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Stability of space-time Petrov-Galerkin discretizations for parabolic evolution equations

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Acknowledgment

Karl Schwarzschild "took a stroll in the Idealand" of Albert Einstein, while facing warfront gunfire, as he writes in a letter, dated Dec 22, 1915, to produce the first nontrivial exact solution to the equations of general relativity. Einstein's emphatic postcard reply ends with "best regards and wishes for the new year" [Sch92, p. 36ff]. In order to describe the trajectory of a test particle, the gravitational field of a point mass is subjected to a nonlinear side condition of the "Determinantengleichung". In his communiqué to the Akademie der Wissenschaften in January 1916, the year of his painful disease and death, Schwarzschild accomplishes this by an ingenious choice of coordinates, which now bear his name, commenting:

> Ein einfacher Kunstgriff gestattet jedoch, diese Schwierigkeit zu umgehen.

The story is heavy with irony, both amusing and sad. Can I amend by confessing that the essence of this dissertation is but an observation or two, on top of a Kunstgriff, that surrendered in exchange for discussions, debates, reviews, presentations, sleepless hours, frustrations, let alone the unconditional support of innocent others?

For the support and fruitful discussions, I would like to express my gratitude to Ch. Schwab for providing many research directions and opportunities; to R. Stevenson and D. Schötzau for agreeing to review the thesis; to my fellow SAMies and friends for providing all sorts of impulses, among others: Marcel B., Claude G., Paolo C., Oleg R., Sohrab K., Julia S., Christine T. (for her contribution to Lemma 4.2.4), Markus H., Martin P., Holger Sr. and Holger Jr., Kristin K., Sevi T., Ulrik F., Sid M., Annika L., Andreas H., C∃ci P.; to Yuki for sharing my anxiety and exhilaration for so long; to my family for not asking too many questions...

Abstract

This Ph.D. thesis was prepared under the supervision of Prof. Ch. Schwab, ETH Zürich. Support by the Swiss NSF Grant No. 127034 and the ERC AdG No. 247277 is gratefully acknowledged. The main contributions of the thesis are summarized in the following.

The minimal residual Petrov-Galerkin framework

A Petrov-Galerkin framework for the stable solution of linear operator equations is developed. The main feature is the admissibility of discrete test subspaces that have larger dimension than the trial subspaces. This renders stable discrete trial and test subspaces (i.e. that satisfy the discrete inf-sup condition) easier to design for well-posed non-symmetric problems. The discrete solution is then defined as the minimizer of the functional residual. The discrete inf-sup condition is shown to imply quasi-optimality of the discrete solution. Following a choice of bases on the discrete trial and test subspaces, the minimization procedure can be equivalently formulated as an algebraic minimization problem by transporting the norm on the continuous test space to the discrete test subspace, or as the corresponding generalized normal equations. The latter can be efficiently preconditioned by transporting a norm on the continuous trial space to the discrete trial subspace.

Space-time Petrov-Galerkin discretizations of parabolic evolution equations

A space-time variational formulation for abstract linear parabolic evolution equations is considered. Lower bounds on the discrete inf-sup constant for general discrete trial and test subspaces equipped with certain subspace-dependent norms are derived. These lower bounds are in terms of a parameter describing compatibility of the discrete trial and test subspaces.

Using these results, it is then found that continuous Galerkin time-stepping methods may be interpreted as stable space-time Petrov-Galerkin methods, provided a CFL condition – a restriction on the time step size – is satisfied. Novel families of discrete trial and test subspaces of space-time sparse tensor product type which do not suffer from this restriction are constructed. Using the latter in the minimal residual Petrov-Galerkin framework leads to stable, fully parallellizable, space-time compressive algorithms. Such algorithms are of significant interest in e.g. optimal control with parabolic PDE constraints.

Parabolic BPX preconditioners

In addition to the stable discrete trial and test subspaces discussed in the previous paragraph, norm inducing operators on the continuous trial and test spaces are needed in the minimal residual Petrov-Galerkin framework. These should be such that their algebraic counterparts are easily invertible. To that end, a pair of operators, called "parabolic BPX preconditioners", is constructed. These are based on multilevel norm equivalences (in the temporal, as well as in the spatial domain) that are already known to play a central role in the multilevel and multigrid methods for elliptic problems.

Zusammenfassung

Diese Dissertation wurde unter der Leitung von Prof. Ch. Schwab erstellt. Der Author bedankt sich für die Unterstützung durch den Swiss NSF Grant No. 127034 und den ERC AdG No. 247277. Die Hauptbeiträge dieser Dissertation sind im Folgenden zusammengefasst.

Residuum-minimierende Petrov-Galerkin Verfahren

Es wird ein Petrov-Galerkin Verfahren zur Lösung wohlgestellter abstrakter linearer Operatorgleichungen, das auf der Minimierung des Residuums über einen diskreten Testraum basiert, vorgestellt. Das Hauptmerkmal des Verfahrens ist die Zulässigkeit von Testräumen mit grösserer Dimension als der Ansatzräume. Für nicht symmetrische Probleme wird dadurch die diskrete inf-sup Bedingung, die die Quasioptimalität der diskreten Lösung impliziert, wesentlich leichter erfüllbar. Sind Basen auf den Ansatz- und Testräumen gegeben, kann das Minimierungsproblem unter Berücksichtigung der Normen auf den diskreten Ansatz- und Testräumen in ein äquivalentes algebraisches Minimierungsproblem umgeschrieben werden, was schliesslich auf ein gutkonditioniertes verallgemeinertes Gauß'sches System von Normalengleichungen führt.

Raum-Zeit Petrov-Galerkin Diskretisierung parabolischer Evolutionsgleichungen

Es wird eine variationelle Formulierung für abstrakte lineare parabolische Evolutionsgleichungen in Bochner Räumen untesucht und es werden Abschätzungen für die diskrete inf-sup Konstante für allgemeine diskrete Ansatz- und Testräume hergeleitet, wenn diese mit unterraumabhängigen Normen versehen sind.

Mithilfe dieser Resultate wird gezeigt, dass sogenannte "continuous Galerkin" Zeitschrittverfahren als stabile Raum-Zeit Petrov-Galerkin Verfahren interpretiert werden können, sofern eine CFL Bedingung erfüllt ist. Neuartige stabile Familien von diskreten Ansatz- und Testräumen für das Residuum-minimierende Petrov-Galerkin Verfahren, die keine solche Bedingung voraussetzen und zudem auf dünnen Gittern in Raum und Zeit basieren, werden eingeführt. Das führt auf stabile, parallelisierbare, Raum-Zeit komprimierende Lösungsalgorithmen. Anwendungen finden sich etwa bei Optimierung mit parabolischen Evolutionsgleichungen als Nebenbedingung.

Parabolischer BPX Vorkonditionierer

Zusätzlich zu den stabilen diskreten Ansatz- und Testräumen sind für das eingeführte Residuum-minimierende Petrov-Galerkin Verfahren norminduzierende Operatoren auf diesen Räumen, für welche die zugehörige Gram-Matrix möglichst leicht invertiert werden kann, notwendig. Zu diesem Zweck werden soganannte parabolische BPX Operatoren eingeführt. Diese bauen auf Normäquivalenzen gewisser hierarchischer Zerlegungen in Sobolevräumen auf, gleichzeitig in Raum und in Zeit, die bereits für Multilevel- und Multigrid-Verfahren für elliptische Differentialgleichungen von zentraler Bedeutung sind.

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1 Introduction

1.1 Overview

The central objective of this thesis is the derivation of novel algorithms for the solution of parabolic evolution equations. Such equations may describe for instance the heat conduction in a component or the distribution of bacteria in the liver. Given the ubiquity of the problem, the spectrum of existing numerical methods algorithms is rather broad. The novelty of the methods proposed here is that they are compressive simultaneously in space and time, practical, fully parallelizable, and possess interesting mathematical optimality properties that can be summarized by saying that a stable family of discrete linear projectors in the natural space-time solution space is obtained. By consequence, the discrete solution is quasi-optimal in that space, the regularity requirements on the input data are extremely low, there is no restriction on the mesh resolution in the temporal direction, and non-linear problems can be analysed in a space-time Galerkin setting. Such methods may further be used as an integral part in parabolic evolution equations depending on a possibly very large, or even countable number of parameters, and optimization problems constrained by such parametric evolution equations.

The thesis is structured as follows. In Chapter 2, the essential notation is established and several notions required subsequently in the study of abstract parabolic evolution equations are recapitulated. The main contributions of the thesis are contained in chapters 3, 4, 5 and 6, and are summarized below. Chapter 7 contains several admissible temporal discretizations that can be used in a modular fashion for the abstract theory of stable space-time discretizations developed in Chapter 5. In Chapter 8 we discuss one application to the solution of semi-linear parabolic evolution equations. Finally, Chapter 9 concludes by summarizing the results of this thesis, identifying further applications, and exposing some open questions. The essential material of chapters 4 and 5 is also contained in [And10; And12].

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1.2 Summary of main contributions

Chapter 3: Linear parabolic evolution equations

This chapter introduces the abstract linear parabolic evolution equation as a problem posed in "space-time". The purpose of the chapter is twofold. The first purpose is to recapitulate some important notions of a solution and to put the subsequent developments into context. The approaches given here are the energy method of J.-L. Lions, semigroup theory, maximal regularity, and self-dual Lagrangians. Three different space-time variational formulations of the abstract linear parabolic evolution equation are presented, and in the thesis we will focus on the variational formulation given in [SS09]. The second purpose is to discuss regularity of solutions that comes from the smoothing property of the flow, which is a characteristic property for parabolic equations. As was shown in [Sch99; SS00], the resulting high regularity for positive times allows the development of high-order numerical methods. This type of regularity can be concisely formulated as membership of the solution to certain countably normed spaces that are used in the study of problems (usually, elliptic) with local singularities. Based on the framework of maximal regularity in weighted Sobolev spaces we show this membership for problems with time-dependent coefficients with smooth/analytic time-dependence of the input data.

Chapter 4: Minimal residual FEM for operator equations

The parabolic evolution equation has rarely been considered in the numerical analysis literature as a well-posed equation between Hilbert spaces in a space-time variational form. This, however, seems necessary in order to obtain space-time compressive numerical solution algorithms that possess certain optimality properties and are fully parallelizable. Such algorithms, in turn, are necessary in e.g. parabolic PDE constrained optimization, and the solution of high-dimensional parametric parabolic evolution equations.

