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Master Thesis

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The Wess-Zumino-Witten model on $\mathfrak{psu}(2|2)_1$

Master's Thesis

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Abstract

In this thesis we develop the representation theory of the affine Lie superalgebra $\mathfrak{su}(2|2)_1$ with a purely algebraic method, and in particular we find all the highest weight representations. We give a criterion for the inclusion of a bosonic subalgebra in a Lie superalgebra being a conformal embedding, and prove that this is the case for $\mathfrak{psu}(2|2)_1$ and $\mathfrak{u}(2|2)_1$. We then present the free field realisation of $\mathfrak{u}(2|2)_1$ and the spectral flow action. The free field representations are then decomposed in terms of affine representations of the bosonic subalgebra, and explicit branching rules are presented. The corresponding characters are computed, and for the case of $\mathfrak{psu}(2|2)_1$ also their modular behaviour. In this case, it turns out that the modular S-matrix contains terms that are linear in the conformal parameter τ, which suggests that the $\mathfrak{psu}(2|2)_1$-WZW model is a logarithmic conformal field theory.

Contents

1 Introduction

One of the gems of string theory in the last decades is the Anti-de Sitter/conformal field theory (AdS/CFT) correspondence and the application of the holographic duality to understand strongly coupled quantum field theories (QFTs). The key aspect of the correspondence stems from the fact that it allows to compute quantum corrections in the non-perturbative regime of a field theory using a classical gravity theory. This has a wide spectrum of applications, that go far beyond string theory: AdS/CFT finds fruitful application in the study of condensed matter systems, non-equilibrium phenomena in strongly coupled plasmas, and gives an explanation for confinement and chiral symmetry braking in non-conformal field theories.

The first connection between string theory and Yang Mills gauge theories was proposed by 't Hooft [tH84], who was searching for a small parameter in the strong coupling regime of quantum chromodynamics in order to obtain valid expansions. Later, the first concrete conjecture of the AdS/CFT correspondence was stated by Maldacena [Mal99] and claimed that $\mathcal{N} = 4$ superconformal Yang-Mills (SYM) theory in 4 spacetime dimensions is dual to Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. An important aspect of this duality, is that since $\mathcal{N} = 4$ SYM theory is a conformal field theory (CFT), one can consider different limits of the 't Hooft parameter, relating the large radius limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$, which is well-approximated by supergravity (low energy limit), to the strongly coupled regime in the field theory.

It is thus of great interest to derive the AdS/CFT correspondence, also because it is likely to enlighten several aspects of this holographic duality. Recently, progress in this direction has been made in a special case which relates the small radius or tensionless limit of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, with $k = 1$ units of Neveu-Schwartz-Neveu-Schwartz (NS-NS) flux, to the free symmetric product orbifold CFT, $\mathrm{Sym}^N(\mathbb{T}^4)$. The first evidence for this duality was the agreement of the full spectrum in the large N limit, see [EGG19] and [GG18]. Then, it was shown in [EGG20] that the correlators in the two descriptions agree manifestly. Moreover, since the string background has pure (and minimal $k = 1$) NS-NS flux, it can be described be an exactly solvable worldsheet Wess-Zumino-Witten (WZW) model. The best description relies on the so-called hybrid formalism of Berkovits Vafa and Witten [BVW99], where the relevant WZW model is based on the superalgebra $\mathfrak{psu}(1,1|2)_k$. This picture, in addition to having manifest spacetime supersymmetry, allows to avoid a limitation of the Ramond-Neveu-Schwartz (RNS) formulation, which is not a priori well defined for $k < 2$. The supergroup sigma model is, on the other hand, well-defined at $k = 1$ and has special features. The tensionless limit corresponds to $k = 1$, and is where the supergroup WZW model admits a free field description in terms of two canonically conjugate pairs of fermions, together with four symplectic bosons (each of spin one half).

For what concerns the case of $\mathrm{AdS}_5 \times \mathrm{S}^5$, the quantisation of a sigma model with this target space is challenging. First of all, the presence of Ramond-Ramond (R-R) five-form flux renders a RNS description impossible [CCY18]. Secondly, the Green-Schwarz (GS) description [MT98] is fairly intractable except for some special cases, see [MT01] and [AF09]. Lastly, pure spinor descriptions [Ber00], which avoid some difficulties of both the above approaches, are not yet technically developed to the point where they can serve a calculational framework. However, a sigma model was recently proposed in [GG21a] as a natural generalisation of the similar free field sigma model proposed for the tensionless limit of $\mathrm{AdS}^3 \times \mathrm{S}^3 \times \mathbb{T}^4$. This is based on the superalgebra $\mathfrak{psu}(2,2|4)$ and more precisely, in the tensionless limit on $\mathfrak{psu}(2,2|4)_1$. As for the case of $\mathfrak{psu}(1,1|2)$, at level $k = 1$ also this affine superalgebra has a free field realisation, which is obtained basically by doubling the oscillators of $\mathfrak{psu}(1,1|2)$. In [GG21a] it is then argued that the corresponding worldsheet gauge constraints reduce the degrees of freedom to a finite number of oscillators in each spectrally flowed sector. Imposing a set of residual gauge constraints on this reduced oscillator Fock space then determines the physical spectrum of the string theory. Remarkably, there is evidence that this prescription reproduces precisely the entire planar spectrum of single trace operators of the free SYM theory.

However, when considering interactions in the SYM theory, the picture is more involved. It is likely that, just as for the spectrum, the integrability approach to correlators [BAA+11] will enter the picture. In this context, upon the choice of vacuum, the $\mathfrak{psu}(2,2|4)$ symmetry of $\mathcal{N} = 4$ SYM is broken down to $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$. In particular, it is very interesting to note that the hexagon approach to SYM correlators [BKV15] is most naturally formulated in terms of bilinears of $\mathfrak{su}(2|2)$ bits, which seem to be closely related to the covariant twistorial wedge modes in [GG21b].

Moreover, an actual WZW model on $\mathfrak{psu}(2,2|4)$ yields a spectrum generating algebra which consists of two commuting copies of $\mathfrak{psu}(2,2|4)$, which does not directly match the symmetries of $\mathcal{N} = 4$ SYM, consisting of one single copy. It thus seems natural to look at a WZW containing fewer degrees of freedom, and a good candidate is the Lie supergroup $\mathfrak{su}(2|2)$. Indeed, there is an inclusion of Lie superalgebras $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \subset \mathfrak{spu}(2,2|4)$, that could be interpreted as an embedding of the (finite) spectrum generating algebra of the WZW model on $\mathfrak{su}(2|2)$ into the symmetry algebra of $\mathcal{N} = 4$ SYM theory. Moreover, as mentioned above, in the integrability context the two copies of $\mathfrak{su}(2|2)$ can be obtained from $\mathfrak{psu}(2,2|4)$ by symmetry braking.

We also point out that the WZW model on $\mathfrak{psu}(2|2)_1$ is strictly related to that on $\mathfrak{psu}(1,1|2)_1$, where the latter was the central object in the tensionless worldsheet description of $\mathrm{AdS}_3 \times \mathrm{S}^3$. One of the differences between the two models lies in the signature of the metric on the corresponding supergroup, which for $\mathfrak{su}(2|2)$ is $(- - - + ++)$ and $(- + + + ++)$ for $\mathfrak{psu}(1,1|2)$. From the global (topological) perspective of the corresponding Lie supergroups, the former has a compact bosonic subgroup, while the latter a non-compact one. This suggests, that in contrast with $\mathfrak{psu}(1,1|2)_1$ which has a continuous spectrum, we expect the $\mathfrak{psu}(2|2)_1$ spectrum to be discrete.

The thesis is structured as follows.

We start with an introduction to affine Kac-Moody Lie algebras, their representations and singular vectors, and we give the Kac-Kazhdan determinant formula for singular vectors in affine highest weight Verma modules. Then, we present Wess-Zumino-Witten models from both a geometric and an algebraic perspective and explain how they can be defined in terms of affine Lie algebras. We explain how the Sugawara construction shows that they are examples of conformal field theories, and we identify WZW primary fields with highest weight representations of the underlying affine algebra. A special emphasis is given to the integrable models, which we argue to be rational CFTs, and we illustrate this concept with the example of $\mathfrak{su}(2)_k$ at $k \in \mathbb{Z}_{>0}$. The chapter ends with a brief overview on the definitions and elementary properties of Lie superalgebras.

In the second chapter, we start with the discussion of the WZW model on the Lie superalgebras $\mathfrak{su}(2|2)$ and $\mathfrak{psu}(2|2)$, which is the main goal of this thesis. First, these superalgebras are defined in terms of generators and commutator relations. We then illustrate the $\mathfrak{sl}(2,\mathbb{R})$-representation theory, which comes into play in virtue of the fact that we have to consider also non-unitary representations of $\mathfrak{su}(2)$. Then, the representation theory of $\mathfrak{su}(2|2)_k$ at level $k = 1$ is worked out; in particular, the shortening of the allowed multiplets are derived with a purely algebraic method.

The third chapter is dedicated to conformal embeddings, starting from the definitions and then introducing the coset construction. We briefly present some results in meromorphic CFTs, in particular about the uniqueness of vertex operators. We need this result, in order to then give a criterion for knowing whether the inclusion of the bosonic subalgebra into the Lie superalgebra is a conformal embedding. We then apply this criterion, showing that such a conformal embedding exists for $\mathfrak{psu}(N|N)_1$ and $\mathfrak{u}(N|N)_1$ for every $N > 1$, and for $N = 2$ we analyse the implications on the allowed representations and their Casimir.

Chapter four is devoted to the free field realisation of $\mathfrak{u}(2|2)_1$, which is explicitely presented in connection to that of $\mathfrak{psu}(2,2|4)_1$. We consider the free field representations and discuss separately the Neveu-Schwartz (NS) and the Ramond (R) sectors, and for the latter we show how different definitions of the symplectic boson zero modes action yields different type of $\mathfrak{su}(2)$-representations. We then define the spectral flow action at the free field level and deduce from it the action on the affine superalgebra generators. This, for instance, allows us to identify the R-sector singlet representation of $\mathfrak{u}(2|2)_1$ with the NS vacuum representation through one unit of a specific spectral flow.

The fifth chapter starts with a review on affine characters and modular invariance in WZW models, with particular attention to the integrable case. This introduction terminates with the suggestive example of $\widehat{\mathfrak{u}}(1)$ modular invariants, which we used to draw some analogies to the $\mathfrak{su}(2)_{-1}$ scenario. We then recall the characters and modular matrices of the $\mathfrak{su}(2)_1$ theory and their relation to the corresponding free field realisation in terms of four real fermions. Then, the Kac-Kazhdan determinant formula is applied to $\mathfrak{su}(2)_{-1}$ and all the singular vectors of the theory are established. The characters of the corresponding affine irreducible modules are given, together with their modular behaviour, and the issue of non-holomorphicity of the character functions is pointed out.

In the last chapter, we consider first the free field realisation of $\mathfrak{su}(2)_{-1}$ in terms of four symplectic bosons. In particular, we compute the free field characters and manage to decompose them in infinite sums of irreducible $\mathfrak{su}(2)_{-1}$-characters thanks to a denominator identity for Lie superalgebras. This is done for the NS sector, as well as for the different R sectors. We then calculate the free field characters of $\mathfrak{psu}(2|2)_1$ and decompose the vacuum module in affine highest weight representations of the bosonic subalgebra. Luckily, this allows to express the vacuum character in terms of theta functions and derivatives of those. We compute the spectrally flowed characters and their modular behaviour, and find that for the modular S-matrix involving every spectrally flowed versions of the vacuum, an explicit linear dependence on the conformal parameter τ is present. Finally, the free field characters of $\mathfrak{su}(2|2)_1$ are computed, and explicit affine branching functions for the embedding of the bosonic subalgebra are found.

2 Wess-Zumino-Witten models and affine Lie algebras

For this section we will mainly follow chapters 14 and 15 of [DFMS97]. We try to present the material that is relevant to our discussion; several results and calculations are just presented as facts, for proofs and more details we refer to the corresponding bibliography. The reader is assumed to be familiar with the theory of simple finite-dimensional Lie algebras and their representations, for which a good review is given in chapter 13 of [DFMS97].

2.1 Affine Lie algebras

Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra generated by elements $J^a \in \mathfrak{g}$ for $1 \leq a \leq \dim \mathfrak{g}$ satisfying $[J^a, J^b] = i \sum_c f_c^{ab} J^c$ for $f_c^{ab} \in \mathbb{C}$, and denote by $\mathfrak{h}$ the Cartan subalgebra whose generator are denoted by $H^1, \dots, H^r \in \mathfrak{g}$. We consider an infinite-dimensional generalisation of this algebra, called the **affine Kac-Moody algebra** or **affine Lie algebra** over $\mathfrak{g}$, denoted by $\widehat{\mathfrak{g}}$ or later by $\mathfrak{g}_k$. This is generated by elements J_n^a, L_0 and $\hat{k}$ for $n \in \mathbb{Z}$, satisfying the commutation relations

$$\begin{aligned} [J_n^a, J_m^b] &= i \sum_c f_c^{ab} J_{n+m}^c + \hat{k} n \delta_{ab} \delta_{n+m,0} \,, \\ [L_0, J_n^a] &= -n J_{n+m}^a \,, \\ [\hat{k}, J_n^a] &= [\hat{k}, L_0] = 0 \,. \end{aligned} \tag{1}$$

In particular, the zero modes J_0^a of $\widehat{\mathfrak{g}}$ generate a Lie subalgebra isomorphic to $\mathfrak{g}$, which we sometimes refer to as the **finite** subalgebra. Note that Eq. (1) implies that $\hat{k}$ is a central element; indeed, one can first define the so called loop algebra consisting only of the modes J_n^a, satisfying the commutation relations given by the first line of Eq. (1) without the term involving $\hat{k}$, and then show that there exists a unique central extension of this algebra given by the introduction of $\hat{k}$. Similarly, the generator L_0 is introduced in order to extend the abelian subalgebra $\{H_0^1, \dots, H_0^r, \hat{k}\}$ to a maximal abelian subalgebra $\widehat{\mathfrak{h}}$. One can then define a Killing form on $\widehat{\mathfrak{g}}$ extending that of $\mathfrak{g}$, which yields an isomorphism between the Cartan subalgebra $\widehat{\mathfrak{h}}$ and its dual $\widehat{\mathfrak{h}}^*$ inducing a scalar product on the latter. We choose the ordered basis $(H^1, \dots, H^r, \hat{k}, -L_0)$ of $\widehat{\mathfrak{h}}$ and we call **affine weights** the elements

$$\hat{\lambda} = (\lambda; k_\lambda; n_\lambda) \in \widehat{\mathfrak{h}}^* \,.$$

The induced scalar product on affine weights takes the form

$$\langle \hat{\lambda}, \hat{\mu} \rangle = \langle \lambda, \mu \rangle + k_\lambda n_\mu + k_\mu n_\lambda \,,$$

where on the right hand side we used the induced scalar product on $\mathfrak{h}^*$, denoted in the same way.

Similarly to the finite case, affine weights in the adjoint representation are called **affine roots**. We denote by Δ the set of roots of $\mathfrak{g}$ and by Δ_+ the subset of positive roots in which we fix a basis $\alpha_1, \ldots, \alpha_r$ of simple roots and the associated coroots $\alpha_i^\vee := 2\alpha_i/|\alpha_i|^2$. For a root $\alpha \in \Delta$ we define $\alpha \cdot H := \sum_i \alpha^i H^i$, where $[H^i, E^\alpha] = \alpha^i E^\alpha$ for every $1 \leq i \leq r$ and $E^\alpha \in \mathfrak{g}$ is the generator associated to the root α. We also denote the Cartan generators in the Chevelley basis by $h^i := \alpha_i^\vee \cdot H$. Then, the affine roots take the form of

$$\hat{\alpha} = (\alpha; 0; n) \quad \text{for } \ n \in \mathbb{Z}\,, \qquad \text{and} \qquad \delta = (0; 0; 1)\,,$$

where $(\alpha; 0; n)$ is associated with the generator $E_n^\alpha \in \widehat{\mathfrak{g}}$ and $n\delta$ with $H_n^i \in \widehat{\mathfrak{g}}$. We write simply α instead of $(\alpha; 0; 0)$; then we can write $\hat{\alpha} = \alpha + n\delta$. A basis of affine roots is obtained from $\{\alpha_i\}$ by adding the extra simple root

$$\alpha_0 := (-\theta; 0; 1) = -\theta + \delta\,,$$

where θ is the highest root of Δ, that is, the unique root $\sum_i m_i \alpha_i$ whose expansion maximizes $\sum_i m_i$. The coefficients of the decomposition of θ in the bases $\{\alpha_i\}$ and $\{\alpha_i^\vee\}$ bear special names, being called, respectively, the **marks** $\{a_i\}$ and the **comarks** $\{a_i^\vee\}$:

$$\theta = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee\,.$$

We define the **affine coroots** as

$$\hat{\alpha}^\vee := \frac{2}{|\hat{\alpha}|^2}(\alpha; 0; n) = \frac{2}{|\alpha|^2}(\alpha; 0; n) = (\alpha^\vee; 0; \frac{2}{|\alpha|^2} n)\,,$$

and for finite roots we omit the hat,

$$\alpha_0^\vee := \alpha_0\,, \qquad \alpha_i^\vee := (\alpha_i^\vee; 0; 0)\,.$$

Then, we have the following equations

$$\delta = \sum_{i=0}^{r} a_i \alpha_i = \sum_{i=0}^{r} a_i^r \alpha_i^\vee\,, \qquad h^\vee := 1 + \sum_{i=1}^{r} a_i^\vee = \sum_{i=0}^{r} a_i^\vee\,,$$

where $h^\vee$ is the **dual Coxeter number** of $\mathfrak{g}$, which only depends on the finite Lie algebra. The full set of affine roots is

$$\hat{\Delta} = \{\alpha + n\delta : n \in \mathbb{Z}, \alpha \in \Delta\} \cup \{n\delta : n \in \mathbb{Z}, n \neq 0\}\,.$$

The root δ is called an **imaginary** since

$$\langle \delta, \delta \rangle = 0 \,,$$

and likewise all the roots in $\{n\delta\}$, which all have multiplicity equal to r. The other roots are called **real** and they all have multiplicity equal to 1. A set of **positive affine roots** is given by

$$\hat{\Delta}_+ = \{\alpha + n\delta : n > 0, \alpha \in \Delta\} \cup \{\alpha : \alpha \in \Delta_+\} \,.$$

One can the define the **extended Cartan matrix** and the **extended Dynkin diagrams** as in the classification of complex simple finite-dimensional Lie algebras, see [DFMS97] for more details.

In the finite case, the fundamental wights $\omega_1, \ldots, \omega_r \in \mathfrak{h}^*$ are defined as the elements of the basis dual to the simple coroots, that is,

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij} \quad \forall\, 1 \le i, j \le r \,.$$

Similarly, the **affine fundamental weights** are defined as elements of $\widehat{\mathfrak{h}}^*$ by

$$\hat{\omega}_i := (\omega_i; a_i^\vee; 0) \quad \text{for} \;\; 1 \le i \le r \,, \qquad \hat{\omega}_0 := (0; 1; 0) \,.$$

Writing $\omega_i = (\omega_i; 0; 0)$ it follows that $\hat{\omega}_i = a_i^\vee \hat{\omega}_0 + \omega_i$. Affine weights can thus be expanded as

$$\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + h\delta \,,$$

where we call λ_i the **Dynkin labels** of $\hat{\lambda}$ and $h \in \mathbb{R}$ its **conformal dimension**. We have that

$$k := \hat{\lambda}(\hat{k}) = \sum_{i=0}^{r} a_i^\vee \lambda_i = \langle \hat{\lambda}, \delta \rangle$$

called the **level**. Since in most applications of interest, the level is fixed from the outset, from now on we will identify $\hat{k}$ with its eigenvalue k and denote $\widehat{\mathfrak{g}}$ by $\mathfrak{g}_k$. Note that the zeroth Dynkin label λ_0 is related to the finite Dynkin labels λ_i for $1 \le i \le r$ by

$$\lambda_0 = k - \langle \lambda, \theta \rangle = k - \sum_{i=1}^{r} a_i^\vee \lambda_i \,. \tag{2}$$

Affine weights will therefore be generally given in terms of Dynkin labels under the form

$$\hat{\lambda} = [\lambda_0, \lambda_1, \ldots, \lambda_r] \,,$$

where we stress that this notation does not keep track of the eigenvalue of L_0, namely of the conformal dimension of $\hat{\lambda}$. However, as we will see this will not be a problem for WZW models, since the conformal dimension of an affine highest weight is determined by its finite part. Finally, the **affine Weyl vector** is defined as

$$\hat{\rho} := \sum_{i=0}^{r} \hat{\omega}_i = [1, 1, \ldots, 1]\,,$$

thus $\hat{\rho}(\hat{k}) = h^\vee$.

2.1.1 Affine highest weight representations

In this section we discuss one type of affine representations, that is, Lie algebra representations of $\mathfrak{g}_k$ with arbitrary but fixed level $k \in \mathbb{R}$. This is called a **highest weight representation** and it is characterised by a unique highest weight state $|\hat{\lambda}\rangle$ annihilated by the action of all ladder operators of positive roots

$$E_0^\alpha\,|\hat{\lambda}\rangle = E_n^{\pm\alpha}\,|\hat{\lambda}\rangle = H_n^i\,|\hat{\lambda}\rangle = 0\,, \qquad \forall\, n > 0,\ \alpha \in \Delta_+\,. \tag{3}$$

The eigenvalue $\hat{\lambda}$ of this state is the affine highest weight of the representation, namely

$$H_0^i\,|\hat{\lambda}\rangle = \lambda^i|\hat{\lambda}\rangle \ \text{ for } i \neq 0\,, \quad \hat{k}\,|\hat{\lambda}\rangle = k|\hat{\lambda}\rangle\,, \quad L_0\,|\hat{\lambda}\rangle = h\,|\hat{\lambda}\rangle\,. \tag{4}$$

All the states in the module are then generated by the action of the lowering operators on $|\hat{\lambda}\rangle$. We denote by $\Omega_{\hat{\lambda}}$ the set of all affine weights in the highest weight representation of $\hat{\lambda}$. From an algebraic point of view, one usually sets h equal to zero by redefining L_0. Note the position of the i-label in λ^i differentiates this value from the Dynkin label $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle$.

The analogues of irreducible finite-dimensional representations of $\mathfrak{g}$ are representations whose projections onto the $\mathfrak{su}(2)$ algebra associated with any real root are finite. One can reduce the analysis on simple roots. Then, for any $\hat{\lambda}' \in \Omega_{\hat{\lambda}}$ one has that

$$\langle \hat{\lambda}', \alpha_i^\vee \rangle = -(p_i - q_i) \qquad \forall\, 0 \leq i \leq r$$

for some positive integers p_i, q_i, which thereby implies that

$$\lambda'_i \in \mathbb{Z} \qquad \forall\, 0 \leq i \leq r\,.$$

For the highest weight $\hat{\lambda} \in \mathbb{Z}$, all p_i are zero, and therefore

$$\lambda_i \in \mathbb{Z}_+ \qquad \forall\, 0 \leq i \leq r\,.$$

This requires in particular that

$$\lambda_0 = k - \langle \lambda, \theta \rangle \in \mathbb{Z}_{>0} \,.$$

Since $\langle \lambda, \theta \rangle \in \mathbb{Z}_{>0}$, this immediately implies that

$$k \in \mathbb{Z}_{>0} \quad \text{and} \quad k \geq \langle \lambda, \theta \rangle \,. \tag{5}$$

An affine weight for which all Dynkin labels are non-negative integers is called **dominant**, and the set of all dominant weights at level k is denoted by P_+^k. A consequence of Eq. (5) is that for fixed value of k, there can be only finitely many dominant highest weight representations. For instance, at $k = 1$, the only such representations are those with highest weight $\hat{\omega}_i$ such that the corresponding simple root α_i has unit comark. Since $a_0^\vee = 0$ independently on the algebra, $\hat{\omega}_0$ is always dominant and the level-1 highest weight representation associated to it is called the **basic representation**. For $\mathfrak{su}(N)$, all comarks are one and hence there are N dominant highest weight representations at level 1 whose highest weights are $\hat{\omega}_i$ for $0 \leq i \leq r$.

Representations that decompose further into finite irreducible representations of $\mathfrak{su}(2)$ and can further be written as a direct sum of finite-dimensional weight spaces are said to be **integrable**. Even though the adjoint representation is not a highest weight representation, it is integrable. The first condition is clearly satisfied, while the second condition is equivalent to the root-space decomposition, that is, the decomposition of the root space into a sum of finite roots and imaginary roots. Dominant highest-weight representations are also integrable. Moreover, if

$$(J_n^a)^\dagger = J_{-n}^a \,, \qquad \text{or} \qquad (H_n^i)^\dagger = H_{-n}^i \quad (E_n^a)^\dagger = E_{-n}^a \,, \tag{6}$$

then dominant highest-weight representations are easily checked to be unitary provided Eq. (5) holds true.

2.1.2 Singular Vectors and Kac-Kazhdan determinant

For dominant highest weights, Eq. (5) is equivalent to the existence of the following singular vectors in the Verma module of the highest weight state $|\hat{\lambda}\rangle$:

$$E_0^{\alpha_i} \, |\hat{\lambda}\rangle = E_1^{\theta} \, |\hat{\lambda}\rangle = 0 \,,$$

and

$$(E_0^{-\alpha_i})^{\lambda_i + 1} \, |\hat{\lambda}\rangle = (E_{-1}^{\theta})^{k - \langle \lambda, \theta \rangle + 1} \, |\hat{\lambda}\rangle = 0 \,, \tag{7}$$

for $1 \leq i \leq r$. In sharp contrast with simple Lie algebras, when these singular vectors are quotiented out from the dominant highest-weight Verma module (modulo their possible intersections), the resulting irreducible module is not finite-dimensional. The imaginary root can be subtracted from any weight without leaving the representation. The source of infinity clearly lies in the absence of a singular vector related to the imaginary root δ, that is, a singular vector that would involve H^i_n for $n < 0$. In the following, we will call the **grade** or **level** the L_0 eigenvalue, shifted such that $L_0 |\hat{\lambda}\rangle = 0$ on the highest weight state $|\hat{\lambda}\rangle$.

The discussion above about singular vectors, and in particular Eq. (7), holds only for integrable highest weight representations. However, Kac and Kazhdan [KK79] showed that for any affine Lie algebra $\mathfrak{g}_k$ with symmetrisable generalised Cartan matrix[1], there exists a formula that identifies all the affine singular weights in an highest weight module for $\mathfrak{g}_k$. We will not explain what does it mean for an affine Lie algebra to have a symmetrisable Cartan matrix, since we will apply this result only to the specific case of $\mathfrak{su}(2)_k$ which is known to satisfy this condition.

More concretely, Kac and Kazhdan showed that highest weight Verma modules can be equipped with a unique (up to normalisation) invariant inner product, the *Shapovalov form*, and in particular they gave a formula for the determinant of the Shapovalov form of the Verma module with affine highest weight $\hat{\lambda}$ restricted to the weight space $\hat{\mu} \in \Omega_{\hat{\lambda}}$, which we denote by $\det_{\hat{\lambda}}(\hat{\mu})$.

Theorem 2.1 (Kac-Kazhdan determinant, [KK79]). *Let $\mathfrak{g}_k$ be an affine Lie algebra with a symmetrisable Cartan matrix, $\hat{\lambda} \in \mathfrak{h}^*_k$ and $\hat{\mu} \in \Omega_{\hat{\lambda}}$. Then, up to a non-zero factor (depending on the choice of basis) one has*

$$det_{\hat{\lambda}}(\hat{\mu}) = \prod_{\hat{\alpha} \in \hat{\Delta}_+} \prod_{l=1}^{\infty} \left(\langle \hat{\lambda} + \hat{\rho}, \hat{\alpha} \rangle - l \frac{\langle \hat{\alpha}, \hat{\alpha} \rangle}{2} \right)^{P(\hat{\mu} - l\hat{\alpha})} , \tag{8}$$

where $P(\hat{\mu})$ denotes the multiplicity of $\hat{\mu}$ in the vacuum Verma module of $\mathfrak{g}_k$, that is, the affine highest weight module generated by the highest affine weight $k\hat{\omega}_0$.

The non-trivial singular vectors and their descendants are all null with respect to the Shapovalov form, meaning that their norm is zero. The presence of such null states can then be detected by computing the determinant of the Shapovalov form in each affine weight space $\hat{\mu}$. Indeed, the presence of a singular vector in the Verma module of $\hat{\lambda}$ is signalled by the vanishing of one of the factors appearing in Eq. (8) and the vanishing of the arguments of the function P occurring in the corresponding exponent (non-vanishing arguments of this P in general correspond to descendants of the singular vector). We will refer to weights which admit a singular vector as **singular weights**. Another consequence

[1]Actually, the result in [KK79] is proven for a class of objects called *contragredient Lie algebras*, which include affine Lie algebras, still under the condition that the associated Cartan matrix is symmetrisable

from [KK79] is that if a weight is singular, than the null vector of that weight is unique up to normalisation.

2.2 Wess-Zumino-Witten models

In this section we follow chapter 15 of [DFMS97]. We assume the reader being familiar with the concept of conformal field theory, for which good references are [BP09] and [DFMS97].

2.2.1 WZW from a geometrical perspective

Wess-Zumino-Witten (WZW) models are examples of two-dimensional conformal field theories with a symmetry algebra given by an affine Lie algebra, which thus generates the spectrum of the theory. What is peculiar of these models is that they have a nice geometric interpretation in terms of a sigma model on a (semisimple) Lie group; in particular they can be formulated directly by an action functional. We will thus introduce them by means of this action and show how to extract from it their algebraic structure, which provides them with an alternative algebraic definition.

We start by considering a quantum field theory on a Riemann surface Σ, such as the sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$, with complex local coordinates $(z, \bar{z})$. We work in Euclidean signature, such that we can apply the tools from complex geometry. Let G be a semisimple Lie group and denote by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ a non-degenerate invariant inner product on $\mathfrak{g}$[2], which by translation can be extended to an invariant Riemannian metric on G and on the cotangent bundle T^*G, that we both denote by $\langle \cdot, \cdot \rangle$. The fields of this model are smooth maps $g \in C^\infty(\Sigma, G)$ whose action functional is defined as a sigma model

$$S[g] = S_0[g] + kS^{WZ}[g] \,, \tag{9}$$

where

$$S_0[g] := \frac{1}{4a^2} \int_\Sigma \langle dg, dg \rangle = \frac{1}{4a^2} \int_\Sigma dz^2 \langle g^{-1}\partial_\mu g \, g^{-1}\partial^\mu g \rangle_{\mathfrak{g}} \,,$$

for some constant $a \in \mathbb{R} \setminus \{0\}$ to be determined later, and the so called *Wess-Zumino term* is given by

$$S^{WZ}[g] := \int_B g^* H = -\frac{i}{12\pi} \int_B \langle g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg \rangle_{\mathfrak{g}} \,,$$

where B is a three-dimensional manifold whose boundary is Σ and H is a multiple of the harmonic three form of G, see [LW22]. This makes sense only if there exists an

[2] Since G is semisimple, the Killing form provides such scalar product. Note that in a matrix representation, this is given by $\langle X, Y \rangle_{\mathfrak{g}} = \operatorname{tr} XY$ for $X, Y \in \mathfrak{g}$, normalised such that $\operatorname{tr}(J^a J^b) = \frac{1}{2}\delta^{ab}$ on the generators of the Lie algebra.

extension of g to B; this requires $\pi_2(G) \cong 1$, which is true for example if G is compact. Then, an extension is in general not unique. For the WZW model to be well-defined, the path integral and hence e^{iS} should not depend on the choice of the extension. The Wess–Zumino term is invariant under continuous deformations of g, and only depends on its homotopy class. Possible homotopy classes are controlled by the homotopy group $\pi_3(G)$. If G is compact connected simple, then $\pi_3(G) \cong \mathbb{Z}$ and different extensions of g lead to values of $S^{WZ}[g]$ that differ by integers. Therefore, they lead to the same value of the path integral provided the level obeys

$$k \in \mathbb{Z}\,. \tag{10}$$

Thus, the classical theory is defined for any $k \in \mathbb{R}$ but quantisation requires the level to be an integer. We remark that this topological argument is true for compact connected simple groups G, but dropping one of these condition might modify the quantisation condition Eq. (10). One can show that Eq. (9) has a $G \times G$ symmetry given by

$$G \times G \times C^\infty(\Sigma, G) \to C^\infty(\Sigma, G),\ (g_L, g_R, g) \mapsto g_L g g_R\,,$$

and derive the equation of motions of Eq. (9). These reflect this symmetry, and can be expressed in local coordinates $(z, \bar{z})$ as

$$\left(1 + \frac{a^2 k}{\pi}\right) \partial(g^{-1}\bar{\partial} g) + \left(1 - \frac{a^2 k}{\pi}\right) \bar{\partial}(g^{-1}\partial g) = 0\,,$$

where $\partial := \partial_z$ and $\bar{\partial} := \partial_{\bar{z}}$. For a given k, we can then choose a such that one of the two terms vanishes; without loss of generality we choose $k \in \mathbb{Z}_{>0}$ and $a^2 = \pi/k$. This choice determines the WZW action Eq. (9). We obtain the equation of motion

$$\partial(g^{-1}\bar{\partial} g) = 0\,, \tag{11}$$

which is solved by $g(z, \bar{z}) = g(z)\bar{g}(\bar{z})$ for every $g, \bar{g} \in C^\infty(\Sigma, G)$. Also, Eq. (11) implies the conservation of the antiholomorphic current $\bar{J} := kg^{-1}\partial g$, which in turn implies that the current $J := -k\partial g g^{-1}$ is holomorphic. Thus, the theory presents a holomorphic and an antiholomorphic current; this property enhances the $G \times G$ symmetry to a local $G(z) \times G(\bar{z})$ symmetry acting by

$$g(z, \bar{z}) \mapsto g_L(z) g(z, \bar{z}) g_R(\bar{z})^{-1}\,, \tag{12}$$

where $g_L, g_R \in C^\infty(\Sigma, G)$ are holomorphic and antiholomorphic respectively.

2.2.2 WZW from an algebraic perspective

We now move to quantizing the theory. An efficient way of doing it is to consider the operator product expansions (OPE) of the currents. Note that the currents are maps $J(z), \bar{J}(\bar{z}) \in C^\infty(\Sigma, \mathfrak{g})$, hence we may write $J(z) = \sum_a J^a(z) T^a$, where $1 \leq a \leq \dim \mathfrak{g}$ are adjoint indices and T^a generators of $\mathfrak{g}$, and similarly for $\bar{J}$. In Eq. (12) we consider infinitesimal variations $g_L(z) = \exp_G(\omega(z))$ with $\exp_G \colon \mathfrak{g} \to G$ being the Lie group exponential and $\omega \in C^\infty(\Sigma, \mathfrak{g})$, and similarly for g_R. Then, the variation of the holomorphic current is given by

$$\delta_\omega J = [\omega, J] - \partial\omega \,,$$

or equivalently

$$\delta_\omega J^a = i \sum_{b,c} f^a_{bc}\, \omega^b J^c - k \partial \omega^a \,, \tag{13}$$

where the indices in f^{ab}_c are raised and lowered with the Killing form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Also, the Ward identity for the local symmetry Eq. (12) is given by

$$\delta_{\omega,\bar{\omega}} \langle X \rangle = -\frac{1}{2\pi i} \oint dz \sum_a \langle J^a X \rangle + \frac{1}{2\pi i} \oint d\bar{z} \sum_a \bar{\omega}^a \langle \bar{J}^a x \rangle \,. \tag{14}$$

Then, substituting Eq. (13) into Eq. (14) leads to the OPE

$$J^a(z) J^b(w) \sim \frac{k \delta^{ab}}{(z-w)^2} + i \sum_c f^{ab}_c \frac{J^c(w)}{(z-w)} \,. \tag{15}$$

We call this OPE structure that of a *current algebra*. By introducing the modes J^a_n from the Laurent expansion

$$J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J^a_n \,,$$

one checks that Eq. (15) is equivalent to Eq. (1) for the current modes at level $\hat{k} = k$. We can repeat the same argument for the antiholomorphic current, which yields an other copy of the affine algebra $\mathfrak{g}_k$ commuting with that generated by the holomorphic modes, since the OPE $J^a(z)\bar{J}^b(\bar{w}) \sim 0$ implies that $[J^a_n, \bar{J}^b_m] = 0$ for every a, b and $n, m \in \mathbb{Z}$.

Remark 2.2. We remark that even though we considered the field $g(z, \bar{z})$ to transform in the fundamental representation of G, the WZW model is defined for g transforming in any unitary representation of G. Neither we need to specify the representations of the left and right G transformations in Eq. (12). The the full spectrum of the theory is uniquely fixed by the group G and it can in principle be obtained by canonical quantization, and global considerations determine the combinations of left and right representations that can appear. However, we will turn to an algebraic formulation of WZW models in terms of the affine algebra $\mathfrak{g}_k$. We stress that $\mathfrak{g}_k$ is not the symmetry algebra of the theory, since

J^a_n do not commute with the Hamiltonian for $n \neq 0$ but only the zero modes do. For this reason, the affine $\mathfrak{g}_k$ is referred to as the spectrum-generating algebra. We will shortly show that WZW models are indeed examples of CFT; this is due to the occurrence of two independent conserved currents generating independent affine algebras, which leads to conformal invariance via the Sugawara construction. Then, we will identify primary fields with dominant weights and physical spectra will be obtained by modular invariance.

2.2.3 The Sugawara construction

To show that the theory is conformal, we argue that the Virasoro algebra embeds in the universal enveloping algebra of $\mathfrak{g}_k$: this is the *Sugawara construction*. This can be done for both left and right movers, that is for the holomorphic and antiholomorphic currents, hence we actually obtain an embedding of two copies of the Virasoro algebra in the enveloping algebra of the the two independent copies of $\mathfrak{g}_k$. In the following we concentrate on the holomorphic case and we assume that $\mathfrak{g}$ is simple. From Eq. (9) one can derive the holomorphic classical energy-momentum tensor $T(z) = \gamma \sum_a (J^a J^a)(z)$, which after quantising the theory requires a normal ordering prescription that we fix to be the conventional one:

$$:A_m B_n: := \begin{cases} A_m B_n & \text{if } m \leq -1\,, \\ B_n A_m & \text{if } m \geq 0\,, \end{cases}$$

for the modes A_m and B_n of any field in the theory and $n, m \in \mathbb{Z}$. The constant prefactor $\gamma \in \mathbb{C}$ in the normal ordered energy momentum tensor is fixed by the quantum theory, requiring that the OPE $T(z)J^a(w)$ reflects the fact that $J^a(z)$ has conformal weight one for every a. This fixes $T(z)$ to be the so called **Sugawara energy-momentum tensor**,

$$T(z) = \frac{1}{2(k+h^\vee)} \sum_a :J^a J^a:(z)\,, \tag{16}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, which is the quadratic Casimir of the adjoint representation, that is, $\sum_{b,c} f^{ab}_{\ c} f^{bc}_{\ d} = 2h^\vee \delta^{ab}$. For instance, for $\mathfrak{su}(N)$ we have that $h^\vee = N$. Then one computes the OPE

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,,$$

which is that one of the Virasoro algebra, with a specific central charge

$$c = c(\mathfrak{g}_k) = \frac{k \dim \mathfrak{g}}{k+h^\vee}\,. \tag{17}$$

This construction proves that on the quantum level, WZW models define CFTs. Actually, Eq. (16) may be regarded as an alternative definition of the $\mathfrak{g}_k$-WZW model. One can compute that the modes of $T(z)$,

$$L_m = \frac{1}{2(k+h^\vee)} \sum_{n\in\mathbb{Z}} :J^a_n J^a_{m-n}: = \frac{1}{2(k+h^\vee)} \left(\sum_{n\le -1} J^a_n J^a_{m-n} + \sum_{n\ge 0} J^a_{m-n} J^a_n \right)$$

satisfy the Virasoro algebra

$$\begin{aligned} [L_m, J^a_n] &= -nJ^a_{m+n}\,, \\ [L_m, L_n] &= \frac{c}{12} m(m^2-1)\delta_{m+n,0} + (m-n)L_{m+n}\,, \end{aligned} \tag{18}$$

with the central charge as in Eq. (17). Eq. (18) implies that the affine Kac-Moody modes and the Virasoro modes are combined in a semidirect product structure and in particular the first line in Eq. (18) shows that $\mathfrak{g}_k$ is a Lie ideal inside the combined algebra. Moreover, in Eq. (18) the commutativity of the zero modes of the affine algebra with the Virasoro generators and with L_0 reflects the built-in $\mathfrak{g}$-invariance. However, the full affine Lie algebra is not a symmetry algebra, since its generators do not all commute with L_0. It will turn out to be the spectrum-generating algebra of the theory. We remark that the Sugawara construction has been presented in terms of the particular currents $J^a(z)$, whose modes are orthonormal with respect to the Killing form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, that is

$$\langle J^a_m, J^b_n\rangle_{\mathfrak{g}} = \delta^{ab}\delta_{m+n,0}\,.$$

In a generic basis $\tilde{J}^a_m$, the affine Lie commutator is changed to

$$[\tilde{J}^a_m, \tilde{J}^a_n] = i\sum_c \tilde{f}^{ab}_c \tilde{J}^c_{m+n} + kn\langle \tilde{J}^a_m, \tilde{J}^b_n\rangle \delta_{n+m,0}$$

where $[\tilde{J}^a_0, \tilde{J}^b_0] = i\sum_c \tilde{f}^{ab}_c \tilde{J}^c_0$. Then, the energy-momentum tensor in this basis reads

$$T(z) = \frac{1}{2(k+h^\vee)} \sum_{a,b} \frac{1}{\langle \tilde{J}^a_0, \tilde{J}^b_0\rangle} :\tilde{J}^a\tilde{J}^a:(z)\,. \tag{19}$$

Indeed, L_0 contains the normal-ordered quadratic Casimir operator of the finite $\mathfrak{g}$, itself defined in terms of the inverse of the Killing form; this directly implies the above generalization.

When $\mathfrak{g}$ is semisimple, that is

$$\mathfrak{g} \cong \bigoplus_i \mathfrak{g}^i$$

for a finite set of simple Lie algebras $\mathfrak{g}^i$ (where the suffix i is simply a counting index, not an exponent in any sense), then we define the Sugawara energy-momentum tensor as

$$T^{\mathfrak{g}_k} := \sum_i T^{\mathfrak{g}^i_{k_i}} , \tag{20}$$

where one can choose the levels k_i independently, but every choice determines k. Eq. (20) is easily shown to satisfy the Virasoro algebra with central charge

$$c(\mathfrak{g}_k) = \sum_i c(\mathfrak{g}^i_{k_i}) ,$$

where $c(\mathfrak{g}^i_{k_i})$ is as in Eq. (17).

2.2.4 WZW primary fields as highest weight states

We established that the Kac-Moody algebra acts on the Hilbert space of the theory, hence all states of the CFT will transform in a representation of $\mathfrak{g}_k$. More precisely, analogously to the purely conformal case where primary fields transform covariantly with respect to scale transformations, a WZW primary field is defined as a field that transforms covariantly with respect to local $G(z) \times G(\bar{z})$ transformations. By Eq. (14) we can reformulate this property for a solution $\Phi(z, \bar{z}) = \Phi_\lambda(z)\Phi_\mu(\bar{z})$ of Eq. (11) consisting in fields Φ_λ, Φ_μ transforming in representations λ, μ of $\mathfrak{g}$, that is, taking values in the respective representation spaces, see Remark 2.2, in terms of the OPE

$$\begin{aligned} J^a(z)\Phi_\lambda(w) &\sim \frac{-T^a_\lambda \Phi_\lambda(w)}{z-w} , \\ \bar{J}^a(\bar{z})\Phi_\mu(\bar{w}) &\sim \frac{\Phi_\mu(\bar{w})T^a_\mu}{\bar{z}-\bar{w}} , \end{aligned} \tag{21}$$

where T^a_λ, T^a_μ are matrices of the representations λ, μ of $\mathfrak{g}$. By expanding the currents as

$$J^a(z) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-1} J^a_n(w)$$

we have that Eq. (21) yields

$$\begin{aligned} J^a_0 \,|\lambda\rangle &= -T^a_\lambda |\lambda\rangle , \\ J^a_n \,|\lambda\rangle &= 0 \quad \forall\, n > 0 , \end{aligned} \tag{22}$$

after introducing the state $|\lambda\rangle := \Phi_\lambda(0)\,|0\rangle$, where $|0\rangle$ denotes the vacuum of the theory. A remarkable aspect of WZW models is that WZW primary fields are also Virasoro primaries. Indeed, one computes

$$L_n \,|\lambda\rangle = 0 \quad \forall\, n > 0 , \tag{23}$$

and

$$L_0\,|\lambda\rangle = h_\lambda|\lambda\rangle \quad \text{with} \;\; h_\lambda = \frac{C_\lambda}{2(k+h^\vee)}\,, \tag{24}$$

where $C_\lambda = \sum_a T^a_\lambda T^a_\lambda$ is the quadratic Casimir of the representation λ of $\mathfrak{g}$. Eq. (23) confirms that every WZW primary is a Virasoro primary, but the converse is not true: a Virasoro primary can be a WZW descendant, as it is the example for the field J^a. Eq. (22) implies that WZW primaries are associated with highest weight representations λ of $\mathfrak{g}$, and all the other states of the theory have the form

$$J^{a_1}_{-n_1}\dots J^{a_N}_{-n_N}\,|\lambda\rangle \quad \text{for} \;\; n_i > 0\,, \tag{25}$$

that is, they belong to the affine highest weight representation $\hat{\lambda} = (\lambda; k; h_\lambda)$ of $\mathfrak{g}_k$, where h_λ is as in Eq. (24). The states in Eq. (25) are associated with descendants fields. The application of negative Virasoro modes needs not to be taken into account separately, since the energy momentum tensor already belongs to the enveloping algebra of $\mathfrak{g}_k$.

