}_\mathrm{v}} (-1)^{|\alpha|}\, |\alpha) \otimes_{\mathfrak{g}} (\alpha|. \tag{B.8f}$$

By construction, both $Q(b)$ and Y are of even parity. Moreover, since $\Big(|\alpha_j| + |\sigma_j| + |\beta_j|\Big)$ is equal to 1 modulo 2 we have $\left(A^{(0)}_{\sigma_j}\right)_{\alpha_j\,\beta_j} = 0$.

We are going to give an alternative representation of this even-parity FMPS under the assumption that the virtual dimension $\mathcal{D}_\mathrm{v}$ obeys the partition $\mathcal{D}_\mathrm{v} = M_\mathrm{e} + M_\mathrm{o}$ where $M_\mathrm{e} \equiv M$ and $M_\mathrm{o} \equiv M$ are the numbers of even- and odd-parity virtual basis vectors, respectively. Parity evenness of $\mathsf{A}[j]$ implies that the $\mathcal{D}_\mathrm{v} \times \mathcal{D}_\mathrm{v}$ dimensional matrices $A^{(0)}_{\sigma_j}$ with the matrix elements $\left(A^{(0)}_{\sigma_j}\right)_{\alpha_j\,\beta_j}$ is either block diagonal

$$A^{(0)}_{\sigma_j} = \begin{pmatrix} B_{\sigma_j} & 0 \\ 0 & C_{\sigma_j} \end{pmatrix}, \quad \text{if } |\sigma_j| = 0, \tag{B.9a}$$

when the physical state is of even parity [as follows from Eq. (B.8c)] or block off diagonal

$$A^{(0)}_{\sigma_j} = \begin{pmatrix} 0 & D_{\sigma_j} \\ F_{\sigma_j} & 0 \end{pmatrix}, \quad \text{if } |\sigma_j| = 1, \tag{B.9b}$$

when the physical state is of odd parity [as follows from Eq. (B.8c)]. All the blocks are here $M \times M$-dimensional. Parity evenness of $Q(b)$ with matrix elements $\big(Q(b)\big)_{\alpha_1\,\beta_1}$ and Y with matrix elements $Y_{\alpha_1\,\beta_1}$ implies that

$$Y = Q(b=0) = \begin{pmatrix} \mathbb{1}_M & 0 \\ 0 & \mathbb{1}_M \end{pmatrix}, \qquad Q(b=1) = \begin{pmatrix} \mathbb{1}_M & 0 \\ 0 & -\mathbb{1}_M \end{pmatrix} := P. \tag{B.9c}$$

Hereby, we introduced the parity matrix P that satisfies

$$P\, A^{(0)}_{\sigma_j}\, P = (-1)^{|\sigma_j|}\, A^{(0)}_{\sigma_j}. \tag{B.9d}$$

Inserting these explicit representations of $Q(b)$ and Y in Eq. (B.8a) delivers

$$|\Psi\rangle^b_0 \equiv |\{A^{(0)}_{\sigma_j}\}; b\rangle := \sum_{\boldsymbol{\sigma}} \operatorname{tr}\left[P^{b+1}\, A^{(0)}_{\sigma_1}\, A^{(0)}_{\sigma_2} \cdots A^{(0)}_{\sigma_N}\right] |\Psi_{\boldsymbol{\sigma}}\rangle, \tag{B.10a}$$

where we used the shorthand notation $|\Psi_{\boldsymbol{\sigma}}\rangle := |\psi_{\sigma_1}\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_2}\rangle \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} |\psi_{\sigma_N}\rangle$. The appearance of the matrix P when $b = 0$ is counterintuitive. It is needed to eliminate from the sum over all physical basis states $\{|\Psi_{\boldsymbol{\sigma}}\rangle\}$ those physical basis states of odd parity. The state $|\{A^{(0)}_{\sigma_j}\}; b\rangle$ has even parity since

$$\left(\sum_{j=1}^{N} |\sigma_j|\right) \bmod 2 = \left[\sum_{j=1}^{N} (|\alpha_j| + |\beta_j|)\right] \bmod 2 = \left(\sum_{j=1}^{N} 2|\alpha_j|\right) \bmod 2 = 0, \tag{B.10b}$$

where we used condition (B.8c) to establish the first equality and the condition $|\beta_j| = |\alpha_{j+1}|$ that is imposed by the contractions of virtual indices to establish the second equality.

B.2.2 *Odd-Parity FMPS*

The FMPS

$$|\Psi\rangle^b_1 := \mathcal{C}_{\mathrm{v}}\left(Q(b)\, Y\, \mathsf{A}[1] \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} \mathsf{A}[N]\right), \tag{B.11a}$$

is an odd-parity FMPS obeying periodic ($b = 0$) or antiperiodic ($b = 1$) boundary conditions if, for any $j = 1, \cdots, N$,

$$\mathsf{A}[j] := \sum_{\alpha_j=1}^{\mathcal{D}_{\mathrm{v}}} \sum_{\sigma_j=1}^{\mathcal{D}} \sum_{\beta_j=1}^{\mathcal{D}_{\mathrm{v}}} \left(A^{(1)}_{\sigma_j}\right)_{\alpha_j\,\beta_j} |\alpha_j) \otimes_{\mathfrak{g}} |\psi_{\sigma_j}\rangle \otimes_{\mathfrak{g}} (\beta_j|, \tag{B.11b}$$

$$|\mathsf{A}[j]| = \left(|\sigma_j| + |\alpha_j| + |\beta_j|\right) \mod 2 = 0, \tag{B.11c}$$

$$Y := \sum_{\alpha,\beta=1}^{\mathcal{D}_{\mathrm{v}}} Y_{\alpha\,\beta}\, |\alpha) \otimes_{\mathfrak{g}} (\beta|, \qquad Y_{\alpha\,\beta} = 0 \text{ if } (|\alpha| + |\beta|) \mod 2 = 0, \tag{B.11d}$$

$$Q(b=0) := \sum_{\alpha=1}^{\mathcal{D}_{\mathrm{v}}} |\alpha) \otimes_{\mathfrak{g}} (\alpha|, \tag{B.11e}$$

$$Q(b=1) := \sum_{\alpha=1}^{\mathcal{D}_{\mathrm{v}}} (-1)^{|\alpha|}\, |\alpha) \otimes_{\mathfrak{g}} (\alpha|. \tag{B.11f}$$

By construction, $Q(b)$ is of even parity while Y is of odd parity. Moreover, the condition $\left(|\alpha_j| + |\sigma_j| + |\beta_j|\right) \mod 2 = 1$ implies that $\left(A^{(0)}_{\sigma_j}\right)_{\alpha_j\,\beta_j} = 0$.

We note that the only difference between definitions (B.8) and (B.11) is the choice for Y. In the former case its parity is even, in the latter case its parity is odd. Analogously to the even FMPS case, we define $2M \times 2M$ dimensional matrices $A^{(1)}_{\sigma_j}$ and Y with the matrix elements $\left(A^{(1)}_{\sigma_j}\right)_{\alpha_j\,\beta_j}$ and $Y_{\alpha_1\,\beta_1}$. The parity $|Y| = 1$ implies that

$$Y = \begin{pmatrix} 0 & Y_1 \\ Y_2 & 0 \end{pmatrix}, \tag{B.12a}$$

where Y_1 and Y_2 are $M \times M$ and dimensional matrices, respectively. Imposing translation symmetry requires that

$$Y\, A^{(1)}_{\sigma_j} = A^{(1)}_{\sigma_j}\, Y. \tag{B.12b}$$

We choose

$$Y := \begin{pmatrix} 0 & \mathbb{1}_M \\ -\mathbb{1}_M & 0 \end{pmatrix}, \qquad P\,Y\,P = -Y, \tag{B.12c}$$

which implies

$$A^{(1)}_{\sigma_j} = \begin{pmatrix} G_{\sigma_j} & 0 \\ 0 & G_{\sigma_j} \end{pmatrix}, \quad \text{if } |\sigma_j| = 0, \tag{B.12d}$$

$$A^{(1)}_{\sigma_j} = \begin{pmatrix} 0 & G_{\sigma_j} \\ -G_{\sigma_j} & 0 \end{pmatrix}, \quad \text{if } |\sigma_j| = 1, \tag{B.12e}$$

where G_{σ_j} are $M \times M$ dimensional matrices. Inserting these explicit representations of $Q(b)$ and Y in Eq. (B.11a) delivers

$$|\Psi\rangle^b_1 \equiv |\{A^{(1)}_{\sigma_j}\}; b\rangle := \sum_{\boldsymbol{\sigma}} \operatorname{tr}\left[P^b\, Y\, A^{(1)}_{\sigma_1}\, A^{(1)}_{\sigma_2} \cdots A^{(1)}_{\sigma_N}\right] |\Psi_{\boldsymbol{\sigma}}\rangle. \tag{B.13a}$$

The state $|\{A^{(1)}_{\sigma_j}\}; b\rangle$ has odd parity since

$$\left(\sum_{j=1}^{N} |\sigma_j|\right) \bmod 2 = \left[\sum_{j=1}^{N} (|\alpha_j| + |\beta_j|)\right] \bmod 2 = \left(|\alpha| + |\beta| + \sum_{j=2}^{N-1} 2|\alpha_j|\right) \bmod 2 = 1, \tag{B.13b}$$

where we used condition (B.11c) to establish the first equality. For the second equality we used the conditions $|\alpha| = |\beta_N|$ and $|\beta| = |\alpha_1|$ where $|\alpha|$, $|\beta|$ are the parities of the virtual indices corresponding to matrix elements $Y_{\alpha\,\beta}$, and $|\beta_j| = |\alpha_{j+1}|$ for $j = 2, \cdots, N-1$ that is imposed by the contractions of virtual indices to establish the second equality.

C

PROOF OF THEOREM 2

In this Appendix, we provide the proofs of Theorem 2, first in $d = 1$ dimension for any symmetry group G_f, and then for any d dimension when G_f is Abelian and unitarily represented.

C.1 PROOF BY FMPS FORMALISM IN 1D

We will prove Theorem 2 for one dimensional systems within the FMPS framework. Our proof follows closely that for the bosonic case [1] introduced in Ref. [158]. We will show that a even-parity or odd-parity injective FMPS necessarily requires the local projective representation $\hat{u}_j$ of the symmetry group G_f to have trivial second cohomology class $[\phi] \in H^2\left(G_f, \mathrm{U}(1)_{\mathfrak{c}}\right)$. In other words, when this cohomology class is nontrivial there is no compatible injective FMPS with even or odd parity. The general forms (B.10a) and (B.13a) as well as the injectivity conditions 1 and 2 are distinct for even and odd parity FMPS. The proofs for the even- and the odd-parity cases are thus treated successively. For conciseness, we are going to supress the symbol $\otimes_{\mathfrak{g}}$ when working with the orthonormal and homogeneous basis

$$\left\{ |\Psi_{\boldsymbol{\sigma}}\rangle \equiv |\psi_{\sigma_1}\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_2}\rangle \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} |\psi_{\sigma_N}\rangle \right\} \tag{C.1a}$$

of the Fock space

$$\mathfrak{F}_{\Lambda} \equiv \mathfrak{F}_1 \otimes_{\mathfrak{g}} \mathfrak{F}_2 \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} \mathfrak{F}_1. \tag{C.1b}$$

1 Bosonic matrix products states presume that the local Fock space $\mathfrak{F}_{\boldsymbol{j}}$ has no more than the trivial $\mathbb{Z}_2$ grading, i.e., $\mathfrak{F}_{\boldsymbol{j}} \equiv \mathfrak{F}_{\boldsymbol{j}\,0} \oplus \mathfrak{F}_{\boldsymbol{j}\,1}$ with $\mathfrak{F}_{\boldsymbol{j}\,0} \equiv \mathfrak{F}_{\boldsymbol{j}}$ and $\mathfrak{F}_{\boldsymbol{j}\,1} \equiv \emptyset$.

C.1.1 *Even-Parity FMPS*

Let [see Eq. (B.9)]

$$|\{A^{(0)}_{\sigma_j}\};b\rangle \equiv \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left(P^{b+1}A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\,\cdots A^{(0)}_{\sigma_N}\right)|\psi_{\sigma_1}\rangle\,|\psi_{\sigma_2}\rangle\cdots|\psi_{\sigma_N}\rangle \tag{C.2}$$

be a translation-invariant, G_f-symmetric, even-parity, and injective FMPS obeying periodic boundary conditions when $b=0$ or antiperiodic boundary conditions when $b=1$. For any $g\in G_f$, the global representation $\widehat{U}(g)$ of g is defined in Eq. (3.12a). By assumption, $|\{A^{(0)}_{\sigma_j}\};b\rangle$ is a nondegenerate gapped ground state of some local fermionic Hamiltonian in one-dimensional space. Hence, for any $g\in G_f$, there exists a phase $\eta(g;b)\in[0,2\pi)$ such that

$$\widehat{U}(g)\,|\{A^{(0)}_{\sigma_j}\};b\rangle = e^{\mathrm{i}\eta(g;b))}|\{A^{(0)}_{\sigma_j}\};b\rangle. \tag{C.3}$$

The action of the transformation $\widehat{U}(g)$ on the right-hand side of Eq. (C.2) gives

$$\begin{aligned}\widehat{U}(g)\,|\{A^{(0)}_{\sigma_j}\};b\rangle &= \sum_{\boldsymbol{\sigma}} \mathsf{K}_g\left[\mathrm{tr}\left(P^{b+1}\,A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\,\cdots A^{(0)}_{\sigma_N}\right)\right]\\ &\qquad\times\left(\hat{u}_1(g)\,|\psi_{\sigma_1}\rangle\right)\left(\hat{u}_2(g)\,|\psi_{\sigma_2}\rangle\right)\cdots\left(\hat{u}_N(g)\,|\psi_{\sigma_N}\rangle\right)\\ &= \sum_{\boldsymbol{\sigma}}\left\{\sum_{\boldsymbol{\sigma}'}\mathrm{tr}\left[P^{b+1}\,\mathsf{K}_g\left[A^{(0)}_{\sigma'_1}\,A^{(0)}_{\sigma'_2}\,\cdots A^{(0)}_{\sigma'_N}\right]\right]\right.\\ &\qquad\left.\times\prod_{j=1}^{N}\langle\psi_{\sigma_j}|\left(\hat{u}_j(g)|\psi_{\sigma'_j}\rangle\right)\right\}|\psi_{\sigma_1}\rangle\,|\psi_{\sigma_2}\rangle\cdots|\psi_{\sigma_N}\rangle\end{aligned} \tag{C.4}$$

after using N times the resolution of the identity, one for each local Fock space $\mathfrak{F}_j$. We use the notation $\left(\hat{u}_j(g)|\psi_{\sigma'_j}\rangle\right)$ to indicate that the operator $\hat{u}_j(g)$ acts on the right, an important fact to keep track of when $\hat{u}_j(g)$ is an antiunitary operator. The right-hand side can be written more elegantly with the definition of the g-dependent $2M\times 2M$ matrix

$$A^{(0)}_{\sigma_j}(g) := \sum_{\sigma'_j}\langle\psi_{\sigma_j}|\left(\hat{u}_j(g)|\psi_{\sigma'_j}\rangle\right)\mathsf{K}_g\left[A^{(0)}_{\sigma'_j}\right]$$

$$\equiv \sum_{\sigma'_j} [\mathcal{U}(g)]_{\sigma_j\,\sigma'_j}\ \mathsf{K}_g\left[A^{(0)}_{\sigma'_j}\right], \tag{C.5a}$$

where the $\mathcal{D}\times\mathcal{D}$ matrix $\mathcal{U}(g)$, whose matrix elements are the complex-valued coefficients weighting the sum over the $2M\times 2M$ matrices $\mathsf{K}_g\left[A^{(0)}_{\sigma'_j}\right]$, acts on the local Fock space $\mathfrak{F}_j$ and we have defined

$$\mathsf{K}_g\left[A^{(0)}_{\sigma_j}\right] := \begin{cases} A^{(0)}_{\sigma_j}, & \text{if } \mathfrak{c}(g)=0,\\ \\ \mathsf{K}\,A^{(0)}_{\sigma_j}\,\mathsf{K}, & \text{if } \mathfrak{c}(g)=1. \end{cases} \tag{C.5b}$$

As usual, K denotes complex conjugation. Equation (C.4) becomes

$$\widehat{U}(g)\,|\{A^{(0)}_{\sigma_j}\};b\rangle = \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left[P^{b+1}\,A^{(0)}_{\sigma_1}(g)\,A^{(0)}_{\sigma_2}(g)\,\cdots A^{(0)}_{\sigma_N}(g)\right]|\psi_{\sigma_1}\rangle\,|\psi_{\sigma_2}\rangle\cdots|\psi_{\sigma_N}\rangle, \tag{C.5c}$$

which is nothing but the FMPS (C.2) with $A^{(0)}_{\sigma_j}$ substituted for $A^{(0)}_{\sigma_j}(g)$. Equating the right-hand sides of Eqs. (C.3) and (C.5c) implies

$$\mathrm{tr}\left[P^{b+1}\,A^{(0)}_{\sigma_1}(g)\,A^{(0)}_{\sigma_2}(g)\,\cdots A^{(0)}_{\sigma_N}(g)\right] = e^{\mathrm{i}\eta(g;b)}\,\mathrm{tr}\left[P^{b+1}\,A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\,\cdots A^{(0)}_{\sigma_N}\right]. \tag{C.6a}$$

This equation is satisfied by the Ansatz

$$A^{(0)}_{\sigma_j}(g) = e^{\mathrm{i}\theta(g)}\,V^{-1}(g)\,A^{(0)}_{\sigma_j}\,V(g), \tag{C.6b}$$

$$P\,V(g)\,P = (-1)^{\kappa(g)}\,V(g), \tag{C.6c}$$

$$\theta(g) := \frac{1}{N}\left[\eta(g;b) - \pi(b+1)\,\kappa(g)\right], \tag{C.6d}$$

where $\kappa(g)=0,1$ dictates if the $2M\times 2M$ unitary matrix $V(g)$ commutes or anticommutes with the $2M\times 2M$ parity matrix P defined in Eq. (B.9c), since

$$\begin{aligned} \mathrm{tr}\left[P^{b+1}\,A^{(0)}_{\sigma_1}(g)\,A^{(0)}_{\sigma_2}(g)\,\cdots A^{(0)}_{\sigma_N}(g)\right] &= e^{\mathrm{i}\theta(g)N}\,\mathrm{tr}\left[P^{b+1}V^{-1}(g)\,A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\,\cdots A^{(0)}_{\sigma_N}\,V(g)\right] \\ \text{cyclicity of the trace}\qquad &= e^{\mathrm{i}\theta(g)N}\,\mathrm{tr}\left[V(g)\,P^{b+1}V^{-1}(g)\,A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\,\cdots A^{(0)}_{\sigma_N}\right] \end{aligned}$$

$$\begin{aligned}
\text{Eq. (C.6d)} \quad &= e^{\mathrm{i}\theta(g)N}\,(-1)^{(b+1)\kappa(g)}\,\mathrm{tr}\left[P^{b+1}A^{(0)}_{\sigma_1}A^{(0)}_{\sigma_2}\cdots A^{(0)}_{\sigma_N}\right]\\
&= e^{\mathrm{i}\theta(g)N+\mathrm{i}\pi(b+1)\kappa(g)}\,\mathrm{tr}\left[P^{b+1}A^{(0)}_{\sigma_1}A^{(0)}_{\sigma_2}\cdots A^{(0)}_{\sigma_N}\right]\\
&\equiv e^{\mathrm{i}\eta(g;b)}\,\mathrm{tr}\left[P^{b+1}A^{(0)}_{\sigma_1}A^{(0)}_{\sigma_2}\cdots A^{(0)}_{\sigma_N}\right].
\end{aligned} \tag{C.6e}$$

The existence of the $2M \times 2M$ invertible matrix $V(g)$ is guaranteed because of the injectivity of the FMPS. In an injective even-parity FMPS, the matrices $A^{(0)}_{\sigma_1}, \cdots, A^{(0)}_{\sigma_\ell}$ span the simple algebra of all $2M \times 2M$ matrices for any $\ell > \ell^\star$ for some nonvanishing integer $\ell^\star$. Hence, provided N is sufficiently large, the family of matrices $\{A^{(0)}_{\sigma_j}(g)\}$ is related to the family of matrices $\{e^{\mathrm{i}\eta(g;b)/N}\,A^{(0)}_{\sigma_j}\}$ that give the *same* FMPS (C.2) by the similarity transformation [see Eqs. (3.35) and (3.40)]

$$A^{(0)}_{\sigma_j}(g) = e^{\mathrm{i}\varphi^{(b)}_g}\,V^{-1}(g)\left[e^{\mathrm{i}\eta(g;b)/N}\,A^{(0)}_{\sigma_j}\right]V(g), \tag{C.7a}$$

for some phase $\varphi^{(b)}_g = [0, 2\pi)$ and some invertible $2M \times 2M$ matrix $V(g)$ that must also obey

$$V(p) = P, \qquad P\,V(g)\,P = (-1)^{\kappa(g)}\,V(g). \tag{C.7b}$$

Here, the map $\kappa\colon G_f \to \{0,1\}$ specifies the algebra between the similarity transformation $V(g)$ corresponding to element $g \in G_f$ and the fermion parity P. The effect of the factor $(-1)^{\kappa(g)}$ is nothing but the phase

$$\varphi^{(b)}_g = -\frac{1}{N}\pi\,(b+1)\,\kappa(g), \tag{C.8}$$

as follows from Eq. (C.6d).