Two noticeable exceptions, which partly motivated this thesis, are the publications [BJ89; BJ90] and [SS09]. In [BJ89; BJ90], space-time discretizations of *hp* type in time are constructed and shown to satisfy the discrete inf-sup condition, leading to a quasi-optimality property (Céa's lemma). However, the stability constant obtained there is not uniform in the structure of temporal discretization and precludes the use of low-order splines in the temporal direction; the same reason makes the scheme unsuitable for problems with low temporal regularity or non-linear problems. Moreover, the main proofs rely in an essential way on the existence of a time-invariant spectral basis of the generator. Rather differently, in [SS09], stability of compressive space-time discretizations is achieved by adaptivity in the framework of so-called adaptive wavelet methods [CDD02]. However, considerable practical difficulties in the construction of suitable wavelet bases are associated with the implementation of this scheme, especially for parabolic partial differential equations posed on non-trivial domains.

These disadvantages are circumvented if the requirement of a standard a priori Petrov-Galerkin discretization is relaxed: we allow discrete test spaces of dimension larger than that of the trial space. This aspect is, in fact, similar to [CDD02] where the discrete test space is chosen adaptively to resolve the action of the operator on a given vector sufficiently well. Thus, the discrete test space is merely required to approximate the "optimal" test functions up to some relative tolerance. In this chapter we therefore develop a conforming "minimal residual Petrov-Galerkin" discretization framework for abstract linear operator equations where the said standard requirement is not present. We show that the discrete inf-sup condition, which is now far easier to achieve due to the freedom gained in the choice of the discrete test space, still implies the essential properties of a standard Galerkin discretization, namely quasi-optimality of the discrete minimal residual solution in the trial space and its continuous dependence on the input data. We then address the derivation, the effective preconditioning, and the iterative solution of the corresponding algebraic equations. Given the importance of the discrete inf-sup condition, we elaborate on several characterizations of it and derive a stability result for trial and test spaces of sparse tensor type, which will subsequently be used in the construction of a priori stable sparse space-time trial and test spaces for parabolic evolution equations in a space-time variational form. This will lead to a priori stable, compressive space-time discretizations and fully parallelizable solution algorithms.

Chapter 5: Stability of space-time Petrov-Galerkin discretizations

This chapter contains the core contributions of the thesis on stable compressive space-time discretizations of parabolic evolution equations. Throughout this chapter we work with the space-time variational formulation given in [SS09]. The main results are derived as follows. First, we equip the abstract discrete trial and test subspaces with certain subspace-dependent (i.e., mesh-dependent) norms and derive lower bounds on the discrete inf-sup constant in terms of structural parameters of the discrete trial and test subspaces. Here, two slightly different sets of subspace-dependent norms and corresponding structural parameters are suggested, which differ in the way non-symmetric generators are treated. These subspace-dependent norms are defined so as to simplify the proofs of stability and to obtain sharper bounds. In the second step, we investigate how the structural parameters and the subspace-dependent norms behave for more concrete discrete trial and test subspaces. These comprise **a)** the known family of continuous Galerkin time-stepping schemes (including Crank-Nicolson time-stepping, interpreted as a space-time method), for which we show that their stability is coupled to the validity of a CFL condition, and **b)** novel families of stable sparse space-time discrete trial and test spaces, see Theorem 5.2.18. The latter, however, are to be used within the minimal residual framework of Chapter 4, since we admit test spaces of dimension larger than that of the trial spaces.

Chapter 6: Parabolic BPX preconditioning

The minimal residual framework of Chapter 4 depends on the availability of norm-inducing operators on the trial and test spaces. This is no restriction in theory, and, in fact, is an integral part of the wavelet adaptive methods, where such operators are given by isomorphisms of the trial/test spaces with sequence spaces via the Riesz basis property of wavelets. To circumvent the already mentioned difficulties in the construction of such wavelets for parabolic evolution problems, we elaborate in this chapter on space-time multilevel subspace splittings that give rise to computationally accessible norm-generating operators. We call the resulting operator the parabolic BPX preconditioner for its relation with the so-called BPX preconditioner, known for its optimality for a certain class of elliptic problems [BPX90; BY93].

2 Preliminaries

The first purpose of this chapter is to establish the notation in Section 2.1. The second purpose is to provide some reference material that will be useful in the study of parabolic evolution equations (but can be skipped on first reading). Hence, the subsequent subsections briefly address vector valued functions and the Bochner integral; tensor product spaces and their relations to Bochner spaces and linear maps; the real method of interpolation for Banach spaces; Bochner-Sobolev spaces, i.e., vector valued functions with weak derivatives, which are the natural spaces for solutions of parabolic evolution equations; notions of analytic functions between real or complex Banach spaces; several results, in particular on embeddings into spaces of continuous functions, for Bochner-Sobolev spaces; countably normed Bochner-Sobolev spaces which we will use to describe high-order regularity of solutions of parabolic evolution equations.

2.1 Notation

For a function f mapping an element $x \in \mathcal{X}$ to an element $y \in \mathcal{Y}$ we write $f : \mathcal{X} \to \mathcal{Y}$, $x \mapsto y = f(x)$, sometimes $f : \mathcal{X} \ni x \mapsto y \in \mathcal{Y}$. If $\mathcal{X} \subset \mathcal{Y}$ are topological spaces we write $\mathcal{X} \hookrightarrow \mathcal{Y}$ if the natural embedding $\mathcal{X} \ni x \mapsto x \in \mathcal{Y}$ is continuous; this embedding is linear if $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces. We write $\mathcal{X} \overset{d}{\hookrightarrow} \mathcal{Y}$ if, in addition, $\mathcal{X}$ is a dense subset of $\mathcal{Y}$. We write $\mathrm{Id}_{\mathcal{X}}$ for the identity on $\mathcal{X}$, or simply Id; the latter may also denote the natural embedding mapping or the unit matrix, when clear from the context.

For two Banach spaces $\mathcal{X}$ and $\mathcal{Z}$ over a field $\mathbb{K}$, the space of continuous linear operators $\mathcal{X} \to \mathcal{Z}$ is denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Z})$. It is endowed with the operator norm $\|\cdot\|_{\mathcal{L}(\mathcal{X},\mathcal{Z})}$. We set $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}, \mathcal{X})$ and $\mathcal{X}' := \mathcal{L}(\mathcal{X}, \mathbb{K})$. We will have $\mathbb{K} = \mathbb{R}$ unless specified otherwise. The space of continuous n-linear maps $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Z}$ between Banach spaces is denoted by $\mathcal{L}_n(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Z})$, and $\mathcal{L}_n(\mathcal{X}, \mathcal{Z})$ if $\mathcal{X} = \mathcal{X}_1 = \ldots \mathcal{X}_n$. We let

$$\mathrm{Iso}(\mathcal{X}, \mathcal{Z}) := \{\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Z}) : \ \mathcal{B} \text{ is bijective and } \mathcal{B}^{-1} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})\} \tag{2.1.1}$$

denote the space of isomorphisms between the Banach spaces $\mathcal{X}$ and $\mathcal{Z}$ (the condition $\mathcal{B}^{-1} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$ is redundant by the Banach open mapping theorem, [Yos95, Section II.5] or [Rud73, Corollary 2.12]), and set $\mathrm{Iso}(\mathcal{X}) := \mathrm{Iso}(\mathcal{X}, \mathcal{X})$. We let $\mathcal{D}(\mathcal{B})$ denote the domain of definition of an operator $\mathcal{B}$.

The complexification of a real Banach space $\mathcal{X}$ is $\widetilde{\mathcal{X}} := \{x + iy : x, y \in X\}$ with the norm $\|x + iy\|_{\widetilde{\mathcal{X}}} = \sup_{0 \le \theta \le 2\pi} \|x \cos\theta + y \sin\theta\|_{\mathcal{X}}$, which turns $\widetilde{\mathcal{X}}$ into a complex Banach space [MST99]. The complexification of an operator $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subseteq \mathcal{X} \to \mathcal{X}$ is the operator $\widetilde{\mathcal{B}} : x + iy \mapsto \mathcal{B}x + i\mathcal{B}y$ with domain $\mathcal{D}(\widetilde{\mathcal{B}}) = \mathcal{D}(\mathcal{B}) + i\mathcal{D}(\mathcal{B})$. The tildes are dropped in the notation. The resolvent set of (the complexification of) an operator $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subseteq \mathcal{X} \to \mathcal{X}$ is defined by

$$\rho(\mathcal{B}) := \{z \in \mathbb{C} : (z\,\mathrm{Id} - B)^{-1} \in \mathcal{L}(\mathcal{X})\}. \tag{2.1.2}$$

The duality pairing on $\mathcal{X} \times \mathcal{X}'$ i.e., $(x, f) \mapsto f(x)$, is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X} \times \mathcal{X}'}$ or ($\langle \cdot, \cdot \rangle_{\mathcal{X}' \times \mathcal{X}}$ for $(f, x) \mapsto f(x)$). A linear operator $\mathcal{M} : \mathcal{X} \to \mathcal{X}'$ is called symmetric if it satisfies $\langle \mathcal{M}x, \widetilde{x} \rangle_{\mathcal{X}' \times \mathcal{X}} = \langle x, \mathcal{M}\widetilde{x} \rangle_{\mathcal{X} \times \mathcal{X}'}$ for all $x, \widetilde{x} \in \mathcal{X}$; it is called positive semi-definite if $\langle \mathcal{M}x, x \rangle_{\mathcal{X} \times \mathcal{X}'} \ge 0$ for all $x \in \mathcal{X}$, in which case $\|\cdot\|_{\mathcal{M}}$ denotes the map

$$\|\cdot\|_{\mathcal{M}} : \mathcal{X} \to \mathbb{R}, \quad x \mapsto \|x\|_{\mathcal{M}} := \sqrt{\langle \mathcal{M}x, x \rangle_{\mathcal{X} \times \mathcal{X}'}}.$$

A symmetric positive semi-definite (s.p.semi-d.) linear operator $\mathcal{M}$ is called symmetric positive definite (s.p.d.) if $\|x\|_{\mathcal{M}} = 0 \Leftrightarrow x = 0$ for any $x \in \mathcal{X}$. If $\mathcal{X}$ is a Hilbert space, its scalar product is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X}\times\mathcal{X}}$, or by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$; if two subspaces $\mathcal{X}_0, \mathcal{X}_1 \subseteq \mathcal{X}$ are orthogonal w.r.t. the scalar product on $\mathcal{X}$, we may write $\mathcal{X}_0 \perp_{\mathcal{X}} \mathcal{X}_1$; the orthogonal complement in $\mathcal{X}$ is denoted by $\mathcal{X}_0^{\perp_{\mathcal{X}}}$.

