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FLOER THEORY OF REAL DEHN TWISTS, LAGRANGIAN HOFER METRIC AND BARCODES

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To my parents.

Zusammenfassung

Diese Doktorarbeit besteht aus zwei voneinander unabhängigen Teilen.

Im ersten Teil studieren wir die Floer Kohomologie des Dehn Twists entlang einer Lagrangen Sphäre in einer symplektischen Mannigfaltigkeit, welche mit einer antisymplektischen Involution ausgestattet ist. Wir betrachten den Dehn Twist als Monodromie in einer reellen Lefschetz Faserung und zeigen, dass die Involution einen Automorphismus auf der Floer Kohomologie induziert. Desweiteren finden wir einen speziellen Fixpunkt von diesem Automorphismus. Die Beweismethoden basieren auf dem Mak–Wu-Kobordismus und Floer-theoretischen Überlegungen.

Der zweite Teil beschäftigt sich mit der Lagrange Hofer-Metrik und Invarianten, welche durch den Barcode von persistenter Floer Homologie definiert sind. Es ist allgemein bekannt, dass die Längen der endlichen Intervalle und die Spektralmetrik untere Schranken der Lagrange Hofer-Metrik sind. Unser Resultat besteht aus einer umgekehrten Ungleichung: Wir geben eine obere Schranke vom Lagrange Hofer Abstand zwischen Äquatoren im Zylinder durch den Barcode der persistenten Floer Homologie. Die obere Schranke ist gegeben durch eine gewichtete Summe von Längen der endlichen Intervalle und der Spektralmetrik.

Abstract

This thesis consists of two independent parts.

In the first part we study Floer cohomology of the Dehn twist along a real Lagrangian sphere in a symplectic manifold endowed with an anti-symplectic involution. We view the Dehn twist as a monodromy map in a real Lefschetz fibration and prove that the involution induces an automorphism in Floer cohomology. Moreover, we identify a distinguished element that is a fixed point of this automorphism. Our methods of proof are based on Mak–Wu's cobordism and Floer-theoretic considerations.

The second part deals with the Lagrangian Hofer distance and invariants coming from the barcodes of persistent Floer homology. It is well-known that the lengths of the finite bars and the spectral distance are lower bounds on the Lagrangian Hofer metric. Our result consists of a reverse inequality: We provide an upper bound on the Lagrangian Hofer distance between equators in the cylinder in terms of the barcode of persistent Floer homology. The bound consists of a weighted sum of the lengths of the finite bars and the spectral distance.

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Contents

Introduction

This thesis contains work about two topics in the field of symplectic geometry. The basic object of study in symplectic geometry is a smooth manifold M endowed with a closed non-degenerate 2-form ω. The pair (M, ω) is called a symplectic manifold. Symplectic manifolds contain an important class of submanifolds: the Lagrangian submanifolds. These are half-dimensional smooth submanifolds $L \subset M$ satisfying $\omega|_{TL} \equiv 0$. Diffeomorphisms $\phi\colon M \to M$ preserving the symplectic structure are called symplectomorphisms. Various flavours of Floer theory give rise to algebraic invariants of these basic objects. In this thesis we work with both, the Floer homology of symplectomorphisms, as well as the Floer homology of pairs of Lagrangians.

The main object in the first part is a special isotopy class of symplectomorphisms, called the *Dehn twist* τ, which can be associated to any parametrised Lagrangian sphere. In our setting, where (M, ω) is a real fiber of a real Lefschetz fibration, the monodromy around the singularity defines a Dehn twist along the vanishing sphere. For this particular τ, we describe an automorphism on the Floer homology of τ and identify special fixed points.

In the second part of the thesis we study various metrics on the space of Lagrangians in cotangent bundles of circles. We establish a relation between the *Lagrangian Hofer metric*, the spectral metric and persistent Lagrangian Floer homology. Persistent Lagrangian Floer homology is a persistence module associated to any pair of exact Lagrangians and gives rise to an algebraic object called *barcode*. The main result consists of an upper bound on the Lagrangian Hofer metric in terms of an invariant that can be read off from the barcode.

Subsequently, we elaborate further on the ingredients and main results of the two subjects.

1 Floer theory of real Dehn twists

Chapter 1 is an extended version of [Die23b]. We study *Floer theory* of the *Dehn twist* along a real Lagrangian sphere in a symplectic manifold endowed with a *real structure*. We prove that there exists a distinguished element in the Floer group that is a fixed point of the automorphism induced by the real structure.

1.1 Floer theory Floer theory is a powerful tool in modern symplectic geometry. Under some technical conditions, it gives rise to algebraic invariants $\mathrm{HF}^*(\phi)$, $\mathrm{HF}^*(L_0, L_1)$ that can be associated to symplectomorphisms ϕ and pairs of Lagrangians (L_0, L_1). While Floer theory can be defined in much more general situations, we assume that (M, ω) is a closed, symplectically aspherical symplectic manifold ($\omega|_{\pi_2(M)} \equiv 0$ and $c_1|_{\pi_2(M)} \equiv 0$) and the Lagrangian submanifolds are assumed to be closed and relatively symplectically aspherical ($\omega|_{\pi_2(M,L)} \equiv 0$). The Floer cohomology groups will be $\mathbb{Z}_2$-graded vectorspaces over the universal Novikov field

$$\Lambda = \left\{ \sum_{k \in \mathbb{N}} a_k q^{\omega_k} \;\middle|\; a_k \in \mathbb{Z}_2, \omega_k \in \mathbb{R} \text{ and } \lim_{k \to \infty} \omega_k = \infty \right\}.$$

For transversely intersecting Lagrangians L_0 and L_1 the chain complex $\mathrm{CF}^*(L_0, L_1)$ underlying $\mathrm{HF}^*(L_0, L_1)$ is generated by the intersection points of L_0 and L_1. The differential comes from counting pseudo-holomorphic strips with boundary on L_0 and L_1 connecting two intersection points. $\mathrm{HF}^*(\phi)$ can be viewed as a special case of Lagrangian Floer cohomology via

$$\mathrm{HF}^*(\phi) = \mathrm{HF}^*(\Gamma_{\mathrm{id}}, \Gamma_\phi),$$

where Γ_{id} and Γ_ϕ are the graphs of id and ϕ in

$$M \times M^- := (M \times M, \omega \oplus -\omega).$$

1.2 Dehn twist Let $\iota\colon S^n \xrightarrow{\approx} S \subset M$ be an embedding whose image in M is a Lagrangian sphere S. The pair (S, ι) is called a parametrised Lagrangian sphere, which often is abbreviated by S. To any such parametrised Lagrangian sphere S, one can associate a symplectomorphism τ_S on M, that is constructed as follows. Choose a tubular neighbourhood $\mathcal{N}$ of S and identify it with the cotangent bundle T^*S^n of S^n. Using geodesic flow one defines an automorphism on T^*S^n, which is compactly supported and equals the antipodal map on S^n. We can then transport it back into $\mathcal{N}$ and extend it via the identity to M. This procedure can be done in such a way that the resulting automorphism is a symplectomorphism. It is called the Dehn twist τ_S along S.

There is an alternative viewpoint for the Dehn twist in the context of Lefschetz fibrations. A Lefschetz fibration is a singular fibration $\pi\colon E \longrightarrow \mathbb{C}$ for a symplectic manifold (E, Ω_E), where all of the critical points are ordinary double points locally modelled by

$$\begin{aligned} \pi\colon \qquad \mathbb{C}^{n+1} \quad &\longrightarrow \mathbb{C}, \\ (z_1, \dots, z_{n+1}) &\longmapsto z_1^2 + \dots + z_{n+1}^2. \end{aligned}$$

The symplectic structure Ω_E on E defines a parallel transport along paths contained in the set of regular values of π. Fix a smooth fiber M of π. Then for any critical point p, consider a small sphere $\epsilon S^n \subset \mathbb{C}^{n+1}$ ($\epsilon > 0$ small enough) from the local model near p. Move this sphere by parallel transport along a regular path to M. The result is a Lagrangian sphere $S \subset M$ endowed with a canonical parametrisation. S is called the vanishing sphere associated to the critical point p. The Picard-Lefschetz theorem ([Sei08a, Section (16c)]) states that the parallel transport along a closed loop that circles once around the critical value $\pi(p)$ (and around no other critical value) is symplectically isotopic to the Dehn twist along the vanishing sphere S.

The Dehn twist has its origin in the study of mapping class groups of surfaces and has been used in singularity theory. Its importance in symplectic geometry was initialized by the work of Seidel [Sei97a]: He proved that, in some cases, the square of the Dehn twist is a symplectomorphism which is smoothly isotopic, but not symplectically isotopic to the identity. The invariant that allowed him to prove this result is the Floer cohomology group $\mathrm{HF}^*(\tau_S)$. In order to study this group, Seidel established a long exact sequence of Floer cohomology groups [Sei03]. The most general version is due to Mak–Wu ([MW18, Theorem 6.4]) and relates Lagrangian Floer cohomology groups of certain pairs of Lagrangians in the product manifold $M \times M^-$. The long exact sequence reads

$$\cdots \to \mathrm{HF}^k(K, S{\times}S) \longrightarrow \mathrm{HF}^k(K, \Delta) \longrightarrow \\ \longrightarrow \mathrm{HF}^k(K, \Gamma_{\tau_S^{-1}}) \longrightarrow \mathrm{HF}^{k+1}(K, S \times S) \to \dots$$

for any admissible Lagrangian submanifold $K \subset M \times M^-$. This long exact sequence is based on the cobordism theory studied by Biran–Cornea [BC13, BC14, BC17] applied to a surgery cobordism constructed by Mak–Wu. The middle map in the long exact sequence is characterized by an element

$$A \in \mathrm{HF}^0(\Delta, \Gamma_{\tau_S^{-1}}) \cong \mathrm{HF}^0(\tau_S^{-1}).$$

1.3 Real structures A real structure on a symplectic manifold (M, ω) is an anti-symplectic involution $c\colon M \longrightarrow M$. This means that c is a diffeomorphism satisfying $c^*\omega = -\omega$ and $c^2 = \mathrm{id}$. We are interested into real structures that preserve a Lagrangian sphere $S \subset M$.

This situation arises naturally in the context of real Lefschetz fibrations. A real Lefschetz fibration is a Lefschetz fibration $\pi\colon E \to \mathbb{C}$, where the total space E is endowed with an anti-symplectic involution $c_E\colon E \longrightarrow E$. Assume that c_E covers complex conjugation $c_{\mathbb{C}}$ on $\mathbb{C}$ and the unique critical point has value $\pi(p) = 0$. If M is the smooth fiber at 1, then c_E induces a real structure c on M. In this case $c(S) = S$ for the vanishing sphere S. Examples of real Lefschetz fibrations can be obtained by real algebraic methods: In [BC17, Section 6.5] the authors explain how to construct real Lefschetz fibrations out of Lefschetz pencils arising in real algebraic geometry.

In [Sal10], Salepci shows that the monodromy in a real Lefschetz fibration is a composition of two anti-symplectic involutions and one of them is c itself. Using the monodromy viewpoint on the Dehn twist, it therefore follows that the Dehn twist is the composition of two anti-symplectic involutions, i.e.

$$\tau_S = c \circ \tilde{c},$$

for another anti-symplectic involution $\tilde{c}$ on M. An equivalent formulation is

$$c\tau_S c = \tau_S^{-1}.$$

This fact is crucial for the proof of the main result.

1.4 Main result The main result Theorem A establishes an automorphism on Floer cohomology induced by c and states that the element $A \in \mathrm{HF}^*(\tau_S^{-1})$, that occurs in the Mak–Wu long exact sequence, is a fixed point of it.

Theorem A (Theorem 1.1.1)**.** *Let (M, ω) be a closed symplectically aspherical symplectic manifold that arises as a real fiber in a real Lefschetz fibration as explained above. Let $c\colon M \longrightarrow M$ be the induced real structure on M and $S \subset M$ the vanishing sphere. Then c induces an automorphism*

$$c_*\colon \mathrm{HF}^*(\tau_S^{-1}) \longrightarrow \mathrm{HF}^*(\tau_S^{-1})$$

and it satisfies $c_*(A) = A$.

In some examples this fixed-point property for A enables us to compute A. Due to the importance of A for Seidel's long exact sequence, this gives new insights into $\mathrm{HF}^*(\tau)$.

The main idea of the proof is to show that the Mak–Wu cobordism is preserved under a symmetry coming from the real structure. This can be shown by an explicit calculation and the theorem then follows from algebraic properties of Floer theory.

2 Lagrangian Hofer metric and barcodes

Chapter 2 is an extended version of [Die23a]. We study the relation between the *Lagrangian Hofer metric* and invariants coming from *persistent Lagrangian Floer homology.* The main result concerns Lagrangians in the cotangent bundle of a circle. As a tool in this two-dimensional setting, we use *combinatorial Floer homology* for curves on surfaces.

2.1 Lagrangian Hofer metric Any compactly supported Hamiltonian function

$$H\colon \mathbb{R}\times M \longrightarrow \mathbb{R}$$

induces an isotopy of symplectomorphisms as follows: Denote by $H_t := H(t, -)$ the time-t Hamiltonian and let $\{X_t^H\}_{t\in\mathbb{R}}$ be the family of vector fields on M defined by the equation

$$\omega\left(X_t^H(x), v\right) = -\mathrm{d}H_t(v),$$

for all $v \in T_xM$. The Hamiltonian flow $\left\{\psi_t^H\right\}_{t\in\mathbb{R}}$ is uniquely defined via

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^H(x) = X_t^H\left(\psi_t^H(x)\right), \\ \psi_0^H = \mathrm{id}. \end{cases}$$

The maps ψ_1^H (or equivalently all ψ_t^H's) are called Hamiltonian diffeomorphisms. The set $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms is a subgroup of the group of diffeomorphisms on M. In 1990, Hofer [Hof90] introduced a bi-invariant Finsler metric on $\mathrm{Ham}(M)$. It is defined by infimizing the oscillation norm of all Hamiltonians that generate a Hamiltonian flow connecting two Hamiltonian diffeomorphisms:

$$d_H(\varphi,\psi) = \inf\left\{\int_0^1 \max_{x\in M} H_t(x) - \min_{x\in M} H_t(x)\,\mathrm{d}t \;\middle|\; \varphi\psi_1^H = \psi\right\}.$$

It is a deep result that d_H defines a non-degenerate metric: This was first proven by Hofer in [Hof90] for $M = \mathbb{R}^{2n}$, later generalised to a wide class of symplectic manifolds by Polterovich [Pol93], and finally proved in full generality by Lalonde-McDuff [LM95].

Hofer's distance is conjectured to be unbounded. While this is not proven in full generality, there have been several developments over the years. We mention two of them: McDuff [McD10] showed that $\mathrm{Ham}(M)$ has infinite diameter under a certain condition on spectral invariants, Usher [Ush13] proved the same under the condition that there exists a non-constant autonomous Hamiltonian, whose contractible closed Hamiltonian orbits are all constant. Usher's work is based on the boundary depth, which is a special invariant that we will encounter later as the length of the longest bar in a barcode. For a detailed survey on these developments, we refer to [Ush13]. A nice survey on various topics in Hofer's geometry can be found in [Pol01].

In [Che00], Chekanov introduced a Lagrangian version of the Hofer metric. Let L and L' be closed connected Lagrangian submanifolds of M that are Hamiltonian isotopic. The *Lagrangian Hofer distance* between L and L' is defined by

$$d_H(L, L') = \inf \left\{ d_H(\mathrm{id}, \varphi) \,\middle|\, \varphi(L) = L' \right\}.$$

Chekanov proved that this is a non-degenerate metric for geometrically bounded symplectic manifolds. Very little is known about Lagrangian Hofer geometry compared to ordinary Hofer geometry. The Lagrangian Hofer distance is not always unbounded. For example, the Lagrangian Hofer distance on the space of Lagrangians Hamiltonian isotopic to a fixed circle in $\mathbb{R}^2$ is bounded by the area enclosed by the circle [Ush13]. On the other hand, Khanevsky [Kha09] showed that Hofer's distance is unbounded on equators in the disc and the cylinder. Other cases with unbounded Lagrangian Hofer metric have been studied in [Lec08, Zap13] (weakly exact cases) and [Ush13] (some products with T^2). It is an open question, whether the Lagrangian Hofer distance on equators of S^2 is unbounded or not [MS95, Problem 32].

2.2 Persistent Floer homology A major tool in many of the previously mentioned results are spectral invariants. They have been introduced by Viterbo [Vit92] on $\mathbb{R}^{2n}$ via generating functions, and later have been defined through Hamiltonian Floer homology on all closed manifolds [Sch00, Oh05]. The Lagrangian spectral invariants have been introduced by Leclercq [Lec08] in the weakly exact case and Leclercq–Zapolsky [LZ18] in the monotone case. (Lagrangian) spectral invariants give rise to the (Lagrangian) spectral metric defined as the difference between the highest and the lowest spectral invariants. The Lagrangian spectral metric is shown to be non-degenerate in [LZ18]. All these invariants can be seen through the eyes of persistence theory. We will adopt this view for the rest of the thesis and proceed with an introduction of persistence theory before coming back to spectral invariants.

The technique of persistence modules has its origin in topological data science [CZ05] and recently it has been applied successfully to symplectic geometry. We refer to [PRSZ20] for a survey. In a nutshell, *filtered* Floer theory gives rise to a persistence module. The Structure Theorem from persistence theory associates to it a multiset of intervals, called a barcode. The barcode contains many interesting invariants that behave continuously with respect to the Hofer metric. This continuity property is a consequence of the Isometry Theorem from persistence theory. Figure 1 below exemplifies these ideas in the case of Lagrangian Floer theory.

Figure 1: Persistence Floer Theory Pipeline

The barcode recovers many of the previously mentioned invariants: Spectral invariants are the endpoints of the infinite bars, the spectral metric $\gamma(L, L')$ is exactly the largest distance between two infinite bars, and the boundary depth $\beta(L, L')$ is the length of the largest finite bar. Due to the isometry property, it holds ([PS16, KS21])

$$\beta(L, L') \leq \gamma(L, L') \leq d_H(L, L'). \tag{1}$$

What makes persistence theory powerful, is that the barcode contains much more information in an accessible way. For example, it gives rise to an invariant called multiplicity sensitive spread which was used in [PS16] to give a lower bound to Hofer's distance to powers. In [She22a], Shelukhin used the information of all the finite bars to prove the Hofer-Zehnder conjecture on the existence of contractible fixed points. The finite bars will play an important role in this thesis. Apart from [She22a] and this thesis, the author is not aware of any other work that uses the finite bars in the barcode in a substantial way.

2.3 Filtered combinatorial Floer homology The main result will apply to Lagrangians in the cotangent bundle of a circle. In this two-dimensional case it is suitable to work with combinatorial Floer homology. This is a purely combinatorial description of Floer homology developed by de Silva–Robbin–Salamon [dSRS14]. Instead of counting pseudo-holomorphic strips to define the differential of $\mathrm{CF}(L, L')$, one can simply work with smooth immersions of half-discs, called *smooth lunes*, or more combinatorially with *combinatorial lunes*. If L and L' are exact Lagrangians then the chain complex $\mathrm{CF}(L, L')$ admits a function $\mathcal{A}\colon L \cap L' \longrightarrow \mathbb{R}$ on the set of generators of $\mathrm{CF}(L, L')$ such that

$$\mathcal{A}(q) - \mathcal{A}(p) = Area(u),$$

whenever there exists a smooth lune u from q to p. This relation determines $\mathcal{A}$ uniquely up to a shift and therefore fits nicely into the combinatorial framework. The function $\mathcal{A}$ defines a filtration of the chain complex $\mathrm{CF}(L, L')$ by the subcomplexes $\mathrm{CF}^{\leq \alpha}(L, L')$ generated by those intersection points p with $\mathcal{A}(p) \leq \alpha$. Applying homology to the resulting filtered chain complex results in a persistence module $\mathrm{HF}^{\leq \bullet}(L, L')$.

2.3 Main result Consider $\Sigma := T^*S^1$ with its standard symplectic structure and let L and L' be two transversely intersecting Lagrangians that are Hamiltonian isotopic to the zero-section. Note that the number of intersection points between L and L' is even, i.e. $\#(L \cap L') = 2n$ for some $n \geq 1$. The barcode $\mathcal{B}(L, L')$ associated to the persistence Floer module $\mathrm{HF}^{\leq \bullet}(L, L')$ contains exactly 2 infinite bars and $n-1$ finite bars. The spectral metric $\gamma(L, L')$ is precisely the difference between the endpoints of the two infinite bars. We denote by

$$\beta_1(L, L') \geq \beta_2(L, L') \geq \cdots \geq \beta_{n-1}(L, L')$$

the lengths of the finite bars. We are now ready to state the main theorem of the second part of the thesis: Theorem B establishes an *upper* bound on the Lagrangian Hofer metric in terms of invariants coming the barcode.

Theorem B (Theorem 2.1.1 and Corollary 2.1.2)**.**

$$d_H(L, L') \leq \sum_{j=1}^{n-1} 2^j \beta_j(L, L') + \gamma(L, L') \leq 2^n \gamma(L, L').$$

The second inequality follows from (1). The proof of the first inequality makes use of Khanevsky's procedure to *delete a leaf* [Kha09]. This is the core of his proof of the following upper bound on the Lagrangian Hofer metric:

$$d_H(L, L') \leq kn + c \tag{2}$$

for some constants c and k (independent of L, L'). The wording *deletion of a leaf* refers to a certain Hamiltonian diffeomorphism φ that removes two intersection points p, q from $L' \cap L$ that are connected by a smallest lune (which corresponds to a leaf in a certain tree). That is,

$$\varphi(L') \cap L = \big(L' \cap L\big) \setminus \{p, q\}.$$

Using a sequence of such φ's, at each time removing two intersection points and controlling carefully the Hofer energy of each φ, Khanevsky arrives at the linear bound (2) in the number of intersection points.

For us deletion of a leaf is also key to prove Theorem B. It allows us to apply induction on the number of intersection points. The main difficulty in the proof is to choose the correct leaf, so that we can keep control over the changes in the barcodes. It turns out that the smallest bar corresponds to a smallest lune. This is the correct leaf to delete.

Chapter 1

Floer theory of real Dehn twists

1.1 Introduction and main result

Let (M, ω) be a closed symplectic manifold and $S \subset M$ a Lagrangian sphere with a parametrisation $\iota\colon S^n \xrightarrow{\approx} S$. Associated to (S, ι) there exists a distinguished symplectic isotopy class represented by the *Dehn twist*. The Dehn twist τ_S is a symplectomorphism compactly supported in a neighbourhood of S. Seidel proved that the square of the Dehn twist, in some cases, is not symplectically, but only smoothly isotopic to the identity [Sei97a], [Sei08b]. To prove this result, Seidel established a Floer homology exact sequence

$$\begin{aligned} \dots \to (\mathrm{HF}^*(S,N) \otimes \mathrm{HF}^*(Q,S))^k \to &\mathrm{HF}^k(Q,N) \to \\ &\to \mathrm{HF}^k(Q, \tau_s(N)) \to \dots \end{aligned} \tag{1.1}$$

for admissible Lagrangian submanifolds Q and N in M [Sei03],[Sei08a]. There is a distinguished element $A \in \mathrm{HF}^*(\tau_S^{-1})$ that characterises the map $\mathrm{HF}^k(Q,N) \to HF^k(Q, \tau_S(N))$ that occurs in the sequence.

Due to the relevance of the above exact sequence it is thus natural to investigate properties of the element A. The goal of this paper is to study the element A in the situation, where there exists an anti-symplectic involution that preserves S.

Throughout the whole exposition (M, ω) is a closed symplectically aspherical symplectic manifold, i.e. $\omega|_{\pi_2(M)} \equiv 0$ and for the first Chern class $c_1|_{\pi_2(M)} = 0$. Unless otherwise explicitely stated, all involved Lagrangian submanifolds are assumed to be closed, oriented and relatively symplectically aspherical, i.e. $\omega|_{\pi_2(M,L)} \equiv 0$). Floer cohomology groups are $\mathbb{Z}_2$-graded with coefficients in the universal Novikov field over $\mathbb{Z}_2$. More details about these assumptions are given in section 1.4.1.

Assume (M, ω) is a real fiber of a real Lefschetz fibration $\pi\colon E \to \mathbb{C}$ with one critical point and vanishing sphere $S \subset M$. See Definitions 1.2.3 and 1.2.5 for the notion of a real Lefschetz fibration. Consider the anti-symplectic involution $c\colon M \longrightarrow M$ induced from the real structure on E. Our main result is

Theorem 1.1.1. *c induces an automorphism*

$$c_*\colon \mathrm{HF}^*(\tau_S^{-1}) \longrightarrow \mathrm{HF}^*(\tau_S^{-1})$$

and it satisfies $c_*(A) = A$.

Remark 1.1.2. $c_*\colon \mathrm{HF}^*(\tau_S^{-1}) \longrightarrow \mathrm{HF}^*(\tau_S^{-1})$ is an involution of a vector space over a field with characteristic 2. Any such map has a fixed point because $(c_* - \mathrm{id})^2 = 0$, hence $\ker(c_* - \mathrm{id}) \neq 0$. The relevance of the second part of Theorem 1.1.1 is therefore not merely the existence of a fixed point. It should rather be understood as a special property of the element A.

It turns out that Theorem 1.1.1 is a special case of a more general result. Let (M, ω) be a symplectic manifold (not necessarily a real fiber of a real Lefschetz fibration) and $\iota\colon S^n \xrightarrow{\approx} S \subset M$ a parametrised Lagrangian sphere. Consider the reflection

$$r(x_1, x_2, \dots, x_{n+1}) = (-x_1, x_2, \dots, x_{n+1})$$

on the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$. Let $g\colon M \longrightarrow M$ be an anti-symplectic map satisfying

$$\begin{aligned} &g(S) = S, \\ &\iota^* g := \iota^{-1} g \iota \text{ is either diffeotopic to } \mathrm{id}_{S^n} \text{ or to } r. \end{aligned} \tag{$\star$}$$

The assumption on the mapping class of $\iota^* g$ is automatically satisfied when M has dimension $2, 4$ or 6.

Theorem 1.1.3. *g induces an automorphism*

$$g_*\colon \mathrm{HF}^*(\tau_S^{-1}) \longrightarrow \mathrm{HF}^*(\tau_S^{-1})$$

and it satisfies $g_*(A) = A$

The anti-symplectic involution c considered in Theorem 1.1.1 satisfies assumption ($\star$). We prove this in Lemma 1.2.6. In particular, Theorem 1.1.1 follows from Theorem 1.1.3.

1.1.1 Outline of the proof. We outline the proof of Theorem 1.1.3. The following Proposition is inspired from [Sal10] in the case of real Lefschetz fibrations. She proves that the monodromy splits into a product of two anti-symplectic involutions. It follows that the monodromy is conjugate to its inverse via the anti-symplectic involution. An analogous property is true for any anti-symplectic map satisfying $(\star)$:

Proposition 1.1.4. *Let $g\colon M \longrightarrow M$ be an anti-symplectic map satisfying $(\star)$. Then $g\tau_S g^{-1}$ is Hamiltonian isotopic to τ_S^{-1}. In fact, there exists a Hamiltonian isotopy $\{\psi_t\}_{t\in[0,1]}$ such that $g' := g\psi_1$ is an anti-symplectic map satisfying $g'\tau_S(g')^{-1} = \tau_S^{-1}$ for some representant τ_S of the Dehn twist.*

Floer-theoretic considerations and Proposition 1.1.4 yield a homomorphism

$$g_*\colon \mathrm{HF}^*(\tau_S^{-1}) \longrightarrow \mathrm{HF}^*(g\tau_S g^{-1}) \cong \mathrm{HF}^*(\tau_S^{-1}). \tag{1.2}$$

This is the automorphism on $\mathrm{HF}^*(\tau_S^{-1})$ induced by g as stated in the first part of Theorem 1.1.3.

To show that $g_*(A) = A$, the second part of Proposition 1.1.4 allows us to assume that $g\tau_S g^{-1} = \tau_S^{-1}$. The rest of the proof is based on the framework of Biran–Cornea [BC13], [BC14], [BC17] and Mak–Wu [MW18] about Lagrangian cobordisms.

Let M^- be the symplectic manifold $(M, -\omega)$. We denote by $\Gamma_\phi \subset M \times M^-$ the graph of ϕ for a symplectomorphism ϕ on M. This is a Lagrangian submanifold of $M \times M^-$. For $\phi = \mathrm{id}$ it is the diagonal and we write $\Delta := \Gamma_{\mathrm{id}}$. Mak–Wu construct a Lagrangian cobordism $V_{MW} \subset M \times M^- \times \mathbb{C}$ that has three ends: $S \times S, \Delta$ and $\Gamma_{\tau_S^{-1}}$. By general results on Lagrangian cobordisms due to Biran–Cornea this cobordism induces an exact triangle in $D\mathcal{F}uk(M \times M^-)$:

$$\begin{tikzcd}
S \times S \arrow[dr] & \\
& \Delta \arrow[dl] \\
\Gamma_{\tau_S^{-1}} \arrow[uu, "{[1]}"] &
\end{tikzcd}$$

The associated long exact sequence is

$$\begin{aligned}
\cdots \to \mathrm{HF}^k(K, S \times S) \to \mathrm{HF}^k(K, \Delta) \to \\
\to \mathrm{HF}^k(K, \Gamma_{\tau_S^{-1}}) \to \mathrm{HF}^{k+1}(K, S \times S) \to \dots,
\end{aligned} \tag{1.3}$$

where K is an admissible Lagrangian submanifold in $M \times M^-$. For the special case $K = Q \times N$, this sequence reduces to Seidel's long exact sequence (1.1). The middle map in sequence (1.3) can be understood as $\mu^2(\alpha_{V_{MW}}, -)$ for an element $\alpha_{V_{MW}} \in \mathrm{HF}^0(\Delta, \Gamma_{\tau_S^{-1}})$. The previously mentioned element $A \in \mathrm{HF}^0(\tau_S^{-1})$ corresponds to $\alpha_{V_{MW}}$ under the isomorphism

$$\mathrm{HF}^0(\Delta, \Gamma_{\tau_S^{-1}}) \cong \mathrm{HF}^0(\tau_S^{-1}). \tag{1.4}$$

Consider the symplectomorphism

$$\begin{aligned}
\Phi^g \colon M \times M^- &\longrightarrow M \times M^-, \\
(x, y) &\longmapsto (g(y), g(x)).
\end{aligned}$$

The property $g\tau_S g^{-1} = \tau_S^{-1}$ implies that Φ^g preserves $\Gamma_{\tau_S^{-1}}$. In fact, Φ^g preserves all the ends of the cobordism V_{MW}. The cobordism $\widetilde{V}_{MW} := (\Phi^g \times \mathrm{id})(V_{MW})$ has therefore the same ends as V_{MW}. The element $\alpha_{\widetilde{V}_{MW}} \in \mathrm{HF}^0(\Delta, \Gamma_{\tau_S^{-1}})$ associated to $\widetilde{V}_{MW}$ is related to $\alpha_{V_{MW}}$ via

$$\alpha_{\widetilde{V}_{MW}} = \Phi^g_*(\alpha_{V_{MW}}),$$

where Φ^g_* is the automorphism

$$\Phi^g_* \colon \mathrm{HF}^*(\Delta, \Gamma_{\tau_S^{-1}}) \longrightarrow \mathrm{HF}^*(\Phi^g(\Delta), \Phi^g(\Gamma_{\tau_S^{-1}})) = \mathrm{HF}^*(\Delta, \Gamma_{\tau_S^{-1}})$$

induced by Φ^g. This automorphism corresponds to the map g_* on $\mathrm{HF}^*(\tau_S^{-1})$, in the sense that the following diagram commutes

$$\begin{tikzcd} \mathrm{HF}^*(\tau_S^{-1}) \arrow[r, "g_*"] \arrow[d, "\cong"] & \mathrm{HF}^*(\tau_S^{-1}) \arrow[d, "\cong"] \\ \mathrm{HF}^*(\Delta, \Gamma_{\tau_S^{-1}}) \arrow[r, "\Phi^g_*"] & \mathrm{HF}^*(\Delta, \Gamma_{\tau_S^{-1}}). \end{tikzcd} \tag{1.5}$$

A major step in the proof is the following

Theorem 1.1.5. *There exists a Hamiltonian isotopy* $\{\psi_t\}_{t\in[0,1]}$ *such that the anti-symplectic map* $g' = g\psi_1$ *satisfies*

$$\left(\Phi^{g'} \times \mathrm{id}\right)(V_{MW}) = V_{MW}.$$

After replacing g by g' from Theorem 1.1.5 we get

$$\alpha_{V_{MW}} = \alpha_{\widetilde{V}_{MW}} = \Phi^g_*(\alpha_{V_{MW}}).$$

Hence $g_*(A) = A$ by commutativity of diagram (1.5). This shows how Theorem 1.1.5 implies Theorem 1.1.1.