Equating the right-hand sides of Eqs. (C.7a) and (C.5a) implies

$$e^{\mathrm{i}\theta(g)}\,V^{-1}(g)\,A^{(0)}_{\sigma_j}\,V(g) = \sum_{\sigma'_j}\left[\mathcal{U}(g)\right]_{\sigma_j\,\sigma'_j}\,\mathsf{K}_g\left[A^{(0)}_{\sigma'_j}\right], \tag{C.9}$$

We would like to isolate $\mathsf{K}_g\left[A^{(0)}_{\sigma'_j}\right]$ on the right-hand side. To this end, we write

$$e^{\mathrm{i}\theta(g)}\sum_{\sigma_j}\left[\mathcal{V}^{\dagger}(g)\right]_{\sigma''_j\sigma_j}V^{-1}(g)\,A^{(0)}_{\sigma_j}\,U(g)=\sum_{\sigma_j,\sigma'_j}\left[\mathcal{U}^{\dagger}(g)\right]_{\sigma''_j\sigma_j}\left[\mathcal{U}(g)\right]_{\sigma_j\,\sigma'_j}\,\mathsf{K}_g\left[A^{(0)}_{\sigma'_j}\right]. \tag{C.10a}$$

To evaluate the right-hand side we use the identities

$$\begin{aligned}\sum_{\sigma_j}\left[\mathcal{U}^{\dagger}(g)\right]_{\sigma''_j\sigma_j}\left[\mathcal{U}(g)\right]_{\sigma_j\,\sigma'_j}&=\sum_{\sigma_j}\langle\psi_{\sigma''_j}|\mathsf{K}_g\left[\left(\hat{u}^{\dagger}_j(g)|\psi_{\sigma_j}\rangle\right)\right]\,\langle\psi_{\sigma_j}|\left(\hat{u}_j(g)|\psi_{\sigma'_j}\rangle\right)\\&=\sum_{\sigma_j}\left(\langle\psi_{\sigma''_j}|\hat{u}_j(g)\right)\,|\psi_{\sigma_j}\rangle\langle\psi_{\sigma_j}|\left(\hat{u}_j(g)|\psi_{\sigma'_j}\rangle\right)\\&=\left(\langle\psi_{\sigma''_j}|\hat{u}_j(g)\right)\left(\hat{u}_j(g)|\psi_{\sigma'_j}\rangle\right)\\&=\mathsf{K}_g\left[\langle\psi_{\sigma''_j}|\left(\hat{u}^{\dagger}_j(g)\hat{u}_j(g)|\psi_{\sigma'_j}\rangle\right)\right]\\&=\mathsf{K}_g\left[\langle\psi_{\sigma''_j}|\psi_{\sigma'_j}\rangle\right]\\&=\delta_{\sigma''_j,\sigma'_j},\end{aligned} \tag{C.10b}$$

which delivers

$$\mathsf{K}_g\left[A^{(0)}_{\sigma''_j}\right]=e^{\mathrm{i}\theta(g)}\sum_{\sigma_j}\left[\mathcal{U}^{\dagger}(g)\right]_{\sigma''_j\sigma_j}V^{-1}(g)\,A^{(0)}_{\sigma_j}\,V(g). \tag{C.10c}$$

By applying K_g to both sides of this equation, we obtain the selfconsistency condition

$$\begin{aligned}A^{(0)}_{\sigma_j}&=\mathsf{K}_g\left[e^{\mathrm{i}\theta(g)}\sum_{\sigma'_j}\left[\mathcal{U}^{\dagger}(g)\right]_{\sigma_j\sigma'_j}V^{-1}(g)\,A^{(0)}_{\sigma'_j}\,V(g)\right]\\&=e^{\mathrm{i}(-1)^{\mathfrak{c}(g)}\,\theta(g)}\sum_{\sigma'_j}\langle\psi_{\sigma_j}|\left(\hat{u}^{\dagger}_j(g)|\psi_{\sigma'_j}\rangle\right)U^{-1}(g)\,A^{(0)}_{\sigma'_j}\,U(g),\end{aligned} \tag{C.11a}$$

with $\hat{u}_j(g)$ defined in Eq. (3.12a), and

$$U(g) := \begin{cases} V(g), & \text{if } \mathfrak{c}(g) = 0, \\ V(g)\,\mathsf{K}, & \text{if } \mathfrak{c}(g) = 1. \end{cases} \tag{C.11b}$$

Had we chosen the elements $h \in G_f$ and $g\,h \in G_f$, Eq. (C.11a) would give the selfconsistency conditions

$$A^{(0)}_{\sigma''_j} = e^{\mathrm{i}(-1)^{\mathfrak{c}(h)}\,\theta(h)} \sum_{\sigma_j} \langle\psi_{\sigma''_j}| \left(\hat{u}^{\dagger}_j(h)|\psi_{\sigma_j}\rangle\right) U^{-1}(h)\, A^{(0)}_{\sigma_j}\, U(h), \tag{C.11c}$$

and

$$A^{(0)}_{\sigma_j} = e^{\mathrm{i}(-1)^{\mathfrak{c}(g\,h)}\,\theta(g\,h)} \sum_{\sigma'_j} \langle\psi_{\sigma_j}| \left(\hat{u}^{\dagger}_j(g\,h)|\psi_{\sigma'_j}\rangle\right) U^{-1}(g\,h)\, A^{(0)}_{\sigma'_j}\, U(g\,h), \tag{C.11d}$$

respectively.

Inserting the selfconsistency condition (C.11a) into the the selfconsistency condition (C.11c) gives

$$\begin{aligned}
A^{(0)}_{\sigma''_j} &= e^{\mathrm{i}(-1)^{\mathfrak{c}(h)}\,\theta(h)} \sum_{\sigma_j} \langle\psi_{\sigma''_j}| \left(\hat{u}^{\dagger}_j(h)|\psi_{\sigma_j}\rangle\right) U^{-1}(h) \\
&\qquad \times \left(e^{\mathrm{i}(-1)^{\mathfrak{c}(g)}\,\theta(g)} \sum_{\sigma'_j} \langle\psi_{\sigma_j}| \left(\hat{u}^{\dagger}_j(g)|\psi_{\sigma'_j}\rangle\right) U^{-1}(g)\, A^{(0)}_{\sigma'_j}\, U(g) \right) U(h) \\
&= e^{\mathrm{i}(-1)^{\mathfrak{c}(h)}\,\theta(h) + \mathrm{i}(-1)^{\mathfrak{c}(h)+\mathfrak{c}(g)}\,\theta(g)} \sum_{\sigma_j,\sigma'_j} \langle\psi_{\sigma''_j}| \left(\hat{u}^{\dagger}_j(h)|\psi_{\sigma_j}\rangle\right) \mathsf{K}_h \left[\langle\psi_{\sigma_j}| \left(\hat{u}^{\dagger}_j(g)|\psi_{\sigma'_j}\rangle\right)\right] \\
&\qquad \times U^{-1}(h)\, U^{-1}(g)\, A^{(0)}_{\sigma'_j}\, U(g)\, U(h) \\
&= e^{\mathrm{i}(-1)^{\mathfrak{c}(h)}\,\theta(h) + \mathrm{i}(-1)^{\mathfrak{c}(h)+\mathfrak{c}(g)}\,\theta(g)} \sum_{\sigma'_j} \langle\psi_{\sigma''_j}|\hat{u}^{\dagger}_j(h) \left(\hat{u}^{\dagger}_j(g)\,|\psi_{\sigma'_j}\rangle\right) \\
&\qquad \times U^{-1}(h)\, U^{-1}(g)\, A^{(0)}_{\sigma'_j}\, U(g)\, U(h) \\
&= e^{\mathrm{i}(-1)^{\mathfrak{c}(h)}\,\theta(h) + \mathrm{i}(-1)^{\mathfrak{c}(h)+\mathfrak{c}(g)}\,\theta(g) - \mathrm{i}(-1)^{\mathfrak{c}(g\,h)}\,\phi(g,h)} \sum_{\sigma'_j} \langle\psi_{\sigma''_j}| \left(\hat{u}^{\dagger}_j(g\,h)\,|\psi_{\sigma'_j}\rangle\right)
\end{aligned}$$

$$\times\, U^{-1}(h)\, U^{-1}(g)\, A^{(0)}_{\sigma'_j}\, U(g)\, U(h). \tag{C.12}$$

In reaching the penultimate and last equalities, we used two identities. First,

$$\sum_{\sigma_j} \langle\psi_{\sigma''_j}|\, \left(\hat{u}^{\dagger}_j(h)\, |\psi_{\sigma_j}\rangle\right)\, \mathsf{K}_h \left[\langle\psi_{\sigma_j}|\, \left(\hat{u}^{\dagger}_j(g)\, |\psi_{\sigma'_j}\rangle\right)\right] = \langle\psi_{\sigma''_j}|\, \left(\hat{u}^{\dagger}_j(h)\, \hat{u}^{\dagger}_j(g)\, |\psi_{\sigma'_j}\rangle\right) \tag{C.13}$$

is obviously true when $\mathfrak{c}(h) = 0$ since $\sum_{\sigma_j} |\psi_{\sigma_j}\rangle\, \langle\psi_{\sigma_j}|$ is the resolution of the identity on $\mathfrak{F}_j$. When $\mathfrak{c}(h) = 1$, $\hat{u}^{\dagger}_j(h)$ is antiunitary so that

$$\begin{aligned}
&\sum_{\sigma_j} \langle\psi_{\sigma''_j}|\, \left(\hat{u}^{\dagger}_j(h)\, |\psi_{\sigma_j}\rangle\right)\, \mathsf{K}_h \left[\langle\psi_{\sigma_j}|\, \left(\hat{u}^{\dagger}_j(g)\, |\psi_{\sigma'_j}\rangle\right)\right] \\
&\qquad = \sum_{\sigma_j} \langle\psi_{\sigma''_j}|\, \left(\hat{u}^{\dagger}_j(h)|\psi_{\sigma_j}\rangle\right) \left[\langle\psi_{\sigma_j}|\, \left(\hat{u}^{\dagger}_j(g)\, |\psi_{\sigma'_j}\rangle\right)\right]^{*} \\
&\qquad = \sum_{\sigma_j} \left[\left(\langle\psi_{\sigma''_j}|\, \hat{u}_j(h)\right) |\psi_{\sigma_j}\rangle\langle\psi_{\sigma_j}|\, \left(\hat{u}^{\dagger}_j(g)|\psi_{\sigma'_j}\rangle\right)\right]^{*} \\
&\qquad = \left[\left(\langle\psi_{\sigma''_j}|\hat{u}_j(h)\right) \left(\hat{u}^{\dagger}_j(g)|\psi_{\sigma'_j}\rangle\right)\right]^{*} \\
&\qquad = \langle\psi_{\sigma''_j}|\, \left(\hat{u}^{\dagger}_j(h)\, \hat{u}^{\dagger}_j(g)|\psi_{\sigma'_j}\rangle\right).
\end{aligned} \tag{C.14}$$

Second, we used the projective representation (3.7) to obtain

$$\begin{aligned}
\hat{u}^{\dagger}_j(h)\, \hat{u}^{\dagger}_j(g) &= \left[\hat{u}_j(g)\, \hat{u}_j(h)\right]^{\dagger} \\
&= \left[e^{+\mathrm{i}\phi(g,h)}\, \hat{u}_j(g\,h)\right]^{\dagger} \\
&= \hat{u}^{\dagger}_j(g\,h)\, e^{-\mathrm{i}\phi(g,h)} \\
&= e^{-\mathrm{i}(-1)^{\mathfrak{c}(g\,h)}\,\phi(g,h)}\, \hat{u}^{\dagger}_j(g\,h).
\end{aligned} \tag{C.15}$$

Equating the right-hand sides of Eqs. (C.12) and (C.11d) gives the condition

$$U^{-1}(h)\, U^{-1}(g)\, A^{(0)}_{\sigma'_j}\, U(g)\, U(h) = e^{\mathrm{i}\chi}\, U^{-1}(g\,h)\, A^{(0)}_{\sigma'_j}\, U(g\,h), \tag{C.16a}$$

$$\chi := (-1)^{\mathfrak{c}(g\,h)}\phi(g,h) - (-1)^{\mathfrak{c}(h)}\,\theta(h) - (-1)^{\mathfrak{c}(h)+\mathfrak{c}(g)}\,\theta(g) + (-1)^{\mathfrak{c}(g\,h)}\,\theta(g\,h). \tag{C.16b}$$

Upon using the fact that $\mathfrak{c}$ is a homomorism so that $\mathfrak{c}(g\,h)=\mathfrak{c}(g)\,\mathfrak{c}(h)$ holds, we arrive at

$$W^{-1}(g,h)\,A^{(0)}_{\sigma_j}\,W(g,h)=e^{-\mathrm{i}\delta(g,h;b)}\,A^{(0)}_{\sigma_j}, \tag{C.17a}$$

where

$$W(g,h):=U(g)\,U(h)\,U^{-1}(g\,h), \tag{C.17b}$$

$$\delta(g,h;b):=(-1)^{\mathfrak{c}(g)}\,\theta(h)+\theta(g)-\phi(g,h)-\theta(g\,h). \tag{C.17c}$$

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$$W^{-1}(g,h)\,A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\cdots A^{(0)}_{\sigma_\ell}\,W(g,h)=e^{-\mathrm{i}\ell\,\delta(g,h;b)}\,A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\cdots A^{(0)}_{\sigma_\ell} \tag{C.18}$$

holds for any positive integer ℓ.

Injectivity of a FMPS implies that for some integer $\ell^\star>1$ and any $\ell\geq\ell^\star$ all the products of the form $A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\cdots A^{(0)}_{\sigma_\ell}$ span the space of all $2M\times 2M$ matrices. Therefore, Eq. (C.18) combined with injectivity implies that the $2M\times 2M$ matrix $W(g,h)$ is an element from the center of the algebra defined by the vector space of all $2M\times 2M$ matrices, i.e., $\{\mathbb{1}_{2M}\}$. Condition (C.18) thus simplifies to

$$A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\cdots A^{(0)}_{\sigma_\ell}=e^{-\mathrm{i}\ell\,\delta(g,h;b)}\,A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\cdots A^{(0)}_{\sigma_\ell}, \tag{C.19}$$

for any $\ell\geq\ell^\star$. Choosing a linar combination of $A^{(0)}_{\sigma_1}\,A^{(0)}_{\sigma_2}\cdots A^{(0)}_{\sigma_\ell}$ equating the identity matrix $\mathbb{1}_{2M}$, delivers the constraint

$$\ell\,\delta(g,h;b)=0,\quad \forall\ell>\ell^\star \implies \delta(g,h;b)=0. \tag{C.20a}$$

Inserting the value of $\delta(g,h;b)$ given in Eq. (C.17) implies the final constraint

$$\phi(g,h)=(-1)^{\mathfrak{c}(g)}\,\theta(h)+\theta(g)-\theta(g\,h). \tag{C.20b}$$

This is the coboundary condition (3.10) when $\phi'=0$. In other words, the local representation $\hat{u}_j$ is equivalent to the trivial projective representation.

C.1.2 *Odd-Parity FMPS*

Let [see Eq. (B.12)]

$$|\{A^{(1)}_{\sigma_j}\};b\rangle = \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left[P^b\, Y\, A^{(1)}_{\sigma_1}\, A^{(1)}_{\sigma_2}\, \cdots A^{(1)}_{\sigma_N}\right] |\psi_{\sigma_1}\rangle\, |\psi_{\sigma_2}\rangle \cdots |\psi_{\sigma_N}\rangle \tag{C.21}$$

be a translation-invariant, G_f-symmetric, odd-parity (each matrix $A^{(1)}_{\sigma_j}$ commutes with the matrix Y), and injective FMPS obeying periodic boundary conditions when $b=0$ or antiperiodic boundary conditions when $b=1$. For any $g\in G_f$, the global representation $\widehat{U}(g)$ of g is defined in Eq. (3.12a). By assumption, $|\{A^{(1)}_{\sigma_j}\};b\rangle$ is a nondegenerate gapped ground state of some local fermionic Hamiltonian in one-dimensional space. Hence, for any $g\in G_f$, there exists a phase $\eta(g;b)\in[0,2\pi)$ such that

$$\widehat{U}(g)\,|\{A^{(1)}_{\sigma_j}\};b\rangle = e^{\mathrm{i}\eta(g;b))}|\{A^{(1)}_{\sigma_j}\};b\rangle. \tag{C.22}$$

The counterpart to Eq. (C.5) is

$$\widehat{U}(g)|\{A^{(1)}_{\sigma_j}\};b\rangle = \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left[P^b\, Y\, A^{(1)}_{\sigma_1}(g)\, A^{(1)}_{\sigma_2}(g)\, \cdots A^{(1)}_{\sigma_N}(g)\right] |\psi_{\sigma_1}\rangle\, |\psi_{\sigma_2}\rangle \cdots |\psi_{\sigma_N}\rangle, \tag{C.23a}$$

$$A^{(1)}_{\sigma_j}(g) := \sum_{\sigma'_j} \langle\psi_{\sigma_j}|\,\hat{v}_j(g)\,|\psi_{\sigma'_j}\rangle\, \mathsf{K}_g\left[A^{(1)}_{\sigma'_j}\right] = \sum_{\sigma'_j} \mathcal{U}(g)_{\sigma_j,\sigma'_j}\, A^{(1)}_{\sigma'_j}, \tag{C.23b}$$

$$\mathsf{K}_g\left[A^{(1)}_{\sigma_j}\right] := \begin{cases} A^{(1)}_{\sigma_j}, & \text{if } \mathfrak{c}(g)=0,\\ \\ \mathsf{K}\, A^{(1)}_{\sigma_j}\, \mathsf{K}, & \text{if } \mathfrak{c}(g)=1. \end{cases} \tag{C.23c}$$

Odd-parity injective FMPS differ from the even ones in one crucial way. There exists a positive integer $\ell^\star \geq 1$ such that for any $\ell \geq \ell^\star$ the products of the form $A^{(1)}_{\sigma_1}\, A^{(1)}_{\sigma_2} \cdots A^{(1)}_{\sigma_\ell}$ span the $\mathbb{Z}_2$-graded algebra of $2M\times 2M$ matrices with the center $\left\{\mathbb{1}_{2M}, Y\right\}$. Consequently, there exists a $2M\times 2M$ invertible matrix $V(g)$ and a phase $\theta(g)\in[0,2\pi)$ such that [recall Eq. (3.35)]

$$V(g) = P\,V(g)\,P, \qquad V(g) = (-1)^{\zeta(g)}\, Y\, V(g)\, Y, \qquad \zeta(g) = 0,1, \tag{C.24a}$$

with $\zeta\colon G_f \to \{-1,+1\}$ a group homomorphism and

$$A^{(1)}_{\sigma_j}(g) = e^{\mathrm{i}\theta(g)}\, V^{-1}(g)\, A^{(1)}_{\sigma_j}\, V(g). \tag{C.24b}$$

The same steps that lead to Eq. (C.6) then give

$$\operatorname{tr}\left[P^b\, Y\, A^{(1)}_{\sigma_1}(g)\, A^{(1)}_{\sigma_2}(g)\, \cdots A^{(1)}_{\sigma_N}(g)\right] = e^{\mathrm{i}\eta(g;b)} \operatorname{tr}\left[P^b\, Y\, A^{(1)}_{\sigma_1}\, A^{(1)}_{\sigma_2}\, \cdots A^{(1)}_{\sigma_N}\right] \tag{C.25a}$$

with the solution

$$A^{(1)}_{\sigma_j}(g) = e^{\mathrm{i}\theta(g)}\, V^{-1}(g)\, A^{(1)}_{\sigma_j}\, V(g), \tag{C.25b}$$

$$Y\, V(g)\, Y = (-1)^{\zeta(g)}\, V(g), \tag{C.25c}$$

$$\theta(g) := \frac{1}{N}\left[\eta(g;b) - \pi\,\kappa(g)\right]. \tag{C.25d}$$

All the steps leading to Eq. (C.17) deliver

$$W^{-1}(g,h)\, A^{(1)}_{\sigma_j}\, W(g,h) = e^{-\mathrm{i}\delta(g,h;b)}\, A^{(1)}_{\sigma_j}, \qquad \sigma_j = 1,\cdots,\mathcal{D}, \qquad j = 1,\cdots,N, \tag{C.26a}$$

where

$$W(g,h) := U(g)\, U(h)\, U^{-1}(g\,h), \tag{C.26b}$$

$$\delta(g,h;b) := (-1)^{\mathfrak{c}(g)}\,\theta(h) + \theta(g) - \phi(g,h) - \theta(g\,h), \tag{C.26c}$$

and $U(g) = V(g)$ if $\mathfrak{c}(g) = 0$ and $U(g) = V(g)\,\mathsf{K}$ if $\mathfrak{c}(g) = 1$. Because $V(g)$ commutes with P so does $W(g,h)$. Because all possible products of the form $A^{(1)}_{\sigma_1}\, A^{(1)}_{\sigma_2} \cdots A^{(1)}_{\sigma_\ell}$ span the $\mathbb{Z}_2$-graded algebra of $2M \times 2M$ matrices with the center $\left\{\mathbb{1}_{2M}, Y\right\}$, $W(g,h)$ is, up to a phase factor, proportional to $\mathbb{1}_{2M}$. The counterpart to the even-parity coboundary condition (C.20) then follows, thereby completing the proof of Theorem 2 for the parity-odd FMPS.