We write $\mathbb{N} := \{1, 2, \ldots\}$ for the positive integers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. If $\mathcal{X} = \mathbb{R}^M$, where $M \in \mathbb{N} \cup \{\infty\}$, and $\mathbf{M} \in \mathcal{X} \times \mathcal{X}$ is a symmetric matrix with $\mathbf{x}^\top \mathbf{M}\mathbf{x} \geq 0$ for all finitely supported $\mathbf{x} \in \mathcal{X}$, then for all $\mathbf{x} \in \mathcal{X}$ we define $\|\mathbf{x}\|_{\mathbf{M}} := \sqrt{\mathbf{x}^\top \mathbf{M}\mathbf{x}}$ if the convergence of the double sum $\mathbf{x}^\top \mathbf{M}\mathbf{x}$ is absolute, and $\|\mathbf{x}\|_{\mathbf{M}} := \infty$ otherwise. For $M \in \mathbb{N}_0 \cup \{\infty\}$ we define

$$[M] := \begin{cases} \emptyset, & M = 0, \\ \{1, \ldots, M\}, & 1 \leq m < \infty, \\ \mathbb{N}, & M = \infty. \end{cases} \tag{2.1.3}$$

For $M \in \mathbb{N} \cup \{\infty\}$ we set $\ell^2_M := \ell^2([M])$ for the space of square summable sequences indexed by $[M]$ with the natural norm and $\ell^2_{\mathbf{M}} := \ell^2_{\mathbf{M}}([M]) := \{\mathbf{x} \in \mathbb{R}^M : \|\mathbf{x}\|_{\mathbf{M}} < \infty\}$ with the norm $\|\cdot\|_{\ell^2_{\mathbf{M}}([M])} = \|\cdot\|_{\mathbf{M}}$; the support of a vector $\mathbf{u} \in \mathbb{R}^M$ is the set of indices defined by $\operatorname{supp} \mathbf{u} := \{m \in [M] : \mathbf{u}_m \neq 0\}$ and $\mathbf{u} \in \mathbb{R}^M$ is said to be finite if $M < \infty$, infinite if $M = \infty$ and finitely supported if $\#\operatorname{supp} \mathbf{u} < \infty$; if $\mathbf{A} \in \operatorname{Iso}(\ell^2_{\mathbf{M}}, \ell^2_{\mathbf{M}})$ then the condition number of $\mathbf{A}$ is defined by $\kappa_2(\mathbf{A}) := \|\mathbf{A}\|_{\mathcal{L}(\ell^2_{\mathbf{M}}, \ell^2_{\mathbf{M}})} \|\mathbf{A}^{-1}\|_{\mathcal{L}(\ell^2_{\mathbf{M}}, \ell^2_{\mathbf{M}})}$; by definition, $\kappa_2(\mathbf{A}) < \infty$ for some $\mathbf{A} : \ell^2_{\mathbf{M}} \to \ell^2_{\mathbf{M}}$ if and only if $\mathbf{A} \in \operatorname{Iso}(\ell^2_{\mathbf{M}}, \ell^2_{\mathbf{M}})$. For $0 < p \leq \infty$, the space of p-summable sequences indexed by a set S of at most countable cardinality $\#S$ is denoted by $\ell^p(S)$.

We write $|D|$ for the Lebesgue measure of a measurable (w.r.t. the Lebesgue measure) set $D \subseteq \mathbb{R}^d$.

2.2 The Bochner integral

For the following measure theoretic notions we refer to [Loè78, Vol. I, Chapter I], [Hal50] or to the references given in this section: a σ-finite complete measure space $(\mathcal{S}, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a σ-algebra of measurable subsets of $\mathcal{S}$, and μ is a (positive, extended-real valued) measure; the Banach space $L^1(\mathcal{S}, d\mu)$ of Lebesgue μ-measurable and μ-integrable scalar(-valued) functions on $\mathcal{S}$; the Borel σ-algebra $\mathcal{B}(\mathcal{S})$ generated by open subset of $\mathcal{S}$ if $\mathcal{S}$ of a topological space; the product measure of σ-finite measures.

Let $(\mathcal{S}, \mathcal{F}, \mu)$ be a σ-finite complete measure space. Let X be a Banach space. A function $f : \mathcal{S} \to X$ is called **μ-measurable** if the two conditions are fulfilled

1. the scalar function $\langle \varphi, f(\cdot) \rangle_{X' \times X}$ is Lebesgue μ-measurable for every $\varphi \in X'$,
2. $f(A)$ is contained in a separable subspace of X for *some* $A \in \mathcal{F}$ with $\mu(\mathcal{S} \setminus A) = 0$,

cf. [Gra08, Section 4.5.3] for this direct approach. The first condition can be replaced by the requirement $f^{-1}(O) \in \mathcal{F}$ for any open $O \subseteq X$ [Rya02, Proposition 2.15]. A μ-measurable function $f : \mathcal{S} \to X$ is called **Bochner μ-integrable** if the scalar function $\|f(\cdot)\|_X$ is Lebesgue μ-integrable. Two such functions f and g are identified if $\|(f - g)(\cdot)\|_X = 0$ μ-a.e. For $1 \leq p \leq \infty$ the space $L^p(\mathcal{S}, d\mu; X)$ of (equivalence classes of) Bochner μ-measurable functions $f : \mathcal{S} \to X$ is equipped with the norm

$$\|f\|_{L^p(\mathcal{S}, d\mu; X)} := \|\|f(\cdot)\|_X\|_{L^p(\mathcal{S}, d\mu)}. \tag{2.2.1}$$

The space of simple functions $\operatorname{span}\{\chi_A x : x \in X, A \in \mathcal{F}\}$, where χ_A is the indicator function of $A \subseteq \mathcal{S}$, is dense in $L^p(\mathcal{S}, d\mu; X)$ by the theorems going back to Pettis [Pet38] and Bochner [Boc33]. It follows that $L^p(\mathcal{S}, d\mu; X)$ is a Banach space and that there exists a unique linear continuous map, called the **Bochner integral**,

$$L^1(\mathcal{S}, d\mu; X) \to X, \quad f \mapsto \int_{\mathcal{S}} f d\mu, \tag{2.2.2}$$

such that $\int_{\mathcal{S}} \chi_A x d\mu = \mu(A)x$ for any $A \in \mathcal{F}$, $x \in X$. Further, we set $\int_A f d\mu := \int_{\mathcal{S}} \chi_A f d\mu$ for any $A \in \mathcal{F}$. For more on the Bochner integral see e.g. [DU77, Chapter II], [Yos95, Chapter V, Section 4 and 5] or [Rya02, Section 2], as well as [Hil53] for an overview.

The following theorem is specific to integration of *vector*-valued functions.

Theorem 2.2.1 (Hille)**.** *Let X and Y be Banach spaces and $(\mathcal{S}, \mathcal{F}, \mu)$ be a σ-finite complete measure space. Let $f \in L^1(\mathcal{S}, d\mu; X)$. Let $T : X \to Y$ be a closed linear operator. Then:*

1. *If $T \in \mathcal{L}(X, Y)$ then $Tf \in L^1(\mathcal{S}, d\mu; Y)$.*
2. *If $Tf \in L^1(\mathcal{S}, d\mu; Y)$ then $T \int_A f d\mu = \int_A Tf d\mu$ for any $A \in \mathcal{F}$.*

Corollary 2.2.2. *Let $f, g : \mathcal{S} \to X$ be μ-measurable. Then $f = g$ μ-a.e. if and only if for all $\chi' \in X'$ there holds $\langle f(\cdot), \chi' \rangle_{X \times X'} = \langle g(\cdot), \chi' \rangle_{X \times X'}$ μ-a.e.*

Proof of Theorem 2.2.1 and Corollary 2.2.2. For Theorem 2.2.1 see either [DU77, Chapter II, Theorem 6] or [Rya02, Proposition 2.18] or [Yos95, Section V.5, Corollary 2]. For Corollary 2.2.2 see [DU77, Chapter II, Corollary 7]. □

2.3 Tensor product spaces

Let X and Y be Banach spaces. For any $x \in X$ and $y \in Y$, the linear functional $x \otimes y$ on the space of bilinear mappings is defined by $(x \otimes y)(B) := B(x, y)$ for all bilinear $B : X \times Y \to \mathbb{R}$. A functional of the form $x \otimes y$ is called a **simple tensor**. The **algebraic tensor product** space $X \widetilde{\otimes} Y$ is the space of functionals u having the form $u = \sum_{i=1}^n (x_i \otimes y_i)$ with $\{(x_i, y_i)\}_{i=1}^n \subset X \times Y$, $n \in \mathbb{N}$. Then, the expression

$$\pi(u) := \inf \left\{ \sum_{i=1}^n \|x_i\|_X \|y_i\|_Y : u = \sum_{i=1}^n x_i \otimes y_i \right\}, \quad u \in X \widetilde{\otimes} Y, \tag{2.3.1}$$

defines a norm on $X \widetilde{\otimes} Y$, called the **projective norm** [Rya02, Chapter 2]. It satisfies $\pi(x \otimes y) = \|x\|_X \|y\|_Y$ for all $x \in X$, $y \in Y$. The Banach space $X \otimes_\pi Y$ is defined as the completion of $X \widetilde{\otimes} Y$ w.r.t. the projective norm π. We will use the following canonical isometric identifications [Rya02, Section 2.2]

$$\mathcal{L}_2(X \times Y, \mathbb{R}) \cong (X \otimes_\pi Y)' \cong \mathcal{L}(X, Y') \cong \mathcal{L}(Y, X') \tag{2.3.2}$$

and [Rya02, Section 2.3]

$$L^1(\mathcal{S}, \mu) \otimes_\pi X \cong L^1(\mathcal{S}, \mu; X) \tag{2.3.3}$$

where $(\mathcal{S}, \mathcal{F}, \mu)$ is any σ-finite complete measure space. If X and Y are Hilbert spaces then the bilinear mapping $\langle \cdot, \cdot \rangle_{X \otimes Y}$, given by

$$\langle x \otimes y, \widetilde{x} \otimes \widetilde{y} \rangle_{X \otimes Y} := \langle x, \widetilde{x} \rangle_X \langle y, \widetilde{y} \rangle_Y, \quad x, \widetilde{x} \in X, \quad y, \widetilde{y} \in Y, \tag{2.3.4}$$

defines an inner product on $X \widetilde{\otimes} Y$ [LC85, Lemma 1.31 – Lemma 1.33]. We write $X \otimes Y$ for the completion of $X \widetilde{\otimes} Y$ w.r.t. this inner product, which makes $X \otimes Y$ again a Hilbert space. If both, $L^2(\mathcal{S}, \mu)$ and X are separable Hilbert spaces, then [RS72, Chapter II, Theorem 10]

$$L^2(\mathcal{S}, \mu) \otimes X \cong L^2(\mathcal{S}, \mu; X). \tag{2.3.5}$$

More on tensor product spaces can be found in e.g. [Sch50], [DU77, Chapter VIII], [LC85], [DF93].