One can derive the conformal Ward identities for correlator functions of n WZW primary fields Φ_{λ_i} for some $\mathfrak{g}$-representations λ_i. These are related to the Virasoro primary nature of the fields, and they take the form

$$\sum_{i=1}^{n} T^a_{\lambda_i}\langle\Phi_{\lambda_1}(z_1)\dots\Phi_{\lambda_n}(z_n)\rangle = 0\,.$$

These identities fix the structure of the two- and three-point functions. Further constraints arise from the null fields in the primary representation, that is, the affine singular vectors and also from the definition of the Sugawara energy-momentum tensor. In particular, by inserting the zero vector

$$\left(L_{-1} + \frac{1}{k+h^\vee}\sum_a (J^a_{-1}T^a_{\lambda_i})\right)|\lambda_i\rangle$$

inside the correlation function of a set of primary fields, one obtains the celebrated *Knizhnik-Zamolodchikov equation*:

$$\left(\partial_{z_i} + \frac{1}{k+h^\vee}\sum_{j\neq i}\frac{\sum_a T^a_{\lambda_i}\otimes T^a_{\lambda_j}}{z_i - z_j}\right)\langle\Phi_{\lambda_1}(z_1)\dots\Phi_{\lambda_n}(z_n)\rangle = 0\,.$$

The solutions of this equation are the correlation functions of primary fields. As in the purely Virasoro case, the correlation functions involving descendant fields can be obtained directly from those of primary fields.

Note that Eq. (22) implies that the primary fields of a WZW model, that is, the fundamental fields from which all other fields can be obtained by the application of the

Virasoro or affine Lie generators, are those associated with the highest weight states of $\mathfrak{g}$-representations. This means that a state $|\hat{\lambda}\rangle$ associated to a primary field as above, satisfies Eq. (3) and Eq. (4) with $h = h_\lambda$ as in Eq. (24). A special class of primary fields is formed by highest weight states of integrable representations $\hat{\lambda} \in P_+^k$. Note that there could be none; for instance if $k < 0$, then P_+^k is empty by Eq. (5). These states generate finite representations with respect to any $\mathfrak{su}(2)$ subalgebra of $\mathfrak{g}_k$, which as we have seen implies the presence of the singular vectors of Eq. (7) in the Verma module of highest weight state $|\hat{\lambda}\rangle$. The presence of these singular vectors further constraints the structure of the correlation functions in the theory. Remarkably, it turns out that all the states in non-integrable representations decouple from the theory, that is, their correlations with arbitrary fields vanish. We stress that this derivation is general, but it requires the existence of at least one field corresponding to an integrable representation, that means

$$P_+^k \neq \emptyset\,. \tag{26}$$

Note that, as explained before, for $k \geq 1$ condition Eq. (26) is always true. We call a WZW model on an affine Lie algebra $\mathfrak{g}_k$ **integrable** if Eq. (26) holds true. So, in integrable WZW models the only physically relevant fields are those in integrable representations. Therefore, primary fields are in correspondence with the affine dominant weights $\hat{\lambda} \in P_+^k$, the highest weights of integrable representations. Since there is a finite number of such weights for a fixed positive integer k, it follows that there is a finite number of primary fields. We thus conclude, that integrable WZW models are **rational** CFTs.

2.3 The example of $\mathfrak{su}(2)_k$

We consider the compact simple real Lie algebra $\mathfrak{su}(2)$ with generators K^a for $a = 3, \pm$ in the usual spin basis, that is

$$[K^3, K^\pm] = \pm K^\pm\,, \qquad [K^+, K^-] = 2K^3\,.$$

Then the Cartan subalgebra is one-dimensional and generated by K^3, and we take the generator dual to the positive root to be K^+. The affine $\mathfrak{su}(2)_k$ algebra is defined by the commutation rules

$$\begin{aligned}
[K^3_m, K^3_n] &= \frac{k}{2} m \delta_{m+m,0}\,,\\
[K^3_m, K^\pm_n] &= \pm K^\pm_{m+n}\,,\\
[K^+_m, K^-_n] &= 2K^3_{m+n} + km\delta_{m+n,0}\,.
\end{aligned} \tag{27}$$

Eq. (27) are equivalent to the OPEs

$$\begin{aligned} K^3(z)K^3(w) &\sim \frac{k/2}{(z-w)^2}\,, \\ K^3(z)K^\pm(w) &\sim \frac{\pm K^\pm(w)}{(z-w)}\,, \\ K^+(z)K^-(w) &\sim \frac{k}{(z-w)^2} + \frac{2K^3(z)}{(z-w)}\,. \end{aligned}$$

The Sugawara energy-momentum tensor reads

$$T^{\mathfrak{su}(2)_k} = \frac{1}{k+2}\left[:\!K^3K^3\!: + \frac{1}{2}\left(:\!K^+K^-\!: + :\!K^-K^+\!:\right)\right].$$

In the notation introduced before, we have that $\theta = \alpha_1$ is the only positive root of the finite $\mathfrak{su}(2)$ and it corresponds to the generator K^3. Also,

$$\langle \alpha_0, \alpha_1^\vee\rangle = \langle \alpha_1, \alpha_0^\vee\rangle = \langle \alpha_1, \alpha_0\rangle = -\alpha_1^2 = -2\,.$$

In Dynkin labels, the simple roots are

$$\alpha_0 = [2,-2]\,, \qquad \alpha_1 = [-2,2]\,.$$

The complete set of roots is

$$\hat{\Delta} = \{\pm\alpha_1,\ \pm\alpha_1 + n\delta,\ n\delta : n\in\mathbb{Z},\ n\neq 0\}\,,$$

corresponding to the generators $K_0^\pm$, $K_n^\pm$ and K_n^3 respectively. Let $\hat{\lambda} = [\lambda_0, \lambda_1]$ be an affine weight, then by Eq. (2) we have that

$$\lambda_0 = k - \lambda_1\,,$$

and one can check that $\lambda_1 = 2\ell$, where $\ell \in \frac{1}{2}\mathbb{N}$ is the $\mathfrak{su}(2)$ spin of the highest weight state $|\hat{\lambda}\rangle$, that is $K_0^3\,|\hat{\lambda}\rangle = \ell|\hat{\lambda}\rangle$. Since the affine weight is completely determined by ℓ, we adopt the notation $|\ell\rangle := |\hat{\lambda}\rangle$. Then, by scaling the $\mathfrak{su}(2)$ Casimir by a factor 1/2, such that it is equal to $\ell(\ell+1)$ on the finite highest weight representation of spin ℓ, by Eq. (24) we have that the conformal dimension of $|\ell\rangle$ is

$$h_\ell = \frac{\ell(\ell+1)}{k+2}\,. \tag{28}$$

All together, the affine $\mathfrak{su}(2)_k$ highest weight representation generated by $|\ell\rangle$, that is the affine weight with finite weight equal to $2\ell \in \mathbb{N}$, is defined by

$$
\begin{aligned}
&K_0^+ \,|\ell\rangle = K_n^\pm \,|\ell\rangle = 0 \qquad \forall\, n > 0\,, \\
&K_0^3 \,|\ell\rangle = \ell \,|\ell\rangle\,, \\
&L_0 \,|\ell\rangle = \frac{\ell(\ell+1)}{k+2} \,|\ell\rangle\,.
\end{aligned}
\tag{29}
$$

The canonical norm on the Verma module $\mathcal{V}_\ell^{(k)}$ of highest weight state $|\ell\rangle$ is given by Eq. (6), which specifies a real form on the Kac-Moody algebra and implies that we are indeed considering the compact form $\mathfrak{su}(2)$.

For $k \in \mathbb{Z}_{>0}$, the singular vectors given by Eq. (7) are all encoded in the following states:

$$
\begin{aligned}
&(K_0^-)^{2\ell+1} \,|\ell\rangle = 0\,, \\
&\mathcal{N} := (K_{-1}^+)^{k+1-2\ell} \,|\ell\rangle = 0\,,
\end{aligned}
\tag{30}
$$

where the first state simply means that $|\ell\rangle$ transforms in the finite-dimensional spin-ℓ representation of the finite $\mathfrak{su}(2)$, which is given by the zero modes K_0^a. On the other hand, one can explicitely show using the commutator relations that $\mathcal{N}$ is a singular vector, in the sense that it is again an affine highest weight state. It turns out that the singular vectors Eq. (30) together with their descendants are the only null-vectors in the Verma module $\mathcal{V}_\ell^{(k)}$ and hence we obtain the corresponding irreducible representation by taking the quotient by the null-vector relations Eq. (30). We denote this irreducible representation of $\mathfrak{su}(2)_k$ by $\mathcal{H}_\ell^{(k)}$.

The presence of these null-vectors constrains the possible representations severely. Since integrable WZW models on compact groups are unitary, it is crucial that the integrable representations are free of negative norm states. For that, one can show by induction on $N \in \mathbb{N}$ that for arbitrary $k \in \mathbb{R}$ we have that

$$
\big|(K_{-1}^+)^N |\ell\rangle\big|^2 = \prod_{n=1}^{N} n(k+1-n-2\ell)\,,
\tag{31}
$$

assuming that $|\ell\rangle$ is normalised. This shows algebraically that for unitarity we must require

$$
k \in \mathbb{Z}_{>0} \quad \text{and} \quad 0 \le \ell \le \frac{k}{2}\,,
$$

which are exactly Eq. (5). This follows from the fact that $\mathcal{N} = 0$ together with the formula for its norm, given by Eq. (31) with $N = k+1-2\ell$. In particular, it follows that in the integrable case there are only finitely many representations as expected.

There is something peculiar that is present at level $k = 1$, which is that $\mathfrak{su}(2)_1$ possesses a free field realisation in terms of four real (or two complex) free fermions ψ^i

for $1 \leq i \leq 4$ satisfying the anti-commutation relation

$$\{\psi^i, \psi^j\} = \delta^{ij} .$$

Then, the generators K^a can be constructed by taking bilinear combinations of the free fermions; we will come back to this construction later when we discuss the free field realisation of $\mathfrak{u}(2|2)_1$. Analogously, one can show that N real fermions ψ^i realise the WZW model on $\mathfrak{so}(N)_1$, whose central charge is $c = N/2$. Indeed, such free field theory has Lagrangian

$$S = \int_\Sigma dz^2 \bar{\psi}^i \gamma^\mu \partial_\mu \psi^i ,$$

where there is an implicit sum over i. It is easy to see that S has an $SO(N)$-symmetry acting by rotating the fermions. The corresponding currents are

$$J^a(z) = \frac{1}{2} T^a_{ij} (\psi^i \psi^j)(z) ,$$

where T^a are $\mathfrak{so}(N)$ generators in the fundamental representation. By general arguments, these currents have to satisfy an $\mathfrak{so}(N)_k$ affine Kac-Moody algebra, which turn out to have level $k = 1$.

2.4 Lie superalgebras

We have now introduced the basic ingredients for studying WZW models on Lie groups or algebras. However, the actual model we will consider is built on a Lie superalgebra. We thus define this concept, following [Kac77]. Throughout this chapter we will work with objects defined over the field $\mathbb{R}$ or $\mathbb{C}$ and we assume all the algebras to be associative.

2.4.1 Basic definitions and properties

A **superalgebra** $\mathfrak{a}$ is a $\mathbb{Z}_2$-graded algebra $\mathfrak{a} = \mathfrak{a}^{(0)} \oplus \mathfrak{a}^{(1)}$, that is

$$X \in \mathfrak{a}^{(i)},\, Y \in \mathfrak{a}^{(j)} \implies XY \in \mathfrak{a}^{(ij)} \qquad \forall\, i, j \in \mathbb{Z}_2 .$$

We call elements in $\mathfrak{a}^{(0)}$ **bosonic** or **even**, while those in $\mathfrak{a}^{(1)}$ **fermionic** or **odd**. We call an element $X \in \mathfrak{a}$ **homogeneus** if $X \in \mathfrak{a}^{(i)}$ for some $i \in \mathbb{Z}_2$. For a homogeneous element $X \in \mathfrak{a}$ we define the **degree** or **parity** as

$$|X| := \begin{cases} 0 & \text{if } \ X \in \mathfrak{a}^{(0)} , \\ 1 & \text{if } \ X \in \mathfrak{a}^{(1)} , \end{cases}$$

as an element of $\mathbb{Z}_2$. A linear subspace $\mathfrak{b} \subset \mathfrak{a}$ is called $\mathbb{Z}_2$-graded if

$$\mathfrak{b} = \bigoplus_{i \in \mathbb{Z}_2} \left(\mathfrak{b} \cap \mathfrak{a}^{(i)} \right) .$$

A **subalgebra** of $\mathfrak{a}$ is a $\mathbb{Z}_2$.graded subalgebra; the same is true for **ideals**. We can then define the quotient of superalgebra by an ideal, which one can check to be again a superalgebra. A **superhomomorphism** of superalgebras $\mathfrak{a}$ and $\mathfrak{a}'$ is an algebra homomorphism $\Phi \colon \mathfrak{a} \to \mathfrak{a}'$ that preserves the grading, in the sense that

$$\Phi(\mathfrak{a}^{(i)}) \subset \mathfrak{a}^{(\varphi(i))} \qquad \forall\, i \in \mathbb{Z}_2 \,,$$

where φ is an automorphism of $\mathbb{Z}_2$, which we take to be the identity unless differently stated. A **superisomorphism** is a superhomomorphism which is an algebra isomorphism on the underlying algebras. Direct and semidirect sums of superalgebras are defined in the usual way, while the tensor product $\mathfrak{a} \otimes \mathfrak{b}$ of two superalgebras consist in the tensor product of the underlying vector spaces with the induced $\mathbb{Z}_2$-grading and the operation defined by

$$(X_1 \otimes Y_1)(X_2 \otimes Y_2) := (-1)^{|Y_1||X_2|} X_1 X_2 \otimes Y_1 Y_2 \qquad \forall\, X_n \in \mathfrak{a},\ Y_m \in \mathfrak{b} \,.$$

A **Lie superalgebra** is a superalgebra $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ together with a bilinear map $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ called the **Lie superbracket** of **Lie supercommutator** satisfying

$$\begin{aligned} [X,Y] &= -(-1)^{|X||Y|}[Y,X] \,, \\ [X,[Y,Z]] &= [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]] \,, \end{aligned}$$

where every element is homogeneous and so will be assumed later in any equation involving the degrees of the elements. The second condition is equivalent to the so called **super Jacobi identity**

$$(-1)^{|X||Z|}[X,[Y,Z]] + (-1)^{|Y||X|}[Y,[Z,X]] + (-1)^{|Z||Y|}[Z,[X,Y]] = 0 \,. \tag{32}$$

In particular, this implies that

$$[X,YZ] = [X,Y]Z + (-1)^{|X||Y|} Y[X,Z] \,.$$

We summarise some properties that follow from the super Jacobi identity.

Lemma 2.3. *Let $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ be a Lie superalgebra. Then the following are true.*

1. *$\mathfrak{g}^{(0)}$ is a Lie subalgebra called the **bosonic subalgebra** of $\mathfrak{g}$.*
2. *$\mathfrak{g}^{(1)}$ is a $\mathfrak{g}^{(0)}$-module under the adjoint action defined by the Lie superbracket.*

3. *The restriction of the Lie superbracket induces a symmetric* $\mathfrak{g}^{(1)}$*-equivariant linear map*

$$\{\cdot,\cdot\}\colon \mathfrak{g}^{(1)} \otimes \mathfrak{g}^{(1)} \to \mathfrak{g}^{(0)}$$

satisfying

$$[\{X,Y\},Z] + [\{Y,Z\},X] + [\{Z,X\},Y] = 0 \quad \forall\, X,Y,Z \in \mathfrak{g}^{(1)}\,.$$

4. *We have that* $[Y,[Y,Y]] = 0\,, \quad \forall\, Y \in \mathfrak{g}^{(1)}$.

In particular, every Lie superalgebra can be specified by three objects: two vector spaces $\mathfrak{g}^{(0)}$, $\mathfrak{g}^{(1)}$ and a linear map $\{\cdot,\cdot\}\colon \mathfrak{g}^{(1)} \otimes \mathfrak{g}^{(1)} \to \mathfrak{g}^{(0)}$ satisfying conditions $1-3$ of Lemma 2.3. It is common in the physics literature to denote the Lie superbracket between two fermions with $\{\cdot,\cdot\}$ and we will stick to this convention later.

There is a natural way of defining a Lie superbracket on a superalgebra $\mathfrak{a}$, namely by

$$[X,Y] := XY - (-1)^{|X||Y|}YX\,. \tag{33}$$

A superalgebra is then called **commutative** if $[X,Y]=0$ for every $X,Y \in \mathfrak{a}$.

Let $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ be a Lie superalgebra. We call the **universal enveloping algebra** of $\mathfrak{g}$ a pair $(U(\mathfrak{g}),\iota)$ consisting of a Lie superalgebra $U(\mathfrak{g})$ and a Lie superhomomorphism $\iota\colon \mathfrak{g} \to U(\mathfrak{g})$, if for any other pair $(U'(\mathfrak{g}),\iota')$ there exists a unique superhomomorphism $\Phi\colon U(\mathfrak{g}) \to U'(\mathfrak{g})$ such that $\iota' = \Phi \circ \iota$. If such pair exists, then it is unique up to superisomorphism. Existence is given for instance by an explicit construction. Let $T(\mathfrak{g})$ denote the tensor superalgebra over $\mathfrak{g}$ with the induced $\mathbb{Z}_2$ grading, and R the ideal of $T(\mathfrak{g})$ generated by the elements

$$[X,Y] - X\otimes Y + (-1)^{|X||Y|}Y \otimes X \quad \forall\, X,Y \in \mathfrak{g} \text{ homogeneous.}$$

We set $U(\mathfrak{g}) := T(\mathfrak{g})/R$. The natural embedding $\iota\colon \mathfrak{g} \to U(\mathfrak{g})$ is checked to be a superhomomorphism, and the pair $(U(\mathfrak{g}),\iota)$ is the enveloping superalgebra of $\mathfrak{g}$. The following theorem generalises the homonymous result for Lie algebras.

Theorem 2.4 (Poincaré-Birkhoff-Witt)**.** *Let* $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ *be a Lie superalgebra,* $X_1,\dots,X_m$ *be a basis of* $\mathfrak{g}^{(0)}$ *and* $Y_1,\dots,Y_n$ *a basis of* $\mathfrak{g}^{(1)}$*. Then the elements of the form*

$$X_1^{k_1}\dots X_m^{k_m} Y_{l_1}\dots Y_{l_n} \quad \textit{for } k_i \geq 0 \;\; \textit{and} \;\; 1 \leq l_1 < \dots < l_n \leq n\,,$$

form a basis of $U(\mathfrak{g})$*.*

Example 2.5. Let $V = V^{(0)} \oplus V^{(1)}$ be a $\mathbb{Z}_2$ graded vector space. Then, the associative algebra $\mathrm{End}(V)$ is equipped with the induced $\mathbb{Z}_2$ grading $\mathrm{End}(V) = \mathrm{End}(V)^{(0)} \oplus \mathrm{End}(V)^1$

with

$$\mathrm{End}(V)^{(j)} := \{X \in \mathrm{End}(V) : X(V^{(j)}) \subset V^{i+j} \ \ \forall\, i \in \mathbb{Z}_2\} \quad \forall\, j \in \mathbb{Z}_2 \,.$$

Then, $\mathrm{End}(V)$ equipped with the superbracket of Eq. (33) is a Lie superalgebra, denoted by $\mathfrak{gl}(V^{(0)}|V^1)$ and by $\mathfrak{gl}(m|n)$ when $V^{(0)} = \mathbb{C}^m$ and $V^{(1)} = \mathbb{C}^n$, in which case we also write $\mathbb{C}^{m|n} := V = \mathbb{C}^m \oplus \mathbb{C}^n$.

Let $\mathfrak{a} = \mathfrak{a}^{(0)} \oplus \mathfrak{a}^{(1)}$ be a superalgebra. A **derivation of degree** $d \in \mathbb{Z}_2$ is an endomorphism $\delta \in \mathrm{End}(\mathfrak{a})^{(d)}$ with the property

$$\delta(XY) = \delta(X)Y + (-1)^{d|X|} X\delta(Y) \,.$$

We denote the space of derivations of any degree by $\mathrm{Der}(\mathfrak{a}) \subset \mathrm{End}(\mathfrak{a})$ which can be seen to be a Lie superalgebra itself with the induced $\mathbb{Z}_2$-grading from $\mathrm{End}(\mathfrak{a})$, which we call the **superalgebra of derivations** of $\mathfrak{a}$.

Example 2.6. Let $\mathfrak{g}$ be a Lie superalgebra. From the super Jacobi identity Eq. (32) it follows that

$$\mathrm{ad}_X \colon \mathfrak{g} \to \mathfrak{g} \,, \quad Y \mapsto [X, Y]$$

is a derivation of $\mathfrak{g}$ of degree $|X|$ for every $X \in \mathfrak{g}$. These derivations are called **inner** and they form an ideal in $\mathrm{Der}(\mathfrak{g})$.

2.4.2 Lie superalgebras of type A and the supertrace

We now consider the Lie superalgebra $\mathfrak{gl}(m, n)$ for $m, n \in \mathbb{N}$, which is called the **general linear supergroup** of degree $m|n$. Note that $\mathfrak{gl}(0|m) \cong \mathfrak{gl}(m|0) \cong \mathfrak{gl}(m)$, the usual Lie algebra. We fix a basis $e_1, \ldots e_m, e_{m+1}, \ldots, e_{m+n}$ of $\mathbb{C}^{m+n}$. Then, elements $X \in \mathfrak{gl}(m, n)$ can be written in block $m|n$-form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} . \tag{34}$$

The bosonic (or even) subalgebra consists of elements of the form

$$X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} , \tag{35}$$

hence, it is isomorphic to $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. We define the **supertrace** of an element $X \in \mathfrak{gl}(m|n)$ as in Eq. (34) by

$$\mathrm{str}(X) := \mathrm{tr}(A) - \mathrm{tr}(D) \,.$$

Note that the supertrace is independent on the choice of basis and it satisfies the following properties.

Lemma 2.7. *1. The bilinear form* $\langle X, Y\rangle := \mathrm{str}(XY)$ *on* $\mathfrak{gl}(m|n)$ *is*

- ***consistent:*** $\langle X, Y\rangle = 0$ *for* X *even and* Y *odd,*
- ***supersymmetric:*** $\langle X, Y\rangle = (-1)^{|X||Y|}\langle Y, X\rangle$,
- ***invariant:*** $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$.

2. $\mathrm{str}([X, Y]) = 0$ *for every* $X, Y \in \mathfrak{gl}(m|n)$.

The **special linear supergroup** of degree $m|n$ to be

$$\mathfrak{sl}(m|n) := \left\{X \in \mathfrak{gl}(m|n) : \mathrm{str}(X) = 0\right\}.$$

One can check that $\mathfrak{sl}(m|n) = [\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)]$ and that $\mathfrak{sl}(m|n)$ is a Lie subalgebra of $\mathfrak{gl}(m|n)$. Moreover, for $m \neq n$ we have that $\mathfrak{sl}(m|n)$ is **simple**, that is, it does not contain any non-trivial ideals, and it has bosonic subalgebra isomorphic to $\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathfrak{u}(1)$. On the other hand, $\mathfrak{sl}(n|n)$ has a one-dimensional center generated by $\mathbb{C}I_{2n}$ and so the quotient

$$\mathfrak{psl}(n|n) := \mathfrak{sl}(n|n)/\mathbb{C}I_{2n}$$

is simple for $n \geq 2$ with bosonic subalgebra isomorphic to $\mathfrak{sl}(n) \oplus \mathfrak{sl}(n)$. Also, $\mathfrak{gl}(n|n)$ is obtained by $\mathfrak{sl}(n|n)$ by adding the generator

$$\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}. \tag{36}$$

We now consider a metric

$$\eta := \begin{pmatrix} \eta^{(p,q)} & 0 \\ 0 & \eta^{(n,0)} \end{pmatrix},$$

where $\eta^{(r,s)}$ denotes a metric of signature (r, s) on $\mathbb{R}^{r+s}$, and define

$$\mathfrak{u}(p, q|n) := \left\{X \in \mathfrak{gl}(p + q|n) : X\eta + \eta X^{\dagger} = 0\right\},$$

and

$$\mathfrak{su}(p, q|n) := \left\{X \in \mathfrak{u}(p, q|n) : \mathrm{str}(X) = 0\right\}.$$

For $m \neq n$ we have that $\mathfrak{su}(p, q|n)$ is simple with bosonic subalgebra isomorphic to $\mathfrak{su}(p, q) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1)$, whilst for $n = m$ the quotient

$$\mathfrak{psu}(n|n) := \mathfrak{su}(n|n)/\mathbb{C}I_{2n} \tag{37}$$

is simple with bosonic subalgebra isomorphic to $\mathfrak{su}(n) \oplus \mathfrak{su}(n)$. As for Lie algebras, the unitary version of the general linear superalgebras can be seen as real forms of the latter.

We mention that Kac gave a classification of Lie superalgebras and studied their representation theory, see [Kac77]. In general Lie superalgebras present analogies with Lie algebras, but also several results valid in the context of Lie algebras are not true or not yet known for Lie superalgebras. Also, one can define the notion of Lie supergroup and naturally associate to each a Lie superalgebra. As we discussed, from the geometric point of view WZW models are defined in terms of Lie groups, but it turns out that all the information of the theory, except for global considerations, can be captured by the corresponding Lie algebra. We thus refrain from introducing Lie supergroups, even if our main case of study is supersymmetric, because we will work directly at the level of the superalgebra and the representation theory of the latter.

3 The affine superalgebra $\mathfrak{psu}(2|2)_k$

3.1 Commutation relations

We introduce the Lie superalgebra $\mathfrak{u}(2|2)_k$ by explicitely giving the commutation relations. The finite Lie superalgebra $\mathfrak{u}(2|2)$ consists in the bosonic subalgebra

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)\,,$$

whose generator we denote by J_0^a, K_0^a for $a = 3, \pm$, U_0 and V_0 respectively, together with eight fermionic generators $S_0^{\alpha\beta\gamma}$ transforming in the $(\mathbf{2}, \overline{\mathbf{2}}) \oplus (\overline{\mathbf{2}}, \mathbf{2})$ representation of the bosonic subalgebra under the adjoint action. From the affine point of view, the bosonic subalgebra of $\mathfrak{u}(2|2)_k$ is thus

$$\mathfrak{su}(2)_{-k} \oplus \mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_{-1/2} \oplus \mathfrak{u}(1)_{1/2}\,, \tag{38}$$

generated by the respective modes. In particular, we have that K_n^a satisfy Eq. (27), while J_n^a the same by replacing k with $-k$. We denote by U_n and V_n the generators of the two affine $\widehat{\mathfrak{u}}(1)$ factors satisfying

$$[U_m, U_n] = -\frac{1}{2} m \delta_{m+n,0}\,, \qquad [V_m, V_n] = \frac{1}{2} m \delta_{m+n,0}\,. \tag{39}$$

We define the combinations

$$Z_n = U_n + V_n\,, \qquad Y_n = U_n - V_n\,,$$

such that

$$\begin{aligned}
&[Z_m, J_n^a] = [Z_m, K_n^a] = 0 = [Y_m, J_n^a] = [Y_n, K_n^a]\,,\\
&[Z_m, Z_n] = [Y_m, Y_n] = 0\,,\\
&[Z_m, S_n^{\alpha\beta\gamma}] = 0\,,\\
&[Y_m, Z_n] = -m\delta_{m+n,0}\,, \qquad [Y_m, S_n^{\alpha\beta\gamma}] = \gamma S_{n+m}^{\alpha\beta\gamma}\,.
\end{aligned} \tag{40}$$

Note that the modes Z_n are central. We can then identify the modes Y_n as those that extend the algebra $\mathfrak{su}(2|2)_k$ to $\mathfrak{u}(2|2)_k$ and the modes of Z_n as those that are quotiented out from $\mathfrak{su}(2|2)_k$ in order to obtain $\mathfrak{psu}(2|2)_k$. In particular, in the fundamental matrix representation of $\mathfrak{u}(2|2)$ we can identify Y_0 with the generator of Eq. (36) for $n = 2$, and Z_0 with I_4, see Eq. (37). In this representation, the bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ corresponds to block diagonal matrices of the form as in Eq. (35) and the eight fermionic generators $S_0^{\alpha\beta\gamma}$ to off-diagonal elements. For the fermionic modes $S_n^{\alpha\beta\gamma}$, the first two indices label the fundamental representations of the two $\mathfrak{su}(2)$ factors, whilst γ the action of the outer automorphism group of $\mathfrak{psu}(2|2)$, which is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

In particular, we have

$$
\begin{aligned}
&[J^3_m, S^{\pm\beta\gamma}_n] = \pm\tfrac{1}{2}\gamma S^{\pm\beta\gamma}_{m+n}\,, && [K^3_m, S^{\alpha\pm\gamma}_n] = \mp\tfrac{1}{2}\gamma S^{\alpha\pm\gamma}_{m+n}\,,\\
&[J^\pm_m, S^{\mp\beta+}_n] = S^{\pm\beta+}_{m+n}\,, && [K^\pm_m, S^{\alpha\pm+}_n] = -S^{\alpha\mp+}_{m+n}\,,\\
&[J^\pm_m, S^{\pm\beta-}_n] = -S^{\mp\beta-}_{m+n}\,, && [K^\pm_m, S^{\alpha\mp-}_n] = S^{\alpha\pm-}_{m+n}\,,
\end{aligned}
$$

which can also be written as

$$
[J^a_m, S^{\alpha\beta\gamma}_n] = \frac{1}{2}\gamma(\sigma^{\gamma a})^\alpha{}_\nu\, S^{\nu\beta\gamma}_{m+n}\,, \qquad [K^a_m, S^{\alpha\beta\gamma}_n] = -\frac{1}{2}\gamma(\sigma^{-\gamma a})^\beta{}_\nu\, S^{\alpha\nu\gamma}_{m+n}\,,
$$

where σ^a, for $a = \pm, 3$ and with the convention $-3 \equiv 3$, are the Pauli matrices, which in this basis take the form

$$
\sigma^+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

The anti-commutators between fermions are

$$
\begin{aligned}
&\{S^{\pm\beta+}_m, S^{\mp\beta-}_n\} = J^\pm_{m+n}\,, && \{S^{\alpha\pm+}_m, S^{\alpha\mp-}_n\} = K^\mp_{m+n}\,,\\
&\{S^{\pm\beta-}_m, S^{\mp\beta+}_n\} = J^\mp_{m+n}\,, && \{S^{\alpha\pm-}_m, S^{\alpha\mp+}_n\} = K^\pm_{m+n}\,,
\end{aligned}
$$

$$
\{S^{\alpha\beta\pm}_m, S^{\alpha\beta\mp}_n\} = \alpha J^3_{m+n} + \beta K^3_{m+n} + Z_{m+n} \mp km\delta_{m+n,0}\,, \tag{41}
$$

and hence, all together

$$
\begin{aligned}
\{S^{\alpha\beta\gamma}_m, S^{\mu\nu\rho}_n\} = {}& \delta^{\beta\nu}\delta^{\gamma,-\rho}\tau_{-\gamma a}{}^{\alpha\mu}\, J^a_{m+n} + \delta^{\alpha\mu}\delta^{\gamma,-\rho}\tau_{\gamma a}{}^{\beta\nu} K^a_{m+n} \\
& + \delta^{\alpha\mu}\delta^{\beta\nu}\delta^{\gamma,-\rho} Z_{m+n} - \delta^{\alpha\mu}\delta^{\beta\nu}\epsilon^{\gamma\rho} km\delta_{m+n,0}\,,
\end{aligned} \tag{42}
$$

where $\tau_3 = \sigma^3$, $\tau_\pm = \frac{1}{2}\sigma^\pm$ and as above we adopt the convention that $-3 \equiv 3$ for the index a of Pauli matrices. Note that in Eq. (41) and Eq. (42) the term involving the modes of Z have to be dropped when considering the commutators of the superalgebra $\mathfrak{psu}(2|2)_k$ since these are quotiented out. On the other hand, when passing from $\mathfrak{u}(2|2)_k$ to $\mathfrak{su}(2|2)_k$, the generator Y and all its modes are simply ignored.

3.2 Non-unitary representations of $\mathfrak{su}(2)$

In order to analyze the affine $\mathfrak{psu}(2|2)_k$-representations, it is necessary to develop first the representation theory of the finite $\mathfrak{psu}(2|2)$ Lie superalgebra. Since the bosonic subalgebra is $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we can decompose representations of $\mathfrak{psu}(2|2)$ in multiplets of $\mathfrak{su}(2)$-representations. Note that since $\mathfrak{su}(2)$ is compact, every unitary irreducible representation is finite-dimensional. As we have seen in Section 2.3, for $k \in \mathbb{Z}_{>0}$ the $\mathfrak{su}(2)_k$-model is integrable and it possesses only $k+1$ unitary integrable highest weight representations, which are characterised by the finite-dimensional spin $\ell \in \frac{1}{2}\mathbb{N}$ representations of $\mathfrak{su}(2)$ lying at the top of the affine module. However, by Eq. (5) it follows that the $\mathfrak{su}(2)_{-k}$-model is non-integrable; indeed, it possesses no unitary highest weight representations and, as we will see in Section 6.4, the spectrum is continuous and parameterised it by the J_0^3-eigenvalue $j \in \mathbb{R}$ (and for continuous representations by an additional parameter $\lambda \in [0,1)$). For this reason, we refer to $\mathfrak{su}(2)_k$ and $\mathfrak{su}(2)_{-k}$ as the *compact* and *non-compact* factor, respectively, of the bosonic subalgebra of $\mathfrak{psu}(2|2)_k$, and analogously for their affine versions.

Even though the spectrum of the $\mathfrak{su}(2)_{-k}$-theory is continuous, from the perspective of the Lie group, we expect the compactness of SU(2) to constrain the set of allowed representations to a discrete subset of spins j. Nevertheless, the finite $\mathfrak{su}(2)$ representations lying at the highest weight $\mathfrak{su}(2)_{-1}$-representations are allowed to be non-unitary, that is, infinite-dimensional. We thus look at all possible representations of $\mathfrak{su}(2)$, or equivalently (if disregarding unitarity), of $\mathfrak{sl}(2,\mathbb{R})$. These are classified in the following three families.

- The *finite-dimensional* representations H_j of spin $j \in \frac{1}{2}\mathbb{N}$. These are the usual unitary representations of dimension $2j+1$, characterised by the J^3-eigenvalue j, which we call *spin*. The Casimir of these representations is

$$C^{\mathfrak{su}(2)}(H_j) = j(j+1)\,.$$

- The *highest/lowest weight discrete* representations $D_j^{\pm}$ of spin $j \in \mathbb{R} \setminus \pm\frac{1}{2}\mathbb{N}$. These are infinite-dimensional non-unitary representations defined by an highest/lowest weight state $|j\rangle$ such that

$$D_j^{\pm} \;:\qquad J^{\pm}\,|j\rangle = 0 \quad \text{ and } \quad J^3\,|j\rangle = j|j\rangle\,,$$

and with Casimir equal to

$$C^{\mathfrak{su}(2)}(D_j^{\pm}) = j(j\pm 1)\,.$$

- The *continuous* representations C_j^{λ}, for $j \in \mathbb{R}$ and $\lambda \in \mathbb{R}/\mathbb{Z}$. These are infinite-dimensional non-unitary representations that neither contain a highest nor a lowest weight state, and they are characterised by their Casimir

$$C^{\mathfrak{su}(2)}(C_j^{\lambda}) = j(j-1) \in \mathbb{R},$$

as well as the fractional part of the J^3-eigenvalues $\lambda \in \mathbb{R}/\mathbb{Z}$. More specifically, the representation C_j^λ is defined by states $|m\rangle$ with $m \in \mathbb{Z} + \lambda$ such that

$$
\begin{aligned}
J^3 \, |m\rangle &= m|m\rangle \, , \\
J^+ \, |m\rangle &= |m+1\rangle \, , \\
J^- \, |m\rangle &= (j(j-1) - m(m-1)) \, |m-1\rangle \, .
\end{aligned}
\tag{43}
$$

Notice that for $j - \lambda \in \mathbb{Z}$, from Eq. (43) we have that $J^- \, |j\rangle = 0$ and hence, there is a subrepresentation

$$
\{|j+m\rangle : m \in \mathbb{N}\} \cong D_j^- \, . \tag{44}
$$

In this case the module C_j^λ is reducible but indecomposable, since the complement of Eq. (44) does not form a subrepresentation. However, the corresponding quotient does:

$$
C_j^\lambda \, / \, \{|j+m\rangle : m \in \mathbb{N}\} \cong D_{j-1}^+ \, .
$$

From now on, we will denote by $\mathbf{n}$ for $n \in \mathbb{Z}_{>0}$ the n-dimensional representation of $\mathfrak{su}(2)$, namely that of spin $j \in \mathbb{N}$ such that $2j+1 = n$. Later we will need the Clebsh-Goardan coefficients of the tensor product of $D_j^\pm$ and C_j^λ with $\mathbf{2}$. An explicit calculation shows that

$$
C_j^\lambda \otimes \mathbf{2} \cong C_{j+1/2}^{\lambda+1/2} \oplus C_{j-1/2}^{\lambda+1/2} \qquad \text{and} \qquad D_j^\pm \otimes \mathbf{2} \cong D_{j+1/2}^\pm \oplus D_{j-1/2}^\pm \, . \tag{45}
$$

3.3 Representations of $\mathfrak{su}(2|2)$

In this chapter we find all the irreducible highest weight representations of $\mathfrak{su}(2|2)$. In the following, we omit all the labels indicating zero modes of the generators defined above, since we will be concerned only with the finite Lie superalgebra. The representations of $\mathfrak{u}(2|2)$ are characterised by an additional parameter, which is the Y_0-eigenvalue; this value is an arbitrary real number, but after quantisation we require that $Y \in \frac{1}{2}\mathbb{Z}$, or when considering $\mathfrak{su}(2|2)$ and $\mathfrak{psu}(2|2)$ there will be the selection rule $Y - Z \in \mathbb{Z}$. Moreover, note that since Z is central, its eigenvalue is constant on any irreducible representation of $\mathfrak{su}(2|2)$ and the representations of $\mathfrak{psu}(2|2)$ correspond to those for which $Z = 0$. Then, given the form of the bosonic subalgebra, every representation of the superalgebra comes in the form of a multiplet of $\mathfrak{su}(2)$-representations. In fact, the eight supercharges $S^{\alpha\beta\gamma}$ of $\mathfrak{su}(2|2)$ generate a 16-dimensional Clifford module and we can find a highest-weight state which is annihilated by half of them. Let us assume that the highest weight state transforms in the representation $\big(j \, , \mathbf{n}\big)$ with respect to the bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, where we allow $j \in \frac{1}{2}\mathbb{Z}$ to denote a spin-j representation of $\mathfrak{su}(2)$ from the whole list of Section 3.2, that is, either finite-dimensional, discrete or continuous (where for continuous representation we should specify also the parameter $\lambda \in \mathbb{R}/\mathbb{Z}$). Since the supercharges transform in the bispinor representation $\big(\mathbf{2} \, , \overline{\mathbf{2}}\big) \oplus \big(\overline{\mathbf{2}} \, , \mathbf{2}\big)$ of the bosonic subalgebra, by Eq. (45) we conclude that a typical multiplet takes the form

$$\begin{array}{ccccc} & & (j\,,\mathbf{n}) & & \\ (j+\tfrac{1}{2}\,,\mathbf{n+1}) & (j+\tfrac{1}{2}\,,\mathbf{n-1}) & & (j-\tfrac{1}{2}\,,\mathbf{n+1}) & (j-\tfrac{1}{2}\,,\mathbf{n-1}) \\ (j+1\,,\mathbf{n}) \quad (j\,,\mathbf{n+2}) & & 2\,(j\,,\mathbf{n}) & & (j\,,\mathbf{n-2}) \quad (j-1\,,\mathbf{n}) \\ (j+\tfrac{1}{2}\,,\mathbf{n+1}) & (j+\tfrac{1}{2}\,,\mathbf{n-1}) & & (j-\tfrac{1}{2}\,,\mathbf{n+1}) & (j-\tfrac{1}{2}\,,\mathbf{n-1}) \\ & & (j\,,\mathbf{n})\,, & & \end{array} \tag{46}$$

where the multiplet is also characterised by the Z-eigenvalue $Z \in \mathbb{R}$, when viewed as a representation of $\mathfrak{su}(2|2)$. Here, the top state is the highest weight state of the Clifford module, and the action of the supercharges moves between the different bosonic representations. For the important cases of $\mathbf{n} = \mathbf{1}$ and $\mathbf{n} = \mathbf{2}$ some shortenings occur. For $\mathbf{n} = \mathbf{2}$, the representation involving $\mathbf{n} - \mathbf{2}$ is absent, that is,

$$\begin{array}{ccccc} & & (j\,,\mathbf{2}) & & \\ (j+\tfrac{1}{2}\,,\mathbf{3}) & (j+\tfrac{1}{2}\,,\mathbf{1}) & & (j-\tfrac{1}{2}\,,\mathbf{3}) & (j-\tfrac{1}{2}\,,\mathbf{1}) \\ (j+1\,,\mathbf{2}) \quad (j\,,\mathbf{4}) & & 2\,(j\,,\mathbf{2}) & & (j-1\,,\mathbf{2}) \\ (j+\tfrac{1}{2}\,,\mathbf{3}) & (j+\tfrac{1}{2}\,,\mathbf{1}) & & (j-\tfrac{1}{2}\,,\mathbf{3}) & (j-\tfrac{1}{2}\,,\mathbf{1}) \\ & & (j\,,\mathbf{2})\,, & & \end{array} \tag{47}$$

while for $\mathbf{n} = \mathbf{1}$ even more representations are missing,

$$\begin{array}{cccc} & (j\,,\mathbf{1}) & & \\ & (j+\tfrac{1}{2}\,,\mathbf{2}) \quad (j-\tfrac{1}{2}\,,\mathbf{2}) & & \\ (j+1\,,\mathbf{1}) & (j\,,\mathbf{3}) \quad (j\,,\mathbf{1}) & & (j-1\,,\mathbf{1}) \\ & (j+\tfrac{1}{2}\,,\mathbf{2}) \quad (j-\tfrac{1}{2}\,,\mathbf{2}) & & \\ & (j\,,\mathbf{1})\,. & & \end{array} \tag{48}$$

Below we will be interested in the affine algebra $\mathfrak{su}(2|2)_k$ at level $k = 1$. Then the second bosonic $\mathfrak{su}(2)_k$ factor also has level $k = 1$, and as a consequence, the affine highest weight states are only allowed to transform in the $\mathbf{1}$ and $\mathbf{2}$ representations of $\mathfrak{su}(2)$[3]. Hence, it is clear that all of the long representations we have presented above are not allowed at $k = 1$. Let us therefore look systematically for short multiplets. Specifically, we will consider shortening conditions for the multiplets Eq. (47) and Eq. (48) for different type of representations j.