C.2 PROOF BY TWISTED BOUNDARY CONDITIONS FOR UNITARY GROUPS

The lattice is $\Lambda = \{1,\cdots,N\} \equiv \mathbb{Z}_N$ with N an integer. The global fermionic Fock space $\mathfrak{F}_\Lambda$ is of dimension 2^{mN} with $n = 2m$ an even number of local Majorana flavors. The local $\mathfrak{F}_j$ and global $\mathfrak{F}_\Lambda$ Fock spaces are generated by the Hermitian Majorana operators $\hat{\chi}_{j,a}$ obeying the Clifford alegbra

$$\left\{\hat{\chi}_{j,a},\hat{\chi}_{j',a'}\right\}=2\,\delta_{j,j'}\,\delta_{a,a'},\qquad j,j'=1,\cdots,N,\qquad a,a'=1,\cdots,n=2m. \tag{C.27}$$

The local and global fermion parity operators are

$$\hat{p}_j:=\prod_{a=1}^{m}\mathrm{i}\hat{\chi}_{j,2a-1}\,\hat{\chi}_{j,2a},\qquad \widehat{P}_\Lambda:=\prod_{j=1}^{N}\hat{p}_j, \tag{C.28}$$

respectively. Any polynomial $\hat{h}_j$ in the Majorana operators that is of finite order, of finite range r (the integer r is the maximum separation between the space labels of the Majorana operators entering $\widehat{h}_j$) of even parity ($\widehat{P}_\Lambda\,\hat{h}_j\,\widehat{P}_\Lambda=\hat{h}_j$), and Hermitian ($\hat{h}_j^\dagger=\hat{h}_j$) is a local Hamiltonian. We define the unitary operator $\widehat{T}_1$ by its action

$$\widehat{T}_1\,\hat{\chi}_{j,a}\,\widehat{T}_1^{-1}=\begin{cases}\hat{\chi}_{j+1,a}, & \text{if } j=1,\cdots,N-1 \text{ and } a=1,\cdots,n=2m,\\[2ex] \hat{\chi}_{1,a}, & \text{if } j=N \text{ and } a=1,\cdots,n=2m.\end{cases} \tag{C.29}$$

It follows that

$$\widehat{T}_1^N=\widehat{\mathbb{1}}_{2mN}, \tag{C.30}$$

i.e., $\widehat{T}_1$ is a unitary representation of the generator of the cyclic group $\mathbb{Z}_N$. For any Abelian central extension G_f of G by $\mathbb{Z}_2^F$ and for any $g\in\ G_f$, we assume the projective representation (3.7) with

$$\hat{u}_j(g)=\hat{v}_j(g)\qquad [\text{as } \mathfrak{c}(g)=0 \text{ always hold by hypothesis}] \tag{C.31}$$

a polynomial in $\hat{\chi}_{j,a}$ with $a=1,\cdots,n=2m$. We make the identifications

$$\begin{aligned}&\hat{v}_j(e)\equiv\widehat{\mathbb{1}}_{2m},\qquad \hat{v}_j(p)\equiv\hat{p}_j,\qquad \hat{p}_j\,\hat{v}_j(g)\,\hat{p}_j=(-1)^{\rho(g)}\,\hat{v}_j(g),\qquad j=1,\cdots,N,\\ &\widehat{U}(e)\equiv\widehat{\mathbb{1}}_{2mN},\qquad \widehat{U}(p)\equiv\widehat{P}_\Lambda,\qquad \widehat{U}(g):=\prod_{j=1}^{N}\hat{v}_j(g),\qquad \forall g\in G_f.\end{aligned} \tag{C.32}$$

We assume that the Hamiltonian $\widehat{h}_j$ is G_f invariant (symmetric), i.e.,

$$\widehat{h}_j = \widehat{U}(g)\,\widehat{h}_j\,\widehat{U}^{-1}(g), \qquad \forall g \in G_f. \tag{C.33}$$

By construction, the Hamiltonian defined by [recall Eq. (3.58)]

$$\widehat{H}_{\mathrm{pbc}} := \sum_{n=1}^{N} \left(\widehat{T}_1\right)^n \hat{h}_j \left(\widehat{T}_1^{\dagger}\right)^n, \qquad \widehat{U}(g)\,\hat{h}_j\,\widehat{U}^{-1}(g), \qquad \forall g \in G_f, \tag{C.34a}$$

is translation invariant (symmetric),

$$\widehat{T}_1\,\widehat{H}_{\mathrm{pbc}}\,\widehat{T}_1^{-1} = \widehat{H}_{\mathrm{pbc}}, \tag{C.34b}$$

and G_f invariant (symmetric),

$$\widehat{U}(g)\,\widehat{H}_{\mathrm{pbc}}\,\widehat{U}^{-1}(g) = \widehat{H}_{\mathrm{pbc}}, \qquad \forall g \in G_f. \tag{C.34c}$$

We define the family of twisted translation operators

$$\widehat{T}_1(g) := \hat{v}_1(g)\,\widehat{T}_1, \qquad g \in G_f, \qquad \mathfrak{c}(g) = 0. \tag{C.35}$$

Their action on the Majorana spinor

$$\hat{\chi}_j := \begin{pmatrix} \hat{\chi}_{j,1} & \cdots & \hat{\chi}_{j,n} \end{pmatrix}^{\mathsf{T}} \tag{C.36a}$$

differ from that in Eq. (C.29),

$$\widehat{T}_1(g)\,\hat{\chi}_j\,\widehat{T}_1^{-1}(g) = \begin{cases} (-1)^{\rho(g)}\,\hat{\chi}_{j+1}, & \text{if } j \neq N, \\ \\ \hat{v}_1(g)\,\hat{\chi}_1\,\hat{v}_1^{-1}(g), & \text{if } j = N. \end{cases} \tag{C.36b}$$

We have the identity

$$\begin{aligned} \left[\widehat{T}_1(g)\right]^N &= \left[\hat{v}_1(g)\,\widehat{T}_1\right]\left[\hat{v}_1(g)\,\widehat{T}_1\right]\cdots\left[\hat{v}_1(g)\,\widehat{T}_1\right]\left[\hat{v}_1(g)\,\widehat{T}_1\right] \\ &= \hat{v}_1(g)\left[\widehat{T}_1\,\hat{v}_1(g)\,\widehat{T}_1\right]\cdots\hat{v}_1(g)\left[\widehat{T}_1\,\hat{v}_1(g)\,\widehat{T}_1\right] \\ &= \hat{v}_1(g)\left[\widehat{T}_1\,\hat{v}_1(g)\,\widehat{T}_1\right]\cdots\hat{v}_1(g)\left[\widehat{T}_1\,\hat{v}_1(g)\,\widehat{T}_1^{-1}\right]\widehat{T}_1^{2} \end{aligned}$$

$$\begin{aligned}
\text{Eq. (C.29)} \quad &= \hat{v}_1(g)\left[\widehat{T}_1\,\hat{v}_1(g)\,\widehat{T}_1\right]\cdots\hat{v}_1(g)\,\hat{v}_2(g)\,\widehat{T}_1^2 \\
&= \widehat{U}(g)\,\widehat{T}_1^N \\
\text{Eq. (C.30)} \quad &= \widehat{U}(g).
\end{aligned} \tag{C.37}$$

Finally, we define the family of twisted Hamiltonians

$$\widehat{H}^{\mathrm{tlt}}_{\mathrm{tw}}(g) := \sum_{j=1}^{N}\left[\widehat{T}_1(g)\right]^j\,\hat{h}^{\mathrm{tlt}}_1\,\left[\widehat{T}_1^{-1}(g)\right]^j, \qquad \widehat{h}^{\mathrm{tlt}}_1 = \widehat{U}(h)\,\widehat{h}^{\mathrm{tlt}}_1\,\widehat{U}^{-1}(h), \qquad \forall h\in G_f. \tag{C.38}$$

By design,

$$\begin{aligned}
\widehat{T}_1(g)\,\widehat{H}^{\mathrm{tlt}}_{\mathrm{tw}}(g)\,\widehat{T}_1^{-1}(g) &= \sum_{j=1}^{N-1}\left[\widehat{T}_1(g)\right]^{j+1}\,\hat{h}^{\mathrm{tlt}}_1\,\left[\widehat{T}_1^{-1}(g)\right]^{j+1} + \left[\widehat{T}_1(g)\right]^{N+1}\,\hat{h}^{\mathrm{tlt}}_1\,\left[\widehat{T}_1^{-1}(g)\right]^{N+1} \\
\text{Eq. (C.37)} \quad &= \sum_{j=1}^{N-1}\left[\widehat{T}_1(g)\right]^{j+1}\,\hat{h}^{\mathrm{tlt}}_1\,\left[\widehat{T}_1^{-1}(g)\right]^{j+1} + \widehat{T}_1(g)\left[\widehat{U}(g)\,\hat{h}^{\mathrm{tlt}}_1\,\widehat{U}^{-1}(g)\right]\widehat{T}_1^{-1}(g) \\
G_f\ \text{symmetry} \quad &= \sum_{j=1}^{N-1}\left[\widehat{T}_1(g)\right]^{j+1}\,\hat{h}^{\mathrm{tlt}}_1\,\left[\widehat{T}_1^{-1}(g)\right]^{j+1} + \widehat{T}_1(g)\,\hat{h}^{\mathrm{tlt}}_1\,\widehat{T}_1^{-1}(g) \\
&= \widehat{H}^{\mathrm{tlt}}_{\mathrm{tw}}(g).
\end{aligned} \tag{C.39}$$

We are going to derive the important identity

$$\widehat{U}(h)\,\widehat{T}_1(g)\,\widehat{U}^{-1}(h) = e^{\mathrm{i}\chi(g,h)}\,\widehat{T}_1(g), \qquad \forall g,h\in G_f, \tag{C.40a}$$

with the phase

$$\chi(g,h) := \phi(h,g) - \phi(g,h) + \pi\,\rho(h)\,[\rho(g)+1]\,(N-1), \qquad \forall g,h\in G_f. \tag{C.40b}$$

We shall then specify the conditions under which the algebra defined by Eqs. (C.39) and (C.40) guarantees that the spectrum of the twisted Hamiltonian is degenerate.

Proof. We begin with the proof of Eq. (C.40). We choose two elements $g,h\in G_f$ with the local representations $\hat{v}_1(g)$ and $\hat{v}_1(h)$, respectively, both of which are unitary.

Step 1. We observe that

$$\widehat{U}(h)\,\hat{v}_1(g) = \hat{v}_1(h)\,\hat{v}_2(h)\cdots\hat{v}_N(h)\;\hat{v}_1(g). \tag{C.41}$$

We can then interchange the local operator $\hat{v}_j(h)$ and $\hat{v}_{j'}(g)$ pairwise at the cost of the fermionic phase $(-1)^{\rho(h)\,\rho(g)}$ for any $j, j' = 1, \cdots, N$. This is done $(N-1)$ times

$$\widehat{U}(h)\,\hat{v}_1(g) = (-1)^{\rho(g)\,\rho(h)(N-1)}\hat{v}_1(h)\,\hat{v}_1(g)\,\hat{v}_2(h)\cdots\hat{v}_N(h). \tag{C.42}$$

We conclude with

$$\widehat{U}(h)\,\hat{v}_1(g) = (-1)^{\rho(g)\,\rho(h)(N-1)}\hat{v}_1(h)\,\hat{v}_1(g)\,\hat{v}_2(h)\cdots\hat{v}_N(h). \tag{C.43}$$

Step 2. We begin with

$$\begin{aligned}\widehat{T}_1\,\widehat{U}^{-1}(h) &= \widehat{T}_1\,\hat{v}_N^{-1}(h)\,\hat{v}_{N-1}^{-1}(h)\,\cdots\,\hat{v}_1^{-1}(h)\\ &= \left[\widehat{T}_1\,\hat{v}_N^{-1}(h)\,\widehat{T}_1^{-1}\right]\left[\widehat{T}_1\,\hat{v}_{N-1}^{-1}(h)\widehat{T}_1^{-1}\right]\,\cdots\,\left[\widehat{T}_1\,\hat{v}_1^{-1}(h)\widehat{T}_1^{-1}\right]\widehat{T}_1\\ \overset{\text{Eq. (3.57a)}}{} &= \hat{v}_1^{-1}(h)\,\hat{v}_N^{-1}(h)\,\cdots\,\hat{v}_2^{-1}(h)\,\widehat{T}_1.\end{aligned} \tag{C.44}$$

Hence,

$$\widehat{T}_1\,\widehat{U}^{-1}(h) = (-1)^{\rho(h)(N-1)}\hat{v}_N^{-1}(h)\,\hat{v}_{N-1}^{-1}(h)\cdots\hat{v}_1^{-1}(h)\,\widehat{T}_1, \tag{C.45}$$

where we have reordered the factors $\hat{v}_j^{-1}(h)$ and, in doing so, obtained the coefficient $(-1)^{\rho(h)(N-1)}$ that encodes the fermionic algebra.

Step 3. We combine Eqs. (C.43) and (C.45) into

$$\begin{aligned}\widehat{U}(h)\,\widehat{T}_1(g)\,\widehat{U}^{-1}(h) &= (-1)^{\rho(h)\,[\rho(g)+1](N-1)}\hat{v}_1(h)\,\hat{v}_1(g)\,\hat{v}_2(h)\cdots\hat{v}_N(h)\\ &\quad\times\hat{v}_N^{-1}(h)\,\cdots\hat{v}_1^{-1}(h)\,\widehat{T}_1\\ &= (-1)^{\rho(h)\,[\rho(g)+1](N-1)}\hat{v}_1(h)\,\hat{v}_1(g)\,\hat{v}_1^{-1}(h)\,\widehat{T}_1.\end{aligned} \tag{C.46}$$

Step 4. We need to massage $\hat{v}_1(h)\,\hat{v}_1(g)\,\hat{v}_1^{-1}(h)$. To this end, we use the fact that the group G_f is Abelian to obtain

$$\begin{aligned}\hat{v}_1(h)\,\hat{v}_1(g)\,\hat{v}_1^{-1}(h) &= e^{\mathrm{i}\phi(h,g)}\,\hat{v}_1(h\,g)\,\hat{v}_1^{-1}(h)\\ &= e^{\mathrm{i}\phi(h,g)}\,\hat{v}_1(g\,h)\,\hat{v}_1^{-1}(h)\end{aligned}$$

$$
\begin{aligned}
&= e^{\mathrm{i}\phi(h,g)} \left[e^{-\mathrm{i}\phi(g,h)}\, \hat{v}_1(g)\, \hat{v}_1(h)\right] \hat{v}_1^{-1}(h) \\
&= e^{\mathrm{i}\phi(h,g)-\mathrm{i}\phi(g,h)}\, \hat{v}_1(g).
\end{aligned}
\tag{C.47}
$$

Insertion into the right-hand side of Eq. (C.46) delivers the result

$$
\begin{aligned}
\widehat{U}(h)\, \widehat{T}_1(g)\, \widehat{U}^{-1}(h) &= (-1)^{\rho(h)\,[\rho(g)+1]\,(N-1)} e^{\mathrm{i}\phi(h,g)-\mathrm{i}\phi(g,h)}\, \hat{v}_1(g)\, \widehat{T}_1 \\
&\equiv e^{\mathrm{i}\chi(g,h)}\, \widehat{T}_1(g),
\end{aligned}
\tag{C.48a}
$$

with the definition

$$
\chi(g,h) = \phi(h,g) - \phi(g,h) + \pi\, \rho(h)\, [\rho(g)+1]\, (N-1). \tag{C.48b}
$$

□

Step 5. It is instructive to derive the transformation law of the phase (C.48b) under the global U(1) gauge transformation generated by

$$
\hat{v}_j(g) =: e^{\mathrm{i}\xi(g)}\, \hat{v}'_j(g), \qquad j = 1, \cdots, N, \qquad \forall g \in G_f. \tag{C.49}
$$

Under this transformation,

$$
\phi'(g,h) = \phi(g,h) - \xi(g) - \xi(h) + \xi(g\,h), \qquad \forall g, h \in G_f, \tag{C.50}
$$

is the phase entering the projective algebra obeyed by the operators $\{\hat{v}'_j(g) \mid g \in G_f\}$ according to Eq. (3.10b). Hence, if we define

$$
\chi'(g,h) := \phi'(h,g) - \phi'(g,h) + \pi\, \rho'(h)\, [\rho'(g)+1]\, (N-1), \qquad \forall g,\, h \in G_f, \tag{C.51}
$$

we then have the relation

$$
\begin{aligned}
\chi(g,h) &= \phi'(h,g) - \phi'(g,h) + \pi\, \rho(h)\, [\rho(g)+1]\, (N-1) \\
&= \chi'(g,h) + \underline{\xi(h)} + \underline{\underline{\xi(g)}} - \underline{\underline{\underline{\xi(h\,g)}}} - \underline{\underline{\xi(g)}} - \underline{\xi(h)} + \underline{\underline{\underline{\xi(g\,h)}}} \\
&= \chi'(g,h), \qquad \forall g,\, h \in G_f.
\end{aligned}
\tag{C.52}
$$

Hence, $\chi(g,h)$ is gauge invariant under the U(1) gauge transformation (C.49). The pair of cocyles ϕ' and ϕ are equivalent if and only if they have the same second cohomology class $[\phi] = [\phi'] \in H^2\big(G_f, \mathrm{U}(1)\big)$, i.e., if and only if they are related by the U(1) gauge transformation (C.50). The gauge invariance of χ implies that it is independent of the choice made of ϕ within the equivalence class $[\phi] \in H^2\big(G_f, \mathrm{U}(1)\big)$. For example, $\chi(g,h) = 0$ holds for all $g, h \in G_f$ for any ϕ belonging to the trivial second cohomology class $[\phi] = 0$ since the function $\phi = 0$ belongs to $[\phi] = 0$. As a corollary, there exists a pair $g, h \in G_f$ for which $\chi(g,h)$ is nonvanishing if and only if $[\phi] \neq 0$.

Step 6. The twisted Hamiltonian $\widehat{H}^{\mathrm{tlt}}_{\mathrm{tw}}(g)$ is constructed so as to commute with the generator $\widehat{T}_1(g)$ of twisted translations and with the representation $\widehat{U}(h)$ of any group element $h \in G_f$, whereby passing $\widehat{U}(h)$ from the left through $\widehat{T}_1(g)$ produces the phase $\exp\big(\mathrm{i}\chi(g,h)\big)$. If it is possible to find a pair (g,h) such that $\chi(g,h)$ is not 0 modulo 2π, then the spectrum of $\widehat{H}^{\mathrm{tlt}}_{\mathrm{tw}}(g)$ must be degenerate. Indeed, any simultaneous eigenstate $|E(g), \exp\big(\mathrm{i}K(g)\big)\rangle$ of $\widehat{H}^{\mathrm{tlt}}_{\mathrm{tw}}(g)$ and $\widehat{T}_1(g)$ must be orthogonal to the state $\widehat{U}(h)\,|E(g), \exp\big(\mathrm{i}K(g)\big)\rangle$, which is also an eigenstate of $\widehat{H}^{\mathrm{tlt}}_{\mathrm{tw}}(g)$ with the energy $E(g)$ but has the eigenvalue $\exp\big(\mathrm{i}[K(g) + \chi(g,h)]\big) \neq \exp\big(\mathrm{i}K(g)\big)$ with respect to $\widehat{T}_1(g)$.

D

COMPLEMENT TO CHAPTER 7

In this Appendix, we complement Chapter 7 by presenting the details of derivations.