Remark 1.1.6. (i) The assumption that (M, ω) is symplectically aspherical is used at two places: First, it guarantees that the group $\mathrm{HF}^*(g\tau_S g^{-1})$ can be canonically identified with $\mathrm{HF}^*(\tau_S^{-1})$ in (1.2), independently of a choice of a Hamiltonian isotopy from $g\tau_S g^{-1}$ to τ_S^{-1}. Secondly, it holds that $g_* = g'_*$, whenever $g' = g\psi_1$ for a Hamiltonian isotopy $\{\psi_t\}$. We explain this further in sections 1.4.2 and 1.4.3.

(ii) The assumption that M is closed is important for our arguments: The version of Floer cohomology we use only works for compactly supported symplectomorphisms. In general however, the monodromy in a Lefschetz fibration with non-compact fibers, if it exists, is not compactly supported. We expect that the results generalise to a non-compact framework, when working with an appropriate version of Floer theory.

(iii) If $g(S) = S$ for an anti-symplectic map g, we have a Hamiltonian isotopy

$$g \circ \tau_{(S,\iota)} \circ g^{-1} \simeq \tau^{-1}_{(S,g\circ\iota)}.$$

However, it is unknown how the Dehn twist depends on the parametrisation of the sphere. Only if $\iota^* g$ is in the mapping class of an isometry, it is known that the Dehn twist associated to $g \circ \iota$ is Hamiltonian isotopic to the Dehn twist associated to ι [Sei97a, Remark 3.1]. This explains why we impose the condition $(\star)$ on the mapping class of $\iota^* g$.

(iv) The second map in the long exact sequence (1.1) is

$$\mu^2(a_N, -)\colon \mathrm{HF}^k(Q, N) \to \mathrm{HF}^k(Q, \tau_S(N))$$

for some element $a_N \in \mathrm{HF}^0(N, \tau_S(N))$. This element is related to $A \in \mathrm{HF}^*(\tau_S^{-1})$ as follows. There is an operation

$$*\colon \mathrm{HF}^*(\tau_S^{-1}) \otimes \mathrm{HF}^*(N, N) \to \mathrm{HF}^*(N, \tau_S(N)).$$

If $e_N \in HF^0(N, N)$ denotes the unit, we have $A * e_N = a_N$. Assume the setting of Theorem 1.1.1. The fixed point property $c_*(A) = A$ then implies

$$\gamma(a_N) = a_{c(N)}, \tag{1.6}$$

where γ is the isomorphism

$$\mathrm{HF}^*(N, \tau_S(N)) \cong \mathrm{HF}^*(c\tau_S(N), c(N)) \cong \mathrm{HF}^*(c(N), \tau_S(c(N))).$$

(The first isomorphism is induced by c and the second isomorphism is induced by τ_S.) The construction of a_N is explained in [Sei08a, Sections 17a-17c]. a_N comes from counting the number of holomorphic sections of a Lefschetz fibration with moving boundary condition coming from moving N via parallel transport. The invariance property (1.6) can be proven directly in Seidel's framework applied to real Lefschetz fibrations, by observing that the holomorphic sections for boundary conditions coming from N and $c(N)$ are in bijection.

1.1.2 Organization of chapter 1. The rest of this chapter is organised as follows. In section 1.2 we recall the definition of a Dehn twist and its interpretation as a monodromy in a Lefschetz fibration. We define real Lefschetz fibrations and explain why Theorem 1.1.1 is a special case of Theorem 1.1.3. Section 1.3 contains a proof of Proposition 1.1.4. In section 1.4 we collect prerequisites from Floer theory. In particular, we explain the automorphism (1.2) related to conjugation invariance, the isomorphism (1.4) between Lagrangian and absolute Floer cohomology and the diagram (1.5). In section 1.5 we recall Biran–Cornea's Lagrangian cobordism framework and how cobordisms induce cone decompositions. The construction of the Mak–Wu cobordism is reviewed in section 1.6. In section 1.7 we prove Theorem 1.1.5 about the symmetry of the Mak–Wu cobordism. Section 1.8 discusses some two dimensional examples. The appendix contains some algebraic background on Fukaya categories that is relevant for Biran–Cornea's cobordism theory.

1.2 Dehn twist and Lefschetz fibrations

There are two points of view for the Dehn twist. One way is to first define a model Dehn twist in T^*S^n and then glue it into a Weinstein neighbourhood of S. This point of view is useful for the Mak–Wu framework because their flow surgery can easily be compared to this version of the Dehn twist.

The other point of view occurs in the framework of Lefschetz fibrations. Given a Lefschetz fibration with one critical point and vanishing sphere S, the monodromy around the singularity turns out to be the Dehn twist along S. This setting can be enhanced with real structures, allowing to apply Salepic's results [Sal10] on monodromies in real Lefschetz fibrations to the Dehn twist. This motivates Proposition 1.1.4, which plays an important role for the proof of Theorem 1.1.1.

In this section we review both, the model Dehn twist, as well as the monodromy viewpoint on Dehn twists. This is due to Seidel [Sei97a, Sei03].

1.2.1 The Dehn twist glued from a local model. Let (M, ω) be a closed symplectic manifold of dimension $2n$, $n \geq 1$. Let $S \subset M$ be a Lagrangian sphere together with an embedding $\iota\colon S^n \longrightarrow M$ of the n-dimensional standard sphere $S^n \subset \mathbb{R}^{n+1}$ with image $\iota(S^n) = S$. We refer to (S, ι) as a parametrised Lagrangian sphere.[1] The Dehn twist $\tau_{(S,\iota)}$ will be associated to (S, ι). Usually we only write τ_S and omit ι from the notation. However, as we explain further below, it is sometimes important to keep track of ι.

We now recall the construction of the Dehn twist τ_S along S, following closely the exposition in [MW18].

Definition 1.2.1. Let $\epsilon > 0$. A *Dehn twist profile function* is a smooth function

$$\nu_\epsilon^{\mathrm{Dehn}}\colon \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$$

satisfying

$$\begin{cases} \nu_\epsilon^{\mathrm{Dehn}}(r) = \pi - r & \text{for } 0 \leq r << \epsilon, \\ 0 < \nu_\epsilon^{\mathrm{Dehn}}(r) < \pi \text{ and strictly decreasing} & \text{for } 0 < r < \epsilon, \\ \nu_\epsilon^{\mathrm{Dehn}}(r) = 0 & \text{for } r \geq \epsilon. \end{cases}$$

Consider the canonical Riemannian metric on S^n and the canonical isomorphism $T_*S^n \cong T^*S^n$. Denote by $\|\xi\|$ the norm of the tangent vector identified with $\xi \in T^*S$. Let

$$T_r^*S^n = \{\xi \in T^*S^n \mid \|\xi\| < r\}$$

be the open subset of T^*S^n consisting of cotangent vectors of norm strictly less than r. We endow T^*S^n with the symplectic structure given by $\mathrm{d}p \wedge \mathrm{d}q$ in local coordinates (q, p), $q \in S^n, p \in T_qS^n$.

Let $V \subset M$ be a Weinstein neighbourhood of S together with a symplectic embedding

$$\varphi\colon V \longrightarrow T^*S^n$$

that identifies $S \subset V$ with the zero-section $S^n \subset T^*S^n$ via ι^{-1} and $\varphi(V) = T_\epsilon^*S^n$ for some $\epsilon > 0$.

[1]Seidel uses the word "framed sphere" for this situation in [Sei08a].

Consider the function

$$\begin{aligned} \mu\colon T^*S^n &\longrightarrow \mathbb{R}, \\ \xi &\longmapsto \|\xi\|. \end{aligned}$$

While μ is not smooth on the zero-section S^n, it does have a well-defined Hamiltonian flow

$$\psi_t^\mu\colon\ (T^*S^n)\setminus S^n \longrightarrow (T^*S^n)\setminus S^n.$$

Definition 1.2.2. The *model Dehn twist* on T^*S^n is defined by

$$\begin{aligned} \tau_{S^n}: T^*S^n &\longrightarrow T^*S^n, \\ \xi &\longmapsto \begin{cases} \psi^\mu_{\nu_\epsilon^{\mathrm{Dehn}}(\mu(\xi))}(\xi) & \text{for } \xi \notin S^n, \\ -q & \text{for } \xi = q \in S^n. \end{cases} \end{aligned}$$

The *Dehn twist* in M along S is then given by copying the model Dehn twist into M via the Weinstein embedding φ:

$$\tau_S = \begin{cases} \varphi^{-1}\circ\tau_{S^n}\circ\varphi & \text{on } V, \\ \mathrm{id} & \text{on } M\setminus V. \end{cases}$$

The resulting diffeomorphism τ_S is a symplectomorphism and its support is contained in $\overline{V}$. Figure 1.1 illustrates the Dehn twist in dimension 2.

The precise map depends on the Dehn twist profile function $\nu_\epsilon^{\mathrm{Dehn}}$ and on the Weinstein neighbourhood (V,φ) of (S,ι). However, different choices of $\nu_\epsilon^{\mathrm{Dehn}}$ and (V,φ) lead to Hamiltonian isotopic symplectomorphisms [Sei03],[Sei97a, Proposition 2.3]. The dependence on the parametrisation ι is unknown. It is only known that if $\sigma\in\mathrm{Diff}(S^n)$ is diffeotopic to an isometry on S^n, then $\tau_{(S,\iota\circ\sigma)}$ is Hamiltonian isotopic to $\tau_{(S,\iota)}$. Since the mapping class group of S^n is trivial for $n=1,2$ and 3, the Hamiltonian isotopy class of τ_S is independent of ι in dimensions $2,4$ and 6 [Sei08a, Remarks 16.1 and 16.6].

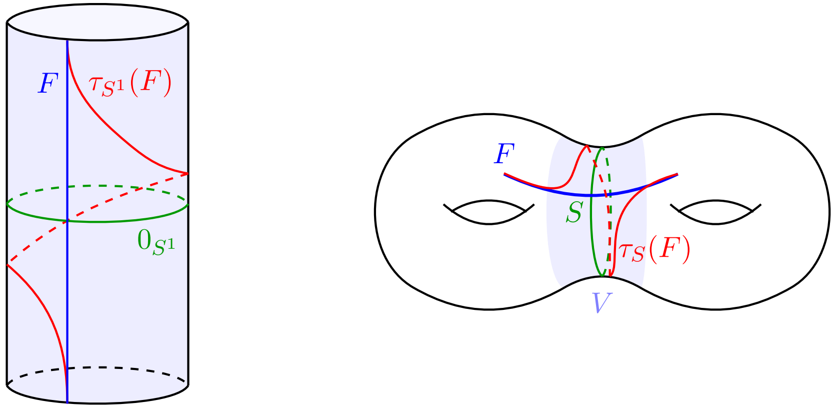


Figure 1.1: The model Dehn twist τ_{S^1} on the left and its gluing into a Weinstein neighbourhood V of the Lagrangian sphere S in a genus 2 surface.

1.2.2 Lefschetz fibrations. For a detailed treatment of Lefschetz fibrations we refer the reader to [Sei08a, BC17, Kea14]. We adopt the definition of a Lefschetz fibration used in [BC17]. Let $n \geq 1$.

Definition 1.2.3. A *Lefschetz fibration* with base $\mathbb{C}$ consists of

- a $2n+2-$dimensional symplectic manifold (E, Ω_E) without boundary endowed with a compatible almost complex structure J_E,
- a proper (J_E, i)-holomorphic map $\pi \colon E \longrightarrow \mathbb{C}$,

such that

(i) π has only finitely many critical points and all critical values are distinct,

(ii) all the critical points of π are ordinary double points: for every critical point $p \in E$, there exist J_E-holomorphic coordinates $(z_1, \ldots, z_{n+1})$ in a neighbourhood U_p of p such that

$\pi(z_1, \dots, z_{n+1}) = \pi(p) + z_1^2 + \cdots + z_{n+1}^2$ holds on U_p. (In particular, J_E is integrable on U_p.)

For $p \in \mathbb{C}$ we denote by $E_p := \pi^{-1}(\{p\})$ the fiber above p. Every regular fiber of π is a closed $2n$-dimensional symplectic manifold whose symplectic form is induced from Ω_E. We call the critical points $p_1, \dots, p_l$ and the corresponding critical values $s_1, \dots, s_l$. The restricted map

$$\pi \colon E \backslash \{p_1, \dots, p_l\} \longrightarrow \mathbb{C} \backslash \{s_1, \dots, s_l\}$$

is a (locally trivial) symplectic fibration. For background material on symplectic fibrations we refer the reader to [MS95, Section 6]. Ω_E determines a connection as follows. Let $x \in E$. Denote by V_x the vertical subspace $\ker \mathrm{d}\pi_x \subset T_x M$. The connection is then given by the horizontal subspaces

$$\mathrm{Hor}_x = \{\xi \in T_x E \mid \forall \eta \in \mathrm{Vert}_x \colon \Omega_E(\xi, \eta) = 0\}.$$

Let $z_0 \in \mathbb{C} \backslash \{s_1, \dots, s_l\}$ be any regular value. We identify the fiber E_{z_0} with a symplectic manifold (M, ω). For any smooth path

$$\lambda \colon [0,1] \longrightarrow \mathbb{C} \backslash \{s_1, \dots, s_l\}$$

with $\lambda(0) = \lambda(1) = z_0$, let $h_\lambda \colon M \longrightarrow M$ be the monodromy map coming from parallel transport along λ. Then h_λ is a symplectomorphism of M.

Consider a smooth path $\gamma \colon [0,1] \longrightarrow \mathbb{C}$ starting at a critical value $\gamma(0) = s_i$, ending at $\gamma(1) = z_0$ and satisfying $\gamma(t) \in \mathbb{C} \backslash \{s_1, \dots, s_l\}$ for $t \neq 0$. To simplify notation, assume that $p_i = 0$ and $\gamma(t) = t$ for small enough t. Consider local holomorphic coordinates $(z_1, \dots, z_{n+1})$ near p_i as in the definition. Then for small enough t,

$$S(t) := \{(x_1, \dots, x_{n+1}) \mid x_i \in \mathbb{R}, x_1^2 + \cdots + x_{n+1}^2 = t\}$$

is a n-dimensional sphere contained in the neighbourhood U_{p_i} of p_i. Applying parallel transport to $S(t_0)$ for small enough $t_0 \neq 0$ we obtain a family $S(t) \subset E_{\gamma(t)}$. $S(1) \subset M$ is called the *vanishing sphere* associated to γ and

$$T_\gamma := \bigcup_{t\in[0,1]} S(t)$$

the *Lefschetz thimble* of p_i along γ. $S(1)$ is a Lagrangian sphere in (M, ω) and T_γ is a Lagrangian disc in (E, Ω_E). Moreover, $S(1)$ comes with a canonical parametrisation, obtained from parallel transport of the standard sphere $S(t_0)$ to $S(1)$.

1.2.3 The Dehn twist as a monodromy. Suppose (M, ω) is the fiber of a Lefschetz fibration $\pi\colon E \longrightarrow \mathbb{C}$ as in the previous section. Let γ be as above a path connecting s_i to z_0 and $S \subset M$ the associated vanishing cycle. Suppose λ is a loop with base point z_0 that goes once around s_i and around no other critical value. The situation is shown in Figure 1.2. Consider the monodromy h_λ.

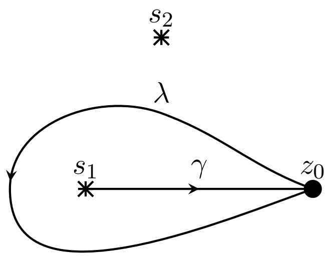


Figure 1.2: The base of the fibration and paths γ and λ.

Theorem 1.2.4 (Symplectic Picard-Lefschetz Theorem [Sei08a, Section (16c)])**.** *The symplectomorphism h_λ is symplectically isotopic to the Dehn twist τ_S.*

Conversely, any Dehn twist can be realized as a monodromy in a Lefschetz fibration [Sei97a, Section 19],[Sei08a, Section (16e)]. More precisely, given a symplectic manifold (M,ω) and a parametrised Lagrangian sphere $S \subset M$, there exists a Lefschetz fibration $\pi\colon E \longrightarrow \mathbb{C}$ satisfying the following properties:

- $M = E_1$.
- There is exactly one critical point $p \in E$ and $\pi(p) = 0$.
- The vanishing sphere associated to $\gamma(t) = t$ is $S \subset M$.
- The monodromy along $\lambda(t) = e^{2\pi i t}$ is symplectically isotopic to τ_S.

We refer the reader to [Sei03, Proposition 1.11] for a full proof of this. We include here an outline of the construction following closely [Sei03].

We first construct a model Lefschetz fibration $\pi^0\colon E^0_\epsilon \longrightarrow \mathbb{C}$ whose smooth fiber is $T^*_\epsilon S^n$. Equip $\mathbb{C}^{n+1}$ with the standard symplectic form Ω_0 and standard complex structure J_0. We use the models

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{C}^{n+1}$$

and

$$T^*S^n = \{(u,v) \in S^n \times \mathbb{R}^{n+1} \mid \langle u, v\rangle = 0\}.$$

Consider the holomorphic map

$$\begin{aligned} Q\colon \mathbb{C}^{n+1} &\longrightarrow \mathbb{C}, \\ (z_1, \ldots, z_{n+1}) &\longmapsto z_1^2 + \cdots + z_{n+1}^2. \end{aligned}$$

The only critical point of Q is $0 \in \mathbb{C}^{n+1}$ and it has critical value 0. Clearly, conditions (i) and (ii) in Definition 1.2.3 are satisfied. The smooth fibers are symplectomorphic to T^*S^n. For example, the fiber of $z_0 = 1$ can be identified with T^*S^n via the symplectomorphism

$$\begin{aligned} \Phi_1\colon Q^{-1}(\{1\}) &\longrightarrow T^*S^n \\ z &\longmapsto \left(\frac{\mathrm{re}(z)}{\|\mathrm{re}(z)\|}, -\,\mathrm{im}(z)\|\mathrm{re}(z)\| \right). \end{aligned} \tag{1.7}$$

In particular, the smooth fibers are not compact and therefore Q is not proper. However, parallel transport is still well-defined and everything from the last section applies.

Consider the path $\gamma(t) = t$ from the critical value 0 to the base point 1. Consider the family of Lagrangian spheres

$$S(t) = \sqrt{t}S^n = \{(\sqrt{t}z_1, \ldots, \sqrt{t}z_{n+1}) \mid z \in S^n \subset \mathbb{R}^{n+1}\} \subset Q^{-1}(\{t\})$$

for $t \in [0,1]$. These are the vanishing cycles for Q. The corresponding Lefschetz thimble is

$$T_\gamma = \bigcup_{t>0} S(t) \cup \{0\}.$$

Let λ be the path $\lambda(t) = e^{2\pi it}$ that parametrises the unit circle. Consider the monodromy h_λ along λ. Then

$$\Phi_1 \circ h_\lambda \circ \Phi_1^{-1} \colon T^*S^n \longrightarrow T^*S^n$$

is given by

$$(u,v) \longmapsto \begin{cases} \psi^\mu_{\nu(\|v\|)}(u,v) & \text{for } v \neq 0, \\ -u & \text{for } v = 0, \end{cases}$$

where ν is a function similar to $\nu_\epsilon^{\text{Dehn}}$ except that ν is not compactly supported but only satisfies $\lim_{r\to\infty} \nu(r) = 0$.

By changing the symplectic structure Ω_0 and the complex structure J_0 on $\mathbb{C}^{n+1}$ one can arrange that the monodromy becomes a model Dehn twist supported in $T^*_\epsilon S^n$. By restricting the resulting fibration to

$$E^0_\epsilon := \left\{ z \in \mathbb{C}^{n+1} \;\middle|\; \frac{|z|^4 - |Q(z)|^2}{4} < \epsilon^2 \right\}$$

we get a Lefschetz fibration $\pi^0 \colon E^0_\epsilon \longrightarrow \mathbb{C}$, whose fiber is $T^*_\epsilon S^n$. The map π^0 is still not proper, but it is trivial at infinity: There exist neighbourhoods $W_\infty \subset E^0_\epsilon$ of ∂E^0_ϵ and $T_\infty \subset T^*_\epsilon S^n$ of $\partial T^*_\epsilon S^n$ such that there is a trivialisation

$$\Phi \colon W_\infty \longrightarrow \mathbb{C} \times T_\infty.$$

As a next step we use the model fibration E^0_ϵ to construct the Lefschetz fibration $\pi\colon E \longrightarrow \mathbb{C}$ satisfying all the properties in the above list. Take a Weinstein neighbourhood $\varphi\colon V \xrightarrow{\cong} T^*_\epsilon S^n$ of S. We set $V_\infty := \varphi^{-1}(W_\infty) \subset V$. Define

$$E := E^0_\epsilon \cup_\sim (\mathbb{C} \times M \backslash (V \backslash V_\infty)),$$

where $\sim$ identifies $w \in W_\infty \subset E^0_\epsilon$ with $(z,x) \in \mathbb{C} \times V_\infty \subset M\backslash(V\backslash V_\infty)$ via $w = \Phi^{-1}(z, \varphi(x))$. π is defined by π^0 on E^0_ϵ and $\pi_{\mathbb{C}}$ on the trivial part. Similarly Ω_E and J_E are obtained by pulling back $\Omega_{E^0_\epsilon}$ and $J_{E^0_\epsilon}$ from E^0_ϵ and the standard structure from the trivial part.

1.2.4 Real Lefschetz fibrations.

A *real structure* on a symplectic manifold (M, ω) is an anti-symplectic involution $c\colon M \longrightarrow M$.

Definition 1.2.5. A *real Lefschetz fibration* is a Lefschetz fibration $\pi\colon E \longrightarrow \mathbb{C}$ together with an anti-symplectic involution $c_E\colon E \longrightarrow E$ on the total space E that covers complex conjugation $c_{\mathbb{C}}\colon \mathbb{C} \longrightarrow \mathbb{C}$. By this we mean that the diagram

$$\begin{tikzcd} E \arrow[r, "c_E"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \mathbb{C} \arrow[r, "c_{\mathbb{C}}"] & \mathbb{C} \end{tikzcd} \tag{1.8}$$

commutes. c_E is called a *real structure* on the Lefschetz fibration E.

Real Lefschetz fibrations have been studied by [Sal10, Sal12] in the smooth category. In the symplectic context they appeared in [BC17]. Real Lefschetz fibrations occur naturally in the context of real algebraic surfaces in complex projective space. We refer the reader to [BC17, Section 6] for examples of real Lefschetz fibrations.

Assume that $p \in E$ is the unique critical point of π and $\pi(p) = 0$. Let $M = E_1$ be the fiber over 1 and let $S \subset M$ be the vanishing sphere associated to the path $\gamma(t) = t$. We prove that Theorem 1.1.1 is a special case of Theorem 1.1.3.

Lemma 1.2.6. *The real structure $c = c_E|_M$ satisfies $c(S) = S$ and $\iota^* c$ is diffeotopic to id_{S^n} or to r, where ι is the canonical framing of the vanishing sphere S. In other words, c satisfies ($\star$).*

Proof. First we note that c_E commutes with parallel transport: For a smooth curve $\gamma\colon [0,1] \to \mathbb{C}\backslash\{0\}$ we denote by $\overline{\gamma} = c_{\mathbb{C}} \circ \gamma$ its complex conjugation. We claim that

$$c_E \circ P_\gamma = P_{\overline{\gamma}} \circ c_E.$$

Indeed, let $v \in E_{\gamma(0)}$. Consider the parallel lift $v(t) \in E_{\gamma(t)}$ of γ starting with $v(0) = v$. Since $Dc_E(H_{v(t)}) = H_{c_E(v(t)}$ we see that $w(t) := c_E(v(t))$ is the parallel lift of $\overline{\gamma}$ starting at $c_E(v)$. Therefore

$$c_E \circ P_\gamma(v) = c_E(v(1)) = w(1) = P_{\overline{\gamma}}(c_E(v)),$$

which proves the claim. Hence $c(S) = S$ follows from $c_E(0) = 0$. For the second part, note that it is enough to consider the model $Q\colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$. In that case, $S = S^n \subset \mathbb{C}^{n+1}$ is a standard sphere. Note that c_E restricted to the thimble $T_\gamma = B^{n+1}(0)$ is a smooth extension of the sphere $c|_{S^n}$ to the ball. Moreover, since parallel transport commutes with c_E, it is a linear extension, in the sense that

$$c_E(x) = c\left(\frac{x}{\|x\|}\right)\|x\|,$$

for $x \in \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$. It follows that $c_E|_{\mathbb{R}^{n+1}}$ is an orthogonal linear transformation and hence $c|_{S^n}$ is an isometry. In particular, $c|_{S^n}$ is diffeotopic to id_{S^n} or to r. □

The following result is due to [Sal10] in the smooth category. Here we adapt it to the symplectic framework.

Lemma 1.2.7. *Let $\tau\colon M \longrightarrow M$ be the monodromy along the boundary loop $\gamma(t) = e^{2\pi i t}, t \in [0,1]$. τ splits into a product of two anti-symplectic involutions on M. More concretely, $\tau = c \circ c_-$ for $c = c_E|_M$ and another anti-symplectic involution $c_-\colon M \longrightarrow M$. In particular, $c\tau c = \tau^{-1}$.*

Proof. Denote by
$$P_{\gamma(s);t} \colon E_{\gamma(s)} \to E_{\gamma(s+t)}$$
the parallel transport for time t along γ starting at $\gamma(s)$. As we have seen in the proof of Lemma 1.2.6, parallel transport commutes with c. Therefore,
$$\left(P_{1;\frac{1}{2}}\right)^{-1} \circ c_E = c_E \circ P_{-1;\frac{1}{2}}.$$
It follows
$$\tau = P_{-1;\frac{1}{2}} \circ P_{1;\frac{1}{2}} = c_E \circ (P_{1;\frac{1}{2}})^{-1} \circ c_E \circ P_{1;\frac{1}{2}} = c \circ c_-,$$
where $c = c_E|_M$ and $c_- = (P_{1;\frac{1}{2}})^{-1} \circ c_E \circ P_{1;\frac{1}{2}}$. $\square$

1.3 Anti-symplectic maps preserving a Lagrangian sphere

In this section we study anti-symplectic maps $g \colon M \longrightarrow M$ that preserve a Lagrangian sphere $S \subset M$. The goal is to prove Proposition 1.1.4. We first study the local behavior of g near S.

Any diffeomorphism $\sigma \colon S^n \longrightarrow S^n$ induces an anti-symplectic map
$$\begin{aligned} T^*\sigma \colon T^*S^n &\longrightarrow T^*S^n \\ (u,v) &\longmapsto (\sigma(u), -D\sigma_u(v)) \,. \end{aligned}$$
In fact, $(T^*\sigma)^*\theta = -\theta$ for the canonical 1-form θ on T^*S^n. The following result shows that any anti-symplectic map on M that preserves S can be modelled in a Weinstein neighbourhood of S by a map of the form $T^*\sigma$.

Proposition 1.3.1. *Let $\iota \colon S^n \longrightarrow M$ be a parametrisation of a Lagrangian sphere $S = \iota(S^n)$. Let $g \colon M \longrightarrow M$ be an anti-symplectic map satisfying $g(S) = S$. Then for any Weinstein neighbourhood $V \subset M$ of S with an embedding*
$$\varphi \colon V \xrightarrow{\cong} T^*_\epsilon S^n \subset T^*S^n,$$

extending ι, there exists a Hamiltonian isotopy $\psi_t\colon M \longrightarrow M$ supported in V with $\psi_0 = \mathrm{id}$ such that $g' := g \circ \psi_1$ satisfies the following property: There exists $0 < \delta < \epsilon$ such that

(a) $g'\left(\varphi^{-1}\left(T_\delta S^n\right)\right) \subseteq V$. In particular, g' induces a map

$$g'_T\colon T^*_\delta S^n \longrightarrow T^*_\epsilon S^n$$

defined via $g'_T := \varphi \circ g' \circ \varphi^{-1}$.

(b) g'_T is equal to $T^*(\iota^* g)$ on $T^*_\delta S^n$.

To prove this result we first consider the model T^*S^n endowed with an anti-symplectic map g preserving the zero-section: $g(S^n) = S^n$. The following result is a model version of Proposition 1.3.1.