Starting with Eq. (47), we assume that $j \in \frac{1}{2}\mathbb{N}$ labels the spin-j finite dimensional $\mathfrak{su}(2)$ representation and we require that the two representations with a $\mathbf{n} - \mathbf{1}$ in the

[3]In this section we are discussing the representations of the finite-dimensional Lie superalgebra $\mathfrak{su}(2|2)$. The affine highest weight states of the corresponding affine algebra will therefore transform in representations of this algebra.

second line are null. This will remove also all other representations that appear further below in the multiplet and fix the eigenvalue of Z to $j+\frac{1}{2}$ or $-j-\frac{1}{2}$. Then, for $j>0$ the multiplet reduces to

$$\begin{array}{ccc} & Z=\pm\left(j+\tfrac{1}{2}\right): & \\ & & \\ & \big(j\,,\mathbf{2}\big) & \\ \big(j+\tfrac{1}{2}\,,\mathbf{1}\big) & & \big(j-\tfrac{1}{2}\,,\mathbf{1}\big)\,, \end{array} \tag{49}$$

whilst for $j=0$ we have

$$\begin{array}{cc} & Z=\pm\tfrac{1}{2}: \\ & \\ & \big(0\,,\mathbf{2}\big) \\ \big(\tfrac{1}{2}\,,\mathbf{1}\big) & \end{array} \tag{50}$$

Similarly, for the multiplet Eq. (48), the only way to eliminate the representation involving the $\mathbf{3}$ is to require it to be null. This gives then the following two possibilities depending on the value of Z, on the left is the configuration obtained for $Z=j\geq 1$ while on the right the one for $Z=-j-1\leq -1$:

$$\begin{array}{ccc} Z=j\geq 1: & & \\ \big(j\,,\mathbf{1}\big) & & \\ & \big(j-\tfrac{1}{2}\,,\mathbf{2}\big) & \\ & & \big(j-1\,,\mathbf{1}\big)\,, \end{array} \qquad \begin{array}{ccc} & Z=-j-1\leq -1: & \\ & & \big(j\,,\mathbf{1}\big) \\ & \big(j+\tfrac{1}{2}\,,\mathbf{2}\big) & \\ \big(j+1\,,\mathbf{1}\big) & & \end{array} \tag{51}$$

where the multiplet on the left is even shorter for $j=0$ and $j=\frac{1}{2}$. However, by redefining $j\to j\pm\frac{1}{2}$ and rearranging the picture[4] the multiplets in Eq. (51) become equivalent to that of Eq. (49), except for the exceptional cases corresponding to $Z=0,\pm\frac{1}{2}$. Indeed, the multiplet on the left in Eq. (51) for $j=\frac{1}{2}$ reduces to

$$\begin{array}{cc} & Z=\tfrac{1}{2}: \\ & \\ \big(\tfrac{1}{2}\,,\mathbf{1}\big) & \\ & \big(0\,,\mathbf{2}\big)\,, \end{array} \tag{52}$$

which is Eq. (50) with $Z=\frac{1}{2}$, and for $j=0$ to

$$\begin{array}{c} Z=0: \\ \\ \big(0\,,\mathbf{1}\big)\,. \end{array} \tag{53}$$

We now consider analogous shortenings when $-j\in\frac{1}{2}\mathbb{Z}_{<0}$ labels the highest weight

[4] "Rearranging" means that we change which state we regard as the highest weight state of the Clifford module.

discrete representation D^+_{-j} of $\mathfrak{su}(2)$. In this case, from Eq. (48) we obtain for $j \neq 0$ the multiplet

$$Z = \pm j \neq 0 : \tag{54}$$

$$\begin{array}{ccc}
\big(D^+_{-j}\,,\mathbf{1}\big) & & \\
& \big(D^+_{-j-\frac{1}{2}}\,,\mathbf{2}\big) & \\
& & \big(D^+_{-j-1}\,,\mathbf{1}\big)\,,
\end{array}$$

while for $Z = 0$ the multiplet

$$Z = 0 : \tag{55}$$

$$\begin{array}{cc}
& \big(D^+_{-1}\,,\mathbf{1}\big) \\
\big(D^+_{-\frac{1}{2}}\,,\mathbf{2}\big) & \\
& \big(D^+_{-1}\,,\mathbf{1}\big)\,.
\end{array}$$

For what concerns the case of $j \in \frac{1}{2}\mathbb{Z}_{>0}$ denoting the lowest weight discrete representation D^-_j, we obtain the multiplets

$$Z = \pm j \neq 0 : \tag{56}$$

$$\begin{array}{ccc}
& & \big(D^-_{j}\,,\mathbf{1}\big) \\
& \big(D^-_{j+\frac{1}{2}}\,,\mathbf{2}\big) & \\
\big(D^-_{j+1}\,,\mathbf{1}\big) & &
\end{array}$$

and

$$Z = 0 : \tag{57}$$

$$\begin{array}{cc}
\big(D^-_{1}\,,\mathbf{1}\big) & \\
& \big(D^-_{\frac{1}{2}}\,,\mathbf{2}\big) \\
\big(D^-_{1}\,,\mathbf{1}\big)\,. &
\end{array}$$

We can then use the results for the multiplets containing discrete representations to deduce those containing continuous representations. We find

$$Z = \pm j \neq 0 : \tag{58}$$

$$\begin{array}{ccc}
\big(C^j_j\,,\mathbf{1}\big) & & \\
& \big(C^{j+\frac{1}{2}}_{j+\frac{1}{2}}\,,\mathbf{2}\big) & \\
& & \big(C^j_{j+1}\,,\mathbf{1}\big)\,,
\end{array}$$

together with

$$Z = 0 : \tag{59}$$

$$\begin{array}{ccc} \big(C_1^0\,,\mathbf{1}\big) & & \\ & \big(C_{\frac{1}{2}}^{\frac{1}{2}}\,,\mathbf{2}\big) & \\ & & \big(C_0^0\,,\mathbf{1}\big)\,. \end{array}$$

We now prove the shortening conditions for the case of j labelling the finite-dimensional representations; the other cases are obtained by an analogous analysis. We start by Eq. (49). We first remark that one has to fix a set of annihilators of the highest weight states of the Clifford module, namely those sitting at the top of Eq. (47). These operators will be four fermionic generators, and there is only one set of them[5] that yields the Clifford module structure of Eq. (51) or Eq. (49). Also. such set depends on the representation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ in which the highest weight states transform. An educated guess is made by looking at the representation appearing in the second row of Eq. (51) and Eq. (49). For Eq. (49), this suggests that we should declare

$$S^{\alpha\pm\mp}\,|j\,,\uparrow\rangle := 0 \qquad \text{for } \alpha = \pm\,, \tag{60}$$

where $|j\,,\uparrow\rangle$ is the highest weight state in $\big(j\,,\mathbf{2}\big)$. The generators in Eq. (47) are in fact all those that raise the compact $\mathfrak{su}(2)$, thus from Eq. (60) it follows that the representations in the second line of Eq. (47) containing $\mathbf{3}$ are not there. However, we have to explicitly impose that the states corresponding to the missing representation in the third line of Eq. (47), that is those corresponding to $\mathbf{n}-\mathbf{2}$, are null. This requirement will fix the value of Z as we see now. We claim that the state $\mathcal{N} := S^{-++}S^{+++}\,|j\,,\uparrow\rangle$ is the highest weight state in $\big(j\,,\mathbf{n}-\mathbf{2}\big)$. This follows since $\mathcal{N}$ has the right spins and it is annihilated by both J^+ and K^+, which is a short computation done using the commutation relations and Eq. (60). Knowing this, we can declare $\mathcal{N}$ to be null by requiring that the state obtained from it by applying any two raising fermionic operators is zero. The only non-vanishing combination is

$$S^{++-}S^{-+-}\,\mathcal{N} = \big(Z - j - \tfrac{1}{2}\big)\,\big(Z + j + \tfrac{1}{2}\big)\,\,|j\,,\uparrow\rangle\,,$$

where we identified Z with its eigenvalue. It then follows that

$$Z = \pm\big(j + \tfrac{1}{2}\big)\,. \tag{61}$$

In the following we discuss the two cases in Eq. (61) in parallel. We start by identifying the

[5] Actually there are two sets of annihilators that yields a solution, but one is the complement of the other. That means, that once found a set $\mathcal{S}$ of four fermionic zero modes that yields a solution, also its complement $\mathcal{S}^\dagger$ yields a solution, namely the one where the highest weight states of the Clifford module are the lowest weight states of the first solution. Since the module Eq. (46) is symmetric under exchanging Clifford-highest with -lowest weight states, we can consider only one solution.

highest weight state in $\big(j+\frac{1}{2}\,,\mathbf{1}\big)$, which we claim to be either $S^{+++}\,|j\,,\uparrow\rangle$ or $S^{---}\,|j\,,\uparrow\rangle$; indeed, one checks that for $Z=j+\frac{1}{2}$ the latter is null and the former is the desired highest weight state, whilst for $Z=-j-\frac{1}{2}$ the situation is reversed. Then, for $Z=j+\frac{1}{2}$ we have that

$$\big(j+\tfrac{1}{2}\,,\mathbf{1}\big)\cong\big\langle\,(J^-)^kS^{+++}\,|j\,,\uparrow\rangle:0\leq k\leq 2j+1\,\big\rangle\,,$$

and similarly for $Z=-j-\frac{1}{2}$ with S^{---} instead. This follows from

$$(J^-)^kS^{+++}\,|j\,,\uparrow\rangle=k\,S^{-++}(J^-)^{k-1}\,|j\,,\uparrow\rangle+S^{+++}(J^-)^k\,|j\,,\uparrow\rangle\,,$$

which vanishes for $k=2j+2$. We proceed by identifying the highest weight state in $\big(j-\frac{1}{2}\,,\mathbf{1}\big)$ which we claim to be

$$\mathcal{N}_-:=S^{+--}\,|j\,,\uparrow\rangle+S^{---}\,|j-1\,,\uparrow\rangle\quad\text{or}\quad\mathcal{N}_+:=S^{-++}\,|j\,,\uparrow\rangle-S^{+++}\,|j-1\,,\uparrow\rangle\,,$$

depending on the value of Z. Indeed, for $Z=j+\frac{1}{2}$, the state $\mathcal{N}_+$ is null and the desired highest weight state is $\mathcal{N}_-$ and for $Z=-j-\frac{1}{2}$ the situation is reversed. As above, if $j\geq\frac{1}{2}$ then we can identify

$$\big(j-\tfrac{1}{2}\,,\mathbf{1}\big)\cong\big\langle\,(J^-)^k\,\mathcal{N}_-:0\leq k\leq 2j-1\,\big\rangle\,,$$

and similarly for $Z=-j-\frac{1}{2}$. This follows from

$$2\,(J^-)^k\,\mathcal{N}_-=(2j-k)\,S^{+--}(J^-)^{k-1}\,|j-1\,,\uparrow\rangle+S^{---}(J^-)^k\,|j-1\,,\uparrow\rangle\,,$$

which vanishes for $k=2j$. This proves the shortening Eq. (49). Moreover, as we claimed above, a further shortening happens when $j=0$, that is when $Z=\pm\frac{1}{2}$. For $Z=\frac{1}{2}$ we have the following representations:

$$\big(j=0\,,\mathbf{2}\big)\cong\big\langle\,|0,\uparrow\rangle\,,|0,\downarrow\rangle\,\big\rangle\quad\text{and}\quad\big(j=\tfrac{1}{2}\,,\mathbf{1}\big)\cong\big\langle\,S^{+++}\,|0,\uparrow\rangle\,,S^{-++}\,|0,\uparrow\rangle\,\big\rangle\,,$$

and one computes that $S^{\mu-+}S^{---}\,|0,\uparrow\rangle=S_0^{\mu-+}S_0^{+--}\,|0,\uparrow\rangle=0$, which implies that representation $\big(j-\frac{1}{2}\,,\mathbf{1}\big)$ drops out. Analogously, for $Z=-\frac{1}{2}$ we have:

$$\big(j=0\,,\mathbf{2}\big)\cong\big\langle\,|0,\uparrow\rangle\,,|0,\downarrow\rangle\,\big\rangle\quad\text{and}\quad\big(j=\tfrac{1}{2}\,,\mathbf{1}\big)\cong\big\langle\,S^{---}\,|0,\uparrow\rangle\,,S^{+--}\,|0,\uparrow\rangle\,\big\rangle\,,$$

and $S^{+++}\,|0,\uparrow\rangle=S^{-++}\,|0,\uparrow\rangle=0$, which implies that again the representation $\big(j-\frac{1}{2}\,,\mathbf{1}\big)$ drops out also in this case. Hence, we also proved the shortening given by Eq. (50).

We now analyze the shortening of Eq. (48). For that, we need to declare a different set of annihilators of the highest weight state $|j\,,0\rangle\in\big(j\,,\mathbf{1}\big)$, namely we impose

$$S^{\alpha\beta+}\,|j\,,0\rangle:=0\qquad\text{for }\alpha,\beta=\pm\,.$$

In this case the condition that fixes Z arises slightly differently than above, namely by requiring that the representation $\big(j\,,\mathbf{3}\big)$ drops out of Eq. (48). We thus identify the highest weight state in $\big(j\,,\mathbf{3}\big)$, which is $\mathcal{N} := S^{++-}S^{-+-}\,|j\,,0\rangle$. As before, we require that the application of two raising fermionic generators annihilates $\mathcal{N}$. The only non-trivial combination is

$$S^{-++}S^{+++}\,\mathcal{N} = (Z-j)(Z+j+1)\,|j\,,0\rangle\,, \tag{62}$$

which fixes the value of Z to either j or $-j-1$. Then the vanishing of Eq. (62) implies that $\mathcal{N}$ is null and hence that all the states obtained from it by the application of one fermionic raising operator are also null. The only non-trivial such states are

$$\begin{aligned} S^{+++}\,\mathcal{N} &= (Z+j+1)\,S^{-+-}\,|j\,,0\rangle\,,\\ S^{-++}\,\mathcal{N} &= -(Z-j+1)\,S^{++-}\,|j\,,0\rangle + S^{-+-}J^-\,|j\,,0\rangle\,. \end{aligned} \tag{63}$$

We proceed now the analysis by distinguishing between the two possible values of Z.

- $Z=j$: the states in Eq. (63) being null translates into

  $$S^{-+-}\,|j\,,0\rangle = 0\,, \qquad S^{++-}\,|j\,,0\rangle = S^{-+-}J^-\,|j\,,0\rangle\,. \tag{64}$$

  As one can check, that the highest weight state in $\big(j+\frac{1}{2}\,,\mathbf{2}\big)$ is $S^{-+-}\,|j\,,0\rangle$, which is now null, hence this representation drops out from Eq. (48). We identify the highest weight state in $\big(j-\frac{1}{2}\,,\mathbf{2}\big)$ to be $\mathcal{M} := S^{++-}\,|j\,,0\rangle = S^{-+-}J^-\,|j\,,0\rangle$ and we compute $K^-\,\mathcal{M} = S^{+--}\,|j\,,0\rangle$. Hence, for $j \geq \frac{1}{2}$ we identify

  $$\big(j-\tfrac{1}{2}\,,\mathbf{2}\big) \cong \big\langle\,(J^-)^k\mathcal{M}\,,\ (J^-)^kS^{+--}\,|j\,,0\rangle : 0 \leq k \leq 2j-1\,\big\rangle\,,$$

  which follows from a computation similar to the ones showed above. Note that, for $j=0$ the state $\mathcal{M}$ is null, hence $\big(j-\frac{1}{2}\,,\mathbf{2}\big)$ drops out and so do all the other representations except $\big(0\,,\mathbf{1}\big)$: this proves Eq. (53). Now, one can check that all fermionic lowering operators except S^{+--} annihilate $\mathcal{M}$. For S^{---} this follows from

  $$S^{---}\,|j\,,0\rangle = [K^-, S^{-+-}]\,|j\,,0\rangle = K^-S^{-+-}\,|j\,,0\rangle = 0\,, \tag{65}$$

  by Eq. (64). It follows that for $j \geq 1$ there is only one additional representation in the Clifford module, which is $\big(j-1\,,\mathbf{1}\big)$ whose highest weight state $\mathcal{L} := S^{+--}\,\mathcal{M}$. Indeed, for $j \geq 1$ we identify

  $$\big(j-1\,,\mathbf{1}\big) \cong \big\langle\,(J^-)^k\mathcal{L} : 0 \leq k \leq 2j-2\,\big\rangle\,.$$

On the other hand, for $j = \frac{1}{2}$ we have that $\mathcal{M} \in \big(0\,,\mathbf{2}\big)$ and thus

$$\mathcal{L} = S^{+--}\,\mathcal{M} = -[J^-, S^{---}] = -J^- S^{---}\,\mathcal{M} = 0\,, \tag{66}$$

by Eq. (65), leaving us with the multiplet as in Eq. (52). We have thus proven the left part of Eq. (51), including the special shortening corresponding to $Z = 0, \frac{1}{2}$.

- $Z = -j - 1$: in this case, the state on the left of Eq. (63) is zero, hence the only non-trivial null vector is

$$2jS^{++-}\,|j\,,0\rangle - S^{-+-}J^-\,|j\,,0\rangle = 0\,, \tag{67}$$

for $j \neq 0$. The highest weight state in $\big(j - \frac{1}{2}\,,\mathbf{2}\big)$ is

$$\mathcal{N} := S^{-+-}\,|j-1\,,0\rangle + S^{++-}\,|j\,,0\rangle\,,$$

which one can check to be null by the choice of Z. Thus the representation $\big(j-\frac{1}{2}\,,\mathbf{2}\big)$ drops out from Eq. (48). We then look at $\big(j + \frac{1}{2}\,,\mathbf{2}\big)$ whose highest weight state is $\mathcal{M} := S^{-+-}\,|j\,,0\rangle$ and compute $K^-\mathcal{M} = S^{---}\,|j\,,0\rangle$. Hence,

$$\big(j + \tfrac{1}{2}\,,\mathbf{2}\big) \cong \big\langle\, (J^-)^k\mathcal{M}\,,\ (J^-)^k S^{---}\,|j\,,0\rangle : 0 \leq k \leq 2j+1 \,\big\rangle\,.$$

The highest weight state in $\big(j + 1\,,\mathbf{1}\big)$ is $\mathcal{L} := S^{---}S^{-+-}\,|j\,,0\rangle$ and thus

$$\big(j+1\,,\mathbf{1}\big) \cong \big\langle\, (J^-)^k\,\mathcal{L} : 0 \leq k \leq 2j+2 \,\big\rangle\,. \tag{68}$$

In order to prove that this is all there is in the Clifford module for $Z = -j-1$, we compute

$$S^{-+-}\mathcal{M} = 0\,, \qquad S^{++-}\mathcal{M} = -S^{-+-}S^{++-}\,|j\,,0\rangle = (S^{-+-})^2 J^-\,|j\,,0\rangle = 0\,,$$

and

$$2jS^{+--}\mathcal{M} = J^-\mathcal{L}\,,$$

where we used Eq. (67). With these equations, we see that the only representation that can be obtained from $\mathcal{M}$ by application of fermionic creation operators is the one generated by $\mathcal{L}$, which is $\big(j+1\,,\mathbf{1}\big)$. Moreover, since all fermionic creation operators annihilates $\mathcal{L}$, there are indeed no more representations in the Clifford module. This proves the right part of Eq. (51).

All together, this concludes the proof of the shortening Eq. (49) for the case of finite-dimensional representations of the non-compact $\mathfrak{su}(2)$. As we will see in the next chapter, these are exactly the multiplets obtained with the free field realisations, namely Eq. (116)

and Eq. (120), including the ultrashort multiplets corresponding to $Z = 0, \pm\frac{1}{2}$. Also the multiplets arising with j labelling discrete or continuous representations can be obtained with the free field realisation.

4 Conformal Embedding

In this chapter we introduce the notion of affine and conformal embedding, following the lines of [DFMS97]. In particular, we see that in order for an affine embedding to exist, the levels of the two algebras must be related by a positive integer factor called the embedding index. On the other hand, for a conformal embedding to happen we need the central charges to agree. Conformal embeddings are important because they allow to study the WZW model on an affine Lie algebra in terms of the WZW model on an affine subalgebra, or viceversa, in terms of a bigger affine algebra. This is relevant for the superalgebra setting, since one can ask when does the bosonic subalgebra conformally embed into a Lie superalgebra. In particular, we present a simple criterion for this, which allows us to prove that there is such bosonic conformal embedding for $\mathfrak{psu}(2|2)_1$ and for $\mathfrak{u}(2|2)_1$. In order to present this criterion, we need to make a short detour in the world of meromorphic conformal field theories and their vertex operators. For that we will follow [God89].

4.1 Affine and conformal embeddings

We start at the level of Lie algebras and present different characterisations of an embedding $\widetilde{\mathfrak{g}} \hookrightarrow \mathfrak{g}$ of a simple Lie algebra $\widetilde{\mathfrak{g}}$ into a semisimple Lie algebra $\mathfrak{g}$.

- **Branching rules**. An irreducible representation of $\mathfrak{g}$ viewed as a representation of $\widetilde{\mathfrak{g}}$ is usually reducible. The corresponding decomposition is called a branching rule and it is denoted by

$$\lambda \mapsto \bigoplus_{\mu \in \widetilde{P}_+} b_{\lambda,\mu}\, \mu$$

where $\widetilde{P}_+$ denotes the set of dominant weights of $\widetilde{\mathfrak{g}}$, that is, of all the weights whose Dynkin labels are all non-negative, and $b_{\lambda,\mu} \in \mathbb{N}$ gives the multiplicity of the irreducible representation μ in $\widetilde{\mathfrak{g}}$ in the decomposition of the irreducible representation λ of $\mathfrak{g}$. The decomposition of the lowest-dimensional non-trivial representation is sufficient to characterise an embedding and to each of its inequivalent branching rules corresponds a distinct embedding. A useful tool for the computation of branching rules uses tensor products. Namely, if

$$\lambda \mapsto \bigoplus_{\mu \in \widetilde{P}_+} b_{\lambda,\mu}\, \mu \quad \text{and} \quad \xi \mapsto \bigoplus_{\nu \in \widetilde{P}_+} b_{\lambda,\nu}\, \nu\,,$$

then

$$\lambda \otimes \xi \mapsto \bigoplus_{\mu,\nu} b_{\lambda,\mu} b_{\xi,\nu}\, \mu \otimes \nu\,.$$

- **Projection matrix**. An explicit projection of every weight of $\mathfrak{g}$ onto a weight of

$\widetilde{\mathfrak{g}}$ is given by a projection matrix $\mathcal{P}\colon \mathfrak{h}^* \to \widetilde{\mathfrak{h}}^*$, where $\mathfrak{h}$ and $\widetilde{\mathfrak{h}}$ denote the Cartan subalgebras of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ respectively. Hence, to compute the branching rules one first projects all the weights of a given irreducible representation of $\mathfrak{g}$ into $\widetilde{\mathfrak{g}}$-weights, and then reorganizes them into irreducible representations. Note that projection matrices are in general not unique.

- **Embedding index**. The embedding index x_e is defined as the ratio of the square length of the projection of the longest root θ of $\mathfrak{g}$, to the square length of the longest root $\tilde{\theta}$ of $\widetilde{\mathfrak{g}}$, that is,
$$x_e := \frac{|\mathcal{P}\theta|^2}{|\tilde{\theta}|^2} \in \mathbb{Z}_{>0}\,.$$

We know move to the affine setting. The embedding $\widetilde{\mathfrak{g}} \hookrightarrow \mathfrak{g}$ has a natural affine extention $\widetilde{\mathfrak{g}}_{\tilde{k}} \hookrightarrow \mathfrak{g}_k$ for some levels $\tilde{k}$ and k. In order for such embedding to exist, it turns out that the levels have to satisfy

$$\tilde{k} = k x_e \geq k\,, \tag{69}$$

We call an embedding $\widetilde{\mathfrak{g}}_{\tilde{k}} \hookrightarrow \mathfrak{g}_k$ such that Eq. (69) an **affine embedding**. In particular, form Eq. (69) it follows that if $k \in \mathbb{Z}_{>0}$ then also $\tilde{k} \in \mathbb{Z}_{>0}$ and both the $\mathfrak{g}_k$- and the $\widetilde{\mathfrak{g}}_{\tilde{k}}$-theory are integrable. The determination of the affine branching rules

$$\hat{\lambda} \mapsto \bigoplus_{\hat{\mu}\,\in \widetilde{P}^k_+} b_{\hat{\lambda},\hat{\mu}}\,\hat{\mu}\,, \tag{70}$$

namely of the branching coefficients $b_{\hat{\lambda},\hat{\mu}} \in \mathbb{N}$, is straightforward but tedious. One decomposes grade by grade the affine module of $\hat{\lambda}$ into irreducible representations of $\widetilde{\mathfrak{g}}$, and then reorganizes the result into a direct sum of affine $\widetilde{\mathfrak{g}}_{\tilde{k}}$ modules $\hat{\lambda}$. To proceed, it is convenient to express a module decomposition into irreducible representations of the corresponding finite Lie algebra:

$$\hat{\lambda} \mapsto \sum_n q^n \bigoplus_i \lambda^{(i,n)}\,,$$

where the powers of q keep track of the grade and $\lambda^{(i,n)}$'s at fixed n denotes the irreducible representations of $\mathfrak{g}$ at grade n.

In what follows we will be interested in a subclass of affine embeddings that preserves the conformal invariance; these are called **conformal embeddings**. Concretely, an affine embedding $\widetilde{\mathfrak{g}}_{\tilde{k}} \hookrightarrow \mathfrak{g}_k$ is called conformal if it satisfies

$$T^{\mathfrak{g}_k} = T^{\widetilde{\mathfrak{g}}_{\tilde{k}}}\,. \tag{71}$$

In particular, this requires

$$c(\mathfrak{g}_k) = c(\widetilde{\mathfrak{g}}_{\tilde{k}}) \quad \Longleftrightarrow \quad \frac{k \dim \mathfrak{g}}{k + h^\vee_{\mathfrak{g}}} = \frac{x_e k \dim \widetilde{\mathfrak{g}}}{x_e k + h^\vee_{\widetilde{\mathfrak{g}}}}\,, \tag{72}$$

where we used Eq. (17) and Eq. (69), and $h^\vee_{\mathfrak{g}}$, $h^\vee_{\widetilde{\mathfrak{g}}}$ denote the respective dual Coxeter numbers. Moreover, for the case of integrable WZW models, Eq. (72) is equivalent to Eq. (71). Indeed, if two theories have the same central charge, their difference (in the sense of the coset construction, see Section 4.2) has zero central charge. Since both theories under consideration are unitary by construction, their difference is also unitary and, having zero central charge, it is trivial. A remarkable fact about conformal embeddings is that they exist only when $k = 1$, which can be deduced by an elementary analysis of Eq. (72). Thus, there is a finite number of possible conformal embeddings, and they have been fully classified.

Above, we looked at affine branching rules. We now consider the branching of Eq. (70) associated to a conformal embedding. We observe that the non-vanishing of $b_{\hat{\lambda},\hat{\mu}}$ means that the finite weight μ can be found at some grade n in the infinite-dimensional highest weight representation to $\hat{\lambda}$ at level 1. By Eq. (71) we can compare the conformal dimensions of the corresponding fields, namely

$$h_\lambda + n = h_\mu\,,$$

or equivalently

$$\frac{C^{\mathfrak{g}}_\lambda}{2(1 + h^\vee_{\mathfrak{g}})} + n = \frac{C^{\widetilde{\mathfrak{g}}}_\mu}{2(x_e + h^\vee_{\widetilde{\mathfrak{g}}})}\,, \tag{73}$$

where $C^{\mathfrak{g}}_\lambda$ denotes the Casimir of λ as a $\mathfrak{g}$-representations and $C^{\widetilde{\mathfrak{g}}}_\mu$ the Casimir of μ as a $\widetilde{\mathfrak{g}}$-representation. A simple way of obtaining the branching rules is to compute the conformal dimension of every integrable representation of the two algebras under consideration and find the triplets (λ, μ, n) satisfying Eq. (73). Then, we look at the decomposition of $\hat{\lambda}$ at grade n in terms of irreducible representations of $\mathfrak{g}$ and write down all their finite branching rules into irreducible representations of $\widetilde{\mathfrak{g}}$. Note that this is a finite process since the difference in the conformal dimensions is always bounded. The number of times that μ appears in all these branching rules at grade n is precisely the coefficient $b_{\hat{\lambda},\hat{\mu}}$.

4.2 The coset construction

The conformal field theories based on affine Lie algebras contain currents, that is, fields of conformal dimension equal to one, which is in contrast to the minimal unitary models with central charge $0 < c < 1$ not containing any such fields. We will now present the so-called coset construction, also known as the *Goddard–Kent–Olive (GKO) construction*, that provides many minimal model CFTs from affine Kac-Moody algebras.

Let $\widetilde{\mathfrak{g}}_{\tilde{k}} \hookrightarrow \mathfrak{g}_k$ be an affine embedding and assume that the $\mathfrak{g}_k$-WZW model is integrable; in particular, the levels are related by Eq. (69) and hence also the $\widetilde{\mathfrak{g}}_{\tilde{k}}$-model is integrable. In the following we denote by $T^{\widetilde{\mathfrak{g}}}_{\tilde{k}}$ and $T^{\mathfrak{g}}_{k}$ the corresponding energy-momentum tensors and similarly their modes by $L_n^{\widetilde{\mathfrak{g}}_{\tilde{k}}}$ and $L_n^{\mathfrak{g}_k}$. Let $\widetilde{J}^a_n$ be generators of $\widetilde{\mathfrak{g}}_{\tilde{k}}$, which are linear combinations of generators J^a_n of $\mathfrak{g}_k$. Then, since

$$[L_n^{\widetilde{\mathfrak{g}}_{\tilde{k}}}, \widetilde{J}^a_n] = -n\widetilde{J}^a_n = [L_n^{\mathfrak{g}_k}, \widetilde{J}^a_n] \quad \forall\, n \in \mathbb{Z}\,,$$

it follows that

$$[L_m^{\mathfrak{g}_k} - L_m^{\widetilde{\mathfrak{g}}_{\tilde{k}}}, L_n^{\widetilde{\mathfrak{g}}_{\tilde{k}}}] = 0 \quad \forall\, m,n \in \mathbb{Z}\,.$$

Defining

$$T^{\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}} := T^{\mathfrak{g}_k} - T^{\widetilde{\mathfrak{g}}_{\tilde{k}}} \quad \textit{and} \quad L_m^{\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}} := L_m^{\mathfrak{g}_k} - L_m^{\widetilde{\mathfrak{g}}_{\tilde{k}}}$$

leads to the commutation relations

$$[L_m^{\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}}, L_n^{\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}}] = (m-n)L_{m+n}^{\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}} + \frac{c(\mathfrak{g}_k) - c(\widetilde{\mathfrak{g}}_{\tilde{k}})}{12} m(m^2-1)\delta_{m+n,0}\,.$$

Therefore, the modes $L_m^{\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}}$ satisfies the Virasoro algebra with central charge equal to the difference of the central charges of the constituent models:

$$c(\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}) := c(\mathfrak{g}_k) - c(\widetilde{\mathfrak{g}}_{\tilde{k}}) \tag{74}$$

From now on, the quotient $\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}$ characterised by the energy momentum tensor $T^{\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}}$ will be referred to as the **coset** or **quotient theory**. We also state that the coset theory contains all fields of $\mathfrak{g}_k$ which have a non-singular OPE with the fields of $\widetilde{\mathfrak{g}}_{\tilde{k}}$. In the present context, this property just means that the two algebras commute.

4.3 Uniqueness of vertex operators

Let $\mathcal{F}$ be a dense subspace of a Hilbert space $\mathcal{H}$, the space of states of a conformal field theory. Assume that there exists a preferred state $\Omega \in \mathcal{F}$ called the **vacuum** and a preferred operator $L\colon \mathcal{H} \to \mathcal{H}$ which annihilates the vacuum,

$$L\,\Omega = 0\,. \tag{75}$$

A **vertex operator** for a given state $\psi \in \mathcal{F}$ is an operator $V(\psi,z)\colon \mathcal{H} \to \mathcal{H}$ defined for every $z \in \mathbb{C}$, such that

$$V(\psi,z)\,\Omega = e^{zL}\psi\,, \tag{76}$$

and the matrix elements

$$z \mapsto \langle \phi_1, V(\psi, z)\phi_2 \rangle$$

are meromorphic functions for every $\phi_i \in \mathcal{F}$.

Two field operators $\xi(z), \eta(z) \colon \mathcal{H} \to \mathcal{H}$ are said to be **local** with respect to each other if for every $\phi_i \in \mathcal{F}$, the function

$$f_{\xi,\eta}(z, w) := \langle \phi_1, \xi(z)\eta(w)\phi_2 \rangle$$

is holomorphic for $|z| > |w|$, with a meromorphic continuation to $(z, w) \in \mathbb{C}^2$, which we still denote by $f_{\xi,\eta}$, satisfying

$$f_{\xi,\eta}(z, w) = \epsilon(\xi, \eta)\, f_{\eta,\xi}(w, z)$$

where $\epsilon(\xi, \eta) = 1$ if either ξ or η is a bosonic field, and $\epsilon(\xi, \eta) = -1$ if both are fermionic.

A **local system** of vertex operators is a family of vertex operators

$$\mathcal{V} = \{V(\psi, \cdot) : \psi \in \mathcal{F}\}\,,$$

which are local with respect to each other. From now on we assume that the theory admits a local system of vertex operators and we fix one denoted by $\mathcal{V}$.

The strength of the locality assumption is shown in the following uniqueness theorem.

Theorem 4.1. *Let $U_\phi(z) \colon \mathcal{H} \to \mathcal{H}$ be an operator defined for $z \in \mathbb{C}$ and one particular $\phi \in \mathcal{F}$. Assume that U_ϕ is local with respect to every vertex operator in $\mathcal{V}$, then*

$$U_\phi(z) = V(\phi, z) \qquad \forall\, z \in \mathbb{C}\,.$$

Proof. Let $\psi \in \mathcal{F}$ and $z \in \mathbb{C}$. Then, by Eq. (76) and locality we have that

$$\begin{aligned} U_\phi(z)e^{wL}\psi &= U_\phi(z)V(\psi, w)\,\Omega \\ &= \epsilon(\phi, \psi)\, V(\psi, w)U_\phi\,\Omega \\ &= \epsilon(\phi, \psi)\, V(\psi, w)e^{zL}\,\phi\,, \end{aligned}$$

hence

$$U_\phi(z)e^{wL}\,\psi = V(\phi, z)e^{wL}\psi\,.$$

Since $\psi \in \mathcal{F}$ was arbitrary and $\mathcal{F} \subset \mathcal{H}$ is dense, the claim follows. □

In particular, from this result it follows that the map $\psi \mapsto V(\psi, z)$ is linear and that $V(\Omega, z) = 1$, for every $z \in \mathbb{C}$.

Let $u \colon \mathcal{H} \to \mathcal{H}$ be an operator. We say that u **acts locally** with respect to $\mathcal{V}$, if $uV(\psi, z)u^{-1}$ is local for every $V \in \mathcal{V}$ and $\psi \in \mathcal{F}$. Then, the following result is a straightforward application of Theorem 4.1.

Proposition 4.2. *Let $\lambda \in \mathbb{C}$ and assume that $e^{\lambda L}$ acts locally with respect to $\mathcal{V}$. Then*

$$e^{\lambda L} V(\psi, z) e^{-\lambda L} = V(\psi, z + \lambda) \,. \tag{77}$$

By differentiating Eq. (77) we obtain

$$\frac{dV(\psi, z)}{dz} = [L, V(\psi, z)] \,, \tag{78}$$

and by applying this equation on the vacuum and using Theorem 4.1 we obtain

$$\frac{dV(\psi, z)}{dz} = V(L\psi, z) \,.$$

Combining Theorem 4.1 with Proposition 4.2 one can easily prove the following duality theorem.

Theorem 4.3. *Assume that e^{zL} acts locally with respect to $\mathcal{V}$ for every $z \in \mathbb{C}$. Then*

$$V(\psi, z) V(\phi, w) = V(V(\psi, z - w), w) \,.$$

Note that until now we did not assume that the theory is conformally invariant. The vacuum expectation value $\langle V(\psi, z)\rangle := \langle \Omega | V(\psi, z)\Omega\rangle$ is translation invariant if and only if $L^\dagger \, \Omega = 0$, in which case

$$\langle V(\psi, z)\rangle = \langle \Omega | \psi \rangle \,,$$

that is, it vanishes for vertex operators of states orthogonal to the vacuum. We extend the translation symmetry Eq. (75) by a global conformal symmetry of the Möbius group, generated by

$$L_{-1} := L \,, \qquad L_0 := \frac{1}{2}[L^\dagger, L] \;\text{ and }\; L_1 := L^\dagger \,, \tag{79}$$

satisfying the $\mathfrak{su}(1,1)$ commutator relations:

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1} \,, \qquad [L_1, L_{-1}] = 2L_0 \,. \tag{80}$$

From now on, we assume the vacuum to be invariant under the $\mathfrak{su}(1,1)$ algebra of Eq. (80), which means that

$$L_n \, \Omega = 0 \quad \text{ for } \; n = -1, 0, 1 \,.$$

Note that the operator L_0 satisfies

$$[L_0, L] \qquad L_0^\dagger = L_0 \;\text{ and }\; L_0 \, \Omega = 0 \,.$$

The following result is another consequence of Theorem 4.1.

Proposition 4.4. *Let $x \in \mathbb{C}$ and assume that x^{L_0} acts locally with respect to $\mathcal{V}$. Let $\psi \in \mathcal{F}$ be such that $L_0\psi = h\psi$ for some $h \in \mathbb{R}$, then*

$$x^{L_0} V(\psi, z) x^{-L_0} = x^h V(\psi, xz) \qquad \forall\, z \in \mathbb{C}\,. \tag{81}$$

Differentiating Eq. (81) we obtain

$$[L_0, V(\psi, z)] = \left(z\frac{d}{dz} + h \right) V(\psi, z)\,. \tag{82}$$

We call a state $\psi \in \mathcal{F}$ satisfying

$$L_0\psi = h\psi \qquad \text{and} \qquad L_1\psi = 0$$

an $\mathfrak{su}(1,1)$ **highest weight** state of conformal dimension $h \in \mathbb{R}$.

The following statement summarizes Eq. (78), Eq. (82) and the analogous result for L_1.

Proposition 4.5. *Assume that L_n for $n = -1, 0, 1$ act locally with respect to $\mathcal{V}$ and let ψ be an $\mathfrak{su}(1,1)$ highest weight state of conformal dimension h. Then*

$$[L_n, V(\psi, z)] = z^n \left(z\frac{d}{dz} + (n+1)h \right) V(\psi, z) \qquad \textit{for} \;\; n = -1, 0, 1\,. \tag{83}$$

For many purposes it is convenient to expand the fields $V(\psi, z) \in \mathcal{V}$ for $L_0\psi = h\psi$ in modes

$$V(\psi, z) = \sum_{r \in \mathbb{Z} - h} V_r(\psi) z^{-r-h}\,.$$

From Eq. (76) it follows that

$$\psi = V_{-h}(\psi)\,\Omega \qquad \text{and} \;\; V_r(\psi)\,\Omega = 0 \;\; \forall\, r > -h\,. \tag{84}$$

Moreover, Eq. (83) is equivalent to the commutation relations

$$[L_n, V_r(\psi)] = (n(h-1) - r)\, V_{r+n}(\psi) \tag{85}$$

for the modes of $V(\psi, z)$. In particular, $[L_0, V_r(\psi)] = -rV_r(\psi)$.

We can also rewrite Theorem 4.3 in the so called operator product expansion form.

Proposition 4.6. *Let $\psi, \phi \in \mathcal{F}$ be such that $L_0\psi = h_\psi\psi$ and $L_0\phi = h_\phi\phi$. Then,*

$$V(\psi, z)V(\phi, w) = \sum_{r \geq 0} (z - w)^{r - h_\psi - h_\phi} V(\chi_r, w)\,,$$

where $|z| > |w|$, $r \in \mathbb{Z} + h_\psi + h_\phi$ and $\chi_r = V_{h_\phi - r}(\psi)\phi$.

We can rewrite Eq. (80) as

$$[L_m, L_n] = (m-n)L_{m+n}\,, \tag{86}$$

for $m, n = -1, 0, 1$. Comparing this with Eq. (85) we see that Eq. (86) is compatible with L_n for $n = -1, 0, 1$ being three of the components of a vertex operator for an $\mathfrak{su}(1,1)$ highest weight state ψ_C with conformal dimension $h = 2$. Let us assume this and write

$$T(z) := V(\psi_C, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,. \tag{87}$$

Then Eq. (86) holds for every $m = -1, 0, 1$ and $n \in \mathbb{Z}$. Moreover, by Eq. (84) we have that

$$\psi_C = L_{-2}\,\Omega\,, \tag{88}$$

and it follows from Eq. (86) that indeed $L_1\psi_C = 0$. We can use Proposition 4.6 to compute

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)} \quad \text{for } \; |z| > |w|\,,$$

where $c = 2\|\psi_C\|^2 \in \mathbb{R}_{>0}$. From this, it follows that

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,.$$

Thus, we conclude that $T(z)$ is the Virasoro energy-momentum tensor of the theory.

4.4 A criterion for bosonic embeddings in superalgebras

In this section we will give a simple condition under which the inclusion of the bosonic affine subalgebra of an affine Lie superalgebra is a conformal embedding. We start with an affine superalgebra $\mathfrak{g}_k$ generated by bosonic modes J^a_n and fermionic modes S^β_n, for some finite set of indices a and β, and $n \in \mathbb{Z}$. We assume that the finite bosonic subalgebra $\tilde{\mathfrak{g}}$ is semisimple. Then, the fermionic zero modes form a completely reducible $\tilde{\mathfrak{g}}$-module under the adjoint action of the bosonic zero modes. In particular, we can write

$$\left[J^a_m, S^\beta_n\right] = (\mathcal{J}^a)^\beta{}_\mu\, S^\mu_{m+n}\,, \tag{89}$$

where $\mathcal{J}^a$ are semisimple complex matrices. We then consider the Sugawara stress-energy tensor of $\tilde{\mathfrak{g}}_{\tilde{k}}$, which has the form

$$T^{\tilde{\mathfrak{g}}_{\tilde{k}}} = N g_{ab} :J^a J^b:\,, \tag{90}$$

where $N = (\tilde{k} + \tilde{h}^\vee)^{-1}$ and $g_{ab} = \langle J_0^a, J_0^b \rangle_{\tilde{\mathfrak{g}}}^{-1}$ is the inverse of the Killing form[6] on $\tilde{\mathfrak{g}}$, see Eq. (19), so in particular $g_{ab} = g_{ba}$. In Eq. (90) and in the following we use the Einstein summation convention. Let $\psi := |\hat{\lambda}\rangle$ be any highest weight state of an affine $\mathfrak{g}_k$-module, then we compute

$$\begin{aligned}
[L_{-2}^{\tilde{\mathfrak{g}}_{\tilde{k}}}, S_n^\gamma]\,\psi &= N g_{ab} \sum_{m\in\mathbb{Z}} [J^a_{-2-m} J^b_m, S_n^\gamma]\,\psi \\
&= N g_{ab} \sum_{m\in\mathbb{Z}} \big([J^a_{-2-m}, S_n^\gamma] J^b_m + J^a_{-2-m}[J^b_m, S_n^\gamma] \big)\,\psi \\
&= N g_{ab} \sum_{m\in\mathbb{Z}} \big((\mathcal{J}^a)^\gamma{}_\mu S^\mu_{n-2-m} J^b_m + J^a_{-2-m} (\mathcal{J}^b)^\gamma{}_\mu S^\mu_{m+n} \big)\,\psi \\
&= N g_{ab} \left(\sum_{m<0} (\mathcal{J}^a)^\gamma{}_\mu [S^\mu_{n-2-m}, J^b_m] + \sum_{m<-n} (\mathcal{J}^b)^\gamma{}_\mu [J^a_{-2-m}, S^\mu_{n+m}] \right) \psi \\
&= N g_{ab} \left(\sum_{m<0} -(\mathcal{J}^a)^\gamma{}_\mu (\mathcal{J}^b)^\mu{}_\nu + \sum_{m<-n} (\mathcal{J}^b)^\gamma{}_\mu \, (\mathcal{J}^a)^\mu{}_\nu \right) S^\nu_{n-2}\,\psi \\
&= N \left(\sum_{m<-n} g_{ab}\,[\mathcal{J}^b, \mathcal{J}^a]^\gamma{}_\nu - \sum_{m=-n}^{-1} g_{ab} (\mathcal{J}^a \mathcal{J}^b)^\gamma{}_\nu \right) S^\nu_{n-2}\,\psi \\
&= -NCnS^\gamma_{n-2}\,\psi = -\frac{C}{\tilde{k} + \tilde{h}^\vee}\,[S_n^\gamma, L_{-2}^{\mathfrak{g}_k}]\,\psi\,,
\end{aligned}$$

where we used that $g_{ab}(\mathcal{J}^a \mathcal{J}^b)^\gamma{}_\nu = C\delta^\gamma_\nu$ is the Casimir of the bosonic representation $J^a \mapsto \mathcal{J}^a$, and $L_n^{\mathfrak{g}_k}$ are the modes of the stress-energy tensor of $\mathfrak{g}_k$. Since $\tilde{\mathfrak{g}}$ is semisimple, we can write

$$\tilde{\mathfrak{g}}_{\tilde{k}} = \bigoplus_i \mathfrak{g}^i_{k_i}\,, \tag{91}$$

where $\mathfrak{g}^i$ are finitely many simple affine algebras. Then, the representation in Eq. (89) can be block-diagonalised with respect to Eq. (91) and so does the Casimir, whose corresponding block components we denote by C_i. We consider the state

$$\mathcal{N} = \left(L_{-2}^{\mathfrak{g}} - \sum_i L_{-2}^{\mathfrak{g}^i} \right) |0\rangle\,,$$

where $|0\rangle$ denotes the vacuum of $\mathfrak{g}_k$. Then $\tilde{\mathfrak{g}}_{\tilde{k}}$ conformally embeds into $\mathfrak{g}_k$, that is,

$$T^{\mathfrak{g}_k} = T^{\tilde{\mathfrak{g}}_{\tilde{k}}} = \sum_i T^{\mathfrak{g}^i_{k_i}}\,, \tag{92}$$

[6]Note that g_{ab} is well-defined since we assumed that $\tilde{\mathfrak{g}}$ is semisimple, which is equivalent to its Killing form being non-degenerate. This is true, for example, for the bosonic subalgebras of $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(2|2)$ but not for those of $\mathfrak{su}(1,1|2)$ and $\mathfrak{su}(2|2)$, since they contain a central $\mathfrak{u}(1)$ factor which renders the Killing form degenerate. However, one can still construct a stress-energy tensor using the Halpern-Kiritsis construction [HK89] as it was done in [GNS22] for $\mathfrak{u}(1,1|2)_1$; then, the same criterion we are presenting also applies.

if and only if

$$\sum_i \frac{C_i}{k_i + h_i^\vee} = 1\,, \tag{93}$$

where k_i and $h_i^\vee$ denote the level and the dual Coxeter number of $\mathfrak{g}^i_{k_i}$ respectively. This assertion directly follows from the computation above; indeed Eq. (93) is equivalent to $S_n^\beta \mathcal{N} = 0$ for every $n > 0$ and fermionic generator S^β. Then, that $J_n^a \mathcal{N} = 0$ for every $n > 0$ and bosonic generator J^a, follows directly from the action of the Virasoro modes[7] on the J_n^a and the fact that the generators of $\mathfrak{g}^i_{k_i}$ commute with those of $\mathfrak{g}^j_{k_j}$ for $i \neq j$. Thus Eq. (93) is equivalent to $\mathcal{N}$ being a null-vector. We now argue that $\mathcal{N} = 0$ is equivalent to Eq. (92). Indeed, by uniqueness of vertex operators, see Theorem 4.1, it follows that $\mathcal{N} = 0$ if and only if

$$0 = V(\mathcal{N}, z) = V(L^{\mathfrak{g}}_{-2}\,|0\rangle, z) - \sum_i V(L^{\mathfrak{g}^i}{}_{-2}\,|0\rangle, z) \qquad \forall\, z \in \mathbb{C}\,.$$

By Eq. (87) and Eq. (88), this is equivalent to Eq. (92).

