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Holography and the Tensionless String

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Presented by

Bob Knighton
Master en Physique des Hautes Energies
Université Paris-Saclay

Born on 11.01.1996, citizen of the USA

Accepted on the recommendation of

Matthias R. Gaberdiel (examiner)
Niklas Beisert (co-examiner)

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To my parents,

for their constant love and support,

and to Alan,

who taught me so much.

Abstract

In this thesis we explore the holographic duality between string theory on AdS_3 spacetimes and two-dimensional conformal field theories through worldsheet methods. Specifically, we gather analytic evidence for the exact duality between tensionless string theory on $\text{AdS}_3\times\text{S}^3\times\mathbb{T}^4$ and the symmetric product orbifold $\text{Sym}^N(\mathbb{T}^4)$. The worldsheet sigma model simplifies drastically in this limit, and the string sigma model is described by a simple free field theory. After reviewing the worldsheet sigma model and deriving its free field realization, we utilize this simplification to compute various worldsheet amplitudes both in global AdS_3 and in certain D-brane backgrounds, and show that the results reproduce the dual CFT answer. We also introduce a mechanism whereby the symmetric orbifold CFT can reorganize itself into a semiclassical string theory on AdS_3, by considering correlation functions in a certain 'large-twist' limit.

Zussamenfassung

In dieser Arbeit untersuchen wir die holographische Dualität zwischen Stringtheorie auf drei-dimensionialer anti-de Sitter Raumzeit und zwei-dimensionalen konformen Feldtheorien durch Weltflächentechniken. Insbesondere sammeln wir Beweise dafür, dass die symmetrische Orbifaltigkeit $\text{Sym}^N(\mathbb{T}^4)$ genau dual zur sogenannten 'spannungslosen' Stringtheorie auf $\text{AdS}_3\times\text{S}^3\times\mathbb{T}^4$ ist. Die konforme Weltflächentheorie erweist sich in diesem Limes als erstaunlich einfach, und lässt sich sogar als eine freie Feldtheorie beschreiben. Nachdem wir die Weltflächentheorie einführen, benutzen wir sie um Amplituden zu berechnen, sowohl auf dem globalen AdS_3 Hintergrund, als auch auf Hintergründen mit D-Branen, und zeigen dass diese Amplituden genau den Amplituden der dualen CFT entsprechen. Außerdem führen wir einen Mechanismus ein, der erklärt, wie sich die Amplituden der symmetrischen Orbifaltigkeit in eine semiklassische Stringtheorie auf AdS_3 organisieren, indem wir einen bestimmten 'großen Twist' Limes betrachten.

Contents

Chapter 1

Introduction

1.1 Quantum gravity

The near-simultaneous discoveries of quantum physics and relativity paved way throughout the 20^{th} century to an unprecedented revolution in our understanding of the physical world. The mathematical frameworks of Quantum Field Theory (QFT) and General Relativity (GR) have provided us with a computational foundation for understanding nearly every process that occurs in nature. The standard model, the shining jewel of quantum field theory, describes with incredible accuracy the electromagnetic and nuclear forces at sub-atomic distances, while General Relativity accurately predicts the orbits of celestial bodies, the collapse of stars, and the expansion of the universe.

Despite their individual success, QFT and GR resist every attempt to be unified into a single consistent framework. While QFT accurately describes the electromagnetic and nuclear forces, it simply fails when applied to the gravitational force as described by general relativity. When computing, say, the scattering between two charged particles, the framework of the standard model can compute the interaction to arbitrary precision and at arbitrarily high energies, assuming that one is mathematically powerful enough to work out the appropriate integrals. When computing, on the other hand, the gravitational interaction between the same pair of particles, the resulting integrals diverge in a way that cannot be consistently regulated. At low energies, this does not pose a problem, and one can treat GR as an effective description of some complete high-energy theory. But as soon as one tries to compute the result of experiments done at higher energies, the results diverge. Gravity is *non-renormalizable.*

It is also not enough to simply ignore the effects of the gravitational field. Classical gravity is universal in the sense that every type of field in the standard model interacts with the gravitational field (there is no such thing as an *uncharged* particle under gravity). Thus, any process which contains standard model particles will eventually break down at high enough energies, at which the effects of quantum gravity will begin to dominate. The energy scale at which this occurs is roughly given by the Planck scale:

$$M_p = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{29} \, \frac{\text{GeV}}{c^2} \, .$$

Above this energy scale, quantum gravity dominates every scattering process, and thus without a renormalizable description of quantum gravity, we cannot describe events at these energies. Inversely, the Planck length

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35} \text{ m}$$

provides a rough bound on distances that we can understand without the full machinery of quantum gravity.

The necessity of the quantization of the gravitational field follows from a simple thought experiment. Consider a non-relativistic particle of mass m whose wavefunction ψ is sharply peaked at two points x_1, x_2 in space, with no real preference for one over the other. The wavefunction ψ can be modeled to a good approximation as

$$|\psi\rangle \approx \frac{1}{\sqrt{2}} |x_1\rangle + \frac{1}{\sqrt{2}} |x_2\rangle \ .$$

If an experiment is done to determine the position of the particle, then the particle will be found at position x_1 or x_2, with equal probability. Now, since the particle has some mass m, it will produce a gravitational field. If the particle is found at position x_1, then it will produce a gravitational field $\vec{g}_1$. On the other hand, if the particle is found at position x_2, it will produce a gravitational field $\vec{g}_2$. Assuming that our hypothetical scientist have the ability to measure gravitational fields with arbitrary precision, they will be able to deduce the position of the particle simply by measuring the gravitational field at several points.

What was the gravitational field before measurement? Clearly, the answer must depend on the result of the measurement of the quantum particle, since particles at different positions generate measurably different gravitational fields. Thus, the gravitational field must be, at least approximately, entangled with the position of the particle. That is, the wavefunction of the particle and the gravitational field, taken as one system, is

$$|\Psi\rangle = \frac{1}{\sqrt{2}} |x_1\rangle \otimes |\vec{g}_1\rangle + \frac{1}{\sqrt{2}} |x_2\rangle \otimes |\vec{g}_2\rangle \ .$$

It follows that the gravitational field $\vec{g}$ should be taken as a quantum object in its own right and, just as the electromagnetic field, must be quantized.

Since gravitational fields in general relativity describe the local geometry of spacetime, this implies that spacetime itself must, in some sense, be a quantum object. For small spacetime fluctuations, this is fine, since we can parametrize the metric tensor $g_{\mu\nu}$ as a small fluctuation around some background metric $g^{(0)}_{\mu\nu}$, i.e.

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \,.$$

If the metric $g^{(0)}_{\mu\nu}$ satisfies the classical equations of motion (the Einstein equations), then the gravitational action can be expanded in $h_{\mu\nu}$ with no linear term. We can formally treat the path integral then as a perturbative series describing a particle $h_{\mu\nu}$, which we call a *graviton*, propagating on a fixed background. We can then carry out the usual prescription of calculating S-matrix elements, decay rates, scattering amplitudes, and any other quantity of interest in perturbative quantum field theory. In principle, there will be more than one gravitational background $g^{(0)}_{\mu\nu}$ which solve the Einstein field equations. In principle, the gravitational path integral could be described by enumerating all such solutions $g^{(i)}_{\mu\nu}$ subject to some asymptotic boundary

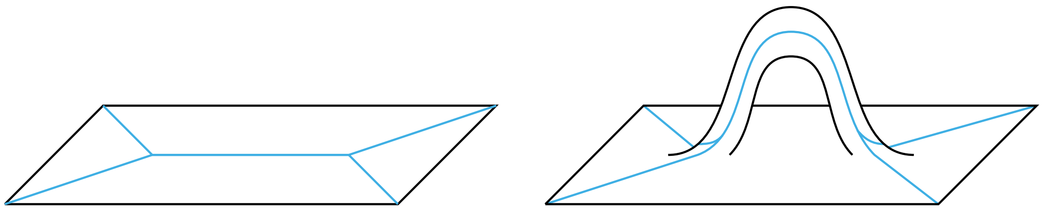

Figure 1.1: A 2-2 scattering process on two different spacetime topologies. In quantum gravity, both processes are in principle allowed.

conditions, and summing over them:

$$Z_{\text{gravity}} = \sum_i \int \mathcal{D}h_{\mu\nu}\, e^{-S[g^{(i)}+h]}\,,$$

where each path integral is then treated perturbatively around the saddle point $g^{(i)}_{\mu\nu}$. As mentioned above, however, this perturbative quantization will eventually break down due to the non-renormalizability of the gravitational interactions.

The problem of quantum gravity, however, is much deeper than the theory simply being non-renormalizable. Gravity does not just describe the local structure of spacetime, but also its *global* structure, i.e. the topology of spacetime. In principle, given some asymptotic boundary conditions, there are many different spacetime topologies which admit solutions to Einstein's equations, and a given quantum gravitational theory should sum over these topologies. These changes in spacetime topology can have measurable effects on the scattering of gravitationally-interacting particles. Consider, for example, the 2-2 scattering of particles in quantum field theory. These particles are prepared at asymptotic infinity, and scattered. At tree-level, we could have some kind of s-channel scattering, as on the left of Figure 1.1. In the limit that these particles are massive, their trajectories will approximately follow geodesics in the spacetime. However, since spacetime is not a fixed, rigid object in quantum gravity, we could also in principle consider the scattering event in the right of Figure 1.1. In such a process, the intermediate particle propagates along a handle which is not present in the original spacetime.

To understand quantum gravity fully, we need to abandon our semiclassical picture of a low-energy effective theory and explore quantum gravity near or below the Planck scale. At this scale, the concept of spacetime itself may not be a useful one anymore, and we will not be able to rely on the simple picture offered to us by general relativity.

The holographic principle

Nowhere is our lack of understanding in quantum gravity more prevalent than in the study of black holes. A black hole, formed form the collapse of dense matter, has a horizon beyond which all events become causally separated from those outside of the horizon, and from which no matter can escape. Classically, black holes also contain a singularity, where the curvature of spacetime becomes arbitrarily large. Since the classical description of gravity is only valid for curvatures much smaller than the Planck scale, the singularity of a black hole represents a location in the physical world where the effects of quantum gravity become important.

Figure 1.2: Left: two point-particles undergoing a fusion process. The resulting worldline is not smooth, and the Feynman rules dictate the strength of the interaction at the vertex. Right: two closed strings undergoing the same process. The singularity at the interaction vertex is 'smoothed out' by the finite size of the string.

Despite the classical fact that no matter or radiation can escape the horizon of a black hole, things are different in the quantum theory. Hawking [1] showed that quantum fields near the horizon of a black hole can undergo pair creation, in which one particle falls into the horizon, never to be seen again, while its twin propagates safely off to asymptotic infinity. From the point of view of a distant observer, the black hole therefore emits particles in the form of radiation. Furthermore, the resulting radiation is completely featureless, and is described only through its temperature

$$T = \frac{\hbar c^3}{8\pi G M k_B}\,.$$

Since radiation carries energy away with it, the black hole will eventually evaporate.

Since a black hole radiates completely featureless radiation with a specific temperature, it is effectively a thermodynamic system, one can assign to it an entropy. Remarkably, the entropy of a black hole is given by

$$S = \frac{k_B A}{4\ell_p^2}\,,$$

where A is the area of the black hole's event horizon. In typical thermodynamic systems, the entropy is an extensive variable, and grows linearly with the volume of the system. The fact that the entropy of a black hole is proportional to the area of the horizon then paints a startling picture: it implies that the information of the black hole is somehow completely stored at its event horizon. Such a system, in which the full information is stored completely at the boundary, is called a holographic system, and the inescapable conclusion is that, somehow, black holes are holographic.

The so-called *holographic principle* suggests that this is not true just for black holes, but rather for any system described by quantum gravity – the dynamics of a quantum-gravitational system in a volume V is completely encoded at the boundary ∂V. As we will discuss in more detail below, the holographic principle is completely realized in our best candidate for a theory of quantum gravity, namely string theory, in the context of the AdS/CFT correspondence.

1.2 String theory

To date, the most promising candidate for a theory of quantum gravity is given by string theory. Roughly speaking, string theory replaces the zero-dimensional particle degrees of freedom of usual quantum field theories with extended one-dimensional

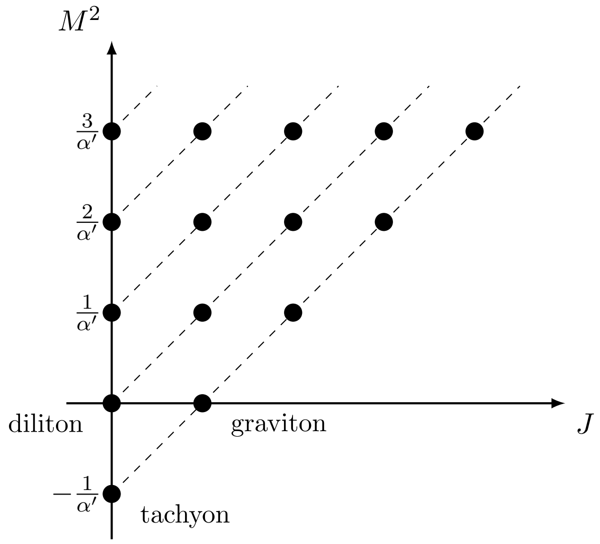

Figure 1.3: The spectrum of closed bosonic string theory. The tachyon state with $M^2 = -1/\alpha'$ is unphysical, and is a signal that bosonic string theory is formulated around the wrong vacuum. Such a state does not exist in a proper treatment of supersymmetric string theory, while the graviton state remains.

degrees of freedom, called strings, whose length ℓ_s is on the order of the Planck length. The nonzero length of these strings leads to a natural resolution of the divergences which arise in quantum field theory: the particle picture is only valid for lengths much larger than ℓ_s, and for smaller distances, the point-particles of field theory get 'smoothed out' into strings.

In a first-quantized approach, the particles of quantum field theory are described by world-lines: embeddings $\mathbb{R} \to \mathcal{M}$ of a particle's trajectory in spacetime. When allowed to propagate freely, such particles are described by an action principle which minimizes their proper length in $\mathcal{M}$. Interactions are included when a worldline is allowed to undergo a sudden change in topology, for example one particle splitting into two, or two particles fusing into one. The worldlines of interacting particles then become significantly more complicated, are no longer smooth, and are described by Feynman diagrams.

In string theory, the world-line is replaced by the embedding of a two-dimensional surface $\Sigma \to \mathcal{M}$, a *worldsheet*. While classical point particles are restricted classically to follow paths of least spacetime distance, strings are restricted to follow paths of least spacetime *area*. Since there are just two smooth connected topologies in one dimension (the interval and the circle), strings come in two flavors: open strings and closed strings. Just as point-particles can decay and fuse in quantum field theory, strings are also allowed to undergo such topology changing operations. One string can split into two, two strings can fuse into one, a closed string can become an open string, and an open string can become a closed string. In contrast to the case of point particles, however, all of these processes still result in a smooth worldsheet. For example, a single closed-string splitting into two is described by the pair-of-pants worldsheet shown in Figure 1.2.

The richness of string theory lies in the nonzero length of the string, since the motion of a string in spacetime has an infinite number of degrees of freedom. The

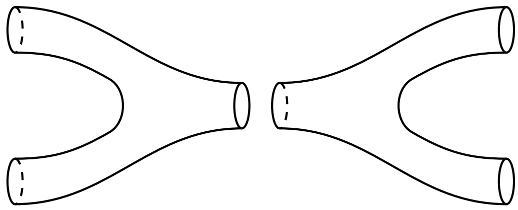

Figure 1.4: A four-point string interaction can be broken up into the union of two three-point string interactions.

simplest are the center-of-mass position and momenta x, p of the string, which survive in the limit that the string is taken to zero length. However, the string is also allowed to vibrate arbitrarily around its center-of-mass, and these vibrations are composed of an infinite tower of harmonic modes. From the point of view of an observer much larger than the string scale, these modes appear as individual species of particles, carrying different spins J and mass M. For the closed bosonic string, the spectrum of the particle-like excitations are shown in Figure 1.3. The mass and spin eigenstates of the theory lie on lines called *Regge trajectories*, which have slope $1/\alpha'$ and intercept n/α'. Remarkably, this spectrum contains a massless symmetric spin-2 state, which behaves precisely like a graviton [2, 3]. It turns out that more is true: in order for a string theory on a background manifold $(\mathcal{M}, g)$ to admit a consistent quantization, the background must satisfy the Einstein field equations [4]. This is the first hint that string theory is a theory of quantum gravity.

String interactions are, as we mentioned above, mediated by processes like the splitting and joining of strings. Sticking to closed strings for now, a basic 3-point interaction can be described by the pair-of-pants interaction in Figure 1.2. Just as in quantum field theory, we can assign a coupling constant g_s to such a three-point interaction. It is a basic topological fact that we can decompose any sufficiently complicated worldsheet[1] into a union of such three-point interactions. For example, a four-point string interaction can be decomposed into two three-point interactions, as shown in Figure 1.4. As an immediate consequence, the strength of a four-point vertex in string theory is g_s^2. This is in extreme contrast to quantum field theories, where the three- and four-point interaction vertices are, in principle, allowed to be completely independent of each other. In string theory, there is exactly one coupling constant, the three-point coupling, and all higher-point interactions are given by powers of the three-point coupling. This observation extends to loop level, where the string worldsheet is allowed to have nontrivial genus. The coupling associated to any string interaction is simply g_s, raised to the power of the number of pairs-of-pants required to construct the worldsheet. This number is given by $n+2g-2 = -\chi$, the negative of the Euler characteristic. As a consequence, the strength of any string 'Feynman diagram' is of the order:

$$\text{string intereraction} \sim g_s^{-\chi}.$$

The above statement is an example of a much more general principle, which is that string theories, in contrast to standard quantum field theories, have very few free parameters. In the case of closed bosonic string theory, the only free parameters are the string coupling g_s and the string length $\ell_s = 1/\sqrt{\alpha'}$. Even the number of

[1]Read: worldsheets with negative Euler characteristic.

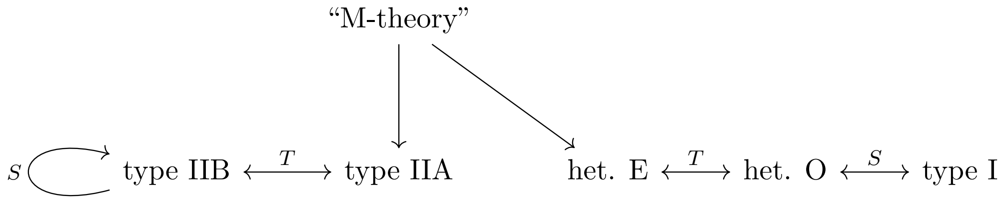

Figure 1.5: The web of superstring dualities.

spacetime dimensions is not free, but is fixed by consistency conditions in the quantum theory: one needs $D = 26$ in bosonic string theory and $D = 10$ in superstring theory for consistent quantization.

Beyond perturbation theory

String theory provides a promising description of quantum gravity, as it contains in its spectrum a massless spin-2 graviton. However, the downside of string theory is that, as of yet, we do not posess a nonperturbative description, as the worldsheet description is only valid for small string coupling.

However, there are many hints for the existence of a non-perturbative completion of string theory. After the development of superstring theories [5–12] it was realized that, perturbatively, there are five superstring theories which exist in ten dimensions: type I, type IIA, type IIB, and the two heterotic strings with gauge groups $E_8 \times E_8$ (*Heterotic E*) and $\mathrm{Spin}(32)/\mathbb{Z}_2$ (*Heterotic O*) [13–16]. One of the most striking features of superstring theories is that these five models all seem to be related to each other through a web of *dualities* [17–25], shown schematically in Figure 1.5. This web of dualities is unified by an 11-dimensional theory, known as M-theory, whose low-energy description is $\mathcal{N} = 1$ supergravity in eleven dimensions.

The so-called S-dualities are particularly interesting, as they relate string theories with different values of the string coupling. For example, the S-duality on type IIB string theory relates the string coupling g_s with its inverse $1/g_s$. Thus, by studying string theory at small values of the coupling, one can tease out information about the strong-coupling behavior of the theory. Thus, not only does string theory provide a tentative perturbative description of full quantum gravity, but also a non-perturbative one.

More than just strings

As the name suggests, string theory is a theory of strings. However, nonperturbatively string theory contains a much larger zoo of extended objects called branes. The role in life of a brane is to be an object on which an open string can end. The motion of an open string can be constrained to some $(p+1)$-dimensional sublocus of spacetime, and this boundary condition is perfectly consistent, both classically and upon quantization. However, as opposed to being simple geometric loci on which strings can end, these so-called Dp-branes taken on their own dynamics nonperturbatively [26]. The tension of a D-brane (i.e. the amount of energy required to perturb a unit of p-dimensional area) is inversely proportional to the string coupling g_s, and so in the weakly-coupled limit of string theory, branes behave like infinitely massive, non-dynamical objects.

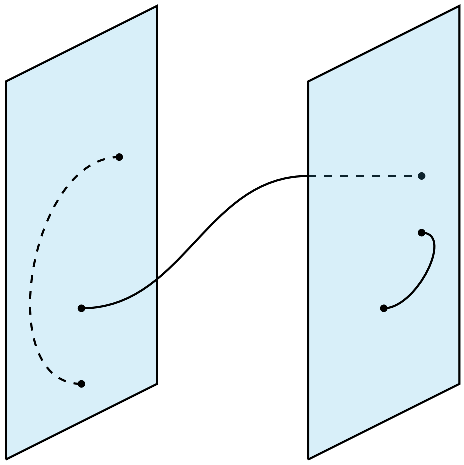

Figure 1.6: Open strings ending on a pair of D2-branes.

Branes can be used to realize non-abelian symmetry in string theory. Given N branes stacked upon each other, we can consider a string whose endpoints end on two of these branes, say i and j. Let us denote this string state by $|i, j\rangle$, where we keep in mind that strings are oriented, and so the labels i, j are ordered. Two strings $|i, j\rangle$ and $|j, k\rangle$ can fuse into a string $|i, k\rangle$. Thus, we can treat the endpoints of the strings as components of $N \times N$ matrices $M^i{}_j$. Fusion of strings is given by matrix multiplication. Furthermore, since all of the branes are identical and located at the same point, there is an overall $\mathrm{U}(N)$ symmetry rotating the labels i into one another. In string theory, this $\mathrm{U}(N)$ symmetry becomes promoted to a gauge symmetry. In fact, it can be shown that the 'world-volume' theory describing the low-energy dynamics of the brane (equivalently the dynamics of open strings moving on the brane) is given by $\mathrm{U}(N)$ (super) Yang-Mills theory.

1.3 The AdS/CFT correspondence

As we discussed above, the holographic principle posits that a gravitational system formulated on a volume $\mathcal{M}$ can be completely described by the dynamics of the boundary $\partial\mathcal{M}$. Maldacena [27] promoted this vague principle to a concrete proposal, known as the AdS/CFT correspondence (see [28] for an extensive review). Roughly, this correspondence relates a UV complete theory of quantum gravity (however that may be defined) on d-dimensional anti-de Sitter spacetime to a conformal field theory whose degrees of freedom completely live on the boundary $\partial\mathrm{AdS}_d$.

The most famous example of this duality is the one originally proposed by Maldacena, which relates IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ to $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions with gauge group $\mathrm{U}(N)$. To motivate it, one considers a stack of N D3 branes in (flat) ten dimensional space. These branes are taken to be completely coincident. Fluctuations of these branes are described at low energies by the dynamics of open strings which end on them. If there are N such branes, the low-energy dynamics of the open strings are described by a four-dimensional gauge theory, specifically $\mathcal{N} = 4$ super Yang-Mills theory.

On the other hand, we can consider the same brane setup from the point of view of the low-energy description of closed strings, namely IIB supergravity in ten dimensions. From the point of view of supergravity, a stack of N branes will backreact on the local geometry, and source a nontrivial gravitational background. Specifically, these branes will source a black hole-like background metric

$$\mathrm{d}s^2 = H^{-1/2}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + H^{1/2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{\mathrm{S}^5}\right) \,, \quad H = 1 + \frac{L^4}{r^4}$$

where we have split the coordinates into Minkowski-like coordinates x^μ with $\mu = 0, \ldots, 3$ and hyperspherical-like coordinates r, Ω_{S^5}. This geometry is very similar to extremal black hole geometries in four-dimensions, and has a horizon at $r = 0$. The near-horizon geometry is given by

$$\mathrm{d}s^2 \xrightarrow{r \ll L} \frac{r^2}{L^2}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{L^2}{r^2}\mathrm{d}r^2 + L^2\mathrm{d}\Omega_{\mathrm{S}^5} \,.$$

This is the metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$, where both factors have the same radius of curvature L. The insight of Maldacena was that the open string picture of strings ending on the stack of D-branes should be completely captured by the description of the closed string picture of the near-horizon geometry of the branes in supergravity. What follows is a proposed duality in the low-energy limit:

$$\mathrm{U}(N)\ \mathcal{N} = 4 \text{ super Yang-Mills}$$
$$\Longleftrightarrow$$
$$\text{IIB string theory on } \mathrm{AdS}_5 \times \mathrm{S}^5 \,.$$

Quantities computable on either side of the duality are related by the so-called *holographic dictionary* [29]. For the case of the $\mathrm{AdS}_5 \times \mathrm{S}^5/\mathcal{N} = 4$ SYM correspondence, the holographic dictionary relates

$$\frac{1}{N} = \frac{g_s}{4\pi^2}\frac{\ell_s^2}{L^2} \,,$$

where g_s is the string coupling, ℓ_s is the string length, and L is the radius of curvature of both the AdS_5 and S^5 geometry. Furthermore, the 't Hooft coupling $\lambda = g_{\mathrm{YM}}N^2$ is given by

$$\lambda = 4\pi^2\frac{L^2}{\ell_s^2} \,.$$

The holographic dictionary also instructs us on how to compare things like scattering amplitudes on both sides of the correspondence. A correlation function of operators $\mathcal{O}_i$ should be computed by a bulk quantum gravity partition function for which the bulk fields have specified boundary conditions:

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = Z_{\mathrm{grav}}[\mathcal{M}] \,.$$

In string theory, the gravitational path integral is computed from the scattering of strings which extend toward the boundary and scatter in the bulk. This is schematically shown in Figure 1.7.

The AdS/CFT correspondence is an example of a *strong/weak* duality. If we take N to be large while keeping the 't Hooft coupling λ fixed, then the holographic dictionary tells us that the string theory in the bulk is weakly coupled ($g_s \to 0$), and that the ratio L/ℓ_s is determined by the 't Hooft coupling. If the 't Hooft coupling λ is large, then the boundary CFT is stringly coupled, while the strings are small

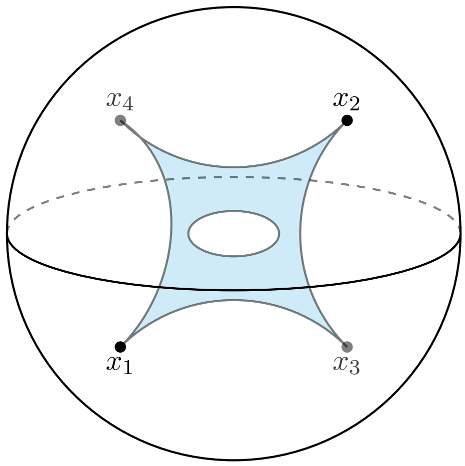

Figure 1.7: The holographic dictionary for computing correlators in the dual CFT.

compared to the curvature of the bulk geometry, and we can thus treat strings as point particles, effectively giving us semiclassical supergravity. On the other hand, if the ‘t Hooft coupling is small, the boundary CFT is weakly coupled, but the bulk string theory is populated by very large strings, and thus the supergravity approximation is completely unusable. The strong/weak nature is the primary reason for why the AdS/CFT correspondence, to this day, is still not completely understood: when one side is tractible, the other side is a mess, and vice-versa.

Because of the strong/weak property of the AdS/CFT correspondence, proving the validity of the duality seems an extremely difficult task at any point of the moduli space, since both sides are never under complete analytic control at the same time. One possiblity to circumvent this problem would be to consider more simple examples of AdS/CFT for which the amount of symmetry on both sides is greatly enhanced. This occurs, for example, in the case of AdS_3/CFT_2. Two dimensional conformal field theories stand out among their higher-dimensional cousins due in large part to an infinite enhancement of the symmetry algebra. While the global conformal group in two dimensional flat space is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, the algebra of *local* conformal transformations is infinite-dimensional, since every holomorphic map f is conformal (similarly for anti-holomorphic maps). Thus, even if the boundary theory in the AdS_3/CFT_2 is strongly coupled, it is generically under much more analytic control than higher-dimensional examples, simply due to this infinite enhancement of symemtry.

On the gravity side, AdS_3 also offers a simplification as quantum gravity in three-dimensions is, in a certain sense, topological. In a semiclassical limit, quantum gravity on AdS_3 is governed by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,,$$

with negative cosmological constant. Taking traces implies $R = 6\Lambda$, and so

$$R_{\mu\nu} = -2\Lambda g_{\mu\nu}\,.$$

Now, in three dimensions, the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ is determined completely by the metric g and the Ricci tensor via

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho} R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\rho} - g_{\nu\rho} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\rho} - \frac{R}{2} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)\,.$$

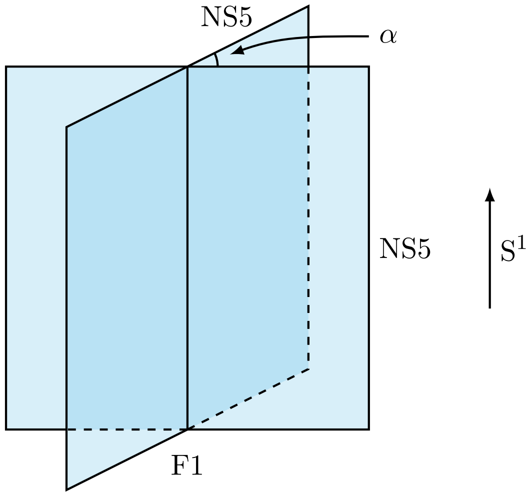

Figure 1.8: A cartoon representation of the NS5/F1 system in type IIB supergravity.

As a consequence, solutions to the Einstein equations satisfy

$$R_{\mu\nu\rho\sigma} = \Lambda \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) .$$

That is, all solutions to Einstein's equations have constant negative curvature everywhere. All manifolds of constant negative curvature in three-dimensions are locally isometric, and in particular there are no propagating degrees of freedom in 3D classical gravity, and in particular there are no long-distance interactions between massive particles, and also no gravitational waves. However, if the bulk manifold $\mathcal{M}$ has an asymptotic boundary $\partial\mathcal{M}$, there are *large* diffeomorphisms that survive as true degrees of freedom in the gravitational theory. In the classical analysis of Brown and Henneaux [30], it was shown that these 'boundary gravitons' generate an asymptotic symmetry algebra which is identical to the algebra of local conformal symmetries in a 2D CFT living on the boundary, namely the Virasoro algebra. The central charge of the boundary Virasoro algebra is given by the famous Brown-Henneaux formula:

$$c = \frac{3L}{2G} ,$$

where $L = \sqrt{-1/\Lambda}$ is the radius of curvature of the AdS_3 spacetime and G is the three-dimensional Newton constant.

Just as one can construct an example of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence by stacking a set of D3 branes together, one can similarly construct an example of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence by considering the so-called F1-NS5 system in type IIB string theory, shown in Figure 1.8. In this system, a stack of Q_5^+ NS5-branes is intersected along a with a stack of Q_5^- NS5-branes at an angle α. Along their intersection, one places Q_1 F1-strings. Just as in the stack of D3 branes, we can consider this system either from the point of view of the open strings living on the intersection, or from the point of view of the closed strings describing gravity in the near-horizon limit of the geometry sourced by the brane system. From the closed string side, the near-horizon geometry is of the form $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$, where the radii R, R_1, R_2 of AdS_3 and the two three-spheres are related via

$$R_1 = \frac{R}{\cos\alpha} , \quad R_2 = \frac{R}{\sin\alpha} ,$$

and specifically satisfy the pythagorean relation

$$\frac{1}{R^2} = \frac{1}{R_1^2} + \frac{1}{R_2^2} \,.$$

Thus, the theory of closed strings in the near horizon geometry should be described by IIB string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$. Of specific interest is the limit $\alpha \to 0$, for which the radius R_2 of the second 3-sphere diverges, effectively degenerating into three flat directions. These directions can be compactified on circles, and the result is that the spacetime becomes

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1 \xrightarrow{\alpha \to 0} \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4 .$$

On the open string side, the dynamics of the F1-NS5 system in the $\alpha \to 0$ limit has been explored extensively, and there is a large body of evidence that the dual CFT lives in the conformal moduli space of the *symmetric orbifold theory* $\mathrm{Sym}^N(\mathbb{T}^4)$, where $N = Q_1 Q_5$ [31–37] (see also [38] for a review). The proposed duality in three-dimensions is then

$$\begin{gathered} \text{IIB string theory on } \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4 \\ \Longleftrightarrow \\ \text{CFT in the moduli space of } \mathrm{Sym}^N(\mathbb{T}^4) \,. \end{gathered}$$

This duality admits a similar holographic dictionary to that of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. The relationship between various parameters in the theory is given by

$$g_s = \frac{1}{\sqrt{Q_1 Q_5}} \,, \quad Q_5 = \frac{R^2}{\ell_s^2} \,.$$

1.4 The tensionless limit

The string length ℓ_s sets the fundamental scale of string theory. Observers much larger than the string length can approximate strings as an infinite tower of particles following Regge trajectories (Figure 1.3). Particles in this tower are either massless or have extremely large masses of order $1/\ell_s$. Thus, an observer much larger than the string scale will only see the massless states of string theory: the supergravity sector. However, as we mentioned above, in order to truly understand quantum gravity, we need to be able to move beyond the supergravity regime, and study string theory at sub-Planckian scales.

For extremely large strings (or extremely small observers), all of the string states become important. In the ‘tensionless’ limit $\ell_s \to \infty$, every state in the string spectrum becomes massless, and the particle picture of string theory breaks down spectacularly. The effective action describing string interactions at low energies will include interactions between states of arbitrary spin, none of which are suppressed, since the string coupling g_s is universal for all string states. This is a tell-tale sign that the tensionless limit is highly nonlocal – indeed, it is described by highly nonlocal objects, namely infinitely long strings.

However, there are some indications that the tensionless limit actually represents a simplification of string theory. The appearance of massless states at special points in parameter space usually signals an enhancement of symmetry. Indeed, the enhancement of the massless spectrum in the tensionless limit has been argued to be a sign of the emergence of a higher-spin symmetry [39–43].

In the context of holography, the string length is related to the ‘t Hooft coupling of the dual CFT. In the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality, the string length is given by

$$\ell_s^2 = \frac{4\pi^2 L^2}{\lambda} \,.$$

Thus, the tensionless limit of string theory in anti-de Sitter spacetime is holographically dual to a *free* conformal field theory on the boundary. This is another explanation for the enhancement of symmetry in the tensionless limit: free field theories simply have more symmetries than their interacting counterparts.

In the near-horizon geometry of the F1-NS5 system, the string length is given by the amount Q_5 of NS-NS flux in the background in units of the AdS_3 radius:

$$\ell_s^2 = Q_5 R^2 \,.$$

In contrast to the five-dimensional example, the above relation implies that the string length is *quantized*, since the amount of NS-NS flux Q_5 is an integer.[2] It we take the amount of NS-NS flux to zero, then the resulting supergravity background has no horizon, and so we do not end on anti-de Sitter space. Thus, the closest thing to a tensionless limit in $\mathrm{AdS}_3/\mathrm{CFT}_2$ with pure NS-NS flux is to take Q_5 to its smallest value, namely $Q_5 = 1$. In this limit, the string length and the AdS_3 radius coincide.

Holographically, we know that the dual of the tensionless limit should lie somewhere in the moduli space of the symmetric orbifold $\mathrm{Sym}^N(\mathbb{T}^4)$ with $N = Q_5 Q1 = Q_1$. The symmetric orbifold $\mathrm{Sym}^N(\mathbb{T}^4)$ is thought to be most most symmetric CFT in its moduli space [44], and so it is natural to guess that the tensionless limit of string theory, whereby the worldsheet theory should admit an enhancement of its massless spectrum, is dual precisely to the symmetric orbifold itself. It was shown in [45] that this is indeed the case, and there is an exact duality

$$\text{tensionless string theory on } \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$$
$$\Longleftrightarrow$$
$$\text{the symmetric orbifold } \mathrm{Sym}^N(\mathbb{T}^4) \,.$$

The evidence put forward by [45] is that the full spectrum of physical states on the worldsheet matches the so-called *single-particle* sector of the symmetric orbifold theory, which is the part one would expect to be dual to a single worldsheet on global AdS_3. Since this original observation, the amount of evidence for this correspondence has only grown. Not only have the spectra of the two theories been shown to agree, the correlation functions, higher-genus partition functions, and individual low-lying states have been shown to be in perfect correspondence.

The reason that so many observables have been calculated on both sides of this correspondence is owed in large part to the relative simplicity of the string theory worldsheet description. As it turns out, precisely at $Q_5 = 1$ units of NS-NS flux,

[2] The difference between these two cases is that the AdS_5 background is supported by Ramond-Ramond flux. While Ramond-Ramond flux is quantized, it is quantized in integer units of the string coupling, and so is effectively continuous in weakly-coupled string theory.

the worldsheet theory simplifies enormously – the spectrum of physical states on the worldsheet is drastically decreased, and the theory itself can be recast from a highly-interacting sigma model on a curved background to a *free* theory. Furthermore, the calculation of any observable in the theory seems to be perturbatively exact. In fact, the perturbative expansion of most observables of interest *truncates* into a polynomial in g_s. Thus, not only does the above duality give us a rare example of an explicitly computable AdS/CFT correspondence, but also gives us a description of quantum gravity in a deeply stringy regime with no non-perturbative corrections.

1.5 Outline of this thesis

The purpose of this thesis is two-fold. First, and most practically, it serves as a report of the research done by the author during the four years of his PhD. Secondly, it is intended to serve as an introduction to some of the progress made in our understanding of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence in the last few years. As such, the thesis is roughly divided into two parts.

The first part, spanning Chapters 2-4, is intended to be a self-contained (but in no sense complete) introduction to aspects of conformal field theory, string theory, and a review of the construction of the tensionless limit of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. Specifically, in Chapter 2, orbifold CFTs (specifically the symmetric orbifold) and Wess-Zumino-Witten models are introduced. In Chapter 3, we review the worldsheet description of strings on AdS_3 backgrounds supported by pure NS-NS flux, making constant use of the $\mathrm{SL}(2,\mathbb{R})$ WZW model. We also introduce the RNS formalism of AdS_3 superstrings, as well as the so-called 'hybrid' formalism, which is necessary to describe string theory at the tensionless point. In Chapter 4, we introduce the tensionless string on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, and review the derivation that the worldsheet spectrum reproduces the spectrum of the dual symmetric orbifold CFT. Throughout these chapters, a basic knowledge of string theory and 2D conformal field theory is assumed.

The second part, spanning Chapters 5-7, reports the progress made by the author in attempting to prove and more deeply understand the tensionless limit of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. In Chapter 5, we explore correlation functions in the tensionless worldsheet theory, and show that they reproduce the boundary result, both at tree-level and at higher-genus. In Chapter 6, we explore the role that D-brane backgrounds in AdS_3 play in the tensionless worldsheet theory, and show, through an explicit calculation of both the open-string worldsheet spectrum and closed-string disk correlation functions, that D-branes which intersect the boundary of AdS_3 are dual to boundary states in the dual orbifold theory. In Chapter 7, we change gears slightly and shift focus from the bulk theory to the boundary CFT. We explore the limit of CFT correlators in which the twists of the symmetric orbifold operators become large. We find that, in this limit, the correlation functions naturally reorganize themselves into integrals over the moduli space of curves, weighted by a Nambu-Got-like action. We also find, through a semiclassical analysis of strings in AdS_3, that the resulting Nambu-Goto action can indeed be recovered by the motion of a classical string in the bulk spacetime. This effectively amounts to a refinement of the 't Hooft argument [46] that large N field theories rearrange themselves into string theories.

Finally, in Chapter 8, we provide a brief summary, as well as comments about unanswered questions and potential future directions of research.

1.6 List of Publications

The present thesis consists of pedagogical exposition based on established literature, as well as new results coming from original published work by the author. The original results are based on the publications listed below.

- [47] A. Dei, M. R. Gaberdiel, R. Gopakumar, and B. Knighton
  Free field world-sheet correlators for AdS_3
  JHEP **02** (2021), arXiv:2009.11306

- [48] M. R. Gaberdiel, R. Gopakumar, B. Knighton, P. Maity
  From Symmetric Product CFTs to AdS_3
  JHEP **05** (2021), arXiv:2011.10038

- [49] B. Knighton
  Higher genus correlators for tensionless AdS_3 *strings*
  JHEP **04** (2021), arXiv:2012.01445

- [50] M. R. Gaberdiel, B. Knighton, J. Vošmera
  D-branes in $AdS_3 \times S^3 \times \mathbb{T}^4$ *at* $k = 1$ *and their holographic duals*
  JHEP **12** (2021), arXiv:2110.05509

- [51] B. Knighton
  Classical geometry from the tensionless string
  JHEP **3** (2023), arXiv:2207.01293

Certain parts of this thesis are taken freely from the above publications. In addition, some parts of the thesis, specifically the discussion of symmetric orbifold partition functions in Chapter 2 and the discussion of the hybrid formalism in Chapter 3, borrow snippets from as-of-yet unpublished work, specifically:

- A. Kanagrias, J. Kames-King, B. Knighton, M. Usatyuk
  The Lion, the Witch, and the Wormhole:
  Ensemble averaging the symmetric product orbifold
  (to appear)

- B. Knighton
  Lectures on the hybrid formalism of superstring theory
  (contribution to YRISW 2022: Taming the String Worldsheet, to appear)

1.7 Acknolwedgements

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Miscellaneous

In addition to the above categories, there are also various people and non-people who I would like to thank, again in no particular order: an abandoned red bridge in the middle of nowhere, the people of tEp, Pembroke College Cambridge, Simone, a particular Chilean owl, Ilse, and Alfred the Uber driver.

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Chapter 2

Conformal field theory

In our study of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, the most useful tool will be the formalism of two-dimensional conformal field theory. Both the worldsheet theory of the string in AdS_3 and the dual field theory on the boundary are studied through the lens of this formalism. The duality we are specifically interested in is

$$\begin{gathered}\text{tensionless IIB strings on } \mathrm{AdS}_3\times\mathrm{S}^3\times\mathbb{T}^4\\ \Longleftrightarrow\\ \text{the symmetric orbifold } \mathrm{Sym}^N(\mathbb{T}^4)\,.\end{gathered} \tag{2.1}$$

On the string theory side, the worldsheet field theory is a sigma model on the spacetime $\mathrm{AdS}_3\times\mathrm{S}^3$ supported by pure NS-NS flux. The geometries AdS_3 and S^3 are special among supergravity backgrounds because they both admit the structure of group manifolds. Specifically,

$$\mathrm{AdS}_3\cong\widetilde{\mathrm{SL}(2,\mathbb{R})}\,,\quad \mathrm{S}^3\cong\mathrm{SU}(2)\,, \tag{2.2}$$

where $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. The above equivalence is not just a topological fact, but a geometric one: the metrics on AdS_3 and S^3 are isometric to the canonical bilinear form on $T\mathrm{SL}(2,\mathbb{R})$ and $T\mathrm{SU}(2)$, namely the Killing forms on these groups. The equivalence between the spacetime geometry on which the strings propagate and a group manifold turns out to represent a massive simplification in the quantization of the theory, and the symmetry algebra on the worldsheet becomes greatly enhanced as a result. The motion of strings on group manifolds is a widely studied subject, and the models describing these strings (with pure NS-NS flux) are called Wess-Zumino-Witten models, or WZW models.

On the boundary, the dual CFT is described by a 'symmetric orbifold' theory. Briefly, such theories are constructed by considering a 'seed' theory X and taking N copies to obtain $X^{\otimes N}$. This new theory has many more degrees of freedom, and enjoys a discrete S_N symmetry permuting the copies of X. The symmetric orbifold theory

$$\mathrm{Sym}^N(X):=X^{\otimes N}/S_N \tag{2.3}$$

is the result of gauging this global S_N symmetry. Conformal field theories for which a discrete symmetry group G is gauged are known as *orbifold theories*, and their structures are extremely rich. The gauging of the discrete group G not only projects the Hilbert space onto states which are invariant under the action of G but also, if

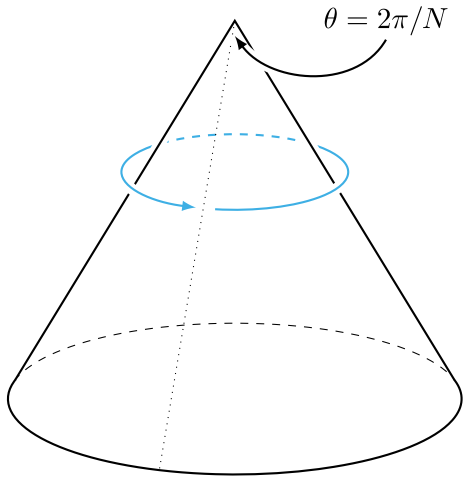

Figure 2.1: The orbifold $\mathbb{R}^2/\mathbb{Z}_N$ as a cone.

the theory is quantized on a circle, gives rise to new 'twisted' states which did not exist in the original, un-gauged CFT. These twisted states are the reason for the richness of orbifold theories, and the twisted states of the symmetric orbifold will be a central object of study in this thesis.

In this chapter, we introduce the basic properties of both orbifold theories (specifically the symmetric orbifold) and WZW models necessary for the study of $\mathrm{AdS}_3/\mathrm{CFT}_2$ from a worldsheet approach. While aiming to be as pedagogical as possible, the treatment will be far from complete. As mentioned in the introduction, we assume that the reader has at least a basic knowledge of conformal field theory. See, however, [52, 53] for canonical introductions, and also [54] for a slightly more mathematical exposition.

2.1 Orbifolds

Given a conformal field theory X, it is often the case that X enjoys some discrete symmetries parametrized by some group G. For example, if X is a CFT of D free scalars, then there is always a natural $\mathbb{Z}_2$ symmetry which acts as $\Phi \to -\Phi$.[1] Given such a theory with symmetry group G, we can construct a new theory X/G obtained by *gauging* the symmetry group G. Such a CFT is known as an *orbifold* CFT.

For concreteness, let's for the moment assume that our CFT X is a sigma model, i.e. that it is a field theory whose fields are maps $\Phi : \Sigma \to M$, where M is some manifold. If M admits some symmetry group G, then the CFT X naturally does as well, since we can just take actions of G on the fields Φ. We 'gauge' the symmetry G by identifying points p_1, p_2 on M to be physically equivalent if $p_1 = g \cdot p_2$. The resulting target space is the topological space

$$M/G := M/(p \sim g \cdot p)\,. \tag{2.4}$$

Now, since we have not specified that G acts freely, the action of G on M may have fixed points. Because of this, the space M/G is potentially not a smooth manifold, but may have isolated 'conical' singularities. The canonical example is the plane $\mathbb{R}^2$

[1]This is a symmetry regardless of whether the scalars are compact.

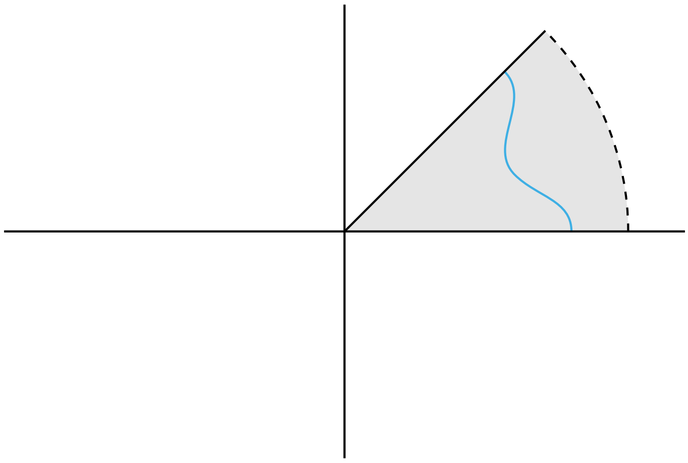

Figure 2.2: A string moving in the orbifold $\mathbb{R}^2/\mathbb{Z}_n$. The string is open when considered as a string on $\mathbb{R}^2$, but is closed in $\mathbb{R}^2/\mathbb{Z}_n$. From the point of view of the $\mathbb{R}^2$, this string obeys a twisted boundary condition, and must be included as a valid closed-string configuration in the orbifold path integral.

equipped with the $\mathbb{Z}_N$ action which rotates points around the origin by an angle of $2\pi/N$. The only fixed point is the origin itself, and the quotient space $\mathbb{R}^2/\mathbb{Z}_N$ looks like a cone, as in Figure 2.1.

In the field theory, we implement the gauging $M \to M/G$ by identifying the fields

$$\Phi(z) \sim g \cdot \Phi(z) \tag{2.5}$$

as being physically equivalent, since they represent the same point in the orbifold M/G. This can lead to some interesting features which are unique to orbifold CFTs. Assume that the worldsheet Σ is parametrized by coordinates (σ, τ), and assume that σ is compact, i.e. $\sigma \sim \sigma + 2\pi$. This is the case, for example, with closed strings. Then in a typical CFT, one imposes the periodicity condition

$$\Phi(\sigma, \tau) = \Phi(\sigma + 2\pi, \tau)\,. \tag{2.6}$$

However, in an orbifold, the *twisted boundary condition*

$$\Phi(\sigma + 2\pi, \tau) = g \cdot \Phi(\sigma, \tau) \tag{2.7}$$

is completely possible. From the point of view of the parent target space M, such a configuration would represent an open string, but on the orbifold M/G, such a string is closed and therefore the above boundary condition is consistent in the orbifold theory (see Figure 2.2). We will discuss in a moment what the existence of these boundary conditions implies for the path integral of the orbifold theory.

The existence of twisted boundary conditions are single-handedly responsible for the richness of orbifold CFTs, and they play a major role in the allowed spectrum of the theory. We will not explore this in detail both in the path integral approach (for which the twisted boundary conditions play the role of homotopy classes of principal G-bundles), and in the algebraic/operator approach (in which the twisted boundary conditions correspond to 'sectors' of the theory, which can be understood via the representation theory of G). While the above discussion was motivated by consider a CFT X which is a sigma model on some manifold M, orbifold field theories do

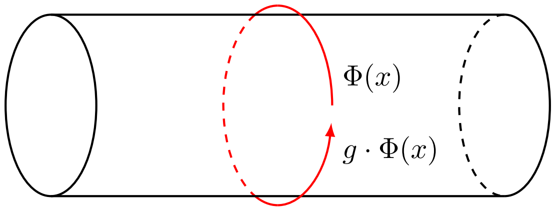

Figure 2.3: An orbifold CFT M/G allows for twisted boundary conditions of fundamental fields around non-contractible loops.

not have to have a geometric origin, and can occur in any CFT which has a finite (or, indeed, infinite) symmetry group G.

Path integral approach

In the path integral approach to field theory, we consider the space $\mathcal{A}$ of all field configurations Φ on Σ equipped with a suitable measure, and integrate over this space, weighted by a factor of $e^{-S[\Phi]}$ for each field configuration. Schematically, the path integral of the CFT X is then given by

$$Z_X = \int_{\mathcal{A}} \mathcal{D}\Phi \, e^{-S[\Phi]} \,, \tag{2.8}$$

Naively, then, the path integral of the orbifold theory X/G should be given as an integral over the quotient space $\mathcal{A}/G$ of equivalence classes of field configurations $[\Phi]$ under the action of G, i.e.

$$Z_{X/G} \stackrel{?}{=} \int_{\mathcal{A}/G} \mathcal{D}\Phi \, e^{-S[\Phi]} \,. \tag{2.9}$$

Since G is a symmetry of the path integral measure and of the action, we would then be able to obtain the path integral of X/G by dividing the path integral of X by the order $|G|$ of the orbifold group as not to over-count physically equivalent states, i.e.

$$Z \stackrel{?}{=} \frac{1}{|G|} \int \mathcal{D}\Phi \, e^{-S[\Phi]} \,. \tag{2.10}$$

It turns out that the above construction is actually incomplete. The main reason is that the space of field configurations on X/G is not the quotient $\mathcal{A}/G$. The basic reason for this is that there are field configurations on X/G which do not lift to single-valued field configurations on X. Indeed, if X has a noncontractible cycle γ based at some point $x \in \Sigma$, then a single-valued field configuration on X/G can obtain a monodromy when transported along γ when viewed as a field configuration on X. Put another way, if we choose a representation $\Phi(x)$ of the orbifold field, then it is completely admissable for $\Phi(\gamma \cdot x)$ to equal $g \cdot \Phi(x)$ for some nontrivial element of g, since both $\Phi(x)$ and $g \cdot \Phi(x)$ lie in the same equivalence class of X/G. An example of such a twisted boundary condition is shown in Figure 2.3.

Twisted boundary conditions like those in Figure 2.3 are characterized by assigning a monodromy g to each loop γ based at a point x, such that transporting the field Φ around γ returns $g(\gamma) \cdot \Phi(x)$. The assignment of an element $g \in G$ to each loop satisfies the following properties:

- If γ is the trivial loop, then $g(\gamma) = \mathrm{id}$.
- The monodromy $g(\gamma)$ depends only on the homotopy class of the loop γ, i.e. $g(\gamma)$ is invariant under smooth deformations of the loop γ.
- Given two loops γ_1, γ_2 based at the same point, $g(\gamma_1 \circ \gamma_2) = g(\gamma_1)g(\gamma_2)$, where $\gamma_1 \circ \gamma_2$ is the composition of the loops γ_1 and γ_2.

The above three properties are equivalent to specifying a group homomorphism $g : \pi_1(\Sigma) \to G$.[2] Each such twisted boundary condition should in principle appear in the path integral of M/G, and so the path integral should sum over them. The result is that the path integral of M/G on Σ can be expressed as

$$Z = \frac{1}{|G|} \sum_{g:\pi_1(\Sigma)\to G} \int_g \mathcal{D}\Phi \, e^{-S[\Phi]} \,, \tag{2.11}$$

where the subscript g in the path integral instructs us to integrate over field configurations which obey the twisted boundary conditions. The factor of $1/|G|$ is again included so that physically equivalent fields are not overcounted.

Now, if two boundary conditions are related by an overall conjugation by an element $h \in G$, then they represent the same field configuration on the orbifold, and will yield the same result in the path integral. Let $g(\gamma)$ and $h^{-1}g(\gamma)h$ denote two boundary conditions on a multivalued field configuration Φ. We have

$$\Phi(\gamma \cdot x) = (h^{-1}g(\gamma)h) \cdot \Phi(x) \implies (h \cdot \Phi)(\gamma \cdot x) = g(\gamma) \cdot (h \cdot \Phi)(x) \,, \tag{2.12}$$

i.e. choosing the boundary condition $h^{-1}g(\gamma)h$ instead of $g(\gamma)$ has the same effect as replacing Φ by $h \cdot \Phi$ globally. Since this is a global symmetry of the theory, these two boundary conditions will yield the same result in the path integral. Thus, to uniquely specify contributions to the path integral, we want to count twisted boundary conditions *up to conjugation*. The space of boundary conditions up to conjugation is given by the so-called 'representation variety'

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G \,. \tag{2.13}$$

Taking into account the conjugation of homomorphisms $g : \pi_1(\Sigma) \to G$, we finally arrive at an expression for the path integral of an orbifold theory, namely

$$Z = \sum_{g\in \mathrm{Hom}(\pi_1(\Sigma),G)/G} \frac{1}{|\mathrm{Aut}(g)|} \int_g \mathcal{D}\Phi \, e^{-S[\Phi]} \,. \tag{2.14}$$

The weighting factor $|\mathrm{Aut}(g)|$ is the order of the automorphism group of the boundary condition g, which can be thought of as the set of homomorphisms $\phi : G \to G$ which leave g invariant. Intuitively, it is the set of overall global transformations of the fields Φ which preserve the boundary conditions. It is worth noting that the above expression has a conceptual resemblance to the standard Feynman diagrammatic expansion in field theories, in which each diagram is weighted by its so-called 'symmetry factor'.

We can exemplify the above discussion by letting Σ be a torus $\mathbb{T}^2$. In this case, there are two non-contractible cycles A, B, and the fundamental group is

$$\pi_1(\mathbb{T}^2) = \langle A, B \mid AB = BA \rangle \cong \mathbb{Z}^2 \,. \tag{2.15}$$

[2]We assume that Σ is connected, so that $\pi_1(\Sigma)$ is independent of the chosen basepoint.

Now, given twisted boundary conditions $g(A) := g$ and $g(B) := h$, we $gh = hg$ in order for these boundary conditions to be consistent, since the A and B loops commute in $\pi_1(\mathbb{T}^2)$. Thus, the orbifold partition function on $\mathbb{T}^2$ is given by

$$Z = \frac{1}{|G|} \sum_{gh=hg} \int_{g,h} \mathcal{D}\Phi \, e^{-S[\Phi]} := \frac{1}{|G|} \sum_{gh=hg} Z_{g,h} \,, \tag{2.16}$$

where $Z_{g,h}$ is the path integral of X evaluated with twisted boundary conditions g, h. We will later derive this formula in the state-operator approach.

The orbifold Hilbert space

In the above path integral approach to orbifolds, we were able to derive the torus partition function of such theories on general grounds. However, this argument tells us little about the algebraic structure of the theory, in particular about the Hilbert space. We follow the treatment of [54].

In order to discuss the Hilbert space of the theory, we consider the theory quantized on a circle S^1. The seed theory X is described, in canonical quantization, by a Hilbert space $\mathscr{H}_0$ of wavefunctionals $\Psi[\Phi]$ which take in a field configuration Φ on the circle and spit out a complex number. Time evolution is described by the Hamiltonian H, and translation along the circle is described by the momentum operator P. In the orbifold theory, we want to consider wavefunctionals which are invariant under the group action on the fields Φ. That is, we define the *invariant Hilbert space* to be the subspace of $\mathscr{H}_0$ given by

$$\mathrm{Inv}_G(\mathscr{H}_0) = \{\Psi \in \mathscr{H}_0 : \Psi[g \cdot \Phi] = \Psi[\Phi]\} \,. \tag{2.17}$$

Algebraically, we can define the action of G on $\mathscr{H}_0$ as $g \cdot \Psi[\Phi] := \Phi[g \cdot \Phi]$, which defines a linear action on $\mathscr{H}_0$, i.e. a representation of G on $\mathscr{H}_0$. Then $\mathrm{Inv}_G(\mathscr{H}_0)$ is, fittingly, the subspace of $\mathscr{H}_0$ invariant under the linear action of G. Formally, since the action of G on $\mathscr{H}_0$ is linear, we can define the *projection operator*

$$\pi_G := \frac{1}{|G|} \sum_{g \in G} g \,, \tag{2.18}$$

which projects onto the invariant subspace of $\mathscr{H}_0$, i.e.

$$\mathrm{Inv}_G(\mathscr{H}_0) = \pi_G(\mathscr{H}_0) \,. \tag{2.19}$$

Now, since we are quantizing X/G on a circle, the invariant subspace $\mathrm{Inv}_G(\mathscr{H}_0)$ is not the full Hilbert space. Since $\Phi \sim g \cdot \Phi$ are physically equivalent field configurations, we should allow field configurations which are 'twisted' by G, i.e. for which $\Phi(x + 2\pi) = g \cdot \Phi(x)$. These field configurations are not allowed in the original CFT, and so their wavefunctionals $\Psi[\Phi]$ will not be elements of the original Hilbert space. Let us denote by $\mathscr{H}_g$ the Hilbert space of states described by fields twisted by g. The action of an element $h \in G$ is that it takes a field twisted by g and maps it into a field twisted by hgh^{-1}. Indeed,

$$(h \cdot \Phi)(x + 2\pi) = hg \cdot \Phi(x) = hgh^{-1} \cdot (h \cdot \Phi)(x) \,. \tag{2.20}$$

Thus, $h : \mathscr{H}_g \to \mathscr{H}_{hgh^{-1}}$, and the group G does not admit an action on $\mathscr{H}_g$, unless g is in the center of G. However, the stabilizer subgroup

$$N_g := \{h \in G : hgh^{-1} = g\} \tag{2.21}$$

does admit an action on $\mathscr{H}_g$. Thus, just as we had to project onto G-invariant states of $\mathscr{H}_0$ in the untwisted-sector, in the g-twisted sector, we are instructed to project onto the space $\text{Inv}_{N_g}(\mathscr{H}_g)$ of N_g-invariant states.

Now, we have a twisted sector for each element g of G. However, by the above discussion, the Hilbert spaces $\mathscr{H}_g$ and $\mathscr{H}_{hgh^{-1}}$ are gauge-equivalent, and should not be individually included in the Hilbert space of the theory. Let C be *any* set of representatives of the conjugacy classes of G, such that each conjugacy class is represented precisely once. Then the Hilbert space of the orbifold theory is given by

$$\mathscr{H} = \bigoplus_{g \in C} \text{Inv}_{N_g}(\mathscr{H}_g)\,, \tag{2.22}$$

where we understand that $\mathscr{H}_{\text{id}} = \mathscr{H}_0$.

Alternatively, we can describe the Hilbert space in a more invariant way, which does not rely on artificially choosing a representative g of each conjugacy class. Instead, we can define for each conjugacy class ρ of G the Hilbert space

$$\mathscr{H}_\rho = \bigoplus_{g \in \rho} \mathscr{H}_g\,. \tag{2.23}$$

The Hilbert space $\mathscr{H}_\rho$ admits then an action of the group G, since conjugation by group elements leaves the conjugacy class invariant by definition. We can thus define the invariant subspace

$$\text{Inv}_G(\mathscr{H}_\rho) \tag{2.24}$$

in a well-defined manner. The orbit-stabilizer theorem then guarantees the isomorphism

$$\text{Inv}_G(\mathscr{H}_\rho) \cong \text{Inv}_{N_g}(\mathscr{H}_g) \tag{2.25}$$

for each representative $g \in \rho$, and so the invariant subspace $\text{Inv}_G(\mathscr{H}_\rho)$ is equivalent to the gauge-invariant states in the twisted-sector we defined above. Putting every twisted sector together, we can thus write the orbifold Hilbert space as

$$\mathscr{H} \cong \text{Inv}_G\left(\bigoplus_{\rho} \mathscr{H}_\rho\right)\,. \tag{2.26}$$

2.2 The symmetric orbifold

While the above discussion of orbifold CFTs is extremely general, in this thesis we are mostly interested in a particular type of orbifold, known as a symmetric orbifold. The primary motivation for studying such theories is that they provide a prototypical example of large N field theories that can be constructed in two dimensions out of relatively simple data. These theories also exhibit many properties that make them interesting candidates for holographic CFTs (see [55] and references therein), and are also known to exhibit an enormous enhanced gauge symmetry which is related to higher-spin symmetry in three-dimensions [56]. In addition to their richness, symmetric orbifold theories are *simple*, and their study boils down to studying the representation theory of the symmetric group S_N. This is similar to the case of free field theories: in order to calculate correlation functions, one needs only to know the combinatorics of Wick contractions. Because of this analogy, symmetric orbifolds

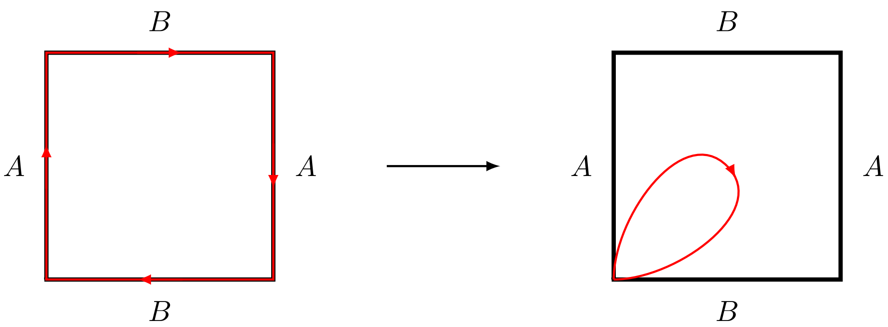

Figure 2.4: The loop $A \cdot B \cdot A^{-1} \cdot B^{-1}$ on a torus is contractible. Thus, twisted boundary conditions π_A, π_B must satisfy $\pi_A \pi_B = \pi_B \pi_A$.

are often thought of as being extremely similar to *free* large N gauge theories, and thus are naturally related to the tensionless limit of AdS/CFT.

Consider a 2D CFT X whose fundamental fields are labeled collectively by Φ. If X has central charge c, we can construct a CFT with arbitrarily large central charge Nc by considering the N^{th} tensor power of X

$$X^{\otimes N} := \underbrace{X \otimes \cdots \otimes X}_{N \text{ times}} . \tag{2.27}$$

The theory $X^{\otimes N}$ contains, as its fundamental fields, N-tuples $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_N)$ of fundamental fields of X. Since $X^{\otimes N}$ is constructed from N copies of an identical seed theory the Hilbert space is an N times tensor product of the original Hilbert space, and there is an obvious symmetry of permuting the individual copies. Let $\Omega \subset S_N$ be a permutation group acting on the letters $\{1, \ldots, N\}$. Then for any permutation $\pi \in \Omega$, there is a natural action on the fundamental fields of $X^{\otimes N}$ given by

$$\pi \cdot \mathbf{\Phi} = (\Phi_{\pi(1)}, \ldots, \Phi_{\pi(N)}) . \tag{2.28}$$

We can define a new CFT, known as the permutation orbifold or $X \wr \Omega$, by gauging the action of Ω on $X^{\otimes N}$. That is, we define the orbifold theory

$$X \wr \Omega := X^{\otimes N} / \Omega . \tag{2.29}$$

The use of the wreath product symbol $\wr$ to denote permutation orbifolds is motivated by the wreath product in group theory and appears frequently in the literature, see [57–60].

A special case of a permutation orbifold comes from taking the permutation group Ω to be the full symmetric group S_N. In this case, the permutation orbifold $X \wr S_N$ is called the *symmetric orbifold* and is often denoted by $\text{Sym}^N(X)$. We will mostly focus on symmetric orbifold theories in this thesis, but most statements we make generalize in a straightforward manner to generic permutation orbifolds.

Partition functions

Let Σ be a Riemann surface of genus one. Its cycles are denoted by A and B, and given a point $z \in \Sigma$, we let $A \cdot z$ denote the operation of transporting z along a cycle homotopic to A. Within the permutation orbifold, as with any orbifold, we impose the gauging of the discrete group Ω by allowing the fundamental fields $\mathbf{\Phi}$ to have non-trivial monodromies when transported around non-contractible loops on Σ. That is, given permutations π_A and π_B, we allow the twisted boundary conditions

$$\begin{aligned} \mathbf{\Phi}(A \cdot z) &= \pi_A \cdot \mathbf{\Phi}(z)\,, \\ \mathbf{\Phi}(B \cdot z) &= \pi_B \cdot \mathbf{\Phi}(z)\,. \end{aligned} \tag{2.30}$$

Now, given that $A \cdot B \cdot A^{-1} \cdot B^{-1}$ is a contractible cycle on the torus (see Figure 2.4), we should not pick up a monodromy when traversing it. Thus, we have

$$\mathbf{\Phi}(z) = \mathbf{\Phi}(A \cdot B \cdot A^{-1} \cdot B^{-1} \cdot z) = (\pi_A \pi_B \pi_A^{-1} \pi_B^{-1}) \cdot \mathbf{\Phi}(z)\,, \tag{2.31}$$

which is only consistent if

$$\pi_A \pi_B = \pi_B \pi_A\,, \tag{2.32}$$

i.e. if the permutations π_A and π_B commute. This is precisely the requirement that the permutations π_A, π_B define a group homomorphism $\pi_1(\Sigma) \cong \mathbb{Z} \oplus \mathbb{Z} \to S_N$. Now, from the general theory of orbifolds, we know that in the path integral of $X \wr \Omega$ on Σ we are required to sum over all twisted boundary conditions π_A, π_B which commute. That is, the partition function is given by

$$Z_\Omega(\Sigma) = \frac{1}{|\Omega|} \sum_{[\pi_A, \pi_B]=0} Z_{\pi_A, \pi_B}(\Sigma)\,, \tag{2.33}$$

where $Z_{\pi_A,\pi_B}(\Sigma)$ is the path integral of $X^{\otimes N}$ on Σ with the twisted boundary conditions imposed by π_A, π_B. For a general choice of π_A, π_B the fields are not single valued, they permute amongst themselves as we travel around different cycles of the torus.

Let us now specialize to the case $\Omega = S_N$. In order to evaluate the individual summands in equation (2.33), we use a standard trick in the theory of orbifolds by considering a covering space $\widetilde{\Sigma}$ on which the fields become single-valued [61]. The space $\widetilde{\Sigma}$ [3] is constructed by taking N copies of Σ, letting the field Φ_i live on the i^{th} copy of Σ, and 'stitching' together the copies of Σ via the twisted boundary conditions π_A, π_B. The resulting surface $\widetilde{\Sigma}$ is an N-fold covering space of Σ in the topological sense (this process is easy to visualize in the case of a one-dimensional theory on a circle, see Figure 2.5). The partition function with twisted boundary conditions reduces to a partition function of the seed theory X on $\widetilde{\Sigma}$, i.e.

$$Z_{\pi_A,\pi_B}(\Sigma) = Z(\widetilde{\Sigma})\,, \tag{2.34}$$

where Z denotes the partition function of X. Therefore, we naively write

$$Z_{S_N}(\Sigma) \overset{?}{=} \frac{1}{N!} \sum_{\widetilde{\Sigma} \to \Sigma} Z(\widetilde{\Sigma})\,, \tag{2.35}$$

where $Z(\widetilde{\Sigma})$ is the partition function of X on $\widetilde{\Sigma}$.

The above equation is actually not quite right. This is because not all pairs of boundary conditions (π_A, π_B) give topologically inequivalent covering spaces. Indeed, if we define $(\pi'_A, \pi'_B) = (\pi\pi_A\pi^{-1}, \pi\pi_B\pi^{-1})$, the resulting covering space is the

[3] Strictly speaking, the covering space $\widetilde{\Sigma}$ depends on the pair (π_A, π_B) but to avoid clutter we omit this in the notation.

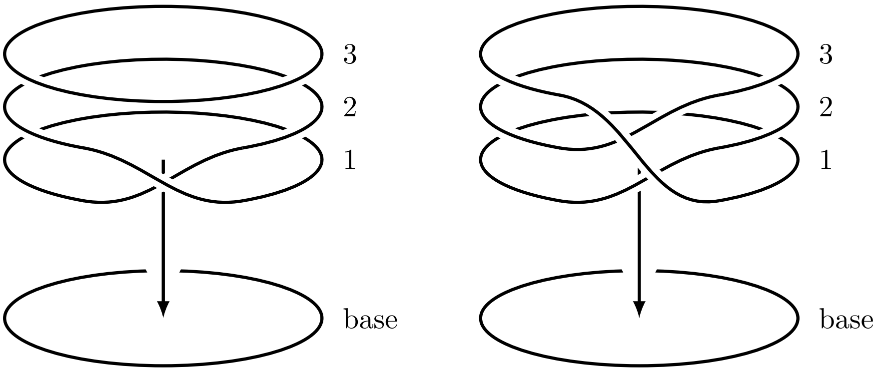

Figure 2.5: Twisted boundary conditions on the circle as 3-fold covering spaces. Left: the fields $\{\Phi_1, \Phi_2, \Phi_3\}$ satisfy twisted boundary conditions $\Phi_1(2\pi) = \Phi_2(0)$, while Φ_3 is single-valued. Right: The fields satisfy boundary conditions $\Phi_1(2\pi) = \Phi_2(0)$, $\Phi_2(2\pi) = \Phi_3(0)$, and $\Phi_3(2\pi) = \Phi_1(0)$.

same, since the effect of conjugating by π simply permutes the sheets of $\widetilde{\Sigma} \to \Sigma$ (the copies of Σ), which is a homeomorphism. If we let

$$\mathcal{O}_{\pi_A,\pi_B} = \left\{(\pi\pi_A\pi^{-1}, \pi\pi_B\pi^{-1})|\pi \in S_N\right\}, \tag{2.36}$$

where we do not double-count equal pairs of permutations, then each covering space $\widetilde{\Sigma} \to \Sigma$ occurs precisely $|\mathcal{O}_{\pi_A,\pi_B}|$ times in the sum (2.33).[4] We define the 'symmetry factor' of a covering space $\widetilde{\Sigma} \to \Sigma$ to be the quotient $|\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)| = N!/|\mathcal{O}_{\pi_A,\pi_B}|$. Thus,

$$Z_{S_N}(\Sigma) = \sum_{\widetilde{\Sigma}\to\Sigma} \frac{Z(\widetilde{\Sigma})}{|\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)|}. \tag{2.37}$$

The factor $|\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)|$ is precisely the degree of the group of *deck transformations*: homeomorphisms of the covering space $\widetilde{\Sigma}$ which preserve the projection $\widetilde{\Sigma} \to \Sigma$.[5]

For a base space which is a torus, it is a topological fact that the covering spaces $\widetilde{\Sigma} \to \Sigma$ considered above are always given by disjoint unions of tori. That is, each covering space we want to consider is given by

$$\widetilde{\Sigma} = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n \to \Sigma, \tag{2.38}$$

where Σ_i is a torus with modular parameter τ_i that is not necessarily equal to the initial modular parameter. Since the partition function of a CFT on a disjoint union

[4] The set $\mathcal{O}_{\pi_A,\pi_B}$ is the orbit set of the element $(\pi_A, \pi_B) \in \mathrm{Hom}(\pi_1(\Sigma), S_N)$ under the S_N action which acts as conjugation. Covering spaces are in one-to-one correspondence with the coset $\mathrm{Hom}(\pi_1(\Sigma), S_N)/S_N$ of this action. Readers familiar with mathematical aspects of gauge theory will recognize this coset as the space of principle S_N bundles over Σ, which is just a fancy word for a covering space.

[5] Since a covering space can be considered a homomorphism $\phi : \pi_1(\Sigma) \to S_N$, we can equivalently define a deck transformation to be an automorphism $\psi : S_N \to S_N$ which leaves ϕ invariant, i.e. for which $\psi \circ \phi = \phi$. Group theoretically, conjugation by elements of S_N defines an action on $\mathrm{Hom}(\pi_1(\Sigma), S_N)$. The group of deck transformations of a covering space $\phi \in \mathrm{Hom}(\pi_1(\Sigma), S_N)$ is the stabilizer $\mathrm{Stab}(\phi)$ under the S_N action. By the orbit-stabilizer theorem, we have $|\mathrm{Stab}(\phi)||\mathcal{O}(\phi)| = |S_N| = N!$, or $N!/|\mathcal{O}(\phi)| = |\mathrm{Stab}(\phi)| = |\mathrm{Deck}(\widetilde{\Sigma} \to \Sigma)|$. Note that we do not require Σ to be a torus, and this statement works for any topological space Σ.

of spaces is the product of the partition functions, we have, for each covering space $\widetilde{\Sigma} \to \Sigma$,

$$Z(\widetilde{\Sigma}) = \prod_{i=1}^{n} Z(\Sigma_i)\,. \tag{2.39}$$

Thus, in order to calculate the partition function $Z_{S_N}(\Sigma)$ of the symmetric orbifold theory $X \wr S_N$ on a torus Σ, one only needs to know the generic torus partition function $Z(\Sigma)$ for the seed theory X – all of the other data is contained in the combinatorics of the covering spaces. This simplification does not occur for partition functions of $X \wr S_N$ on higher-genus surfaces: as we will see later, if Σ has genus g, calculating the partition function $Z_{S_N}(\Sigma)$ requires knowing the partition functions of the seed theory X on surfaces of many different genera.

Example: $N = 2$

For $N = 2$, the above discussion can be made very concrete. The only two permutations in S_2 are the identity e and the two-cycle π. Since S_2 is abelian, all permutations commute among each other, and we can immediately write down the sum (2.33) as

$$Z_2(\Sigma) = \frac{1}{2}\left(Z_{e,e}(\Sigma) + Z_{\pi,e}(\Sigma) + Z_{e,\pi}(\Sigma) + Z_{\pi,\pi}(\Sigma)\right)\,. \tag{2.40}$$

If we realize the torus Σ as a parallelogram in the complex plane $\mathbb{C}/\{m + n\tau\}$, we can choose the A-cycle to act as $A \cdot z = z + \tau$ and the B-cycle to act as $B \cdot z = z + 1$. Then $Z_{\pi,e}$ is the partition function of fields (Φ_1, Φ_2) with $\Phi_1(z + \tau) = \Phi_2(z)$. This is single valued on the torus obtained by making the B-cycle twice as long, i.e.

$$\mathbb{C}/\{m + 2n\tau\}\,. \tag{2.41}$$

This is a torus with modular parameter 2τ, and so

$$Z_{\pi,e}(\tau) = Z(2\tau)\,. \tag{2.42}$$

Similarly,

$$Z_{e,\pi}(\tau) = Z(\tfrac{\tau}{2})\,, \quad Z_{\pi,\pi}(\tau) = Z(\tfrac{\tau+1}{2})\,. \tag{2.43}$$

All of the above covering tori are depicted in Figure 2.6. The partition function $Z_{e,e}(\tau)$ is just the partition function of $X^{\otimes 2}$ with no twisted boundary conditions, and so

$$Z_{e,e}(\tau) = Z(\tau)^2\,. \tag{2.44}$$

Putting the above discussion together, we can write the full $X \wr S_2$ symmetric orbifold partition function (2.40) as

$$Z_2(\tau) = \frac{1}{2}Z(\tau)^2 + \frac{1}{2}Z(\tfrac{\tau}{2}) + \frac{1}{2}Z(2\tau) + \frac{1}{2}Z(\tfrac{\tau+1}{2})\,. \tag{2.45}$$

Each term in this sum can be seen as the partition function of the seed theory X evaluated on a covering space $E \to \Sigma$, see Figure 2.6. A similar, more complicated discussion can be done for $N = 3$, which requires 18 pairs of commuting permutations in S_3. The end result is

$$\begin{aligned} Z_3(\tau) =& \frac{1}{6}Z(\tau)^3 + \frac{1}{2}Z(\tau)Z(2\tau) + \frac{1}{2}Z(\tau)Z(\tfrac{\tau}{2}) + \frac{1}{2}Z(\tau)Z(\tfrac{\tau+1}{2}) \\ &+ \frac{1}{3}Z(3\tau) + \frac{1}{3}Z(\tfrac{\tau}{3}) + \frac{1}{3}Z(\tfrac{\tau+1}{3}) + \frac{1}{3}Z(\tfrac{\tau+2}{3})\,. \end{aligned} \tag{2.46}$$

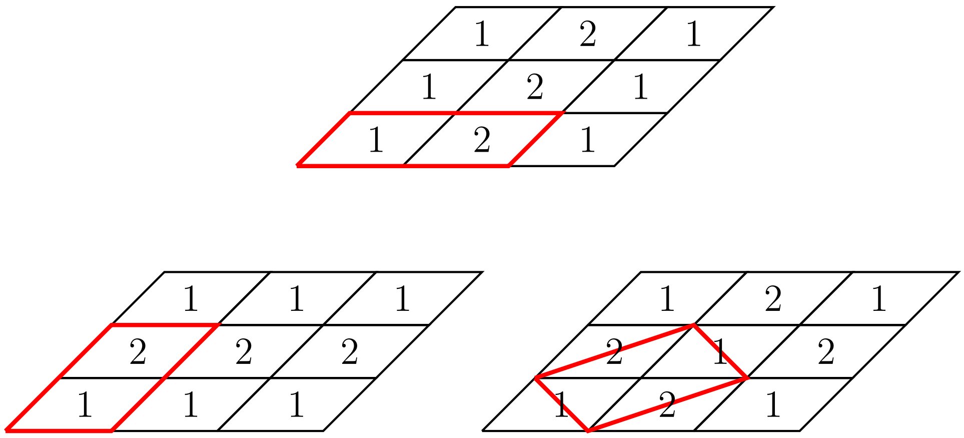

Figure 2.6: The connected covering spaces for the torus $N = 2$ symmetric orbifold. Individual cells represent the base torus (with modular parameter τ), and the numbers label which copy of the seed theory lives on which sheet of the base torus. The permutations π_A, π_B prescribe how to stitch together the copies of the seed theory onto the covering space. The covering space itself is the fundamental domain for which the arrangement of labels $1, 2$ is periodic (shown in red). The fundamental domains in the above examples have modular parameter $\tau/2$, 2τ, and $(\tau+1)/(1-\tau) \sim (\tau+1)/2$, respectively.

Hilbert space structure

The Hilbert space of a generic orbifold CFT consists of both untwisted states and twisted states. The untwisted states live in the Hilbert space $\text{Inv}_G(\mathscr{H}_0)$ of states of the parent theory which are invariant under the action of the orbifold group. For the symmetric orbifold, the parent theory is N copies of X, and so the parent Hilbert space is N copies of the Hilbert space $\mathscr{H}_X$. The invariant subgroup is given by states of the form

$$\sum_{\pi \in S_N} |\psi_{\pi(1)}\rangle \otimes \cdots \otimes |\psi_{\pi(N)}\rangle \ , \tag{2.47}$$

for some set of N vectors $|\psi_i\rangle \in X$. Thus, elements of the untwisted sector are in one-to-one correspondence with states in X.

In the twisted sector, we start by picking some group element g which represents a conjugacy class of G. In the case of the symmetric orbifold, all conjugacy classes are labeled by the cycle types. That is, given a partition

$$\sum_k w_k = N \tag{2.48}$$

of N, we can define a permutation

$$\pi = (1 \cdots w_1) \circ (w_1 + 1 \cdots w_2 + w_1 + 1) \circ \cdots \ , \tag{2.49}$$

i.e. which has one cycle of length w_1, one cycle of length w_2, etc. For such a permutation, the Hilbert space is occupied by states which are twisted by the permutation π, and which are invariant under the stabilizer N_π.