Proposition 1.3.2. *Let $g\colon T^*S^n \longrightarrow T^*S^n$ be an anti-symplectic map restricting to $\sigma\colon S^n \longrightarrow S^n$. Then for every $\eta_2 > 0$ there exists $0 < \eta_1 < \eta_2$ and a Hamiltonian isotopy*

$$\psi^H_t : T^*S^n \longrightarrow T^*S^n$$

with $\operatorname{supp} H \subset T^*_{\eta_2} S^n$ such that

$$g\psi^H_1 = T^*\sigma \quad \textit{on} \quad T^*_{\eta_1} S^n.$$

Proof. Consider the symplectomorphism $\psi := g^{-1} \circ T^*\sigma$. Write in local coordinates $\psi(q,p) = (u(q,p), v(q,p))$ with $u(q,p) \in S^n$ and $v(q,p) \in T_{u(q,p)}S^n$. Since $g = T^*\sigma$ on S^n, we have $u(q,0) = q$. Consider the following isotopy of symplectomorphisms $\psi_t\colon T^*S^n \longrightarrow T^*S^n$ between $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$:

$$\psi_t(q,p) = \begin{cases} (u(q,tp), \frac{v(q,tp)}{t}) & t \neq 0, \\ (u(q,0), (\partial_p v(q,0))p) & t = 0. \end{cases}$$

We show that $\psi_0 = \mathrm{id}$. Write $D\psi_{(q,0)}$ in local coordinates

$$D\psi_{(q,0)} = \begin{pmatrix} \mathrm{id} & \partial_q v(q,0) \\ \partial_p u(q,0) & \partial_p v(q,0) \end{pmatrix}.$$

Using that $D\psi_{(q,0)}$ is a symplectic matrix, we get

$$\partial_p v(q,0) = \mathrm{id}.$$

In particular, $\psi_0(q,p) = (q,p)$ as claimed. Moreover, ψ_t is a Hamiltonian isotopy: For $n \geq 2$ this is automatic. For $n = 1$ it follows from $\psi_t(S^1) = S^1$ [MS95, Theorem 10.2.5, Exercise 10.2.6]. Finally cut off the Hamiltonian so that the resulting Hamiltonian H has support in $T^*_{\eta_2}S^n$. Then $\psi_1^H = \psi = g^{-1} \circ T^*\sigma$ on $T^*_{\eta_1}S^n$ for η_1 small enough. In particular, $g\psi_1^H = T^*\sigma$ on $T^*_{\eta_1}S^n$. □

This model case leads to a proof of Proposition 1.3.1.

Proof of Proposition 1.3.1. Let $\varphi\colon V \xrightarrow{\cong} T^*_\epsilon S^n$ be a Weinstein embedding extending ι. Choose a number $0 < \eta_2 < \epsilon$ such that $g\left(\varphi^{-1}\left(T_{\eta_2}S^n\right)\right) \subseteq V$. Then g induces a map $g_T\colon T^*_{\eta_2}S^n \longrightarrow T^*_\epsilon S^n$ defined via $g_T := \varphi \circ g \circ \varphi^{-1}$. By Proposition 1.3.2 there exists a Hamiltonian isotopy $\psi_t^H\colon T^*S^n \longrightarrow T^*S^n$ with compact support contained in $T^*_{\eta_2}S^n$ such that

$$g_T\psi_1^H = T^*(\iota^* g) \;\text{ on }\; T^*_{\eta_1}S^n$$

for η_1 small enough. (Note that the proof of Proposition 1.3.2 carries over to g_T even though it's only defined on $T^*_{\eta_2}S^n$.) Take $\delta = \eta_1$ and let ψ_t^K be the Hamiltonian isotopy on M obtained by extending $\varphi^{-1} \circ \psi_t^H \circ \varphi$ on V by the identity to all of M. Put $g' = g\psi_1^K$. Then $g'_T = g_T\psi_1^H$ and Proposition 1.3.1 is satisfied. □

Recall the assumption $(\star)$ for Theorem 1.1.3 about the mapping class of the restriction of g to S^n. We only allow two mapping classes: the identity id_{S^n} and the reflection $r\colon S^n \longrightarrow S^n$ given by

$$r(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1}).$$

Proposition 1.1.4 states that then $g\tau_S g^{-1} = \tau_S^{-1}$. Before turning to a proof of this we need the following observation.

Lemma 1.3.3. *Let $\sigma_0, \sigma_1 \colon S^n \longrightarrow S^n$ be two diffeotopic smooth maps. Then there exists a Hamiltonian isotopy ψ_t^H on $T^*S^n \longrightarrow T^*S^n$ such that*

$$(T^*\sigma_0) \circ \psi_1^H = T^*\sigma_2.$$

Proof. Let σ_t be a diffeotopy between σ_0 and σ_1. Consider the smooth isotopy $T^*\sigma_t$ of anti-symplectic maps. We claim that the symplectic isotopy

$$\psi_t := (T^*\sigma_0)^{-1} \circ (T^*\sigma_t)$$

is Hamiltonian. Clearly $\psi_t^*\theta = \theta$. Let X_t be the symplectic vector field generated by the flow ψ_t. Then the Hamiltonian $H_t = \theta(X_t)$ generates ψ_t [MS95, Proposition 9.3.1]. □

We are now ready for a proof of Proposition 1.1.4.

Proof of Proposition 1.1.4. First consider the model T^*S^n endowed with the anti-symplectic involution $c = T^*\sigma$ for $\sigma = \mathrm{id}_{S^n}$ or $\sigma = r$. We show that

$$c\tau_{S^n}c = \tau_{S^n}^{-1}$$

for a model Dehn twist τ_{S^n}. Let $\xi \in T_x^*S^n$. Then $\|c(\xi)\| = \|\xi\|$. Moreover, if γ is the geodesic in S^n with

$$\gamma(0) = x \text{ and } \gamma'(0) = \xi$$

then $\overline{\gamma} := \sigma \circ \gamma$ is a geodesic satisfying

$$\overline{\gamma}(0) = \sigma(x) \text{ and } \overline{\gamma}'(0) = \mathrm{D}\sigma_x(\xi) = -c(\xi).$$

Therefore,

$$\begin{aligned} c\psi_t^\mu(\xi) = c(\gamma'(t)) &= -D\sigma(\gamma'(t)) \\ &= -\overline{\gamma}'(t) \\ &= -\psi_t^\mu(-c(\xi)) = (\psi_t^\mu)^{-1}(c(\xi)). \end{aligned}$$

We conclude

$$c\psi^\mu_{\nu_\epsilon(\|\xi\|)}(\xi) = \left(\psi^\mu_{\nu_\epsilon(\|c(\xi)\|)}\right)^{-1} c(\xi)$$

and therefore $c\tau_{S^n} = \tau_{S^n}^{-1} c$. This settles the model case.

Now let M, S and g be as in the statement. Choose a Weinstein neighbourhood (V, φ) as before. Let ψ_t be a Hamiltonian isotopy as in Proposition 1.3.1. By Lemma 1.3.3 and assumption $(\star)$ we can change the Hamiltonian isotopy so that $g' = g \circ \psi_1$ satisfies $g'_T = c$ on $T^*_\delta S^n$ for $\delta > 0$ small enough and c as in the above model case. Let τ_S be the Dehn twist obtained from a model Dehn twist supported in $T^*_\delta S^n$ via the Weinstein neighbourhood (V, φ). Then on $\varphi^{-1}(T^*_\delta S^n)$ we have

$$g' \tau_S (g')^{-1} = \varphi^{-1} g'_T \tau_{S^n} (g'_T)^{-1} \varphi = \varphi^{-1} c \tau_{S^n} c \varphi = \varphi^{-1} \tau_{S^n}^{-1} \varphi = \tau_S^{-1}.$$

On $M \backslash \varphi^{-1}(T^*_\delta S^n)$, we have $\tau_S = \tau_S^{-1} = \mathrm{id}$ and hence

$$g' \tau_S (g')^{-1} = \tau_S^{-1}$$

holds everywhere. This proves the second part of Proposition 1.1.4. The first part follows from the second part. □

1.4 Floer cohomology

In this section we explain the Floer cohomology machinery we are using. After outlining the construction, we collect the main properties of Floer cohomology that are relevant in the proof of Theorems 1.1.1 and 1.1.3.

1.4.1 Setting. We assume that M is symplectically aspherical, that is

$$\omega|_{\pi_2(M)} \equiv 0 \quad \text{and} \quad c_1|_{\pi_2(M)} \equiv 0.$$

Moreover, we assume that all involved Lagrangian submanifolds are relatively symplectically aspherical, that is

$$\omega|_{\pi_2(M,L)} \equiv 0$$

for a Lagrangian $L \subset M$. In particular, $S \subset M$ is relatively symplectically aspherical. This is automatic if M is symplectically aspherical, unless S has dimension 1. In the latter case, the condition is equivalent to S being a non-contractible circle. In this situation, Floer cohomology $\mathrm{HF}^*(f)$ for a symplectomorphism $f \in \mathrm{Symp}(M)$, and Lagrangian Floer cohomology $\mathrm{HF}^*(L, K)$ for Lagrangians L, K as above can be defined over the universal Novikov field

$$\Lambda = \left\{ \sum a_k q^{\omega_k} \middle| |a_k \in \mathbb{Z}_2, \omega_k \in \mathbb{R}, \lim_{k \to \infty} \omega_k = \infty \right\}.$$

$\mathrm{HF}^*(f)$ and $\mathrm{HF}^*(L, K)$ are $\mathbb{Z}_2$-graded, whenever L and K are oriented. We outline the definitions of these groups in the next sections.

1.4.2 Floer cohomology for symplectomorphisms. We briefly collect the basic ideas and notation for Floer cohomology of a symplectomorphism following [DS94]. For more detailed expositions, we refer the reader to [DS94] for the monotone case, and to [Sei97b] and [Lee05] for W^+-symplectic manifolds.

Let (M, ω) be a closed symplectically aspherical symplectic manifold. Let $f \in \mathrm{Symp}(M)$ be a symplectomorphism. We first need to choose a Hamiltonian perturbation, namely a family of Hamiltonian functions $\{H_s \colon M \longrightarrow \mathbb{R}\}_{s \in \mathbb{R}}$. It should be f-periodic, in the sense that

$$H_s = H_{s+1} \circ f.$$

Roughly speaking, Floer cohomology of f is Morse cohomology on the twisted loop space

$$\Omega_f := \{x \in C^\infty(\mathbb{R}, M) \mid x(s+1) = f(x(s))\}$$

with the closed 1-form

$$\lambda_H(x)(\xi) = \int_0^1 \omega\left(\dot{x}(s) - X_s^H(x(s)), \xi(s)\right) ds.$$

Here, X_s^H denotes the Hamiltonian vector field of H_s. We write $P_f(H)$ for the set of $x \in \Omega_f$ satisfying $\dot{x}(s) = X_s^H(x(s))$. For a generic choice of H, $P_f(H)$ is a finite set. The vector space underlying the Floer complex is the Λ-vector space generated by $P_f(H)$:

$$\mathrm{CF}^*(f; H) = \bigoplus_{x \in P_f(H)} \Lambda x.$$

$\mathrm{CF}^*(f;H)$ is $\mathbb{Z}/2$-graded as follows. A generator $x \in P_f(H)$ corresponds to a fixed point $x(0)$ of $f_H := (\psi_1^H)^{-1} f$. The degree $\deg(x) \in \mathbb{Z}/2$ of x is related to the index of $x(0)$ by

$$(-1)^{\deg(x)} = \operatorname{sign}\left(\det\left(\mathrm{id} - (Df_H)_{x(0)}\right)\right).$$

To define the differential on $\mathrm{CF}^*(f;H)$, we need to choose a family of almost complex structures $\mathcal{J} = \{J_s\}_{s\in\mathbb{R}}$ on M, compatible with ω and f-periodic, meaning that $J_s = f^*(J_{s+1})$. One considers finite-energy solutions

$$u\colon \mathbb{R}\times\mathbb{R} \longrightarrow M, (s,t) \longmapsto u(s,t)$$

of Floer's equation

$$\frac{\partial u}{\partial t} + J_s(u)\left(\frac{\partial u}{\partial s} - X_s^H(u)\right) = 0,$$

which are f-periodic in s, $u(s+1,t) = f(u(s,t))$, and satisfy the asymptotic conditions

$$\lim_{t\to-\infty} u(s,t) = x(s) \text{ and } \lim_{t\to\infty} u(s,t) = y(s)$$

for some Hamiltonian chords $x, y \in \Omega_f$. Consider the moduli space $\mathcal{M}(x,y;\mathcal{J},H)$ of all such solutions u. For regular $(\mathcal{J},H)$, the moduli space is a smooth manifold. $\mathbb{R}$ acts on the one-dimensional component $\mathcal{M}^1(x,y;\mathcal{J},H)$ by translation, and the quotient set

$$\widehat{\mathcal{M}}^1(x,y;\mathcal{J},H) = \mathcal{M}^1(x,y;\mathcal{J},H)/\mathbb{R}$$

is discrete.

The Floer differential $\partial\colon \mathrm{CF}^*(f;\mathcal{J},H) \longrightarrow \mathrm{CF}^*(f;\mathcal{J},H)$ is defined by

$$\partial(y) = \sum_{x\in P_f(H)} \sum_{u\in\widehat{\mathcal{M}}^1(x,y;\mathcal{J},H)} q^{w(u)} x.$$

The homology of the chain complex $\mathrm{CF}^*(f)$ is called the Floer cohomology of f with Floer data $(\mathcal{J},H)$ and denoted by $\mathrm{HF}^*(f;\mathcal{J},H)$.

There are graded continuation maps for different choices of Floer data: Suppose $(\mathcal{J}, H)$ and $(\mathcal{J}', H')$ are regular Floer data as above. Choose a family $(J_{s,t}, H_{s,t})$ that satisfies the periodicity assumptions

$$H_{s,t} = H_{s+1,t} \circ f \text{ and } J_{s,t} = f^*(J_{s+1,t})$$

and interpolates between (J_s, H_s) and (J'_s, H'_s), i.e.

$$\begin{aligned} H_{s,t} = H'_s,\, J_{s,t} = J'_t \qquad & \text{for } t \text{ near } -\infty, \\ H_{s,t} = H_s,\, J_{s,t} = J_t \qquad & \text{for } t \text{ near } \infty. \end{aligned}$$

We denote by $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ the moduli space of solutions to the 1-parametric Floer equation

$$\frac{\partial u}{\partial t} + J_{s,t}(u)\left(\frac{\partial u}{\partial s} - X^H_{s,t}(u)\right) = 0$$

that are f-periodic in s and tend to x and y as $t \to \pm\infty$. For generic choice of $(J_{s,t}, H_{s,t})$ the moduli space is a manifold and its zero-dimensional component $\mathcal{M}^0(x, y; J_{s,t}, H_{s,t})$ is discrete. The chain-level continuation map is the chain map

$$\begin{aligned} C_{J_{s,t},H_{s,t}} \colon \mathrm{CF}^*(f; \mathcal{J}, H) &\longrightarrow \mathrm{CF}^*(f, \mathcal{J}', H') \\ y &\longmapsto \sum_{x \in P_f(H')} \sum_{u \in \mathcal{M}^0(x,y;J_{s,t},H_{s,t})} q^{\omega(u)} x. \end{aligned}$$

The map induced in cohomology is independent of the homotopy $(J_{s,t}, H_{s,t})$. This allows us to identify the two Floer cohomology groups $\mathrm{HF}^*(f; \mathcal{J}, H)$ and $\mathrm{HF}^*(f; \mathcal{J}', H')$ and simply write $\mathrm{HF}^*(f)$ for the cohomology group of f.

Suppose f and f' are related by a Hamiltonian isotopy. Choose a Hamiltonian isotopy $h_t = \psi^H_t$ such that $f' = \psi^H_1 f$. This isotopy induces an isomorphism

$$\Phi^{f,f'}_h \colon \mathrm{HF}^*(f) \longrightarrow \mathrm{HF}^*(f'). \tag{1.9}$$

In general, this map might depend on the Hamiltonian isotopy. However, when (M, ω) is symplectically aspherical, the isomorphism is independent of h: Consider first the case of Hamiltonian Floer homology. Let $h_t := \psi_t^H$ be a Hamiltonian loop, i.e. $h_0 = h_1 = \mathrm{id}$. Consider the induced automorphism $\Phi_h \colon \mathrm{HF}^*(\mathrm{id}) \longrightarrow \mathrm{HF}^*(\mathrm{id})$. In [Sei97b], Seidel constructed a group homomorphism

$$q\colon \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow QH^\times(M,\omega).$$

$QH^\times(M,\omega)$ acts on $\mathrm{HF}^*(\mathrm{id})$ via the quantum cup product $*$. Therefore, q determines an action of $\pi_1(\mathrm{Ham}(M,\omega))$ on $\mathrm{HF}^*(\mathrm{id})$. Seidel shows that

$$\Phi_h(a) = q(h) * a.$$

When (M,ω) is symplectically aspherical, we have a ring isomorphism $QH(M,\omega) \cong H^*(M;\Lambda)$ and $QH^\times(M,\omega) = \Lambda^\times$. In [Sch00], Schwarz proves that q is in fact trivial in symplectically aspherical manifolds. In particular it follows that $\Phi_h = \mathrm{id}$ on $\mathrm{HF}^*(\mathrm{id})$ [Sch00, Section 3].

To transport this result to $\mathrm{HF}^*(f)$, we use that $\mathrm{HF}^*(\mathrm{f})$ is a module over $\mathrm{HF}^*(\mathrm{id})$. We denote this module action by

$$*\colon \mathrm{HF}^*(\mathrm{id}) \otimes \mathrm{HF}^*(\mathrm{f}) \longrightarrow \mathrm{HF}^*(f)$$

Then

$$\Phi_h^{f,f}(a * x) = \Phi_h(a) * x,$$

for any $a \in \mathrm{HF}^*(\mathrm{id})$ and $x \in \mathrm{HF}^*(f)$ [Sei97b, Proposition 6.3], [Sei08b, Section (2a)]. In particular, it follows from $\Phi_h = \mathrm{id}$ that also $\Phi_h^{f,f} = \mathrm{id}$.

The independence of the isomorphism (1.9) follows from the case of a loop and the property

$$\Phi_{h\cdot h'}^{f,f''} = \Phi_h^{f',f''} \circ \Phi_h^{f,f'}$$

for any two Hamiltonian isotopies h and h'. Here, $h \cdot h'$ denotes the concatenation of the paths $h(t)$ and $h'(t) \circ h_1$ in $\mathrm{Ham}(M,\omega)$.

1.4.3 Conjugation invariance. Let f be a symplectomorphism on X and g be an anti-symplectic diffeomorphism on X. We will make substantial use of the following fact, which is an anti-symplectic version of the well-known conjugation invariance of Floer cohomology (see e.g. [Sei08b, section 3]).

Proposition 1.4.1. *There exists a canonical graded isomorphism*

$$g_* \colon \mathrm{HF}^*(f^{-1}) \to \mathrm{HF}^*(gfg^{-1}).$$

We apply this result to $f = \tau_S$ in the situation where $g\tau_S g^{-1}$ is Hamiltonian isotopic to τ_S^{-1}. We then get an automorphism

$$g_* \colon \mathrm{HF}^*(\tau_S^{-1}) \longrightarrow \mathrm{HF}^*(g\tau_S g^{-1}) \cong \mathrm{HF}^*(\tau_S^{-1}).$$

This is induced by the chain-level map

$$\mathrm{CF}(\tau_S^{-1}; H_s) \longrightarrow \mathrm{CF}(g\tau_S g^{-1}; H_{1-s} \circ g^{-1})$$

sending a generator x to $g(x)$, concatenated with the *canonical* isomorphism

$$\mathrm{HF}(g\tau_S g^{-1}; H_{1-s} \circ g^{-1}) \cong \mathrm{HF}(\tau_S^{-1}; H_s).$$

As we explained above, this isomorphism does not depend on a choice of Hamiltonian isotopy between τ_S^{-1} and $g\tau_S g^{-1}$.

Remark 1.4.2. If $g' = \psi_1^H g$ then $g_* = g'_*$. To see this, consider the following diagram:

$$\begin{tikzcd}
 & \mathrm{HF}(g\tau_S g^{-1}; H_{1-s} \circ g^{-1}) \arrow[dr, "\cong"] & \\
\mathrm{HF}(\tau_S^{-1}; H_s) \arrow[ur, "g_*"] \arrow[dr, "g'_*"'] & & \mathrm{HF}(\tau_S^{-1}; H_s) \\
 & \mathrm{HF}(\psi_1^H g\tau_S g^{-1}(\psi_1^H)^{-1}; H_{1-s} \circ g^{-1} \circ (\psi_1^H)^{-1}) \arrow[uu, "(\psi_1^H)_*"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The left triangle is commutative on the chain level. The right triangle commutes because all the Seidel elements are the identity, as we explained in the last section. This allows us in the proof of $g_*(A) = A$ to replace g by any g' which is Hamiltonian isotopic to g.

We outline a proof for the statement on conjugation by an anti-symplectic map.

Proof of Proposition 1.4.1. Let $(\mathcal{J}, H)$ be a Floer datum for f^{-1}. Consider the Hamiltonian function

$$K_s := H_{1-s} \circ g^{-1}$$

and the almost complex structure

$$J'_s := -(g^{-1})^* J_{1-s}.$$

The Hamiltonian vector field associated to K_s and its flow are given by

$$\begin{aligned} X_t^K(x) &= -\mathrm{D}g\left(X_{1-t}^H(g^{-1}(x))\right), \\ \psi_t^K(x) &= g\psi_{1-t}^H(\psi_1^H)^{-1}g^{-1}(x). \end{aligned}$$

It follows that $(\mathcal{J}', K)$ is an admissible Floer datum for gfg^{-1}. Moreover, the bijection

$$\begin{aligned} P_{f^{-1}}(H) &\longrightarrow P_{gfg^{-1}}(K) \\ x(t) &\longmapsto \tilde{x}(t) := g(x(1-t)) \end{aligned}$$

extends to a Λ-linear isomorphism

$$g_* \colon \mathrm{CF}^*(f^{-1}; \mathcal{J}, H) \longrightarrow \mathrm{CF}^*(gfg^{-1}; \mathcal{J}', K). \tag{1.10}$$

There is also a bijection between the moduli spaces

$$\begin{aligned} \mathcal{M}^1(x, y; \mathcal{J}, H) &\longrightarrow \mathcal{M}^1(\tilde{x}, \tilde{y}; \mathcal{J}', K) \\ u(s,t) &\longmapsto v(s,t) = g(u(1-s,t)). \end{aligned}$$

Therefore, g_* in (1.10) is an isomorphism of chain complexes. Moreover, the degree is preserved because

$$\begin{aligned}\det\big(\mathrm{Id} - \mathrm{D}((\psi_1^K)^{-1} g f g^{-1})_{\tilde{x}(0)}\big) &= \det\big(\mathrm{Id} - \mathrm{D}(g\psi_1^H f g^{-1})_{g(x(1))}\big)\\ &= \det\big(\mathrm{Id} - \mathrm{D}(\psi_1^H f)_{f^{-1}(x(0))}\big)\\ &= \det\big(\mathrm{Id} - \mathrm{D}(f\psi_1^H)_{x(0)}\big)\\ &= \det\big(\mathrm{Id} - \mathrm{D}((\psi_1^H)^{-1} f^{-1})_{x(0)}\big).\end{aligned}$$

The second equality follows from factoring out $\mathrm{D}g$ and $\mathrm{D}g^{-1}$ and using multiplicativity of the determinant, the third follows from multiplying with $\mathrm{D}f$ and $\mathrm{D}f^{-1}$ from left and right, the fourth used that $\mathrm{D}(f\psi_1^H)_{x(0)}$ is symplectic. This clearly implies $\deg(\tilde{x}) = \deg(x)$, which means that the isomorphism g_* preserves the degree.

Concatenation of this chain isomorphism with a continuation map

$$\mathrm{CF}^*(gfg^{-1}; \mathcal{J}', K) \longrightarrow \mathrm{CF}^*(gfg^{-1}; \mathcal{J}, H)$$

shows Proposition 1.4.1. □

1.4.4 Lagrangian Floer cohomology. Let $L_0, L_1 \subset M$ be two closed relatively symplectically aspherical Lagrangians. Let

$$H\colon M \times [0,1] \longrightarrow \mathbb{R}$$

be a Hamiltonian function such that $\psi_1^H(L_0) \cap L_1$ is a transverse intersection. Choose also a 1-parametric family of ω-compatible almost complex structures $\mathcal{J} = \{J_t\}_{t\in\mathbb{R}}$ on M. Let $P_{L_0,L_1}(H)$ be the set of paths $x \in C^\infty([0,1], M)$ such that $x(0) \in L_0$, $x(1) \in L_1$ and $x(t) = \psi_t^H(x(0))$. The underlying Λ-vectorspace of the Floer complex $\mathrm{CF}(L_0, L_1; \mathcal{J}, H)$ is

$$\mathrm{CF}(L_0, L_1; \mathcal{J}, H) = \bigoplus_{x \in P_{L_0,L_1}(H)} \Lambda x.$$

The differential is defined as follows. For $x, y \in P_{L_0,L_1}(H)$ consider the 0-dimensional moduli space $\mathcal{M}(x, y; \mathcal{J}, H)$ of solutions to Floer's equation

$$\begin{cases} \frac{\partial u}{\partial t} + J_s(u)\left(\frac{\partial u}{\partial s} - X_s^H(u)\right) = 0, \\ u(0,t) \in L_0, \qquad u(1,t) \in L_1, \\ \lim_{t\to-\infty} u(s,t) = x, \\ \lim_{t\to\infty} u(s,t) = y, \end{cases}$$

modulo the $\mathbb{R}$-action by translation. Then for $y \in P_{L_0,L_1}(H)$ the differential is

$$\partial y = \sum_{x\in P_{L_0,L_1}(H)} \sum_{u\in\mathcal{M}(x,y;\mathcal{J},H)} q^{\omega(u)} x.$$

If L_0 and L_1 are oriented, $\mathrm{CF}(L_0, L_1; \mathcal{J}, H)$ is graded as follows. The degree of $x \in P_{L_0,L_1}(H)$ is defined by

$$(-1)^{\deg(x)} = (-1)^{\frac{n(n+1)}{2}} \nu(x(1); \psi_1^H(L_0), L_1),$$

where $\nu(z; K_0, K_1) \in \{\pm 1\}$ denotes the intersection index of K_0 and K_1 at z. This number is defined to be $+1$ if $v_1, \ldots, v_{2n}$ is a positive basis for T_xM whenever $v_1, \ldots, v_n$ is a positive basis for T_zK_0 and $v_{n+1}, \ldots, v_{2n}$ is a positive basis for T_zK_1. See [Sei00, Section 2d] for the grading, and [RS22] for the intersection index.

1.4.5 Lagrangian Floer cohomology in the product manifold Note that for any symplectomorphism f on a symplectically aspherical manifold M, the graph Γ_f is a relatively symplectically aspherical Lagrangian submanifold in the product $M \times M^-$. Also, products of relatively symplectically aspherical Lagrangians in M are relatively symplectically aspherical Lagrangians in $M \times M^-$.

We endow the graph Γ_f with the following orientation: Given a positive basis $v_1, \ldots, v_{2n}$ of T_xM, then the basis

$$(v_1, Df_x(v_1)), \ldots, (v_{2n}, Df_x(v_{2n}))$$

of $T_x\Delta \subset T_xM \oplus T_xM$ is defined to be positive if $(-1)^{\frac{n(n-1)}{2}} = 1$ and negative otherwise, see [WW10]. Moreover, given an oriented Lagrangian N, note that $f(N)$ has a canonical orientation.

Let Q and N be oriented Lagrangians in M. There are the following canonical graded isomorphisms between Floer cohomology groups for Lagrangians in $M \times M^-$ and Lagrangians in M:

(i) $\mathrm{HF}^*(Q \times N, \Gamma_{f^{-1}}) \cong \mathrm{HF}^*(Q, f(N))$

(ii) $\mathrm{HF}^*(Q \times N, Q' \times N') \cong \mathrm{HF}^*(Q, Q') \otimes \mathrm{HF}^*(N', N)$

Moreover, Floer cohomology of a symplectomorphism f can be viewed as Lagrangian Floer cohomology of the pair (Δ, Γ_f). This isomorphism is well-known, see for instance [WW10], [MW18] and [LZ18, section 2.7].

Proposition 1.4.3. *There is a canonical graded isomorphism*

$$\Psi_f \colon \mathrm{HF}(f) \longrightarrow \mathrm{HF}(\Delta, \Gamma_f).$$

Moreover, for any anti-symplectic map $g\colon M \longrightarrow M$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{HF}^*(f^{-1}) \arrow[r, "g_*"] \arrow[d, "\Psi_{f^{-1}}"] & \mathrm{HF}^*(gfg^{-1}) \arrow[d, "\Psi_{gfg^{-1}}"] \\
\mathrm{HF}^*(\Delta, \Gamma_{f^{-1}}) \arrow[r, "\Phi^g_*"] & \mathrm{HF}^*(\Delta, \Gamma_{gfg^{-1}}),
\end{tikzcd}$$

where Φ^g is the symplectomorphism

$$\begin{aligned} \Phi^g \colon M \times M^- &\longrightarrow M \times M^- \\ (x, y) &\longmapsto (g(y), g(x)). \end{aligned}$$

As a special case, we recover the commutative diagram (1.5) when $f = \tau_S$ and $g\tau_S g^{-1} = \tau_S^{-1}$.

Proof of Proposition 1.4.3. Choose a Floer datum $(\mathcal{J}, H)$ for f as in Section 1.4.2. The generators of $\mathrm{CF}(f; \mathcal{J}, H)$ correspond to fixed points of $(\psi_1^H)^{-1} f$. For the Lagrangian Floer complex, we choose the following Floer datum:

$$K_s(x, y) = -\frac{1}{2} H_{\frac{1-s}{2}}(x) - \frac{1}{2} H_{\frac{s+1}{2}}(y).$$

and

$$\tilde{J}_s := \tilde{J}_s^{(1)} \oplus \tilde{J}_s^{(2)} := J_{\frac{1-s}{2}} \oplus \left(-J_{\frac{s+1}{2}}\right).$$

Generators of $\mathrm{CF}(\Delta, \Gamma_f; \widetilde{\mathcal{J}}, K)$ correspond to intersection points between Δ and $(\psi_1^K)^{-1}(\Gamma_f)$. The flows of H and K are related by

$$\psi_s^K(x,y) = \left(\psi^H_{\frac{1-s}{2}}(x), \psi^H_{\frac{s+1}{2}}(y)\right).$$

Consider the bijection

$$\begin{aligned} \Delta \cap (\psi_1^K)^{-1}(\Gamma_f) &\longleftrightarrow \mathrm{Fix}\left((\psi_1^H)^{-1} f\right) \\ \tilde{x} = (x,x) &\longleftrightarrow x \end{aligned} \tag{1.11}$$

between the generators of the two Floer complexes. There is a bijection between the corresponding moduli spaces

$$\begin{aligned} \mathcal{M}(\tilde{x}, \tilde{y}; \widetilde{\mathcal{J}}, K) &\longleftrightarrow \mathcal{M}(x, y; \mathcal{J}, H) \\ (v_1(s,t), v_2(s,t)) &\longleftrightarrow u(s,t) = \begin{cases} v_1(1-2s, -2t) & s \in [0, \frac{1}{2}] \\ v_2(2s-1, -2t) & s \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

Therefore, the bijection (1.11) extends to a chain isomorphism

$$\mathrm{CF}(\Delta, \Gamma_f; \widetilde{\mathcal{J}}, K) \longrightarrow \mathrm{CF}(f; \mathcal{J}, H).$$

We show that this chain isomorphism preserved the grading. To simplify notation, assume that $H = K = 0$. Let $(x,x) \in \Delta \cap \Gamma_f$. Let $\mathcal{B}^M$ be a basis of $T_x M$ and consider the bases $\mathcal{B}^\Delta$ and $\mathcal{B}^{\Gamma_f}$ of $T_{(x,x)}\Delta$ and $T_{(x,x)}\Gamma_f$ associated to $\mathcal{B}^M$. Note that $\mathcal{B}^\Delta$ and $\mathcal{B}^{\Gamma_f}$ are either both positive or both negative. Hence $\nu(x,x) = 1$ if and only if the basis $\mathcal{B} = \left(\mathcal{B}^\Delta, \mathcal{B}^{\Gamma_f}\right)$ is a positive basis of $T_{(x,x)}(M \times M^-)$. A computation shows that

$$\mathcal{B} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & Df_x \end{pmatrix} \mathcal{B}_0,$$

where $\mathcal{B}_0 = \big((\mathcal{B}^M, 0), (0, \mathcal{B}^M)\big)$. $\mathcal{B}_0$ is positively oriented if and only if n is even. The determinant of the matrix is $\det(Df_x - \mathrm{Id}) = \det(\mathrm{Id} - Df_x)$. Hence,

$$\nu(x, x) = (-1)^n \mathrm{sign}\det(\mathrm{Id} - Df_x)$$

and

$$\begin{aligned}
(-1)^{\deg(x,x)} &= (-1)^n (-1)^{\frac{2n(2n+1)}{2}} \nu(x, x) \\
&= (-1)^n (-1)^{\frac{2n(2n+1)}{2}} (-1)^n \mathrm{sign}\det(\mathrm{Id} - Df_x) \\
&= (-1)^n (-1)^{\frac{2n(2n+1)}{2}} (-1)^n (-1)^{\deg(x)} \\
&= (-1)^{\deg(x)}.
\end{aligned}$$

This shows that the isomorphism above indeed preserves the grading. The commutativity of the diagram follows from chain-level commutativity. □

1.5 Lagrangian cobordisms

In this section we briefly recall some ingredients from the cobordism theory developed by Biran–Cornea in the series of papers [BC13, BC14, BC17]. This is background material which is relevant for the Mak–Wu long exact sequence, the definition of the element A, as well as the proof of Theorem 1.1.1.