We mention that Eq. (93) can be adapted to take into account additional fermionic indices γ of $S_n^{\beta\gamma}$, which may label the action of non-trivial outer automorphisms of $\mathfrak{g}_k$, as we will see in the following examples. In particular, instead of Eq. (89) we may write for every γ:

$$[J^a_m, S_n^{\beta\gamma}] = (\mathcal{J}^{(\gamma)a})^\beta{}_\mu\, S^{\mu\gamma}_{m+n}\,,$$

where the representation $J^a \mapsto \mathcal{J}^{(\gamma)a}$ of $\{S^{\beta\gamma}\}_\beta$ depends on γ, and we denote the corresponding Casimir by $C^{(\gamma)}$ and by $C_i^{(\gamma)}$ that associated to the decomposition in Eq. (91). Then, the conformal embedding condition becomes

$$\sum_i \frac{C_i^{(\gamma)}}{k_i + h_i^\vee} = 1 \qquad \forall\, \gamma\,.$$

We now look at some examples for which this criterion can be applied.

Example 4.7. Using the same conventions as in [EGG19], we consider $\mathfrak{psu}(1,1|2)_1$ with its bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{su}(2)_1$. The eight fermionic generators $S_0^{\alpha\beta\gamma}$ transform in the $2\,(\mathbf{2},\mathbf{2})$ representation of the bosonic subalgebra[8]. The corresponding Casimirs are $C_1 = -j_1(j_1-1) = -\frac{3}{4}$ (since $j_1 = -\frac{1}{2}$ in this convention is the spin characterising the lowest weight representation of $\mathfrak{sl}(2,\mathbb{R})$) and $C_2 = j_2(j_2+1) = \frac{3}{4}$ (since $j_2 = \frac{1}{2}$ denotes the spin characterising the highest weight representation of $\mathfrak{su}(2)$). The levels are $k_1 = k_2 = 1$

[7] Recall that for for every $J^a \in \mathfrak{g}^i$ we have that $[L_n, J^a_m] = -mJ^a_{m+n}$, which is true both for $L^{\mathfrak{g}_k}$ and $L^{\mathfrak{g}^i_{k_i}}$, whilst $[L_n^{\mathfrak{g}^j_{k_j}}, J^a_m] = 0$ for $i \neq j$.

[8] This is an example where the index $\gamma = \pm$ encodes the transformation behaviour under the outer automorphisms $\mathfrak{su}(2)$. Nevertheless, the bosonic representations are independent of γ, since both copies transform under the same representation $(\mathbf{2},\mathbf{2})$ of the bosonic subalgebra. Hence, in this case it is enough to check that Eq. (93) holds for one γ.

and the dual Coxeter numbers $h_1^\vee = -2$ and $h_2^\vee = 2$. With this information, one can easily verify that (93) holds true and hence that there is a conformal embedding

$$\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{su}(2)_1 \hookrightarrow \mathfrak{psu}(1,1|2)_1 \,. \tag{94}$$

Knowing Eq. (94) is useful because it can be used to constraint the Casimir of $\mathfrak{psu}(1,1|2)$, and in turn the allowed representations. In particular, in [GNS22] it is argued, in a way analogous to as we did for the shortening of $\mathfrak{su}(2|2)$ multiplets, that the only highest weight states allowed in the $\mathfrak{psu}(1,1|2)_1$-spectrum are the vacuum

$$\big(j = 0\,, \mathbf{1}\big)\,, \tag{95}$$

where $j = 0$ denotes the one-dimensional trivial representation of $\mathfrak{sl}(2,\mathbb{R})$, and those that form the multiplet

$$\begin{array}{ccc} & \big(C^j_\lambda\,, \mathbf{2}\big) & \\ \big(C^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}\,, \mathbf{1}\big) & & \big(C^{j-\frac{1}{2}}_{\lambda-\frac{1}{2}}\,, \mathbf{1}\big)\,, \end{array} \tag{96}$$

and the same multiplet with the replacement $C^j_\lambda \to D^\pm_j$. Here, C^j_λ denotes the continuous $\mathfrak{sl}(2,\mathbb{R})$ representation with Casimir $C^{\mathfrak{sl}(2,\mathbb{R})} = -j(j-1)$ and λ is the fractional part of the J^3_0-eigenvalue, while $D^\pm_j$ denotes the spin j highest/lowest weight discrete representation, see [GNS22]. Now, since we proved that the bosonic subalgebra conformally embeds in the whole superalgebra, we have the following relation between the Casimirs:

$$C^{\mathfrak{psu}(1,1|2)} = -C^{\mathfrak{sl}(2,\mathbb{R})} + \tfrac{1}{3}C^{\mathfrak{su}(2)}\,,$$

and since $C^{\mathfrak{psu}(1,1|2)}$ has the same value on the whole superalgebra representation generated by the multiplet Eq. (96), we can relate for instance

$$j(j-1) + \tfrac{1}{4} = C^{\mathfrak{psu}(1,1|2)}\big(C^j_\lambda\,, \mathbf{2}\big) = C^{\mathfrak{psu}(1,1|2)}\big(C^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}\,, \mathbf{1}\big) = j^2 - \tfrac{1}{4}\,,$$

which directly fixes $j = \frac{1}{2}$ and $C^{\mathfrak{psu}(1,1|2)} = 0$ on every representation Eq. (96). Also the vacuum Eq. (95) is an allowed representation and it also has $C^{\mathfrak{psu}(1,1|2)} = 0$. Notice that in order to arrive at this conclusion, we had to know already the structure of the multiplet in which the highest weight states transform. This can be achieved by purely algebraic arguments or deduced by the free field realisation, as we have seen for $\mathfrak{psu}(2|2)_1$ and $\mathfrak{su}(2|2)_1$.

Example 4.8. Similarly to the example above, we consider the superalgebra $\mathfrak{psu}(2|2)_1$ with bosonic subalgebra $\mathfrak{su}(2)_{-1} \oplus \mathfrak{su}(2)_1$. The eight fermionic generators $S_0^{\alpha\beta\gamma}$ transform in the representation $(\mathbf{2}, \overline{\mathbf{2}}) \oplus (\overline{\mathbf{2}}, \mathbf{2}) \cong 2(\mathbf{2}, \mathbf{2})$ of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$[9], therefore the Casimirs are

[9]In this case the index γ encodes the outer automorphisms $\mathfrak{sl}(2,\mathbb{R})$, and even though the bosonic

$C_1 = C_2 = \frac{3}{4}$. The levels are $k_1 = -1$, $k_2 = 1$ and the dual Coxeter numbers $h_1^\vee = h_2^\vee = 2$, thus one can verify that also in this case (93) holds true and hence there is a conformal embedding

$$\mathfrak{su}(2)_{-1} \oplus \mathfrak{su}(2)_1 \hookrightarrow \mathfrak{psu}(2|2)_1 \,. \tag{97}$$

Example 4.9. We generalise Example 4.8, and investigate the embedding

$$\mathfrak{su}(N)_{-k} \oplus \mathfrak{su}(N)_k \hookrightarrow \mathfrak{psu}(N|N)_k \tag{98}$$

for $N > 1$ and $k \in \mathbb{R}$. The superalgebra $\mathfrak{psu}(N|N)$ has $2N^2$ fermionic generators that transform in the $(\mathbf{N}, \overline{\mathbf{N}}) \oplus (\overline{\mathbf{N}}, \mathbf{N})$ representations of $\mathfrak{su}(N) \oplus \mathfrak{su}(N)$ and the Casimirs $C_1 = C_2 = \frac{N^2-1}{2N}$ has the same value on the fundamental $\mathbf{N}$ and the anti-fundamental $\overline{\mathbf{N}}$ representation of $\mathfrak{su}(N)$. Also, the dual Coxeter number of $\mathfrak{su}(N)$ is equal to N. Knowing this we compute

$$\frac{C_1}{-k + h_1^\vee} + \frac{C_2}{k + h_2^\vee} = \frac{N^2 - 1}{N^2 - k^2} \,. \tag{99}$$

This proves that for every $N > 1$ the embedding in Eq. (98) is conformal if and only if $k = \pm 1$. Then, for $k = 1$, this gives the following relation between the Casimirs of highest weight representations:

$$C^{\mathfrak{psu}(N|N)_1} = \frac{C_{-1}^{\mathfrak{su}(N)}}{N-1} + \frac{C_1^{\mathfrak{su}(N)}}{N+1} \,,$$

where $C_{\pm 1}^{\mathfrak{su}(N)}$ denotes the Casimir of the finite $\mathfrak{su}(N)$ in $\mathfrak{su}(N)_{\pm 1}$ respectively. Note that there are exactly N integrable highest weight representations of $\mathfrak{su}(N)$ at $k = 1$ but none for $k = -1$.

As we did above for $\mathfrak{psu}(1,1|2)_1$, we want to investigate the implications of the conformal embedding in Eq. (97) on the allowed representations of $\mathfrak{psu}(2|2)_1$. By Section 3.3, we know the structure of the $\mathfrak{psu}(2|2)_1$ highest weight representations. The allowed multiplets are the vacuum

$$\big(j = 0, \mathbf{1}\big) \,, \tag{100}$$

the triplet containing highest weight discrete representations

$$\begin{array}{ccc} & \big(D^+_{-1/2}\,, \mathbf{2}\big) & \\ \big(D^+_{-1}\,, \mathbf{1}\big) & & \big(D^+_{-1}\,, \mathbf{1}\big)\,, \end{array} \tag{101}$$

representations depend on $\gamma = \pm$, they are isomorphic, and hence the Casimirs are still independent of γ and its enough to check Eq. (93) for one γ.

the triplet containing lowest weight representations

$$\begin{array}{ccc} & \left(D^{-}_{1/2}\,,\mathbf{2}\right) & \\ \left(D^{-}_{1}\,,\mathbf{1}\right) & & \left(D^{-}_{1}\,,\mathbf{1}\right)\,, \end{array} \tag{102}$$

and the triplet containing continuous representations,

$$\begin{array}{ccc} & \left(C^{1/2}_{1/2}\,,\mathbf{2}\right) & \\ \left(C^{0}_{0}\,,\mathbf{1}\right) & & \left(C^{1}_{0}\,,\mathbf{1}\right)\,, \end{array} \tag{103}$$

It thus turns out, that the value of Z_0 is directly related to the spin j of the $\mathfrak{su}_{-1}(2)$ representations, hence the condition $Z_0 = 0$ already constrains the allowed spins. However, we can use the conformal embedding of Eq. (97) to compute the Casimir of these representations in a very simple way. Indeed, Eq. (99) for $N = 2$ becomes

$$C^{\mathfrak{psu}(2|2)_1} = j(j+1) + \frac{\ell(\ell+1)}{3}\,,$$

where j and ℓ are the spin of the $\mathfrak{su}(2)_{-1}$ and $\mathfrak{su}(2)_1$ factor respectively. Thus, we find that $C^{\mathfrak{psu}(2|2)_1} = 0$ for both Eq. (100) and Eq. (101). The other multiplets, Eq. (102) and Eq. (103), the Casimir of the non-compact $\mathfrak{su}(2)$ is different, and Eq. (99) takes instead the form

$$C^{\mathfrak{psu}(2|2)_1} = j(j-1) + \frac{\ell(\ell+1)}{3}\,,$$

which gives again $C^{\mathfrak{psu}(2|2)_1} = 0$ on both multiplets.

Example 4.10. We now turn to the superalgebra $\mathfrak{u}(2|2)_1$. we claim that

$$T^{\mathfrak{u}(2|2)_1} = T^{\mathfrak{su}(2)_{-1}} + T^{\mathfrak{su}(2)_1} - {:}ZY{:}\,, \tag{104}$$

where $-\,{:}ZY{:} = -\,{:}U^2{:} + {:}V^2{:}$ is the stress energy tensor of the system $\widehat{\mathfrak{u}}(1)_U \oplus \widehat{\mathfrak{u}}(1)_V$, see Eq. (39); hence, Eq. (104) is equivalent to the existence of the conformal embedding

$$\mathfrak{su}(2)_{-1} \oplus \mathfrak{su}(2)_1 \oplus \widehat{\mathfrak{u}}(1) \oplus \widehat{\mathfrak{u}}(1) \hookrightarrow \mathfrak{u}(2|2)_1\,. \tag{105}$$

We prove Eq. (104) by showing that state

$$\mathcal{N} := \left(L^{\mathfrak{u}(2|2)_1}_{-2} - L^{\mathfrak{su}(2)_{-1}}_{-2} - L^{\mathfrak{su}(2)_1}_{-2} + {:}ZY{:}_{-2}\right)|0\rangle$$

is null, namely that $X_n\mathcal{N} = 0$ for every $n > 0$ and generator X of $\mathfrak{u}(2|2)$. For the bosonic generators J^a and K^a the assertion is clear, since their modes commute with those of Z and Y, see (40). For the fermionic generators $S^{a\beta\gamma}$ this follows from Example 4.8 together with the following computation,

$$[S_n^{\alpha\beta\gamma}, {:}ZY{:}_{-2}]\,|0\rangle = \gamma\left(Z_{-1}S_{n-1}^{\alpha\beta\gamma} + Z_0 S_{n-2}^{\alpha\beta\gamma}\right)|0\rangle = \gamma Z_{-1}S_{n-1}^{\alpha\beta\gamma}\,|0\rangle\,, \tag{106}$$

where we used that $Z_0\,|0\rangle = 0$. Note that Eq. (106) vanishes for every $n > 0$ since $S_m^{\alpha\beta\gamma}\,|0\rangle = 0$ for every $m \geq 0$. Finally, that the positive modes of Z and Y annihilate $\mathcal{N}$ follows from

$$\begin{aligned}[Z_n, {:}ZY{:}_{-2}] &= -nZ_{n-2} = -[Z_n, L_{-2}^{\mathfrak{u}(2|2)_1}]\,,\\ [Y_n, {:}ZY{:}_{-2}] &= -nY_{n-2} = -[Y_n, L_{-2}^{\mathfrak{u}(2|2)_1}]\,.\end{aligned}$$

From Eq. (104) we deduce that

$$C^{\mathfrak{u}(2|2)_1} = j(j+1) + \frac{\ell(\ell+1)}{3} - ZY\,, \tag{107}$$

holds for every representation of $\mathfrak{u}(2|2)_1$, where Z and Y are the eigenvalues of Z_0 and Y_0 respectively. As a consistency check, one can compute the value of the Casimir $C^{\mathfrak{u}(2|2)_1}$ on each component of the allowed multiplets using Eq. (107) and show that it is constant along each multiplet for every value of Y. Moreover, for each allowed multiplet we may find the values of Y such that $C^{\mathfrak{u}(2|2)_1} = 0$; these are exactly the multiplets obtained in the free field realisation.

We also mention that one can prove Eq. (104) using Eq. (93), knowing that $h^\vee = 0$ for $\mathfrak{u}(1)$ and that the Casimirs of both $\widehat{\mathfrak{u}}(1)$ factors are equal. Then, the fact that the two algebras possess levels of opposite sign, see Eq. (38), shows that Eq. (104) holds true.

For what concerns $\mathfrak{su}(2|2)_1$, the embedding

$$\mathfrak{su}(2)_{-1} \oplus \mathfrak{su}(2)_1 \oplus \widehat{\mathfrak{u}}(1)_Z \hookrightarrow \mathfrak{su}(2|2)_1 \tag{108}$$

is **not** conformal. Indeed, since the modes of Z commute among themselves, a factor proportional to ${:}Z{:}^2$ in the null vector $\mathcal{N}$ as above cannot compensate $[Z_n, L_{-2}^{\mathfrak{u}(2|2)_1}] = -nZ_{n-2}$. Also, a relation between the $\mathfrak{su}(2)$-Casimirs and the Z_0 does not hold between different components of the allowed multiplets found in Section 3.3. Lastly, Eq. (93) is not applicable since the affine algebra $\widehat{\mathfrak{u}}(1)_Z$ has level $k = 0$, so Eq. (93) is undefined.

5 The free field realisation at $k = 1$

Luckily, there is a free field realisation of the Lie superalgebra $\mathfrak{u}(2|2)_1$, which allows to study the highest weight representations and their characters. For this construction, we followed [GG21b] and used the fact that there is an embedding of affine superalgebras $\mathfrak{u}(2|2)_1 \oplus \mathfrak{u}(2|2)_1 \subset \mathfrak{u}(2,2|4)_1$.

We first recall that the superalgebra $\mathfrak{psu}(1,1|2)_1$ has a free field realisation in terms of two pairs of symplectic bosons, which is a first order system of bosons of spin half, see [EGG19], and two pairs of complex fermions, modulo two $\mathfrak{u}(1)$ fields. Doubling the degrees of freedom leads to a free field realisation of $\mathfrak{psu}(2,2|4)_1$, see [GG21b]. More specifically, we consider two pairs of symplectic boson fields $(\lambda^\alpha, \mu^\dagger_\alpha)$ and $(\mu^{\dot\alpha}, \lambda^\dagger_{\dot\alpha})$ with $\alpha, \dot\alpha = 1, 2$, as well as four complex fermions $(\psi^a, \psi^\dagger_a)$ with $a = 1, 2, 3, 4$, satisfying commutation relations

$$[\lambda^\alpha_r, (\mu^\dagger_\beta)_s] = \delta^\alpha_\beta \delta_{r,-s} \,, \qquad [\mu^{\dot\alpha}_r, (\lambda^\dagger_{\dot\beta})_s] = \delta^{\dot\alpha}_{\dot\beta} \delta_{r,-s} \,, \qquad \{\psi^a_r, (\psi^\dagger_\beta)_s\} = \delta^a_b \delta_{r,-s} \,.$$

We combine these fields as $Y_J = (\mu^\dagger_\alpha, \lambda^\dagger_{\dot\alpha}, \psi^\dagger_a)$ and $X^I = (\lambda^\alpha, \mu^{\dot\alpha}, \psi^a)$, and then consider the normal ordered bilinears

$$J^I{}_J = Y_J X^I \,.$$

These fields generate the superalgebra $\mathfrak{u}(2,2|4)_1$, see [GG21b]. The generator $\mathcal{C} = Y_I\, Z^I$ of $\mathfrak{u}(2,2|4)_1$ plays an important role since its modes $\mathcal{C}_n$ are central and in order to obtain $\mathfrak{psu}(2,2|4)_1$ one needs to quotient them out. In the following we will use the Einstein summation convention.

We begin by identifying the subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4)$, that is generated by

$$\begin{aligned}
\mathcal{L}^\alpha{}_\beta &= \mu^\dagger_\beta \lambda^\alpha - \delta^\alpha_\beta U \,, \\
\dot{\mathcal{L}}^{\dot\alpha}{}_{\dot\beta} &= \mu^\dagger_{\dot\beta} \lambda^{\dot\alpha} - \delta^{\dot\alpha}_{\dot\beta} \dot U \,, \\
\mathcal{R}^a{}_b &= \psi^\dagger_b \psi^a - \tfrac{1}{2} \delta^a_b \mathcal{V} \,,
\end{aligned}$$

where we introduced the generators[10]

$$U = \tfrac{1}{2} \mu^\dagger_\gamma \lambda^\gamma \,, \qquad \dot U = \tfrac{1}{2} \lambda^\dagger_{\dot\gamma} \mu^{\dot\gamma} \,, \qquad \mathcal{V} = \tfrac{1}{2} \psi^\dagger_c \psi^c \,,$$

which commute with $\mathcal{L}^\alpha{}_\beta$, $\dot{\mathcal{L}}^{\dot\alpha}{}_{\dot\beta}$ and $\mathcal{R}^a{}_b$. We set

$$\mathcal{B} = U + \dot U \,, \qquad \mathcal{C} = U + \dot U + \mathcal{V} \,, \qquad \mathcal{D} = U - \dot U \,,$$

[10]Note that there are some slight differences with respect to [GG21b] in the definition of these generators and also in notation; this is has been done in order to render some later expressions more cumbersome.

and the define the “off-diagonal” generators

$$\begin{aligned}
\mathcal{Q}^{a}{}_{\alpha} &= \psi^a \mu^\dagger_\alpha\,, & \mathcal{S}^{\alpha}{}_{a} &= \lambda^\alpha \psi^\dagger_a\,,\\
\dot{\mathcal{Q}}^{\dot\alpha}{}_{a} &= \mu^{\dot\alpha} \psi^\dagger_a\,, & \dot{\mathcal{S}}^{a}{}_{\dot\alpha} &= \psi^a \lambda^\dagger_{\dot\alpha}\,,\\
\mathcal{P}^{\dot\alpha}{}_{\beta} &= \mu^{\dot\alpha} \mu^\dagger_\beta\,, & \mathcal{K}^{\alpha}{}_{\dot\beta} &= \lambda^\alpha \lambda^\dagger_{\dot\beta}\,.
\end{aligned}$$

For what concerns the $\mathfrak{su}(4)$ algebra, we use the convention that the zero modes $(\mathcal{R}^a{}_b)_0$ with $a<b$ are the positive roots, and define the Cartan generators of $\mathfrak{su}(4)$ to be

$$H_i = (\mathcal{R}^{i+1}{}_{i+1})_0 - (\mathcal{R}^i{}_i)_0\,, \qquad i=1,2,3\,.$$

The bosonic generators satisfy

$$[(\mathcal{L}^\alpha{}_\beta)_m, (\mathcal{L}^\gamma{}_\delta)_n] = \delta^\alpha_\delta (\mathcal{L}^\gamma{}_\beta)_{m+n} - \delta^\gamma_\beta (\mathcal{L}^\alpha{}_\delta)_{m+n} + m\left(-\delta^\gamma_\beta\delta^\alpha_\delta + \tfrac{1}{2}\delta^\alpha_\beta\delta^\gamma_\delta\right)\delta_{m+n,0}\,,$$

and likewise for $(\dot{\mathcal{L}}^{\dot\alpha}{}_{\dot\beta})_m$. The modes of the generators

$$J^+ = \mathcal{L}^1{}_2\,, \qquad J^- = \mathcal{L}^2{}_1\,, \qquad J^3 = \tfrac{1}{2}\left(\mathcal{L}^2{}_2 - \mathcal{L}^1{}_1\right)\,,$$

satisfy the $\mathfrak{su}(2)_{-1}$ relations, that is, Eq. (27) for $k=-1$. The same construction also applies to the $(\dot{\mathcal{L}}^{\dot\alpha}{}_{\dot\beta})_m$ generators, which therefore lead to another copy of $\mathfrak{su}(2)_{-1}$ that we denote by dotted generators $\dot{J}^a_n$. On the other hand, for the $(\mathcal{R}^a{}_b)_m$ generators we find

$$[(\mathcal{R}^a{}_b)_m, (\mathcal{R}^c{}_d)_n] = \delta^a_d (\mathcal{R}^c{}_b)_{m+n} - \delta^c_b (\mathcal{R}^a{}_d)_{m+n} + m\left(\delta^a_d\delta^c_b - \tfrac{1}{4}\delta^a_b\delta^c_d\right)\delta_{m+n,0}\,,$$

which are the commutation relations of $\mathfrak{su}(4)_1$. The $\mathfrak{u}(1)$ currents satisfy

$$\begin{aligned}
&[\mathcal{D}_m, \mathcal{B}_n] = [\mathcal{D}_m, \mathcal{C}_n] = [\mathcal{C}_m, \mathcal{C}_n] = 0\,,\\
&[\mathcal{B}_m, \mathcal{B}_n] = [\mathcal{B}_m, \mathcal{C}_n] = [\mathcal{D}_m, \mathcal{D}_n] = -m\delta_{m+n,0}\,.
\end{aligned} \tag{109}$$

Finally, the fermionic generators satisfy the anti-commutation

$$\begin{aligned}
\{(\mathcal{S}^\alpha{}_a)_m, (\mathcal{Q}^b{}_\beta)_n\} &= \delta^a_b (\mathcal{L}^\alpha{}_\beta)_{m+n} + \delta^\alpha_\beta (\mathcal{R}^a{}_b)_{m+n} + \tfrac{1}{2}\delta^b_a\delta^\alpha_\beta(\mathcal{D}_{m+n} + \mathcal{C}_{m+n} + 2m\delta_{m+n,0})\,,\\
\{(\dot{\mathcal{S}}^{\dot\alpha}{}_a)_m, (\dot{\mathcal{Q}}^b{}_{\dot\beta})_n\} &= \delta^a_b (\dot{\mathcal{L}}^{\dot\alpha}{}_{\dot\beta})_{m+n} + \delta^{\dot\alpha}_{\dot\beta} (\mathcal{R}^a{}_b)_{m+n} - \tfrac{1}{2}\delta^b_a\delta^{\dot\alpha}_{\dot\beta}(\mathcal{D}_{m+n} - \mathcal{C}_{m+n} + 2m\delta_{m+n,0})\,.
\end{aligned}$$

We now identify the affine subalgebra $\mathfrak{u}(2|2)_1 \oplus \mathfrak{u}(2|2)_1$. One copy is generated by the bilinears constructed from the fields $(\lambda^\alpha, \mu^\dagger_\alpha)$ and $(\psi^\dagger_a, \psi^a)$ with $a=1,2$, whilst the other one from $(\mu^{\dot\alpha}, \lambda^\dagger_{\dot\alpha})$ and $(\psi^\dagger_b, \psi^b)$ with $b=3,4$. We introduce

$$R^a{}_b = \psi^\dagger_b\psi^a - \delta^a_b V \;\text{ for }\; a,b=1,2 \quad\text{ and }\quad \dot{R}^a{}_b = \psi^\dagger_b\psi^a - \delta^a_b \dot{V} \;\text{ for }\; a,b=3,4\,,$$

where

$$V = \tfrac{1}{2}(\psi_1^\dagger\psi^1 + \psi_2^\dagger\psi^2) \quad \text{and} \quad \dot{V} = \tfrac{1}{2}(\psi_3^\dagger\psi^3 + \psi_4^\dagger\psi^4)\,.$$

Note that then $\mathcal{V} = V + \dot{V}$. We also define

$$Z = U + V\,, \qquad Y = U - V\,,$$

and their dotted analogues. Then, we identify

$$K^+ = R^1{}_2\,, \qquad K^- = R^2{}_1\,, \qquad K^3 = \tfrac{1}{2}\left(R^2{}_2 - R^1{}_1\right)\,, \tag{110}$$

which satisfy the commutator relations of $\mathfrak{su}(2)_1$, which are Eq. (27) with $k = 1$. The dotted copy of $\mathfrak{u}(2|2)_1$ is obtained by the replacement $1 \mapsto 3$ and $2 \mapsto 4$ in Eq. (110) and we denote the corresponding generators by $\dot{K}^a_n$. In order to obtain the algebra $\mathfrak{u}(2)_1$ from Eq. (110), one has to add the $\mathfrak{u}(1)$ generator V, and analogously $\dot{V}$ for the dotted copy. We identify

$$S^{\alpha\beta\gamma} = \begin{cases} \mathcal{Q}^\beta{}_\alpha & \text{if } \gamma = +\,, \\ \mathcal{S}^\alpha{}_\beta & \text{if } \gamma = -\,, \end{cases}$$

and write $- = 1$, $+ = 2$ for the indices α and β. Then one superalgebra $\mathfrak{u}(2|2)_1$ is generated by U_n, V_n, J^a_n, K^a_n for $a = \pm, 3$ and $S^{\alpha\beta\gamma}_n$ for $\alpha, \beta, \gamma = +, -$, since the satisfy the same commutation relations presented in Section 3.1. Analogously, the dotted copy arises by the dotted version of the bosonic and fermionic generators.

5.1 The Neveu-Schwarz sector

The vacuum representation of $\mathfrak{u}(2|2)_1$ arises from the Neveu-Schwarz (NS) sector where both the symplectic bosons and the fermions are half-integer moded[11], and it is generated from a ground state $|0\rangle$, which we call **vacuum**, satisfying

$$\lambda^\alpha_r\,|0\rangle = (\mu^\dagger_\alpha)_r\,|0\rangle = \psi^\alpha_r\,|0\rangle = (\psi^\dagger_\alpha)_r\,|0\rangle = 0 \qquad \text{for } \ r \geq \tfrac{1}{2}\,,\ \alpha = 1, 2\,.$$

Thus, this state has then the property that

$$U_0\,|0\rangle = V_0\,|0\rangle = Z_0\,|0\rangle = Y_0\,|0\rangle\; = 0 \quad \text{ and } \quad J^3_0\,|0\rangle = K^3_0\,|0\rangle = 0\,.$$

While for Y_0 this is a matter of convention (that is, it depends on the normal ordering prescription), though the natural one, this is imposed for Z_0 by the commutation relations of Eq. (41). In particular, this shows that the NS sector defines indeed the vacuum representation of $\mathfrak{u}(2|2)_1$, that we will denote by $\mathcal{V}$. Note that since $Z_0 = 0$, this

[11]Since the supercurrent generators involve one fermion and one symplectic boson, the moding of all generators has to be the same in order for the $\mathfrak{u}(2|2)_1$ generators to be integer moded.

representation descend to one of $\mathfrak{psu}(2|2)_1$, which we denote by $\mathcal{L}$.

5.2 The Ramond sector

Following the lines of [GG21b] and [DGGK21] we construct the Ramond (R) sector. We label the states by the symplectic boson occupation numbers $|m_1, m_2\rangle$. Since in the Lie supergroup perspective the bosonic subgroup $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is compact, we can assume that the spins, namely the J_0^3 and K_0^3 eigenvalues, are half-integers, see [GG21b]. There is freedom in defining the action of the symplectic bosons zero modes, also because we are dealing with two pairs of them. We will consider four different R sector representations; as we will see, the first two differ only by the Z_0 and Y_0 eigenvalues, whose sign is reversed, and they both contain only finite-dimensional representations of the non-compact $\mathfrak{su}(2)$ factor, whilst the third and fourth representation will contain infinite-dimensional highest and lowest weight discrete representations. In every case, we require without loss of generality that

$$m_1 \in \tfrac{1}{2}\mathbb{N} \quad \text{and} \quad m_2 \in \tfrac{1}{2}\mathbb{Z}\,.$$

The subspace generated by the vectors with $m_1, m_2 \in \frac{1}{2}\mathbb{N}$ always form an irreducible subrepresentations, and we shall in the following concentrate on this subspace. This highest weight space is annihilated by the modes X_n^I and $(Y_J)_n$ with $n > 0$, and the full affine representation is generated from it by the action of the non-positive modes. The zero mode action of the symplectic bosons is already encoded in the above occupation numbers, but we also have the action of the fermionic zero modes. They generate a Clifford algebra representation, and with respect to the zero modes of the $\mathfrak{su}(2)_1$ generators, that are bilinears in the fermions, the states for fixed values of m_i transform as

$$(2 \cdot \mathbf{1}) \oplus \mathbf{2}\,.$$

We now explicitly express the possible actions of the $\mathfrak{u}(2|2)_1$ generators. We start by the bosonic zero modes, which on highest weight states can act as

$$\begin{aligned}
\lambda_0^1\,|m_1, m_2\rangle &:= 2m_1|m_1 - \tfrac{1}{2}, m_2\rangle\,, & (\mu_1^\dagger)_0\,|m_1, m_2\rangle &:= |m_1 + \tfrac{1}{2}, m_2\rangle\,, \\
\lambda_0^2\,|m_1, m_2\rangle &:= 2m_2|m_1, m_2 - \tfrac{1}{2}\rangle\,, & (\mu_2^\dagger)_0\,|m_1, m_2\rangle &:= |m_1, m_2 + \tfrac{1}{2}\rangle\,.
\end{aligned} \tag{111}$$

From this one can compute the action of the $\mathfrak{u}(2)_{-1}$ generators[12]

$$\begin{aligned}
J_0^3\,|m_1,m_2\rangle &= (m_2-m_1)|m_1,m_2\rangle\,,\\
J_0^+\,|m_1,m_2\rangle &= 2m_1|m_1-\tfrac{1}{2},m_2+\tfrac{1}{2}\rangle\,,\\
J_0^-\,|m_1,m_2\rangle &= 2m_2|m_1+\tfrac{1}{2},m_2-\tfrac{1}{2}\rangle\,,\\
U_0\,|m_1,m_2\rangle &= (m_1+m_2+\tfrac{1}{2})|m_1,m_2\rangle\,.
\end{aligned} \tag{112}$$

Moreover, one can compute the $\mathfrak{su}(2)_{-1}$-Casimir

$$C^{\mathfrak{su}(2)} = J_0^3\,J_0^3 + \tfrac{1}{2}\,(J_0^+\,J_0^- + J_0^-\,J_0^+) = j\,(j+1) = (m_1+m_2)\,(m_1+m_2+1)\,,$$

thus the associated spin is $j = m_1+m_2$. For the action of the fermionic zero modes we define

$$\psi_0^a\,|m_1,m_2\rangle := 0 \qquad \text{for } a=1,2\,. \tag{113}$$

Then the action of the creation operators $\psi_a^\dagger$ with $a = 1,2$ leads to a 4-dimensional Clifford module; with respect to $\mathfrak{su}(2)_1$, it decomposes into two singlet states

$$2\,\cdot\,\mathbf{1}: \qquad |m_1,m_2\rangle \quad \text{and} \quad (\psi_2^\dagger)_0(\psi_1^\dagger)_0\,|m_1,m_2\rangle\,,$$

as well as a doublet spanned by

$$\mathbf{2}: \qquad (\psi_2^\dagger)_0\,|m_1,m_2\rangle \quad \text{and} \quad (\psi_1^\dagger)_0\,|m_1,m_2\rangle\,.$$

For $a \neq b$ we can compute,

$$(\psi_a^\dagger\psi^b)_0\,|m_1,m_2\rangle = 0 \quad \text{and} \quad (\psi_a^\dagger\psi^a)_0\,|m_1,m_2\rangle = -\tfrac{1}{2}\,|m_1,m_2\rangle\,,$$

hence

$$V_0 = \begin{cases} -\frac{1}{2} & \text{on the singlet } |m_1,m_2\rangle\,,\\ \;\;0 & \text{on the doublet,}\\ +\frac{1}{2} & \text{on the singlet } (\psi_2^\dagger)_0(\psi_1^\dagger)_0\,|m_1,m_2\rangle\,. \end{cases}$$

[12]The action of U_0 is fixed only up to a constant which depends on the normal ordering prescription; this in turn implies that the value of Y_0 depends on the convention adopted. However, the value of Z_0 is determined by Eq. (41) once the action of the fermionic and bosonic zero modes is defined, hence it is independent of the convention.

Recalling that $Z_0 = U_0 + V_0$ and $Y_0 = U_0 - V_0$, one computes

$$Z_0 = \begin{cases} j & \text{on the first singlet,} \\ j+\frac{1}{2} & \text{on the doublet,} \\ j+1 & \text{on the second singlet.} \end{cases} \qquad Y_0 = \begin{cases} j+1 & \text{on the first singlet,} \\ j+\frac{1}{2} & \text{on the doublet,} \\ j & \text{on the second singlet,} \end{cases} \tag{114}$$

where as above $j = m_1 + m_2$. We now analyze the irreducible representations of $\mathfrak{u}(2|2)$ characterized by the fixed value of $Z_0 \in \frac{1}{2}\mathbb{N}$, since Z_0 is central. We write such representations as multiplets $(j\,,\mathbf{n})_{Y,Z}$ of the bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_Y \oplus \mathfrak{u}(1)_Z$, where $\mathbf{n}$ and j denote the n- and $(2j+1)$-dimensional representation of $\mathfrak{su}(2)$ respectively and Y, Z the eigenvalues of Y_0 and Z_0 respectively.

- $Z_0 = 0$: this translates into

$$0 = Z_0 \,|m_1, m_2\rangle = j\,|m_1, m_2\rangle = (m_1 + m_2)\,|m_1, m_2\rangle\,,$$

and since $m_i \in \frac{1}{2}\mathbb{N}$, it follows that $m_1 = m_2 = 0$. From this, one easily sees that $|0,0\rangle$ is annihilated by all $\mathfrak{u}(2|2)$ generators, except for

$$U_0\,|0,0\rangle = \tfrac{1}{2}\,|0,0\rangle\,, \qquad V_0\,|0,0\rangle = -\tfrac{1}{2}\,|0,0\rangle\,, \qquad Y_0\,|0,0\rangle = |0,0\rangle\,.$$

Indeed, the application of any fermionic creation operator annihilates $|0,0\rangle$, since each involve a mode λ_0^α. This shows that the representation of $\mathfrak{u}(2|2)$ arising from this construction is the one-dimensional (ultrashort) representation generated by the single state $|0,0\rangle$, which we label by

$$(j = 0, \mathbf{1})_{1,0}\,. \tag{115}$$

Note that, except that for the Y_0 eigenvalue, this coincides with the vacuum representation.

- $Z_0 = \frac{1}{2}$: the highest weight states $|m_1, m_2\rangle$ satisfy $m_1 + m_2 = \frac{1}{2}$. The allowed states are then $|\frac{1}{2}, 0\rangle$, $|0, \frac{1}{2}\rangle$ which generate a $(j = \frac{1}{2}\,, \mathbf{1})$ representation, and $\psi_a^\dagger|0,0\rangle$ for $a = 1, 2$ which generate a $(j = 0\,, \mathbf{2})$ representation. Note that the application of two fermionic creation operators results in a zero state, since such would contain two λ_0^α's, thus annihilating each state with $m_1 + m_2 = \frac{1}{2}$. All together, we have

$$(j = 0\,, \mathbf{2})_{\frac{1}{2},\frac{1}{2}} \oplus (j = \tfrac{1}{2}\,, \mathbf{1})_{\frac{3}{2},\frac{1}{2}}\,.$$

- $Z_0 = j+1\,$, for $j \in \frac{1}{2}\mathbb{N}$: in this case, we have the following representations

$$
\begin{array}{lll}
(j+1\,,\mathbf{1}) : & |m_1, m_2\rangle & \text{with } \; m_1 + m_2 = j+1\,, \\
(j+\frac{1}{2}\,,\mathbf{2}) : & (\psi^\dagger_a)_0\,|m_1, m_2\rangle & \text{with } \; m_1 + m_2 = j+\frac{1}{2} \; \text{ and } \; a = 1,2\,, \\
(j\,,\mathbf{1}) \;\; : & (\psi^\dagger_2)_0(\psi^\dagger_1)_0\,|m_1, m_2\rangle & \text{with } \; m_1 + m_2 = j\,.
\end{array}
$$

All together, this gives

$$
(j\,,\mathbf{1})_{j,\,j+1} \oplus (j+\tfrac{1}{2}\,,\mathbf{2})_{j+1,\,j+1} \oplus (j+1\,,\mathbf{1})_{j+2,\,j+1}\,. \tag{116}
$$

We denote by $\mathcal{R}$ the whole affine module defined by Eq. (111) and Eq. (113), and the subrepresentations with fixed $Z = Z_0 \in \frac{1}{2}\mathbb{N}$ by $\mathcal{R}_Z$.

As we mentioned at the beginning of the section, we may define the action of the symplectic bosons zero modes differently and this yields different representations of $\mathfrak{u}(2|2)$. Hence, we define

$$
\begin{aligned}
\lambda^1_0\,|m_1, m_2\rangle &:= |m_1 + \tfrac{1}{2}, m_2\rangle\,, \qquad & (\mu^\dagger_1)_0\,|m_1, m_2\rangle &:= -2m_1|m_1 - \tfrac{1}{2}, m_2\rangle\,, \\
\lambda^2_0\,|m_1, m_2\rangle &:= |m_1, m_2 + \tfrac{1}{2}\rangle\,, \qquad & (\mu^\dagger_2)_0\,|m_1, m_2\rangle &:= -2m_2|m_1, m_2 - \tfrac{1}{2}\rangle\,,
\end{aligned} \tag{117}
$$

from which we compute

$$
\begin{aligned}
J^3_0\,|m_1, m_2\rangle &= (m_1 - m_2)|m_1, m_2\rangle\,, \\
J^+_0\,|m_1, m_2\rangle &= -2m_2|m_1 + \tfrac{1}{2}, m_2 - \tfrac{1}{2}\rangle\,, \\
J^-_0\,|m_1, m_2\rangle &= -2m_1|m_1 - \tfrac{1}{2}, m_2 + \tfrac{1}{2}\rangle\,, \\
U_0\,|m_1, m_2\rangle &= -(m_1 + m_2 + \tfrac{1}{2})|m_1, m_2\rangle\,,
\end{aligned}
$$

and the $\mathfrak{su}(2)$-Casimir

$$
C^{\mathfrak{su}(2)} = j\,(j+1) = (m_1 + m_2)\,(m_1 + m_2 + 1),
$$

thus the associated spin is again $j = m_1 + m_2$. For the action of the fermionic zero modes we define instead[13]

$$
(\psi^\dagger_a)_0\,|m_1, m_2\rangle := 0\,, \quad \text{for } \; a = 1,2\,. \tag{118}
$$

Then the Clifford module decomposes with respect to $\mathfrak{su}(2)$ into two singlet states

$$
2\,\cdot\,\mathbf{1} : \qquad |m_1, m_2\rangle \quad \text{and} \quad \psi^2_0\psi^1_0\,|m_1, m_2\rangle\,,
$$

[13] Note that there is freedom in the definition of the fermionic zero modes action, which however does not affect the structure of the whole Clifford module. Here, we choose a different action from Eq. (113) in order to obtain the full conjugate spectrum of $Z_0 \in -\frac{1}{2}\mathbb{N}$, since Eq. (113) would give a gap corresponding to $Z_0 = -\frac{1}{2}$.

and a doublet

$$\mathbf{2}: \qquad \psi_0^1\,|m_1, m_2\rangle \quad \text{and} \quad \psi_0^2\,|m_1, m_2\rangle\,.$$

As above we compute

$$V_0 = \begin{cases} +\frac{1}{2} & \text{on the singlet } |m_1, m_2\rangle, \\ 0 & \text{on the doublet,} \\ -\frac{1}{2} & \text{on the singlet } \psi_0^2\psi_0^1\,|m_1, m_2\rangle\,. \end{cases}$$

and consequently

$$Z_0 = \begin{cases} -j & \text{on the first singlet,} \\ -j-\frac{1}{2} & \text{on the doublet,} \\ -j-1 & \text{on the second singlet.} \end{cases} \qquad Y_0 = \begin{cases} -j-1 & \text{on the first singlet,} \\ -j-\frac{1}{2} & \text{on the doublet,} \\ -j & \text{on the second singlet.} \end{cases}$$

By an analogous analysis to the one above, we obtain the following $\mathfrak{u}(2|2)$-representations.

- $Z_0 = 0$:
$$(j=0, \mathbf{1})_{-1,0}\,. \tag{119}$$

- $Z_0 = -\frac{1}{2}$:
$$(j=0\,, \mathbf{2})_{-\frac{1}{2},-\frac{1}{2}} \oplus (j=\tfrac{1}{2}\,, \mathbf{1})_{-\frac{3}{2},-\frac{1}{2}}\,.$$

- $Z_0 = -j-1$, for $j \in \frac{1}{2}\mathbb{N}$:
$$(j\,, \mathbf{1})_{-j,-j-1} \oplus (j+\tfrac{1}{2}\,, \mathbf{2})_{-j-1,-j-1} \oplus (j+1\,, \mathbf{1})_{-j-2,-j-1}\,. \tag{120}$$

Note that these multiplets are the same of those found above with the replacement $Z_0 \mapsto -Z_0$ and $Y_0 \mapsto -Y_0$. As representations of $\mathfrak{su}(2)_1$ they differ only by the sign of the Z_0-eigenvalue; in particular, the representations with $Z_0 = 0$ coincide with the NS vacuum as $\mathfrak{su}(2|2)_1$- and $\mathfrak{psu}(2|2)_1$-representation. We denote by $\overline{\mathcal{R}}$ the full affine module generated by Eq. (117) and Eq. (118) and by $\overline{\mathcal{R}}_Z$ the subrepresentations to fixed $Z = Z_0 \in -\frac{1}{2}\mathbb{N}$.