For holographic applications, we will mostly be interested in twisted sectors for which π has only one cycle of length $w > 1$. We call such permutations *single-cylce.*

For each single-cycle conjugacy class, we can find the simple representative

$$\pi_w := (1 \cdots w)\,. \tag{2.50}$$

The stabilizer group N_{π_w} is a product $\mathbb{Z}_w \times S_{N-w}$, where the first factor is generated by π_w, while the second factor simply permutes the other $N-w$ letters. A generic element of the twisted sector is given by

$$\sum_{\pi \in S_{N-w}} |\psi\rangle^{(w)} \otimes \underbrace{|\varphi_{\pi(w+1)}\rangle \otimes \cdots \otimes |\varphi_{\pi(N)}\rangle}_{N-w \text{ states}}\,, \tag{2.51}$$

where $|\psi\rangle^{(w)}$ is an element in the twisted sector of $X^{\otimes w}$ whose wave functional $\psi[\Phi]$ picks up a monodromy of π_w when transported around the spatial circle, and $|\psi_i\rangle$, $i = w+1, \ldots, N$ are states in the seed theory X (here, we think of S_{N-w} as acting on the set $\{w+1, \ldots, N\}$). We will mostly be interested in states for which the 'extra' states $|\varphi_i\rangle$ are the ground state $|0\rangle$, and we will call such states *single-particle* states. We will simply write $|\psi\rangle^{(w)}$ for the resulting state when there is no confusion. By the operator-product-correspondence, we can promote $|\psi\rangle^{(w)}$ to a field $\mathcal{O}^{(w)}$. Since $|\psi\rangle^{(w)}$ lies in the twisted sector, it will not be exactly local, but will induce a monodromy on the nontrivial fields $\mathbf{\Phi}$ of the symmetric orbifold theory:

$$\mathbf{\Phi}(e^{2\pi i}z)\,\mathcal{O}^{(w)}(0) = \pi_w \cdot \mathbf{\Phi}(z)\,\mathcal{O}^{(w)}(0)\,. \tag{2.52}$$

Such fields are called 'twist fields', and their correlation functions are of fundamental interest in holography. Schematically, we will see in later chapters that a twist field of length w will correspond holographically to a string which winds the asymptotic boundary of AdS_3 w times. We will come back to twist fields and their correlators in Chapter 5, and specifically in Appendix 5.A.

The grand canonical ensemble

Symmetric orbifold theories can be thought of as field theories of N identical particles, each described by an individual field theory X. Gauging the permutation symmetry equates to being ignorant of the precise ordering of these particles. In many-body systems, it is common to let the number N of particles vary, specifying a chemical potential μ, i.e. the energy cost of adding one new particle to the system. The full configuration/Hilbert space of such systems is defined as the direct sum over all of the individual configuration spaces for fixed N.

Motivated by this analogy, it is natural to also let the value of N in the symemtric orbifold vary. We can formally sum over all values of N and consider the *grand canonical* symmetric orbifold, formally defined by the sum

$$\text{Sym}(X) := \bigoplus_{N=0}^{\infty} \text{Sym}^N(X)\,. \tag{2.53}$$

Working with $\text{Sym}(X)$ allows us to work with all symmetric orbifold theories at once. We can keep track of the specific orbifold $\text{Sym}^N(X)$ by introducing a formal variable p which keeps track of N. In analogy to second-quantized statistical mechanics, we define the 'grand canonical' partition function $\mathfrak{Z}(p,\tau)$ by

$$\mathfrak{Z}(p,\tau) = \sum_{N=0}^{\infty} p^N Z_N(\tau)\,, \tag{2.54}$$

where $p = e^{-\mu}$ is the chemical potential in our multi-particle analogy.

In the grand canonical ensemble, partition functions of the symmetric orbifold admit an analytic simplification [59, 62]. In view of the above discussion. We can write the grand canonical partition function as

$$\mathfrak{Z}(p,\tau) = \sum_{\substack{\text{covering spaces} \\ \widetilde{\Sigma}\to\Sigma}} \frac{p^{\deg(\widetilde{\Sigma})}}{|\mathrm{Aut}(\widetilde{\Sigma})|} Z(\widetilde{\Sigma})\,, \tag{2.55}$$

where Σ is the torus of modular parameter τ, and the degree $\deg(\widetilde{\Sigma})$ is the number of times $\widetilde{\Sigma}$ covers Σ. The above sum is over *all* covering spaces, both connected and disconnected. This sum is completely analogous to the sum of vacuum diagrams in a quantum field theory, and as such we can use the standard combinatorial trick of writing $\mathfrak{Z}$ as the exponential of the sum over all *connected* covering spaces, i.e.

$$\mathfrak{Z}(p,\tau) = \exp\left(\sum_{\substack{\widetilde{\Sigma}\to\Sigma \\ \text{connected}}} \frac{p^{\deg(\widetilde{\Sigma})}}{|\mathrm{Aut}(\widetilde{\Sigma})|} Z(\widetilde{\Sigma})\right). \tag{2.56}$$

Since all connected covering spaces of a torus are again tori, it follows that the sum in the exponential is a sum over all tori which cover Σ. If we define $\Sigma = \mathbb{C}/\Lambda$, then every covering torus is given by $\widetilde{\Sigma} = \mathbb{C}/\widetilde{\Lambda}$ such that $\widetilde{\Lambda}$ is a sublattice of Λ. The degree of this covering is just the index $[\widetilde{\Lambda}:\Lambda]$ and the automorphism group has size $|\mathrm{Aut}(\widetilde{\Sigma})| = [\widetilde{\Lambda}:\Lambda]$. Thus,

$$\mathfrak{Z}(p,\tau) = \exp\left(\sum_{\widetilde{\Lambda}\subset\Lambda} \frac{p^{[\widetilde{\Lambda}:\Lambda]}}{[\widetilde{\Lambda}:\Lambda]} Z(\mathbb{C}/\widetilde{\Lambda})\right). \tag{2.57}$$

For a given value of $k := [\widetilde{\Lambda}:\Lambda]$, the sum in the exponential is exactly the definition of the k^{th} *Hecke operator*, defined by

$$T_k Z(\tau) = \sum_{[\widetilde{\Lambda}:\Lambda]=k} \frac{p^k}{k} Z(\mathbb{C}/\widetilde{\Lambda})\,. \tag{2.58}$$

If the original partition function Z is a modular function, then the Hecke operator takes a particularly simple form

$$T_k Z(\tau) = \frac{1}{k}\sum_{ad=k}\sum_{b=0}^{d-1} Z\left(\frac{a\tau+b}{d}\right), \tag{2.59}$$

and so we have

$$\mathfrak{Z}(p,\tau) = \exp\left(\sum_{k=1}^{\infty} p^k T_k Z(\tau)\right), \tag{2.60}$$

The sum (2.59) has a nice geometric interpretation. The integers a, d in the sum indicate how many times the covering space wraps around the A and B cycles, respectively. The integer b then indicates a Dehn twist on the base torus, which is a Dehn twist around the B cycle of angle $2\pi b/d$ on the covering torus.

Flavor and spin structure

Often we will be interested in orbifolds of CFTs X which have either degrees of freedom with half-integer spin (such as supersymmetric theories), or extra U(1) symmetries, or both. For example, in the main text we will be interested in symmetric orbifolds of a specific $\mathcal{N} = 4$ superconformal theory (the $\mathbb{T}^4$ sigma model), which has an SU(2) R-symmetry as well as half-integer spin degrees of freedom. In this discussion we will be rather abstract to begin with, and near the end provide explicit formulae which will be useful later on. See Appendix 5.B for a primer on Riemann surface theory that we will use in this discussion.

In order to define the partition function Z of a theory X which has spinors, we need to define a spin structure. Let Σ be a surface of genus g and ψ a spinor living on it. Just as in an orbifold theory, ψ is allowed to obey twisted boundary conditions when transported around a loop $\gamma \in \pi_1(\Sigma)$. Specifically, ψ is allowed to either come back to itself under the action of γ or come back to itself with a minus sign. We call such an assignment of phases ± 1 for each loop $\gamma \in \pi_1(\Sigma)$ a *spin structure*. Algebraically, a spin structure is a homomorphism $S : \pi_1(\Sigma) \to \mathbb{Z}_2$. However, since $\mathbb{Z}_2$ is abelian, it suffices to consider a homomorphism from the abelianization of $\pi_1(\Sigma)$ to $\mathbb{Z}_2$, i.e.

$$S : H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}_2 \,. \tag{2.61}$$

Since $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, there are 2^{2g} possible spin structures. Each choice of spin structure provides a different set of boundary conditions, and thus a different answer for the path integral. However, unlike in the case of orbifolds, we do not sum over all spin structures in a generic CFT.[6]

If our CFT X has some extra U(1) charges, we can also introduce $2g$ chemical potentials taking values in U(1) into a path integral, known as *flavor*. The flavors are introduced in the following way. Let Φ be a field in the CFT X which has charge r under the U(1) symmetry. Then the *flavored* path integral is defined by allowing the fields Φ to satisfy the boundary conditions

$$\Phi(\gamma \cdot x) = y(\gamma)^r \Phi(x) \,, \tag{2.62}$$

where $y(\gamma)$ is some U(1) phase. Such a choice y is equivalent to a homomorphism $y : \pi_1(\Sigma, \mathbb{Z}) \to \mathrm{U}(1)$. Again, since U(1) is abelian, we can replace the fundamental group with the homology group and we find

$$y : H_1(\Sigma, \mathbb{Z}) \to \mathrm{U}(1) \,. \tag{2.63}$$

Given a seed theory with both a spin structure and a flavor, we write

$$Z(\Sigma, S, y) \tag{2.64}$$

for the path integral on Σ with flavor y and spin structure S.

In the symmetric orbifold theory, the inclusion of both spin structure and flavor are now straightforward. In the symmetric orbifold the path integral is defined via the sum over covering spaces $\Gamma : \widetilde{\Sigma} \to \Sigma$. Note that the existence of a covering map induces a map between homologies:

$$\Gamma_* : H_1(\widetilde{\Sigma}, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z}) \,. \tag{2.65}$$

[6] We do, however, sum over the spin structures in the RNS formalism of string theory. This leads to what is known as the *GSO projection*.

Thus, given a spin structure S and/or a flavor y defined on the surface Σ, we can define a canonical spin structure and/or flavor on $\widetilde{\Sigma}$ via composition with Γ_*, which we call the pullbacks Γ^*S, Γ^*y of S, y by Γ, i.e.

$$\begin{aligned} \Gamma^*S &:= S \circ \Gamma_* : H_1(\widetilde{\Sigma}, \mathbb{Z}) \to \mathbb{Z}_2\,, \\ \Gamma^*y &:= y \circ \Gamma_* : H_1(\widetilde{\Sigma}, \mathbb{Z}) \to \mathrm{U}(1) \end{aligned} \tag{2.66}$$

define a spin structure and flavor on the covering surface. We note in passing that each choice of spin structure defines a *spin bundle* S over Σ whose square is the canonical bundle K, and a choice of flavor y defines a flat U(1) bundle over Σ. The above definition of pullback coincides with the usual notion of the pullback of a bundle.

The symmetric orbifold theory with spin structure S and flavor y thus has partition function

$$Z_N(\Sigma, S, z) = \sum_{\substack{\Gamma:\widetilde{\Sigma}\to\Sigma \\ \text{degree } N}} \frac{Z(\widetilde{\Sigma}, \Gamma^*S, \Gamma^*y)}{|\mathrm{Aut}(\widetilde{\Sigma})|}\,. \tag{2.67}$$

As a down-to-earth example, and one that will show up later in Chapter 4, let us consider a CFT X on a torus with modular parameter τ with spin structure S and a single flavor y. The spin structure on a torus is defined by a pair $\alpha, \beta \in \{0, 1/2\}$ such that a spinor picks up the phase $(-1)^{2\alpha}$ when transported around the A cycle and $(-1)^{2\beta}$ when transported around the B cycle. For simplicity, let us assume that our flavor only has a nontrivial component along the A cycle. As we know from the discussion of the grand canonical partition function, we can write the partition function as the exponential of the sum over connected graphs. The end result in the case of nontrivial spin structure and chemical potential y is that we have

$$\mathfrak{Z}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(p, \tau, y) = \exp\left(\sum_{k=1}^{\infty} p^k \mathfrak{T}_k Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau, y)\right)\,, \tag{2.68}$$

where $\mathfrak{T}_k$ is a supersymmetric, flavored version of the usual Hecke operator T_k, given by[7]

$$\mathfrak{T}_k Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau, y) = \frac{1}{k}\sum_{ad=k}\sum_{b=0}^{k-1} Z\begin{bmatrix}a\alpha + b\beta\\ d\beta\end{bmatrix}\left(\frac{a\tau+b}{d}, y^a\right)\,. \tag{2.69}$$

Here, the spin structure subscripts are added modulo 1.

The single-particle spectrum

Let us assume that the partition function of the seed theory has an expansion

$$Z(\tau, \bar{\tau}) = \sum_{h,\bar{h}} c_{h,\bar{h}} q^h \bar{q}^{\bar{h}}\,, \tag{2.70}$$

where the sum runs over all states in the seed theory X. For the moment we ignore the usual $q^{-c/24}\bar{q}^{-c/24}$ contribution for the sake of simplicity, but it can be added

[7] While the usual Hecke operator T_k acts as a linear operator on the space of modular forms, the generalized Hecke operator $\mathfrak{T}_k$ acts on the space of *vector-valued Jacobi forms*.

in easily. To compute the Sym(X) partition function, we first apply the Hecke operators T_k to the seed theory partition function. We have

$$\begin{aligned} T_k Z(\tau,\bar\tau) &= \frac{1}{k}\sum_{ad=k}\sum_{b=0}^{d-1} Z\left(\frac{a\tau+b}{d},\frac{a\bar\tau+b}{d}\right) \\ &= \frac{1}{k}\sum_{ad=k}\sum_{b=0}^{d-1}\sum_{h,\bar h} c_{h,\bar h}\, q^{ah/d}\bar q^{a\bar h/d} e^{2\pi i(h-\bar h)b/d}\,. \end{aligned} \tag{2.71}$$

Summing over b restricts the sum over conformal weights with $h-\bar h\in d\mathbb{Z}$, and so

$$T_k Z(\tau,\bar\tau) = \sum_{ad=k}\frac{1}{a}\sum_{h-\bar h\in d\mathbb{Z}} c_{h,\bar h}\, q^{ah/d}\bar q^{a\bar h/d}\,. \tag{2.72}$$

The second step is to sum over k, weighted by p^k. This gives

$$\sum_{k=1}^{\infty} p^k T_k Z(\tau,\bar\tau) = \sum_{k=1}^{\infty} p^k \sum_{ad=k}\frac{1}{k}\sum_{h-\bar h\in d\mathbb{Z}} c_{h,\bar h}\, q^{ah/d}\bar q^{a\bar h/d}\,. \tag{2.73}$$

Now, we can replace the sum over k and $ad=k$ for a sum over all positive integers a,d. This gives

$$\begin{aligned} \sum_{k=1}^{\infty} p^k T_k Z(\tau,\bar\tau) &= \sum_{a,d=1}^{\infty}\sum_{h-\bar h\in d\mathbb{Z}} \frac{p^{ad}}{a} c_{h,\bar h}\, q^{ah/d}\bar q^{a\bar h/d} \\ &= -\sum_{d=1}^{\infty}\sum_{h-\bar h\in d\mathbb{Z}} c_{h,\bar h}\log\left(1-p^d q^{h/d}\bar q^{\bar h/d}\right)\,, \end{aligned} \tag{2.74}$$

where we have performed the sum over a explicitly. Finally, we can write down the grand canonical partition function as

$$\begin{aligned} \mathfrak{Z}(p,\tau) &= \exp\left(\sum_{k=1}^{\infty} p^k T_k Z(\tau,\bar\tau)\right) \\ &= \prod_{d=1}^{\infty}\prod_{h-\bar h\in d\mathbb{Z}}\left(1-p^d q^{h/d}\bar q^{\bar h/d}\right)^{-c_{h,\bar h}}\,. \end{aligned} \tag{2.75}$$

This is the so-called *multi-particle form* of the grand canonical partition function.

In analogy with the statistical physics of Bose-Einstein gases, the multi-particle form of the grand canonical partition function $\mathfrak{Z}$ is suggestive as the second-quantized form of of a *single-particle* subsector of the symmetric product theory. The partition function of this single-particle sector is read off as

$$Z_{\text{s.p.}}(\tau,\bar\tau) = \sum_{d=1}^{\infty}\sum_{h-\bar h\in d\mathbb{Z}} q^{h/d}\bar q^{\bar h/d} = \sum_{d=1}^{\infty}\frac{1}{d}Z'\left(\frac{\tau}{d},\frac{\bar\tau}{d}\right)\,, \tag{2.76}$$

where the prime on the seed theory partition function implies that we are keeping only states with $h-\bar h\in d\mathbb{Z}$. We note that we can get rid of the prime by re-introducing the sum over b in the Hecke operators, namely

$$Z'\left(\frac{\tau}{d},\frac{\bar\tau}{d}\right) = \sum_{b=0}^{d-1} Z\left(\frac{\tau+b}{d},\frac{\bar\tau+b}{d}\right)\,, \tag{2.77}$$

so that the single-particle spectrum takes its final form:

$$Z_{\mathrm{s.p.}}(\tau,\overline{\tau}) = \sum_{d=1}^{\infty}\sum_{b=0}^{d-1} Z\left(\frac{\tau+b}{d},\frac{\overline{\tau}+b}{d}\right) . \tag{2.78}$$

Note specifically that each summand formally resembles a Hecke operator for which the sum over a is restricted to $a=1$ and without the $1/k$ normalization.

We also note in passing that in the case of a partition function which is flavored and has a spin structure, we have (assuming that the flavor is only nontrivial along the A-cycle)

$$Z_{\mathrm{s.p.},(\alpha,\beta)}(\tau,y) = \sum_{d=1}^{\infty}\sum_{b=0}^{d-1} Z_{(\alpha+b\beta,d\beta)}\left(\frac{\tau+b}{d},y\right) , \tag{2.79}$$

where we have suppressed the right-moving dependence.

For a CFT on a torus, the four choices of spin structures have names:

$$(0,\tfrac{1}{2}) \to \widetilde{\mathrm{NS}}\,, \quad (0,0)\to\widetilde{\mathrm{R}}\,, \quad (\tfrac{1}{2},\tfrac{1}{2})\to \mathrm{NS}\,, \quad (\tfrac{1}{2},0)\to\mathrm{R}\,. \tag{2.80}$$

For example, if we start with a seed theory whose partition function is evaluated in the NS-sector $(\frac{1}{2},\frac{1}{2})$, then the single-particle partition function of its symmetric orbifold theory will be given by

$$\begin{aligned} Z^{\mathrm{NS}}_{\mathrm{s.p.}}(\tau,y) &= \sum_{d=1}^{\infty}\sum_{b=0}^{d-1} Z\begin{bmatrix}\frac{b}{2}+\frac{1}{2}\\ \frac{d}{2}\end{bmatrix}\left(\frac{\tau+b}{d},y\right) ,\\ &= \sum_{d=1}^{\infty}\sum_{b=0}^{d-1}\begin{cases} Z^{\mathrm{R}}\left(\frac{\tau+b}{d},y\right) , & b,d \text{ even}\\ Z^{\widetilde{\mathrm{R}}}\left(\frac{\tau+b}{d},y\right) , & b \text{ odd, } d \text{ even}\\ Z^{\mathrm{NS}}\left(\frac{\tau+b}{d},y\right) , & b \text{ even, } d \text{ odd}\\ Z^{\widetilde{\mathrm{NS}}}\left(\frac{\tau+b}{d},y\right) , & b,d \text{ odd}\,. \end{cases} \end{aligned} \tag{2.81}$$

2.3 Wess-Zumino-Witten models

Conformal field theories find one of their most natural homes in the description of strings moving in some target manifold $\mathcal{M}$. The conformal field theory in question will be a sigma model whose fundamental fields are maps $X:\Sigma\to\mathcal{M}$. If $\mathcal{M}$ has a metric g, then we can write down an action

$$S_{\mathrm{kin}} \propto \int_{\Sigma} g_{\mu\nu}(X)\mathrm{d}X^{\mu}\wedge\star\mathrm{d}X^{\nu}\,. \tag{2.82}$$

In addition to the metric tensor, we can also include a two-form B living on the manifold $\mathcal{M}$. Its effect on the sigma model is to include a term $X^{*}B$ to the action, and we are left with

$$S\propto\int_{\Sigma}\left(g_{\mu\nu}(X)\mathrm{d}X^{\mu}\wedge\star\mathrm{d}X^{\nu} + B_{\mu\nu}(X)\mathrm{d}X^{\mu}\wedge\mathrm{d}X^{\nu}\right) . \tag{2.83}$$

The field theoretic description of a sigma model on a manifold $(\mathcal{M},g,B)$ is in general extremely difficult to quantize. For flat spaces, the above action reduces to $\dim(\mathcal{M})$ copies of a free boson, and the resulting theories are the famous Narain CFTs, but for more general sigma models the quantization is simply intractable.

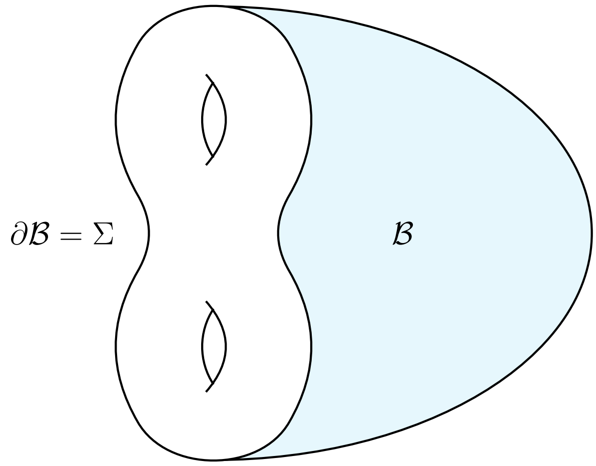

Figure 2.7: The three-manifold $\mathcal{B}$ is constructed so that its boundary is the worldshet Σ. Different choices of continuation of g as a function on $\mathcal{B}$ should be physically equivalent.

We will be interested later on in the motion of strings in a specific set of curved backgrounds, namely AdS_3 and S^3, both supported by a B field whose field strength $\mathrm{d}B$ is a constant multiple of the volume form. Both of these backgrounds are curved, and so a sigma model description seems unlikely to lead to a solvable worldsheet theory. However, the manifolds AdS_3 and S^3 have the property that they are isometric to the group manifolds $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$. As such, the worldsheet sigma model description admits an enhanced symmetry induced by the group structure of these manifolds. In fact, just as the global conformal symmetry in two dimensions extends to an infinite-dimensional Virasoro symmetry, the symmetry of a sigma model on a group manifold G is promoted from a finite symmetry to an infinite-dimensional symmetry algebra (a so-called current algebra) on the worldsheet. The existence of this infinite dimensional algebra allows the quantization of the worldsheet theory to be carried out in a much more controlled fashion than for more generic sigma models, and in many cases leads to an exactly solvable worldsheet theory. In this section, we review the construction and quantization of such models, known as *Wess-Zumino-Witten* models, or simply WZW models.

Given a compact group G, we can describe the motion of a string worldsheet Σ in G by a map $g : \Sigma \to G$. To define a field theory with g as the fundamental field, we start by writing down a kinetic term:

$$S_{\text{kin}}[g] = \frac{1}{4\pi f^2}\int_\Sigma \mathrm{Tr}[g^{-1}\partial g\, g^{-1}\overline{\partial} g]\,. \tag{2.84}$$

The fields $g^{-1}\partial g$ and $g^{-1}\overline{\partial} g$ define $\mathfrak{g}$-valued one-forms on Σ, and the trace Tr is taken in the adjoint representation of $\mathfrak{g}$. As a (familiar) example, let us take $G = \mathrm{U}(1)$. Any point in G is expressed as $e^{i\alpha}$ for some real α. Taking the derivative of g gives

$$g^{-1}\partial g = i\partial\alpha\,, \quad g^{-1}\overline{\partial} g = i\overline{\partial}\alpha\,. \tag{2.85}$$

And so the kinetic term for $G = \mathrm{U}(1)$ can be written in terms of the real field α as

$$S_{\text{kin}}[\alpha] = -\frac{1}{4\pi f^2}\int_\Sigma \partial\alpha\overline{\partial}\alpha\,, \tag{2.86}$$

whic is just the kinetic term of a string in one flat dimension.

The equations of motion of the above action are of the form

$$\overline{\partial}\left(g^{-1}\partial g\right)+\partial\left(g^{-1}\overline{\partial}g\right)=0\,. \tag{2.87}$$

This is equivalent to the conservation of the current

$$J_z=g^{-1}\partial g\,,\quad J_{\bar z}=g^{-1}\overline{\partial}g\,. \tag{2.88}$$

This current is associated to the global symmetry of the theory associated to left-translations $g\to gg_R^{-1}$. In the language of differential forms, J is a $\mathfrak{g}$-valued one-form on the worldsheet Σ, which is the pullback of the so-called Maurer-Cartan form:

$$J=g^{-1}\mathrm{d}g\in\Omega^1(\Sigma)\otimes\mathfrak{g}\,. \tag{2.89}$$

In a conformal field theory, we want currents J for which the components are (anti)holomorphic. For this to be the case, we would like, for example, that J_z is holomorphic and $J_{\bar z}$ is anti-holomorphic. However, for non-abelian groups G, we cannot have both $\partial J_{\bar z}=0$ and $\overline{\partial}J_z=0$. To see this, we note that the Maurer-Cartan form $J=g^{-1}\mathrm{d}g$ satisfies the flatness condition

$$\mathrm{d}J+J\wedge J=0\,. \tag{2.90}$$

However, if $\overline{\partial}J_z=\partial J_{\bar z}=0$, then expanding in components $J=J_z\mathrm{d}z+J_{\bar z}\mathrm{d}\bar z$, we have

$$\mathrm{d}J=\overline{\partial}J_z\,\mathrm{d}\bar z\wedge\mathrm{d}z+\partial J_{\bar z}\,\mathrm{d}\bar z\wedge\mathrm{d}z=0\,, \tag{2.91}$$

since both terms vanish individually. However, we know that

$$\mathrm{d}J=-J\wedge J \tag{2.92}$$

does not vanish unless G is abelian. Thus, the components of J cannot be holomorphic and anti-holomorphic unless G is abelian.

In order to improve the kinetic term so that the conserved currents are chiral, we introduce the so-called *Wess-Zumino* term

$$S_{\mathrm{WZ}}[g]=-\frac{ik}{2\pi}\int_{\mathcal{B}}\mathrm{Tr}[g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g]\,. \tag{2.93}$$

Here, $\mathcal{B}$ is a 3-manifold with boundary $\partial\mathcal{B}=\Sigma$. For abelian groups, this term identically vanishes due to the anti-symmetry of the wedge product. For non-abelian groups, it is necessary for the existence of a holomorphic stress tensor on the worldsheet. The equations of motion for the total action $S_{\mathrm{kin}}+S_{\mathrm{WZ}}$ are given by

$$(1+f^2k)\partial\left(g^{-1}\overline{\partial}g\right)+(1-f^2k)\overline{\partial}(g^{-1}\partial g)=0\,. \tag{2.94}$$

Thus, for $f^2=1/k$, we have that the right-moving current $J=g^{-1}\overline{\partial}g$ is anti-holomorphic. One can also show that the current $\mathrm{d}g\,g^{-1}$ associated to left-translations $g\to g_Lg$ has a holomorphic conserved component if $f^2=1/k$. We call this choice of parameters the *Wess-Zumino-Witten point* and, for most choices of group G the WZW model is not conformal unless we are at the WZW point. Thus, the action of the WZW model is

$$S_{\mathrm{WZW}}[g]=\frac{k}{4\pi}\int_{\Sigma}\mathrm{Tr}[g^{-1}\mathrm{d}g\wedge\star g^{-1}\mathrm{d}g]-\frac{ik}{2\pi}\int_{\mathcal{B}}\mathrm{Tr}[g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g]\,. \tag{2.95}$$

The above construction should not depend on how we choose to extend the field $g : \Sigma \to G$ to a field $g : \mathcal{B} \to G$. It turns out that, if $g, \tilde{g}$ are two separate such extensions, then

$$S_{\mathrm{WZ}}[g] - S_{\mathrm{WZ}}[\tilde{g}] \in 2\pi k\mathbb{Z}\,. \tag{2.96}$$

Thus, in order for the path integrand $e^{iS_{\mathrm{WZ}}}$ to be independent of the choice $g : \Sigma \to \mathcal{B}$, we demand that k is an integer.[8]

From now on, we focus only on the left-moving part of the theory. The current J, as Lie-algebra valued one-forms, admits an expansion in the basis of the Lie algebra $\mathfrak{g}$. Let us take such a basis T_a to have

$$\mathrm{Tr}(T_a T_b) = \kappa_{ab}\,, \tag{2.97}$$

where κ_{ab} is the Killing form on $\mathfrak{g}$, whose inverse we denote by κ^{ab}. Expanding in this basis, we define the currents J^a as

$$J(z) = J^a(z) T_a\,. \tag{2.98}$$

As an example, let us take $g : \Sigma \to G$ close to the identity in G. Then

$$g = e^{T_a x^a} \implies J = g^{-1}\partial g = T_a \partial x^a \tag{2.99}$$

so that J^a is the holomorphic derivative of the local coordinate x^a on G around the identity.

Upon quantization of the theory, we impose canonical OPEs between fields and their conjugate momenta. Doing this for the WZW model, we find the following OPEs between the currents J^a:

$$J^a(z)J^b(w) \sim \frac{k\,\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c\,J^c(w)}{z-w}\,. \tag{2.100}$$

The above OPE algebra is actually the most general algebra that can be satisfied between two fields of conformal weight $h = 1$ in a unitary CFT, and is known as a *Kac-Moody* algebra, and is denoted by $\mathfrak{g}_k$ or sometimes $\widehat{\mathfrak{g}}_k$. The current algebra is the primary object of study in Wess-Zumino-Witten models, and an understanding of the unitary representations of $\mathfrak{g}_k$ is paramount to understanding the spectrum of the above CFT.

The Sugawara construction

Given the WZW model, there is a standard method of constructing a stress tensor on the worldsheet given only the currents J^a. Using the flat space string as an example, we had $J^\mu = \partial X^\mu$ and the stress tensor was simply

$$T = \frac{1}{2}\partial X^\mu \partial X_\mu = \frac{1}{2}\delta_{\mu\nu} J^\mu J^\nu\,. \tag{2.101}$$

That is, the stress tensor is built purely as a bilinear in the holomorphic conserved currents J^μ.

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[8] Here, it is important that G is compact, for which the above statement is equivalent to $H_3(G,\mathbb{Z}) \cong \mathbb{Z}$. If G is non-compact, then the action is truly independent of the continuation $g : \mathcal{B} \to G$, and there is no need to quantize k.

A natural guess for the stress tensor of an arbitrary WZW model with current algebra $\mathfrak{g}_k$ would be

$$T \overset{?}{=} \gamma \kappa_{ab} J^a J^b \,, \tag{2.102}$$

i.e. as the natural bilinear combination of the currents J^a. Since we know that the stress tensor has to be of weight $h = 1$, we can fix the coefficient γ by demanding the OPE

$$T(z) J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \,. \tag{2.103}$$

This calculation yields

$$\gamma = \frac{1}{2(k+h^\vee)} \,, \tag{2.104}$$

where $h^\vee$ is the *dual Coxeter number* of the Lie algebra $\mathfrak{g}$, defined by

$$f^d_{\ bc} f^{abc} = h^\vee \kappa^{ad} \,. \tag{2.105}$$

Two important examples which we will use repeatedly throughout this thesis are $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{g} = \mathfrak{su}(2)$, for which

$$h^\vee(\mathfrak{sl}(2,\mathbb{R})) = -2 \,, \quad h^\vee(\mathfrak{su}(2)) = 2 \,. \tag{2.106}$$

Once we have constructed a stress tensor T for the WZW model, we can compute its central charge by demanding

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \,. \tag{2.107}$$

This calculation is again quite tedious, but can be done, and at the end of the day the WZW model on the Lie group G with level k is given by

$$c(\mathfrak{g}_k) = \frac{k \dim(\mathfrak{g})}{k+h^\vee} \,. \tag{2.108}$$

For example,

$$c(\mathfrak{sl}(2,\mathbb{R})_k) = \frac{3k}{k-2} \,, \quad c(\mathfrak{su}(2)_k) = \frac{3k}{k+2} \,. \tag{2.109}$$

Chapter 3

Strings in AdS_3

As mentioned in the introduction, throughout this thesis we will be interested primarily in string theory on the spacetime $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$. We will assume that all fluxes considered will factor through the three factors of the spacetime manifold, so that we can consider the AdS_3, S^3, and $\mathbb{T}^4$ theories individually. The $\mathbb{T}^4$ theory is easy to describe: it is simply given by four compact free bosons and four free fermions (since we are considering $\mathcal{N} = 1$ worldsheet supersymmetry). The AdS_3 and S^3 terms are described by more complicated sigma models. If we assume in addition that our background has only NS-NS flux and no Ramond-Ramond flux, then, as we mentioned in the previous chapter, we can treat the quantization of the worldsheet sigma model through the study of the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ WZW models.

In this chapter, we review the study of string theory on AdS_3. Specifically, following the approach of [63–65], we study the $\mathrm{SL}(2,\mathbb{R})$ WZW model, both classically and as a quantum theory. We then introduce the supersymmetric generalization of bosonic AdS_3 string theory, both in the RNS formalism and in the 'hybrid' formalism of [66]. It is assumed that the reader has a basic understanding of the RNS superstring in flat spacetime.[1]

3.1 Classical strings in AdS_3

One approach to study string theory on AdS_3 would be to choose some local coordinates, write the metric and the B-field of a supergravity solution, and write the corresponding Polyakov action. For AdS_3 backgrounds this is a simple task. The metric and B-field of AdS_3 with pure NS-NS flux in type IIB supergravity is given by

$$\mathrm{d}s^2 = \frac{L^2}{r^2}\left(\mathrm{d}r^2 + \mathrm{d}\gamma\,\mathrm{d}\overline{\gamma}\right)\,, \quad B = \frac{L^2}{r^2}\mathrm{d}\overline{\gamma}\wedge\mathrm{d}\gamma\,, \tag{3.1}$$

where r is a radial coordinate such that the boundary of AdS_3 lies at $r = 0$, and $\gamma, \overline{\gamma}$ parametrize the boundary coordinates of AdS_3. In Euclidean signature, γ and $\overline{\gamma}$ are complex conjugates of each other, while for Lorentzian signature, they are real and independent. In the context of string theory on AdS_3, we will find it more convenient to define $r = e^{-\Phi}$ such that

$$\mathrm{d}s^2 = L^2\left(\mathrm{d}\Phi^2 + e^{2\Phi}\mathrm{d}\gamma\,\mathrm{d}\overline{\gamma}\right)\,, \quad B = L^2 e^{2\Phi}\mathrm{d}\overline{\gamma}\wedge\mathrm{d}\gamma\,. \tag{3.2}$$

[1] See [53, 67–70] for standard introductions to worldsheet string theory.

The Polyakov action on this background can be read off easily from the metric and the B-field, and we find

$$S = \frac{L^2}{2\pi\ell_s^2}\int_\Sigma \mathrm{d}^2z\left(\partial\Phi\,\overline{\partial}\Phi + e^{2\Phi}\overline{\partial}\gamma\,\partial\overline{\gamma}\right)\,. \tag{3.3}$$

This action defines a classical sigma model on AdS_3, and the solutions to the classical equations of motion describe classical string propagation on this background. However, although the theory itself looks quite simple, it has the drawback that it is nonlinear, and thus naively extremely difficult to quantize.

A second option is to note that AdS_3, as a (pseudo-)Riemannian manifold, is equivalent to the universal cover of the group $\mathrm{SL}(2,\mathbb{R})$ (or the coset $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$). To see this, we can define the 2×2 matrix

$$g = \begin{pmatrix}1 & 0\\ \bar{\gamma} & 1\end{pmatrix}\begin{pmatrix}e^{\Phi} & 0\\ 0 & e^{-\Phi}\end{pmatrix}\begin{pmatrix}1 & \gamma\\ 0 & 1\end{pmatrix} = \begin{pmatrix}e^{\Phi} & e^{\Phi}\gamma\\ e^{\Phi}\bar{\gamma} & e^{-\Phi}+e^{\Phi}\gamma\bar{\gamma}\,.\end{pmatrix}\,. \tag{3.4}$$

In Lorentzian signature, this matrix has real entries and unit determinant. Furthermore, the WZW model action

$$S_{\mathrm{WZW}} = \frac{k}{4\pi}\int_\Sigma \mathrm{Tr}[g^{-1}\mathrm{d}g\wedge\star g^{-1}\mathrm{d}g] - \frac{ik}{2\pi}\int_{\mathcal{B}}\mathrm{Tr}[g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g] \tag{3.5}$$

reproduces the sigma model action (3.3) under the identification

$$k = \frac{L^2}{\ell_s^2}\,. \tag{3.6}$$

As we discussed in Chapter 2, string theory on a group manifold with pure NS-NS flux, described by a Wess-Zumino-Witten model, admits an enhanced symmetry algebra which can be exploited to quantize the theory exactly. Thus, in order to study string theory on AdS_3, it is worthwhile to study in more detail the geometry of the group manifold $\mathrm{SL}(2,\mathbb{R})$, as well as the representation theory of the global algebra $\mathfrak{sl}(2,\mathbb{R})$ and its affine extension $\mathfrak{sl}(2,\mathbb{R})_k$. It turns out that the representation theory of the affine algebra is extremely rich, and, as a consequence of the non-compactness of the group $\mathrm{SL}(2,\mathbb{R})$, each highest-weight representation admits an infinite set of non-equivalent non-highest-weight representations related to each other by the so-called *spectral flow* operation. These 'exotic' representations in turn are invaluable in the study of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, as they are dual to the twist fields of the symmetric orbifold theory.

The rest of this section is dedicated to the study of the $\mathrm{SL}(2,\mathbb{R})$ WZW model, and is largely a review of the seminal analysis of [63–65] (specifically [63]).

The geometry of AdS_3

A useful way to represent Anti-de Sitter space in d dimensions is by embedding it as a hyperboloid in $d+1$ dimensions. Let $X_0,\ldots,X_d$ be a set of coordinates in $\mathbb{R}^{d+1}$, and consider the subspace of points which satisfy the equation

$$X_{-1}^2 + X_0^2 - X_1^2 - \cdots - X_{d=1}^2 = 1\,. \tag{3.7}$$

This equation defines a hyperboloid in $\mathbb{R}^{d+1}$ and is invariant under the symmetry group $\mathrm{SO}(2,d-1)$ (where we take the metric to have signature $(++-\cdots-)$). The resulting submanifold is isometric to AdS_d.

Now let us specialize to $d = 3$. In this case, the equation becomes

$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 = 1 \,. \tag{3.8}$$

If we reorganize the coordinates $X_0, \ldots, X_3$ into a 2×2 matrix

$$\begin{pmatrix} X_{-1} + X_1 & X_0 - X_2 \\ -X_0 - X_2 & X_{-1} - X_1 \end{pmatrix} . \tag{3.9}$$

This identification maps $\mathbb{R}^4$ onto the set $M_2(\mathbb{R})$ of real 2×2 matrices, and (3.8) is precisely the set of matrices with determinant 1. That is, we have[2]

$$\mathrm{AdS}_3 \cong \mathrm{SL}(2, \mathbb{R}) \,. \tag{3.10}$$

Furthermore, the metric on AdS$_3$ is precisely the natural metric on $\mathrm{SL}(2,\mathbb{R})$ induced by the Killing form on $\mathfrak{sl}(2,\mathbb{R})$. As we will discuss below, the realization of AdS$_3$ as a group manifold means that we can easily quantize string theory on this background.

First, let us relate the realization of AdS$_3$ as the group manifold $\mathrm{SL}(2,\mathbb{R})$ to a more familiar representation of AdS$_3$. The solution set of (3.8) can be parametrized via three coordinates (t, ρ, ϕ). Specifically, let

$$\begin{aligned} X_{-1} &= \cos t \, \cosh\rho \,, \quad X_0 = \sin t \, \cosh\rho \,, \\ X_1 &= \cos\phi \, \sinh\rho \,, \quad X_2 = \sin\phi \, \sinh\rho \,. \end{aligned} \tag{3.11}$$

In these coordinates, the metric $\mathrm{d}X_{-1}^2 + \mathrm{d}X_0^2 - \mathrm{d}X_1^2 - \mathrm{d}X_2^2$ pulls back to the metric

$$\mathrm{d}s^2 = -\cosh^2\rho \, \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho \, \mathrm{d}\phi^2 \,. \tag{3.12}$$

This is the standard metric of AdS$_3$ in so-called global coordinates. Note that in equation (3.11), the domain of t is implicitly taken to take values on the circle. Of course, we can extend the validity of this coordinate by simply allowing t to take values $t \in \mathbb{R}$. This corresponds to taking the *universal cover* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, which is the manifold that is isometric to global AdS$_3$.[3] An algebraic fact which will become useful later is that we can decompose any $\mathrm{SL}(2,\mathbb{R})$ element into a product

$$g = e^{iu\sigma_2} e^{\rho\sigma_3} e^{iv\sigma_2} \,, \tag{3.13}$$

where σ_i are the Pauli matrices[4] and

$$u = \frac{1}{2}(t + \phi) \,, \quad v = \frac{1}{2}(t - \phi) \,. \tag{3.15}$$

The global coordinates of AdS$_3$ describe a space which has a conformal boundary at $\rho \to \infty$. Indeed, if we take ρ large (and fixed), then the metric is

$$\mathrm{d}s^2 \approx \frac{e^{2\rho}}{4} \left(-\mathrm{d}t^2 + \mathrm{d}\phi^2 \right) \,, \tag{3.16}$$

---

[2]Strictly speaking, this is not fully true, since AdS$_3$ is simply-connected and $\pi_1(\mathrm{SL}(2,\mathbb{R})) = \mathbb{Z}$. We will come back to this point later.

[3]One might think that we could also extend the validity of the coordinate ϕ, but note that near $\rho \to 0$, the metric (for fixed t) becomes $\mathrm{d}s^2 \approx \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2$. This is the metric of Euclidean $\mathbb{R}^2$, and it is well-known that we need $\phi \sim \phi + 2\pi$ in order for this metric to not have a conical singularity.

[4]We use the convention

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.14}$$

which is conformally equivalent to the cylinder. Thus, we can think of AdS$_3$ as a spacetime which is a solid cylinder, but for which the boundary is at an infinite distance to every other point.

Because of the isometry between (global) AdS$_3$ and (the universal cover of) $\mathrm{SL}(2,\mathbb{R})$, the propagation of strings on AdS$_3$ is most conveniently described in terms of a Wess-Zumino-Witten model. Recall that a WZW model on a Lie group G is via maps $g:\Sigma\to G$ given by the action

$$S = k\,S_{\text{kin}}[g] + k\,S_{\text{WZ}}[g]\,, \tag{3.17}$$

where S_{kin} is the action of the principal chiral model (the usual quadratic kinetic action of a sigma model), and S_{WZ} is the Wess-Zumino term that ensures the conformal invariance of the full action.

Classical geodesics

For the moment, let's treat the $\mathrm{SL}(2,\mathbb{R})$ model as a classical conformal field theory in order to gain some intuition, and discuss its quantization later. The basic fields are the global coordinates t,ρ,ϕ, which are now taken to be functions on the worldsheet. We can organize these three fields into a single $\widetilde{\mathrm{SL}}(2,\mathbb{R})$-valued field

$$g = \begin{pmatrix} \cos t\cosh\rho + \cos\phi\sinh\rho & \sin t\cosh\rho - \sin\phi\sinh\rho \\ -\sin t\cosh\rho - \sin\phi\sinh\rho & \cos t\cosh\rho - \cos\phi\sinh\rho \end{pmatrix}\,. \tag{3.18}$$

From g, one can build left- and right-moving currents

$$J := k\partial g\,g^{-1}\,,\quad \bar{J} := kg^{-1}\bar{\partial}g\,, \tag{3.19}$$

so that the equations of motion read $\overline{\partial}J = \partial\bar{J} = 0$. Solutions to these equations of motion can always be written in the form

$$g = g_L(z)g_R(\bar{z})\,, \tag{3.20}$$

and there is an overall $G_L\times G_R$ symmetry given by

$$g_L(z)\to h_L g_L(z)\,,\quad g_R(\bar{z})\to g_R(\bar{z})h_R^{-1}\,, \tag{3.21}$$

for any group elements $h_L,h_R\in\mathrm{SL}(2,\mathbb{R})$. Moreover, if we demand that the strings we consider are closed, then g_L,g_R can be defined up to the monodromy

$$g_L(e^{2\pi i}z) = g_L(z)M\,,\quad g_R(e^{2\pi i}\bar{z}) = M^{-1}g_R(\bar{z})\,, \tag{3.22}$$

so that the overall field $g(z,\bar{z})$ is globally defined.

However, although any holomorphic and antiholomorphic $\mathrm{SL}(2,\mathbb{R})$-valued functions g_L,g_R generate a solution to the equations of motion, in string theory we still need to impose the Virasoro constraint $T(z)=\overline{T}(\overline{z})=0$ coming from the gauge-fixing of the Polyakov action. Since our string theory is a WZW model, the stress-tensor is given via the Sugawara construction, i.e.

$$T_{\text{tot}} = \underbrace{\frac{1}{2k}\mathrm{Tr}[JJ]}_{T_{\text{AdS}}} + T_{\text{rest}}\,, \tag{3.23}$$

and similarly for the right-moving stress tensor, where $T_{\rm rest}$ is the 'rest' of the stress tensor. That is, if the string propagates on $\mathrm{AdS}_3 \times X$, then $T_{\rm rest}$ is the stress tensor of X. Thus, the Virasoro constraints require

$$\begin{aligned} T_{\rm AdS} &= \frac{1}{2k}\mathrm{Tr}[JJ] = k\mathrm{Tr}[\partial g\, g^{-1}\partial g\, g^{-1}] = -T_{\rm rest}\,, \\ \overline{T}_{\rm AdS} &= \frac{1}{2k}\mathrm{Tr}[\overline{J}\,\overline{J}] = k\mathrm{Tr}[g^{-1}\overline{\partial} g\, g^{-1}\overline{\partial} g] = -\overline{T}_{\rm rest}\,. \end{aligned} \tag{3.24}$$

Because of the simplicity of the solutions to the equations of motion, we can already start to write down classical trajectories. For example, we can consider the solution

$$g_L(z) = Ue^{iv(z)\sigma_2}\,, \quad g_R(\overline{z}) = e^{i\overline{u}(\overline{z})\sigma_2}V\,, \tag{3.25}$$

where v and $\overline{u}$ are generic holomorphic and antiholomorphic functions, and U, V are any $\mathrm{SL}(2,\mathbb{R})$ matrices. For this solution, we can automatically calculate the currents and the stress tensor as

$$\begin{aligned} J = k\partial g\, g^{-1} = ik\partial v\,(U\sigma_2 U^{-1})\,, \quad \overline{J} = kg^{-1}\overline{\partial} g = ik\overline{\partial}\overline{u}\,(V^{-1}\sigma_2 V)\,, \\ T_{\rm AdS} = -k(\partial v)^2\,, \quad \overline{T}_{\rm AdS} = -k(\overline{\partial}\overline{u})^2\,. \end{aligned} \tag{3.26}$$

Now, let us assume our string satisfies $T_{\rm rest} = \overline{T}_{\rm rest} = h$, where h is a constant. Then the Virasoro constraints require

$$(\partial v)^2 = \frac{h}{k}\,, \quad (\overline{\partial}\overline{u})^2 = \frac{k}{k}\,. \tag{3.27}$$

That is, we require that v and $\overline{u}$ are *linear* functions of z and $\overline{z}$. If we take the coefficients of the linear functions to have the same sign, we obtain

$$g = U\begin{pmatrix} \cos(\alpha\tau) & \sin(\alpha\tau) \\ -\sin(\alpha\tau) & \cos(\alpha\tau) \end{pmatrix} V\,, \tag{3.28}$$

where $\alpha = \sqrt{2kh}$ and $\tau = (z - \overline{z})/2$. If we take a Lorentzian signature on the worldsheet (so that z and $\overline{z}$ are real and independent), then we can write $z = \sigma + \tau$, $\overline{z} = \sigma - \tau$.

If $U = V = 1$, the above solution to the equations of motion and Virasoro constrains describes a string which is stationary and at the center of AdS_3. Specifically, the string does not extend in the spatial directions, and so describes the worldline of a time-like particle. For $U, V \neq 1$, we can obtain the trajectory by applying an $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ transformation on the $U = V = 1$ trajectory. The resulting trajectory still has a one-dimensional worldsheet, and the trajectory is still spacelike.

We can also construct spacelike geodesics in a very similar way to the construction of timelike geodesics. Specifically, the trajectory

$$g = U\begin{pmatrix} e^{\alpha\tau} & 0 \\ 0 & e^{-\alpha\tau} \end{pmatrix} V \tag{3.29}$$

defines a solution to the equations of motion and the Virasoro constraints, and describes a spacelike geodesic which approaches the boundary at $\tau = -\infty$ and $\tau = \infty$. Both the timelike and spacelike geodesics are represented in Figure 3.1.

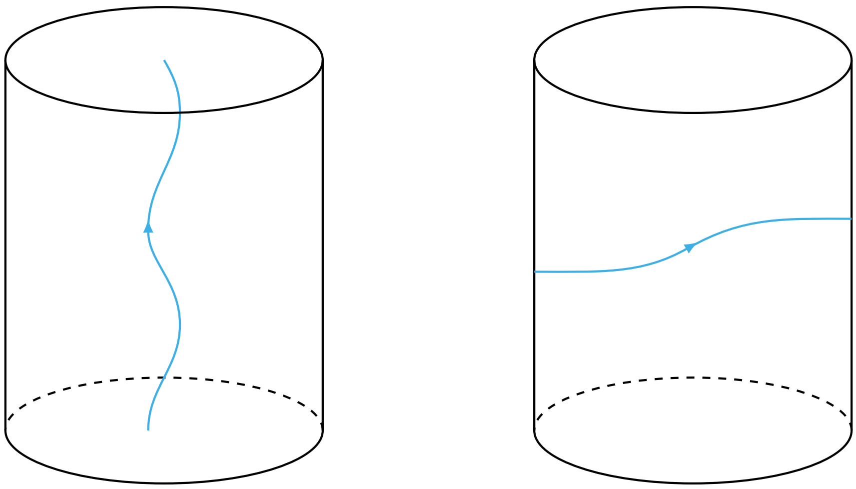

Figure 3.1: Timelike (left) and spacelike (right) trajectories of particle-like strings moving in AdS$_3$.

Spectral flow

The classical string solutions we derived above indeed satisfy the Virasoro constraints, but are not particularly interesting, since they describe particle-like strings. However, given a classical solution to the equations of motion, we can generate a new solution via the operation

$$g \mapsto \exp\left(\frac{iw_L}{2}z\sigma_2\right) g(z,\overline{z}) \exp\left(\frac{iw_R}{2}\overline{z}\sigma_2\right) . \tag{3.30}$$

This does not break the chiral splitting required for solutions to the equations of motion. Using the decomposition

$$g = e^{i(t+\phi)\sigma_2} e^{\rho\sigma_3} e^{i(t-\phi)\sigma_2} \tag{3.31}$$

relating $\mathrm{SL}(2,\mathbb{R})$ matrices to global coordinates on AdS$_3$, we see that the above operation has the effect of shifting t,ϕ via

$$t+\phi \mapsto t+\phi+\frac{w_L}{2}z\,, \quad t-\phi \mapsto t-\phi+\frac{w_R}{2}\overline{z}\,. \tag{3.32}$$

Adding and subtracting these two expressions gives

$$\begin{aligned} t &\mapsto t+\left(\frac{w_L+w_R}{2}\right)\tau+\left(\frac{w_L-w_R}{2}\right)\sigma\,,\\ \phi &\mapsto \phi+\left(\frac{w_L-w_R}{2}\right)\tau+\left(\frac{w_L+w_R}{2}\right)\sigma\,, \end{aligned} \tag{3.33}$$

where we are again working in Euclidean signature such that $z=\sigma+\tau$ and $\overline{z}=\sigma-\tau$. Since we are considering closed strings, we identify σ with $\sigma+2\pi$. Since we are considering global AdS$_3$, which is the universal cover of $\mathrm{SL}(2,\mathbb{R})$, we require that t is a single-valued function of σ,τ. This requires us to demand $w_L=w_R$. Furthermore, since ϕ must be single-valued up to a shift of $2\pi n$ for some integer n, we see that we require w_L and w_R to be integers.

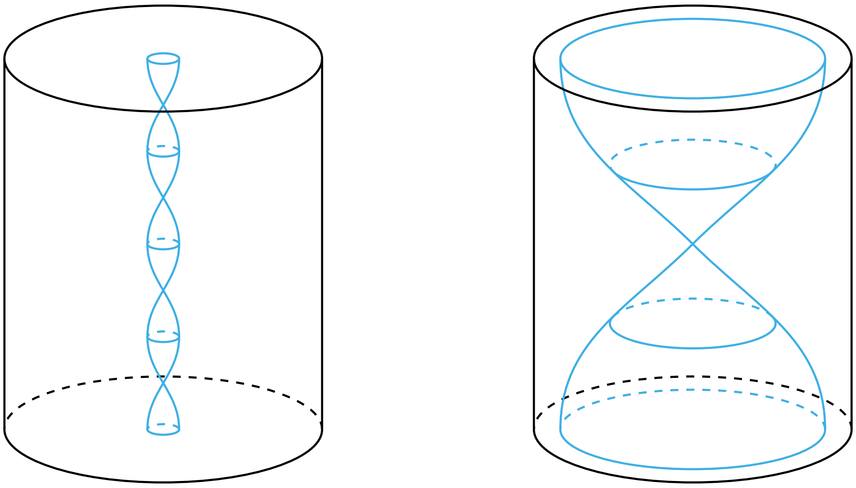

Figure 3.2: Spectrally-flowed images of timelike (left) and spacelike (right) geodesics in AdS$_3$. The spectrally-flowed timelike geodesics oscillate near the center of AdS$_3$ and are dubbed *short strings*, while the spectrally flowed images of spacelike geodesics approach the boundary of AdS$_3$ at early and late times, and are dubbed *long strings*.

With this in mind, we define the *spectral flow* operation to be the above operation when $w_L = w_R = w \in \mathbb{Z}$, i.e.

$$\sigma^w(g) = \exp\left(\frac{iw}{2} z\,\sigma_2\right) g \exp\left(\frac{iw}{2}\bar{z}\,\sigma_2\right) . \tag{3.34}$$

Spectral flow has the property that it maps classical solutions of the equations of motion to other classical solutions to the equations of motion. Furthermore, it acts on the global coordinates of AdS$_3$ as

$$\sigma^w(t) = t + w\tau\,, \quad \sigma^w(\phi) = \phi + w\sigma\,. \tag{3.35}$$

Thus, intuitively, spectral flow acts on classical solutions by winding them w times around the origin of AdS$_3$ and by stretching the worldsheet out in the time direction.

For example, we can start with the spacelike and timelike trajectories of the previous section and act upon them with the spectral flow operation. The resulting worldsheets are shown in Figure 3.2. The spectrally-flowed timelike geodesics oscillate near the center of AdS$_3$ and are dubbed *short strings*, while the spectrally flowed images of spacelike geodesics approach the boundary of AdS$_3$ at early and late times, and are dubbed *long strings*.

Spectral flow is not just a method of generating new classical string solutions from old ones, but rather, as we will discuss in the context the quantum theory on AdS$_3$, is an invaluable method of classifying the allowed representations on the worldsheet. Recall that the currents J^a in the quantized theory will form a current algebra, and the string theory spectrum falls into representations of this algebra. Focusing on the left-moving sector, the currents J are defined as

$$J = k\partial g\, g^{-1}\,. \tag{3.36}$$

Since we know how spectral flow acts on the fields g, we can deduce how it will act on the currents. We have

$$\sigma^w(J) = \left(e^{iwz\sigma_2/2} J e^{-iwz\sigma_2/2} + \frac{ikw}{2}\sigma_2\right) . \tag{3.37}$$

Now, it is convenient to pick a basis of $\mathfrak{sl}(2,\mathbb{R})$. We can do this by defining the generators

$$T^{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix} , \quad T^3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{3.38}$$

In terms of these generators, we can define

$$J^a = \mathrm{Tr}[T^a J] . \tag{3.39}$$

On the components of J, the spectral flow operation acts in a particularly simple way:

$$\sigma^w(J^{\pm}) = e^{\mp iwz} J^{\pm} , \quad \sigma^w(J^3) = J^3 + \frac{kw}{2} , \tag{3.40}$$

and similarly for the right-moving currents.[5] Furthermore, from the Sugawara construction, we can calculate the action of spectral flow on the stress tensor to be

$$\begin{aligned} \sigma^w(T) &= \frac{1}{2k}\mathrm{Tr}[\sigma^w(J)\sigma^w(J)] \\ &= \frac{1}{2k}\mathrm{Tr}\left[JJ + ikw\sigma_2 J - \frac{k^2w^2}{4}\sigma_2^2\right] \\ &= T - wJ^3 - \frac{kw^2}{4} . \end{aligned} \tag{3.41}$$

Thus, it is not guaranteed that, after spectrally flowing a classical solution, the resulting spectrally-flowed solution will satisfy the Virasoro constraints, and one will have to impose the Virasoro constraint

$$\sigma^w(T_{\mathrm{AdS}}) + T_{\mathrm{rest}} = 0 , \tag{3.42}$$

along with the right-moving counterpart, by hand.

3.2 Quantum strings on AdS$_3$

Now that we have discussed classical strings on AdS$_3$, we are ready to venture into the quantum theory. Again, the starting point of the analysis is to consider the string theory as a Wess-Zumino-Witten model on the group $\mathrm{SL}(2,\mathbb{R})$.

For compact simple Lie groups G, the level k is quantized for consistency of the quantum theory. We argued in Section 2.3 that this quantization is related to the nontriviality of the cohomology group

$$H^3(G,\mathbb{Z}) \cong \mathbb{Z} , \tag{3.43}$$

along with the condition that the path integral is well-defined (i.e. the action is well-defined up to an addition of the form $2\pi i n$). The group $\mathrm{SL}(2,\mathbb{R})$, however, is not compact, nor is its universal cover. In fact, we have

$$H^3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z}) = 0 . \tag{3.44}$$

[5] The fact that we took $w_L = w_R$ earlier implies that spectral flow acts the same on the left- and right-moving currents.

As a consequence, the usual argument that the level k is quantized does not hold in the case of the $\mathrm{SL}(2,\mathbb{R})$ WZW model. Since the level k is related to the AdS$_3$ radius and the string length by $k = L^2/\ell_s^2$, this means that the CFT of a string propagating in AdS$_3$ is consistent for any value of L. As we will discuss later, however, we are usually interested in backgrounds of the form $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}$. For $\mathcal{M} = \mathbb{T}^4$ or $\mathcal{M} = \mathrm{K3}$, we will show that the radii of the AdS$_3$ and S^3 factors have to match. Since the radius of the S^3 factor is determined (in the case of pure NS-NS flux) by the level of the underlying SU(2) WZW model, the radius of the S^3 is quantized, and as such the AdS$_3$ radius will become quantized as well. We will return to this point later.

For the moment, let us ignore the compact directions, and assume that k can be any (positive) number. The currents of the theory are still defined as before to be

$$J^a(z) = k\mathrm{Tr}[T^a \partial g\, g^{-1}]\,, \quad \overline{J}^a(\overline{z}) = k\mathrm{Tr}[T^{a*} g^{-1}\overline{\partial} g]\,, \tag{3.45}$$

except now we consider them as operators living on the worldsheet. While we did the classical analysis on the cylinder and in Lorentzian signature, it will be much easier to work with the quantum theory in Euclidean signature on the complex plane. In the plane, the currents J^a satisfy the OPEs

$$J^a(z)J^b(w) \sim \frac{k\,\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{z-w}\,, \tag{3.46}$$

which in the case of $\mathrm{SL}(2,\mathbb{R})$ takes the explicit form

$$\begin{gathered} J^3(z)J^\pm(w) \sim \pm\frac{J^\pm(w)}{z-w}\,, \quad J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} - \frac{2J^3(w)}{z-w}\,, \\ J^3(z)J^3(w) \sim -\frac{k/2}{(z-w)^2}\,. \end{gathered} \tag{3.47}$$

We will later find it convenient to decompose J^a into modes

$$J^a(z) := \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}\,, \tag{3.48}$$

for which the above current algebra reads

$$\begin{gathered} [J^3_m, J^\pm_n] = \pm J^\pm_{m+n}\,, \quad [J^+_m, J^-_n] = -2J^3_{m+n} + km\delta_{m+n,0}\,, \\ [J^3_m, J^3_n] = -\frac{km}{2}\delta_{m+n,0}\,. \end{gathered} \tag{3.49}$$

Now, we can construct the stress tensor of the theory via the Sugawara construction, which has to be corrected in the quantum case. The correction is to shift the level k to $k + h^\vee$, where $h^\vee$ is the dual coxeter number of the Lie algebra. The dual coxeter number of $\mathfrak{sl}(2,\mathbb{R})$ is $h^\vee = -2$, and so the stress-energy tensor is

$$T_{\mathrm{AdS}} = \frac{1}{2(k-2)}\mathrm{Tr}[JJ] = \frac{1}{2(k-2)}\left(J^+J^- + J^-J^+ - 2J^3J^3\right)\,. \tag{3.50}$$

Here, and in all quantities following, we implicitly assume that all products of operators are normal-ordered. The central charge of the theory is computed by taking the OPE of the stress-tensor with itself and the result is

$$c_{\mathrm{AdS}} = \frac{3k}{k-2}\,, \tag{3.51}$$

which we also derived in Chapter 2. Note that in the limit $k \to \infty$, we have $c_{\text{AdS}} \to 3$, which is simply the number of dimensions of the spacetime.

In order to define a consistent string theory, we have to consider a background given by AdS$_3$ $\times$ X, where X is some 'internal' CFT (which is not necessarily a sigma model). Consistency requires that the sum of the central charges of the AdS$_3$ and X factors is 26, in order to cancel the contribution of the conformal b, c ghosts. Furthermore, physical states in the theory are required to satisfy the Virasoro constraints

$$(L_0^{\text{AdS}} + L_0^X)\,|\psi\rangle = |\psi\rangle\ , \quad (L_n^{\text{AdS}} + L_n^X)\,|\psi\rangle = 0\,, \quad n > 0\,. \tag{3.52}$$

Highest-weight representations

States in the SL$(2,\mathbb{R})$ model will fall into representations of the current algebra generated by J^a. Let us start by considering highest-weight representations, i.e. representations built from descendants of highest-weight states. A highest-weight state $|\psi\rangle$ satisfies

$$J_n^a\,|\psi\rangle = 0\,, \quad n > 0\,. \tag{3.53}$$

Since the zero modes J_0^a act nontrivially on $|\psi\rangle$, highest-weight states will fall into representations of the zero mode algebra generated by J_0^a. This algebra is nothing more than the global $\mathfrak{sl}(2,\mathbb{R})$ symmetry algebra, and so in order to understand highest-weight states, we need to understand the representation theory of $\mathfrak{sl}(2,\mathbb{R})$.

The representation theory of $\mathfrak{sl}(2,\mathbb{R})$ is very similar to that of $\mathfrak{su}(2)$, since both are simply different real forms of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. The main difference is that the dimensions of the representations need not be finite-dimensional, since SL$(2,\mathbb{R})$ is not compact. However, in complete analogy to representations of $\mathfrak{su}(2)$, we can label states by pairs $|m,j\rangle$ of quantum numbers. The number m labels the eigenvalue of the Cartan generator J_0^3, while j labels the quadratic Casimir of the representation. Due to the commutation relations

$$[J_0^3, J_0^\pm] = \pm J_0^\pm\,, \tag{3.54}$$

we can choose $J_0^\pm$ as ladder operators, which raise and lower the value of m by one unit. A convenient choice of action of $J_0^\pm$ on the states $|m,j\rangle$ is

$$J_0^\pm\,|m,j\rangle = (m \pm j)\,|m \pm 1, j\rangle\ . \tag{3.55}$$

From this action, the quadratic Casimir is readily computed to be

$$\begin{aligned} C_2\,|\psi\rangle &= \frac{1}{2}\left(J_0^+ J_0^- + J_0^- J_0^+ - 2 J_0^3 J_0^3\right)|\psi\rangle \\ &= j(1-j)\,|\psi\rangle\ , \end{aligned} \tag{3.56}$$

which formally resembles the quadratic Casimir $C_2 = j^2 + j$ of the spin-j representation of $\mathfrak{su}(2)$. Once we have constructed a representation of $\mathfrak{sl}(2,\mathbb{R})$, we can use it to construct a highest-weight representation of the full affine algebra $\mathfrak{sl}(2,\mathbb{R})_k$ by taking the linear span of all descendants of the states $|m,j\rangle$, i.e.

$$\widehat{R} := \text{Span}_{\mathbb{C}}\left\{J_{-n_\ell}^{a_\ell} \cdots J_{-n_1}^{a_1}\,|m,j\rangle \mid |m,j\rangle \in R\,,\ \text{and } n_i > 0\right\}\ . \tag{3.57}$$

The $\mathfrak{sl}(2,\mathbb{R})$ representations we will be interested in here are known as the *principal series* representations. We summarize them now:

- **Discrete principal series:** The discrete principal series $\mathcal{D}_j^\pm$ are defined as follows. For $\mathcal{D}_j^+$, one starts with the state $|j,j\rangle$, which is annihilated by J_0^- and defines
$$\mathcal{D}_j^+ = \text{Span}\left\{|j,j\rangle\,,\,|j+1,j\rangle\,,\,|j+2,j\rangle\,,\,\ldots\right\}\,. \tag{3.58}$$
For $\mathcal{D}_j^-$, the construction is similar, except we start with the state $|-j,j\rangle$, which is annihilated by J_0^+ and define
$$\mathcal{D}_j^- = \text{Span}\left\{|-j,j\rangle\,,\,|-j-1,j\rangle\,,\,|-j-2,j\rangle\,,\,\ldots\right\}\,. \tag{3.59}$$
Discrete principal series representations are recognized by the fact that they have a lowest/highest-weight state with respect to the J_0^3 Cartan generator. These representations are unitary if and only if j is real and positive.

- **Continuous principal series:** The continuous principal series $\mathcal{C}_j^\lambda$ have two labels: the spin j and a continuous parameter $\lambda \in [0,1)$. The representation $\mathcal{C}_j^\lambda$ is given by
$$\mathcal{C}_j^\lambda = \text{Span}\left\{\ldots\,,\,|\lambda-1,j\rangle\,,\,|\lambda,j\rangle\,,\,|\lambda+1,j\rangle\,,\,\ldots\right\}\,. \tag{3.60}$$
Continuous principal series representations do not have highest/lowest-weight states with respect to J_0^3. These representations are unitary if and only if $j \in \frac{1}{2} + i\mathbb{R}$.

In addition to the principal series representations, $\mathfrak{sl}(2,\mathbb{R})$ also admits the trivial representation ρ_0 and so-called 'complimentary' representations $\mathcal{E}_j^\lambda$, which we will not discuss here, as they do not play much of a role in the string spectrum.[6] Once we have a representation, we can promote it to an affine representation by taking descendants. We denote affine representations with a hat, and so our highest-weight representations on the worldsheet are given by
$$\widehat{\rho}_0\,,\quad \widehat{\mathcal{D}}_j\,,\quad \widehat{\mathcal{C}}_j^\lambda\,. \tag{3.61}$$
Of course, these representations are only representations of the left-moving sector of the theory, and the full string spectrum will include right-moving versions of all of these representations.

Spectral flow and the string spectrum

It turns out that, since $\text{SL}(2,\mathbb{R})$ is non-compact, the representations of $\mathfrak{sl}(2,\mathbb{R})_k$ are not simply the highest-weight representations. However, we can construct all of the representations which appear in the full string spectrum from the highest-weight representations we described above. Just as in the case of the classical string, there is an operation σ^w which allows us to map highest-weight representations to new, non highest-weight representations, which we will also call spectral flow. Spectral flow is defined by its action on the currents J^a, namely
$$\sigma^w(J^\pm)(z) = z^{\pm w}J^\pm(z)\,,\quad \sigma^w(J^3)(z) = J^3(z) + \frac{kw}{2z}\,. \tag{3.62}$$
The discrepency between these transformation laws and the transformation laws (3.40) is that we are working on the complex plane as opposed to the cylinder, and

[6] These representations are defined in [63].

the two are related via a Wick rotation and the conformal mapping $z \mapsto i\log z$. Decomposing the currents into modes, we find

$$\sigma^w(J_n^\pm) = J_{n\mp w}^\pm\,, \quad \sigma^w(J_n^3) = J_n^3 + \frac{kw}{2}\delta_{n,0}\,. \tag{3.63}$$

It is easy to check that this operation is an automorphism of the mode algebra $\mathfrak{sl}(2,\mathbb{R})_k$ defined in (3.49), and thus spectral flow is a symmetry of our theory.

Given a representation R of $\mathfrak{sl}(2,\mathbb{R})_k$, we can use the spectral flow operation to define a representation $\sigma^w(R)$. The action of σ^w on representations can be defined implicitly: the underlying vector spaces of R and $\sigma^w(R)$ are the same, but the action of $\mathfrak{sl}(2,\mathbb{R})_k$ on these vectors is different. Let $|\psi\rangle \in R$ and its corresponding vector in $\sigma^w(R)$ be denoted by $|\psi\rangle^{(w)}$ (formally, we define a map $(\cdot)^{(w)} : R \to \sigma^w(R)$ which acts as the identity on the underlying vector spaces). Then the action of the $\mathfrak{sl}(2,\mathbb{R})_k$ modes on $|\psi\rangle^{(w)}$ are

$$J_n^a\,|\psi\rangle^{(w)} = \left(\sigma^w(J_n^a)\,|\psi\rangle\right)^{(w)}\,. \tag{3.64}$$

Now, let us take R to be a highest-weight representation, and $|m,j\rangle$ a highest-weight state (here we have in mind $R = \widehat{\mathcal{D}}_j^\pm$ or $\widehat{\mathcal{C}}_j^\lambda$). The spectrally-flowed image $|m,j\rangle^{(w)}$ satisfies

$$J_n^\pm\,|m,j\rangle^{(w)} = 0\,, \quad n > \pm w\,, \quad J_n^3\,|m,j\rangle^{(w)} = 0\,, \quad n > 0\,, \tag{3.65}$$

and

$$\begin{aligned} J_w^+\,|m,j\rangle^{(w)} &= (m+j)\,|m,j\rangle^{(w)}\,, \\ J_{-w}^-\,|m,j\rangle^{(w)} &= (m-j)\,|m,j\rangle^{(w)}\,, \\ J_0^3\,|m,j\rangle^{(w)} &= \left(m + \frac{kw}{2}\right)|m,j\rangle^{(w)}\,. \end{aligned} \tag{3.66}$$

Thus, the spectrally-flowed representation $\sigma^w(R)$ is *not* a highest-weight representation of $\mathfrak{sl}(2,\mathbb{R})_k$.

Just as classical spectral flow takes simple string trajectories like those in Figure 3.1 to more interesting worldsheet configurations like those in Figure 7.4, quantum spectral flow takes simple (highest-weight) representations to more interesting (non-highest-weight) representations, namely

$$\sigma^w(\widehat{\mathcal{D}}_j^+)\,, \quad \sigma^w(\widehat{\mathcal{D}}_j^-)\,, \quad \sigma^w(\widehat{\mathcal{C}}_j^\lambda)\,. \tag{3.67}$$

In [63], it was proposed that the full string spectrum is furnished by the spectrally-flowed continuous representations $\sigma^w(\widehat{\mathcal{C}}_j^\lambda) \times \overline{\sigma^w(\widehat{\mathcal{C}}_j^\lambda)}$ with $j = \frac{1}{2} + is$ and spectrally-flowed discrete representations $\sigma^w(\widehat{\mathcal{D}}_j^+) \times \overline{\sigma^w(\widehat{\mathcal{D}}_j^+)}$ with j satisfying the bound

$$\frac{1}{2} < j < \frac{k-1}{2}\,. \tag{3.68}$$

The latter bound was found by a generalization of the no-ghost theorem, and this string spectrum is free of negative-norm states.

In Section 3.1, we explored the classical trajectories of strings and argued that spectral flow corresponds, classically, to 'winding' a string w times around the cylinder at asymptotic infinity. We found that the spectral flowed images of timelike and spacelike geodesics resulted in worldsheet configurations of 'short strings' and 'long strings', respectively (see Figure 3.2). In [63], it was argued from a semiclassical evaluation analysis of the long and short string trajectories described above that the spectrally-flowed discrete representations $\sigma^w(\widehat{\mathcal{D}}_j^+)$ correspond to short string excitations, while the spectrally-flowed continuous representations $\sigma^w(\widehat{\mathcal{C}_j^\alpha})$ correspond to long-string exictations.

3.3 Superstrings in the RNS formalism

In the previous sections we introduced string theory on AdS_3 as a WZW model on $\mathrm{SL}(2,\mathbb{R})$, and discussed its spectrum. Ultimately, however, we want to study *superstring* theories living on an AdS_3 background. This can be achieved in the RNS formalism by introducing a superpartner ψ^a to each current J^a in the $\mathrm{SL}(2,\mathbb{R})$ model. In this section we will discuss how precisely such a theory is constructed.

We begin by taking a step back and replacing AdS_3 with a generic group manifold, and then specializing to $\mathrm{SL}(2,\mathbb{R})$ in the end. Let G be a simply-connected semi-simple Lie group and $\mathfrak{g}$ its Lie algebra. A bosonic WZW model on G at level k is described by the representation theory of the affine algebra $\mathfrak{g}_k$, which is generated by affine currents $J^a(z)$ satisfying the OPEs

$$J^a(z)J^b(w) \sim \frac{k\,\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c\,J^c(w)}{z-w}\,. \tag{3.69}$$

Just as in the flat space string, we can extend the bosonic WZW model to include $\mathcal{N}=1$ worldsheet supersymmetry[7] by introducing worldsheet fermions ψ^a which are superpartners to J^a. Since ψ^a and J^a should live in a supermultiplet, they should transform in the same way under the symmetry group G, i.e. they should transform in the adjoint representation of $\mathfrak{g}$. This implies

$$\oint_w \frac{\mathrm{d}z}{2\pi i} J^a(z)\psi^b(w) = f^{ab}{}_c\,\psi^c(w)\,, \tag{3.70}$$

which is equivalent to the OPE

$$J^a(z)\psi^b(w) \sim \frac{f^{ab}{}_c\,\psi^c(w)}{z-w}\,. \tag{3.71}$$

Furthermore, the OPEs between two fermions should be covariant under G-transformations, and thus proportional to κ^{ab}. A natural normalization is

$$\psi^a(z)\psi^b(w) \sim \frac{k\,\kappa^{ab}}{z-w}\,. \tag{3.72}$$

Combining all of the OPEs, we have a worldsheet theory with fields ψ^a, J^a satisfying the current algebra

$$\begin{aligned} J^a(z)J^b(w) &\sim \frac{k\,\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c\,J^c(w)}{z-w}\,,\\ J^a(z)\psi^b(w) \sim \frac{f^{ab}{}_c\,\psi^c(w)}{z-w}\,,&\quad \psi^a(z)\psi^b(w) \sim \frac{k\,\kappa^{ab}}{z-w}\,. \end{aligned} \tag{3.73}$$

[7] Since we are only working in the holomorphic sector, we strictly speaking only have $\mathcal{N}=(1,0)$ SUSY. Adding in anti-holomorphic fermions will give $\mathcal{N}=(1,1)$.

This algebra has a name: $\mathfrak{g}_k^{(1)}$ and should be thought of as the $\mathcal{N}=1$ supersymmetric version of $\mathfrak{g}_k$. It possesses an outer automorphism of supersymmetry transformations:

$$\delta_\varepsilon \psi^a = \varepsilon J^a \,, \quad \delta_\varepsilon J^a = \varepsilon \partial \psi^a \,, \tag{3.74}$$

which leaves the form of the algebra invariant.

In principle, solving the worldsheet theory is now a matter of studying the representations of $\mathfrak{g}_k^{(1)}$. However, due to the interaction between the fermions and the bosonic currents, these representations are typically quite complicated. A standard trick in the theory of supersymmetric WZW models is to define new currents $\mathcal{J}^a$ such that the OPE between $\mathcal{J}^a$ and the fermions is trivial, so that we can treat the fermionic part of the theory as dim($\mathfrak{g}$) *free* fermions. The correct definition for the currents $\mathcal{J}^a$ turns out to be [71]

$$\mathcal{J}^a = J^a + \frac{1}{2k} f^a{}_{bc} (\psi^b \psi^c) \,. \tag{3.75}$$

The OPEs of the redefined $\psi^a, \mathcal{J}^a$ fields can be computed, and the end result is

$$\begin{aligned} \mathcal{J}^a(z)\mathcal{J}^b(w) &\sim \frac{(k-h^\vee)\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c \mathcal{J}^c(w)}{z-w} \,, \\ \mathcal{J}^a(z)\psi^b(w) \sim \text{regular} \,, &\quad \psi^a(z)\psi^b(w) \sim \frac{k\,\kappa^{ab}}{z-w} \,, \end{aligned} \tag{3.76}$$

where $h^\vee$ is the dual coxeter number of the Lie algebra $\mathfrak{g}$. That is, the currents $\mathcal{J}^a$ satisfy the current algebra $\mathfrak{g}_{k-h^\vee}$ and the fermions ψ^a can now be treated as free fields. Thus, at the level of the operator algebra defining the $\mathcal{N}=1$ worldsheet theory, we have

$$\mathfrak{g}_k^{(1)} \cong \mathfrak{g}_{k-h^\vee} \oplus (\text{dim}(\mathfrak{g}) \text{ free fermions}) \,. \tag{3.77}$$

The stress tensor and supercurrent of this theory can be found by demanding that $\mathcal{J}^a$ and ψ^a have conformal weight $h=1$ and $h=1/2$, respectively. We find

$$\begin{aligned} T &= \frac{1}{2k}\kappa_{ab}\Big((\mathcal{J}^a\mathcal{J}^b) - (\psi^a\partial\psi^b) \Big) \,, \\ G &= \frac{1}{k}\left(\kappa_{ab}\mathcal{J}^a\psi^b - \frac{1}{6k} f_{abc}(\psi^a\psi^b\psi^c) \right) \,. \end{aligned} \tag{3.78}$$

The stress tensor and supercurrent can be found to generate an $\mathcal{N}=(1,0)$ superconformal algebra with central charge

$$c(\mathfrak{g}_k^{(1)}) = \text{dim}(\mathfrak{g}) \left(\frac{k-h^\vee}{k} + \frac{1}{2} \right) \,. \tag{3.79}$$

This agrees with the central charge of a purely bosonic $\mathfrak{g}_{k-h^\vee}$ current algebra plus dim($\mathfrak{g}$) free bosons.

Now, let us return to the case of interest: string theory in AdS$_3$. The algebra $\mathfrak{sl}(2,\mathbb{R})$ has $h^\vee = -2$ and so

$$\mathfrak{sl}(2,\mathbb{R})_k^{(1)} \cong \mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus (3 \text{ free fermions}) \,, \tag{3.80}$$

and the stress tensor and supercurrent are as described above. The central charge is then

$$c(\mathfrak{sl}(2,\mathbb{R})_k^{(1)}) = 3\left(\frac{k+2}{k} + \frac{1}{2} \right) \,. \tag{3.81}$$

In superstring theory, we require that the full central charge of the full worldsheet theory vanishes. The full worldsheet theory has the form of a target space sigma model, coupled to the (b,c) conformal and (β,γ) superconformal ghost system. The ghost system has central charge $c=-15$, and so the sigma model must have central charge $c=15$ to define a consistent (i.e. anomaly-free) worldsheet theory. If we start with a supersymmetric worldsheet theory on the space $\mathrm{AdS}_3\times\mathcal{M}$, with pure NS-NS flux in the AdS_3 component, the anomaly cancellation requires

$$c(\mathfrak{sl}(2,\mathbb{R})_k^{(1)})+c(\mathcal{M})=15\,. \tag{3.82}$$

As explained in the introduction, there are three such choices of $\mathcal{M}$ which are of holographic interest, namely

$$\mathcal{M}=\mathrm{S}^3\times\mathbb{T}^4\,,\quad \mathcal{M}=\mathrm{S}^3\times\mathrm{K3}\,,\quad \mathcal{M}=\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1\,. \tag{3.83}$$

Since both $\mathbb{T}^4$ and K3 sigma models have the same central charge ($c=6$), we can treat them as one case to consider. The factors of S^3 can conveniently be described by a WZW model on SU(2) in the case of pure NS-NS flux, which we will always assume.

Let's start with the case $\mathcal{M}=\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$. If k_1 and k_2 are the levels of the two SU(2) WZW models, we have (noting that $h^\vee=-2$ for $\mathfrak{su}(2)$)

$$c(\mathfrak{sl}(2,\mathbb{R})_k^{(1)})+c(\mathcal{M})=3\left(\frac{k+2}{k}+\frac{1}{2}\right)+3\left(\frac{k_1-2}{k_1}+\frac{1}{2}\right)+3\left(\frac{k_2-2}{k_2}+\frac{1}{2}\right)+\frac{3}{2}\,. \tag{3.84}$$

The last term came from the S^1 factor, which has central charge $c=3/2$. Demanding that the total central charge equal 15, we find that not all values of the levels k,k_1,k_2 are allowed, and the consistent string backgrounds satisfy

$$\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}\,. \tag{3.85}$$

Recalling that the level of a WZW model is related to its characteristic scale R via $k=R^2/\ell_s^2$, the above equation requires that the radii R_1,R_2 of the two three-spheres are related to the radius R of AdS_3 via the relation

$$\frac{1}{R^2}=\frac{1}{R_1^2}+\frac{1}{R_2^2}\,. \tag{3.86}$$

As we mentioned in Section 1.3 that this is precisely the condition that $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$ defines a solution to type IIB supergravity with pure NS-NS flux, and also arises naturally as the near-horizon geometry of the F1-NS5 system.