D.1 DETAILS FOR EVEN-EVEN STACKING

Proof of Eq. (7.13). For the identity $e \in \mathbb{Z}_2^F$, we have

$$\widehat{V}_1(e) = \hat{\mathbb{1}}_1, \qquad \widehat{V}_2(e) = \hat{\mathbb{1}}_2, \qquad \rho_1(e) = \rho_2(e) = 0, \tag{D.1a}$$

which delivers when inserted in Eq. (7.12)

$$\widehat{U}_\wedge(e) = \hat{\mathbb{1}}_1\, \hat{\mathbb{1}}_2 \equiv \hat{\mathbb{1}}_\wedge. \tag{D.1b}$$

For the fermion parity $p \in \mathbb{Z}_2^F$, we have

$$\widehat{V}_1(p) = \widehat{U}_1(p), \qquad \widehat{V}_2(p) = \widehat{U}_2(p), \qquad \rho_1(p) = \rho_2(p) = 0, \tag{D.2a}$$

which delivers when inserted in Eq. (7.12)

$$\widehat{U}_\wedge(p) = \widehat{U}_1(p)\, \widehat{U}_2(p). \tag{D.2b}$$

□

Proof that definition (7.12) *satisfies Eq.* (7.4) *and, a forteriori, Eq.* (6.17). Without loss of generality, we only consider the action of $\widehat{U}_\wedge(g)$ on a Majorana operator $\hat{\gamma}_i^{(1)}$ associated with representation $\widehat{U}_1$ for any $i = 1, \cdots, n_1$. We have

$$\begin{aligned}\widehat{U}_\wedge(g)\, \hat{\gamma}_i^{(1)}\, \widehat{U}_\wedge^\dagger(g) = &\, \widehat{V}_1(g)\, \widehat{V}_2(g)\, \left[\widehat{U}_1(p)\right]^{\rho_2(g)} \left[\widehat{U}_2(p)\right]^{\rho_1(g)} \mathsf{K}_\wedge^{\mathfrak{c}(g)} \\ &\times \hat{\gamma}_i^{(1)} \\ &\times \mathsf{K}_\wedge^{\mathfrak{c}(g)} \left[\widehat{U}_2^\dagger(p)\right]^{\rho_1(g)} \left[\widehat{U}_1^\dagger(p)\right]^{\rho_2(g)} \widehat{V}_2^\dagger(g)\, \widehat{V}_1^\dagger(g)\end{aligned}$$

$$
\begin{aligned}
&= \widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)} \\
&\times \overline{\hat{\gamma}^{(1)}_i}^{1,g} \\
&\times \left[\widehat{U}^\dagger_2(p)\right]^{\rho_1(g)}\,\left[\widehat{U}^\dagger_1(p)\right]^{\rho_2(g)}\,\widehat{V}^\dagger_2(g)\,\widehat{V}^\dagger_1(g), \qquad \text{(D.3a)}
\end{aligned}
$$

where we used Eq. (7.6) to trade complex conjugation by $\mathsf{K}_\wedge$ with complex conjugation by K_1. As Majorana operator $\overline{\hat{\gamma}^{(1)}_i}^{1,g}$ commutes with $\widehat{U}_2(p)$, while it anticommutes with $\widehat{U}_1(p)$, one finds

$$
\widehat{U}_\wedge(g)\,\hat{\gamma}^{(1)}_i\,\widehat{U}^\dagger_\wedge(g) = (-1)^{\rho_2(g)}\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\overline{\hat{\gamma}^{(1)}_i}^{1,g}\,\widehat{V}^\dagger_2(g)\,\widehat{V}^\dagger_1(g). \qquad \text{(D.3b)}
$$

Passing $\widehat{V}_2(g)$ to the right of $\hat{\gamma}^{(1)}_i$ brings a second multiplicative phase factor of $(-1)^{\rho_2(g)}$. Hence,

$$
\widehat{U}_\wedge(g)\,\hat{\gamma}^{(1)}_i\,\widehat{U}^\dagger_\wedge(g) = \widehat{V}_1(g)\,\overline{\hat{\gamma}^{(1)}_i}^{1,g}\,\widehat{V}^\dagger_1(g). \qquad \text{(D.3c)}
$$

The definition (6.19) then implies that

$$
\begin{aligned}
\widehat{U}_\wedge(g)\,\hat{\gamma}^{(1)}_i\,\widehat{U}^\dagger_\wedge(g) &= \widehat{V}_1(g)\,\mathsf{K}_1^{\mathfrak{c}(g)}\,\hat{\gamma}^{(1)}_i\,\mathsf{K}_1^{\mathfrak{c}(g)}\,\widehat{V}^\dagger_1(g) \\
&= \widehat{U}_1(g)\,\hat{\gamma}^{(1)}_i\,\widehat{U}^\dagger_1(g), \qquad \text{(D.3d)}
\end{aligned}
$$

which is nothing but the condition (7.4). □

Proof of Eq. (7.14a). When the representations $\widehat{U}_\wedge(g)$ and $\widehat{U}_\wedge(h)$ of two elements g and h of G_f are composed, we obtain from definition (7.12)

$$
\begin{aligned}
\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) &= \widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g)} \\
&\times \widehat{V}_1(h)\,\widehat{V}_2(h)\,\left[\widehat{U}_1(p)\right]^{\rho_2(h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(h)} \\
&= \widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)} \\
&\times \overline{\widehat{V}_1(h)}^{\wedge,g}\,\overline{\widehat{V}_2(h)}^{\wedge,g}\,\left[\widehat{U}_1(p)\right]^{\rho_2(h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}, \qquad \text{(D.4a)}
\end{aligned}
$$

we used the reality condition obeyed by $\widehat{U}_1(p)$ and $\widehat{U}_2(p)$, and the fact that

$$\mathfrak{c}(g) + \mathfrak{c}(h) = \mathfrak{c}(g\,h) \bmod 2. \tag{D.4b}$$

According to definition (7.6), we can trade $\mathsf{K}_\wedge$ with K_1 and K_2 when $\mathsf{K}_\wedge$ acts on operators $\widehat{V}_1$ and $\widehat{V}_2$, respectively. We therefore find

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) =& \widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)}\\ &\times \overline{\widehat{V}_1(h)}^{1,g}\,\overline{\widehat{V}_2(h)}^{2,g}\,\left[\widehat{U}_1(p)\right]^{\rho_2(h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}.\end{aligned} \tag{D.5}$$

We can bring $\overline{\widehat{V}_1(h)}^{1,g}$ to the right of $\widehat{V}_1(g)$ at the cost of the multiplicative phase factor $(-1)^{2\rho_1(h)\rho_2(g)} = 1$. This multiplicative phase factor arises from two multiplicative phase factors. In turn, each multiplicative phase factor arises from the identity

$$\widehat{V}_i(g)\,\widehat{V}_j(h) = (-1)^{\rho_i(g)\,\rho_j(h)}\,\widehat{V}_j(h)\,\widehat{V}_i(g), \qquad i \neq j = 1,2, \tag{D.6}$$

that holds for any pair $\widehat{V}_i(g)$ and $\widehat{V}_j(h)$ of unitary operators with $i \neq j = 1,2$. This identity simply states that $\widehat{V}_1(g)$ and $\widehat{V}_2(h)$ either commute when none of them are simultaneously fermionic or anticommute otherwise. Now, $\overline{\widehat{V}_1(h)}^{1,g}$ commutes with $\left[\widehat{U}_2(p)\right]^{\rho_1(g)}$ for $\rho_2(p) = 0$ while it shares the same parity as the $\widehat{U}_1(p)$ parity of $\widehat{U}_1(h)$. Commuting $\overline{\widehat{V}_1(h)}^{1,g}$ across $\left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)}$ produces the multiplicative phase factor $(-1)^{\rho_1(h)\rho_2(g)}$. Commuting $\overline{\widehat{V}_1(h)}^{1,g}$ with $\widehat{V}_2(g)$ brings another factor of $(-1)^{\rho_1(h)\rho_2(g)}$. One is left with

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = & \widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\,\widehat{V}_2(g)\,\left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)}\\ &\times \overline{\widehat{V}_2(h)}^{2,g}\,\left[\widehat{U}_1(p)\right]^{\rho_2(h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}.\end{aligned} \tag{D.7}$$

The next step consists in passing $\left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)}$ to the right of $\overline{\widehat{V}_2(h)}^{2,g}$ at the cost of the multiplicative phase factor $(-1)^{\rho_1(g)\rho_2(h)}$. One is left with

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) =& (-1)^{\rho_1(g)\rho_2(h)}\,\widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\,\widehat{V}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g}\\ &\times \left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)}\,\left[\widehat{U}_1(p)\right]^{\rho_2(h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}.\end{aligned} \tag{D.8}$$

To proceed, we combine the manipulations

$$\begin{aligned}\widehat{V}_i(g)\,\overline{\widehat{V}_i(h)}^{i,g} &= \widehat{V}_i(g)\,\mathsf{K}_i^{\mathfrak{c}(g)}\,\widehat{V}_i(h)\,\mathsf{K}_i^{\mathfrak{c}(h)}\,\mathsf{K}_i^{\mathfrak{c}(h)}\,\mathsf{K}_i^{\mathfrak{c}(g)}\\ &= \widehat{U}_i(g)\,\widehat{U}_i(h)\,\mathsf{K}_i^{\mathfrak{c}(h)}\,\mathsf{K}_i^{\mathfrak{c}(g)}\\ &= e^{\mathrm{i}\phi_i(g,h)}\,\widehat{U}_i(g\,h)\,\mathsf{K}_i^{\mathfrak{c}(g\,h)}, \qquad i=1,2,\end{aligned} \tag{D.9a}$$

with the manipulations

$$\begin{aligned}\left[\widehat{U}_1(p)\right]^{\rho_2(g\,h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g\,h)} &= \left[\widehat{U}_1(p)\right]^{\rho_2(g)+\rho_2(h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)+\rho_1(h)}\\ &= \left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_1(p)\right]^{\rho_2(h)}\\ &\quad\times\left[\widehat{U}_2(p)\right]^{\rho_1(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(h)}\\ &= \left[\widehat{U}_1(p)\right]^{\rho_2(g)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g)}\\ &\quad\times\left[\widehat{U}_1(p)\right]^{\rho_2(h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(h)}.\end{aligned} \tag{D.9b}$$

One is left with

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) &= e^{\mathrm{i}\phi_1(g,h)+\mathrm{i}\phi_2(g,h)+\mathrm{i}\pi\,\rho_1(g)\,\rho_2(h)}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\\ &\quad\times\left[\widehat{U}_1(p)\right]^{\rho_2(g\,h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g\,h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}\\ &\equiv e^{\mathrm{i}\phi_\wedge(g,h)}\,\widehat{U}_\wedge(g\,h),\end{aligned} \tag{D.10a}$$

where

$$\phi_\wedge(g,h) := \phi_1(g,h)+\phi_2(g,h)+\pi\,\rho_1(g)\,\rho_2(h) \tag{D.10b}$$

and

$$\widehat{U}_\wedge(g\,h) := \widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\,\left[\widehat{U}_1(p)\right]^{\rho_2(g\,h)}\,\left[\widehat{U}_2(p)\right]^{\rho_1(g\,h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}. \tag{D.10c}$$

□

D.2 DETAILS FOR EVEN-ODD STACKING

Proof of Eq. (7.19). For the identity $e\in\mathbb{Z}_2^F$, we have

$$\widehat{V}_1(e)=\hat{\mathbb{1}}_1,\qquad \widehat{V}_2(e)=\hat{\mathbb{1}}_2,\qquad \widehat{Q}_2(e)=\hat{\mathbb{1}}_2,\qquad \rho_1(e)=\rho_2(e)=0, \tag{D.11a}$$

which delivers when inserted in Eq. (7.18)

$$\widehat{U}_{\wedge}(e) = \hat{\mathbb{1}}_1 \, \hat{\mathbb{1}}_2 \equiv \hat{\mathbb{1}}_{\wedge}. \tag{D.11b}$$

For the fermion parity $p \in \mathbb{Z}_2^F$, we have

$$\widehat{V}_1(p) = \widehat{U}_1(p), \qquad \widehat{V}_2(p) = \widehat{P}_2 \, \mathrm{i}\hat{\gamma}^{(2)}_{n_2}, \qquad \widehat{Q}_2(p) = \hat{\gamma}^{(2)}_{\infty}, \tag{D.12a}$$

$$\rho_1(p) = 0, \quad \rho_2(p) = 1, \tag{D.12b}$$

where, by definition (6.24), $\widehat{P}_2$ is the fermion parity operators constructed from the generators of the Clifford algebra $C\ell_{n_2-1}$. When these definitions are inserted in Eq. (7.18), one finds

$$\widehat{U}_{\wedge}(p) = \widehat{U}_1(p)\, \widehat{P}_2 \, \mathrm{i}\hat{\gamma}^{(2)}_{n_2} \, \hat{\gamma}^{(2)}_{\infty} = \widehat{U}_1(p)\, \widehat{U}_2(p). \tag{D.12c}$$

□

Proof that definition (7.18) *satisfies Eq.* (7.4) *and, a forteriori, Eq.* (6.17). We begin with the action of $\widehat{U}_{\wedge}(g)$ on a Majorana operator $\hat{\gamma}^{(1)}_i$ associated with representation $\widehat{U}_1$ for any $i = 1, \cdots, n_1$. We have

$$\begin{aligned}\widehat{U}_{\wedge}(g)\, \hat{\gamma}^{(1)}_i \, \widehat{U}^{\dagger}_{\wedge}(g) &= \widehat{V}_{\wedge}(g)\, \widehat{Q}_{\wedge}(g)\, \mathsf{K}^{\mathfrak{c}(g)}_{\wedge} \, \hat{\gamma}^{(1)}_i \, \mathsf{K}^{\mathfrak{c}(g)}_{\wedge} \, \widehat{Q}^{\dagger}_{\wedge}(g)\, \widehat{V}^{\dagger}_{\wedge}(g) \\ &= \widehat{V}_{\wedge}(g)\, \widehat{Q}_{\wedge}(g)\, \overline{\hat{\gamma}^{(1)}_i}^{1,g} \, \widehat{Q}^{\dagger}_{\wedge}(g)\, \widehat{V}^{\dagger}_{\wedge}(g),\end{aligned} \tag{D.13a}$$

where we have used the definition (7.6) to trade complex conjugation by $\mathsf{K}_{\wedge}$ with complex conjugation by K_1. We seek to interchange the order between $\widehat{Q}_{\wedge}(g)$ with $\overline{\hat{\gamma}^{(1)}_i}^{1,g}$. By definition (7.18), $\widehat{Q}_{\wedge}(g)$ only contains the Majorana operator $\widehat{\gamma}^{(2)}_{\infty}$, which anticommutes with $\hat{\gamma}^{(1)}_i$ for any $i = 1, \cdots, n_1$. One finds

$$\begin{aligned}\widehat{U}_{\wedge}(g)\, \hat{\gamma}^{(1)}_i \, \widehat{U}^{\dagger}_{\wedge}(g) &= (-1)^{q_2(g)+\rho_1(g)} \widehat{V}_{\wedge}(g)\, \overline{\hat{\gamma}^{(1)}_i}^{1,g} \, \widehat{V}^{\dagger}_{\wedge}(g) \\ &= (-1)^{q_2(g)+\rho_1(g)} \widehat{V}_1(g)\, \widehat{V}_2(g) \, \left[\widehat{U}_1(p)\right]^{\rho_1(g)} \\ &\quad \times \overline{\hat{\gamma}^{(1)}_i}^{1,g} \\ &\quad \times \left[\widehat{U}^{\dagger}_1(p)\right]^{\rho_1(g)} \widehat{V}^{\dagger}_2(g)\, \widehat{V}^{\dagger}_1(g).\end{aligned} \tag{D.13b}$$

As the fermion parity operator $\widehat{U}_1(p)$ anticommutes with $\hat{\gamma}_i^{(1)}$, interchanging their order results in the phase factor $(-1)^{\rho_1(g)}$. The phase factor $(-1)^{q_2(g)}$ dictates if the unitary operator $\widehat{V}_2^\dagger(g)$ commutes or anticommutes with $\hat{\gamma}_i^{(1)}$ [indeed $q_2(g)$ is the fermion parity of the unitary operator $\widehat{V}_2(g)$ according to Eq. (6.27)]. Hence, interchanging their order delivers

$$\begin{aligned}
\widehat{U}_\wedge(g)\,\hat{\gamma}_i^{(1)}\,\widehat{U}_\wedge^\dagger(g) &= (-1)^{2q_2(g)+2\rho_1(g)}\,\widehat{V}_1(g)\,\overline{\hat{\gamma}_i^{(1)}}^{1,g}\,\widehat{V}_1^\dagger(g) \\
&= \widehat{U}_1(g)\,\hat{\gamma}_i^{(1)}\widehat{U}_1^\dagger(g),
\end{aligned} \tag{D.13c}$$

for any $i=1,\cdots,n_1$. This is nothing but Eq. (7.4) for representation $\widehat{U}_1$.

Second, we need to evaluate the action of $\widehat{U}_\wedge(g)$ on the Majorana operator $\hat{\gamma}_i^{(2)}$ associated with representation $\widehat{U}_2$ for any $i=1,\cdots,n_2$. We have

$$\begin{aligned}
\widehat{U}_\wedge(g)\,\hat{\gamma}_i^{(2)}\,\widehat{U}_\wedge^\dagger(g) &= \widehat{V}_\wedge(g)\,\widehat{Q}_\wedge(g)\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}\,\hat{\gamma}_i^{(2)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}\,\widehat{Q}_\wedge^\dagger(g)\,\widehat{V}_\wedge^\dagger(g) \\
&= \widehat{V}_\wedge(g)\,\widehat{Q}_\wedge(g)\,\overline{\hat{\gamma}_i^{(2)}}^{2,g}\,\widehat{Q}_\wedge^\dagger(g)\,\widehat{V}_\wedge^\dagger(g),
\end{aligned} \tag{D.14a}$$

where we have used the definition (7.6) to trade complex conjugation by $\mathsf{K}_\wedge$ with complex conjugation by K_2. By definition (7.18), $\widehat{Q}_\wedge(g)$ contains only the Majorana operator $\hat{\gamma}_\infty^{(2)}$, which anticommutes with $\hat{\gamma}_i^{(2)}$ for any $i=1,\cdots,n_2$. One finds

$$\begin{aligned}
\widehat{U}_\wedge(g)\,\hat{\gamma}_i^{(2)}\,\widehat{U}_\wedge^\dagger(g) &= (-1)^{q_2(g)+\rho_1(g)}\widehat{V}_\wedge(g)\,\overline{\hat{\gamma}_i^{(2)}}^{2,g}\,\widehat{V}_\wedge^\dagger(g) \\
&= (-1)^{q_2(g)+\rho_1(g)}\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_1(p)\right]^{\rho_1(g)} \\
&\quad\times\overline{\hat{\gamma}_i^{(2)}}^{2,g} \\
&\quad\times\left[\widehat{U}_1^\dagger(p)\right]^{\rho_1(g)}\,\widehat{V}_2^\dagger(g)\,\widehat{V}_1^\dagger(g) \\
&= (-1)^{q_2(g)+\rho_1(g)}\,\widehat{V}_2(g)\,\widehat{V}_1(g)\,\left[\widehat{U}_1(p)\right]^{\rho_1(g)} \\
&\quad\times\overline{\hat{\gamma}_i^{(2)}}^{2,g} \\
&\quad\times\left[\widehat{U}_1^\dagger(p)\right]^{\rho_1(g)}\,\widehat{V}_1^\dagger(g)\,\widehat{V}_2^\dagger(g),
\end{aligned} \tag{D.14b}$$

where, in reaching the last line, we have interchanged $\widehat{V}_1(g)$ and $\widehat{V}_2(g)$ on the left and $\widehat{V}_1^\dagger(g)$ and $\widehat{V}_2^\dagger(g)$ on the right. Both interchanges cost the same multiplicative phase factor ± 1 and cancel each other. The fermion parity operator $\widehat{U}_1(p)$ itself carries even fermion parity and therefore commute with $\hat{\gamma}_i^{(2)}$ for any $i = 1, \cdots, n_2$, i.e.,

$$\widehat{U}_\wedge(g)\,\hat{\gamma}_i^{(2)}\,\widehat{U}_\wedge^\dagger(g) = (-1)^{q_2(g)+\rho_1(g)}\,\widehat{V}_2(g)\,\widehat{V}_1(g)\,\overline{\hat{\gamma}_i^{(2)}}^{2,g}\,\widehat{V}_1^\dagger(g)\,\widehat{V}_2^\dagger(g), \tag{D.14c}$$

Passing operator $\widehat{V}_1(g)$ through $\hat{\gamma}_i^{(2)}$ costs the phase factor $(-1)^{\rho_1(g)}$. Hence, one obtains

$$\begin{aligned}\widehat{U}_\wedge(g)\,\hat{\gamma}_i^{(2)}\,\widehat{U}_\wedge^\dagger(g) &= (-1)^{q_2(g)}\,\widehat{V}_2(g)\,\overline{\hat{\gamma}_i^{(2)}}^{2,g}\,\widehat{V}_2^\dagger(g)\\ &= (-1)^{q_2(g)}\,\widehat{V}_2(g)\,\overline{\hat{\gamma}_i^{(2)}}^{2,g}\left[\hat{\gamma}_\infty^{(2)}\right]^{q_2(g)}\left[\hat{\gamma}_\infty^{(2)}\right]^{q_2(g)}\widehat{V}_2^\dagger(g)\\ &= \widehat{V}_2(g)\left[\hat{\gamma}_\infty^{(2)}\right]^{q_2(g)}\overline{\hat{\gamma}_i^{(2)}}^{2,g}\left[\hat{\gamma}_\infty^{(2)}\right]^{q_2(g)}\widehat{V}_2^\dagger(g),\end{aligned} \tag{D.14d}$$

where we have used the fact that $\hat{\gamma}_\infty^{(2)}$ squares to identity to reach the second equality and that $\left[\hat{\gamma}_\infty^{(2)}\right]^{q_2(g)}$ anticommutes with $\hat{\gamma}_i^{(2)}$ to reach the last equality. Recalling Eq. (6.27), one observes

$$\begin{aligned}\widehat{U}_\wedge(g)\,\hat{\gamma}_i^{(2)}\,\widehat{U}_\wedge^\dagger(g) &= \widehat{V}_2(g)\left[\hat{\gamma}_\infty^{(2)}\right]^{q_2(g)}\overline{\hat{\gamma}_i^{(2)}}^{2,g}\left[\hat{\gamma}_\infty^{(2)}\right]^{q_2(g)}\widehat{V}_2^\dagger(g)\\ &= \widehat{V}_2(g)\,\widehat{Q}_2(g)\,\mathsf{K}_2^{\mathfrak{c}(g)}\,\hat{\gamma}_i^{(2)}\,\mathsf{K}_2^{\mathfrak{c}(g)}\,\widehat{Q}_2^\dagger(g)\,\widehat{V}_2^\dagger(g)\\ &= \widehat{U}_2(g)\,\hat{\gamma}_i^{(2)}\,\widehat{U}_2^\dagger(g),\end{aligned} \tag{D.14e}$$

which is nothing but Eq. (7.4) for representation $\widehat{U}_2$. □

Remark. For any $g \in G_f$, the definition (7.18) guarantees that $\widehat{U}_\wedge(g)$ is of even fermion parity. This property is inherited from the facts that $\widehat{U}_2(g)$ is of even fermion parity according to Eq. (6.27) and by the presence of the factor $\left[\widehat{U}_1(p)\,\hat{\gamma}_\infty^{(2)}\right]^{\rho_1(g)}$ that compensates for the fermion parity of the operator $\widehat{V}_1(g)$.