2.4 Notions from Banach space interpolation theory

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces such that $Y \hookrightarrow X$, both real or both complex. We require a few results from the theory of real interpolation for intermediate spaces $Y \subseteq (X,Y)_{\theta,p} \subseteq X$ in this particular situation. Details and proofs may be found [Tri78; BL76; DL93] in a comprehensive form; see e.g. [Ama95, Chapter I, Section 2] or [Lun09, Chapter 1] for a concise exposition. The intermediate spaces $Y \subseteq (X,Y)_{\theta,p} \subseteq X$ are defined by means of the K**-functional**,

$$K(t,x) := \inf\{\|x-y\|_X + t\|y\|_Y : y \in Y\}, \quad t > 0, \quad x \in X. \tag{2.4.1}$$

For any $x \in X$ set

$$\|x\|_{(X,Y)_{\theta,p}} := \left(\int_0^\infty \left(t^{-\theta}K(t,x)\right)^p \frac{dt}{t}\right)^{1/p}, \quad 0 < \theta < 1, \quad 1 \le p \le \infty, \tag{2.4.2}$$

with the modification $\|x\|_{(X,Y)_{\theta,p}} := \operatorname{ess\,sup}_{t>0} t^{-\theta}K(t,x)$ for $p = \infty$. Define the subspace $(X,Y)_{\theta,p} \subseteq X$ as

$$(X,Y)_{\theta,p} := \{x \in X : \|x\|_{(X,Y)_{\theta,p}} < \infty\}, \tag{2.4.3}$$

which is a Banach space with the norm $\|\cdot\|_{(X,Y)_{\theta,p}}$. For $0 < \theta < 1$ and $1 \le p_1 \le p_2 \le \infty$ one has the continuous inclusions $Y \hookrightarrow (X,Y)_{\theta,p_1} \hookrightarrow (X,Y)_{\theta,p_2} \hookrightarrow \overline{Y}^{\|\cdot\|_X}$, and for $0 < \theta_1 < \theta_2 < 1$ also $(X,Y)_{\theta_2,\infty} \hookrightarrow (X,Y)_{\theta_1,1}$. For any $0 < \theta < 1$ and $1 \le p \le \infty$, there holds the interpolation inequality

$$\|x\|_{(X,Y)_{\theta,p}} \le c_{\theta,p}\|x\|_X^{1-\theta}\|x\|_Y^{\theta} \quad \forall x \in Y, \tag{2.4.4}$$

where $c_{\theta,p} > 0$, see [BL76, Theorem 3.1.2] and [Lun09, Theorem 1.6 and Corollary 1.7].

2.5 Bochner-Sobolev spaces

For the definition of Bochner-Sobolev spaces we follow [DL92, Chapter XVI, §2]. Let X be a real Banach space and $J \subseteq \mathbb{R}$ an open interval. The space of functions $J \to X$ having all continuous derivatives and with compact support in J is denoted by $C_0^\infty(J;X)$. If X is the scalar field $\mathbb{R}$ or $\mathbb{C}$, then $C_0^\infty(J) := C_0^\infty(J;X)$. The topology on $C_0^\infty(J;X)$ is that of uniform convergence on compacta. The space $\mathcal{L}(C_0^\infty(J);X)$ of continuous linear mappings $C_0^\infty(J) \to X$ is called the space of X**-valued distributions over** J. For any such distribution $f \in \mathcal{L}(C_0^\infty(J);X)$ and $k \in \mathbb{N}$ we define its k**-th distributional derivative** $f^{(k)} \in \mathcal{L}(C_0^\infty(J);X)$ by $f^{(k)}(\varphi) := (-1)^k f(\varphi^{(k)})$, $\varphi \in C_0^\infty(J)$, where $\varphi^{(k)}$ denotes the (classical) k-th derivative of φ. To any $u \in L^p(J;X)$, $1 \le p \le \infty$, and any $k \in \mathbb{N}_0$ we may associate an X-valued distribution $\widetilde{u}$ by $\widetilde{u}(\varphi) := \int_J u(t)\varphi(t)dt$, $\varphi \in C_0^\infty(J)$. If there exists a (necessarily unique) $v \in L^p(J;X)$ such that $\widetilde{v} = \widetilde{u}^{(k)}$ then we set $u^{(k)} := v$ and write $u^{(k)} \in L^p(J;X)$. We will usually write $\partial_t u := u^{(1)}$. For $1 \le p \le \infty$ and $k \in \mathbb{N}_0$ we define the Bochner-Sobolev space

$$W^{k,p}(J;X) := \{u \in L^p(J;X) : u^{(\ell)} \in L^p(J;X), \ell = 1,\dots,k\}. \tag{2.5.1}$$

For $1 \le p < \infty$, the expression

$$\|u\|^p_{W^{k,p}(J;X)} := \sum_{\ell=0}^k \|u^{(\ell)}\|^p_{L^p(J;X)}, \quad u \in W^{k,p}(J;X), \tag{2.5.2}$$

defines a norm on $W^{k,p}(J;X)$ with the usual modification for $p = \infty$, and renders $W^{k,p}(J;X)$ a Banach space. As usual, we write $H^k(J;X) := W^{k,2}(J;X)$.

2.6 Analyticity in Banach spaces

The notions of analyticity or holomorphy of a function $f : X \to Z$ between real or complex Banach spaces, as well as their origins are sketched in [Tay43, Section 8]. A more general theory of analytic functions in locally convex spaces is available [Her89], but will not be required here. For the proof of regularity of solution to parabolic evolution equations in Theorem 3.3.6 we will require the case where both, X and Z, are (real) Banach spaces. There are two main approaches to the definition of analyticity: via the power series expansion and via the Fréchet derivative. The main results using the first approach can be formulated without reference to the dimension or the scalar field [Whi65]. This requires some preparation.

In this section, X, Y and Z will denote Banach spaces over the same scalar field $\mathbb{R}$ or $\mathbb{C}$, and E denotes an open subset of X. Unless explicitly stated otherwise, the results in this section hold in either case. The following definition generalizes the notion of a monomial.

Definition 2.6.1. A continuous, n-linear map $a_n \in \mathcal{L}_n(X, Z)$ is called **symmetric** if $a_n(x_1, \dots, x_n) = a_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any $x_1, \dots, x_n \in X$ and any permutation σ of $\{1, \dots, n\}$. For convenience of notation we set $\mathcal{L}_0(X, Z) := Z$, and any element $a_0 \in \mathcal{L}_0(X, Z)$ shall be symmetric by definition. For any $x \in X$, $h \in X$, and non-negative integers $p + q = n \in \mathbb{N}$ we abbreviate

$$a_n x^p h^q := a_n(x_1, \dots, x_p, h_1, \dots, h_q). \tag{2.6.1}$$

Before defining analyticity and weak analyticity in Banach spaces, we state the following theorem.

Theorem 2.6.2. *Let X and Z be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $E \subseteq X$ a ball. Let $f : E \subseteq X \to Z$. Equivalent are*

1. *there exist symmetric $a_n \in \mathcal{L}_n(X, Z)$, $n \in \mathbb{N}_0$, such that the series $\sum_{n\in\mathbb{N}_0} a_n h^n$ converges to $f(h)$ for all $h \in E$,*
2. *for each $z' \in Z'$ there exist symmetric $b_n \in \mathcal{L}_n(X, \mathbb{K})$, $n \in \mathbb{N}_0$, such that the series $\sum_{n\in\mathbb{N}_0} b_n h^n$ converges to $\langle z', f(h)\rangle_{Z'\times Z}$ for all $h \in E$.*

Proof. Let $\{z_n\}_{n\in\mathbb{N}} \subset Z$ and assume that $\sup_{z'} \langle z', z_n\rangle_{Z'\times Z} < \infty$ for all $n \in \mathbb{N}$, where the supremum is over all $z' \in Z'$ with $\|z'\|_{Z'} \le 1$. Viewing z_n as an element of the bidual Z'', the Banach-Steinhaus theorem implies that $\sup_{n\in\mathbb{N}} \|z_n\|_Z < \infty$. Thus, the set $B' := \{z' \in Z' : \|z'\|_{Z'} \le 1\}$ is *strictly fundamental* in the sense of [AO53]. It suffices to consider B' instead of Z' in the claim, since any $z' \in Z'$ may be rescaled to satisfy $z' \in B'$. Therefore, the claim follows from [AO53, Theorem 6.5] (some necessary definitions are in Section 3 of that article, where it is also argued that the qualifier "symmetric" in the claim is redundant). □

It is typical for a characterization like Theorem 2.6.2, cf. Theorem 2.6.6, to rely in a fundamental way on the Hahn-Banach extension theorem or the Banach-Steinhaus theorem, i.e., the principle of uniform boundedness (see e.g. [Bré83, Chapter I and Theorem II.1] for those theorems). Consequently, similar characterizations exist in locally convex spaces, cf. [Tay72], [Her89, Proposition 3.1.2]. Note that neither separability nor reflexivity is required.