For the case of $\mathcal{M}=\mathrm{S}^3\times\mathbb{T}^4$, the analysis is similar. Let k' be the level of the S^3 factor. Then the full central charge is

$$c(\mathfrak{sl}(2,\mathbb{R})_k^{(1)})+c(\mathcal{M})=3\left(\frac{k+2}{k}+\frac{1}{2}\right)+3\left(\frac{k'-2}{k'}+\frac{1}{2}\right)+6\,. \tag{3.87}$$

Demanding again that the total central charge is $c=15$ results in a restriction of the allowed values of k,k'. Specifically, we find that consistent string backgrounds satisfy

$$k=k'\,. \tag{3.88}$$

Geometrically, this means that the radii of AdS_3 and S^3 in this background are identical. This can be thought of as the $R_2 \to \infty$ limit of $\mathcal{M} = \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$, where the second S^3 factor becomes large its curvature becomes small. In this limit, $\mathrm{S}^3 \times \mathrm{S}^1$ effectively becomes four flat directions, i.e. a manifold which is locally equivalent to $\mathbb{T}^4$. In the F1-NS5 system, this corresponds to taking the angle between the two stacks of NS5 branes (see Figure 1.8) to zero.

To conclude, the RNS description of the background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ is described by

$$\begin{aligned} &\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \oplus \mathfrak{su}(2)^{(1)}_k \oplus \mathbb{T}^4 \\ &\qquad \cong \mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k-2} \oplus \mathbb{T}^4 \oplus (6 \text{ free fermions}) \end{aligned} \tag{3.89}$$

along with the usual b, c, β, γ ghost systems. The spectrum of this theory can now be computed by using known results about the spectra of the bosonic $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ WZW models.

3.4 The hybrid formalism

An alternative approach to the study of superstrings on backgrounds like $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ is the hybrid formalism of Berkovits, Vafa, and Witten [66]. The essential idea is to replace the worldsheet sigma model on $\mathrm{AdS}_3 \times \mathrm{S}^3$ with a sigma model on the superspace $\mathrm{Super}(\mathrm{AdS}_3 \times \mathrm{S}^3)$. This is achieved via a series of field redefinitions on the RNS variables of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ worldsheet theory, and the resulting description has manifest target space supersymmetry, while still admitting a covariant quantization. In this section we will review the construction of the hybrid formalism on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, but before doing so, let us motivate the end result.

String theory on the target space $\mathrm{AdS}_3 \times \mathrm{S}^3$ with pure NS-NS flux is given by a WZW model on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ with some specified level.[8] The resulting worldsheet description has a left- and right-moving sector, both described by $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$. However, in order for the closed string theory to be consistent, states in the left- and right-moving sectors are not completely independent, and must lie in the same representation of the current algebra $\mathfrak{sl}(2,\mathbb{R})_k \times \mathfrak{su}(2)_k$. This picture is corroborated by the geometric fact that

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \cong \frac{(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2))_L \times (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2))_R}{(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2))_{\mathrm{diag}}} \,. \tag{3.90}$$

In order to define a string theory on the superspace of $\mathrm{AdS}_3 \times \mathrm{S}^3$, one could imagine writing $\mathrm{Super}(\mathrm{AdS}_3 \times \mathrm{S}^3)$ as a similar quotient. Indeed, such a quotient exists:

$$\mathrm{Super}(\mathrm{AdS}_3 \times \mathrm{S}^3) \cong \frac{\mathrm{PSU}(1,1|2)_L \times \mathrm{PSU}(1,1|2)_R}{(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2))_{\mathrm{diag}}} \,. \tag{3.91}$$

Here, $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ is the target-space supersymmetry algebra of $\mathrm{AdS}_3 \times \mathrm{S}^3$ backgrounds with equal AdS_3 and S^3 radii. The supergroup $\mathrm{PSU}(1,1|2)$ has bosonic subgroup $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$, and 8 supercharges. As a matrix supergroup, it has schematic form

$$\mathrm{PSU}(1,1|2) \cong \left(\begin{array}{c|c} \mathrm{SL}(2,\mathbb{R}) & \text{supercharges} \\ \hline \text{supercharges} & \mathrm{SU}(2) \end{array}\right) . \tag{3.92}$$

[8]As usual, strictly speaking we are working with the universal cover $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ since global AdS_3 is simply connected.

Given the analogy to the usual bosonic string construction on $\text{AdS}_3\times\text{S}^3$, it seems logical to conjecture that a manifestly target-space supersymmetric description of pure NS-NS strings on $\text{AdS}_3 \times \text{S}^3$ could be achieved by considering a WZW model on the supergroup $\text{PSU}(1,1|2)$. This is effectively the conclusion of the hybrid formalism, however the full description is a bit more subtle, as we will discuss below.

From RNS to hybrid

The hybrid formalism on $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ can be obtained as a field redefinition of the standard RNS formalism. First, let us set up some notation and recall the field content of the RNS theory.

- The AdS_3 component is described by an $\mathcal{N} = 1$ supersymmetric WZW model on $\text{SL}(2,\mathbb{R})$, and is furnished by (decoupled) currents $\mathcal{J}^a$, which satisfy the $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ algebra, and 3 free fermions ψ^a.
- The S^3 component is similarly described by currents $\mathcal{K}^a$ satisfying the $\mathfrak{su}(2)_{k-2}$ algebra, and 3 free fermions χ^a.
- We will be somewhat agnostic about the details of the compactified directions $\mathbb{T}^4$, since it suffices to require that it have $\mathcal{N} = 2$ worldsheet supersymmetry.[9] We denote the generators of the $\mathcal{N} = 2$ superconformal algebra by $(T_c, G_c^{\pm}, J_c)$.
- The ghost content of the theory is given by the conformal b, c and superconformal β, γ ghost systems.

The idea in translating the RNS variables to hybrid variables is to consider a set of complicated field redefinitions which replaces the fermions ψ^a, χ^a as well as some of the ghost degrees of freedom with target-space supersymmetry generators $S^{\alpha\beta\gamma}$.

The first step in these field redefinitions is to consider the bosonization of several worldsheet fields, specifically $b, c, \beta, \gamma, \psi^a, \chi^a, J_c$. For the b, c ghost system, we define a scalar σ such that

$$b = e^{-\sigma}\,, \quad c = e^{\sigma}\,, \tag{3.93}$$

and for the β, γ system we can define scalars φ and χ such that

$$\beta = e^{-\varphi}e^{\chi}\partial\chi\,, \quad e^{-\chi}e^{\varphi}\,. \tag{3.94}$$

For the worldsheet fermions ψ^a and χ^a, a convenient basis for bosonization is given by defining scalars $\sigma_1, \sigma_2, \sigma_3$ such that

$$\frac{1}{k}\psi^+\psi^- = \partial\sigma_1\,, \quad \frac{1}{k}\chi^+\chi^- = \partial\sigma_2\,, \quad \frac{2}{k}\psi^3\chi^3 = \partial\sigma_3\,. \tag{3.95}$$

Given the OPEs between the worldsheet fermions, one can readily calculate the OPEs of the scalars σ_i and find

$$\sigma_i(z)\sigma_j(w) \sim \delta_{ij}\log(z-w)\,. \tag{3.96}$$

From the scalars H_i, it is possible to recover the original RNS fermions via

$$\psi^{\pm} = \sqrt{k}\,e^{\pm\sigma_1}\,, \quad \chi^{\pm} = \sqrt{k}\,e^{\pm\sigma_2}\,, \quad \psi^3 \mp \chi^3 = \sqrt{k}\,e^{\pm\sigma_3}\,. \tag{3.97}$$

[9] This requires the manifold to be hyper-Kähler, and the only hyper-Kähler four-manifolds are $\mathbb{T}^4$ and K3. Thus, the following derivation of the hybrid formalism applies equally well to strings on $\text{AdS}_3 \times \text{S}^3 \times \text{K3}$.

Finally, since we assume that the internal manifold $\mathcal{M}_4$ has at least $\mathcal{N} = (2,2)$ worldsheet supersymmetry and has a central charge $c = 6$ (as is the case for $\mathcal{M}_4 = \mathbb{T}^4, \mathrm{K3}$). If this is the case, the internal $\mathcal{M}_4$ theory contains a U(1) current J_c which generates the R-symmetry of the $\mathcal{N} = 2$ algebra. We also bosonize this current by including a scalar H such that

$$\partial H_c = J_c \,. \tag{3.98}$$

Here, J_c is normalized so that its self-OPE is

$$J_c(z)J_c(w) \sim \frac{c}{3(z-w)^2} = \frac{2}{(z-w)^2} \,, \tag{3.99}$$

so that H_c satisfies

$$H_c(z)H_c(w) \sim 2\log(z-w) \,. \tag{3.100}$$

Now, using these scalars, we can write down physical ($h = 1$) Ramond-sector vertex operators which act as candidates for target-space supersymmetry generators. A natural guess is

$$q^{\alpha\beta\gamma} = \exp\left(\frac{\alpha}{2}\sigma_1 + \frac{\beta}{2}\sigma_2 + \frac{\alpha\beta\gamma}{2}\sigma_3 + \frac{\gamma}{2}H_C - \frac{1}{2}\varphi\right) \,, \tag{3.101}$$

where $\alpha, \beta, \gamma \in \{+,-\}$. The $e^{-\varphi/2}$ is again chosen so that the operator $q^{\alpha\beta\gamma}$ is in the canonical $q = -\frac{1}{2}$ picture of R-sector operators. The conformal weight of this state is given by

$$h(q^{\alpha\beta\gamma}) = \frac{\alpha^2+\beta^2+2\gamma^2+\alpha^2\beta^2\gamma^2}{8} - \frac{1}{8} + \frac{1}{2} = \frac{5}{8} - \frac{1}{8} + \frac{1}{2} = 1 \,. \tag{3.102}$$

Similar to in the flat space case, these generators are designed so that the number of minus signs is even (excluding the $-\frac{1}{2}\varphi$ term), which is why the scalar σ_3 has a funny coefficient of $\alpha\beta\gamma$. Also like in the flat space case, all of these operators are in the $q = -\frac{1}{2}$ picture, and thus there is no chance of their anti-commutators satisfying the supersymmetry algebra.

The trick to move from the RNS formalism to the hybrid formalism in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}_4$ is now to define half of the supersymmetry generators to live in the $q = -\frac{1}{2}$ picture, while the other half is defined through picture changing to be in the $q = \frac{1}{2}$. Let us pick $q^{\alpha\beta+}$ to be in the $q = -\frac{1}{2}$, while we picture-change $q^{\alpha\beta-}$. That is, we define

$$S^{\alpha\beta+} = q^{\alpha\beta+} \,, \quad S^{\alpha\beta-} = Z \cdot q^{\alpha\beta-} \,. \tag{3.103}$$

The naming $S^{\alpha\beta\gamma}$ is conventional.

Now we can search for superspace variables conjugate to the above supersymmetry generators. For the supercharges $S^{\alpha\beta+}$ this is achieved by just reversing the quantum numbers and defining

$$\theta_{\alpha\beta} = \exp\left(-\frac{\alpha}{2}\sigma_1 - \frac{\beta}{2}\sigma_2 - \frac{\alpha\beta}{2}\sigma_3 - \frac{1}{2}H_c + \frac{1}{2}\varphi\right) \tag{3.104}$$

This field has weight $h = 0$ and satisfies

$$\theta_{\alpha\beta-}(z)S^{\gamma\delta+}(w) \sim \frac{\delta_\alpha{}^\beta\delta_\beta{}^\delta}{z-w} \,. \tag{3.105}$$

Thus, the supersymmetry generators $Q^{\alpha\beta+}$ act on the θ coordinates geometrically by translations. Writing the explicit form of $S^{\alpha\beta-}$, we might also look for coordinates $\widetilde{\theta}_{\alpha\beta}$ which transform geometrically under $Q^{\alpha\beta-}$. However, a careful analysis shows that one cannot introduce $\theta_{\alpha\beta+}$ in a way that is independent of $\theta_{\alpha\beta}$, and in particular we cannot choose both to be free fields. This is in stark contrast to the case of four flat spacetime dimensions, where all of the superspace variables could be introduced and chosen independently to form free field theories.

The (unfortunate) conclusion of the above discussion is that the hybrid formalism on $\mathrm{AdS}_3 \times \mathrm{S}^3$ can only make half of the spacetime supersymmetry completely manifest. Letting $p^{\alpha\beta}$ be the conjugate momenta of $\theta_{\alpha\beta}$, we can immediately write down

$$p^{\alpha\beta} = \exp\left(\frac{\alpha}{2}\sigma_1 + \frac{\beta}{2}\sigma_2 + \frac{\alpha\beta}{2}\sigma_3 + \frac{1}{2}H_c - \frac{1}{2}\varphi\right) . \tag{3.106}$$

Together, the fields θ, p form a set of four first-order fermionic systems with OPEs

$$\theta_{\alpha\beta}(z)p^{\gamma\delta}(w) \sim \frac{\delta_\alpha{}^\gamma \delta_\beta{}^\delta}{z-w} . \tag{3.107}$$

These four systems form four manifest super-coordinates in the target space. Bosonizing gives four scalars. Since we started with six scalars (the three bosonized fermion systems σ_i, the φ of the $\beta\gamma$ system, and the R-symmetry H_c of the compactified theory), we simply do not have enough scalars to generate a second set of super-coordiantes which make $Q^{\alpha\beta-}$ manifest.

Having introduced the system p, θ, however, we can write all of the supersymmetry generators purely in terms of these free fields and $\mathcal{J}$,$\mathcal{K}$. First, we define the 'fermionic' currents

$$J^a_{(\mathrm{f})} = \frac{1}{2}(\widetilde{\sigma}^a)_\alpha{}^\beta(p^{\alpha\gamma}\theta_{\beta\gamma}) , \quad K^a_{(\mathrm{f})} = \frac{1}{2}(\sigma^a)_\alpha{}^\beta(p^{\gamma\alpha}\theta_{\gamma\beta}) . \tag{3.108}$$

It can be checked that these satisfy the algebras $\mathfrak{sl}(2,\mathbb{R})_{-2}$ and $\mathfrak{su}(2)_2$, respectively. In fact, these are simply the fermion bilinears we subtracted from J^a, K^a in order to define the 'decoupled' currents $\mathcal{J}^a, \mathcal{K}^a$, just expressed in the hybrid variables p, θ. Thus, we have

$$J^a = \mathcal{J}^a + J^a_{(\mathrm{f})} , \quad K^a = \mathcal{K}^a + K^a_{(\mathrm{f})} . \tag{3.109}$$

Second, we can use θ, p to write down the supersymmetry generators $S^{\alpha\beta\gamma}$. By the definition of $p^{\alpha\beta}$, we have

$$S^{\alpha\beta+} = p^{\alpha\beta} . \tag{3.110}$$

Far less obviously, however, is the following expression for $S^{\alpha\beta-}$, namely

$$S^{\alpha\beta-} = k\partial\theta^{\alpha\beta} + (\widetilde{\sigma}_a)^\alpha{}_\gamma\left(\mathcal{J}^a + \frac{1}{2}J^a_{(\mathrm{f})}\right)\theta^{\gamma\beta} - (\sigma_a)^\beta{}_\gamma\left(\mathcal{K}^a + \frac{1}{2}K^a_{(\mathrm{f})}\right)\theta^{\alpha\gamma} , \tag{3.111}$$

where we have raised the indices on θ using the usual ε symbol. Defining the supersymmetry charges

$$Q^{\alpha\beta\gamma} = \oint \frac{\mathrm{d}z}{2\pi i} S^{\alpha\beta\gamma}(z) , \tag{3.112}$$

a direct computation shows that the supercharges $Q^{\alpha\beta\gamma}$, along with the charges

$$J^a_0 = \oint \frac{\mathrm{d}z}{2\pi i} J^a(z) , \quad K^a_0 = \oint \frac{\mathrm{d}z}{2\pi i} K^a(z) \tag{3.113}$$

satisfy the supersymmetry algebra $\mathfrak{psu}(1,1|2)$.

In fact, we can do better. Taking the fields $J^a, K^a, S^{\alpha\beta\gamma}$ as being currents on the worldsheet, we can determine their current algebra. They satisfy the current algebra

$$\begin{aligned}
J^a(z)J^b(w) &\sim \frac{k\widetilde{\kappa}^{ab}}{(z-w)^2} + \frac{\widetilde{f}^{ab}{}_c J^b(w)}{z-w}\,, \\
K^a(z)K^b(w) &\sim \frac{k\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c K^c(w)}{z-w}\,, \\
J^a(z)S^{\alpha\beta\gamma}(w) &\sim \frac{(\widetilde{\sigma}^a)^\alpha{}_\delta S^{\delta\beta\gamma}(w)}{z-w}\,, \\
K^a(z)S^{\alpha\beta\gamma}(w) &\sim \frac{(\sigma^a)^\beta{}_\delta S^{\alpha\delta\gamma}(w)}{z-w}\,, \\
S^{\alpha\beta+}(z)S^{\gamma\delta-}(w) &\sim \frac{k\,\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}}{(z-w)^2} + \frac{\varepsilon^{\alpha\gamma}(\sigma_a)^{\beta\delta}K^a(w) - \varepsilon^{\beta\delta}(\widetilde{\sigma}_a)^{\alpha\gamma}J^a(w)}{z-w}\,,
\end{aligned} \tag{3.114}$$

where κ, f, σ (resp. $\widetilde{\kappa}, \widetilde{f}, \widetilde{\sigma}$) are the Killing form, structure constants, and spinor generators of $\mathfrak{su}(2)$ (resp. $\mathfrak{sl}(2,\mathbb{R})$). This is the current algebra $\mathfrak{psu}(1,1|2)_k$ which would be obtained from a Wess-Zumino-Witten model on the supergroup $\mathrm{PSU}(1,1|2)$ at level k. A more careful analysis of the worldsheet fields shows that the action of the hybrid variables p, θ along with J, K indeed describes a WZW model on $\mathrm{PSU}(1,1|2)$ [66].

As a quick aside, we mention some of the group theoretic properties of the supergroup $\mathrm{PSU}(1,1|2)$. As was mentioned above, $\mathrm{PSU}(1,1|2)$ is a supergroup with bosonic subgroup $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ and with eight odd dimensions, corresponding to the supercharges of $\mathrm{AdS}_3 \times \mathrm{S}^3$. As such, we can assign its 'superdimension' to be

$$\mathrm{Sdim}(\mathrm{PSU}(1,1|2)) = 6 - 8 = -2\,, \tag{3.115}$$

where the superdimension counts the number of even dimensions minus the number of odd dimensions of the supergroup. Moreover, the superalgebra $\mathfrak{psu}(1,1|2)$ has the surprising property that its dual Coxeter number $h^\vee$ vanishes.[10] Thus, a naive calculation for the central charge of the $\mathfrak{psu}(1,1|2)_k$ WZW model is

$$c(\mathfrak{psu}(1,1|2)_k) = \frac{k\,\mathrm{Sdim}(\mathfrak{psu}(1,1|2))}{k+h^\vee} = -2\,. \tag{3.116}$$

This agrees with the counting of central charges from the bosonic $\mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k-2}$ WZW models and the four first-order (p,θ) systems with $c = -2$ each:

$$c(\mathfrak{sl}(2,\mathbb{R})_{k+2}) + c(\mathfrak{su}(2)_{k-2}) - 4\cdot 2 = \frac{3(k+2)}{k} + \frac{3(k-2)}{k} - 8 = -2\,. \tag{3.117}$$

In order to quantize the worldsheet theory, we need something of central charge $+2$ to cancel the Weyl anomaly of the $\mathfrak{psu}(1,1|2)_k$ theory. As we will see, this will be the total central charge of the left-over ghost system in the hybrid formalism.

[10] The supergroups with vanishing dual Coxeter number have been classified, and they are $\mathfrak{psl}(n|n)$, $\mathfrak{osp}(2n+2|2n)$, and $\mathfrak{d}(2,1;\alpha)$.

Decoupling the internal CFT

Since used the R-symmetry current J_c of $\mathcal{M}_4$ to define the hybrid variables in $\mathrm{AdS}_3 \times \mathrm{S}^3$, there is no way that the hybrid variables can decouple from the RNS variables on $\mathcal{M}_4$. However, we can perform a similarity transformation

$$\Phi_c^{\mathrm{new}} = e^{\mathcal{W}} \Phi_c e^{-\mathcal{W}} \,, \tag{3.118}$$

where

$$\mathcal{W} = \oint \frac{\mathrm{d}z}{2\pi i} (H_c \mathcal{P})(z) \,, \tag{3.119}$$

where $\mathcal{P} = \partial\chi - \partial\varphi$ is the picture current. The effect on the R-symmetry current is

$$J_c^{\mathrm{new}} = J_c - 2\mathcal{P} \,, \tag{3.120}$$

which decouples from the hybrid fields. Just like in the flat space case, the $c = 9$ $\mathcal{N} = (2,0)$ generators are modified to

$$G_c^{\pm,\mathrm{new}} = e^{\pm(\varphi-\chi)} G_c^{\pm} \,, \quad T_c^{\mathrm{new}} = T_c - \mathcal{P} J_c + \frac{3}{2}\mathcal{P}^2 \,, \tag{3.121}$$

which, together with J_c^{new}, still satisfy an (untwisted) $c = 6$ $\mathcal{N} = (2,0)$ superconformal algebra.

The $\boldsymbol{\sigma}, \boldsymbol{\rho}$ ghosts

In the RNS formalism, we started with the six scalars

$$\varphi, \chi, \sigma_1, \sigma_2, \sigma_3, H_c \tag{3.122}$$

from which we defined four first order systems (p, θ) and a new R-symmetry current J_c^{new}. Thus, a scalar is missing. The p, θ system and the new R-symmetry are bosonized by the five scalars

$$\frac{\alpha}{2}\sigma_1 + \frac{\beta}{2}\sigma_2 + \frac{\alpha\beta}{2}\sigma_3 + \frac{H_C}{2} - \frac{\varphi}{2} \,, \quad H_c - 2\chi + 2\varphi \,. \tag{3.123}$$

The unique linear combination which is orthogonal to all five of these currents and has OPE $\rho(z)\rho(w) = -\log(z-w)$ is given by

$$\rho = 2\varphi - H_c - \chi \,. \tag{3.124}$$

This scalar has background charge $Q_\rho = 3$ and central charge

$$c(\rho) = (1 + 3Q_\rho^2) = 28 \,. \tag{3.125}$$

Furthermore, we never used the b, c system when constructing the hybrid fields, and so we still have the σ ghost with $c = -26$. Together, ρ and σ define the full ghost content of the hybrid formalism on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}_4$, and we refer to them collectively as the (ρ, σ) system, which has central charge

$$c(\rho, \sigma) = 28 - 26 = 2 \,. \tag{3.126}$$

The topological twist

In the process of decoupling the internal CFT, we wound up introducing supercharges $G_C^{\pm,\text{new}}$ and R-symmetry J_C^{new} which, together with T_c^{new} still satisfy the $c = 6$ $\mathcal{N} = (2,0)$ superconformal algebra. However, in the full theory, these redefined generators have the following two properties:

- $G_c^{\pm,\text{new}}$ have conformal weights $h = \frac{3}{2} \pm \frac{1}{2}$ with respect to the stress tensor T of the full 10-dimensional theory.
- J_c^{new} has a background charge of $Q = 2$.

This indicates that the stress tensor of the compactified theory has been 'twisted' relative to the original stress tensor T_c of the RNS formalism. Indeed, writing out the full stress tensor of the RNS theory in hybrid variables gives

$$\begin{aligned} &T_{\text{RNS}}(\mathcal{J},\psi,\mathcal{K},\chi,b,c,\beta,\gamma,\Phi_c) \\ &\quad = \underbrace{T_{\text{hybrid}}(J,K,S)}_{\mathfrak{psu}(1,1|2)_k} + T_{\text{ghosts}}(\rho,\sigma) + \underbrace{T_c^{\text{new}}(\Phi_c^{\text{new}}) + \frac{1}{2}\partial J_c^{\text{new}}}_{\text{topological twisted } \mathcal{M}_4} \, . \end{aligned} \tag{3.127}$$

Thus, in order for the hybrid theory to be equivalent to the original RNS description, we need to include the topological twisting term to the compact theory $\mathcal{M}_4$, so that the stress tensors of the two theories match. Therefore, the full hybrid formalism is given by the quantum equivalence:

$$\begin{gathered} \text{hybrid strings on } \text{AdS}_3 \times \text{S}^3 \times \mathcal{M}_4 \\ \Longleftrightarrow \\ \mathfrak{psu}(1,1|2)_k \oplus [(\rho,\sigma) \text{ ghosts}] \oplus [\text{topologically twisted } \mathcal{M}_4] \, . \end{gathered} \tag{3.128}$$

As a sanity check, we can calculate the full central charge of the theory. Since topologically twisted theories have vanishing central charge, we have

$$c_{\text{tot}} = c(\mathfrak{psu}(1,1|2)_k) + c(\sigma,\rho) = -2 + 2 = 0 \, , \tag{3.129}$$

and so the hybrid string on $\text{AdS}_3 \times \text{S}^3 \times \mathcal{M}_4$ defines a consistent string theory.

Chapter 4

The tensionless string

We are now ready to introduce the central player in this thesis: the tensionless string. As discussed in the last chapter, string theory on supersymmetric AdS_3 backgrounds like $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ can be described in the RNS formalism by considering quantum $\mathcal{N}=1$ WZW models with levels chosen as to appropriately cancel the worldsheet anomalies. Let us focus on the case of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, since it is the case we will deal with for the rest of this thesis. The $\mathcal{N}=1$ $\mathbb{T}^4$ sigma model has central charge $c=6$. Anomaly cancellation requires that the levels of the AdS_3 and S^3 components are equal. Thus, the full theory is described by the model

$$\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \oplus \mathfrak{su}(2)^{(1)}_k \oplus \mathbb{T}^4 \,. \tag{4.1}$$

Now, we know that in the RNS formalism, we can consistently decouple the fermions from the affine currents by shifting the level of the models by the dual coxeter number $h^\vee$ of the corresponding group. Since $h^\vee(\mathfrak{sl}(2,\mathbb{R})) = -2$ and $h^\vee(\mathfrak{su}(2)) = 2$, we have

$$\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \oplus \mathfrak{su}(2)^{(1)}_k \cong \mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k-2} \oplus (6 \text{ free fermions}) \,. \tag{4.2}$$

This description of the $\mathrm{AdS}_3 \times \mathrm{S}^3$ worldsheet model provides a perfectly consistent description for most values of the level k. In particular, values of large k (corresponding to the supergravity limit) lead to a completely well-defined worldsheet theory, if an analytically complicated one. However, there is a glaring problem when taking k to its minimal value $k=1$: the bosonic $\mathfrak{su}(2)$ algebra obtains a negative level. As such, its central charge becomes

$$c(\mathfrak{su}(2)_{-1}) = -3 \,. \tag{4.3}$$

Conformal field theories with negative central charges are problematic, as they do not possess unitary representations and their quantization is subtle. Naively, it would seem that $k=1$ should then somehow lie outside of the allowed space of theories. From a phenomenological perspective, however, the level $k=1$ should be perfectly allowed, since it corresponds to the near-horizon geometry of an NS5-F1 system with one NS5 brane.

One way around the apparent problem at $k=1$ is to treat the $\mathfrak{su}(2)_{-1}$ model as a ghost system which, along with the usual b,c and β,γ ghost systems, simply conspire to remove degrees of freedom from the remaining factors of the theory in a consistent manner, see for instance [72].

Another way around the problems at $k=1$ is to abandon the RNS approach altogether in favor of another approach which is well-behaved in this limit. Recall that the hybrid formalism of string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, introduced in the previous Chapter, trades in the worldsheet supersymmetry present in the RNS formalism for (a subset of) the spacetime supersymmetry of the $\mathrm{AdS}_3 \times \mathrm{S}^3$ supergravity background. This is achieved by replacing the bosonic group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ with a corresponding supergroup $\mathrm{PSU}(1,1|2)$. The price that one ways for rewriting the worldsheet theory in a manifestly spacetime supersymmetric way is that one has to 'topologically twist' the internal $\mathbb{T}^4$ in order to end up with a worldsheet theory which is quantum-equivalent to the original RNS description. The resulting worldsheet theory is given schematically by

$$\mathfrak{psu}(1,1|2)_k \oplus (\text{topologically twisted } \mathbb{T}^4) \oplus \text{ghosts}\,. \tag{4.4}$$

The hybrid formalism is useful for describing the limit $k=1$ since the super-affine algebra $\mathfrak{psu}(1,1|2)_k$ is perfectly defined at $k=1$. More than that, the spectrum of representations of $\mathfrak{psu}(1,1|2)_k$ is actually simpler at $k=1$ than for any higher integer value of k. The first reason for this simplification is that the unitary (and thus physical) representations of the bosonic subalgebra $\mathfrak{su}(2)_1 \subset \mathfrak{psu}(1,1|2)_1$ truncate, and so the full representations of $\mathfrak{psu}(1,1|2)_1$ (which must be decomposible into representations of $\mathfrak{su}(2)_1$) have a much simpler structure, as we will discuss further in Section 4.1 below. The second, and equivalent, reason is that the worldsheet $\mathrm{PSU}(1,1|2)$ WZW model with $k=1$ units of flux is actually completely equivalent to a certain *free* theory on the worldsheet, which we will introduce and discuss in Section 4.3.

4.1 The spectrum at $k = 1$

In the hybrid formalism at level k, the worldsheet spectrum is organized into representations of the affine algebra $\mathfrak{psu}(1,1|2)_k$. Let us first consider highest-weight representations, which stem from representations of the global algebra $\mathfrak{psu}(1,1|2)$. Representations of $\mathfrak{psu}(1,1|2)$ can in turn be decomposed in terms of representations of the bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$. Using the notation of Chapter 3, we denote the continuous and discrete representations of $\mathfrak{sl}(2,\mathbb{R})$ as $\mathcal{C}^j_\lambda$ and $\mathcal{D}^j_\pm$, respectively. Irreducible representations of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$ can then be written as

$$\big(\mathcal{C}^j_\lambda, \underline{n}\big) \quad \text{or} \quad \big(\mathcal{D}^j_\pm, \underline{n}\big)\,, \tag{4.5}$$

where $\underline{n}$ is the n-dimensional representation of $\mathfrak{su}(2)$. In addition to the bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$, the superalgebra $\mathfrak{psu}(1,1|2)$ also has eight fermionic generators $S^{\alpha\beta\gamma}$. Using the commutation relations of $\mathfrak{psu}(1,1|2)$, it is not difficult to show that if the state $|\psi\rangle$ transforms in the $\big(\mathcal{C}^j_\lambda, \underline{n}\big)$ of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$, then $S^{\alpha\beta\gamma}\,|\psi\rangle$ transforms in the $\big(\mathcal{C}^{j+\frac{\alpha}{2}}_{\lambda+\frac{\alpha}{2}}, \underline{n+\beta}\big)$. That is to say

$$S^{\alpha\beta\gamma} : \big(\mathcal{C}^j_\lambda, \underline{n}\big) \to \big(\mathcal{C}^{j+\frac{\alpha}{2}}_{\lambda+\frac{\alpha}{2}}, \underline{n+\beta}\big)\,. \tag{4.6}$$

Now, as is standard when working with Clifford algebras, we can choose half of the generators, say $S^{\alpha\beta+}$ to act as 'raising' operators, and half of the generators, say $S^{\alpha\beta-}$ to act as 'lowering' operators. This is a consistent choice, since the $S^{\alpha\beta+}$ all anticommute among themselves. We can thus decompose an irreducible representation of $\mathfrak{psu}(1,1|2)$ by considering a 'Clifford module' of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$ representations. A typical Clifford module then takes the form

$$\begin{array}{ccccc} & & (\mathcal{C}^j_\lambda, \underline{n}) & & \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{n+1}) & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{n-1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{n+1}) & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{n-1}) \\ (\mathcal{C}^{j+1}_\lambda, \underline{n}) \quad (\mathcal{C}^{j+1}_\lambda, \underline{n+2}) & & 2\cdot(\mathcal{C}^j_\lambda, \underline{n}) & & (\mathcal{C}^j_\lambda, \underline{n-2}) \quad (\mathcal{C}^{j-1}_\lambda, \underline{n}) \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{n+1}) & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{n-1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{n+1}) & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{n-1}) \\ & & (\mathcal{C}^j_\lambda, \underline{n})\,, & & \end{array} \tag{4.7}$$

where we use the operators $S^{\alpha\beta+}$ to move down a horizontal line and the operators $S^{\alpha\beta-}$ to move back up. Note that we have used the fact that, since $\lambda \in \mathbb{R}/\mathbb{Z}$, the representations labelled by $\lambda + \frac{1}{2}$ and $\lambda - \frac{1}{2}$ are the same. Similar expressions exist for discrete representations $\mathcal{D}^j_\pm$, but we will not need them.

Once we have the representations of the global $\mathfrak{psu}(1,1|2)$ algebra, we can immediately construct highest-weight representations of the algebra by acting with oscillator modes. The resulting highest-weight representations have precisely the structure as the Clifford modules in (4.7), and so we will not write them again.

For large values of k, the number of possible representations allowed in the $\mathfrak{psu}(1,1|2)_k$ model is rather large. However, the condition that the $\mathfrak{su}(2)_k$ representations are unitary imposes a restriction on the allowed values of $\underline{n}$ appearing in the Clifford module (4.7). Specifically, for $k = 1$, the only allowed representations of $\mathfrak{su}(2)_1$ are the $\underline{1}$ and $\underline{2}$, and we should throw away any representations with larger values of n. This leads to a so-called *shortening condition* of the Clifford modules considered in (4.7). Starting with $n = 2$, the Clifford module will now terminate at the second line, and we have

$$\begin{array}{cc} \multicolumn{2}{c}{(\mathcal{C}^j_\lambda, \underline{2})} \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{1}) & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \underline{1}) \end{array} \tag{4.8}$$

This turns out to be the only allowed structure of a Clifford module at $k = 1$. In fact, more is true, and it turns out that only representations with $j = \frac{1}{2}$ are allowed (i.e. unitary) in the $\mathfrak{psu}(1,1|2)_k$ model [45]. Thus, the allowed highest-weight representations of the $\mathfrak{psu}(1,1|2)$ model take the form

$$\begin{array}{cc} \multicolumn{2}{c}{(\mathcal{C}^{\frac{1}{2}}_\lambda, \underline{2})} \\ (\mathcal{C}^{1}_{\lambda+\frac{1}{2}}, \underline{1}) & (\mathcal{C}^{0}_{\lambda+\frac{1}{2}}, \underline{1}) \end{array} \tag{4.9}$$

for $\lambda \in [0,1]$, and we label these representations $\mathscr{F}_\lambda$. Moreover, the discrete representations $\mathcal{D}^j_\pm$ do not survive the shortening at $k = 1$, and so only continuous representations exist.

Just as in the case of bosonic strings on AdS_3, the full worldsheet spectrum is much larger than just the highest-weight representations. There are furthermore *spectrally-flowed* representations. Just as the spectral flow operator σ^w acts on the affine modes J^a_n of the $\mathfrak{sl}(2,\mathbb{R})_k$ algebra, we can define a spectral flow operator which acts on the full $\mathfrak{psu}(1,1|2)_k$ algebra. The action can be written explicitly as

$$\begin{gathered}\sigma^w(J^3_n) = J^3_n + \frac{kw}{2}\delta_{n,0}\,, \quad \sigma^w(J^\pm_n) = J^\pm_{n\mp w}\,,\\ \sigma^w(K^3_n) = K^3_n + \frac{kw}{2}\delta_{n,0}\,, \quad \sigma_w(K^\pm_n) = K^\pm_{n\pm w}\,,\\ \sigma^w(S^{\alpha\beta\gamma}_n) = S^{\alpha\beta\gamma}_{n+\frac{1}{2}w(\beta-\alpha)}\,.\end{gathered} \tag{4.10}$$

The operation σ^w can be easily shown to be an automorphism of $\mathfrak{psu}(1,1|2)_k$, and thus can be used to build new representations from old ones. Following the logic of Section 3.2, the full spectrum is thus the sum of all highest-weight representations $\mathscr{F}_\lambda$ and their spectrally-flowed images $\sigma^w(\mathscr{F}_\lambda)$. Put algebraically, the Hilbert space of the theory is given by[1]

$$\mathscr{H} = \int_0^1 \mathrm{d}\lambda \bigoplus_{w\in\mathbb{Z}} \sigma^w(\mathscr{F}_\lambda) \oplus \sigma^w(\overline{\mathscr{F}}_\lambda)\,. \tag{4.11}$$

The partition function

Now that we have decomposed the Hilbert space of the $\mathfrak{psu}(1,1|2)_1$ WZW model, we can compute its partition function. The essential ingredient will be the characters

$$\mathrm{ch}[\sigma^w(\mathscr{F}_\lambda)](\tau;t,z) = \mathrm{Tr}_{\sigma^w(\mathscr{F}_\lambda)}\left[q^{L_0}x^{J_0}y^{K_0}\right]\,, \tag{4.12}$$

where $q = e^{2\pi i\tau}$, $x = e^{2\pi it}$, and $y = e^{2\pi iz}$. In terms of these characters, the partition function is then easily computed as

$$\begin{aligned}Z_{\mathfrak{psu}(1,1|2)_k}(\tau;t,z) &= \mathrm{Tr}_{\mathscr{H}}\left[q^{L_0}x^{J_0}y^{K_0}\right]\\ &= \int_0^1 \mathrm{d}\lambda \sum_{w\in\mathbb{Z}} \left|\mathrm{ch}[\sigma^w(\mathscr{F}_\lambda)](\tau;t,z)\right|^2\,.\end{aligned} \tag{4.13}$$

The chemical potentials t, z are introduced both in order to refine the counting of states. Note that J^3_0 is holographically dual to L_0 in the boundary CFT, and so the chemical potential t should play the role of the boundary L_0 chemical potential. Furthermore, the zero mode K^3_0 plays the role of the Cartan generator of the $\mathfrak{su}(2)$ R-symmetry algebra in the boundary CFT.

The evaluation of these characters can either be done via an explicit sum over states in the $\mathfrak{psu}(1,1|2)_1$ model or, more simply, by using the free field realization we will introduce in the next section. The calculation is done explicitly in [45], and we will simply quote the result:

$$\mathrm{ch}[\sigma^w(\mathscr{F}_\lambda)](\tau;t,z) = q^{\frac{w^2}{2}} \sum_{r\in\mathbb{Z}+\lambda} x^r q^{-wr} \frac{\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)}{\eta(\tau)^4}\,. \tag{4.14}$$

Interestingly, the sum over r can be explicitly evaluated via Poisson resummation as a Dirac comb. Specifically,

$$\sum_{r\in\mathbb{Z}+\lambda} x^r q^{-wr} = \sum_{m\in\mathbb{Z}} e^{-2\pi i\lambda m}\delta(t - w\tau + m)\,, \tag{4.15}$$

[1] Strictly speaking, the representation $\mathscr{F}_{\frac{1}{2}}$ is problematic and needs to be individually dealt with. This subtlety will not worry us, but see [45] for a full discussion.

and so the character is

$$\mathrm{ch}[\sigma^w(\mathscr{F}_\lambda)](\tau;t,z) = q^{\frac{w^2}{2}} \sum_{m\in\mathbb{Z}} e^{-2\pi i\lambda m}\delta(t-w\tau+m)\frac{\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)}{\eta(\tau)^4}\,. \tag{4.16}$$

In order to compute the full partition function of $\mathfrak{psu}(1,1|2)_1$, the form (4.11) of the Hilbert space instructs us to first square the $\sigma^w(\mathscr{F}_\lambda)$ character, and then integrate over λ and sum over w. Squaring the character gives

$$|q|^{w^2} \sum_{m,m'\in\mathbb{Z}} e^{-2\pi i\lambda(m-m')}\delta(t-w\tau+m)\delta(\bar{t}-w\bar{\tau}+m')\left|\frac{\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)}{\eta(\tau)^4}\right|^2\,. \tag{4.17}$$

Integrating over λ simply imposes the constraint $m = m'$, and so after summing over w, we find

$$Z_{\mathfrak{psu}(1,1|2)_1}(\tau;t,z) = \sum_{w,m\in\mathbb{Z}} |q|^{w^2}\left|\frac{\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)}{\eta(\tau)^4}\right|^2 \delta^{(2)}(t-w\tau+m)\,. \tag{4.18}$$

The existence of a delta function in the $\mathfrak{psu}(1,1|2)_1$ partition function is a strange one – it suggests that the worldsheet modulus, and thus the complex structure of the worldsheet, cannot be arbitrary, but rather *localizes* to specific points in the moduli space. These points in the moduli space in turn have a geometric interpretation. They correspond to configurations where the worldsheet wraps the angular cycle of AdS_3 w times, and wraps the thermal cycle precisely once. Put another way, these worldsheet configurations are realized by maps $\Sigma \to \partial(\mathrm{AdS}_3)$ such that the worldsheet torus *covers* the boundary torus. This localization property is a hallmark of the tensionless string, and will appear again and again in this thesis.

The full string spectrum

Now that we have written down the partition function of the $\mathfrak{psu}(1,1|2)_1$ model, we can write down the corresponding partition function for the full $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathbb{T}^4$ string theory. Schematically, we have

$$Z_{\text{string}}(\tau;t,z) = Z_{\mathfrak{psu}(1,1|2)_1}(\tau;t,z)Z_{\mathbb{T}^4}(\tau)Z_{\text{ghosts}}(\tau)\,. \tag{4.19}$$

The partition function of the ρ,σ ghost system can be worked out via analogy with the RNS formalism. We simply quote the result (see [45])[2]

$$Z_{\text{ghosts}} = \left|\frac{\eta(\tau)^4}{\vartheta_2(0;\tau)\vartheta_2(0;\tau)}\right|^2\,. \tag{4.20}$$

Furthermore, the $\mathbb{T}^4$ contribution can be calculated easily as well. The topological twist required to move to the hybrid formalism effectively results in computing the $\mathbb{T}^4$ partition function in the R-sector of the fermions, and the result is

$$Z^{\mathbb{T}^4}\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} = \left|\frac{\vartheta_2(0;\tau)\vartheta_2(0;\tau)}{\eta(\tau)^6}\right|^2\Theta_{\mathbb{T}^4}(\tau)\,, \tag{4.21}$$

[2]The denominator of this result technically vanishes, due to the Ramond-sector fermionic zero mode. The vanishing factor will later cancel against another vanishing factor, leading to a well-defined result. A more careful treatment can be done by letting the ghosts and the fermions of the $\mathbb{T}^4$ be charged under the J^3_0 and K^3_0 currents of the $\mathfrak{psu}(1,1|2)_1$ factor of the theory, see [45].

where $\Theta_{\mathbb{T}^4}$ is the Narain theta function which counts the winding and momentum modes of the torus, i.e.

$$\Theta_{\mathbb{T}^4}(\tau) = \sum_{(p,\bar{p})\in\Gamma_{4,4}} q^{p^2/2}\bar{q}^{\bar{p}^2/2} . \tag{4.22}$$

Putting everything together, we find the full string partition function to be

$$Z_{\text{string}}(\tau; t, z) = \sum_{w,m\in\mathbb{Z}} q^{\frac{w^2}{2}} \left| \frac{\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)}{\eta(\tau)^6} \right|^2 \Theta_{\mathbb{T}^4}(\tau)\, \delta^{(2)}(t - w\tau + m) . \tag{4.23}$$

Using the delta-function, we can remove all of the t-dependence of the partition function, and we find

$$Z_{\text{string}}(\tau; t, z) = \sum_{w,m\in\mathbb{Z}} |x|^w \left| \frac{\vartheta_2(\frac{z+w\tau-m}{2};\tau)\vartheta_2(\frac{w\tau-m-z}{2};\tau)}{\eta(\tau)^6} \right|^2 \Theta_{\mathbb{T}^4}(\tau)\; \delta^{(2)}(t-w\tau+m) . \tag{4.24}$$

The theta function terms can be simplified as follows. Let us write

$$\vartheta_{(0,0)} = \vartheta_3 , \quad \vartheta_{(0,1/2)} = \vartheta_2 , \quad \vartheta_{(1/2,0)} = \vartheta_4 , \quad \vartheta_{(1/2,1/2)} = \vartheta_1 . \tag{4.25}$$

Then the theta functions satisfy

$$\vartheta_{(\alpha,\beta)}\left(\pm\frac{z}{2} + \frac{w\tau}{2} - \frac{m}{2};\tau\right) = q^{-\frac{w^2}{8}} y^{\pm\frac{w}{2}} e^{-i\pi w\alpha} \vartheta_{(\alpha+b/2,\beta+w/2)}\left(\pm\frac{z}{2};\tau\right) , \tag{4.26}$$

and so specifically

$$\vartheta_2\left(\pm\frac{z}{2} + \frac{w\tau}{2} - \frac{m}{2};\tau\right) = q^{-\frac{w^2}{8}} y^{\pm\frac{w}{2}} \vartheta_{(m/2,1/2+w/2)}\left(\pm\frac{z}{2};\tau\right) , \tag{4.27}$$

Plugging this into the string partition function, we thus have

$$\begin{aligned} &Z_{\text{string}}(\tau; t, z) \\ &= \sum_{w,m\in\mathbb{Z}} |q|^{\frac{w^2}{2}} \left| \frac{\vartheta_{(m/2,1/2+w/2)}(-\frac{z}{2};\tau)\vartheta_{(m/2,1/2+w/2)}(\frac{z}{2};\tau)}{\eta(\tau)^6} \right|^2 \Theta_{\mathbb{T}^4}(\tau)\, \delta^{(2)}(t - w\tau + m) , \end{aligned} \tag{4.28}$$

where we have used the delta function to set $|x| = |q|^w$. This can be written in a much simpler fashion by noting that the summand is simply the $\mathbb{T}^4$ partition function with spin structure $(m/2 + 1/2, w/2)$ and chemical potential z for the R-symmetry, so that

$$Z_{\text{string}}(\tau; t, z) = \sum_{m,w\in\mathbb{Z}} |q|^{\frac{w^2}{2}} Z^{\mathbb{T}^4} \begin{bmatrix} \frac{m}{2} + \frac{1}{2} \\ \frac{w}{2} \end{bmatrix} (z;\tau)\, \delta^{(2)}(t - w\tau + m) , \tag{4.29}$$

where we have defined

$$Z^{\mathbb{T}^4} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z;\tau) := \left| \frac{\vartheta_{(\alpha+1/2,\beta+1/2)}(-\frac{z}{2};\tau)\vartheta_{(\alpha+1/2,\beta+1/2)}(\frac{z}{2};\tau)}{\eta(\tau)^6} \right|^2 \Theta_{\mathbb{T}^4}(\tau) \tag{4.30}$$

The partition function in equation (4.29) bares a striking resemblance to the single-particle partition function in the symmetric orbifold (2.81). In [45], it was shown that this resemblance can be made concrete: by imposing physical state conditions on the worldsheet, one effectively removes the delta function in the sum and one is left with

$$\sum_{w=1}^{\infty}\sum_{m=0}^{w-1} |x|^{\frac{w}{2}} Z^{\mathbb{T}^4}\begin{bmatrix}\frac{m}{2}+\frac{1}{2}\\ \frac{w}{2}\end{bmatrix}\left(z;\frac{t+m}{w}\right), \tag{4.31}$$

which is indeed the single-particle spectrum of the symmetric orbifold in the NS-sector. The restriction to $m \in 0,\dots,w-1$ is due to the symmetry of the $\mathbb{T}^4$ partition function under $\tau \to \tau+1$, so that the contributions in the above sum are not over-counted. The conclusion of this analysis is:

> *The one-loop string partition function of tensionless IIB string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ exactly reproduces the single-particle spectrum of the symmetric orbifold $Sym(\mathbb{T}^4)$.*

This result shows that the *full* single-string spectrum reproduces the *full* single-particle spectrum of the proposed dual CFT.

While we will not show the full argument of [45] which recovers the single-particle spectrum, we will take a look below at another, slightly more complicated analysis: the partition function on *thermal* AdS_3. We will find, there, that the complete symmetric orbifold partition function is recovered from the worldsheet theory, and not just the single-particle sector. This effectively serves as a proof of the correspondence between the symmetric orbifold theory and the tensionless string, at the level of the spectra.

4.2 The thermal AdS partition function

In [73], it was shown how to use the string partition function derived in the previous section to compute the dual CFT torus partition function, i.e. the full spectrum of the symmetric orbifold. We briefly review this analysis here.

By the general rules of the AdS/CFT correspondence, if we want to compute a torus partition function in a CFT, this is done by considering gravitational observables in a bulk spacetime whose boundary has the topology of a torus. For the moment, let us work in Euclidean signature. The statement 'locally AdS_3' can be understood as requiring that the bulk manifold $\mathcal{M}$ admits a metric g which has constant negative curvature, i.e. $\mathcal{M}$ is a hyperbolic 3-manifold. Such manifolds are always topologically quotients $\mathbb{H}^3/\Gamma$ of global hyperbolic 3-space $\mathbb{H}^3$ (i.e. global Euclidean AdS_3) by some Kleinian group Γ (i.e. a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$).

Specifically, let us take coordinates $(r,x,\overline{x})$ on $\mathbb{H}^3$, where $x,\overline{x}$ are complex coordinates on the boundary and r is a radial coordinate so that the boundary lies at $r\to 0$. The constant-curvature metric is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2+\mathrm{d}x\,\mathrm{d}\overline{x}}{r^2}\,. \tag{4.32}$$

The action of Γ on $\mathbb{H}^3$ is best understood by defining a quaternionic coordiante $y = x + ju$. Then if $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$, the action of γ on a point $y \in \mathbb{H}^3$ is

$$\gamma\cdot y = (ay+b)(cy+d)^{-1}\,. \tag{4.33}$$

One can check that the k component of $\gamma \cdot y$ vanishes, so that $\gamma \cdot y$ indeed defines an element of $\mathbb{H}^3$.

Now, the action of Γ on the asymptotic boundary $\partial\mathbb{H}^2 \cong \mathbb{CP}^1$ reduces to the standard action of Möbius transformations on the Riemann sphere. However, given a Kleinian group Γ, it is not guaranteed that Γ acts proper discontinuously on the asymptotic boundary of $\mathbb{H}^3$. This does not stop us from defining $\mathbb{H}^3/\Gamma$, since $\mathbb{H}^3$ is defined without its boundary, but if we want to know what the asymptotic boundary of $\mathbb{H}^3/\Gamma$ is, we need to study the action of Γ on $\mathbb{CP}^1$ a bit more carefully. Given a Kleinian group Γ, let $\Omega \subset \mathbb{CP}^1$ be the maximal subset on which Γ acts 'badly'. Then define $U = \mathbb{CP}^1 \setminus \Omega$. Then Γ acts freely on U, and we have

$$\partial(\mathbb{H}^3/\Gamma) \cong U/\Gamma\,, \tag{4.34}$$

where, as always, we mean the asymptotic boundary. See [74] for more details.

In order to find hyperbolic 3-manifold $\mathcal{M}$ with torus boundary, then, it is sufficient to find Kleinian groups Γ such that U/Γ is a torus. This is equivalent to saying that $\pi_1(U/\Gamma) \cong \mathbb{Z} \oplus \mathbb{Z}$. It turns out, see [74], that there are three possibilities for Γ:

- **Thermal AdS$_3$:** $\Gamma \cong \mathbb{Z}$ is generated by the matrix

$$\begin{pmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix}\,, \tag{4.35}$$

  with q some complex number, which we can assume without loss of generality satisfies $|x| < 1$. The action of Γ on $\mathbb{CP}^1$ has fixed points at 0 and ∞, and so $U = \mathbb{CP}^1 \setminus \{0, \infty\}$. The resulting geometry $\mathbb{H}^3/\Gamma$ is a solid torus whose boundary has modular parameter t with $x = e^{2\pi i t}$.

- **The conical defect geometry:** $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}_w$ for some integer w. This is generated by the two matrices

$$\begin{pmatrix} x^{1/2w} & 0 \\ 0 & x^{-1/2w} \end{pmatrix}\,, \quad \begin{pmatrix} e^{\pi i/w} & 0 \\ 0 & e^{-\pi i/w} \end{pmatrix}\,. \tag{4.36}$$

  For this geometry, we still have $U = \mathbb{CP}^1 \setminus \{0, \infty\}$, and the boundary is still a torus of modular parameter t, but the action of Γ has a fixed point at $r \to \infty$, and the resulting geometry $\mathbb{H}^3/\Gamma$ has a conical defect in the center.

- **The cusp geometry:** $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$ and is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\,. \tag{4.37}$$

  In this case, $U = \mathbb{CP}^1 \setminus \{\infty\} \cong \mathbb{C}$, and U/Γ is just the representation of a torus with modular parameter t as the quotient of $\mathbb{C}$ by a lattice. The resulting bulk geometry, however, has a very bad singularity at $r \to \infty$, which should be thought of as the $w \to \infty$ limit of the conical defect geometry.

Of these three options, thermal AdS$_3$ will be the one we are interested in. We will not concern ourselves with the conical defect geometry nor with the cusp geometry, but they are both of physical interest and were considered in [73].

Let us pick a basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ given by

$$J_0^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,, \quad J_0^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \,, \quad J_0^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \,. \tag{4.38}$$

Note that these generators are different than those of Chapter 3, but we will use them nonetheless since they will be convenient for this discussion. By the form of the $\mathfrak{sl}(2,\mathbb{C})$ generators, we can write the Kleinian group Γ acting on $\mathbb{H}^3$ as

$$\Gamma = \left\langle x^{J_0^3} \right\rangle \,. \tag{4.39}$$

Now, we are interested in calculating the worldsheet partition function of the tensionless string on $\mathbb{H}^3/\Gamma$ with $\Gamma = \langle q^{J_0^3} \rangle$. For the sake of simplicity, we don't keep track of the SU(2) R-symmetry chemical potential z, but it is not difficult to incorporate, and was done in [73]. Of course, if we know the worldsheet model on $\mathbb{H}^3$, then the worldsheet model of $\mathbb{H}^3/\Gamma$ is simply given by considering an orbifold of the $\mathbb{H}^3$ worldsheet theory. As in any orbifold, we want to sum over all possible twisted boundary conditions on the worldsheet. Since the orbifold group for thermal AdS_3 is $\Gamma \cong \mathbb{Z}$, each twisted boundary condition on the worldsheet torus is given by a pair of integers k, ℓ. For each choice k, ℓ we can calculate the partition function of the worldsheet theory with fields satisfying

$$\Phi(z+\tau) = x^{kJ_0^3}\Phi(z) \,, \quad \Phi(z+1) = x^{\ell J_0^3}\Phi(x) \,. \tag{4.40}$$

Let us denote the partition function associated with the integers k, ℓ by $Z^{(k,\ell)}$. Then the string spectrum on thermal AdS_3 should be given by

$$\frac{1}{|\Gamma|} \sum_{k,\ell\in\mathbb{Z}} Z^{(k,\ell)}(\tau; t) \,. \tag{4.41}$$

Here, the order $|\Gamma|$ of the gauge group is technically infinite, but it was argued in [73] that it is natural to make the replacement

$$\frac{1}{|\Gamma|} \to \text{Im}(t) \,. \tag{4.42}$$

Intuitively, this replacement is motivated by the fact that the boundary of thermal AdS_3 is the boundary of global AdS_3, modulo the group Γ. If Γ were a finite group, we would have the relationship

$$\text{Vol}(\partial\text{TAdS}_3) = \text{Vol}(\partial\text{AdS}_3)/|\Gamma| \,. \tag{4.43}$$

Since Γ is an infinite group, this relationship is strictly not true, but is regularized by the fact that the boundary of AdS_3 has infinite volume. The volume of the boundary of thermal AdS_3 is just the area of a torus with modular parameter t, which is $\text{Im}(t)$. Thus, it is somewhat natural to consider the replacement of $1/|\Gamma|$ with $\text{Vol}(\partial\text{TAdS}_3)$ by taking the volume $\text{Vol}(\partial\text{AdS}_3)$ as an infinite irrelevant constant. A first principles derivation of this replacement, however, is not known, and we will simply take it as an assumption.

Now, fortunately, we do not have to calculate the above partition functions from scratch. The partition function with $\ell = 1$ and $k = 0$ is calculated by taking the fields to be periodic along the B-cycle of the torus and to pick up a monodromy of the form $x^{J_0^3}$ along the A-cycle. In canonical quantization, this is automatically performed by the trace

$$\mathrm{Tr}_{\mathscr{H}}\left[q^{L_0}\overline{q}^{\overline{L}_0}x^{J_0^3}\overline{x}^{\overline{J}_0^3}\right]\,. \tag{4.44}$$

This, however, is precisely the string partition function Z_{string} calculated in the previous section. For the purposes of this section, however, we will find it useful to consider instead the specialized trace

$$\mathrm{Tr}_{\mathscr{H}}\left[(-1)^F q^{L_0}\overline{q}^{\overline{L}_0}x^{J_0^3}\overline{x}^{\overline{J}_0^3}\right]\,. \tag{4.45}$$

The inclusion of the fermion number operator $(-1)^F$ forces the fermions to be periodic along the A-cycle, and so we are essentially calculating the string partition function in the $\widetilde{\mathrm{R}}$ sector on the worldsheet. We choose to calculate the worldsheet partiiton function in the $\widetilde{\mathrm{R}}$ sector, since it is invariant under modular transformations.[3] The result is extremely similar to the result without the $(-1)^F$, except that the spin structure of the $\mathbb{T}^4$ partition function is changed. The result is

$$\begin{aligned} Z^{(1,0)}(\tau;t) &:= \mathrm{Tr}_{\mathscr{H}}\left[(-1)^F q^{L_0}\overline{q}^{\overline{L}_0}x^{J_0^3}\overline{x}^{\overline{J}_0^3}\right] \\ &= \sum_{m,w\in\mathbb{Z}} |x|^{\frac{kw}{2}} Z^{\mathbb{T}^4}\begin{bmatrix}\frac{m}{2}\\ \frac{w}{2}\end{bmatrix}(\tau)\delta^{(2)}(t-w\tau+m) \end{aligned} \tag{4.46}$$

Furthermore, we can calculate the partition function $Z^{(k,0)}$ simply by replacing $t\to kt$. Thus,

$$\begin{aligned} Z^{(k,0)}(\tau;t) &= Z_{\text{string}}(\tau;kt) \\ &= \sum_{m,w\in\mathbb{Z}} |x|^{\frac{kw}{2}} Z^{\mathbb{T}^4}\begin{bmatrix}\frac{m}{2}\\ \frac{w}{2}\end{bmatrix}(\tau)\delta^{(2)}(kt-w\tau+m)\,. \end{aligned} \tag{4.47}$$

Now, we can deduce the form of the rest of the partition functions $Z^{(k,\ell)}$ by utilizing modular transformations. Since modular transformations mix the A- and B-cycles, a generic modular transformation will not leave the integers (k,ℓ) fixed, and indeed they transform under $\mathrm{SL}(2,\mathbb{Z})$ as a doublet. Specifically,

$$\begin{pmatrix}k\\ \ell\end{pmatrix}\to\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}k\\ \ell\end{pmatrix}\,. \tag{4.48}$$

The partition function $Z^{(k,\ell)}$ then transforms like

$$Z^{(ak+b\ell,ck+d\ell)}\left(\frac{a\tau+b}{c\tau+d};t\right) = |c\tau+d|^2 Z^{(k,\ell)}(\tau;t)\,, \tag{4.49}$$

where the $|c\tau+d|^2$ factor comes from the fact that Z_{string} for the global AdS_3 model transforms with weight $(1,1)$ under modular transformations. Inverting this relation, we have

$$Z^{(k,\ell)}\left(\frac{a\tau+b}{c\tau+d};t\right) = |c\tau+d|^2 Z^{(dk-b\ell,a\ell-ck)}(\tau;t)\,. \tag{4.50}$$

[3] It is possible to work directly in the R sector, so long as one makes sure to keep track of the spin structure under modular transformations.

Thus, given only $(k=1,\ell=0)$, we can perform various modular transformations to obtain the partition function for $(k=d,\ell=-c)$. Since the a,c entries of an $\mathrm{SL}(2,\mathbb{Z})$ matrix satisfy $(c,d)=1$, we can thus find the partition function for any two numbers which are coprime by doing modular transformations on $Z^{(1,0)}$. Similarly, if we take $k\neq 1$, we can perform modular transformations to obtain the answer for $Z^{(kd,-kc)}$, with c,d coprime. In this way, we can obtain every twisted partition function, just by knowledge of the string partition function on global AdS_3.

Let us compute the partition function $Z^{(dk,-ck)}$. We have

$$\begin{aligned} Z^{(dk,-ck)}(\tau;t) &= |c\tau+d|^{-2} Z^{(k,0)}\left(\frac{a\tau+b}{c\tau+d};t\right) \\ &= |c\tau+d|^{-2}\sum_{w,m\in\mathbb{Z}} |x|^{\frac{kw}{2}} Z^{\mathbb{T}^4}\begin{bmatrix}\frac{m}{2}\\ \frac{w}{2}\end{bmatrix}\left(\frac{a\tau+b}{c\tau+d}\right) \\ &\qquad\qquad \times \delta^{(2)}\left(kt-w\frac{a\tau+b}{c\tau+d}+m\right). \end{aligned} \tag{4.51}$$

Now, the $\mathbb{T}^4$ partition function is invariant under a simultaneous transformation of the complex structure and spin structure, namely

$$Z^{\mathbb{T}^4}\begin{bmatrix}a\mu+b\nu\\ c\mu+d\nu\end{bmatrix}\left(\frac{a\tau+b}{c\tau+d}\right) = Z^{\mathbb{T}^4}\begin{bmatrix}\mu\\ \nu\end{bmatrix}(\tau)\,, \tag{4.52}$$

and so we have

$$Z^{(dk,-ck)}(\tau;t) = |c\tau+d|^{-2}\sum_{w,m\in\mathbb{Z}} |x|^{\frac{kw}{2}} Z^{\mathbb{T}^4}\begin{bmatrix}\frac{dm-bw}{2}\\ \frac{aw-cm}{2}\end{bmatrix}(\tau)\delta^{(2)}\left(kt-w\frac{a\tau+b}{c\tau+d}+m\right). \tag{4.53}$$

We can simplify this further by noting the delta function identity

$$|c\tau+d|^{-2}\delta^{(2)}\left(kt-w\frac{a\tau+b}{c\tau+d}+m\right) = \delta^{(2)}\left((c\tau+d)kt-w(a\tau+b)+m(c\tau+d)\right). \tag{4.54}$$

This can be seen from noting that $|c\tau+d|^2$ is the Jacobian of the integration measure d^2z on $\mathbb{C}$ under the transformation $z\to(c\tau+d)z$. Thus, we have

$$\begin{aligned} Z^{(dk,-ck)}(\tau;t) = &\sum_{w,m\in\mathbb{Z}} |x|^{\frac{kw}{2}} Z^{\mathbb{T}^4}\begin{bmatrix}\frac{dm-bw}{2}\\ \frac{aw-cm}{2}\end{bmatrix}(\tau) \\ &\times \delta^{(2)}((c\tau+d)kt-w(a\tau+b)+m(c\tau+d))\,. \end{aligned} \tag{4.55}$$

The thermal AdS_3 partition function should then be computed by summing over all k and pairs of coprime integers (c,d). This seems rather messy, but it can be simplified by trading in the sums over w,m,k,c,d for a single sum over four integers $\alpha,\beta,\gamma,\delta$. The mapping between the two sums is given by

$$\alpha = dk\,,\quad \beta = md-wb\,,\quad \gamma=-ck\,,\quad \delta = wa-mc\,, \tag{4.56}$$

or, more succinctly,

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}k & m\\ 0 & w\end{pmatrix}. \tag{4.57}$$

This relationship is invertible – namely, every integer matrix with elements $\alpha, \beta, \gamma, \delta$ can be written as a product of an $\mathrm{SL}(2,\mathbb{Z})$ matrix with an upper-triangular integer matrix in the above fashion. Thus, we can safely swap the sum over w, m, k, c, d with a sum over $\alpha, \beta, \gamma, \delta$. The result is

$$\begin{aligned}\sum_{k,\ell\in\mathbb{Z}} Z^{(k,\ell)}(\tau;t) &= \sum_{\alpha,\beta,\gamma,\delta\in\mathbb{Z}} |x|^{\frac{\alpha\delta-\beta\gamma}{2}} Z^{\mathbb{T}^4}\begin{bmatrix}\frac{\beta}{2}\\ \frac{\delta}{2}\end{bmatrix}(\tau)\delta^{(2)}((\alpha-\gamma\tau)t-\delta\tau+\beta)\\ &= \sum_{\alpha,\beta,\gamma,\delta} |x|^{\frac{\alpha\delta-\beta\gamma}{2}} Z^{\mathbb{T}^4}\begin{bmatrix}\frac{\beta}{2}\\ \frac{\delta}{2}\end{bmatrix}\left(\frac{\alpha t+\beta}{\gamma t+\delta}\right)|\gamma t+\delta|^{-2}\delta^{(2)}\left(\tau-\frac{\alpha t+\beta}{\gamma t+\delta}\right).\end{aligned} \tag{4.58}$$

Now, in order to compute the full thermal AdS_3 string amplitude, we are instructed multiply by $\mathrm{Im}\tau$ to integrate over the fundamental domain. The integral in the fundamental domain has to have a chosen measure to reflect the modular properties of the integrand, namely that it has modular weight $(1,1)$. Thus, we want to calculate

$$\mathrm{Im}(t)\int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\mathrm{Im}(\tau)}\sum_{k,\ell\in\mathbb{Z}} Z^{(k,\ell)}(\tau;t)\,. \tag{4.59}$$

This integral is easily performed by use of the delta support, but we have to be careful to only integrate over things that appear in the fundamental domain, and not just in the upper half-plane. For each point of support $(\alpha t+\beta)/(\gamma t+\delta)$, there is a unique modular transformation $\rho\in\mathrm{SL}(2,\mathbb{Z})$ which brings it into the fundamental domain. Thus, instead of summing over all integer matrices, we can sum over all integer matrices modulo the (left) action of $\mathrm{SL}(2,\mathbb{Z})$. That is, instead of summing over the set $\mathrm{Mat}(\mathbb{Z})$ of all integer matrices we sum over the quotient space

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}(\mathbb{Z}) := \left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{Mat}(\mathbb{Z})\middle|\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\sim\rho\cdot\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\forall\rho\in\mathrm{SL}(2,\mathbb{Z})\right\}. \tag{4.60}$$

The result of the integral is now completely straightforward. We assume we can find a representative in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}(\mathbb{Z})$, and we sum over all of them. The end result is

$$\sum_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}(\mathbb{Z})} |x|^{\frac{\alpha\delta-\beta\gamma}{2}} Z^{\mathbb{T}^4}\begin{bmatrix}\frac{\beta}{2}\\ \frac{\delta}{2}\end{bmatrix}\left(\frac{\alpha t+\beta}{\gamma t+\delta}\right)\cdot|\gamma t+\delta|^{-2}\frac{\mathrm{Im}(t)}{\mathrm{Im}(\tau)}\bigg|_{\tau=(\alpha t+\beta)/(\gamma t+\delta)}. \tag{4.61}$$

We note that, due to the modular properties of the $\mathbb{T}^4$ partition function, the above expression is well-defined, and doesn't depend on the representative of the equivalence classes chosen in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}(\mathbb{Z})$. Specifically, we can always choose a representative of the form

$$\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\,,\quad \delta>0\,,\quad \beta=0,\ldots,\delta-1\,, \tag{4.62}$$

Since both $\mathrm{Im}(t)$ and $\mathrm{Im}(\tau)$ are positive, we cannot have $a/\delta<0$, and so we only include matrices with $\alpha>0$. For these representatives, we have

$$\delta^{-2}\frac{\mathrm{Im}(t)}{\mathrm{Im}(\tau)}\bigg|_{\tau=(\alpha t+\beta)/\delta}=\frac{1}{\alpha\delta}\,, \tag{4.63}$$

and so the one-loop string partition function on thermal AdS_3 gives

$$\sum_{\alpha,d=1}^{\infty}\sum_{\beta=0}^{\delta-1}\frac{1}{\alpha\delta}|x|^{\frac{\alpha\delta}{2}}Z^{\mathbb{T}^4}\begin{bmatrix}\frac{\beta}{2}\\ \frac{\delta}{2}\end{bmatrix}\left(\frac{\alpha t+\beta}{\delta}\right). \tag{4.64}$$

We can trade the sum over α,δ for a sum over $k=\alpha\delta$ and $\delta|k$, and we have

$$\sum_{k=1}^{\infty}\sum_{\alpha\delta=k}\sum_{\beta=0}^{\delta-1}\frac{1}{k}|x|^{\frac{k}{2}}Z^{\mathbb{T}^4}\begin{bmatrix}\frac{\beta}{2}\\ \frac{\delta}{2}\end{bmatrix}\left(\frac{\alpha t+\beta}{\delta}\right)=\sum_{k=1}^{\infty}|x|^{\frac{k}{2}}\mathfrak{T}_k Z^{\mathbb{T}^4}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(t). \tag{4.65}$$

This is simply the connected part of the symmetric orbifold partition function in the $\widetilde{\mathrm{NS}}$-sector (see Chapter 2), if one makes the identification

$$p=|x|^{1/2}=e^{-\pi\mathrm{Im}(t)}. \tag{4.66}$$

Since this calculation was only performed with respect to a single string worldsheet, we are instructed to exponentiate the result. This gives

$$\begin{aligned}\mathfrak{Z}_{\text{Thermal AdS}_3}(t)&=\exp\left(\sum_{\alpha,d=1}^{\infty}\sum_{\beta=0}^{\delta-1}\frac{1}{\alpha\delta}p^{\alpha\delta}Z^{\mathbb{T}^4}\begin{bmatrix}\frac{\beta}{2}\\ \frac{\delta}{2}\end{bmatrix}\left(\frac{\alpha t+\beta}{\gamma t+\delta}\right)\right)\\&=\exp\left(\sum_{k=1}^{\infty}p^k\mathfrak{T}_kZ^{\mathbb{T}^4}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(t)\right),\end{aligned} \tag{4.67}$$

which is the grand canonical partition function of the symmetric orbifold in the $\widetilde{\mathrm{NS}}$-sector.

In [73], not only was the string partition function computed on thermal AdS_3, but it was also computed on the Euclidean BTZ black hole (any image of thermal AdS_3 under the action of $\mathrm{SL}(2,\mathbb{Z})$ on its boundary), as well as the 'conical defect' geometry. All of these computations produce the same result: the one-loop partition function of the symmetric orbifold. The conclusion is that the tensionless string demonstrates a kind of "background independence", and the result of any calculation depends only on the precise boundary conditions taken, and not on the details of the bulk. This idea was taken further in [75], where the background independence of the tensionless string was essentially proven, and it was argued that the tensionless string formulated on one background $\mathcal{M}$ with boundary Σ automatically 'knows' about all other backgrounds $\mathcal{M}'$ with the same boundary.

4.3 The free field realization

The simplicity of the string spectrum of the $\mathrm{PSU}(1,1|2)$ WZW model at level $k=1$ is fundamentally due to a special feature enjoyed my certain affine algebras at low levels [76]: the algebra $\mathfrak{psu}(1,1|2)_1$ is actually realizable as a free field theory [45].

Let us first explain what we mean in the context of a simple example [52]. Let us consider a theory of N free chiral fermions with action

$$S=\frac{1}{2\pi}\int\mathrm{d}^2z\,\delta_{ij}\psi^i\overline{\partial}\psi^j. \tag{4.68}$$

This action defines a chiral CFT with central charge $c = N/2$. Furthermore, it admits a global SO(N) symmetry

$$\psi^i \to M^i{}_j \psi^j \, . \tag{4.69}$$

Letting T^a be the generators of the Lie algebra $\mathfrak{so}(N)$ in the fundamental representation, the conserved currents associated to the global SO(N) symmetry of the theory are given by

$$J^a = \frac{1}{2} T^a_{ij} (\psi^i \psi^j) \, . \tag{4.70}$$

Taking the OPEs of these currents, they satisfy an affine Kac-Moody algebra, specifically

$$J^a(z) J^b(w) \sim \frac{\delta^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{z-w} \, , \tag{4.71}$$

which is the algebra $\mathfrak{so}(N)_1$.

Not only does the free fermion theory contain a current algebra $\mathfrak{so}(N)_1$ in its current algebra: the free fermion theory is actually quantum equivalent to (the chiral part of) a SO(N) WZW model at level $k = 1$. This can be shown by calculating the stress tensor of the two theories and showing that they are indeed equivalent at the quantum level. Another check for this correspondence is that the central charge of the SO(N) WZW model at level k is

$$c(\mathfrak{so}(N)_k) = \frac{k \dim(\mathfrak{so}(N))}{k + h^\vee(\mathfrak{so}(N))} = \frac{k \, N(N-1)/2}{k + N - 2} \, , \tag{4.72}$$

where we have used the fact that $h^\vee(\mathfrak{so}(N)) = N - 2$. At $k = 1$, we have $c = N/2$, which is precisely the central charge of N free fermions.

The free field construction of $\mathfrak{psu}(1,1|2)$

We can write down a similar, yet more complicated, version of the free field construction of $\mathfrak{so}(N)_1$ for the algebra $\mathfrak{psu}(1,1|2)_1$. Consider four free bosons of conformal weight $h = \frac{1}{2}$ and OPEs

$$\lambda^\dagger(z) \mu(w) \sim \frac{1}{z-w} \, , \quad \lambda^\dagger(z) \mu(w) \sim \frac{1}{z-w} \, . \tag{4.73}$$

Here, we emphasize that † is not the Hermitian or complex conjugate, but is simply a notational choice, inspired by the fact that, say, $\lambda^\dagger$ and μ are canonical conjugates in the Lagrangian formalism. In the terminology of [76], we call each pair $(\mu^\dagger, \lambda)$ and $(\lambda^\dagger, \mu)$ a pair of *symplectic bosons*, although they are really just $\beta\gamma$ systems with $h = \frac{1}{2}$. In terms of these fields, we can construct currents

$$J^+ = (\lambda^\dagger \lambda) \, , \quad J^- = (\mu^\dagger \mu) \, , \quad J^3 = \frac{1}{2} \left(\lambda^\dagger \mu - \mu^\dagger \lambda \right) \, . \tag{4.74}$$

It is readily checked that these currents satisfy the algebra $\mathfrak{sl}(2,\mathbb{R})_1$, namely

$$\begin{gathered} J^3(z) J^\pm(w) \sim \pm \frac{J^\pm(w)}{z-w} \, , \quad J^3(z) J^3(w) \sim -\frac{1}{2} \frac{1}{(z-w)^2} \, , \\ J^+(z) J^-(w) \sim \frac{1}{(z-w)^2} - \frac{2 J^3(w)}{z-w} \, . \end{gathered} \tag{4.75}$$

Thus, one might be led to believe that the two pairs of symplectic bosons are equivalent to the level 1 WZW model on $\mathfrak{sl}(2,\mathbb{R})$. However, each pair of symplectic bosons has central charge $c=-1$, and so the full symplectic boson theory has $c=-2$. The $\mathfrak{sl}(2,\mathbb{R})_k$ theory has central charge

$$c\left(\mathfrak{sl}(2,\mathbb{R})_k\right)=\frac{3k}{k-2}\,, \tag{4.76}$$

which gives $c=-3$ for $k=1$. The mismatch is due to the fact that the symplectic bosons generate a much larger algebra than just $\mathfrak{sl}(2,\mathbb{R})_1$. In fact, the full algebra of currents they can generate is $\mathfrak{sp}(4)_1$ (thus the name *symplectic* bosons), which is the affinization of the linear symmetries of the OPEs (4.73).

In order to reduce the current algebra to $\mathfrak{sl}(2,\mathbb{R})_1$, we need to eliminate all other currents. One way to do this is to consider the current

$$U=\frac{1}{2}\left(\lambda^\dagger\mu+\mu^\dagger\lambda\right)\,. \tag{4.77}$$

Under this current, every field with a dagger has charge $-1/2$ and every field without a dagger has charge $1/2$. Thus, all currents constructed from bilinears of the free fields which do not have an equal number of daggered and undaggered fields is charged under U. The only currents not charged under U are the J^a and U itself (although the OPE of U with itself does have a central term). Thus, we can eliminate all currents except for J^a by 'gauging' U. The process of gauging U reduces the central charge of the symplectic boson theory by one unit, and so we end up with a theory of central charge $c=-3$, exactly the central charge of the $\mathfrak{sl}(2,\mathbb{R})_1$ theory.

Now, in order to construct $\mathfrak{psu}(1,1|2)$, we also have to find a construction of the other bosonic subalgebra $\mathfrak{su}(2)_1$. This can be done by introducing four free fermions $\psi^\dagger_a,\psi^a$ with $a=1,2$ and with OPEs

$$\psi^\dagger_a(z)\psi^b(w)\sim\frac{\delta_a{}^b}{z-w}\,. \tag{4.78}$$

Using these fields, we can construct the currents

$$\begin{gathered}K^+=(\psi^\dagger_1\psi^2)\,,\quad K^-=-(\psi^\dagger_2\psi^1)\,,\\ K^3=\frac{1}{2}\left(\psi^\dagger_1\psi^1-\psi^\dagger_2\psi^2\right)\,,\quad V=\frac{1}{2}\left(\psi^\dagger_1\psi^1+\psi^\dagger_2\psi^2\right)\,.\end{gathered} \tag{4.79}$$

which together satisfy the current algebra $\mathfrak{u}(1)_1$. The currents K^a generate the subalgebra $\mathfrak{su}(2)_1\subset\mathfrak{u}(1)_1$, while V generates the trace component of $\mathfrak{u}(2)$. Thus, just as we 'gauged out' the current U in the symplectic boson theory to get a $\mathfrak{sl}(2,\mathbb{R})_1$ algebra, we can perform a similar procedure by removing all degrees of freedom charged under V. This eliminates all other possible bilinears in the fermions, and we are left with only the currents K^a. The free fermions have a total central charge of $c=4/2=2$, and after gauging V this reduces to $c=1$. In comparison, we have

$$c\left(\mathfrak{su}(2)_k\right)=\frac{3k}{k+2}\,, \tag{4.80}$$

which gives $c=1$ when $k=1$.

Thus, we have a realization of the current algebras $\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{su}(2)_1$. However, we want a representation of the 'full' current algebra $\mathfrak{psu}(1,1|2)_1$, which we wrote in equation (3.114). To do this, let us consider the linear combination

$$Z = U + V\,. \tag{4.81}$$

If we gauge this current, the only surviving bilinears in the symplectic bosons are the currents J^a, and the only surviving bilinears in the free fermions are the K^a. In addition to those, we also have the following eight mixed bilinears

$$\begin{aligned} S^{+++} &= \psi_1^\dagger \lambda\,, \quad S^{+-+} = \psi_1^\dagger \lambda\,, \quad S^{-++} = -\psi_1^\dagger \mu\,, \quad S^{--+} = -\psi_2^\dagger \mu\,, \\ S^{++-} &= -\lambda^\dagger \psi^2\,, \quad S^{+--} = \lambda^\dagger \psi^1\,, \quad S^{-+-} = -\mu^\dagger \psi^2\,, \quad S^{---} = \mu^\dagger \psi^1\,. \end{aligned} \tag{4.82}$$

which are also uncharged under Z. In addition to Z, there is another linear combination

$$Y = U - V \tag{4.83}$$

which can be constructed.

The currents J^a, K^a, $S^{\alpha\beta\gamma}$, Y and Z form the algebra $\mathfrak{u}(1,1|2)_1$:

$$\begin{aligned} J^a(z)J^b(w) &\sim \frac{\widetilde{\kappa}^{ab}}{(z-w)^2} + \frac{\widetilde{f}^{ab}{}_c J^b(w)}{z-w}\,, \\ K^a(z)K^b(w) &\sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c K^c(w)}{z-w}\,, \\ J^a(z)S^{\alpha\beta\gamma}(w) &\sim \frac{(\widetilde{\sigma}^a)^\alpha{}_\delta S^{\delta\beta\gamma}(w)}{z-w}\,, \\ K^a(z)S^{\alpha\beta\gamma}(w) &\sim \frac{(\sigma^a)^\beta{}_\delta S^{\alpha\delta\gamma}(w)}{z-w}\,, \\ S^{\alpha\beta+}(z)S^{\gamma\delta-}(w) &\sim \frac{\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}}{(z-w)^2} + \frac{\varepsilon^{\alpha\gamma}(\sigma_a)^{\beta\delta}K^a(w) - \varepsilon^{\beta\delta}(\widetilde{\sigma}_a)^{\alpha\gamma}J^a(w)}{z-w} \\ &\qquad + \frac{\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}Z(w)}{z-w}\,, \\ Y(z)S^{\alpha\beta\gamma}(w) &\sim \frac{\gamma S^{\alpha\beta\gamma}(w)}{z-w}\,. \\ Z(z)Y(w) &\sim -\frac{1}{(z-w)^2}\,. \end{aligned} \tag{4.84}$$

This is nearly identical to the $\mathfrak{psu}(1,1|2)_1$ algebra we saw in equation (3.114), except that the $\{S, S\}$ anti-commutator has the current Z appearing as a central term, and the currents $S^{\alpha\beta\gamma}$ are charged under the current Y. This algebra is known as $\mathfrak{u}(1,1|2)_1$. In order to reduce to $\mathfrak{psu}(1,1|2)_1$, we need to 'gauge' the two currents Y and Z. The mechanism proposed by [45] and later refined in [47] is to perform this gauging in three steps:

1) Consider the coset of the $\mathfrak{u}(1,1|2)_1$ algebra with respect to the current Z. That is, we consider only the subspace of the Hilbert space of the $\mathfrak{u}(1,1|2)_1$ algebra which satisfies

$$Z_n\,|\psi\rangle = 0\,, \quad n \geq 0\,. \tag{4.85}$$

2) Once we have removed states which are not annihilated by non-negative modes of Z_n, if we consider a physical state $|\psi\rangle$, states of the form $Z_{-n}|\psi\rangle$ are orthogonal to all other physical states, and furthermore have zero norm. Thus, these states are null, and we can consider the quotient of the Hilbert space under the equivalence

$$|\varphi\rangle \sim |\varphi\rangle + Z_{-n}|\psi\rangle \ . \tag{4.86}$$

3) Finally, since the supercharges $S^{\alpha\beta\gamma}$ are charged under the current Y, we have to sum over all possible Y-charges in order to obtain a Hilbert space which is closed under the action of the $\mathfrak{psu}(1,1|2)_1$ currents.

The highest-weight representations

Now that we have a free field theory whose algebra of spin-1 currents is $\mathfrak{psu}(1,1|2)_1$, we need to check if it is actually equivalent to the PSU$(1,1|2)$ WZW model. One way to do this is to show that the representations of $\mathfrak{psu}(1,1|2)_1$ are reproduced by the free fields. Let us start with the highest-weight representations.