Proof of Eq. (7.20a). When representations $\widehat{U}_\wedge(g)$ and $\widehat{U}_\wedge(h)$ of two elements $g, h \in G_f$ are composed, we obtain from definition (7.18)

$$\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = \widehat{V}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\left[\widehat{U}_1(p)\,\hat{\gamma}_\infty^{(2)}\right]^{\rho_1(g)}\mathsf{K}_\wedge^{\mathfrak{c}(g)}$$

$$
\begin{aligned}
&\times \widehat{V}_1(h)\,\widehat{V}_2(h)\,\widehat{Q}_2(h)\,\left[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(h)}\,\mathsf{K}^{\mathfrak{c}(h)}_{\wedge}\\
&=\widehat{V}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\left[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}\\
&\times \overline{\widehat{V}_1(h)}^{1,g}\,\overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g}\,\left[\widehat{U}_1(p)\,\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)}\,\mathsf{K}^{\mathfrak{c}(g\,h)}_{\wedge}.
\end{aligned}
\tag{D.15}
$$

We can safely commute the operator $\overline{\widehat{V}_1(h)}^{1,g}$ to the left of the operator $\widehat{V}_2(g)$. This is so for two reasons. First, $\widehat{V}_2(g)\,\widehat{Q}_2(g)$ has even fermion parity by definition and thus commutes with $\overline{\widehat{V}_1(h)}^{1,g}$. Second, the product $\left[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}$ commutes with $\overline{\widehat{V}_1(h)}^{1,g}$ since both $\widehat{U}_1(p)$ and $\hat{\gamma}^{(2)}_{\infty}$ anticommute with $\overline{\widehat{V}_1(h)}^{1,g}$ if $\rho_1(g)=1$. One is left with

$$
\begin{aligned}
\widehat{U}_{\wedge}(g)\,\widehat{U}_{\wedge}(h) &= \widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\left[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}\\
&\times \overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g}\,\left[\widehat{U}_1(p)\,\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)}\,\mathsf{K}^{\mathfrak{c}(g\,h)}_{\wedge}\\
&= e^{\mathrm{i}\phi_1(g,h)}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\left[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}\\
&\times \overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g}\,\left[\widehat{U}_1(p)\,\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)}\,\mathsf{K}^{\mathfrak{c}(g\,h)}_{\wedge},
\end{aligned}
\tag{D.16}
$$

where in reaching the last line we made use of Eq. (D.9a) to trade $\widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}$ with $\widehat{V}_1(g\,h)$. We can further safely commute the operator $\widehat{U}_1(p)$ to the left of $\mathsf{K}^{\mathfrak{c}(g\,h)}_{\wedge}$ since $\widehat{U}_1(p)$ is an even parity operator from representation $\widehat{U}_1$ and therefore commutes with all operators from representation $\widehat{U}_2$. We find

$$
\begin{aligned}
\widehat{U}_{\wedge}(g)\,\widehat{U}_{\wedge}(h) &= e^{\mathrm{i}\phi_1(g,h)}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}\\
&\times \overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g}\,\left[\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)}\,\left[\widehat{U}_1(p)\right]^{\rho_1(g)+\rho_1(h)}\,\mathsf{K}^{\mathfrak{c}(g\,h)}_{\wedge}\\
&= e^{\mathrm{i}\phi_1(g,h)}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}
\end{aligned}
$$

$$\times \overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g}\,\left[\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)}\,\left[\widehat{U}_1(p)\right]^{\rho_1(g\,h)}\,\mathsf{K}_{\wedge}^{\mathfrak{c}(g\,h)}, \tag{D.17}$$

where the fact that ρ_1 is a group homomorphism is used in reaching the last line. We shall pass the operator $\left[\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}$ to the right of $\overline{\widehat{Q}_2(h)}^{2,g}$. Doing so costs the phase factor $(-1)^{\rho_1(g)\,q_2(h)}$ since $\hat{\gamma}^{(2)}_{\infty}$ commutes with $\widehat{Q}_2(g)$ for any g and the operator $\overline{\widehat{V}_2(h)}^{2,g}$ has fermion parity $q(h)$ [recall definition (6.27)]. We obtain the expression

$$\begin{aligned}
\widehat{U}_{\wedge}(g)\,\widehat{U}_{\wedge}(h) =&\, e^{\mathrm{i}\phi_1(g,h)+\mathrm{i}\pi\rho_1(g)\,q_2(h)}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g}\\
&\times\left[\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}\left[\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)}\left[\widehat{U}_1(p)\right]^{\rho_1(g\,h)}\mathsf{K}_{\wedge}^{\mathfrak{c}(g\,h)}\\
=&\, e^{\mathrm{i}\phi_1(g,h)+\mathrm{i}\phi_2(g,h)+\mathrm{i}\pi\rho_1(g)\,q_2(h)}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\,\widehat{Q}_2(g\,h)\\
&\times\left[\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}\left[\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)}\left[\widehat{U}_1(p)\right]^{\rho_1(g\,h)}\mathsf{K}_{\wedge}^{\mathfrak{c}(g\,h)},
\end{aligned} \tag{D.18a}$$

where in reaching the last line we did the manipulations

$$\begin{aligned}
\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g} &= \widehat{V}_2(g)\,\widehat{Q}_2(g)\,\mathsf{K}_2^{\mathfrak{c}(g)}\,\widehat{V}_2(h)\,\widehat{Q}_2(h)\,\mathsf{K}_2^{\mathfrak{c}(g)}\\
&= \widehat{U}_2(g)\,\widehat{U}_2(h)\,\mathsf{K}_2^{\mathfrak{c}(h)}\,\mathsf{K}_2^{\mathfrak{c}(g)}\\
&= e^{\mathrm{i}\phi_2(g,h)}\,\widehat{U}_2(g\,h)\,\mathsf{K}_2^{\mathfrak{c}(g\,h)}\\
&= e^{\mathrm{i}\phi_2(g,h)}\,\widehat{V}_2(g\,h).
\end{aligned} \tag{D.18b}$$

The identity (6.30b) allows one to trade $\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}$ for $(-1)^{\left(\mathfrak{c}(g)+q_2(g)+\rho_2(g)\right)}\,\hat{\gamma}^{(2)}_{\infty}$. Because the fermion parity of $\widehat{U}_1(p)$ is even, $\left[\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}\left[\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(h)}\left[\widehat{U}_1(p)\right]^{\rho_1(g\,h)}$ can be replaced with $\left[\widehat{U}_1(p)\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g\,h)}$. One is left with

$$\begin{aligned}
\widehat{U}_{\wedge}(g)\,\widehat{U}_{\wedge}(h) =&\, e^{\mathrm{i}\phi_1(g,h)+\mathrm{i}\phi_2(g,h)+\mathrm{i}\pi\rho_1(g)\,q_2(h)+\mathrm{i}\pi\left[\mathfrak{c}(g)+q_2(g)+\rho_2(g)\right]\rho_1(h)}\\
&\times\widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\,\widehat{Q}_2(g\,h)\,\left[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g\,h)}\mathsf{K}_{\wedge}^{\mathfrak{c}(g\,h)}\\
\equiv&\, e^{\mathrm{i}\phi_{\wedge}(g,h)}\,\widehat{U}_{\wedge}(g\,h),
\end{aligned} \tag{D.19a}$$

where

$$\phi_{\wedge}(g,h) := \phi_1(g,h) + \phi_2(g,h) + \pi\rho_1(g)\, q_2(h) + \pi\rho_1(h)\, [\mathfrak{c}(g) + q_2(g) + \rho_2(g)]\,, \tag{D.19b}$$

and

$$\widehat{U}_{\wedge}(g\,h) := \widehat{V}_1(g\,h)\, \widehat{V}_2(g\,h)\, \widehat{Q}_2(g\,h)\, \left[\widehat{U}_1(p)\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g\,h)} \mathsf{K}_{\wedge}^{\mathfrak{c}(g\,h)}. \tag{D.19c}$$

□

Proof of (7.21). We shall show that the expression (D.19b) for $\phi_{\wedge}(g,h)$ is gauge equivalent to

$$\phi'_{\wedge}(g,h) := \phi_1(g,h) + \phi_2(g,h) + \pi\rho_1(g)\, \rho_2(h) + \pi\rho_1(g)\, \mathfrak{c}(h). \tag{D.20}$$

In order to transform Eq. (D.19b) to Eq. (D.20), we shall trade the 1-cochain q_2 with the 1-cochain ρ_2. We will make use of the identity (6.30b). First, note that the Majorana operator $\hat{\gamma}^{(2)}_{\infty}$ may be odd or even under complex conjugation by K_2. We introduce an auxiliary index ζ to label its eigenvalue under conjugation by K_2 through

$$\overline{\hat{\gamma}^{(2)}_{\infty}}^{\,2} := r(-1)^{\zeta}\, \hat{\gamma}^{(2)}_{\infty}. \tag{D.21}$$

Second, depending on whether an element $g \in G_f$ is unitary $[\mathfrak{c}(g) = 0]$ or antiunitary $[\mathfrak{c}(g) = 1]$, the identity (6.30b) becomes

$$\hat{\gamma}^{(2)}_{\infty} = (-1)^{q_2(g)+\rho_2(g)}\, \hat{\gamma}^{(2)}_{\infty}, \tag{D.22a}$$

or

$$\hat{\gamma}^{(2)}_{\infty} = (-1)^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)+\zeta}\, \hat{\gamma}^{(2)}_{\infty}, \tag{D.22b}$$

respectively. Since both sides contain the same Majorana operator $\hat{\gamma}^{(2)}_{\infty}$, the phase factors must be equal. In other words, we have the identification

$$q_2(g) = \mathfrak{c}(g)(1+\zeta) + \rho_2(g) \bmod 2. \tag{D.22c}$$

We are going to show that the last four terms on the right-hand side of Eq. (D.19b), namely

$$\pi\rho_1(g)\,q_2(h)+\pi\rho_1(h)\,[\mathfrak{c}(g)+q_2(g)+\rho_2(g)]\,, \tag{D.23a}$$

are gauge equivalent to the last two terms on the right-hand side of Eq. (D.20), namely

$$\pi\rho_1(g)\,\rho_2(h)+\pi\rho_1(g)\,\mathfrak{c}(h). \tag{D.23b}$$

Upon replacing $q_2(g)$ with the right-hand side of Eq. (D.22c) in Eq. (D.23a), one finds

$$\begin{aligned}\pi\rho_1(g)\,q_2(h)+\pi\rho_1(h)\,[\mathfrak{c}(g)+q_2(g)+\rho_2(g)] &= \pi\rho_1(g)\,\rho_2(h)+\pi\rho_1(g)\,\mathfrak{c}(h)\\ &\quad+\pi\zeta\,\left[\rho_1(g)\,\mathfrak{c}(h)+\rho_1(h)\,\mathfrak{c}(g)\right] \mod 2\pi.\end{aligned} \tag{D.24}$$

Modulo 2π, the expression (D.24) differs from the expression (D.23b) by

$$\pi\zeta\,\left[\rho_1(g)\,\mathfrak{c}(h)+\rho_1(h)\,\mathfrak{c}(g)\right]. \tag{D.25}$$

The final step of the proof consists in showing that $\rho_1(g)\,\mathfrak{c}(h)+\rho_1(h)\,\mathfrak{c}(g)$ is a coboundary (i.e., a pure gauge), i.e., there exist a 1-cochain $\alpha: G_f\to\{0,1\}$ such that

$$\rho_1(g)\,\mathfrak{c}(h)+\rho_1(h)\,\mathfrak{c}(g)=\alpha(g)+\alpha(h)-\alpha(g\,h). \tag{D.26a}$$

This is achieved by observing that both ρ_1 and $\mathfrak{c}$ are group homomorphisms and as such satisfy

$$\rho_1(g)+\rho_1(h)=\rho_1(g\,h), \qquad \mathfrak{c}(g)+\mathfrak{c}(h)=\mathfrak{c}(g\,h). \tag{D.26b}$$

Hence, we can write

$$\begin{aligned}\rho_1(g)\,\mathfrak{c}(h)+\rho_1(h)\,\mathfrak{c}(g) &= \underbrace{\rho_1(g)\,\mathfrak{c}(h)+\rho_1(h)\,\mathfrak{c}(g)+\rho_1(g)\,\mathfrak{c}(g)+\rho_1(h)\,\mathfrak{c}(h)}_{=\rho_1(g\,h)\,\mathfrak{c}(g\,h)}\\ &\quad-\rho_1(g)\,\mathfrak{c}(g)-\rho_1(h)\,\mathfrak{c}(h)\\ &=\rho_1(g\,h)\mathfrak{c}(g\,h)-\rho_1(g)\,\mathfrak{c}(g)-\rho_1(h)\,\mathfrak{c}(h).\end{aligned} \tag{D.26c}$$

By choosing the 1-cochain

$$\alpha(g):=\rho_1(g)\,\mathfrak{c}(g), \tag{D.26d}$$

the expression (D.25) is indeed proportional to a coboundary, i.e., it can be gauged away. As promised, $\phi_{\wedge}(g,h)$ from Eq. (D.19b) is gauge equivalent to $\phi'_{\wedge}(g,h)$ from Eq. (D.20). □

D.3 DETAILS FOR ODD-ODD STACKING

Proof of Eq. (7.26). For the identity $e \in \mathbb{Z}_2^F$, we have

$$\widehat{V}_1(e) = \hat{\mathbb{1}}_1, \quad \widehat{Q}_1(e) = \hat{\mathbb{1}}_1, \quad \widehat{V}_2(e) = \hat{\mathbb{1}}_2, \quad \widehat{Q}_2(e) = \hat{\mathbb{1}}_2, \quad \rho_1(e) = \rho_2(e) = 0, \tag{D.27a}$$

which delivers when inserted in Eq. (7.25)

$$\widehat{U}_{\wedge}(e) = \hat{\mathbb{1}}_1\, \hat{\mathbb{1}}_2 \equiv \hat{\mathbb{1}}_{\wedge}. \tag{D.27b}$$

For the fermion parity $p \in \mathbb{Z}_2^F$, we have

$$\begin{aligned} &\widehat{V}_1(p) = \widehat{P}_1\, \mathrm{i}\hat{\gamma}^{(1)}_{n_1}, \quad \widehat{Q}_1(p) = \hat{\gamma}^{(1)}_{\infty}, \\ &\widehat{V}_2(p) = \widehat{P}_2\, \mathrm{i}\hat{\gamma}^{(2)}_{n_2}, \quad \widehat{Q}_2(p) = \hat{\gamma}^{(2)}_{\infty}, \quad \rho_1(p) = \rho_2(p) = 1. \end{aligned} \tag{D.28a}$$

When these definitions are inserted in Eq. (7.25), one finds

$$\widehat{U}_{\wedge}(p) = -\mathrm{i}\widehat{P}_1\, \mathrm{i}\hat{\gamma}^{(1)}_{n_1}\, \widehat{P}_2\, \mathrm{i}\hat{\gamma}^{(2)}_{n_2} = \widehat{P}_1\, \widehat{P}_2\, \mathrm{i}\hat{\gamma}^{(1)}_{n_1}\, \hat{\gamma}^{(2)}_{n_2}\,. \tag{D.28b}$$

The choice of the multiplicative phase factor $(-\mathrm{i})^{\delta_{g,p}}$ in Eq. (7.25) is not unique since representation $\widehat{U}(g)$ of any element $g \in G_f$ is defined up to a multiplicative U(1) phase. We observe that the multiplicative factor $(-\mathrm{i})^{\delta_{g,p}}$ in Eq. (7.25) ensures that the stacked representation $\widehat{U}_{\wedge}(p)$ is Hermitian in compliance with the “gauge” choice made in definition (6.24). □

Remark. The Majorana operators $\hat{\gamma}^{(1)}_{\infty}$ and $\hat{\gamma}^{(2)}_{\infty}$ do not enter the definition (7.25) of the stacked representation $\widehat{U}_{\wedge}$. This is expected as the stacked representation $\widehat{U}_{\wedge}$ has $[\mu_{\wedge}] = 0$. Accordingly, $\widehat{U}_{\wedge}$ should be constructed solely out of the even number $n_1 + n_2$ of Majorana operators spanning the fermionic Fock space of the stacked boundary [recall definition (6.19)].

Remark. The definition (7.25) is not symmetric under exchange of the labels 1 and 2, as is to be expected by inspection of Eq. (7.7).

Proof that definition (7.25) satisfies Eq. (7.4) *and, a forteriori, Eq.* (6.17). To see this, we shall consider for any $g \in G_f$ four cases, namely conjugation by $\widehat{U}_\wedge(g)$

(i) of Majorana operators $\hat{\gamma}_i^{(1)}$ with $i = 1, \cdots, n_1 - 1$,

(ii) of Majorana operators $\hat{\gamma}_i^{(2)}$ with $i = 1, \cdots, n_2 - 1$,

(iii) of Majorana operator $\hat{\gamma}_{n_1}^{(1)}$,

(iv) and of Majorana operator $\hat{\gamma}_{n_2}^{(2)}$.

We will verify that for each of these cases the consistency condition (7.4) is satisfied.

(i) Conjugating any Majorana operator $\hat{\gamma}_i^{(1)}$ with $i = 1, \cdots, n_1 - 1$ by $\widehat{U}_\wedge(g)$ gives

$$\begin{aligned}
\widehat{U}_\wedge(g)\,\hat{\gamma}_i^{(1)}\,\widehat{U}_\wedge^\dagger(g) = &(-\mathrm{i})^{\delta_{g,p}}\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)} \\
&\times \left[\hat{\gamma}_{n_1}^{(1)}\right]^{q_1(g)+\rho_1(g)} \left[\hat{\gamma}_{n_2}^{(2)}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \\
&\times \mathsf{K}_\wedge^{\mathfrak{c}(g)}\,\hat{\gamma}_i^{(1)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g)} \\
&\times \left[\hat{\gamma}_{n_2}^{(2)}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \left[\hat{\gamma}_{n_1}^{(1)}\right]^{q_1(g)+\rho_1(g)} \\
&\times \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\widehat{V}_2^\dagger(g)\,\widehat{V}_1^\dagger(g)\,(+\mathrm{i})^{\delta_{g,p}} \\
= &\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)} \\
&\times \left[\hat{\gamma}_{n_1}^{(1)}\right]^{q_1(g)+\rho_1(g)} \left[\hat{\gamma}_{n_2}^{(2)}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \\
&\times \overline{\hat{\gamma}_i^{(1)}}^{1,g} \\
&\times \left[\hat{\gamma}_{n_2}^{(2)}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \left[\hat{\gamma}_{n_1}^{(1)}\right]^{q_1(g)+\rho_1(g)} \\
&\times \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\widehat{V}_2^\dagger(g)\,\widehat{V}_1^\dagger(g),
\end{aligned} \tag{D.29}$$

where we used the fact that action of $\mathsf{K}_\wedge$ is identical to that of K_1 for $\hat{\gamma}_i^{(1)}$ with $i = 1, \cdots, n_1 - 1$ [definition (7.7)]. We shall pass all the remaining terms except $\widehat{V}_1(g)$ to the right of $\overline{\hat{\gamma}_i^{(1)}}^{1,g}$. Operators $\widehat{U}_\wedge(p)$, $\hat{\gamma}_{n_1}^{(1)}$, and $\hat{\gamma}_{n_2}^{(2)}$ all anticommute with $\hat{\gamma}_i^{(1)}$ for $i = 1, \cdots, n_1 - 1$. Therefore, the induced multiplicative phase factor is $(-1)^{q_1(g)+q_2(g)}$. As the operator $\widehat{V}_2(g)$ may either anticommute or commute with $\hat{\gamma}_i^{(1)}$ depending on the value taken by $q_2(g)$, another multiplicative phase factor $(-1)^{q_2(g)}$ is induced. We thus find

$$\begin{aligned}\widehat{U}_{\wedge}(g)\,\hat{\gamma}^{(1)}_{i}\,\widehat{U}^{\dagger}_{\wedge}(g) &= (-1)^{q_1(g)}\,\widehat{V}_1(g)\,\overline{\hat{\gamma}^{(1)}_{i}}^{1,g}\,\widehat{V}^{\dagger}_1(g)\\ &= (-1)^{q_1(g)}\,\widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\overline{\hat{\gamma}^{(1)}_{i}}^{1,g}\,\widehat{V}^{\dagger}_1(g). \end{aligned} \tag{D.30}$$

The identity was represented by the square of $\hat{\gamma}^{(1)}_{\infty}$ in order to reach the last equality. Passing $\hat{\gamma}^{(1)}_{\infty}$ to the right of $\overline{\hat{\gamma}^{(1)}_{i}}^{1,g}$ induces the multiplicative phase factor $(-1)^{q_1(g)}$ i.e.,

$$\begin{aligned}\widehat{U}_{\wedge}(g)\,\hat{\gamma}^{(1)}_{i}\,\widehat{U}^{\dagger}_{\wedge}(g) &= \widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\overline{\hat{\gamma}^{(1)}_{i}}^{1,g}\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\widehat{V}^{\dagger}_1(g)\\ &= \widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\mathsf{K}^{\mathfrak{c}(g)}_1\,\hat{\gamma}^{(1)}_{i}\,\mathsf{K}^{\mathfrak{c}(g)}_1\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\widehat{V}^{\dagger}_1(g). \end{aligned} \tag{D.31}$$

According to the definition (6.27), we conclude that

$$\widehat{U}_{\wedge}(g)\,\hat{\gamma}^{(1)}_{i}\,\widehat{U}^{\dagger}_{\wedge}(g) = \widehat{U}_1(g)\,\hat{\gamma}^{(1)}_{i}\,\widehat{U}^{\dagger}_1(g). \tag{D.32}$$

The consistency condition (7.4) for $\hat{\gamma}^{(1)}_{i}$ with $i=1,\cdots,n_1-1$, thus holds.