In Eq. (111) and Eq. (117) we have seen two possible definitions of the symplectic bosons action. Note that in both cases the two pairs $(\lambda_0^\alpha, (\mu_\alpha^\dagger)_0)$ for $\alpha = 1, 2$ act in the same way. It is then natural to look also at the R representation where the two pairs act in opposite ways. This yields

$$\lambda_0^1\,|m_1, m_2\rangle := 2m_1|m_1-\tfrac{1}{2}, m_2\rangle\,, \qquad (\mu_1^\dagger)_0\,|m_1, m_2\rangle := |m_1+\tfrac{1}{2}, m_2\rangle\,, \tag{121}$$
$$\lambda_0^2\,|m_1, m_2\rangle := |m_1, m_2+\tfrac{1}{2}\rangle\,, \qquad (\mu_2^\dagger)_0\,|m_1, m_2\rangle := -2m_2|m_1, m_2-\tfrac{1}{2}\rangle\,.$$

from which we compute

$$
\begin{aligned}
J_0^3\,|m_1,m_2\rangle &= -(m_1+m_2+\tfrac{1}{2})|m_1,m_2\rangle\,,\\
J_0^+\,|m_1,m_2\rangle &= -4m_1m_2|m_1-\tfrac{1}{2},m_2-\tfrac{1}{2}\rangle\,,\\
J_0^-\,|m_1,m_2\rangle &= |m_1+\tfrac{1}{2},m_2+\tfrac{1}{2}\rangle\,,\\
U_0\,|m_1,m_2\rangle &= (m_1-m_2)|m_1,m_2\rangle\,,
\end{aligned}
\tag{122}
$$

and the $\mathfrak{su}(2)$-Casimir

$$
C^{\mathfrak{su}(2)} = j\,(j+1) = (m_1-m_2-\tfrac{1}{2})\,(m_1-m_2+\tfrac{1}{2}), \tag{123}
$$

with spin $j = m_1 - m_2 - \frac{1}{2}$. We define the action of the fermionic zero modes as in Eq. (113), which gives Eq. (114) with $j = m_1 - m_2 - \frac{1}{2}$. We now analyze the irreducible representations of $\mathfrak{u}(2|2)$ characterized by the fixed value of $Z_0 \in \frac{1}{2}\mathbb{Z}$. For that, we write $Z_0 = j - \frac{1}{2}$ for $j \in \frac{1}{2}\mathbb{Z}$ such that

$$
Z_0\,|m_1,m_2\rangle = (m_1-m_2-\tfrac{1}{2})|m_1,m_2\rangle = (j-\tfrac{1}{2})|m_1,m_2\rangle\,,
$$

implying that $m_1 = m_2 + j$. We distinguish between three cases and find the following multiplets, where we always have $m \in \frac{1}{2}\mathbb{N}$.

- $j \in -\frac{1}{2}\mathbb{N}$:

$$
\begin{aligned}
(D^+_{j-\frac{1}{2}}\,,\mathbf{1}) &: \qquad |m,m-j\rangle\,,\\
(D^+_{j-1}\,,\mathbf{2}) &: \qquad (\psi^\dagger_a)_0\,|m,m-j+\tfrac{1}{2}\rangle\,,\\
(D^+_{j-\frac{3}{2}}\,,\mathbf{1}) &: \qquad (\psi^\dagger_2)_0(\psi^\dagger_1)_0\,|m,m-j+1\rangle\,.
\end{aligned}
$$

- $j \in \frac{1}{2}\mathbb{N}$ and $j \geq 1$:

$$
\begin{aligned}
(D^+_{-j-\frac{1}{2}}\,,\mathbf{1}) &: \qquad |m+j,m\rangle\,,\\
(D^+_{-j}\,,\mathbf{2}) &: \qquad (\psi^\dagger_a)_0\,|m+j-\tfrac{1}{2},m\rangle\,,\\
(D^+_{-j+\frac{1}{2}}\,,\mathbf{1}) &: \qquad (\psi^\dagger_2)_0(\psi^\dagger_1)_0\,|m+j-1,m\rangle\,.
\end{aligned}
$$

All together, these can be summarised by

$$
(D^+_{-j}\,,\mathbf{1})_{\pm j-1,\,\pm j} \oplus (D^+_{-j-\frac{1}{2}}\,,\mathbf{2})_{\pm j,\,\pm j} \oplus (D^+_{-j-1}\,,\mathbf{1})_{\pm j+1,\,\pm j}\,, \tag{124}
$$

for every $j \in \frac{1}{2}\mathbb{Z}_{>0}$, which is Eq. (54).

- $j = \frac{1}{2}$, $Z_0 = 0$:

$$\begin{aligned}
(D^+_{-1}, \mathbf{1}) &: \qquad |m + \tfrac{1}{2}, m\rangle\,, \\
(D^+_{-1/2}, \mathbf{2}) &: \qquad (\psi^\dagger_a)_0\, |m, m\rangle\,, \\
(D^+_{-1}, \mathbf{1}) &: \qquad (\psi^\dagger_2)_0 (\psi^\dagger_1)_0\, |m, m + \tfrac{1}{2}\rangle\,,
\end{aligned}$$

  which in particular yields an additional highest weight representation of $\mathfrak{psu}(2|2)$ characterised by

$$(D^+_{-1}, \mathbf{1}) \oplus (D^+_{-1/2}, \mathbf{2}) \oplus (D^+_{-1}, \mathbf{1})\,, \tag{125}$$

  which is exactly Eq. (55).

We denote by $\mathcal{R}^+$ the whole affine module defined by Eq. (121) and Eq. (113) and by $\mathcal{R}^+_Z$ the subrepresentations with fixed $Z = Z_0 \in \frac{1}{2}\mathbb{Z}$.

We consider the action of the symplectic bosons zero modes that is dual to that of Eq. (121), namely

$$\begin{aligned}
\lambda^1_0\, |m_1, m_2\rangle &:= |m_1 + \tfrac{1}{2}, m_2\rangle\,, & (\mu^\dagger_1)_0\, |m_1, m_2\rangle &:= -2m_1 |m_1 - \tfrac{1}{2}, m_2\rangle\,, \\
\lambda^2_0\, |m_1, m_2\rangle &:= 2m_2 |m_1, m_2 - \tfrac{1}{2}\rangle\,, & (\mu^\dagger_2)_0\, |m_1, m_2\rangle &:= |m_1, m_2 + \tfrac{1}{2}\rangle\,,
\end{aligned} \tag{126}$$

from which we compute

$$\begin{aligned}
J^3_0\, |m_1, m_2\rangle &= m_1 + m_2 + \tfrac{1}{2} |m_1, m_2\rangle\,, \\
J^+_0\, |m_1, m_2\rangle &= |m_1 + \tfrac{1}{2}, m_2 + \tfrac{1}{2}\rangle\,, \\
J^-_0\, |m_1, m_2\rangle &= -4m_1 m_2 |m_1 - \tfrac{1}{2}, m_2 - \tfrac{1}{2}\rangle\,, \\
U_0\, |m_1, m_2\rangle &= (m_2 - m_1) |m_1, m_2\rangle\,,
\end{aligned}$$

and the $\mathfrak{su}(2)$-Casimir

$$C^{\mathfrak{su}(2)} = j\,(j-1) = (m_2 - m_1 + \tfrac{1}{2})\,(m_2 - m_1 - \tfrac{1}{2}),$$

thus the spin is $j = m_2 - m_1 + \frac{1}{2}$. We define the action of the fermionic zero modes as in Eq. (118) and we obtain

$$Z_0 = \begin{cases} j & \text{on the first singlet,} \\ j - \frac{1}{2} & \text{on the doublet,} \\ j - 1 & \text{on the second singlet,} \end{cases} \qquad Y_0 = \begin{cases} j - 1 & \text{on the first singlet,} \\ j - \frac{1}{2} & \text{on the doublet,} \\ j & \text{on the second singlet,} \end{cases}$$

with $j = m_2 - m_1 + \frac{1}{2}$. We write $Z_0 = j + \frac{1}{2}$ for $j \in \frac{1}{2}\mathbb{Z}$ such that

$$Z_0\, |m_1, m_2\rangle = (m_2 - m_1 + \tfrac{1}{2}) |m_1, m_2\rangle = (j + \tfrac{1}{2}) |m_1, m_2\rangle\,,$$

implying that $m_2 = m_1 + j$. We then find the following multiplets, where $m \in \frac{1}{2}\mathbb{N}$.

- For $j \in \frac{1}{2}\mathbb{Z}_{>0}$:

$$\begin{aligned} (D^-_{j+\frac{1}{2}}\,, \mathbf{1}) \ &: \qquad |m, m+j\rangle\,, \\ (D^-_{j+1}\,, \mathbf{2}) \ &: \qquad \psi^a_0\,|m, m+j+\tfrac{1}{2}\rangle\,, \\ (D^-_{j+\frac{3}{2}}\,, \mathbf{1}) \ &: \qquad \psi^2_0\psi^1)_0\,|m, m+j+1\rangle\,. \end{aligned}$$

- $j \in \frac{1}{2}\mathbb{Z}$ and $j \leq -1$:

$$\begin{aligned} (D^-_{-j+\frac{1}{2}}\,, \mathbf{1}) \ &: \qquad |m-j, m\rangle\,, \\ (D^-_{-j+1}\,, \mathbf{2}) \ &: \qquad \psi^a_0\,|m-j-\tfrac{1}{2}, m\rangle\,, \\ (D^-_{-j+\frac{3}{2}}\,, \mathbf{1}) \ &: \qquad \psi^2_0\psi^1_0\,|m-j-1, m\rangle\,. \end{aligned}$$

  All together, these can be summarised by

$$(D^-_j\,, \mathbf{1})_{\pm j-1,\, \pm j} \oplus (D^-_{j+\frac{1}{2}}\,, \mathbf{2})_{\pm j,\, \pm j} \oplus (D^-_{j+1}\,, \mathbf{1})_{\pm j+1,\, \pm j}\,, \tag{127}$$

  for $j \in \frac{1}{2}\mathbb{Z}_{>0}$. Note that Eq. (127) is exactly the conjugate of Eq. (124) and reproduced Eq. (56)-

- $j = -\frac{1}{2}$, $Z_0 = 0$:

$$\begin{aligned} (D^-_1\,, \mathbf{1}) \ &: \qquad |m+\tfrac{1}{2}, m\rangle\,, \\ (D^-_{1/2}\,, \mathbf{2}) \ &: \qquad \psi^a_0\,|m, m\rangle\,, \\ (D^-_1\,, \mathbf{1}) \ &: \qquad \psi^2_0\psi^1_0\,|m, m+\tfrac{1}{2}\rangle\,, \end{aligned}$$

  which in particular yields Eq. (57).

We denote by $\mathcal{R}^-$ the whole affine module defined by Eq. (126) and Eq. (118) and by $\mathcal{R}^-_Z$ the subrepresentations with fixed $Z = Z_0 \in \frac{1}{2}\mathbb{Z}$.

The only representations found in Section 3.3 that we are missing in the free field construction are those with highest weight states transforming in multiplets of $\mathfrak{su}(2|2)_1$ containing continuous representations of $\mathfrak{su}(2)$. These can be obtained by defining the R sector as in Eq. (111) and Eq. (113) but allowing $m_1, m_2 \in \frac{1}{2}\mathbb{Z}$. Then, the $\mathfrak{su}(2)$-modules that appear are generally reducible but indecomposable, which is an expected feature of continuous representations C^j_j with $j \in \frac{1}{2}\mathbb{Z}$, see Section 3.2. Indeed, one can carry out the same analysis as above, finding exactly the multiplets Eq. (58) and Eq. (59) with specific Y_0-eigenvalues for each component in the multiplets, which can be deduced by Eq. (114). We denote this free field representation by $\widetilde{\mathcal{R}}$ and by $\widetilde{\mathcal{R}}_Z$ that to fixed $Z_0 = Z \in \frac{1}{2}\mathbb{Z}$.

5.3 The spectral flow

For the actual world-sheet theory it is to be expected that we do not just need these highest weight representations, but also the representations that are obtained from them by spectral flow. Analogously to [GG21b] and [DGGK21], there are two spectral flow actions that can be defined on the free fields, namely

$$\sigma^{(\alpha)}(\lambda^\alpha_r) := \lambda^\alpha_{r+\frac{1}{2}}\,, \qquad \sigma^{(\alpha)}((\mu^\dagger_\alpha)_r) := (\mu^\dagger_\alpha)_{r-\frac{1}{2}}\,,$$

on the symplectic bosons, and

$$\sigma^{(\alpha)}(\psi^\alpha_r) := \psi^\alpha_{r+\frac{1}{2}}\,, \qquad \sigma^{(\alpha)}((\psi^\dagger_\alpha)_r) := (\psi^\dagger_\alpha)_{r-\frac{1}{2}}\,,$$

on the fermions, where $\alpha = 1, 2$. The combination

$$\sigma_Y := \sigma^{(1)} \circ \sigma^{(2)}$$

leaves the subalgebra $\mathfrak{su}(2)_{-1} \oplus \mathfrak{su}(2)_1$ and the fermionic generators $S^{\alpha\beta\gamma}_m$ invariant and acts on the $\mathfrak{u}(1)$ generators as

$$\sigma^w_Y(U_m) = U_m + \tfrac{w}{2}\delta_{m,0}\,, \qquad \sigma^w_Y(V_m) = V_m - \tfrac{w}{2}\delta_{m,0}\,,$$

which implies

$$\sigma^w_Y(Z_m) = Z_m \qquad \sigma^w_Y(Y_m) = Y_m + w\delta_{m,0}\,.$$

and by Eq. (104)

$$\sigma^w_Y(L_0^{\mathfrak{u}(2|2)_1}) = L_0^{\mathfrak{u}(2|2)_1} - wZ_0\,,$$

where we specified that the energy-momentum tensor is that of $\mathfrak{u}(2|2)_1$, because the $\mathfrak{u}(1)_Y$ algebra is absent for $\mathfrak{su}(2|2)_1$ and $\mathfrak{psu}(2|2)_1$ and thus the automorphism σ_Y is non-trivial only for $\mathfrak{u}(2|2)_1$.

The other natural combination is

$$\sigma := \sigma^{(1)} \circ \left(\sigma^{(2)}\right)^{-1}$$

and it acts on the generators as

$$\begin{aligned} \sigma^w(J^3_m) &= J^3_m - \tfrac{w}{2}\delta_{m,0}\,, & \sigma^w(U_m) &= U_m\,, \\ \sigma^w(J^\pm_m) &= J^\pm_{m\pm 1}\,, & \sigma^w(V_m) &= V_m\,, \\ \sigma^w(K^3_m) &= K^3_m + \tfrac{w}{2}\delta_{m,0}\,, & \sigma^w(Z_m) &= Z_m\,, \\ \sigma^w(K^\pm_m) &= K^\pm_{m\pm 1}\,, & \sigma^w(Z_m) &= Z_m\,, \\ \sigma^w(S^{\alpha\beta\gamma}_m) &= S^{\alpha\beta\gamma}_{m+\frac{w}{2}\gamma(\alpha-\beta)}\,, & \sigma^w(L_0) &= L_0 + w(J^3_0 + K^3_0)\,. \end{aligned} \tag{128}$$

We can obtain one additional independent spectral flows by declaring $\tilde{\sigma}^{(\alpha)}$ to act as $\sigma^{(\alpha)}$ on the symplectic bosons, and

$$\tilde{\sigma}^{(\alpha)}(\psi^\alpha_r) := \psi^\alpha_{r-\frac{1}{2}}\,, \qquad \tilde{\sigma}^{(\alpha)}((\psi^\dagger_\alpha)_r) := (\psi^\dagger_\alpha)_{r+\frac{1}{2}}\,,$$

on the fermions. The combination

$$\sigma_Z := \tilde{\sigma}^{(1)} \circ \tilde{\sigma}^{(2)}$$

then leaves again the subalgebra $\mathfrak{su}(2)_{-1} \oplus \mathfrak{su}(2)_1$ invariant and acts on the other generators as

$$\sigma_Z^w(Z_m) = Z_m + w\delta_{m,0} \qquad \sigma_Z^w(Y_m) = Y_m \qquad \sigma_Z^w(S^{\alpha\beta\gamma}_m) = S^{\alpha\beta\gamma}_{m-\gamma w}\,,$$

and by Eq. (104) we have

$$\sigma_Z^w(L_0^{\mathfrak{u}(2|2)_1}) = L_0^{\mathfrak{u}(2|2)_1} - wY_0\,.$$

Note that since the $\widehat{\mathfrak{u}}(1)_Z$ algebra extends $\mathfrak{psu}(2|2)_1$ to $\mathfrak{su}(2|2)_1$, the automorphism σ_Z is non-trivial only for $\mathfrak{su}(2|2)_1$ and $\mathfrak{u}(2|2)_1$ and it transforms L_0 differently depending on which algebra we are considering.

The other combination

$$\tilde{\sigma} := \tilde{\sigma}^{(1)} \circ \left(\tilde{\sigma}^{(2)}\right)^{-1}$$

acts on the generators as

$$\begin{aligned}
\tilde{\sigma}^w(J^3_m) &= J^3_m - \tfrac{w}{2}\delta_{m,0}\,, & \tilde{\sigma}^w(U_m) &= U_m\,,\\
\tilde{\sigma}^w(J^\pm_m) &= J^\pm_{m\pm1}\,, & \tilde{\sigma}^w(V_m) &= V_m\,,\\
\tilde{\sigma}^w(K^3_m) &= K^3_m - \tfrac{w}{2}\delta_{m,0}\,, & \tilde{\sigma}^w(Z_m) &= Z_m\,,\\
\tilde{\sigma}^w(K^\pm_m) &= K^\pm_{m\mp1}\,, & \tilde{\sigma}^w(Z_m) &= Z_m\,,\\
\tilde{\sigma}^w(S^{\alpha\beta\gamma}_m) &= S^{\alpha\beta\gamma}_{m+\frac{w}{2}\gamma(\alpha+\beta)}\,, & \tilde{\sigma}^w(L_0) &= L_0 + w(J^3_0 - K^3_0)\,.
\end{aligned}$$

In particular, the action of $\tilde{\sigma}$ is equivalent to that of σ.

Finally we need to fix our conventions for how to describe spectrally flowed representations. Suppose that ρ is some spectral flow automorphism, that is some combination of $\sigma^{(\alpha)}$ and $\tilde{\sigma}^{(\alpha)}$ for $\alpha = 1, 2$. Then we define the ρ-spectrally flowed representation, denoted by $\rho(\mathcal{H})$, to be spanned by the states $[\Phi]^\rho$, where Φ is a state in a highest weight representation $\mathcal{H}$ of the kind described in above, and the "twisted" action of any generator X_n on $[\Phi]^\rho$ is defined by

$$X_n\,[\Phi]^\rho := [\rho(X_n)\,\Phi]^\rho\,,$$

and we write $|0\rangle_w^\rho := [|0\rangle]^{\rho^w}$. We call the modules $\rho^w(\mathcal{H})$ with $w \in \mathbb{Z}$ the **spectrally flowed** modules of $\mathcal{H}$.

Typically, spectrally flowed representations are not highest weight representations. Also recall that in general the classification of irreducible $\mathfrak{g}$-modules yields a classification of the affine highest-weight $\mathfrak{g}_k$ modules. In order to include also the spectrally flowed versions of the highest weight modules, one needs to enlarge the representation category of $\mathfrak{g}_k$ by the following definition, see [CR12]. We define a $\mathfrak{g}_k$-module $\mathcal{H}$ to be **admissible** if it satisfies the following conditions:

- $\mathcal{H}$ is finitely generated,
- the Cartan generators h^i act semisimply on $\mathcal{H}$ (though L_0 need not),
- for every $|v\rangle \in \mathcal{H}$, there exists an $N > 0$ such that

$$X_n\,|v\rangle = 0 \qquad \forall\, X \in \mathfrak{g}\ \forall\, n > N\,.$$

Example 5.1. We said that in general spectrally flowed highest weight representations are non-highest weight. However, as representations of $\mathfrak{u}(2|2)_1$, the R sector singlet representation in Eq. (115) is the image of the NS sector highest weight representation under one unit of a specific spectral flow, namely

$$|0\rangle_1^Y := |0\rangle_{w=1}^{\sigma_Y} = |0,0\rangle \iff \sigma_Y(\mathcal{V}) \cong \mathcal{R}_0\,, \tag{129}$$

Note that this distinction is relevant only when we consider representations of $\mathfrak{u}(2|2)_1$, otherwise with respect to $\mathfrak{su}(2|2)_1$ the spectral flow of Eq. (129) is trivial and the two representations coincide.

In order to prove Eq. (129), we note that all positive modes, as well as the zero modes λ_0^α, ψ_0^α for $\alpha = 1,2$ annihilate both sides of the first equation. Furthermore, one checks that also the eigenvalues of all the Cartan generators agree, that is,

$$K_0^3\,|0\rangle_1^Y = J_0^3\,|0\rangle_1^Y = Z_0\,|0\rangle_1^Y = 0\,, \qquad Y_0\,|0\rangle_1^Y = |0\rangle_1^Y\,.$$

Analogously, one shows that

$$|0\rangle_{-1}^Y := |0\rangle_{w=-1}^{\sigma_Y} = |0,0\rangle \iff \sigma_Y^{-1}(\mathcal{V}) \cong \overline{\mathcal{R}}_0\,.$$

Example 5.2. We consider the spectral flow automorphism of $\mathfrak{u}(2|2)_1$ given by $\rho := \sigma_Y^{-1} \circ \sigma_Z$ and claim that

$$\rho(\mathcal{R}_j) \cong \mathcal{R}_{j+1} \qquad \forall\, j \in \tfrac{1}{2}\mathbb{N}\,. \tag{130}$$

Indeed, recall that the full affine $\mathfrak{u}(2|2)_1$ representation $\mathcal{R}_j$ is characterised by the highest weight multiplet

$$(j-1\,,\mathbf{1})_{j-1,\,j}\oplus(j-\tfrac{1}{2}\,,\mathbf{2})_{j,\,j}\oplus(j\,,\mathbf{1})_{j+1,\,j}\,,\tag{131}$$

with additional shortening occurring for $j=0,\frac{1}{2}$. In any case, let us denote by $|j\rangle$ the highest weight state in $(j\,,\mathbf{1})_{j+1,\,j}$, so that in particular $S_0^{\alpha\beta+}\,|j\rangle=0$, and its image under the spectral flow ρ by $|j\rangle^\rho:=|j\rangle^\rho_{w=1}$. It is then immediate to check that $|j\rangle^\rho$ generates the representation $(j\,,\mathbf{1})_{j,\,j+1}$ and that

$$S_n^{\alpha\beta\pm}\,|j\rangle^\rho=0\begin{cases}\text{if}\;\;n\geq 1\,,\\ \text{if}\;\;n\geq 0\,.\end{cases}$$

Thus, $|j\rangle^\rho$ generates the multiplet

$$(j\,,\mathbf{1})_{j,\,j+1}\oplus(j+\tfrac{1}{2}\,,\mathbf{2})_{j+1,\,j+1}\oplus(j+1\,,\mathbf{1})_{j+2,\,j+1}\,,\tag{132}$$

corresponding to $\mathcal{R}_{j+1}$. For instance, the representation $(j+\frac{1}{2}\,,\mathbf{2})_{j+1,\,j+1}$ is generated by the states $S_0^{\alpha\beta+}\,|j\rangle^\rho$. Indeed, one computes

$$\rho(L_0)=L_0+Z_0-Y_0+1\,,$$

where L_0 refers to the energy-momentum tensor of $\mathfrak{u}(2|2)_1$, and consequently

$$L_0\,|j\rangle^\rho=L_0S_0^{\alpha\beta+}\,|j\rangle^\rho=0\,,$$

where we used that by the conformal embedding in Eq. (104), the $\mathfrak{u}(2|2)_1$ Casimir vanishes on all the multiplets found in the various free field R sectors, and that the central charge of $\mathfrak{u}(2|2)_1$ is zero. By induction, it follows that

$$\rho^j(\mathcal{R}_0)\cong\mathcal{R}_j\qquad\text{and}\qquad\rho^j(L_0)=L_0+j(j+Z_0-Y_0)=L_0+j(j-Y_0)\,,\tag{133}$$

where in the last equality we used that $Z_0=0$ on $\mathcal{R}_0$.

Moreover, one can show analogously that $\rho^{-1}(\overline{\mathcal{R}}_Z)\cong\overline{\mathcal{R}}_{Z-1}$ for every $Z\in-\frac{1}{2}\mathbb{N}$. Also, similar results hold for the other R sector representations found above. Note that this result also holds for $\mathfrak{su}(2|2)_1$ in which case $\rho=\sigma_Z$ and the L_0 transformation is different.

6 Characters and Modular Invariants

We review the theory of affine characters and in particular those of integrable affine highest weight modules following chapter 14.4 of [DFMS97]. We then discuss modular invariance of partition functions following chapter 4 of [BP09].

6.1 Affine characters of affine integrable modules

Let $\mathfrak{g}_k$ be an affine Lie algebra at fixed level $k \in \mathbb{R}$ with finite Lie algebra $\mathfrak{g}$ of rank r. Let $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ and set $q := \mathrm{e}^{2\pi i\tau}$, such that $|q| < 1$. Let also $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ be variables, which we will call **chemical potentials**. Then, the **affine character** of the $\mathfrak{g}_k$-module of affine highest weight $\hat{\lambda}$ is

$$\mathrm{ch}_{\hat{\lambda}}(z;\tau;t) := \mathrm{e}^{2\pi ikt}\,\mathrm{Tr}_{\hat{\lambda}}\,\mathrm{e}^{2\pi i\tau L_0}\mathrm{e}^{2\pi i\sum_j z_j h^j}\,, \tag{134}$$

where the trace is taken over an orthonormal basis of $\hat{\lambda}$ and in the following we will often set $t = 0$ and dropping the t-dependence, since at fixed level k the corresponding term does not contain any particular information. If the highest weight representation $\hat{\lambda}$ is integrable, then Eq. (134) can be shown to be equal to

$$\mathrm{ch}_{\hat{\lambda}}(z;\tau) = \sum_{\hat{\lambda}'\in\Omega_{\hat{\lambda}}} \mathrm{mult}_{\hat{\lambda}}(\hat{\lambda}')e^{\hat{\lambda}'}(\hat{\xi})\,,$$

where $\hat{\xi} = -2\pi i(\zeta;\tau;t)$ with $\zeta = \sum_{i=1}^r z_i\alpha_i^\vee$ and the sum runs over all the weights in the affine representation, $\mathrm{mult}_{\hat{\lambda}}(\hat{\lambda}') \in \mathbb{N}$ denotes the multiplicity of $\hat{\lambda}'$ in the representation $\hat{\lambda}$ and $e^{\hat{\lambda}'}$ denotes a formal exponential satisfying

$$e^{\hat{\lambda}'}e^{\hat{\mu}} = e^{\hat{\lambda}'+\hat{\mu}} \quad \text{and} \quad e^{\langle\hat{\lambda}',\hat{\mu}\rangle} = \mathrm{e}^{\hat{\lambda}'(\hat{\mu})} \quad \forall\,\hat{\lambda}',\hat{\mu} \in \mathfrak{h}_k^*\,.$$

Eq. (134) can be rewritten in the form called *Weyl-Kac character formula*, which make use of the affine Weyl group action, see [DFMS97, Eq. 14.148]. Then, thanks to the *Macdonald-Weyl denominator identity* it can be written in terms of **generalised theta functions**

$$\Theta_{\hat{\lambda}}^{(k)}(z;\tau;t) = \mathrm{e}^{2\pi ikt}\sum_{\alpha^\vee\in Q^\vee}\mathrm{e}^{2\pi i\langle\alpha^\vee+\lambda/k,\zeta\rangle}\mathrm{e}^{\pi ik\tau|\alpha^\vee+\lambda/k|^2}\,, \tag{135}$$

where $Q^\vee = \oplus_{i=1}^r\mathbb{Z}\alpha_i^\vee$ is the **coroot lattice**. More precisely, we have that

$$\mathrm{ch}_{\hat{\lambda}}(t;\tau;t) = \mathrm{e}^{-2\pi i\tau m_{\hat{\lambda}}}\frac{\sum_{w\in W}\epsilon(w)\Theta^{(k)}_{w(\hat{\lambda}+\hat{\rho})}(z;\tau;t)}{\sum_{w\in W}\epsilon(w)\Theta^{(k)}_{w(\hat{\rho})}(z;\tau;t)}\,, \tag{136}$$

where W is the Weyl group of $\mathfrak{g}$ and $\epsilon(w) := (-1)^{|w|}$ is the signature of $w \in W$, see [DFMS97, Chapter 13.1.8]. As we will, the fact that the affine characters cane be expressed in terms of generalised theta functions is a key feature of integrable affine representations, since Eq. (135) have nice modular properties, as we will see. In Eq. (136) there is a special quantity called the **modular anomaly**

$$m_{\hat{\lambda}} = \frac{|\lambda + \rho|^2}{2(k + h^\vee)} - \frac{|\rho|^2}{2h^\vee} = h_\lambda - \frac{c}{24}\,,$$

where h_λ and c are defined as in Eq. (24) and Eq. (17) respectively. As one can see in Eq. (136), it is natural to normalise the character with respect to the modular anomaly; therefore, we introduce the **normalised affine character** (and we will often omit the epithet "normalised") as

$$\chi_{\hat{\lambda}}(z;\tau) := \mathrm{Tr}_{\hat{\lambda}}\, \mathrm{e}^{2\pi i \tau (L_0 - c/24)} \mathrm{e}^{2\pi i \sum_j z_j h^j}\,, \tag{137}$$

which is the form of the character we will always use. When evaluated at $z = 0$, the character of Eq. (137) is said to be **specialised** and takes the form

$$\chi_{\hat{\lambda}}(\tau) := \chi_{\hat{\lambda}}(0;\tau) = q^{h_\lambda - c/24} \sum_{n \geq 0} d(n) q^n\,,$$

where $d(n)$ gives the total number of states at grade n.

Example 6.1. We want to look at the characters of integrable $\mathfrak{su}(2)_k$-modules for $k \in \mathbb{Z}_{>0}$. In this case, the affine dominant weights are of the form

$$[k - \lambda_1, \lambda_1] \quad \text{for} \;\; 0 \leq \lambda_1 \leq k\,.$$

In the following, we write $\lambda_1 = 2\ell$, so that $0 \leq \ell \leq k/2$, and we call the spin ℓ $\mathfrak{su}(2)_k$-module the highest weight integrable affine representation generated by the highest weight state $|\ell\rangle$, see Section 2.3. Recall that ℓ completely characterises the representation for fixed k, and the state $|\ell\rangle$ transforms in the finite dimensional spin ℓ representation under the finite $\mathfrak{su}(2)$.

The generalised theta functions 135 in this case take the form

$$\Theta^{(k)}_m(z;\tau) := \sum_{n \in \mathbb{Z} + \frac{m}{2k}} q^{kn^2} z^{kn}\,,$$

and with these we can express the integrable spin ℓ $\mathfrak{su}(2)_k$-characters as

$$\begin{aligned}\chi_\ell^{(k)}(z;\tau) &:= \frac{\Theta^{(k+2)}_{2\ell+1}(z;\tau)-\Theta^{(k+2)}_{-2\ell-1}(z;\tau)}{\Theta^{(2)}_{1}(z;\tau)-\Theta^{(2)}_{-1}(z;\tau)} \\ &= \frac{\Theta^{(k+2)}_{2\ell+1}(z;\tau)-\Theta^{(k+2)}_{-2\ell-1}(z;\tau)}{q^{\frac{1}{8}}(z^{\frac{1}{2}}-z^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^n)}\,,\end{aligned} \tag{138}$$

where for the second equality we used *Jacobi's triple product* identity

$$\sum_{n\in\mathbb{Z}}(-1)^n z^{n+\frac{1}{2}}q^{\frac{1}{2}(n+\frac{1}{2})^2} = q^{\frac{1}{8}}(z^{\frac{1}{2}}-z^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q)\,,$$

which is true for every $q,z\in\mathbb{C}$ with $|q|<1$ and $z\neq 0$. The oscillator contribution in the infinite product in the last line of Eq. (138) comes from the modes J^3_{-n} and $J^{\pm}_{-n}$ for $n>0$ which are charged under the finite $\mathfrak{su}(2)$ by $z=0$, $z^{\pm 1}$ respectively. Indeed, since these modes are bosonic, they contribute to the character by

$$\sum_{i=0}^{\infty}(z^a q)^i = (1-z^a q^n)^{-1} \quad \text{for} \quad a=0,\pm\,.$$

The rest of the expression accounts for the fact that the ground states form a spin ℓ $\mathfrak{su}(2)$-representation and for the null vectors in the module. The numerator of Eq. (138) at lowest orders in q looks like

$$\Theta^{(k+2)}_{2\ell+1}(z;\tau)-\Theta^{(k+2)}_{-2\ell-1}(z;\tau) = q^{\frac{(\ell+\frac{1}{2})^2}{k+2}}\left(z^{\ell+\frac{1}{2}}-z^{-\ell-\frac{1}{2}}\right)-q^{\frac{(k-\ell+\frac{3}{2})^2}{k+2}}\left(z^{k-\ell+\frac{3}{2}}-z^{-k+\ell-\frac{3}{2}}\right)+\dots\,.$$

Recall that the character of the finite-dimensional spin ℓ $\mathfrak{su}(2)$-representation is given by

$$\chi_\ell^{\mathfrak{su}(2)}(\zeta) := \sum_{m=-\ell,\,m+\ell\in\mathbb{Z}}^{\ell} z^m = \frac{z^{\ell+\frac{1}{2}}-z^{-\ell-\frac{1}{2}}}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}} = \frac{\sin((2\ell+1)\pi\zeta)}{\sin(\pi\zeta)} \quad \text{for} \quad z=\mathrm{e}^{2\pi i\zeta}\,. \tag{139}$$

Thus, the oscillator contribution is multiplied by

$$\begin{aligned}&q^{\frac{(\ell+\frac{1}{2})^2}{k+2}-\frac{1}{8}}\chi_\ell^{\mathfrak{su}(2)}(\zeta)-q^{\frac{(k-\ell+\frac{3}{2})^2}{k+2}-\frac{1}{8}}\chi_{k+1-\ell}^{\mathfrak{su}(2)}(\zeta)+\dots \\ &\qquad = q^{\frac{\ell(\ell+1)}{k+2}-\frac{k}{8(k+2)}}\left(\chi_\ell^{\mathfrak{su}(2)}(\zeta)-q^{k+1-2l}\chi_{k+1-\ell}^{\mathfrak{su}(2)}(\zeta)+\dots\right)\,.\end{aligned}$$

We recognize on the right hand side the factor $q^{h-\frac{c}{24}}$, where the conformal dimension and the central charge have been described in Section 2.3. We also understand the second term in the series as subtracting out the null vector $(J^+_{-1})^{k+1-2\ell}\,|\ell\rangle$, see Eq. (30). The next term in the series correspond to the fact that we have also subtracted all the descendants of the null-vector. However, some descendants are actually not there and have to be put in again into the character. This pattern continues and yields an alternating sum. This is made even more explicit when looking at the specialised character, that is, letting $z \to 0$, which after some manipulations yields

$$\chi_\ell^{(k)}(\tau) := \lim_{z\to 0} \chi_\ell^{(k)}(z;\tau) = \frac{q^{\frac{(\ell+\frac{1}{2})^2}{k+2}}}{\eta(\tau)^3} \sum_{n\in\mathbb{Z}} (2\ell+1+2n(k+2))q^{n(2\ell+1+(k+2)n)} .$$

Remark 6.2. In this chapter we mainly considered characters of integrable modules and we ignored issues related to the convergence of the characters of Eq. (134) and Eq. (137). There are in general two points of view when considering characters; one can define them as complex formal power series (or formal distributions) in the variables q and z, which are then called **formal characters**, or as meromorphic functions in z expanded on a specific convergence region dictated by q (where we always assume that $|q| < 1$), which are then called **character functions**. It turns out that the characters of integrable representations are holomorphic in z, thus one can identify formal characters with formal distributions. However, even in the "simplest" non-integrable setting, namely that of affine Lie algebras at *admissible fractional levels*, see [DFMS97, Chapter 18], the characters usually have poles in z and hence one must distinguish between formal characters and character functions. In particular, when working with character functions one loses the correspondence between modules and characters. More precisely, the $\mathbb{Z}$-linear map which assigns to each module in the fusion ring its character is not injective as in the integrable setting, see for instance [CR13].

However, for what concerns modular invariance of WZW models at fractional levels, Kac and Wakimoto [KW88] observed that for a given admissible level, there is a finite number of admissible representations transforming linearly among themselves under modular transformations. This readily leads to a formal modular invariant which is simply the diagonal invariant built out of the admissible character functions.

In this regard, one of the most studied case is that of $\mathfrak{sl}(2)_k$ and $\mathfrak{su}(2)_k$ for admissible fractional levels k, see [CR13] and [LMRS04]. In this setting, the spectral flow action on the character functions has finite orbits, meaning that even though there are infinitely many inequivalent spectrally flowed modules, there are only finitely many linearly independent character functions. In particular, one finds recursive relations between character functions and their spectrally flowed versions, which allow to characterise the kernel of the map from modules to characters. If such kernel contains modules that close under fusion, namely if forms an ideal, then one can consistently define fusion at the level of characters and the resulting ring over $\mathcal{Z}$ is called the **Grothendieck ring of characters**. Assigning modules to their characters therefore define a surjective $\mathbb{Z}$-linear map from the fusion ring to the Grothendieck ring.

As mentioned above, this has a peculiar effect when considering modular invariance.

Specifically, one expects from rational theories that pairing each module with itself under the holomorphic and antiholomorphic affine actions leads to a modular invariant partition function. However, the coincidence of characters means that there are infinitely many modules all contributing the same amount to the partition function, which therefore diverges. One can then regularise this divergence by only allowing the linearly independent characters to contribute, effectively dividing the modular invariant by the infinite multiplicity of each independent character, and in this way one recovers the modular invariant of Kac and Wakimoto [KW88]. This is indeed invariant under the modular action of $\mathrm{PSL}(2, \mathbb{Z})$, but one should be careful in interpreting it as a *physical* partition function, since technically it does not refer to a complete set of modules of the theory. In particular, there is no set of modules corresponding to this partition function which is closed under fusion. In essence however, it determines a modular invariant partition function in the Grothendieck ring of characters. This is no different to what one does in rational theories, and evidence is in favor that this is what one should do in logarithmic theories as well. However, determining a modular invariant in this way does not answer the fundamental question of how the holomorphic and antiholomorphic sectors of the theory are glued together. Applications hence require a justification of why such a partition function is appropriate.

The proposal of [LMRS02] in interpreting the modular invariant on the Grothendieck ring of characters in relation to the Kac-Wakimoto invariant, is to regard the character functions as being defined only on the respective annulus of convergence. Summing them to get a partition function is therefore viewed as summing over the different annuli in order to have a finite meromorphic partition function on the z-plane with $|q| < 1$. Presumably this means each character should take value zero outside its given annulus, in contrast with analytic continuation. This proposal seems however unlikely [Rid09] since modular transformations do not preserve this annuli structure, hence one is forced to analytically continue the characters into the rest of the z-plane.

In fact, it would be better to extend the definition of partition function so that every module contributes. In [Rid09] it is suggested to introduce an additional quantum number to distinguish representations with the same character, namely same character functions with different annuli of convergence. It is not clear however that such a quantum number need exist and it seems plausible that modular invariants for fractional level models can only be defined at the level of Grothendieck ring.

With these comments, one should also be careful when interpreting the Verlinde formula. The potential problem linked to the fact that the modular transformations relate different regions of convergences, is not taken into account in the derivation of the Verlinde formula. Therefore, it is well established only for integrable representations, or equivalently, for holomorphic character functions. On the other hand, if spectrally flowed modules appear under the modular transformations, this means that we do not flow back onto the original set of fields when going around cycles on the torus. This clearly indicates that the Verlinde formula does not apply in general to admissible fractional level WZW models. In particular, a naive application of the Verlinde formula to some admissible fractional level theories [Rid09] has lead to negative fusion coefficients. In [CR12] it is argued how this arises because the characters of the infinitely many irreducible modules are not all linearly independent as meromorphic functions.

6.2 Modular Invariance

In CFTs higher order corrections in perturbation theory are equivalent to defining the fields on higher genus Riemannian surfaces. At one-loop, that is genus one, such surface is the torus. It turns out that the partition function on the torus has a symmetry under the *modular group* $\mathrm{PSL}(2,\mathbb{Z})$; we say that it is **modular invariant**. This feature heavily constrains the field content of the theory and so the allowed representations.

Starting from a tree-level CFT defined on the Riemann sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$ we can obtain a CFT on the torus $T^2(\tau) \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ by taking the quotient of the punctured sphere with a two dimensional integral lattice specified by a number $\tau \in \mathbb{H}$ called **modular parameter**, where we denote the upper half plane by

$$\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\} .$$

Since we wish to work in a differential geometric setting, instead of a topological one, for different values of the modular parameter we obtain in general inequivalent tori, where by inequivalent we mean non-isometric or with non-isomorphic complex structures. There is an action of the modular group on $\mathbb{H}$ given by Möbius transformations

$$\mathrm{PSL}(2,\mathbb{Z}) \times \mathbb{H} \to \mathbb{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d} . \tag{140}$$

This action preserves the torus $T^2(\tau)$ for fixed $\tau \in \mathbb{Z}$. The modular group $\mathrm{PSL}(2,\mathbb{Z})$ is generated by two elements:

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \ \tau \mapsto \tau + 1 \quad \text{and} \quad S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \ \tau \mapsto -\frac{1}{\tau} ,$$

with relations $S^2 = I$ and $(ST)^3 = I$ as elements of $\mathrm{PSL}(2,\mathbb{Z})$.

The partition function of a CFT on the torus turns out to be

$$\mathcal{Z}(\tau,\bar{\tau}) := \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) , \tag{141}$$

where $q = \mathrm{e}^{2\pi i \tau}$ and $\bar{q} = \mathrm{e}^{-2\pi i \bar{\tau}}$. Then, by the discussion above Eq. (141) has a $\mathrm{PSL}(2,\mathbb{Z})$

symmetry. In the following we will be mainly concerned with integrable WZW models.