Because all of the free fields are of half-integer spin, highest-weight representations of the free field theory are split into two kinds: the NS sector and the R sector. Similarly, since all of the fields live on the same surface with the same spin structure, each field has to be in the same sector simultaneously. The highest-weight representations of the NS-sector is completely trivial, and just consists of a single state $|\Omega\rangle$, annihilated by all positive modes of all of the free fields, along with all of its descendants. In terms of the $\mathfrak{psu}(1,1|2)_1$ currents, $|\Omega\rangle$ is annihilated by all non-negative modes, and thus the highest-weight representation in the NS sector just gives us the vacuum representation of $\mathfrak{psu}(1,1|2)_1$.

The interesting representations occur in the Ramond sector. Here, highest-weight states live in a nontrivial representation of the zero mode algebra of the free fields. For the bosons, we have

$$[\lambda^\dagger_0, \mu_0] = 1\,, \quad [\mu^\dagger_0, \lambda] = 1\,, \tag{4.87}$$

and so the zero-mode algebra is just two copies of the harmonic oscillator algebra. For the fermions, we have

$$\{(\psi^\dagger_a)_0, \psi^a_0\} = \delta^a{}_b\,, \tag{4.88}$$

which is a simple Clifford algebra. Thus, we can determine representations by defining quantum numbers m_1, m_2 such that m_1 is lowered (resp. raised) by μ_0 (resp. $\lambda^\dagger_0$) and m_2 is raised (resp. lowered) by λ_0 (resp. $\mu^\dagger_0$). It is convenient to define the linear combinations

$$m := m_1 + m_2\,, \quad j := m_1 - m_2\,. \tag{4.89}$$

With these quantum numbers, we can write down a basis for the R-sector highest-weight states of the symplectic boson theory to be

$$\begin{aligned} \lambda_0\,|m,j\rangle &= |m+\tfrac{1}{2}, j-\tfrac{1}{2}\rangle\ , \quad \mu^\dagger_0\,|m,j\rangle = -(m-j)\,|m-\tfrac{1}{2}, j+\tfrac{1}{2}\rangle\ , \\ \mu_0\,|m,j\rangle &= |m-\tfrac{1}{2}, j-\tfrac{1}{2}\rangle\ , \quad \lambda^\dagger_0\,|m,j\rangle = (m+j)\,|m+\tfrac{1}{2}, j+\tfrac{1}{2}\rangle\ . \end{aligned} \tag{4.90}$$

It is straightforward to check, then, that m, j correspond precisely to the quantum numbers of highest-weight representations of $\mathfrak{sl}(2,\mathbb{R})_1$. Indeed, we have

$$J^\pm_0\,|m,j\rangle = (m\pm j)\,|m\pm 1, j\rangle\ , \quad J^3_0\,|m,j\rangle = m\,|m,j\rangle\ . \tag{4.91}$$

Now, we need to specify how these states transform in the zero-mode representation of the fermions. A simple choice would be to demand that they are annihilated by ψ^1_0 and $(\psi^\dagger_1)_0$. Then we can use ψ^2_0 $(\psi^\dagger_2)_0$ to generated new states. For a given value of m, j, we have the states

$$|m,j\rangle \ , \quad (\psi^\dagger_2)_0\psi^2_0\,|m,j\rangle \ , \quad \psi^2_0\,|m,j\rangle \ , \quad (\psi^\dagger_2)_0\,|m,j\rangle \ . \tag{4.92}$$

The first two states together form a doublet under the $\mathfrak{su}(2)_1$ generators, while the last two states form two singlets.

Finally, we have to impose the condition that Z annihilates all physical states. We can read off the eigenvalues of Z_0 rather easily by noting that the eigenvalue of U_0 is $j-1/2$ and the daggered (resp. undaggered) fermions carry V charge $-1/2$ (resp. $+1/2$). Thus, we have

$$\begin{aligned} Z_0\,|m,j\rangle &= \left(j-\frac{1}{2}\right)|m,j\rangle \ , \\ Z_0\left((\psi^\dagger_2)_0\psi^2_0\,|m,j\rangle\right) &= \left(j-\frac{1}{2}\right)(\psi^\dagger_2)_0\psi^2_0\,|m,j\rangle \ , \\ Z_0\left(\psi^2_0\,|m,j\rangle\right) &= j\,\psi^2_0\,|m,j\rangle\,|m,j\rangle \ , \\ Z_0\left((\psi^\dagger_2)_0\,|m,j\rangle\right) &= (j-1)(\psi^\dagger_2)_0\,|m,j\rangle \ . \end{aligned} \tag{4.93}$$

The conclusion is that the Z_0 condition requires the doublet states under $\mathfrak{su}(2)_1$ to have spin $j=1/2$, while the singlets have spin $j=0$ and $j=1$. Thus, we conclude that the highest-weight R-sector representation of the free field realization decomposes into representations of $\mathfrak{sl}(2,\mathbb{R})_1\oplus\mathfrak{su}(2)_1$ as

$$\begin{array}{cc} & (\mathcal{C}^{\frac{1}{2}}_{\lambda},\underline{2}) \\ (\mathcal{C}^{1}_{\lambda+\frac{1}{2}},\underline{1}) & (\mathcal{C}^{0}_{\lambda+\frac{1}{2}},\underline{1})\ , \end{array} \tag{4.94}$$

where here λ is the fractional part of m. This is *precisely* the form of the short representation $\mathscr{F}_\lambda$ of $\mathfrak{psu}(1,1|2)$ that we have already encountered, when we argued that it is the only allowed highest-weight representation of $\mathfrak{psu}(1,1|2)_1$ (other than the trivial representation). Indeed, one can check that the action of the supercharges in the free field realization preicsely reproduce the Clifford module structure of the short representation of $\mathfrak{psu}(1,1|2)$, and the selection rule $j=1/2$ is now clear from the point of view of the free fields [45].

Spectrally-flowed representations

In addition to highest-weight representations, the worldsheet spectrum of $\mathfrak{psu}(1,1|2)_1$ contains spectrally-flowed representations, corresponding to strings asymptotically winding the asymptotic AdS_3 boundary. These can be described in the free field theory in the following way.

Starting with the symplectic boson theory, we note that, since the $(\mu^\dagger,\lambda)$ and $(\lambda^\dagger,\mu)$ pairs are completely independent before gauging Z, we can define the two automorphisms $\sigma^{(1,0)}$ and $\sigma^{(0,1)}$ which act as

$$\begin{aligned} \sigma^{(1,0)}(\lambda^\dagger_r) &= \lambda^\dagger_{r-\frac{1}{2}}\ , \quad \sigma^{(1,0)}(\mu_r) = \mu_{r+\frac{1}{2}}\ , \\ \sigma^{(0,1)}(\mu^\dagger_r) &= \mu^\dagger_{r+\frac{1}{2}}\ , \quad \sigma^{(0,1)}(\lambda_r) = \lambda_{r-\frac{1}{2}}\ . \end{aligned} \tag{4.95}$$

The combinations $\sigma^{(p,q)} := (\sigma^{(1,0)})^p \circ (\sigma^{(0,1)})^q$ define an automorphism of the symplectic boson aglebra for all $p, q \in \mathbb{Z}$. Specifically interesting are the combinations

$$\sigma^w := \sigma^{(w,w)} , \quad \widehat{\sigma}^w := \sigma^{(w,-w)} . \tag{4.96}$$

To get a feel for what these automorphisms do physically, let us evaluate them on the currents J^a, U. We perform the calculation explicitly in Appendix 4.A. The result is

$$\sigma^w(J^{\pm}_n) = J^{\pm}_{n\mp w} , \quad \sigma^w(J^3_n) = J^3_n + \frac{w}{2}\delta_{n,0} , \quad \sigma^w(U_n) = U_n , \tag{4.97}$$

and

$$\widehat{\sigma}^w(J^a_n) = J^a_n , \quad \widehat{\sigma}^w(U_n) = U_n + \frac{w}{2}\delta_{n,0} . \tag{4.98}$$

That is, σ^w acts as the standard $\mathfrak{sl}(2,\mathbb{R})_1$ spectral flow on the currents J^3 and acts trivially on U, while the 'orthogonal' operator $\widehat{\sigma}^w$ acts trivially on J^a, but raises the U_0 charge of states by $w/2$.

This construction can be extended on the full $\mathfrak{psu}(1,1|2)$ algebra by defining the appropriate spectral flow on the free fermions. The appropriate definition is

$$\begin{aligned} \sigma^{(1,0)}((\psi^\dagger_1)_r) &= (\psi^\dagger_1)_{r+1/2} , \quad \sigma^{(1,0)}(\psi^1_r) = \psi^1_{r-1/2} \\ \sigma^{(0,1)}((\psi^\dagger_2)_r) &= (\psi^\dagger_2)_{r-1/2} , \quad \sigma^{(0,1)}(\psi^2_r) = \psi^2_{r+1/2} , \end{aligned} \tag{4.99}$$

with all other actions trivial. The resulting spectral flow operator σ can now be shown to exactly reproduce the standard spectral flow on $\mathfrak{psu}(1,1|2)_1$, namely

$$\begin{aligned} \sigma^w(J^3_m) &= J^3_m + \frac{w}{2}\delta_{m,0} , \\ \sigma^w(J^{\pm}_m) &= J^{\pm}_{m\mp w} , \\ \sigma^w(K^3_m) &= K^3_m + \frac{w}{2}\delta_{m,0} , \\ \sigma^w(K^{\pm}_m) &= K^{\pm}_{m\pm w} , \\ \sigma^w(S^{\alpha\beta\gamma}_m) &= S^{\alpha\beta\gamma}_{m+\frac{w}{2}(\beta-\alpha)} . \end{aligned} \tag{4.100}$$

Furthermore, σ acts trivially on the gauged current Z, while $\widehat{\sigma}$ acts trivially on $\mathfrak{psu}(1,1|2)_1$ and shifts the value of Y. Once we gauge Z, then, we cannot act on highest-weight physical states with $\widehat{\sigma}$ to generate new representations, as these representations would not satisfy the physical state conditions. Thus, in order to generate new physical representations in the free field theory, we can only act with the spectral flow operator σ. Put another way, the highest-weight plus spectrally-flowed spectrum of the free field realization of $\mathfrak{psu}(1,1|2)_1$ precisely reproduces the representations $\sigma^w(\mathscr{F}_\lambda)$ of $\mathfrak{psu}(1,1|2)_1$. Once we glue the left- and right-movers together we thus have the Hilbert space

$$\mathscr{H} = \int_0^1 \mathrm{d}\lambda \bigoplus_{w\in\mathbb{Z}} \sigma^w(\mathscr{F}_\lambda) \otimes \overline{\sigma^w(\mathscr{F}_\lambda)} . \tag{4.101}$$

Finally, we note that, by use of the free field realization, the characters $\mathrm{ch}[\sigma^w(\mathscr{F}_\lambda)]$ are also computed in a straightforward manner, starting from the characters of the full $\mathfrak{u}(1,1|2)_1$ algebra, and imposing the steps to reduce $\mathfrak{u}(1,1|2)_1$ to $\mathfrak{psu}(1,1|2)_1$. For sake of brevity, we do not repeat the derivation of these characters here, but a detailed calculation can be found in Appendix C of [45].

Twisting the free field theory

In the above discussion of the free field realization of the $\mathfrak{psu}(1,1|2)_1$ WZW model, we demanded that the all of the free fields, both bosonic and fermionic, had conformal weight $h = \frac{1}{2}$. We then gauged the null current Z and obtained, as an end result, the current algebra $\mathfrak{psu}(1,1|2)_1$.

However, conformal weight of the free fields being $h = \frac{1}{2}$ is actually somewhat artificial. To see this, let us define a one-parameter family of new CFTs by defining a new stress tensor $T^{(\alpha)}$ which is 'twisted' by ∂Z:

$$T^{(\alpha)} = T + \alpha \partial Z \,. \tag{4.102}$$

Here, T is the stress tensor of $\mathfrak{psu}(1,1|2)_1$ written in terms of the free fields. We note that after gauging Z, the physical state conditions for T are equivalent to those for $T^{(\alpha)}$. This is straightfoward to see, since

$$L_n^{(\alpha)} = L_n - \alpha n Z_n \,. \tag{4.103}$$

Furthermore, since Z is null, the central charge of the theory does not change under the twist. This is not to say that twisting the stress tensor has absolutely no effect on the theory. Firstly, the conformal weights of the free fields are not invariant under the twist, since the free fields are all charged under Z. The effect of the twist is that the fields now have conformal weights:

$$\begin{aligned} h(\lambda, \mu, \psi^a) &= \frac{1-\alpha}{2}\,, \\ h(\lambda^\dagger, \mu^\dagger, \psi^\dagger_a) &= \frac{1+\alpha}{2}\,. \end{aligned} \tag{4.104}$$

Secondly, the current $Y = U - V$ now has an anomalous conservation law, due to the OPE (see, for instance, equation (13.28) of [53])

$$T^{(\alpha)}(z)Y(w) \sim \frac{2\alpha}{(z-w)^3} + \frac{Y(w)}{(z-w)^2} + \frac{\partial Y(w)}{z-w}\,. \tag{4.105}$$

Specifically, the 'background charge' of Y is $Q_Y = 2\alpha$. We can also consider the currents U, V individually, and we find that they both have background charge $Q_U = Q_V = \alpha$.

Later, when we discuss correlation functions, we will find that the 'twisted' description of the theory will be somewhat more convenient to work with. Specifically, we will find that taking $\alpha = 2$ will be particularly convenient. The conformal weights of the free fields under this choice of α are

$$h(\lambda, \mu, \psi^a) = -\frac{1}{2}\,, \quad h(\lambda^\dagger, \mu^\dagger, \psi^\dagger_a) = \frac{3}{2}\,. \tag{4.106}$$

The anomalous conservation law of U will require us to always consider correlation functions of operators whose U charge adds to $(1-g)Q_U = 2 - 2g$ when evaluated on a surface of genus g.

4.A Spectral flow in the free field realization

In this appendix, we show in detail how the spectral flow of the free field realization reproduces the usual spectral flow of $\mathfrak{sl}(2,\mathbb{R})_1$. Recall that the spectral flow operator

σ^w acts on the free fields $\lambda, \mu^\dagger, \mu, \lambda^\dagger$ as

$$\begin{aligned} \sigma^w(\lambda_r) &= \lambda_{r-\frac{w}{2}}\,, \quad \sigma^w(\mu^\dagger_r) = \mu^\dagger_{r+\frac{w}{2}} \\ \sigma^w(\mu_r) &= \mu_{r+\frac{w}{2}}\,, \quad \sigma^w(\lambda^\dagger_r) = \lambda^\dagger_{r-\frac{w}{2}}\,. \end{aligned} \tag{4.107}$$

Now, we can translate this statement into an action on the local fields $\mu^{(\dagger)}(z)$ and $\lambda^{(\dagger)}(z)$ as

$$\sigma^w(\mu^{(\dagger)}(z)) = z^{-\frac{w}{2}}\mu^{(\dagger)}(z)\,, \quad \sigma^w(\lambda^{(\dagger)}(z)) = z^{\frac{w}{2}}\lambda^{(\dagger)}(z)\,. \tag{4.108}$$

Here, by $\mu^{(\dagger)}$, we mean either μ or $\mu^\dagger$, and similarly for $\lambda^{(\dagger)}$.

To compute the effect of σ^w on the currents J^a, we need to use the definition of normal ordering. We take the normal ordering $(AB)(y)$ to be the coefficient of $(z-y)^0$ in the OPE $A(z)B(y)$. For example,

$$\mu^\dagger(z)\lambda(y) = \frac{1}{z-y} + (\mu^\dagger\lambda)(y) + \cdots\,, \quad \lambda^\dagger(z)\mu(y) = \frac{1}{z-y} + (\lambda^\dagger\mu)(y) + \cdots \tag{4.109}$$

and so on. We define the spectral flow of the normal-ordered product $(AB)(y)$ to be the $(z-y)^0$ coefficient of $\sigma^w(A(z))\sigma^w(B(y))$. Using this definition, we have, for example,

$$\begin{aligned} \sigma^w(\lambda^\dagger(z))\sigma^w(\mu(y)) &= \left(\frac{z}{y}\right)^{\frac{w}{2}} \lambda^\dagger(z)\mu(y) \\ &\sim \left(\frac{z}{y}\right)^{\frac{w}{2}} \left(\frac{1}{z-y} + (\lambda^\dagger\mu)(y) + \cdots\right) \\ &\sim \left(1 + \frac{w}{2}\frac{z-y}{y} + \cdots\right)\left(\frac{1}{z-y} + (\lambda^\dagger\mu)(y) + \cdots\right) \\ &\sim \frac{1}{z-w} + (\lambda^\dagger\mu)(y) + \frac{w}{2y} + \cdots\,. \end{aligned} \tag{4.110}$$

Thus, we can read off

$$\sigma^w((\lambda^\dagger\mu)(y)) = (\lambda^\dagger\mu)(y) + \frac{w}{2y}\,. \tag{4.111}$$

A similar calculation yields

$$\sigma^w((\mu^\dagger\lambda)(y)) = (\mu^\dagger\lambda)(y) - \frac{w}{2y}\,. \tag{4.112}$$

The calculations for $(\lambda^\dagger\lambda)$ and $(\mu^\dagger\mu)$ are even simpler, since there is no singular term in the OPE, and we simply find

$$\sigma^w((\lambda^\dagger\lambda)(y)) = y^w(\lambda^\dagger\lambda)(y)\,, \quad \sigma^w((\mu^\dagger\mu)(y)) = y^{-w}(\mu^\dagger\mu)(y)\,. \tag{4.113}$$

Now, the $\mathfrak{sl}(2,\mathbb{R})_1$ currents are given by

$$J^+ = (\lambda^\dagger\lambda)\,, \quad J^- = (\mu^\dagger\mu)\,, \quad J^3 = \frac{1}{2}\left(\lambda^\dagger\mu - \mu^\dagger\lambda\right)\,. \tag{4.114}$$

Thus, we can immediately calculate the effect of σ^w on these fields, and we find

$$\sigma^w(J^\pm(z)) = z^{\pm w}J^\pm(z)\,, \quad \sigma^w(J^3(z)) = J^3(z) + \frac{w}{2z}\,. \tag{4.115}$$

Expanding the currents in terms of modes

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} \tag{4.116}$$

we find

$$\sigma^w(J^\pm_n) = J^\pm_{n\mp w}\,, \quad \sigma^w(J^3_n) = J^3_n + \frac{w}{2}\delta_{n,0}\,. \tag{4.117}$$

This is precisely the spectral flow of the $\mathfrak{sl}(2,\mathbb{R})_1$ algebra introduced in Chapter 3. This approach can be used to easily derive the spectral flow of all $\mathfrak{psu}(1,1|2)_1$ currents derived from the free field realization.

Chapter 5

Correlators of the tensionless string

In the previous section, we reviewed the argument that the one-loop string partition function of tensionless string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ matches the connected symmetric orbifold partition function. This, in turn, shows that the spectrum of second-quantized string theory naturally reproduces the spectrum of the symmetric orbifold. The matching of the perturbative spectra is extremely convincing evidence that these two theories are, in fact, equivalent.

However, in order to show that the tensionless string is truly dual to the symmetric orbifold theory, we need to not only match the perturbative spectra, but also the correlation functions of the theories. Since the symmetric orbifold Hilbert space decomposes into twisted sectors, a generic correlation function in the symmetric orbifold theory will be of the form

$$\left\langle \mathcal{O}_1^{(w_1)}(x_1) \cdots \mathcal{O}_n^{(w_n)}(x_n) \right\rangle , \tag{5.1}$$

where w_i labels the twist of the operator – operators in the w-twisted sector stem from conjugacy classes of S_N whose cycle-type is $(1 \cdots w)$. As we saw in the previous section in the discussion of the string spectrum, the twisted sectors of the symmetric orbifold are naturally reproduced in string theory by spectrally-flowed states. Indeed, in the matching of the single-particle partition functions in Chapter 4, the w-cycle twisted sector was reproduced by spectrally-flowed characters of $\mathfrak{psu}(1,1|2)_1$. In the context of correlation functions, then, it would be natural to assume that one could reproduce correlators of w-twisted sector states via correlation functions of spectrally flowed states in the $\mathfrak{psu}(1,1|2)_1$ model. Schematically, we want

$$\begin{aligned} &\left\langle \mathcal{O}_1^{(w_1)}(x_1) \cdots \mathcal{O}_n^{(w_n)}(x_n) \right\rangle_c \\ &\qquad = \sum_{g=0}^{\infty} g_s^{2g-2+n} \int_{\mathcal{M}_{g,n}} \left\langle V_{h_1}^{w_1}(x_1, z_1) \cdots V_{h_n}^{w_n}(x_n, z_n) \right\rangle , \end{aligned} \tag{5.2}$$

where $V_h^w(x,z)$ are schematically vertex operators in the w-spectrally-flowed sector which are tentatively dual to the CFT states $\mathcal{O}^{(w)}$. The subscript 'c' refers to the connected part of the symmetric orbifold correlator, which is the part of the correlation function which is expressible in terms of a single worldsheet. We define this more precisely in Appendix 5.A, where we carefully review the construction of correlation functions in the symmetric orbifold theory.

The genus expansion on the right-hand-side of (5.2) should be reproduced by the $1/N$ expansion in the symmetric orbifold theory. Thus, in order to understand how the symmetric orbifold correlators might be recovered from the genus expansion in string theory, it is important to understand the $1/N$ expansion of symmetric orbifolds. For simplicity, let us take the twist fields to be twisted-sector ground states σ_w. Now, let $\Gamma : \Sigma \to \mathbb{CP}^1$ be a holomorphic map from some Riemann surface Σ to the sphere $\mathbb{CP}^1$ on which the correlator of the twist fields is defined. We say that Γ is a *branched covering* of $\mathbb{CP}^1$, branched over x_i with ramification w_i, if there exist n points z_i on Σ such that Γ has the local behavior

$$\Gamma(z) \sim x_i + a_i^{\Gamma}(z - z_i)^{w_i} + \cdots\,, \quad z \to z_i\,. \tag{5.3}$$

It can be shown (see Appendix 5.A) that the connected part of the symmetric orbifold correlator can be expressed as a sum over all such covering maps. Specifically,

$$\langle \sigma_{w_1}(x_1) \cdots \sigma_{w_n}(x_n) \rangle_c = \sum_{g=0}^{\infty} N^{1-g-n/2} \sum_{\Gamma:\Sigma\to\mathbb{CP}^1} C_\Gamma Z_X(\Sigma)\,, \tag{5.4}$$

where $Z_X(\Sigma)$ is the partition function of the seed theory X on the surface Σ. Note that we sum only over covering maps such that Σ is connected. The prefactors C_Γ are functions of the covering map data Γ and the conformal weights h_i of the symmetric orbifold fields.

In [77], it was conjectured that in the string theory holographically dual to $\text{Sym}^N(X)$ at large N, the worldsheet should be given precisely by the covering surfaces Σ. This interpretation, however, would require the dual string theory to have some rather strange features. To see this, note that the sum over covering maps $\Gamma : \Sigma \to \mathbb{CP}^1$ in (5.4) is truly a sum, and the complex structure of Σ is determined purely via the pullback of the complex structure on $\mathbb{CP}^1$. Thus, in order for the integral over the moduli space $\mathcal{M}_{g,n}$ in (5.2) to reproduce the discrete sum in (5.4), we would require that the worldsheet correlators *localize* to discrete points on the moduli space $\mathcal{M}_{g,n}$. Specifically, the worldsheet correlators should schematically take the form

$$\left\langle V_{h_1}^{w_1}(x_1, z_1) \cdots V_{h_n}^{w_n}(x_n, z_n) \right\rangle_{\Sigma_{\text{ws}}} \propto \sum_{\Gamma:\Sigma\to\mathbb{CP}^1} C_\Gamma\, \delta^{(n+3g-3)}(\Sigma_{\text{ws}}, \Sigma)\, Z_{\mathbb{T}^4}(\Sigma)\,, \tag{5.5}$$

where the delta function is formally a distribution on $\mathcal{M}_{g,n}$ which vanishes unless Σ_{ws} and Σ have the same complex structure.

Analytically, the above localization behavior seems very strange from the point of view of a 2D CFT. However, geometrically, a rather beautiful picture emerges. In the $\mathfrak{psu}(1,1|2)_1$ worldsheet theory, the only representations which are allowed are the continuous representations at $j = 1/2$, their supersymmetric descendants, as well as their spectrally-flowed images. As we discussed in Section 3.2, the continuous representations correspond semiclassically to the 'long strings' of [63], and so the worldsheet theory of the tensionless string is described purely in terms of long strings. The nature of long strings is that their classical trajectories approach the asymptotic boundary of AdS_3 at late times. The emergent picture in the tensionless limit is that these long strings do not just approach the boundary at late times, but are *always* located at the boundary of AdS_3. Furthermore, the configurations of the worldsheet are extremely rigid, and are effectively glued to those configurations which cover the

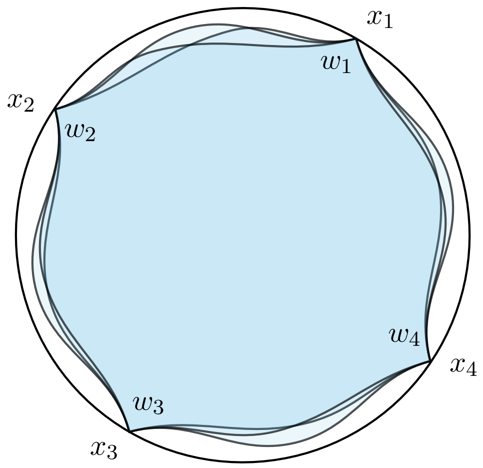

Figure 5.1: A worldsheet configuration in AdS_3 corresponding to the localized configurations contributing to the four-point function $\langle \sigma_{w_1}(x_1)\cdots\sigma_{w_4}(x_4)\rangle$.

boundary of AdS_3. The branching behavior of Γ near $z = z_i$ can be thought of as the string asymptotically winding around the point x_i on the boundary of AdS_3. The corresponding worldsheet is schematically drawn in Figure 5.1.

It was pointed out in [78] that the above worldsheet contribution can furthermore be understood from the semiclassical sigma-model of a string on AdS_3. Writing the AdS_3 metric and B-field as

$$\mathrm{d}s^2 = k\left(\mathrm{d}\Phi^2 + e^{2\Phi}\mathrm{d}\gamma\,\mathrm{d}\overline{\gamma}\right)\,, \quad B = ke^{2\Phi}\mathrm{d}\overline{\gamma}\wedge\mathrm{d}\gamma\,, \tag{5.6}$$

the gauge-fixed Polyakov action takes the form[1]

$$S_{\mathrm{AdS}_3} = \frac{k}{2\pi}\int_\Sigma \mathrm{d}^2z\left(\partial\Phi\,\overline{\partial}\Phi + e^{2\Phi}\overline{\partial}\gamma\,\partial\overline{\gamma}\right)\,. \tag{5.8}$$

In this parametrization, the boundary of AdS_3 is given by the limit $\Phi\to\infty$. This limit is somewhat singular from the point of view of the sigma model, but can be studied more easily by introducing auxiliary fields $\beta,\overline{\beta}$ and writing the action as

$$S_{\mathrm{AdS}_3} = \frac{k}{4\pi}\int_\Sigma \mathrm{d}^2z\left(2\partial\Phi\,\overline{\partial}\Phi + \beta\overline{\partial}\gamma + \overline{\beta}\partial\overline{\gamma} - \beta\overline{\beta}e^{-2\Phi}\right)\,. \tag{5.9}$$

In this form, the worldsheet action at $\Phi\to\infty$ simply becomes a free field theory composed of a scalar field Φ and a first-order system (β,γ) of conformal weight $h_\beta = 1$ and $h_\gamma = 0$. Remarkably, given a covering map $\Gamma:\Sigma\to\mathbb{CP}^1$ branched over x_i with ramifications w_i, we can construct a solution to the equations of motion of this model via

$$\gamma(z,\overline{z}) = \Gamma(z)\,, \quad \overline{\gamma}(z,\overline{z}) = \overline{\Gamma}(\overline{z})\,, \quad \Phi = -\frac{1}{2}\log|\partial\Gamma|^2 - \log\varepsilon\,, \tag{5.10}$$

[1]This action can also be recovered from the $\mathrm{SL}(2,\mathbb{R})$ WZW model by parametrizing elements of $\mathrm{SL}(2,\mathbb{R})$ as

$$g = \begin{pmatrix}1 & 0\\ \bar{\gamma} & 1\end{pmatrix}\begin{pmatrix}e^{\Phi} & 0\\ 0 & e^{-\Phi}\end{pmatrix}\begin{pmatrix}1 & \gamma\\ 0 & 1\end{pmatrix} = \begin{pmatrix}e^{\Phi} & e^{\Phi}\gamma\\ e^{\Phi}\bar{\gamma} & e^{-\Phi}+e^{\Phi}\gamma\bar{\gamma}.\end{pmatrix}\,. \tag{5.7}$$

where ε is a formal regularization parameter. The above string configuration is a solution to the equations of motion in the limit $\varepsilon \to 0$, i.e. in the limit $\Phi \to \infty$. Since $(\gamma, \overline{\gamma})$ parametrizes the boundary of AdS_3, this solution describes a string glued to the boundary of AdS_3 and which winds around the boundary insertion points w_i times. Furthermore, since the worldsheet theory becomes free in the limit $\Phi \to \infty$, these solutions are also valid at the quantum level.

The intuitive picture of the tensionless string, then, is that *only* the above solutions are valid at the quantum level, and so the correlation functions vanish if the string is not situated such that it holomorphically covers the boundary of AdS_3. While the above semiclassical picture is nice, it is not, however, a proof. In this Chapter we will explore in detail the correlation functions of spectrally-flowed states in the $\mathfrak{psu}(1,1|2)_1$ WZW model and show that the above localization principle is actually a consistency condition for worldsheet correlators, which can be derived purely based on local Ward identities. We begin in Section 5.1 by reviewing the analysis of [78] (later generalized to higher-genus worldsheets in [79]), which argues for the consistency of localized solutions in the context of the bosonic $\text{SL}(2,\mathbb{R})$ model. We then cover the analysis of [47, 49] in Section 5.2 and 5.3, which uses the full machinery of the $\mathfrak{psu}(1,1|2)_1$ WZW model to prove that the localization of the worldsheet correlators is not only consistent with local Ward identities, but that it is in fact the *only* consistent solution.

Crucial to this analysis is the free field realization of $\mathfrak{psu}(1,1|2)_1$ which we introduced in Section 4.3. In terms of the free fields, the localization principle can be stated extremely compactly as a relationship between the free fields and the covering map, which we dub the *incidence relation*

$$\mu(z) - \Gamma(z)\lambda(z) = 0\,. \tag{5.11}$$

The interpretation of this relation is that if we think of the free fields μ, λ as homogeneous coordinates $[\mu : \lambda]$ on $\mathbb{CP}^1$, then the map $\Sigma \to \mathbb{CP}^1$ given by $z \mapsto [\mu(z), \lambda(z)]$ precisely realizes the covering map $\Gamma : \Sigma \to \mathbb{CP}^1$ since

$$[\mu(z) : \lambda(z)] \sim [\mu(z)/\lambda(z) : 1] = [\Gamma(z) : 1]\,. \tag{5.12}$$

In Section 5.4, we explore this relation further, and argue that the free field theory is closely related to a twistor string theory. Finally, in Appendix 5.A, we review in detail the analysis of [80] for the computation of symmetric orbifold correlators, and in Appendix 5.B we collect various facts and conventions related to higher-genus Riemann surfaces, which are used in Section 5.3.

5.1 Correlators in the bosonic model

Before jumping into the calculation of correlators in the supersymmetric tensionless string, let us briefly review the analysis of [78], which outlines a strategy for constraining correlation functions in the $\text{SL}(2,\mathbb{R})$ WZW model.

States in the bosonic WZW model on $\text{SL}(2,\mathbb{R})$ fall into representations of the algebra $\mathfrak{sl}(2,\mathbb{R})_k$. This algebra is given by the OPEs

$$\begin{gathered} J^3(z)J^{\pm}(w) \sim \pm\frac{J^{\pm}(w)}{z-w}\,, \quad J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} - \frac{2J^3(w)}{z-w}\,, \\ J^3(z)J^3(w) \sim -\frac{k/2}{(z-w)^2}\,. \end{gathered} \tag{5.13}$$

Expanding the currents into modes gives the mode algebra

$$\begin{gathered}[J^3_m, J^\pm_n] = \pm J^\pm_{m+n}\,, \quad [J^+_m, J^-_n] = km\delta_{m+n,0} - 2J^3_{m+n}\,, \\ [J^3_m, J^3_n] = -\frac{k}{2}m\delta_{m+n,0}\,. \end{gathered} \tag{5.14}$$

Higest-weight states of this algebra are formed by first considering representations of the zero-mode algebra $\mathfrak{sl}(2,\mathbb{R})$ and acting on these states with the oscillator modes J^a_{-n}. We can pick a specific set of states in $\mathfrak{sl}(2,\mathbb{R})$ written as $|m,j\rangle$ such that

$$J^3_0\,|m,j\rangle = m\,|m,j\rangle\ , \quad J^\pm_0\,|m,j\rangle = (m\pm j)\,|m\pm 1,j\rangle\ . \tag{5.15}$$

The quantum number j labels the representation of $\mathfrak{sl}(2,\mathbb{R})$ while m labels the charge of the state under the Cartan generator J^3_0.

From a highest-weight state $|m,j\rangle$ we can construct a vertex operator $V_{m,j}(z)$ via the operator-state correspondence. The action of the $\mathfrak{sl}(2,\mathbb{R})$ generators can be rephrased into OPEs between the currents and the vertex operators as

$$J^3(z)V_{m,j}(w) \sim \frac{m\,V_{m,j}(w)}{z-w}\,, \quad J^\pm(z)V_{m,j}(w) \sim \frac{(m\pm j)V_{m\pm1,j}(w)}{z-w}\,. \tag{5.16}$$

From a highest-weight representation of $\mathfrak{sl}(2,\mathbb{R})_k$, recall that we can construct a new representation by composing the representation with the spectral flow operation

$$\sigma^w(J^3_n) = J^3_n + kw\delta_{n,0}\,, \quad \sigma^w(J^\pm_n) = J^\pm_{n\mp w}\,. \tag{5.17}$$

The spectrally-flowed states $|m,j\rangle^{(w)}$, which are the images of the highest-weight states $|m,j\rangle$ under spectral flow, are acted on by $\mathfrak{sl}(2,\mathbb{R})_k$ currents as

$$J^a_n\,|m,j\rangle^{(w)} = \left(\sigma^w(J^a_n)\,|m,j\rangle\right)^{(w)}\,. \tag{5.18}$$

That is to say, the action of J^a_n on the spectrally-flowed state is the spectrally-flowed image of the action of $\sigma^w(J^a_n)$ on the highest-weight state $|m,j\rangle$. Specifically, we have

$$\begin{aligned} J^3_0\,|m,j\rangle^{(w)} &= \left(m + \frac{kw}{2}\right)|m,j\rangle^{(w)}\ , \\ J^\pm_{\pm w}\,|m,j\rangle^{(w)} &= (m\pm j)\,|m\pm1,j\rangle^{(w)}\ . \end{aligned} \tag{5.19}$$

Furthermore, since $|m,j\rangle$ is a highest-weight state, we have

$$\begin{aligned} J^3_n\,|m,j\rangle^{(w)} &= 0\,, \quad n>0\,, \\ J^\pm_n\,|m,j\rangle^{(w)} &= 0\,, \quad n>\pm w\,. \end{aligned} \tag{5.20}$$

Just as in the case of the highest-weight states, we can introduce vertex operators $V^w_{h,j}$ which are dual to $|m,j\rangle^{(w)}$, where we have introduced a label $h = m + kw/2$, which is the eigenvalue of J^3_0 in the spectrally-flowed sector. The above action of $\mathfrak{sl}(2,\mathbb{R})_k$ on the states $|m,j\rangle^{(w)}$ now tells us that the simple poles in the $J^\pm$ OPEs are replaced with a pole of order $w+1$ for J^+ and a zero of order $w-1$ for J^-. Specifically,

$$\begin{aligned} J^3(z)V^w_{h,j}(w) &\sim \frac{h\,V^w_{h,j}(w)}{z-w}\,, \\ J^+(z)V^w_{h,j}(w) &\sim \frac{(h-\frac{kw}{2}+j)V^w_{h+1,j}(w)}{(z-w)^{w+1}}\,, \\ J^-(z)V^w_{h,j}(w) &\sim (h-\tfrac{kw}{2}-j)(z-w)^{w-1}V^w_{h-1,j}(w)\,. \end{aligned} \tag{5.21}$$

Now, we should think of the operators $V^w_{h,j}$ with positive spectral flow as corresponding to the emission of a string at the infinite past in AdS_3. Mapping the boundary of AdS_3 to the Riemann sphere, this would correspond to emitting a string at $x = 0$. However, in general, we want to calculate the scattering of strings being emitted from any point on the boundary of AdS_3. In order to do this, we recall that the generators of conformal transformations on the boundary of AdS_3 are precisely the zero modes J^a_0. Specifically, we have the mapping

$$J^+_0 \longleftrightarrow L^{\mathrm{CFT}}_{-1}\,, \quad J^3_0 \longleftrightarrow L^{\mathrm{CFT}}_0\,, \quad J^-_0 \longleftrightarrow L^{\mathrm{CFT}}_1\,. \tag{5.22}$$

Thus, to move an operator on the boundary of AdS_3 from $x = 0$ to a generic point $x \neq 0$, we use the translation operator L^{CFT}_{-1}. Motivated by the duality between the boundary Virasoro modes and the $\mathfrak{sl}(2,\mathbb{R})_k$ zero modes, we are led to define the x-translated vertex operators

$$V^w_{h,j}(x,z) := e^{xJ^+_0} V^w_{h,j}(z) e^{-xJ^+_0}\,. \tag{5.23}$$

The above vertex operators are precisely those which appear in the right-hand-side of (5.2), and correspond to w-twisted sector states of conformal weight h. In general we are interested in computing (or at least constraining) the correlation functions

$$\left\langle V^{w_1}_{h_1,j_1}(x_1,z_1)\cdots V^{w_n}_{h_n,j_n}(x_n,z_n)\right\rangle\,. \tag{5.24}$$

The local Ward identities

The strategy of [78] is to compute the above correlation function by utilizing the structure of the local $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ current algebra, which describes the AdS_3 part of the RNS formalism of string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$. We will not go into detail on their analysis, but we review the salient points, as they will serve as a guiding hand when considering the problem in the tenionless limit.

The trick of [78] is to consider the meromorphic functions

$$\mathscr{J}^a(z) = \left\langle J^a(z) V^{w_1}_{h_1,j_1}(x_1,z_1)\cdots V^{w_n}_{h_n,j_n}(x_n,z_n)\right\rangle\,, \tag{5.25}$$

and uses the various known OPEs between currents and vertex operators to derive a set of identities relating various correlation functions. The first step in this analysis is to compute the OPEs between currents J^a and the x-shifted vertex operators $V^w_{h,j}(x,z)$. This can be done in a straightforward manner by defining

$$J^{a(x)}(z) := e^{-xJ^+_0} J^a(z) e^{xJ^+_0}\,. \tag{5.26}$$

With this definition, the OPEs between J^a and $V^w_{h,j}$ are given by

$$\begin{aligned} J^a(\zeta)V^w_{h,j}(x,z) &= J^a(\zeta) e^{xJ^+_0} V^w_{h,j}(0,z) e^{-xJ^+_0} \\ &= e^{xJ^+_0}\left(J^{a(x)}(\zeta)V^w_{h,j}(0,z)\right) e^{-xJ^+_0}\,. \end{aligned} \tag{5.27}$$

That is, we can find the x-shifted OPEs by consider the OPEs of the unshifted vertex operators with $J^{a(x)}$. A simple calculation gives

$$J^{+(x)} = J^+\,, \quad J^{3(x)} = J^3 + xJ^+\,, \quad J^{-(x)} = J^- + 2xJ^3 + x^2 J^+\,. \tag{5.28}$$

Thus, the J^a OPEs with x-shifted vertex operators are just linear combinations of the J^a OPEs with the vertex operators at $x = 0$.

A specific consequence of this representation of the x-shifted currents is the identity

$$J^{-(x)} - 2xJ^{3(x)} + x^2 J^{+(x)} = J^- \,, \tag{5.29}$$

which implies that the OPE of $J^- - 2xJ^3 + x^2 J^+$ has a regular OPE with $V^w_{h,j}(x,z)$. Indeed,

$$\left(J^-(\zeta) - 2xJ^3(\zeta) + x^2 J^+(\zeta)\right) V^w_{h,j}(x,z) = e^{xJ_0^+}\left(J^-(\zeta)V^w_{h,j}(0,z)\right)e^{-xJ_0^+} \,, \tag{5.30}$$

which has a zero of order $w-1$ at $\zeta \to z$. Now, if we assume a covering map Γ exists which satisfies (5.3), then the combination

$$\mathscr{J}^-(z) - 2\Gamma(z)\mathscr{J}^3(z) + \Gamma(z)^2\mathscr{J}^+(z) \tag{5.31}$$

has zeroes of order $w_i - 1$ at $x = x_i$ and double poles at the poles of Γ. Thus, as an analytic function,

$$\mathscr{J}^-(z) - 2\Gamma(z)\mathscr{J}^3(z) + \Gamma(z)^2\mathscr{J}^+(z) \propto \sqrt{\partial\Gamma(z)} \,. \tag{5.32}$$

The significance of the above identity is that, along with the knowledge of the OPEs between the currents J^a and the spectrally-flowed vertex operators, one can use it to derive a series of powerful constraining identities that recursively identify correlators with various values of h_i, j_i. These strong analytic constraints, known as the *local Ward identities*, essentially require that correlators be consistent with the local $\mathfrak{sl}(2,\mathbb{R})_k$ algebra on the worldsheet. It was shown in [78] through a very non-trivial analysis that localizing correlators of the form

$$\left\langle \prod_{i=1}^{n} V^{w_i}_{h_i,j_i}(x_i,z_i) \right\rangle = \sum_{\Gamma} W_\Gamma \prod_{i=1}^{n} |a_i^\Gamma|^{-h_i-\bar{h}_i}\delta^{(n-3)}(\Gamma(z_i)-x_i) \tag{5.33}$$

solve the local Ward identities (although they are far from being the only solution). The factors a_i^Γ are the coefficients in the local Taylor expansion (5.3), and appear in the same way in the correlation functions of twist fields in the symmetric orbifold (see Appendix 5.A). Furthermore, these correlators satisfy the local Ward identities if and only if

$$\sum_{i=1}^{n}\left(j_i - \frac{1}{2}\right) = \frac{(k-1)(n-2)}{2} \,. \tag{5.34}$$

Satisfyingly, for $k = 1$ (corresponding to the tensionless limit), this constraint is satisfied when we take $j_i = 1/2$ for all vertex operators. As we saw in Chapter 4, the only representations which survive in the worldsheet theory at $k = 1$ are those with $j_i = 1/2$. Thus, the analysis of [78] gave very strong evidence that the correlation functions of the tensionless string indeed reproduce the correlators of the symmetric product orbifold.[2]

In [79], the analysis of [78] was extended to higher-genus worldsheets, where one expects the worldsheet correlators to be delta-function localized to a discrete subspace of $\mathcal{M}_{g,n}$. Just as in the genus 0 case, the localized correlators solve the

[2]See also [81–83] for progress in utilizing this approach to compute correlators beyond $k = 1$.

local Ward identities if and only if a certain constraint on the representations j_i is imposed. For a genus g worldsheet, this constraint takes the form

$$\sum_{i=1}^{n}\left(j_i-\frac{1}{2}\right)=\frac{(k-1)(n+2g-2)}{2}\,, \tag{5.35}$$

which again is satisfied for $k=1$ and $j_i=1/2$.

While the analysis of [78, 79] was a crucial step forward in showing that the correlation functions of worldsheet vertex operators in AdS_3 can reproduce correlation functions in the symmetric orbifold, the solutions (5.33) are not unique, and so the precise form of the correlation functions could not be completely nailed down. Furthermore, the analysis was done in the RNS formalism, which has conceptual problems at the tensionless limit. In the next Section, we will provide an argument for localization using a similar strategy, but which is done directly in the hybrid formalism at $k=1$. In that analysis, the free field realization of Chapter 4 will turn out to be an invaluable tool, and we find that the solutions to the local Ward identity are completely unique, and always localize onto the covering maps.

5.2 Correlators in the free field realization

As we discussed in the Chapter 4, the worldsheet theory for tensionless type IIB string theory on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathbb{T}^4$ is nicely described in the hybrid formalism as

$$\mathfrak{psu}(1,1|2)_1\oplus(\text{topologically twisted }\mathbb{T}^4)\oplus\text{ghosts}\,. \tag{5.36}$$

Furthermore, the WZW model on $\mathfrak{psu}(1,1|2)$ at level $k=1$ admits a realization in terms of two bosonic and two fermionic $(-\frac{1}{2},\frac{3}{2})$ first-order systems with the total 'ghost current'

$$Z=\frac{1}{2}\left(\mu^\dagger\lambda+\lambda^\dagger\mu+\psi^\dagger_a\psi^a\right) \tag{5.37}$$

gauged.

In this section, we will use the free field realization to constrain correlators of the form (5.24). In order to do this, we first need to explore how the vertex operators $V^w_{h,j}(x,z)$ are constructed in the free field realization. We will find that it is convenient to use the 'twisted' version of the free field realization defined in Section 4.3, so that μ,λ have weight $h=-\frac{1}{2}$, while $\mu^\dagger,\lambda^\dagger$ have conformal weight $h=\frac{3}{2}$. It is possible to compute correlation functions in the untwisted theory, but the price one pays is the introduction of fields with nonzero Z charge but which live in the vacuum representation of $\mathfrak{psu}(1,1|2)_1$, see [47].

As we described in Chapter 4, the unflowed states of the free field theory are taken to live in the Ramond-sector. In the twisted free theory, the highest-weight Ramond-Sector takes the form

$$\begin{aligned}\lambda_1\,|m,j\rangle&=|m+\tfrac{1}{2},j-\tfrac{1}{2}\rangle\,, & \mu^\dagger_{-1}\,|m,j\rangle&=-(m-j)\,|m-\tfrac{1}{2},j+\tfrac{1}{2}\rangle\,,\\ \mu_1\,|m,j\rangle&=|m-\tfrac{1}{2},j-\tfrac{1}{2}\rangle\,, & \lambda^\dagger_{-1}\,|m,j\rangle&=(m+j)\,|m+\tfrac{1}{2},j+\tfrac{1}{2}\rangle\,,\end{aligned} \tag{5.38}$$

while all higher modes annihilating $|m,j\rangle$. Dual to these states are vertex operators

$V_{m,j}(z)$ which satisfy the OPEs

$$\begin{aligned}\lambda(z)V_{m,j}(0) &\sim z^{-\frac{1}{2}}V_{m+\frac{1}{2},j-\frac{1}{2}}(0)\,, \quad \mu^\dagger(z)V_{m,j}(0) \sim -(m-j)z^{-\frac{1}{2}}V_{m-\frac{1}{2},j+\frac{1}{2}}(0)\,,\\ \mu(z)V_{m,j}(0) &\sim z^{-\frac{1}{2}}V_{m-\frac{1}{2},j-\frac{1}{2}}(0)\,, \quad \lambda^\dagger(z)V_{m,j}(0) \sim (m+j)z^{-\frac{1}{2}}V_{m+\frac{1}{2},j+\frac{1}{2}}(0)\,.\end{aligned} \tag{5.39}$$

In addition to these vertex operators, we can define an infinite family $V_{m,j}^{(p,q)}$ of vertex operators which are defined by applying the spectral flow $\sigma^{(p,q)} = \sigma^{(p)} \circ \widehat{\sigma}^{(q)}$, where σ is the usual $\mathfrak{psu}(1,1|2)_1$ spectral flow, and $\widehat{\sigma}^{(q)}$ is the 'orthogonal' spectral flow defined in Section 4.3, which acts trivially on $\mathfrak{psu}(1,1|2)_1$, but which raises the Z-charge of a state by one half unit. Since the physical worldsheet theory is obtained after gauging Z, it should be noted that $V_{m,j}^{(p,q)}$ and $V_{m,j}^{(p,q')}$ should represent the same physical field in the worldsheet theory for $q \neq q'$. This is extremely reminiscent to the concept of *picture changing* in the RNS formalism in string theory, we refer to the worldsheet vertex operator $V_{m,j}^{(p,q)}$ as being in the p^{th} spectrally-flowed sector and in the q picture (a similar observation has been made before in the context of the free worldsheet theory in the untwisted description, see (2.4) of [84]). The OPEs of the free fields and the flowed vertex operators take the form

$$\begin{aligned}\lambda(z)V^{(p,q)}(0) &\sim \mathcal{O}\left(z^{\frac{q-p-1}{2}}\right)\,, \quad \mu^\dagger(z)V^{(p,q)}(0) \sim \mathcal{O}\left(z^{\frac{p-q-1}{2}}\right)\,,\\ \mu(z)V^{(p,q)}(0) &\sim \mathcal{O}\left(z^{\frac{p+q-1}{2}}\right)\,, \quad \lambda^\dagger(z)V^{(p,q)}(0) \sim \mathcal{O}\left(z^{-\frac{p+q+1}{2}}\right)\,,\end{aligned} \tag{5.40}$$

where we have stripped off the m, j dependence, as it does not change under spectral flow.

Now that we have defined the vertex operators $V_{m,j}^{(p,q)}(z)$, we can proceed in a similar fashion to the $\mathfrak{sl}(2,\mathbb{R})_k$ analysis of [78] and introduce an x-dependence via

$$V_{m,j}^{(p,q)}(x,z) := e^{xJ_0^+}V_{m,j}^{(p,q)}(z)e^{-J_0^+}\,. \tag{5.41}$$

Here, we are to understand J^+ as the bilinear $\lambda^\dagger\lambda$. In order to study correlation functions, it will be useful to find the OPEs of the free fields with the x-shifted vertex operators. For the moment, we are only interested in the OPEs of the undaggered bosons μ, λ, but similar expressions will exist for the daggered versions. Just as in the previous section, we can define

$$\lambda^{(x)} = e^{-xJ_0^+}\lambda e^{xJ_0^+}\,, \quad \mu^{(x)} = e^{-xJ_0^+}\mu e^{xJ_0^+}\,. \tag{5.42}$$

so that the OPE of λ, μ with $V(x,z)$ can be computed by considering the OPE of $\lambda^{(x)}, \mu^{(x)}$ with the unshifted vertex operator $V(0,z)$. A simple calculation shows that

$$\lambda^{(x)} = \lambda\,, \quad \mu^{(x)} = \mu + x\lambda\,. \tag{5.43}$$

Thus, the OPEs of the free fields with the x-shifted vertex operators is

$$\begin{aligned}\lambda(\zeta)V_{m,j}^{(p,q)}(x,z) &\sim (\zeta-z)^{\frac{q-p-1}{2}}V_{m+\frac{1}{2},j-\frac{1}{2}}^{(p,q)}(x,z)\,,\\ \mu(\zeta)V_{m,j}^{(p,q)}(x,z) &\sim x(\zeta-z)^{\frac{p+q-1}{2}}V_{m+\frac{1}{2},j-\frac{1}{2}}^{(p,q)}(x,z)\,.\end{aligned} \tag{5.44}$$

Furthermore, the linear combination $\mu - x\lambda$ satisfies

$$\mu^{(x)} - x\lambda^{(x)} = \mu\,, \tag{5.45}$$

and so, assuming $p+q > 0$, the OPE of $\mu - x\lambda$ with an x-shifted vertex operator will have the same type of regular behavior as μ does with an unshifted vertex operator. Specifically,

$$(\mu(\zeta) - x\lambda(\zeta))V^{(p,q)}_{m,j}(x,z) \sim (\zeta - z)^{\frac{p+q-1}{2}} V^{(p,q)}_{m-\frac{1}{2},j-\frac{1}{2}}(x,z)\,. \tag{5.46}$$

The canonical picture

As we mentioned above, vertex operators in different pictures q should be physically equivalent. At the end of the day, however, we would like to compute correlation functions of the form

$$\left\langle \prod_{i=1}^{n} V^{(p_i,q_i)}_{m_i,j_i}(x_i,z_i) \right\rangle\,. \tag{5.47}$$

Now, we know that, in the twisted description of the free fields, the current U is not completely conserved, but has a background charge $Q_U = 2$. Thus, in a genus g correlation function, the sum of all U charges of the vertex operators must be $2-2g$. We can easily calculate the U-charge of the states in the above correlator, and we find

$$Q_U\left(V^{(p,q)}_{m,j}\right) = \left(j - \frac{1}{2}\right) + \frac{q}{2}\,. \tag{5.48}$$

Thus, the requirement that the above correlator is nonzero is that the sum of U-charges is $2-2g$, i.e.

$$\sum_{i=1}^{n}\left(j_i - \frac{1}{2} + q_i\right) = 2 - 2g\,. \tag{5.49}$$

Naively, one would then assume that physical states should satisfy $j = 1/2$ (see Section 4.1), and so we would require

$$\sum_{i=1}^{n} q_i \stackrel{?}{=} 2 - 2g\,. \tag{5.50}$$

However, this is not quite the full story. In the hybrid formalism of [66], integrated correlators of the states Φ_i take the schematic form

$$\int_{\mathcal{M}_{g,n}} \left\langle \prod_{a=1}^{n+3g-3} \widehat{G}^-(\mu_a)\left[\int \widetilde{G}^+\right]^{g-1} \int J \prod_{i=1}^{n}\Phi_i \right\rangle\,, \tag{5.51}$$

where μ_a are the $n+3g-3$ Beltrami differentials parametrizing the tangent space of $\mathcal{M}_{g,n}$ at some reference surface, and $\widehat{G}^-$ are either G^- or $\widetilde{G}^-$. It was claimed in [47] that the appropriate assignment of $\widehat{G}^-$ operators is

$$(n+2g-2)\times\widetilde{G}^-\,,\quad (g-1)\times G^-\,. \tag{5.52}$$

The effect of choosing $n+2g-2$ of the operators to be $\widetilde{G}^-$ instead of G^- has a very simple effect: for all intents and purposes, we can place the operators $\widetilde{G}^-$ at the locations of $n+2g-2$ of the vertex operators[3], and the effect of placing $\widetilde{G}^-$ at the vertex operator $V^{(p_i,q_i)}_{m_i,j_i}$ is to lower its j-eigenvalue by 1. See [47] for a more complete account.

The end effect is that instead of considering correlators of the form (5.47), we have to modify our definition a bit and consider correlators of the form

$$\left\langle \prod_{i=1}^{n} V^{(p_i,q_i)}_{m_i,\tilde{j}_i}(x_i,z_i) \right\rangle , \tag{5.53}$$

such that

$$\sum_{i=1}^{n} \tilde{j}_i + n + 2g - 2 = \sum_{i=1}^{n} j_i \,. \tag{5.54}$$

If we now impose the (non)-conservation of the U current on these states, we find

$$\sum_{i=1}^{n}\left(\tilde{j}_i - \frac{1}{2}\right) + \sum_{i=1}^{n}\frac{q_i}{2} = \sum_{i=1}^{n}\left(j_i - \frac{1}{2}\right) + \sum_{i=1}^{n}\frac{q_i}{2} - (n+2g-2) = 2-2g \,. \tag{5.55}$$

Rearranging, we find

$$\sum_{i=1}^{n}\left(j_i - \frac{1}{2}\right) + \sum_{i=1}^{n}\frac{q_i}{2} - n = 0 \,. \tag{5.56}$$

We can now impose the physical state condition $j_i = 1/2$, and we find that the pictures q_i have to satisfy

$$\sum_{i=1}^{n}(q_i - 2) = 0 \,. \tag{5.57}$$

This constraint is most easily satisfied by choosing a *canonical* picture $q=2$ for all vertex operators. With this in mind, we define the vertex operators of interest as

$$V^{w_i}_{h_i,j_i}(x_i,z_i) := V^{(w_i,2)}_{m_i,\tilde{j}_i}(x_i,z_i) \,. \tag{5.58}$$

Here, the quantum number h_i is given by

$$h_i = m_i + \frac{w_i}{2} \,, \tag{5.59}$$

and is the eigenvalue of J^3_0 on the corresponding (unshifted) state. The vertex operators in the canonical picture satisfy the following OPEs with the free fields:

$$\begin{aligned} \lambda(z)\, V^{w_i}_{h_i,j_i}(x_i,z_i) &\sim (z-z_i)^{-\frac{w_i-1}{2}}\, V^{w_i}_{h_i+\frac{1}{2},j_i+\frac{1}{2}}(z_i) + \cdots \\ \mu(z)\, V^{w_i}_{h_i,j_i}(x_i,z_i) &\sim x_i(z-z_i)^{-\frac{w_i-1}{2}}\, V^{w_i}_{h_i+\frac{1}{2},j_i+\frac{1}{2}}(z_i) + \cdots \end{aligned} \tag{5.60}$$

and, as we noted before, the combination $\mu - x_i\lambda$ has a zero near $z = z_i$. Specifically,

$$(\mu(z) - x_i\lambda(z))\, V^{w_i}_{h_i,j_i}(x_i,z_i) \sim (z-z_i)^{\frac{w_i+1}{2}}\, V^{w_i}_{h_i-\frac{1}{2},j_i+\frac{1}{2}}(z_i) + \cdots \,. \tag{5.61}$$

[3] By this we mean that some vertex operators can be acted on by $\widetilde{G}^-$ more than one time – the precise locations of the vertex operators does not matter.

The localization argument

With the above definitions in mind, we are ready to tackle the study of correlation functions in the free field realization of $\mathfrak{psu}(1,1|2)_1$. We are interested in correlators of the form

$$\left\langle V^{w_1}_{h_1,j_1}(x_1,z_1)\cdots V^{w_n}_{h_n,j_n}(x_n,z_n)\right\rangle \,, \tag{5.62}$$

where $V^w_{h,j}(x,z)$ are defined as above. In Section 5.1, we argued that correlation functions of spectrally-flowed states can be constrained by considering meromorphic functions $\mathscr{J}^a$ which are defined by inserting the fields J^a into the basic correlator (5.24). In the free fieldrealization, we will employ a similar trick. Consider the functions

$$\mathscr{L}(z) := \left\langle \lambda(z)\prod_{i=1}^n V^{w_i}_{h_i,j_i}(x_i,z_i)\right\rangle \,, \quad \mathscr{M}(z) := \left\langle \mu(z)\prod_{i=1}^n V^{w_i}_{h_i,j_i}(x_i,z_i)\right\rangle \,. \tag{5.63}$$

As we will argue below, the local Ward identities obtained for these functions can be massaged in such a way as to be algebraically equivalent to the requirements for the existence of a covering map $\Gamma:\Sigma\to\mathbb{CP}^1$ branched over x_i, so that the functions $\mathscr{L},\mathscr{M}$ are only nonzero if a covering map exists, and vise-versa. This will in turn directly prove that the correlation functions (5.62) are localized to the points on the moduli space $\mathcal{M}_{g,n}$ for which such covering maps exist. Furthermore, the prefactor of (5.33) (the product of the coefficients a_i^Γ) will pop out of the analysis almost for free. The fundamental result we will use will be the so-called 'incidence relation' [47]

$$\mathscr{M}(z)-\Gamma(z)\mathscr{L}(z)=0 \tag{5.64}$$

which directly relates the free fields $\mathscr{L},\mathscr{M}$ to the covering map Γ.

The proof of the above relation as well as the proof for localization of the correlators is surprisingly straightfoward. By the OPEs (5.60), we know that both of these functions have poles of order $(w_i-1)/2$ near $z=z_i$. We will find it more convenient to work with the 'renormalized' functions $Q(z),P(z)$ defined by

$$Q(z)=\prod_{i=1}^n(z-z_i)^{\frac{w_i-1}{2}}\mathscr{M}(z)\,,\quad P(z)=\prod_{i=1}^n(z-z_i)^{\frac{w_i-1}{2}}\mathscr{L}(z)\,. \tag{5.65}$$

These functions have the following properties:

- Both $Q(z)$ and $P(z)$ are holomorphic and globally-defined (i.e. have no branch cuts) on $\mathbb{C}$.
- Near $z=\infty$, both $P(z)$ and $Q(z)$ grow like $\mathcal{O}(z^N)$ with $N=1+\sum_i(w_i-1)/2$.[4]

By Liouville's theorem, we conclude that that Q,P are *polynomials* in z. Now, what of the other Ward identities? By the OPE (5.61), we know that $\mathscr{M}-x_i\mathscr{L}$ has a zero of order $(w_i+1)/2$ at $z=z_i$. In terms of the polynomaials Q,P, this condition becomes

$$Q(z)-x_iP(z)\sim\mathcal{O}((z-z_i)^{w_i})\,. \tag{5.66}$$

Remarkably, the above conditions are completely algebraically equivalent to the condition for the ration Q/P to be a covering map ramified over x_i. Indeed, the

[4]The 1 comes from the fact that $\mathscr{L},\mathscr{M}$ grow like $\mathcal{O}(z)$ near $z=\infty$, which is a result of transforming like conformal primaries of weight $h=-1/2$ under the conformal transformation $z\mapsto -1/z$.

defining properties of a covering map $\Gamma : \mathbb{CP}^1 \to \mathbb{CP}^1$ are that Γ is a rational function of degree $N = 1 + \sum_i (w_i - 1)/2$ such that

$$\Gamma(z) - x_i \sim \mathcal{O}((z - z_i)^{w_i}) \tag{5.67}$$

near $z = z_i$. That Γ is a rational function of degree N implies that $\Gamma = Q/P$ for polynomials Q, P of degree N, and the local behavior of Γ near $z = z_i$ is equivalent to (5.66). Thus, we have the following statement:

> *Solutions to the genus-zero local Ward identities on the worldsheet are in one-to-one correspondence with covering maps $\Gamma : \mathbb{CP}^1 \to \mathbb{CP}^1$ ramified over x_i.*

Since such covering maps only exist at discrete points in the moduli space $\mathcal{M}_{0,n}$ for $n \geq 3$, we conclude that the local Ward identities only have nontrivial solutions if the conformal structure of the worldsheet lies on these points in $\mathcal{M}_{0,n}$.

Strictly speaking, we have only shown that the functions $\mathscr{M}, \mathscr{L}$ vanish when a covering map exists. We can use this statement to immediately infer that the correlation functions (5.24) also vanish. We can do this by considering the expansion of $\mathscr{L}$ near a point $z = z_i$ by virtue of the OPE (5.60). We have

$$\mathscr{L}(z) \sim (z - z_i)^{-\frac{w_i - 1}{2}} \left\langle V^{w_i}_{h_i + \frac{1}{2}, j_i + \frac{1}{2}}(x_i, z_i) \prod_{j \neq i} V^{w_j}_{h_j, j_j}(x_j, z_j) \right\rangle . \tag{5.68}$$

This statement holds true regardless of the value of j, h, and so if $\mathscr{L}$ identically vanishes (which occurs if a covering map Γ does not exist), then the correlators (5.24) also identically vanish. This proves that the correlators of spectrally-flowed ground states completely localize to the sublocus of $\mathcal{M}_{0,n}$ where a covering map exists.

In non-canonical pictures

As a side-remark, we note that the above argument also holds for vertex operators which are not necessarily in the canonical $q = 2$ picture. If we allow the pictures q_i of the vertex operators to vary, we will find that the functions $\mathscr{L}, \mathscr{M}$ has poles of order $(w_i - q_i + 1)$, and so we can define the polynomials Q, P as

$$Q(z) = \prod_{i=1}^{n} (z - z_i)^{\frac{w_i - q_i + 1}{2}} \mathscr{M}(z) , \quad P(z) = \prod_{i=1}^{n} (z - z_i)^{\frac{w_i - q_i + 1}{2}} \mathscr{L}(z) . \tag{5.69}$$

These will again be globally well-defined and will have no poles away from $z = \infty$. The growth near $z \to \infty$ is easily calculated to be z^N with

$$N = 1 + \sum_{i=1}^{n} \frac{w_i - 1}{2} + \sum_{i=1}^{n} \left(1 - \frac{q_i}{2}\right) . \tag{5.70}$$

However, the last term vanishes by anomalous U-conservation, see equation (5.57), and so N just reproduces the Riemann-Hurwitz relation. Furthermore, since

$$\mathscr{M}(z) - x_i \mathscr{L}(z) \sim \mathcal{O}\left((z - z_i)^{\frac{w_i + q_i - 1}{2}}\right) , \quad z \to z_i \tag{5.71}$$

we instantly see that

$$Q(z) - x_i P(z) \sim \mathcal{O}((z-z_i)^{w_i}) , \quad z \to z_i . \tag{5.72}$$

Since $Q(z)$ and $P(z)$ are polynomials of degree N satisfying the above critical relations at $z = z_i$, we instantly see by the above arguments that Q/P is a covering map Γ, and so the localization argument works for any choice of picture, so long as the U-constraint $\sum_i (q_i - 2) = 0$ is satisfied.

Constraining the correlators

We have now used the free field realization to prove that correlators of spectrally-flowed ground states in the tensionless string localize. However, in order to compare the result to the symmetric orbifold answer, we have to do a bit more work. In particular, symmetric orbifold correlators have, for each covering map Γ, a prefactor which depends on the conformal weights h_i of the states in the correlator, as well as the analytic data of Γ. Specifically, the dependence of h_i on the contribution from each covering map is given by

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i^{(w_i)}(x_i) \right\rangle_\Gamma \propto \prod_{i=1}^{n} (a_i^\Gamma)^{-h_i} . \tag{5.73}$$

Here, a_i^Γ are the constants defined in (5.3). One can think of these prefactors as the Jacobian arising from pulling back the twist fields $\mathcal{O}_i^{(w_i)}$ by the covering map Γ, see Appendix 5.A.

From the worldsheet, we can derive this behavior using the incidence relation

$$\mathscr{M}(z) - \Gamma(z)\,\mathscr{L}(z) = 0 . \tag{5.74}$$

Let us expand this relation near $z = z_i$. The covering map satisfies (5.3), and so

$$\mathscr{M}(z) - \big(x_i + a_i^\Gamma (z-z_i)^{w_i} + \cdots\big)\,\mathscr{L}(z) = 0 . \tag{5.75}$$

Now, using (5.61), we know the behavior of $\mathscr{M} - x_i \mathscr{L}$, and by (5.60), we know the behavior of $\mathscr{L}$. Specifically,

$$\begin{aligned} \mathscr{M}(z) - x_i \mathscr{L}(z) &\sim (z-z_i)^{\frac{w_i+1}{2}} \left\langle V^{w_i}_{h_i-\frac{1}{2}, j_i+\frac{1}{2}}(x_i,z_i) \prod_{j\neq i} V^{w_j}_{h_j,j_j}(x_j,z_j) \right\rangle + \cdots \\ a_i^\Gamma (z-z_i)^{w_i} \mathscr{L}(z) &\sim a_i^\Gamma (z-z_i)^{\frac{w_i+1}{2}} \left\langle V^{w_i}_{h_i+\frac{1}{2}, j_i+\frac{1}{2}}(x_i,z_i) \prod_{j\neq i} V^{w_j}_{h_j,j_j}(x_j,z_j) \right\rangle + \cdots \end{aligned} \tag{5.76}$$

Comparing coefficients, we immediately obtain the recursion relation

$$\left\langle V^{w_i}_{h_i-\frac{1}{2}, j_i+\frac{1}{2}}(x_i,z_i) \prod_{j\neq i} V^{w_j}_{h_j,j_j}(x_j,z_j) \right\rangle = a_i^\Gamma \left\langle V^{w_i}_{h_i+\frac{1}{2}, j_i+\frac{1}{2}} \prod_{j\neq i} V^{w_j}_{h_j,j_j}(x_j,z_j) \right\rangle , \tag{5.77}$$

whose solution is uniquely given by the right-hand-side of (5.73).

The conclusion is that the correlators of spectrally-flowed ground states of tensionless string theory structurally have the form

$$\left\langle \prod_{i=1}^{n} V^{w_i}_{h_i,j_i}(x_i,z_i) \right\rangle \sim \sum_{\Gamma} W_{\Gamma} \prod_{i=1}^{n} (a_i^{\Gamma})^{-h_i} f_{\Gamma}^{(n-3)}(z_i)\,, \tag{5.78}$$

where $f_{\Gamma}^{(n-3)}(z_i)$ is some distribution on the moduli space $\mathcal{M}_{0,n}$ which has support only for those moduli for which covering maps exist. We conclude this section with a few comments.

- The above analysis was performed only with the left-moving degrees of freedom in mind. However, since our worldsheet theory has identical left- and right-moving sectors, the above argument will hold for the right-movers as well. The resulting prefactor will include not just $(a_i^{\Gamma})^{-h_i}$ but also $(\overline{a}_i^{\Gamma})^{-\overline{h}_i}$. This is precisely the full structre expected from the symmetric orbifold, see Appendix 5.A.

- The distribution $f^{(n-3)}$ is in principle any distribution with support on the appropriate subset of the moduli space $\mathcal{M}_{0,n}$. Ideally, one would want to show that f is indeed a delta-function, as to agree with the proposal of (5.2). For simple examples at small w_i (see [47]), and for 4-point functions (see [85]) it has been shown that f is indeed a delta function for those special cases.

- The correlation functions contain a prefactor W_{Γ} for each covering map Γ which is independent of $h_i, \overline{h}_i$, but which could depend on the spins j_i, the spectral flow w_i, and the locations x_i, z_i. For simple cases of small w_i, these prefactors can mostly be worked out by considering the local Ward identities for the fields $\mu^{\dagger}$ and $\lambda^{\dagger}$. These Ward identities are not as constraining as those for μ and λ, but they can be used to tease out some extra structure of the correlators, see the appendix of [47].

- The above argument only holds to leading order in string perturbation theory, i.e. for genus zero covering spaces. In the next section, we will see that it in fact holds for all genera.

5.3 Localization at higher genus

In the previous section, we only discussed the contributions to the correlation function

$$\left\langle \prod_{i=1}^{n} V^{w_i}_{h_i,j_i}(x_i,z_i) \right\rangle \tag{5.79}$$

when the worldsheet is a sphere. However, perturbative string theory in general contains contributions at all orders in perturbation theory. Specifically, the final object we would like to compute in perturbative string theory is given by a sum over all genera, namely

$$\sum_{g=0}^{\infty} g_s^{n+2g-2} \int_{\mathcal{M}_{g,n}} \left\langle \prod_{i=1}^{n} V^{w_i}(x_i,z_i) \right\rangle_g \,, \tag{5.80}$$

where the g subscript suggests that we are calculating the correlator on a surface of genus g.

The primary difficulty of repeating the above localization argument for higher-genus surfaces is that the insertion of $\lambda(z)$ and $\mu(z)$ into correlation functions does not produce a function on the surface Σ_g, but rather sections of line bundles. Otherwise, the arguments in this section are largely similar to those for $g = 0$. We collect some basic facts about line bundles on Riemann surfaces in Section 5.B.

We can set up the problem of computing (5.79) by considering n points z_i on some fixed Riemann surface Σ_g. We also fix a complex structure on Σ_g, so that we can do complex analysis on it, and we will integrate over the moduli space $\mathcal{M}_{g,n}$ of complex structures and marked points $\{z_1, \ldots, z_n\}$ afterwards. Since the OPEs between fields in a CFT are specified by local data (i.e. close-distance behavior), the OPEs we derived in the previous section are still completely valid in the case of higher-genus surfaces. That is, we still have

$$\begin{aligned}\lambda(z)\, V^{w_i}_{h_i,j_i}(x_i, z_i) &\sim (z - z_i)^{-\frac{w_i-1}{2}}\, V^{w_i}_{h_i+\frac{1}{2},j_i+\frac{1}{2}}(z_i) + \cdots \\ \mu(z)\, V^{w_i}_{h_i,j_i}(x_i, z_i) &\sim x_i(z - z_i)^{-\frac{w_i-1}{2}}\, V^{w_i}_{h_i+\frac{1}{2},j_i+\frac{1}{2}}(z_i) + \cdots\end{aligned} \tag{5.81}$$

as we take $z \to z_i$. Furthermore, the linear combination $\mu(z) - x_i\lambda(z)$ satisfies

$$(\mu(z) - x_i\lambda(z))\, V^{w_i}_{h_i,j_i}(x_i, z_i) \sim (z - z_i)^{\frac{w_i+1}{2}}\, V^{w_i}_{h_i-\frac{1}{2},j_i+\frac{1}{2}}(z_i) + \cdots\,. \tag{5.82}$$

The above OPEs can again be combined into the properties of two analytic objects $\mathscr{L}$ and $\mathscr{M}$ defined by

$$\mathscr{L}(z) := \left\langle \lambda(z) \prod_{i=1}^{n} V^{w_i}_{h_i,j_i}(x_i, z_i) \right\rangle , \quad \mathscr{M}(z) := \left\langle \mu(z) \prod_{i=1}^{n} V^{w_i}_{h_i,j_i}(x_i, z_i) \right\rangle . \tag{5.83}$$

The 'functions' $\mathscr{L}, \mathscr{M}$ now have poles of order $\frac{w_i-1}{2}$ at the punctures $z = z_i$, and the linear combination $\mathscr{L} - x_i\mathscr{M}$ has a zero of order $\frac{w_i+1}{2}$ at $z = z_i$. However, as alluded above, these objects are not simply functions on Σ_g, but sections of a meromorphic line bundle over Σ_g. Since λ, μ are defined as conformal primaries of weight $h = -\frac{1}{2}$, the objects $\mathscr{L}, \mathscr{M}$ are sections of the line bundle $\mathcal{L} = S^{-1}$, where S is a 'spin bundle' on Σ_g. Informally, the sections of the bundle $\mathcal{L}$ are locally described by meromorphic half-differentials

$$f(z)(\mathrm{d}z)^{-1/2}\,. \tag{5.84}$$

The exponent on $\mathrm{d}z$ is precisely needed to reproduce the transformation properties of λ, μ under local coordinate transformations.

Given a section of a line bundle $\mathcal{L}$, it is natural to associate a divisor D, which is a formal linear sum of its zeroes, counting multiplicity (a pole is considered in this language to be a zero of negative multiplicity). It is a basic fact about line bundles that the 'degree'

$$\deg(D) = \#\text{zeroes} - \#\text{poles} \tag{5.85}$$

is the same for every section of $\mathcal{L}$, and so it makes sense to talk about the degree $\deg(\mathcal{L})$ of the bundle $\mathcal{L}$, without making reference to a specific section. The degree of $\mathcal{L} = S^{-1}$ is

$$\deg(S^{-1}) = 1 - g\,. \tag{5.86}$$

This follows from the fact that $S^2 = K$ is the *canonical bundle* of meromorphic 1-forms on Σ_g, which has degree $\deg(K) = 2g - 2$.

Given that we know all of the zeroes and poles of $\mathscr{L}, \mathscr{M}$, we know that their divisors are given by

$$D(\mathscr{L}) = D_+(\mathscr{L}) - \sum_{i=1}^{n} \frac{w_i - 1}{2} z_i \,, \quad D(\mathscr{M}) = D_+(\mathscr{M}) - \sum_{i=1}^{n} \frac{w_i - 1}{2} z_i \,, \tag{5.87}$$

where $D_+(\mathscr{L})$ and $D_+(\mathscr{M})$ are the divisors which include only the zeroes of $\mathscr{L}$ and $\mathscr{M}$. Since the degree of divisors are linear, we have

$$\deg(D_+(\mathscr{L})) = \deg(D_+(\mathscr{M})) = 1 - g + \sum_{i=1}^{n} \frac{w_i - 1}{2} \,. \tag{5.88}$$

Let us call this number N.

It is another basic fact that the divisor of the section of a line bundle uniquely determines that section, up to an overall scaling factor. That is, if we know the N zeroes of $\mathscr{L}$, we can uniquely determine it up to a scale factor, and similarly for $\mathscr{M}$. However, only the relative scale factor between $\mathscr{L}, \mathscr{M}$ matters, since the overall scale factor can be absorbed by a conformal transformation. Thus, naively, we can say that the total number of 'free' parameters we have for $\mathscr{L}$ and $\mathscr{M}$ is given by

$$\text{d.o.f.}(\mathscr{L}, \mathscr{M}) \stackrel{?}{=} 2N + 1 \,. \tag{5.89}$$

However, this is not exactly right, since the divisors of $\mathscr{L}, \mathscr{M}$ cannot be chosen completely independently of each other. It is another fact (see Appendix 5.B) that the image of the divisors $D(\mathscr{L})$ and $D(\mathscr{M})$ must be mapped to the same point in the Jacobian $\text{Jac}(\Sigma_g)$. Effectively, this requirement imposes another g complex conditions, and we find

$$\text{d.o.f.}(\mathscr{L}, \mathscr{M}) = 2N + 1 - g = 3 - 3g + \sum_{i=1}^{n} (w_i - 1) \,. \tag{5.90}$$

Now, we can impose the constraints of the OPE (5.82), i.e.

$$\mathscr{M}(z) - x_i \mathscr{L}(z) \sim \mathcal{O}\left((z - z_i)^{(w_i + 1)/2} \right) \,. \tag{5.91}$$

This imposes $(w_i + 1)/2$ conditions for each marked point z_i, and so the number of constrained degrees of freedom is

$$\text{d.o.f.}(\mathscr{L}, \mathscr{M}) - \text{constraints} = 3 - 3g - n \,. \tag{5.92}$$

Finally, letting the complex structure and the marked points z_i vary, we have

$$\text{d.o.f.}(\mathscr{L}, \mathscr{M}) - \text{constraints} + \text{moduli} = 0 \,. \tag{5.93}$$

This tells us that the local Ward identities (encoded in the local properties of $\mathscr{L}$ and $\mathscr{M}$ near the marked points z_i) are enough to completely constrain the form of $\mathscr{L}, \mathscr{M}$ up to an irrelevant overall scaling constant. Put another way, the sections $\mathscr{L}, \mathscr{M}$ of $\mathcal{L}$ can only be thought to exist on a dimension-zero sublocus of the moduli space $\mathcal{M}_{g,n}$. By an argument similar to the one for $g = 0$, this automatically implies that the correlation functions (5.79) localize to the same sublocus.

Now what can we say about this sublocus? To answer this, let us assume that the moduli of the surface Σ_g are tuned in such a way that the sections $\mathscr{L}, \mathscr{M}$ do indeed exist. Then we can consider the ratio

$$\Gamma(z) = \frac{\mathscr{M}(z)}{\mathscr{L}(z)} \,. \tag{5.94}$$

Since $\mathscr{L}$ is not identically zero, $\Gamma(z)$ is a globally-defined meromorphic function on Σ_g. Furthermore, its divisor is

$$D(\Gamma) = D_+(\mathscr{M}) - D_+(\mathscr{L}) \,. \tag{5.95}$$

Since $\deg(D_+(\mathscr{M})) = \deg(D_+(\mathscr{M})) = N$, we see that Γ has N poles and N zeroes. Furthermore, we have

$$\Gamma(z) - x_i = \frac{\mathscr{M}(z) - x_i \mathscr{L}(z)}{\mathscr{L}(z)} \sim \mathcal{O}((z - z_i)^{w_i}) \,. \tag{5.96}$$

Thus, Γ has critical points at $z = z_i$ of order w_i. Furthermore, since

$$\deg(\Gamma) = N = 1 - g + \sum_{i=1}^{n} \frac{w_i - 1}{2} \,, \tag{5.97}$$

the Riemann-Hurwitz formula tells us that Γ has no other critical points, and so, assuming $\mathscr{L}, \mathscr{M}$ exist, we conclude that Γ is precisely a branched covering map $\Sigma_g \to \mathbb{CP}^1$ of degree N with critical points at $z = z_i$ of order w_i.

Similarly, we can run the same argument in reverse. Let Γ be a covering map $\Gamma : \Sigma_g \to \mathbb{CP}^1$ branched at $z = z_i$, with complex structure on Σ_g induced by that of $\mathbb{CP}^1$. Let Q_a, P_a for $a = 1, \ldots, N$ be the set of zeroes and poles, respectively, of Γ. Then Γ can be uniquely written down as

$$\Gamma(z) = C \prod_{a=1}^{N} \vartheta(z, Q_a) \vartheta(z, P_a)^{-1} \,, \tag{5.98}$$

where C is some constant and $\vartheta(x, y)$ is the *prime form* on Σ_g, a certain type of higher-genus generalization of the usual torus theta functions (see Appendix 5.B). Let us then define

$$\begin{aligned} \mathscr{L}(z) &= \left(\prod_{i=1}^{n} \vartheta(z, z_i)^{-\frac{w_i - 1}{2}} \right) \left(\prod_{a=1}^{N} \vartheta(z, P_a) \right) \sigma(z)^{-1} \,, \\ \mathscr{M}(z) &= C \left(\prod_{i=1}^{n} \vartheta(z, z_i)^{-\frac{w_i - 1}{2}} \right) \left(\prod_{a=1}^{N} \vartheta(z, Q_a) \right) \sigma(z)^{-1} \,, \end{aligned} \tag{5.99}$$

where σ is a multivalued $g/2$-form whose role is to make sure that $\mathscr{L}, \mathscr{M}$ are globally-defined $-\frac{1}{2}$ forms, and is formally defined in Appendix 5.B.[5]

[5]The condition of being globally-defined can be relaxed so that $\mathscr{L}, \mathscr{M}$ can pick up a minus sign when transported around the homology cycles of Σ_g. This just corresponds to a choice of spin structure on Σ_g, which is a choice we always leave free.