Remark 2.6.3 (Adapted from [Whi65])**.** Consider the formal series

$$f(x) = \sum_{n\in\mathbb{N}} a_n x^n \tag{2.6.2}$$

for $x \in X$, where $a_n \in \mathcal{L}_n(X, Z)$. Let $h \in X$. Assume that $C := \sup_{n\in\mathbb{N}_0} \|a_n h^n\|_Z$ is finite; this is e.g. the case if the series $f(h)$ converges. Let $0 < r < 1$, set $B := \{x \in X : \|x\|_X \le r\|h\|_X\}$. Then the series $f(x)$ converges absolutely (indeed, for all $x \in B$ we have $\|a_n x^n\|_Z \le r^n \|a_n h^n\|_Z \le Cr^n$) and uniformly on the closed ball B. Therefore, $f : B \to Z$ is continuous.

Definition 2.6.4. Let $f : E \subseteq X \to Z$, where $E \subseteq X$ is open and $x_0 \in E$. Then f is is called

1. **analytic at** x_0 if there exist symmetric $a_n \in \mathcal{L}_n(X, Z)$, $n \in \mathbb{N}_0$, such that $f(x_0+h) = \sum_{n\in\mathbb{N}_0} a_n h^n$ whenever $\sum_{n\in\mathbb{N}_0} \|a_n\|_{\mathcal{L}_n(X,Z)} \|h\|_X^n$ converges for h near 0 and $x_0 + h \in E$. It is called **analytic on** E if it is analytic at every $x_0 \in E$.
2. **weakly analytic at** x_0 (**on** E) if the function $\langle z', f(\cdot)\rangle_{Z'\times Z}$ is analytic at x_0 (on E) for each $z' \in Z'$.

By Theorem 2.6.2, analyticity and weak analyticity in Banach spaces are equivalent.

Definition 2.6.5. Let X and Z be Banach spaces and $x_0 \in X$. A map $f : X \to Z$ is called **Fréchet differentiable at** x_0 if there exists $A \in \mathcal{L}(X, Z)$ such that

$$\lim_{X\ni h\to 0} \frac{\|f(x_0 + h) - f(x_0) - Ah\|_Z}{\|h\|_X} = 0. \tag{2.6.3}$$

In this case we may write $D_x f(x_0) = Df(x_0) = f'(x_0) = f^{(1)}(x_0) = A$, and $f'(x_0)$ is called the **Fréchet derivative of** f **at** x_0. Let $E \subseteq X$ open. For $k \in \mathbb{N}$, the spaces $C^k(E; Z)$ of k times continuously Fréchet differentiable functions are defined by induction, i.e.,

$$C^{k+1}(E; X) := \{f \in C^k(E; X) : Df^{(k)} \in C^0(E; \mathcal{L}_k(X, Z))\} \tag{2.6.4}$$

where $f^{(k)}$ denotes the k-th Fréchet derivative, $f^{(\ell)} := Df^{(\ell-1)}$, $\ell = 1, \dots, k$. Further, $C^\infty(E; Z) := \bigcap_{k\in\mathbb{N}} C^k(E; Z)$.

For a function $f : X \times Y \to Z$, the partial Fréchet derivative w.r.t. the i-th variable (or w.r.t. x) is denoted by D_i (or D_x). For instance, $D_1 f(x_0, y_0) = D_x f(x_0, y_0)$ is the Fréchet derivative of the map $f(\cdot, y_0) : X \to Z$ at x_0.

On finite dimensional spaces we have the following characterization of (real-)analytic functions. We recall the multi-index notation $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\alpha! = \prod_{i=1}^n \alpha_i!$ for $\alpha \in \mathbb{N}_0^n$, $n \in \mathbb{N}$.

Theorem 2.6.6. *Assume $X = \mathbb{R}^n$, $n \in \mathbb{N}$, and Z a real Banach space. Let $E \subseteq X$ be open. For $f \in C^\infty(E; Z)$ the following are equivalent*

1. *f is analytic on E,*
2. *for each $x_0 \in E$ there exists an open ball $B \subseteq E$ with $x_0 \in B$, and constants $C, d \geq 0$ such that*
$$\|D^\alpha f(x)\|_Z \leq C d^{|\alpha|} \alpha! \quad \forall x \in B \quad \forall \alpha \in \mathbb{N}_0^n. \tag{2.6.5}$$
3. *for each $x_0 \in E$ and each $h \in X$ of unit norm, the function $t \mapsto f(x_0 + th)$ is analytic around zero.*

Proof. We prove 1. $\Leftrightarrow$ 2., for 1. $\Leftrightarrow$ 3. see [Sic70; Boc70]. If $Z = \mathbb{R}$ then [KP02, Proposition 2.2.10] is what is claimed here. The case of a general Banach space Z follows by means of the Banach-Steinhaus theorem: let $B \subseteq E$ be an open ball, $d > 0$; then we have (2.6.5) if and only if

$$\sup_{x\in B} \sup_{\alpha\in\mathbb{N}_0^n} \langle z', D^\alpha f(x) d^{-|\alpha|}/\alpha!\rangle_{Z'\times Z} < \infty \quad \forall z' \in Z'. \tag{2.6.6}$$

Indeed, (2.6.5) obviously implies (2.6.6), while the converse follows from the Banach-Steinhaus theorem applied to the family $D^\alpha f(x) d^{-|\alpha|}/\alpha! \in Z''$ of linear functionals on Z' indexed by $x \in B$ and $\alpha \in \mathbb{N}_0^n$. Recalling from Theorem 2.6.2 that the mapping f is analytic if and only if it is weakly analytic, the claim thus reduces to the scalar-valued case of [KP02, Proposition 2.2.10]. $\square$

For the following theorem, known as the implicit function theorem, see [Whi65, p. 1081] and references therein.

Theorem 2.6.7. *Let X, Y and Z be Banach spaces, all real or all complex. Let $F : X \times Y \to Z$ and $(x_0, y_0) \in X \times Y$. Assume*

1. *$(x_0, y_0) \in X \times Y$ satisfies $F(x_0, y_0) = 0$,*
2. *$(x, y) \mapsto F(x, y)$ is analytic in an open neighborhood N of (x_0, y_0),*
3. *the Fréchet derivative $D_x F(x_0, y_0)$ (w.r.t. the first component x) exists,*
4. *$D_x F(x_0, y_0) : X \to Z$ is an isomorphism.*

Then there exists an open neighborhood $E \subseteq Y$ of y_0 and a unique continuous map $\bar{x} : E \subseteq Y \to X$ satisfying $\bar{x}(y_0) = x_0$ and $F(\bar{x}(y), y) = 0$ for all $y \in E$. Moreover, $\bar{x}$ is analytic on E, and the chain rule $D\bar{x}(y) = -\left[D_x F(\bar{x}(y), y)\right]^{-1} \circ D_y F(\bar{x}(y), y)$ holds for all $y \in E$.

If $N \ni (x, y) \mapsto F(x, y)$ is merely of class C^k, $k \in \mathbb{N} \cup \{\infty\}$, then so is $\bar{x}$.

2.7 More on function spaces

Let X be a real Banach space. Let $J \subseteq \mathbb{R}$ be a non-trivial interval. Recall the definition of the Bochner space $L^p(J; X)$ from Section 2.2, where J is understood to be equipped with the Borel σ-algebra $\mathcal{B}(J)$ and the Lebesgue measure $|\cdot|$.

Proposition 2.7.1. *For $1 \leq p < \infty$, $|J| < \infty$, any of the following are dense in $L^p(J; X)$:*

1. *$\{x \in L^p(J, X) : \#x(J) < \infty\}$,*
2. *$\{x \in L^p(J; X) : x$ is pw. constant w.r.t. an equidistant partition of $J\}$,*
3. *$\{x \in C^\infty(J; X) : x$ has compact support in the interior of $J\}$.*

Proof. The first property is the statement on density of simple functions, see Section 2.2. The second is [Rou05, Proposition 1.36], and follows from the first. The third may be obtained by mollification. □

If $J \subseteq (0, \infty)$ is an interval, for $1 \leq p < \infty$ and $\nu \in \mathbb{R}$ we define the weighted L^p space

$$L^p_\nu(J; X) := \{g : J \to X \text{ with } t^\nu g \equiv (t \mapsto t^\nu g(t)) \in L^p(J; X)\} \tag{2.7.1}$$

with the natural norm $\|g\|_{L^p_\nu(J;X)} := \|t^\nu g\|_{L^p(J;X)}$, and

$$W^{1,p}_\nu(J; X) := \{x \in L^p_\nu(J; X) : \partial_t x \in L^p_\nu(J; X)\} \tag{2.7.2}$$

with the norm given by $\|x\|^p_{W^{1,p}_\nu(J;X)} := \|x\|^p_{L^p_\nu(J;X)} + \|\partial_t x\|^p_{L^p_\nu(J;X)}$. We will often consider the case $0 \leq \nu < 1/p'$, where $p' := p/(p-1)$ is the *dual index* of p.

We collect here several useful results.

Lemma 2.7.2. *Let $1 \leq p < \infty$, let $J = (0, b) \subset \mathbb{R}$ be a bounded non-trivial interval. For any $g \in L^p_\nu(J; X)$, $g' \in L^{p'}_{-\nu}(J; X')$ and $\nu \in \mathbb{R}$ set*

$$\langle g, g' \rangle_{L^p_\nu(J;X) \times L^{p'}_{-\nu}(J;X')} := \int_J \langle g(t), g'(t) \rangle_{X \times X'} dt. \tag{2.7.3}$$

If X' is reflexive and/or separable then, for any $\nu \in \mathbb{R}$, $(L^p_\nu(J; X))'$ can be identified with $L^{p'}_{-\nu}(J; X')$ with (2.7.3) as the duality pairing.