(ii) Conjugating any Majorana operator $\hat{\gamma}^{(2)}_{i}$ with $i=1,\cdots,n_2-1$ by $\widehat{U}_{\wedge}(g)$ is achieved by repeating all the steps between Eqs. (D.29) and (D.32) with $\hat{\gamma}^{(1)}_{i}$ substituted by $\hat{\gamma}^{(2)}_{i}$ with the intermediary steps

$$\begin{aligned}\widehat{U}_{\wedge}(g)\,\hat{\gamma}^{(2)}_{i}\,\widehat{U}^{\dagger}_{\wedge}(g) &= (-1)^{q_2(g)}\,\widehat{V}_2(g)\,\overline{\hat{\gamma}^{(2)}_{i}}^{2,g}\,\widehat{V}^{\dagger}_2(g)\\ &= (-1)^{q_2(g)}\,\widehat{V}_2(g)\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\overline{\hat{\gamma}^{(2)}_{i}}^{2,g}\,\widehat{V}^{\dagger}_2(g)\\ &= \widehat{U}_2(g)\,\hat{\gamma}^{(2)}_{i}\,\widehat{U}^{\dagger}_2(g). \end{aligned} \tag{D.33}$$

The consistency condition (7.4) for $\hat{\gamma}^{(2)}_{i}$ with $i=1,\cdots,n_2-1$ thus holds.

(iii) Conjugating $\hat{\gamma}^{(1)}_{n_1}$ by $\widehat{U}_{\wedge}(g)$ delivers

$$\begin{aligned}\widehat{U}_{\wedge}(g)\,\hat{\gamma}^{(1)}_{n_1}\,\widehat{U}^{\dagger}_{\wedge}(g) = (-\mathrm{i})^{\delta_{g,p}}\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\\ \times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\end{aligned}$$

$$
\begin{aligned}
&\times \mathsf{K}_{\wedge}^{\mathfrak{c}(g)}\,\hat{\gamma}^{(1)}_{n_1}\,\mathsf{K}_{\wedge}^{\mathfrak{c}(g)}\\
&\times \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\\
&\times \left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\widehat{V}^{\dagger}_2(g)\,\widehat{V}^{\dagger}_1(g)\,(+\mathrm{i})^{\delta_{g,p}}\\
=&\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\
&\times \hat{\gamma}^{(1)}_{n_1}\\
&\times \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\\
&\times \left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\widehat{V}^{\dagger}_2(g)\,\widehat{V}^{\dagger}_1(g),
\end{aligned}
\tag{D.34a}
$$

where we used the fact that, according to definition (7.7), $\hat{\gamma}^{(1)}_{n_1}$ is even under complex conjugation by $\mathsf{K}_{\wedge}$.

We shall reorder the terms. The operators $\widehat{U}_{\wedge}(p)$ and $\hat{\gamma}^{(2)}_{n_2}$ anticommute with $\hat{\gamma}^{(1)}_{n_1}$, while $\hat{\gamma}^{(1)}_{n_1}$ commutes with itself. Hence, conjugation of $\hat{\gamma}^{(1)}_{n_1}$ by $\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\times\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}$ amounts to multiplying $\hat{\gamma}^{(1)}_{n_1}$ by the phase factor $(-1)^{q_2(g)+\rho_1(g)}$. Conjugating $\hat{\gamma}^{(1)}_{n_1}$ by $\widehat{V}_2(g)$ amounts to multiplying $\hat{\gamma}^{(1)}_{n_1}$ by the phase factor $(-1)^{q_2(g)}$. We thus arrive at

$$
\begin{aligned}
\widehat{U}_{\wedge}(g)\,\hat{\gamma}^{(1)}_{n_1}\,\widehat{U}^{\dagger}_{\wedge}(g) &= (-1)^{\rho_1(g)}\,\widehat{V}_1(g)\,\hat{\gamma}^{(1)}_{n_1}\,\widehat{V}^{\dagger}_1(g)\\
&= (-1)^{\rho_1(g)}\,\widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\hat{\gamma}^{(1)}_{n_1}\,\widehat{V}^{\dagger}_1(g)\\
&= (-1)^{\rho_1(g)+q_1(g)}\,\widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\hat{\gamma}^{(1)}_{n_1}\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\widehat{V}^{\dagger}_1(g).
\end{aligned}
\tag{D.34b}
$$

Here, we traded the identity by the the square of $\hat{\gamma}^{(1)}_{\infty}$ and used the fact that $\hat{\gamma}^{(1)}_{\infty}$ anticommutes with $\hat{\gamma}^{(1)}_{n_1}$.

With the help of the identity (6.30c), we may trade $(-1)^{\rho_1(g)+q_1(g)}\,\hat{\gamma}^{(1)}_{n_1}$ for $\overline{\hat{\gamma}^{(1)}_{n_1}}^{1,g}$. Doing so delivers

$$\begin{aligned}
\widehat{U}_{\wedge}(g)\,\hat{\gamma}^{(1)}_{n_1}\,\widehat{U}^{\dagger}_{\wedge}(g) &= (-1)^{\rho_1(g)+q_1(g)}\,\widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\hat{\gamma}^{(1)}_{n_1}\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\widehat{V}^{\dagger}_1(g)\\
&= \widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\overline{\hat{\gamma}^{(1)}_{n_1}}^{1,g}\,\left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}\,\widehat{V}^{\dagger}_1(g)\\
&= \widehat{U}_1(g)\,\hat{\gamma}^{(1)}_{n_1}\,\widehat{U}^{\dagger}_1(g),
\end{aligned} \tag{D.34c}$$

where we used the definition (6.27). The consistency condition (7.4) for $\hat{\gamma}^{(1)}_{n_1}$ thus holds.

(iv) Conjugating $\hat{\gamma}^{(2)}_{n_2}$ by $\widehat{U}_{\wedge}(g)$ delivers

$$\begin{aligned}
\widehat{U}_{\wedge}(g)\,\hat{\gamma}^{(2)}_{n_2}\,\widehat{U}^{\dagger}_{\wedge}(g) = (-\mathrm{i})^{\delta_{g,p}}\,&\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\
&\times \mathsf{K}^{\mathfrak{c}(g)}_{\wedge}\,\hat{\gamma}^{(2)}_{n_2}\,\mathsf{K}^{\mathfrak{c}(g)}_{\wedge}\\
&\times \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\\
&\times \left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\widehat{V}^{\dagger}_2(g)\,\widehat{V}^{\dagger}_1(g)\,(+\mathrm{i})^{\delta_{g,p}}\\
= \widehat{V}_1(g)\,&\widehat{V}_2(g)\,\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\\
&\times \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\
&\times (-1)^{\mathfrak{c}(g)}\,\hat{\gamma}^{(2)}_{n_2}\\
&\times \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\\
&\times \left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\widehat{V}^{\dagger}_2(g)\,\widehat{V}^{\dagger}_1(g),
\end{aligned} \tag{D.35a}$$

where we used the fact that, according to definition (7.7), $\hat{\gamma}^{(2)}_{n_2}$ is odd under complex conjugation by $\mathsf{K}_{\wedge}$.

We shall reorder terms. The operators $\widehat{U}_{\wedge}(p)$ and $\hat{\gamma}^{(1)}_{n_1}$ anticommute with $\hat{\gamma}^{(2)}_{n_2}$, while $\hat{\gamma}^{(2)}_{n_2}$ commutes with itself. Hence, conjugating $\hat{\gamma}^{(2)}_{n_2}$ by $\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)} \times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}$

$\times \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}$ returns $\hat{\gamma}^{(2)}_{n_2}$ multiplied by the phase factor $(-1)^{q_1(g)+\rho_2(g)+\mathfrak{c}(g)}$. We can simultaneously bring $\widehat{V}_1(g)$ to the right of $\widehat{V}_2(g)$ and $\widehat{V}^\dagger_1(g)$ to the left of $\widehat{V}^\dagger_2(g)$ at no cost of a multiplicative factor. Conjugation of $\hat{\gamma}^{(2)}_{n_2}$ by $\widehat{V}_1(g)$ amounts to multiplying $\hat{\gamma}^{(2)}_{n_2}$ by the phase factor $(-1)^{q_1(g)}$. We thus arrive at

$$\begin{aligned}
\widehat{U}_\wedge(g)\,\hat{\gamma}^{(2)}_{n_2}\,\widehat{U}^\dagger_\wedge(g) &= (-1)^{\rho_2(g)}\,\widehat{V}_2(g)\,\hat{\gamma}^{(2)}_{n_2}\,\widehat{V}^\dagger_2(g)\\
&= (-1)^{\rho_2(g)}\,\widehat{V}_2(g)\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\hat{\gamma}^{(2)}_{n_2}\,\widehat{V}^\dagger_2(g)\\
&= (-1)^{\rho_2(g)+q_2(g)}\,\widehat{V}_2(g)\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\hat{\gamma}^{(2)}_{n_2}\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\widehat{V}^\dagger_2(g).
\end{aligned} \tag{D.35b}$$

Here, we traded the identity by the the square of $\hat{\gamma}^{(2)}_{\infty}$ and used the fact $\hat{\gamma}^{(2)}_{\infty}$ anticommutes with $\hat{\gamma}^{(2)}_{n_2}$. With the help of the identity (6.30c), we may trade $(-1)^{\rho_2(g)+q_2(g)}\hat{\gamma}^{(2)}_{n_2}$ for $\overline{\hat{\gamma}^{(2)}_{n_2}}^{2,g}$. Doing so delivers

$$\begin{aligned}
\widehat{U}_\wedge(g)\,\hat{\gamma}^{(2)}_{n_2}\,\widehat{U}^\dagger_\wedge(g) &= (-1)^{\rho_2(g)+q_2(g)}\widehat{V}_1(g)\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\hat{\gamma}^{(2)}_{n_2}\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\widehat{V}^\dagger_2(g)\\
&= \widehat{V}_2(g)\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\overline{\hat{\gamma}^{(2)}_{n_2}}^{2,g}\,\left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}\,\widehat{V}^\dagger_2(g)\\
&= \widehat{U}_2(g)\,\hat{\gamma}^{(2)}_{n_2}\,\widehat{U}^\dagger_2(g),
\end{aligned} \tag{D.35c}$$

where we used the definition (6.27). The consistency condition (7.4) for $\hat{\gamma}^{(2)}_{n_2}$ thus hold. □

Before proving Eq. (7.27a) we are going to derive two useful identities.

Claim 14. Under the assumption that Eqs. (6.17) and (7.4) hold for the stacked representation $\widehat{U}_\wedge$ defined by Eq. (7.25), the pair of identities

$$\overline{\widehat{V}_1(h)}^{\wedge,g} = (-1)^{q_1(h)\left[\rho_1(g)+q_1(g)\right]}\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\,\overline{\widehat{V}_1(h)}^{1,g}\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}, \tag{D.36a}$$

$$\begin{aligned}
\overline{\widehat{V}_2(h)}^{\wedge,g} = (-1)^{q_2(h)\left[\mathfrak{c}(g)+\rho_2(g)+q_2(g)\right]}\,&\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\
\times\,\overline{\widehat{V}_2(h)}^{2,g}\,&\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}.
\end{aligned} \tag{D.36b}$$

relates complex conjugation by $\mathsf{K}_\wedge$ to complex conjugation by K_1 and K_2 for any pair $g,h\in G_f$.

Proof. Consistency conditions (6.17) and (7.4) imply the identity

$$\widehat{U}_\wedge(g)\,\widehat{V}_1(h)\,\widehat{U}^\dagger_\wedge(g)=\widehat{U}_{\mathrm{bulk}}(g)\,\widehat{V}_1(h)\,\widehat{U}^\dagger_{\mathrm{bulk}}(g)=\widehat{U}_1(g)\,\widehat{V}_1(h)\,\widehat{U}^\dagger_1(g). \tag{D.37}$$

In order to relate conjugation by $\mathsf{K}_\wedge$ with conjugation by K_1 we are going to insert the definition of $\widehat{U}_1(g)$ on the right-hand side of Eq. (D.37) and the definition of $\widehat{U}_\wedge(g)$ on the left-hand side of Eq. (D.37) and compare the resulting expressions.

The right-hand side of Eq. (D.37) upon insertion of the definition of $\widehat{U}_1(g)$ is

$$\begin{aligned}\widehat{U}_1(g)\,\widehat{V}_1(h)\,\widehat{U}^\dagger_1(g)&=\widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_\infty\right]^{q_1(g)}\,\mathsf{K}_1^{\mathfrak{c}(g)}\,\widehat{V}_1(h)\,\mathsf{K}_1^{\mathfrak{c}(g)}\,\left[\hat{\gamma}^{(1)}_\infty\right]^{q_1(g)}\,\widehat{V}^\dagger_1(g)\\&=\widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_\infty\right]^{q_1(g)}\,\overline{\widehat{V}_1(h)}^{1,g}\,\left[\hat{\gamma}^{(1)}_\infty\right]^{q_1(g)}\,\widehat{V}^\dagger_1(g)\\&=(-1)^{q_1(g)\,q_1(h)}\,\widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\,\widehat{V}^\dagger_1(g),\end{aligned} \tag{D.38}$$

where in reaching the last line we have interchanged $\hat{\gamma}^{(1)}_\infty$ with $\overline{\widehat{V}_1(h)}^{1,g}$ at the cost of the phase factor $(-1)^{q_1(g)\,q_1(h)}$ [recall that $q_1(g)$ is the fermion parity of the unitary operator $\widehat{V}_1(g)$ by definition (6.27)].

The left-hand side of Eq. (D.37) upon insertion of the definition of $\widehat{U}_\wedge(g)$ is

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{V}_1(h)\,\widehat{U}^\dagger_\wedge(g)=&\,(-\mathrm{i})^{\delta_{g,p}}\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\\&\times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\&\times\mathsf{K}_\wedge^{\mathfrak{c}(g)}\,\widehat{V}_1(h)\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}\\&\times\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\\&\times\left[\widehat{U}^\dagger_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\widehat{V}^\dagger_2(g)\,\widehat{V}^\dagger_1(g)\,(+\mathrm{i})^{\delta_{g,p}}\\=&\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\\&\times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\&\times\overline{\widehat{V}_1(h)}^{\wedge,g}\end{aligned}$$

$$\times \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}$$
$$\times \left[\widehat{U}^{\dagger}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)} \widehat{V}^{\dagger}_2(g)\,\widehat{V}^{\dagger}_1(g), \tag{D.39}$$

where the multiplicative phases $(\pm\mathrm{i})^{\delta_{g,p}}$ cancel each other. In order to pass all operators from the representation $\widehat{U}_2$ to the right of $\overline{\widehat{V}_1(h)}^{\wedge,g}$, we observe that:

1. interchanging $\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}$ with $\overline{\widehat{V}_1(h)}^{\wedge,g}$ induces the multiplicative phase factor
$(-1)^{\left[\mathfrak{c}(g)+q_2(g)+\rho_2(g)\right]q_1(h)}$,

2. interchanging $\widehat{V}_2(g)\,\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}$ with $\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}$ twice does not cost any overall multiplicative phase factors,

3. interchanging $\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}$ with $\overline{\widehat{V}_1(h)}^{\wedge,g}$ induces the multiplicative phase factor
$(-1)^{\left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)\right]q_1(h)}$,

4. interchanging $\widehat{V}_2(g)$ with $\overline{\widehat{V}_1(h)}^{\wedge,g}$ induces the multiplicative phase factor $(-1)^{q_2(g)\,q_1(h)}$,

We thus find

$$\widehat{U}_{\wedge}(g)\,\widehat{V}_1(h)\,\widehat{U}^{\dagger}_{\wedge}(g) = (-1)^{q_1(h)\rho_1(g)}\widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}$$
$$\times\,\overline{\widehat{V}_1(h)}^{\wedge,g}\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\widehat{V}^{\dagger}_1(g). \tag{D.40}$$

Equating the right-hand sides of Eqs. (D.40) and (D.38) in view of Eq. (D.37) and solving for $\overline{\widehat{V}_1(h)}^{\wedge,g}$ delivers

$$\overline{\widehat{V}_1(h)}^{\wedge,g} = (-1)^{q_1(h)\left[\rho_1(g)+q_1(g)\right]}\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\overline{\widehat{V}_1(h)}^{1,g}\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}. \tag{D.41}$$

The same strategy is to be repeated for the representation $\widehat{U}_2$. The only difference as compared to the steps leading to Eq. (D.41) occurs with the manipulations that follow the counterpart to Eq. (D.39), as we shall see.

Consistency conditions (6.17) and (7.4) imply the identity

$$\widehat{U}_{\wedge}(g)\,\widehat{V}_2(h)\,\widehat{U}^{\dagger}_{\wedge}(g) = \widehat{U}_{\mathrm{bulk}}(g)\,\widehat{V}_2(h)\,\widehat{U}^{\dagger}_{\mathrm{bulk}}(g) = \widehat{U}_2(g)\,\widehat{V}_2(h)\,\widehat{U}^{\dagger}_2(g). \tag{D.42}$$

The counterpart to Eq. (D.38) is

$$\widehat{U}_2(g)\,\widehat{V}_2(h)\,\widehat{U}^{\dagger}_2(g) = (-1)^{q_2(g)q_2(h)}\,\widehat{V}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g}\,\widehat{V}^{\dagger}_2(g). \tag{D.43}$$

It follows simply by interchanging labels 1 and 2. The counterpart to Eq. (D.39) is

$$\begin{aligned}
\widehat{U}_{\wedge}(g)\,\widehat{V}_2(h)\,\widehat{U}^{\dagger}_{\wedge}(g) =& (-\mathrm{i})^{\delta_{g,p}}\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)} \\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)} \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \\
&\times \mathsf{K}^{\mathfrak{c}(g)}_{\wedge}\,\widehat{V}_2(h)\,\mathsf{K}^{\mathfrak{c}(g)}_{\wedge} \\
&\times \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)} \\
&\times \left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\widehat{V}^{\dagger}_2(g)\,\widehat{V}^{\dagger}_1(g)\,(+\mathrm{i})^{\delta_{g,p}} \\
=& \widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)} \\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)} \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \\
&\times \overline{\widehat{V}_2(h)}^{\wedge,g} \\
&\times \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)} \\
&\times \left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\,\widehat{V}^{\dagger}_2(g)\,\widehat{V}^{\dagger}_1(g).
\end{aligned} \tag{D.44}$$

In contrast to Eq. (D.39), the total phase factor that arises from reordering terms is different. This is so because the operators $\hat{\gamma}^{(1)}_{n_1}$ and $\hat{\gamma}^{(2)}_{n_2}$ in the definition (7.25) of the stacked representation carry different exponents.