A necessary condition for the sections (5.99) being globally-defined is that the divisor of, say, $\mathscr{L}$, lies in the image of the bundle S^{-1} under the Abel-Jacobi map. That is,

$$\sum_{a=1}^{N}\mu(P_a)-\sum_{i=1}^{n}\frac{w_i-1}{2}\mu(z_i)=\mu(S^{-1})=-\frac{1}{2}\mu(K)\,, \tag{5.100}$$

where K is the canonical bundle $K=S^2$. Multiplying this expression by -2 gives

$$\sum_{i=1}^{n}(w_i-1)\mu(z_i)-2\sum_{a=1}^{N}\mu(P_a)=\mu(K)\,. \tag{5.101}$$

This is precisely the same condition that one would find if one demanded that $\partial\Gamma$ is a globally-defined one-form, since its zeroes are $z=z_i$ with multiplicity w_i-1, and its poles are $z=P_a$ with multiplicity 2. Finally, the requirement that $\mathscr{M}$ is globally-defined is

$$\sum_{i=1}^{n}(w_i-1)\mu(z_i)-2\sum_{a=1}^{N}\mu(Q_a)=\mu(K)\,. \tag{5.102}$$

This is equivalent to the condition for $\mathscr{L}$ since, if Γ is globally-defined, then

$$\sum_{a=1}^{N}\mu(Q_a)=\sum_{a=1}^{N}\mu(P_a)\,. \tag{5.103}$$

Finally, the condition that Γ has a critical point at $z=z_i$ with $\Gamma(z_i)=x_i$ can easily be shown to be equivalent to the OPEs (5.82). Therefore, the sections $\mathscr{L},\mathscr{M}$ constructed above constitute a solution to the Ward identities on the worldsheet, and we are left with the result:

> *The correlation functions* (5.79) *have support only on the dimension-zero sublocus of $\mathcal{M}_{g,n}$ for which branched covering maps $\Gamma:\Sigma_g\to\mathbb{CP}^1$ exist.*

Furthermore, by the same arguments shown at genus $g=0$, we see that the dependence of the correlation function on h_i takes the form

$$\left\langle\prod_{i=1}^{n}V^{w_i}_{h_i,j_i}(x_i,z_i)\right\rangle_{\Gamma}\propto\prod_{i=1}^{n}(a_i^{\Gamma})^{-h_i}\,, \tag{5.104}$$

where we have ignored right-moving contributions.

Non-renormalization

Finally, we end this section by commenting on an important feature of the correlation functions in the tensionless string. Given a set w_i of spectral flows, we know that the correlator only has nonzero contribution if the moduli of the surface Σ_g are chosen so that a covering map $\Gamma:\Sigma_g\to\mathbb{CP}^1$ exists. However, such covering maps do not exist for every choice of topology g. To see this, note that w_i cannot be greater than the degree N of the covering map for all i, since w_i is the number of *local* preimages of a point x near x_i, and the total number of preimages N cannot be smaller. Using the Riemann-Hurwitz relation, we find

$$w_j\leq 1-g+\sum_{i=1}^{n}\frac{w_i-1}{2}\,. \tag{5.105}$$

Rearranging, we can use this simple fact to find an upper-bound on the genus g:

$$g \leq \sum_{i\neq j} \frac{w_i - 1}{2} - \frac{w_j - 1}{2} \,. \tag{5.106}$$

Since we chose j to be arbitrary, we can obtain a tight upper bound by minimizing the right-hand-side over j, i.e.

$$g \leq \min_{j=1,\ldots,n} \left(\sum_{i\neq j} \frac{w_i - 1}{2} - \frac{w_j - 1}{2} \right) \,. \tag{5.107}$$

Practically, the above bound has a deep implication on the physics of the tensionless string [78, 79]. For any correlator of spectrally-flowed states, there is a maximum genus which contributes. Given that the full amplitude in string theory is given by

$$\mathcal{A}(x_1, \ldots, x_n) = \sum_{g=0}^{\infty} g_s^{2g-2} \int_{\mathcal{M}_{g,n}} \left\langle \prod_{i=1}^{n} V^{w_i}_{h_i,j_i}(x_i, z_i) \right\rangle \,, \tag{5.108}$$

the conclusion of the above bound on g is that there is a maximal order which contributes in g_s. The conclusion is that calculations in the tensionless string are *perturbatively exact*! This feature sets tensionless string theory apart from most examples of physical string theories, as there do not seem to be nontrivial non-perturbative objects like branes that appear in the calculation of any correlation function.[6]

5.4 Relationship to twistor theory

The free field realization of the PSU(1, 1|2) WZW model at level $k = 1$ has proven extremely computationally effective at constraining the correlation functions of tensionless strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$. However, the free field realization is somewhat lacking in geometric intuition. While the affine-current formulation of the PSU(1, 1|2) WZW model has an extremely geometric origin (as a sigma model on the superspace of $\mathrm{AdS}_3 \times \mathrm{S}^3$), in the free field variables, it is not at all clear what the target space of the string is. In this section, we make some statements in this regard.

Let us start with the purely bosonic part of the free field theory. We start with two $(-\frac{1}{2}, \frac{3}{2})$ first-order systems $(\lambda, \mu^\dagger)$ and $(\mu, \lambda^\dagger)$ with action[7]

$$S = \frac{1}{2\pi} \int_\Sigma \mathrm{d}^2 z \left(\mu^\dagger \overline{\partial} \lambda + \lambda^\dagger \overline{\partial} \mu \right) \,. \tag{5.109}$$

Furthermore, we gauge the overall current

$$U = \frac{1}{2} \left(\mu^\dagger \lambda + \lambda^\dagger \mu \right) \,. \tag{5.110}$$

[6] Strictly speaking, we can only make this conclusion for nonperturbative objects which do not alter the boundary conditions of the bulk AdS_3 geometry. As we will explore in the next chapter, there are indeed branes which can be considered in the tensionless string, but these branes have nontrivial interaction with the boundary conditions of AdS_3.

[7] We currently focus only on the holomorphic sector of the theory. Every statement we make has an obvious anti-holomorphic counterpart.

This current is generated by the global symmetry

$$(\lambda, \mu) \to \alpha(\lambda, \mu)\,, \quad (\lambda^\dagger, \mu^\dagger) \to \alpha^{-1}(\lambda^\dagger, \mu^\dagger)\,, \tag{5.111}$$

which clearly keeps the action invariant.

Now, from a Hamiltonian point of view, the conjugate momenta of (λ, μ) are $(\mu^\dagger, \lambda^\dagger)$. Thus, we can think of the free fields as parametrizing a 'phase space' whose 'position' coordinates are (λ, μ) and whose 'momentum' coordinates are $(\mu^\dagger, \lambda^\dagger)$. However, since we are gauging the current U, we need to think of these coordinates up to equivalence under the transformations (5.111). Thus, we can loosely think of the free field realization as describing a string theory moving in the target space

$$\mathbb{CP}^1 \cong (\lambda, \mu)/\sim\,, \tag{5.112}$$

i.e. the Riemann sphere. Implementing the equivalence relation $\sim$ at the level of a worldsheet theory equates naturally to the gauging of the current U.

It is tempting to interpret the projective space $\mathbb{CP}^1$ as being the boundary of AdS_3. Indeed, the semiclassical description of the tensionless string is that the string is 'glued' to the boundary. This gluing is in turn described by the covering map

$$\Gamma : \Sigma \to \partial(\mathrm{AdS}_3) \cong \mathbb{CP}^1 \tag{5.113}$$

which describes the motion of the string near the boundary. Since both the covering map Γ and the free fields (λ, μ) describe maps from the worldsheet to $\mathbb{CP}^1$, one might think that these two maps are equivalent. This is indeed the case, and is an immediate consequence of the incidence relation

$$\mu(z) - \Gamma(z)\lambda(z) = 0\,. \tag{5.114}$$

This relation tells us that the pair (λ, μ) describes the map Γ on $\mathbb{CP}^1$ in homogeneous coordinates, since

$$[\lambda(z) : \mu(z)] = \left[1 : \frac{\mu(z)}{\lambda(z)}\right] = [1 : \Gamma(z)] \tag{5.115}$$

as points on $\mathbb{CP}^1$.

The relation between homogeneous coordinate $[\lambda : \mu]$ in some complex projective space and a point $\Gamma(z) = x$ in two-dimensional Minkowski space is highly reminiscent of twistor theory. In the usual treatment of twistor theory in four dimensions, one consideres two spinorial fields $\mu^{\dot\alpha}, \lambda^\alpha$ with $\alpha = 1, 2$ being spinor indices in $\mathfrak{su}(2)$. The four components of these fields can be brought together into homogeneous coordiantes on $\mathbb{CP}^3$:

$$[\lambda^1 : \lambda^2 : \mu^{\dot 1} : \mu^{\dot 2}] \in \mathbb{CP}^3\,. \tag{5.116}$$

These variables are known as twistors. On the other hand, one can define a point in four-dimensional Minkowski space as a 2×2 Hermitian matrix via

$$x^{\dot\alpha}{}_\beta := x^\mu (\sigma_\mu)^{\dot\alpha}{}_\beta\,. \tag{5.117}$$

The fundamental relationship of twistor theory is that the twistors $\lambda^{\dot\alpha}, \mu^\alpha$ are related to points in Minkowski space via the *incidence relation*

$$\mu^{\dot\alpha} = x^{\dot\alpha}{}_\beta \lambda^\beta\,. \tag{5.118}$$

In our case, λ, μ therefore play the role of twistor variables that coordinatise the $\mathbb{CP}^1$. Indeed, we can include the fermions ψ^a in this analogy, and we find that the quadruple $[\lambda : \mu : \psi^1 : \psi^2]$, up to scaling, provide coordinates on the $\mathcal{N} = 2$ supertwistor space $\mathbb{CP}^{1|2}$ – the lower dimensional analogue of the 4D supertwistor space $\mathbb{CP}^{3|4}$. This is in fact not an accident, which can be seen by comparing the form of the action for the free-fields λ, μ to the form of the closed twistor string theory proposed by Berkovits [86] for 4D Yang-Mills theory. Defining

$$Z^I = (\lambda, \mu, \psi^1, \psi^2) \,, \quad Y_I = (\mu^\dagger, \lambda^\dagger, \psi_1^\dagger, \psi_2^\dagger) \,, \tag{5.119}$$

the action of the (left-moving) worldsheet theory is given by

$$S = \int \mathrm{d}^2 z \, Y_I \nabla Z^I \,, \tag{5.120}$$

where

$$\nabla = \overline{\partial} + \overline{A} \tag{5.121}$$

is the covariant antiholomorphic derivative with respect to an antiholomorphic gauge field $\overline{A}$. Indeed, expanding everything out, we see

$$S = \int \mathrm{d}^2 z \, \left(\mu^\dagger \overline{\partial} \lambda + \lambda^\dagger \overline{\partial} \mu + \psi_a^\dagger \overline{\partial} \psi^a + 2Z \, \overline{A}\right) \,. \tag{5.122}$$

The existence of the gauge field $\overline{A}$ is included so that the current

$$2Z = \mu^\dagger \lambda + \lambda^\dagger \mu + \psi_a^\dagger \psi^a \tag{5.123}$$

is gauged. This current is precisely the one we needed to gauge in order to reduce the free field worldsheet theory to the free field realization of $\mathfrak{psu}(1,1|2)_1$, and appears in an identical fashion in the worldsheet theory of twistor string theory [86]. The twistor string of [86] also admits a delta-function localization to the solutions of the incidence relation, making this analogy even stronger.[8] It would be interesting to understand this analogy deeper.[9]

5.A Correlators of the symmetric orbifold

In this section, we provide a derivation of correlation functions of the symmetric orbifold theory $\mathrm{Sym}^N(X)$ in terms of the data of the CFT X. In order to compute these correlation functions, we use the covering space approach of [80, 91], which instructs us to compute correlation functions in symmetric orbifold theories by summing over analytic maps $\Gamma : \Sigma \to \mathbb{CP}^1$ from (possibly disconnected) surfaces Σ to the CFT sphere $\mathbb{CP}^1$, with special 'branching' properties over certain critical points. In this appendix, we primarily consider the contributions to correlation functions which arise from connected spaces Σ, as these are meant to be dual to quantities computed from single worldsheets [77],[10] although we will make some comments about the disconnected contributions.

[8] A similar statement also applies to the twistor string theory of Witten [87]. See for instance, eq. (1) of [86] or eq. (1.1) of [88]. However, in that context the delta-function localization happens on the worldvolume of a D-string.

[9] A similar construction of the tensionless dual to free $\mathcal{N} = 4$ super Yang-Mills theory also admits a nearly identical analogy to twistor string theory, see [89, 90].

[10] This is similar to the case of AdS_5/CFT_4, for which single-trace operators in $\mathcal{N} = 4$ SYM are meant to be dual to single string worldsheets.

Twist fields in orbifold CFTs

Before we can explore the correlation functions of twist fields, we first need to define them. We follow the exposition of [92].

Recall that in an orbifold theory in the path integral formalism, the nontrivial field configurations Φ are those which satisfy twisted boundary conditions when transported around nontrivial cycles on the surface Σ on which the field theory lives. If, in addition to compact surfaces, we consider surfaces with punctures, each puncture acts as a generator of the fundamental group $\pi_1(\Sigma)$, and thus we can consider twisted boundary conditions around these punctures.

Consider for concreteness the Riemann sphere $\mathbb{CP}^1$ with punctures at points $x_1, \ldots, x_n$. We can use conformal symmetry to place one of the points, say x_1, at the origin. Now, let γ be a small loop circling $x = 0$ based at some point x. Since we are in an orbifold theory, we can consider the boundary condition

$$\Phi(\gamma \cdot x) = g \cdot \Phi(x) \,, \tag{5.124}$$

where $g \in G$ is some element of the orbifold group. For some fixed element $g \in G$, the above monodromy relation defines boundary conditions of the path integral which describe how fundamental fields behave near the point $x = 0$. In the path integral formalism, this procedure (defining boundary conditions near a puncture) is one way of constructing local fields. Thus, we can instead consider this boundary condition as stemming from a field σ_g placed at $x = 0$ which satisfies the condition

$$\Phi(\gamma \cdot x)\, \sigma_g(0) = g \cdot \Phi(x)\, \sigma_g(0) \,. \tag{5.125}$$

Such fields are known as *twist fields*, and they make up the interesting field content of an orbifold CFT. In the Hilbert space formulation of orbifolds (see Section 2.1), the twist fields are the local operators dual to the elements of the twisted sector of the orbifold Hilbert space.

It should be emphasized that the twist fields σ_g are not themselves local fields, since the induce nontrivial monodromies in the fundamental fields of the theory. They are more analogous to spin fields in fermionic CFTs [61], and their correlation functions will generically involve branch cuts. See, for example, [93] for an introduction to this phenomenon.

In addition to not being local, twist fields themselves are also not gauge-invariant objects. Recall that the gauge symmetries in a discrete orbifold are global symmetries of the form $\Phi \to h \cdot \Phi$ for some $h \in G$. Under this transformation, the monodromy condition between Φ and σ_g transforms to

$$\Phi'(\gamma \cdot x)\, \sigma_g(0) = (hgh^{-1}) \cdot \Phi'(x)\, \sigma_g(0) \,, \tag{5.126}$$

where $\Phi' = h \cdot \Phi$. Thus, σ_g acts as $\sigma_{hgh^{-1}}$ after a gauge transformation, and is therefore not gauge invariant unless g lies in the center of G. However, we can construct a gauge-invariant operator $\sigma_{[g]}$ associated to the *conjugacy class* of g in G. Specifically, we define

$$\sigma_{[g]} = \frac{1}{\sqrt{|[g]|}} \sum_{h \in [g]} \sigma_h \,, \tag{5.127}$$

where the sum is over all elements of the conjugacy class $[g]$. Since gauge transformations act on twist fields via conjugation, the field $\sigma_{[g]}$ is clearly left invariant. Furthermore, the prefactor is chosen so that the state $|\sigma_{[g]}\rangle$ is normalized.

Twist fields in the symmetric orbifold

In the symmetric orbifold theory, we can be a bit more concrete. Gauge-invariant twist fields will be labelled by their conjugacy class of S_N. Furthermore, it is well-known that the conjugacy classes of S_N are in one-to-one correspondence with partitions of N. Concretely, let

$$\sum_{i=1}^{N} i \cdot N_i = N \tag{5.128}$$

be a partition of N. Then the corresponding conjugacy class consists of those permutations whose cycle type contains N_1 cycles of length 1, N_2 cycles of length 2, etc. Let us write $\boldsymbol{N}$ as the shorthand for this partition. The size of the conjugacy class associated to $\boldsymbol{N}$ is given by

$$|[\boldsymbol{N}]| = \frac{N!}{\prod_{i=1}^{N} i^{N_i} N_i!}\,, \tag{5.129}$$

which can be derived via an application of the orbit-stabilizer theorem. We conclude that the gauge-invariant twist fields of the symmetric orbifold are given by

$$\sigma_{\boldsymbol{N}} = \sqrt{\frac{\prod_{i=1}^{N} i^{N_i} N_i!}{N!}} \sum_{g\in[\boldsymbol{N}]} \sigma_g\,. \tag{5.130}$$

The special case of most interest to us are the *single-cycle* twist fields. These are fields whose conjugacy class is composed of permutations of cycle type $(1\cdots w)$, i.e. which act as a single-cycle. For $w \geq 2$, the corresponding permutations have $N_w = 1$ and $N_1 = N - w$, and so

$$|[(1\cdots w)]| = \frac{N!}{w(N-w)!}\,, \tag{5.131}$$

and thus the twist fields are of the form

$$\sigma_w := \sqrt{\frac{w(N-w)!}{N!}} \sum_{g\in[(1\cdots w)]} \sigma_g\,. \tag{5.132}$$

One has to be careful about applying the above expression for $w = 1$. The permutations with cycle type $w = 1$ are simply the identity permutation, and so we have $\sigma_1 = \mathrm{id}$, which is the ground state of the symmetric orbifold theory. A naive application of the above formula, however, would give

$$\sigma_1 \overset{?}{=} \frac{1}{\sqrt{N}}\mathrm{id}\,, \tag{5.133}$$

which is not canonically normalized.

Correlators of twist fields

The primary quantities we are interested in are correlation functions of twist fields, specifically of single-cycle twist fields. For concreteness, we work on the Riemann sphere $\mathbb{CP}^1$, but this discussion follows for higher-genus correlators as well. The correlator

$$\langle \sigma_{w_1}(x_1)\cdots\sigma_{w_n}(x_n)\rangle \tag{5.134}$$

of n twisted-sector ground states with $w_i \geq 2$ is computed in the following way. We first expand the definition of each twist operator in terms of representatives of their conjugacy class, obtaining

$$\langle \sigma_{w_1}(x_1) \cdots \sigma_{w_n}(x_n) \rangle = \prod_{i=1}^{n} \left(\mathcal{N}_{w_i} \sum_{g \in [(1\dots w_i)]} \right) \langle \sigma_{g_1}(x_1) \cdots \sigma_{g_n}(x_n) \rangle \ , \tag{5.135}$$

where $\mathcal{N}_{w_i} = \sqrt{w_i(N-w_i)!/N!}$ is the normalization factor for σ_{w_i}. Next, we compute the correlators of non-gauge-invariant twist fields on the right-hand side, and sum up the result. The computation of $\langle \sigma_{g_1}(x_1) \cdots \sigma_{g_n}(x_n) \rangle$ can be now done using the same trick we used in Chapter 2 to compute partition functions of the symmetric orbifold theory: we move to a covering space where the fundamental fields Φ^i of the seed theory are single-valued.

To get an idea for what such a covering space might look like, let us take an example of $\sigma_{(1\cdots w)}(0)$. In the path integral, the fundamental fields Φ^i, when transported around 0 counter-clockwise, will have the following monodromy property

$$\Phi^1 \mapsto \Phi^2 \, , \quad \Phi^2 \mapsto \Phi^3 \, , \quad \dots \, , \quad \Phi^w \mapsto \Phi^1 \, , \quad \Phi^i \mapsto \Phi^i \, , \quad \text{if } i > w \, . \tag{5.136}$$

Thus, the covering space on which the fields are single-valued will locally look like a branched cover of $\mathbb{CP}^1$ around $x = 0$. Let Σ be the covering space and let $\Gamma : \Sigma \to \mathbb{CP}^1$ be the (branched) cover map. Σ naturally inherits a complex structure from $\mathbb{CP}^1$, so we can do complex analysis on Σ. Locally, we want the inverse of Γ to have a branch cut of the form

$$\Gamma^{-1}(x) \sim z_0 + x^{1/w} \, , \quad x \to 0 \, , \tag{5.137}$$

so that going around the z_0 w times takes us back to the original preimage. Inverting this relationship, we demand that, near $z = z_0$, $\Gamma(z) \sim (z - z_0)^w$.

Returning to our correlation function, we require that the appropriate covering map Γ satisfies

$$\Gamma(z) \sim x_i + a_i^{\Gamma}(z - z_i)^{w_i} + \cdots \ , \tag{5.138}$$

where a_i^{Γ} are some constants, and $z_i \in \Sigma$ are some points on the covering space over which Γ is branched. It turns out that if $\Sigma \setminus \{z_1, \dots, z_n\}$ is stable (i.e. has a finite number of automorphisms), then the above local requirements for Γ are enough to determine Γ *globally*, up to a discrete choice. Put another way, there are generically only a finite number of covering maps Γ satisfying the above local analytic conditions.

Generically, Σ does not have to be a connected space. For example, the correlation function

$$\langle \sigma_{(123)}(x_1)\sigma_{(321)}(x_2)\sigma_{(45)}(x_3)\sigma_{(45)}(x_4) \rangle \tag{5.139}$$

is evaluated by moving to a covering space which has $N-3$ disconnected components, each of which are topologically a sphere. The first sphere is branched over x_1, x_2 with $w = 3$, the second sphere is branched over x_3, x_4 with $w = 2$, and the other $N-5$ spheres are not branched, and we can consider the covering map Γ restricted to these spheres to be the identity. In this case, we say that the number of *active* colors n_c is 5, since only 5 elements of $\{1, \dots, N\}$ are involved in the above permutations. For a correlator with n_c active colors, the covering space will decompose as

$$\Sigma \cong \Sigma_0 \times \underbrace{(\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1)}_{N-n_c \text{ times}} \, , \tag{5.140}$$

where the covering map Γ is trivial (i.e. the identity map) on the $N - n_c$ Riemann spheres, and is nontrivial on Σ_0. In our example above, $\Sigma_0 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ with a nontrivial covering map branched over x_1, x_2, x_3, x_4. Since we normalize the partition function of the sphere $\mathbb{CP}^1$ to be $Z(\mathbb{CP}^1) = 1$, we can safely ignore the $N - n_c$ trivial covering spaces and focus entirely on the nontrivial component Σ_0.

We are most interested in the case where Σ_0 is connected. This is only possible if the permutations $g_1, \ldots, g_n$ in (5.135) form a *transitive subgroup* of $S_{n_c} \subset S_N$. For example, the correlator

$$\left\langle \sigma_{(1345)}(x_1)\sigma_{(475)}(x_2)\sigma_{(57)}(x_3)\sigma_{(153)}(x_4)\right\rangle \tag{5.141}$$

defines a connected covering space Σ_0 and has $n_c = 5$ active colors. In the case of connected covering spaces Σ_0, we can compute the genus of the covering space easily by the Riemann-Hurwitz formula, which states

$$g = 1 - n_c + \sum_{i=1}^{n} \frac{w_i - 1}{2}\,, \tag{5.142}$$

and a careful analysis of the large-N behavior of the above expansion shows that a covering map of genus g gives a contribution of the form

$$N^{1-g-n/2} = N^{-\chi(g,n)/2}\,. \tag{5.143}$$

Thus, the large N expansion of correlators in the symmetric orbifold theory has precisely the form of a genus expansion in a string theory!

The conformal anomaly

Based on the above discussion, it would naively make sense to conclude that the connected part of correlation functions of the symmetric orbifold take the form

$$\langle \sigma_{w_1}(x_1) \cdots \sigma_{w_n}(x_n)\rangle_{\text{conn.}} \overset{?}{=} \sum_{g=0}^{\infty} \sum_{\Gamma:\Sigma\to\mathbb{CP}^1} N^{-\chi(g,n)/2} Z_X(\Sigma)\,, \tag{5.144}$$

where the sum is over *connected* covering spaces Γ of genus g branched over x_i, and $Z_X(\Sigma)$ is the partition function of the seed theory on Σ. However, there is a caveat which has to be addressed: the partition function $Z_X(\Sigma)$ in the above formula is computed with respect to the pullback metric under Γ. Let us pick the standard almost-everywhere-flat metric $\mathrm{d}s^2 = \mathrm{d}x\,\mathrm{d}\overline{x}$ on $\mathbb{CP}^1$, which has a delta-function singularity of curvature at $x = \infty$. Then the pullback metric under $\Gamma : \Sigma \to \mathbb{CP}^1$ is given by

$$\mathrm{d}s^2_\Sigma = \mathrm{d}\Gamma\,\mathrm{d}\overline{\Gamma} = |\partial\Gamma|^2 \mathrm{d}z\,\mathrm{d}\overline{z}\,. \tag{5.145}$$

However, we would like to compute partition functions of the seed theory X on Σ with respect to the almost flat metric,[11] which locally takes the form

$$\mathrm{d}\tilde{s}^2_\Sigma = \mathrm{d}z\,\mathrm{d}\overline{z}\,. \tag{5.146}$$

These two metrics are related to each other by a Weyl transformation

$$\mathrm{d}s^2_\Sigma = e^{\Phi}\mathrm{d}\tilde{s}^2_\Sigma\,, \quad \Phi = \log\partial\Gamma + \log\overline{\partial\Gamma}\,. \tag{5.147}$$

[11]The main reason for this is so that when the covering space Σ is a sphere, we can normalize $Z_X(\Sigma) = 1$.

Although the CFT X is classically ignorant of the choice of conformal class of metric, the trace anomaly implies that the partition function evaluated with respect to the metric $\mathrm{d}s^2_\Sigma$ is going to be related to that computed with respect to $\mathrm{d}\tilde{s}^2_\Sigma$ via the conformal anomaly

$$Z_X(\Sigma, \mathrm{d}s^2_\Sigma) = e^{-cS_L[\Phi]/6} Z_X(\Sigma, \mathrm{d}\tilde{s}^2_\Sigma)\,, \tag{5.148}$$

where $S_L[\Phi]$ is the Liouville action

$$S_L[\Phi] = \frac{1}{8\pi}\int_\Sigma \mathrm{d}^2z\sqrt{\tilde{g}}\left(-\frac{1}{2}\Phi\widetilde{\Delta}\Phi + \widetilde{R}\Phi\right)\,, \tag{5.149}$$

where $\sqrt{\tilde{g}}, \widetilde{\Delta}, \widetilde{R}$ are the volume element, Laplacian, and scalar curvature of the metric $\mathrm{d}\tilde{s}^2_\Sigma$, respectively.

Finally, there is one more subtlety which has to be taken into account. The twisted-sector ground states are conformal primaries of weight

$$h_w = \bar{h}_w = \frac{c(w^2-1)}{24w}\,, \tag{5.150}$$

and thus when pulled back to the covering surface Σ to not reproduce exactly the ground state of X, but rather

$$(\Gamma^*\sigma_w)(z_i) = |\partial\Gamma(z_i)|^{-h_w-\bar{h}_w}\mathbf{1} \propto |w_i a_i^\Gamma|^{-h_w-\bar{h}_w}\mathbf{1}\,, \tag{5.151}$$

where $\mathbf{1}$ is the ground state of X. The $\propto$ symbol is used since, strictly speaking, $\partial\Gamma$ vanishes at $z = z_i$, and the pullback of σ_w is thus formally divergent. However, this divergence can just be absorbed into the correlation function, since it is universal for all covering maps Γ.

Thus, we can put everything together and we find that the connected part of the twisted sector correlator is given by

$$\begin{aligned}&\langle\sigma_{w_1}(x_1)\cdots\sigma_{w_n}(x_n)\rangle_{\text{conn.}}\\ &\quad= \sum_{g=0}^{\infty} N^{-\chi(g,n)/2}\sum_{\Gamma:\Sigma\to\mathbb{CP}^1} e^{-cS_L[\Phi_\Gamma]/6}\prod_{i=1}^{n}|w_i a_i^\Gamma|^{-h_w-\bar{h}_w}Z_X(\Sigma)\,.\end{aligned} \tag{5.152}$$

Evaluating the conformal anomaly

Given a covering space $\Gamma : \Sigma \to \mathbb{CP}^1$, the conformal anomaly we want to calculate is determined by the Liouville action

$$S_L = \frac{1}{8\pi}\int_\Sigma \mathrm{d}^2z\,\sqrt{\tilde{g}}\left(-\frac{1}{2}\Phi\widetilde{\Delta}\Phi + \widetilde{R}\Phi\right)\,, \tag{5.153}$$

where, as mentioned above, we take the metric $\mathrm{d}\tilde{s}^2_\Sigma$ to be the pullback $\Gamma^*(\mathrm{d}s^2)$ of the metric on $\mathbb{CP}^1$, multiplied by the Weyl factor $e^{-\Phi}$ with

$$\Phi = \log\partial\Gamma + \log\overline{\partial\Gamma}\,. \tag{5.154}$$

Our task is to explicitly calculate the Liouville action in terms of the covering map data Γ.

Before, performing the calculation, we will find it somewhat simpler to work with the pullback metric $\mathrm{d}s^2_\Sigma = \Gamma^*(\mathrm{d}s^2)$. Since

$$\mathrm{d}s^2_\Sigma = e^\Phi \mathrm{d}\tilde{s}^2_\Sigma \,, \tag{5.155}$$

we can easily write down the volume element, laplacian, and Ricci curvature with respect to $\mathrm{d}s^2_\Sigma$ as

$$\sqrt{\widetilde{g}} = e^{-\Phi}\sqrt{g} \,, \quad \widetilde{\Delta} = e^\Phi \Delta \,, \quad \widetilde{R} = e^\Phi R + e^\Phi \Delta\Phi \,. \tag{5.156}$$

Thus, we can write the conformal anomaly as

$$S_L = \frac{1}{8\pi}\int_\Sigma \mathrm{d}^2 z \sqrt{g}\left(\frac{1}{2}\Phi\Delta\Phi + R\Phi\right) . \tag{5.157}$$

That is, working with $\mathrm{d}s^2_\Sigma$ instead of $\mathrm{d}\tilde{s}^2_\Sigma$ has the effect of reversing the kinetic term in the Liouville action.

The computation of this anomaly is somewhat tricky, since the metric g is singular at the insertion points z_i (as well as at the poles λ_a of Γ, assuming one picks the specific metric on $\mathbb{CP}^1$ that we choose below), and as such the curvature R, as well as the Laplacian $\Delta\Phi$, has delta-function singularities at these points. Nevertheless, the form of the conformal anomaly was calculated carefully in [80], by considering a regulating process defined by considering small disks $\mathbb{D} = (\cup_i \mathbb{D}_i) \cup (\cup_a \mathbb{D}_a)$ surrounding the zeroes z_i and the poles λ_a of $\partial\Gamma$. These disks contain singularities in the metric g, and so removing them acts as a regulator. One then 'replaces' the metric $g_\mathbb{D}$ on these disks with a smooth metric $\widetilde{g}_\mathbb{D}$ which has the same total curvature as $g_\mathbb{D}$ (as to satisfy the Gauss-Bonnet theorem), but 'smoothed out' uniformly over the areas of the components of $\mathbb{D}$. One then computes the conformal anomaly with the new smooth metric $\widetilde{g}$, and takes the radii of these disks to zero while keeping track of various divergences, and obtains, as an end result, a finite answer. Here, we will present a much less careful, heuristic method for computing the conformal anomaly, which nevertheless gives the correct answer.

We can calculate the conformal anomaly by explicitly choosing the metric on $\mathbb{CP}^1$ to be flat everywhere except for a singularity at $z = \infty$ which contains 2 units of Ricci curvature. That is, we take the curvature on the sphere to be delta-function localized at infinity. The pullback of this metric is given by

$$\mathrm{d}s^2 = |\partial\Gamma|^2 \,\mathrm{d}z\,\mathrm{d}\bar{z} = e^\Phi \mathrm{d}z\,\mathrm{d}\bar{z} \,. \tag{5.158}$$

From this metric, we can easily calculate the curvature to be

$$R = -\Delta\Phi \,, \tag{5.159}$$

and thus, the full Liouville action can be reduced to

$$S_L = \frac{1}{16\pi}\int_\Sigma \mathrm{d}^2 z \,\sqrt{g}\,R\,\Phi \,. \tag{5.160}$$

Now, we can actually calculate this expression rather easily. We do this by noting that the curvature R is the pullback of the curvature on $\mathbb{CP}^1$ on Γ. This uniquely determines the curvature to be of the form

$$\sqrt{g}\,R = 8\pi\sum_{a=1}^{N}\delta^{(2)}(z,\lambda_a) - 4\pi\sum_{i=1}^{n}(w_i - 1)\delta^{(2)}(z,z_i) \,. \tag{5.161}$$

Here, λ_a are the poles of the covering map Γ. Intuitively, this expression tells us that, since λ_a is the preimage of $x = \infty$ on $\mathbb{CP}^1$, each pole should carry 8π units of curvature on Σ. The branch points then z_i carry $-4\pi(w_i - 1)$ units of curvature. This can also be explicitly verified by calculating the Laplacian of Φ. The result in the end is that we have a delta-function supported curvature, and so the Liouville action can be directly evaluated to be

$$S_L = \frac{1}{2}\sum_{a=1}^{N}\Phi(\lambda_a) - \frac{1}{4}\sum_{i=1}^{n}(w_i - 1)\Phi(z_i)\,. \tag{5.162}$$

The conformal anomaly associated to the correlation functions of $\text{Sym}^N(X)$ for a seed theory X with central charge c is then

$$e^{-cS_L/6} = \prod_{i=1}^{n} e^{c(w_i-1)\Phi(z_i)/24}\prod_{a=1}^{N} e^{-c\Phi(\lambda_a)/12}\,. \tag{5.163}$$

Now, strictly speaking, the above expression is poorly defined, since Φ is singular at z_i and λ_a. A careful regulization of this expression is possible, see [80]. However, we can be cavalier about the various infinite constants arising in the expansion and attempt to evaluate the expression naively anyway. Let us parametrize

$$\partial\Gamma(z) = C\prod_{i=1}^{n}(z - z_i)^{w_i-1}\prod_{a=1}^{N}(z - \lambda_a)^{-2}\,. \tag{5.164}$$

Then the Liouville field Φ is given, up to an overall constant, by

$$\Phi = \sum_{i=1}^{n}(w_i - 1)\log|z - z_i|^2 - 2\sum_{a=1}^{N}\log|z - \lambda_a|^2\,. \tag{5.165}$$

Thus, we have

$$\Phi(z_i) = \sum_{j\neq i}^{n}(w_j - 1)\log|z_i - z_j|^2 - 2\sum_{a=1}^{N}\log|z_i - \lambda_a|^2 + \cdots\,, \tag{5.166}$$

where the $\cdots$ represents irrelevant constants (which are formally infinite). Exponentiating gives

$$\begin{aligned} e^{\Phi(z_i)} &\propto \left|\prod_{j\neq i}^{n}(z_i - z_j)^{w_j-1}\prod_{a=1}^{N}(z_i - \lambda_a)^{-2}\right|^2 \\ &\propto \left|\lim_{z\to z_i}\frac{\partial\Gamma(z)}{(z - z_i)^{w_i-1}}\right|^2 = w_i^2\left|a_i^\Gamma\right|^2\,, \end{aligned} \tag{5.167}$$

where a_i^Γ is the first nontrivial coefficient of Γ when expanded around $z = z_i$, i.e.

$$\Gamma(z) \sim x_i + a_i^\Gamma(z - z_i)^{w_i}\,. \tag{5.168}$$

Moreover, we have

$$\begin{aligned} e^{\Phi(\lambda_a)} &\propto \left|\prod_{i=1}^{n}(\lambda_a - z_i)^{w_i-1}\prod_{b\neq a}^{N}(\lambda_a - \lambda_b)^{-2}\right|^2 \\ &\propto \left|\lim_{z\to\lambda_a}(z - \lambda_a)^2\partial\Gamma(z)\right|^2 = \left|C_a^\Gamma\right|^2\,, \end{aligned} \tag{5.169}$$

where C_a^Γ is the residue of Γ at $z=\lambda_a$.

Putting everything together, we find that, at least according to a naive calculation (with little regards for convergence issues), the conformal anomaly takes the form

$$e^{-cS_L/6} \propto \prod_{i=1}^{n} \left|w_i a_i^\Gamma\right|^{c(w_i-1)/12} \prod_{a=1}^{N} \left|C_a^\Gamma\right|^{-c/6} . \tag{5.170}$$

We can now finally compute the correlation function of twisted-sector operators in the symmetric orbifold. We have

$$\begin{aligned} \left\langle \prod_{i=1}^{n} \mathcal{O}_i^{(w_i)}(x_i) \right\rangle_\Gamma &= e^{-cS_L/6} \left\langle \prod_{i=1}^{n} \Gamma^* \mathcal{O}_i^{(w_i)}(z_i) \right\rangle \\ &\propto \prod_{i=1}^{n} \left|w_i a_i^\Gamma\right|^{c(w_i-1)/12} \prod_{a=1}^{N} \left|C_a^\Gamma\right|^{-c/6} \prod_{i=1}^{n} |\partial\Gamma(z_i)|^{-h_i-\bar{h}_i} \left\langle \prod_{i=1}^{n} \mathcal{O}_i(z_i) \right\rangle \\ &\propto \prod_{i=1}^{n} \left|w_i a_i^\Gamma\right|^{c(w_i-1)/12-h_i-\bar{h}_i} \prod_{a=1}^{N} |C_a^\Gamma|^{-c/6} \left\langle \prod_{i=1}^{n} \mathcal{O}_i(z_i) \right\rangle , \end{aligned} \tag{5.171}$$

where the Γ subscript picks out the contribution to the correlator stemming from the covering map Γ. The result is simply an analytic prefactor multiplied by a correlator of the seed theory X. As an explicit example, let us take $\mathcal{O}_i^{(w_i)}$ to be a twisted-sector ground state σ_{w_i}. Its conformal weight is

$$h_i = \bar{h}_i = \frac{c(w_i^2-1)}{24 w_i} , \tag{5.172}$$

and the corresponding operator in the seed theory is just the identity operator. Thus,

$$\left\langle \prod_{i=1}^{n} \sigma_{w_i}(x_i) \right\rangle_\Gamma = W_\Gamma \prod_{i=1}^{n} \left|w_i a_i^\Gamma\right|^{-c(w_i-1)/12w_i} \prod_{a=1}^{N} |C_a^\Gamma|^{-c/6} Z_X(\Sigma) , \tag{5.173}$$

where $Z_X(\Sigma)$ is the partition function of the seed theory X on the covering space Σ,[12] and W_Γ is an overall coefficient that can be determined by a more careful analysis of the divergent contributions to the Liouville action. As noted in [92], this pre-factor is expected to be independent of the specific covering map Γ, but we leave it in for completeness. This result is consistent with those of [77, 80, 92], up to an overall prefactor which depends only on the twists w_i.

Putting everything together

We can now put everything together and write down the complete (connected) symmetric orbifold correlation function. Formally, we are instructed to sum over all covering maps $\Gamma : \Sigma \to \mathbb{CP}^1$, weighted by a factor of $N^{1-g-n/2}$. Thus, we can write

$$\left\langle \prod_{i=1}^{n} \sigma_{w_i}(x_i) \right\rangle_c = \sum_{\Gamma:\Sigma\to\mathbb{CP}^1} N^{1-g-n/2} W_\Gamma \prod_{i=1}^{n} \left|w_i a_i^\Gamma\right|^{-c(w_i-1)/12w_i} \prod_{a=1}^{N} |C_a^\Gamma|^{-c/6} Z_X(\Sigma) . \tag{5.174}$$

[12] If Σ is the sphere, it is common to normalize $Z_X(\Sigma)=1$. Although all of our analysis was done at $g=0$, the results extend easily to higher genus.

5.B Some Riemann surface theory

In this appendix, we introduce some of the main results of the study of compact Riemann surfaces. In particular, we introduce meromorphic functions and forms, divisors, spin structures, Abel's theorem, and the construction of meromorphic functions from their divisors. We will assume that the reader has a basic understanding of Riemann surfaces as one dimensional complex manifolds. For an excellent introduction to the theory of Riemann surfaces, see [94]. For the most part, we follow the notation and conventions of [95].

Before diving into the theory, let us first fix our conventions. Consider a genus g Riemann surface Σ. We can choose a basis for the homology group $H_1(\Sigma, \mathbb{Z})$ as in Figure 5.2, consisting of cycles α_μ, β_μ with the intersection numbers

$$\alpha_\mu \cap \alpha_\nu = 0 \ , \quad \beta_\mu \cap \beta_\nu = 0 \ , \quad \alpha_\mu \cap \beta_\nu = \delta_{\mu\nu} \ . \tag{5.175}$$

This choice of homology basis is essentially unique, up to a $\mathrm{Sp}(2g, \mathbb{Z})$ transformation which leaves these intersections invariant. All of the analysis we will do in this text is independent of the particular basis α_μ, β_μ, but we fix one for concreteness. A Riemann surface Σ together with a choice of homology basis α_μ, β_μ is called a *marked* Riemann surface. A marked Riemann surface Σ is moreover homeomorphic to a $4g$-gon whose edges are α_μ and β_μ, glued together as in Figure 5.2.

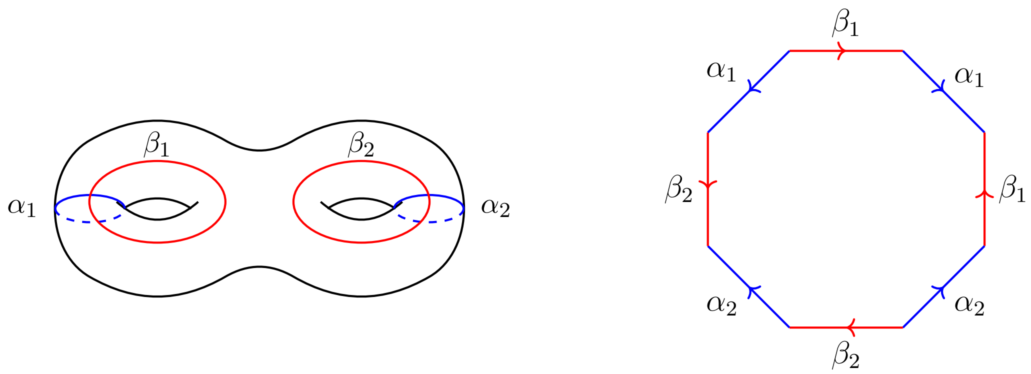

Figure 5.2: Left: A genus 2 surface with canonical homology basis $\alpha_1, \alpha_2, \beta_1, \beta_2$. Right: The same surface modelled as an $4g = 8$-gon with identified edges representing the α and β cycles.

Furthermore, the surface Σ admits a basis of g linearly independent holomorphic one-forms $\omega_1, \ldots, \omega_g$. One can choose this basis such that

$$\oint_{\alpha_\mu} \omega_\nu = \delta_{\mu\nu} \ . \tag{5.176}$$

That is, such that ω_μ is dual to α_μ. Once such a basis is picked, the integral of ω_ν over the cycle β_μ is left unconstrained. We thus define a complex $g \times g$ matrix Ω, the so-called *period matrix*, via

$$\Omega_{\mu\nu} = \oint_{\beta_\mu} \omega_\nu \ . \tag{5.177}$$

A fundamental result in the theory of Riemann surfaces, the Riemann bilinear relations, implies that Ω is a symmetric matrix and that $\mathrm{Im}(\Omega)$ is a positive-definite matrix.

Meromorphic forms

With our basic notions out of the way, we can now introduce the primary object of study – meromorphic forms on the Riemann surface Σ.

On a generic manifold, a differential 1-form is a linear combination $A_\mu(x)\mathrm{d}x^\mu$ of the basis vectors $\mathrm{d}x^\mu$ with coefficients being functions on the manifold. On a Riemann surface, the appropriate definition would be either $f(z,\overline{z})\,\mathrm{d}z$, $f(z,\overline{z})\,\mathrm{d}\overline{z}$, or some linear combination thereof. A *meromorphic form of weight 1* on the surface Σ is a form which can be written locally as $f(z)\,\mathrm{d}z$ for some meromorphic function $f(z)$. Similarly, an anti-meromorphic form of weight 1 would be expressed as $f(\overline{z})\,\mathrm{d}\overline{z}$. In geometric terms, one can consider a meromorphic form to be a section of a certain line bundle over Σ, which we will call K (often called the *canonical line bundle*).[13] The space of sections $\Gamma(K)$ on K is often written as $\mathcal{M}^1(\Sigma)$. That is, $\mathcal{M}^1(\Sigma)$ is the space of meromorphic forms of weight 1 on Σ.

We can construct meromorphic forms of other weights as well. In general, a meromorphic form of weight h is an object locally expressible as $f(z)\,(\mathrm{d}z)^h$. Such an object is a section of the line bundle K^h, and we call the space of such forms $\mathcal{M}^h(\Sigma)$. The space $\mathcal{M}^0(\Sigma)$ is just the space of meromorphic functions, and will often be abbreviated as $\mathcal{M}(\Sigma)$ (without the superscript). We say a meromorphic form is *holomorphic* if it is locally expressable as a holomorphic function $f(z)\,(\mathrm{d}z)^h$ – i.e. if it has no poles. We denote the space of holomorphic forms of weight h by $\mathcal{O}^h(\Sigma)$.

Now, more abstractly, one can specify a meromorphic form to be an object $\omega(z)$ which, under the conformal coordinate transformation $z \to \widetilde{z}$, transforms according to the rule

$$\omega(\widetilde{z}) = \left(\frac{\mathrm{d}\widetilde{z}}{\mathrm{d}z}\right)^h \omega(z)\ . \tag{5.178}$$

Thus, the relationship between meromorphic forms on Σ and conformal field theory becomes quite evident – the transformation law of a meromorphic form of weight h is precisely the data of a chiral field of conformal weight h. That is,

$$\text{meromorphic forms of weight } h \Longleftrightarrow \text{conformal fields of weight } h\ . \tag{5.179}$$

This relationship is precisely why the meromorphic forms are useful in the study of conformal field theory on Riemann surfaces.

Divisors

Now that we have defined meromorphic forms, we turn our head to divisors. A *divisor* is a formal (finite) linear sum of points on the surface Σ, with coefficients in $\mathbb{Z}$ (or $\mathbb{Z}/2$, for spinor forms). That is, given a collection of points $a_i \in \Sigma$, we define the divisor D to be

$$D = \sum_i n_i a_i \tag{5.180}$$

for some integers n_i. Given two divisors D and D', their sum is given by adding their coefficients (where the coefficient of a point is taken to be zero if it does not contribute to the sum). Thus, the set of divisors on Σ form a free Abelian group $\mathrm{Div}(\Sigma)$ generated by points on Σ. The degree $\deg(D)$ of a divisor is defined to be the sum of its coefficients.

[13] We will later also use the letter K for the divisor class of this bundle. It is unfortunately quite common to denote line bundles and divisor classes with the same letters.

A divisor's place in life is to represent the set of zeroes and poles of some meromorphic function $f(z)$. That if, given a meromorphic function $f(z)$ with zeroes of order n_i at points a_i and poles of order m_i at the points b_i, we can associate a divisor (f) to $f(z)$ given by

$$(f) = \sum_i n_i a_i - \sum_i m_i b_i \ . \tag{5.181}$$

That is, the divisor of f is the set of its zeroes and poles, counting multiplicities, where a pole is treated as a zero of negative order. If f has no poles, the divisor (f) has all positive coefficients. We call a divisor with no negative coefficients a *positive divisor*, for obvious reasons.

An immediate consequence of the definition of the divisor of a meromorphic function is that the divisor of the function $f(z)g(z)$ is the sum of the divisors of $f(z)$ and $g(z)$, i.e.

$$(f \cdot g) = (f) + (g) \ . \tag{5.182}$$

That is, the map $(\cdot) : \mathcal{M}(\Sigma) \to \mathrm{Div}(\Sigma)$ is a homomorphism of abelian groups.

We can also talk about divisors of meromorphic forms of weight h in exactly the same way, by counting the zeroes and poles weighted by their multiplicity. That is, there is a map $(\cdot) : \mathcal{M}^h(\Sigma) \to \mathrm{Div}(\Sigma)$ in the same way as there is for meromorphic functions.

It is often too cumbersome to study the space of divisors itself, since it is an enormous Abelian group. A much smaller group can be generated by quotienting $\mathrm{Div}(\Sigma)$ by the subset of divisors which are the vanishing sets of some meromorphic function. Specifically, we call a divisor $P \in \mathrm{Div}(\Sigma)$ a *principal divisor* if $P = (f)$ for some meromorphic function f. We define an equivalence relation $\sim$ by

$$D \sim D' \iff D = D' + P \text{ for some principal divisor } P \ . \tag{5.183}$$

The space $\mathrm{Div}(\Sigma)/\sim$ is called the *divisor class group*, and is typically denoted $\mathrm{Cl}(\Sigma)$. One particularly nice feature of the divisor class group is that any two meromorphic forms of the same (integer) weight h have the same divisor, up to equivalence. This is simple to see since, given meromorphic forms $f(z)\,(\mathrm{d}z)^h$ and $g(z)\,(\mathrm{d}z)^h$, their ratio is a well-defined meromorphic function. Thus, to each bundle K^h, we can associate a unique element of $\mathrm{Cl}(\Sigma)$.[14] For this reason, we denote the divisor class of differential forms of weight 1 by K, the so-called *canonical class* on Σ.

A central result is that the elements of the trivial divisor class (i.e. the divisors of meromorphic functions) have degree zero. Put plainly, if a meromorphic function has zeroes at a_i of order n_i and poles at b_i of order m_i, then

$$\sum_i n_i = \sum_i m_i \ . \tag{5.184}$$

Put another way, if f is a meromorphic function, then $\deg((f)) = 0$.[15] As a consequence, the elements of any divisor class all have the same degree, since they differ by the divisor of a meromorphic function, whose degree vanishes. Since the divisor classes we care about are of the form $h \cdot K$ for some integer h, we know the degree

[14] As it turns out, this correspondence is bijective. Specifically, the divisor class group is isomorphic to the *Picard group* $\mathrm{Pic}(\Sigma)$ of equivalence classes of holomorphic line bundles on Σ.

[15] This is not true for non-compact Riemann surfaces. For example, on $\mathbb{C}$ the function $f(z) = z$ has one zero but no poles, and thus $\deg((f)) = 1$.

of all meromorphic forms if we know the degree of those forms with weight $h = 1$. This degree is dependent on the genus g of the surface Σ, and is known to be

$$\deg(K) = 2g - 2\ . \tag{5.185}$$

Thus, since the degree of the sum of two divisors is just the sum of the degrees, the degree of a meromorphic form of weight h is simply

$$\deg(h \cdot K) = (2g - 2)h\ . \tag{5.186}$$

Existence of forms and Abel's theorem

Given a merormophic form $\omega \in \mathcal{M}^h(\Sigma)$ of weight h, we know that its divisor $D = (\omega)$ must have degree $\deg D = (2g - 2)h$. However, not every divisor of that degree can be expressed as the divisor of some meromorphic form. The question, then, is what are the necessary and sufficient conditions on D such that $D = (\omega)$ for some form ω?

To start, a simpler question is to ask which divisors define a meromorphic function $f \in \mathcal{M}(\Sigma)$. To answer this question, consider the map $\widetilde{\mu} : \Sigma \to \mathbb{C}^g$ defined by

$$\widetilde{\mu}(z) = \int_p^z \vec{\omega}\ , \tag{5.187}$$

where p is some base-point and $\vec{\omega} = (\omega_1, \ldots, \omega_g)$ is the vector of canonically normalized holomorphic one-forms on Σ. However, note that the integration contour is not uniquely defined, and $\widetilde{\mu}$ is multivalued on $\mathbb{C}^g$. Indeed, one could wrap the contour around the α_μ cycle m_μ times and around the β_μ cycle n_μ times and the integral would change by a factor

$$\widetilde{\mu}(z) \to \widetilde{\mu}(z) + \sum_\mu m_\mu \vec{e}_\mu + \sum_\mu n_\mu \left(\sum_\nu \Omega_{\mu\nu} \vec{e}_\nu \right)\ . \tag{5.188}$$

That is, $\widetilde{\mu}$ is ambiguous up to the addition of an element of the lattice $\mathbb{Z}^g + \Omega\mathbb{Z}^g$. Thus, $\widetilde{\mu}(z)$ is unambiguous as an element of the quotient space $\mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$, known as the *Jacobian* of Σ, denoted by $\mathrm{Jac}(\Sigma)$. We can therefore define a new function $\mu : \Sigma \to \mathrm{Jac}(\Sigma)$ given by

$$\mu(z) = \int_p^z \vec{\omega} \quad (\mathrm{mod}\ \mathbb{Z}^g + \Omega\mathbb{Z}^g)\ , \tag{5.189}$$

which is independent of the integration path chosen. The function μ is known as the *Abel-Jacobi map*. We can also extend the definition of μ to a linear homomorphism $\mu : \mathrm{Div}(\Sigma) \to \mathrm{Jac}(\Sigma)$. If $D = \sum_i n_i p_i$ is a divisor on Σ, then we can define

$$\mu(D) = \int_p^D \vec{\omega} = \sum_i n_i \int_p^{p_i} \vec{\omega_i} \quad (\mathrm{mod}\ \mathbb{Z}^g + \Omega\mathbb{Z}^g)\ , \tag{5.190}$$

which maps the linear structure on $\mathrm{Div}(\Sigma)$ to the linear structure on $\mathrm{Jac}(\Sigma)$.

We can now provide a definitive answer to whether a meromorphic function f with $(f) = D$ exists for any divisor D. Given a divisor D with $\deg D = 0$, there exists a unique meromorphic function (up to scalar multiplication) with $D = (f)$ if and only if $\mu(D)=0$. This is *Abel's theorem.* A compact and elegant way to state Abel's theorem is that the short sequence of vector spaces

$$\mathcal{M}(\Sigma) \xrightarrow{(\cdot)} \mathrm{Div}_0(\Sigma) \xrightarrow{\mu} \mathrm{Jac}(\Sigma) \tag{5.191}$$

is exact, where $\mathrm{Div}_0(\Sigma)$ is the subspace of $\mathrm{Div}(\Sigma)$ of divisors with degree zero, and $(\cdot) : \mathcal{M}(\Sigma) \to \mathrm{Div}_0(\Sigma)$ is the map taking a meromorphic function f to its divisor (f). Note also that Abel's theorem implies that $\mu(D)$ depends only on the equivalence class of the divisor D, since if $D' = D + (f)$ for any meromorphic function f, we have $\mu(D) = \mu(D')$.

Now we can answer the question for forms of generic weight h. Given two forms ω_1, ω_2 of degree h, note that their ratio $f(z) = \omega_1(z)/\omega_2(z)$ defines a meromorphic function,[16] and thus we have

$$\mu((\omega_1)) - \mu((\omega_2)) = \mu((\omega_1/\omega_2)) = \mu((f)) = 0 \ . \tag{5.192}$$

That is, the divisors (ω_1) and (ω_2) have the same image under μ. For $h=1$, recall that the divisor (ω) is equivalent to the canonical class K. The divisor of a meromorphic form of degree h is equivalent to hK, and thus for any $\omega \in \mathcal{M}^h(\Sigma)$, we have

$$\mu((\omega)) = h\mu(K) \ . \tag{5.193}$$

Note that the definition of $\mu(K)$ depends on the choice of basepoint p in (5.187). Abel's theorem now extends to the construction of meromorphic forms, stating that a divisor D with degree $\deg D = (2g-2)h$ is of the form (ω) if and only if $\mu(D) = h\mu(K)$. We will use this result extensively.

Spinors and spin structures

So far, we have discussed meromorphic forms of weight h, with h integer. However, in field theory, we also deal with conformal fields of weight $1/2$ (for example, the symplectic Bosons $\xi^\pm, \eta^\pm$ of the main text). Furthermore, the prime form $\vartheta(x,y)$, which we discuss later, is defined as a form of weight $-1/2$ in both its arguments. Thus, we need to discuss forms of half-integer weight, as well as their divisors.

Let us start with meromorphic forms of weight $1/2$. Such a form can be written formally as $f(z)\sqrt{\mathrm{d}z}$. More formally, it is an object $\omega(z)$ which transforms like a conformal field of weight $1/2$ under coordinate transformations. The important subtlety is that we must choose a particular branch of the square root. This can be done unambiguously locally, but we must choose how to stitch together chosen branches globally in a consistent way. Such a stitching of branch choices is called a *spin structure*, and on a surface of genus g there are 2^{2g} such inequivalent spin structures, consisting of all the different choices of monodromy when transporting around the $2g$ cycles α_μ, β_μ.

Equivalently, we can think of a spin bundle on Σ as a bundle S such that $S^2 = K$ – i.e., a bundle whose sections square to meromorphic forms of degree 1. A choice of such a bundle is equivalent to a choice of spin structure, and there are again 2^{2g} such non-isomorphic choices. To such a spin bundle S, we associate a divisor class

[16] This is true for integer h. For half-integer h, this is true if ω_1 and ω_2 are spinors on the same spin structure on Σ, so that the resulting function f has no branch cuts.

Δ, such that 2Δ is the canonical class on Σ. The degree of such a divisor class Δ is simply half the degree of the canonical class. That is,

$$\deg(\Delta) = g - 1 \ . \tag{5.194}$$

Put concretely, a meromorphic form of weight 1/2 will always satisfy $Z(\omega) - P(\omega) = g - 1$, where Z counts the number of zeroes and P counts the number of poles.

Finally we present one more way to think of spin structures in terms of the Jacobian lattice. A spin structure is represented by a divisor class Δ such that $2\Delta = K$, and thus under the Abel-Jacobi map we must have

$$2\mu(\Delta) \equiv 0 \pmod{\mathbb{Z} + \Omega\mathbb{Z}} . \tag{5.195}$$

Thus, we can write $\mu(\Delta)$ as

$$\mu(\Delta) = \frac{1}{2}\left(\vec{m} + \Omega\vec{n}\right) \ , \tag{5.196}$$

where $\vec{m}, \vec{n} \in \mathbb{Z}^g$ are vectors of integers. Such an element of the lattice $\mathbb{Z}^g + \Omega\mathbb{Z}^g$ is called a *half-period.* There are 2^{2g} such half-periods, one for each spin structure on Σ.

Constructing functions and the prime form

Given a divisor D with $\deg D = 0$ and $\mu(D) = 0$, Abel's theorem tells us that there is a unique meromorphic function $f : \Sigma \to \mathbb{C}$ with divisor D, up to scalar multiplication. It is then natural to ask if there is a generic way to construct such a function out of elementary components. On the sphere, we know this is possible. If $D = \sum_i n_i z_i$ of points z_i with multiplicities n_i such that $\sum_i n_i = 0$, the expression

$$f(z) = \prod_i (z - z_i)^{n_i} \tag{5.197}$$

defines such a meromorphic function, built from the fundamental building block $(x - y)$.

One could then ask whether a function analogous to $(x - y)$ exists on a Riemann surface of genus g. The answer is, in fact, yes, and it is given by the so-called "prime form" $\vartheta(x, y)$. The prime form is constructed using objects called theta functions, but we will not need the explicit form of its construction here (the interested reader is encouraged to consult Chapter 3 of [95]). For our purposes, it suffices to know that the prime form satisfies the following properties:[17]

- $\vartheta(x, y)$ is a meromorphic form of weight $-\frac{1}{2}$ in both x and y.
- $\vartheta(x, y)$ has a simple zero only when $x = y$.
- $\vartheta(x, y) = -\vartheta(y, x)$.
- $\vartheta(x, y)$ is periodic around the α-cycles, but is only quasi-periodic around the β-cycles, picking up the monodromy

$$\vartheta(x + \beta_\mu, y) = \exp\left(-i\pi\Omega_{\mu\mu} - 2\pi i \int_x^y \omega_\mu\right) \vartheta(x, y) \ . \tag{5.198}$$

[17]Technically, the construction of the prime form requires one to pick a spin structure on Σ. We will ignore this subtlety, since it does not enter in our analysis. See [95] for more details.

Assuming that such an object with these properties exists, we claim that we can use it to construct meromorphic forms from their divisors. Given a divisor $D = \sum_i n_i z_i$ with $\mu(D) = 0$ and $\deg D = \sum_i n_i = 0$, the expression

$$f(z) = \prod_i \vartheta(z, z_i)^{n_i} \tag{5.199}$$

is a globally-defined meromorphic function on Σ with divisor D. It is clear that the function $f(z)$ defined above has divisor D. Furthermore, the function $f(z)$ is well-defined on the surface Σ (i.e. it is periodic around the cycles of Σ). Indeed, we have

$$\begin{aligned} f(z+\beta_\mu) &= \prod_i \vartheta(z+\beta_\mu, z_i)^{n_i} \\ &= \prod_i \left(\exp\left(-i\pi\Omega_{\mu\mu} - 2\pi i \int_z^{z_i} \omega_\mu \right)^{n_i} \right) f(z) \\ &= \exp\left(-i\pi\Omega_{\mu\mu} \sum_i n_i - 2\pi i \sum_i n_i \int_z^{z_i} \omega_\mu \right) f(z) \\ &= \exp\left(-i\pi \deg D\, \Omega_{\mu\mu} - 2\pi i \mu(D) \right) f(z) \ , \end{aligned} \tag{5.200}$$

and the monodromy vanishes if $\deg D = 0$ and $\mu(D) = 0$. Periodicity around the α cycles is trivial since $\vartheta(x, y)$ is periodic around the α cycles. Thus, (5.199) is a globally defined meromorphic function with the desired divisor. By Abel's theorem, this function is unique.

Chapter 6

Holographic duals for D-branes

The AdS/CFT correspondence, in its most generic form, describes a duality between a bulk theory of quantum gravity on AdS_{d+1} and a CFT which is formulated on a d-dimensional manifold which is equivalent to the boundary of the bulk manifold. Crucial to this statement is that the boundary conditions of the bulk gravitational theory are *fixed*. One cannot alter the boundary conditions of the bulk theory without altering properties of the boundary CFT. For example, in the 3D/2D correspondence, the bulk manifold $\mathcal{M}$ is taken to be locally hyperbolic, and its boundary $\partial\mathcal{M}$ has a natural conformal structure, given by the boundary value of the metric tensor near the boundary. Changing the boundary conditions of the metric tensor thus changes the conformal structure of the boundary theory.

On general grounds, string theory from the worldsheet is defined only perturbatively, but we know that non-perturbative objects, such as D-branes, should also exist in the theory. In the context of AdS/CFT, these non-perturbative objects could manifest themselves as branes which are to be included in the bulk theory, and which contribute some non-perturbative correction in the $1/N$ expansion of the boundary theory. However, if D-branes in the bulk have a nontrivial backreaction on the boundary, then they will manifestly spoil the dictionary of the AdS/CFT correspondence between the bulk theory and the boundary theory.

This does not mean that one cannot ask what the roles of D-branes which have nontrivial boundary backreactions are. In fact, this question has led to the development of the field of the anti-de Sitter/boundary CFT (AdS/BCFT) correspondence. In this paradigm, one studies the effects that bulk branes which intersect the boundary in nontrivial ways effect the dual CFT. See [96] and references therein.

In the case of the tensionless string, we have seen in the previous chapter that the bulk theory seemingly has no non-perturbative corrections in $1/N$. Intuitively, this means that there are no non-perturbative objects in the string theory which lie purely in the bulk of AdS_3. However, in AdS_3 there are other types of branes which extend all the way to the boundary of spacetime and which preserve half of the global $\mathrm{SL}(2,\mathbb{R})$ symmetries [97]. A natural question is: to what do these branes correspond in the AdS/BCFT correspondence?

In this chapter, we explore the tensionless string formulated on a background with D-branes which extend to the boundary. We find that the above question has a very simple answer: the branes in the bulk which backreact on the AdS_3 boundary are exactly dual to boundary states in the dual CFT. We confirm this by computing spectrum of open strings stretching between two such branes and comparing it with the (single-particle) spectrum of the symmetric orbifold on a cylinder. We also compute disk correlation functions on both sides, and show that they match to leading order in the genus expansion. This provides a nontrivial example of an exact AdS/BCFT correspondence that can be explicitly verified in string theory.

6.1 D-branes and twined conjugacy classes

Before jumping into the D-branes of the tensionless string, let us consider the more simple case of D-branes in the bosonic $\mathrm{SL}(2,\mathbb{R})$ WZW model. These were originally explored in [97], and this section is largely a summary of those results. We begin by mentioning some generalities of symmetry-preserving boundary states in generic WZW models, see [98–102] for detailed discussions.

In a WZW model on a semi-simple Lie group G, the chiral algebra is generated by the currents $J^a(z)$. In order to study boundary states in this theory, we consider the model on a worldsheet whose topology is the upper half-plane. Boundary states of this theory are then classified by boundary conditions near the real line $z = \overline{z}$. Since we only want to consider *conformal* boundary states (i.e. those boundary states which preserve conformal symmetry), we want the boundary to be invisible to the operator $T(z) - \overline{T}(\overline{z})$ that generates translations in the $\mathrm{Im}(z)$-direction. That is, we consider boundary conditions in the theory which satisfy

$$T(z) = \overline{T}(\overline{z}) , \quad \text{at } z = \overline{z} . \tag{6.1}$$

Another way of explaining this condition is that it is precisely the one that breaks the conformal symmetry as little as possible. Specifically, the global $\mathrm{SL}(2,\mathbb{C})$ symmetry generated by the modes L_{-1}, L_0, L_1 and $\overline{L}_{-1}, \overline{L}_0, \overline{L}_1$ gets broken down to the $\mathrm{SL}(2,\mathbb{R})$ symmetry which maps the real line to itself and is generated by the diagonal generators

$$\frac{1}{2}(L_{-1} + \overline{L}_{-1}) , \quad \frac{1}{2}(L_0 + \overline{L}_0) , \quad \frac{1}{2}(L_1 + \overline{L}_1) . \tag{6.2}$$

Now, in a WZW model, the stress-tensor is constructed from the chiral currents $J^a(z)$ through the Sugawara construction, i.e.

$$T(z) = \frac{\kappa_{ab}}{2(k + h^\vee)} (J^a J^b)(z) . \tag{6.3}$$

Thus, the boundary conditions for the currents J^a must satisfy the consistency condition

$$\kappa_{ab}(J^a J^b)(z) = \kappa_{ab}(\bar{J}^a \bar{J}^b)(\overline{z}) , \quad \text{at } z = \overline{z} \tag{6.4}$$

in order for the boundary conditions to preserve the residual $\mathrm{SL}(2,\mathbb{R})$ conformal symmetry. In addition to preserving the diagonal conformal symmetry, however, it is convenient to demand that the boundary states preserve some form of residual $G_L \times G_R$ symmetry of the WZW model. This can be achieved by specifying the boundary conditions

$$J^a(z) = \omega(\bar{J})^a(\overline{z}) , \quad \text{at } z = \overline{z} , \tag{6.5}$$

where ω is some automorphism of $\mathfrak{g}$. The resulting boundary condition breaks the full $G_L \times G_R$ symmetry down to a (twisted) diagonal subgroup generated by the zero modes

$$\frac{1}{2}(J_0^a + \omega(\bar{J})_0^a) . \tag{6.6}$$

As was discussed in [99–101], these boundary conditions have a nice geometric meaning in terms of the geometry of the target space G in the WZW model. They correspond to D-branes whose worldvolumes lie on the *twined conjugacy class*

$$\mathcal{W}_g^\omega := \left\{\omega(h) g h^{-1} | h \in G\right\} . \tag{6.7}$$

This worldvolume corresponds to the orbit of a group element g under the action of the diagonal generators (6.6), and thus is preserved under the residual symmetry of the theory. We also note that if ω is an inner automorphism, i.e. $\omega(h) = g_0 h g_0^{-1}$ for some fixed element $g_0 \in G$, then the resulting worldvolume is

$$\begin{aligned} \mathcal{W}_g^\omega &= \left\{g_0 h g_0^{-1} g h^{-1} | h \in G\right\} \\ &= \left\{g_0 \cdot (h g_0^{-1})(g g_0^{-1})(h g_0^{-1})^{-1} | h \in G\right\} \\ &= \left\{g_0 \cdot h(g g_0^{-1}) h^{-1} | h \in G\right\} \\ &= g_0 \cdot \mathcal{W}^{\mathrm{id}}_{g g_0^{-1}} . \end{aligned} \tag{6.8}$$

That is, the worldvolume corresponding to an inner automorphism can be obtained by a left-translation of a worldvolume corresponding to the trivial automorphism. Thus, to understand the worldvolumes of D-branes in WZW models, it suffices to classify worldvolumes corresponding to outer automorphisms ω.

Conversely, if ω is any automorphism (not necessarily inner), then we have

$$\mathcal{W}_g^\omega = g \cdot \mathcal{W}_1^{g^{-1}\omega g} , \tag{6.9}$$

where $g^{-1}\omega g$ is the automorphism ω composed with conjugation by g^{-1}. Thus, we can also study conjugacy classes of the identity element if we are willing to admit nontrivial inner automorphisms.

D-branes in $\mathrm{SL}(2,\mathbb{R})$

Now that we have some of the generalities out of the way, we can move on to the specific case of D-branes in $\mathrm{SL}(2,\mathbb{R})$.[1] The two types automorphisms we can consider are the trivial automorphism $\omega = \mathrm{id}$ and nontrivial automorphisms. Up to conjugation, there is only one nontrivial automorphism of $\mathrm{SL}(2,\mathbb{R})$, which acts on group elements h as

$$\omega(h) \to \omega_0 h \omega_0 , \tag{6.10}$$

where

$$\omega_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{6.11}$$

This automorphism is outer, since ω_0 it does not belong to $\mathrm{SL}(2,\mathbb{R})$, but rather $\mathrm{GL}(2,\mathbb{R})$.

For the trivial automorphism, the worldvolume is the conjugacy class of the element g. Now, it is a fact about $\mathrm{SL}(2,\mathbb{R})$ that conjugacy classes are uniquely determined by the trace of the group element. That is, the conjugacy class of g is the set of all $g' \in \mathrm{SL}(2,\mathbb{R})$ such that $\mathrm{Tr}\, g' = \mathrm{Tr}\, g$. In the parametrization (3.11) of AdS_3, the trace of an $\mathrm{SL}(2,\mathbb{R})$ element is simply

$$\mathrm{Tr}\, g = 2X_0 = \sin t \cosh \rho := 2C , \tag{6.12}$$

[1] As always, we are implicitly studying the universal cover of $\mathrm{SL}(2,\mathbb{R})$.

where $C \in \mathbb{R}$ is some constant. We can get some insight into the geometry of the geometry of these worldvolumes by using the defining relation $X_{-1}^2+X_0^2-X_1^2-X_2^2 = 1$ of AdS_3. The worldvolumes with fixed C satisfy

$$X_{-1}^2 - X_1^2 - X_2^2 = 1 - C^2. \tag{6.13}$$

Clearly, the geometry of the worldvolume then depends on whether C^2 is less than, equal to, or greater than 1. For $C^2 < 1$, the resulting geometry is Euclidean and hyperbolic. For $C^2 > 1$, it is two-dimensional de Sitter space. For $C^2 = 1$, it is a light-cone.

For the outer automorphism ω, we can describe the resulting geometry as a fixed-point set under the involution $\iota(h) := \omega(h^{-1})$. Indeed, the worldvolume $\mathcal{W}_1^\omega$ is fixed by ι:

$$\iota(\omega_0 h \omega_0 h^{-1}) = \omega_0(\omega_0 h \omega_0 h^{-1})^{-1}\omega_0 = \omega_0 h \omega_0 h^{-1}\,. \tag{6.14}$$

Furthermore, by (6.9), we can view the worldvolume for an element $g \neq 1$, as the fixed-point set $\mathcal{W}_1^{g^{-1}\omega g}$ of the involution $\iota_g(h) := g^{-1}\omega(h^{-1})g$, multiplied on the left by g.

On generic $\mathrm{SL}(2,\mathbb{R})$ elements parametrized by (3.11), ι acts as

$$X_2 \to -X_2\,, \tag{6.15}$$

and so the fixed-point set is $X_2 = 0$, or $\sin\phi = 0$. This defines a slice of two-dimensional anti-de Sitter space out of AdS_3. A similar analysis shows that the worldvolume $\mathcal{W}_g^\omega$ has AdS_2 geometry for any $g \in \mathrm{SL}(2,\mathbb{R})$.

Gluing conditions

Given that we now have a classification of D-branes in the $\mathrm{SL}(2,\mathbb{R})$ WZW model, we are ready to consider their associated boundary states on the worldsheet. However, there is one subtlety that we need to clarify. In the analysis of [63] (and, by extension, of Chapter 3) the left- and right-moving currents are not treated on the same footing, and are defined with a relative 'twist' with respect to each other. This twist is precisely the auter automorphism ω of $\mathrm{SL}(2,\mathbb{R})$, which acts on the generators of $\mathfrak{sl}(2,\mathbb{R})$ as

$$J^3 \mapsto -J^3\,, \quad J^\pm \mapsto J^\pm\,. \tag{6.16}$$

Thus, when defining gluing conditions, we need to take into account that all conditions should be additionally twisted by ω.

The first case is that of the 'trivial' gluing conditions, i.e. for which there is no auter automorphism. The gluing conditions are given by

$$J^3(z) = -\bar{J}^3(\bar{z})\,, \quad J^\pm(z) = \bar{J}^\mp(\bar{z})\,, \quad \text{at } z = \bar{z}\,. \tag{6.17}$$

Furthermore, their associated Ishibashi states satisfy

$$(J_n^3 - \bar{J}_{-n}^3)|\psi\rangle\!\rangle_S = (J_n^\pm + \bar{J}_{-n}^\mp)|\psi\rangle\!\rangle_S = 0\,. \tag{6.18}$$

We will refer to the branes which satisfy these boundary conditions as *spherical branes.*

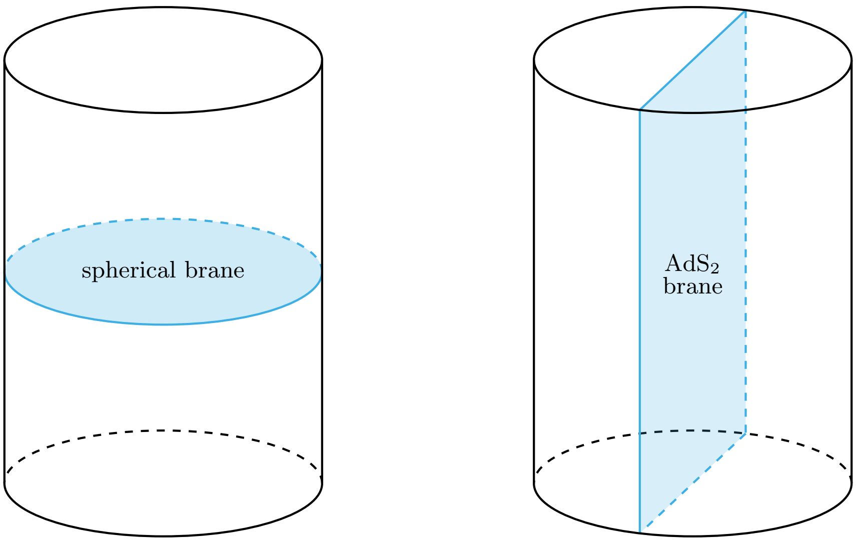

Figure 6.1: Geometry of the spherical and AdS_2 branes.

The second case is when there is a nontrivial outer automorphism. In this case, the gluing conditions are twisted by ω. Since the definition of the right-moving current already includes a twist by ω, this implies the gluing conditions

$$J^a(z) = \bar{J}^a(\bar{z}) \,, \quad \text{at } z = \bar{z} \,. \tag{6.19}$$

The associated Ishibashi states satisfy

$$(J^a_n + \bar{J}^a_{-n})|\psi\rangle\rangle_A = 0 \,. \tag{6.20}$$

Since the worldvolumes of these branes have the geometry of AdS_2, we refer to the branes corresponding to these boundary conditions as *AdS_2 branes*. The geometry of both the spherical and AdS_2 branes are shown in Figure 6.1.

6.2 Branes in the PSU(1,1|2) model

Now that we have discussed a bit the boundary states in the $\mathfrak{sl}(2,\mathbb{R})$ model, we can turn our attention to the calculation of boundary states and their overlaps in the $\mathfrak{psu}(1,1|2)_1$ model, which we hope to compare to the symmetric orbifold theory on the boundary. Just as in the case of $\mathfrak{sl}(2,\mathbb{R})$, we would like to impose boundary conditions which preserve a diagonal part of the $\mathfrak{psu}(1,1|2)_1$ symmetry algebra. Just as in $\mathfrak{sl}(2,\mathbb{R})$, there are two cases of interest, corresponding to whether the associated automorphism is ‘trivial’ or ‘twisted’. These will again describe the spherical and AdS_2 branes, but now in a full supersymmetric background.

Since we are interested in computing Ishibashi states as well as their overlaps, we will need to recall some facts about the bulk $\mathfrak{psu}(1,1|2)_1$ theory. Specifically, as we reviewed in Chapter 4, the bulk spectrum of the theory is given by

$$\mathscr{H} = \int_0^1 \mathrm{d}\lambda \bigoplus_{w\in\mathbb{Z}} \sigma^w(\mathscr{F}_\lambda) \otimes \overline{\sigma^w(\mathscr{F}_\lambda)} \,, \tag{6.21}$$

where $\mathscr{F}_\lambda$ is the Clifford module of equation (4.9), and σ^w is the spectral flow operator. Moreover, to compute overlaps of boundary states, we will need the form of the character of the representation $\sigma^w(\mathscr{F}_\lambda)$. This was also given in Chapter 4, but in this chapter we will be interested in the so-called *specialized* character, which differs from the usual character by an insertion of the fermion number $(-1)^F$, and is given by

$$\widetilde{\text{ch}}[\sigma^w(\mathscr{F}_\lambda)] = (-1)^w q^{\frac{w^2}{2}} \sum_{m\in\mathbb{Z}} e^{-2\pi i\lambda m}\delta(t-w\tau+m)\frac{\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)}{\eta(\tau)^4} . \tag{6.22}$$

Under the modular S-transformation, we have

$$\begin{aligned} &e^{\frac{\pi i}{2\tau}(t^2-z^2)}\widetilde{\text{ch}}[\sigma^w(\mathscr{F}_\lambda)](\tfrac{t}{\tau},\tfrac{z}{\tau};-\tfrac{1}{\tau}) \\ &\qquad = \sum_{w'\in\mathbb{Z}}\int_0^1 d\lambda'\, S_{(w,\lambda),(w',\lambda')}\,\widetilde{\text{ch}}[\sigma^{w'}(\mathscr{F}_{\lambda'})](t,z;\tau) \, , \end{aligned} \tag{6.23}$$

where we have introduced the modular S-matrix

$$S_{(w,\lambda),(w',\lambda')} = \frac{|\tau|}{-i\tau}\, e^{2\pi i[w'(\lambda-\frac{1}{2})+w(\lambda'-\frac{1}{2})]} . \tag{6.24}$$

Note that $S_{(w,\lambda),(w',\lambda')}$ depends on τ, as is typical for logarithmic CFTs. This dependence will however drop out of all physical quantities such as the fusion coefficients that can be determined via the Verlinde formula as in [45].

Spherical branes

As explained above, we take the "trivial" gluing conditions to be

$$J^3(z) = -\bar{J}^3(\bar{z}) \ , \qquad J^\pm(z) = \bar{J}^\mp(\bar{z}) \tag{6.25}$$

at $z=\bar{z}$, where the right-movers are denoted by a bar. On the other hand, since $\mathfrak{su}(2)$ does not admit any outer automorphisms, we may always, without loss of generality, assume that the $\mathfrak{su}(2)$ gluing conditions are trivial

$$K^a(z) = \bar{K}^a(\bar{z}) . \tag{6.26}$$

Finally, we need to impose suitable gluing conditions on the fermionic currents so as to preserve the full $\mathfrak{psu}(1,1|2)_1$ algebra. It is easy to see that these are

$$S^{\alpha\beta\gamma}(z) = \varepsilon(i\sigma^2)^\alpha{}_\mu\, \bar{S}^{\mu\beta\gamma}(\bar{z}) \ , \tag{6.27}$$

where $\varepsilon=\pm$ plays the role of the worldsheet spin structure. At $k=1$ where the radii of both the AdS_3 and S^3 are comparable to the string length, we expect the worldvolumes of these branes to be fuzzy, but generally localized around a particular value in time (since formally they satisfy a Dirichlet boundary condition along the AdS_3 time direction). Adopting the terminology of [103], we will call them spherical branes.

Translating the above gluing conditions into the closed string language we are thus looking for Ishibashi states [104] that are characterized by

$$(J^3_n - \bar{J}^3_{-n})|w,\lambda,\varepsilon\rangle\!\rangle_{\text{S}} = 0 \ , \tag{6.28a}$$

$$(J^\pm_n + \bar{J}^\mp_{-n})|w,\lambda,\varepsilon\rangle\!\rangle_{\text{S}} = 0 \ , \tag{6.28b}$$

$$(K^a_n + \bar{K}^a_{-n})|w,\lambda,\varepsilon\rangle\!\rangle_{\text{S}} = 0 \ , \tag{6.28c}$$

$$(S^{\alpha\beta\gamma}_n + \varepsilon(i\sigma^2)^\alpha{}_\mu\bar{S}^{\mu\beta\gamma}_{-n})|w,\lambda,\varepsilon\rangle\!\rangle_{\text{S}} = 0 \ . \tag{6.28d}$$

The label "S" here stands for "spherical" and is there to distinguish these states from the Ishibashi states for the AdS_2 branes, which we introduce below and label by "AdS".

Let us first consider the unflowed representations. The condition (6.28a) with $n = 0$ implies that the left- and right-moving J_0^3 eigenvalues must be equal. In order for this to be possible we need $\lambda = \bar{\lambda}$, and this is indeed the case for all representations appearing in the bulk spectrum (6.21). Thus in the unflowed sector all $\mathcal{F}_\lambda$ sectors have an Ishibashi state which we shall denote by $|0,\lambda,\varepsilon\rangle\!\rangle_{\rm S}$.