1.5.1 Definitions. In this section we recall the definition of Lagrangian cobordisms [BC13, BC14, BC17]. Let (M, ω) be a symplectic manifold. Consider the product symplectic manifold $(M \times \mathbb{R}^2, \omega \oplus \omega_{\mathrm{std}})$. Here, $\omega_{\mathrm{std}} = \mathrm{d}x \wedge \mathrm{d}y$ denotes the standard symplectic form on $\mathbb{R}^2$. We denote by $\pi \colon M \times \mathbb{R}^2 \to \mathbb{R}^2$ the projection to the plane. For subsets $V \subset M \times \mathbb{R}^2$ and $Z \subset \mathbb{R}^2$, we write $V|_Z := V \cap \pi^{-1}(Z)$ for the restriction of V over Z. A Lagrangian submanifold $V \subset M \times \mathbb{R}^2$ is called a *Lagrangian cobordism* [BC14, Definition 2.2.1] if there exists $R > 0$ such that

(i)

$$V|_{(-\infty,-R]\times\mathbb{R}} = \bigcup_{j=1}^{k_-} L_j \times (-\infty,-R] \times \{j\}$$

for some closed Lagrangian submanifolds $L_1,\ldots,L_{k_-} \subset M$,

(ii)

$$V|_{[R,\infty)\times\mathbb{R}} = \bigcup_{j=1}^{k_+} L'_j \times [R,\infty) \times \{j\}$$

for some closed Lagrangian submanifolds $L'_1,\ldots,L'_{k_+} \subset M$,

(iii) $V|_{[-R,R]\times\mathbb{R}} \subset \mathbb{R}^2 \times M$ is compact.

V is called a Lagrangian cobordism from the Lagrangian family $(L'_j)_{j=1,\ldots,k_+}$ to the Lagrangian family $(L_i)_{i=1,\ldots,k_-}$, denoted by

$$(L'_j)_{j=1,\ldots,k_+} \rightsquigarrow (L_i)_{i=1,\ldots,k_-}.$$

Denote by $E_R^+ = V|_{[R,\infty)\times\mathbb{R}}$ the positive cylindrical ends and similarly by $E_{-R}^- = V|_{(-\infty,-R]\times\mathbb{R}}$ the negative cylindrical ends. We recall the notion of horizontal isotopy from [BC14, Definition 2.2.3]. A horizontal isotopy with respect to a Lagrangian cobordism V is a not necessarily compactly supported Hamiltonian isotopy $\{\psi_t^H\}_{t\in[0,1]}$ of $M \times \mathbb{R}^2$ such that

(i) There exists a real number $R' > R$ such that for all $t \in [0,1]$,

$$\psi_t^H\left(E_{-R'}^-\right) \subseteq E_{-R'}^- \text{ and } \psi_t^H\left(E_{R'}^+\right) \subseteq E_{R'}^+.$$

(ii) There exists a constant $K > 0$ such that for all $x \in E_R^+ \cup E_R^-$,

$$|\mathrm{d}\pi_x(X_t^H(x))| < K.$$

In what follows we work with oriented cobordisms. Suppose the families $(L'_j)_{j=1,\ldots,k_+}$ and $(L_i)_{i=1,\ldots,k_-}$ consist of oriented Lagrangians. Then an *oriented cobordism* V from the family $(L'_j)_{j=1,\ldots,k_+}$ to the family $(L_i)_{i=1,\ldots,k_-}$ is a cobordism V endowed with an orientation that restricts to the given orientations on the ends.

1.5.2 Lagrangian cobordisms induce cone decompositions.

We recall here how a cobordism gives rise to cone decompositions of its ends in the derived Fukaya category $\mathcal{DF}uk(M)$. For a brief explanation of $\mathcal{DF}uk(M)$ we refer the reader to the Appendix. Since we work with cohomology, rather than homology, we write here a cohomological reformulation of Theorem A from [BC14]. We work with a $\mathbb{Z}_2$-grading.

Theorem 1.5.1 (Theorem A in [BC14]). *Let V be an oriented cobordism from L to the family $(L_1[l-1], L_2[l-2], \ldots, L_l)$. Assume that all Lagrangians involved (including V) are relatively symplectically aspherical.* [2] *Then there exists a graded quasi-isomorphism*

$$L \cong \mathrm{Cone}\,(\ldots \mathrm{Cone}(\mathrm{Cone}(L_1 \to L_2) \to L_3) \to \cdots \to L_l)$$

in the derived Fukaya category $\mathcal{DF}uk(M)$.

Here, we denote by $L[k], k \in \mathbb{Z}$ the Lagrangian L with the same orientation for even k, and with oppostite orientation for odd k. The theorem also holds in the context of $\mathbb{Z}$-gradings, see also [MW18, Theorem 5.2].

A special case occurs when there are only three Lagrangians involved, namely V has one right end, L, and two left ends, $L_1[1]$ and L_2 as shown in Figure 1.3. In this situation we get a quasi-

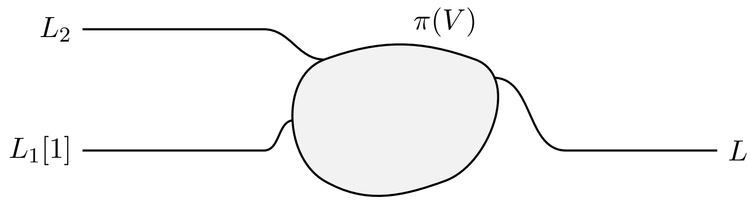


Figure 1.3: A cobordism with three ends.

isomorphism

$$L \cong Cone(L_1 \longrightarrow L_2).$$

[2]The original statement only assumes monotonicity conditions.

In other words there is an exact triangle

$$\begin{tikzcd}
L_1 \arrow[dr] & \\
& L_2 \arrow[dl, "\varphi_V"] \\
L \arrow[uu, "{[1]}"] &
\end{tikzcd}$$

in $\mathcal{DF}uk(M)$. As we explain further in Appendix A.2, the morphism φ_V can be understood as a μ^2-operation with a unique element $\alpha_V \in HF^0(L_2, L)$. The associated long exact sequence in cohomology reads

$$\cdots \longrightarrow \mathrm{HF}^{k-1}(K, L) \longrightarrow \mathrm{HF}^k(K, L_1) \longrightarrow$$
$$\longrightarrow \mathrm{HF}^k(K, L_2) \xrightarrow{\mu^2(\alpha_V, -)} \mathrm{HF}^k(K, L) \longrightarrow \dots$$

The morphism $\varphi_V \in Mor_{\mathcal{DF}uk(M)}(L_2, L)$ is determined by the cobordism V and it only depends on the horizontal isotopy type of V [BC14, Theorem B]. There are various descriptions of φ_V as summarized in [BC14, Section 4.8]. We recall one possibe view on the element $\alpha_V \in \mathrm{HF}^0(L_2, L)$. Consider the Lagrangian $\gamma \times L_2$ colored in blue in Figure 1.4. Consider the map φ_V obtained from count-

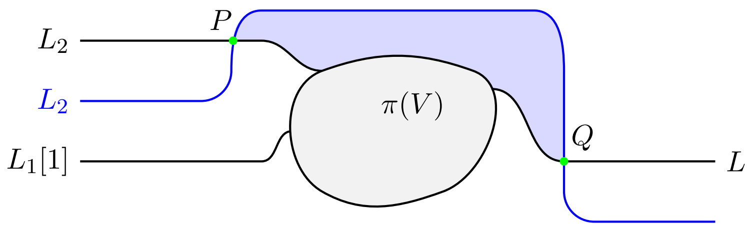


Figure 1.4: Counting holomorphic strips.

ing holomorphic strips with boundary on $\gamma \times L_2$ and V, output in $\mathrm{CF}(L_2, L_2)$ over P and input in $\mathrm{CF}(L_2, L)$ over Q. Such strips project to the area shaded in blue in Figure 1.4. The resulting map is

$$\varphi_V : \mathrm{HF}^*(L_2, L_2) \longrightarrow \mathrm{HF}^*(L_2, L)$$

and $\alpha_V = \varphi_V(e)$, where $e \in HF^0(L_2, L_2)$ is the unit.

Consider a symplectomorphism $\Phi\colon M \longrightarrow M$ and the cobordism $(\Phi \times \mathrm{id})(V)$ with ends $\Phi(L_1)$, $\Phi(L_2)$ and $\Phi(L)$. It can be seen from the above description of α_V that $\alpha_{(\Phi\times\mathrm{id})(V)} = \Phi_*(\alpha_V)$, where

$$\Phi_*\colon \mathrm{HF}^*(L_2, L) \longrightarrow \mathrm{HF}^*(\Phi(L_2), \Phi(L))$$

is the induced map in cohomology. Indeed, strips u occuring in the count for φ_V are in 1:1-correspondence with strips v occuring in the count for $\varphi_{(\Phi\times\mathrm{id})(V)}$ via $v = (\Phi \times \mathrm{id})(u)$. In particular, if $(\Phi \times \mathrm{id})(V)$ and V are horizontally isotopic then $\Phi_*(\alpha_V) = \alpha_V$.

Cobordisms with three ends occur in the situation of surgery [BC13, Section 6]: Let L_1 and L_2 be two Lagrangians intersecting transversely in a single point $p \in L_1 \cap L_2$. Then there is a surgery construction yielding a third Lagrangian $L_1 \#_p L_2$. Another surgery construction applied to $L_1 \times \mathbb{R}$ and $L_2 \times i\mathbb{R}$ allows to construct a cobordism V with the three ends L_1, L_2 and $L_1 \#_p L_2$. The Mak-Wu cobordism will be obtained from a variant of this construction. It is explained in detail in the next section.

1.6 Mak-Wu cobordism

We consider a symplectic manifold (M, ω) and a parametrised Lagrangian sphere $S \subset M$. Mak-Wu [MW18] constructed a Lagrangian cobordism V_{MW} in the product manifold $M \times M^-$ with three ends: the product Lagrangian $S \times S$, the diagonal Δ and the graph $\Gamma_{\tau_S^{-1}}$. In this section, we will recall the construction of this cobordism following closely [MW18].

1.6.1 The graph of the Dehn twist. Following the principle that surgeries provide cobordisms with three ends [BC13, Section 6], the Mak-Wu cobordism also arises as the trace of a surgery. The first step therefore is to understand $\Gamma_{\tau_S^{-1}}$ as the result of a surgery between $S \times S \subset M \times M^-$ and the diagonal $\Delta \subset M \times M^-$ along the clean intersection $\Delta_S := (S \times S) \cap \Delta$.

The surgery construction takes place locally in a Weinstein neighbourhood of $S \times S$. We choose a very specific neighbourhood, so that we can later compare it to the graph of a model Dehn twist. Let $\varphi\colon V \longrightarrow T^*_\epsilon S^n$ be a Weinstein neighbourhood of S as in section 1.2.1 that extends the parametrisation of S. Then consider the symplectic embedding

$$\begin{aligned}\widetilde{\varphi}\colon V \times V &\longrightarrow T^*_\epsilon S^n \oplus T^*_\epsilon S^n \subset T^*(S^n \times S^n)\\ (x,y) &\longmapsto (\varphi(x), -\varphi(y))\end{aligned}$$

that identifies $S \times S$ with the zero-section in $T^*(S^n \times S^n)$. Note that

$$\widetilde{\varphi}^{-1}(N^*_{\Delta_S}) = \Delta \cap (V \times V),$$

where

$$N^*_{\Delta_S} := \{\alpha \in T^*(S^n \times S^n) \mid \forall v \in \Delta_S\colon \alpha(v) = 0\}.$$

We will define a surgery model in $T^*(S^n \times S^n)$ for surgery of the zero-section and $N^*_{\Delta_S}$ along their intersection Δ_S. Then we will glue the surgery model into $V \times V$ via $\tilde{\varphi}$. To define the surgery model, we need some auxiliary functions similar to the Dehn twist profile functions.

Definition 1.6.1. A λ-*admissible* function $\nu_\lambda\colon \mathbb{R}_{\geq 0} \longrightarrow [0,\lambda]$ is a smooth function satisfying

$$\begin{cases}\nu_\lambda(0) = \lambda, & \\ \nu_\lambda^{-1} \text{ has vanishing derivatives of all orders at } \lambda, & \\ 0 < \nu_\lambda(r) < \lambda \text{ and strictly decreasing} & \text{for } 0 < r < \epsilon,\\ \nu_\lambda(r) = 0 & \text{for } r \geq \epsilon.\end{cases}$$

Let

$$\pi_2\colon T^*(S^n \times S^n) \cong T^*S^n \oplus T^*S^n \to T^*S^n$$

be the projection to the second summand. Consider its norm

$$\begin{aligned}\mu_\pi\colon T^*(S^n \times S^n) &\longrightarrow \mathbb{R}\\ \xi &\longmapsto \|\pi_2(\xi)\|.\end{aligned}$$

This has a well-defined Hamiltonian flow on $T^*(S^n \times S^n)\backslash T^*S^n$. Let $\lambda < \pi$ and let $\nu = \nu_\lambda$ be a λ-admissible function. We define the following *flow handle*:

$$H_\nu = \left\{ \psi^{\mu_\pi}_{\nu(\mu_\pi(\xi))}(\xi) \in T^*(S^n \times S^n) \,\middle|\, \xi \in N^*_{\Delta_S}\backslash\Delta_S,\, \mu_\pi(\xi) < \epsilon \right\}.$$

H_ν can be glued to a part of $S \times S$ and $N^*_{\Delta_S}$, resulting in a smooth Lagrangian in $T^*(S^n \times S^n)$ that coincides with $N^*_{\Delta_S}$ outside of $T^*_\epsilon S \oplus T^*_\epsilon S^n$. We denote the resulting Lagrangian by

$$(S^n \times S^n)\#^\nu_{\Delta_S} N^*_{\Delta S}.$$

This is the model surgery. We finally glue it into $V \times V$ via the embedding $\widetilde{\varphi}$:

$$(S \times S)\#^\nu_{\Delta_S}\Delta := \widetilde{\varphi}^{-1}\left((S^n \times S^n)\#^\nu_{\Delta_S} N^*_{\Delta_S}\right) \cup \left(\Delta\backslash(V \times V^-)\right).$$

Mak-Wu [MW18, Lemma 3.4] show that all such surgeries are Hamiltonian isotopic for different choices of ν. Moreover, the same construction works for $\nu = \nu^{\mathrm{Dehn}}_\epsilon$ (even though this is *not* admissible) and the result is again Hamiltonian isotopic to any of the other surgeries. For $\nu = \nu^{\mathrm{Dehn}}_\epsilon$ the model surgery coincides with the graph of the model Dehn twist. It follows that

$$(S \times S)\#^{\nu^{\mathrm{Dehn}}_\epsilon}_{\Delta_S}\Delta = \Gamma_{\tau_S^{-1}}$$

and so any of the above surgeries is Hamiltonian isotopic to $\Gamma_{\tau_S^{-1}}$. In particular, since $\Gamma_{\tau_S^{-1}}$ is relatively symplectically aspherical, so are the surgeries $(S \times S)\#^\nu_{\Delta_S}\Delta$.

Remark 1.6.2. This version of surgery is a special case of E_2-flow surgery, introduced in [MW18, section 2.3] in more general situations.

1.6.2 The cobordism. The Lagrangian $(S \times S)\#^\nu_{\Delta_S}\Delta$ obtained from surgery is related to $S \times S$ and Δ via a cobordism. This follows from a construction called "trace of a surgery", which is a surgery construction in one dimension higher. It was first introduced in

[BC13] for the case of a transverse surgery in a point. As shown in [MW18], exactly the same construction works for the E_2-surgery along clean intersections. We recall the construction in our special case.

As before we first construct a local model. We will then glue the model back into $M \times M^- \times T^*\mathbb{R}$ via the symplectomorphism

$$\widetilde{\varphi} \times \mathrm{id} \colon V \times V \times T^*\mathbb{R} \longrightarrow T^*_\epsilon S^n \oplus T^*_\epsilon S^n \oplus T^*\mathbb{R} \subset T^*(S^n \times S^n \times \mathbb{R}).$$

Let $\nu = \nu_\lambda$ be a λ-admissible function, as defined in Definition 1.6.1. This time we will use the flow of the function

$$\begin{aligned}\mu_{\hat{\pi}} \colon T^*(S^n \times S^n \times \mathbb{R}) &\longrightarrow \mathbb{R},\\ (\xi_1, \xi_2, p) &\longmapsto \|(\xi_2, p)\| = \sqrt{\|\xi_2\|^2 + p^2}.\end{aligned}$$

The handle in the model $T^*(S^n \times S^n \times \mathbb{R})$ is defined as follows:

$$\widehat{H}_\nu = \left\{ \psi^{\mu_{\hat{\pi}}}_{\nu(\mu_{\hat{\pi}}(\xi))}(\xi) \in T^*(S^n \times S^n \times \mathbb{R}) \middle| \begin{array}{l} \xi \in N^*_{\Delta_S \times \{0\}} \setminus (\Delta_S \times \{0\}) \\ \mu_{\hat{\pi}}(\xi) < \epsilon \end{array} \right\}.$$

This model handle $\widehat{H}_\nu$ glues to a part of $(S^n \times S^n \times \mathbb{R}) \setminus \partial \widehat{H}_\nu$, which yields the model surgery trace

$$(S^n \times S^n \times \mathbb{R}) \#_{\Delta_S \times \{0\}} N^*_{\Delta_S \times \{0\}}.$$

Near the boundary of $T^*_\epsilon S^n \oplus T^*_\epsilon S^n \oplus T^*\mathbb{R}$ it coincides with $N^*_{\Delta_S \times \{0\}}$. Moreover $\widetilde{\varphi} \times \mathrm{id}$ identifies

$$N^*_{\Delta_S \times \{0\}} = N^*_{\Delta_S} \times T^*_0\mathbb{R}$$

with $\Delta \times T^*_0\mathbb{R}$, where $T^*_0\mathbb{R}$ is the cotangent fiber over 0. Therefore we can glue the model surgery trace into $M \times M^- \times T^*\mathbb{R}$ via $\widetilde{\varphi} \times \mathrm{id}$ by

$$V := (\widetilde{\varphi} \times \mathrm{id})^{-1} \left((S^n \times S^n \times \mathbb{R}) \#_{\Delta_S \times \{0\}} N^*_{\Delta S \times \{0\}} \right) \cup (\Delta \times T^*_0\mathbb{R}) \,.$$

The result is a Lagrangian submanifold $V \subset M \times M^- \times T^*\mathbb{R}$. We identify $T^*\mathbb{R}$ and $\mathbb{C}$ via $(q, p) \leftrightarrow q - ip$. In particular, the fiber $T^*_0\mathbb{R}$

over 0 is identified with the imaginary axis $i\mathbb{R}$. This justifies the notation

$$V =: (S \times S \times \mathbb{R})\#_{\Delta_S \times \{0\}} (\Delta \times i\mathbb{R})\,.$$

The Lagrangian V satisfies

$$\begin{aligned} V \cap \pi_{\mathbb{C}}^{-1}(\epsilon) &= S \times S \times \{\epsilon\}, \\ V \cap \pi_{\mathbb{C}}^{-1}(i\epsilon) &= \Delta \times \{i\epsilon\}, \\ V \cap \pi_{\mathbb{C}}^{-1}(0) &= \big((S \times S)\#^{\nu}_{\Delta_S}\Delta\big) \times \{0\}. \end{aligned}$$

By taking half of V, extending it by a ray of $(S \times S)\#^{\nu}_{\Delta_S}\Delta$ at 0 and smoothing it, and bending the ends, as explained in [BC13, Section 6.1], we get a cobordism

$$\widetilde{V}\colon (S \times S)\#^{\nu}_{\Delta_S}\Delta \rightsquigarrow (S \times S, \Delta).$$

As discussed in section 1.6.1, $(S \times S)\#^{\nu}_{\Delta_S}\Delta$ is Hamiltonian isotopic to $\Gamma_{\tau_S^{-1}}$. Gluing a corresponding suspension to $\widetilde{V}$ finally gives us the claimed cobordism

$$V_{MW}\colon \Gamma_{\tau_S^{-1}} \rightsquigarrow (S \times S, \Delta).$$

Figure 1.5 illustrates the cobordism V_{MW}.

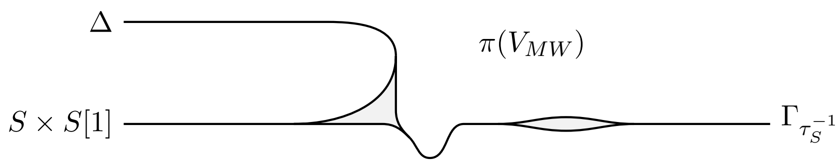


Figure 1.5: The Mak-Wu cobordism.

1.6.3 Floer theory.

Mak–Wu [MW18] explains how to put $\mathbb{Z}$-gradings on $S \times S$, Δ, $\Gamma_{\tau_S^{-1}}$ and on V_{MW} such that V_{MW} becomes a $\mathbb{Z}$-graded cobordism from $\Gamma_{\tau_S^{-1}}$ to $(S \times S[1], \Delta)$. Here, we only use

$\mathbb{Z}/2$-gradings. Their results immediately descend to the $\mathbb{Z}_2$-graded setting and V_{MW} becomes an oriented cobordism.

If M and $S \subset M$ are (relatively) symplectically aspherical, then V_{MW} is a relatively symplectically aspherical Lagrangian in the product $M \times M^- \times \mathbb{C}$ with relatively symplectically aspherical ends [MW18, Lemma 6.2, 6.3]. Floer theory for V_{MW} and its ends is therefore well-defined. We can apply Theorem 1.5.1 and get a long exact sequence of graded Lagrangian Floer cohomology groups [MW18, Theorem 6.4]:

$$\begin{aligned} \ldots \longrightarrow \mathrm{HF}^k(K, S \times S) &\xrightarrow{\mu^2(B,-)} \mathrm{HF}^k(K, \Delta) \xrightarrow{\mu^2(A,-)} \\ &\xrightarrow{\mu^2(A,-)} \mathrm{HF}^k(K, \Gamma_{\tau^{-1}}) \xrightarrow{\mu^2(C,-)} \mathrm{HF}^{k+1}(K, S \times S) \longrightarrow \ldots \end{aligned}$$

for any admissible Lagrangian submanifold $K \subset (M \times M, \omega \oplus -\omega)$. This is precisely the sequence (1.3). As indicated, the maps are given by μ^2 operations with elements $A \in \mathrm{HF}^0(\Delta, \Gamma_{\tau^{-1}})$, $B \in \mathrm{HF}^0(S \times S, \Delta)$ and $C \in \mathrm{HF}^1(\Gamma_{\tau_S^{-1}}, S \times S)$. A, B and C are independent of K.

Proposition 1.6.3. *If $2c_1(M) = 0$ in $\mathrm{H}^2(M; \mathbb{Z})$ and $2c_1(M, S) = 0$ in $\mathrm{H}^2(M, S; \mathbb{Z})$ then $A \neq 0$.* [3]

Proof. Under the condition on the Chern class, everything becomes $\mathbb{Z}$-graded, see [Sei00]. For $K = \Delta$, the sequence becomes

$$\begin{aligned} \ldots \longrightarrow \mathrm{HF}^k(S, S) \longrightarrow \mathrm{HF}^k(\mathrm{id}) &\xrightarrow{\Psi} \\ &\xrightarrow{\Psi} \mathrm{HF}^k(\Delta, \Gamma_{\tau^{-1}}) \longrightarrow \mathrm{HF}^{k+1}(S, S) \longrightarrow \ldots. \end{aligned}$$

Assume by contradiction that $A = 0$. Then $\Psi = 0$ and hence we get $\mathbb{Z}$-graded isomorphisms

$$\mathrm{H}^*(S; \Lambda) \cong QH^*(S) \cong \mathrm{HF}^*(S, S) \cong \mathrm{HF}^*(\mathrm{id}) \cong \mathrm{QH}^*(M) \cong \mathrm{H}^*(M; \Lambda).$$

This is not possible. We conclude that $A \neq 0$. $\square$

[3] The condition $2c_1(M, S) = 0$ is automatic for $n \geq 2$. For $n = 1$ it is equivalent to S being a non-contractible circle.

1.7 Symmetry of the Mak-Wu cobordism

Let $g\colon M \longrightarrow M$ be an anti-symplectic map satisfying ($\star$). In this section, we prove Theorem 1.1.5. It states that g is Hamiltonian isotopic to some g' such that $\Phi^{g'} \times \mathrm{id}$ preserves V_{MW}, where $\Phi^{g'}(x,y) = (g'(y), g'(x))$.

1.7.1 Local Model We first consider the local model. Consider T^*S^n together with an anti-symplectic involution c of the form $T^*\sigma$ for $\sigma = \mathrm{id}_{S^m}$ or $\sigma = r$. Recall from the proof of Proposition 1.1.4 that

$$\|c(\xi)\| = \|\xi\| \tag{1.12}$$

and

$$c\psi_t^{\mu}c = \psi_{-t}^{\mu}. \tag{1.13}$$

Consider the symplectomorphism

$$\begin{aligned}\Psi = \Phi^c \times \mathrm{id}\colon T^*S^n \times T^*S^n \times T^*\mathbb{R} &\longrightarrow T^*S^n \times T^*S^n \times T^*\mathbb{R}\\ (\xi_1, \xi_2, z) &\longmapsto (c(-\xi_2), -c(\xi_1), z).\end{aligned}$$

We first show that the model surgery cobordism is preserved under Ψ.

Proposition 1.7.1. *Let ν be an admissible function. Then*

$$\begin{aligned}&\Psi\big((S^n \times S^n \times \mathbb{R})\#^{\nu}_{\Delta_S\times\{0\}} N^*_{\Delta_S\times\{0\}}\big)\\ &\qquad = (S^n \times S^n \times \mathbb{R})\#^{\nu}_{\Delta_S\times\{0\}} N^*_{\Delta_S\times\{0\}}.\end{aligned}$$

Proof. It is enough to show that

$$\Psi(\widehat{H}_\nu) = \widehat{H}_\nu.$$

Let us first unwrap the definition of $\widehat{H}_\nu$. The flow of $\mu_{\hat{\pi}}$ can be rewritten in terms of component-wise flows. For this, we introduce the norm function on $T^*\mathbb{R}$ given by

$$\mu^{\mathbb{R}}\colon T^*\mathbb{R} \longrightarrow \mathbb{R},\ \mu^{\mathbb{R}}(q,p) = |p|.$$

Recall that $\mu\colon T^*S \longrightarrow \mathbb{R}$ is the Hamiltonian function $\mu(\xi) = \|\xi\|$ we used earlier to define τ_S. Then for $(\xi_1, \xi_2, p) \in T^*S^n \oplus T^*S^n \oplus T^*\mathbb{R}$ we have

$$\psi_t^{\mu_{\hat{\pi}}}(\xi_1, \xi_2, p) = \left(\xi_1, \psi^{\mu}_{\frac{t\|\xi_2\|}{\sqrt{\|\xi_2\|^2+p^2}}}(\xi_2), \psi^{\mu^{\mathbb{R}}}_{\frac{t|p|}{\sqrt{\|\xi\|^2+p^2}}}(p)\right).$$

We introduce the following abbreviations:

$$s(\|\xi\|, |p|) = \nu\left(\sqrt{\|\xi\|^2 + p^2}\right)\frac{\|\xi\|}{\sqrt{\|\xi\|^2 + p^2}}$$

and

$$r(\|\xi\|, |p|) = \nu\left(\sqrt{\|\xi\|^2 + p^2}\right)\frac{|p|}{\sqrt{\|\xi\|^2 + p^2}}.$$

With this notation $\widehat{H}_\nu$ is the set

$$\left\{\left(\xi, \psi^{\mu}_{s(\|\xi\|,|p|)}(-\xi), \psi^{\mu^{\mathbb{R}}}_{r(\|\xi\|,|p|)}(0,p)\right) \,\middle|\, \begin{array}{c} \xi \in T^*_\epsilon S, p \in \mathbb{R}, \\ 0 < \sqrt{\|\xi\|^2 + p^2} < \epsilon \end{array}\right\}.$$

Let $\xi \in T^*S$ and $p \in \mathbb{R}$ such that

$$\sqrt{\|\xi\|^2 + p^2} < \epsilon.$$

Elements of the handle $\widehat{H}_\nu$ are of the form

$$\alpha := (\xi, \psi^{\mu}_{s(\|\xi\|,|p|)}(-\xi), (r(\|\xi\|, |p|), p)).$$

Therefore $\Phi(\widehat{H}_\nu)$ consists of the elements

$$\Phi(\alpha) := (c(-\psi^{\mu}_{s(\|\xi\||p|)}(-\xi)), -c(\xi), (r(\|\xi\|, |p|), p)).$$

Renaming $\zeta := c(-\psi^{\mu}_{s(\|\xi\||p|)}(-\xi))$ and using (1.12) and (1.13) we compute

$$\begin{aligned}\Phi(\alpha) &= (\zeta, -c(-\psi^{\mu}_{-s(\|c(\zeta)\|,|p|)}(-c(\zeta))), (r(\|c(\zeta)\|, |p|), p)) \\ &= (\zeta, c(\psi^{\mu}_{-s(\|\zeta\|,|p|)}(c(-\zeta))), (r(\|\zeta\|, |p|), p)) \\ &= (\zeta, \psi^{\mu}_{s(\|\zeta\|,|p|)}(-\zeta), (r(\|\zeta\|, |p|), p)).\end{aligned}$$

These are precisely the elements of $\widehat{H}_\nu$. Therefore $\Phi(\widehat{H}_\nu) = \widehat{H}_\nu$ and the Proposition follows. $\square$

Next we turn to the suspension part of the cobordism. Let $(S^n \times S^n)\#^{\nu_t}_{\Delta_S}\Delta$, $t \in [0,1]$ be a Hamiltonian isotopy, where all ν_t are admissible, except for ν_1 which coincides with $\nu_\epsilon^{\mathrm{Dehn}}$. The Hamiltonian $K_t\colon T^*(S^n \times S^n) \longrightarrow T^*(S^n \times S^n)$ generating this isotopy can be chosen to be of the form $K_t(\xi_1, \xi_2) = K_t(\|\xi_1\|, \|\xi_2\|)$, see [MW18, Lemma 3.6]. Moreover, K_t can be chosen to be zero for t close to 0 or 1. The suspension cobordism associated to this Hamiltonian isotopy is the cylindrical extension of the Lagrangian

$$\mathcal{S} := \left\{ (\psi_t^K(x), t - iK_t(\psi_t^K(x))) \in T^*S^n \times T^*S^n \times \mathbb{C} \left| \begin{matrix} x \in H^{\nu_0}, \\ t \in [0,1] \end{matrix} \right. \right\}.$$

Proposition 1.7.2. $\Psi(\mathcal{S}) = \mathcal{S}$.