Proof. Set $q = p'$. For $\nu = 0$, this is a standard result, see the account of [DU76; DU77], more precisely [DU77, Section IV.1, Theorem 1]: if X' has the Radon-Nikodým property then there exists an isomorphism $I : (L^p(J;X))' \to L^q(J;X')$ such that $(I^{-1}g')(g) = \int_J \langle g(t), g'(t)\rangle_{X\times X'} dt$; the Radon-Nikodým property for X' holds if X' is reflexive or if X' is separable [DU77, Section III.3, Corollary 4 and Theorem 1], also [Rya02, Section 5.4]. To pass to the case $\nu \in \mathbb{R}$ note that $I_\nu : L^p(J;X) \to L^p_\nu(J;X)$, $g \mapsto t^{-\nu}g$, and $I_{-\nu} : L^q(J;X') \to L^q_{-\nu}(J;X')$, $g' \mapsto t^\nu g'$, are isometric isomorphisms. The desired isomorphism is now given by the composition $I_{-\nu} \circ I \circ I'_\nu$. □

Lemma 2.7.3. *Let $J = (0,b) \subseteq \mathbb{R}$ be an interval. Let $Y \hookrightarrow X$ be Banach spaces. Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Set $W(t) := (\nu p + 1)^{-1} t^{\nu p+1}$ for $t > 0$, where $\nu > -1/p$. Then the map $\xi \mapsto x$, $x(t) := \xi(W(t))$ (a.e.) $t \in J$, is an isomorphism*

$$\text{from} \quad \{\xi \in L^p(W(J);Y) : \partial_t \xi \in L^q(W(J);X)\} \quad \text{onto} \quad \{x \in L^p_\nu(J;Y) : \partial_t x \in L^q_{-\nu}(J;X)\}. \tag{2.7.4}$$

Proof. We use the variable substitution $\tau = W(t)$. Then $\frac{d\tau}{dt} = t^{\nu p}$. Thus,

$$\int_J \|t^\nu x(t)\|_Y^p dt = \int_{W(J)} \|\xi(\tau)\|_Y^p t^{\nu p} t^{-\nu p} d\tau = \int_{W(J)} \|\xi(\tau)\|_Y^p d\tau, \tag{2.7.5}$$

and

$$\|\partial_t x\|^q_{L^q_{-\nu}(J;X)} = \int_{W(J)} \|t^{-\nu} c(\nu p + 1) t^{\nu p} \partial_\tau \xi(\tau)\|_X^q \frac{t^{-\nu p} d\tau}{c(\nu p + 1)} = \|\partial_\tau \xi\|^q_{L^q(W(J);X)}, \tag{2.7.6}$$

where the relation $q(p-1) = p$ was used in the last step. Hence the claim. □

Lemma 2.7.4. *Let $J = (0,b) \subseteq \mathbb{R}$ be an interval, $1 < p, q < \infty$ with $1/p + 1/q = 1$. Let $\nu > -1/p$. Let $V \hookrightarrow H \cong H' \hookrightarrow V'$ be a Gelfand triple of densely embedded Hilbert spaces, where H is identified with its dual H' via the scalar product on H. Then*

$$\{x \in L^p_\nu(J;V) : \partial_t x \in L^q_{-\nu}(J;V')\} \hookrightarrow C^0(\overline{J};H), \tag{2.7.7}$$

i.e., for any x in the space on the left there exists a continuous function $\overline{J} \to H$ which agrees with x almost everywhere on J. Moreover, for any $x, \widetilde{x} \in L^p_\nu(J;V)$ with distributional derivatives in $L^q_{-\nu}(J;V')$, and any $s, t \in \overline{J}$, $s \le t$, there holds ***the integration-by-parts formula***

$$\langle x(t), \widetilde{x}(t)\rangle_H - \langle x(s), \widetilde{x}(s)\rangle_H = \int_s^t \{\langle \partial_t x(t), \widetilde{x}(t)\rangle_{V'\times V} + \langle x(t), \partial_t \widetilde{x}(t)\rangle_{V\times V'}\}\, dt. \tag{2.7.8}$$

Proof. The embedding (2.7.7) and the formula (2.7.8) hold in the unweighted case $\nu = 0$, see e.g. [Rou05, Lemma 7.3]. The weighted case follows by means of the isomorphism of Lemma 2.7.3. □

Remark 2.7.5. Let $V \hookrightarrow H \cong H' \hookrightarrow V'$ be a Gelfand triple of densely embedded Hilbert spaces. Lemma 2.7.4 with $p = q = 2$ and $\nu = 0$ implies that the trace $x(0) \in H$ of a function $x \in H^1(J;V') \cap L^2(J;V)$ is well-defined in H and the trace map $x \mapsto x(0)$ is continuous.

In the remainder of the section, given Banach spaces $Y \hookrightarrow X$, we define the space

$$\mathcal{X}^p_\nu(J;X,Y) := W^{1,p}_\nu(J;X) \cap L^p_\nu(J;Y)$$

and endow it with the norm $[\|\cdot\|^p_{L^p_\nu(J;Y)} + \|\partial_t \cdot\|^p_{L^p_\nu(J;X)}]^{1/p}$.

Remark 2.7.6. Let $J = (0,b) \subseteq \mathbb{R}$ a nonempty interval. Let $Y \hookrightarrow X$ be Banach spaces, $1 < p, q < \infty$ with $1/p + 1/q = 1$. Let $0 \le \nu < 1/q$. Then $\mathcal{X}^p_\nu(J;X,Y)$ is precisely the set of (equivalence classes of) functions $x : J \to X$ such that $x \in W^{1,p}((\alpha,\beta);X)$ for every $\inf J < \alpha < \beta < \sup J$, and

$$t^{\nu+1/p} x(t) \in L^p(J, t^{-1}dt; Y) \quad \text{and} \quad t^{\nu+1/p} \partial_t x(t) \in L^p(J, t^{-1}dt; X). \tag{2.7.9}$$

The following lemma concerning the characterization of the trace space of $\mathcal{X}^p_\nu(J;X,Y)$ as the interpolation space $(X,Y)_{\theta,p}$ was motivated by [PS04, Proposition 3.1].

Lemma 2.7.7. *Let $J=(0,b)\subseteq\mathbb{R}$ be a nonempty interval. Let X and Y be Banach spaces with $Y\hookrightarrow X$. Let $1<p,q<\infty$ with $1/p+1/q=1$, and $0\leq\nu<1/q$. Then*

$$(X,Y)_{1/q-\nu,p}=\{x(0):x\in\mathcal{X}^p_\nu(J;X,Y)\} \tag{2.7.10}$$

and, moreover, $\chi\mapsto\inf\{\|x\|_{\mathcal{X}^p_\nu(J;X,Y)}:\chi=x(0),x\in\mathcal{X}^p_\nu(J;X,Y)\}$ defines a norm that is equivalent to $\|\cdot\|_{(X,Y)_{1/q-\nu,p}}$.

Proof. This follows from the trace method of interpolation: observing Remark 2.7.6, the proof of [Lun09, Proposition 1.13] for $J=\mathbb{R}_+$ is valid mutatis mutandis if $J=(0,b)$ is a nonempty interval. □

The following Theorem 2.7.9 is a characterization that will be important in our applications, see Lemma 8.1.1 and Remark 8.1.2, and requires a definition of certain Banach valued function spaces that are different from Bochner spaces. The proof of Theorem 2.7.9 may also be found in [Fat99, Theorem 12.2.11], cf. the remarks in [Fat05, Section 4.1].

Definition 2.7.8. For any Banach space X let $L^\infty_{w\star}(J;X')$ denote the space of functions $x:J\to X'$ such that

1. $J\ni t\mapsto\langle x(t),\chi\rangle_{X'\times X}$ is measurable for every $\chi\in X$,
2. the norm $\|x\|_{L^\infty_{w\star}(J;X')}$, given by
$$\inf\{C>0:\forall\chi\in X:|\{t\in J:|\langle x(t),\chi\rangle_{X'\times X}|>C\|\chi\|_X\}|=0\}, \tag{2.7.11}$$
is finite,
3. two such functions x and $\widetilde{x}$ are identified if and only if $\|x-\widetilde{x}\|_{L^\infty_{w\star}(J;X')}=0$.

Theorem 2.7.9. *Let X be a Banach space, $J\subseteq\mathbb{R}$ a bounded interval. Then the map $\Phi:L^\infty_{w\star}(J;X')\to L^1(J;X)'$, $g\mapsto\int_J\langle g(t),\cdot\rangle_{X'\times X}dt$ is an isometric isomorphism, i.e., $L^1(J;X)'\cong L^\infty_{w\star}(J;X')$. Further, if X is reflexive then $L^p(J;X)'\cong L^q(J;X')$ for all $1\leq p<\infty$, $1/p+1/q=1$.*

Proof. [TT69, Chapter VII, Theorem 7, Theorem 8, Theorem 10]. □

Corollary 2.7.10. *If $(\mathcal{S},\mathcal{F},\mu)$ is a σ-finite measure space then*

$$L^\infty(J\times\mathcal{S},d\nu)\cong L^1(J\times\mathcal{S},d\nu)'\cong L^1(J;L^1(\mathcal{S},d\mu))'\cong L^\infty_{w\star}(J;L^\infty(\mathcal{S},d\mu)), \tag{2.7.12}$$

where $\nu=|\cdot|\otimes\mu$ denotes the the natural product measure on $J\times\mathcal{S}$.