Next we observe that the Ishibashi state in the w-spectrally flowed sector can be simply obtained from the unflowed Ishibashi state via

$$|w,\lambda,\varepsilon\rangle\!\rangle_{\rm S} = \big[|0,\lambda,\varepsilon\rangle\!\rangle_{\rm S}\big]^w \ . \tag{6.29}$$

Thus there exists a (non-trivial) Ishibashi state in each sector, and we shall label it by

$$|w,\lambda,\varepsilon\rangle\!\rangle_{\rm S} \qquad \text{for all } w\in\mathbb{Z} \text{ and } \lambda\in[0,1). \tag{6.30}$$

For the following we shall mainly need the elementary overlaps

$${}_{\rm S}\langle\!\langle w',\lambda',\mp|\hat{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}\hat{x}^{\frac{1}{2}(J_0^3+\bar{J}_0^3)}|w,\lambda,\pm\rangle\!\rangle_{\rm S} = \delta_{w,w'}\,\delta_{\lambda,\lambda'}\,\widetilde{\rm ch}[\sigma^w(\mathscr{F}_\lambda)](\hat{t};\hat{\tau}) \ , \tag{6.31}$$

where the character appearing on the right-hand-side involves $(-1)^F$, see eq. (6.22), since

$$(-1)^F|w,\lambda,\varepsilon\rangle\!\rangle_{\rm S} = |w,\lambda,-\varepsilon\rangle\!\rangle_{\rm S} \ . \tag{6.32}$$

Here $\hat{q}$ and $\hat{x}$ are defined by

$$\hat{q} = e^{2\pi i\hat{\tau}} \ , \qquad \hat{x} = e^{2\pi i\hat{t}} \ , \tag{6.33}$$

where $\hat{\tau}$ and $\hat{t}$ are real. Identifying $\hat{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}$ with the worldsheet propagator, the overlap (6.31) computes the zero-point worldsheet conformal block on a cylinder with length $\hat{\tau}$. The $\mathfrak{sl}(2;\mathbb{R})$ chemical potential $\hat{t}$, on the other hand, will be interpreted as the length of the corresponding spacetime cylinder, which is cut out on the boundary of AdS_3 by the two spherical branes. Similarly, we could have introduced a chemical potential $\hat{\zeta}$ for the $\mathfrak{su}(2)$ currents K_n^a by inserting $\hat{y}^{\frac{1}{2}(K_0^3+\bar{K}_0^3)}$ into the overlap with $\hat{y} = e^{2\pi i\hat{\zeta}}$ — we will suppress this in order not to clutter the formulae.

The inclusion of the spacetime propagator $\hat{x}^{\frac{1}{2}(J_0^3+\bar{J}_0^3)}$ can be thought of either as a way of 'unspecialising' the cylinder conformal blocks, or, as computing a twisted overlap between Ishibashi states of the form

$$|w,\lambda,\varepsilon;\hat{t}\,\rangle\!\rangle_{\rm S} = \hat{x}^{\frac{1}{2}(J_0^3+\bar{J}_0^3)}|w,\lambda,\varepsilon\rangle\!\rangle_{\rm S} = \hat{x}^{J_0^3}|w,\lambda,\varepsilon\rangle\!\rangle_{\rm S} \ , \tag{6.34}$$

shifted by a distance $\hat{t}$ along the spacetime cylinder. (Here we have used the gluing condition (6.28a) in the final equation.) These shifted Ishibashi states then satisfy the gluing conditions

$$(J_n^3 - \bar{J}_{-n}^3)|w,\lambda,\varepsilon;\hat{t}\rangle\!\rangle_{\rm S} = 0 \ , \tag{6.35a}$$

$$(e^{\pm\pi i\hat{t}}J_n^\pm + e^{\mp\pi i\hat{t}}\bar{J}_{-n}^\mp)|w,\lambda,\varepsilon;\hat{t}\rangle\!\rangle_{\rm S} = 0 \ , \tag{6.35b}$$

$$(K_n^a + \bar{K}_{-n}^a)|w,\lambda,\varepsilon;\hat{t}\rangle\!\rangle_{\rm S} = 0 \ , \tag{6.35c}$$

$$(e^{\pm\frac{1}{2}\alpha\pi i\hat{t}}S_n^{\alpha\beta\gamma} + \varepsilon e^{\frac{1}{2}\beta\pi i\hat{t}}(i\sigma^2)^\alpha{}_\mu\bar{S}_{-n}^{\beta\gamma})|w,\lambda,\varepsilon;\hat{t}\rangle\!\rangle_{\rm S} = 0 \ . \tag{6.35d}$$

We note that for $\hat{t} = k \in \mathbb{Z}$, the shifted Ishibashi states (6.34) again obey the Ishibashi conditions (6.28), with $\varepsilon \mapsto (-1)^k\varepsilon$, i.e.

$$|w,\lambda,\varepsilon;k\rangle\!\rangle_{\rm S} = (-1)^{kw} e^{2\pi i k\lambda} |w,\lambda,(-1)^k\varepsilon\rangle\!\rangle_{\rm S} \ . \tag{6.36}$$

This reflects the periodicity $\hat{t} \to \hat{t}+2$ of the $\mathrm{SL}(2;\mathbb{R})$ group manifold, which was promoted to global AdS_3 by considering its universal covering.

Consistent boundary states

Next we want to assemble these Ishibashi states into consistent boundary states. Since we have an Ishibashi state from each closed string sector, we would expect that there exists a D-brane whose open string spectrum will just consist of the 'vacuum' representation $\mathscr{F}_{1/2}$. In fact, this property seems to hold for any boundary state of the form

$$\|W,\Lambda,\varepsilon\rangle\!\rangle_{\rm S} = \sum_{w\in\mathbb{Z}} \int_0^1 d\lambda\, e^{2\pi i[w(\Lambda-\frac{1}{2})+(\lambda-\frac{1}{2})W]} |w,\lambda,\varepsilon\rangle\!\rangle_{\rm S} \ , \tag{6.37}$$

where W and Λ are some parameters whose physical interpretation will be discussed shortly. Indeed, the unspecialized $\mathfrak{psu}(1,1|2)_1$ overlap equals

$$\hat{Z}^{\rm S}_{(W_1,\Lambda_1)|(W_2,\Lambda_2)}(\hat{t};\hat{\tau}) \equiv {}_{\rm S}\langle\!\langle \Lambda_2, W_2, \mp\| \hat{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})} \hat{x}^{\frac{1}{2}(J_0^3+\bar{J}_0^3)} \|\Lambda_1, W_1, \pm\rangle\!\rangle_{\rm S} \tag{6.38a}$$

$$= \sum_{w\in\mathbb{Z}} \int_0^1 d\lambda\, e^{2\pi i w(\Lambda_1-\Lambda_2)}\, e^{2\pi i(\lambda-\frac{1}{2})(W_1-W_2)} \widetilde{\rm ch}[\sigma^w(\mathscr{F}_\lambda)](\hat{t};\hat{\tau}) \ , \tag{6.38b}$$

where, as before, taking the opposite values of the parameter ε for the two boundary states effectively inserts $(-1)^F$ into the overlap, and thus yields the $\mathfrak{psu}(1,1|2)_1$ supercharacter $\widetilde{\rm ch}$, as appropriate when discussing boundary states of supergroups. Performing the modular S-transformation to the open-string channel, we obtain the corresponding boundary superpartition function

$$Z^{\rm S}_{(W_1,\Lambda_1)|(W_2,\Lambda_2)}(t;\tau) = e^{-\frac{\pi i\tau}{2t^2}}\, \widetilde{\rm ch}[\sigma^{W_2-W_1}(\mathscr{F}_{\frac{1}{2}-\Lambda_1+\Lambda_2})](-\tfrac{\tau}{t};\tau) \ , \tag{6.39}$$

where we define the open-string variables t and τ via

$$\hat{t} = -\frac{1}{t} \ , \qquad \hat{\tau} = -\frac{1}{\tau} \ . \tag{6.40}$$

The prefactor $e^{-\frac{\pi i\tau}{2t^2}}$ is familiar from the modular transformation of Jacobi forms; it reflects the fact that the 'open string' spectrum is twisted by the insertion of the chemical potential in the closed string overlap, see eq. (6.34). For $W_1 = W_2$ and $\Lambda_1 = \Lambda_2$ the open string spectral flow is trivial, and we just get $\mathscr{F}_{1/2}$ as anticipated.

In order to understand the interpretation of W and Λ we observe that

$$(-1)^{W(F+1)} e^{\pi i W(J_0^3+\bar{J}_0^3)} \|0,\Lambda-\tfrac{W}{2},\varepsilon\rangle\!\rangle_{\rm S} = \|W,\Lambda,\varepsilon\rangle\!\rangle_{\rm S} \ . \tag{6.41}$$

Thus up to twisting by $(-1)^F$ and suitably adjusting Λ, the parameter W can be identified with the shift of the boundary state in the time direction of AdS_3, see eq. (6.35). On the other hand, Λ describes the Wilson line in the angular direction of AdS_3 along which our boundary state satisfies a Neumann boundary condition; indeed, the relative Wilson lines Λ_1 and Λ_2 introduce the factors $e^{2\pi i w(\Lambda_1-\Lambda_2)}$ into the w-flowed sectors of the boundary state overlap, see eq. (6.38b). For the following these parameters will not play an important role, and in order to simplify our expressions we shall set them to $W = 0$ and $\Lambda = \frac{1}{2}$ from now on; then the boundary state simplifies to

$$\|\varepsilon\rangle\rangle_{\mathrm{S}} \equiv \|W = 0, \Lambda = \tfrac{1}{2}, \varepsilon\rangle\rangle_{\mathrm{S}} = \sum_{w\in\mathbb{Z}} \int_0^1 d\lambda \, |w, \lambda, \varepsilon\rangle\rangle_{\mathrm{S}} \ . \tag{6.42}$$

AdS_2 branes

For completeness let us also discuss the boundary states describing the AdS_2 branes. Their gluing conditions are 'twined' relative to the gluing conditions for the above spherical branes, and hence look trivial in our conventions, i.e. we have at $z = \bar{z}$

$$J^a(z) = \bar{J}^a(\bar{z}) \ , \tag{6.43a}$$
$$K^a(z) = \bar{K}^a(\bar{z}) \ , \tag{6.43b}$$
$$S^{\alpha\beta\gamma}(z) = \varepsilon\, \bar{S}^{\alpha\beta\gamma}(\bar{z}) \ , \tag{6.43c}$$

where ε is again a sign. This time, the corresponding Ishibashi states are characterized by

$$(J^a_n + \bar{J}^a_{-n})|0, \lambda, \varepsilon\rangle\rangle_{\mathrm{AdS}} = 0 \ , \tag{6.44a}$$
$$(K^a_n + \bar{K}^a_{-n})|0, \lambda, \varepsilon\rangle\rangle_{\mathrm{AdS}} = 0 \ , \tag{6.44b}$$
$$(S^{\alpha\beta\gamma}_n + \varepsilon\, \bar{S}^{\alpha\beta\gamma}_{-n})|0, \lambda, \varepsilon\rangle\rangle_{\mathrm{AdS}} = 0 \ . \tag{6.44c}$$

Now, the first line of (6.44) implies $m = -\bar{m}$, and hence only has a solution in the unflowed sector if $\lambda = -\bar{\lambda}$ (mod 1). Given the structure of the bulk spectrum (6.21), this is only possible for $\lambda \in \{0, \frac{1}{2}\}$, and thus we only have the two Ishibashi states

$$|0, \lambda, \varepsilon\rangle\rangle_{\mathrm{AdS}} \qquad \text{with } \lambda \in \{0, \tfrac{1}{2}\} \ . \tag{6.45}$$

It is also not difficult to convince oneself that no Ishibashi states exist for non-trivial spectral flow.

The elementary overlap relevant for the case of the AdS_2 branes now reads

$${}_{\mathrm{AdS}}\langle\langle 0, \lambda', \mp| \hat{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})} \hat{\xi}^{\frac{1}{2}(J^3_0-\bar{J}^3_0)} |0, \lambda, \pm\rangle\rangle_{\mathrm{AdS}} = \delta_{\lambda,\lambda'}\, \widetilde{\mathrm{ch}}[\mathscr{F}_\lambda](\hat{\theta}; \hat{\tau}) \ , \tag{6.46}$$

where $\hat{\xi} = e^{2\pi i \hat{\theta}}$ implements the translation by $\hat{\theta}$ in the angular direction of the AdS_3 boundary cylinder. (This is again the direction in which the branes satisfy a Dirichlet boundary condition on the AdS_3 boundary.) Alternatively, we may describe this translation in terms of shifted Ishibashi states (c.f. the analysis of eq. (6.35))

$$|0, \lambda, \varepsilon; \hat{\theta}\rangle\rangle_{\mathrm{AdS}} = \hat{\xi}^{\frac{1}{2}(J^3_0-\bar{J}^3_0)}|0, \lambda, \varepsilon\rangle\rangle_{\mathrm{AdS}} = \hat{\xi}^{J^3_0}|0, \lambda, \varepsilon\rangle\rangle_{\mathrm{AdS}} \ , \tag{6.47}$$

which satisfy the twisted Ishibashi conditions

$$(J^3_n + \bar{J}^3_{-n})|0, \lambda, \varepsilon; \hat{\theta}\rangle\rangle_{\mathrm{AdS}} = 0 \ , \tag{6.48a}$$
$$(e^{\pm\pi i\hat{\theta}} J^\pm_n + e^{\mp\pi i\hat{\theta}} \bar{J}^\pm_{-n})|0, \lambda, \varepsilon; \hat{\theta}\rangle\rangle_{\mathrm{AdS}} = 0 \ , \tag{6.48b}$$
$$(K^a_n + \bar{K}^a_{-n})|0, \lambda, \varepsilon; \hat{\theta}\rangle\rangle_{\mathrm{AdS}} = 0 \ , \tag{6.48c}$$
$$(e^{+\frac{1}{2}\alpha\pi i\hat{\theta}} S^{\alpha\beta\gamma}_n + \varepsilon e^{-\frac{1}{2}\alpha\pi i\hat{\theta}} \bar{S}^{\alpha\beta\gamma}_{-n})\, |0, \lambda, \varepsilon; \hat{\theta}\rangle\rangle_{\mathrm{AdS}} = 0 \ . \tag{6.48d}$$

For $\hat{\theta} = k \in \mathbb{Z}$, these again reduce to the Ishibashi conditions (6.44) with the sign in the supercurrent condition being $(-1)^k \varepsilon$; explicitly, we have

$$|0, \lambda, \varepsilon; k\rangle\!\rangle_{\text{AdS}} = e^{2\pi i k \lambda} |0, \lambda, (-1)^k \varepsilon\rangle\!\rangle_{\text{AdS}} \ . \tag{6.49}$$

For $\hat{\theta} \to \hat{\theta} + 2$, we recover the periodicity of the AdS_3 bulk in its angular coordinate, where the Ishibashi states pick up a phase of $e^{4\pi i \lambda} = +1$ (since $\lambda \in \{0, \frac{1}{2}\}$) upon fully encircling the center of AdS_3. On the other hand, when translating the Ishibashi states to an antipodal position by shifting $\hat{\theta} \to \hat{\theta} + 1$, they acquire a factor of $(-1)^F$, apart from picking up a phase of $e^{2\pi i \lambda} = \pm 1$.

Consistent boundary states

For the AdS_2 branes the situation is in a sense opposite to what we had for the spherical branes: now we only have two Ishibashi states, and thus we expect to find a relatively 'big' open string spectrum. In any case, given that there are only two Ishibashi states, there is essentially only one ansatz we can make for the boundary states, namely

$$\|\Theta, \varepsilon\rangle\!\rangle_{\text{AdS}} = \frac{1}{\sqrt{2}} \sum_{\lambda \in \{0, \frac{1}{2}\}} e^{2\pi i (\lambda - \frac{1}{2})\Theta} |0, \lambda, \varepsilon\rangle\!\rangle_{\text{AdS}} \ , \tag{6.50}$$

where Θ is an integer, which can be restricted to $\Theta \in \{0, 1\}$. As in the case of the spherical branes, calculating the overlaps of these boundary states reveals the spectrum of boundary operators, and we find in this case

$$\mathcal{H}_{\Theta_1|\Theta_2} = \bigoplus_{w \in 2\mathbb{Z} + \Theta_1 - \Theta_2} \int_0^1 d\lambda \, \sigma^w(\mathscr{F}_\lambda) \ . \tag{6.51}$$

Specifically, the string beginning and ending on the same brane consists of all w-even flowed representations, while the string stretching between branes differing in Θ consists of all the w-odd flowed representations, see also [105, 106]. Since the two boundary states are related via

$$(-1)^{\Theta(F+1)} e^{\pi i \Theta (J_0^3 - \bar{J}_0^3)} \|0, \varepsilon\rangle\!\rangle_{\text{AdS}} = \|\Theta, \varepsilon\rangle\!\rangle_{\text{AdS}} \ , \tag{6.52}$$

the two AdS_2 branes are mapped into one another upon a rotation by π in the angular direction. That is to say, they stretch between two fixed antipodal points on the AdS_3-boundary, but have opposite orientation. (The branes that start and end at a different pair of antipodal points can be obtained as in eq. (6.47); they then satisfy the modified boundary conditions of eq. (6.48), see also [97].)

6.3 Boundary states in the symmetric orbifold

In the previous section we have constructed the symmetry preserving boundary states of the $\text{PSU}(1,1|2)$ WZW model. They can be combined with suitable boundary states from the torus factor to describe the D-branes of the full worldsheet theory, see the discussion in Section 6.4 below. As we will see, at least for the spherical branes, these D-branes can be directly identified with suitable D-branes in the dual symmetric orbifold theory. In order to make this identification more precise we now need to construct the corresponding boundary states of the symmetric orbifold theory. This will be the subject of this section.

We shall first briefly review the bulk spectrum of a symmetric orbifold theory. We shall then consider the boundary states which satisfy the same gluing condition in each individual tensor factor of the symmetric orbifold. The corresponding Ishibashi states exist in each twisted sector, and they give rise to what one may call the maximally-fractional boundary states. In the main part we shall explain the construction for a generic bosonic symmetric orbifold; the modifications that arise for the supersymmetric $\mathbb{T}^4$ theory are described in Appendix 6.B.

Bulk spectrum

Let us start by reviewing briefly the (bulk) spectrum of a symmetric orbifold CFT; for simplicity we shall here focus on the case where the seed theory X is a (diagonal) quasirational bosonic CFT with spectrum

$$\mathcal{H}^{\mathrm{X}} = \bigoplus_{\alpha} \mathcal{H}_\alpha \otimes \bar{\mathcal{H}}_\alpha \ , \tag{6.53}$$

where α runs over the different irreducible representations of the chiral algebra of X.[2]

We are interested in the torus amplitude (partition function) $\mathcal{Z}^{S_N}(t,\bar{t}\,)$ of the symmetric orbifold theory $\mathrm{X}^{\otimes N}/S_N$. Let us denote the holomorphic and anti-holomorphic torus moduli by t and $\bar{t}$, with $x = e^{2\pi i t}$ and $\bar{x} = e^{-2\pi i \bar{t}}$, and let $Z(t,\bar{t}\,)$ be the partition function of the seed theory X. The simplest method to compute the symmetric orbifold partition function is by going to the grand canonical ensemble [62] since the generating function (the grand canonical partition function) can be expressed in terms of the partition function $Z(t,\bar{t}\,)$ of the seed theory as

$$\mathfrak{Z}(p,t,\bar{t}\,) = \sum_{N=1}^{\infty} p^N \mathcal{Z}^{S_N}(t,\bar{t}\,) = \exp\left(\sum_{k=1}^{\infty} p^k T_k Z(t,\bar{t}\,)\right) , \tag{6.54}$$

where T_k is the Hecke operator

$$T_k Z(t,\bar{t}\,) = \frac{1}{k}\sum_{w|k}\sum_{\kappa=0}^{w-1} Z\big(\tfrac{(k/w)t+\kappa}{w}, \tfrac{(k/w)\bar{t}+\kappa}{w}\big) \ . \tag{6.55}$$

Here w denotes the length of any individual cycle by which we are twisting, while the sum over κ projects onto $\mathbb{Z}_w$-invariant states. (Recall that the twisted sectors of the symmetric orbifold theory are labelled by the conjugacy classes of S_N, which in turn are described by the cycle shapes, i.e. the partitions of N.)

Denoting by $d(h,\bar{h})$ the non-negative integer multiplicities of the states in the seed CFT, it is possible to rewrite the grand canonical partition function in a manifestly multiparticle form [62]

$$\mathfrak{Z}(p,t,\bar{t}\,) = \prod_{w=1}^{\infty}\prod_{h,\bar{h}} \Big[1 - p^w x^{\frac{h}{w}+\frac{c}{24}(w-\frac{1}{w})-\frac{cw}{24}} \bar{x}^{\frac{\bar{h}}{w}+\frac{c}{24}(w-\frac{1}{w})-\frac{cw}{24}}\Big]^{-d(h,\bar{h})}\Bigg|_{h-\bar{h}\in w\mathbb{Z}} \ , \tag{6.56}$$

[2] If X is rational, the sum over α is finite; in the quasirational case, there are countably many representations that appear. For example, the torus theory $\mathrm{X} = \mathbb{T}^4$ is quasirational.

where c denotes the central charge of the seed CFT. In particular, the ground-state conformal dimension Δ_w in w-twisted sector can be read off to equal

$$\Delta_w = \frac{c}{24}\Big(w - \frac{1}{w}\Big) \ . \tag{6.57}$$

We can also see from (6.56) that the single-particle states come only from conjugacy classes containing just one cycle, while generic permutations yield multiparticle states as per their cycle shape. Furthermore, the actual single-particle partition function equals

$$\mathcal{Z}_{\text{s.p.}}(t,\bar{t}\,) = \sum_{w=1}^{\infty} Z(\tfrac{t}{w},\tfrac{\bar{t}}{w})\Big|_{h-\bar{h}\in w\mathbb{Z}} \ . \tag{6.58}$$

Finally, the bulk grand canonical partition function $\mathfrak{Z}(p,t,\bar{t}\,)$ admits a geometric interpretation in terms of (unramified) covering spaces of the base torus [57]. Indeed, if we think of (6.54) as a diagrammatic expansion, then the 'connected part' would be

$$\log \mathfrak{Z}(p,t,\bar{t}\,) = \sum_{k=1}^{\infty} p^k T_k Z(t,\bar{t}\,) \ . \tag{6.59}$$

Now, the Hecke operator can be thought of as summing the partition function $Z(t,\bar{t}\,)$ over each distinct torus which holomorphically covers the base torus k times. The modulus of each covering torus is the argument appearing in (6.55), and the factor of $1/k$ corrects for the size of the automorphism group $\mathbb{Z}_w \times \mathbb{Z}_{k/w}$ of the corresponding covering, as is typical in diagrammatic expansions. Since the only (unramified) covering space of a torus is a disjoint union of tori, equation (6.54) then tells us that the partition function of the symmetric orbifold $\mathrm{X}^{\otimes N}/S_N$ is calculated as a sum over the X partition function on all disjoint products of tori which cover the base torus N times. Holographically, this sum over covering spaces arises from a localization principle in which the integral over the worldsheet torus modulus τ localizes to only those values for which the worldsheet holomorphically covers the boundary of thermal AdS_3, as was shown in [45, 73], see also [64].

Boundary states of the seed theory

In order to describe the boundary states of the symmetric orbifold theory, we first need to understand those of the seed theory itself. We shall always assume that these boundary states have already been constructed (and e.g. for the torus theory we have primarily in mind, this is the case). The main aim of our analysis here is to explain how these (known) boundary conditions of the seed theory can be 'lifted' to the symmetric orbifold theory.

We shall only be interested in boundary states of the seed theory that preserve the full chiral algebra of X. Let us denote the chiral fields of X collectively by $W(z)$, then the branes of interest will satisfy the boundary condition

$$W(z) = (\Omega \bar{W})(\bar{z}) \ , \qquad \text{at } z = \bar{z}, \tag{6.60}$$

where Ω is an automorphism of the chiral algebra of X. In terms of boundary states we then consider the Ishibashi states

$$|\beta\rangle\rangle \in \mathcal{H}_\beta \otimes \bar{\mathcal{H}}_\beta \ , \tag{6.61}$$

that are characterized by

$$\left(W_n + (-1)^h (\Omega \bar{W})_{-n}\right) |\beta\rangle\rangle = 0 \ , \tag{6.62}$$

where h is the conformal dimension of W, and β runs over those representations for which $\Omega(\beta) = \beta$. (If Ω is inner, then this will be the case for all representations, but if Ω is outer, the β's will only run over some subset of representations.) The corresponding boundary states will then be of the form

$$\|u\rangle\rangle = \sum_\beta B_\beta(u) \, |\beta\rangle\rangle \ , \tag{6.63}$$

where $B_\beta(u)$ are some suitable coupling constants that depend on the structure of X (and the choice of Ω — if Ω is trivial, then they can be expressed in terms of the usual S-matrix elements).

In particular, we shall assume that they satisfy the Cardy condition [107], which means that the overlap takes the form

$$\langle\langle u \| e^{\pi i \hat{t}(L_0 + \bar{L}_0 - \frac{c}{12})} \, \|v\rangle\rangle = \sum_\beta \bar{B}_\beta(u) B_\beta(v) \, \chi_\beta(\hat{t}\,) = \sum_\alpha n^\alpha_{u|v} \, \chi_\alpha(t) \ , \tag{6.64}$$

where in the last step we have performed the S-modular transformation to the open string, writing $\hat{t} = -1/t$. In particular, the relative open string spectrum consists of the representations α appearing with multiplicity $n^\alpha_{u|v} \in \mathbb{N}_0$.

Lifting to the symmetric orbifold

We can lift these seed theory branes directly to the tensor product theory $\mathrm{X}^{\otimes N}$ by imposing the 'factorized' gluing conditions

$$W^{(i)}(z) = (\Omega \bar{W})^{(i)}(\bar{z}) \ , \qquad i = 1, \ldots, N \tag{6.65}$$

at $z = \bar{z}$. Here the $W^{(i)}(z)$ are the chiral fields in the i'th copy of $\mathrm{X}^{\otimes N}$, and Ω is the same for all factors, i.e. does not depend on i. The corresponding boundary states are then essentially just the tensor products of the above seed theory branes,

$$\|\mathbf{u}\rangle\rangle = \sum_{\underline{\beta}} B_{\underline{\beta}}(\mathbf{u}) \, |\underline{\beta}\rangle\rangle \ , \qquad B_{\underline{\beta}}(\mathbf{u}) = \prod_{i=1}^N B_{\beta_i}(u_i) \ , \tag{6.66}$$

where $\mathbf{u} = (u_1, \ldots, u_N)$ and $\underline{\beta} = (\beta_1, \ldots, \beta_N)$ are the obvious multi-indices, and the $B_{\beta_i}(u_i)$ are the coefficients appearing in eq. (6.63).

In the next step we now need to impose the S_N orbifold projection. For general $\mathbf{u}$, the boundary states are not orbifold invariant, and we need to sum over the images $\sigma(\mathbf{u})$, where σ acts on the multi-index $\mathbf{u}$ by permuting the entries. The simplest branes, however, arise provided we choose an S_N invariant $\mathbf{u}$, i.e. we take $\mathbf{u} = (u, \ldots, u)$. Then the above boundary state is by itself orbifold invariant, and it will give rise to a 'maximally-fractional' D-brane once we have added in the corresponding twisted sector Ishibashi states, as we are about to describe.[3]

[3]Obviously there are also in-between possibilities, e.g. the boundary state is invariant under some subgroup of S_N, and these branes can be constructed similarly. In what follows we shall concentrate on the 'maximally-fractional' case that is directly relevant for what we have in mind.

In order to explain the structure of the twisted sector Ishibashi states we need to introduce some notation. Let us consider the conjugacy class labelled by $[\sigma]$, where σ has the cycle shape corresponding to the partition

$$N = \sum_{j=1}^{r} l_j \ , \tag{6.67}$$

i.e. it consists of r cycles of length l_j, $j = 1, \ldots, r$. (Here we include also cycles of length one.) In the $[\sigma]$ twisted sector, the modes are then of the form

$$W_p^{[j]} \ , \qquad p \in \tfrac{1}{l_j}\mathbb{Z} \ , \quad (j = 1, \ldots, r) \ . \tag{6.68}$$

(This is to say, for each cycle we have one set of modes, but their mode numbers are now not in general integer, but rather run over $\frac{1}{l_j}\mathbb{Z}$.) The Ishibashi states that corresponds to the gluing condition (6.65) are then characterized by

$$\Big(W_p^{[j]} - (-1)^h (\Omega \bar{W}^{[j]})_{-p}\Big)\, |\underline{\beta}\rangle\rangle_{[\sigma]} = 0 \ , \qquad p \in \tfrac{1}{l_j}\mathbb{Z} \ , \quad (j = 1, \ldots, r) \ , \tag{6.69}$$

and they are now labelled by $\underline{\beta} = (\beta_1, \ldots, \beta_r)$, i.e. there is one β parameter for each cycle. The relevant overlap between two such Ishibashi states is

$${}_{[\sigma]}\langle\langle \underline{\beta}|\, e^{\pi i \hat{t}(L_0 + \bar{L}_0 - \frac{Nc}{12})}\, |\underline{\beta}\rangle\rangle_{[\sigma]} = \prod_{j=1}^{r} \chi_{\beta_j}(\tfrac{\hat{t}}{l_j}) \tag{6.70}$$

provided that the multi-indices (and the twisted sectors) agree, and zero otherwise.

With these preparations in place we can now write down the maximally fractional symmetric orbifold D-branes: they are labelled by a pair (u, ρ), where u labels the D-branes of the seed theory as in (6.63), while ρ is a representation of S_N, and they are explicitly given by

$$\|u,\rho\rangle\rangle = \sum_{(\underline{\beta},[\sigma])} B_{(\underline{\beta},[\sigma])}(u,\rho)\, |\underline{\beta}\rangle\rangle_{[\sigma]} \ , \qquad B_{(\underline{\beta},[\sigma])}(u,\rho) = \Big(\frac{|[\sigma]|}{N!}\Big)^{\frac{1}{2}} \chi_\rho([\sigma]) \prod_{j=1}^{r} B_{\beta_j}(u) \ , \tag{6.71}$$

where $|[\sigma]|$ denotes the number of elements in the conjugacy class $[\sigma]$, while $\chi_\rho([\sigma])$ is the character of the conjugacy class $[\sigma]$ in the representation ρ. Indeed, the overlap between two such boundary states equals then

$$\begin{aligned}
&\langle\langle u,\rho_1\|\, e^{\pi i \hat{t}(L_0 + \bar{L}_0 - \frac{|\omega| c}{12})}\, \|v,\rho_2\rangle\rangle \\
&\qquad = \frac{1}{N!} \sum_{\sigma \in S_N} \bar{\chi}_{\rho_1}(\sigma)\, \chi_{\rho_2}(\sigma) \prod_{j=1}^{r} \sum_{\beta_j} \bar{B}_{\beta_j}(u)\, B_{\beta_j}(v) \chi_{\beta_j}(\tfrac{\hat{t}}{l_j}) \\
&\qquad = \frac{1}{N!} \sum_{\sigma \in S_N} \bar{\chi}_{\rho_1}(\sigma)\, \chi_{\rho_2}(\sigma) \prod_{j=1}^{r} \sum_{\alpha} n^{\alpha}_{u|v}\, \chi_\alpha(l_j t) \\
&\qquad = \sum_{\alpha} n^{\alpha}_{u|v} \Big[\frac{1}{N!} \sum_{\sigma \in S_N} \bar{\chi}_{\rho_1}(\sigma)\, \chi_{\rho_2}(\sigma)\, \mathrm{Tr}_{\alpha^{\otimes N}}\big(\sigma e^{2\pi i t (L_0 - \frac{Nc}{24})}\big)\Big] \ ,
\end{aligned} \tag{6.72}$$

where in going to the middle line we have used (6.64) for each j. Finally, we note that the square bracket in the last line simply projects onto those states in $\alpha^{\otimes N}$ that transform in the representation $\rho_1 \otimes \rho_2^*$ of S_N; in particular, our D-branes therefore satisfy the Cardy condition [107].

The grand canonical ensemble

In our application to holography we will be interested in the D-branes for which $\rho_1 = \rho_2 = \text{id}$ is the trivial representation. In this case the group projection in (6.72) is to the S_N invariant states. It is then convenient not to work with a fixed S_N orbifold, but rather to introduce a fugacity p, and work in the grand canonical ensemble, for which the boundary partition function is defined to be

$$\mathfrak{Z}_{u|v}(p,t) = \sum_{N=1}^{\infty} p^N \mathcal{Z}^{S_N}_{(u,\text{id})|(v,\text{id})}(t) \ . \tag{6.73}$$

In order to massage this into a simpler form, we consider eq. (6.72) with $\rho_1 = \rho_2 = \text{id}$, which reads

$$Z^{S_N}_{(u,\text{id})|(v,\text{id})}(t) = \frac{1}{N!} \sum_{\sigma \in S_N} \prod_{j=1}^{r} Z_{u|v}(l_j t) \ . \tag{6.74}$$

The sum over elements $\sigma \in S_N$ depends only on the conjugacy class of the permutation, characterized by the partition $\sum_j l_j = N$. It is more convenient to write such a partition as a sum $\sum_j k_j n_j$, where k_j are *distinct* integers, and n_j are their multiplicities. In terms of these partitions, the size of the corresponding conjugacy class is given by

$$|C_{k_j,n_j}| = \frac{N!}{\prod_j k_j^{n_j} n_j!} \ , \tag{6.75}$$

and so the symmetric orbifold partition function takes the form

$$Z^{S_N}_{(u,\text{id})|(v,\text{id})}(t) = \sum_{\text{partitions of } N} \prod_j \frac{1}{k_j^{n_j} n_j!} Z_{u|v}(k_j t)^{n_j} \ . \tag{6.76}$$

From this, the grand canonical partition function is immediately calculated, and we find

$$\begin{aligned} \mathfrak{Z}_{u|v}(p,t) &= \sum_{N=1}^{\infty} p^N \sum_{\text{partitions of } N} \prod_j \frac{1}{k_j^{n_j} n_j!} Z_{u|v}(k_j t)^{n_j} \\ &= \prod_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{p^{nk}}{k^n n!} Z_{u|v}(kt)^n \right) = \exp\left(\sum_{k=1}^{\infty} \frac{p^k}{k} Z_{u|v}(kt) \right) \ . \end{aligned} \tag{6.77}$$

Moreover, if we expand the seed CFT partition function in terms of multiplicities, i.e.

$$Z_{u|v}(t) = \sum_h d_{u|v}(h) x^{h-\frac{c}{24}} \ , \tag{6.78}$$

then we can write $\mathfrak{Z}_{u|v}(p,t)$ in a manifestly multiparticle form analogous to (6.56). Indeed,

$$\begin{aligned} \mathcal{Z}_{u|v}(t) &= \exp\left(\sum_h d_{u|v}(h) \sum_{k=1}^{\infty} \frac{p^k}{k} x^{k(h-\frac{c}{24})} \right) \\ &= \exp\left(-\sum_h d_{u|v}(h) \log\left(1 - p x^{h-\frac{c}{24}}\right) \right) \\ &= \prod_h \left(1 - p\, x^{h-\frac{c}{24}}\right)^{-d_{u|v}(h)} \ . \end{aligned} \tag{6.79}$$

Just as in the bulk case, the grand canonical partition function computed between maximally fractional branes also admits a nice geometric interpretation in terms of covering spaces of the cylinder. Since a cylinder has vanishing Euler characteristic, its only (unramified) covering spaces are disjoint unions of cylinders whose moduli are integer multiples of the modulus t. Considering only the connected component of (6.77), namely

$$\log \mathfrak{Z}_{u|v}(p,t) = \sum_{k=1}^{\infty} \frac{p^k}{k} Z_{u|v}(kt) \ , \tag{6.80}$$

we see that the coefficient of p^k in this expansion is then the seed-CFT partition function evaluated on an cylinder with modulus kt, which is indeed the only connected covering space of the base cylinder. Furthermore, the factor of $1/k$ again accounts for the size of the automorphism group $\mathbb{Z}_k$ of the covering. In analogy with the bulk calculation, one would expect that the mechanism through which (6.77) is reproduced in AdS_3 is via the worldsheet modulus integral localizing to only those worldsheets which holomorphically cover the cylinder on the boundary of AdS_3. This is indeed the case, as we will see in Section 6.4.

6.4 Matching of the cylinder amplitudes

In this section we return to the worldsheet perspective. We begin by constructing the full D-branes of the worldsheet theory. This is to say we combine a brane from $\mathfrak{psu}(1,1|2)_1$ with a boundary state for the $\mathbb{T}^4$ sector, as well as one for the ghosts. We then determine their cylinder amplitudes. On the worldsheet we need to integrate over the modular parameter τ, and as we shall see, this integral can actually be done explicitly. For the case of the spherical branes on AdS_3, the resulting cylinder amplitude agrees then precisely with a cylinder amplitude in the dual symmetric orbifold CFT. This suggests that there is a direct correspondence between the spherical D-branes of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ and the maximally fractional D-branes in the symmetric orbifold of $\mathbb{T}^4$.

We also do a similar computation in the open string channel (both on the worldsheet and in the dual CFT), and not surprisingly, but somewhat nontrivially, the result also agrees. This seems to suggest that the open-closed duality (Cardy condition) on the worldsheet is essentially equivalent to that in the dual CFT, at least for these spherical branes.

Closed-string channel calculation

Let us consider the worldsheet boundary states of the form

$$\|u,\varepsilon\rangle\!\rangle_{\mathrm{S}} = \|\varepsilon\rangle\!\rangle_{\mathrm{S}}\, \|u,\mathrm{R},\varepsilon\rangle\!\rangle_{\mathbb{T}^4}\, \|\mathrm{ghost},\varepsilon\rangle\!\rangle \ , \tag{6.81}$$

where $\|\varepsilon\rangle\!\rangle_{\mathrm{S}}$ is the $\mathfrak{psu}(1,1|2)_1$ boundary state (6.42), while $\|u,\mathrm{R},\varepsilon\rangle\!\rangle_{\mathbb{T}^4}$ and $\|\mathrm{ghost},\varepsilon\rangle\!\rangle$ are boundary states for the $\mathbb{T}^4$ and the $\rho\sigma$ ghost system, respectively. Since the $\mathbb{T}^4$ is topologically twisted, the torus boundary states just consist of the RR part of a usual supersymmetric $\mathbb{T}^4$ boundary state. Note that, as usual, we have aligned the spin structure ε across the three factors.

We are interested in the worldsheet cylinder amplitude,

$$\hat{Z}^{\mathrm{w}}_{u|v}(\hat{t},\hat{\zeta};\hat{\tau}) = {}_{\mathrm{S}}\langle\!\langle v,\mp\| \hat{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}\hat{x}^{\frac{1}{2}(J_0^3+\bar{J}_0^3)}\hat{y}^{\frac{1}{2}(K_0^3-\bar{K}_0^3)}\|u,\pm\rangle\!\rangle_{\mathrm{S}} \ , \tag{6.82}$$

where we have inserted a $(-1)^F$ factor (i.e. considered opposite spin structures for the two boundary states), as is convenient for supergroup CFTs. Here $\hat{q} = e^{2\pi i\hat{\tau}}$ describes the propagation along the worldsheet time, while $\hat{x} = e^{2\pi i\hat{t}}$ with $\hat{t}$ real measures the separation between the two branes along the boundary of AdS_3, i.e. it separates the two spherical branes by the distance $\hat{t}$ along the boundary of AdS_3, see eq. (6.34).

Given that the boundary state in (6.81) factorizes, the same will be true for the cylinder diagram (6.82). The explicit form of the $\mathfrak{psu}(1,1|2)_1$ overlap was already derived above, except that we now also want to introduce the $\mathfrak{su}(2)$ chemical potential parametrized by $\hat{\zeta}$. Using eq. (6.22), this modifies (6.38b) to

$$\hat{Z}^{\text{S}}(\hat{t},\hat{\zeta};\hat{\tau}) = \sum_{w\in\mathbb{Z}} \int_0^1 d\lambda (-1)^w \hat{q}^{\frac{w^2}{2}} \sum_{r\in\mathbb{Z}+\lambda} \hat{x}^r \hat{q}^{-rw} \frac{\vartheta_1(\frac{\hat{t}+\hat{\zeta}}{2};\hat{\tau})\vartheta_1(\frac{\hat{t}-\hat{\zeta}}{2};\hat{\tau})}{\eta(\hat{\tau})^4} , \qquad (6.83)$$

where relative to eq. (6.38b) we have set $W_1 = W_2 = 0$ and $\Lambda_1 = \Lambda_2 = \frac{1}{2}$. Next we use

$$\int_0^1 d\lambda \sum_{r\in\mathbb{Z}+\lambda} e^{2\pi i r(\hat{t}-w\hat{\tau})} = \delta(\hat{t}-w\hat{\tau}) = \frac{1}{w}\delta(\tfrac{\hat{t}}{w}-\hat{\tau}) , \qquad (6.84)$$

as well as the theta function identity

$$\vartheta_1(\tfrac{w\hat{\tau}\pm\hat{\zeta}}{2};\hat{\tau}) = \begin{cases} \hat{q}^{-\frac{w^2}{8}} e^{\frac{\pi i w}{2}} \vartheta_1(\pm\frac{\hat{\zeta}}{2};\hat{\tau}) & \text{if } w\in 2\mathbb{Z} \\ \hat{q}^{-\frac{w^2}{8}} e^{\frac{\pi i w}{2}} \vartheta_4(\pm\frac{\hat{\zeta}}{2};\hat{\tau}) & \text{if } w\in 2\mathbb{Z}+1, \end{cases} \qquad (6.85)$$

to rewrite (6.83) as

$$\begin{aligned}\hat{Z}^{\text{S}}(\hat{t},\hat{\zeta};\hat{\tau}) = &\sum_{\substack{w=1\\ w \text{ even}}}^{\infty} \frac{1}{w}\hat{x}^{\frac{w}{4}}\delta(\tfrac{\hat{t}}{w}-\hat{\tau})\frac{\vartheta_1(+\frac{\hat{\zeta}}{2};\hat{\tau})\vartheta_1(-\frac{\hat{\zeta}}{2};\hat{\tau})}{\eta(\hat{\tau})^4} + \\ &+\sum_{\substack{w=1\\ w \text{ odd}}}^{\infty} \frac{1}{w}\hat{x}^{\frac{w}{4}}\delta(\tfrac{\hat{t}}{w}-\hat{\tau})\frac{\vartheta_4(+\frac{\hat{\zeta}}{2};\hat{\tau})\vartheta_4(-\frac{\hat{\zeta}}{2};\hat{\tau})}{\eta(\hat{\tau})^4} . \end{aligned} \qquad (6.86)$$

Here we have used that, because of the delta function $\delta(\hat{t}-w\hat{\tau})$ in the $\mathfrak{psu}(1,1|2)_1$ overlap, only positive values of w contribute. Geometrically, the delta function means that the spacetime cylinder modulus $\hat{t}$ localizes to those values $w\hat{\tau}$, for which a holomorphic covering map from the worldsheet cylinder exists, see the discussion in Section 6.3; this is obviously the analogue of what was found in [45, 47, 78].

As regards the ghost part overlap, we postulate that it leads to

$$\hat{Z}_{\text{ghost}}(\hat{\tau}) = \frac{\eta(\hat{\tau})^4}{\vartheta_1(0;\hat{\tau})\vartheta_1(0;\hat{\tau})} , \qquad (6.87)$$

since the ghosts do not carry any chemical potential.[4] This is somewhat different than what was proposed in [45], but it ties in naturally with [84]; in particular, the $\mathfrak{su}(2)$ quantum numbers are already correctly accounted for in terms of the $\mathfrak{su}(2)$

[4]Strictly speaking the denominator of eq. (6.87) is divergent because of the zero modes in the ϑ_1 factor. This will cancel against a similar contribution from the numerator in eq. (6.88) below.

subalgebra of $\mathfrak{psu}(1,1|2)$ (and there is no need to involve any chemical potential from the $\mathbb{T}^4$ or the ghost sector). Finally, the $\mathbb{T}^4$ part gives

$$\hat{Z}^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{R}}}(\hat{\tau}) = \frac{\vartheta_1(0;\hat{\tau})\vartheta_1(0;\hat{\tau})}{\eta(\hat{\tau})^6}\hat{\Theta}^{\mathbb{T}^4}_{u|v}(\hat{\tau}) \ , \tag{6.88}$$

where the $\hat{\Theta}$ function describes the winding and momentum modes. Now the numerator of eq. (6.88) cancels precisely with the denominator of eq. (6.87), while the numerator of eq. (6.87) cancels, say the bosonic oscillators in eq. (6.86). Finally, putting all of these pieces together and performing the $\hat{\tau}$ integral we find

$$\int_0^\infty d\hat{\tau}\, \hat{Z}^{\mathrm{w}}_{u|v}(\hat{t},\hat{\zeta};\hat{\tau}) = \sum_{\substack{w=1\\ w\text{ even}}}^{\infty} \frac{1}{w}\hat{x}^{\frac{w}{4}}\hat{Z}^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{R}}}(\hat{\zeta},\tfrac{\hat{t}}{w}) + \sum_{\substack{w=1\\ w\text{ odd}}}^{\infty} \frac{1}{w}\hat{x}^{\frac{w}{4}}\hat{Z}^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{NS}}}(\hat{\zeta},\tfrac{\hat{t}}{w}) \ , \tag{6.89}$$

where

$$\hat{Z}^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{R}}}(\hat{\zeta},\tfrac{\hat{t}}{w}) = \frac{\vartheta_1(+\frac{\hat{\zeta}}{2};\frac{\hat{t}}{w})\vartheta_1(-\frac{\hat{\zeta}}{2};\frac{\hat{t}}{w})}{\eta(\frac{\hat{t}}{w})^6}\,\hat{\Theta}^{\mathbb{T}^4}_{u|v}(\tfrac{\hat{t}}{w}) \ , \tag{6.90}$$

$$\hat{Z}^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{NS}}}(\hat{\zeta},\tfrac{\hat{t}}{w}) = \frac{\vartheta_4(+\frac{\hat{\zeta}}{2};\frac{\hat{t}}{w})\vartheta_4(-\frac{\hat{\zeta}}{2};\frac{\hat{t}}{w})}{\eta(\frac{\hat{t}}{w})^6}\,\hat{\Theta}^{\mathbb{T}^4}_{u|v}(\tfrac{\hat{t}}{w}) \ . \tag{6.91}$$

In order to compare this to the dual CFT answer we now need to recall that the worldsheet theory only sees the single particle sector of the symmetric orbifold theory. Thus in order to match with the dual CFT we need to include string field theoretic multi-worldsheet states. This is most conveniently done as in [73], i.e. by introducing a chemical potential σ (with the associated fugacity $p = e^{2\pi i\sigma}$) for the number of F-strings wound around the bulk of the AdS_3, and working in a grand canonical ensemble,[5]

$$\hat{\mathfrak{Z}}_{u|v}(\hat{\zeta};p,\hat{t}) = \exp\Bigg(\sum_{\substack{w=1\\ w\text{ even}}}^{\infty} \frac{p^w}{w}\hat{Z}^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{R}}}(\hat{\zeta},\tfrac{\hat{t}}{w}) + \sum_{\substack{w=1\\ w\text{ odd}}}^{\infty} \frac{p^w}{w}\hat{Z}^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{NS}}}(\hat{\zeta},\tfrac{\hat{t}}{w})\Bigg) \ . \tag{6.92}$$

This then matches precisely with the symmetric orbifold answer, see eq. (6.178).

Two comments are in order at this point. First, note that the difference $\Lambda_1 - \Lambda_2$ of the worldsheet $\mathfrak{psu}(1,1|2)_1$ Wilson lines would introduce the phase factor $e^{2\pi i w(\Lambda_1-\Lambda_2)}$ into the sum over w in (6.92). This is therefore equivalent to multiplying the w-twisted part of the $\mathrm{Sym}(\mathbb{T}^4)$ overlap by the same phase, which in turn can be reproduced by multiplying the $\mathrm{Sym}_N(\mathbb{T}^4)$ boundary states at fixed N by the global phase $e^{2\pi i\Lambda_i}$. This is therefore invisible at fixed N — multiplying all the boundary states by a fixed phase is invisible in all cylinder diagrams.

Second, we note that the whole analysis would also go through (with minor modifications) had we not twisted the worldsheet overlaps with $(-1)^F$. In that case, we would instead end up with an NS cylinder correlator in $\mathrm{Sym}(\mathbb{T}^4)$.

[5] In analogy with the sphere contribution to the closed string partition function of [45], it is natural to assume that the contribution to the path integral due to two disconnected discs accounts for the ground-state shift $\hat{x}^{-\frac{w}{4}}$ in the individual w-twisted sectors.

Open-string channel

We can also redo the calculation of the cylinder string amplitude in the open-string channel, where, by virtue of the open-closed duality, we expect to obtain the same result as in the closed-string channel. The key ingredient here is the $(-1)^F$-twisted worldsheet boundary partition function

$$Z^{\mathrm{w}}_{u|v}(t,\zeta;\tau) = \mathrm{Tr}_{\mathcal{H}_{u|v}}\left((-1)^F\, q^{L_0-\frac{c}{24}}\, e^{-2\pi i\frac{\tau}{t}J^3_0}\, e^{2\pi i\frac{\tau}{t}\zeta K^3_0}\right) , \tag{6.93}$$

which can be obtained by modular S-transforming the boundary states overlap (6.82) with the relations

$$\hat{\tau} = -\tfrac{1}{\tau} , \qquad \hat{t} = -\tfrac{1}{t} , \qquad \hat{\zeta} = -\hat{t}\,\zeta \tag{6.94}$$

between the open- and closed-string channel parameters. The trace (6.93) again factorizes into the $\mathfrak{psu}(1,1|2)_1$ WZW part, the ghost part and the $\mathbb{T}^4$ part. First, starting from (6.39) with $\Lambda_1=\Lambda_2=\frac{1}{2}$, $W_1=W_2=0$ and substituting the explicit form (6.22) for the $\mathscr{F}_{1/2}$ character, we obtain the $\mathfrak{psu}(1,1|2)_1$ part of the boundary partition function

$$\begin{aligned} Z^{\mathrm{S}}(t,\zeta;\tau) = \tau \sum_{\substack{k=1\\ k\text{ even}}}^{\infty} \frac{1}{k}\hat{x}^{\frac{k}{4}} e^{\frac{\pi i k\zeta^2}{2t}}\,\delta(\tau-kt)\,\frac{\vartheta_1(+\frac{k\zeta}{2};\tau)\vartheta_1(-\frac{k\zeta}{2};\tau)}{\eta(\tau)^4} \\ -\tau \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty} \frac{1}{k}\hat{x}^{\frac{k}{4}} e^{\frac{\pi i k\zeta^2}{2t}}\,\delta(\tau-kt)\,\frac{\vartheta_2(+\frac{k\zeta}{2};\tau)\vartheta_2(-\frac{k\zeta}{2};\tau)}{\eta(\tau)^4} , \end{aligned} \tag{6.95}$$

where the sum over k corresponds to the sum over w in (6.86). In the process, we have also used the θ-function identity

$$\vartheta_1(-\tfrac{k}{2};\tau) = \begin{cases} (-1)^{\frac{k}{2}}\vartheta_1(0;\tau) & \text{if } k\in 2\mathbb{Z} \\ (-1)^{\frac{k-1}{2}}\vartheta_2(0;\tau) & \text{if } k\in 2\mathbb{Z}+1\,. \end{cases} \tag{6.96}$$

Note that as for the closed-string overlap (6.86), the boundary partition function (6.95) is δ-function localized at $\tau = kt$ for $k\in\mathbb{Z}$; this corresponds precisely to those values of the spacetime cylinder modulus t where a holomorphic covering map from the worldsheet cylinder exists. Furthermore, since $t,\tau\geqslant 0$, we can restrict again to $k\geqslant 1$. The modular transformation of the ghost and $\mathbb{T}^4$ factors, see eq. (6.87) and (6.88) respectively, are standard,

$$Z^{\mathrm{ghost}}(\tau) = -\frac{\eta(\tau)^4}{\vartheta_1(0;\tau)\vartheta_1(0;\tau)} , \qquad Z^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{R}}}(\tau) = -\frac{\vartheta_1(0;\tau)\vartheta_1(+0;\tau)}{\eta(\tau)^6}\Theta^{\mathbb{T}^4}_{u|v}(\tau) , \tag{6.97}$$

where the overall minus signs are included to compensate for the properties of the ϑ_1 functions under the modular S-transformation.

Altogether, multiplying (6.95) and (6.97) and integrating over the period modulus of the worldsheet cylinder, we obtain the amplitude[6]

$$\int_0^\infty \frac{d\tau}{\tau}\, Z^{\mathrm{w}}_{u|v}(t,\zeta;\tau) = \sum_{\substack{k=1\\ k\text{ even}}}^{\infty} \frac{1}{k}\hat{x}^{\frac{k}{4}} e^{\frac{\pi i k\zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{R}}}(k\zeta;kt) + \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty} \frac{1}{k}\hat{x}^{\frac{k}{4}} e^{\frac{\pi i k\zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v,\mathrm{R}}(k\zeta;kt) , \tag{6.98}$$

[6]Here we have also included the usual factor of τ in the worldsheet measure that comes from the modular transformation of the ghosts.

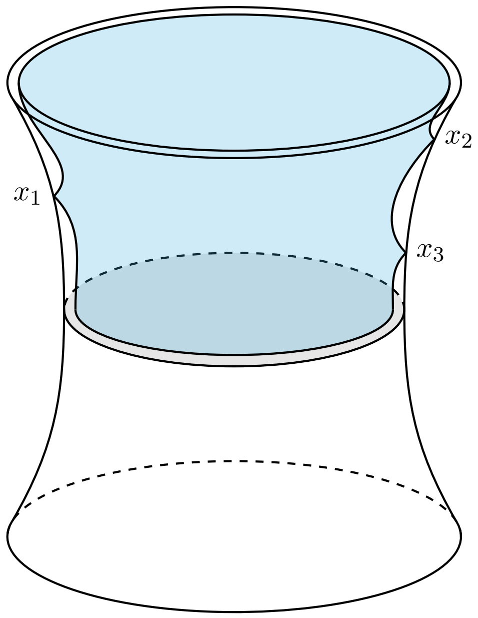

Figure 6.2: The worldsheet configuration associated to the tree-level scattering of states $V^{w_i}_{h_i,j_i}(x_i,z_i)$ in the spherical-brane background.

where the $\mathbb{T}^4$ boundary partition function $Z^{\mathbb{T}^4}_{i|j,\mathrm{R}}$ can be obtained from (6.97) by replacing $\vartheta_1 \to i\vartheta_2$.

As before, the worldsheet analysis only captures the single string spectrum, and in order to relate this to the full dual CFT we need to go to the grand canonical ensemble by introducing a chemical potential p for the number of times k the worldsheet cylinder covers the boundary spacetime cylinder. Eliminating the $\hat{x}^{k/4}$ factor as before by taking into account the contribution to the string amplitude due to two disconnected discs, we obtain the 1-loop open-string partition function

$$\mathfrak{Z}_{u|v}(\zeta;p,t) = \exp\left(\sum_{\substack{k=1 \\ k \text{ even}}}^{\infty} \frac{p^k}{k} e^{\frac{\pi i k \zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{R}}}(k\zeta;kt) + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{p^k}{k} e^{\frac{\pi i k \zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v,\mathrm{R}}(k\zeta;kt) \right). \tag{6.99}$$

This is exactly equal to the partition function (6.179) for the Ramond boundary spectrum of the $\mathrm{Sym}(\mathbb{T}^4)$ maximally-fractional boundary states.

6.5 Matching correlation functions

We saw above the the worldsheet overlap of two spherical brane states in the $\mathfrak{psu}(1,1|2)_1$ model, tensored with specific boundary states in the $\mathbb{T}^4$ theory, reproduced the (single-particle part of the) overlap of maximally spherical branes in the dual symmetric orbifold theory. This essentially showed that the cylinder diagram on the worldsheet reproduces the cylinder diagram on the boundary theory.

It is also natural to consider the correlation function of asymptotic states $V^w_{h,j}(x,z)$ in the presence of a spherical D-brane, such as in Figure 6.2. As we saw in Chapter 5, it is convenient to consider correlation functions in the free field realization. Thus, in order to study disk correlators on the worldsheet, it is important to discuss how the spherical branes function from the point of view of the free fields.

The boundary conditions (6.28) can be written in terms of the free fields as[7]

$$\begin{aligned} \lambda(z) &= \overline{\mu}(\overline{z})\,, \quad \mu(z) = \overline{\lambda}(\overline{z})\,, \\ \lambda^\dagger(z) &= \overline{\mu}^\dagger(\overline{z})\,, \quad \mu^\dagger(z) = \overline{\lambda}^\dagger(\overline{z})\,, \end{aligned} \tag{6.100}$$

Indeed, this is simply the trivial gluing condition for the free fields twisted by the outer automorphism

$$\begin{pmatrix} \mu \\ \lambda \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \tag{6.101}$$

of the free field theory, which acts on the $\mathfrak{sl}(2,\mathbb{R})$ generators as conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Furthermore, the boundary conditions for AdS_2 branes are given by

$$\begin{aligned} \lambda(z) &= \overline{\lambda}(\overline{z})\,, \quad \mu(z) = \overline{\mu}(\overline{z})\,, \\ \lambda^\dagger(z) &= \overline{\lambda^\dagger}(\overline{z})\,, \quad \mu^\dagger(z) = \overline{\mu}^\dagger(\overline{z})\,. \end{aligned} \tag{6.102}$$

The worldsheet doubling trick

We now turn our attention to the correlation function

$$\Big\langle \prod_{i=1}^n V^{w_i}_{m_i,j_i}(x_i,z_i) \Big\rangle_B \equiv \Big\langle \prod_{i=1}^n V^{w_i}_{m_i,j_i}(x_i,z_i) \, \Big\| B \Big\rangle\Big\rangle \,, \tag{6.103}$$

where the index B on the left-hand-side indicates that this correlator is evaluated in the presence of the boundary state $\|B\rangle\rangle$. Since we take the boundary state $\|B\rangle\rangle$ to be located on the real line, the above correlator is taken on the upper half-plane $\mathbb{H}$, which is conformally equivalent to the unit disk $\mathbb{D}$. To start off we shall consider the case where the boundary state B corresponds to a spherical D-brane; the situation for the AdS_2 branes is discussed in Section 6.5.

In order to compute the correlation function (6.103), we now use the same trick as for the correlators on the sphere. That is, we define the correlator-valued complex functions

$$\mathcal{L}(z) = \Big\langle \lambda(z) \prod_{i=1}^n V^{w_i}_{m_i,j_i}(x_i,z_i) \Big\rangle_{(u,\varepsilon)_\mathrm{S}} \,, \quad \mathcal{M}(z) = \Big\langle \mu(z) \prod_{i=1}^n V^{w_i}_{m_i,j_i}(x_i,z_i) \Big\rangle_{(u,\varepsilon)_\mathrm{S}} \tag{6.104}$$

where $(u,\varepsilon)_\mathrm{S}$ indicates that the boundary condition is a spherical brane. We now want to explore their complex-analytic properties, and for this we can use a doubling trick to consider a function on the full Riemann sphere instead of on the disk.

To do this, note that the boundary conditions (6.100), along with the left/right symmetry of the worldsheet theory, tell us that if we define

$$\begin{aligned} \mathscr{L}(z) &:= \begin{cases} \mathcal{L}(z)\,, & \Im(z) > 0 \\ \overline{\mathcal{M}}(\overline{z})\,, & \Im(z) < 0\,, \end{cases} \\ \mathscr{M}(z) &:= \begin{cases} \mathcal{M}(z)\,, & \Im(z) > 0 \\ \overline{\mathcal{L}}(\overline{z})\,, & \Im(z) < 0\,, \end{cases} \end{aligned} \tag{6.105}$$

[7] Here, we focus only on the boundary conditions for the bosons. The boundary conditions for the fermions are listed in Section 3 of [50].

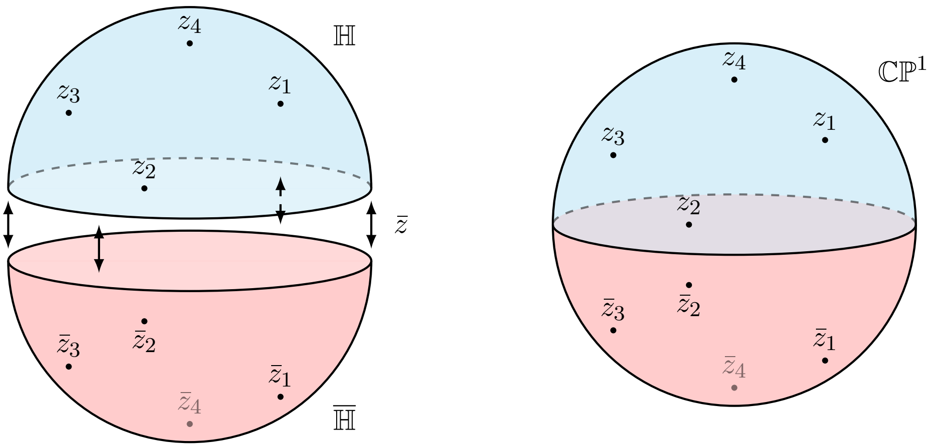

Figure 6.3: The doubling trick. We compute correlators on the upper-half-plane $\mathbb{H}$ by considering a correlator on the 'doubled' worldsheet, given by the Riemann sphere.

to be meromorphic functions on the full Riemann sphere, then they satisfy the functional equation

$$\mathscr{L}(z) = (\mathscr{M}(\overline{z}))^* \,. \tag{6.106}$$

Now, we can use complex analysis on the sphere to determine the form of $\mathscr{M}, \mathscr{L}$. Based on the OPEs (5.60) and (5.61) between the free fields μ, λ and $V^w_{m,j}$, we find that the functions $\mathscr{M}, \mathscr{L}$ have the following properties:

1. Since μ, λ have conformal weight $h = -1/2$, the functions $\mathscr{M}, \mathscr{L}$ grow as $\mathcal{O}(z)$ as $z \to \infty$.
2. $\mathscr{M}, \mathscr{L}$ have poles of order $\frac{w_i - 1}{2}$ at $z = z_i$ and $z = \bar{z}_i$.
3. The linear combination $\mathscr{M} - x_i \mathscr{L}$ has a zero of order $\frac{w_i+1}{2}$ at $z = z_i$.
4. The linear combination $\mathscr{L} - x_i \mathscr{M}$ has a zero of order $\frac{w_i+1}{2}$ at $z = \bar{z}_i$.

Given these properties of the functions $\Omega^{\pm}(z)$, we can immediately implement the tricks we used in Chapter 5 to constrain the form of the worldsheet correlators. We define polynomials

$$\begin{aligned} Q(z) &= \prod_{i=1}^{n} (z - z_i)^{\frac{w_i-1}{2}} \prod_{i=1}^{n} (z - \overline{z}_i)^{\frac{w_i-1}{2}} \mathscr{M}(z) \,, \\ P(z) &= \prod_{i=1}^{n} (z - z_i)^{\frac{w_i-1}{2}} \prod_{i=1}^{n} (z - \overline{z}_i)^{\frac{w_i-1}{2}} \mathscr{L}(z) \,. \end{aligned} \tag{6.107}$$

Then Q, P are meromorphic functions on the sphere whose only poles are at $z = \infty$, and so they are polynomials of degree

$$N = 1 + \sum_{i=1}^{n} (w_i - 1) \,. \tag{6.108}$$

No, if we define a map

$$\Gamma(z) := \frac{Q(z)}{P(z)} \,, \tag{6.109}$$

then $\Gamma(z)$ satisfies the functional equation $\Gamma(z) = 1/\overline{\Gamma(\bar{z})}$, or specifically

$$|\Gamma(z)|^2 = 1\,, \quad z \in \mathbb{R}\,. \tag{6.110}$$

Moreover, by properties 3. and 4., one can show that Γ has critical points at $z = z_i$ and $z = \bar{z}_i$, i.e.

$$\begin{aligned} \Gamma(z) &\sim x_i + \mathcal{O}\big((z-z_i)^{w_i}\big)\,, && z \to z_i\,,\\ \Gamma(z) &\sim \frac{1}{\bar{x}_i} + \mathcal{O}\big((z-\bar{z}_i)^{w_i}\big)\,, && z \to \bar{z}_i\,. \end{aligned} \tag{6.111}$$

Finally, Γ, as a topological map, has degree

$$N = 1 + \sum_{i=1}^{n} (w_i - 1)\,, \tag{6.112}$$

which is precisely the degree of a disk covering map with critical points of order w_i. Together, (6.111) and (6.112) demonstrate that the function Γ has precisely the form of a covering map from the upper half-plane to the disk (see Appendix 6.C). Thus, by the same arguments that apply in the bulk case, we see that the correlation functions (6.103) localize to those points in the moduli space where a covering map $\Gamma : \mathbb{H} \to \mathbb{D}$ exists. Furthermore, as shown there, see eq. (6.191), the corresponding locus has the correct codimension to turn the worldsheet moduli space integral into a finite sum.

The case for AdS_2 branes

As an aside we can also preform a similar analysis for the boundary states $\|\mathrm{AdS}_2\rangle\!\rangle$ corresponding to the AdS_2 branes. In this case, the boundary conditions (6.102) instruct us to define

$$\begin{aligned} \mathscr{L}(z) &:= \begin{cases} \mathcal{L}(z)\,, & \Im(z) > 0\\ \overline{\mathcal{L}}(\overline{z})\,, & \Im(z) < 0\,, \end{cases}\\ \mathscr{M}(z) &:= \begin{cases} \mathcal{M}(z)\,, & \Im(z) > 0\\ \overline{\mathcal{M}}(\overline{z})\,, & \Im(z) < 0\,. \end{cases} \end{aligned} \tag{6.113}$$

With this definition, the functions $\mathscr{M}, \mathscr{L}$ have poles of degree $\frac{w_i-1}{2}$ at $z = z_i$ and $z = \overline{z}_i$, and satisfy the regularity conditions

$$\begin{aligned} \mathscr{M}(z) - x_i \mathscr{L}(z) &\sim \mathcal{O}\left((z-z_i)^{\frac{w_i+1}{2}}\right)\,,\\ \mathscr{M}(z) - \overline{x}_i \mathscr{L}(z) &\sim \mathcal{O}\left((z-\overline{z}_i)^{\frac{w_i+1}{2}}\right)\,. \end{aligned} \tag{6.114}$$

As usual, defining polynomials

$$\begin{aligned} Q(z) &= \prod_{i=1}^{n} (z-z_i)^{\frac{w_i-1}{2}} \prod_{i=1}^{n} (z-\overline{z}_i)^{\frac{w_i-1}{2}} \mathscr{M}(z)\,,\\ P(z) &= \prod_{i=1}^{n} (z-z_i)^{\frac{w_i-1}{2}} \prod_{i=1}^{n} (z-\overline{z}_i)^{\frac{w_i-1}{2}} \mathscr{L}(z)\,. \end{aligned} \tag{6.115}$$

we see that the corresponding covering map constructed on the worldsheet as $\Gamma = Q/P$ then satisfies the functional equation

$$\Gamma(z) = \overline{\Gamma(\bar{z})} \,, \tag{6.116}$$

and has appropriate critical points at $z = z_i$ and $z = \bar{z}_i$ with $\Gamma(z_i) = x_i$ and $\Gamma(\bar{z}_i) = \bar{x}_i$. Therefore, by an argument identical to the one for spherical branes, the function Γ satisfies precisely the properties of a covering map $\Gamma : \mathbb{H} \to \mathbb{H}$ (see Appendix 6.C). This implies that the correlation functions of n bulk strings in the presence of an AdS_2 brane localize to the points in the moduli space where these covering maps exist. This gives strong support to the idea that also the AdS_2 brane must correspond to a D-brane in the symmetric orbifold theory. However, so far, we have not managed to identify the corresponding D-brane explicitly.

Comparison to the symmetric orbifold

As we have mentioned before, the symmetric orbifold correlation functions on the sphere can be calculated by employing ramified covering maps $\Gamma : \Sigma \to \mathbb{CP}^1$ to lift the twisted sector states to a surface on which they are single-valued [80]. Such correlation functions can be expressed as a sum over such covering maps in a Feynman diagrammatic expansion [77]

$$\langle \mathcal{O}_1^{w_1}(x_1) \cdots \mathcal{O}_n^{w_n}(x_n) \rangle = \sum_{\Gamma} C_\Gamma \, \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle_{\text{seed}} \,, \tag{6.117}$$

where C_Γ is the conformal factor that is obtained by pulling back from the base space by Γ, and the correlators on the right-hand-side are seed theory correlators evaluated on the covering space Σ.

One would expect that for the symmetric product correlators on a disk a similar construction should work, where now the appropriate covering maps are of the form $\Gamma : \Sigma \to \mathbb{D}$. For the maximally fractional branes (u, id) constructed in Section 6.3, one would expect the natural generalisation of equation (6.117) to be

$$\langle \mathcal{O}_1^{w_1}(x_1) \cdots \mathcal{O}_n^{w_n}(x_n) \rangle_{(u,\mathrm{id})} = \sum_{\Gamma:\Sigma \to \mathbb{D}} C_\Gamma \, \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle_{u,\text{seed}} \,, \tag{6.118}$$

where the right-hand-side is a sum over seed theory correlators evaluated on the covering space Σ and with boundary condition u along $\partial\Sigma$. For the genus zero contribution, i.e. for the case that Σ is a disk, the correlation functions on the right-hand-side can then be calculated as $2n$-point functions on the sphere using the appropriate doubling trick, in complete parallel to the worldsheet computation from above. In this way, the correlation functions calculated on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ naturally reproduce the structure that is expected from the symmetric orbifold perspective.

In particular, this allows us to compare the bulk-boundary coefficients of the two sets of branes. In the symmetric orbifold CFT, these are given by the second equation of (6.71), where we consider $\rho = \mathrm{id}$, and focus on the contribution of a single w-cycle σ; the combinatorial prefactor $\sqrt{|[\sigma]|/N!}$ equals then

$$\frac{1}{\sqrt{w}} \frac{1}{\sqrt{(N-w)!}} \,. \tag{6.119}$$

The second factor, $1/\sqrt{(N-w)!}$, is just the symmetry factor associated with the $N-w$ identical 1-cycle states, and thus the 'single-particle' one-point function from the w-cycle twisted sector equals

$$\frac{1}{\sqrt{w}}\, B_\beta(u) \ . \tag{6.120}$$

This now agrees precisely with the worldsheet bulk-boundary coefficient associated to the brane of the form (6.81). In particular, the factor $B_\beta(u)$ just comes from the $\mathbb{T}^4$ boundary state $\|u,\mathrm{R},\varepsilon\rangle\!\rangle_{\mathbb{T}^4}$ in (6.81),[8] while the prefactor $1/\sqrt{w}$ reflects the relative normalisation of the worldsheet and spacetime disk correlators, since the worldsheet disk covers the spacetime disk w times.

6.A Some boundary CFT

In this appendix, we review some of the basic ideas of boundary conformal field theory. Standard references on the subject include the seminal papers of Cardy [107, 108], as well as the textbook [109] and references therein.

The idea of boundary conformal field theory is to generalize the standard construction of conformal field theories on, say, compact Riemann surfaces Σ, to field theories on surfaces which have non-empty boundaries. Since the resulting field theory is no longer defined on a compact space, a well-defined path integral requires fixing some set of boundary conditions. In the operator language of CFT, the boundary data imposed at $\partial\Sigma$ corresponds to preparing a so-called *boundary state*, often denoted with a doubled ket $\|\psi\rangle\!\rangle$, at each of the components of $\partial\Sigma$. Boundary states are inherently nonlocal, and are described implicitly through their so-called *gluing conditions*.

As a simple example, let us take a single free boson Φ, and formulate the theory on the upper half-plane $\mathbb{H}$, whose boundary is the real line.[9] There are many possible boundary conditions which can be chosen, but the simplest are:

$$\begin{aligned} \text{Dirichlet:} \quad & \Phi(x) = \Phi_0 \,, \\ \text{Neumann:} \quad & (\partial - \overline{\partial})\Phi(x) = 0 \,, \end{aligned} \tag{6.121}$$

for all $x \in \mathbb{R}$. In the context of string theory, Dirichlet boundary conditions correspond to a string whose endpoints are *fixed* at some location Φ_0, while Neumann boundary conditions allow the motion of the string to be free, with the condition that no momentum flows out of the endpoints.

In order to impose the above boundary condition, we consider the mode expansion of Φ. Under the equations of motion, Φ can be decomposed into left- and right-moving parts, and we have

$$\begin{aligned} \Phi_L(z) &= \frac{x_0}{2} + \alpha_0 \log(z) + \sum_{n\neq 0} \frac{1}{n}\alpha_n z^{-n} \,, \\ \Phi_R(\overline{z}) &= \frac{x_0}{2} + \overline{\alpha}_0 \log(\overline{z}) + \sum_{n\neq 0} \frac{1}{n}\overline{\alpha}_n \overline{z}^{-n} \,. \end{aligned} \tag{6.122}$$

[8] The boundary state in (6.81) is always evaluated in the RR sector (since the $\mathbb{T}^4$ is topologically twisted), while for the symmetric orbifold $B_\beta(u)$ labels the coefficient in the RR sector if w is even, and in the NSNS sector if w is odd. However, because of spectral flow these coefficients are always the same.

[9] Now and in the following, we implicitly take the conformal closure $\overline{\mathbb{H}} = \mathbb{H}\cup\{i\infty\}$ and $\overline{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ unless otherwise stated.

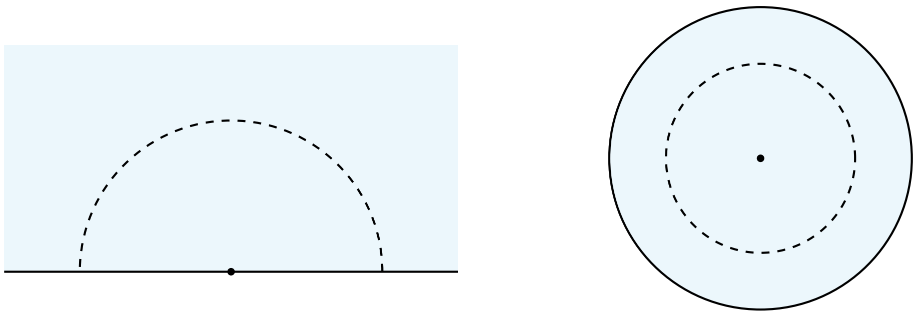

Figure 6.4: BCFTs on the upper half-plane and the disk. In radial quantization, (semi) circles around the origin (dashed lines) play the role of constant time slices. On the upper half-plane constant time slices have the topology of intervals and thus describe open strings, while on the disk constant time slices and circles and thus describe closed strings. The two descriptions are equivalent since the Möbius transformation $(i-z)/(i+z)$ maps between the two spaces.

In terms of the mode expansion, the Dirichlet boundary condition can be expressed as the requirement

$$\alpha_n + \overline{\alpha}_n = 0\,, \quad x_0 = \Phi_0\,, \tag{6.123}$$

while the Neumann boundary condition is

$$\alpha_n - \overline{\alpha}_n = 0\,. \tag{6.124}$$

The above relations on the mode expansion of Φ has a direct consequence: it breaks the independence of the left- and right-moving sectors of the theory. This is a hallmark of boundary CFT – while the theory formulated on a compact surface enjoys both a holomorphic and an anti-holomorphic sector which are independent, the inclusion of boundary conditions spoils this independence, and the left- and right-moving sectors have to be *glued*. We will come back to this point below when we discuss quantization.

We should mention that in the context of quantization, where we will impose the above relations as conditions on boundary states $\|\psi\rangle\rangle$, it is actually more convenient to work on the unit disk $\mathbb{D}$ as opposed to the upper half-plane. We should mention that, in the context of radial quantization, equal time slices in the upper half-plane $\mathbb{H}$ describe open strings, while constant time slices in the disk $\mathbb{D}$ describe closed strings. Of course, the two treatments are equivalent, since there is a Möbius transformation which maps between the two spaces. On the disk, the appropriate boundary conditions are

$$\begin{aligned} \text{Dirichlet:} \quad & \Phi(e^{i\phi}) = \Phi_0\,, \\ \text{Neumann:} \quad & \partial_r\Phi(e^{i\phi}) := (e^{i\phi}\partial + e^{-i\phi}\overline{\partial})\Phi(e^{i\phi}) = 0\,, \end{aligned} \tag{6.125}$$

where we have written $z = re^{i\phi}$ and $\overline{z} = re^{-i\phi}$. In terms of the mode expansions, the conditions are now slightly different:

$$\begin{aligned} \text{Dirichlet:} \quad & \alpha_n - \overline{\alpha}_{-n} = 0\,, \\ \text{Neumann:} \quad & \alpha_n + \overline{\alpha}_{-n} = 0\,. \end{aligned} \tag{6.126}$$

Note specifically that while boundary conditions on the upper half-plane relate α_n and $\overline{\alpha}_n$, boundary conditions on the disk relate α_n and $\overline{\alpha}_{-n}$.

The above discussion is not unique to free bosons, but can be completely generalized. Let us consider a generic CFT with boundary, and let us denote its overall field content by Φ (we now take Φ to be completely general, not to be confused with the free boson above). For concreteness, again, let us take the example of the upper half-plane $\mathbb{H}$, where we impose some kind of boundary condition on the real line $\mathbb{R} = \partial\mathbb{H}$. Intuitively, the existence of a boundary automatically breaks the conformal symmetry of the theory down to the subgroup of conformal transformations which preserve the boundary. Specifically, the Riemann sphere $\mathbb{CP}^1$ enjoys a global $\mathrm{SL}(2,\mathbb{C})$ conformal symmetry generated by Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}\,. \tag{6.127}$$

However, the upper-half plane enjoys only a subset of Möbius transformations, namely those with real coefficients. As such, the existence of a boundary breaks the conformal symmetry down to the subgroup to $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{C})$. In terms of the Virasoro generators $L_n, \overline{L}_n$, the $\mathrm{SL}(2,\mathbb{R})$ transformations are generated by the diagonal subgroup

$$L_n^{(\mathrm{diag})} := L_n + \overline{L}_n\,, \quad n = -1, 0, 1\,. \tag{6.128}$$

Furthermore, the 'off-diagonal' generators $L_n - \overline{L}_n$ are precisely those which deform the boundary of $\mathbb{H}$. The above statement can also be generalized to local conformal transformations, and we conclude that the Virasoro generators $L_n - \overline{L}_n$ deform the boundary of $\mathbb{H}$ for all $n \in \mathbb{Z}$, when considering a local patch of $\mathbb{H}$ that includes the boundary.

In terms of the stress-tensor of the CFT, we can formulate the above symmetry breaking by requiring that the stress tensor satisfies the boundary condition

$$T(x) = \overline{T}(x)\,, \quad x \in \mathbb{R}\,. \tag{6.129}$$

Since the stress tensor is generically a composite operator of the fundamental fields Φ, we must demand that the boundary conditions of Φ have to be compatible with the above boundary condition for T. If Φ satisfies this requirement, we say that our CFT has *conformal* boundary conditions. We will not consider any other type of boundary condition in this thesis.

We note again that if we were to impose conformal boundary conditions on the disk rather than the upper half-plane, the conformal boundary conditions take the form

$$e^{2i\phi}T(z = e^{i\phi}) - e^{-2\pi i}\overline{T}(\overline{z} = e^{-i\phi}) = 0\,, \tag{6.130}$$

which in terms of Virasoro modes reads

$$L_n = \overline{L}_{-n}\,. \tag{6.131}$$

Before we move on to quantization, we mention one more type of boundary condition. In the case that our CFT is a WZW model on some group G, we need to consider boundary conditions for the affine currents J^a and $\overline{J}^a$. Generically, we can choose any boundary condition we like, so long as the stress tensor satisfies (6.129). However, in practice, we want to preserve some of the $G_L \times G_R$ global symmetry of the theory. This is achieved by choosing boundary conditions of the form

$$J^a(x) = \omega(\overline{J})^a(x)\,, \tag{6.132}$$

where ω is some automorphism of the algebra $\mathfrak{g}$. If these so-called 'symmetry-preserving' boundary conditions are satisfied, then the (twisted) diagonal generators

$$J_n^{a,(\mathrm{diag})} = J_n^a + \omega(\overline{J})_n^a \tag{6.133}$$

are preserved by the boundary conditions and form a residual $\mathfrak{g}_k$ algebra of the theory. We discuss these types of boundary conditions and their relationships to branes more above in Section 6.1. More generally, if our CFT admits a Chiral algebra $\mathcal{W}$, then we can impose boundary conditions

$$W(x) = \Omega(\overline{W})(x)\,, \quad x \in \mathbb{R}\,, \tag{6.134}$$

where $\Omega : \mathcal{W} \to \mathcal{W}$ is any automorphism of the chiral algebra which preserves the Virasoro algebra. If W is a conformal primary of weight $h(W)$, then we can translate these boundary conditions on the upper half-plane into conditions on the modes:

$$W_n = (-1)^{h(W)}\Omega(\overline{W})_{-n}\,. \tag{6.135}$$

Ishibashi states

As mentioned above, we ultimately want to promote the classical boundary conditions to boundary conditions in the path integral, which, on the level of the operator formalism, is implemented via a boundary state $\|\psi\rangle\rangle$. Let $\mathcal{W}$ denote the full chiral algebra of the theory. Then if we consider some gluing condition

$$W(x) = \Omega(\overline{W})(x)\,, \quad x \in \mathbb{R} \tag{6.136}$$

where $\Omega : \mathcal{W} \to \mathcal{W}$ is some automorphism, the resulting condition on the modes is

$$W_n = (-1)^{h(W)}\Omega(\overline{W})_{-n}\,, \tag{6.137}$$

which can be seen from performing a conformal transformation from the upper half-plane to the disk. Here, h is the conformal weight of the operator W. The above boundary condition can be implemented in the Hilbert space by defining boundary states which satisfy

$$\left(W_n - (-1)^{h(W)}\Omega(\overline{W})_{-n}\right)\|\psi\rangle\rangle = 0\,, \quad W \in \mathcal{W}\,. \tag{6.138}$$

Generically, the above constraint will have a large number of solutions, and a general boundary state will be given by a linear combination of these solutions. Let us assume that the our bulk CFT has a spectrum which decomposes into diagonal representations of $\mathcal{W}$. Letting ρ_i denote the irreducible representations of $\mathcal{W}$ that appear in the theory, we write

$$\mathscr{H} = \bigoplus_i \mathscr{H}_i \oplus \overline{\mathscr{H}}_i\,, \tag{6.139}$$

where $\mathscr{H}_i$ is the vector space of ρ_i in the left-moving sector, and $\overline{\mathscr{H}}_i$ is its right-moving counterpart. In such a CFT, it was shown by Ishibashi [104, 110] that for each representation ρ_i, a solution $|i\rangle\rangle$ to the boundary conditions can be found. Specifically, letting $|i,N\rangle$ be a formal basis for $\mathscr{H}_i$, we have

$$|i\rangle\rangle := \sum_N |i,N\rangle \otimes U\overline{|i,N\rangle}\,. \tag{6.140}$$

Here, U is a unitary operator acting on $\overline{\mathscr{H}}_i$ which satisfies the property

$$U\overline{W}_n = (-1)^{h(W)}\Omega(\overline{W})_n U\,. \tag{6.141}$$

Intuitively, one can think of U as being the operator which acts like Ω on states in some representation. If we are given such an operator U, then it is easy to check that $|i\rangle\rangle$ satisfies the boundary conditions (6.138).