Proof. Elements of $\mathcal{S}$ can be written as

$$\alpha = \Big(\xi, \psi^\mu_{\nu_t(\|\xi\|)}(-\xi), t - iK_t(\|\xi\|, \|\xi\|)\Big)$$

for some $\xi \in T^*_\epsilon S$. Elements of the corresponding part of $\Psi(\mathcal{S})$ are of the form

$$\Big(c(-\psi^\mu_{\nu_t(\|\xi\|)}(-\xi), -c(\xi), t - iK_t(\|\xi\|, \|\xi\|)\Big)$$

for $\xi \in T^*_\epsilon S$. As before, we see that these elements are precisely the elements of $\mathcal{S}$ via the transformation $\zeta = c(-\psi^\mu_{\nu_t(\|\xi\|)}(-\xi))$. Therefore $\Psi(\mathcal{S}) = \mathcal{S}$. $\square$

1.7.2 The symmetry of the Mak-Wu cobordism. Let (V, φ) be a Weinstein neighbourhood of S with $\varphi(V) = T^*_\epsilon S$ for some $\epsilon > 0$. As in the proof of Proposition 1.1.4 let $\psi_t\colon M \longrightarrow M$ be a Hamiltonian isotopy such that $g' = g\psi_1$ is an anti-symplectic map satisfying

$$g'_T := \varphi g' \varphi^{-1} = T^*\sigma \text{ on } T^*_\delta S^n$$

for $\sigma = \mathrm{id}_{S^n}$ or $\sigma = r$ and some $\delta > 0$. Let ν be an admissible function such that the handle $\widehat{H}_\nu$ is contained in $T^*_\delta S^n \oplus T^*_\delta S^n \oplus T^*_\delta \mathbb{R}$. Moreover, we choose an isotopy ν_t between ν and $\nu_1 = \nu_\delta^{\mathrm{Dehn}}$, where all ν_t are supported in $T^*_\delta S^n$.

We show that Theorem 1.1.5 is satisfied for the cobordism V_{MW} constructed from these choices and $\Phi^{g'}$. That is, we show

$$\left(\Phi^{g'} \times \mathrm{id}\right)(V_{MW}) = V_{MW}.$$

Proof of Theorem 1.1.5. On the neighbourhood

$$(\tilde{\varphi} \times \mathrm{id})^{-1}(T^*_\delta S^n \oplus T^*_\delta S^n \oplus T^*_\delta \mathbb{R})$$

of the surgery we have

$$\Phi^{g'} \times \mathrm{id} = (\tilde{\varphi} \times \mathrm{id})^{-1} \Psi (\tilde{\varphi} \times \mathrm{id}),$$

where Ψ is the symplectomorphism from the previous section associated to $T^*\sigma$. Therefore, Proposition 1.7.1 implies that $\Phi^{g'} \times \mathrm{id}$ preserves

$$(S \times S \times \mathbb{R}) \,\#_{\Delta_S \times \{0\}} \,(\Delta \times i\mathbb{R}).$$

Hence, the surgery part of the cobordism V_{MW} is mapped to itself under $\Phi^{g'} \times \mathrm{id}$.

Similarly, Proposition 1.7.2 shows that the suspension part of V_{MW} is preserved under $\Phi^{g'} \times \mathrm{id}$. The symmetry of the surgery part and the symmetry of the suspension part together prove Theorem 1.1.5. □

Remark 1.7.3. For any anti-symplectic map g satisfying assumption $(\star)$ and $g\tau_S g^{-1} = \tau_S^{-1}$, it can be shown that $(\Phi^g \times \mathrm{id})(V_{MW})$ is *almost* horizontally isotopic to V_{MW} via a Hamiltonian isotopy Ψ_t that preserves the ends $S \times S$ and Γ_{id}, but acts by a Hamiltonian loop on the third end $\Gamma_{\tau_S^{-1}}$. We expect that in the symplectically aspherical case, it follows

$$\alpha_{(\Phi^g \times \mathrm{id})(V_{MW})} = \alpha_{\Psi_1(V_{MW})} = \alpha_{V_{MW}},$$

even though Ψ_t is not a genuine horizontal isotopy. This provides an alternative viewpoint on Theorem 1.1.1.

1.8 Examples.

In [Sei96], Seidel computed Floer cohomology of products of disjoint Dehn twists on surfaces of genus ≥ 2. As a special case, his result yields a $\mathbb{Z}$-graded isomorphism

$$\mathrm{HF}^*(\tau_S^{-1}) \cong \mathrm{H}^*(M \backslash S; \Lambda). \tag{1.14}$$

Later, Gautschi [Gau03] generalised Seidel's result to diffeomorphisms of finite type, still on surfaces. Recently Pedrotti [Ped22] suggested $\mathbb{Z}_2$-graded version of (1.14) for rational, W^+-monotone symplectic manifolds of dimension at least 4. The W^+-condition is explained in Seidel [Sei97b]. It is immediate that symplectically aspherical manifolds are W^+-monotone.

It turns out that the automorphism c_* on $HF(\tau_S^{-1})$ corresponds to the (topologically induced) map c^* on singular cohomology. More precisely, the following diagram commutes:

$$\begin{tikzcd} \mathrm{HF}^*(\tau_S^{-1}) \arrow[r, "\cong"] \arrow[d, "c_*"] & \mathrm{H}^*(M \backslash S; \Lambda) \arrow[d, "c^*"] \\ \mathrm{HF}^*(\tau_S^{-1}) \arrow[r, "\cong"] & \mathrm{H}^*(M \backslash S; \Lambda). \end{tikzcd} \tag{1.15}$$

Together with Theorem 1.1.1 this allows us to deduce topological restrictions on the element $A \in \mathrm{HF}^*(\tau_S^{-1})$ and sometimes enables us to compute A. We end this chapter with some concrete examples in two dimensions.

Example 1.8.1 (Genus 2 surface)**.** Consider the genus 2 surface Σ_2. Take S to be a separating curve, going once around between the two holes, as in Figure 1.6. Consider the Dehn twist τ_S along S.

As in [Sei96] we can work over $\mathbb{Z}_2$ instead of the Novikov field, and the Floer cohomology groups are $\mathbb{Z}$-graded.

τ_S splits into the product of two anti-symplectic involutions: Take c to be the anti-symplectic involution which is a reflection along S. It is straight-forward to check that $c\tau_S c$ is Hamiltonian isotopic to τ_S^{-1}.

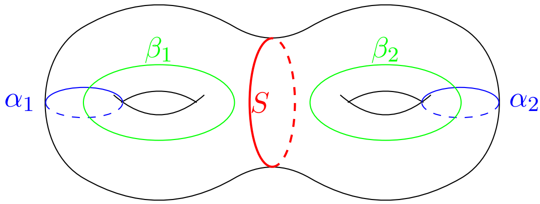


Figure 1.6: Genus 2 surface with Lagrangian sphere S.

We compute $c_*\colon HF^*(\tau_S^{-1}) \to HF^*(\tau_S^{-1})$. By the isomorphism (1.14) we get

$$\begin{aligned} HF^*(\tau_S^{-1}) &\cong H^*(\Sigma\backslash S;\mathbb{Z}_2) \\ &\cong H^*(S^1\vee S^1;\mathbb{Z}_2)\oplus H^*(S^1\vee S^1;\mathbb{Z}_2) \\ &\cong \mathbb{Z}_2[pt_1]\oplus\mathbb{Z}_2\alpha_1\oplus\mathbb{Z}_2\beta_1\oplus\mathbb{Z}_2[pt_2]\oplus\mathbb{Z}_2\alpha_2\oplus\mathbb{Z}_2\beta_2. \end{aligned}$$

In degree 0, the matrix representing c^* on $H^0(\Sigma\backslash S;\mathbb{Z}_2)$ with respect to the basis $[pt_1],[pt_2]$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows from Theorem 1.1.1 that $A = [pt_1] + [pt_2]$.

Example 1.8.2 (Higher genus surfaces)**.** Similarly, we can consider any surface Σ of genus $g \geq 2$ and a separating circle S in it that is the fixed point set of a reflection. Then $HF^0(\tau_S^{-1}) \cong \mathbb{Z}_2\oplus\mathbb{Z}_2$, where each of the two summands corresponds to one of the connected components of $\Sigma\backslash S$. Theorem 1.1.1 implies $A = (1,1)$.

Example 1.8.3 (Torus)**.** Let S be any non-contractible embedded circle in the torus T^2. Using the long exact sequence (1.3) applied to $K = \Delta$ one computes

$$HF^0(\tau_S^{-1}) \cong H^{even}(T^2;\Lambda)/H^0(S;\Lambda) \cong H^2(T^2;\Lambda) \cong \Lambda.$$

For any anti-symplectic involution $c\colon T^2 \to T^2$ satisfying $c(S) = S$, it follows that $c_* = \mathrm{id}$.

Chapter 2

Lagrangian Hofer metric and barcodes

2.1 Introduction and main results

Let $(M, \omega = -\mathrm{d}\lambda)$ be an exact symplectic manifold. Consider the group $\mathrm{Ham}(M)$ of compactly supported Hamiltonian diffeomorphisms. Any compactly supported Hamiltonian function $H \in C^\infty([0,1] \times M)$ generates a Hamiltonian flow $\{\psi_t^H\}$. The Hofer norm of a Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(M)$ is given by

$$\|\varphi\|_H = \inf \left\{ \int_0^1 \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \,\mathrm{d}t \,\middle|\, \psi_1^H = \varphi \right\}.$$

Let L and L' be closed connected Lagrangian submanifolds in M that are Hamiltonian isotopic. The Lagrangian Hofer distance between L and L' is defined by

$$d_H(L, L') = \inf \left\{ \|\varphi\|_H \,\middle|\, \varphi(L) = L' \right\}.$$

For transversely intersecting and exact Lagrangians L and L' we consider the Floer complex $\mathrm{CF}(L, L')$ over $\mathbb{Z}_2$. A choice of primitives of the exact 1-forms $\lambda|_L$ and $\lambda|_{L'}$ gives rise to an action functional $\mathcal{A}$ that induces a filtration on the Floer complex $\mathrm{CF}(L, L')$. Therefore, the homology group $\mathrm{HF}(L, L')$ of $\mathrm{CF}(L, L')$ becomes a persistence module $\mathrm{HF}^{\leq \bullet}(L, L')$. The barcode $\mathcal{B}(L, L')$ associated with it gives rise to a number of invariants for the pair (L, L'). One of them is the Lagrangian spectral metric $\gamma(L, L')$, which is the largest distance between two infinite bars in $\mathcal{B}(L, L')$ [Vit92, KS21]. The boundary depth $\beta_1(L, L')$, studied in [Ush13], is the length of the longest finite bar. In this thesis, we also consider the lengths of the other finite bars. We denote by

$$\beta_1(L,L') \geq \beta_2(L,L') \geq \cdots \geq \beta_k(L,L')$$

the lengths of the finite bars ordered by their size. While the barcode $\mathcal{B}(L,L')$ actually depends on the choice of primitives of $\lambda|_L$ and $\lambda|_{L'}$, the numbers $\gamma(L,L')$ and $\beta_i(L,L')$, $i \in \{1,\ldots,k\}$, are independent of it. Kislev-Shelukhin [KS21] proved the following inequalities

$$\beta_1(L,L') \leq \gamma(L,L') \leq d_H(L,L'). \tag{2.1}$$

In this thesis we prove a converse inequality for equators in the cylinder. From now on, we work in $\Sigma := S^1 \times (-1,1)$. In local coordinates (q,p), the standard symplectic form on Σ is $\omega = \mathrm{d}q \wedge \mathrm{d}p = -\mathrm{d}\lambda$ for $\lambda = p\mathrm{d}q$. Let $L_0 = S^1 \times \{0\} \subset \Sigma$ denote the zero-section. We are interested into the set $\mathcal{L}(L_0)$ of all Lagrangians $L \subset \Sigma$ which are Hamiltonian isotopic to L_0.

Theorem 2.1.1. *Suppose that $L, L' \in \mathcal{L}(L_0)$ intersect transversely in $2n$ points. Then*

$$d_H(L,L') \leq \sum_{j=1}^{n-1} 2^j \beta_j(L,L') + \gamma(L,L').$$

Using the inequalities (2.1) we get the following bound:

Corollary 2.1.2. *For any L, L' as above,*

$$\gamma(L,L') \leq d_H(L,L') \leq 2^n \gamma(L,L').$$

2.1.1 Relation to previous work.

Theorem 2.1.1 was inspired by the following result of Khanevsky.

Theorem 2.1.3 ([Kha11][1]). *There exist constants k and c such that for any transversely intersecting $L, L' \in \mathcal{L}(L_0)$,*

$$d_H(L,L') \leq k \cdot \#(L \cap L') + c.$$

[1] Khanevsky proved this result for a wider class of surfaces and Lagrangians, not just the cylinder.

Moreover, Khanevsky proved that Hofer's distance on $\mathcal{L}(L_0)$ is unbounded [Kha09]. Let $L \in \mathcal{L}(L_0)$ and consider a sequence $\{L_n\} \subset \mathcal{L}(L_0)$ of Lagrangians, transverse to L, such that $d_H(L, L_n) \xrightarrow{n\to\infty} \infty$. It follows from Khanevsky's result that $\#(L \cap L_n) \xrightarrow{n\to\infty} \infty$. In contrast to Hofer's metric d_H, the spectral distance γ is bounded [She22c]. Therefore, Corollary 2.1.2 shows that the number of bars in $\mathcal{B}(L, L_n)$ tends to ∞. This recovers $\#(L \cap L_n) \xrightarrow{n\to\infty} \infty$ because the intersection points are in bijection with the endpoints of the bars in $\mathcal{B}(L, L_n)$.

It is known that γ is C^0-continuous [BHS22], and even Hausdorff continuous [She22b]. In contrast, d_H is not expected to be C^0-continuous. Our result gives the following insight into convergence in Hofer's metric.

Corollary 2.1.4. *Suppose $\{L_n\}_{n\in\mathbb{N}}$ is a sequence of Lagrangians, transverse to L, that C^0-converges to L. If $k_n := \#(L \cap L_n)$ is bounded, then $\{L_n\}$ converges to L in the Lagrangian Hofer metric.*

2.1.2 Outline of the proof. We explain the strategy to prove Theorem 2.1.1. Let $L, L' \in \mathcal{L}(L_0)$ be two transverse Lagrangians that intersect in $2n$ points. The action spectrum $\{\mathcal{A}(q) | q \in L \cap L'\}$ coincides with the endpoints of the bars in $\mathcal{B}(L, L')$. Therefore, the barcode $\mathcal{B}(L, L')$ consists of $n-1$ finite bars and 2 infinite bars. We prove the Theorem by induction on the number of intersection points.

Base case: If there are only two intersection points, say q and p, then Hofer's distance is equal to the area of one of the Floer strips connecting q and p. This is also the difference between the action values of the two infinite bars in $\mathcal{B}(L, L')$, hence $d_H(L, L') = \gamma(L, L')$.

Induction Step: Suppose Theorem 2.1.1 holds for Lagrangians transversely intersecting in $2(n-1)$ points. Let L and L' be as above intersecting in $2n \geq 4$ many points. We assume that any two intersection points $q \neq p \in L \cap L'$ satisfy $\mathcal{A}(q) \neq \mathcal{A}(p)$. In order to use the induction hypothesis, we construct a Lagrangian L'' using Khanevsky's construction for *deleting a leaf* [Kha09]. A *leaf* is a connected component of $\Sigma \backslash (L \cup L')$, which is bounded by one connected component of $L \backslash (L \cap L')$ and one connected component of $L' \backslash (L \cap L')$. See Figure 2.1 for an illustration and section 2.4 for more details.

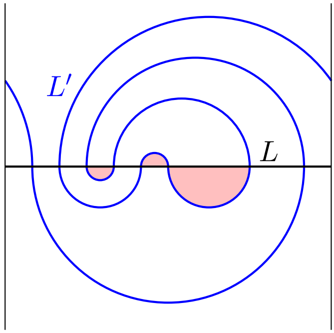


Figure 2.1: A pair of equators (L, L') in the cylinder (the left and right vertical lines are identified). The 3 leaves are coloured in red.

In section 2.5 we show the following

Proposition 2.1.5. *Let $[a, b)$ be a shortest finite bar in $\mathcal{B}(L, L')$. Let $\bar{q}, \bar{p} \in L \cap L'$ be the intersection points satisfying $\mathcal{A}(\bar{q}) = b$ and $\mathcal{A}(\bar{p}) = a$. Then $\bar{q}$ and $\bar{p}$ are connected by a leaf.*

This leaf has area $\mathcal{A}(\bar{q}) - \mathcal{A}(\bar{p}) = b - a = \beta_{n-1}(L, L')$.

We can therefore remove the intersection points $\bar{q}$ and $\bar{p}$ using Khanevsky's construction for deleting a leaf: For any $\epsilon > 0$ there exists a Hamiltonian diffeomorphism φ of Hofer norm

$$\|\varphi\|_H \leq \beta_{n-1}(L, L') + \epsilon$$

such that $L'' := \varphi(L')$ intersects L transversely and the number of intersection points is $2(n-1)$.

The Floer complexes $\mathrm{CF}(L, L')$ and $\mathrm{CF}(L, L'')$ can be endowed with action filtrations such that their persistence homologies are $\frac{\beta_{n-1}(L,L')+\epsilon}{2}$-interleaved. It follows that

$$|\beta_j(L, L') - \beta_j(L, L'')| \leq \beta_{n-1}(L, L') + \epsilon \tag{2.2}$$

for all $1 \leq j \leq n-2$ and

$$|\gamma(L, L') - \gamma(L, L'')| \leq \beta_{n-1}(L, L') + \epsilon. \tag{2.3}$$

The theorem holds true for (L, L'') by the induction hypothesis. We therefore get

$$\begin{aligned}
d_H(L, L') &\leq d_H(L, L'') + d_H(L'', L') \\
&\leq \left(\sum_{j=1}^{n-2} 2^j \beta_j(L, L'') + \gamma(L, L'') \right) + \big(\beta_{n-1}(L, L') + \epsilon\big) \\
&\leq \sum_{j=1}^{n-2} 2^j \left(\beta_j(L, L') + \beta_{n-1}(L, L') + \epsilon\right) \\
&\qquad + \big(\gamma(L, L') + \beta_{n-1}(L, L') + \epsilon\big) + \beta_{n-1}(L, L') + \epsilon \\
&= \sum_{j=1}^{n-2} 2^j \beta_j(L, L') + \gamma(L, L') \\
&\qquad\qquad + \left(\left(\sum_{j=1}^{n-2} 2^j \right) + 2 \right) (\beta_{n-1}(L, L') + \epsilon) \\
&= \sum_{j=1}^{n-1} 2^j \beta_j(L, L') + \gamma(L, L') + 2^{n-1}\epsilon,
\end{aligned}$$

where we used (2.2), (2.3) in the third inequality. Taking the limit as $\epsilon \to 0$ finishes the induction step.

Remark 2.1.6. (i) The interleaving between the persistence modules $\mathrm{HF}(L, L')$ and $\mathrm{HF}(L, L'')$, as well as the inequalities (2.2) and (2.3) are well-known and hold in wide generality under the name of *stability*, see for example [KS21, Ush13]. Instead of directly applying the general theory to our special case, we include in section 2.4 a combinatorial proof for equators in the cylinder.

(ii) The bound in Theorem B is not expected to be tight. The weights are coming from induction and the bounds (2.2) and

(2.3). It is possible to get a better control over the change of the barcode when deleting a leaf. While this attempt does not yield a better bound in Theorem B we include an explanation of these ideas in section 2.6.

(iii) The bound in Theorem B can be computed algorithmically. We elaborate more on this in Remark 2.3.5.

2.1.3 Organization of chapter 2. In section 2.2 we explain filtered combinatorial Floer homology for Lagrangians in the cylinder. The machinery of persistence modules and barcodes is reviewed in section 2.3. In section 2.4 we recall Khanevsky's construction of deletion of a leaf and analyse its effect on persistent Floer homology and its barcodes. In particular, we give a combinatorial proof for inequalities (2.2) and (2.3). Proposition 2.1.5 is proved in section 2.5. Section 2.6 establishes a better control of the change in the barcode when deleting a leaf.

2.2 Filtered combinatorial Floer homology

In this section we introduce the filtered Floer complex for transversely intersecting Lagrangians in $\mathcal{L}(L_0)$. We focus on our object of interest, the cylinder, even though the concepts make sense in much more generality. We use combinatorial Floer homology for curves in surfaces as developed by de Silva–Robbin–Salamon. We follow closely the exposition in [dSRS14]. [2]

2.2.1 Lagrangian Floer complex. Let $L, L' \in \mathcal{L}(L_0)$ such that L intersects L' transversely. The Floer complex of the pair (L, L') is a chain complex whose underlying $\mathbb{Z}_2$-vector space is generated by the intersection points. More concretely, denoting $P = L \cap L'$ we

[2] The setting in [dSRS14] does not always cover the case of two isotopic curves. However, the proofs for most of the statements we use in this thesis carry over to our setting. Whenever not, we indicate a proof.

define

$$\mathrm{CF}(L, L') = \bigoplus_{p \in P} \mathbb{Z}_2 p.$$

The differential is obtained from counting the number of so-called smooth lunes connecting two intersection points. We formalise this as follows. Let

$$\mathbb{D} := \{z \in \mathbb{C} \,|\, \mathrm{Im} z \geq 0, |z| \leq 1\}$$

be the standard half disc. Let $q, p \in P$. A *smooth lune* from q to p is a smooth orientation-preserving immersion $u \colon \mathbb{D} \longrightarrow \Sigma$ satisfying the boundary conditions

$$u(\mathbb{D} \cap \mathbb{R}) \subseteq L, \qquad u(\mathbb{D} \cap S^1) \subseteq L', \qquad u(-1) = q, \qquad u(1) = p.$$

Figure 2.2 below shows an example of a smooth lune from q to p.

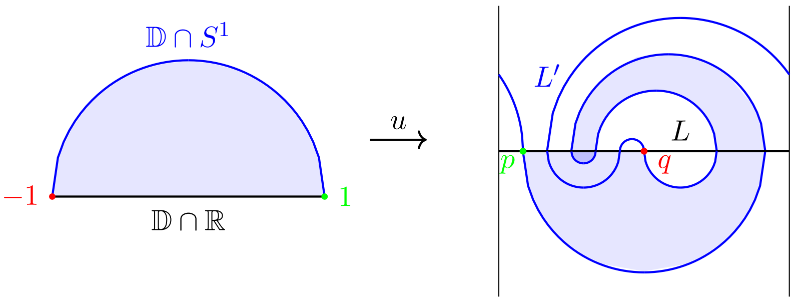


Figure 2.2: A smooth lune from q to p.

Two smooth lunes $u, u' \colon \mathbb{D} \longrightarrow \Sigma$ are called equivalent if there exists an orientation-preserving diffeomorphism $\varphi \colon \mathbb{D} \longrightarrow \mathbb{D}$ such that

$$\varphi(1) = 1, \ \varphi(-1) = -1 \ \text{ and } \ u' = u \circ \varphi.$$

Define $n(q, p) \in \mathbb{Z}_2$ to be the number modulo 2 of equivalence classes of smooth lunes from q to p. The differential is defined on the generators $q \in P$ by

$$\partial(q) = \sum_{p \in P} n(q, p) p.$$

The linear extension to $\partial\colon \mathrm{CF}(L, L') \longrightarrow \mathrm{CF}(L, L')$ satisfies $\partial^2 = 0$. [3] Therefore, $(\mathrm{CF}(L, L'), \partial)$ is a chain complex.

The following property of smooth lunes will be useful.

Lemma 2.2.1. *Let $u\colon \mathbb{D} \to \Sigma$ be a smooth lune from q to p. Let*

$$q = x_0, x_1, \ldots, x_l, x_{l+1} = p$$

be the points in $P \cap u(\mathbb{D} \cap S^1)$ ordered by their ordering on L' when following L' from q towards p along $u(\mathbb{D} \cap S^1)$ (see Figure 2.3). Assume $x_1, x_l \notin \{q, p\}$. Then x_1 and x_l are not contained in $u(\mathbb{D} \cap \mathbb{R})$.

Proof. If not, u is not an immersion at $u^{-1}(x_1)$ or $u^{-1}(x_l)$. $\square$

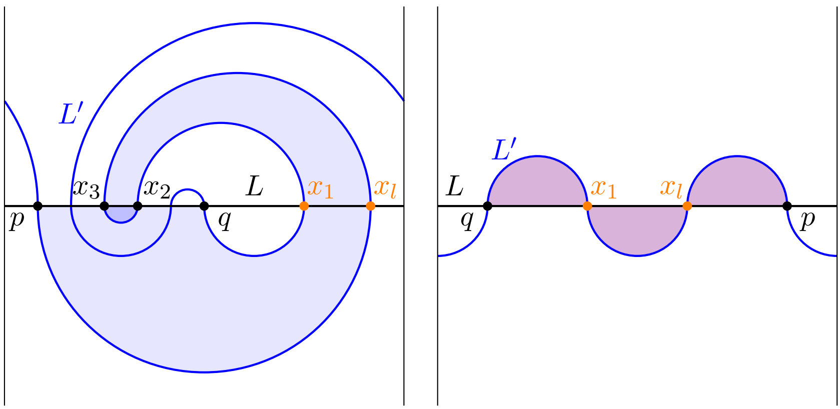


Figure 2.3: A smooth lune on the left, where x_1 and x_l do not lie in $u(\mathbb{D} \cap \mathbb{R})$. The violet area on the right can not be obtained by a smooth lune.

Remark 2.2.2. [dSRS14, Theorem 6.7] characterises smooth lunes in terms of their boundary behavior. Here are some consequences.

[3]The proof in [dSRS14] carries over to this setting. It is based on studying *broken hearts*, which are immersed discs with one non-convex corner. A quick proof can be found in [Abo08, Lemma 2.11].

In a small enough neighbourhood U of q we may choose coordinates (x, y) such that $L \cap U$ coincides with the x-axis and $L' \cap U$ coincides with the y-axis. If a smooth lune u leaves q, $u(\mathbb{D}) \cap U$ lies entirely either in the first quadrant $(x \geq 0, y \geq 0)$ or in the third quadrant $(x \leq 0, y \leq 0)$. Similarly, if a lune u enters p, then locally $u(\mathbb{D})$ lies either in the second quadrant $(x \leq 0, y \geq 0)$ or in the forth quadrant $(x \geq 0, y \leq 0)$. For any $q, p \in L \cap L'$, there are at most 2 lunes from q to p. Figure 2.4 shows an example with 2 lunes.

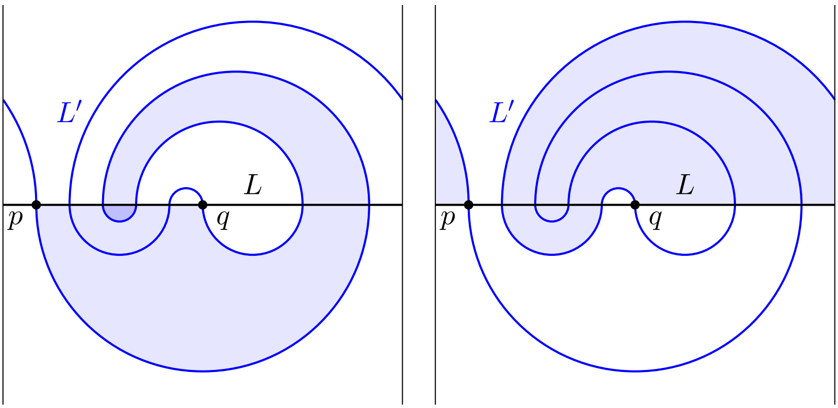


Figure 2.4: Two lunes from q to p.

2.2.2 Filtration. Any $L \in \mathcal{L}(L_0)$ is exact, meaning that the 1-form $\lambda|_L \in \Omega^1(L)$ is exact. We call a function

$$h_L : L \longrightarrow \mathbb{R}$$

such that $\mathrm{d}h_L = \lambda|_L$ a *marking* of L. The marking h_L is unique up to an additive constant because L is connected. The filtration on $\mathrm{CF}(L, L')$ we introduce below will depend on a choice of markings of L and L'.

Fix two markings h_L and $h_{L'}$ of L and L' respectively. Consider the space of paths from L to L', namely

$$\Omega_{L,L'} := \left\{\gamma \in C^\infty([0,1]) \,|\, \gamma(0) \in L, \gamma(1) \in L'\right\}.$$

We identify tangent vectors ξ at $\gamma \in \Omega_{L,L'}$ with vector fields $\xi(t) \in T_{\gamma(t)}\Sigma$ along γ. We define the action functional $\mathcal{A}\colon \Omega_{L,L'} \longrightarrow \mathbb{R}$ by

$$\mathcal{A}(\gamma) = h_L(\gamma(1)) - h_{L'}(\gamma(0)) - \int_0^1 \lambda_{\gamma(t)}(\dot\gamma(t))\, \mathrm{d}t.$$

The exterior derivative of $\mathcal{A}$ is

$$\mathrm{d}\mathcal{A}_\gamma(\xi) = \int_0^1 \omega(\dot\gamma(t), \xi(t))\, \mathrm{d}t$$

for any $\xi \in T_\gamma\Omega_{L,L'}$. We view P as a subset of $\Omega_{L,L'}$ by viewing $p \in P$ as a constant path. Let $q, p \in P$ and let u be a smooth lune from q to p. It follows

$$\int_{\mathbb{D}} u^*\omega = \mathcal{A}(q) - \mathcal{A}(p). \tag{2.4}$$

Remark 2.2.3. Any two neighbouring intersection points are connected by a smooth lune. More precisely, choose an orientation on L and order the intersection points $s_1, s_2, \ldots, s_{2n}$ according to their order on L. Then for each $1 \le i \le 2n$, there exists a smooth lune from s_i to s_{i+1}, or a smooth lune from s_{i+1} to s_i. (Here, we use cyclic notation for the indices, i.e. $s_{2n+1} = s_1$ etc.) We explain these smooth lunes further in section 2.4.3. It follows that equation (2.4) determines the action functional on P uniquely up to an additive constant.