6.2.1 Modular covariance of integrable affine characters

The modular group action of Eq. (140) can be extended on affine weights of $\mathfrak{g}_k$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z;\tau;t) := \left(zc\tau + d; \frac{a\tau+b}{c\tau+d}; t + \frac{c|z|^2}{2(c\tau+d)} \right) ,$$

where we identify $z = (z_1, \ldots, z_r)$ with $\zeta = \sum_{i=1}^r z_i \alpha_i^\vee$. In particular,

$$T \cdot (z;\tau;t) = (z;\tau+1;t) \quad \text{and} \quad S \cdot (z;\tau;t) = \left(\frac{z}{\tau}; -\frac{1}{\tau}; t + \frac{|z|^2}{2\tau} \right) .$$

This action then descend on affine characters as

$$A \cdot \chi_{\hat{\lambda}}(z;\tau;t) := \chi_{\hat{\lambda}}(A \cdot (z;\tau;t)) \quad \forall\, A \in \mathrm{PSL}(2,\mathbb{Z}) .$$

The result that is at the core of modular covariance in unitary rational WZW models is that the characters of dominant highest weight representations at some fixed level k transform into each other under the modular group action, that is

$$\begin{aligned} \chi_{\hat{\lambda}}(z;\tau+1;t) &= \sum_{\hat{\mu} \in P_+^k} \mathcal{T}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(z;\tau;t) , \\ \chi_{\hat{\lambda}}(z/\tau; -1/\tau; t + |z|^2/2\tau) &= \sum_{\hat{\mu} \in P_+^k} \mathcal{S}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(z;\tau;t) , \end{aligned} \tag{142}$$

where

$$\mathcal{T}_{\hat{\lambda}\hat{\mu}} = \delta_{\hat{\lambda}\hat{\mu}} \mathrm{e}^{2\pi i m_{\hat{\lambda}}} , \tag{143}$$

which means that the T-transformation induces only a phase change, and

$$\mathcal{S}_{\hat{\lambda}\hat{\mu}} = i^{|\Delta_+|} |P/Q^\vee|^{-\frac{1}{2}} (k+h^\vee)^{-r/2} \sum_{w \in W} \epsilon(w) \mathrm{e}^{-2\pi i \langle w(\lambda+\rho), \mu+\rho \rangle/(k+h^\vee)} , \tag{144}$$

where $P := \oplus_{i=1}^r \mathbb{Z}\omega_i$ denotes the **weight lattice** of $\mathfrak{g}$ and $P/Q^\vee$ the set of lattice points of P lying in an elementary cell of $Q^\vee$. It is very important for the derivation of Eq. (143) and Eq. (144) that the affine characters can be expressed in terms of generalised theta functions, see Eq. (136). Then, Eq. (143) is straightforward, while Eq. (144) follows from the *Poisson resummation formula*

$$\sum_{x \in \Gamma} f(x) = \frac{1}{|\Gamma|} \sum_{p \in \Gamma^*} \hat{f}(p) , \tag{145}$$

where $\Gamma = \oplus_{i=1}^d \mathbb{Z}\epsilon_i \subset \mathbb{R}^d$ is a d-dimensional lattice with volume $|\Gamma| := \sqrt{\det \langle \epsilon_i, \epsilon_j \rangle}$, $\Gamma^* := \oplus_{i=1}^d \mathbb{Z}\epsilon_i^*$ is the dual lattice with $\langle \epsilon_i, \epsilon_j^* \rangle = \delta_{ij}$ for every $1 \leq i, j \leq d$, and $f \colon \mathbb{R}^d \to \mathbb{C}$ is a Schwartz function with Fourier transform $\hat{f}$. In particular, for $d = 1$, $\varepsilon_1 = 1$ and $f(x) := \mathrm{e}^{-\pi a x^2 + bx}$ with $a \in \mathbb{H}$ and $b \in \mathbb{C}$, one obtains the useful identity

$$\sum_{n \in \mathbb{Z}} \mathrm{e}^{-\pi a n^2 + bn} = \frac{1}{\sqrt{a}} \sum_{m \in \mathbb{Z}} \mathrm{e}^{-\frac{\pi}{a}\left(m - \frac{b}{2\pi i}\right)^2} \tag{146}$$

We investigate the properties of the modular T- and S-matrices. A very important fact is that both matrices are unitary

$$\mathcal{T}^{-1} = \mathcal{T}^\dagger \quad \text{and} \quad \mathcal{S}^{-1} = \mathcal{S}^\dagger .$$

Note that $S^2 = 1$ but $\mathcal{S}^2 \neq 1$. Indeed, at the level of affine weights

$$S^2 \cdot (z; \tau; t) = (-z; \tau; t) ,$$

which implies that at the level of characters

$$S^2 \cdot \chi_{\hat{\lambda}}(z; \tau; t) = \chi_{\hat{\lambda}}(-z; \tau; t) = \chi_{\hat{\lambda}^*}(z; \tau; t) , \tag{147}$$

where $\hat{\lambda}^*$ the affine highest weight conjugate to $\hat{\lambda}$, see [DFMS97, Chapter 14]. This in particular shows that $\mathcal{S}^2$ is the **charge conjugation** matrix

$$\mathcal{C} := \mathcal{S}^2 \quad \text{with} \quad \mathcal{C} \cdot \chi_{\hat{\lambda}} = \chi_{\hat{\lambda}^*} .$$

On $\mathcal{S}$, the action of $\mathcal{C}$ is simply the usual complex conjugation

$$\bar{\mathcal{S}} = \mathcal{C}\mathcal{S} = \mathcal{S}\mathcal{C} .$$

An interesting result, is that when looking at the specialised form of the second line in Eq. (142) and in particular at its asymptotic behaviour for $\tau \to i0^+$, one finds that

$$\mathcal{S}_{\hat{\lambda}0} \geq \mathcal{S}_{00} > 0 ,$$

where the label 0 stands for $k\hat{\omega}_0$, the vacuum representation. This means that the S-transformation of the vacuum character is a non-trivial linear combination of every integrable highest weight character present in the model.

6.2.2 Modular invariance in WZW models

For an integrable WZW model, the full Hilbert space decomposes as

$$\mathcal{H} = \bigoplus_{\hat\lambda,\hat\mu\in P_+^k} \mathcal{M}_{\hat\lambda\hat\mu}\mathcal{H}_{\hat\lambda}\otimes\overline{\mathcal{H}}_{\hat\mu} \tag{148}$$

where the tensor product reflects the separation into *holomorphic* and *antiholomorphic* sectors. Also, $\mathcal{M}_{\hat\lambda\hat\mu}\in\mathbb{N}$ gives the multiplicity of the combined modules $\mathcal{H}_{\hat\lambda}\otimes\overline{\mathcal{H}}_{\hat\mu}$ in the Hilbert space of the theory. Since there are finitely many integrable highest weight representations, say $N>0$, the coefficients of $\mathcal{M}$ can be assembled into a matrix which we call **mass matrix** since it specifies the physical spectrum of the model. The partition function Eq. (141) can thus be expressed in terms of the affine integrable characters as

$$\mathcal{Z}(\tau,\bar\tau) = \sum_{\hat\lambda,\hat\mu\in P_+^k}\chi_{\hat\lambda}(\tau)\mathcal{M}_{\hat\lambda\hat\mu}\,\overline{\chi}_{\hat\mu}(\bar\tau)\,, \tag{149}$$

where now $\mathcal{M}_{\hat\lambda\hat\mu}$ can be interpreted as the multiplicity of the primary field which under $G(z)\times G(\bar z)$ transforms with respect to the λ and μ representations of $\mathfrak{g}$ respectively. This form of the partition function for WZW models does not fully take into account the Lie algebra symmetry; in other words, even though the parameter t in Eq. (137) in unnecessary, the chemical potential $z\in\mathbb{C}^r$ is required for the full characterisation of the spectrum and is missing in Eq. (149). For instance, the z dependence is needed to distinguish between conjugate characters, see Eq. (147). However, in order to lighten the notation, we omit these parameters[14].

With Eq. (142) we have that modular invariance of Eq. (149) is equivalent to the following conditions:

$$\mathcal{T}^\dagger\mathcal{M}\mathcal{T} = \mathcal{S}^\dagger\mathcal{M}\mathcal{S} = \mathcal{M} \qquad\Longleftrightarrow\qquad [\mathcal{M},\mathcal{S}] = [\mathcal{M},\mathcal{T}] = 0\,, \tag{150}$$

that is, $\mathcal{M}$ must be in the centraliser of $\mathcal{S}$ and $\mathcal{T}$.

In addition to being modular invariant, the partition functions must satisfy the following physical conditions:

1. $\mathcal{M}_{\hat\lambda\hat\mu}\in\mathbb{N}$,
2. $\mathcal{M}_{00}=1$ for the vacuum state to be unique.

[14] The complete partition function Eq. (141) including the chemical potentials of the Cartan generators is often written under the form

$$\mathcal{Z}(z,\bar z;\tau,\bar\tau) = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0-c/24}x_i^{h_0^i}\bar q^{L_0-c/24}\bar x_i^{h_0^i}\,,$$

where $x_i = \mathrm{e}^{2\pi i z_i}$, $\bar x_i = \mathrm{e}^{-2\pi i\bar z_i}$ and there is an implicit product over $i=1,\ldots,r$.

An $N \times N$-matrix $\mathcal{M}$ satisfying Eq. (150) and conditions 1, 2 is said to be a **physical invariant**. It appears though that the physical conditions are not quite sufficient to fully specify well-defined theories. For example, a physical invariant could lead to a theory with non-integer fusion coefficients, which should not be acceptable. At this time, a complete set of conditions that must be satisfied by a physical invariant to qualify it as a genuine rational CFT is not known.

There is a natural choice for the mass matrix, namely

$$\mathcal{M}_{\hat{\lambda}\hat{\mu}} = \delta_{\hat{\lambda}\hat{\mu}} \,.$$

Then the partition function is simply

$$\mathcal{Z}(\tau,\bar{\tau}) = \sum_{\hat{\lambda}\in P_+^k} \chi_{\hat{\lambda}}(\tau)\overline{\chi}_{\hat{\lambda}}(\bar{\tau}) \,,$$

which is modular invariant by unitarity of both $\mathcal{T}$ and $\mathcal{S}$. Such theory is called **diagonal**. In this case, all integrable representations appear exactly once and all the fields have equal holomorphic and anti-holomorphic conformal dimensions. Moreover, the diagonal theory with $\mathcal{M} = I$ is a physical invariant.

The problem of finding modular invariance has lead to mainly three approaches: the method of *outer automorphisms* (which is an abelian orbifold construction), *conformal embeddings* (immersion into a larger theory, see Section 4) and *Galois permutations* (modular-invariant permutation of the fields associated with an automorphism of the fusion rules). These methods always produce physical theories but none of these prove to be complete; that is, all known physical invariants cannot be generated by only one of these techniques. For a more detailed discussion we refer to [DFMS97, Chapter 17].

Example 6.3. In Example 6.1 we gave an explicit expression for the integrable $\hat{\mathfrak{su}}(2)_k$ characters. Recall that for fixed level $k \in \mathbb{Z}_{>0}$ there are $k+1$ integrable highest weight representations which we label by the spin $\ell \in \mathbb{N}$ such that $0 \leq \ell \leq k/2$. We choose the ordered basis $(\chi_0^{(k)}, \chi_{1/2}^{(k)}, \ldots, \chi_{k/2}^{(k)})$ for the vector space of integrable affine characters of $\hat{\mathfrak{su}}(2)_k$. Then, by Eq. (143) the modular T-matrix is given by

$$\mathcal{T}_{\ell\ell'}^{(k)} = \mathrm{e}^{2\pi i\left(\frac{\ell(\ell+1)}{k+2} - \frac{k}{8(k+2)}\right)}\delta_{\ell,\ell'} \,.$$

Since $|\Delta_+| = 1$, $|P/Q^\vee| = 1$, $h^\vee = 2$ and $|\omega_1|^2 = \frac{1}{2}$, the modular S-matrix of Eq. (144) is given by

$$\mathcal{S}_{\ell\ell'}^{(k)} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(2\ell+1)(2\ell'+1)}{k+2}\right) \qquad \text{for} \quad 0 \leq \ell,\ell' \leq k/2 \,, \tag{151}$$

which is clearely symmetric and one can check explicitely that it is also unitary. Eq. (151)

can be obtained explicitely with Eq. (138) and the Poisson summation formula, Eq. (146).

A remarkable fact is that for the $\mathfrak{su}(2)_k$-WZW model there exists a complete classification of modular invariants; this is due to Cappelli, Itzykson and Zuber [CIZ87]. There are three types of modular invariants: the A-type, corresponding to the diagonal invariants, which exists for every level $k \in \mathbb{Z}_{>0}$, the D-type, which exists for even level, and the E-type, which exist at the three exceptional levels $k = 10$, $k = 16$ and $k = 28$. The terminology ADE comes from the fact that the classification problem for $\mathfrak{su}(2)_k$ can be mapped to the classification of simply-laced Lie algebras. The modular invariants have also a physical interpretation. The A-type modular invariant defines the SU(2)-WZW model at level $k \in \mathbb{Z}_{>0}$; it contains every integrable representation exactly once. The D-type corresponds to the SO(3)-WZW model, which turns out to have a quantisation condition $k \in 2\mathbb{Z}$ instead of Eq. (10) because it is not simply-connected. This is an instance where the global topology of the Lie group affects the spectrum of the theory.

Remark 6.4. We stress that in the discussion above we considered WZW models on an affine algebra $\mathfrak{g}_k$ that admit at least one integrable representation, which then implies that all the relevant representation of the theory consist in the finite set of all integrable highest weight representations of $\mathfrak{g}_k$. At the level of the corresponding Lie group G, this requires that G is compact.

As we mentioned before, global properties of G also affects the spectrum. Indeed, if G is compact and simply-connected, then the theory possess a diagonal invariant. On the other hand, if G is compact but not simply connected, then the WZW model is still rational but not necessarily diagonal. As an example, the SO(3)-WZW model exists for even integer levels $k \in 2\mathbb{Z}_{>0}$ and its spectrum is a non-diagonal combination of finitely many integrable highest weight representations corresponding to the D-type invariants in the $\mathfrak{su}(2)_k$ classification.

If G is not compact, then the WZW is non-rational. Moreover, its spectrum may include non-highest weight representations. This is the example for the $SL(2,\mathbb{R})$-WZW model, whose spectrum is built from highest weight representations, plus their images under the spectral flow automorphisms of the affine Lie algebra, see [MO01].

If G is a Lie supergroup, and correspondingly $\mathfrak{g}$ is a Lie superalgebra, then the spectrum may involve representations that do not factorize as tensor products of representations of the holomorphic and antiholomorphic symmetry algebras. This occurs for example in the case of $\mathfrak{gl}(1|1)$ [SS06] and also in more complicated supergroups such as $\mathfrak{psu}(1,1|2)$ [GQS06]. Non-factorizable representations are responsible for the fact that the corresponding WZW models are logarithmic conformal field theories.

6.2.3 Modular transformations and invariants for coset theories

We come back to the setting of Section 4.1 and consider an affine embedding, following the lines of [DFMS97, Chapter 18.7]. In this context, the branching coefficients satisfy a simple rule, which can also be used for their determination. More specifically, the branching functions correspond to the normalised character identity

$$\chi_{\mathcal{P}\hat{\lambda}}(z;\tau) = \sum_{\hat{\mu}\in\widetilde{P}_+^{x_e k}} \mathrm{e}^{2\pi i\tau(m_{\hat{\lambda}}-m_{\hat{\mu}})} b_{\hat{\lambda},\hat{\mu}}(\tau)\chi_{\hat{\mu}}(z;\tau)\,, \tag{152}$$

which simplifies further for conformal embeddings, since $m_{\hat{\lambda}} = h_\lambda - \frac{c}{24}$, and one can use Eq. (73). Moreover, for characters of integrable representations there is an asymptotic relation given by

$$\chi_{\hat{\lambda}}(\tau\to i0^+) \sim \mathcal{S}_{\hat{\lambda},0}\,\mathrm{e}^{i\pi c/12\tau}\,.$$

All together, we obtain

$$\mathcal{S}_{\hat{\lambda},0} = \sum_{\hat{\mu}} b_{\hat{\lambda},\hat{\mu}}\,\widetilde{\mathcal{S}}_{\hat{\mu},0}\,,$$

where $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are the S-matrices of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ respectively. Note that this relation holds true in the case of integrable theories.

We now move our attention to coset theories, continuing the discussion of Section 4.2. To extract the coset $\mathfrak{g}_k/\widetilde{\mathfrak{g}}_{\tilde{k}}$ conformal theory from the $\mathfrak{g}_k$-WZW model, we must strip off its $\widetilde{\mathfrak{g}}_{\tilde{k}}$ content. In practice, this means that if we consider the affine branching rules of Eq. (70), then the various characters of the coset model should emerge from this decomposition. In other words, the branching functions are the natural candidates for the coset characters. However, this is not quite exact and we must first consider the precise relationship between characters and branching functions. For that, we look at Eq. (152) and define the **normalised branching functions** as

$$\chi_{\{\hat{\lambda};\,\hat{\mu}\}}(\tau) := \mathrm{e}^{2\pi i\tau(m_{\hat{\lambda}}-m_{\hat{\mu}})} b_{\hat{\lambda},\hat{\mu}}(\tau)\,. \tag{153}$$

By Eq. (152) we identify the coset characters with normalised branching functions. However, an immediate consequence of this identification is that not all pairs of fields (or weights) can be combined into coset fields. Indeed, for the branching function Eq. (153) to be non-zero, the so called **selection rule** must be satisfied:

$$\mathcal{P}\lambda - \mu \in \mathcal{P}Q\,, \tag{154}$$

where Q is the root lattice of $\mathfrak{g}$ and $\mathcal{P}$ is the projection matrix of the embedding $\widetilde{\mathfrak{g}} \hookrightarrow \mathfrak{g}$.

We now consider the modular transformation properties of the branching functions.

The modular transformations of affine integrable characters Eq. (142) leads to

$$\begin{aligned}\chi_{\{\hat\lambda;\,\hat\mu\}}(-1/\tau) &= \sum_{\hat\lambda'\in P_+^k,\,\hat\mu'\in\widetilde{P}_+^{kxe}} \mathcal{S}_{\hat\lambda\hat\lambda'}\,\widetilde{\mathcal{S}}^{-1}_{\hat\mu\hat\mu'}\,\chi_{\{\hat\lambda';\,\hat\mu'\}}(\tau)\,,\\ \chi_{\{\hat\lambda;\,\hat\mu\}}(\tau+1) &= \mathrm{e}^{2\pi i(m_{\hat\lambda}-m_{\hat\mu})}\,\chi_{\{\hat\lambda;\,\hat\mu\}}(\tau)\,,\end{aligned} \tag{155}$$

where $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are the modular S-matrices of $\mathfrak{g}_k$ and $\widetilde{\mathfrak{g}}_{\tilde k}$ respectively, and we omitted the projection operator from the $\mathcal{S}$ matrix indices since $\chi_{\mathcal{P}\hat\lambda}$ and $\chi_{\hat\lambda}$ have identical modular transformation properties. Then, Eq. (155) shows that the transformation matrices for the normalised branching functions are

$$\begin{aligned}\mathcal{S}_{\{\hat\lambda;\,\hat\mu\},\{\hat\lambda';\,\hat\mu'\}} &:= \mathcal{S}_{\hat\lambda\hat\lambda'}\,\widetilde{\mathcal{S}}^{*}_{\hat\mu\hat\mu'}\\ \mathcal{T}_{\{\hat\lambda;\,\hat\mu\},\{\hat\lambda';\,\hat\mu'\}} &:= \mathcal{T}_{\hat\lambda\hat\lambda'}\,\widetilde{\mathcal{T}}^{*}_{\hat\mu\hat\mu'}\end{aligned}$$

where the * here means complex conjugation. In particular, unitarity of the branching function modular matrices is inherited from the unitarity of the WZW modular matrices. Note that the $\mathcal{T}$ transformation matrix for $\chi_{\{\hat\lambda;\,\hat\mu\}}$ in Eq. (155) is given by

$$\chi_{\{\hat\lambda;\,\hat\mu\}}(\tau+1) = \mathrm{e}^{2\pi i(h_\lambda-h_\mu-c/24)}\,\chi_{\{\hat\lambda;\,\hat\mu\}}(\tau)\,,$$

where c is the coset central charge as in Eq. (74). Moreover, there is a simple expression for the fractional part of the conformal dimension for the coset field $\{\hat\lambda;\hat\mu\}$. If the tip of the $\hat\mu$ representation of $\widetilde{\mathfrak{g}}_{\tilde k}$ lies at grade n in the $\hat\lambda$ representation of $\mathfrak{g}_k$, then

$$h_{\{\hat\lambda;\,\hat\mu\}} = h_\lambda - h_\mu + n\,,$$

whose fractional part is just $h_\lambda - h_\mu$. Thus,

$$\chi_{\{\hat\lambda:\,\hat\mu\}}(\tau+1) = \mathrm{e}^{2\pi i(h_\lambda-h_\mu-c/24)}\,\chi_{\{\hat\lambda:\,\hat\mu\}}(\tau)\,.$$

Finally, we mention that from the modular transformations of the coset characters in Eq. (155), one can construct modular invariants for the coset theory from invariants of $\mathfrak{g}_k$ and $\widetilde{\mathfrak{g}}_{\tilde k}$. At first sight, a straightforward way of constructing the coset mass matrix $\mathcal{M}_{coset}$ is the product

$$\mathcal{M}_{coset} = \mathcal{M}\,\widetilde{\mathcal{M}}\,, \tag{156}$$

of the mass matrices $\mathcal{M}$ of $\mathfrak{g}_k$ and $\widetilde{\mathcal{M}}$ of $\widetilde{\mathfrak{g}}_{\tilde k}$. Then, the modular invariance of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ ensures automatically the invariance of their product. However, this simple product matrix does not give the coset partition function, since the selection rules of Eq. (154), which impose constraints on the summations, are not taken into account. In general, the true partition function has the form

$$\mathcal{Z}_{coset}(\tau,\bar{\tau}) \sim \sum_{\substack{\hat{\lambda},\hat{\lambda}'\in P_+^k\ \hat{\mu},\hat{\mu}'\in \widetilde{P}_+^{kx_e} \\ \mathcal{P}\lambda-\mu=\mathcal{P}\lambda'-\mu'=0 \bmod Q}} \chi_{\{\hat{\lambda};\hat{\mu}\}}(\tau)\,\mathcal{M}_{\hat{\lambda},\hat{\lambda}'}\,\overline{\mathcal{M}}_{\hat{\mu},\hat{\mu}'}\,\overline{\chi}_{\{\hat{\lambda}';\hat{\mu}'\}}\,, \tag{157}$$

where the proportionality factor depends on the length of fields identification orbits under the outer automorphism group, which we did not discuss here and we refer to [DFMS97, Chapter 18.3]. The partition function Eq. (157) no longer has the simple product form Eq. (156), and modular invariance is not guaranteed from the onset but it turns out that it still holds true. Moreover, note that by construction, the coset models we considered are rational conformal field theories. Indeed, since there is a finite number of primary fields in both the $\mathfrak{g}_k$- and the $\widetilde{\mathfrak{g}}_{\tilde{k}}$-WZW model, there is a finite number of branching rules, and thus a finite number of coset primary fields.

6.2.4 Modular invariants of $\widehat{\mathfrak{u}}(1)$

We now consider the affine Lie algebra $\mathfrak{u}(1)_k$, which is also called the *Heisenberg algebra*. This affine Lie algebra is generated by modes J_n for $n \in \mathbb{Z}$ and L_0 satisfying the commutation relations

$$[J_m, J_n] = km\delta_{m+n,0} \quad \text{and} \quad [L_0, J_n] = -nJ_n\,.$$

Note that by redefinition of the generators, we can assume without loss of generality that $k = 1$; this is also why when considering the affine algebra $\widehat{\mathfrak{u}}(1)$ the specification of the level is often omitted. The module of the Heisenberg algebra is simply the Fock space of a free boson and it is specified by an highest weight state $|s\rangle$ for $s \in \mathbb{R}$ such that

$$J_0\,|s\rangle = s|s\rangle \quad \text{and} \quad J_n\,|s\rangle = 0 \ \ \forall\, n > 0\,.$$

Such modules are always irreducible, and the states are of the form

$$J_{-1}^{n_1}J_{-2}^{n_2}\dots|s\rangle = |s;n_1,n_2,\dots\rangle \quad \text{with } n_i \in \mathbb{N}\,,$$

which holds up to a normalisation constant and only finitely many n_i's are nonzero. Then

$$L_0\,|s\rangle = |s;n_1,n_2,\dots\rangle = \left(\frac{s^2}{2} + \sum_{i=0}^{\infty} n_i\right)|s;n_1,n_2,\dots\rangle\,,$$

and hence the number of states at fixed grade n is given by $p(n) \in \mathbb{N}$, the number of partitions of n. Also, since the dual Coxeter number of $\mathfrak{u}(1)$ is zero, this theory has central charge $c = 1$. Thus, the specialised character of the Heisemberg module is thus equal to the inverse of the Euler function times a factor related to the modular anomaly:

$$\chi_s(\tau) := q^{\frac{s^2}{2}-\frac{1}{24}} \sum_{n=0}^{\infty} p(n) q^n = \frac{q^{\frac{s^2}{2}}}{\eta(\tau)} .$$

Note that these characters are holomorphic in $\tau \in \mathbb{H}$ and integrable as functions of $s \in \mathbb{R}$. From this, we can compute the modular transformations

$$\begin{aligned} \chi_s(\tau+1) &= \mathrm{e}^{2\pi i\left(\frac{s^2}{2}-\frac{1}{24}\right)} \chi_s(\tau) , \\ \chi_s(-1/\tau) &= \int_{\mathbb{R}} dt\, \mathrm{e}^{-2\pi i st} \chi_t(\tau) , \end{aligned} \tag{158}$$

by using the modular behaviour of $\eta(\tau)$, see Appendix A, and the *Fresnel-integral*:

$$\int_{\mathbb{R}} \mathrm{e}^{iax^2+ibx} = \sqrt{\frac{i\pi}{a}} \mathrm{e}^{-i\frac{b^2}{4a}} \quad \forall\, a \in \mathbb{H},\ b \in \mathbb{C} .$$

From Eq. (158) which we read of the modular matrices

$$\mathcal{T}_{st} = \mathrm{e}^{2\pi i\left(\frac{s^2}{2}-\frac{1}{24}\right)} \delta(s-t) \quad \text{and} \quad \mathcal{S}_{st} = \mathrm{e}^{-2\pi i st} \tag{159}$$

for $s, t \in \mathbb{R}$. Note that we are in a more general setting than Eq. (142) since we have a continuous spectrum and the $\mathcal{T}$ and $\mathcal{S}$ matrices are now distribution valued functions of $(s,t) \in \mathbb{R}^2$. The matrices in Eq. (159) are still unitary in the sense that

$$\int_{\mathbb{R}} dt\, \mathcal{T}_{st} \mathcal{T}^{\dagger}_{tr} = \delta(s-r) \quad \text{and} \quad \int_{\mathbb{R}} dt\, \mathcal{S}_{st} \mathcal{S}^{\dagger}_{tr} = \delta(s-r) . \tag{160}$$

This suggests that a diagonal modular invariant should be applicable in some sense. Let $\bar{\tau}$ be the complex conjugate of $\tau \in \mathbb{H}$ and $\overline{\chi}_{\bar{s}}(\bar{\tau})$ for $\bar{s} \in \mathbb{R}$ be the affine character corresponding to the antiholomorphic copy of $\widehat{\mathfrak{u}}(1)$. Then, we consider a partition function of the form

$$\mathcal{Z}(\tau, \bar{\tau}) = \int_{\mathbb{R}} \int_{\mathbb{R}} ds d\bar{s}\, \chi_s(\tau) \mathcal{M}_{s\bar{s}} \overline{\chi}_{\bar{s}}(\bar{\tau}) ,$$

for some generalised mass matrix $\mathcal{M}_{s\bar{s}}$, a distribution valued function of $(s, \bar{s}) \in \mathbb{R}^2$. For the special case of a diagonal mass matrix

$$\mathcal{M}^{diag}_{s\bar{s}} := \delta(s-\bar{s}) ,$$

the associated partition function is

$$\mathcal{Z}_{diag}(\tau, \bar{\tau}) = \int_{\mathbb{R}} ds\, \chi_s(\tau) \overline{\chi}_s(\bar{\tau}) = \int_{\mathbb{R}} ds\, \frac{\mathrm{e}^{\pi i(\tau-\bar{\tau})s^2}}{|\eta(\tau)|^2} = \frac{1}{\sqrt{2\,\mathrm{Im}(\tau)}} \frac{1}{|\eta(\tau)|^2} , \tag{161}$$

which is exactly the partition function of a free boson. This reflects the fact that the modes of the conserved current of a free boson theory satisfy the affine $\widehat{\mathfrak{u}}(1)$ commutation rules. One can easily verify that Eq. (161) is indeed modular invariant, since

$$S\colon\ \mathrm{Im}(\tau) \mapsto \frac{\mathrm{Im}(\tau)}{|\tau|^2}\,.$$

In the field-theoretic setting, the boson can be compactified on a circle of radius $R \in \mathbb{R}_{>0}$ which causes windings that must be taken into account at the level of the partition function. In particular, the periodicity condition on the boson field imposed by the compactification constrains the J_0 and the $\bar{J}_0$ eigenvalues to

$$p_R(m,n) := \frac{m}{R} + \frac{Rn}{2} \quad \text{and} \quad \overline{p}_R(m,n) := \frac{m}{R} - \frac{Rn}{2} \quad \text{for } m,n \in \mathbb{Z} \tag{162}$$

respectively. Thus, the corresponding mass matrix is given by

$$\mathcal{M}^{(R)}_{s\bar{s}} := \sum_{m,n\in\mathbb{Z}} \delta(s - p_R(m,n))\delta(\bar{s} - \overline{p}_R(m,n))\,,$$

which gives the familiar partition function of a boson compactified on a circle of radius R:

$$\mathcal{Z}_R(\tau;\bar{\tau}) := \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{p_R(m,n)^2}{2}} \bar{q}^{\frac{\overline{p}_R(m,n)^2}{2}}\,. \tag{163}$$

One can check that Eq. (163) is modular invariant for every $R > 0$ by applying twice the Poisson summation formula Eq. (146). We show this with a *formal* computation using Eq. (158):

$$\begin{aligned}
\mathcal{Z}_R\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) &= \sum_{m,n\in\mathbb{Z}} \int_{\mathbb{R}}\int_{\mathbb{R}} dt d\bar{t}\, \mathrm{e}^{-2\pi i (t p_R(m,n) - \bar{t}\,\overline{p}_R(m,n))} \chi_t(\tau)\overline{\chi}_{\bar{t}}(\bar{\tau}) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} dt d\bar{t} \sum_{m,n\in\mathbb{Z}} \mathrm{e}^{2\pi i \left(m\frac{t-\bar{t}}{R} + nR\frac{t+\bar{t}}{2}\right)} \chi_t(\tau)\overline{\chi}_{\bar{t}}(\bar{\tau}) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} dt d\bar{t} \sum_{m,n\in\mathbb{Z}} \delta\left(\frac{t-\bar{t}}{R} - m\right) \delta\left(\frac{t+\bar{t}}{R} - n\right) \chi_t(\tau)\overline{\chi}_{\bar{t}}(\bar{\tau}) \\
&= \int_{\mathbb{R}} dt \sum_{m,n\in\mathbb{Z}} \delta\left(\frac{2t}{R} - \frac{2n}{R^2} - m\right) \chi_t(\tau)\overline{\chi}_{\frac{2n}{R}-t}(\bar{\tau}) \\
&= \sum_{m,n\in\mathbb{Z}} \chi_{p_R(n,m)}(\tau)\overline{\chi}_{\overline{p}_R(n,m)}(\bar{\tau}) = \mathcal{Z}_R(\tau,\bar{\tau})\,,
\end{aligned} \tag{164}$$

where for the third equality we used

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi i n x} = \sum_{m\in\mathbb{Z}} \delta(x-n) \quad \forall\, x \in \mathbb{R}\,. \tag{165}$$

This calculation is only formal since the interchange of summation and integration in the first equality of Eq. (164) is actually not allowed, since

$$\int_{\mathbb{R}}\int_{\mathbb{R}} dt d\bar{t}\, \left|\mathrm{e}^{-2\pi i(tp_R(m,n)-\bar{t}\bar{p}_R(m,n))}\chi_t(\tau)\overline{\chi}_{\bar{t}}(\bar{\tau})\right| = \frac{1}{\mathrm{Im}(\tau)}$$

is not summable over $\mathbb{Z}^2$, and hence the Fubini's theorem is not applicable.

We thus found that the $\widehat{\mathfrak{u}}(1)$ theory has at least two types of modular invariants: the diagonal invariant of Eq. (161) and the continuous family of Eq. (163) parametrised by a positive real number $R > 0$. Moreover, these invariants can be understood in a field-theoretic context. We point out that even if at the level of the affine Lie algebra the theory possess a continuous spectrum parametrised by $s \in \mathbb{R}$, once considering the compactness properties coming from the corresponding Lie group $\mathrm{U}(1) \cong S^1_R$ we obtain a spectrum which consists only of a discrete subset of these representations parametrised by two integers, see Eq. (162). This is similar to what we wish to happen for $\mathfrak{su}(2)_{-1}$, which has no integrable representations and a continuous spectrum. Nevertheless we expect the compactness of SU(2) to reduce the set of representations appearing in the spectrum to a discrete subset.

6.3 Characters of $\mathfrak{su}(2)_1$

We begin by recalling the character formulae of the integrable unitary representations of $\mathfrak{su}(2)_1$. Using the notation as in Example 6.1 and the conventions on the theta functions given in Appendix A, we have that

$$\chi_0^{(1)}(z;\tau) = \left[\frac{\vartheta_3(\frac{z+\nu}{2};\tau)\,\vartheta_3(\frac{z-\nu}{2};\tau)}{\eta(\tau)}\right]_{\nu=0} = \left[\frac{\vartheta_3(z;2\tau)\vartheta_3(\nu;2\tau)+\vartheta_2(z;2\tau)\vartheta_2(\nu;2\tau)}{\eta(\tau)}\right]_{\nu=0}, \tag{166}$$

and

$$\chi_{1/2}^{(1)}(z;\tau) = \left[\frac{\vartheta_2(\frac{z+\nu}{2};\tau)\,\vartheta_2(\frac{z-\nu}{2};\tau)}{\eta(\tau)}\right]_{\nu=0} = \left[\frac{\vartheta_2(z;2\tau)\vartheta_3(\nu;2\tau)+\vartheta_3(z;2\tau)\vartheta_2(\nu;2\tau)}{\eta(\tau)}\right]_{\nu=0}, \tag{167}$$

where $[\cdot]_{\nu=r}$ extracts the coefficient of the factor $(e^{2\pi i\nu})^r$. We remark that for $\mathfrak{su}(2)_1$, the free field realisation in terms of two complex fermions actually yields[15]

$$\mathfrak{so}(4)_1 \cong \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_1\,. \tag{168}$$

[15]This reflects the fact that the fermions naturally lead to $\mathfrak{u}(2)_1 \cong \mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$ and the $\mathfrak{u}(1)$ current V can actually be extended to another $\mathfrak{su}(2)_1$ algebra - we can equivalently think of this construction in terms of four real fermions generating $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ - by considering the charged generators $\tilde{K}^+ = \psi^1\,\psi^2$ and $\tilde{K}^- = \psi_1^\dagger\,\psi_2^\dagger$.

This then accounts for the full central charge of $c = 2$ coming from four real fermions, each contributing $c = \frac{1}{2}$. The representations of $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_1$ are also naturally described in this language: in the NS sector we have

$$\mathcal{H}^{(1)}_{NS} \cong \left(\mathcal{H}^{(1)}_{\ell=0} \otimes \mathcal{H}^{(1)}_{\ell=0}\right) \oplus \left(\mathcal{H}^{(1)}_{\ell=1/2} \otimes \mathcal{H}^{(1)}_{\ell=1/2}\right) , \tag{169}$$

where $\mathcal{H}^{(1)}_{\ell}$ denotes the spin ℓ irreducible $\mathfrak{su}(2)_1$-module. Note that Eq. (169) is equivalent to the second equality in Eq. (166). On the other hand, the R sector leads to

$$\mathcal{H}^{(1)}_{R} \cong \left(\mathcal{H}^{(1)}_{\ell=1/2} \otimes \mathcal{H}^{(1)}_{\ell=0}\right) \oplus \left(\mathcal{H}^{(1)}_{\ell=0} \otimes \mathcal{H}^{(1)}_{\ell=1/2}\right) , \tag{170}$$

which is equivalent to the second equality in Eq. (167). The modular-behaviour of these characters is well-known and encoded in the modular T- and S-matrices. Note that at level $k = 1$ the ordered set $(\chi^{(1)}_0, \chi^{(1)}_{1/2})$ constitutes a basis for the set of unitary irreducible representations of $\mathfrak{su}(2)_1$. With respect to this basis, the modular transformations are

$$\mathcal{T}^{(1)} = \begin{pmatrix} \mathrm{e}^{-\frac{\pi i}{12}} & 0 \\ 0 & \mathrm{e}^{-\frac{5\pi i}{12}} \end{pmatrix} ,$$

and

$$\mathcal{S}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} ,$$

which are both symmetric and unitary, see Example 6.1.

6.4 Characters of $\mathfrak{su}(2)_{-1}$

In relation to Remark 6.2, we point out that the $\mathfrak{su}(2)_{-1}$ theory, in addition to being non-integrable it does not even belong to the realm of admissible fractional level WZW models. Indeed, as we will see, there are infinitely many admissible representations, whose characters present analogous convergence issues as in [Rid09], so we expect the discussion of Remark 6.2 to at least partially apply also to this setting. In particular, even though there are some formal similarities between the modular transformations presented in Section 6.2.4 and those of $\mathfrak{su}(2)_{-1}$, the author was not able to find a partition function for this theory that is at least *formally* modular invariant. Moreover, even if such exists, there would be probably technical difficulties in interpreting it as a *physical* invariant, see Remark 6.2.

6.4.1 Singular vectors in the Verma modules

We begin by analyzing the structure of the $\mathfrak{su}(2)_{-1}$-Verma module with highest weight state $|j\rangle$ of $\mathfrak{su}(2)$-spin $j \in \mathbb{R}$, where we allow all the $\mathfrak{su}(2)$-representations presented in

Section 3.2. In particular, we prove that there are no singular vectors except for the discrete highest weight representations of spin $j \in \frac{1}{2}\mathbb{Z}_{<0}$ and for the lowest weight discrete representation of spin $j \in \frac{1}{2}\mathbb{Z}_{>0}$, where in both cases the singular submodule is generated by a single vector. In turn, this implies that also the continuous representations with $j \in \frac{1}{2}\mathbb{N}$ and $\lambda - j \in \mathbb{Z}$ contain singular vectors.

That the vacuum Verma module is free of singular vectors can be elegantly argued from Eq. (193) through the following reasoning: if it contained a non-trivial null vector $\mathcal{N}$, then by Theorem 4.1 the vertex operator associated to it is zero, namely $V(\mathcal{N}, z) = 0$ for every $z \in \mathbb{C}$. In particular, the zero mode $V_0(\mathcal{N})$, applied on any highest-weight state $|j\rangle$ of the $\mathfrak{su}(2)_{-1}$-theory vanishes, that is

$$V_0(\mathcal{N})\,|j\rangle = P(j)\,|j\rangle = 0\,, \tag{171}$$

where $P(j)$ is a polynomial in j. This follows from the fact that $V_0(\mathcal{N})$ applied on $|j\rangle$ has grade zero and hence it can be obtained from $|j\rangle$ by application of $\mathfrak{su}(2)_{-1}$-zero modes and using the commutator rules it can be expressed as a polynomial in j and the conformal dimension $h_j = C_j$ of $|j\rangle$, which is equal to the Casimir C_j of the spin j $\mathfrak{su}(2)$-representation in which $|j\rangle$ transforms, see Eq. (28). Since $P(j)$ posses finitely many roots, we conclude that the existence of a null vector would restrict the set of allowed representations to a finite subset of spins j, corresponding to the roots of $P(j)$. However, this would contradict Eq. (193), which is an admissible representation of $\mathfrak{su}(2)_{-1}$ containing infinitely many j's. It thus follows that there is no null vector in the vacuum Verma module of $\mathfrak{su}(2)_{-1}$.

We now consider the Verma module corresponding to the highest weight representation of generic spin $j \in \mathbb{R}$ of $\mathfrak{su}(2)_{-1}$. We start by recalling our conventions. For completeness, we report here the commutation relations of the affine algebra $\mathfrak{su}(2)_k$,

$$\begin{aligned}
[J^3_m, J^3_n] &= \frac{k}{2} m \delta_{m+n,0}\,, \\
[J^3_m, J^\pm_n] &= \pm J^\pm_{m+n}\,, \\
[J^+_m, J^-_n] &= 2J^3_{m+n} + km\delta_{m+n,0}\,.
\end{aligned}$$

The automorphisms of $\mathfrak{su}(2)_k$ which preserve the Cartan subalgebra are generated by the **conjugation** automorphism τ and the **spectral flow** automorphism σ, which we already defined above through the free field representation. These automorphisms leave the level k invariant and their action is given by

$$\begin{aligned}
&\tau(J^\pm_n) = J^\mp_n\,, && \tau(J^3_n) = -J^3_n\,, && \tau(L_0) = L_0\,, \\
&\sigma^w(J^\pm_n) = J^\pm_{n\pm w}\,, && \sigma^w(J^3_n) = J^3_n + \frac{k}{2} w \delta_{n,0}\,, && \sigma(L_n) = L_n + wJ^3_n + \frac{k}{4} w^2 \delta_{n,0}\,.
\end{aligned} \tag{172}$$

Note that $\tau\sigma = \sigma^{-1}\tau$ and according to the convention introduced in 5.3, the module $\tau(\mathcal{H})$ is the conjugate of $\mathcal{H}$.

We proceed with the singular vector analysis. It will be convenient to denote by $\lambda = 2j$ the $\mathfrak{su}(2)$-weight of the affine weights, where j is the usual spin. As we have seen in Section 2.3, at fixed level k an affine $\mathfrak{su}(2)_k$-weight $\hat{\lambda} = (\lambda, k, h_\lambda)$ is completely determined by the finite $\mathfrak{su}(2)$-weight λ, since its conformal dimension is given by

$$h_\lambda = \frac{\lambda(\lambda+2)}{4(k+2)} = \frac{j(j+1)}{k+2}\,,$$

and similarly for a lowest weight by the replacement $\lambda \mapsto -\lambda$. In the following we will focus on the case $k = -1$, which is non-integrable, that is, it contains only non-unitary affine representations. Hence, the null vector relations Eq. (7), namely Eq. (30), do not hold and we shall instead use the Kac-Kazhdan determinant formula, Eq. (8). For a highest weight $\mathfrak{su}(2)_k$-module with highest weight state $|\lambda\rangle$, this takes the form [Rid09]

$$\begin{aligned}\det{}_\lambda(\mu, m) = \prod_{l=1}^{\infty}\Bigg\{&(\lambda+1+l)^{P(-\mu+2l,\,m)}\prod_{n=1}^{\infty}(\lambda+1+n(k+2)-l)^{P(-\mu+2l,\,m-nl)}\\ &\cdot(-\lambda-1+n(k+2)-l)^{P(-\mu-2l,\,m-nl)}\,(n(k+2))^{P(-\mu,m-nl)}\Bigg\}\,,\end{aligned} \tag{173}$$

where $P(\mu, m)$ denotes the multiplicity of $(\mu, 0, m) \in \Omega_{\hat{\lambda}}$ in the vacuum Verma module (this is independent of k). The presence of a singular vector in the Verma module to λ is signalled by the vanishing of one of the factors appearing in this formula **and** the vanishing of the arguments of the function P occurring in the corresponding exponent. Moreover, recall that if a weight is singular, than the null vector of that weight is unique up to normalisation [KK79].

We now specialise to $k = -1$ and we omit the k-label in the specification of weights. Note that this theory has central charge $c = -3$, see Eq. (17). The Verma module to $\lambda = 2j$ is characterised by an highest weight state $|j\rangle$ satisfying Eq. (29) with $k = -1$. We than see that Eq. (173) vanishes when

$$l = \lambda+1\,, \qquad l = \lambda+1+n\,, \qquad l = -\lambda-1+n \quad \text{with } n \in \mathbb{Z}_{>0}\,. \tag{174}$$

Since $l \in \mathbb{Z}_{>0}$, we see that the first equation has a solution if and only if $\lambda \in \mathbb{N}$; moreover, a necessary condition for the other two equations two have a solution is $\lambda \in \mathbb{Z}$. We distinguish between two cases: $\lambda \in \mathbb{N}$ and $\lambda \in \mathbb{Z}_{<0}$.

For $\lambda \in \mathbb{N}$ the first equation always has a solution and the arguments of P in the corresponding exponent vanish if $\mu = 2l = 2(\lambda+1)$ and $m = 0$, indicating that the singular vector has weight $(-2(\lambda+1), h_\lambda)$, corresponding to the state $(J_0^-)^{2j+1}|j\rangle$, which means that the state $|j\rangle$ transforms in the finite-dimensional spin j representations H_j of $\mathfrak{su}(2)$.

By repeating the same reasoning for the other two equations one finds correspondingly the singular weights

$$\begin{aligned}&(-\lambda-2m, h_\lambda+m(m-\lambda-1)) \quad \text{for} \quad m \geq \max\{1, \lambda+2\}, \\ &(\lambda+2m, h_\lambda+m(m+\lambda+1)) \quad \text{for} \quad m \geq \max\{1, -\lambda\}\,,\end{aligned} \tag{175}$$

which is true for every $\lambda \in \mathbb{Z}$. These seem at first sight to be additional singular vectors, however they all actually belong to the submodule generated by the singular vector $(-2(\lambda+1), h_\lambda)$. Indeed, by repeating the above Kac-Kazhdan analysis for $(-2(\lambda+1), h_\lambda)$ we find exactly the singular vectors of Eq. (175). As a consistency check, one can also repeat the analysis for all the singular weights in Eq. (175) and find that the so obtained singular vectors are are again of the form as in Eq. (175), which confirms that they all lie in the same singular submodule. From this we conclude, that the Verma module to **highest** weight λ is irreducible for all $\lambda \in \mathbb{R} \setminus \mathbb{Z}_{<0}$ if we naturally supply the definition of the module by requiring that for $\lambda = 2j \in \mathbb{N}$ the highest-weight state also satisfies

$$(J_0^-)^{2j+1}\,|j\rangle = 0\,,$$

that is, that $|j\rangle$ lies in the finite dimensional $\mathfrak{su}(2)$-representation of dimension $2j+1$.

We now consider the case where $\lambda \in \mathbb{Z}_{<0}$. It follows that the first equation in Eq. (174) has no solution, which in turn implies that there is no singular vector at level zero (where by level we refer to the shifted eigenvalue of $L_0 - h_\lambda$). Indeed, this is true for all $\lambda \notin \mathbb{N}$, meaning that the corresponding highest weight states lie in a (semi)-infinite dimensional representation of $\mathfrak{su}(2)$, namely D_j^+. The first (with lowest level) singular vector obtained from Eq. (175), is $(-\lambda, h_{-\lambda}) = (-\lambda, h_\lambda - \lambda)$ corresponding to the state

$$(J_{-1}^+)^{-2j}\,|j\rangle\,, \tag{176}$$

which one can explicitly compute to be a highest weight state, that is singular, by using the commutation relations. Moreover, this singular vector generates a submodule that contains all the others of Eq. (175), as one can confirm by repeating the Kac-Kazhdan analysis for the weight $(-\lambda, h_\lambda - \lambda)$. It follows that for $\lambda = 2j \in \mathbb{Z}_{<0}$ the quotient of the Verma module to λ by the submodule generated by the vector Eq. (176) is irreducible.