Given two Ishibashi states $|i\rangle\rangle, |j\rangle\rangle$, it is also simple to compute their overlap. Let ℓ be some positive number. Then

$$\langle\langle j|e^{-\pi\ell\left(L_0+\overline{L}_0-\left(\frac{c+\overline{c}}{24}\right)\right)}|i\rangle\rangle = \delta_{ij}\chi_i(i\ell)\,, \tag{6.142}$$

where χ_i is the character of the chiral algebra $\mathcal{W}$ in the representation ρ_i. This expression is extremely useful, as it allows one to compute overlaps of boundary states, and thus compute the open string spectrum between two branes.

The Cardy condition

Ishibashi states provide us with a complete basis of vectors $|i\rangle\rangle$ which satisfy the gluing condition imposed via a particular choice of boundary condition, corresponding to some automorphism Ω of the symmetry algebra of the CFT. As we mentioned above, generic boundary states are given by linear combinations of Ishibashi states. Let $\|\alpha\rangle\rangle$ be such a boundary state. Then we can write

$$\|\alpha\rangle\rangle = \sum_i B_i^\alpha|i\rangle\rangle\,, \tag{6.143}$$

for some constants B_i^α.

However, not all linear combinations $\|\alpha\rangle\rangle$ define a consistent boundary. To see this, consider the overlap

$$\langle\langle\beta\|e^{-\pi\ell\left(L_0+\overline{L}_0-\left(\frac{c+\overline{c}}{24}\right)\right)}\|\alpha\rangle\rangle\,. \tag{6.144}$$

Using the overlap (6.142), we can compute the above matrix element in terms of the coefficients B_i^α, B_i^β as

$$\sum_i B_i^\alpha\overline{B_i^\beta}\chi_i(i\ell)\,. \tag{6.145}$$

Now, the operator between the states $\|\alpha\rangle\rangle, \|\beta\rangle\rangle$ is the propagator of a closed string, and so their matrix element describes the tree-level amplitude of a closed string being emitted from the boundary $\|\alpha\rangle\rangle$ and being absorbed by the boundary $\|\beta\rangle\rangle$. However, we can also view the same string diagram as describing an open string stretched between the two branes $\|\alpha\rangle\rangle, \|\beta\rangle\rangle$ and propagating in a closed loop. The latter description would define the spectrum

$$\mathrm{Tr}_{\alpha,\beta}[e^{-2\pi t(L_0-\frac{c}{24})}] = \sum_i n^i_{\alpha\beta}\chi_i(it) \tag{6.146}$$

of an open string stretched between the two branes. As an expansion in the parameter $q = e^{-2\pi\ell}$, we know that this trace must have integer coefficients, assuming that the spectrum of the open string is discrete. The relationship between the overlap

(6.144) and the partition function (6.146) is that they are related via a modular S transformation, i.e. $\ell = 1/t$. Recalling the definition of the modular S-matrix:

$$\chi_i(i/t) = \sum_j S_{ij}\chi_j(it)\,, \tag{6.147}$$

we can write the equivalence of the closed- and open-string channels as

$$\sum_{i,j} S_{ij} B_i^\alpha \overline{B_i^\beta} \chi_j(it) = \sum_j n^j_{\alpha\beta}\chi_j(it)\,, \tag{6.148}$$

which implies the so-called *Cardy condition*

$$\sum_i S_{ij} B_i^\alpha \overline{B_i^\beta} \in \mathbb{Z}_{\geq 0}\,. \tag{6.149}$$

This condition places an enormous constraint on the form of the coefficients B_i^α, as it needs to hold for every representation ρ_j, and every pair of boundary states $\|\alpha\rangle\!\rangle, \|\beta\rangle\!\rangle$.

The Cardy condition seems, in general, somewhat difficult to satisfy. However, if the underlying CFT has a charge conjugate modular invariant partition function, then we know that the multiplicities N^i_{jk} of the fusion of two representations is given by the Verlinde formula as

$$N^i_{jk} = \frac{S_{j\ell}S_{k\ell}\overline{S_{i\ell}}}{S_{0\ell}}\,, \tag{6.150}$$

where 0 stands for the vacuum character. Thus, if we define

$$B_i^\alpha = \frac{S_{\alpha i}}{\sqrt{S_{0i}}}\,, \tag{6.151}$$

where we have now taken α to label a representation of $\mathcal{W}$, then we have

$$\sum_i S_{ij} B_i^\alpha \overline{B_i^\beta} = \sum_i \frac{S_{ij}S_{\alpha i}\overline{S_{\beta i}}}{S_{0i}} = N^\beta_{j\alpha}\,, \tag{6.152}$$

which is a positive integer, and so the Cardy condition is satisfied. Thus, we can define a boundary state $\|\alpha\rangle\!\rangle$ for each representation ρ_α in terms of the modular S-matrix. We note that, while we strictly required that the CFT to have a charge conjugate modular invariant partition function, the ansatz (6.151) is often applicable for far more generic CFTs.

The doubling trick

In the path integral approach to boundary CFT, we define a CFT on a surface Σ with nonempty boundary $\partial\Sigma$ and impose some fixed set of boundary conditions on $\partial\Sigma$. For the moment, let us continue to use the example of the upper half-plane $\mathbb{H}$, and we will comment on the more generic structure later. As discussed above, the most generic conformal boundary conditions at $\partial\mathbb{H}$ 'glue' the left- and right-moving parts of the Virasoro generators along the boundary. Specifically, conformal boundary conditions imply

$$T(z) = \overline{T}(\overline{z})\,, \quad z \in \mathbb{R}\,. \tag{6.153}$$

A standard trick in boundary CFT is the so-called 'doubling trick', which amounts to trading the holomorphic and antiholomorphic operators $T(z)$ and $\overline{T}(\overline{z})$ living on the upper half-plane $\mathbb{H}$ for one *holomorphic* operator $\mathscr{T}(z)$ living on the full complex plane. The operator $\mathscr{T}$ is defined via

$$\mathscr{T}(p) = \begin{cases} T(p)\,, & z \in \mathbb{H} \\ \overline{T}(\overline{p})\,, & z \in \mathbb{C} \setminus \mathbb{H}\,, \end{cases} \tag{6.154}$$

Here, we write p instead of z or $\overline{z}$ to emphasize that we are defining the dependence of $\mathscr{T}$ on the *point* $p \in \mathbb{C}$. In particular, a function written as $f(p)$ is not implied to be holomorphic, and $f(\overline{p})$ is implied to mean f evaluated at the complex conjugate point to p. The operator $\mathscr{T}(z)$ is well-defined on the real line, due to the conformal boundary condition $T = \overline{T}$ imposed there.

The new stress-tensor $\mathscr{T}(z)$ can be thought of as an analytic continuation of $T(z)$ to the full complex plane. Note that we could also define a right-moving stress-tensor by

$$\overline{\mathscr{T}}(p) = \begin{cases} \overline{T}(p)\,, & z \in \mathbb{H}\,, \\ T(\overline{p})\,, & z \in \mathbb{C} \setminus \mathbb{H}\,. \end{cases} \tag{6.155}$$

However, the right-moving stress tensor is related directly to the left-moving stress tensor by the functional equation

$$\mathscr{T}(p) = \overline{\mathscr{T}}(\overline{p})\,. \tag{6.156}$$

That is, the left- and right-movers are completely coupled. The gist of the doubling trick boils down to the following sentence: a boundary CFT on $\mathbb{H}$ is equivalent to a CFT on $\mathbb{C}$ for which the left- and right-moving quantities are related via functional equations like (6.156).

Note specifically that because of this functional equation, there is no sense in which the left- and right-moving Virasoro generators in the doubled theory are independent, and so we have traded and left- and right-moving Virasoro symmetry on $\mathbb{H}$ for only a left-moving Virasoro symmetry on $\mathbb{C}$, and the modes of $\mathscr{T}$ will generate one copy of the Virasoro algebra.

The doubling trick and be imposed on more than just the stress tensor. For example, in a WZW model, in order to obtain some diagonal G symmetry, we impose the boundary condition

$$J(z) = \omega\left(\bar{J}\right)(\overline{z})\,, \tag{6.157}$$

where $\omega : \mathfrak{g} \to \mathfrak{g}$ is some automorphism of the Lie algebra of G.[10] This, again, 'glues' the left- and right-moving current algebras together, which can be made manifest by applying the doubling trick and defining

$$\mathscr{J}(p) = \begin{cases} J(p)\,, & z \in \mathbb{H}\,, \\ \omega(\bar{J})(\overline{p})\,, & z \in \mathbb{C} \setminus \mathbb{H}\,. \end{cases} \tag{6.158}$$

The new current $\mathscr{J}(z)$ is holomorphic everywhere on the complex plane and again is well-defined for real z due to the gluing conditions. Again, we can define a right-moving current $\overline{\mathscr{J}}$ on all of $\mathbb{C}$, which would be related to its left-moving counterpart via the functional equations

$$\mathscr{J}(p) = \omega(\overline{\mathscr{J}})(\overline{p}) \tag{6.159}$$

[10]For now we drop the Lie algebra indices on currents.

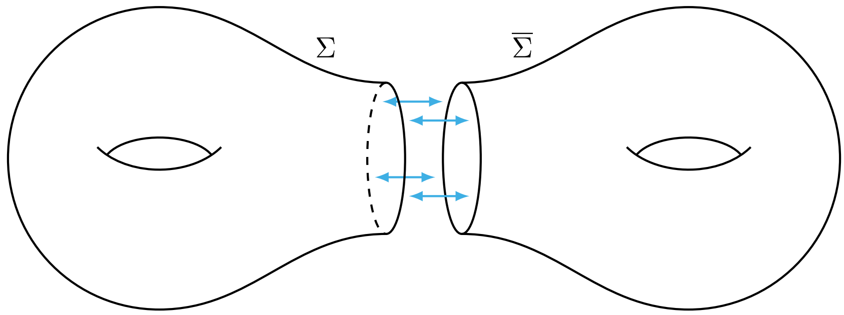

Figure 6.5: The doubling trick: a surface Σ with boundary can be constructed by considering its 'compact double' $\Sigma_c := \Sigma \cup_{\partial\Sigma} \overline{\Sigma}$, obtained by gluing Σ and its orientation reversal $\overline{\Sigma}$ along their boundaries.

The doubling trick in general

The doubling trick for fields on the upper half-plane is a special case of a more general technique which applies to surfaces of all genera and for an arbitrary number of boundary components. Let Σ be some surface with non-empty boundary. Then there is a natural surface Σ_c associated to Σ known as the *compact double* of Σ. It is defined by considering a second copy $\overline{\Sigma}$ with opposite orientation to that of Σ, and gluing Σ and $\overline{\Sigma}$ together along their boundary components, as shown in Figure 6.5.

Now, on Σ_c, there is a natural orientation-reversing involution $\iota : \Sigma_c \to \Sigma_c$ defined by mapping points originally from Σ to their corresponding points on $\overline{\Sigma}$. Clearly, the fixed-point set of ι is the set of points in Σ_c which were originally points on $\partial\Sigma$. Thus, we have

$$\Sigma \cong \Sigma_c/\iota\,. \tag{6.160}$$

For example, in the case that Σ is the closure $\overline{\mathbb{H}}$ of the upper half-plane $\mathbb{H}$,[11] its compact double is the Riemann sphere $\mathbb{CP}^1$, and the involution ι is simply complex conjugation, which fixes the (extended) real line $\partial\overline{\mathbb{H}}$. Similarly, if we take Σ to be the closed unit disk $\overline{\mathbb{D}} := \{|z| \leq 1\}$, then the compact double of Σ is again the Riemann sphere $\mathbb{CP}^1$, but now the involution ι is given by

$$\iota : z \longrightarrow \frac{1}{\bar{z}}\,, \tag{6.161}$$

which fixes the unit circle $\partial\overline{\mathbb{D}}$.

As we have seen in the above examples, the involution ι is the appropriate generalization of complex conjugation to a compact double. As such, we can define the doubling trick on an arbitrary surface by defining a holomorphic stress tensor $\mathscr{T}$ on Σ_c in terms of the left- and right-moving stress tensors on Σ as

$$\mathscr{T}(p) = \begin{cases} T(p)\,, & p \in \Sigma\,, \\ (\iota^*\overline{T})(p)\,, & p \in \Sigma_c \setminus \Sigma\,. \end{cases} \tag{6.162}$$

for all $p \in \Sigma_c$. The operator ι^* is understood as the pullback of the operator $\overline{T}$ by the map ι. Concretely, if φ is a conformal primary of weight $(h, \bar{h})$, then in a local

[11] That is, $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$.

coordinate system, the pullback looks like

$$(\iota^*\varphi)(p) = \left(\frac{\partial\iota}{\partial z}\bigg|_{\iota(p)}\right)^h \left(\frac{\partial\iota}{\partial\bar{z}}\bigg|_{\iota(p)}\right)^{\bar{h}} \varphi(\iota(p))\,. \tag{6.163}$$

For example, when $\Sigma = \overline{\mathbb{H}}$ and ι simply acts as conjugation, we have $\iota^*\mathscr{T} = \mathscr{T}$, and we recover the usual doubling trick on the sphere. When $\Sigma = \overline{\mathbb{D}}$ is the disk, this gluing condition becomes

$$\mathscr{T}(p) = \frac{1}{\bar{p}^4}\overline{\mathscr{T}}\left(\frac{1}{\bar{p}}\right) \tag{6.164}$$

We can do the same for WZW models on arbitrary surfaces with boundary, by considering sections $\mathscr{J}$ of K satisfying the functional equation

$$\mathscr{J}(p) = -\frac{1}{\bar{p}^2}\omega(\overline{\mathscr{J}})\left(\frac{1}{\bar{p}}\right) \tag{6.165}$$

6.B The D-branes of the symmetric orbifold of $\mathbb{T}^4$

In this appendix we explain how the construction of the maximally-fractional boundary states for a bosonic symmetric orbifold theory, see Section 6.3, can be generalized to the situation with fermions. We shall mainly focus on the case where the seed CFT is the superconformal theory of four free bosons and fermions on a 4-torus $\mathbb{T}^4$. We will furthermore restrict ourselves to the NS sector in the closed-string channel. Our results can, however, straightforwardly be extended to more general cases.

The main new ingredient we need to take care of comes from the fact that we may pick up minus signs from permuting fermions (because of Fermi statistics). More specifically, if $\sigma \in S_N$ has a cycle shape as described in (6.67), we find that, see e.g. [111, Section 4.3] [72, Appendix A]

$$\mathrm{tr}_{\mathcal{H}}\left[\sigma e^{2\pi i t(L_0 - \frac{Nc}{24})}\right] = \prod_{j=1}^{r} Z^{(\epsilon_j)}(l_j t)\ , \qquad \epsilon_j = (-1)^{(l_j-1)}\ , \tag{6.166}$$

where $Z^{(\epsilon)}$ is the trace with the insertion of $(-1)^{\epsilon F}$.

Boundary states in the seed $\mathbb{T}^4$

In the seed $\mathbb{T}^4$ SCFT, we have 4 free bosons and 4 free fermions. We group them into two complex pairs as $\partial X^{j\pm}$, $\psi^{j\pm}$, where $j = 1, 2$. They then satisfy the nontrivial OPEs

$$\partial X^{j+}(x)\partial X^{k-}(y) \sim \frac{\delta^{jk}}{(x-y)^2}\,, \qquad \psi^{j+}(x)\psi^{k-}(y) \sim \frac{\delta^{jk}}{x-y}\ . \tag{6.167}$$

Let us assume that we impose standard Dp-brane boundary conditions directly on the free bosons and fermions; for example, for a D0-brane, they would read

$$(\alpha_n^{j\pm} - \bar{\alpha}_{-n}^{j\pm})|\beta,\varepsilon,s\rangle\rangle = 0\ , \tag{6.168a}$$

$$(\psi_r^{j\pm} - i\varepsilon\bar{\psi}_{-r}^{j\pm})|\beta,\varepsilon,s\rangle\rangle = 0\ , \tag{6.168b}$$

where $s \in \{\mathrm{NS}, \mathrm{R}\}$ denotes the closed string sector, and we have $n \in \mathbb{Z}$, while $r \in \mathbb{Z} + 1/2$ for $s = \mathrm{NS}$ and $r \in \mathbb{Z}$ for $s = \mathrm{R}$. (Here the $\alpha_n^{j\pm}$ are the modes for the bosons, and the $\psi_r^{j\pm}$ those of the fermions.)

The construction of the corresponding boundary states is standard, see e.g. [112], and we denote them by $\|u, \varepsilon, s\rangle\!\rangle$, where $s \in \{\mathrm{NS}, \mathrm{R}\}$; note that in superstring theory one usually combines the NS-NS and R-R contributions in order to guarantee that the open string will be GSO-projected. However, in the current setup this is not appropriate since the symmetric orbifold of $\mathbb{T}^4$ is not the worldsheet theory of a string theory. Instead it describes the dual CFT which does not have a GSO-projection. In any case, we expand these boundary states in terms of the Ishibashi states as

$$\|u, \varepsilon, s\rangle\!\rangle = \sum_\beta B_\beta(u, s)\, |\beta, \varepsilon, s\rangle\!\rangle \ , \tag{6.169}$$

where here β runs over those momentum/winding sectors that are compatible with the gluing conditions (6.168a) with $n = 0$; for example, for a D0 brane, only the pure momentum sectors contribute. Their relative overlaps are then given by — as in the main part we choose opposite values of ε for the two boundary states, and hence denote them by $\widetilde{\mathrm{NS}}$ and $\widetilde{\mathrm{R}}$

$$\begin{aligned} \hat{Z}^{\mathbb{T}^4}_{u|v,\widetilde{\mathrm{NS}}}(\hat\zeta;\hat t\,) &= \frac{\vartheta_4(\frac{\hat\zeta}{2};\hat t\,)\vartheta_4(-\frac{\hat\zeta}{2};\hat t\,)}{\eta(\hat t\,)^6}\, \hat\Theta^{\mathbb{T}^4}_{u|v}(\hat t\,) \ , \\ \hat{Z}^{\mathbb{T}^4}_{u|v,\widetilde{\mathrm{R}}}(\hat\zeta;\hat t\,) &= \frac{\vartheta_1(\frac{\hat\zeta}{2};\hat t\,)\vartheta_1(-\frac{\hat\zeta}{2};\hat t\,)}{\eta(\hat t\,)^6}\, \hat\Theta^{\mathbb{T}^4}_{u|v}(\hat t\,) \ , \end{aligned} \tag{6.170}$$

where $\hat\Theta^{\mathbb{T}^4}_{u|v}(\hat t\,)$ depends on which momentum/winding sectors contribute, and $\hat\zeta$ is the $\mathfrak{su}(2)$ chemical potential.

Given that we have chosen opposite values of ε for the two boundary states, the corresponding open string is then in the R or $\tilde{\mathrm{R}}$ sector, and we find from the S-modular transformation, writing $\hat t = -1/t$ and $\hat\zeta = -\hat t\zeta$, the two expressions

$$\begin{aligned} Z^{\mathbb{T}^4}_{u|v,\mathrm{R}}(\zeta;t) &= \frac{\vartheta_2(\frac{\zeta}{2};t)\vartheta_2(-\frac{\zeta}{2};t)}{\eta(t)^6}\, \Theta^{\mathbb{T}^4}_{u|v}(t)\,, \\ Z^{\mathbb{T}^4}_{u|v,\widetilde{\mathrm{R}}}(t\,) &= -\frac{\vartheta_1(\frac{\zeta}{2};t)\vartheta_1(-\frac{\zeta}{2};t)}{\eta(t)^6}\, \Theta^{\mathbb{T}^4}_{u|v}(t) \ , \end{aligned} \tag{6.171}$$

where $\Theta^{\mathbb{T}^4}_{u|v}(t)$ describes the momentum/winding modes from the open string perspective; they are determined via the relation

$$\frac{\Theta^{\mathbb{T}^4}_{u|v}(t)}{\eta(t)^4} = \frac{\hat\Theta^{\mathbb{T}^4}_{u|v}(\hat t\,)}{\eta(\hat t\,)^4} \ . \tag{6.172}$$

Sym($\mathbb{T}^4$) boundary states

Next we want to generalize the symmetric orbifold construction of Section 6.3 to incorporate correctly the signs from eq. (6.166). We shall only concentrate on the boundary states that come from the $s = \mathrm{NS}$ sector; recall that the $s = \mathrm{R}$ sector has central charge proportional to N, and hence does not correspond to perturbative string degrees of freedom from the AdS perspective. Following (6.71), we make the ansatz

$$\|u,\rho,\varepsilon,\mathrm{NS}\rangle\!\rangle = \sum_{(\underline{\beta},[\sigma])} B_{(\underline{\beta},[\sigma])}(u,\rho,\mathrm{NS})\,|\underline{\beta},\varepsilon,\mathrm{NS}\rangle\!\rangle_{[\sigma]}\,, \tag{6.173}$$

where the boundary state coefficients $B_{(\underline{\beta},[\sigma])}(u,\rho,\mathrm{NS})$ can be written as

$$B_{(\underline{\beta},[\sigma])}(u,\rho,\mathrm{NS}) = \left(\frac{|[\sigma]|}{N!}\right)^{\frac{1}{2}} \chi_\rho([\sigma]) \prod_{j=1}^{r} B_{\beta_j}(u,s(l_j))\ , \tag{6.174}$$

with $s(l_j)=\mathrm{NS}$ when $l_j\in 2\mathbb{Z}+1$, while $s(l_j)=\mathrm{R}$ when $l_j\in 2\mathbb{Z}$. Correspondingly, the Ishibashi states $|\underline{\beta},\varepsilon,\mathrm{NS}\rangle\!\rangle_{[\sigma]}$ satisfy the fermionic conditions (for $j=1,\ldots,r$)

$$\left(\psi^{[j]}_p + i\varepsilon(\bar{\psi}^{[j]})_{-p}\right)|\underline{\beta},\varepsilon,\mathrm{NS}\rangle\!\rangle_{[\sigma]} = 0\ , \tag{6.175}$$

where $p\in\frac{1}{l_j}\mathbb{Z}$ for $l_j\in 2\mathbb{Z}$, and $p\in\frac{1}{l_j}(\mathbb{Z}+1/2)$ for $l_j\in 2\mathbb{Z}+1$. Thus the individual single-cycle twisted sectors look as though they are in the Ramond sector for even cycle-length. Choosing opposite spin structures as before, the corresponding overlaps are then

$$\hat{\mathscr{Z}}^{S_N}_{(u,\rho_1)|(v,\rho_2),\tilde{\mathrm{NS}}}(\hat\zeta;\hat t\,) = \frac{1}{|S_N|}\sum_{\sigma\in S_N} \bar\chi_{\rho_1}([\sigma])\chi_{\rho_2}([\sigma]) \prod_{j=1}^{r} \hat Z^{\mathbb{T}^4}_{u|v,\tilde s(l_j)}\big(\hat\zeta;\tfrac{\hat t}{l_j}\big)\ . \tag{6.176}$$

Upon the S-modular transformation, going to the open string description, $\tilde{\mathrm{NS}}\to\mathrm{R}$, while $\tilde{\mathrm{R}}\to\tilde{\mathrm{R}}$, and hence the open string is always in the Ramond sector, but there is a sign depending on whether l_j is even or odd, reflecting precisely the signs in eq. (6.166). Thus the open string is in the $\mathrm{R}^{\otimes N}$ sector — this is because we took the two ε values of the boundary states opposite — and the group factors imply again that we project onto those states that transform in the representation $\rho_1\otimes\rho_2^*$ with respect to S_N.

For the simple case $\rho_1=\rho_2=\mathrm{id}$, so that $\bar\chi_{\rho_1}([\sigma])=\chi_{\rho_2}([\sigma])=1$ for all $\sigma\in S_N$, it is again convenient to go to the grand canonical ensemble

$$\hat{\mathfrak{Z}}_{u|v,\tilde{\mathrm{NS}}}(p,\hat\zeta;\hat t\,) = \sum_{N=1}^{\infty} p^N\,\hat{\mathscr{Z}}^{S_N}_{(u,\mathrm{id})|(v,\mathrm{id}),\tilde{\mathrm{NS}}}(\hat\zeta;\hat t\,)\ , \tag{6.177}$$

which can be rewritten, as in (6.72), as

$$\hat{\mathfrak{Z}}_{u|v,\tilde{\mathrm{NS}}}(p,\hat\zeta;\hat t\,) = \exp\Bigg(\sum_{\substack{w=1\\ w\ \mathrm{odd}}}^{\infty}\frac{p^w}{w}\hat Z^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{NS}}}\big(\hat\zeta;\tfrac{\hat t}{w}\big) + \sum_{\substack{w=1\\ w\ \mathrm{even}}}^{\infty}\frac{p^w}{w}\hat Z^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{R}}}\big(\hat\zeta;\tfrac{\hat t}{w}\big)\Bigg). \tag{6.178}$$

Finally, modular transforming into the open string channel and keeping track of the elliptic prefactors (which we did not include in (6.171)), we obtain the grand canonical Ramond boundary partition function

$$\mathfrak{Z}_{u|v,\mathrm{R}}(p,\zeta;t) = \exp\Bigg(\sum_{\substack{k=1\\ k\ \mathrm{odd}}}^{\infty}\frac{p^k}{k}e^{\frac{\pi i k\zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v,\mathrm{R}}(k\zeta;kt) + \sum_{\substack{k=1\\ k\ \mathrm{even}}}^{\infty}\frac{p^k}{k}e^{\frac{\pi i k\zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v,\tilde{\mathrm{R}}}(k\zeta;kt)\Bigg), \tag{6.179}$$

where we have relabelled $w \mapsto k$.

We should mention that also the spin structures in (6.178) arise naturally from the interpretation of the grand canonical ensemble as a sum over covering spaces of the cylinder. Indeed, given a covering map of degree w, a fermion which picks up a (-1) monodromy on the base cylinder will pick up a $(-1)^w$ monodromy on the covering cylinder — intuitively, the compact cycle on the covering cylinder maps to w copies of the compact cycle on the base cylinder. That is, if we have a fermion in the NS sector on the base space, then it will be in the NS sector on the covering space if w is odd, but in the R sector if w is even.

6.C Planar coverings of the disk

In the main text, we argued that correlation functions of spectrally flowed highest-weight states with a a spherical brane boundary condition are calculable in terms of holomorphic covering maps $\Gamma : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, where we denote by $\overline{\mathbb{D}}$ be the closed unit disk $\{z \in \mathbb{C} : |z|^2 \leq 1\}$. Such a map should have the following properties:

- On the interior $\mathbb{D}$ of the unit disk, Γ should be a holomorphic function taking values on $\mathbb{D}$. That is,
$$\bar{\partial}\Gamma(z) = 0 \ , \quad |\Gamma(z)| < 1 \ , \quad z \in \mathbb{D} \ . \tag{6.180}$$
- Γ should map the boundary of the unit disk to itself. That is,
$$|\Gamma(z)|^2 = 1 \ , \quad |z|^2 = 1 \ . \tag{6.181}$$
- Finally, for Γ to be a branched covering, there should exist n marked points z_i such that, near $z = z_i$, Γ has a critical point of order w_i. That is,
$$\Gamma(z) \sim x_i + \mathcal{O}\big((z - z_i)^{w_i}\big) \ , \qquad z \to z_i \ . \tag{6.182}$$

Below, we detail the algebraic constraints arising from these properties, and demonstrate that such a covering map only exists given that the insertion points z_i are chosen appropriately.

Algebraic constraints

Given any branched covering between two surfaces X and Y, the degree N of the covering map $\Gamma : X \to Y$ (i.e. the number of preimages of Γ at a generic point) is determined by the Riemann-Huwritz formula
$$\chi(Y)\, N = \chi(X) + \sum_{i=1}^{n} (w_i - 1) \ . \tag{6.183}$$

Since the disk has Euler characteristic $\chi(\overline{\mathbb{D}}) = 1$, the degree of Γ thus turns out to be
$$N = 1 + \sum_{i=1}^{n} (w_i - 1) \ , \tag{6.184}$$
where w_i is the degree of the critical point at $z_i \in \mathbb{D}$. Note that we do not allow critical points at the boundary.

Now, it is well known that all holomorphic functions from the disk to itself are rational. Thus, we can assume that Γ takes the form $\Gamma(z) = Q_N(z)/P_N(z)$, where Q_N and P_N are polynomials of order N. The requirement that Γ maps the circle to itself can be rephrased as

$$|Q_N(e^{i\phi})|^2 = |P_N(e^{i\phi})|^2 \ . \tag{6.185}$$

If we write

$$Q_N(z) = \sum_{a=0}^{N} q_a z^a \ , \qquad P_N(z) = \sum_{a=0}^{N} p_a z^a \ , \tag{6.186}$$

then we have

$$|Q_N(e^{i\phi})|^2 = \sum_{a,b=0}^{N} q_a q_b^* e^{i(a-b)\phi} = \sum_{a=0}^{N} |q_a|^2 + 2\sum_{a<b} \Re\left(q_a q_b^* e^{i(a-b)\phi}\right) \ , \tag{6.187}$$

and similarly for $|P_N|^2$. Requiring $|Q_N|^2 = |P_N|^2$ along the unit circle means matching coefficients of $e^{i(a-b)\phi}$ for each value of $a-b$. This in the end gives $2N+1$ real constraints. Since the covering map Γ had $2N+1$ complex degrees of freedom originally (the coefficients of Q_N and P_N), we see that requiring Γ to map the unit circle to itself reduces the number to $2N+1$ *real* degrees of freedom. This reflects the fact that an open string has only half the chiral degrees of freedom of a closed string.

Next, we demand that Γ has an appropriate critical point at each z_i. That is, we demand

$$\Gamma(z) - x_i \sim \mathcal{O}\big((z-z_i)^{w_i}\big) \ . \tag{6.188}$$

In terms of the polynomials Q_N and P_N, this constraint takes the form

$$Q_N(z) - x_i P_N(z) \sim \mathcal{O}\big((z-z_i)^{w_i}\big) \ . \tag{6.189}$$

This introduces w_i (complex) constraints at each critical point, and so overall there are $2\sum_i w_i$ *real* constraints. Thus, the moduli space of such maps has dimension

$$\dim_{\mathbb{R}}(\Gamma : \mathbb{D} \to \mathbb{D},\ z_i, x_i \text{ fixed}) = 2N + 1 - 2\sum_{i=1}^{n} w_i = -2n + 3 \ . \tag{6.190}$$

Note that $2n-3$ is exactly the dimension of the moduli space of a disk with n marked points. Indeed, there are $2n$ (real) moduli for the n points z_i, while we have three real conformal Killing vectors which can be used to fix $z_1 = 0$ and $z_2 \in (0,1)$. Thus,

$$\dim_{\mathbb{R}}(\Gamma : \mathbb{D} \to \mathbb{D},\ x_i \text{ fixed}) = 0 \ , \tag{6.191}$$

and we conclude that constructing such a map is a rigid problem.

Example: $w_1 = w_2 = 1$

The simplest example of such a covering map is to consider the case where $w_1 = w_2 = 1$. The degree of such a map is $N = 1$, i.e. Γ is a rational linear function. Requiring that Γ maps the unit circle onto itself restricts it to be of the form

$$\Gamma(z) = e^{i\phi}\frac{z-a}{1-\bar{a}z} \ , \tag{6.192}$$

where $a \in \mathbb{D}$ and $\phi \in \mathbb{R}$.

In order to simplify our life, we can use the three real moduli of the disk, in order to fix z_i and x_i to be of the form $z_1 = x_1 = 0$ and $z_2, x_2 \in (0,1)$. The resulting covering map is simply the identity function $\Gamma(z) = z$, and we find that this only maps z_2 to x_2 if $z_2 = x_2$, which gives one real constraint on the insertion points z_i. This constraint, in turn, corresponds to the one real modulus of a disk with two marked points.

The doubling trick

We can also characterize Γ in a different way by analytically continuing it. That is, instead of treating Γ as a map from the disk to itself, we can instead treat Γ as a holomorphic map on the full Riemann sphere. Indeed, if we define

$$\Gamma(z) = \frac{1}{\overline{\Gamma(1/\bar{z})}} \ , \qquad |z| > 1 \ , \tag{6.193}$$

then the induced map $\Gamma : \mathbb{CP}^1 \to \mathbb{CP}^1$ satisfies $|\Gamma(z)|^2 = 1$ along the unit circle by construction. Furthermore, Γ now has double the critical points as its restriction to the disk. If we expand around $z = 1/\bar{z}_i$, we find that Γ has the critical behavior

$$\Gamma\left(\frac{1}{\bar{z}_i} + \varepsilon\right) \sim \frac{1}{\bar{x}_i} + \mathcal{O}(\varepsilon^{w_i}) \ . \tag{6.194}$$

Thus, $1/\bar{z}_i$ is also a critical point of order w_i with $\Gamma(1/\bar{z}_i) = 1/\bar{x}_i$.

The doubling trick allows us to treat our covering map as a map on the sphere, as opposed to a covering map on the disk. The resulting map has critical points at z_i and $1/\bar{z}_i$. We can thus calculate the resulting degree from Riemann-Hurwitz for sphere coverings, and we find

$$N = 1 + \sum_{i=1}^{n} \frac{w_i - 1}{2} + \sum_{i=1}^{n} \frac{w_i - 1}{2} = 1 + \sum_{i=1}^{n} (w_i - 1) \ , \tag{6.195}$$

which agrees with the argument from the covering of the disk. The fact that $\chi(\mathbb{CP}^1) = 2\chi(\mathbb{D})$ is compensated by the fact that $\Gamma : \mathbb{CP}^1 \to \mathbb{CP}^1$ has twice as many critical points as $\Gamma : \mathbb{D} \to \mathbb{D}$. Furthermore, just as on the disk, since any holomorphic map $\Gamma : \mathbb{CP}^1 \to \mathbb{CP}^1$ is rational, we can again write $\Gamma(z) = Q_N(z)/P_N(z)$ for some polynomials Q_N and P_N of degree N.

The moduli counting argument also works in the context of the doubling trick. Note that each pole λ_a of Γ is accompanied by a corresponding zero at $1/\bar{\lambda}_a$, by definition of the analytic continuation of Γ. Thus, we can write Γ as

$$\Gamma(z) = C \prod_{a=1}^{N} (z - \lambda_a)^{-1} \left(z - 1/\bar{\lambda}_a\right) \ . \tag{6.196}$$

This parametrisation gives us $N+1$ complex degrees of freedom. Furthermore, the requirement that $|\Gamma(z)|^2 = 1$ along the unit circle removes one real degree of freedom. Indeed, we have

$$|\Gamma(e^{i\phi})|^2 = |C|^2 \prod_{a=1}^{N} \frac{|e^{i\phi} - 1/\bar{\lambda}_a|^2}{|e^{i\phi} - \lambda_a|^2} = |C|^2 \prod_{a=1}^{N} \frac{1}{|\lambda_a|^2} \ , \tag{6.197}$$

and so requiring Γ to map the unit circle to itself simply fixes the norm of C. All of the remaining constraints come from simply demanding that Γ has critical points at $z = z_i$. The criticality of the points at $z = 1/\bar{z}_i$ is then automatic.

Covering by the UHP

In the context of boundary conformal field theory, it is often more convenient to consider the worldsheet as the upper half-plane (UHP) instead of the disk, so that the gluing conditions are implemented along the real line. In this case, the relevant covering maps are of the form $\Gamma : \mathbb{H} \to \mathbb{D}$ or $\Gamma : \mathbb{H} \to \mathbb{H}$. The UHP (with a point at infinity) and the unit disk are conformally equivalent under, for example, the map $f : \mathbb{H} \to \mathbb{D} \cup \{\infty\}$ given by

$$f(z) = \frac{z - i}{z + i} \; . \tag{6.198}$$

Thus, by composing with f, one can easily turn a covering map $\Gamma : \mathbb{D} \to \mathbb{D}$ into $\Gamma \circ f : \mathbb{H} \to \mathbb{D}$ or $f^{-1} \circ \Gamma \circ f : \mathbb{H} \to \mathbb{H}$. In this way, the theory of branched coverings between the disk and UHP, and between the UHP and itself, is entirely equivalent to that of coverings from the disk to the disk. That said, it is convenient to review the algebraic properties of these types of maps.

UHP to disk

A map $\Gamma : \mathbb{H} \to \mathbb{D}$ can be analytically extended to a branched covering of the Riemann sphere by the functional equation

$$\Gamma(z) = \frac{1}{\overline{\Gamma(\bar{z})}} \; . \tag{6.199}$$

The resulting function is rational of order N and has the form

$$\Gamma(z) = C \prod_{a=1}^{N} (z - \lambda_a)^{-1} (z - \bar{\lambda}_a) \; , \tag{6.200}$$

where $|C|^2 = 1$. Finally, Γ satisfies the algebraic conditions

$$\Gamma(z) - x_i \sim \mathcal{O}((z - z_i)^{w_i}) \, , \qquad \Gamma(z) - \frac{1}{\bar{x}_i} \sim \mathcal{O}\left((z - \bar{z}_i)^{w_i}\right) \; , \tag{6.201}$$

near $z = z_i$ and $z = \bar{z}_i$, respectively, where x_i are marked points on the image disk.

UHP to UHP

A branched covering of the form $\Gamma : \mathbb{H} \to \mathbb{H}$ can similarly be extended to a branched covering of the Riemann sphere via the analytic continuation

$$\Gamma(z) = \overline{\Gamma(\bar{z})} \; . \tag{6.202}$$

The resulting function is rational of order N and so generically has the form

$$\Gamma(z) = C \, \frac{\prod_{a=1}^{N} (z - Q_a)}{\prod_{a=1}^{N} (z - P_a)} \; . \tag{6.203}$$

The functional equation (6.202), along with the fact that the boundary of the UHP is $\mathbb{R} \cup \{\infty\}$, tells us that the normalisation C, the zeroes Q_a, and the poles P_a all have to be real. Finally, Γ satisfies the algebraic conditions

$$\Gamma(z_i) - x_i \sim \mathcal{O}\left((z - z_i)^{w_i}\right) , \qquad \Gamma(z) - \bar{x}_i \sim \mathcal{O}\left((z - \bar{z}_i)^{w_i}\right) \ , \tag{6.204}$$

near $z = z_i$ and $z = \bar{z}_i$, where x_i are marked points on the UHP.

Chapter 7

The large-twist limit

7.1 Motivation: refining the ‘t Hooft argument

One of the earliest indications that large N gauge theories could be dual to string theories was given by ‘t Hooft in 1974 [46]. He argued that the free energy $\mathcal{F}$ of a large N gauge theory could be organized into an expansion in $1/N$. Schematically, the free energy, as a function of N and the ‘t Hooft coupling $\lambda = g^2 N$ expands as

$$\mathcal{F}(\lambda, N) = \sum_{g=0}^{\infty} N^{2-2g} F_g(\lambda)\,, \tag{7.1}$$

where $F_g(\lambda)$ are some functions of the ‘t Hooft coupling, but not of N. The functions $F_g(\lambda)$ are computed by summing over connected Feynman diagrams which have genus g (i.e. such that $F - E + V = 2 - 2g$, where F, E, V are the number of faces, edges, and vertices of the graph). If we write $g_s = 1/N$ and define for every Feynman graph of genus g we define a surface Σ_g, we can write

$$\mathcal{F}(\lambda, g_s) = \sum_{g=0}^{\infty} g_s^{2g-2} \sum_{\substack{\text{diagrams} \\ \text{on } \Sigma_g}} \mathcal{A}_{\text{diagram}}(\lambda)\,. \tag{7.2}$$

This sum over surfaces weighted by g_s^{2g-2} is strongly remeniscent of the genus expansion in string theory, and so it is natural to postulate that large N gauge theories are (approximately) dual to string theories.

While the above argument shows a tentative relationship between large N gauge theories and string theory, it is rather schematic. In string theory, we would compute the free energy as a sum over genera, followed by an integral over the moduli space $\mathcal{M}_g$ of genus g curves:

$$\mathcal{F}_{\text{string}}(\lambda, g_s) = \sum_{g=0}^{\infty} g_s^{2g-2} \int_{\mathcal{M}_g} \mathrm{d}\mu\, \mathcal{A}_{\text{string}}(\lambda, \mu)\,, \tag{7.3}$$

where $\mathcal{A}_{\text{string}}(\mu, \lambda)$ is the string CFT partition function on the surface with moduli μ. In order for the string free energy to reproduce the gauge theory free energy, the relation

$$\sum_{\substack{\text{diagrams} \\ \text{on } \Sigma_g}} \mathcal{A}_{\text{diagram}}(\lambda) = \int_{\mathcal{M}_g} \mathrm{d}\mu\, \mathcal{A}_{\text{string}}(\lambda, \mu) \tag{7.4}$$

would have to somehow be satisfied. However, the left-hand-side is a sum over purely topological data – the diagrams which can be placed on the topological space Σ_g, while the right-hand-side is a continuous integral over the $(3g-3)$-dimensional moduli space of complex structures on Σ_g. It seems that, in order for these two procedures to give the same result, we would have to have some kind of relationship between Feynman graphs and moduli spaces of curves.

From free CFTs to strings

In the seminal series of papers [113–115], a mechanism for the equivalence (7.4) was suggested in the free ($\lambda \to 0$) limit of large N matrix theories, as well as for the Gaussian matrix model [116–118] (see also [119] for more recent progress). The idea is to consider correlation functions of the form

$$\left\langle \prod_{i=1}^{n} \mathrm{Tr}[\Phi^{w_i}](x_i) \right\rangle \tag{7.5}$$

in the free limit. The diagrammatic expansion, one sums over all Wick contractions. These Wick contractions are graphs Γ to which one assigns the edge $i \to j$ a number J_{ij} denoting the number of matrix indices contracted between $\mathrm{Tr}[\Phi^{w_i}]$ and $\mathrm{Tr}[\Phi^{w_j}]$. Since we are not considering directed graphs, the numbers J_{ij} are symmetric, and there is one for each edge. Now, the multiplicities J_{ij} must satisfy the n conditions

$$\sum_{j\neq i} J_{ij} = w_i \,, \tag{7.6}$$

since the number of contractions coming from one vertex must equal w_i. We now define the dual graph Γ^* whose vertices lie in the faces of Γ, and whose edges intersect those of Γ. To each edge in Γ^*, we define the *length* of the edge to be the multiplicity J_{ij} of the edge of Γ it intersects with. Let Γ^* have E edges, F faces, and V vertices. Then the number of lengths J_{ij} is the number of edges E, which according to Euler's theorem satisfies

$$F - E + V = 2 - 2g \,. \tag{7.7}$$

The number of faces of Γ^* is just n (since this is the number of vertices in Γ). Now, if we assume that Γ^* is trivalent,[1] then the number of vertices and edges is related by $2E = 3V$, and so we have

$$n - E + \frac{2E}{3} = 2 - 2g \implies E = 3(n + 2g - 2) \,. \tag{7.8}$$

However, the number of 'degrees of freedom' (i.e. the number of independent lengths J_{ij}) is given by $E - n$, since we must impose the restriction (7.6). Thus, the number of independent lengths is

$$\#\text{ of lengths} = 2(n + 3g - 3) \,. \tag{7.9}$$

This is nothing more than the real dimension of $\mathcal{M}_{g,n}$. Thus, it seems reasonable to suspect a relationship between the sum over possible Wick contractions J_{ij} and an 'integral' over the moduli space $\mathcal{M}_{g,n}$.

[1] Γ^* is indeed trivalent for generic Wick contractions. However, the following argument can be modified for non-generic contractions, see [116].

The relationship between Wick contractions in large N free theories and the moduli space $\mathcal{M}_{g,n}$ in fact goes far beyond a simple counting argument. We can generalize the above setup to a more abstract geometric picture. Let us consider a surface Σ with n punctures. Consider a trivalent graph Γ^* such that each face of Γ^* contains precisely one of the punctures of Σ, and to each edge e of Γ^* assign a length $\ell(e) \in \mathbb{R}_+$. The number of independent lengths is precisely given by the number of edges

$$E = 3n + 6g - 6\,. \tag{7.10}$$

It is a theorem of Strebel that each such graph Γ^* and choice of lengths $\ell(e)$ defines a point in $\mathcal{M}_{g,n} \times \mathbb{R}^+$, and that this map is bijective [120–122].

The inverse map, from the moduli space $\mathcal{M}_{g,n} \times \mathbb{R}^n$ to the space of Strebel graphs is given by defining a particular meromorphic quadratic differential φ on the surface $\Sigma \subset \mathcal{M}_{g,n}$ which satisfies the following properties:

- φ has n double poles at the punctures of Σ and satisfies

  $$\varphi(z) \sim \frac{p_i^2}{(z-z_i)^2} + \cdots\,, \tag{7.11}$$

  where $p_i \in \mathbb{R}_+$ for each i. Furthermore, φ has no other poles.

- Since φ is a quadratic differential with $2n$ poles (counting multiplicity), the number of zeroes of φ is $2n + 4g - 4$, call them u_a. The integrals

  $$\ell_{ab} = \int_{u_a}^{u_b} \sqrt{\varphi} \tag{7.12}$$

  are real and positive.

Given the two above conditions, we can define a graph Γ^* as the graph whose vertices are the zeroes of φ and whose edges are the 'critical trajectories' between the zeroes u_a [115]. The length of an edge in Γ^* is the integral of $\sqrt{\varphi}$ along that edge, i.e. the number ℓ_{ab}. For a fixed set of residues p_i, such a quadratic differential is unique for each marked surface $\Sigma \in \mathcal{M}_{g,n}$, and so the above quadratic differential denotes a unique point in $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$, and constructs a graph Γ^* (see Figure 7.4 below for an example on $\mathbb{CP}^1$ with four marked points). Such a quadratic differential φ is called a *Strebel differential.*

In view of the above generalization, the proposal of [116] is that the worldsheet dual to the Gaussian matrix model (a large N gauge theory in zero dimensions) should localize to those points in the moduli space $\mathcal{M}_{g,n} \times \mathbb{R}_+$ such that the lengths ℓ_{ab} and the residues p_i^2 are *integer*. Indeed, the quadratic differential dual to the graph Γ^* in a free gauge theory should satisfy

$$\ell_{ab} \longleftrightarrow J_{ij}\,, \quad p_i = w_i\,. \tag{7.13}$$

In this chapter, we explore an analogous mechanism to that of free field theories in the context of symmetric orbifold CFTs $\mathrm{Sym}^K(X)$.[2] As we laid out in the previous chapters of this thesis, such theories are dual to (tensionless) string theories in AdS_3. The conceptual benefit of working with symmetric orbifold theories is that their

[2]We use the letter K as opposed to N to denote the number of seed theory copies for the rest of the chapter, as we will use the letter N for something else.

genus expansion can be written as a sum over surfaces Σ_g that are not just topological spaces, but also come equipped with a complex structure (via the pullback under the covering map $\Gamma : \Sigma_g \to \mathbb{CP}^1$), and so the 'sum over diagrams' in the symmetric orbifold theory can be rephrased as a sum over discrete points in the moduli space $\mathcal{M}_{g,n}$ of curves. Specifically, given a correlator

$$\left\langle \mathcal{O}_1^{(w_1)}(x_1)\cdots\mathcal{O}_n^{(w_n)}(x_n)\right\rangle \tag{7.14}$$

the number of points in the moduli space $\mathcal{M}_{g,n}$ which contribute at genus g grows as we take the w_i large. Motivated by this statement, we will study correlation functions with twists $1 \ll w_i \ll K$. Through the analysis of the analytic structure of these correlators, three major observations will emerge, which we will spend the rest of this chapter exploring:

- The discrete sum over surfaces Σ_g with fixed complex structure coallesces into an integral over the full moduli space $\mathcal{M}_{g,n}$.
- For each point in the moduli space, there is an extremely simple construction of the corresponding Strebel differential φ in terms of the covering map $\Gamma : \Sigma_g \to \mathbb{CP}^1$.
- The integral over the moduli space is weighted by a term of the form

$$e^{-A}\,,\quad A \propto \int_\Sigma |\varphi|\,, \tag{7.15}$$

  which is the area of the worldsheet Σ in the so-called Strebel gauge.
- In a semiclassical approximation, the Strebel differential can also be realted to the Nambu-Goto action of a string propagating in AdS_3.

All of these observations, taken together, should be thought of as an extreme refinement of the 't Hooft argument since, not only do symmetric orbifolds have large K expansions which are reminiscent of the genus expansion of string theory, but this large K expansion precisely reproduces an integral over $\mathcal{M}_{g,n}$ weighted by the Nambu-Goto action of a string in AdS_3.

7.2 The large-twist limit of the symmetric orbifold

The scattering equations

Consider a correlator in the symmetric orbifold of the form

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i^{(w_i)}(x_i)\right\rangle . \tag{7.16}$$

As we discussed in previous chapters, such correlation functions enjoy a diagrammatic expansion in terms of covering spaces $\Gamma : \Sigma \to \mathbb{CP}^1$, and the 'connected' component can be expressed as a sum over such maps [80, 123], weighted by an appropriate value of the string coupling and a conformal anomaly

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i^{(w_i)}(x_i)\right\rangle_{\mathrm{c}} = \sum_{\Gamma:\Sigma\to\mathbb{CP}^1} g_s^{2g-2+n} e^{-S_{\mathrm{L}}[\Phi_\Gamma]} \left\langle \prod_{i=1}^{n} \mathcal{O}_i(z_i)\right\rangle_{\Sigma} , \tag{7.17}$$

where g is the genus of the covering surface and z_i are the critical points of Γ on Σ, i.e.

$$\Gamma(z) = x_i + \mathcal{O}((z-z_i)^{w_i}), \quad z \to z_i. \tag{7.18}$$

Here, S_{L} is the Liouville action on the covering surface Σ

$$S_{\mathrm{L}}[\Phi] = \frac{c}{48\pi}\int_{\Sigma}(2\partial\Phi\,\bar{\partial}\Phi + R\,\Phi), \tag{7.19}$$

and Φ_Γ is the 'classical' Liouville field associated to the covering map Γ

$$\Phi_\Gamma = \log|\partial\Gamma|^2. \tag{7.20}$$

The main point to note is that all of the information about the correlation functions of twisted correlators in the symmetric orbifold is contained in the correlator of the seed theory on arbitrary surfaces Σ and in the holomorphic data of the covering map Γ. Thus, assuming the seed theory is under sufficiently good control, we can reduce the problem of computing correlators in the symmetric orbifold to the problem of constructing a covering map with the desired analytic behavior.[3]

Generically, given the data $\{x_i\}$, $\Sigma\setminus\{z_i\}$ and $\{w_i\}$, finding such a covering map is an algebraically difficult problem, and can only be solved when the moduli of $\Sigma\setminus\{z_i\}$ are finely-tuned. In fact, given points on the sphere $\{x_i\}$ and twists $\{w_i\}$, there are only finitely many points in the moduli space $\mathcal{M}_{g,n}$ for which a covering map $\Gamma:\Sigma\setminus\{z_i\}\to\mathbb{CP}^1\setminus\{x_i\}$ exists with the correct critical behavior near z_i.

For now, let us restrict to the case where our covering surface has genus zero, so that we are considering branched covering maps $\Gamma:\mathbb{CP}^1\to\mathbb{CP}^1$ (although the following discussion can be extended to covering spaces of higher genus). In this case, the degree of Γ can be determined by the Riemann-Hurwitz formula as

$$\deg(\Gamma) = N = 1 + \sum_{i=1}^{n}\frac{w_i-1}{2}. \tag{7.21}$$

Thus, since Γ is a holomorphic function on $\mathbb{CP}^1$ (or equivalently a meromorphic function on $\mathbb{C}$), we can write

$$\Gamma(z) = \frac{Q_N(z)}{P_N(z)} \tag{7.22}$$

for polynomials Q_N, P_N of degree N.[4] Let us call the zeroes of P_N λ_a for $a = 1,\ldots,N$.

Now, let us consider the derivative $\partial\Gamma$, which will be a ratio of polynomials of degree $2N$. It is easy to see that Γ must be of the form

$$\partial\Gamma(z) = C\frac{\prod_{i=1}^{n}(z-z_i)^{w_i-1}}{\prod_{a=1}^{N}(z-\lambda_a)^2}, \tag{7.23}$$

since $\partial\Gamma$ has zeroes of order w_i-1 at $z=z_i$ and poles of order 2 at $z=\lambda_a$. The numerator has degree $2N-2$ so that $\partial\Gamma(z)\sim 1/z^2$ as $z\to\infty$. Now, since $\partial\Gamma$ is a total derivative of a meromorphic function, its residue at any pole must vanish, since otherwise Γ would have logarithmic contributions to its Laurent series

[3]Strictly speaking, the Liouville action defined above diverges for $\Phi=\Phi_\Gamma$, and so one also needs to introduce a regularisation scheme, as in [80].

[4]We are assuming none of the x_i and none of the z_i lie at infinity.

expansion around $z = \lambda_a$. Demanding that the residue at λ_a vanishes leads to the so-called 'scattering equations' [124]

$$\sum_{i=1}^{n} \frac{w_i - 1}{\lambda_a - z_i} = \sum_{b \neq a} \frac{2}{\lambda_a - \lambda_b}\,, \tag{7.24}$$

which are the central algebraic constraints for the existence of a covering map (of course, one also has to demand that $\Gamma(z_i) = x_i$ for all i once the scattering equations are solved).

The matrix model and the spectral curve

The insight of [48] was to rewrite (7.24) in terms of the classical equations of motion of an $N \times N$ Hermitian matrix model. Indeed, consider the matrix integral

$$\int \mathrm{d}M \, \exp\left(-V(M)\right)\,, \tag{7.25}$$

where the potential is a so-called Penner-like potential

$$V(M) = N \sum_{i=1}^{n} \alpha_i \log(M - z_i \mathbf{1})\,. \tag{7.26}$$

We can now diagonalise M in terms of eigenvalues λ_a, and recast the matrix integral into the form

$$\int \mathrm{d}^N\lambda \, \Delta(\lambda)^2 \, \exp\left(-N \sum_{a=1}^{N} \sum_{i=1}^{n} \alpha_i \log(\lambda_a - z_i)\right)\,, \tag{7.27}$$

where $\Delta(\lambda)^2 = \prod_{a<b}(\lambda_a - \lambda_b)^2$ is the usual Vandermonde determinant arising upon diagonalisation. The inclusion of the Vandermode determinant induces an effective potential given by

$$V_{\text{eff}}(z) = \sum_{i=1}^{n} \alpha_i \log(z - z_i) - \frac{2}{N} \sum_{a=1}^{N} \log(z - \lambda_a)\,, \tag{7.28}$$

such that the partition function of the matrix model can be written as[5]

$$\int \mathrm{d}^N\lambda \, \exp\left(-N \sum_{a=1}^{N} V_{\text{eff}}(\lambda_a)\right)\,. \tag{7.29}$$

The matrix model integral is generally computed by first considering the tree-level contributions, i.e. the configurations such that $V'_{\text{eff}}(\lambda_a) = 0$ for all a (and indeed for $N \gg 1$, these are the dominant contributions). These classical equations of motion are

$$\sum_{i=1}^{n} \frac{\alpha_i}{\lambda_a - z_i} = \frac{2}{N} \sum_{b \neq a} \frac{1}{\lambda_a - \lambda_b}\,. \tag{7.30}$$

[5]One needs to be careful with the term $\log(\lambda_a - \lambda_a) = -\infty$ in the effective potential; however, this only introduces an (infinite) constant, and thus does not affect the dynamics of the theory.

Thus, we recover precisely the scattering equations (7.24) with $\alpha_i = (w_i - 1)/N$.[6]

For $N \gg 1$, there are powerful methods for computing the solutions to the saddle-point equations for matrix models which we now have at our disposal. In particular, we assume that at large N the eigenvalues λ_a condense into a curve in the complex plane, which potentially has many disconnected components. Let $\mathcal{C} = \bigcup_{\ell=1}^{m} \mathcal{C}_\ell$ be the decomposition of this curve into disjoint segments. Then we can write the scattering equations as

$$\sum_{i=1}^{n} \frac{\alpha_i}{\lambda - z_i} = 2P \int_{\mathcal{C}} \frac{\rho(\lambda')\,\mathrm{d}\lambda'}{\lambda - \lambda'}\,, \tag{7.31}$$

where $\rho(\lambda)$ is the asymptotic density of eigenvalues along the curve $\mathcal{C}$ and P denotes the principle value of the integral.

Equation (7.31) defines a Riemann-Hilbert problem, whose solution can be found with the help of a so-called spectral curve. We define an auxiliary function

$$y(z) = \sum_{i=1}^{n} \frac{\alpha_i}{z - z_i} - 2 \int_{\mathcal{C}} \frac{\rho(\lambda)\,\mathrm{d}\lambda}{z - \lambda} \tag{7.32}$$

which is holomorphic and globally defined on $\mathbb{C} \setminus \mathcal{C}$. By the scattering equations (7.31), we see that $y(z) = 0$ on the endpoints of $\mathcal{C}$. Furthermore, y has square-root branch cuts along $\mathcal{C}$, and is thus not globally defined on $\mathbb{C}$. However, its square $\phi(z) = y(z)^2$ is globally defined on $\mathbb{C}$. We can therefore think of y as being globally defined on a Riemann surface $\tilde{\Sigma}$ defined via the polynomial equation

$$\tilde{\Sigma} : y^2 = \phi(z)\,. \tag{7.33}$$

The curve $\tilde{\Sigma}$ is known as the *spectral curve* of the matrix model, and is a double cover of the Riemann sphere ramified at the points $\partial\mathcal{C}$. A natural basis of the homology group $H_1(\tilde{\Sigma}, \mathbb{Z})$ is one in which cycles which surround branch cuts form the A-cycles of $\tilde{\Sigma}$, while the remaining cycles form the B-cycles. If $\mathcal{C}$ has $2m$ endpoints, then the genus of $\tilde{\Sigma}$ is $m - 1$, since only $m - 1$ of the A-cycles/B-cycles are independent.

Once the spectral curve is known, it can be used to reconstruct the density $\rho(\lambda)$, and thus the solution of the scattering equations. Indeed, since $\mathcal{C}$ is essentially just the branch cut of a function $y = \sqrt{\phi}$, it can be chosen arbitrarily such that its endpoints are the zeroes of the function y. Once we know y, we can then determine the density $\rho(\lambda)$ via the Sokhotski–Plemelj theorem, i.e. by taking the difference in the value of y just above/below a branch cut

$$\rho(\lambda) = \frac{1}{4\pi i}(y(\lambda + i\varepsilon) - y(\lambda - i\varepsilon))\,, \tag{7.34}$$

for $\lambda \in \mathcal{C}$. Furthermore, we can define the so-called *filling fractions* of the density ρ via periods of the differential $\sqrt{\phi}$. That is,

$$c_\ell = \int_{\mathcal{C}_\ell} \rho(\lambda)\,\mathrm{d}\lambda = \frac{1}{2\pi i} \oint_{\mathcal{C}_\ell} \sqrt{\phi(z)}\,\mathrm{d}z\,, \tag{7.35}$$

where the integral domain of the second integral is the A-cycle of $\tilde{\Sigma}$ enclosing the branch cut $\mathcal{C}_\ell$. In terms of the covering map Γ, the filling fractions determine what fraction of the poles condense on to the component $\mathcal{C}_\ell$ of the curve $\mathcal{C}$.

[6] Strictly speaking, the solutions λ_a to the scattering equations that we are interested in are complex, as opposed to real solutions one would expect from a Hermitian matrix model. Since the matrix model defined above is only used as an auxilliary object, this subtlety causes no harm, and we can simply take $z_i, \lambda_a \in \mathbb{C}$.

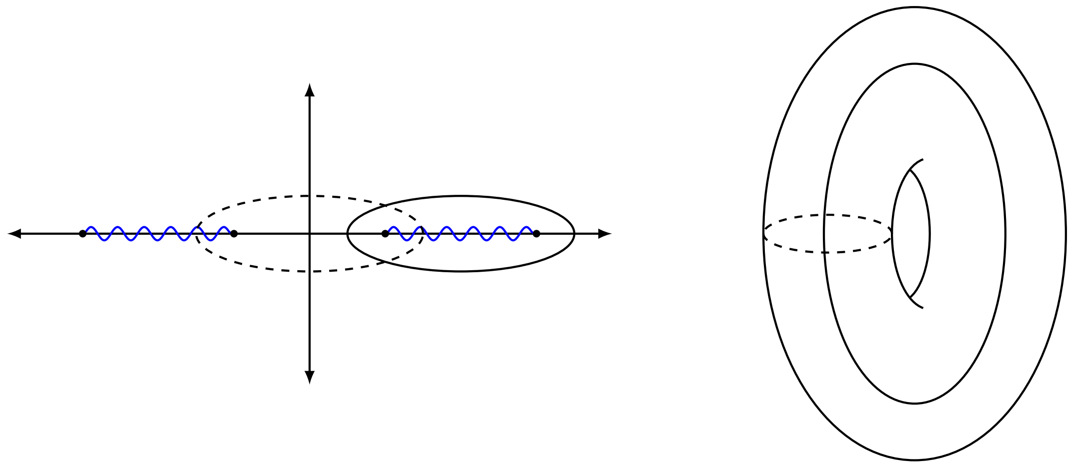

Figure 7.1: Left: Non-contractible cycles of a genus-1 spectral curve $\tilde{\Sigma} : y^2 = \phi(z)$. Squiggly blue lines represent the branch cuts, while solid cycles represent the A-cycles of $\tilde{\Sigma}$ and dashed cycles represent the B-cycles. Right: The lift of the non-contractible cycles onto the genus-1 covering surface.

The Strebel differential

The differential $\phi(z)\,(\mathrm{d}z)^2$ defining the spectral curve above has the following properties:

- By (7.32), near the insertion points $z = z_i$, we have
$$\phi(z)(\mathrm{d}z)^2 \sim \frac{\alpha_i^2(\mathrm{d}z)^2}{(z-z_i)^2} + \cdots . \tag{7.36}$$
- ϕ has simple zeroes precisely at the endpoints of the branch-cuts $\mathcal{C}_\ell$ (let us call them a_ℓ, b_ℓ, with $\partial\mathcal{C}_\ell = \{a_\ell, b_\ell\}$).
- If γ is a path connecting the zeroes a_ℓ and b_ℓ, then
$$\frac{1}{2\pi i}\int_\gamma \sqrt{\phi(z)}\,\mathrm{d}z \in \mathbb{R}\,. \tag{7.37}$$

This last point is due to (7.34), since $\rho(\lambda)$ is taken to be real. As was mentioned briefly above, these properties identify what is known as a *Strebel differential* (see Appendix A of [125] for a gentle introduction). Let us define a quadratic differential on Σ via
$$\varphi(z) = -\phi(z)\,(\mathrm{d}z)^2\,. \tag{7.38}$$
It is a theorem of Strebel that given values $\alpha_i \in \mathbb{R}^+$, there exists a unique quadratic differential φ satisfying the above properties for any given value of the moduli $\{z_i\}$. Thus, for each choice of the moduli $\{z_i\}$, we find a unique spectral curve, and thus we can uniquely solve the scattering equations (7.31).

Let us now return to the problem of constructing the covering map. Note that for finite N, we have

$$\frac{1}{N}\log\partial\Gamma(z) = \sum_{i=1}^{n}\alpha_i\log(z-z_i) - \frac{2}{N}\sum_{a=1}^{N}\log(z-\lambda_a) \tag{7.39}$$

which is the effective potential of our matrix model, and we have ignored a constant contribution. Taking N large, we can write

$$\frac{1}{N}\log\partial\Gamma(z)=\sum_{i=1}^{n}\alpha_i\log(z-z_i)-2\int_{\mathcal{C}}\mathrm{d}\lambda\,\rho(\lambda)\log(z-\lambda)\,. \tag{7.40}$$

Finally, we can take the derivative and we find

$$\frac{1}{N}\frac{\partial^2\Gamma}{\partial\Gamma}=\sum_{i=1}^{n}\frac{\alpha_i}{z-z_i}-2\int_{\mathcal{C}}\frac{\rho(\lambda)\,\mathrm{d}\lambda}{z-\lambda}=y(z)\,. \tag{7.41}$$

The branch cuts of y can be thought of as arising from the coalescence of poles of Γ along the curve $\mathcal{C}$.[7] Thus, we can relate the spectral curve coordinate y directly to the covering map. This allows us to immediately write down the Strebel differential (7.38) as

$$\varphi=-\frac{1}{N^2}\left(\frac{\partial^2\Gamma}{\partial\Gamma}\right)^2\,. \tag{7.43}$$

Finally, as noted in [48], at large N we can approximate φ via the Schwarzian derivative of Γ, i.e.

$$\varphi=\frac{2}{N^2}S[\Gamma]=\frac{2}{N^2}\left(\partial\left(\frac{\partial^2\Gamma}{\partial\Gamma}\right)-\frac{1}{2}\left(\frac{\partial^2\Gamma}{\partial\Gamma}\right)^2\right)\,. \tag{7.44}$$

In the large N limit, the first term in the Schwarzian subdominates, and we are simply left with (7.43). As we will see later, however, writing the Strebel differential in terms of the Schwarzian derivative of the covering map allows for a natural interpretation of the AdS_3 worldsheet theory.

The Strebel-gauge action

Now let us return to the problem of constructing correlation functions of the symmetric orbifold, which is computed in terms of the Liouville action of the field $\Phi_\Gamma=\log|\partial\Gamma|^2$. Note that one can immediately write the derivative of the Liouville field in terms of the spectral curve $y(z)$, i.e.

$$\partial\Phi_\Gamma=Ny\,,\quad\bar{\partial}\Phi_\Gamma=N\bar{y}\,, \tag{7.45}$$

and thus the Liouville action can be expressed as

$$S_L[\Phi_\Gamma]=\frac{cN^2}{24\pi}\int_\Sigma|y|^2=\frac{N^2}{4\pi}\int_\Sigma|\varphi|\,, \tag{7.46}$$

where φ is the Strebel differential, and in the second equality we have specified to the $\mathbb{T}^4$ (or K3) seed theory, i.e. $c=6$. If we are working on a covering surface with genus g and n punctures, there is a unique Strebel differential for each value

[7] For a simple example of this phenomenon, consider a function f defined by

$$f(z)=\frac{1}{N}\sum_{a=1}^{N}\frac{1}{z-a/N}\,. \tag{7.42}$$

As $N\to\infty$, we have $f(z)\to\log(1-z)+\log(z)$, which has a branch cut along $[0,1]$.

of α_i and each value of the moduli of the covering space. Thus, we can express the symmetric product orbifold correlation functions in this limit as an integral over $\mathcal{M}_{g,n}$, namely

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i^{w_i}(x_i) \right\rangle_{\mathrm{c}} \sim \sum_{g=0}^{\infty} g_s^{n+2g-2} \int_{\mathcal{M}_{g,n}} \exp\left(-\frac{N^2}{4\pi} \int_{\Sigma} |\varphi| \right) \left\langle \prod_{i=1}^{n} \mathcal{O}_i(z_i) \right\rangle_{\Sigma} . \tag{7.47}$$

The effective action (7.46) and the integral over $\mathcal{M}_{g,n}$ has an immediate stringy interpretation. Given a Strebel differential φ, one can associate a canonical metric $g_{z\bar{z}} = N^2|\varphi|/4$ such that g is flat except at the poles and zeroes of φ, which give curvature singularities of opposite sign. The action (7.46) then has the interpretation of the Nambu-Goto action of some minimal area string whose pullback metric is g. Since this action arose naturally from a 2D CFT with known AdS_3 dual, one would expect that g is the pullback of the AdS_3 metric on the worldsheet. Indeed, in Section 7.3 we will see that this is the case precisely when one takes N to be large.

Example: the partition function

In the above subsections, we reviewed the construction of the Strebel differential in the large-twist limit of the symmetric orbifold. In order to exemplify how this might work, let us consider the correlation function for which the covering map is as simple as possible: the genus-one partition function. Here, Γ is an unramified covering map between two tori, the worldsheet and the boundary of thermal AdS_3, and its degree can be taken to be as large as possible without introducing analytic difficulties.

To compute the partition function, we follow the approach of [58]. In order to keep track of the twist of the CFT states, we consider the 'grand canonical' partition function

$$\mathfrak{Z}(p,t) = \sum_{K=0}^{\infty} p^K Z_{\mathrm{Sym}^K(\mathcal{M})}(t) , \tag{7.48}$$

where p is a chemical potential counting the order K, and t is the torus modulus. A fundamental result in the theory of permutation orbifolds is that this partition function can be expressed in terms of Hecke operators

$$\mathfrak{Z}(p,t) = \exp\left(\sum_{N=1}^{\infty} p^N T_N Z(t) \right) , \tag{7.49}$$

where $Z(t)$ is the partition function of the seed theory $\mathcal{M}$, and the Hecke operators T_N are defined below. Intuitively, the term in the exponential is generated by connected covering maps of degree w, which are always tori. The part of this partition function which is visible from the worldsheet theory is the connected component $\log \mathcal{Z}(p,t)$, and the full partition function can only be recovered by considering a second quantized theory of strings.

Let us examine the form of the connected component of the partition function at large N. Recall that the Hecke operator T_N acts on modular functions as

$$T_N Z(t) = \frac{1}{N} \sum_{ad=N} \sum_{b=0}^{d-1} Z\left(\frac{at+b}{d} \right) . \tag{7.50}$$

The modulus $\tau = (at+b)/d$ is to be interpreted as the modulus of a torus which covers the original torus with degree N and is therefore interpreted as the modulus of the dual worldsheet. The number of such covering tori is given by the *divisor function* of N

$$\sum_{ad=N}\sum_{b=0}^{d-1} 1 = \sum_{d|N} d = \sigma_1(N)\,. \tag{7.51}$$

The growth rate of this function is known to be asymptotically bounded by as $\mathcal{O}(N\log\log N)$ [126], and so the Hecke operator diverges at worst double logarithmically as $N \to \infty$.[8]

We can thus, naively, consider the limit in which $N \to \infty$, for which the number of covering spaces diverges, and it seems reasonable that the space of allowed τ forms a continuum. To exemplify how this works, we can consider the simple case of N prime. Since N only has two divisors, 1 and itself, the moduli of the covering tori can be written as

$$\left\{Nt, \frac{t}{N}, \dots, \frac{t+N-1}{N}\right\}\,. \tag{7.52}$$

As $N \to \infty$, we can forget the first element of (7.52) without affecting the sum too much, and focus simply on the moduli of the form $\tau = (t+b)/N$. However, these moduli do not all lie in the fundamental domain, and will generically all lie in different fundamental domains of $\mathrm{SL}(2,\mathbb{Z})$. If we bring them into the fundamental domain, their distribution will be effectively random, since the modular transformations bringing each modulus into $\mathcal{F}$ differ wildly as we change b (see Figure 7.2 for an example with $N = 971$). More concretely, the map $\mathbb{H} \to \mathcal{F}$ taking a point in the upper-half-plane into its representative in the fundamental domain exhibits chaotic behavior as $\mathrm{Im}(\tau) \to 0$.

As we increase N (but keeping N prime), we can approximate the Hecke operator T_N as the following integral

$$T_N Z(t) \approx \lim_{\varepsilon\to 0}\int_0^1 \mathrm{d}x\, Z(x+i\varepsilon)\,, \tag{7.53}$$

where we have defined the small number $\varepsilon = t/iN$. The line $x + i\varepsilon$ for $x \in [0, 1)$ equidistributes into the fundamental domain $\mathcal{F}$ as $\varepsilon \to 0$ with measure $\mathrm{d}^2\tau/\mathrm{Im}(\tau)^2$ [128].[9] Thus, we have

$$T_N Z(t) \longrightarrow \frac{3}{\pi}\int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\mathrm{Im}(\tau)^2}\, Z(\tau)\,, \tag{7.54}$$

where the prefactor is simply $\mathrm{Vol}(\mathcal{F})^{-1}$, which we will ignore from now on. Therefore, the symmetric orbifold partition function for fixed N approaches the path integral of a string propagating on the seed theory $\mathcal{M}$ as $N \to \infty$, which we will interpret below as the string propagating only through the origin of AdS_3 and only having fluctuations in the $\mathcal{M}$-direction.

In the language of the Strebel differential, we might try to immediately write down an answer as

$$\int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\mathrm{Im}(\tau)^2}\, Z(\tau) e^{-S_{\mathrm{NG}}(\tau)}\,, \tag{7.55}$$

[8] In fact, assuming the Riemann hypothesis is true, it can be shown that $\sigma_1(N) < e^{\gamma} N \log\log N$ [127]. If the reader is uncomfortable with convergence issues, we can simply take N prime, for which $\sigma_1(N) = N + 1$, and the Hecke operator converges, assuming $Z(\tau)$ is reasonably well-behaved as $\tau \to i\infty$.

[9] We thank Lorenz Eberhardt for pointing this out.

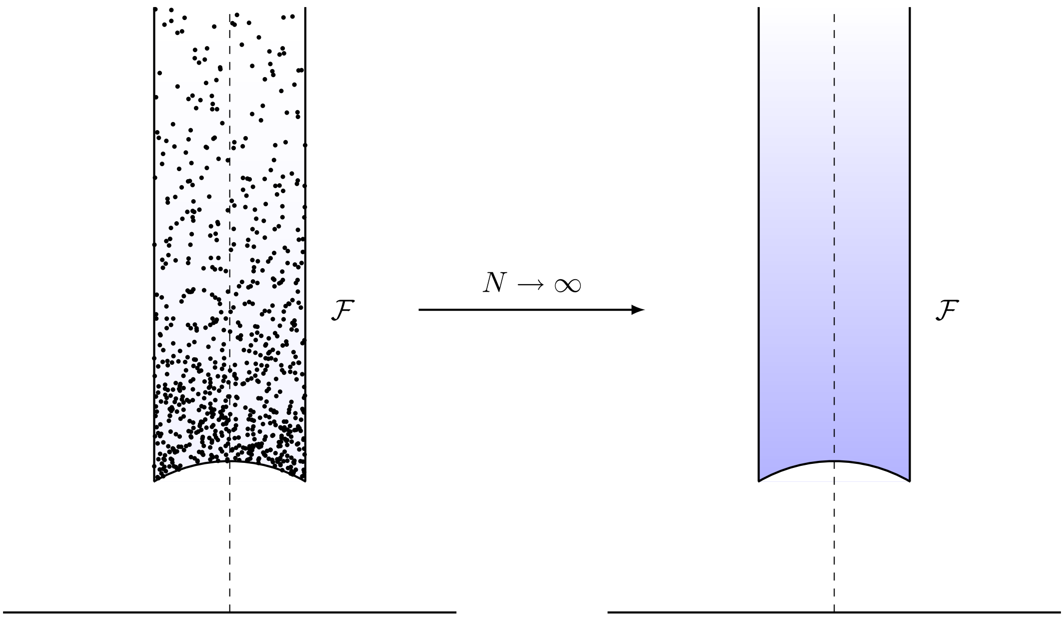

Figure 7.2: Left: the distribution of covering tori for $N = 971$ and $t = 0.2 + 2i$. Moduli with $\text{Im}(\tau) > 4$ have been omitted. Right: the limiting probability distribution as $N \to \infty$.

where the measure $\mathrm{d}\mu$ is the same as above, since it arises from the density of covering maps in $\mathcal{F}$ as $N \to \infty$. The above analysis then seems to suggest that $S_{\text{NG}}(\tau) = 0$ for all τ. The reason for this is quite simple: for the configuration above, the Strebel differential vanishes.[10] Indeed, as we will argue in a moment, the covering map Γ from the worldsheet torus to the AdS boundary torus is linear, and thus its corresponding Strebel differential vanishes.

The covering map from the worldsheet torus to the spacetime boundary is given by $\Gamma(z) = \alpha z$ for some constant α. The periodicity conditions require $\Gamma(z+1) = \Gamma(z) + ct + d$ and $\Gamma(z+\tau) = \Gamma(z) + at + b$ for integers a, b, c, d. This requires $\alpha = ct + d$ and $\tau = (at+b)/(ct+d)$. As above, we can take $c = 0$ and $0 \leq b \leq d - 1$. Thus, the covering map is $\Gamma(z) = dz$. The integer d determines the number of times the worldsheet wraps the contractible cycle of AdS_3, and in general will be large in the sum.

From the point of view of the worldsheet, the vanishing of the Strebel action is quite surprising, since we expect it to reproduce the area of some semiclassical string worldsheet in AdS_3. In fact, in the limit in question, this area vanishes. As we will see in the next section, in the limit of large d this covering map describes a string completely localized in the center of AdS_3, and thus has zero area (see Figure 7.3 below). The only part of the worldsheet theory where the string has proper dynamics, then, is the internal manifold $\mathcal{M}$, and so the string partition function is simply the integral of $Z(\tau)$ over the moduli space $\mathcal{M}_{1,0}$.[11] We should also note that

[10] This naively contradicts the statement that Strebel differential are in one-to-one correspondence to points in the moduli space $\mathcal{M}_{g,n} \times \mathbb{R}^n$. However, this statement only holds for surfaces with $n + 2g - 2 > 0$, i.e. for which the space of holomorphic quadratic differentials is non-vanishing. Mathematically, this corresponds to the stability condition of the moduli space $\mathcal{M}_{g,n}$.

[11] Strictly speaking, the string also propagates in the S^3. The motion of the string in this direction should be apparent if we consider the full supersymmetric generalization of this analysis.

the string being localized to the center of AdS_3 supports the proposal of [75] that a string wrapping a contractible cycle in the bulk generates a conical defect, since a string localized to the center of AdS_3 simply behaves as a massive particle, which sources a defect singularity.

7.3 Semiclassical analysis

In the previous section we reviewed how the correlation functions of the symmetric product orbifold reorganize themselves into a path integral over string worldsheets whose areas are given by the Strebel metric $\mathrm{d}s^2 = N^2|\varphi|\mathrm{d}z\,\mathrm{d}\bar{z}/4$. In this section, we explore the large-twist limit of a classical string sigma model on AdS_3 and show how the Strebel differential arises naturally in this context as the pullback of the AdS_3 metric onto the worldsheet, providing a link between the large-twist limit of the symmetric orbifold and that of the AdS_3 theory.

The worldsheet sigma model

Consider the semiclassical action of (bosonic) string theory on AdS_3, whose first-order form (see, e.g. [78]) can be written as

$$S = \frac{k}{4\pi}\int_\Sigma \mathrm{d}^2 z\,\left(4\partial\Phi\,\bar{\partial}\Phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - e^{-2\Phi}\beta\bar{\beta} - k^{-1}R\Phi\right)\,, \tag{7.56}$$

where R is the worldsheet curvature, and k is the amount of NS-NS flux (the tensionless limit corresponds to $k = 1$). Geometrically, the pair $(\gamma, \bar{\gamma})$ parametrises the motion of the string along the boundary of AdS_3 in complex coordinates, while the scalar Φ is related to the Poincaré radial coordinate as $r^2 = e^{-2\Phi}$, so that $\Phi \to \infty$ is at the boundary of AdS_3.[12] Given a holomorphic map $\Gamma : \Sigma \to \mathbb{CP}^1$, one can construct a solution to the classical equations of motion given by [78]

$$\gamma(z,\bar{z}) = \Gamma(z)\,,\quad \bar{\gamma}(z,\bar{z}) = \bar{\Gamma}(\bar{z})\,,\quad \Phi(z,\bar{z}) = \log\frac{1}{\varepsilon} - \frac{1}{2}\log|\partial\Gamma|^2\,. \tag{7.57}$$

(There is a corresponding expression for β, but we will not need it). Here, ε is a scaling parameter which can be thought of as an infrared cutoff. In the limit that $\varepsilon \to 0$, the corresponding classical solution satisfies $\Phi \to \infty$, and thus the worldsheet is 'pinned' to the boundary. This is precisely the limit in which the action (7.56) becomes free, and so the path integral expanded around the solution (7.57) becomes a Gaussian integral, and we can thus expect (7.57) to be valid quantum mechanically, regardless of the value of k.

In the tensionless ($k = 1$) string theory on $AdS_3 \times S^3 \times \mathcal{M}$, the classical solution can be shown to be the only contribution to the string path integral [47, 78]. In particular, if we consider vertex operators $V^{w_i}(x_i, z_i)$, where w_i labels the spectral flow of the corresponding state, then the correlation function

$$\left\langle \prod_{i=1}^{n} V^{w_i}(x_i, z_i)\right\rangle \tag{7.58}$$

[12]Note that the normalisation of the radial coordinate r is such that $r = 0$ is the asymptotic boundary and $r = \infty$ is the center of AdS.

receives only contributions in the path integral from solutions of the form (7.57), where the holomorphic map Γ satisfies

$$\Gamma(z) \sim x_i + \mathcal{O}((z - z_i)^{w_i}) \tag{7.59}$$

near the worldsheet insertion points. Thus, taking the scaling limit $\varepsilon \to 0$, we are left with the conclusion that *every* contribution to the string theory path integral is given by a string which is completely localised to the boundary of AdS. That is, for tensionless strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}$, we can take seriously the expression (7.57) as the motion of the worldsheet in the target space.

Naively, one would conclude, then, that the tensionless string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}$ knows nothing about the bulk, since for the solutions (7.57) we have $\Phi \to \infty$ and so the string never probes the spacetime geometry beyond the boundary. For finite values of w, this is indeed true. However, note that if w is taken to be very large (and, in particular, parametrically larger than the infrared cutoff $1/\varepsilon$), then we can actually consider worldsheets which probe the interior. To see this, let us expand the solution (7.57) around the point $z = z_i$. We have

$$\Phi(z, \bar{z}) = \log \frac{1}{\varepsilon w_i} - \frac{w_i - 1}{2} \log |z - z_i|^2 + \cdots \tag{7.60}$$

The radial profile $r^2 = e^{-2\Phi}$ then takes the form

$$r^2(z, \bar{z}) = \varepsilon^2 |\partial \Gamma|^2 \sim \varepsilon^2 w_i^2 |z - z_i|^{w_i - 1} \,. \tag{7.61}$$

That is, if we take $w_i \gg 1/\varepsilon$, the radial profile can become non-zero, and the string is allowed to probe the bulk.