Proof. Let $\mathcal{B}(J)$ denote the Borel σ-algebra generated by the open subsets of J. Then $(J\times\mathcal{S},\mathcal{B}(J)\otimes\mathcal{F},\nu)$ is a σ-finite measure space [Loè78, Vol. I, Section 8.2]. Therefore we have $L^1(J\times\mathcal{S},d\nu)'\cong L^\infty(J\times\mathcal{S},d\nu)$, see e.g. [WZ77, Theorem 10.44]. The second identification is due to the Fubini-Tonelli theorem, and the third follows using Theorem 2.7.9 with $X'=L^1(\mathcal{S},d\mu)'\cong L^\infty(\mathcal{S},d\mu)$. □

2.8 Countably normed spaces

Let $J = (0, b) \subseteq \mathbb{R}^1$, $b > 0$, be an interval. Let X be a Banach space. For any $\gamma \in \mathbb{R}$ we define the weight function $\Phi_\gamma : (0, \infty) \to \mathbb{R}$, $\Phi_\gamma(t) := t^\gamma$. For all $0 \leq \beta < 1$, integers $\ell, m, k \in \mathbb{N}_0$, and $1 \leq p < \infty$, we introduce the seminorm

$$|x|_{W^{\ell,p}_{\beta,k}(J;X)} := \|\Phi_{\beta+k}\partial_t^{k+\ell}x\|_{L^p(J;X)} = \|\Phi_k\partial_t^{k+\ell}x\|_{L^p_\beta(J;X)} \tag{2.8.1}$$

for all $x \in C^\infty(J; X)$. By $W^{\ell,p}_{\beta,m}(J; X)$ we denote the completion of the set of functions in $C^\infty(J; X)$ having finite norm

$$\|x\|^p_{W^{\ell,p}_{\beta,m}(J;X)} := \sum_{k=0}^{m} |x|^p_{W^{\ell,p}_{\beta,k}(J;X)} + \begin{cases} 0, & \ell = 0, \\ \|x\|^p_{W^{\ell-1,p}(J;X)}, & \ell \geq 1, \end{cases} \tag{2.8.2}$$

w.r.t. this norm. On the "countably normed space"

$$W^{\ell,p}_{\beta,\infty}(J; X) := \bigcap_{m\in\mathbb{N}_0} W^{\ell,p}_{\beta,m}(J; X), \tag{2.8.3}$$

the expression

$$|x|_{B^{\ell,p}_\beta(J;X)} := \inf_{d>0} \sup_{k\in\mathbb{N}_0} \frac{|x|_{W^{\ell,p}_{\beta,k}(J;X)}}{d^k k!} \tag{2.8.4}$$

defines a seminorm. The countably normed space $B^{\ell,p}_\beta(J; X)$ is then defined as

$$B^{\ell,p}_\beta(J; X) := \left\{x \in W^{\ell,p}_{\beta,\infty}(J; X) : |x|_{B^{\ell,p}_\beta(J;X)} < \infty\right\}. \tag{2.8.5}$$

For the case $k = 0$ we write $W^{\ell,p}_\beta := W^{\ell,p}_{\beta,0}$. If $p = 2$ then we write $H^\ell_{\beta,k} := W^{\ell,2}_{\beta,k}$, $H^\ell_\beta := W^{\ell,2}_\beta$ and $B^\ell_\beta := B^{\ell,2}_\beta$. These definitions largely follow [BD81; GB86a; GB86b].

The following is an "almost characterization" of the spaces $B^{\ell,p}_\beta(J; X)$ that will be used in the proof of higher order regularity of solutions to parabolic evolution equations in Theorem 3.3.6.

Proposition 2.8.1. *Let $\ell \in \mathbb{N}_0$, $0 \leq \beta < 0$, $1 \leq p < \infty$ and $J = (0, b)$ an interval, X a real Banach space. Let $x \in W^{\ell,p}_{\beta,0}(J; X)$, set $y := \partial_t^\ell x \in L^p_\beta(J; X)$. For any $0 < \delta < 1$ define*

1. $\Lambda_\delta := (1 - \delta, 1 + \delta)$,
2. $J_\delta := \{\tau \in J : \tau\Lambda_\delta \subseteq J\} = (0, \frac{b}{1+\delta})$,
3. $y_\lambda(\tau) := y(\lambda\tau)$ *for* $\tau \in J_\delta$.

Then, for any $0 < \delta < 1$, the following are equivalent

1. $x \in B^{\ell,p}_\beta(\lambda J_\delta; X)$ *for all* $\lambda \in \Lambda_\delta$,
2. *the map* $\Lambda_\delta \ni \lambda \mapsto y_\lambda \in L^p_\beta(J_\delta; X)$ *is analytic.*

Proof. Let $0 < \delta < 1$. In the following, the value of the constants $C, d \geq 0$ may change from one statement to another. We have the equivalences: $x \in B^{\ell,p}_\beta(\lambda J_\delta; X)$ for all $\lambda \in \Lambda_\delta$

$\Leftrightarrow$ for all $\lambda \in \Lambda_\delta$ there exist constants $C, d \geq 0$ such that

$$|x|_{W^{\ell,p}_{\beta,k}(\lambda J_\delta;X)} \leq Cd^k k! \quad \forall k \in \mathbb{N}_0, \tag{2.8.6}$$

$\Leftrightarrow$ for all $\lambda_0 \in \Lambda_\delta$ there exist an open interval $B \subseteq \Lambda_\delta$ with $\lambda_0 \in B$, and constants $C, d \geq 0$ such that

$$|x|_{W^{\ell,p}_{\beta,k}(\lambda J_\delta;X)} \leq C d^k k! \quad \forall k \in \mathbb{N}_0 \quad \forall \lambda \in B, \tag{2.8.7}$$

$\Leftrightarrow$ for all $\lambda_0 \in \Lambda_\delta$ there exist B as above, and constants $C, d \geq 0$ such that

$$\left\|\partial_\lambda^k y_\lambda\right\|_{L^p_\beta(J_\delta;X)} = \frac{\|\Phi_k \partial_t^k y\|_{L^p_\beta(\lambda J_\delta;X)}}{\lambda^{\beta+k+1/p}} = \frac{|x|_{W^{\ell,p}_{\beta,k}(\lambda J_\delta;X)}}{\lambda^{\beta+k+1/p}} \leq C d^k k! \tag{2.8.8}$$

holds for all $k \in \mathbb{N}_0$ and $\lambda \in B$,

$\Leftrightarrow$ the map $\Lambda_\delta \ni \lambda \mapsto y_\lambda \in L^p_\beta(J_\delta; X)$ is analytic.

The last equivalence is due to Theorem 2.6.6. This shows the claim. $\square$

3 Linear parabolic evolution equations

In this chapter we give a brief overview on the theory of abstract linear parabolic evolution equations in Banach spaces. We start by formulating the problem in Section 3.1. In Section 3.2 we discuss existence and uniqueness of solutions from different perspectives: the variational (energy) approach, semigroup theory, the framework of maximal regularity, and finally by means of a symmetric variational formulation of the problem derived from the theory of self-dual Lagrangians. Some interrelations between those will be established. We subsequently address in Section 3.3 the temporal regularity of solutions, first using the semigroup theory, then using the framework of maximal regularity; the primary objective there is to show the membership of the solution to certain (countably normed) weighted Bochner-Sobolev spaces of vector valued functions introduced in Section 2.8. In Section 3.4 we briefly address semi-linear evolution equations using the general notions of Section 3.2.3.

3.1 Problem statement

For the following let $0 < \mathsf{T} < \infty$. We set $\mathsf{J} := (0, \mathsf{T}) \subset \mathbb{R}^1$. Let X be a real Banach space. We consider the following **abstract evolutionary equation**, also called **abstract initial value problem** or **abstract Cauchy problem**,

$$\begin{cases} \partial_t u(t) + (Au)(t) = g(t), & \text{(a.e.) } t \in \mathsf{J}, \\ u(0) = u^0. \end{cases} \tag{3.1.1}$$

Here,

- the **source term** $g(t) \in X$, (a.e.) $t \in \mathsf{J}$, and the **initial datum** $u^0 \in X$ are given;
- for (a.e.) $t \in \mathsf{J}$, $A(t) : \mathcal{D}(A(t)) \subseteq X \to X$ are linear, closed operators with a *common* domain $\mathcal{D}(A) = \mathcal{D}(A(t))$ for (a.e.) $t \in \mathsf{J}$; it is endowed with the norm $\|\cdot\|_{\mathcal{D}(A)}$ such that $\|\cdot\|_{\mathcal{D}(A)} \sim \|A(t)\cdot\|_X + \|\cdot\|_X$ for (a.e.) $t \in \mathsf{J}$. We recall that an operator $B : \mathcal{D}(B) \subseteq X \to X$ is closed if and only if $\mathcal{D}(B)$ is a Banach space for the graph norm $\|\cdot\|_{\mathcal{D}(B)} = \|B\cdot\|_X + \|\cdot\|_X$;
- we identify $A(\cdot)$ with the superposition mapping $u \mapsto (t \mapsto A(t)u(t))$;
- the (total) derivative $\partial_t u$ w.r.t. the temporal variable t denotes the **distributional derivative**. It may be interpreted as the X-valued linear operator which is continuous for the weak topology on X and satisfies

$$\langle (\partial_t u)(\varphi), \chi' \rangle_{X \times X'} = -\int_{\mathsf{J}} \frac{d\varphi}{dt}(t) \langle u(t), \chi' \rangle_{X \times X'} dt \quad \forall \chi' \in X' \tag{3.1.2}$$

  for all $\varphi \in C^\infty(\mathsf{J})$ with compact support in the interior of J;
- the meaning of $u(0)$ will be clarified later.

More details will be specified at a later stage. We speak of the **constant generator** case if $A(\cdot)$ in (3.1.1) is a constant operator-valued map, and **non-constant generator** case if this restriction is not present. Different notions of solution of (3.1.1) are available and their presentation will require some preparation. Equation (3.1.1) characterizes a possibly unique solution in a given sense by means of additional continuity or smoothness assumptions on g, A and also u^0, e.g. $u^0 \in \mathcal{D}(A)$. This, and related questions, is discussed in this chapter. We remark that we consider here evolution equations of parabolic type, that is we will assume that for (a.e.) $t \in \mathsf{J}$, the operator $-A(t)$ is sectorial (see Definition 3.2.14)

and densely defined in X, which implies $\mathcal{D}(A) \hookrightarrow X$ densely, see Section 3.2.2. Allowing the domains of $A(t)$ to vary or not be dense in X incurs technical difficulties beyond the scope of this work, see e.g. [Yag10, Chapter 3] and [Ama95, Chapter IV] for more on those topics.

Remark 3.1.1. The following observation will be useful. For (a.e.) $t \in \mathsf{J}$ we have

$$\partial_t u(t) + A(t)u(t) = g(t) \tag{3.1.3}$$

if and only if, with $\check{u}(t) := e^{-a_{\text{shift}}t}u(t)$ and $\check{g}(t) := e^{-a_{\text{shift}}t}g(t)$,

$$\partial_t \check{u}(t) + (A(t) + a_{\text{shift}}\text{Id})\check{u}(t) = \check{g}(t). \tag{3.1.4}$$

The equalities are understood in the sense of distributions, cf. (3.1.2).