We have $\int_{\mathbb{D}} u^*\omega \ge 0$ because u is an orientation-preserving immersion. It follows that

$$\mathcal{A}(q) - \mathcal{A}(p) \ge 0,$$

whenever there is a smooth lune from q to p. In particular, the differential lowers filtration:

$$\mathcal{A}(\partial q) < \mathcal{A}(q).$$

Therefore, for any $\alpha \in \mathbb{R}$,

$$\mathrm{CF}^{\leq \alpha}((L,h_L),(L',h_{L'})) := \bigoplus_{\substack{p\in P,\\ \mathcal{A}(p)\leq\alpha}} \mathbb{Z}_2 p$$

is a subcomplex of $\mathrm{CF}(L,L')$. Moreover, for each $\alpha \leq \beta$ there are inclusions

$$\mathrm{CF}^{\leq \alpha}((L,h_L),(L',h_{L'})) \subseteq \mathrm{CF}^{\leq \beta}((L,h_L),(L',h_{L'})).$$

The collection

$$\mathrm{CF}^{\leq \bullet}((L,h_L),(L',h_{L'})) = \left\{\mathrm{CF}^{\leq \alpha}((L,h_L),(L',h_{L'}))\right\}_{\alpha\in\mathbb{R}}$$

is called the filtered Floer complex of the pair $((L,h_L),(L',h_{L'}))$.

2.3 Persistent Floer homology

2.3.1 Persistence modules. For an overview of the theory of persistent homology and its use in symplectic topology see [PRSZ20]. We follow closely chapters 1 and 2 from [PRSZ20]. [4]

Definition 2.3.1. A *persistence module* over $\mathbb{Z}_2$ consists of an $\mathbb{R}$-indexed family of $\mathbb{Z}_2$-vector spaces $\{V_t\}_{t\in\mathbb{R}}$ and linear maps

$$f_{s,t}\colon V_s \longrightarrow V_t$$

for any $s \leq t$ satisfying

(i) $f_{t,t} = \mathrm{id}$ for any $t \in \mathbb{R}$,

(ii) $f_{s,t} \circ f_{r,s} = f_{r,t}$ for any $r \leq s \leq t$.

[4] The definitions here differ slightly from those in [PRSZ20] in the convention for semicontinuity. Since we consider $\mathrm{CF}^{\leq\alpha}$ and not $\mathrm{CF}^{<\alpha}$ we need to work with intervals of the form $[a,b)$ and not $(a,b]$.

Any filtered chain complex gives rise to a persistence module by applying the homology functor to the subcomplexes and the inclusions. More precisely, let (C, ∂) be a chain complex over $\mathbb{Z}_2$ with an $\mathbb{R}$-indexed filtration given by a family $\{C^\alpha\}_{\alpha \in \mathbb{R}}$ of subcomplexes of C such that for all $\alpha \subset \beta$ we have $C^\alpha \subseteq C^\beta$. Then we get a persistence module with vector spaces $\mathrm{H}(C^\alpha)$ and the persistence maps

$$\mathrm{incl}_* \colon \mathrm{H}(C^\alpha) \longrightarrow \mathrm{H}(C^\beta)$$

for $\alpha \leq \beta$.

Definition 2.3.2. A persistence module V is *of finite type* if

(i) For any $t \in \mathbb{R}$, there exists $\epsilon > 0$ such that $f_{s,t}$ is an isomorphism for any $s \in [t, t+\epsilon)$.

(ii) There exist $t_1, \ldots, t_m \in \mathbb{R}$ such that for all $t \in \mathbb{R} \setminus \{t_1, \ldots, t_m\}$, there exists $\epsilon > 0$ such that $f_{s,r}$ is an isomorphism for any $s \leq r \in (t-\epsilon, t+\epsilon)$.

(iii) There exists s_- such that $V_s = 0$ for any $s < s_-$.

Let (C, ∂) be a complex with finite basis $E = \{q_1, \ldots, q_l\}$. Suppose we are given a function $\mathcal{A} \colon E \longrightarrow \mathbb{R}$. Extend $\mathcal{A}$ to C via

$$\mathcal{A}\left(\sum_{i=1}^{l} a_i q_i\right) = \max\{\mathcal{A}(q_i) \mid a_i \neq 0\}.$$

Here, $\mathcal{A}(0) = -\infty$. If $\mathcal{A}(\partial x) \leq \mathcal{A}(x)$ for all $x \in C$ then $\mathcal{A}$ induces a filtration on C with the subcomplexes

$$C^{\leq \alpha} = \bigoplus_{\mathcal{A}(q_i) \leq \alpha} \mathbb{Z}_2 q_i.$$

The filtered complex C is called a filtered complex with a preferred basis [PRSZ20, Section 6.2]. The resulting persistence module $\mathrm{H}(C)$ is of finite type: (i) is satisfied because the values $\mathcal{A}(q_i)$ form a discrete set. (ii) is satisfied for $t_i = \mathcal{A}(q_i)$ and (iii) is satisfied for $s_- = \min\{\mathcal{A}(q) \mid q \in E\}$.

Alternatively, one could define persistence modules through the language of functors. Let $\mathbb{R}$ be the category with objects the points in $\mathbb{R}$ and non-empty morphism spaces $Mor_{\mathbb{R}}(x, y) = \{*\}$ precisely when $x \leq y$. Then a persistence module is a functor from the category $\mathbb{R}$ to the category Vec of $\mathbb{Z}_2$-vector spaces. Using this language one easily defines the category of persistence modules as the category of functors from $\mathbb{R}$ to Vec. In particular there is a notion of morphism between persistence modules. Using the direct sum construction in Vec we also get a direct sum of persistence modules.

2.3.2 Barcodes. Isomorphism classes of persistence modules of finite type can be classified by their barcodes.

Definition 2.3.3. A *barcode of finite type* $\mathcal{B} = \{I_j\}_{j=1}^n$ is a finite multiset of intervals I_j of two possible types:

(i) Finite bars: $I_j = [a_j, b_j)$, where $a_j < b_j$ are real numbers.

(ii) Infinite bars: $I_j = [c_j, \infty)$, where c_j is a real number.

The correspondence between barcodes and persistence modules is based on interval modules. Given an interval $[a, b)$ define the persistence module $V^{[a,b)}$ by

$$V_t^{[a,b)} = \begin{cases} \mathbb{Z}_2 & \text{if } t \in [a, b), \\ 0 & \text{else} \end{cases}$$

with internal maps $f_{s,t} \colon V_s^{[a,b)} \longrightarrow V_t^{[a,b)}$

$$f_{s,t} = \begin{cases} \mathrm{id} & a \leq s \leq t < b, \\ 0 & \text{else.} \end{cases}$$

$V^{[a,b)}$ is called an interval module. It turns out that any persistence module is a direct sum of interval modules:

Theorem 2.3.4 (Structure Theorem,[CZ05])**.** *For a persistence module V of finite type, there is an isomorphism of persistence modules*

$$V \cong \bigoplus_{I \in \mathcal{B}} V^I$$

for a unique barcode $\mathcal{B} = \mathcal{B}(V)$ of finite type.

For a proof of this theorem, we recommend [PRSZ20, Chapter 2]. [5]

We now explain one way to compute the barcode of persistent homology of "Floer-type" chain complexes. We follow [PRSZ20, Section 6.2]. Consider again a filtered complex C with a preferred basis. A basis

$$F = \{e_1, \ldots, e_k, f_1, \ldots, f_k, g_1, \ldots, g_h\}$$

is called *Jordan basis* for ∂ if

(i) (Orthogonality) For all $a_f \in \mathbb{Z}_2$,

$$\mathcal{A}\left(\sum_{f \in F} a_f f\right) = \max\left\{\mathcal{A}(f) \,|\, a_f \neq 0\right\}.$$

(ii) $\partial e_i = f_i$ for $1 \leq i \leq k$.

(iii) $\partial g_i = 0$ for $1 \leq i \leq h$.

Such a basis always exists ([PRSZ20, Theorem 6.2.1]). The barcode can now be read off the action values of the Jordan basis F [PRSZ20, Theorem 6.2.2]:
$\mathcal{B}(\mathrm{H}(C))$ consists of

- the finite bars $[\mathcal{A}(f_i), \mathcal{A}(e_i))$ for $i = 1, \ldots, k$,
- the infinite bars $[\mathcal{A}(g_i), \infty)$ for $i = 1, \ldots, h$.

It follows from the orthogonality condition that $\mathcal{A}(F) = \mathcal{A}(E)$. In particular, the endpoints of the bars are precisely the action values on the preferred basis E (or equivalently F). This way of computing the barcode has first been used in [Bar94] (without the language of persistence modules) and it has been generalised to more involved situations in [UZ16]. [6]

[5] There are more general versions for persistence modules not necessarily of finite type, see [BCB20].

[6] In [UZ16] a Jordan basis arises from a *singular value decomposition* of $\partial\colon C \to \ker(\partial)$ for a Floer-type chain complex C.

2.3.3 Persistent Floer homology and its barcode.

In our case, we consider the filtered complex $\mathrm{CF}^{\leq\bullet}((L,h_L),(L',h_{L'}))$ with preferred basis P. Spelling it out, the Floer persistence module consists of the $\mathbb{Z}_2$-vector spaces

$$\left\{\mathrm{HF}^{\leq\alpha}((L,h_L),(L',h_{L'}))\right\}_{\alpha\in\mathbb{R}} = \left\{\mathrm{H}_*\left(\mathrm{CF}^{\leq\alpha}((L,h_L),(L',h_{L'}))\right)\right\}_{\alpha\in\mathbb{R}}$$

and the linear maps

$$i_{\alpha,\beta}\colon \mathrm{HF}^{\leq\alpha}((L,h_L),(L',h_{L'})) \longrightarrow \mathrm{HF}^{\leq\beta}((L,h_L),(L',h_{L'}))$$

induced by inclusion for any $\alpha\leq\beta$. We refer to this persistence module as

$$\mathrm{HF}^{\leq\bullet}((L,h_L),(L',h_{L'})).$$

Applying the Structure Theorem to $\mathrm{HF}^{\leq\bullet}((L,h_L),(L',h_{L'}))$ we get a barcode which we denote by $\mathcal{B}((L,h_L),(L',h_{L'}))$. We've seen in the previous section that the endpoints of the bars in the barcode $\mathcal{B}((L,h_L),(L',h_{L'}))$ are exactly the action values on $\mathcal{A}(P)$. Moreover, the infinite bars correspond to classes in $\mathrm{HF}(L,L')\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$. If L intersects L' in $2n$ points, the barcode $\mathcal{B}((L,h_L),(L',h_{L'}))$ therefore consists of $n-1$ finite bars and two infinite bars.

The barcode depends on h_L and $h_{L'}$, but different choices of markings yield the same barcode up to a shift. Therefore, the length of each bar is independent of the markings. We denote by

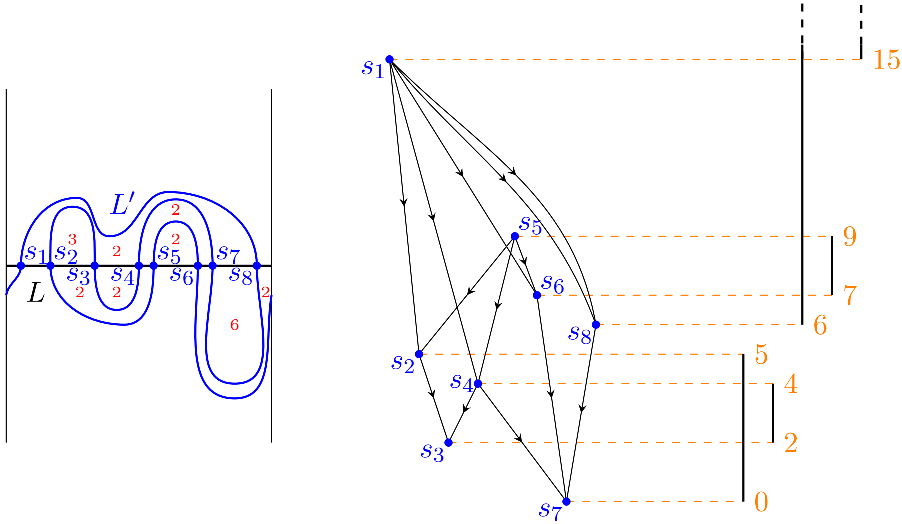
$$\beta_1(L,L')\geq\beta_2(L,L')\geq\cdots\geq\beta_{n-1}(L,L')$$

the lengths of the finite bars. Similarly if $[c_1,\infty),[c_2,\infty)$ are the infinite bars, where $c_1\leq c_2$, then

$$\gamma(L,L')=c_2-c_1$$

is independent of the markings. The quantity $\gamma(L,L')$ is called the spectral distance between L and L'.

Figure 2.5 shows an example of the barcode associated to a pair of Lagrangians.



(L, L') $CF(L, L')$ $\mathcal{B}(L, L')$

Figure 2.5: A barcode (on the right) associated to two Lagrangians (on the left). The red numbers indicate the area of the bounded white regions in the cylinder. The orange numbers indicate the action values of the intersection points.

Remark 2.3.5. There is an algorithm to construct a Jordan basis from any orthogonal basis. In particular, $\beta_i(L, L')$ and $\gamma(L, L')$ can be algorithmically computed from the filtered complex $\mathrm{CF}(L, L')$. The combinatorial characterization of smooth lunes via combinatorial lunes in [dSRS14, Theorem 6.7] suggests an algorithm [dSRS14, Remark 6.11] to compute $\mathrm{CF}(L, L')$. Combining their algorithm with an algorithm to compute barcodes makes the upper bound in Theorem B computable.

2.3.4 Stability. We follow closely [PRSZ20, Sections 1.3 and 2.2]. The Algebraic Stability Theorem states that the isomorphism in the Structure Theorem is an isometry with respect to the *interleaving* *distance* on the set of isomorphism classes of persistence modules and the *bottleneck distance* on the set of barcodes. We briefly recall these distances.

Given a persistence module V and $\delta > 0$ denote by $V[\delta]$ the shifted persistence module with $V[\delta]_t = V_{t+\delta}$ and $f_{st}^{V[\delta]} = f_{s+\delta,t+\delta}^V$. There is a canonical persistence morphism

$$\Phi_V \colon V \longrightarrow V[\delta]$$

given by $\Phi_V(v) = f_{s,s+\delta}(v)$ for $v \in V_s$.

Definition 2.3.6. A δ-*interleaving* between persistence modules V and W consists of two persistence morphisms $f \colon V \longrightarrow W[\delta]$ and $g \colon W \longrightarrow V[\delta]$ such that

$$g[\delta] \circ f = \Phi_V^{2\delta} \text{ and } f[\delta] \circ g = \Phi_W^{2\delta}.$$

The interleaving distance $d_{\text{int}}(V, W)$ is the infimum of $\delta > 0$ such that there exists a δ-interleaving between V and W.

For the bottleneck distance, we need the notion of matchings of barcodes.

Definition 2.3.7. A δ-*matching* of barcodes $\mathcal{B}$ and $\mathcal{B}'$ is a bijection $\mu \colon \mathcal{B}_0 \longrightarrow \mathcal{B}_0'$ of subsets $\mathcal{B}_0 \subseteq \mathcal{B}$ and $\mathcal{B}_0' \subseteq \mathcal{B}'$ satisfying

(i) $\mathcal{B}_0$ contains all bars from $\mathcal{B}$ of length $> \delta$,

(ii) $\mathcal{B}_0'$ contains all bars from $\mathcal{B}'$ of length $> \delta$,

(iii) If $\mu([a, b)) = [a', b')$ then $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$.

The bottleneck distance $d_{\text{bot}}(\mathcal{B}, \mathcal{B}')$ is the infimum of $\delta > 0$ such that there exists a δ-matching between $\mathcal{B}$ and $\mathcal{B}'$.

Theorem 2.3.8 (Isometry Theorem, [CdSGO16, Les15])**.**

$$d_{\text{int}}(V, W) = d_{\text{bot}}(\mathcal{B}(V), \mathcal{B}(W).$$

For a barcode $\mathcal{B}$ denote by $\beta_1(\mathcal{B}) \geq \beta_2(\mathcal{B}) \geq \dots$ the lengths of the finite bars and let $\gamma(\mathcal{B})$ be the distance between the highest and lowest infinite bars. Then

Theorem 2.3.9. *For any two barcodes $\mathcal{B}, \mathcal{B}'$ of finite type*

$$|\beta_i(\mathcal{B}) - \beta_i(\mathcal{B}')| \leq 2d_{\mathrm{bot}}(\mathcal{B}, \mathcal{B}')$$

and

$$|\gamma(\mathcal{B}) - \gamma(\mathcal{B}')| \leq 2d_{\mathrm{bot}}(\mathcal{B}, \mathcal{B}').$$

This follows from work of Usher and Zhang and the statement for the finite bars is the content of [PRSZ20, Theorem 4.2.2]. The statement for γ is very similar and well-known. We include a proof here for completeness.

Proof. Let $\mu\colon \mathcal{B}_0 \longrightarrow \mathcal{B}_0'$ be a δ-matching. Let

$$[c_1, \infty) \supseteq [c_2, \infty) \supseteq \cdots \supseteq [c_k, \infty)$$

be the infinite bars of $\mathcal{B}$ and

$$[c_1', \infty) \supseteq [c_2', \infty) \supseteq \cdots \supseteq [c_k', \infty)$$

the infinite bars of $\mathcal{B}'$. Then

$$\begin{aligned}\gamma(\mathcal{B}) - \gamma(\mathcal{B}') &= (c_k - c_1) - (c_k' - c_1') \\ &= (c_k - c_k') + (c_1 - c_1') \leq 2d_{\mathrm{bot}}(\mathcal{B}, \mathcal{B}')\end{aligned}$$

where the inequality follows from [PRSZ20, Corollary 4.1.2]. □

We finally explain how these results apply to geometry. Let M be an exact symplectic manifold and L, L' be exact Lagrangians as in the introduction. Then any Hamiltonian $H \in C^\infty([0,1] \times M)$ induces an $\frac{\mathrm{osc}(H)}{2}$-interleaving between $\mathrm{HF}^{\leq \bullet}(L, L')$ and $\mathrm{HF}^{\leq \bullet}(L, \psi_1^H(L'))$, for well-chosen filtrations [KS21]. Here,

$$\mathrm{osc}(H) = \int_0^1 \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \mathrm{d}t.$$

In particular it follows from the previous results that

$$|\beta_i(L, L') - \beta_i(L, \varphi(L'))| \leq \|\varphi\|$$

and

$$|\gamma(L, L') - \gamma(L, \varphi(L'))| \leq \|\varphi\|.$$

This shows the inequalities (2.2) and (2.3).

We include a combinatorial construction of the above interleaving for $M = \Sigma$ in section 2.4.4.

2.4 Deletion of a leaf

Throughout this section, let $L \in \mathcal{L}(L_0)$ be a Lagrangian that intersects L_0 transversely. For simplicity, we will only study the pair (L_0, L) in the remainder of the thesis. This is no restriction, because Hofer's distance and persistent Floer homology are invariant under Hamiltonian isotopies. In this section we study the process of *deletion of a leaf* introduced in [Kha09].[7] After recalling this construction, we study its effect on persistence Floer homology and its barcode.

2.4.1 Hamiltonian diffeomorphism. Fix an orientation on L_0 and denote by $s_1, \dots, s_{2n}$ be the intersection points of $L_0 \cap L$, such that the ordering corresponds to their order on L_0. We use cyclic notation for the indices (e.g. $s_{2n+1} = s_1$). Write $[s_i, s_j]$ for the interval on L_0 with left end s_i and right end s_j. Khanevsky associates a graph $T(L)$ to L consisting of two rooted trees, whose vertices carry weights and whose edges are oriented and ordered. Following closely [Kha09] we recall this construction.

The graph $T(L)$ consists of

- one vertex for each connected component of $\Sigma \backslash (L_0 \cup L)$,

[7] We only consider L_0 and $\Sigma = T^*S^1$, while Khanevsky's constructions work for more general curves and surfaces.

- each vertex v carries a weight $a(v) \in (0, \infty]$ equal to the area of the corresponding region in Σ,
- an edge between v_1 and v_2, whenever the corresponding regions have a common boundary along a segment of $L_0 \backslash L$,
- an orientation of the edge from the vertex corresponding to the region in the upper half of Σ to the vertex corresponding to the region in the lower half of Σ,
- an ordering of the edges, by assigning number $j \in \{1, \ldots, 2n\}$ to the edge that corresponds to the segment $[s_j, s_{j+1}]$ on L_0.

In what follows, we often do not distinguish between a vertex and its corresponding component in $\Sigma \backslash (L_0 \cup L)$. Denote the edge with number j by e_j. Figure 2.6 shows an example of $T(L)$.

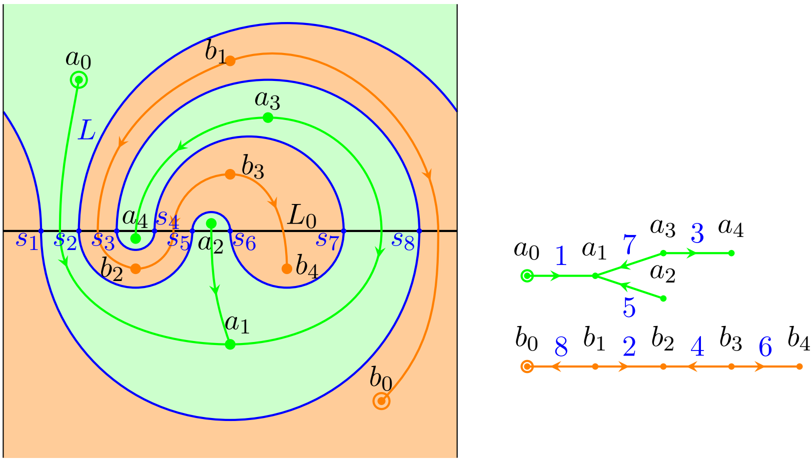


Figure 2.6: A tree associated to a Lagrangian.

The graph $T(L)$ consists of two connected components corresponding to the two regions of $\Sigma \backslash L$. For each of these connected components, the vertex corresponding to the unbounded region is set to be the root. Thus each connected component is a rooted tree. A leaf of $T(L)$, if different from the roots, corresponds to a region in $\Sigma \backslash (L_0 \cup L)$ that is bounded by one connected component of $L_0 \backslash L$ and one connected component of $L \backslash L_0$. It has corners at two neighboured intersection points. In the language of combinatorial Floer theory, this region corresponds to a minimal smooth lune $v \colon \mathbb{D} \longrightarrow \Sigma$ for (L_0, L), in the sense that there is no smooth lune $u \colon \mathbb{D} \longrightarrow \Sigma$ with $\mathrm{Im}(u) \subsetneq \mathrm{Im}(v)$. Denote by $\bar{q}, \bar{p} \in L_0 \cap L$ the corners of the smooth lune v. We call v a *leaf* from $\bar{q}$ to $\bar{p}$ and identify it with the leaf in the tree. In Figure 2.6 the leafs are the regions with weights a_2, a_4 and b_4.

Remark 2.4.1. Lemma 2.1.5 states that a shortest bar gives rise to a leaf: If $[\mathcal{A}(\overline{p}), \mathcal{A}(\overline{q}))$ is a shortest bar in $\mathcal{B}(L_0, L)$ then $\overline{q}$ and $\overline{p}$ are connected by a leaf. Conversely, if $\overline{q}$ and $\overline{p}$ are corners of a leaf *with minimal area* among all leaves, then the interval $[\mathcal{A}(\overline{p}), \mathcal{A}(\overline{q}))$ is a bar in $\mathcal{B}(L_0, L)$. However, in general a (non-minimal) leaf does not correspond to a bar.

The following result is due to Khanevsky:

Proposition 2.4.2 ([Kha09]). *Suppose there is leaf v from $\bar{q}$ to $\bar{p}$ of area $a(v)$. Let w be a vertex at distance 2 from v in $T(L)$. Then for any $\epsilon > 0$ there exists a Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(\Sigma)$ such that $\|\varphi\|_H < a(v) + \epsilon$ and which removes v from the tree by moving its weight $a(v)$ from v to the vertex w. The support of φ is contained in a neighbourhood of v and a curve as shown on the right in Figure 2.7. In particular, $L_0 \cap \varphi(L) = (L_0 \cap L) \backslash \{\bar{q}, \bar{p}\}$ and $\varphi(x) = x$ for $x \in (L \cap L_0) \backslash \{\bar{q}, \bar{p}\}$.*

We refer to this Hamiltonian isotopy by *deletion of a leaf.* Figure 2.7 shows schematically how the deletion of a leaf looks like.

Remark 2.4.3. Khanevsky applied this process only to leaves that are at distance ≥ 2 from the root, by moving weight to the vertex that is at distance 2 and closer to the root. This is of no significance to us. We allow deletion of a leaf different from the root, by moving its weight to any other leaf at distance two. The construction of φ works the same.

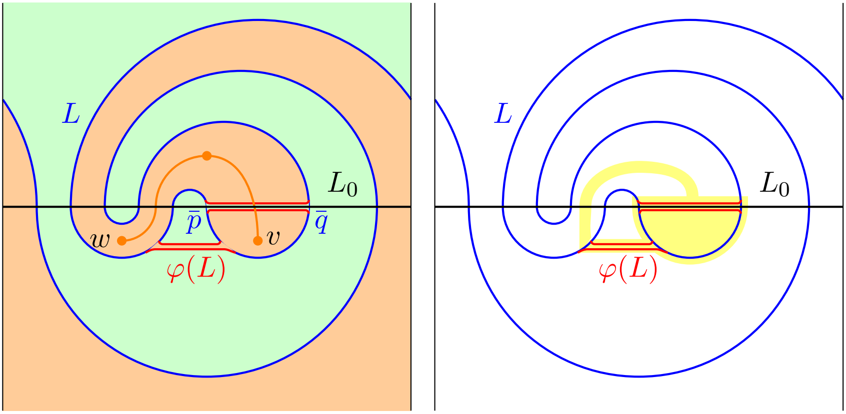


Figure 2.7: The procedure of deleting a leaf. The yellow region sketches the support of φ.

2.4.2 Effect on the Floer complex. We denote the set of intersection points of L and L_0 by P and abbreviate the Floer complex by $C := \mathrm{CF}(L_0, L)$. Suppose we obtain L' by deleting a leaf from $\bar{q}$ to $\bar{p}$ as explained in Proposition 2.4.2 by $L' = \varphi(L)$. Then the set of intersection points of L_0 and L' is $P' = P\backslash\{\bar{q}, \bar{p}\}$ and we denote the Floer complex of the pair (L_0, L') by $C' := \mathrm{CF}(L_0, L')$. Following [dSRS14, Appendix C] the chain complex C' can be expressed in terms of C as follows. Recall that the differentials in C and C' are given by the formulae

$$\partial(q) = \sum_{p \in P} n(q,p)p, \qquad \partial(q) = \sum_{p' \in P'} n'(q',p')p'$$

where $n(q,p) \in \mathbb{Z}_2$ denotes the mod 2 count of smooth lunes for (L_0, L) from q to p and $n'(q',p') \in \mathbb{Z}_2$ denotes the mod 2 count of smooth lunes for (L_0, L') from q' to p'. The following result is proven in [dSRS14, Chapter 11] for *nonisotopic* Lagrangians.

Proposition 2.4.4. *n' is related to n via*

$$n'(q',p') = n(q',p') + n(q',\bar{p})n(\bar{q},p').$$

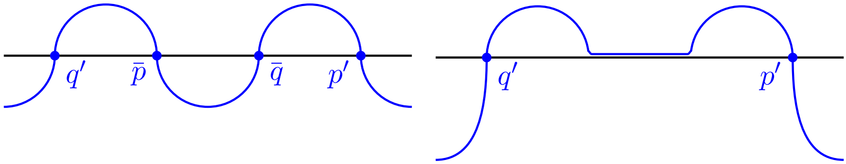


Figure 2.8: A new lune occurs after deletion of the leaf from $\bar{q}$ to $\bar{p}$.

Figure 2.8 below illustrates the situation for $n(q', \bar{p}) = n(\bar{q}, p') = 1$.

We show how to deduce Proposition 2.4.4 from the the nonisotopic case. As a first step, we need the following Lemma.