Moreover, note that $D_j^- = \tau(D_{-j}^+)$, where τ denotes the restriction of the conjugation automorphism on the finite $\mathfrak{su}(2)$ subalgebra, and the analogous relation holds for the corresponding affine $\mathfrak{su}(2)_{-1}$-modules. Hence, from Eq. (176) we infer that the singular submodule of the Verma module with highest weight states transforming in the D_j^- representation for $j \in \frac{1}{2}\mathbb{Z}_{>0}$ is generated by the singular vector

$$(J_{-1}^-)^{2j}\,|j\rangle\,. \tag{177}$$

6.4.2 The affine characters

We can now compute the characters of the $\mathfrak{su}(2)_{-1}$ highest weight representations, which as we have seen are characterised by the spin $j \in \mathbb{R}$ (and the continuous ones also by $\lambda \in \mathbb{R}/\mathbb{Z}$). We denote the chemical potential of $\mathfrak{su}(2)$ by $x := \mathrm{e}^{2\pi i t}$ for $t \in \mathbb{C}$. As we will see, all the character formulae can be written as meromorphic functions expanded on a specific convergence domain in the x-plane, so in general an $\mathfrak{su}(2)_{-1}$ character consists in such a function **and** the specification of the domain. In particular, following the discussion in Remark 6.2, we mainly work with character functions, unless differently stated.

By the null vector analysis, for the highest weight affine representation of spin $j \in \frac{1}{2}\mathbb{N}$ we have

$$\chi_j^{(-1)}(t;\tau) = q^{j(j+1)}\,\chi_j^{\mathfrak{su}(2)}(t)\,\chi_0^{(-1)}(t;\tau) = iq^{(j+\frac{1}{2})^2}\,\frac{x^{j+\frac{1}{2}} - x^{-j-\frac{1}{2}}}{\vartheta_1(t;\tau)}\,, \tag{178}$$

valid on the convergence region $|q| < |x| < |q|^{-1}$, where $\chi_j^{\mathfrak{su}(2)}(t)$ was defined in Eq. (139) and

$$\chi_0^{(-1)}(t;\tau) = q^{\frac{1}{8}}\frac{1}{\prod_{n\geq 1}(1-xq^n)(1-q^n)(1-x^{-1}q^n)}$$

is the character of the irreducible vacuum Verma module. Note that $\vartheta_1(t;\tau)$ has zeros at $x = q^n$ for all $n \in \mathbb{Z}$ and the vanishing of the denominator in Eq. (178) at $x = 1$ is compensated by the vanishing of the numerator, hence the convergence region can be extended from $1 < |x| < |q|^{-1}$ to $|q| < |x| < |q|^{-1}$, feature which is peculiar of characters of affine modules generated by finite-dimensional $\mathfrak{su}(2)$ representations. For the highest weight affine representations of spin $j \in \mathbb{R}\setminus\frac{1}{2}\mathbb{Z}$ we have

$$\chi_{j,+}^{(-1)}(t;\tau) = q^{\frac{1}{8}}\frac{q^{j(j+1)}\sum_{m\leq j} x^m}{\prod_{n\geq 1}(1-xq^n)(1-q^n)(1-x^{-1}q^n)} = \frac{iq^{(j+\frac{1}{2})^2}x^{j+\frac{1}{2}}}{\vartheta_1(t;\tau)}\,, \tag{179}$$

where the second equality holds for $1 < |x| < |q|^{-1}$, and the $+$-sign in the label of the character is there to recall that it corresponds to the highest weight discrete representation D_j^+. The character of the conjugate representation, namely that corresponding to a lowest weight discrete representation, is

$$\chi_{j,-}^{(-1)}(t;\tau) = q^{\frac{1}{8}}\frac{q^{j(j-1)}\sum_{m\geq j} x^m}{\prod_{n\geq 1}(1-xq^n)(1-q^n)(1-x^{-1}q^n)} = -\frac{iq^{(j-\frac{1}{2})^2}x^{j-\frac{1}{2}}}{\vartheta_1(t;\tau)}\,,$$

where the second equality holds on the convergence region $|q| < |x| < 1$. We point out that there is an identity

$$\chi_{j,+}^{(-1)}(t;\tau) = \chi_{-j,-}^{(-1)}(-t;\tau) \quad \forall\, j \in \mathbb{R}\setminus\tfrac{1}{2}\mathbb{Z}$$

as formal power series, whilst as meromorphic functions

$$\chi^{(-1)}_{j,+}(t;\tau) = -\chi^{(-1)}_{j+1,-}(t;\tau) \quad \forall\, j \in \mathbb{R} \setminus \tfrac{1}{2}\mathbb{Z}\,,$$

which at the level of characters is wrong since the two sides of the equation possess disjoint convergence regions. By the above null-vector analysis and in particular by Eq. (176), we can also express the character of the highest weight affine representation with $j \in \frac{1}{2}\mathbb{Z}_{<0}$, which is

$$\chi^{(-1)}_{j,+}(t;\tau) = \widetilde{\chi}^{(-1)}_{j,+}(t;\tau)(1 - q^{-2j}x^{-2j})\,, \tag{180}$$

where $\widetilde{\chi}^{(-1)}_{j,+}(t;\tau)$ is the character of Eq. (179) formally extended to $j \in \frac{1}{2}\mathbb{Z}_{<0}$. Note that since in Eq. (180) the singularity at $x = q^{-1}$ is removable, the convergence region can be extended from $1 < |x| < |q|^{-1}$ to $1 < |x| < |q|^{-2}$. Using Eq. (177), the corresponding result for the lowest weight affine representation with $j \in \frac{1}{2}\mathbb{Z}_{>0}$ is

$$\chi^{(-1)}_{j,-}(t;\tau) = \widetilde{\chi}^{(-1)}_{j,-}(t;\tau)(1 - q^{2j}x^{-2j})\,,$$

with extended convergence region $|q|^2 < |x| < 1$ after removing the singularity at $x = q$. Again, there is an identity of formal power series

$$\chi^{(-1)}_{j,+}(t;\tau) = \chi^{(-1)}_{-j,-}(-t;\tau) \quad \forall\, j \in \tfrac{1}{2}\mathbb{Z}_{<0}\,. \tag{181}$$

We now turn to the characters $\chi^{(-1)}_{j,\lambda}$ of the continuous representation C^λ_j for $j \in \mathbb{R}$ and $\lambda \in \mathbb{R}/\mathbb{Z}$. As we have seen in Section 3.2, for $j - \lambda \notin \mathbb{Z}$ this representation is irreducible, and by the Kac-Kazhdan analysis, the affine Verma module constructed out of it is free of singular vectors if $\lambda \neq 0, \frac{1}{2}$. For $j - \lambda \in \mathbb{Z}$ and $\lambda \neq 0, \frac{1}{2}$, the module C^λ_j is reducible and indecomposable but its Verma module is still free of singular vectors. We conclude that

$$\chi^{(-1)}_{j,\lambda}(t;\tau) = q^{\frac{1}{8}} \frac{q^{j(j-1)} \sum_{m\in\mathbb{Z}} x^{\lambda+m}}{\prod_{n\geq 1}(1-xq^n)(1-q^n)(1-x^{-1}q^n)} = \frac{q^{(j-\frac{1}{2})^2} \sum_{m\in\mathbb{Z}} x^{\lambda+m}}{\eta(\tau)^3} \qquad \text{if } \lambda \neq 0, \tfrac{1}{2}\,, \tag{182}$$

where we rearranged

$$\frac{\sum_{m\in\mathbb{Z}} x^m}{\prod_{n\geq 1}(1-xq^n)(1-x^{-1}q^n)} = \frac{q^{\frac{1}{12}} \sum_{m\in\mathbb{Z}} x^m}{\eta(\tau)^2}\,.$$

Note that Eq. (182) does not converge anywhere on the x-plane, so we have to consider this character as a formal power series. For $j \in \frac{1}{2}\mathbb{Z}$ and $\lambda = j \bmod \mathbb{Z}$, the situation is quite different. Since in this case j and $-j+1$ parameterise the same module, we restrict our attention to $j \in \frac{1}{2}\mathbb{N}$. Then, as we have seen in Section 3.2, the module C^j_j is indecomposable but its structure is given by

$$C_j^j \cong D^+_{-j-1} \oplus H_j \oplus D^-_{j+1} \qquad \forall\, j \in \tfrac{1}{2}\mathbb{N}\,. \tag{183}$$

This decomposition then translates at the level of affine $\mathfrak{su}(2)_{-1}$-modules, implying the presence of non-trivial singular vectors. Hence, we might compute the character of the corresponding affine module $\mathcal{C}_j^j$ in terms of those appearing in Eq. (183) by considering them as formal power series, since some of the character functions involved possess disjoint convergence regions.

We also note that the following identities hold true

$$\begin{aligned} \chi_j^{(-1)} &= \widetilde{\chi}_{j,+}^{(-1)} - \widetilde{\chi}_{-j-1,+}^{(-1)} = -\widetilde{\chi}_{j+1,-}^{(-1)} + \widetilde{\chi}_{-j,-}^{(-1)} && \forall\, j \in \tfrac{1}{2}\mathbb{N}\,, \\ \chi_{j,+}^{(-1)} &= \widetilde{\chi}_{j,+}^{(-1)} - \widetilde{\chi}_{-j,+}^{(-1)} && \forall\, j \in \tfrac{1}{2}\mathbb{Z}_{<0}\,, \\ \chi_{j,-}^{(-1)} &= \widetilde{\chi}_{j,-}^{(-1)} - \widetilde{\chi}_{-j,-}^{(-1)} && \forall\, j \in \tfrac{1}{2}\mathbb{Z}_{>0}\,, \end{aligned} \tag{184}$$

where the expressions are convergent on the regions dictated by the corresponding formulae specified above.

We now turn to the spectral flow σ defined in Eq. (172). For $k = -1$, we find the following isomorphisms

$$\begin{aligned} \sigma(\mathcal{H}_j) &\cong \mathcal{D}^+_{-j-\frac{1}{2}}\,, & \sigma^{-1}(\mathcal{H}_j) &\cong \mathcal{D}^-_{j+\frac{1}{2}} && \forall\, j \in \tfrac{1}{2}\mathbb{N}\,, \\ \sigma(\mathcal{D}^-_j) &\cong \mathcal{D}^+_{j-\frac{1}{2}}\,, & \sigma^{-1}(\mathcal{D}^+_j) &\cong \mathcal{D}^-_{j+\frac{1}{2}} && \forall\, j \notin \tfrac{1}{2}\mathbb{Z}\,. \end{aligned} \tag{185}$$

By Eq. (172), every $\mathfrak{su}(2)_{-1}$ character transforms as

$$\chi^{(-1)}_{\sigma^w(j)}(t;\tau) = x^{\frac{-w}{2}} q^{\frac{-w^2}{4}} \chi_j^{(-1)}(t + w\tau;\tau) \qquad \forall\, w \in \mathbb{Z}\,,$$

which has convergence region in the x-plane scaled by a factor $|q|^{-w}$. Using Eq. (230) one shows that the extended characters transform as

$$\widetilde{\chi}^{(-1)}_{\sigma^w(j),+} = (-1)^w\, \widetilde{\chi}^{(-1)}_{j+\frac{w}{2},+}\,, \qquad \widetilde{\chi}^{(-1)}_{\sigma^w(j),-} = (-1)^w\, \widetilde{\chi}^{(-1)}_{j+\frac{w}{2},-} \quad \forall\, j \in \mathbb{R}\,, \tag{186}$$

with convergence regions $|q|^{-w} < |x| < |q|^{-w-1}$ and $|q|^{-w+1} < |x| < |q|^{-w}$ respectively. Since $\widetilde{\chi}_j^{(-1)}$ for $j \in \mathbb{R}$ formally build a basis for all the characters, see Eq. (184), from Eq. (186) we can also deduce the character of all the spectrally flowed modules. When forgetting about the convergence domain of the characters and treating them as meromorphic functions, we derive recursive relations similar to those in [Rid09], namely

$$\begin{aligned} \chi^{(-1)}_{\sigma^{w+1}(j)} + \chi^{(-1)}_{\sigma^w(j+\frac{1}{2})} + \chi^{(-1)}_{\sigma^{w-1}(j)} + \chi^{(-1)}_{\sigma^w(j-\frac{1}{2})} &= 0 \quad \forall j \in \tfrac{1}{2}\mathbb{Z}_{>0}\,, \\ \chi^{(-1)}_{\sigma^w(j)} + \chi^{(-1)}_{\sigma^{w+1}(j+1)} + \chi^{(-1)}_{\sigma^w(j+1)} + \chi^{(-1)}_{\sigma^{w-1}(j+\frac{1}{2})} &= 0 \quad \forall j \in \tfrac{1}{2}\mathbb{N}\,, \end{aligned}$$

for every $w \in \mathbb{Z}$. These shows explicitly that the association of modules to character functions is not injective and it allows to identify the kernel of this map. It could be interesting to find all the linearly independent character functions, namely to identify the Grothendieck ring of characters. From that, it could be possible to construct a modular invariant partition function, similar in spirit to the Kac-Wakimoto invariant [KW88].

Inspired by [HHRS91], we look if the sums over all spectrally flowed versions of a character at fixed spin yields a useful basis of the Grothendieck ring for considering modular transformations. Thus, using Eq. (186) we compute

$$\sum_{w\in\mathbb{Z}} \widetilde{\chi}^{(-1)}_{\sigma^w(j),+}(t;\tau) = \frac{-i\, q^{j^2} x^j \vartheta_4(t/2+j\tau;\tau/2)}{\vartheta_1(t;\tau)} = \frac{i\, q^{j^2} x^j \big(\vartheta_2(t+2j\tau;2\tau) - \vartheta_3(t+2j\tau;2\tau)\big)}{\vartheta_1(t;\tau)}\,, \tag{187}$$

which for $j \in \frac{1}{2}\mathbb{Z}$ further simplifies to

$$\sum_{w\in\mathbb{Z}} \widetilde{\chi}^{(-1)}_{\sigma^w(j),+}(t;\tau) = i\,(-1)^{2j}\, \frac{\vartheta_2(t;2\tau) - \vartheta_3(t;2\tau)}{\vartheta_1(t;\tau)}\,,$$

where we used Eq. (231). It follows that

$$\begin{aligned} \sum_{w\in\mathbb{Z}} \chi^{(-1)}_{\sigma^w(j)}(t;\tau) &= 0 \qquad \forall\, j \in \tfrac{1}{2}\mathbb{N}\,, \\ \sum_{w\in\mathbb{Z}} \chi^{(-1)}_{\sigma^w(j),+}(t;\tau) &= 0 \qquad \forall\, j \in \tfrac{1}{2}\mathbb{Z}_{<0}\,, \end{aligned} \tag{188}$$

which shows that such sums are not of interest, if the characters are considered as meromorphic functions. At the level of characters and not character functions, Eq. (187) involves a sum of terms with disjoint convergence domains, so we need to be careful in interpreting Eq. (188).

We can compute the modular transformations to be

$$\begin{aligned} \widetilde{\chi}^{(-1)}_{j,\pm}(t;\tau+1) &= \mathrm{e}^{2\pi i\left(j(j\pm1)+\frac{1}{8}\right)}\, \widetilde{\chi}^{(-1)}_{j,\pm}(t;\tau)\,, \\ \widetilde{\chi}^{(-1)}_{j,\pm}\left(\frac{t}{\tau};-\frac{1}{\tau}\right) &= \mathrm{e}^{-\frac{\pi i t^2}{2\tau}} \int_{\mathbb{R}} ds \sqrt{2}i\, \mathrm{e}^{-\pi i(2s\pm1)(2j\pm1)}\, \widetilde{\chi}^{(-1)}_{s,\pm}(t;\tau)\,. \end{aligned} \tag{189}$$

Note the similarity with the generalised modular matrices of the $\widehat{\mathfrak{u}}(1)$-theory, see Eq. (159). In particular, we have that

$$\mathcal{T}_{js} = \mathrm{e}^{2\pi i\left(j(j\pm1)+\frac{1}{8}\right)}\, \delta(j-s) \quad \text{and} \quad \mathcal{S}_{js} = \sqrt{2i}\, \mathrm{e}^{-\pi i(2s\pm1)(2j\pm1)}\,, \tag{190}$$

which satisfy the unitarity relations of Eq. (160). One could then attempt to construct a "naive" diagonal modular invariant analogous to Eq. (161), and this yields

$$\mathcal{Z}_{diag.\,naive}(t,\bar{t};\tau,\bar{\tau}) := \int dj\, \widetilde{\chi}^{(-1)}_{j,+}(t;\tau)\widetilde{\chi}^{(-1)}_{j,+}(\bar{t};\bar{\tau}) = \frac{1}{2}\frac{1}{\sqrt{\mathrm{Im}(\tau)}}\frac{\mathrm{e}^{2\pi\frac{\mathrm{Im}(t)^2}{\mathrm{Im}(\tau)}}}{|\vartheta(t;\tau)|^2}\,,$$

which interestingly agrees with the contribution of the discrete representations D^+_j to the partition function of the SL$(2,\mathbb{R})$ WZW model at level $k>2$, whose allowed range of spins is $1/2<j<(k-1)/2$ [MO01]. This probably correspond to an analogous contribution of the discrete representations D^+_j of $\mathfrak{su}(2)_{-1}$ on the range $-1/2<j<0$, which combines with the spectral flow sum to give the range $j\in\mathbb{R}\setminus\frac{1}{2}\mathbb{Z}$ at the level of character functions, see 186. We could have expected some similarity in such contribution since

$$\mathfrak{su}(2)_{-1}\cong\mathfrak{sl}(2,\mathbb{R})_1\,.$$

However, since the corresponding Lie group is compact, when considering $\mathfrak{su}(2)_{-1}$ we would like to construct a partition function containing only a discrete subset of the allowed representations, namely those with $j\in\frac{1}{2}\mathbb{Z}$. The author tried constructions analogous to that of Eq. (163), where the spins included in the partition function belong to a lattice in $\mathbb{R}^2$, but without success.

An other interesting *formal* computation that we report here, comes from Eq. (189) and it is

$$\chi^{(-1)}_{\sigma^w(j)}\left(\frac{t}{\tau};-\frac{1}{\tau}\right) = \mathrm{e}^{-\frac{\pi i t^2}{2\tau}}\int_{\mathbb{R}} ds\, 2\sqrt{2}\, e^{2\pi i s w}\, \sin(\pi(2s+1)(2j\pm 1))\, \widetilde{\chi}^{(-1)}_{s,\pm}(t;\tau) \qquad \forall\, j\in\tfrac{1}{2}\mathbb{N}\,, \tag{191}$$

which summing over $w\in\mathbb{Z}$ and using Eq. (165) yields

$$\sum_{w\in\mathbb{Z}}\chi^{(-1)}_{\sigma^w(j)}\left(\frac{t}{\tau};-\frac{1}{\tau}\right) = \mathrm{e}^{-\frac{\pi i t^2}{2\tau}}\sum_{j'\in\mathbb{N}} 2\sqrt{2}\,\sin(\pi(2j+1)(2j'+1))\,\chi^{(-1)}_{j'}(t;\tau) \qquad \forall\, j\in\tfrac{1}{2}\mathbb{N}\,.$$

This, together with Eq. (188), shows in particular that the summation over the spectral flow action does not commute with the modular S-transformation. We point out the apparent similarity of the modular S-matrix elements appearing in Eq. (191) and Section 6.4.2 with those of Eq. (151) for $k=-1$.

7 Free field characters

7.1 The free field characters of $\mathfrak{su}(2)_{-1}$

We recall that the affine characters of the integrable $\mathfrak{su}(2)_1$ modules can be easily extracted from the corresponding free field realisation in terms of two complex fermions. This is due to Eq. (168). On the other hand, the situation is quite different for $\mathfrak{su}(2)_{-1}$, since it is not possible to extend the $\mathfrak{u}(1)$ generator U_0 to another commuting $\mathfrak{su}(2)$ algebra as before because the analogues of $\tilde{K}^{\pm}$ in Footnote 15 do not commute with the generators $J^{\pm}$. More precisely, the bilinears formed by two pairs of symplectic bosons generate $\mathfrak{sp}(4)_1$, which contains $\mathfrak{su}(2)_{-1} \oplus \mathfrak{u}(1)_1$ as the U_0-uncharged subalgebra, but it is bigger since at the level of the zero modes

$$\mathfrak{sp}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathbf{3}_1 \oplus \overline{\mathbf{3}}_{-1}\,, \tag{192}$$

where $\mathbf{3}_1$ and $\overline{\mathbf{3}}_{-1}$ denote the 3-dimensional representation of $\mathfrak{su}(2)$ and its conjugate respectively, and the subscript labels the U_0-eigenvalue. The additional two terms in Eq. (192) are formed by the six independent bilinears $\lambda^\alpha\lambda^\beta$ and $\mu^\dagger_\alpha\mu^\dagger_\beta$ respectively, which generate an ideal of $\mathfrak{sp}(4)$ but not a subalgebra. Hence, there is no decomposition analogous to Eq. (168) and the relevant (U_0-uncharged) algebra associated to two pairs of symplectic bosons is $\mathfrak{u}(2)_{-1} \cong \mathfrak{su}(2)_{-1} \oplus \mathfrak{u}(1)$ which has central charge $c = -3+1 = -2$, in agreement with the central charge of the four symplectic bosons (each symplectic boson contributes $c = -\frac{1}{2}$).

As before, there are again two natural sectors: the NS sector in which the symplectic bosons are half-integer moded, as well as the R sector in which the symplectic bosons are integer moded. Because of Eq. (192), we expect the U_0-uncharged part of the free-field representations (both in the NS and R sectors) to be reducible with respect to $\mathfrak{su}(2)_{-1}$. Indeed, the analogous of Eq. (169) turns out to be

$$\mathcal{H}^{(-1)}_{NS} \cong \bigoplus_{j\in\mathbb{N}} \mathcal{H}^{(-1)}_j\,, \tag{193}$$

where $\mathcal{H}^{(-1)}_j$ denotes the spin-j irreducible representation of $\mathfrak{su}(2)_{-1}$, whose structure has been described in the previous section. Eq. (193) can be shown at the level of characters, namely

$$
\begin{aligned}
\chi_{NS}^{(-1)}(t;\tau) &= \left[q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n) \prod_{a,b=\pm\frac{1}{2}} \frac{1}{1-x^a y^b q^{n-\frac{1}{2}}} \right]_{\mu=0} \\
&= \left(q^{\frac{1}{8}} \prod_{n=1}^{\infty} \prod_{a=\pm 1,0} \frac{1}{1-x^a q^n} \right) \left(\sum_{j=0}^{\infty} \sum_{m=-j}^{j} x^m q^{j(j+1)} \right) \\
&= \chi_0^{(-1)}(t;\tau) \sum_{j=0}^{\infty} \chi_j^{\mathfrak{su}(2)}(t) q^{j(j+1)} \\
&= \sum_{j=0}^{\infty} \chi_j^{(-1)}(t;\tau)
\end{aligned}
\tag{194}
$$

where $y = e^{2\pi i \mu}$ is the chemical potential of Z_0[16]. This equation can also be expressed in terms of theta functions as

$$
\left[\frac{\eta(\tau)^3}{\vartheta_4(\frac{t+\mu}{2};\tau)\vartheta_4(\frac{t-\mu}{2};\tau)} \right]_{\mu=0} = \frac{q^{\frac{1}{4}}}{\vartheta_1(t;\tau)} \sum_{j=0}^{\infty} \sin((2j+1)\pi t) q^{j(j+1)} \,.
$$

For what concerns the R sector, the analogous of Eq. (170) turns out to be

$$
\mathcal{H}_R^{(-1)} \cong \bigoplus_{Z\in\frac{1}{2}\mathbb{Z}} \mathcal{H}_{R,Z}^{(-1)} \quad \text{with} \quad \mathcal{H}_{R,\pm Z}^{(-1)} \cong q^{-Z(Z\pm 1)} \bigoplus_{j\in\mathbb{N}+Z} \mathcal{H}_j^{(-1)} \quad \text{for every } Z\in\tfrac{1}{2}\mathbb{N}\,, \tag{195}
$$

where we are considering the Ramond sector corresponding to $\mathcal{R}$ as defined in Eq. (111), and we denote by $\mathcal{H}_{R,Z}^{(-1)}$ the subrepresentation corresponding to fixed $Z_0 = Z$. The conjugate R sector $\overline{\mathcal{R}}$, defined by Eq. (117), yields the conjugate representations and we denote it by $\overline{\mathcal{H}}_R^{(-1)}$ and by $\overline{\mathcal{H}}_{R,Z}^{(-1)}$ the subrepresentation to fixed $Z_0 = Z$. We have an analogous decomposition for the conjugate R sector as

$$
\overline{\mathcal{H}}_R^{(-1)} \cong \bigoplus_{Z\in\frac{1}{2}\mathbb{Z}} \overline{\mathcal{H}}_{R,Z}^{(-1)} \quad \text{with} \quad \overline{\mathcal{H}}_{R,\mp Z}^{(-1)} \cong q^{-Z(Z\pm 1)} \bigoplus_{j\in\mathbb{N}+Z} \mathcal{H}_j^{(-1)} \quad \text{for every } Z\in\tfrac{1}{2}\mathbb{N}\,.
$$

One can deduce Eq. (195) at the level of characters:

$$
\begin{aligned}
\chi_R^{(-1)}(t,\mu;\tau) &= q^{-\frac{1}{8}} \sum_{m_1,m_2\in\frac{1}{2}\mathbb{N}} x^{m_2-m_1} y^{m_1+m_2} \prod_{n=1}^{\infty} (1-q^n) \prod_{a,b=\pm\frac{1}{2}} \frac{1}{1-x^a y^b q^n} \\
&= \sum_{s\in\frac{1}{2}\mathbb{N}} y^s \chi_s^{\mathfrak{su}(2)}(t) \frac{4\sin(\pi\frac{t+\mu}{2})\sin(\pi\frac{t-\mu}{2})\eta(\tau)^3}{\vartheta_1(\frac{t+\mu}{2};\tau)\vartheta_1(\frac{t-\mu}{2};\tau)} \\
&= \sum_{Z\in\frac{1}{2}\mathbb{Z}} y^Z \chi_{R,Z}^{(-1)}(t;\tau)\,,
\end{aligned}
\tag{196}
$$

[16]Note that the Z_0 has the same value as U_0 on the symplectic bosons, since they are V_0-uncharged

where we defined

$$\chi^{(-1)}_{R,\,Z}(t;\tau) := \left[\chi^{(-1)}_{R}(t,\mu;\tau)\right]_{\mu=Z} .$$

Then the following relation holds true:

$$q^{2Z}\chi^{(-1)}_{R,\,Z}(t;\tau) = \chi^{(-1)}_{R,\,-Z}(t;\tau) \quad \forall\, Z \in \tfrac{1}{2}\mathbb{N} . \tag{197}$$

Denoting by $\overline{\chi}^{(-1)}_{R}(t;\tau)$ the character of the conjugate R sector representation $\overline{\mathcal{R}}$ (notice that the bar does not mean complex conjugation), we have that

$$\overline{\chi}^{(-1)}_{R}(t,\mu;\tau) = \chi^{(-1)}_{R}(-t,-\mu;\tau) = \chi^{(-1)}_{R}(t,-\mu;\tau) ,$$

which yields

$$\overline{\chi}^{(-1)}_{R,\,-Z}(t;\tau) = \chi^{(-1)}_{R,\,Z}(t;\tau) \quad \forall\, Z \in \tfrac{1}{2}\mathbb{Z} .$$

Then, the key result is the following relation which holds for every $Z \in \frac{1}{2}\mathbb{N}$ and confirms Eq. (195):

$$q^{(Z+\frac{1}{2})^2}\,\chi^{(-1)}_{R,\,Z}(t;\tau) = \chi^{(-1)}_{0}(t;\tau)\sum_{j\in\mathbb{N}+Z}\chi^{\mathfrak{su}(2)}_{j}(t)\,q^{j(j+1)} = \sum_{j\in\mathbb{N}+Z}\chi^{(-1)}_{j}(t;\tau) , \tag{198}$$

and consequently

$$q^{(Z-\frac{1}{2})^2}\,\chi^{(-1)}_{R,\,-Z}(t;\tau) = \sum_{j\in\mathbb{N}+Z}\chi^{(-1)}_{j}(t;\tau) , \tag{199}$$

and the analogous statement for the conjugate representation follows directly. In particular, for $Z = 0$ we obtain

$$q^{\frac{1}{4}}\chi^{(-1)}_{R,\,0}(t;\tau) = \chi^{(-1)}_{NS}(t;\tau) .$$

Eq. (198) follows easily from this particular denominator identity for Lie superalgebras [KW94], which states that for $|q| < |u|\,, |v| < 1$ we have that

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-uvq^{n-1})(1-u^{-1}v^{-1}q^n)}{(1-uq^{n-1})(1-u^{-1}q^n)(1-vq^{n-1})(1-v^{-1}q^n)} = \left(\sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{-\infty}\right)u^m v^n q^{mn} , \tag{200}$$

and by substituting $u = x^{\frac{1}{2}}y^{-\frac{1}{2}}$, $v = x^{\frac{1}{2}}y^{\frac{1}{2}}$ it yields

$$\begin{aligned}\sum_{m,n=0}^{\infty} x^{\frac{n-m}{2}}y^{\frac{n+m}{2}}\prod_{n=1}^{\infty}&\frac{(1-q^n)^2(1-xq^n)(1-x^{-1}q^n)}{(1-x^{\frac{1}{2}}y^{-\frac{1}{2}}q^n)(1-x^{-\frac{1}{2}}y^{\frac{1}{2}}q^n)(1-x^{\frac{1}{2}}y^{\frac{1}{2}}q^n)(1-x^{-\frac{1}{2}}y^{-\frac{1}{2}}q^n)}\\ &= \left(\sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{-\infty}\right)\frac{x^{\frac{n+m+1}{2}}}{x-1}y^{\frac{m-n-1}{2}}q^{mn} ,\end{aligned}$$

which proves Eq. (198) and Eq. (197). Additionally, by substituting $u = q^{\frac{1}{2}}x^{\frac{1}{2}}y^{-\frac{1}{2}}$,

$v = q^{\frac{1}{2}} x^{\frac{1}{2}} y^{\frac{1}{2}}$ in Eq. (200) one can show that for every $Z \in \frac{1}{2}\mathbb{Z}$:

$$\left[\prod_{n=1}^{\infty} \prod_{a,b=\pm\frac{1}{2}} \frac{1}{1 - x^a\, y^b\, q^{n-\frac{1}{2}}}\right]_{\mu=Z} = q^Z \left[\sum_{s\in\frac{1}{2}\mathbb{N}} y^s \sum_{m=-s}^{s} x^m \prod_{n=1}^{\infty} \prod_{a,b=\pm\frac{1}{2}} \frac{1}{1 - x^a\, y^b\, q^n}\right]_{\mu=Z},$$

which for $Z = 0$, together with Eq. (198), proves Eq. (194).

A result similar to Eq. (198) holds for the character of the symplectic bosons in the R sector $\mathcal{R}^+$ defined by Eq. (121) which contained discrete representations of $\mathfrak{su}(2)$ to spin $j \in -\frac{1}{2}\mathbb{N}$. Indeed, the character is given by

$$\begin{aligned}
\chi^{(-1)}_{R+}(t,\mu;\tau) &= q^{-\frac{1}{8}} \sum_{m_1,m_2\in\frac{1}{2}\mathbb{N}} x^{-m_1-m_2-\frac{1}{2}} y^{m_1-m_2-\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^n) \prod_{a,b=\pm\frac{1}{2}} \frac{1}{1 - x^a\, y^b\, q^n} \\
&= x^{-\frac{1}{2}} y^{-\frac{1}{2}} \sum_{s\in\frac{1}{2}\mathbb{N}} x^{-s} \chi^{\mathfrak{su}(2)}_s(\mu) \frac{4\sin(\pi\frac{t+\mu}{2})\sin(\pi\frac{t-\mu}{2})\eta(\tau)^3}{\vartheta_1(\frac{t+\mu}{2};\tau)\vartheta_1(\frac{t-\mu}{2};\tau)} \\
&= \sum_{Z\in\frac{1}{2}\mathbb{Z}} y^Z\, \chi^{(-1)}_{R+,\,Z}(t;\tau)\,,
\end{aligned} \tag{201}$$

where as above we defined

$$\chi^{(-1)}_{R+,\,Z}(t;\tau) := \left[\chi^{(-1)}_{R+}(t,\mu;\tau)\right]_{\mu=Z}.$$

The key fact is that

$$\chi^{(-1)}_{R+,\,Z-\frac{1}{2}}(t;\tau) = \chi^{(-1)}_{R+,\,-Z-\frac{1}{2}}(t;\tau) \quad \forall\, Z \in \tfrac{1}{2}\mathbb{N}\,, \tag{202}$$

and that

$$q^{(Z+\frac{1}{2})^2}\, \chi^{(-1)}_{R+,\,Z}(t;\tau) = \sum_{j\in\mathbb{N}-Z} \chi^{(-1)}_{-j,+}(t;\tau) \quad \forall\, Z \in -\tfrac{1}{2}\mathbb{N}\,. \tag{203}$$

In particular, for $Z = 0$ one obtains

$$q^{\frac{1}{4}} \chi^{(-1)}_{R+,\,0}(t;\tau) = \sum_{j\in\mathbb{Z}_{<0}} \chi^{(-1)}_{j,+}(t;\tau)\,.$$

Eq. (202) and Eq. (203) can be proven again by applying Eq. (200) with the substitutions $u = x^{-\frac{1}{2}} y^{\frac{1}{2}}$ and $v = x^{-\frac{1}{2}} y^{-\frac{1}{2}}$; note that this changes the convergence region to

$1<|x|^{\frac{1}{2}}|y|^{-\frac{1}{2}},|x|^{\frac{1}{2}}|y|^{\frac{1}{2}}<|q|^{-1}$, which needs to be taken into account in order to obtain

$$\sum_{m,n=0}^{\infty} x^{-\frac{n+m+1}{2}} y^{\frac{n-m-1}{2}} \prod_{n=1}^{\infty} \frac{x^{\frac{1}{2}}(1-q^n)^2(1-xq^n)(1-x^{-1}q^{n-1})}{(1-x^{\frac{1}{2}}y^{-\frac{1}{2}}q^n)(1-x^{-\frac{1}{2}}y^{\frac{1}{2}}q^n)(1-x^{\frac{1}{2}}y^{\frac{1}{2}}q^n)(1-x^{-\frac{1}{2}}y^{-\frac{1}{2}}q^n)}$$
$$= \left(\sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{-\infty}\right) x^{-\frac{n+m}{2}} y^{\frac{m-n-1}{2}} q^{mn}\,.$$

Eq. (203) shows that

$$\mathcal{H}^{(-1)}_{R+} \cong \bigoplus_{Z\in\frac{1}{2}\mathbb{Z}} \mathcal{H}^{(-1)}_{R+,Z} \quad \text{with} \quad \mathcal{H}^{(-1)}_{R+,\pm Z-\frac{1}{2}} \cong q^{-Z^2} \bigoplus_{j\in\mathbb{N}+Z} \mathcal{D}^{+}_{-j-\frac{1}{2}} \quad \text{for every } Z\in\tfrac{1}{2}\mathbb{N}\,. \tag{204}$$

We can transfer these results for $\mathcal{R}^+$ to $\mathcal{R}^-$ defined by Eq. (117) by noticing that

$$\chi^{(-1)}_{R-}(t,\mu;\tau) = \chi^{(-1)}_{R+}(-t,-\mu;\tau)\,,$$

which implies that for the representations with fixed $Z\in\frac{1}{2}\mathbb{Z}$ we have

$$\chi^{(-1)}_{R-,Z}(t;\tau) = \chi^{(-1)}_{R+,-Z}(-t;\tau)\,.$$

Then, Eq. (197) translates into

$$\chi^{(-1)}_{R-,Z+\frac{1}{2}}(t;\tau) = \chi^{(-1)}_{R-,-Z+\frac{1}{2}}(t;\tau) \quad \forall\, Z\in\tfrac{1}{2}\mathbb{N}\,,$$

and we can use Eq. (181) to infer

$$q^{(Z-\frac{1}{2})^2}\chi^{(-1)}_{R-,Z}(t;\tau) = \sum_{j\in\mathbb{N}+Z} \chi^{(-1)}_{j,-}(t;\tau) \quad \forall\, Z\in\tfrac{1}{2}\mathbb{Z}_{>0}\,. \tag{205}$$

In particular, for $Z=0$ we obtain

$$q^{\frac{1}{4}}\chi^{(-1)}_{R+,0}(t;\tau) = \sum_{j\in\mathbb{Z}_{>0}} \chi^{(-1)}_{j,-}(t;\tau)\,.$$

We can write Eq. (205) as

$$\mathcal{H}^{(-1)}_{R-} \cong \bigoplus_{Z\in\frac{1}{2}\mathbb{Z}} \mathcal{H}^{(-1)}_{R-,Z} \;\text{ with }\; \mathcal{H}^{(-1)}_{R-,\pm Z+\frac{1}{2}} \cong q^{-Z^2} \bigoplus_{j\in\mathbb{N}+Z} \mathcal{D}^{-}_{j+\frac{1}{2}} \;\; \forall\, Z\in\tfrac{1}{2}\mathbb{N}\,.$$

Finally, the full affine character corresponding to the free field representation of four symplectic bosons defined in Eq. (111) and allowing $m_1, m_2 \in \frac{1}{2}\mathbb{Z}$ is given by

$$\begin{aligned}\chi^{(-1)}_{\widetilde{R}}(t,\mu;\tau) &= q^{-\frac{1}{8}} \sum_{m_1,m_2\in\frac{1}{2}\mathbb{Z}} x^{m_2-m_1} y^{m_1+m_2} \prod_{n=1}^{\infty}(1-q^n) \prod_{a,b=\pm\frac{1}{2}} \frac{1}{1-x^a y^b q^n} \\ &= \left(\sum_{Z,s\in\mathbb{Z}} + \sum_{Z,s\in\mathbb{Z}+\frac{1}{2}}\right) y^Z x^s \frac{1}{\eta(\tau)^3} = \sum_{Z\in\frac{1}{2}\mathbb{Z}} y^Z \chi^{(-1)}_{\widetilde{R},Z}(t;\tau)\end{aligned} \tag{206}$$

where

$$\chi^{(-1)}_{\widetilde{R},Z}(t;\tau) := \left[\chi^{(-1)}_{\widetilde{R}}(t,\mu;\tau)\right]_{\mu=Z} = \sum_{s\in\mathbb{Z}+Z} x^s \frac{1}{\eta(\tau)^3} \quad \forall\, Z \in \tfrac{1}{2}\mathbb{Z}\,. \tag{207}$$

7.2 The free field characters of $\mathfrak{psu}(2|2)_1$

Following the lines of [EGG19], we note that we can obtain $\mathfrak{psu}(2|2)_1$ from $\mathfrak{u}(2|2)_1$ by a coset construction. More explicitely, there is an isomorphism

$$\begin{aligned}\mathfrak{psu}(2|2)_1 &\cong \frac{\mathfrak{u}(2|2)_1}{\widehat{\mathfrak{u}}(1)_U \oplus \widehat{\mathfrak{u}}(1)_V} \\ &\cong \frac{\text{two pairs of symplectic bosons and two complex fermions}}{\widehat{\mathfrak{u}}(1)_U \oplus \widehat{\mathfrak{u}}(1)_V}\,,\end{aligned} \tag{208}$$

where $\widehat{\mathfrak{u}}(1)_U$ and $\widehat{\mathfrak{u}}(1)_V$ denote the affine algebra generated by the modes of U and V respectively. With the analogous notation for the affine algebra generated by Y and Z, we have the following short exact sequences

$$\begin{aligned}&0 \to \mathfrak{su}(2|2)_1 \to \mathfrak{u}(2|2)_1 \to \widehat{\mathfrak{u}}(1)_Y \to 0\,, \\ &0 \to \widehat{\mathfrak{u}}(1)_Z \to \mathfrak{su}(2|2)_1 \to \mathfrak{psu}(2|2)_1 \to 0\,,\end{aligned} \tag{209}$$

which give

$$\begin{aligned}\mathfrak{u}(2|2)_1 &\cong \mathfrak{su}(2|2)_1 \oplus \widehat{\mathfrak{u}}(1)_Y\,, \\ \mathfrak{su}(2|2)_1 &\cong \mathfrak{psu}(2|2)_1 \oplus \widehat{\mathfrak{u}}(1)_Z\,.\end{aligned}$$

The coset free field representations of Eq. (208) are labelled by

$$(\sigma^w_Y(\mathcal{E}); Y; Z) \tag{210}$$

where $\sigma^w_Y(\mathcal{E})$ denotes a σ^w_Y-spectrally flowed representation of two pairs of symplectic bosons. Since the supercharges are bilinear expressions of one symplectic boson and one fermion, and since we require them to be integer-moded, the moding of the symplectic bosons also fixes that of the fermions. Thus the fermions will be in the R sector if w is even and in the NS sector if w is odd. In particular, this thereby fixes the representations of the $\mathfrak{su}(2)_1$ algebra. Finally, Y and Z denote the eigenvalues of Y_0 and Z_0. With these conventions, the symplectic bosons and free fermions transform with respect to $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_Y \oplus \mathfrak{u}(1)_Z$ as

$$\begin{aligned} \text{symplectic bosons } (\lambda^\alpha, \mu^\dagger_\beta) \quad &: \quad (\overline{\mathbf{2}}, 1)_{-1,-1} \oplus (\mathbf{2}, 1)_{1,1} \,, \\ \text{fermions } (\psi^\alpha, \psi^\dagger_\beta) \quad &: \quad (1, \overline{\mathbf{2}})_{-1,1} \oplus (1, \mathbf{2})_{1,-1} \,. \end{aligned}$$

By Eq. (209), it follows that only the zero-charge sector of the central extension Z descends to a representation of $\mathfrak{psu}(2|2)_1$. Furthermore, in order to obtain complete $\mathfrak{psu}(2|2)_1$-representations, we have to sum over all Y_0-charges, because the fermionic modes carry charge with respect to $\mathfrak{u}(1)_Y$. On the other hand, for irreducible representations of $\mathfrak{su}(2|2)_1$ we have that $Z \in \frac{1}{2}\mathbb{Z}$ and we also have to sum over all allowed Y_0-charges, which are specified by the requirement that $Z - Y \in \mathbb{Z}$. Hence, we have the identifications

$$\begin{aligned} \mathcal{L} &\cong \bigoplus_{Y \in \mathbb{Z}} (\sigma^w_Y(\mathcal{K}); Y; 0) \,, \\ \mathcal{R}_Z &\cong \mathcal{Z} \otimes \bigoplus_{Z - Y \in \mathbb{Z}} (\sigma^w_Y(\mathcal{E}); Y; Z) \,, \\ \overline{\mathcal{R}}_{-Z} &\cong \mathcal{Z} \otimes \bigoplus_{Z - Y \in \mathbb{Z}} (\sigma^w_Y(\overline{\mathcal{E}}); Y; -Z) \,, \end{aligned} \tag{211}$$

for every $Z \in \frac{1}{2}\mathbb{N}$, where $\mathcal{K}$ denote the NS representation of two pairs of symplectic bosons, while $\mathcal{E}$ and $\overline{\mathcal{E}}$ denote the R sector defined by Eq. (111) and Eq. (117) respectively, and $\mathcal{Z}$ denotes the representation of a single free boson associated to the algebra $\widehat{\mathfrak{u}}(1)_Z$. Analogous isomorphisms also hold for the other R sectors defined by Eq. (121) and Eq. (126). Note that here we are considering the representations in the R sectors $\mathcal{R}_Z$ and $\overline{\mathcal{R}}_Z$ as $\mathfrak{su}(2|2)_1$ representations.