In order to pass all operators from the representation $\widehat{U}_1$ to the right of $\overline{\widehat{V}_2(h)}^{\wedge,g}$, We observe that:

1. interchanging $\hat{\gamma}^{(1)}_{n_1}$ or $\widehat{U}_{\wedge}(p)$ or $\widehat{V}_1(g)$ with $\hat{\gamma}^{(2)}_{n_2}$ twice does not produce a multiplicative phase factor,

2. interchanging $\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}$ with $\overline{\widehat{V}_2(h)}^{\wedge,g}$ induces the phase factor $(-1)^{[q_1(g)+\rho_1(g)]q_2(h)}$,

3. interchanging $\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}$ with $\overline{\widehat{V}_2(h)}^{\wedge,g}$ induces the phase factor $(-1)^{\left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)\right]q_2(h)}$,

4. interchanging $\widehat{V}_1(g)$ with $\overline{\widehat{V}_2(h)}^{\wedge,g}$ induces the phase factor $(-1)^{q_1(g)q_2(h)}$.

We thus find

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{V}_2(h)\,\widehat{U}^\dagger_\wedge(g) = (-1)^{q_2(h)\left(\mathfrak{c}(g)+\rho_2(g)\right)}\,\widehat{V}_2(g)\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\ \times\,\overline{\widehat{V}_2(h)}^{\wedge,g}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\,\widehat{V}^\dagger_2(g).\end{aligned}\tag{D.45}$$

Equating the right-hand sides of Eqs. (D.43) and (D.45) in view of Eq. (D.37) and solving for $\overline{\widehat{V}_2(h)}^{\wedge,g}$ delivers

$$\begin{aligned}\overline{\widehat{V}_2(h)}^{\wedge,g} =& (-1)^{q_2(h)\left[\mathfrak{c}(g)+\rho_2(g)+q_2(g)\right]}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\ &\times\,\overline{\widehat{V}_2(h)}^{1,g}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}.\end{aligned}\tag{D.46}$$

□

Equations (D.41) and (D.46) give the prescription to trade complex conjugation $\mathsf{K}_\wedge$ with K_1 and K_2, respectively. We will make use of these equations when deriving the 2-cochain $\phi_\wedge$ of the stacked representation. We are now at a position to compute the 2-cochain $\phi_\wedge(g,h)$ associated with the stacked representation $\widehat{U}_\wedge$.

Proof of Eq. (7.27a). Composing the representations $\widehat{U}_\wedge(g)$ and $\widehat{U}_\wedge(h)$ of any pair $g,h\in G_f$ delivers

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = (-\mathrm{i})^{\delta_{g,p}}\,\widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\\ \times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\mathsf{K}^{\mathfrak{c}(g)}_\wedge\\ \times(-\mathrm{i})^{\delta_{h,p}}\,\widehat{V}_1(h)\,\widehat{V}_2(h)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(h)+\rho_1(h)+\rho_2(h)}\end{aligned}$$

$$
\begin{aligned}
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)} \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)} \mathsf{K}_\wedge^{\mathfrak{c}(h)}\\
&= (-\mathrm{i})^{\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\,\delta_{h,p}}\, \widehat{V}_1(g)\, \widehat{V}_2(g)\, \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)} \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\
&\times \overline{\widehat{V}_1(h)}^{\wedge,g}\, \overline{\widehat{V}_2(h)}^{\wedge,g} \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(h)+\rho_1(h)+\rho_2(h)}\\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)} \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}\\
&\times (-1)^{\chi_1}\, \mathsf{K}_\wedge^{\mathfrak{c}(g\,h)},
\end{aligned}
\tag{D.47a}
$$

where

$$
(-1)^{\chi_1(g,h)} := (-1)^{\mathfrak{c}(g)\left[\mathfrak{c}(h)+q_2(h)+\rho_2(h)\right]}. \tag{D.47b}
$$

Here, we have passed the complex conjugation $\mathsf{K}_\wedge^{\mathfrak{c}(g)}$ to the right. In doing so, operators $\widehat{V}_1(h)$ and $\widehat{V}_2(h)$ are replaced by their complex conjugates $\overline{\widehat{V}_1(h)}^{\wedge,g}$ and $\overline{\widehat{V}_2(h)}^{\wedge,g}$, respectively. The operators $\widehat{U}_\wedge(p)$ and $\hat{\gamma}^{(1)}_{n_1}$ are, by definition, invariant under complex conjugation by $\mathsf{K}_\wedge^{\mathfrak{c}(g)}$. On the other hand, the operator $\hat{\gamma}^{(2)}_{n_2}$ is odd under conjugation by $\mathsf{K}_\wedge$. This is the origin of the multiplicative phase factor $(-1)^{\mathfrak{c}(g)[\mathfrak{c}(h)+q_2(h)+\rho_2(h)]}$. The multiplier $(-1)^{\mathfrak{c}(g)}$ in the phase factor $(-\mathrm{i})^{\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\,\delta_{h,p}}$ arises when the complex conjugation $\mathsf{K}_\wedge^{\mathfrak{c}(g)}$ is passed through $(-\mathrm{i})^{\delta_{h,p}}$.

Our aim is now to bring $\widehat{V}_1(g)$ to the left of $\overline{\widehat{V}_1(h)}^{\wedge,g}$ and $\widehat{V}_2(g)$ to the left of $\overline{\widehat{V}_2(h)}^{\wedge,g}$. To do so, we apply the following steps.

(i) We bring $\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}$ to the left of $\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(h)+\rho_1(h)+\rho_2(h)}$ and make use of the identity

$$
\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)} \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(h)+\rho_1(h)+\rho_2(h)} = \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)} \tag{D.48}
$$

that follows from $\mathfrak{c}$, ρ_1, and ρ_2 being group homomorphisms. Passing $\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}$ through $\hat{\gamma}^{(1)}_{n_1}$, $\hat{\gamma}^{(2)}_{n_2}$, $\overline{\widehat{V}_1(h)}^{\wedge,g}$, and $\overline{\widehat{V}_2(h)}^{\wedge,g}$ induces multiplicative phase

$$
(-1)^{\chi_2(g,h)} := (-1)^{\left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)\right]\left[q_1(g)+\rho_1(g)+\mathfrak{c}(g)+q_2(g)+\rho_2(g)+q_1(h)+q_2(h)\right]}
$$

$$= (-1)^{\left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)\right]\,\left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)+q_1(g\,h)+q_2(g\,h)\right]}, \qquad \text{(D.49a)}$$

owing to the fact that the 1-cochains q_1 and q_2 are group homomorphisms (this follows from both cochains keeping track of the fermion parity of the unitary operators $\widehat{V}_1$ and $\widehat{V}_2$).

We can further simplify the expression by noting that

$$\left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)\right]\,\left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)\right] = \left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)\right] \bmod 2. \qquad \text{(D.49b)}$$

Hence, we have the identity

$$(-1)^{\chi_2(g,h)} := (-1)^{\left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)\right]\,\left[1+q_1(g\,h)+q_2(g\,h)\right]}. \qquad \text{(D.49c)}$$

The order of the operators on the right-hand side of Eq. (D.47a) after these manipulations is

$$\begin{aligned} \widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) \propto\, & \widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \\ & \times \overline{\widehat{V}_1(h)}^{\wedge,g}\,\overline{\widehat{V}_2(h)}^{\wedge,g}\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)} \\ & \times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}. \end{aligned} \qquad \text{(D.49d)}$$

(ii) We interchange the operators $\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}$ and $\overline{\widehat{V}_1(h)}^{\wedge,g}$ which produces the phase factor

$$(-1)^{\chi_3(g,h)} := (-1)^{\left[\mathfrak{c}(g)+q_2(g)+\rho_2(g)\right]\,q_1(h)}, \qquad \text{(D.50a)}$$

while

$$\begin{aligned} \widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) \propto\, & \widehat{V}_1(g)\,\widehat{V}_2(g)\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)} \\ & \times \overline{\widehat{V}_1(h)}^{\wedge,g}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \\ & \times \overline{\widehat{V}_2(h)}^{\wedge,g}\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)} \end{aligned}$$

$$\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)} \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)} \mathsf{K}^{\mathfrak{c}(g\,h)}_{\wedge}. \tag{D.50b}$$

(iii) We bring $\widehat{V}_2(g)$ to the right of $\overline{\widehat{V}_1(h)}^{\wedge,g}$. This produces the multiplicative phase factor

$$(-1)^{\chi_4(g,h)} := (-1)^{\left[q_1(h)+q_1(g)+\rho_1(g)\right]q_2(g)}, \tag{D.51a}$$

while

$$\begin{aligned}\widehat{U}_{\wedge}(g)\,\widehat{U}_{\wedge}(h) \propto\, &\widehat{V}_1(g)\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\\ &\times \overline{\widehat{V}_1(h)}^{\wedge,g}\,\widehat{V}_2(g)\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\ &\times \overline{\widehat{V}_2(h)}^{\wedge,g}\,\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)}\\ &\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)} \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)} \mathsf{K}^{\mathfrak{c}(g\,h)}_{\wedge}.\end{aligned} \tag{D.51b}$$

(iv) We sum (modulo 2) the phases $\chi_1(g,h)$ defined in Eq. (D.47), $\chi_2(g,h)$ defined in Eq. (D.49), $\chi_3(g,h)$ defined in Eq. (D.50), and $\chi_4(g,h)$ defined in Eq. (D.51)

$$\begin{aligned}\chi_{1234}(g,h) :=\, &\chi_1(g,h)+\chi_2(g,h)+\chi_3(g,h)+\chi_4(g,h)\\ =\, &\mathfrak{c}(g)\left[\mathfrak{c}(h)+q_2(h)+\rho_2(h)\right]\\ &+\left[\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)\right]\left[1+q_1(g\,h)+q_2(g\,h)\right]\\ &+\left[\mathfrak{c}(g)+q_2(g)+\rho_2(g)\right]q_1(h)\\ &+\left[q_1(h)+q_1(g)+\rho_1(g)\right]q_2(g).\\ =\, &\mathfrak{c}(g)\left[1+\mathfrak{c}(h)+q_1(g)+q_2(g)+\rho_2(h)\right]\\ &+\rho_1(g)\left[1+q_1(g)+q_1(h)+q_2(h)\right]\\ &+\rho_2(g)\left[1+q_1(g)+q_2(g)+q_2(h)\right]\\ &+q_1(g)q_2(g),\end{aligned} \tag{D.52a}$$

where we used the identities

$$q_1(g) + q_1(h) = q_1(g\,h) \bmod 2, \qquad q_2(g) + q_2(h) = q_2(g\,h) \bmod 2. \tag{D.52b}$$

Collecting steps (i)-(iv) gives

$$\begin{aligned}
\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = \widehat{V}_1(g)\, &\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\, \overline{\widehat{V}_1(h)}^{\wedge,g}\\
&\times \widehat{V}_2(g)\, \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\, \overline{\widehat{V}_2(h)}^{\wedge,g}\\
&\times \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)}\\
&\times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)}\, \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}\\
&\times (-\mathrm{i})^{\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\,\delta_{h,p}}\,(-1)^{\chi_{1234}(g,h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)},
\end{aligned} \tag{D.53}$$

where $\chi_{1234}(g,h)$ is defined in Eq. (D.52).

We shall work on the first line of the right-hand side of Eq. (D.53). We note that the operators $\widehat{V}_1(h)$ and $\widehat{V}_2(h)$ are conjugated by $\mathsf{K}_\wedge$, whose action differs from that of K_1 and K_2. To trade $\overline{\widehat{V}_1(h)}^{\wedge,g}$ and $\overline{\widehat{V}_2(h)}^{\wedge,g}$ with $\overline{\widehat{V}_1(h)}^{1,g}$ and $\overline{\widehat{V}_2(h)}^{2,g}$, respectively, we use the pair of identities (D.41) and (D.46), respectively. One finds using (D.41) that

$$\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\, \overline{\widehat{V}_1(h)}^{\wedge,g} = (-1)^{q_1(h)\left[\rho_1(g)+q_1(g)\right]}\, \overline{\widehat{V}_1(h)}^{1,g}\, \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}. \tag{D.54a}$$

Similarly, one finds using (D.46) that

$$\begin{aligned}
\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\, \overline{\widehat{V}_2(h)}^{\wedge,g} &= (-1)^{q_2(h)\left[\mathfrak{c}(g)+\rho_2(g)+q_2(g)\right]}\\
&\times \overline{\widehat{V}_2(h)}^{1,g}\, \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}.
\end{aligned} \tag{D.54b}$$

Inserting both identities into Eq. (D.53) gives

$$\begin{aligned}
\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = &(-1)^{\chi_{1234}(g,h)}\,(-1)^{\chi_{\mathrm{conj}}(g,h)}\,(-\mathrm{i})^{\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\,\delta_{h,p}}\\
&\times \widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\, \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\, \widehat{V}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g}\, \left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}
\end{aligned}$$

$$\times\left[\widehat{U}_{\wedge}(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)}\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}$$
$$\times\,\mathsf{K}^{\mathfrak{c}(g\,h)}_{\wedge},\tag{D.55a}$$

where we have consolidated the two multiplicative phase factors on the right-hand sides of Eqs. (D.54a) and (D.54b) into the multiplicative phase factor

$$(-1)^{\chi_{\mathrm{conj}}(g,h)}:=(-1)^{q_1(h)\left[\rho_1(g)+q_1(g)\right]+q_2(h)\left[\mathfrak{c}(g)+\rho_2(g)+q_2(g)\right]}.\tag{D.55b}$$

To proceed, we make an interlude that relies on the fact that $\widehat{U}_1$ and $\widehat{U}_2$ are projective representations of the group G_f. On the one hand,

$$\begin{aligned}\widehat{U}_i(g)\,\widehat{U}_i(h)&=\widehat{V}_i(g)\left[\hat{\gamma}^{(i)}_{\infty}\right]^{q_i(g)}\mathsf{K}^{\mathfrak{c}(g)}_1\,\widehat{V}_i(h)\left[\hat{\gamma}^{(i)}_{\infty}\right]^{q_i(h)}\mathsf{K}^{\mathfrak{c}(h)}_i\\&=\widehat{V}_i(g)\left[\hat{\gamma}^{(i)}_{\infty}\right]^{q_i(g)}\overline{\widehat{V}_i(h)}^{\,i,g}\left[\overline{\hat{\gamma}^{(i)}_{\infty}}^{\,i,g}\right]^{q_i(h)}\mathsf{K}^{\mathfrak{c}(g\,h)}_i,\end{aligned}\tag{D.56}$$

where we used the definition (6.27) and brought $\mathsf{K}^{\mathfrak{c}(g)}_1$ to the left of $\mathsf{K}^{\mathfrak{c}(h)}_i$. We may interchange $\left[\hat{\gamma}^{(i)}_{\infty}\right]^{q_i(g)}$ and $\overline{\widehat{V}_i(h)}^{\,1,g}$ at the cost of the phase factor $(-1)^{q_i(g)\,q_i(h)}$ and use the identity (6.30b) to trade $\left[\overline{\hat{\gamma}^{(i)}_{\infty}}^{\,i,g}\right]^{q_i(h)}$ with its complex conjugate. One is left with

$$\begin{aligned}\widehat{U}_i(g)\,\widehat{U}_i(h)&=(-1)^{q_i(g)q_i(h)}\,\widehat{V}_i(g)\,\overline{\widehat{V}_i(h)}^{\,i,g}\left[\hat{\gamma}^{(i)}_{\infty}\right]^{q_i(g)}\left[\overline{\hat{\gamma}^{(i)}_{\infty}}^{\,i,g}\right]^{q_i(h)}\mathsf{K}^{\mathfrak{c}(g\,h)}_i\\ \text{Eq. (6.30)}&=(-1)^{q_i(g)q_i(h)+q_i(h)\left[\mathfrak{c}(g)+q_i(g)+\rho_i(g)\right]}\,\widehat{V}_i(g)\,\overline{\widehat{V}_i(h)}^{\,i,g}\\&\quad\times\left[\hat{\gamma}^{(i)}_{\infty}\right]^{q_i(g\,h)}\mathsf{K}^{\mathfrak{c}(g\,h)}_i\\&=(-1)^{q_i(h)\left[\mathfrak{c}(g)+\rho_i(g)\right]}\,\widehat{V}_i(g)\,\overline{\widehat{V}_i(h)}^{\,i,g}\left[\hat{\gamma}^{(i)}_{\infty}\right]^{q_i(g\,h)}\mathsf{K}^{\mathfrak{c}(g\,h)}_i,\end{aligned}\tag{D.57}$$

where, in reaching the last equality, we have simplified the phase factor by dropping terms that are 0 modulo 2. On the other hand, by definition [recall Eq. (A.68)]

$$\widehat{U}_i(g)\,\widehat{U}_i(h)=e^{\mathrm{i}\phi_i(g,h)}\,\widehat{U}_i(g\,h)$$

$$= e^{\mathrm{i}\phi_i(g,h)}\, \widehat{V}_i(g\,h)\, \left[\hat{\gamma}^{(i)}_{\infty}\right]^{q_i(g\,h)}\, \mathsf{K}_i^{\mathfrak{c}(g\,h)}, \qquad i=1,2. \tag{D.58}$$

Equating the right-hand sides of Eqs. (D.57) and (D.58) gives

$$\widehat{V}_i(g)\, \overline{\widehat{V}_i(h)}^{\,i,g} = (-1)^{q_i(h)\left[\mathfrak{c}(g)+\rho_i(g)\right]}\, e^{\mathrm{i}\phi_i(g,h)}\, \widehat{V}_i(g\,h), \qquad i=1,2. \tag{D.59}$$

Inserting Eq. (D.59) into Eq. (D.55) delivers

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) =\, & e^{\mathrm{i}\pi\chi_{1234}(g,h)+\mathrm{i}\pi\chi_{\mathrm{conj}}(g,h)+\mathrm{i}\phi_{\mathrm{comp}}(g,h)+\mathrm{i}\frac{3\pi}{2}\left(\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\,\delta_{h,p}\right)} \\ & \times \widehat{V}_1(g\,h)\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\,\widehat{V}_2(g\,h)\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)} \\ & \times \left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)}\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)} \\ & \times \mathsf{K}_\wedge^{\mathfrak{c}(g\,h)},\end{aligned} \tag{D.60a}$$

where we have defined the phase factor accumulated from the group composition rule (D.59)

$$\phi_{\mathrm{comp}}(g,h) := \phi_1(g,h)+\phi_2(g,h)+\pi\, q_1(h)\left[\mathfrak{c}(g)+\rho_1(g)\right]+\pi\, q_2(h)\left[\mathfrak{c}(g)+\rho_2(g)\right]. \tag{D.60b}$$

It remains to reorder operators on the right-hand side of Eq. (D.60a) with the goal to isolate the operator $\widehat{U}_\wedge(g\,h)$, whose definition is given by Eq. (7.25). This is done with the following steps.