Lemma 2.4.5. *Suppose there are smooth lunes $u_{q',\bar{p}}$ from q' to $\bar{p}$ and $u_{\bar{q},p'}$ from $\bar{q}$ to p'. Then $u_{q',\bar{p}}$ does not contain p' in its image and $u_{\bar{q},p'}$ does not contain q' in its image.*

Proof. It follows from Proposition 11.1 and Step 5 in the proof of Theorem 9.2 in [dSRS14, Proposition 11.1] that there exists a smooth (L_0, L')-lune u' from q' to p', whose image contains the images of $u_{q',\bar{p}}$ and $u_{\bar{q},p'}$ except for small neighbourhoods of $\bar{p}$ and $\bar{q}$. But u' does not cover q' and p' except at the two corners -1 and $+1$ of $\mathbb{D}$. Therefore, $u_{q',\bar{p}}$ does not contain p' and $u_{\bar{q},p'}$ does not contain q'. □

Proof of Proposition 2.4.4. Fix $q', p' \in P'$. Because $\bar{q}$ and $\bar{p}$ are connected by a leaf it follows from Lemma 2.2.1 that there is at most one smooth lune from q' to $\bar{p}$ and at most one smooth lune from $\bar{p}$ to q'. Consider the surface Σ^0 obtained from Σ by puncturing it in the first quadrant near q' and in the second quadrant near p'. Consider also the surface Σ_0 obtained from Σ by puncturing it in the third quadrant near q' and in the fourth quadrant near p'. The punctures make sure that L_0 and L are not isotopic in Σ^0 and Σ_0. In Σ^0 and Σ_0, $\bar{q}$ and $\bar{p}$ are still connected by a leaf. We can therefore delete it. Let us denote by n^0 and n_0 the number of smooth lunes for (L_0, L) in Σ^0 and Σ_0 and by $(n^0)'$ and n_0' the number of smooth lunes for (L_0, L') in Σ^0 and Σ_0. The formula holds in Σ^0 and Σ_0. We therefore get

$$(n^0)'(q',p') = n^0(q',p') + n^0(q',\bar{p})n^0(\bar{q},p'), \tag{2.5}$$
$$n_0'(q',p') = n_0(q',p') + n_0(q',\bar{p})n_0(\bar{q},p'). \tag{2.6}$$

Note that $n^0(q',p')$ counts exactly the lower lunes leaving q' and entering p'. Similarly $n_0(q',p')$ counts exactly the upper lunes leaving q' and entering p'. Therefore,

$$n(q',p') = n^0(q',p') + n_0(q',p').$$

An analogous argument for n' leads to

$$n'(q',p') = (n^0)'(q',p') + n_0'(q',p').$$

Without loss of generality assume that the leaf from $\bar{q}$ to $\bar{p}$ is an upper leaf. Then

$$n_0(q',\bar{p}) = n_0(\bar{q},p') = 0.$$

Adding the equalities (2.5) and (2.6) therefore yields

$$n'(q',p') = n(q',p') + n^0(q',\bar{p})n^0(\bar{q},p').$$

It is left to show that

$$n^0(q',\bar{p})n^0(\bar{q},p') = n(q',\bar{p})n(\bar{q},p'). \tag{2.7}$$

If the right-hand side is 0 then the left-hand side clearly is also 0. If the right-hand side is 1 then $n(q',\bar{p}) = n(\bar{q},p') = 1$. It follows from Lemma 2.4.5 that the smooth lunes $u_{q',\bar{p}}$ and $u_{\bar{q},p'}$ are also smooth lunes in Σ^0. Hence $n^0(q',\bar{p}) = n^0(\bar{q},p') = 1$ and equation (2.7) follows. □

Having established Proposition 2.4.4 in our setting, we can proceed as in [dSRS14, Appendix C]. There are chain maps

$$\Psi\colon C \longrightarrow C' \text{ and } \Phi\colon C' \longrightarrow C$$

given by

$$\Psi(q) = \begin{cases} q & q \neq \bar{q}, \bar{p}, \\ 0 & q = \bar{q}, \\ \sum\limits_{p' \in P'} n(\bar{q}, p')p' & q = \bar{p} \end{cases}$$

and

$$\Phi(q') = q' + n(q', \bar{p})\bar{q}.$$

These chain maps are chain homotopy inverses to each other: $\Psi \circ \Phi = \mathrm{Id}$ and $\Phi \circ \Psi - \mathrm{Id} = \partial T + T\partial$ for the chain homotopy $T\colon C \to C$ given by

$$T(q) = \begin{cases} \bar{q} & q = \bar{p}, \\ 0 & q \neq \bar{p}. \end{cases}$$

2.4.3 Effect on the filtration. We first study the change of the graph $T = T(L)$ to the graph $T' = T(L')$. There are two cases, one for a leaf above L_0 (Case 1) and one for a leaf below L_0 (Case 2), as shown in Figure 2.9. In Case 1, let $j \in \{1, \ldots, 2n\}$ be the index with $\bar{q} = s_j$ and $\bar{p} = s_{j+1}$. Similarly in Case 2, let $j \in \{1, \ldots, 2n\}$ be the index with $\bar{p} = s_j$ and $\bar{q} = s_{j+1}$. The intersection points of L_0 with L' are $s_1, \ldots, s_{j-1}, s_{j+2}, \ldots s_{2n}$. For the ordering of the edges of $T(L')$ we use the numbering by $1, \ldots, j-1, j+2, \ldots, 2n$. Let k be the index of the edge adjacent to w that lies on the path from w to v. By renumbering if needed, we may arrange that $j + 2 \leq k \leq 2n$. Figure 2.10 illustrates the change of the graphs in Case 1. Case 2 is analogous with reversed arrows.

Let $\mathcal{A}\colon P \longrightarrow \mathbb{R}$ be the filtration on C. (As pointed out before, the filtration is unique up to shift. The shift is not relevant to us. We therefore ignore this ambiguity.)

Lemma 2.4.6. *Up to shift, the filtration on C' is given by*

$$\mathcal{A}'(s_l) = \begin{cases} \mathcal{A}(s_l) + \frac{a(v)+\epsilon}{2} & j + 2 \leq l \leq k \\ \mathcal{A}(s_l) - \frac{a(v)+\epsilon}{2} & \textit{else} \end{cases}$$

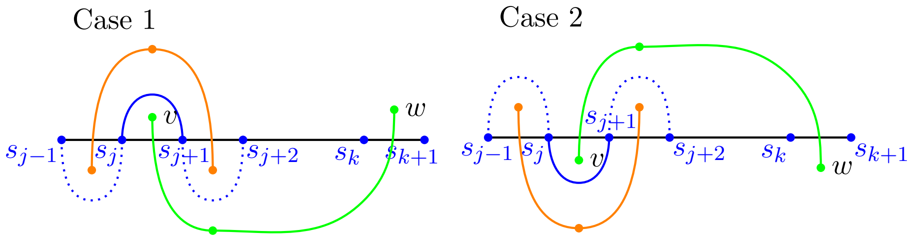


Figure 2.9: Two scenarios for the leaf v.

in Case 1 and by

$$\mathcal{A}'(s_l) = \begin{cases} \mathcal{A}(s_l) - \frac{a(v)+\epsilon}{2} & j+2 \leq l \leq k \\ \mathcal{A}(s_l) + \frac{a(v)+\epsilon}{2} & \textit{else.} \end{cases}$$

in Case 2.

The rest of this section is devoted to a proof of this.

There is always a smooth lune between s_i and s_{i+1}: To see this, we introduce some notation. Given an edge e in a forrest G consisting of rooted trees, we denote by $v(e)$ the vertex of the edge that is further away from the root of its component. The tree G_v is the subgraph of descendents of the vertex v. Consider now the edge e_i in $T = T(L)$ and the subgraph $T_{v(e_i)}$. The corresponding region in Σ is an embedded smooth lune u^i between s_i and s_{i+1}. If e_i points away from the root, it is a lune from s_{i+1} to s_i. If e_i points towards the root, it is a lune from s_i to s_{i+1}. For example consider the edge e_1 in Figure 2.6. Then the regions with weights a_1, a_2, a_3, a_4 make up a smooth lune from s_2 to s_1.

The smooth lune u^i has area $W(T_{v(e_i)})$, where W denotes the total weight of a weighted tree. From equation (2.4) we deduce the following formula for the action difference between two neighboured intersection points:

$$\mathcal{A}(s_{j+1}) - \mathcal{A}(s_j) = s(e_j)W(T_{v(e_j)}), \tag{2.8}$$

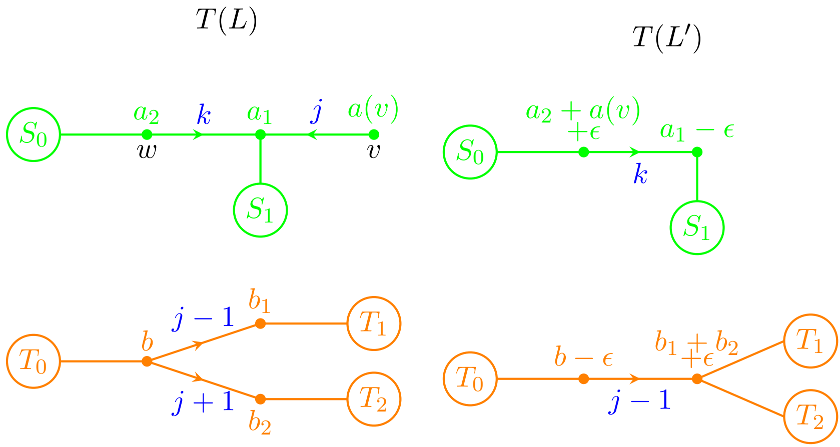


Figure 2.10: Change of trees. The only changes of weights happen at those vertices, where the weights are recorded in green and orange.

where the sign is determined as follows:

$$s(e) = \begin{cases} 1 & \text{edge points away from the root} \\ -1 & \text{edge points towards the root.} \end{cases}$$

Using this formalism we can now prove Lemma 2.4.6.

Proof of Lemma 2.4.6. By Remark 2.2.3 it is enough to prove that $\mathcal{A}'$ defined by the formula in Lemma 2.4.6 satisfies (2.8) for T'. We consider Case 1. First note that $s(e_l) = s(e'_l)$ for all $l \neq j, j+1$. For $l \notin \{j-1, j, j+1, k\}$ we compute

$$\begin{aligned} \mathcal{A}'(s_{l+1}) - \mathcal{A}'(s_l) &= \mathcal{A}(s_{l+1}) - \mathcal{A}(s_l) \\ &= s(e_l) W\left(T_{v(e_l)}\right) \\ &= s(e'_l) W\left(T'_{v(e_l)}\right). \end{aligned}$$

For $l = j-1$ first note that $s(e_{j-1}) = s(e_{j+1}) = s(e'_{j-1}) = 1$ and as it can be seen from Figure 2.10 we have

$$W\left(T'_{v(e'_{j-1})}\right) = W\left(T_{v(e_{j-1})}\right) + W\left(T_{v(e_{j+1})}\right) + \epsilon.$$

Therefore

$$\begin{aligned}
\mathcal{A}'(s_{j+2}) - \mathcal{A}'(s_{j-1}) &= \left(\mathcal{A}(s_{j+2}) + \frac{a(v)+\epsilon}{2}\right) \\
&\qquad - \left(\mathcal{A}(s_{j-1}) - \frac{a(v)+\epsilon}{2}\right) \\
&= \mathcal{A}(s_j) - \mathcal{A}(s_{j-1}) + \mathcal{A}(s_{j+2}) - \mathcal{A}(s_{j+1}) + \epsilon \\
&= s(e_{j-1}) W\left(T_{v(e_{j-1})}\right) \\
&\qquad + s(e_{j+1}) W\left(T_{v(e_{j+1})}\right) + \epsilon \\
&= s(e'_{j-1}) W\left(T'_{v(e'_{j-1})}\right).
\end{aligned}$$

Similarly for $l = k$ we see from Figure 2.10 that $s(e_k) = s(e'_k) = 1$ and

$$W\left(T'_{v(e'_k)}\right) = W\left(T_{v(e_k)}\right) - a(v) - \epsilon.$$

Therefore

$$\begin{aligned}
\mathcal{A}'(s_{k+1}) - \mathcal{A}'(s_k) &= \left(\mathcal{A}(s_{k+1}) - \frac{a(v)+\epsilon}{2}\right) - \left(\mathcal{A}(s_k) + \frac{a(v)+\epsilon}{2}\right) \\
&= s(e_k) W\left(T_{v(e_k)}\right) - a(v) - \epsilon \\
&= s(e'_k) W\left(T'_{v(e'_k)}\right).
\end{aligned}$$

This shows that $\mathcal{A}'$ is an action functional for (L_0, L'). Case 2 works similarly. $\square$

2.4.4 Effect on the barcode. We say that a chain map $\psi\colon D \to E$ between filtered chain complexes D and E *shifts filtration by at most* δ if $\psi(D^{\leq \alpha}) \subset E^{\leq \alpha+\delta}$ for all $\alpha \in \mathbb{R}$. Let C and C' be the Floer complexes from before endowed with the filtrations $\mathcal{A}$ and $\mathcal{A}'$ from Lemma 2.4.6. The chain maps Φ and Ψ are related to the filtrations on C and C' as follows.

Proposition 2.4.7. *The chain maps Φ and Ψ shift action by at most $\frac{a(v)+\epsilon}{2}$. The chain homotopy T shifts action by at most $a(v)+\epsilon$.*

It follows that

$$\Phi_* \colon \mathrm{HF}^{\leq\alpha}(L_0, L') \longrightarrow \mathrm{HF}^{\leq\alpha+\frac{a(v)+\epsilon}{2}}(L_0, L)$$

and

$$\Psi_* \colon \mathrm{HF}^{\leq\alpha}(L_0, L) \longrightarrow \mathrm{HF}^{\leq\alpha+\frac{a(v)+\epsilon}{2}}(L_0, L')$$

satisfy

$$\Psi_* \circ \Phi_* \colon = i'_{\alpha,\alpha+a(v)+\epsilon} \colon \mathrm{HF}^{\leq\alpha}(L_0, L') \longrightarrow \mathrm{HF}^{\leq\alpha+a(v)+\epsilon}(L_0, L')$$

and

$$\Phi_* \circ \Psi_* \colon = i_{\alpha,\alpha+a(v)+\epsilon} \colon \mathrm{HF}^{\leq\alpha}(L_0, L) \longrightarrow \mathrm{HF}^{\leq\alpha+a(v)+\epsilon}(L_0, L).$$

This is the promised combinatorially constructed $\frac{a(v)+\epsilon}{2}$-interleaving claimed in Section 2.3.4. As explained there, the bounds (2.2) and (2.3) now follow from general persistence theory.

For the proof of Proposition 2.4.7 we need the following

Lemma 2.4.8. *Assume Case 1. Then for any $q, p \in P \backslash \{\bar{q}, \bar{p}\}$*

(i) $n(q, \bar{p}) = 1$ implies $q = s_i$ for $j + 2 \leq i \leq k$.

(ii) $n(\bar{q}, p) = 1$ implies $p = s_i$ for $k + 1 \leq i \leq j - 1$.

Proof. Let $\bar{q} \neq s_l \in P$ be the intersection point that is a neighbour of $\bar{p}$ viewed on L. Then $j + 1 \leq l \leq k$. By Lemma 2.2.1 any lune u entering $\bar{p}$ *from the right* must have $s_l \notin u(\mathbb{D} \cap \mathbb{R}) = [\bar{p}, s_i]$. In particular, $j + 2 \leq i \leq l \leq k$. This shows (i). The proof for part (ii) works similarly. □

Proof of Proposition 2.4.7. We only show it for Case 1. The other case is similar. Let $q = s_i$ for some $i \neq j, j + 1$. Suppose $n(q, \bar{p}) = 1$. Then by part (i) of Lemma 2.4.8 $q = s_i$ for some $j+2 \leq i \leq k$. Hence $\mathcal{A}(q) = \mathcal{A}'(q) - \frac{a(v)+\epsilon}{2}$ by Lemma 2.4.6. Therefore for any $q \in P$

$$
\begin{aligned}
\mathcal{A}(\Phi q) &= \mathcal{A}(q - n(q,\overline{p})\overline{q}) \\
&= \begin{cases} \mathcal{A}(q) & \text{if } n(q,\overline{p}) = 0 \\ \max\{\mathcal{A}(q), \mathcal{A}(\overline{q})\} & \text{if } n(q,\overline{p}) = 1 \end{cases} \\
&\leq \begin{cases} \mathcal{A}'(q) + \frac{a(v)+\epsilon}{2} & \text{if } n(q,\overline{p}) = 0 \\ \max\{\mathcal{A}'(q) + \frac{a(v)+\epsilon}{2}, \mathcal{A}(q) + a(v) + \epsilon\} & \text{if } n(q,\overline{p}) = 1 \end{cases} \\
&= \mathcal{A}'(q) + \frac{a(v)+\epsilon}{2}.
\end{aligned}
$$

This shows that Φ shifts action by at most $\frac{a(v)+\epsilon}{2}$. Similarly, Ψ shifts action by at most $\frac{a(v)+\epsilon}{2}$:

$$
\begin{aligned}
\mathcal{A}'(\Psi\overline{p}) &= \max\{\mathcal{A}'(p) \,|\, n(\overline{q}, p) = 1\} \\
&\leq \max\left\{\mathcal{A}(p) - \frac{a(v)+\epsilon}{2} \,\middle|\, n(\overline{q}, p) = 1\right\} \\
&\leq \mathcal{A}(\overline{q}) - \frac{a(v)+\epsilon}{2} \\
&= \mathcal{A}(\overline{p}) + \frac{a(v)+\epsilon}{2},
\end{aligned}
$$

where the first inequality follows from part (ii) of Lemma 2.4.8 and Lemma 2.4.6.

The chain homotopy T shifts action by at most $a(v) + \epsilon$. □

2.5 The shortest bar

The goal of this section is to show Proposition 2.1.5 that identifies the shortest bar with the smallest leaf. As in the previous section, we only consider the pair (L_0, L) for $L \in \mathcal{L}(L_0)$. We assume that L_0 and L intersect in at least $2n \geq 4$ intersection points. Moreover, the action values $\mathcal{A}(q)$, $q \in L_0 \cap L$, are assumed to be distinct. Let $[a, b)$ be the shortest (finite) bar in $\mathcal{B}(L_0, L)$ and let $\bar{q}, \bar{p} \in L_0 \cap L$ be the intersection points satisfying $\mathcal{A}(\bar{q}) = b$ and $\mathcal{A}(\bar{p}) = a$. The goal is to show that there is a leaf from $\bar{q}$ to $\bar{p}$. The proof is based on the following two lemmas whose proofs will be given shortly after.

Lemma 2.5.1. *Let $x, y \in L_0 \cap L$ such that $n(x, y) = 1$. Then $\mathcal{A}(x) - \mathcal{A}(y) \geq b - a$.*

Lemma 2.5.2. *Let $u\colon \mathbb{D} \longrightarrow \Sigma$ be a smooth lune for (L_0, L). Then there exists a leaf $v\colon \mathbb{D} \longrightarrow \Sigma$ such that* $\mathrm{Im}(v) \subseteq \mathrm{Im}(u)$.

Let $\mathcal{S} \subset \mathrm{CF}(L_0, L)$ be a Jordan basis for ∂ (see the definition in section 2.3.2). We explain how Proposition 2.1.5 follows from the previous two lemmas.

Proof of Proposition 2.1.5. As a first step towards the proof, we show $n(\bar{q}, \bar{p}) = 1$: Let

$$e = \bar{q} + \sum_{\mathcal{A}(q) < \mathcal{A}(\bar{q})} n_q q \in \mathcal{S}$$

be the basis element with $\mathcal{A}(e) = b$ and

$$f = \bar{p} + \sum_{\mathcal{A}(p) < \mathcal{A}(\bar{p})} m_p p \in \mathcal{S}$$

be the basis element with $\mathcal{A}(f) = a$. Since $\partial e = f$, there exists $q' \in L_0 \cap L$ with $\mathcal{A}(q') \leq \mathcal{A}(\bar{q}) = b$ such that $n(q', \bar{p}) = 1$. By Lemma 2.5.1 it follows that $\mathcal{A}(q') = \mathcal{A}(\bar{q})$. Under the assumption that all the action values are distinct, we conclude that $\bar{q} = q'$, hence $n(\bar{q}, \bar{p}) = n(q', \bar{p}) = 1$. In particular, there exists a unique smooth lune $u_{\bar{q},\bar{p}}$ from $\bar{q}$ to $\bar{p}$.

Suppose by contradiction that the smooth lune $u_{\bar{q},\bar{p}}$ is not a leaf. By Lemma 2.5.2, there exists a leaf, say from x to y, whose image is strictly contained in $\mathrm{Im}(u_{\bar{q},\bar{p}})$. If $n(x, y) = 1$, then the inequality

$$\mathcal{A}(x) - \mathcal{A}(y) < \mathcal{A}(\bar{q}) - \mathcal{A}(\bar{p}) = b - a$$

is a contradiction to Lemma 2.5.1. If $n(x, y) = 0$, then there are two smooth lunes from x to y, one of them being a leaf. It follows from Lemma 2.2.1 that the only possibility for such a situation is when $L_0 \cap L = \{x, y\}$. This is not the case here because there are at least 4 intersection points. We conclude that $u_{\bar{q},\bar{p}}$ is a leaf from $\bar{q}$ to $\bar{p}$. □

Proof of Lemma 2.5.1. Let $x' \in L_0 \cap L$ be a generator of $CF(L_0, L)$ satisfying

$$\mathcal{A}(x') = \min\{\mathcal{A}(x'')|n(x'', y) = 1\}.$$

By minimality of $\mathcal{A}(x')$, $\partial x'$ is not a boundary in $\mathrm{CF}^{<\mathcal{A}(x')}(L_0, L)$. Hence $\mathcal{A}(x')$ is an upper end of a finite bar $[\tilde{a}, \tilde{b})$. Since $n(x, y) = 1$ one has $\tilde{b} = \mathcal{A}(x') \leq \mathcal{A}(x)$. Let

$$e' := x' + \sum_{\mathcal{A}(x'') < \mathcal{A}(x')} l_{x''} x'' \in \mathcal{S}$$

be the basis element with $\mathcal{A}(e') = \tilde{b}$. Then $\partial e'$ contains y as a summand because $n(x', y) = 1$, but $n(x'', y) = 0$ for all x'' with $\mathcal{A}(x'') < \mathcal{A}(x')$. In particular, $\tilde{a} = \mathcal{A}(\partial e') \geq \mathcal{A}(y)$. By minimality of the bar $[a, b)$ it follows that

$$b - a \leq \tilde{b} - \tilde{a} \leq \mathcal{A}(x) - \mathcal{A}(y).$$

$\square$

Proof of Lemma 2.5.2. Without loss of generality, we may assume that u is a smooth lune from s_1 to s_l, $3 \leq l \leq 2n$ with $u(\mathbb{D} \cap \mathbb{R}) = [s_1, s_l]$. Intuitively, if there were no leaf contained in $\mathrm{Im}(u)$, whenever some part of the graph $T(L)$ enters the region of the lune, it will also leave the lune. Entering and leaving happens only on $[s_1, s_l]$, along which the two components of the graph alternate. This is not possible.

To make this argument rigorous, let us assume that $\mathrm{Im}(u)$ does not contain any leaf. We denote by C_1 the upper, and by C_2 the lower component of $\Sigma \backslash L$. Consider the preimages $R_k := u^{-1}(C_k) \subset \mathbb{D}$ of C_k under u for $k = 1, 2$. See Figure 2.11 for an illustration of these sets.

Then $R_1 \cap R_2 = \emptyset$. Let $0 = \lambda_1 \leq \cdots \leq \lambda_l = 1$ be the points on $\mathbb{D} \cap \mathbb{R}$ with $u(\lambda_i) = s_i$. Then $[\lambda_i, \lambda_{i+1}] \subset \overline{R_1}$ for odd i and $[\lambda_i, \lambda_{i+1}] \subset \overline{R_2}$ for even i. Moreover, $[\lambda_1, \lambda_2]$ is in the same connected component of $\overline{R_1}$ as $[\lambda_{l-1}, \lambda_l]$. We set $j_1 = l-1$. Note that $j_1 \geq 3$. We claim that there exists $4 \leq j_2 < j_1$, such that $[\lambda_2, \lambda_3]$ and $[\lambda_{j_2}, \lambda_{j_2+1}]$

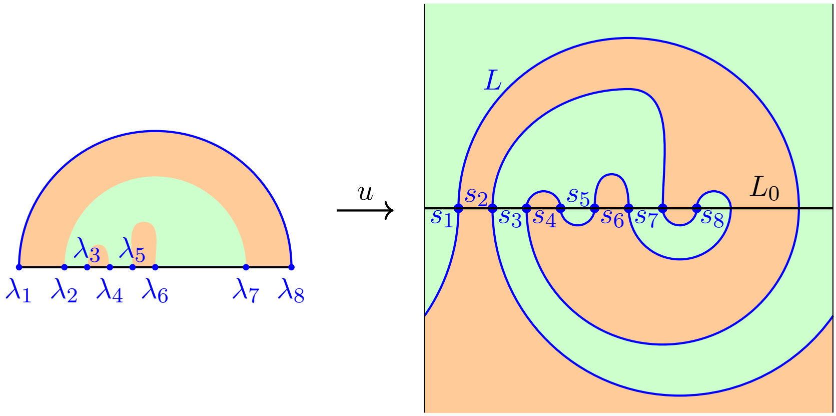


Figure 2.11: On the right, C_1 is the green region and C_2 the orange region. On the left, R_1 is coloured in green and R_2 in orange for the upper smooth lune from s_1 to s_8.

are in the same connected component of $\overline{R_2}$. If not, consider the connected component R of $[\lambda_2, \lambda_3]$ in $\overline{R_2}$. $R \cap \partial \mathbb{D} = [\lambda_2, \lambda_3]$ because $\mathrm{Int}(R) \cap R_1 = \emptyset$. Therefore, u restricts to a diffeomorphism from R to $u(R)$. $\partial R \backslash [\lambda_2, \lambda_3] \subset L$ and hence $u(R)$ is a region in $\Sigma \backslash (L_0 \cup L)$ that is bounded by $[s_2, s_3]$ and part of L. Therefore, $u(R)$ is exactly the region corresponding to the subtree $T_{v(e_j)}$. But then $u(R)$ contains a leaf, which contradicts our assumption. We therefore find a j_2 as claimed.

Proceeding like this, we obtain an infinite sequence $\{j_k\}_{k \in \mathbb{N}}$ of natural numbers with

$$1 < j_k < j_{k-1} < l.$$

This is impossible. We conclude that u does contain a leaf. $\square$

Proposition 2.1.5 was the last missing step in the proof of Theorem 2.1.1 as outlined in section 2.1.2.

The result that the smallest bar corresponds to a smooth lune generalises to all "Floer-type" situations. We formalize this statement as follows. Let (C, ∂) be a filtered complex over $\mathbb{Z}_2$ with preferred basis $E = \{q_1, \dots, q_l\}$ as in section 2.3 and differential ∂ determined by numbers $n(q_i, q_j) \in \mathbb{Z}_2$ such that

$$\partial(q_i) = \sum_{j=1}^{l} n(q_i, q_j) q_j.$$

The following properties hold

Proposition 2.5.3. *Denote by $\mathcal{B} := \mathcal{B}(\mathrm{H}(C))$ the barcode of the homology persistence module of C. Let $q, p \in E$.*

(i) If $[\mathcal{A}(p), \mathcal{A}(q))$ is a bar in $\mathcal{B}$ then there exists $q' \in E$ such that $n(q', p) = 1$ and $\mathcal{A}(q') \leq \mathcal{A}(q)$.

(ii) If $n(q, p) = 1$ then there exists a bar $[a, b) \in \mathcal{B}$ such that $[a, b) \subseteq [\mathcal{A}(p), \mathcal{A}(q))$.

(iii) Let $[a, b) \in \mathcal{B}$ be a smallest bar. Then there exist $p, q \in E$ such that $a = \mathcal{A}(p)$, $b = \mathcal{A}(q)$ and $n(q, p) = 1$. Moreover,

$$\mathcal{A}(q) - \mathcal{A}(p) = \min\{\mathcal{A}(q') - \mathcal{A}(p') \mid n(q', p') = 1\}.$$

Proof. The proof of (i) is the same as the first part of the proof of Proposition 2.1.5. Part (ii) follows from the very same argument as in the proof of Lemma 2.5.1. Part (iii) follows from (i) and (ii): Take $q \in E$ with $\mathcal{A}(q) = b$ and $p \in E$ with $\mathcal{A}(p) = a$. Then we take $q' \in E$ from (i), apply (ii) to q' and p and get

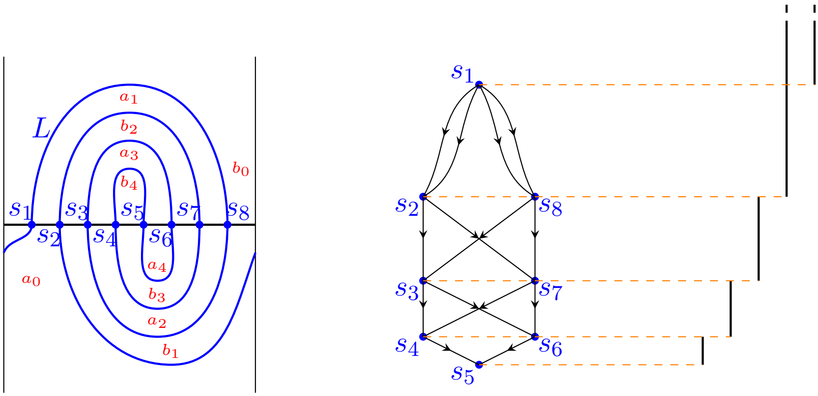
$$[a', b') \subseteq [\mathcal{A}(p), \mathcal{A}(q')) \subseteq [a, b)$$

for a bar $[a', b')$. By minimality of $[a, b)$ it follows $\mathcal{A}(q') = b$, hence (iii) is satisfied by p and q'. □

Remark 2.5.4. Applying part (iii) to Floer complexes allows to deduce that the smallest bar is realized by a Floer strip. However, one has to be careful about the question on minimality, as (iii) does not exclude the existence of smaller strips that occur with multiplicity two. In the case of the cylinder we studied, we could exclude this because every smooth lune contains a leaf and leaves never occur with multiplicity two for $n \geq 2$.

2.6 Towards a better bound

The bound in Theorem 2.1.1 is (at least in some cases) quite weak. In this section we show an example and discuss possible improvements. The theory of Wasserstein metrics, together with the precise analysis of the filtration in section 2.4.3, will enable us to improve the control over the change of the barcode when deleting a leaf.



(L_0, L) $\qquad$ $CF(L_0, L)$ $\qquad$ $\mathcal{B}(L_0, L)$

Figure 2.12: A linear example for $n = 4$ with all finite weights of the same size.

2.6.1 An example. Consider for example a Lagrangian L that intersects L_0 in such a way that the associated graph $T(L)$ consists of two linear components. See Figures 2.12 and 2.13.

Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be the finite weights as in the pictures. We assume in the following that n is even.[8] The condition for L being exact is

$$a_1 + a_3 + \cdots + a_{n-1} = b_1 + b_3 + \cdots + b_{n-1}.$$

[8] This choice is only relevant for a few details in the computation, but not for the qualitative result.

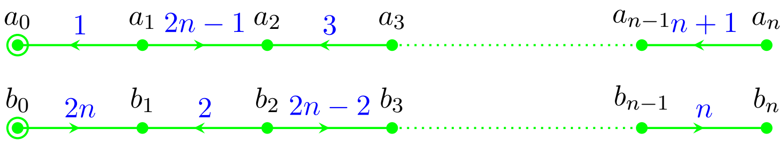


Figure 2.13: Linear trees.