We can thus compute the characters of both the $\mathfrak{su}(2|2)_1$ and the $\mathfrak{psu}(2|2)_1$ representations using the free field realisation. We start with $\mathfrak{psu}(2|2)$ and in particular with the vacuum module $\mathcal{L}$, which arises in the NS sector. Its character is computed by multiplying the NS character of two complex fermions given by Eq. (166) with the NS character of two pairs of symplectic bosons given by Eq. (194), then extracting the coefficient to $Z_0 = 0$, summing over all $Y_0 \in \mathbb{Z}$ and dividing the result by the character of two free bosons, see Eq. (208). Concretely, we have that

$$\begin{aligned} \mathrm{ch}[(\mathcal{K}; Y; 0)](t, z; \tau) &= \eta(\tau)^2 \left[\frac{\vartheta_3(\frac{z+\nu-\mu}{2})\vartheta_3(\frac{z-\nu+\mu}{2})}{\vartheta_4(\frac{t+\nu+\mu}{2})\vartheta_4(\frac{t-\nu-\mu}{2})} \right]_{\nu=0, \mu=Y} \\ &= \begin{cases} \chi^{(1)}_0(z;\tau) \sum_{j \in \mathbb{N} + \frac{|Y|}{2}} \chi^{(-1)}_j(t;\tau) & \text{if } Y \in 2\mathbb{Z} \,, \\ \chi^{(1)}_{1/2}(z;\tau) \sum_{j \in \mathbb{N} + \frac{|Y|}{2}} \chi^{(-1)}_j(t;\tau) & \text{if } Y \in 2\mathbb{Z}+1 \,, \end{cases} \end{aligned}$$

where ν and μ denote the chemical potentials of Y_0 and Z_0 respectively. By Eq. (211) it follows that

$$
\begin{aligned}
\mathrm{ch}[\mathcal{L}](t,z;\tau) &= \sum_{Y\in\mathbb{Z}} \mathrm{ch}[(\mathcal{K};Y;0)](t,z;\tau) \\
&= \chi^{(1)}_0(z;\tau)\sum_{j\in\mathbb{N}}(2j+1)\chi^{(-1)}_j(t;\tau) + \chi^{(1)}_{1/2}(z;\tau)\sum_{j\in\mathbb{N}+\frac{1}{2}}(2j+1)\chi^{(-1)}_j(t;\tau) \\
&= \frac{\partial_t\vartheta_2(t;2\tau)\vartheta_3(z;2\tau)+\partial_t\vartheta_3(t;2\tau)\vartheta_2(z;2\tau)}{\pi\,\eta(\tau)\vartheta_1(t;\tau)} \\
&= \frac{\partial_t\left(\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)\right)}{\pi\,\eta(\tau)\vartheta_1(t;\tau)}\,.
\end{aligned}
\tag{212}
$$

The second line of Eq. (212) gives the decomposition of the vacuum module $\mathcal{L}$ in representations of the affine bosonic subalgebra, that is, the branching rules for the embedding $\mathfrak{su}(2)_{-1}\oplus\mathfrak{su}(2)_1\subset\mathfrak{psu}(2|2)_1$; this is depicted in Table 1. Such decomposition is due to

$L_0-\frac{1}{12}$									
0					$\left(0\,,\mathbf{1}\right)_0$				
1				$\left(\frac{1}{2}\,,\mathbf{2}\right)_{-1}$		$\left(\frac{1}{2}\,,\mathbf{2}\right)_1$			
2			$\left(1\,,\mathbf{1}\right)_{-2}$		$\left(1\,,\mathbf{1}\right)_0$		$\left(1\,,\mathbf{1}\right)_2$		
4		$\left(\frac{3}{2}\,,\mathbf{2}\right)_{-3}$		$\left(\frac{3}{2}\,,\mathbf{2}\right)_{-1}$		$\left(\frac{3}{2}\,,\mathbf{2}\right)_1$		$\left(\frac{3}{2}\,,\mathbf{2}\right)_3$	
6	$\iddots$		$\left(2\,,\mathbf{1}\right)_{-2}$		$\left(2\,,\mathbf{1}\right)_0$		$\left(2\,,\mathbf{1}\right)_2$		$\ddots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 1: Decomposition of the vacuum module $\mathcal{L}$ of $\mathfrak{psu}(2|2)_1$ in affine highest weight representations $(j,\mathbf{n})_Y$ of the bosonic subalgebra $\mathfrak{su}(2)_{-1}\oplus\mathfrak{su}(2)_1$, where the conformal dimensions are displayed on the left and Y indicates the Y_0-eigenvalue. Note that Y_0 is not part of the $\mathfrak{psu}(2|2)_1$ algebra, so the label Y is actually arbitrary is there to simply because the relative Y-value between the representations helps to understand the pattern of the fermionic modes in the module.

the fact that the action of negative fermionic modes $S^{\alpha\beta\gamma}_{-n}$ for $n\in\mathbb{N}$ generates highest weight states with respect to the affine bosonic subalgebra. For instance, one can check that as it is indicated in Table 1, the vectors $S^{\alpha\beta\pm}_{-1}|0\rangle$ are singular with respect to $\mathfrak{su}(2)_{-1}\oplus\mathfrak{su}(2)_1$ and generate the affine modules $\left(\frac{1}{2}\,,\mathbf{2}\right)_{-1}\oplus\left(\frac{1}{2}\,,\mathbf{2}\right)_1$. In general it seems that the pattern of these singular vectors is pretty complicated to deduce with a purely algebraic approach, mostly, as we will see, for the modules $\mathcal{R}_Z$ with $Z\neq 0$, hence we work at the level of characters.

We now wish to compute the spectrally flowed characters and the modular behaviour of Eq. (212). For that, we remark that in order to obtain good modular properties, we shall include a factor $(-1)^F$ in the definition of the character, where F is the fermion number, effectively replacing the character by the **supercharacter**. This modification has the concrete effect of substituting the plus sign in the second and third line of Eq. (212) with a minus sign, and thus substituting ϑ_2 to ϑ_1 in the numerator of the last line, that is,

$$\mathrm{sch}[\mathcal{L}](t,z;\tau)=\frac{\partial_t\left(\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)\right)}{\pi\,\eta(\tau)\vartheta_1(t;\tau)}\,. \tag{213}$$

Note that as for the characters of $\mathfrak{su}(2)_{-1}$, the presence of ϑ_1 in the denominator of Eq. (213) makes this affine character a meromorphic function on the x-plane, and its validity is bound to be true only on the convergence region specified by $|q|<|x|<|q|^{-1}$.

The non-holomorphicity of the character causes some technical issues when we compute the characters of the spectrally flowed vacuum module, since as in [Rid09], for different integer values of spectral flow the flowed characters generally possess disjoint convergence regions, see Remark 6.2. Indeed, using Eq. (230) and Eq. (231), as well as the spectral flow action Eq. (128), we compute

$$\mathrm{ch}[\sigma^w(\mathcal{L})](t,z;\tau)=(-1)^w\frac{-\pi i w\,\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)+\partial_t(\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau))}{\pi\,\eta(\tau)\vartheta_1(t;\tau)}\,, \tag{214}$$

which is valid only in the convergence region $|q|^{1-w}<|x|<|q|^{-w-1}$. Note that Eq. (214) has a convergence domain that intersects that of Eq. (213) only for the values $w=\pm1$. However, if we ignore the issue of the convergence regions, that is, if we consider the characters Eq. (214) as meromorphic functions of x, we obtain the following identities:

$$\begin{aligned}\mathrm{ch}[\mathcal{L}]&=\frac{(-1)^w}{2}\left(\mathrm{ch}[\sigma^w(\mathcal{L})]+\mathrm{ch}[\sigma^{-w}(\mathcal{L})]\right)\,,\\ \mathrm{ch}[\sigma^{w+1}(\mathcal{L})]&=3(-1)^w\mathrm{ch}[\mathcal{L}]+(-1)^w\mathrm{ch}[\sigma(\mathcal{L})]-\mathrm{ch}[\sigma^w(\mathcal{L})]\,,\end{aligned} \tag{215}$$

for every $w\in\mathbb{Z}$. Eq. (215) imply that there are only two independent character functions among all the spectrally flowed versions of the vacuum. This in turn implies that as in [Rid09] the association of $\mathfrak{psu}(2|2)_1$-modules with the corresponding character functions is not injective. Thus, all the discussion of Remark 6.2 is relevant also for the $\mathfrak{psu}(2|2)_1$ theory.

We can now make use of the simple modular behaviour of theta functions, see Appendix A, and compute

$$\mathrm{sch}[\mathcal{L}]\left(\frac{t}{\tau},\frac{z}{\tau};-\frac{1}{\tau}\right)=\mathrm{e}^{\frac{\pi i}{2\tau}(z^2-t^2)}\frac{\pi x\,\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)-i\tau\,\partial_t(\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau))}{\pi\,\eta(\tau)\vartheta_1(t;\tau)}\,, \tag{216}$$

where we recognize the prefactor $\mathrm{e}^{\frac{\pi i}{2\tau}(z^2-t^2)}$ coming from the general transformation property of weak Jacobi forms of index 1 and -1, and the second therm in the fraction which is again the vacuum character. It follows that the component of the modular S-matrix of $\mathfrak{psu}(2|2)_1$ relating the vacuum module to itself, is given by

$$\mathcal{S}_{\mathcal{L},\mathcal{L}}=i\tau\,, \tag{217}$$

and the explicit (linear) τ-dependence suggests that this model is an example of a logarithmic CFT.

More generally, the modular transformation of the spectrally flowed characters is

$$\mathrm{sch}[\sigma^w(\mathcal{L})]\left(\frac{t}{\tau},\frac{z}{\tau};-\frac{1}{\tau}\right)=$$
$$\mathrm{e}^{\frac{\pi i}{2\tau}(z^2-t^2)}(-1)^w\frac{\pi(w+x)\,\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)-i\tau\,\partial_t(\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau))}{\pi\,\eta(\tau)\vartheta_1(t;\tau)}\,,$$

if we treat the characters as meromorphic functions, that is, if we work at the level of character functions.

7.3 The free field characters of $\mathfrak{su}(2|2)_1$

We now look at the free field $\mathfrak{su}(2|2)_1$ characters. Similarly to as we did for $\mathcal{L}$, we first compute the full R character to fixed values Z and Y of Z_0 and Y_0. We begin with the R free field representation $\mathcal{R}$ which yields the representations $\mathcal{R}_Z$ for fixed $Z\in\frac{1}{2}\mathbb{N}$. We obtain

$$\begin{aligned}\eta(\tau)\,\mathrm{ch}[(\mathcal{E};Y;Z)](t,z;\tau)&=\left[\frac{\vartheta_2(\frac{z+\nu-\mu}{2})\vartheta_2(\frac{z-\nu+\mu}{2})}{\eta(\tau)}\sum_{j\in\frac{1}{2}\mathbb{Z}}\mathrm{e}^{2\pi i(\mu+\nu)(j+\frac{1}{2})}\chi^{(-1)}_{R,j}(t;\tau)\right]_{\nu=Z,\,\mu=Y}\\&=\begin{cases}q^{\frac{(Y-Z)^2}{4}}\,\chi^{(1)}_{1/2}(z;\tau)\,\chi^{(-1)}_{R,\,\frac{Y+Z-1}{2}}(t;\tau) & \text{if } Y-Z\in2\mathbb{Z}\,,\\ q^{\frac{(Y-Z)^2}{4}}\,\chi^{(1)}_{0}(z;\tau)\,\chi^{(-1)}_{R,\,\frac{Y+Z-1}{2}}(t;\tau) & \text{if } Y-Z\in2\mathbb{Z}+1\,,\end{cases}\end{aligned}\tag{218}$$

where the eta function on the left hand side comes from the $\widehat{\mathfrak{u}}(1)_Z$ factor generated by Z, see Eq. (209). Using Eq. (211) we compute

$$\begin{aligned}&\eta(\tau)\,\mathrm{ch}[\mathcal{R}_Z](t,z;\tau)=\sum_{Y\in\mathbb{Z}+Z}\mathrm{ch}[(\mathcal{E};Y;Z)](t,z;\tau)\\&=\chi^{(1)}_0(z;\tau)\sum_{n\in\mathbb{Z}}q^{(n+\frac{1}{2})^2}\chi^{(-1)}_{R,\,Z+n}(t;\tau)+\chi^{(1)}_{1/2}(z;\tau)\sum_{n\in\mathbb{Z}}q^{n^2}\chi^{(-1)}_{Z+n-\frac{1}{2}}(t;\tau)\\&=\begin{cases}q^{Z(Z-1)}\left[\chi^{(1)}_0\sum_{Y\in2\mathbb{Z}}q^{ZY}\sum_{j\in\mathbb{N}+\frac{|Y|}{2}}\chi^{(-1)}_j+\chi^{(1)}_{1/2}\sum_{Y\in2\mathbb{Z}+1}q^{ZY}\sum_{j\in\mathbb{N}+\frac{|Y|}{2}}\chi^{(-1)}_j\right]\\ q^{Z(Z-1)}\left[\chi^{(1)}_{1/2}\sum_{Y\in2\mathbb{Z}}q^{ZY}\sum_{j\in\mathbb{N}+\frac{|Y|}{2}}\chi^{(-1)}_j+\chi^{(1)}_0\sum_{Y\in2\mathbb{Z}+1}q^{ZY}\sum_{j\in\mathbb{N}+\frac{|Y|}{2}}\chi^{(-1)}_j\right]\end{cases}\\&=\begin{cases}q^{Z(Z-1)}\left[\chi^{(1)}_0\sum_{j\in\mathbb{N}}\left(\sum_{i=-j}^{j}q^{2Zi}\right)\chi^{(-1)}_j+\chi^{(1)}_{1/2}\sum_{j\in\mathbb{N}+\frac{1}{2}}\left(\sum_{i=-j}^{j}q^{2Zi}\right)\chi^{(-1)}_j\right]\\ q^{Z(Z-1)}\left[\chi^{(1)}_{1/2}\sum_{j\in\mathbb{N}}\left(\sum_{i=-j}^{j}q^{2Zi}\right)\chi^{(-1)}_j+\chi^{(1)}_0\sum_{j\in\mathbb{N}+\frac{1}{2}}\left(\sum_{i=-j}^{j}q^{2Zi}\right)\chi^{(-1)}_j\right]\,,\end{cases}\end{aligned}\tag{219}$$

where in the last two equations the top line holds if $Z\in\mathbb{N}$ and the bottom one if $Z\in\mathbb{N}+\frac{1}{2}$; also, in the last equation the sums in the round brackets run over integers whenever $j\in\mathbb{N}$ and half integers if $j\in\mathbb{N}+\frac{1}{2}$. Note that we dropped the arguments of all the functions in the last two equations for more readability. Then, the last equation explicitly gives the branching rules for the representation associated to the affine embedding of the affine bosonic subalgebra $\mathfrak{su}(2)_{-1} \oplus \mathfrak{su}(2)_1 \oplus \widehat{\mathfrak{u}}(1)_Z$. At this point, we clarify the fact that since the multiplets in Eq. (116) and in Eq. (120) differ only by the substitution $Z_0 \to -Z_0$ and $Y_0 \to -Y_0$ we have the following identities for every $Z \in \frac{1}{2}\mathbb{N}$:

$$\begin{aligned} \mathrm{ch}[(\mathcal{E}; Y; Z)](t, z; \tau) &= \mathrm{ch}[(\overline{\mathcal{E}}; -Y; -Z)](t, z; \tau)\,, \\ \mathrm{ch}[\mathcal{R}_Z](t, z; \tau) &= \mathrm{ch}[\mathcal{R}_{-Z}](t, z; \tau)\,. \end{aligned}$$

Unfortunately, the branching functions in Eq. (219) do not seem to allow for a simpler understanding of the modular behaviour of the affine $\mathfrak{su}(2|2)_1$-character, as it was for the vacuum $\mathcal{L}$. Only for $\mathcal{R}_0$, where one observes that Table 2 is the same as Table 1 but with an overall shifted Y_0-eigenvalue by 1 (which is irrelevant from the perspective of $\mathfrak{su}(2|2)_1$) and by quotienting out the affine $\widehat{\mathfrak{u}}(1)_Z$ factor. These relation is explicit also at level of

$L_0 - \frac{1}{24}$									
0					$(0\,,\mathbf{1})_1$				
1				$(\frac{1}{2}\,,\mathbf{2})_0$		$(\frac{1}{2}\,,\mathbf{2})_2$			
2			$(1\,,\mathbf{1})_{-1}$		$(1\,,\mathbf{1})_1$		$(1\,,\mathbf{1})_3$		
4		$(\frac{3}{2}\,,\mathbf{2})_{-2}$		$(\frac{3}{2}\,,\mathbf{2})_0$		$(\frac{3}{2}\,,\mathbf{2})_2$		$(\frac{3}{2}\,,\mathbf{2})_4$	
6	$\iddots$		$(2\,,\mathbf{1})_{-1}$		$(2\,,\mathbf{1})_1$		$(2\,,\mathbf{1})_3$		$\ddots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 2: Decomposition of the affine module $\mathcal{R}_0$ in affine highest weight representations of the bosonic subalgebra $\mathfrak{su}(2)_{-1} \oplus \mathfrak{su}(2)$. Note that we omitted the representations generated by the affine $\widehat{\mathfrak{u}}(1)_Z$, so each affine bosonic representation is actually tensored with the affine $\widehat{\mathfrak{u}}(1)$ module $\mathcal{Z}$. The conformal dimension is also different from that of Table 1, as the latter is the former shifted by the modular anomaly of a free boson, namely by $-1/24$.

characters, where

$$\mathrm{ch}[(\mathcal{E}, Y + 1, 0)] = \mathrm{ch}[(\mathcal{K}, Y, 0)] \quad \forall\, Y \in \mathbb{Z}\,, \tag{220}$$

or more generally

$$\mathrm{ch}[(\mathcal{E}, Y + 1 - Z, Z)] = q^{Z(Z-1-Y)}\,\mathrm{ch}[(\mathcal{K}, Y, 0)] \quad \forall\, Z \in \mathbb{N}\;\forall\, Y \in \mathbb{Z}\,, \tag{221}$$

from which follows that

$$\mathrm{ch}[(\mathcal{E}, Y - Z, Z)] = q^{Z(Z-Y)}\,\mathrm{ch}[(\mathcal{E}, Y, 0)] \quad \forall\, Z \in \mathbb{N}\;\forall\, Y \in \mathbb{Z}\,, \tag{222}$$

which is Eq. (133) at the level of characters. Then, when summed over $Y \in \mathbb{Z}$, Eq. (220) yields

$$\eta(\tau)\,\mathrm{ch}[\mathcal{R}_0] = \mathrm{ch}[\mathcal{L}]\,,$$

which is exactly Eq. (129). Similarly, summing Eq. (221) over $Y \in \mathbb{Z}+Z$ yields for every $Z \in \mathbb{N}$:

$$\eta(\tau)\,\mathrm{ch}[\mathcal{R}_Z](t,z:\tau) = q^{Z(Z-1)} \sum_{Y\in\mathbb{Z}} q^{ZY} \begin{cases} \chi_0^{(1)}(z;\tau)\, \sum_{j\in\mathbb{N}+\frac{|Y|}{2}} \chi_j^{(-1)}(t;\tau) & \text{if } Y \in 2\mathbb{Z}\,, \\ \chi_{1/2}^{(1)}(z;\tau)\, \sum_{j\in\mathbb{N}+\frac{|Y|}{2}} \chi_j^{(-1)}(t;\tau) & \text{if } Y \in 2\mathbb{Z}+1\,, \end{cases} \tag{223}$$

which agrees with Eq. (219). Eq. (223) allows to understand the affine bosonic structure of $\mathcal{R}_Z$ for $Z \in \mathbb{N}$ in relation to the structure of $\mathcal{L}$ (or $\mathcal{R}_0$) seen in Table 1. In particular, if we ignore for a moment the conformal dimensions at which each affine bosonic module appears, we note that the affine bosonic content of $\mathcal{R}_Z$ is the same as that of $\mathcal{R}_0$. By considering also the conformal dimensions, we obtain the structure of $\mathcal{R}_Z$, up to an overall difference in the conformal dimension, by taking that of $\mathcal{L}$ in Table 1 and "flowing to the right" of Z columns, weighting each column by a different conformal dimension. For instance, the decomposition of $\mathcal{R}_1$ is given in Table 3.

$L_0 - \frac{1}{24}$								
0			$\left(0\,,\mathbf{1}\right)_0$	$\left(\frac{1}{2}\,,\mathbf{2}\right)_1$	$\left(1\,,\mathbf{1}\right)_3$			
1						$\left(\frac{3}{2}\,,\mathbf{2}\right)_4$		
2		$\left(\frac{1}{2}\,,\mathbf{2}\right)_{-1}$	$\left(1\,,\mathbf{1}\right)_0$		$\left(2\,,\mathbf{1}\right)_3$		$\left(2\,,\mathbf{1}\right)_5$	
3				$\left(\frac{3}{2}\,,\mathbf{2}\right)_1$				
4	$\left(1\,,\mathbf{1}\right)_{-2}$	$\left(\frac{3}{2}\,,\mathbf{2}\right)_{-1}$						$\left(\frac{5}{2}\,,\mathbf{2}\right)_6$
5						$\left(\frac{5}{2}\,,\mathbf{2}\right)_4$		
6	$\iddots$		$\left(2\,,\mathbf{1}\right)_0$		$\left(3\,,\mathbf{1}\right)_3$			$\ddots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 3: Decomposition of the affine $\mathfrak{su}(2|2)_1$-module $\mathcal{R}_1$. As in Table 2, also here we omitted the overall tensor product with the affine $\widehat{\mathfrak{u}}(1)$-module $\mathcal{Z}$.

For $Z \in \mathbb{N}+\frac{1}{2}$ a similar reasoning is also true, but one also needs to exchange all the $\mathfrak{su}(2)_1$ representations, namely $\mathbf{1} \leftrightarrow \mathbf{2}$. For instance, the structure of $\mathcal{R}_{1/2}$ is shown in Table 4.

We wish to compute also the characters of the Ramond sectors $\mathcal{R}^{\pm}$. We denote by $\mathcal{E}^{\pm}$ the R sector representations of two pairs of symplectic bosons defined by Eq. (121) and Eq. (126) respectively; then $\mathrm{ch}[\mathcal{E}^{\pm};Y;Z](t,z;\tau)$ is equal to Eq. (218) up to the

$L_0-\frac{1}{24}$								
0				$(0\,,\mathbf{2})_0$	$(\frac{1}{2}\,,\mathbf{1})_1$			
1			$(\frac{1}{2}\,,\mathbf{1})_{-1}$			$(1\,,\mathbf{2})_2$		
2				$(1\,,\mathbf{2})_0$			$(\frac{3}{2}\,,\mathbf{1})_3$	
3		$(1\,,\mathbf{2})_{-2}$			$(\frac{3}{2}\,,\mathbf{1})_1$			
4			$(\frac{3}{2}\,,\mathbf{1})_{-1}$					$(2\,,\mathbf{2})_4$
5	$(\frac{3}{2}\,,\mathbf{1})_{-3}$					$(2\,,\mathbf{2})_2$		
6				$(2\,,\mathbf{2})_0$				
7	$\iddots$	$(2\,,\mathbf{2})_{-2}$					$(\frac{5}{2}\,,\mathbf{1})_3$	$\ddots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 4: Decomposition of the affine $\mathfrak{su}(2|2)_1$-module $\mathcal{R}_{1/2}$.

replacement $\chi^{(-1)}_{R,s} \to \chi^{(-1)}_{R\pm,s}$. More concretely,

$$
\begin{aligned}
\eta(\tau)\,\mathrm{ch}[\mathcal{R}^{\pm}_Z](t,z;\tau) &= \sum_{Y\in\mathbb{Z}+Z} \mathrm{ch}[(\mathcal{E}^{\pm};Y;Z)](t,z;\tau) \\
&= \begin{cases} q^{Z^2}\left[\chi^{(1)}_0 \sum_{Y\in 2\mathbb{Z}+1} q^{ZY} \sum_{j\in\mathbb{N}+\frac{|Y|}{2}} \chi^{(-1)}_{\mp(j+\frac{1}{2}),\pm} + \chi^{(1)}_{1/2} \sum_{Y\in 2\mathbb{Z}} q^{ZY} \sum_{j\in\mathbb{N}_0+\frac{|Y|}{2}} \chi^{(-1)}_{\mp(j+\frac{1}{2}),\pm}\right] \\ q^{Z^2}\left[\chi^{(1)}_{1/2} \sum_{Y\in 2\mathbb{Z}+1} q^{ZY} \sum_{j\in\mathbb{N}+\frac{|Y|}{2}} \chi^{(-1)}_{\mp(j+\frac{1}{2}),\pm} + \chi^{(1)}_0 \sum_{Y\in 2\mathbb{Z}} q^{ZY} \sum_{j\in\mathbb{N}+\frac{|Y|}{2}} \chi^{(-1)}_{\mp(j+\frac{1}{2}),\pm}\right] \end{cases} \\
&= \begin{cases} q^{Z^2}\left[\chi^{(1)}_0 \sum_{j\in\mathbb{N}+\frac{1}{2}} \left(\sum_{i=-j}^{j} q^{2Zi}\right) \chi^{(-1)}_{\mp(j+\frac{1}{2}),\pm} + \chi^{(1)}_{1/2} \sum_{j\in\mathbb{N}} \left(\sum_{i=-j}^{j} q^{2Zi}\right) \chi^{(-1)}_{\mp(j+\frac{1}{2}),\pm}\right] \\ q^{Z^2}\left[\chi^{(1)}_{1/2} \sum_{j\in\mathbb{N}+\frac{1}{2}} \left(\sum_{i=-j}^{j} q^{2Zi}\right) \chi^{(-1)}_{\mp(j+\frac{1}{2}),\pm} + \chi^{(1)}_0 \sum_{j\in\mathbb{N}} \left(\sum_{i=-j}^{j} q^{2Zi}\right) \chi^{(-1)}_{\mp(j+\frac{1}{2}),\pm}\right] \end{cases},
\end{aligned}
\tag{224}
$$

where in the last two equations the top line holds if $Z\in\mathbb{Z}$ and the bottom one if $Z\in\mathbb{Z}+\frac{1}{2}$; also the sums in the round brackets in the last equation run over integers whenever $j\in\mathbb{N}$ and half integers if $j\in\mathbb{N}+\frac{1}{2}$. Again, these are just the branching rules for the affine bosonic embedding $\mathfrak{su}(2)_{-1}\oplus\mathfrak{su}(2)_1\oplus\widehat{\mathfrak{u}}(1)_Z\subset\mathfrak{su}(2|2)_1$.

For the special case of $Z=0$, the representations $\mathcal{R}^{\pm}_0$ descend to representations $\mathcal{L}^{\pm}$ of $\mathfrak{psu}(2|2)_1$. Concretely, the module $\mathcal{L}^{\pm}$ is obtained from $\mathcal{R}^{\pm}_0$ by quotienting out all the states containing at least one mode Z_n with $n<0$, namely $\mathcal{R}^{\pm}_0\cong\mathcal{L}^{\pm}\otimes\mathcal{Z}$. At the level of the characters, this means that the character of $\mathcal{L}^{\pm}$ equals the character to $\mathcal{R}^{\pm}_0$ divided by the character of a free boson. Then, Eq. (224) can be written using Eq. (180) and Eq. (231) in terms of theta functions as follows:

$$
\begin{aligned}
\mathrm{ch}[\mathcal{L}^{\pm}](t,z;\tau) &= \chi^{(1)}_0(z;\tau)\sum_{j\in\mathbb{Z}_{>0}} 2j\chi^{(-1)}_{\mp j,\pm}(t;\tau) + \chi^{(1)}_{1/2}(z;\tau)\sum_{j\in\mathbb{N}}(2j+1)\chi^{(-1)}_{\mp(j+\frac{1}{2}),\pm}(t;\tau) \\
&= \frac{\vartheta_3(z;2\tau)(\pm\pi i\vartheta_2(t;2\tau)-\partial_t\vartheta_2(t;2\tau))+\vartheta_2(z;2\tau)(\pm\pi i\vartheta_3(t;2\tau)-\partial_t\vartheta_3(t;2\tau))}{\pi\,\eta(\tau)\vartheta_1(t;\tau)} \\
&= \frac{\pm\pi i\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)-\partial_t\left(\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)\right)}{\pi\,\eta(\tau)\vartheta_1(t;\tau)},
\end{aligned}
\tag{225}
$$

which agrees with Eq. (214) with $w = \pm 1$ after replacing $\vartheta_2 \to \vartheta_1$ in the last line of Eq. (225), that is, by replacing the character with the supercharacter. Note that the domain of convergence of Eq. (214) for $w = 1$ is $1 < |x| < |q|^{-2}$ and for $w = -1$ is $|q|^2 < |x| < 1$, which is the same as the corresponding domains in Eq. (225), since the latter contains the characters $\chi^{(-1)}_{j,\pm}(t;\tau)$ respectively. The affine bosonic content of these modules is depicted in Table 5. This explicit computation is an incarnation of the fact that

$$\sigma^{\pm 1}(\mathcal{L}) \cong \mathcal{L}^{\pm} \,.$$

$L_0 - \frac{1}{12}$							
0			$\big(\mp 1\,,\mathbf{1}\big)_{-1}$	$\big(\mp \frac{1}{2}\,,\mathbf{2}\big)_0$	$\big(\mp 1\,,\mathbf{1}\big)_1$		
1		$\big(\mp \frac{3}{2}\,,\mathbf{2}\big)_{-2}$		$\big(\mp \frac{3}{2}\,,\mathbf{2}\big)_0$		$\big(\mp \frac{3}{2}\,,\mathbf{2}\big)_2$	
2	$\big(\mp 2\,,\mathbf{1}\big)_{-3}$		$\big(\mp 2\,,\mathbf{1}\big)_{-1}$		$\big(\mp 2\,,\mathbf{1}\big)_1$		$\ddots$
4		$\big(\mp \frac{5}{2}\,,\mathbf{2}\big)_{-2}$		$\big(\mp \frac{5}{2}\,,\mathbf{2}\big)_0$		$\big(\mp \frac{5}{2}\,,\mathbf{2}\big)_2$	
$\vdots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 5: Decomposition of the module $\mathcal{L}^{\pm}$, where all the labels $j \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$ denote the highest weight discrete representation D^+_j of $\mathfrak{su}(2)$ if $j < 0$, or the lowest weight discrete representations D^-_j if $j > 0$. One can identify the states at level zero via the spectral flow action. Indeed, for $\mathcal{L}^+$, the highest weight state in $\big(-\frac{1}{2}\,,\mathbf{2}\big)_0$ corresponds to $|0\rangle^\sigma_{w=1}$, while those in $\big(-1\,,\mathbf{1}\big)_{\pm 1}$ to $S^{\mp\pm\pm}_0\,|0\rangle^\sigma_1 = [S^{\mp\pm\pm}_{-1}\,|0\rangle]^\sigma_1$ correspondingly. Analogously, for $\mathcal{L}^-$, the highest weight state in $\big(\frac{1}{2}\,,\mathbf{2}\big)_0$ corresponds to $|0\rangle^\sigma_{w=-1}$, while those in $\big(1\,,\mathbf{1}\big)_{\pm 1}$ to $S^{\pm\mp\pm}_0\,|0\rangle^\sigma_{-1} = [S^{\pm\mp\pm}_{-1}\,|0\rangle]^\sigma_{-1}$ correspondingly. Note that as we already know from the conformal embedding, the Casimir of $\mathfrak{psu}(2|2)$ of both representations is zero.

Lastly, we compute the characters of the free field realisation $\widetilde{\mathcal{R}}$ defined by Eq. (111) and Eq. (113) allowing $m_1, m_2 \in \frac{1}{2}\mathbb{Z}$. Denoting by $\widetilde{\mathcal{E}}$ the corresponding sector, we obtain

$$\begin{aligned}\eta(\tau)\,\mathrm{ch}[\widetilde{\mathcal{R}}_Z](t,z;\tau) &= \sum_{Y\in\mathbb{Z}+Z} \mathrm{ch}[(\widetilde{\mathcal{E}};Y;Z)](t,z;\tau) \\ &= \sum_{s\in\mathbb{Z}+\frac{1}{2}+Z} x^s \frac{\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)}{\eta(\tau)^4}\,,\end{aligned} \tag{226}$$

which converges nowhere in the x-plane and thus has to be interpreted as a formal power series. Then, by passing to supercharacters, which effectively replaces $\vartheta_2 \to \vartheta_1$, we compute the spectrally flowed characters

$$\begin{aligned}
\eta(\tau)\,\mathrm{ch}[\sigma^w(\widetilde{\mathcal{R}}_Z)](t,z;\tau) &= (-1)^w q^{-\frac{w^2}{2}} \sum_{s\in\mathbb{Z}+\frac{1}{2}+Z} x^s q^{sw} \frac{\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)}{\eta(\tau)^4} \\
&= (-1)^w q^{-\frac{w^2}{2}} \mathrm{e}^{2\pi i(t+w\tau)(\frac{1}{2}+Z)} \sum_{s\in\mathbb{Z}} \mathrm{e}^{2\pi i(t+w\tau)s} \frac{\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)}{\eta(\tau)^4} \\
&= (-1)^w q^{-\frac{w^2}{2}} \mathrm{e}^{2\pi i(t+w\tau)(\frac{1}{2}+Z)} \sum_{m\in\mathbb{Z}} \delta(t+w\tau-m) \frac{\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)}{\eta(\tau)^4} \\
&= (-1)^w q^{-\frac{w^2}{2}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{2\pi i m(\frac{1}{2}+Z)} \delta(t+w\tau-m) \frac{\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)}{\eta(\tau)^4}\,,
\end{aligned} \tag{227}$$

and their modular transformation

$$\begin{aligned}
&\eta(-\tfrac{1}{\tau})\,\mathrm{ch}[\sigma^w(\widetilde{\mathcal{R}}_Z)](\tfrac{t}{\tau},\tfrac{z}{\tau};-\tfrac{1}{\tau}) \\
&= \mathrm{e}^{\frac{\pi i}{2\tau}(z^2-t^2)}(-1)^w \frac{\mathrm{e}^{\frac{\pi i}{\tau}(t^2+w^2)}}{i\tau} \sum_{m\in\mathbb{Z}} \mathrm{e}^{2\pi i m(\frac{1}{2}+Z)} \delta(\tfrac{t-w-m\tau}{\tau}) \frac{\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)}{\eta(\tau)^4} \\
&= \mathrm{e}^{\frac{\pi i}{2\tau}(z^2-t^2)}\,(-i\,\mathrm{sgn}(\mathrm{Re}(\tau)))\,(-1)^w \sum_{m\in\mathbb{Z}} q^{-\frac{m^2}{2}} \mathrm{e}^{2\pi i m(\frac{1}{2}+Z)} \delta(t-w-m\tau) \frac{\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)}{\eta(\tau)^4} \\
&= \mathrm{e}^{\frac{\pi i}{2\tau}(z^2-t^2)} \sum_{w'\in\mathbb{Z}} \int_0^1 d\lambda'\, \mathcal{S}_{(w,Z+\frac{1}{2}),(w',\lambda')}\,\mathrm{sch}[\sigma^{w'}(\mathcal{F}_{\lambda'})](t,z;\tau)\,,
\end{aligned} \tag{228}$$

where in the second equality we used

$$\delta\left(\frac{x}{\tau}\right) = \tau\,\mathrm{sgn}(\mathrm{Re}(\tau))\,\delta(x) \qquad \forall\, x\in\mathbb{R}\ \forall\,\tau\in\mathbb{H}\,,$$

and we used the characters analogous to those in [EGG19] for the continuous multiplets of $\mathfrak{psu}(1,1|2)$ to general $\lambda\in\mathbb{R}/\mathbb{Z}$:

$$\mathrm{sch}[\sigma^w(\mathcal{F}_\lambda)](t,z;\tau) = (-1)^w q^{-\frac{w^2}{2}} \sum_{s\in\mathbb{Z}+\lambda} x^s q^{sw} \frac{\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)}{\eta(\tau)^4}\,.$$

The modular S-matrix in Eq. (228) is given by

$$\mathcal{S}_{(w,\lambda),(w',\lambda')} = -i\,\mathrm{sgn}(\mathrm{Re}(\tau))\,\mathrm{e}^{2\pi i\left[w'(\lambda+\frac{1}{2})+w(\lambda'+\frac{1}{2})\right]}\,. \tag{229}$$

8 Outlook

We mention some of the possible developments that could follow to the results presented in this thesis.

We analyzed all the representations appearing in the $\mathfrak{su}(2)_{-1}$ theory and their characters. It would be interesting to find a modular invariant partition function and in particular one containing only a discrete subset of modules, since the global WZW model is defined on the corresponding compact Lie group. Finding such invariant could in turn be useful in constructing one for the $\mathfrak{psu}(2|2)_1$ and $\mathfrak{su}(2|2)_1$ models. Indeed, because of the conformal embedding of the bosonic subalgebra for the former, and the explicit branching rules for the latter, knowing the mass matrices of all the components of the respective affine bosonic subalgebras should allow for an understanding of the partition function for the superalgebra. However, as already explained in Remark 6.2, the technical issues related to the non-holomorphicity of the $\mathfrak{su}(2)_{-1}$ characters and the fact that the Grothendieck ring of characters is "smaller" than the ring of modules, challenges the physical interpretation of partition functions constructed out of character functions. The same convergence issues appear in the characters of the superalgebras discussed here.

A big step forward in the understanding of the WZW model on $\mathfrak{su}(2|2)_1$ would be to have under control the modular transformation of its characters. Even though we presented explicit brunching rules in terms of affine bosonic representations, they do not seem to simplify this task. However, the relatively close relation between the $\mathfrak{su}(2|2)_1$ vacuum module, namely $\mathcal{R}_0$, and all the representations $\mathcal{R}_Z$ for $Z \in \mathbb{N}$, suggests that it should be possible to relate the issue of understanding the modular behaviour of the latter to the better-understood case of the former. Similar relations should be possible between the other R sector representations with fixed $Z_0 \in \mathbb{Z}$ and $\mathcal{R}_0$. Computing the modular transformations should also clarify if in order to build an invariant partition function one has to consider also spectrally flowed modules.

For what concerns $\mathfrak{psu}(2|2)_1$, it is important to understand the logarithmic nature of the theory, emerging from Eq. (217). In particular, the appearance of reducible but indecomposable modules on which L_0 is not diagonalisable has been observed in [EGG19] for $\mathfrak{psu}(1,1|2)_1$ and in [CR12] for $\mathfrak{sl}(2,\mathbb{R})_k$ at admissible fractional levels k. In the former case, there is a unique such module, which does not appear separately in the spectrum since it is already taken into account from the contribution of irreducible representations. At this point, it is also useful to point out the "duality" of the representation theory of $\mathfrak{psu}(1,1|2)_1$ and that of $\mathfrak{psu}(2|2)_1$. The main difference, is that for the latter we only allowed the spin j of the non-integrable factor $\mathfrak{su}(2)_{-1}$ to be quantised to $j \in \frac{1}{2}\mathbb{Z}$, and in particular we imposed $\lambda = j \bmod \mathbb{Z}$ for the continuous representations. This is due to global considerations regarding the compactness of the bosonic subalgebra. The situation is instead quite different for $\mathfrak{psu}(1,1|2)_1$, since because of the non-compactness of the $\mathfrak{sl}(2,\mathbb{R})_1$ factor in the bosonic subalgebra, the spectrum is continuous and more specifically parametrised by $\lambda \in \mathbb{R}/\mathbb{Z}$. Then, the indecomposable module mentioned before appears at $\lambda = 1/2$, and it is the unique module from the family of continuous $\mathfrak{sl}(2,\mathbb{R})$-representations that appears also in the $\mathfrak{psu}(2|2)_1$-theory. It would be interesting to make this connection more precise.

As a continuation to this project, it would be also relevant to compute the fusion rules and the Verlinde formula for both $\mathfrak{psu}(2|2)_1$ and $\mathfrak{su}(2|2)_1$. We note that the fusion rules of symplectic bosons have been explicitely worked out in [Rid10] and those of $\mathfrak{psu}(1,1|2)_1$ in [EGG19]. It is to be expected that the physical interpretation of the fusion coefficients obtained from a "naive" application of the Verlinde formula is hidden, see Remark 6.2.

A Theta functions

Following the notation of [BLT13], let $q := \mathrm{e}^{2\pi i\tau} \in \mathbb{C}$ for $\tau \in \mathbb{H}$, $z \in \mathbb{C}$, and we define the theta functions as

$$\begin{aligned}\vartheta \begin{bmatrix}\alpha \\ \beta\end{bmatrix} (z;\tau) &:= \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi i(n+\alpha)^2\tau + 2\pi i(n+\alpha)(z+\beta)} \\ &= \mathrm{e}^{2\pi i\alpha(z+\beta)} q^{\frac{\alpha^2}{2}} \prod_{n=1}^{\infty} (1-q^n)(1+q^{n+\alpha-\frac{1}{2}}\mathrm{e}^{2\pi i(z+\beta)})(1+q^{n-\alpha-\frac{1}{2}}\mathrm{e}^{-2\pi i(z+\beta)})\,,\end{aligned}$$

where the second equality holds by applying the *Jacobi triple product*. The Jacobi theta functions are then

$$\vartheta_1 := \vartheta\begin{bmatrix}\frac{1}{2} \\ \frac{1}{2}\end{bmatrix}\,, \quad \vartheta_2 := \vartheta\begin{bmatrix}\frac{1}{2} \\ 0\end{bmatrix}\,, \quad \vartheta_3 := \vartheta\begin{bmatrix}0 \\ 0\end{bmatrix}\,, \quad \vartheta_4 := \vartheta\begin{bmatrix}0 \\ \frac{1}{2}\end{bmatrix}\,.$$

These functions obey the following addition rules:

$$\begin{aligned}\vartheta_1(\tfrac{z+t}{2};\tau)\vartheta_1(\tfrac{z-t}{2};\tau) &= \vartheta_2(z;2\tau)\vartheta_3(t;2\tau) - \vartheta_3(z;2\tau)\vartheta_2(t;2\tau)\,,\\ \vartheta_2(\tfrac{z+t}{2};\tau)\vartheta_2(\tfrac{z-t}{2};\tau) &= \vartheta_2(z;2\tau)\vartheta_3(t;2\tau) + \vartheta_3(z;2\tau)\vartheta_2(t;2\tau)\,,\\ \vartheta_3(\tfrac{z+t}{2};\tau)\vartheta_3(\tfrac{z-t}{2};\tau) &= \vartheta_3(z;2\tau)\vartheta_3(t;2\tau) - \vartheta_2(z;2\tau)\vartheta_2(t;2\tau)\,,\\ \vartheta_4(\tfrac{z+t}{2};\tau)\vartheta_4(\tfrac{z-t}{2};\tau) &= \vartheta_3(z;2\tau)\vartheta_3(t;2\tau) - \vartheta_2(z;2\tau)\vartheta_2(t;2\tau)\,,\end{aligned}$$

and quasi-periodicity relations, such as

$$\begin{aligned}\vartheta_i(z+w\tau;\tau) &= \mathrm{e}^{-2\pi iwz} q^{\frac{-w^2}{2}} \vartheta_i(z;\tau) & i &= 2,3\,,\\ \vartheta_i(z+w\tau;\tau) &= (-1)^w \mathrm{e}^{-2\pi iwz} q^{\frac{-w^2}{2}} \vartheta_i(z;\tau) & i &= 1,4\,,\end{aligned} \tag{230}$$

for every $w \in \mathbb{Z}$, and

$$\begin{aligned}\vartheta_2(z+w\tau;2\tau) &= \mathrm{e}^{-\pi iwz} q^{-\frac{w^2}{4}} \begin{cases}\vartheta_2(z;2\tau) & \forall\, w\in 2\mathbb{Z}\,,\\ \vartheta_3(z;2\tau) & \forall\, w \in 2\mathbb{Z}+1\,,\end{cases}\\ \vartheta_3(z+w\tau;2\tau) &= \mathrm{e}^{-\pi iwz} q^{-\frac{w^2}{4}} \begin{cases}\vartheta_3(z;2\tau) & \forall\, w\in 2\mathbb{Z}\,,\\ \vartheta_2(z;2\tau) & \forall\, w \in 2\mathbb{Z}+1\,.\end{cases}\end{aligned} \tag{231}$$

We also need the modular transformations

$$\begin{aligned}\vartheta_1(\tfrac{z}{\tau};-\tfrac{1}{\tau}) &= -i\sqrt{-i\tau}\,\mathrm{e}^{\frac{\pi iz^2}{\tau}}\vartheta_1(z;\tau)\,,\\ \vartheta_2(\tfrac{z}{\tau};-\tfrac{1}{\tau}) &= \sqrt{-i\tau}\,\mathrm{e}^{\frac{\pi iz^2}{\tau}}\vartheta_4(z;\tau)\,,\\ \vartheta_3(\tfrac{z}{\tau};-\tfrac{1}{\tau}) &= \sqrt{-i\tau}\,\mathrm{e}^{\frac{\pi iz^2}{\tau}}\vartheta_3(z;\tau)\,,\\ \vartheta_4(\tfrac{z}{\tau};-\tfrac{1}{\tau}) &= \sqrt{-i\tau}\,\mathrm{e}^{\frac{\pi iz^2}{\tau}}\vartheta_2(z;\tau)\,,\end{aligned}$$

and

$$\vartheta_2(\tfrac{z}{\tau};-\tfrac{2}{\tau}) = \sqrt{-\tfrac{i\tau}{2}}\,\mathrm{e}^{\frac{\pi i z^2}{2\tau}}\big(\vartheta_3(z;\tau)-\vartheta_2(z;\tau)\big)\,,$$
$$\vartheta_3(\tfrac{z}{\tau};-\tfrac{2}{\tau}) = \sqrt{-\tfrac{i\tau}{2}}\,\mathrm{e}^{\frac{\pi i z^2}{2\tau}}\big(\vartheta_3(z;\tau)+\vartheta_2(z;\tau)\big)\,.$$

We also make use of the Dedekind eta function, which is defined by

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,,$$

and it transform under the modular group as

$$\eta(\tau+1) = \mathrm{e}^{\frac{\pi i}{12}}\,\eta(\tau)\,,$$
$$\eta(-\tfrac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)\,.$$

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