As a simple example, let us consider the classical solutions which arise when computing the one-loop partition function of the string theory. We consider γ to take values on a torus, so that the target space geometry is thermal AdS_3 with modular parameter t. Specifically, that means we consider the identification $\gamma \sim \gamma + 1$ and $\gamma \sim \gamma + t$. Furthermore, we take the worldsheet to be a torus with modular parameter τ. As explained in Section 7.2, the appropriate covering map is now a holomorphic map from the worldsheet torus to the boundary torus of thermal AdS_3 and takes the form $\Gamma(z) = dz$ for some positive integer d. Furthermore, we demand $\tau = (at + b)/d$ for integers a, b in order for this map to be well-defined, i.e. so that $\gamma(z) \sim \gamma(z + 1) \sim \gamma(z + \tau)$ on the AdS_3 boundary torus. Here, a represents the number of times the worldsheet wraps the compact time direction of the AdS_3 boundary, and d counts the number of times the worldsheet wraps the angular direction. For this covering map, the radial profile is given by

$$r^2(z, \bar{z}) = \varepsilon^2 |\partial \Gamma|^2 = \varepsilon^2 d^2 \,. \tag{7.62}$$

If we consider the limit in which the worldsheet wraps the angular direction many times, i.e. $d \gg 1/\varepsilon$, we see that the radial profile vanishes, i.e. the string is completely localised to the center of AdS_3. In this limit, the string behaves like a massive particle in the center of AdS_3, which sources a conical defect in the resulting geometry [75], see Figure 7.3. Note that the existence of conical defects in an effective AdS_3 bulk theory is of interest from a purely gravitational stand-point (see, for instance, [129, 130]), and it has been argued that the inclusion of these singularities is required for the consistency of the gravitational path integral [131]. Thus, although

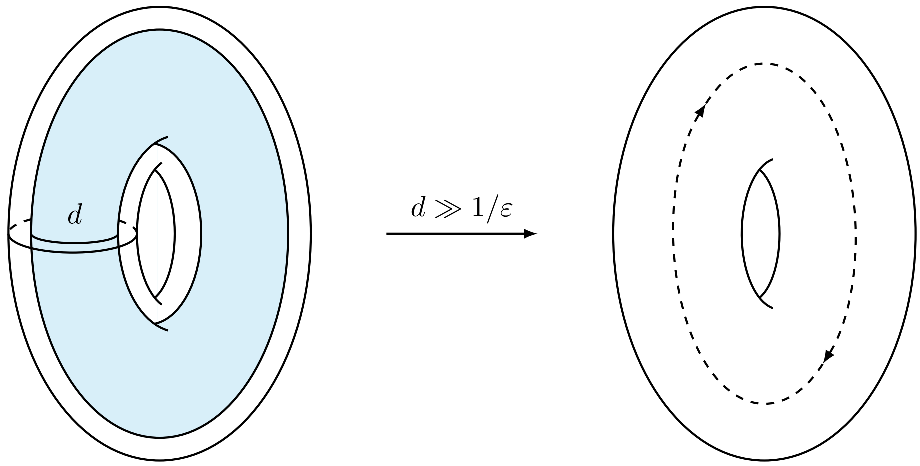

Figure 7.3: A string in thermal AdS_3 which wraps the contractible cycle many times behaves like a massive particle.

our discussion is not related to pure AdS_3 gravity, it is satisfying to see the existence of conical defects arising automatically from a worldsheet theory.

We should emphasize that this limit is very strange from a physical perspective. The parameter $1/\varepsilon$ represents the UV cutoff in the dual CFT, and the duals of strings with winding w are states whose conformal weights grow linearly in w. Thus, taking $w_i \gg 1/\varepsilon$ corresponds in the dual CFT to considering states whose energies are higher than the UV cutoff, which is physically not a meaningful thing to do. Furthermore, the solutions (7.57) only solve the equations of motion in the limit $\varepsilon \to 0$, and we have no reason to expect $w_i \gg 1/\varepsilon$ to be a physically meaningful limiit from the classical sigma model. However, as we will see below, this limit is useful in explaining schematically how the semiclassical Nambu-Goto-like action (7.46) emerges from a string moving in AdS_3. Thus, we will go forward with considering correlators in the limit $w_i \gg 1/\varepsilon$, while remembering that this limit is physically poorly defined, and that we should take it with a rather large grain of salt.

The Strebel metric from classical geometry

Consider the semiclassical solutions (7.57) of the previous section, and consider Poincaré coordinates $(r, \gamma, \bar{\gamma})$ on AdS_3. As we mentioned above, the semiclassical solutions give us an embedding of the string into AdS_3 via

$$r(z,\bar{z}) = e^{-\Phi(z,\bar{z})}\,, \quad \gamma(z,\bar{z}) = \Gamma(z)\,, \quad \bar{\gamma}(z,\bar{z}) = \bar{\Gamma}(\bar{z})\,. \tag{7.63}$$

Now, the Poincaré metric $\mathrm{d}s^2 = (\mathrm{d}r^2 + \mathrm{d}\gamma\,\mathrm{d}\bar{\gamma})/r^2$ can be pulled back to the string worldsheet using this embedding. The result is

$$\mathrm{d}s^2_{\text{worldsheet}} = (\mathrm{d}\Phi)^2 + e^{2\Phi}|\partial\Gamma|^2\mathrm{d}z\,\mathrm{d}\bar{z}\,. \tag{7.64}$$

Using the relation $r^2 = \varepsilon^2|\partial\Gamma|^2$ from (7.57), we end up with

$$\mathrm{d}s^2_{\text{worldsheet}} = \left(\frac{1}{4}\left|\frac{\partial^2\Gamma}{\partial\Gamma}\right|^2 + \frac{1}{\varepsilon^2}\right)\mathrm{d}z\,\mathrm{d}\bar{z}\,. \tag{7.65}$$

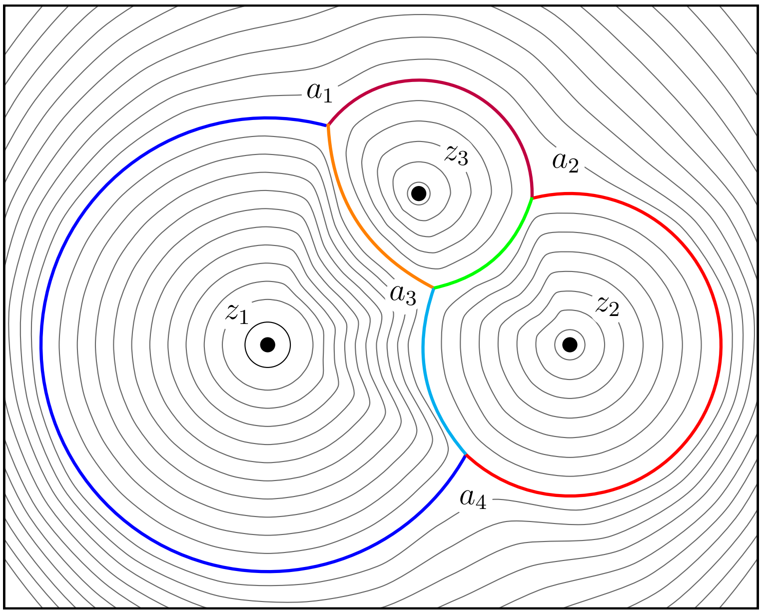

Figure 7.4: The natural coordinate system associated to a Strebel differential φ on the worldsheet. Contours correspond to lines of constant time in the local $(y, \bar{y})$ coordinates near each insertion point. Colored lines correspond to the loci where the local coordinates have to be glued together. The zeroes a_i of the Strebel differential lie on these so-called 'critical' trajectories. Figure taken from [48].

In the usual limit $\varepsilon \to 0$, the worldsheet metric is then conformally equivalent to the flat $\mathrm{d}z\,\mathrm{d}\bar{z}$ metric. However, the Strebel differential

$$\varphi = \frac{2}{N^2} S[\Gamma] \sim -\frac{1}{N^2}\left(\frac{\partial^2\Gamma}{\partial\Gamma}\right)^2 \tag{7.66}$$

is well-defined as $N \to \infty$, and thus we can write the worldsheet metric in this limit as

$$\mathrm{d}s^2_{\text{worldsheet}} \sim \left(\frac{N^2}{4}|\varphi| + \frac{1}{\varepsilon^2}\right)\mathrm{d}z\,\mathrm{d}\bar{z}\,. \tag{7.67}$$

Therefore, if we take the double scaling limit $\varepsilon \to 0$ and $N\varepsilon \to \infty$, we see that the metric is dominated by the radial profile of the string motion, and we are left with the Strebel metric

$$\mathrm{d}s^2_{\text{Strebel}} \sim \frac{N^2}{4}|\varphi|\mathrm{d}z\,\mathrm{d}\bar{z}\,. \tag{7.68}$$

In this sense, the Strebel metric is the pullback of the AdS_3 metric precisely in the large-twist limit. Thus, we see that the expression (7.46) for correlators in the symmetric orbifold can be recovered geometrically as the semiclassical motion of a string moving in AdS_3 in the large-twist limit, which was predicted in [48]. In this limit, the Strebel metric comes entirely from the radial part of the AdS_3 metric, and so we see that the worldsheet moves almost purely in the radial direction of AdS_3.

Interestingly, there is a natural geometric interpretation of the Strebel gauge metric, which has wide application in string theory and string field theory [132, 133]. Given a Strebel differential φ, we can locally define 'natural' worldsheet coordinates

$\mathrm{d}y = \frac{N}{2}\sqrt{\varphi}$ and $\mathrm{d}\bar{y} = \frac{N}{2}\sqrt{\bar{\varphi}}$, which are defined on $\Sigma \setminus \{z_1, \ldots, z_n\}$.[13] In these coordinates, the Strebel metric is given simply by

$$\mathrm{d}s^2_{\mathrm{Strebel}} = \mathrm{d}y\,\mathrm{d}\bar{y}\,, \tag{7.69}$$

i.e. the natural coordinates $(y, \bar{y})$ locally describe a flat worldsheet metric. Near a pole of the Strebel differential located at the insertion point z_i in the original coordinates, we have

$$\mathrm{d}y = \frac{N}{2}\sqrt{\varphi} \sim \frac{w_i/2}{(z - z_i)}\mathrm{d}z\,, \tag{7.70}$$

so that

$$y \sim \frac{w_i}{2}\log(z - z_i)\,. \tag{7.71}$$

Near $z = z_i$, the worldsheet looks like a semi-infinte tube of circumference $w_i/2$, and the $(y, \bar{y})$ coordinates are the natural cylindrical coordinates on this semi-infinite tube. The $(y, \bar{y})$ coordinates are only locally defined near the poles z_i of φ, and away from the poles the different coordinate systems have to be 'glued' together along the so-called 'critical' trajectories of φ (see Figure 7.4).

As was shown in [132], the Strebel-gauge metric can be seen to solve a minimal area problem. In particular, the Strebel metric $\mathrm{d}s^2 = N^2|\varphi|/4$ is the minimal-area metric among those satisfying the following three criteria:

- The metric is induced by some quadratic differential φ.
- Given a puncture z_i and a path γ_i surrounding z_i, the length of γ_i is fixed and given by $2\pi N\alpha_i/4 = \pi(w_i - 1)/2$.
- The metric has at most quadratic poles at $z = z_i$.

The second condition translates in the language of AdS_3 that the worldsheet winds w_i times around its insertion point. The third allows one to define a regularized area of $\Sigma \setminus \{z_1, \ldots, z_n\}$ (so that the minimal area problem is well-defined). Thus, the Strebel gauge metric can be thought of as the minimal area of a string propagating in AdS_3 which winds around its insertion points w_i times. In this way, we can really think of the Strebel-gauge metric as the semiclassical worldsheet minimizing the Nambu-Goto action, thus justifying the stringy interpretation of (7.46).

In the $(y, \bar{y})$ coordinate system, we can calculate the semiclassical motion of the string in the $(r, \gamma, \bar{\gamma})$ AdS_3 coordinates rather easily. In terms of the covering map, we have

$$\mathrm{d}y \sim i\frac{1}{2}\partial\log\partial\Gamma \tag{7.72}$$

and so $\partial\Gamma \sim e^{-2iy}$. Since $r^2 = \varepsilon^2|\partial\Gamma|^2$, we have

$$r^2(y, \bar{y}) = \varepsilon^2 e^{4\mathrm{Im}(y)}\,. \tag{7.73}$$

Thus, as $\mathrm{Im}(y) \to -\infty$, we see that $r^2 \to 0$ and the worldsheet approaches the boundary of AdS_3. As we increase y, r increases and the worldsheet approaches the center of AdS_3. Thus, near $z = z_i$, we see again that the worldsheet resembles a semi-infinite tube which extends from the boundary of AdS_3 deep into the bulk (see Figure 7.5). Each of these tubes corresponds to a face in the graph of Figure 7.4,

[13] The worldsheet coordinate y is not to be confused with the spectral curve of Section 7.2.

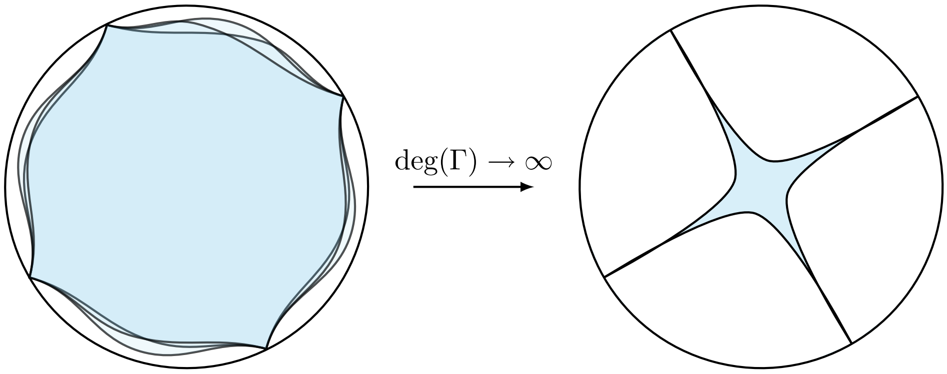

Figure 7.5: Left: The worldsheet 'glued' to the AdS_3 boundary for finite $N = \deg(\Gamma)$. Right: The semiclassical worldsheet corresponding to the Strebel-gauge metric $g_{z\bar{z}} = N^2|\varphi|/4$ in the limit $N \to \infty$. The worldsheet near $z = z_i$ is approximated by a semi-infinite tube situated radially in AdS_3.

and they are glued together in the AdS_3 bulk along the graph of critical trajectories of φ.

It is worth noting that the above coordinate system has a natural place in the context of string field theory, where it is used as a preferred coordinate for the insertion of off-shell states on the worldsheet [133]. It would be interesting, then, to explore whether the large-twist limit of the tensionless worldsheet theory could be used to explore closed-string correlators of off-shell states. This could potentially provide a playground for understanding closed-string field theory for the tensionless string.

7.4 Relation to the worldsheet theory

The free field realization

In the above discussion, we have provided an argument that the large-twist limit of symmetric orbifold theories takes a schematic, stringy, form

$$\left\langle \prod_{i=1}^{n} \mathcal{O}^{(w_i)}(x_i) \right\rangle_{\text{conn}} \sim \sum_{g=0}^{\infty} \int_{\mathcal{M}_{g,n}} \mathrm{d}\mu \, e^{-S_{\text{NG}}} \tag{7.74}$$

and furthermore we have argued that the resulting Nambu-Goto action in this limit can be obtained by the motion of a string moving on semiclassical trajectories in AdS_3.

However, we know what the string theory dual of the symmetric orbifold is, namely the tensionless limit of $AdS_3 \times S^3 \times \mathbb{T}^4$ strings. As we have discussed at length in the previous chapters, this theory is a free worldsheet theory whose bosonic sector is roughly described by fields (μ, λ) and their conjugates $(\lambda^\dagger, \mu^\dagger)$, for which we gauge the simultaneous scaling symmetry

$$\begin{aligned} (\lambda, \mu) &\to \alpha(\lambda, \mu)\,, \\ (\mu^\dagger, \lambda^\dagger) &\to \alpha^{-1}(\mu^\dagger, \lambda^\dagger)\,, \end{aligned} \tag{7.75}$$

A natural question is if it is possible to see this worldsheet structure from the symmetric orbifold analysis.

One key ingredient in answering this question will be the expressions for λ, μ evaluated on correlators. As we showed in Chapter 5, the meromorphic $-\frac{1}{2}$ forms on the worldsheet

$$\mathscr{L}(z) = \left\langle \lambda(z) \prod_{i=1}^{n} V_{h_i,j_i}^{w_i}(x_i, z_i) \right\rangle , \quad \mathscr{M}(z) = \left\langle \mu(z) \prod_{i=1}^{n} V_{h_i,j_i}^{w_i}(x_i, z_i) \right\rangle , \tag{7.76}$$

satisfy the *incidence relation*

$$\mathscr{M}(z) - \Gamma(z)\mathscr{L}(z) = 0 \, . \tag{7.77}$$

This means that if we interpret $[\mathscr{M} : \mathscr{L}]$ as homogeneous coordinates on $\mathbb{CP}^1$, then the map $z \mapsto [\mathscr{M}(z) : \mathscr{L}(z)]$ is exactly the covering map Γ. This, in turn, allows us to identify the worldsheet in AdS_3 with the covering space appearing in the expansion of symmetric orbifold correlators.

In addition to the incidence relation, we can actually constrain the form of the fields $\mathscr{M}, \mathscr{L}$ further. Since we know that they both have poles of order $\frac{w_i-1}{2}$ at $z = z_i$, we can define

$$\mathscr{L} = \frac{f_1(z)}{\sqrt{\partial\Gamma(z)}} \, , \quad \mathscr{M} = \frac{f_2(z)}{\sqrt{\partial\Gamma(z)}} \, . \tag{7.78}$$

The functions f_1, f_2 can have at worst simple poles at the poles of $\Gamma(z)$ (since these will be cancelled by the single-poles of the denominator $\sqrt{\partial\Gamma}$), and must satisfy

$$f_2(z) - x_i f_1(z) \sim \mathcal{O}((z - z_i)^{w_i}) \, . \tag{7.79}$$

These properties uniquely specify $f_1 = 1$ and $f_2 = \Gamma$, up to an overall scaling factor, and so we have

$$\mathscr{L} = \frac{1}{\sqrt{\partial\Gamma}} \, , \quad \mathscr{M} = \frac{\Gamma}{\sqrt{\partial\Gamma}} \, . \tag{7.80}$$

See also [134] for similar expressions.

Twistors from the Strebel differential

As it turns out, the worldsheet free fields are deeply connected to the Strebel differential. Given a covering map Γ, we can consider the holomorphic differential equation

$$\partial^2 f + \frac{1}{2} S[\Gamma] f = 0 \, . \tag{7.81}$$

This equation has a two-dimensional family of solutions, which, under coordinate transformations, transform as conformal primaries of weight $h = -\frac{1}{2}$ (i.e. define sections of the inverse spinor bundle S^{-1} for some given spin structure S). Indeed, one can immediately write down a pair of solutions for f, which are precisely the classical values of the twistor fields, namely

$$\lambda = \frac{1}{\sqrt{\partial\Gamma}} \, , \quad \mu = \frac{\Gamma}{\sqrt{\partial\Gamma}} \, . \tag{7.82}$$

The fact that the twistor fields (7.82) satisfy the differential equation (7.81) can be seen either from directly verifying that (7.82) is a solution, or by factorising (7.81) as

$$\left(\partial^2 + \frac{1}{2}S[\Gamma]\right) f = \left(\partial - \partial \log \sqrt{\partial\Gamma}\right)\left(\partial + \partial \log \sqrt{\partial\Gamma}\right) f \,, \tag{7.83}$$

so that λ is the solution obtained by demanding f is annihilated by $\partial + \partial \log \sqrt{\partial\Gamma}$, and μ is obtained similarly.

Recalling that, in the large-twist limit, the Strebel differential is related to the Schwarzian via

$$\varphi = -\frac{2}{N^2}S[\Gamma]\,, \tag{7.84}$$

we see that the fields λ and μ can be constructed through the solutions of the differential equation

$$\partial^2 f = \frac{N^2}{4}\varphi\, f\,. \tag{7.85}$$

Thus, there is a natural relationship between the worldsheet free fields and the Strebel differential, namely

$$\frac{\partial^2\lambda}{\lambda} = \frac{\partial^2\mu}{\mu} = \frac{N^2}{4}\varphi\,. \tag{7.86}$$

If we know the explicit form of the covering map Γ, this differential equation is easy to solve. However, in the large-twist limit, Γ itself is not a well-defined function, and the best option we have is to implicitly define it through the Strebel differential. We can similarly recover the worldsheet fields μ, λ in this limit purely through the Strebel differential. To see this, note that the holomorphic Schrödinger equation (7.85) can be solved via the WKB approximation in the limit $N \gg 1$. Indeed, in a neighborhood of a regular point z_0 (i.e. z_0 is not a critical point of Γ), one can immediately write down a pair of approximate solutions given by

$$f_\pm = \varphi^{-1/4}\exp\left(\pm\frac{N}{2}\int_{z_0}^{z}\sqrt{\varphi}\right)\,. \tag{7.87}$$

This solution, however, is not valid as we take z very far away from z_0. In particular, if z_0 is within one of the regions bounded by the critical trajectories of φ (see Figure 7.4), then the above approximation will only be valid within that region. Once z crosses one of the critical trajectories, $\sqrt{\varphi}$ crosses a zero or a branch cut, and the WKB approximation breaks down (analogous to when one encounters a 'turning point' in one-dimensional quantum mechanics). That is, the solution to the WKB approximation is only valid locally near $z = z_i$. A full solution would require gluing the separate solutions together.

Within each region surrounding the insertion, we can interpret the two solutions above as the combinations $\lambda(z)$ and $\mu(z) - x_i\lambda(z)$. Indeed, near $z = z_i$ we have

$$f_\pm \sim (z - z_i)^{1/2}\exp\left(\pm\frac{w_i}{2}\int_{z_0}^{z}\frac{1}{z - z_i}\mathrm{d}z\right) \sim (z - z_i)^{\frac{1}{2}\pm\frac{w_i}{2}}\,, \tag{7.88}$$

which is precisely the asymptotic behavior of $\mu(z) - x_i\lambda(z)$ and $\lambda(z)$, respectively.

We note that we can write the above solutions in terms of the natural coordinate system $(y, \bar{y})$ induced by φ. Given a base-point z_0, we have

$$y(z) = \frac{N}{2}\int_{z_0}^{z}\sqrt{\varphi}\,, \qquad \bar{y}(\bar{z}) = \frac{N}{2}\int_{\bar{z}_0}^{\bar{z}}\sqrt{\bar{\varphi}}\,. \tag{7.89}$$

In this coordinate system, the metric is simply $\mathrm{d}s^2 = \mathrm{d}y\,\mathrm{d}\bar{y}$, and is flat unless φ is singular. The solutions $f_\pm$ then take the very simple form

$$f_\pm(y) \sim \frac{e^{\pm y}}{\sqrt{\mathrm{d}y}}\,.$$

Since we think of $f_\pm$ as giving local coordinates on the target space $\mathbb{CP}^1$, we can take their ratio to get a local coordinate on $\mathbb{C}\cup\{\infty\}$, namely

$$\frac{f_+}{f_-} = e^{2y} \sim (z-z_i)^{w_i}\,. \tag{7.90}$$

Of course, this ratio is nothing other that $\Gamma(z) - x_i$, since $f_+ = \mu - x_i\lambda$ and $f_- = \lambda$. The WKB approximation then simply tells us that the coordinates y and the coordinates $\Gamma - x_i$ are related to each other by exponentiation.

Finally, let us comment that there is a suggestive relationship between the twistor fields and the Liouville field Φ. Recall that the relationship between the Strebel differential φ and Φ is given by $\partial\Phi = iN\sqrt{\varphi}$, so that

$$\Phi = iN\int_{z_0}^{z}\sqrt{\varphi}\,. \tag{7.91}$$

This allows us to instantly write down the solutions $f_\pm$ in terms of the Liouville field, namely

$$f_\pm \sim (\partial\Phi)^{-1/2}e^{\pm\Phi/2}\,. \tag{7.92}$$

Since, within the coordinate system near $z = z_i$, the solutions $f_\pm$ are related to the free fields λ, μ, we can think of Φ as the bosonisation of these free fields.[14]

Let us set $x_i = 0$ for convenience, so that we can identify the two solutions $f_\pm$ with μ and λ. Then we have explicitly

$$\lambda \sim \frac{e^{-\Phi/2}}{\sqrt{\partial\Phi}}\,,\quad \mu \sim \frac{e^{\Phi/2}}{\sqrt{\partial\Phi}}\,. \tag{7.93}$$

Now, we can propose a similar relationship between the Liouville field Φ and the daggered fields $\mu^\dagger, \lambda^\dagger$. In particular, it was noted in [78, 134] that Φ can be thought of as the bosonisation of J^3 in the $\mathfrak{sl}(2,\mathbb{R})$ current algebra, i.e. $J^3 = \partial\Phi$, and so naively we can propose

$$\lambda^\dagger\mu - \mu^\dagger\lambda = 2J^3 = 2\partial\Phi\,. \tag{7.94}$$

Furthermore, since we are imposing the gauge symmetry (7.75), we further propose that the current corresponding to this symmetry vanishes [134]. Specifically,

$$\lambda^\dagger\mu + \mu^\dagger\lambda = 0\,. \tag{7.95}$$

Thus, we can immediately write down the semiclassical values of $\mu^\dagger, \lambda^\dagger$ to be

$$\mu^\dagger = -(\partial\Phi)^{3/2}e^{\Phi/2}\,,\quad \lambda^\dagger = (\partial\Phi)^{3/2}e^{-\Phi/2}\,. \tag{7.96}$$

[14]Similar expressions relating the AdS_3 free field to the Liouville field Φ are found in, for example, equation (2.9) of [134]. The difference between their expression and ours is a choice of gauge in the free field realisation.

Furthermore, we can calculate the semiclassical stress-tensor of these expressions, and we find

$$T = \frac{1}{2}\left(\mu^\dagger\partial\lambda - \lambda\partial\mu^\dagger + \lambda^\dagger\partial\mu - \mu^\dagger\partial\lambda\right) + \cdots \sim (\partial\Phi)^2 \,, \tag{7.97}$$

where $\cdots$ represents terms which vanish under the gauge constraint. Since $(\partial\Phi)^2$ is simply the Strebel differential (up to a constant), we reproduce the semiclassical result of [134] that the stress tensor of the free field theory is related to the Strebel differential.[15]

Reconstructing the worldsheet

Above, we argued for a suggestive relationship between the Strebel differential appearing in the large-twist limit of the symmetric orbifold CFT and the twistorial free field description of the worldsheet. However, let us for a moment assume we know nothing of the worldsheet theory. Given nothing more than the knowledge that correlators are naturally expressed in terms of Strebel differentials, how much about the worldsheet theory can we extract?

Let φ be a generic Strebel differential on the surface Σ with quadratic residues α_i^2 at $z = z_i$. Armed with the hindsight of the above discussions, we might simply postulate that the worldsheet theory can be described in terms of the solutions to the holomorphic Schrödinger equation

$$\partial^2 f = \varphi\, f \,. \tag{7.98}$$

What can we say about the solutions f? Near a pole $z = z_i$ we have

$$\partial^2 f \sim \frac{\alpha_i^2 f}{(z - z_i)^2} \,. \tag{7.99}$$

The independent solutions to this differential equation are

$$f_\pm(z) \sim (z - z_i)^{\frac{1\pm r_i}{2}} \,, \quad r = \sqrt{1+\alpha_i^2} \,. \tag{7.100}$$

Thus, since $\alpha_i^2 \in \mathbb{R}_+$ by construction, one of the solutions has a zero of degree $\frac{1+r_i}{2}$ at $z = z_i$ and the other has a pole of degree $\frac{r_i-1}{2}$. If $f_\pm$ are truly to describe worldsheet degrees of freedom, then the asymptotic behaviors (7.100) should describe their OPEs with vertex operators $V_{\alpha_i}(z_i)$ on the worldsheet. In particular, promoting $f_\pm$ to worldsheet fields, we should have

$$f_\pm(z)\, V_{\alpha_i}(z_i) \sim (z - z_i)^{\frac{1\pm r_i}{2}} \,. \tag{7.101}$$

If we assume the existence of a current $\partial\phi$ on the worldsheet under which $f_\pm$ have charges $\pm\frac{1}{2}$, then we can explicitly construct such a vertex operator as

$$V_{\alpha_i}(z) \sim e^{r_i\phi/2} \,, \tag{7.102}$$

The remaining piece of the OPE then tells us that $f_\pm(z)\,|\Omega\rangle \sim z^{\frac{1}{2}}$ as $z \to 0$, where $|\Omega\rangle$ is the CFT vacuum. This is the hallmark of a conformal field of weight $h = -\frac{1}{2}$.

[15] As was pointed out in [134], this is not so surprising, and in general stress-tensors of Liouville theories and Strebel differentials converge in the 'large-twist' limit, c.f. the discussion in Chapter 5 of [95].

The simplest way to build a worldsheet CFT is then to introduce canonical conjugate fields for the two $h = -\frac{1}{2}$ fields, which are of weight $h = \frac{3}{2}$.

Thus, from the (as yet unjustified, but motivated by the example of the tensionless string) postulate that the worldsheet degrees of freedom are given by solutions to the holomorphic Schrödinger equation with φ as the potential, we can deduce the OPEs (7.101) and the fact that the worldsheet fields have conformal weight $h = -\frac{1}{2}$, assuming that the vertex operators can be expressed as exponentials of a scalar ϕ, under whose current $\partial\phi$ the worldsheet fields $f_{\pm}$ come with charge $\pm\frac{1}{2}$.

Of course, these properties are nearly those of the worldsheet dual of the symmetric orbifold CFT.[16] However, it has been argued [113–115] that the worldsheet dual to generic free (holographic) gauge theories should also be expressible naturally in terms of Strebel differentials. Thus, it would not be unreasonable to assume that the above discussion generalizes to the worldsheet dual of *any* holographic free CFT, not just the symmetric orbifold. The conclusion would be that the worldsheet dual of a generic free CFT is, at least partially, describable in terms of worldsheet fields which formally resemble those of the free field realisation of $\mathfrak{psu}(1,1|2)_1$. This is indeed true for the case of the worldsheet dual of free $\mathcal{N} = 4$ super Yang-Mills [89, 90], and has also recently shown to be true for a large class of free $\mathcal{N} = 2$ quiver gauge theories dual to tensionless string theory on $\mathrm{AdS}_5 \times (\mathrm{S}^5)/\mathbb{Z}_N$ [135]. It would thus be interesting to explore further the connection between twistor-like worldsheet theories and the stringy duals to free holographic CFTs.

7.5 Comments on AdS$_2$

In this section, we explore the relationship between the tensionless string on AdS$_3$ in the large-twist limit and gravity on Euclidean AdS$_2$ via a particular dimensional reduction. In particular, we show that the dynamics of a string which ends on a certain type of D-brane in AdS$_3$ is seemingly governed by a one-dimensional Schwarzian action in the large-twist limit, suggesting a potential connection to Jackiw-Teitelboim gravity [136] in two dimensions.

Branes in AdS$_3$

In Chapter 6, the duality between tensionless strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ and the symmetric orbifold theory was extended to include gravitational backgrounds with D-branes. In particular, branes in the AdS$_3$ bulk which are localized at a specific point in time (so-called *spherical branes* in the parlance of [97]), depicted in Figure 7.6, were shown to be dual to boundary states in the symmetric orbifold theory which are 'maximally symmetric', in the sense that they share the same boundary conditions in all copies of the $\mathbb{T}^4$ seed theory.

Let us denote such a boundary state in the symmetric orbifold as $\|\psi\rangle\rangle$. We can consider correlation functions in the presence of such a boundary state

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i^{(w_i)}(x_i) \right\rangle_{\psi,\mathbb{D}} := \left\langle \prod_{i=1}^{n} \mathcal{O}_i^{(w_i)}(x_i) \middle\| \psi \right\rangle\!\!\Bigg\rangle_{\mathbb{D}} . \tag{7.103}$$

Specifically, we model the base space of the CFT as the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and impose the boundary condition corresponding to $\|\psi\rangle\rangle$ on the boundary $|z| = 1$.

[16] It is not obvious, however, how to see the gauging of the scaling symmetry in equation (7.75).

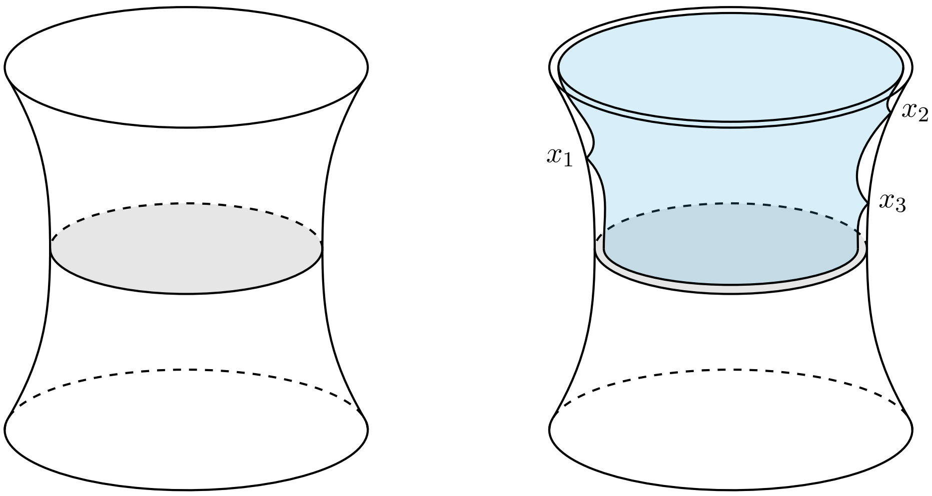

Figure 7.6: Left: A 'spherical' brane localized at time $t = t_0$ in AdS_3. The induced metric on the brane is that of Euclidean AdS_2. Right: The worldsheet configuration associated to the CFT three-point function $\langle \mathcal{O}_1^{(w_1)}(x_1)\mathcal{O}_2^{(w_2)}(x_2)\mathcal{O}_2^{(w_2)}(x_2)\|\psi\rangle\rangle$.

It was argued in [50] that these correlation functions enjoy a similar construction in terms of covering maps $\Gamma : \Sigma \to \mathbb{D}$ which cover the disk $\mathbb{D}$ with some Riemann surface Σ with boundary. Schematically, we can associate to each of these covering surfaces an Euler characteristic $\chi(\Sigma) = 2 - 2g - n - b$, where b is the number of boundary components of Σ, and the correlation function (7.103) can be expressed as

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i^{(w_i)}(x_i) \right\rangle_{\psi,\mathbb{D}} = \sum_{\Gamma:\Sigma\to\mathbb{D}} K^{\chi(\Sigma)} e^{-S_L[\Phi_\Gamma]} \left\langle \prod_{i=1}^{n} \mathcal{O}_i(z_i) \right\rangle_{\psi,\Sigma} , \tag{7.104}$$

where in the right-hand-side z_i are the branch points of Γ with $\Gamma(z_i) = x_i$, $\mathcal{O}_i$ are the seed theory operators defined on the covering surface Σ, and $S_L[\Phi_\Gamma]$ is the same Liouville action we introduced previously, except now defined on a Riemann surface with boundary

$$S_L[\Phi_\Gamma] = \frac{1}{8\pi} \int_\Sigma (2\partial\Phi_\Gamma\bar{\partial}\Phi_\Gamma + R\Phi) , \tag{7.105}$$

and Φ_Γ, as usual, is given by $\Phi_\Gamma = \log|\partial\Gamma|^2$.

Just as in the case of the symmetric orbifold on the sphere, there is a natural construction to associate a Strebel differential φ to each covering space $\Gamma : \Sigma \to \mathbb{D}$. The simplest such method is to consider the compact double cover Σ_c of Σ, for which there exists an orientation preserving diffeomorphism $\iota : \Sigma_c \to \Sigma_c$ such that $\Sigma \cong \Sigma_c/\iota$, where the boundary of Σ is identified with the fixed-point set of ι. We can then consider the covering map $\Gamma_c : \Sigma_c \to \mathbb{CP}^1$ which maps z_i to x_i and $\iota(z_i)$ to $-1/\bar{x}_i$ with branching w_i. Taking the degree of Γ_c to be large, the Schwarzian derivative $S[\Gamma]$ converges to a Strebel differential φ_c defined on Σ_c. Letting $i : \Sigma_c \to \Sigma$ be the projection map associated to ι, we can then define a Strebel differential on Σ by $i^*\varphi_c$.

By the arguments in Section 7.2, we can then relate the symmetric orbifold correlator to an integral over the moduli space $\mathcal{M}_{n,g,b}$ of surfaces of genus g with n

punctures and b boundary components[17] as

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i^{(w_i)}(x_i) \right\rangle_{\psi,\mathbb{D}} \sim \sum_{g,b} K^{2-2g-n-b} \int_{\mathcal{M}_{n,g,b}} e^{-\frac{N^2}{4\pi}\int_\Sigma |\varphi|} \left\langle \prod_{i=1}^{n} \mathcal{O}_i(z_i) \right\rangle_{\Sigma} . \tag{7.106}$$

This has the holographic interpretation of an open string propagating in the background of Figure 7.6, where $N^2|\varphi|/4$ is the pullback of the effective AdS$_3$ metric.

Dimensional reduction and the Schwarzian

Let us now consider a particular dimensional reduction from AdS$_3$ to Euclidean AdS$_2$ in the above setup. We do this by treating the endpoints of the string as individual objects which move only in the plane $t = t_0$ and writing down their dynamics. The effect of the 'dimensional reduction' on the covering map is to consider only the boundary covering map $\Gamma : \partial\Sigma \to \mathrm{S}^1$ which maps the boundary of the worldsheet to the boundary circle of the spherical brane, and as such parametrizes the motion of the string endpoint. For now let us consider $b = 1$ and $g = 0$ (i.e. take the worldsheet topology to be a disk) since this is the leading contribution in $1/N$. Since the boundary circle can be thought of as the complex numbers with unit modulus, we have $\Gamma(\tau) = e^{i\theta(\tau)}$, where τ is the boundary angle on the covering space and θ is the angle on the Euclidean AdS$_2$ boundary circle.

In this limit, we can explicitly calculate the Strebel differential on the boundary circle. We have

$$\partial\Phi_\Gamma\,\bar{\partial}\Phi_\Gamma \to \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\log\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 = \left(\frac{\theta''}{\theta'}\right)^2 . \tag{7.107}$$

Taking the limit in which the degree of θ is large (i.e. for which $\theta : \mathrm{S}^1 \to \mathrm{S}^1$ wraps the circle many times), we can approximate the above kinetic term via

$$\left(\frac{\theta''}{\theta'}\right)^2 \sim -2\{\theta,\tau\} , \tag{7.108}$$

where $\{\theta,\tau\} := S[\theta](\tau)$ is the Schwarzian derivative of θ with respect to τ. Therefore, the Nambu-Goto action of (7.106) can be written schematically as

$$\int_{\partial\Sigma} \mathrm{d}\tau\,\{\theta,\tau\} , \tag{7.109}$$

i.e. the dynamics are governed by a one-dimensional Schwarzian theory. Holographically, since we consider the covering space Σ to be the worldsheet, the above action describes the motion of the endpoints of the string along the brane. The motion of the endpoints defines a function $\theta : \partial\Sigma \to \mathrm{S}^1$, and the above action describes the effective dynamics of the string endpoints in the large-twist limit.[18]

Amazingly, the above action is precisely of the form of the action which defines the boundary dynamics of Jackiw-Teitelboim gravity in two dimensions [139], thus hinting that, in this compactification limit, gravitational dynamics in the effective two-dimensional Euclidean theory has a subsector which is governed by a theory

[17] This is a moduli space of real dimension $\dim_{\mathbb{R}}(\mathcal{M}_{g,n,b}) = 2n + 6g - 6 + 3b$.

[18] The Schwarzian action as the semiclassical Nambu-Goto action of an open string in AdS$_3$ was also recovered in [137, 138].

which formally resembles JT gravity. If we restore constants and take the central charge c of the dual CFT to be generic, the effective action of the string endpoint is

$$S_{\text{eff}} = -\frac{c}{12\pi}\int \mathrm{d}\tau\, \{\theta,\tau\}\,. \tag{7.110}$$

In JT gravity, the constant prefactor out front is given by $a/16\pi G_N^{(2)}$, where $G_N^{(2)}$ is the two-dimensional Newton constant and a is a dimensionless parameter controlling the asymptotics of the diliton (see, for instance, Section 2.3.1 of [140]). Relating this to the prefactor $c/12\pi$ above gives

$$G_N^{(2)} = \frac{3a}{4c}\,. \tag{7.111}$$

Now, the 2D Newton constant in string theory is schematically related to the 3D Newton constant via $G_N^{(2)} = G_N^{(3)}\ell_s/2$, where ℓ_s is the string length. In the tensionless limit of string theory on AdS_3, the string length is equal to the AdS_3 radius R, and so

$$G_N^{(3)} = a\frac{3R}{2c}\,, \tag{7.112}$$

which, up to the overall dimensionless constant a, is the famous Brown-Henneaux formula for the AdS_3 dual of a CFT with central charge c [30]. Thus, not only does the effective action (7.110) schematically reproduce the Schwarzian action for JT gravity, the constant prefactor can also be produced, assuming that the Newton constant $G_N^{(3)}$ is related to the central charge of the boundary theory by the Brown-Henneaux formula.[19]

There is a clear difference between (7.109) and the usual Schwarzian theory. In the context of JT gravity, the Schwarzian field $\theta : \mathrm{S}^1 \to \mathrm{S}^1$ is a one-to-one diffeomorphism from the boundary circle onto itself, whereas in our case θ defines an N-to-one map. Such Schwarzian theories are considered in the literature, and are known to describe the dynamics of JT gravity with an $\mathbb{Z}_N$ conical defect inserted in the bulk [141]. Retrospectively, the appearance of a conical defect in the effective 2D theory is not surprising, since it was argued in [75] and in Section 7.3 that, in the large-twist limit, the string generates a conical defect around any contractible cycle wound by the worldsheet.

Actually, the existence of a relationship between a dimensional reduction of the tensionless string and the Schwarzian theory of JT gravity can be justified in another way. Let us consider JT gravity on the disk. The fundamental degrees of freedom of the theory are boundary gravitons, which correspond to diffeomorphisms of the boundary circle, up to a global Möbius transformation. Such a large diffeomorphism can be expressed in terms of a function $\theta : \mathrm{S}^1 \to \mathrm{S}^1$. In particular, one considers the partition function of JT gravity on the disk with a finite cut-off defined by a curve in the disk given by [142]

$$\theta = \theta(\tau)\,, \quad r = \varepsilon\frac{\mathrm{d}\theta}{\mathrm{d}\tau}(\tau)\,. \tag{7.113}$$

Here, r is the hyperbolic radial coordinate (i.e. $r \to \infty$ is the boundary of the disk). The parameter ε is some infrared cutoff, which can be made arbitrarily small, so that

[19] It is not clear however what the role of the genus-counting parameter S_0 of JT gravity is in this setup.

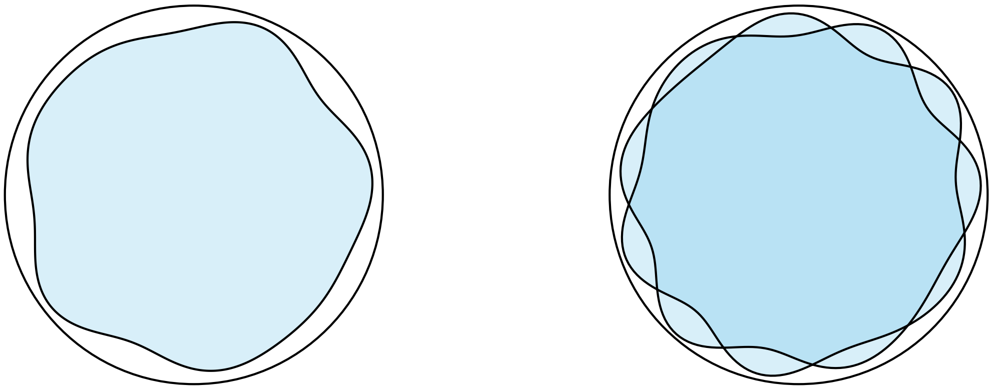

Figure 7.7: Left, a boundary cutoff in JT induced by the boundary diffeomorphism $f : \mathrm{S}^1 \to \mathrm{S}^1$. Right, the boundary cutoff defined by the endpoint motion of a covering map $\Gamma : \partial\Sigma \to \mathrm{S}^1$ with $\deg(\Gamma) = 2$.

the cutoff approaches the boundary. The Schwarzian theory is then considered by evaluating the JT action on the appropriate cutoff geometry, and then considering the path integral over $\mathrm{Diff}(\mathrm{S}^1)/\mathrm{SL}(2,\mathbb{R})$.

The relationship to the string worldsheet in AdS_3 is that (7.113) can be thought of as the projection of the classical worldsheet solution (7.57) into Euclidean AdS_2. That is, we can consider θ to be the value of the covering map Γ along the Euclidean AdS_2 brane in Figure 7.6. The primary difference, however, is that the worldsheet covering map is not a one-to-one map, but rather an N-to-one map (see Figure 7.7).

The realisation of JT gravity as coming from the dimensional reduction of a brane setup in AdS_3 was also recently pointed out in [143–145]. It is thus interesting to see that such a realisation seems to naturally arise in the context of the tensionless string theory on AdS_3. It would be worth exploring this relationship further to see if the Schwarzian form of the action (7.109) is truly an indication of some JT-like behaviour of strings in the presence of the spherical AdS_2 brane, or if it is simply a coincidence (perhaps due to the universality of the Schwarzian theory [146]). This would likely require a much better understanding of the effective bulk field theory describing the tensionless string, which is still poorly understood.

Chapter 8

Epilogue

8.1 Summary

Over the course of the last six chapters, we have introduced and explored tensionless string theory on $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$. Specifically, in Chapters 4-7, we provided strong evidence that the worldsheet theory is exactly dual to the symmetric orbifold $\text{Sym}(\mathbb{T}^4)$ living on the boundary of AdS_3. Let us briefly review the main results:

- In Chapter 4, we reviewed the analysis of [45] (later refined in [73] and [147]) that the spectrum of physical states on the worldsheet matches the spectrum of single-particle states in the symmetric orbifold theory. We also introduced the free field realization of $\mathfrak{psu}(1,1|2)_1$, which is a powerful tool in the calculation of spectra and correlation functions in the worldsheet model.

- In Chapter 5, we showed that the worldsheet correlation functions in the $\mathfrak{psu}(1,1|2)_1$ worldsheet theory have a localization property, first proposed in [77], which reproduces the sum over holomorphic covering maps found in the symmetric orbifold theory. This localization property holds not only at tree-level, but at all orders in string perturbation theory, and is again strong evidence of an exact duality between the bulk worldsheet theory and the boundary CFT.

- In Chapter 6, we explored the role of D-branes in the bulk AdS_3 spacetime. We showed that the spectrum of open strings stretched between two 'spherical' branes in the worldsheet theory precisely reproduces the single-particle spectrum of the symmetric orbifold on an annulus. We also showed that the disk amplitude of a string in a D-brane background reproduces the connected part of the disk correlation function of twist fields in the symmetric orbifold. This provided strong evidence that the D-branes considered are exactly dual to boundary states in the dual CFT.

- Finally, in Chapter 7, we explored the inverse question of starting with the symmetric orbifold theory and reconstructing its bulk dual. We found that correlation functions of twist fields admit a remarkable simplification in the limit that the twists of all fields are taken to be large, and that, in this limit, the correlators can be recast into the form of a semiclassical string moving in AdS_3. This exemplifies the construction of [113–115], and potentially sheds light onto how to derive the string duals of more general large N free field theories.

The content of these chapters do not, however, make any claim of completeness. Certain subjects simply could not be covered in lieu of time, space, and sanity. These include the higher-spin structure of the symmetric orbifold and worldsheet theory [56, 71, 148–152], the construction of DDF operators on the worldhseet [147, 153], the evidence of background independence of the tensionless worldsheet theory [75], and the case of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$, which also has a tensionless limit whose dual appears to be a symmetric orbifold theory [154, 155], as well as many other topics.

8.2 Outlook and unanswered questions

The above collected evidence for the duality between tensionless strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ and the symmetric orbifold of $\mathbb{T}^4$. However, although this particular duality is now on very solid ground, there are several questions that remain to be answered, but whose resolutions seem in reach given our current understanding. We list some of them here, although this list is in no sense complete.

What is the mechanism behind localization?

In order to calculate correlation functions in the tensionless string and compare them to symmetric orbifold correlation functions, we utilized a powerful localization principle, which showed that worldsheet correlation functions only had support at special, discrete points in the moduli space of curves. Such localization properties are ubiquitous in supersymmetric field theories and topological string theories. However, the mechanism behind localization in the tensionless string seems somewhat different than in supersymetric field theories and topological strings. In the tensionless string, the integrand itself only has support on discrete points of the moduli space, whereas in the context of supersymmetric localization, the integrand itself is generically nontrivial on the full moduli space, but the integral over the moduli space only receives contributions from a special, 'equivariant' sublocus.

It would be interesting to determine whether the localization of the tensionless worldsheet theory could also be understood in terms of some kind of equivariant localization principle. Since the localization occurs on a sublocus of $\mathcal{M}_{g,n}$, it may be instructive to consider the full string integral

$$\int_{\mathcal{M}_{g,n}} \mathrm{d}\mu \int \mathcal{D}\Phi \, e^{-S_{\text{string}}[\Phi,h]} \,, \tag{8.1}$$

where Φ collectively denotes the fields on the worldsheet and h is a representative of the equivalence class $[h] \in \mathcal{M}_{g,n}$. Perhaps there is an equivarient localization principle which localizes the *full* integral, over both fields and metrics, to the sublocus of $\mathcal{M}_{g,n}$ for which covering maps exist. Such a localization principle would shed light on precisely *why* the tensionless string localizes, and what makes it so special among the AdS_3 string theories.

What happens in higher dimensions?

While the topics reviewed in this thesis shed light on the nature of the stringy $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, the looming question remains about the nature of this correspondence in higher dimensions. Of particular interest is the case of Ads_5/CFT_4. In the case of IIB strings on $AdS_5 \times S^5$, the tensionless limit should be holographically dual to *free* $\mathcal{N} = 4$ super Yang-Mills [27]. One might hope that a similar worldsheet theory could be used to explore this limit of AdS_5/CFT_4. However, qualitatively, there are two major differences between three and five dimensions. First, AdS_5 is not a group manifold, and so one cannot treat the worldsheet theory directly as a WZW model. Secondly, the background $Ads_5 \times S^5$ is supported by Ramond-Ramond flux, whereas the $AdS_3 \times S^3 \times \mathbb{T}^4$ background is supported purely through NS-NS flux. This means that the worldsheet theories should be qualitatively different in nature.

Nevertheless, a worldsheet theory has been proposed which produces the single-trace spectrum of free $\mathcal{N} = 4$ super Yang-Mills [89, 90] (and generalized to $\mathcal{N} = 2$ quiver theories in [135]). However, to date this worldsheet theory is not completely understood, and it is still not known how to compute, say, its correlation functions (although it seems to admit a similar large-twist limit to the symmetric orbifold, see [134]).

What about non-AdS geometries?

Recently it has become clear that there are deformations of 2D CFTs which still have holographic duals. Specifically, the so-called 'single trace' $T\overline{T}$-deformation of symmetric orbifold theories [156] is meant to be dual to a spacetime which interpolates between AdS_3 in the bulk and a linear dilaton geometry near the boundary. Since the deformation of the boundary theory is irrelevant, the boundary theory is no longer a CFT, but the effect of the deformation on the field theory spectrum is remarkably solvable [157, 158] and retains some notion of modular invariance [159]. Due to the solvability of the deformation, one might expect that the bulk worldsheet description also remains solvable. From the perspective of the worldsheet theory, the single-trace $T\overline{T}$-deformation has been proposed to be dual to a current-current deformation [160–162]. Thus, it would be worthwhile to study this type of deformation in the context of the tensionless string theory on AdS_3. Such an analysis is currently underway, and will appear in a future publication [163].

What is the role of the bulk?

As far as we understand, the tensionless string describes worldsheets propagating in AdS_3 which are completely 'glued' to the boundary. Such strings do not interact with the bulk of AdS_3 in any meaningful way, and it is not even clear whether such a 'bulk' can even be defined, or whether we are simply working too far in the quantum regime for the concept of a classical bulk geometry to make sense. In [75], it was shown that tensionless string theory formulated on two bulk manifolds $\mathcal{M}, \mathcal{M}'$ which have the same boundaries have the same worldsheet partition functions (up to some subtleties regarding boundary spin structure). This is interpreted as a sort of background independence in string theory, where different bulk manifolds are thought of as highly excited string states with respect to another bulk manifold. Typically, such changes in bulk geometry are thought of as non-perturbative effects, and thus are typically not visible in string perturbation theory. However, the tensionless string is a perturbatively exact theory, and so the different bulk geometries are truly perturbative phenomena in the tensionless string. It would be interesting to explore further in what sense the tensionless string can be thought of as having a bulk geometry. Since the tensionless string describes three-dimensional quantum gravity far away from the semiclassical approximation, an understanding of the role of the bulk in this theory could shed light on our understanding of the role of spacetime geometry below the Planck scale.

What can we learn from deformations?

What we do know is that, although the bulk may be a nebulous concept in the tensionless regime, once one deforms the theory toward the supergravity regime, one should expect to recover a smooth bulk geometry. Thus, somewhere in between the tensionless and supergravity regime, there should be a transition from 'stringy' to 'classical' geometry.

One class of such deformations involves, on the symmetric orbifold side, the addition of a twist-2 operator:

$$S \to S + \lambda \int \mathcal{O}^{(2)} \,. \tag{8.2}$$

As $\mathcal{O}^{(2)}$ is a twist field, it effectively introduces a nontrivial *local* interaction (as opposed to the global, topological interaction of orbifolding) between the individual copies of the seed theory. From the point of view of the worldsheet, it introduces a contact interaction between individual disconnected worldsheets.

The worldsheet dual of this deformation is the inclusion of a worldsheet operator in the $w = 2$ spectrally flowed sector [85], and semiclassically it corresponds to introducing a small amount of Ramond-Ramond flux to the $\text{AdS}_3 \times \text{S}^3$ background.[1] This deformation increases the AdS_3 radius by an amount proportional to λ^2, and thus perturbs away from the tensionless limit. It would be worth studying this deformation in more detail. Specifically, it would be interesting to understand exactly how deforming away from the symmetric orbifold point flows between the rigid, localized worldsheet theory of the tensionless string to the much more familiar approximation of supergravity.

[1]A similar deformation also appears in the definition of the expected CFT dual of pure NS-NS string theory on AdS_3 for $k > 1$, see [164].

Bibliography

[1] S. W. Hawking, "Particle Creation by Black Holes," *Commun. Math. Phys.* **43** (1975) 199–220. [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[2] J. Scherk and J. H. Schwarz, "Dual Models for Nonhadrons," *Nucl. Phys. B* **81** (1974) 118–144.

[3] T. Yoneya, "Connection of Dual Models to Electrodynamics and Gravidynamics," *Prog. Theor. Phys.* **51** (1974) 1907–1920.

[4] C. G. Callan, Jr., E. J. Martinec, M. J. Perry, and D. Friedan, "Strings in Background Fields," *Nucl. Phys. B* **262** (1985) 593–609.

[5] P. Ramond, "Dual Theory for Free Fermions," *Phys. Rev. D* **3** (1971) 2415–2418.

[6] A. Neveu and J. H. Schwarz, "Factorizable dual model of pions," *Nucl. Phys. B* **31** (1971) 86–112.

[7] F. Gliozzi, J. Scherk, and D. I. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model," *Nucl. Phys. B* **122** (1977) 253–290.

[8] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory," *Nucl. Phys. B* **181** (1981) 502–530.

[9] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory. 2. Vertices and Trees," *Nucl. Phys. B* **198** (1982) 252–268.

[10] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory. 3. Loops and Renormalization," *Nucl. Phys. B* **198** (1982) 441–460.

[11] M. B. Green and J. H. Schwarz, "Supersymmetrical String Theories," *Phys. Lett. B* **109** (1982) 444–448.

[12] J. H. Schwarz, "Superstrings," *Stud. Nat. Sci.* **19** (1983) 193–212.

[13] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory," *Phys. Lett. B* **149** (1984) 117–122.

[14] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "The Heterotic String," *Phys. Rev. Lett.* **54** (1985) 502–505.

[15] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "Heterotic String Theory. 1. The Free Heterotic String," *Nucl. Phys. B* **256** (1985) 253.

[16] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "Heterotic String Theory. 2. The Interacting Heterotic String," *Nucl. Phys. B* **267** (1986) 75–124.

[17] M. Dine, P. Y. Huet, and N. Seiberg, "Large and Small Radius in String Theory," *Nucl. Phys. B* **322** (1989) 301–316.

[18] J. Dai, R. G. Leigh, and J. Polchinski, "New Connections Between String Theories," *Mod. Phys. Lett. A* **4** (1989) 2073–2083.

[19] A. Sen, "Strong - weak coupling duality in four-dimensional string theory," *Int. J. Mod. Phys. A* **9** (1994) 3707–3750, `arXiv:hep-th/9402002`.

[20] A. Sen, "Dyon - monopole bound states, selfdual harmonic forms on the multi - monopole moduli space, and SL(2,Z) invariance in string theory," *Phys. Lett. B* **329** (1994) 217–221, `arXiv:hep-th/9402032`.

[21] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," *Nucl. Phys. B* **438** (1995) 109–137, `arXiv:hep-th/9410167`.

[22] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven-dimensions," *Nucl. Phys. B* **460** (1996) 506–524, `arXiv:hep-th/9510209`.

[23] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys. B* **443** (1995) 85–126, `arXiv:hep-th/9503124`.

[24] J. Polchinski and E. Witten, "Evidence for heterotic - type I string duality," *Nucl. Phys. B* **460** (1996) 525–540, `arXiv:hep-th/9510169`.

[25] J. H. Schwarz, "An SL(2,Z) multiplet of type IIB superstrings," *Phys. Lett. B* **360** (1995) 13–18, `arXiv:hep-th/9508143`. [Erratum: Phys.Lett.B 364, 252 (1995)].

[26] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, `arXiv:hep-th/9510017`.

[27] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `arXiv:hep-th/9711200`.

[28] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, `arXiv:hep-th/9905111`.

[29] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `arXiv:hep-th/9802150`.

[30] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," *Commun. Math. Phys.* **104** (1986) 207–226.

[31] R. Dijkgraaf, "Instanton strings and hyperKahler geometry," *Nucl. Phys. B* **543** (1999) 545–571, `arXiv:hep-th/9810210`.

[32] J. de Boer, "Large N elliptic genus and AdS / CFT correspondence," *JHEP* **05** (1999) 017, `arXiv:hep-th/9812240`.

[33] J. M. Maldacena, G. W. Moore, and A. Strominger, "Counting BPS black holes in toroidal Type II string theory," `arXiv:hep-th/9903163`.

[34] F. Larsen and E. J. Martinec, "U(1) charges and moduli in the D1 - D5 system," *JHEP* **06** (1999) 019, `arXiv:hep-th/9905064`.

[35] N. Seiberg and E. Witten, "The D1 / D5 system and singular CFT," *JHEP* **04** (1999) 017, `arXiv:hep-th/9903224`.

[36] R. Argurio, A. Giveon, and A. Shomer, "Superstrings on AdS(3) and symmetric products," *JHEP* **12** (2000) 003, `arXiv:hep-th/0009242`.

[37] M. R. Gaberdiel and I. Kirsch, "Worldsheet correlators in AdS(3)/CFT(2)," *JHEP* **04** (2007) 050, `arXiv:hep-th/0703001`.

[38] J. R. David, G. Mandal, and S. R. Wadia, "Microscopic formulation of black holes in string theory," *Phys. Rept.* **369** (2002) 549–686, `arXiv:hep-th/0203048`.

[39] A. Sagnotti and M. Tsulaia, "On higher spins and the tensionless limit of string theory," *Nucl. Phys. B* **682** (2004) 83–116, `arXiv:hep-th/0311257`.

[40] D. J. Gross and P. F. Mende, "String Theory Beyond the Planck Scale," *Nucl. Phys. B* **303** (1988) 407–454.

[41] D. J. Gross and P. F. Mende, "The High-Energy Behavior of String Scattering Amplitudes," *Phys. Lett. B* **197** (1987) 129–134.

[42] D. J. Gross, "High-Energy Symmetries of String Theory," *Phys. Rev. Lett.* **60** (1988) 1229.

[43] B. Sundborg, "Stringy gravity, interacting tensionless strings and massless higher spins," *Nucl. Phys. B Proc. Suppl.* **102** (2001) 113–119, `arXiv:hep-th/0103247`.

[44] C. Ahn, M. R. Gaberdiel, and M. H. Kim, "The small $\mathcal{N} = 4$ superconformal $\mathcal{W}_\infty$ algebra," *J. Phys. A* **53** no. 39, (2020) 395401, `arXiv:2004.07439 [hep-th]`.

[45] L. Eberhardt, M. R. Gaberdiel, and R. Gopakumar, "The Worldsheet Dual of the Symmetric Product CFT," *JHEP* **04** (2019) 103, `arXiv:1812.01007 [hep-th]`.

[46] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," *Nucl. Phys. B* **72** (1974) 461.

[47] A. Dei, M. R. Gaberdiel, R. Gopakumar, and B. Knighton, "Free field world-sheet correlators for AdS_3," *JHEP* **02** (2021) 081, `arXiv:2009.11306 [hep-th]`.

[48] M. R. Gaberdiel, R. Gopakumar, B. Knighton, and P. Maity, "From symmetric product CFTs to AdS_3," *JHEP* **05** (2021) 073, arXiv:2011.10038 [hep-th].

[49] B. Knighton, "Higher genus correlators for tensionless AdS_3 strings," *JHEP* **04** (2021) 211, arXiv:2012.01445 [hep-th].

[50] M. R. Gaberdiel, B. Knighton, and J. Vošmera, "D-branes in $AdS_3 \times S^3 \times \mathbb{T}^4$ at $k = 1$ and their holographic duals," *JHEP* **12** (2021) 149, arXiv:2110.05509 [hep-th].

[51] B. Knighton, "Classical geometry from the tensionless string," arXiv:2207.01293 [hep-th].

[52] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[53] R. Blumenhagen, D. Lüst, and S. Theisen, *Basic concepts of string theory*. Theoretical and Mathematical Physics. Springer, Heidelberg, Germany, 2013.

[54] R. H. Dijkgraaf, "A geometrical approach to two-dimensional conformal field theory," *Ph. D. Thesis* (1989) .

[55] A. Belin, A. Castro, C. A. Keller, and B. Mühlmann, "The Holographic Landscape of Symmetric Product Orbifolds," *JHEP* **01** (2020) 111, arXiv:1910.05342 [hep-th].

[56] M. R. Gaberdiel and R. Gopakumar, "The Higher Spin Square," in *International Workshop on Higher Spin Gauge Theories*, pp. 17–23. 2017.

[57] P. Bantay, "Characters and modular properties of permutation orbifolds," *Phys. Lett. B* **419** (1998) 175–178, arXiv:hep-th/9708120.

[58] P. Bantay, "Permutation orbifolds," *Nucl. Phys. B* **633** (2002) 365–378, arXiv:hep-th/9910079.

[59] P. Bantay, "Permutation orbifolds and their applications," *Fields Inst. Commun.* **39** (2003) 13, arXiv:hep-th/0104203.

[60] P. Bantay, "Symmetric products, permutation orbifolds and discrete torsion," *Lett. Math. Phys.* **63** (2003) 209–218, arXiv:hep-th/0004025.

[61] L. J. Dixon, D. Friedan, E. J. Martinec, and S. H. Shenker, "The Conformal Field Theory of Orbifolds," *Nucl. Phys. B* **282** (1987) 13–73.

[62] R. Dijkgraaf, G. W. Moore, E. P. Verlinde, and H. L. Verlinde, "Elliptic genera of symmetric products and second quantized strings," *Commun. Math. Phys.* **185** (1997) 197–209, arXiv:hep-th/9608096.

[63] J. M. Maldacena and H. Ooguri, "Strings in AdS(3) and SL(2,R) WZW model 1.: The Spectrum," *J. Math. Phys.* **42** (2001) 2929–2960, arXiv:hep-th/0001053.

[64] J. M. Maldacena, H. Ooguri, and J. Son, "Strings in AdS(3) and the SL(2,R) WZW model. Part 2. Euclidean black hole," *J. Math. Phys.* **42** (2001) 2961–2977, `arXiv:hep-th/0005183`.

[65] J. M. Maldacena and H. Ooguri, "Strings in AdS(3) and the SL(2,R) WZW model. Part 3. Correlation functions," *Phys. Rev. D* **65** (2002) 106006, `arXiv:hep-th/0111180`.

[66] N. Berkovits, C. Vafa, and E. Witten, "Conformal field theory of AdS background with Ramond-Ramond flux," *JHEP* **03** (1999) 018, `arXiv:hep-th/9902098`.

[67] M. B. Green, J. H. Schwarz, and E. Witten, *SUPERSTRING THEORY. VOL. 1: INTRODUCTION.* Cambridge Monographs on Mathematical Physics. 7, 1988.

[68] M. B. Green, J. H. Schwarz, and E. Witten, *SUPERSTRING THEORY. VOL. 2: LOOP AMPLITUDES, ANOMALIES AND PHENOMENOLOGY.* 7, 1988.

[69] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[70] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[71] K. Ferreira, M. R. Gaberdiel, and J. I. Jottar, "Higher spins on AdS_3 from the worldsheet," *JHEP* **07** (2017) 131, `arXiv:1704.08667 [hep-th]`.

[72] M. R. Gaberdiel and R. Gopakumar, "Tensionless string spectra on AdS_3," *JHEP* **05** (2018) 085, `arXiv:1803.04423 [hep-th]`.

[73] L. Eberhardt, "Partition functions of the tensionless string," *JHEP* **03** (2021) 176, `arXiv:2008.07533 [hep-th]`.

[74] A. Maloney and E. Witten, "Quantum Gravity Partition Functions in Three Dimensions," *JHEP* **02** (2010) 029, `arXiv:0712.0155 [hep-th]`.

[75] L. Eberhardt, "Summing over Geometries in String Theory," *JHEP* **05** (2021) 233, `arXiv:2102.12355 [hep-th]`.

[76] D. Gaiotto and M. Rapčák, "Vertex Algebras at the Corner," *JHEP* **01** (2019) 160, `arXiv:1703.00982 [hep-th]`.

[77] A. Pakman, L. Rastelli, and S. S. Razamat, "Diagrams for Symmetric Product Orbifolds," *JHEP* **10** (2009) 034, `arXiv:0905.3448 [hep-th]`.

[78] L. Eberhardt, M. R. Gaberdiel, and R. Gopakumar, "Deriving the AdS_3/CFT_2 correspondence," *JHEP* **02** (2020) 136, `arXiv:1911.00378 [hep-th]`.

[79] L. Eberhardt, "AdS_3/CFT_2 at higher genus," *JHEP* **05** (2020) 150, `arXiv:2002.11729 [hep-th]`.

[80] O. Lunin and S. D. Mathur, "Correlation functions for M**N / S(N) orbifolds," *Commun. Math. Phys.* **219** (2001) 399–442, arXiv:hep-th/0006196.

[81] A. Dei and L. Eberhardt, "String correlators on AdS_3: three-point functions," *JHEP* **08** (2021) 025, arXiv:2105.12130 [hep-th].

[82] A. Dei and L. Eberhardt, "String correlators on AdS_3: four-point functions," *JHEP* **09** (2021) 209, arXiv:2107.01481 [hep-th].

[83] A. Dei and L. Eberhardt, "String correlators on AdS_3: Analytic structure and dual CFT," *SciPost Phys.* **13** no. 3, (2022) 053, arXiv:2203.13264 [hep-th].

[84] M. R. Gaberdiel and K. Naderi, "The physical states of the Hybrid Formalism," *JHEP* **10** (2021) 168, arXiv:2106.06476 [hep-th].

[85] M.-A. Fiset, M. R. Gaberdiel, K. Naderi, and V. Sriprachyakul, "Perturbing the symmetric orbifold from the worldsheet," arXiv:2212.12342 [hep-th].

[86] N. Berkovits, "An Alternative string theory in twistor space for N=4 superYang-Mills," *Phys. Rev. Lett.* **93** (2004) 011601, arXiv:hep-th/0402045.

[87] E. Witten, "Perturbative gauge theory as a string theory in twistor space," *Commun. Math. Phys.* **252** (2004) 189–258, arXiv:hep-th/0312171.

[88] R. Roiban, M. Spradlin, and A. Volovich, "On the tree level S matrix of Yang-Mills theory," *Phys. Rev. D* **70** (2004) 026009, arXiv:hep-th/0403190.

[89] M. R. Gaberdiel and R. Gopakumar, "The worldsheet dual of free super Yang-Mills in 4D," *JHEP* **11** (2021) 129, arXiv:2105.10496 [hep-th].

[90] M. R. Gaberdiel and R. Gopakumar, "String Dual to Free N=4 Supersymmetric Yang-Mills Theory," *Phys. Rev. Lett.* **127** no. 13, (2021) 131601, arXiv:2104.08263 [hep-th].

[91] O. Lunin and S. D. Mathur, "Three point functions for M(N) / S(N) orbifolds with N=4 supersymmetry," *Commun. Math. Phys.* **227** (2002) 385–419, arXiv:hep-th/0103169.

[92] A. Dei and L. Eberhardt, "Correlators of the symmetric product orbifold," *JHEP* **01** (2020) 108, arXiv:1911.08485 [hep-th].

[93] L. Bhardwaj and Y. Tachikawa, "On finite symmetries and their gauging in two dimensions," *JHEP* **03** (2018) 189, arXiv:1704.02330 [hep-th].

[94] S. K. Donaldson, *Riemann surfaces.* Oxford University Press, 2011.

[95] B. Eynard, "Lectures notes on compact Riemann surfaces," arXiv:1805.06405 [math-ph].

[96] M. Fujita, T. Takayanagi, and E. Tonni, "Aspects of AdS/BCFT," *JHEP* **11** (2011) 043, arXiv:1108.5152 [hep-th].

[97] C. Bachas and M. Petropoulos, "Anti-de Sitter D-branes," *JHEP* **02** (2001) 025, `arXiv:hep-th/0012234`.

[98] C. Klimcik and P. Severa, "Open strings and D-branes in WZNW model," *Nucl. Phys. B* **488** (1997) 653–676, `arXiv:hep-th/9609112`.

[99] A. Y. Alekseev and V. Schomerus, "D-branes in the WZW model," *Phys. Rev. D* **60** (1999) 061901, `arXiv:hep-th/9812193`.

[100] G. Felder, J. Frohlich, J. Fuchs, and C. Schweigert, "The Geometry of WZW branes," *J. Geom. Phys.* **34** (2000) 162–190, `arXiv:hep-th/9909030`.

[101] S. Stanciu, "D-branes in group manifolds," *JHEP* **01** (2000) 025, `arXiv:hep-th/9909163`.

[102] A. Y. Alekseev, A. Recknagel, and V. Schomerus, "Noncommutative world volume geometries: Branes on SU(2) and fuzzy spheres," *JHEP* **09** (1999) 023, `arXiv:hep-th/9908040`.

[103] B. Ponsot, V. Schomerus, and J. Teschner, "Branes in the Euclidean AdS(3)," *JHEP* **02** (2002) 016, `arXiv:hep-th/0112198`.

[104] N. Ishibashi, "The Boundary and Crosscap States in Conformal Field Theories," *Mod. Phys. Lett. A* **4** (1989) 251.

[105] P. M. Petropoulos and S. Ribault, "Some remarks on anti-de Sitter D-branes," *JHEP* **07** (2001) 036, `arXiv:hep-th/0105252`.

[106] P. Lee, H. Ooguri, J.-W. Park, and J. Tannenhauser, "Open strings on AdS(2) branes," *Nucl. Phys. B* **610** (2001) 3–48, `arXiv:hep-th/0106129`.

[107] J. L. Cardy, "Boundary Conditions, Fusion Rules and the Verlinde Formula," *Nucl. Phys. B* **324** (1989) 581–596.

[108] J. L. Cardy, "Conformal Invariance and Surface Critical Behavior," *Nucl. Phys. B* **240** (1984) 514–532.

[109] A. Recknagel and V. Schomerus, *Boundary Conformal Field Theory and the Worldsheet Approach to D-Branes.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2013.

[110] T. Onogi and N. Ishibashi, "Conformal Field Theories on Surfaces With Boundaries and Crosscaps," *Mod. Phys. Lett. A* **4** (1989) 161. [Erratum: Mod.Phys.Lett.A 4, 885 (1989)].

[111] I. Brunner and M. R. Gaberdiel, "Matrix factorisations and permutation branes," *JHEP* **07** (2005) 012, `arXiv:hep-th/0503207`.

[112] M. R. Gaberdiel, "Lectures on nonBPS Dirichlet branes," *Class. Quant. Grav.* **17** (2000) 3483–3520, `arXiv:hep-th/0005029`.

[113] R. Gopakumar, "From free fields to AdS," *Phys. Rev. D* **70** (2004) 025009, `arXiv:hep-th/0308184`.

[114] R. Gopakumar, "From free fields to AdS. 2.," *Phys. Rev. D* **70** (2004) 025010, arXiv:hep-th/0402063.

[115] R. Gopakumar, "From free fields to AdS: III," *Phys. Rev. D* **72** (2005) 066008, arXiv:hep-th/0504229.

[116] S. S. Razamat, "On a worldsheet dual of the Gaussian matrix model," *JHEP* **07** (2008) 026, arXiv:0803.2681 [hep-th].

[117] S. S. Razamat, "From Matrices to Strings and Back," *JHEP* **03** (2010) 049, arXiv:0911.0658 [hep-th].

[118] R. Gopakumar, "What is the Simplest Gauge-String Duality?," arXiv:1104.2386 [hep-th].

[119] R. Gopakumar and E. A. Mazenc, "Deriving the Simplest Gauge-String Duality – I: Open-Closed-Open Triality," arXiv:2212.05999 [hep-th].

[120] K. Strebel, *Quadratic differentials.* Springer-Verlag, 1984.

[121] M. Mulase and M. Penkava, "Ribbon graphs, quadratic differentials on riemann surfaces, and algebraic curves defined over $\bar{q}$," 1998.

[122] D. Zvonkine, "Strebel differentials on stable curves and kontsevich's proof of witten's conjecture," 2004.

[123] S. Hamidi and C. Vafa, "Interactions on Orbifolds," *Nucl. Phys. B* **279** (1987) 465–513.

[124] K. Roumpedakis, "Comments on the S_N orbifold CFT in the large N-limit," *JHEP* **07** (2018) 038, arXiv:1804.03207 [hep-th].

[125] P. Maity, "Scattering and Strebel graphs," arXiv:2108.09458 [hep-th].

[126] G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers.* Oxford University Press, 6th ed., 1938.

[127] S. Ramanujan, "Highly composite numbers," *The Ramanujan Journal* (1997) 119–153.

[128] M. Young, "Mathoverflow answer." https://mathoverflow.net/questions/115100/numerical-evaluation-of-the-petersson-product-of-elliptic-modular-forms.

[129] I. Y. Arefeva and A. A. Bagrov, "Holographic dual of a conical defect," *Teor. Mat. Fiz.* **182** no. 1, (2014) 3–27.

[130] D. Berenstein, D. Grabovsky, and Z. Li, "Aspects of Holography in Conical AdS_3," arXiv:2205.02256 [hep-th].

[131] N. Benjamin, S. Collier, and A. Maloney, "Pure Gravity and Conical Defects," *JHEP* **09** (2020) 034, arXiv:2004.14428 [hep-th].

[132] B. Zwiebach, "How covariant closed string theory solves a minimal area problem," *Commun. Math. Phys.* **136** (1991) 83–118.

[133] N. Moeller, "Closed bosonic string field theory at quartic order," *JHEP* **11** (2004) 018, arXiv:hep-th/0408067.

[134] F. Bhat, R. Gopakumar, P. Maity, and B. Radhakrishnan, "Twistor Coverings and Feynman Diagrams," arXiv:2112.05115 [hep-th].

[135] M. R. Gaberdiel and F. Galvagno, "Worldsheet dual of free $\mathcal{N} = 2$ quiver gauge theories," arXiv:2206.08795 [hep-th].

[136] R. Jackiw, "Lower Dimensional Gravity," *Nucl. Phys. B* **252** (1985) 343–356.

[137] A. Banerjee, A. Kundu, and R. R. Poojary, "Strings, Branes, Schwarzian Action and Maximal Chaos," arXiv:1809.02090 [hep-th].

[138] A. Banerjee, A. Kundu, and R. Poojary, "Maximal Chaos from Strings, Branes and Schwarzian Action," *JHEP* **06** (2019) 076, arXiv:1811.04977 [hep-th].

[139] L. V. Iliesiu, S. S. Pufu, H. Verlinde, and Y. Wang, "An exact quantization of Jackiw-Teitelboim gravity," *JHEP* **11** (2019) 091, arXiv:1905.02726 [hep-th].

[140] T. G. Mertens and G. J. Turiaci, "Solvable Models of Quantum Black Holes: A Review on Jackiw-Teitelboim Gravity," arXiv:2210.10846 [hep-th].

[141] E. Mefford and K. Suzuki, "Jackiw-Teitelboim quantum gravity with defects and the Aharonov-Bohm effect," *JHEP* **05** (2021) 026, arXiv:2011.04695 [hep-th].

[142] J. Engelsöy, T. G. Mertens, and H. Verlinde, "An investigation of AdS_2 backreaction and holography," *JHEP* **07** (2016) 139, arXiv:1606.03438 [hep-th].

[143] H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas, and S. Shashi, "Jackiw-Teitelboim Gravity from the Karch-Randall Braneworld," arXiv:2206.04695 [hep-th].

[144] H. Geng, "Aspects of AdS_2 Quantum Gravity and the Karch-Randall Braneworld," arXiv:2206.11277 [hep-th].

[145] F. Deng, Y.-S. An, and Y. Zhou, "JT Gravity from Partial Reduction and Defect Extremal Surface," arXiv:2206.09609 [hep-th].

[146] A. Ghosh, H. Maxfield, and G. J. Turiaci, "A universal Schwarzian sector in two-dimensional conformal field theories," *JHEP* **05** (2020) 104, arXiv:1912.07654 [hep-th].

[147] K. Naderi, "DDF operators in the hybrid formalism," *JHEP* **12** (2022) 043, arXiv:2208.01617 [hep-th].

[148] L. Eberhardt, M. R. Gaberdiel, and I. Rienacker, "Higher spin algebras and large $\mathcal{N} = 4$ holography," *JHEP* **03** (2018) 097, arXiv:1801.00806 [hep-th].

[149] M. R. Gaberdiel and R. Gopakumar, "Higher Spins & Strings," *JHEP* **11** (2014) 044, arXiv:1406.6103 [hep-th].

[150] M. R. Gaberdiel and R. Gopakumar, "String Theory as a Higher Spin Theory," *JHEP* **09** (2016) 085, arXiv:1512.07237 [hep-th].

[151] M. R. Gaberdiel, C. Peng, and I. G. Zadeh, "Higgsing the stringy higher spin symmetry," *JHEP* **10** (2015) 101, arXiv:1506.02045 [hep-th].

[152] M. Baggio, M. R. Gaberdiel, and C. Peng, "Higher spins in the symmetric orbifold of K3," *Phys. Rev. D* **92** (2015) 026007, arXiv:1504.00926 [hep-th].

[153] M. R. Gaberdiel and B. Nairz, "BPS correlators for AdS_3/CFT_2," *JHEP* **09** (2022) 244, arXiv:2207.03956 [hep-th].

[154] L. Eberhardt, M. R. Gaberdiel, and W. Li, "A holographic dual for string theory on $AdS_3 \times S^3 \times S^3 \times S^1$," *JHEP* **08** (2017) 111, arXiv:1707.02705 [hep-th].

[155] L. Eberhardt and M. R. Gaberdiel, "Strings on $AdS_3 \times S^3 \times S^3 \times S^1$," *JHEP* **06** (2019) 035, arXiv:1904.01585 [hep-th].

[156] A. Giveon, N. Itzhaki, and D. Kutasov, "$T\overline{T}$ and LST," *JHEP* **07** (2017) 122, arXiv:1701.05576 [hep-th].

[157] A. B. Zamolodchikov, "Expectation value of composite field $t\bar{t}$ in two-dimensional quantum field theory," arXiv:hep-th/0401146 [hep-th].

[158] A. Hashimoto and D. Kutasov, "Strings, symmetric products, $T\bar{T}$ deformations and Hecke operators," *Phys. Lett. B* **806** (2020) 135479, arXiv:1909.11118 [hep-th].

[159] O. Aharony, S. Datta, A. Giveon, Y. Jiang, and D. Kutasov, "Modular invariance and uniqueness of $T\bar{T}$ deformed CFT," *JHEP* **01** (2019) 086, arXiv:1808.02492 [hep-th].

[160] A. Giveon, N. Itzhaki, and D. Kutasov, "A solvable irrelevant deformation of AdS_3/CFT_2," *JHEP* **12** (2017) 155, arXiv:1707.05800 [hep-th].

[161] S. Chakraborty, A. Giveon, and D. Kutasov, "$T\bar{T}$, $J\bar{T}$, $T\bar{J}$ and String Theory," *J. Phys. A* **52** no. 38, (2019) 384003, arXiv:1905.00051 [hep-th].

[162] A. Hashimoto and D. Kutasov, "$T\overline{T}, J\overline{T}, T\overline{J}$ partition sums from string theory," *JHEP* **02** (2020) 080, arXiv:1907.07221 [hep-th].

[163] A. Dei, B. Knighton, and S. Sethi, "The worldsheet dual of single-trace $T\overline{T}$," *to appear* .

[164] L. Eberhardt, "A perturbative CFT dual for pure NS–NS AdS_3 strings," *J. Phys. A* **55** no. 6, (2022) 064001, arXiv:2110.07535 [hep-th].