The barcode $\mathcal{B}(L_0, L)$ consists of the bars

$$
\begin{aligned}
&\big[\mathcal{A}(s_{n+1}), \min\{\mathcal{A}(s_n), \mathcal{A}(s_{n+2})\}\big), \\
&\big[\max\{\mathcal{A}(s_n), \mathcal{A}(s_{n+2})\}, \min\{\mathcal{A}(s_{n-1}), \mathcal{A}(s_{n+3})\}\big), \\
&\big[\max\{\mathcal{A}(s_{n-1}), \mathcal{A}(s_{n+3})\}, \min\{\mathcal{A}(s_{n-2}), \mathcal{A}(s_{n+4})\}\big), \\
&\ldots \\
&\big[\max\{\mathcal{A}(s_3), \mathcal{A}(s_{2n-1})\}, \min\{\mathcal{A}(s_2), \mathcal{A}(s_{2n})\}\big), \\
&\big[\max\{\mathcal{A}(s_2), \mathcal{A}(s_{2n})\}, \infty\big), \\
&\big[\mathcal{A}(s_1), \infty\big).
\end{aligned}
$$

The lengths of the finite bars and the spectral metric are

$$
\begin{aligned}
\beta_{n-1}(L_0, L) &= \min\{a_n, b_n\}, \\
\beta_{n-2}(L_0, L) &= \min\{a_{n-1} + a_n, a_{n-1} + b_n, b_{n-1} + a_n, b_{n-1} + b_n\}, \\
\beta_{n-3}(L_0, L) &= \min\{a_{n-2} + a_{n-1} + a_n, a_{n-2} + b_{n-1} + a_n, \\
&\qquad\quad b_{n-2} + a_{n-1} + b_n, b_{n-2} + b_{n-1} + b_n\}, \\
&\ldots \\
\beta_1(L_0, L) &= \min\{a_2 + a_3 + \cdots + a_{n-1} + a_n, \\
&\qquad\quad a_2 + b_3 + \cdots + b_{n-1} + a_n, \\
&\qquad\quad b_2 + a_3 + \cdots + a_{n-1} + b_n, \\
&\qquad\quad b_2 + b_3 + \cdots + b_{n-1} + b_n\}, \\
\gamma(L_0, L) &= \min\{a_1 + a_2 + \cdots + a_n, b_1 + b_2 + \cdots + b_n\}.
\end{aligned}
$$

Deleting repeatedly the smallest leaf one sees

$$\begin{aligned} d_H(L_0, L) \leq \min\{a_n, b_n\} &+ \min\{a_{n-1}, b_{n-1}\} \\ &+ \min\{a_{n-2} + a_n, b_{n-2} + b_n\} \\ &+ \min\{a_{n-3} + a_{n-1}, b_{n-3} + b_{n-1}\} \\ &+ \cdots + \min\{a_2 + a_4 + \cdots + a_n, b_2 + b_4 + \cdots + b_n\} \\ &+ \min\{a_1 + a_3 + \cdots + a_{n-1}, b_1 + b_3 + \cdots + b_{n-1}\} \\ \leq \beta_1(L_0, L) + \beta_2(L_0, L) &+ \cdots + \beta_{n-1}(L_0, L) + \gamma(L_0, L). \end{aligned}$$

In particular we get a bound that is much smaller than the bound in Theorem 2.1.1. We don't know of an example where the upper bound is actually attained.

2.6.2 Wasserstein distance. One key step in trying to improve the bound in Theorem 2.1.1 lies in the control of the barcode when deleting a leaf. Let us denote

$$\ell(L, L') := \sum_{j=1}^{n-1} \beta_j(L, L') + \gamma(L, L').$$

Suppose L'' is obtained from L' by $\varphi \in \mathrm{Ham}(\Sigma)$ as in Proposition 2.4.2. Then equations (2.2) and (2.3) lead to

$$|\ell(L, L') - \ell(L, L'')| \leq n \|\varphi\|_H.$$

If we had the following much stronger control

$$|\ell(L, L') - \ell(L, L'')| \leq \|\varphi\|_H. \tag{2.9}$$

we would get the following much better bound

$$d_H(L, L') \leq \sum_{j=1}^{n-1} j\beta_j(L, L') + \gamma(L, L').$$

For general L', L'' equation (2.9) is certainly wrong. On the other hand, one might hope that such a statement becomes true for the specific case that L'' is obtained form L' by deleting the smallest leaf. We could find the following improvement of equations (2.2) and (2.3).

Proposition 2.6.1. *Consider the pair* (L_0, L) *and suppose* L' *is obtained from* L *by deleting a leaf between* s_j *and* s_{j+1} *by moving its weight to a vertex across the edge* e_k *(see Figure 2.9). Then*

$$|\ell(L_0, L) - \ell(L_0, L')| \leq (|k-j|-1)\|\varphi\|_H,$$

where $\varphi \in \mathrm{Ham}(\Sigma)$ *is the Hamiltonian diffeomorphism that implements the deletion of the leaf.*

In Proposition 2.6.1 we work with a pair (L_0, L), so that the notation from Figure 2.9 applies. We now prepare for the proof of this result. In doing this, we consider again a more general pair (L, L') of Lagrangians in $\mathcal{L}(L_0)$. In Proposition 2.6.1 we are interested into a control of ℓ. ℓ is closely related to the Wasserstein metric $d_1^{\mathcal{W}}$ on barcodes. It is defined as follows. Let $\mathcal{B}$ and $\mathcal{B}'$ be two barcodes of finite type and $\mu\colon \mathcal{B}_0 \longrightarrow \mathcal{B}'_0$ be a matching. We measure the ℓ^1-defect of μ by

$$r(\mu) = \sum_{\mu([a,b))=[a',b')} \big(|a-a'| + |b-b'|\big) + \sum_{[a,b)\notin \mathcal{B}_0 \cup \mathcal{B}'_0} (b-a)$$

with the convention that $\infty - \infty = 0$. Then the Wasserstein distance $d_1^{\mathcal{W}}$ between $\mathcal{B}$ and $\mathcal{B}'$ is given by

$$d_1^{\mathcal{W}}(\mathcal{B}, \mathcal{B}') = \inf_{\mu} r(\mu).$$

The relation of this measurement to ℓ becomes apparent when comparing $\mathcal{B}(L, L')$ to a specific barcode $\mathcal{B}_c$ which only contains two bars: $[c, \infty), [c, \infty)$. For a good choice of c, the Wasserstein distance from $\mathcal{B}(L, L')$ to $\mathcal{B}_c$ is exactly $\ell(L, L')$. Moreover,

$$|\ell(L, L') - \ell(L, L'')| \leq d_1^{\mathcal{W}}\left(\mathcal{B}(L, L'), \mathcal{B}(L, L'')\right). \tag{2.10}$$

The proof of Proposition 2.6.1 is based on Wasserstein stability as studied in [ST20]. As a preparation for the proof, we first collect some relevant notions and results from their work.

Let V be a persistence module of finite type. The 1-norm of V is the sum of the lengths of the bars of $\mathcal{B}(V)$. That is,

$$\|V\| = \sum_{[a,b)\in\mathcal{B}(V)} (b-a).$$

In particular, it is finite if and only if V has only finite bars. The 1-norm is additive with respect to short exact sequences:

Lemma 2.6.2 ([ST20, Lemma 7.30])**.** *Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of persistence modules. Then*

$$\|V_2\| = \|V_1\| + \|V_3\|.$$

The algebraic 1-Wasserstein distance between two persistence modules V_1 and V_2 is

$$d_1^{\mathcal{W}}(V_1, V_2) = \inf_{(W,g,h)} \|\ker g \oplus \operatorname{coker} g \oplus \ker h \oplus \operatorname{coker} h\|,$$

where the infimum is taken over all diagrams

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "g"'] & V_2 \\ V_1 & \end{tikzcd}$$

of persistence modules. These triples are called *interpolating objects* of V_1 and V_2. [ST20, Theorem 7.27, 7.28] show that the algebraic Wasserstein distance coincides with the Wasserstein distance of barcodes. Namely,

$$d_1^{\mathcal{W}}(V_1, V_2) = d_1^{\mathcal{W}}(\mathcal{B}(V_1), \mathcal{B}(V_2)).$$

2.6.3 Deletion of a leaf and barcodes. We prove Proposition 2.6.1. The proof is inspired by the proof of Wasserstein stability for sublevel set filtrations in [ST20, Theorem 7.33].

Proof of Proposition 2.6.1. Let $a(v)$ be the weight of the leaf we delete. Then the Hofer norm is $\|\varphi\|_H = a(v) + \epsilon$ for some small enough $\epsilon > 0$. Since it has no relevance in the proof, we will ignore ϵ. We assume Case 1 from Figure 2.9 and $k \geq j$. We compare the chain complexes $\mathrm{CF}(L_0, L)$ with filtration $\mathcal{A}$ to $\mathrm{CF}(L_0, L')$ with filtration $\mathcal{A}'$ given by

$$\mathcal{A}'(s_l) = \begin{cases} \mathcal{A}(s_l) + a(v) & j+2 \leq l \leq k, \\ \mathcal{A}(s_l) & \text{else.} \end{cases}$$

Note that we choose here an action functional that is shifted by a constant from Lemma 2.4.6. Consider now the new filtered complex $\widetilde{C} := \mathrm{CF}(L_0, L)$ with filtration induced by the functional

$$\widetilde{\mathcal{A}}(s_l) := \begin{cases} \max\{\mathcal{A}(s_l), \mathcal{A}'(s_l)\} & l \neq j, j+1, \\ \mathcal{A}(s_j) & l = j, j+1. \end{cases}$$

This is a filtered chain complex because whenever $n(q,p) = 1$ then

$$\widetilde{\mathcal{A}}(q) - \widetilde{\mathcal{A}}(p) \geq \mathcal{A}(q) - \mathcal{A}(p) - a(v) \geq 0$$

by Lemma 2.5.1. Note that the filtered chain complex $(\widetilde{C}, \tilde{\mathcal{A}})$ won't be attained as a Floer complex of two transverse Lagrangians. Intuitively it corresponds to the moment, where all the area of the leaf has been moved, but the two intersection points are still there.

There are filtered chain maps

$$g \colon \widetilde{C} \longrightarrow \mathrm{CF}(L_0, L)$$

and

$$h \colon \widetilde{C} \longrightarrow \mathrm{CF}(L_0, L').$$

g is just the identity and h is given by

$$h(s_l) = \begin{cases} s_l & l \neq j, j+1, \\ 0 & l = j, \\ \sum\limits_{i \neq j, j+1} n(s_j, s_i) s_i & l = j+1. \end{cases}$$

We explain why h preserves filtration. Suppose $n(s_j, s_i) = 1$ for some $i \neq j+1$. Then by Lemma 2.4.8$\mathcal{A}'(s_i) = \mathcal{A}(s_i)$. Therefore,

$$\begin{aligned}\mathcal{A}'(h(s_{j+1})) &= \max\{\mathcal{A}'(s_i) \,|\, n(s_j, s_i) = 1\} \\ &= \max\{\mathcal{A}(s_i) \,|\, n(s_j, s_i) = 1\} \\ &\leq \mathcal{A}(s_{j+1}) = \widetilde{\mathcal{A}}(s_{j+1}),\end{aligned}$$

where the last inequality follows from Lemma 2.5.1.

Considering the diagram

$$\begin{tikzcd} \mathrm{H}_*(\widetilde{C}) \arrow[r, "h_*"] \arrow[d, "g_*"'] & \mathrm{HF}_*(L_0, L') \\ \mathrm{HF}_*(L_0, L) & \end{tikzcd}$$

we may view $(\mathrm{H}_*(\widetilde{C}), g_*, h_*)$ as an interpolating object of the persistence modules $\mathrm{HF}_*(L_0, L)$ and $\mathrm{HF}_*(L_0, L')$. We use it to bound $d_1^{\mathcal{W}}(\mathrm{HF}(L_0, L), \mathrm{HF}(L_0, L'))$. Consider the long exact sequences

$$\cdots \longrightarrow \mathrm{H}_*(\widetilde{C}) \xrightarrow{g_*} \mathrm{HF}(L_0, L) \longrightarrow \mathrm{H}_*(\operatorname{coker} g) \xrightarrow{\delta} \mathrm{H}_*(\widetilde{C}) \longrightarrow \ldots$$

$$\cdots \to \mathrm{H}_*(\ker h) \xrightarrow{\iota_*} \mathrm{H}_*(\widetilde{C}) \xrightarrow{h_*} \mathrm{HF}(L_0, L') \longrightarrow \mathrm{H}_*(\ker h) \to \ldots.$$

It follows that

$$\begin{aligned} \operatorname{coker} g_* &\cong \ker \delta, \qquad & \operatorname{coker} h_* &\cong \ker \iota_*, \\ \ker g_* &\cong \mathrm{im}\delta, \qquad & \ker h_* &\cong \mathrm{im}\iota_*. \end{aligned}$$

We estimate

$$\begin{aligned} &\|\ker g_* \oplus \operatorname{coker} g_* \oplus \ker h_* \oplus \operatorname{coker} h_*\| \\ &\qquad = \|\mathrm{im}\, \delta \oplus \ker \delta \oplus \mathrm{im}\, \iota_* \oplus \ker \iota_*\| \\ &\qquad = \|H_*(\operatorname{coker} g)\| + \|H_*(\ker h)\| \\ &\qquad \leq \|\operatorname{coker} g\| + \|H_*(\ker h)\| \\ &\qquad = \sum_{i=1}^{2n} |\widetilde{\mathcal{A}}(s_i) - \mathcal{A}(s_i)| + \widetilde{\mathcal{A}}(s_j) - \widetilde{\mathcal{A}}(s_{j+1}) \\ &\qquad = (k - (j+2) + 1))a(v) + a(v) \\ &\qquad = (k - j + 1)a(v). \end{aligned}$$

The 2nd equality follows from applying Lemma 2.6.2 to the short exact sequences

$$0 \longrightarrow \ker \delta \longrightarrow H_*(\operatorname{coker} g) \xrightarrow{\delta} \operatorname{im} \delta \longrightarrow 0$$

and

$$0 \longrightarrow \ker \iota_* \longrightarrow H_*(\ker h) \xrightarrow{\iota_*} \operatorname{im} \iota_* \longrightarrow 0.$$

The inequality follows from Lemma 2.6.2 applied to the short exact sequence

$$0 \longrightarrow \ker p \longrightarrow \operatorname{coker} i \xrightarrow{p} H_*(\operatorname{coker} i) \longrightarrow 0.$$

It follows that the algebraic Wasserstein distance between $\mathrm{HF}(L_0, L)$ and $\mathrm{HF}(L_0, L')$ is bounded above by $(k - j + 1)a(v)$. Therefore, we get

$$d_1^{\mathcal{W}}(\mathcal{B}(L_0, L), \mathcal{B}(L_0, L')) \leq (k - j - 1)a(v).$$

Together with equation (2.10) this implies Lemma 2.6.1. $\square$

Chapter A

Auxiliary material

The purpose of the appendix is to briefly explain the algebraic background that is relevant for the definition of the main character of Chapter 1 : the element $A \in \mathrm{HF}^*(\tau^{-1})$. All the material here comes from Seidel's book [Sei08a]. We refer the interested reader to it for a thorough treatment of A_∞-categories and Fukaya categories.

A.1 A_∞-categories

In this section we collect the definitions of the basic objects in the theory of A_∞-categories. This includes A_∞-modules and the Yoneda embedding. We follow closely the conventions for the A_∞-machinery from [Sei08a, Sections 1 and 2] with two simplifications: we work over the field $\mathbb{Z}_2$ and we only use a $\mathbb{Z}_2$-grading. This simplifies the formulas significantly and is enough for our purpose.

We call a vector space V *$\mathbb{Z}_2$-graded* if it is enhanced with a decomposition into a direct sum $V = V_0 \oplus V_1$ of vector spaces. Maps between $\mathbb{Z}_2$-graded vector spaces are linear maps that preserve the grading, meaning that $f(V_0) \subseteq V_0'$ and $f(V_1) \subseteq V_1'$. For an integer n we denote by $V[n]$ the $\mathbb{Z}_2$-vector space with the same underlying vector space structure but $V[n]_i = V_{i+n}$ for $i = 0, 1$. That is, $V = V[n]$ stays unchanged for even n and $V[n]$ has a switched order in the direct sum decomposition. In particular, linear maps $f \colon V \to V'$ that interchange the grading can be viewed as graded maps $f \colon V \to V'[1]$.

Definition A.1.1. An *A_∞-category* $\mathcal{A}$ consists of a set of objects $\mathrm{Ob}\mathcal{A}$ and $\mathbb{Z}_2$-graded vector spaces $Mor_{\mathcal{A}}(L_0, L_1)$ for each pair of objects $L_0, L_1 \in \mathrm{Ob}\mathcal{A}$ together with (graded) composition maps

$$\mu^d_{\mathcal{A}} \colon Mor_{\mathcal{A}}(L_{d-1}, L_d) \otimes \cdots \otimes Mor_{\mathcal{A}}(L_0, L_1) \longrightarrow Mor_{\mathcal{A}}(L_0, L_d)[1]$$

for $d \geq 1$ satisfying the A_∞-relations

$$\sum_{\substack{1\leq m\leq d\\ 0\leq n\leq d-m}} \mu_{\mathcal{A}}^{d-m+1}\big(a_d, \ldots, a_{n+m+1}, \mu_{\mathcal{A}}^{m}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1\big) = 0.$$

In particular, $\mu_{\mathcal{A}}^1$ endowes $Mor_{\mathcal{A}}(L_0, L_1)$ with a cochain complex structure. Its cohomology groups $\mathrm{H}^*(Mor_{\mathcal{A}}(L_0, L_1))$ are endowed with an associative product induced by $\mu_{\mathcal{A}}^2$. The resulting category with objects $\mathrm{Ob}\mathcal{A}$ and morphism sets $\mathrm{H}^*\left(Mor_{\mathcal{A}}(L_0, L_1)\right)$ is the cohomology category $\mathrm{H}(\mathcal{A})$. We assume that this category is *unital.*

There are notions of *functors* of A_∞-categories and *natural transformations* between functors. For the definitions we refer the reader to [Sei08a]. We now proceed with the notion of A_∞-modules.

Definition A.1.2. An A_∞*-module* $\mathcal{M}$ over $\mathcal{A}$ consists of $\mathbb{Z}_2$-graded vector spaces $\mathcal{M}(L)$ for each object $L \in \mathrm{Ob}\mathcal{A}$ together with maps

$$\mu_{\mathcal{M}}^d\colon \mathcal{M}(L_{d-1})\otimes Mor_{\mathcal{A}}(L_{d-2}, L_{d-1})\otimes\cdots\otimes Mor_{\mathcal{A}}(L_0, L_1) \longrightarrow \mathcal{M}(L_0)$$

for $d \geq 1$ satisfying the relations

$$\begin{aligned}\sum_{0\leq n\leq d} \mu_{\mathcal{M}}^{n+1}&\left(\mu_{\mathcal{M}}^{d-n}(b, a_{d-1}, \ldots, a_{n+1}), a_n, \ldots, a_1\right)\\ = \sum_{\substack{1\leq m\leq d-1\\ 0\leq n\leq d-m-1}} &\mu_{\mathcal{M}}^{d-m+1}\big(b, a_{d-1}, \ldots, a_{n+m+1}, \mu_{\mathcal{A}}^{m}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1\big).\end{aligned}$$

There is a A_∞-category $mod_{\mathcal{A}}$ whose objects are A_∞-modules over $\mathcal{A}$ and whose morphisms are so-called pre-module homomorphisms. A *pre-module homomorphism* $f\colon \mathcal{M} \longrightarrow \mathcal{M}'$ of $\mathcal{A}$-modules consists of a family of maps

$$f^d\colon \mathcal{M}(L_{d-1})\otimes Mor_{\mathcal{A}}(L_{d-2}, L_{d-1})\otimes\cdots\otimes Mor_{\mathcal{A}}(L_0, L_1) \longrightarrow \mathcal{M}'(L_0)$$

for $d \geq 1$ satisfying again a certain set of equations. We omit here the definition of the composition maps that turn $mod_{\mathcal{A}}$ into an A_∞-module.

The *Yoneda embedding* is a specific A_∞-functor

$$\mathcal{Y}\colon \mathcal{A} \to mod_{\mathcal{A}}$$

that takes an object $L \in \mathrm{Ob}\mathcal{A}$ to the $\mathcal{A}$-module $\mathcal{Y}(L)$ defined by

$$\mathcal{Y}(L)(K) := Mor_{\mathcal{A}}(K, L)$$

for each $K \in \mathrm{Ob}\mathcal{A}$ and

$$\mu^d_{\mathcal{Y}(L)}(b, a_{d-1}, \ldots, a_1) := \mu^d_{\mathcal{A}}(b, a_{d-1}, \ldots, a_1)$$

for $a_i \in Mor_{\mathcal{A}}(K_{i-1}, K_i)$, $i \in \{1, \ldots, d-1\}$ and $b \in \mathcal{Y}(L)(K_{d-1}) = Mor_{\mathcal{A}}(K_{d-1}, L)$. There is more data to $\mathcal{Y}$, namely a whole family $\mathcal{Y}^d$ of maps. We only say what $\mathcal{Y}^1$ is. It is the graded map

$$\mathcal{Y}^1\colon Mor_{\mathcal{A}}(L_0, L_1) \longrightarrow Mor_{mod_{\mathcal{A}}}(\mathcal{Y}(L_0), \mathcal{Y}(L_1))$$

defined by sending $a \in Mor_{\mathcal{A}}(L_0, L_1)$ to the pre-module homomorphism $\mathcal{Y}^1(a)$ sending

$$\mathcal{Y}(L_0)(K_{d-1}) \otimes Mor_{\mathcal{A}}(K_{d-2}, K_{d-1}) \otimes \cdots \otimes Mor_{\mathcal{A}}(K_0, K_1)$$

to $\mathcal{Y}(L_1)(K_0)$ via

$$(b, a_{d-1}, \ldots, a_1) \longmapsto \mu^{d+1}_{\mathcal{A}}(a, b, a_{d-1}, \ldots, a_1).$$

By [Sei08a, Section 2g] the Yoneda embedding induces a unital, full and faithfull embedding

$$\mathrm{H}(\mathcal{Y})\colon \mathrm{H}(\mathcal{A}) \to \mathrm{H}(mod_{\mathcal{A}}).$$

Let $f\colon \mathcal{M} \to \mathcal{M}'$ be a A_∞-module homomorphism. There is a well-defined *mapping cone* for morphisms of $\mathcal{A}$-modules.

Definition A.1.3. The *cone* of f is the A_∞-module

$$\mathcal{C}one(f)(L) = \mathcal{M}(L)[1] \oplus \mathcal{M}'(L)$$

with composition maps

$$\begin{aligned}
&\mu^d_{\mathcal{C}one(f)}((b, b'), a_{d-1}, \ldots, a_1) \\
&= \left(\mu^d_{\mathcal{M}}(b, a_{d-1}, \ldots, a_1), \mu^d_{\mathcal{M}'}(b', a_{d-1}, \ldots, a_1) + f^d(b, a_{d-1}, \ldots, a_1)\right).
\end{aligned}$$

In particular, when we restrict attention to $d = 1$, $\mathcal{C}one(f)$ is the usual mapping cone over a chain map f. Therefore, for each object L there exists a long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^*(\mathcal{M}(L)) \xrightarrow{f^*} H^*(\mathcal{M}'(L)) \longrightarrow \\ \longrightarrow H^*(\mathcal{C}one(f)) \longrightarrow H^*(\mathcal{M}(L))[1] \longrightarrow \dots \end{aligned}$$

A.2 Triangulated categories

The derived cateogory $\mathcal{DA}$ of $\mathcal{A}$ is the category constructed as follows: Consider the image of $\mathcal{Y}$ in $mod_{\mathcal{A}}$. We add all cones to our A_∞-category and call it the triangulated completion of the image of $\mathcal{Y}$ in $mod_{\mathcal{A}}$. $\mathcal{DA}$ is its cohomology category.

The following is an immediate consequence of the properties of the Yoneda embedding.

Corollary A.2.1. *Each $f \in Mor_{D\mathcal{A}}(\mathcal{Y}(L_1), \mathcal{Y}(L_2))$ can be represented by $\mathcal{Y}^1(\alpha)$ for some $\alpha \in Mor_{\mathcal{A}}(L_1, L_2)$. Moreover, $[\alpha] \in Mor_{H(\mathcal{A})}(L_1, L_2)$ is uniquely defined.*

Proof. First, note that

$$Mor_{D\mathcal{A}}(\mathcal{Y}(L_1), \mathcal{Y}(L_2)) \cong \mathrm{H}(Mor_{mod_{\mathcal{A}}}(\mathcal{Y}(L_1), \mathcal{Y}(L_2))).$$

For any object K, $\mathcal{Y}(\alpha)$ determines the map

$$\mathcal{Y}(L_1)(K) \cong Mor(K, L_1) \xrightarrow{\mu^2(\alpha,-)} Mor(K, L_2) \cong \mathcal{Y}(L_2)$$

The existence and uniqueness of α follow immediately from $\mathrm{H}(\mathcal{Y})$ being full and faithful. $\square$

$\mathcal{DA}$ is a so-called triangulated category. We briefly recall this concept. An exact triangle in $\mathcal{DA}$ is any diagram

$$\begin{tikzcd} \mathcal{M} \arrow[dr, "{[f]}"] & \\ & \mathcal{M}' \arrow[dl] \\ \mathcal{M}'' \arrow[uu, "{[1]}"] & \end{tikzcd}$$

which is isomorphic to

$$\begin{tikzcd} \mathcal{M} \arrow[dr, "{[f]}"] & \\ & \mathcal{M}' \arrow[dl] \\ \mathcal{C}one(f) \arrow[uu, "{[1]}"] & \end{tikzcd}$$

in $\mathcal{DA}$. It is shown in [Sei08a] that this endowes $\mathcal{DA}$ with the structure of a triangulated category.

Suppose there exists an exact triangle $L_0 \to L_1 \to L_2$ in $\mathcal{DA}$ for objects L_0, L_1 and L_2 in $\mathcal{A}$ and morphisms in $\mathcal{A}$. Here we identified $\mathcal{A}$ as a subcategory of $mod_{\mathcal{A}}$ via the Yoneda embedding. Then for each object K in $\mathcal{A}$ there is a long exact sequence of chain complexes

$$\cdots \to Mor^k_{\mathrm{H}\mathcal{A}}(K, L_0) \longrightarrow Mor^k_{\mathrm{H}\mathcal{A}}(K, L_1) \longrightarrow$$
$$\longrightarrow Mor^k_{\mathrm{H}\mathcal{A}}(K, L_2) \longrightarrow Mor^{k+1}_{\mathrm{H}\mathcal{A}}(K, L_0) \to \dots$$

A.3 Fukaya categories

Roughly speaking the Fukaya category $\mathcal{F}uk(M)$ is an A_∞-category whose objects are Lagrangian submanifolds and whose morphism spaces are Floer complexes. The compositions μ^d are given by counting pseudo-holomorphic $d+1$-gons with Lagrangian boundary conditions. We give here a brief outline of the construction in the weakly exact case, following the exposition in [BC14].

Let (M, ω) be a weakly exact closed symplectic manifold. Let $\mathcal{L}$ be the set of closed, weakly exact Lagrangians in M. The Fukaya category $\mathcal{F}uk(M)$ has objects $\mathrm{Ob}\mathcal{F}uk(M) = \mathcal{L}$. For any pair (L_0, L_1) of Lagrangians in $\mathcal{L}$ choose Floer data $\mathcal{D}^{L_0,L_1} = (\mathcal{J}, H)$ such that $L_0 \pitchfork \psi_1^H(L_1)$. Then the morphism space is the cochain complex

$$Mor_{\mathcal{F}uk(M)}(L_0, L_1) = \mathrm{CF}^*(L_0, L_1; \mathcal{J}, H).$$

In particular, μ^1 is given by the Floer differential.

The higher order compositions μ^d are defined in an analogous way by counting pseudo-holomorphic $d+1$-gons as follows. Let S_{d+1} be a disc with $d+1$ punctures $\zeta_0, \zeta_1, \dots, \zeta_d$ on the boundary, ordered in counterclockwise direction. Let C_i be the connected component of the boundary between ζ_i, ζ_{i+1} for $i = 0, \dots, d$. See Figure A.1. For a

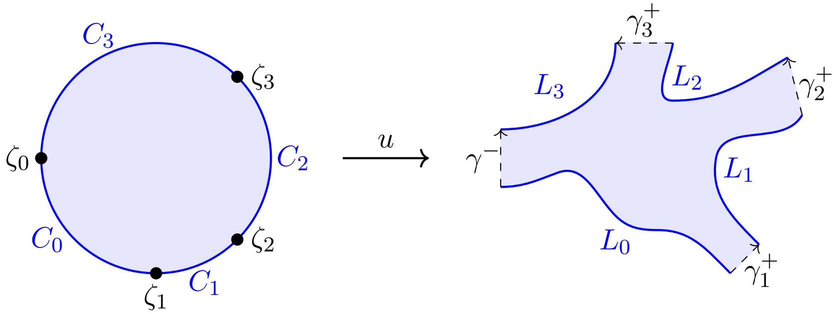


Figure A.1: A $3+1$ punctured disc on the left and its boundary conditions on the right.

$d+1$-tuple $(L_0, \dots, L_d) \in \mathcal{L}$ choose perturbation data $\mathcal{D}^{L_0,\dots,L_d}(K, J)$ consisting of a 1-form K on S_{d+1} with values in $C^\infty(M)$ and a family J_z indexed by $z \in S_{d+1}$. The 1-form K gives rise to a 1-form Y_K with values in the set of Hamiltonian vector fields. Consider $\gamma^-, \gamma_1^+, \dots, \gamma_d^+$ Hamiltonian chords of $H^{L_0,L_d}, H^{L_0,L_1}, \dots, H^{L_{d-1},L_d}$ respectively. The polygons that contribute to $\mu^d(\gamma_1^+, \dots, \gamma_d^+)$ as a multiple of γ^- are smooth maps

$$u\colon S_{r+1} \longrightarrow M$$

satisfying the equation

$$D_z u + J_z(u) \circ Du_z \circ j_{S_{r+1}} = Y_K(u) + J_z(u) \circ Y_K(u) \circ j_{S_{r+1}},$$

where $j_{S_{r+1}}$ is a complex structure on S_{r+1}. The polygons should satisfy the boundary conditions

$$u(C_i) \subseteq L_i$$

Moreover, u should tend to γ^- at the puncture ζ_0 and tend to $\gamma_1^+, \dots, \gamma_d^+$ at the punctures $\zeta_1, \dots, \zeta_d$ respectively. These limits can be made rigorous through the choice of compatible strip-like ends. The details are omitted here.

Such polygons u can be collected to a moduli space

$$\mathcal{M}(\gamma^-; \gamma_1^+, \dots, \gamma_d^+).$$

Then μ^d is defined by

$$\mu^d(\gamma_1^+, \dots, \gamma_d^+) = \sum_{\gamma^-, u} T^{\omega(u)} \gamma^-,$$

where the sum is over all $u \in \mathcal{M}(\gamma^-; \gamma_1^+, \dots, \gamma_d^+)$.

The result of this construction is a whole family of A_∞-categories, indexed by the choice of regular and compatible Floer and perturbation data $\mathcal{D}^{L_0, \dots, L_d}$. They are all quasi-equivalent [Sei08a, Section 10]. In particular, their derived categories are all equivalent. We call it $\mathcal{DF}uk(M)$.

By the generalities on A_∞-categories collected in the previous section, $\mathcal{DF}uk(M)$ is a triangulated category. If there is an exact triangle $L_0 \to L_1 \to L_2$ in $\mathcal{DF}uk(M)$ for Lagrangians $L_0, L_1, L_2 \in \mathcal{L}$ then for any Lagrangian K there is an induced long exact sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{HF}^k(K, L_0) \longrightarrow \mathrm{HF}^k(K, L_1) \longrightarrow & \\ \longrightarrow \mathrm{HF}^k(K, L_2) \longrightarrow \mathrm{HF}^{k+1}(K, L_0) \longrightarrow \dots \end{aligned}$$

Moreover, the maps can be understood as μ^2-operations with unique elements $a \in \mathrm{HF}^0(L_0, L_1), b \in \mathrm{HF}^0(L_1, L_2)$ and $c \in \mathrm{HF}^1(L_2, L_0)$.

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