(i) Bringing $\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}$ to the left of $\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}$ induces the multiplicative phase factor

$$(-1)^{\left[\mathfrak{c}(g)+q_2(g)+\rho_2(g)\right]\,\left[\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)+q_1(h)+\rho_1(h)\right]}, \tag{D.61a}$$

while

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) \propto\, & \widehat{V}_1(g\,h)\,\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\,\widehat{V}_2(g\,h)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)} \\ & \times \left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)}\,\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g\,h)+q_2(g\,h)+\rho_2(g\,h)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}.\end{aligned} \tag{D.61b}$$

(ii) Bringing $\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}$ to the left of $\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)}$ induces the multiplicative phase factor

$$(-1)^{\left[q_1(g)+\rho_1(g)\right]\,\left[\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)+q_2(g\,h)\right]}, \tag{D.62a}$$

while

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) \propto{}& \widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)}\\ &\times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g\,h)+\rho_1(g\,h)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g\,h)+q_2(g\,h)+\rho_2(g\,h)}\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}.\end{aligned} \tag{D.62b}$$

(iii) The total phase that is acuumulated in steps (i) and (ii) is

$$\begin{aligned}\chi_{\mathrm{ord}}(g,h) :={}& \left[\mathfrak{c}(g)+q_2(g)+\rho_2(g)\right]\left[\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)+q_1(h)+\rho_1(h)\right]\\ &+\left[q_1(g)+\rho_1(g)\right]\left[\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)+q_2(g\,h)\right].\end{aligned} \tag{D.63}$$

Combining all the phase factors, one finds

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) ={}& e^{\mathrm{i}\pi\chi_{1234}(g,h)+\mathrm{i}\pi\chi_{\mathrm{conj}}(g,h)+\mathrm{i}\phi_{\mathrm{comp}}(g,h)+\mathrm{i}\pi\chi_{\mathrm{ord}}(g,h)+\chi_{\mathrm{gag}(g,h)}}\\ &\times(-\mathrm{i})^{\delta_{gh,p}}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)}\\ &\times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g\,h)+\rho_1(g\,h)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g\,h)+q_2(g\,h)+\rho_2(g\,h)}\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)},\end{aligned} \tag{D.64a}$$

where we have defined the phase factor

$$\chi_{\mathrm{gag}(g,h)} := \frac{3\pi}{2}\left(\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\,\delta_{h,p}-\delta_{gh,p}\right). \tag{D.64b}$$

We have derived the composition rule

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) ={}& e^{\mathrm{i}\pi\chi_{1234}(g,h)+\mathrm{i}\pi\chi_{\mathrm{conj}}(g,h)+\mathrm{i}\phi_{\mathrm{comp}}(g,h)\mathrm{i}\pi\chi_{\mathrm{ord}}(g,h)+\chi_{\mathrm{gag}(g,h)}}\\ &\times(-\mathrm{i})^{\delta_{gh,p}}\,\widehat{V}_1(g\,h)\,\widehat{V}_2(g\,h)\,\left[\widehat{U}_\wedge(p)\right]^{\mathfrak{c}(g\,h)+\rho_1(g\,h)+\rho_2(g\,h)}\\ &\times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g\,h)+\rho_1(g\,h)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g\,h)+q_2(g\,h)+\rho_2(g\,h)}\mathsf{K}_\wedge^{\mathfrak{c}(g\,h)}\\ &\equiv e^{\mathrm{i}\phi_\wedge(g,h)}\,\widehat{U}_\wedge(g\,h)\end{aligned} \tag{D.65a}$$

for the stacked representation $\widehat{U}_\wedge$, where we used the definitions (7.25) for $\widehat{U}_\wedge(g\,h)$ and

$$\begin{aligned}
\phi_\wedge(g,h) :=& \pi\chi_{1234}(g,h) + \pi\chi_{\mathrm{conj}}(g,h) + \phi_{\mathrm{comp}}(g,h) + \pi\chi_{\mathrm{ord}}(g,h) + \chi_{\mathrm{gag}}(g,h) \\
=& \phi_1(g,h) + \phi_2(g,h) + \chi_{\mathrm{gag}}(g,h) \\
&+ \pi\Big(q_1(h)\Big[\mathfrak{c}(g) + \rho_1(g)\Big] + q_2(h)\Big[\mathfrak{c}(g) + \rho_2(g)\Big] \\
&\quad + \mathfrak{c}(g)\Big[1 + \mathfrak{c}(h) + q_1(g) + q_2(g) + \rho_2(h)\Big] \\
&\quad + \rho_1(g)\Big[1 + q_1(g) + q_1(h) + q_2(h)\Big] \\
&\quad + \rho_2(g)\Big[1 + q_1(g) + q_2(g) + q_2(h)\Big] \\
&\quad + q_1(g)q_2(g) \\
&\quad + q_1(h)\Big[\rho_1(g) + q_1(g)\Big] + q_2(h)\Big[\mathfrak{c}(g) + \rho_2(g) + q_2(g)\Big] \\
&\quad + \Big[\mathfrak{c}(g) + q_2(g) + \rho_2(g)\Big]\Big[\mathfrak{c}(g\,h) + \rho_1(g\,h) + \rho_2(g\,h) + q_1(h) + \rho_1(h)\Big] \\
&\quad + \Big[q_1(g) + \rho_1(g)\Big]\Big[\mathfrak{c}(g\,h) + \rho_1(g\,h) + \rho_2(g\,h) + q_2(g\,h)\Big]\Big) \\
=& \phi_1(g,h) + \phi_2(g,h) + \chi_{\mathrm{gag}}(g,h) + \pi\chi(g,h),
\end{aligned} \tag{D.65b}$$

to reach the last equality in Eq. (D.65a). We have reserved the phase $\chi(g,h)$ for all phases other than the 2-cochains $\phi_1(g,h)$, $\phi_2(g,h)$, and $\chi_{\mathrm{gag}}(g,h)$ in Eq. (D.65b), i.e.,

$$\begin{aligned}
\chi(g,h) :=& q_1(h)\Big[\mathfrak{c}(g) + \rho_1(g)\Big] + q_2(h)\Big[\mathfrak{c}(g) + \rho_2(g)\Big] \\
&+ \mathfrak{c}(g)\Big[1 + \mathfrak{c}(h) + q_1(g) + q_2(g) + \rho_2(h)\Big] \\
&+ \rho_1(g)\Big[1 + q_1(g) + q_1(h) + q_2(h)\Big] \\
&+ \rho_2(g)\Big[1 + q_1(g) + q_2(g) + q_2(h)\Big] \\
&+ q_1(g)q_2(g) \\
&+ q_1(h)\Big[\rho_1(g) + q_1(g)\Big] + q_2(h)\Big[\mathfrak{c}(g) + \rho_2(g) + q_2(g)\Big] \\
&+ \Big[\mathfrak{c}(g) + q_2(g) + \rho_2(g)\Big]\Big[\mathfrak{c}(g\,h) + \rho_1(g\,h) + \rho_2(g\,h) + q_1(h) + \rho_1(h)\Big] \\
&+ \Big[q_1(g) + \rho_1(g)\Big]\Big[\mathfrak{c}(g\,h) + \rho_1(g\,h) + \rho_2(g\,h) + q_2(g\,h)\Big].
\end{aligned} \tag{D.65c}$$

The phase factor $\chi_{\mathrm{gag}}(g,h)$ that appear in Eq. (D.65b) is an artifact of the particular gauge choice we have made when defining an Hermitian representation for the fermion parity operator in Eq. (6.18). Indeed, we observe that $\chi_{\mathrm{gag}}(g,h)$ is nothing but a pure gauge, i.e., $\chi_{\mathrm{gag}}(g,h) = \delta^1_{\mathfrak{c}}\xi(g,h)$ if we choose $\xi(g) = -\frac{3\pi}{2}\delta_{g,h}$. Under such a gauge transformation the representation $\widehat{U}_\wedge(p)$ of fermion parity p is no longer Hermitian.

However, by definition, the equivalence classes. $[\phi_\wedge]$ of the stacked 2-cochain $\phi_\wedge(g,h)$ are invariant under the gauge transformations. Therefore, the stacked 2-cochain $\phi_\wedge(g,h)$ is gauge equivalent to

$$\phi_\wedge(g,h) \sim \phi_1(g,h) + \phi_2(g,h) + \pi\chi(g,h). \tag{D.66}$$

□

Proof of Eq. (7.28). We now show that the phase $\chi(g,h)$ defined in Eq. (D.65c) is gauge equivalent to

$$\chi(g,h) \sim \rho_1(g)\,\rho_2(h). \tag{D.67}$$

In turn, the stacked 2-cochain $\phi_\wedge(g,h)$ is gauge equivalent to

$$\phi_\wedge(g,h) \sim \phi_1(g,h) + \phi_2(g,h) + \pi\rho_1(g)\,\rho_2(h). \tag{D.68}$$

We shall use the fact that for $i = 1,2$, q_i, ρ_i, and $\mathfrak{c}$ are all $\mathbb{Z}_2 = \{0,1\}$-valued group homomorphisms. Hence, for any $g,h \in G_f$ and $i = 1,2$, they satisfy

$$q_i(g) + q_i(h) = q_i(g\,h), \qquad \rho_i(g) + \rho_i(h) = \rho_i(g\,h), \qquad \mathfrak{c}(g) + \mathfrak{c}(h) = \mathfrak{c}(g\,h). \tag{D.69}$$

A consequence of $\mathfrak{c}$ being $\mathbb{Z}_2$-valued group homomorphism is that

$$\mathfrak{c}(g\,h) + \mathfrak{c}(h) = \mathfrak{c}(g) + \mathfrak{c}(h) + \mathfrak{c}(h) = \mathfrak{c}(g) + 2\mathfrak{c}(h) = \mathfrak{c}(g) \mod 2, \tag{D.70a}$$

$$\mathfrak{c}(g\,h) + \mathfrak{c}(g) = \mathfrak{c}(g) + \mathfrak{c}(h) + \mathfrak{c}(g) = \mathfrak{c}(h) + 2\mathfrak{c}(g) = \mathfrak{c}(h) \mod 2. \tag{D.70b}$$

The same identities hold for the homomorphisms q_i and ρ_i with $i = 1,2$.

We start by rewriting the phase factor $\chi(g,h)$ by expanding the last two lines of Eq. (D.65c)

$$\begin{aligned}
\chi(g,h) =\,& q_1(h)\left[\mathfrak{c}(g) + \rho_1(g)\right] + q_2(h)\left[\mathfrak{c}(g) + \rho_2(g)\right] \\
&+ \mathfrak{c}(g)\left[1 + q_1(g) + q_2(g) + q_1(h)\right] \\
&+ \rho_1(g)\left[1 + q_1(g) + q_1(h)\right] \\
&+ \rho_2(g)\left[1 + q_1(g) + q_2(g) + q_2(h) + q_1(h)\right] \\
&+ q_1(g)\,q_2(g) + q_2(g)\,q_1(h)
\end{aligned}$$

$$
\begin{aligned}
&+ q_1(h)\Big[\rho_1(g) + q_1(g)\Big] + q_2(h)\Big[\mathfrak{c}(g) + \rho_2(g) + q_2(g)\Big]\\
&+ \mathfrak{c}(g)\Big[\underline{\mathfrak{c}(h)} + \underline{\mathfrak{c}(g\,h)} + \underline{\underline{\rho_1(h)}} + \underline{\underline{\rho_1(g\,h)}} + \underline{\underline{\underline{\rho_2(h)}}} + \underline{\underline{\underline{\rho_2(g\,h)}}}\Big]\\
&+ \rho_1(g)\Big[\underline{\mathfrak{c}(g\,h)} + \underline{\underline{\rho_1(g\,h)}} + \underline{\underline{\underline{\underline{q_2(h)}}}} + \underline{\underline{\underline{\rho_2(g\,h)}}} + \underline{\underline{\underline{\underline{q_2(g\,h)}}}}\Big]\\
&+ \rho_2(g)\Big[\underline{\mathfrak{c}(g\,h)} + \underline{\underline{\rho_1(g\,h)}} + \underline{\underline{\rho_1(h)}} + \underline{\underline{\underline{\rho_2(g\,h)}}}\Big]\\
&+ q_2(g)\Big[\underline{\mathfrak{c}(g\,h)} + \underline{\underline{\rho_1(g\,h)}} + \underline{\underline{\rho_1(h)}} + \underline{\underline{\underline{\rho_2(g\,h)}}}\Big]\\
&+ q_1(g)\Big[\underline{\mathfrak{c}(g\,h)} + \underline{\underline{\rho_1(g\,h)}} + \underline{\underline{\underline{\rho_2(g\,h)}}} + \underline{\underline{\underline{\underline{q_2(g\,h)}}}}\Big].
\end{aligned}
\tag{D.71}
$$

By using identities (D.69) and (D.70) on the underlined terms, we make sure that the arguments of $\mathfrak{c}$, q_i, and ρ_i with $i = 1, 2$ depend on either g or h but not on their product,

$$
\begin{aligned}
\chi(g,h) = {}& q_1(h)\Big[\mathfrak{c}(g) + \rho_1(g)\Big] + q_2(h)\Big[\mathfrak{c}(g) + \rho_2(g)\Big]\\
&+ \mathfrak{c}(g)\Big[1 + \mathfrak{c}(g) + q_1(g) + q_1(h) + q_2(g) + \rho_1(g) + \rho_2(g)\Big]\\
&+ \rho_1(g)\Big[1 + q_1(g) + q_1(h) + q_2(g) + \mathfrak{c}(g) + \mathfrak{c}(h) + \rho_1(g) + \rho_1(h) + \rho_2(g) + \rho_2(h)\Big]\\
&+ \rho_2(g)\Big[1 + q_1(g) + q_1(h) + q_2(g) + q_2(h) + \mathfrak{c}(g) + \mathfrak{c}(h) + \rho_1(g) + \rho_2(g) + \rho_2(h)\Big]\\
&+ q_1(g)q_2(g)\\
&+ q_1(h)\Big[\rho_1(g) + q_1(g)\Big] + q_2(h)\Big[\mathfrak{c}(g) + \rho_2(g) + q_2(g)\Big]\\
&+ q_2(g)\Big[\mathfrak{c}(g) + \mathfrak{c}(h) + \rho_1(g) + \rho_2(g) + \rho_2(h) + q_1(h)\Big]\\
&+ q_1(g)\Big[\mathfrak{c}(g) + \mathfrak{c}(h) + \rho_1(g) + \rho_1(h) + \rho_2(g) + \rho_2(h) + q_2(g) + q_2(h)\Big].
\end{aligned}
\tag{D.72}
$$

As the right-hand side of Eq. (D.72) is defined modulo 2, every pair of identical terms can be dropped on the right-hand side of Eq. (D.72). All such pairs are identified by being colored in red and numbered. One finds

$$
\begin{aligned}
\chi(g,h) = {}& q_1(h)\Big[\underset{1}{\mathfrak{c}(g)} + \underset{2}{\rho_1(g)}\Big] + q_2(h)\Big[\underset{3}{\mathfrak{c}(g) + \rho_3(g)}\Big]\\
&+ \mathfrak{c}(g)\Big[\underset{4}{1} + \underset{4}{\mathfrak{c}(g)} + \underset{5}{q_1(g)} + \underset{1}{q_1(h)} + \underset{6}{q_2(g)} + \underset{7}{\rho_1(g)} + \underset{8}{\rho_2(g)}\Big]\\
&+ \rho_1(g)\Big[\underset{9}{1} + \underset{10}{q_1(g)} + q_1(h) + \underset{12}{q_2(g)} + \underset{7}{\mathfrak{c}(g)} + \mathfrak{c}(h) + \underset{9}{\rho_1(g)} + \rho_1(h) + \underset{11}{\rho_2(g)} + \rho_2(h)\Big]\\
&+ \rho_2(g)\Big[\underset{13}{1} + \underset{14}{q_1(g)} + q_1(h) + \underset{15}{q_2(g)} + q_2(h) + \underset{8}{\mathfrak{c}(g)} + \mathfrak{c}(h) + \underset{11}{\rho_1(g)} + \underset{13}{\rho_2(g)} + \rho_2(h)\Big]\\
&+ \underset{16}{q_1(g)q_2(g)}
\end{aligned}
$$

$$
\begin{aligned}
&+ q_1(h)\Big[\underset{2}{\rho_1(g)} + q_1(g)\Big] + q_2(h)\Big[\mathfrak{c}(g) + \underset{3}{\rho_2(g)} + q_2(g)\Big] \\
&+ q_2(g)\Big[\underset{6}{\mathfrak{c}(g)} + \mathfrak{c}(h) + \underset{12}{\rho_1(g)} + \underset{15}{\rho_2(g)} + \rho_2(h) + q_1(h)\Big] \\
&+ q_1(g)\Big[\underset{5}{\mathfrak{c}(g)} + \mathfrak{c}(h) + \underset{10}{\rho_1(g)} + \rho_1(h) + \underset{14}{\rho_2(g)} + \rho_2(h) + \underset{16}{q_2(g)} + q_2(h)\Big] \\
= {} & \rho_1(g)\Big[q_1(h) + \mathfrak{c}(h) + \rho_1(h) + \rho_2(h)\Big] \\
&+ \rho_2(g)\Big[q_1(h) + q_2(h) + \mathfrak{c}(h) + \rho_2(h)\Big] \\
&+ q_1(h)\,q_1(g) + q_2(h)\,q_2(g) \\
&+ q_2(g)\Big[\mathfrak{c}(h) + \rho_2(h) + q_1(h)\Big] \\
&+ q_1(g)\Big[\mathfrak{c}(h) + \rho_1(h) + \rho_2(h) + q_2(h)\Big].
\end{aligned}
\tag{D.73}
$$

We can use the gauge equivalence of cochains to make further simplifications. As is done for even-odd stacking in Eq. (D.26c), we can relate products of two 1-cochains by a coboundary. In general, for two $\mathbb{Z}_2$-valued 1-cochains $\alpha \in C^1(G_f, \mathbb{Z}_2)$ and $\beta \in C^1(G_f, \mathbb{Z}_2)$, we have

$$
\alpha(g)\beta(h) \sim \alpha(h)\beta(g) \implies \alpha(g)\,\beta(h) + \alpha(h)\,\beta(g) \sim 0 \bmod 2,
\tag{D.74}
$$

for any $g, h \in G_f$. We use this gauge equivalence in order to simplify further the right-hand side of Eq. (D.73). One finds

$$
\begin{aligned}
\chi(g,h) = {} & \rho_1(g)\Big[\underset{1}{q_1(h)} + \mathfrak{c}(h) + \rho_1(h) + \rho_2(h)\Big] \\
&+ \rho_2(g)\Big[\underset{2}{q_1(h)} + \underset{3}{q_2(h)} + \mathfrak{c}(h) + \rho_2(h)\Big] \\
&+ q_1(h)\,q_1(g) + q_2(h)\,q_2(g) \\
&+ q_2(g)\Big[\mathfrak{c}(h) + \underset{3}{\rho_2(h)} + \underset{4}{q_1(h)}\Big] \\
&+ q_1(g)\Big[\mathfrak{c}(h) + \underset{1}{\rho_1(h)} + \underset{2}{\rho_2(h)} + \underset{4}{q_2(h)}\Big],
\end{aligned}
\tag{D.75}
$$

where we have colored in red and numbered pairs of terms that can be dropped as they are equal to 0 modulo 2. Consequently,

$$
\begin{aligned}
\chi(g,h) = {} & \rho_1(g)\Big[\mathfrak{c}(h) + \rho_1(h) + \rho_2(h)\Big] + \rho_2(g)\Big[\mathfrak{c}(h) + \rho_2(h)\Big] \\
& q_1(g)\,\mathfrak{c}(h) + q_2(g)\,\mathfrak{c}(h) + q_1(h)\,q_1(g) + q_2(h)\,q_2(g)
\end{aligned}
$$

$$= \rho_1(g)\,\rho_2(h) + \sum_{i=1}^{2} \left\{ \rho_i(g)\left[\mathfrak{c}(h) + \rho_i(h)\right] + q_i(g)\,\left[\mathfrak{c}(h) + q_i(h)\right] \right\}, \tag{D.76}$$

where in reaching the last equality, we have isolated the term $\rho_1(g)\,\rho_2(h)$ and reorganized the remaining terms as a sum over $i = 1, 2$.

Finally to prove Eq. (D.68), we are going to show that the each term inside the summation in Eq. (D.76) vanishes. To this end, we apply the identity (D.22c) that was used for the case of even-odd stacking. If we define

$$\overline{\hat{\gamma}^{(1)}_{\infty}}^{1} = (-1)^{\zeta_1}\,\hat{\gamma}^{(1)}_{\infty}, \qquad \overline{\hat{\gamma}^{(2)}_{\infty}}^{2} = (-1)^{\zeta_2}\,\hat{\gamma}^{(2)}_{\infty}, \qquad \zeta_1, \zeta_2 = 0, 1, \tag{D.77a}$$

we obtain Eq. (6.30), two identities that are the equivalents of Eq. (D.22c), namely,

$$q_i(g) = \mathfrak{c}(g)(1 + \zeta_i) + \rho_i(g) \text{ mod } 2, \qquad i = 1, 2, \tag{D.77b}$$

for any $g \in G_f$. If we insert identity (D.77b), the argument of the summation in Eq. (D.76) becomes

$$\begin{aligned} \rho_i(g)\left[\mathfrak{c}(h) + \rho_i(h)\right] + q_i(g)\,\left[\mathfrak{c}(h) + q_i(h)\right] &= \rho_i(g)\left[\mathfrak{c}(h)\,\zeta_i + q_i(h)\right] \\ &\quad + q_i(g)\,\left[\mathfrak{c}(h)\,\zeta_i + \rho_i(h)\right] \\ &= \mathfrak{c}(h)\,\zeta_i\left[\rho_i(g) + q_i(g)\,\right] \\ &\quad + \rho_i(g)\,q_i(h) + q_i(g)\,\rho_i(h) \\ &= \mathfrak{c}(h)\,\zeta_i\left[\rho_i(g) + q_i(g)\,\right], \end{aligned} \tag{D.78}$$

where Eq. (D.74) was used to reach the last equality. Inserting identity (D.77b) once again and the using the fact that product $\zeta_i\left[1 + \zeta_i\right]$ for $\zeta_i = 0, 1$ is vanishing modulo 2, delivers the final result

$$\begin{aligned} \mathfrak{c}(h)\,\zeta_i\left[\rho_i(g) + q_i(g)\,\right] &= \mathfrak{c}(g)\,\mathfrak{c}(h)\,\zeta_i\left[1 + \zeta_i\right] \\ &= 0 \text{ mod } 2, \end{aligned} \tag{D.79}$$

which completes the proof. □

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CURRICULUM VITAE

Name	Ömer Mert Aksoy
Date of Birth	June 20, 1994
Place of Birth	Van, Turkey
Nationality	Turkish

EDUCATION

Since Sep 2019	ETH Zurich and Paul Scherrer Institute, Switzerland: PhD Student in physics; Supervisors: Prof. Dr. Christopher Mudry; Prof. Dr. Manfred Sigrist
Sep 2017 – June 2019	ETH Zurich and Paul Scherrer Institute, Switzerland: M.Sc. in physics; Supervisors: Prof. Dr. Christopher Mudry; Prof. Dr. Manfred Sigrist; Master Thesis: *Topological terms for the symmetry class DIIIR with quartic interactions in two-dimensional space*
Aug 2012 – June 2017	Bilkent University, Turkey: B.Sc. in electrical and electronics engineering; Minor in physics
Aug 2008 – June 2012	Ankara Science High School, Turkey: Mathematics and Science Program

EMPLOYMENT

June 2018 – June 2019	DMTEC, ETH Zurich, Switzerland: Student Research Assistant
June 2016 – Sept 2016	Fraunhofer IIS, Germany: Summer Internship
June 2015 – Aug 2015	GDS Engineering, Turkey: Summer Internship