3.2 Existence and uniqueness

The abstract Cauchy problem (3.1.1) subject to the restrictions stated in Section 3.1 was first investigated by Tanabe [Tan60] and Sobolevskii [Sob61], cf. the historical reviews in [Kat61; LM72; Yag10; Paz83]. Previously, semigroup theory, developed by Hille [Hil48] and Yosida [Yos48], was used in [Sol58; HP57] to study the constant generator case. In the Hilbert space setting, a remarkably general variational method for the abstract Cauchy problem (3.1.1) can be found in [LM72, Volume I, Chapter 3, Section 4.4] or [DL92, Chapter XVIII, §3, Section 2-4], cf. the earlier usage of the "Faedo-Galerkin" method of proof in [Lio56]. The "operational" method of Da Prato and Grisvard [DG75] was used to study abstract parabolic equations, see also [AT87; DV87], and [Prü02] for more recent developments.

As already indicated, several notions of solutions to (3.1.1) can be found in the literature, and the following list is necessarily partially redundant and ambiguous: *strong*, *strict*, *classical*, *almost everywhere*, *weak*, *hyperweak*, *very weak*, *generalized*, *variational*, *mild* solutions, or simply *solutions*. Two particularly important concepts in our context are *generalized solutions* (Definition 3.2.11) and *mild solutions* (Definition 3.2.16). These are the central subjects of Section 3.2.1 and Section 3.2.2, respectively. Subsequently, in Section 3.2.3 we elaborate on the notion of maximal regularity, where some results of Section 3.2.1 are generalized. Sections 3.2.2 and 3.2.3 also set the stage for additional regularity results given in Section 3.3. In Section 3.2.4 we derive a *symmetric* variational formulation of the abstract Cauchy problem (3.1.1) based on notions from convex analysis.

3.2.1 The variational method

Let H be a Hilbert space over $\mathbb{R}$, and $V \subseteq H$ be a dense subspace which is continuously embedded in H. Identifying H with its dual H' via the scalar product $\langle \cdot, \cdot \rangle_H$ on H, by means of the Riesz representation theorem we obtain the **Gelfand triple** (a.k.a. evolution triple)

$$V \overset{d}{\hookrightarrow} H \cong H' \overset{d}{\hookrightarrow} V'. \tag{3.2.1}$$

Then $\langle \cdot, \cdot \rangle_H = \langle \cdot, \cdot \rangle_{V \times V'}$ on $V \times V$ and $\langle \cdot, \cdot \rangle_{V \times V'}$ is in fact the unique continuous extension of $\langle \cdot, \cdot \rangle_H$ by linearity. Therefore, unless for purposes of emphasis, we drop the subscript in the notation and simply write $\langle \cdot, \cdot \rangle$ to refer to either.

Remark 3.2.1. Since V is a Hilbert space, in particular reflexive, with Lemma 2.7.2 we have the canonical identification $(L^2(\mathsf{J}; V))' = L^2(\mathsf{J}; V')$.

Assumption 3.2.2. For (a.e.) $t \in \mathsf{J}$ we are given a bilinear form $a(t; \cdot, \cdot)$ and a linear map $A(t)$,

$$a(t; \cdot, \cdot) : V \times V \to \mathbb{R} \quad \text{and} \quad A(t) : V \to V' \tag{3.2.2}$$

such that $a(t; \nu, \widetilde{\nu}) = \langle A(t)\nu, \widetilde{\nu} \rangle$ for all $\nu, \widetilde{\nu} \in V$.

Of course, each can be defined in terms of the other, and we will use whichever is convenient. For the symmetric and the the anti-symmetric part of A we set

$$\widehat{A}(t) := \frac{1}{2}\left(A(t) + A(t)'\right) \quad \text{and} \quad \widetilde{A}(t) := \frac{1}{2}\left(A(t) - A(t)'\right), \quad \text{(a.e.) } t \in \mathsf{J}, \tag{3.2.3}$$

and abbreviate

$$P(t)u(t) := -\partial_t u(t) - \widetilde{A}(t)u(t), \quad \text{(a.e.) } t \in \mathsf{J}. \tag{3.2.4}$$

We will frequently drop the dependence on t for convenience of notation.

Example 3.2.3. Examples for the spaces V and H that we have in mind are (closed subspaces of) Sobolev or Bochner spaces, such as $V = H_0^1(D)$ and $H = L^2(D)$ for the archetypal diffusion problem in a bounded domain $D \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary condition; or $V = L^2(\Omega, d\mathbb{P}; H_0^1(D))$ and $H = L^2(\Omega, d\mathbb{P}; L^2(D))$ for diffusion problems subject to a parameter $\omega \in \Omega$ in a probability space $(\Omega, \Sigma, \mathbb{P})$.

Assumption 3.2.4. There exist $0 < a_{\min} \leq a_{\max} < \infty$ and $a_{\text{shift}} \geq 0$ such that for all $\nu, \widetilde{\nu} \in V$ and (a.e.) $t \in \mathsf{J}$ there hold

1. $|a(t; \nu, \widetilde{\nu})| \leq a_{\max} \|\nu\|_V \|\widetilde{\nu}\|_V$, (boundedness)
2. $a(t; \nu, \nu) + a_{\text{shift}} \|\nu\|_H^2 \geq a_{\min} \|\nu\|_V^2$, (Gårding inequality)
3. $\mathsf{J} \ni s \mapsto a(s; \nu, \widetilde{\nu}) \in \mathbb{R}$ is measurable. (measurability)

An example of $a(t; \cdot, \cdot)$ satisfying Assumption 3.2.4 is given in (8.1.5). Using Remark 3.1.1, we may w.l.o.g. assume $a_{\text{shift}} = 0$. Note that the Gårding inequality in Assumption 3.2.4 is equivalent to

$$\langle (\widehat{A}(t) + a_{\text{shift}} \operatorname{Id})\nu, \nu\rangle = \langle (A(t) + a_{\text{shift}} \operatorname{Id})\nu, \nu\rangle \geq a_{\min} \|\nu\|_V^2 \quad \forall \nu \in V, \tag{3.2.5}$$

since the anti-symmetric part satisfies $\langle \widetilde{A}(t)\nu, \nu\rangle = 0$ for all $\nu \in V$. Here, Id denotes the embedding $V \hookrightarrow V'$. Consider now the formal mapping

$$(A, v, \widetilde{v}) \mapsto \int_{\mathsf{J}} \langle A(t)v(t), \widetilde{v}(t)\rangle dt, \quad v, \widetilde{v} \in L^2(\mathsf{J}; V). \tag{3.2.6}$$

We can interpret the family of operators $\{A(t)\}_{t \in J}$ as a mapping

1. $A_1 : L^1(\mathsf{J}; V \otimes V) \to \mathbb{R}$ linear,
2. $A_2 : L^2(\mathsf{J}; V) \times L^2(\mathsf{J}; V) \to \mathbb{R}$ bilinear,
3. $A_3 : L^2(\mathsf{J}; V) \to L^2(\mathsf{J}; V')$ linear.

Using Remark 3.2.1 and (2.3.2), A_2 is bounded if and only if A_3 is.

Proposition 3.2.5. *Boundedness and measurability in the sense of Assumption 3.2.4 hold if and only if* $A, A', \widehat{A}, \widetilde{A} \in (L^1(\mathsf{J}; V \otimes V))'$.

Proof. Since $(L^1(\mathsf{J}; V \otimes V))'$ is a vector space, it suffices to consider A and A'. Using Theorem 2.7.9 and the tensor product identification (2.3.2) observe that

$$(L^1(\mathsf{J}; V \otimes V))' \cong L^\infty_{w\star}(\mathsf{J}; (V \otimes V)') \cong L^\infty_{w\star}(\mathsf{J}; \mathcal{L}(V, V')). \tag{3.2.7}$$

Now $A \in L^\infty_{w\star}(\mathsf{J}; \mathcal{L}(V, V'))$ is merely a restatement of boundedness and measurability in Assumption 3.2.4. Further, these hypotheses hold for A if and only if they hold for the adjoint A'. This completes the proof. □

Similarly, the boundedness and measurability hypotheses of Assumption 3.2.4 imply

$$\forall \nu \in V: \quad [t \mapsto a(t;\nu,\cdot)] \in L^\infty_{w\star}(\mathsf{J};V') \cong (L^1(\mathsf{J};V))'. \tag{3.2.8}$$

Therefore, for all $v \in L^2(\mathsf{J};V)$, the integral $\int_{\mathsf{J}} a(t;\widetilde{v}(t),v(t))dt$ is declared for $\widetilde{v} = \chi_J \otimes \nu$ with arbitrary $J \subseteq \mathsf{J}$ measurable and $\nu \in V$, and with this, for all elements $\widetilde{v} \in L^2(\mathsf{J}) \otimes V$ by linearity and continuity. If V is separable, we have the identification $L^2(\mathsf{J}) \otimes V \cong L^2(\mathsf{J};V)$, see (2.3.5), and therefore (3.2.6) is well-defined. This (together with applications in partial differential equations) motivates to the following assumption.

Assumption 3.2.6. V is separable.

V being separable implies that also H and V' are separable by density of V in H. These considerations result in the following Proposition 3.2.7.

Proposition 3.2.7. *Assumption 3.2.4 and Assumption 3.2.6 imply*

$$(\widehat{A} + a_{\text{shift}}\,\text{Id}), (A + a_{\text{shift}}\,\text{Id}) \in \text{Iso}(L^2(\mathsf{J};V), L^2(\mathsf{J};V')). \tag{3.2.9}$$

Proof. Both are well-defined, $L^2(\mathsf{J};V)$-elliptic and bounded. The claim is due to the Lax-Milgram lemma (Theorem 4.1.2). □

Anticipating the following well-posedness results (Theorem 3.2.9 and Theorem 3.2.10) for the abstract Cauchy problem (3.1.1) we define

$$\mathcal{X} := L^2(\mathsf{J};V) \cap H^1(\mathsf{J};V') \quad \text{and} \quad \mathcal{Y} := L^2(\mathsf{J};V) \times H. \tag{3.2.10}$$

The norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$ that we use here are given by

$$\|u\|^2_{\mathcal{X}} := \|u\|^2_{L^2(\mathsf{J};V)} + \|\partial_t u\|^2_{L^2(\mathsf{J};V')} \quad \forall u \in \mathcal{X} \tag{3.2.11}$$

and

$$\|v\|^2_{\mathcal{Y}} := \|v_1\|^2_{L^2(\mathsf{J};V)} + \|v_2\|^2_H \quad \forall v = (v_1,v_2) \in \mathcal{Y}. \tag{3.2.12}$$

Note that $\|\cdot\|^2_{L^2(\mathsf{J};V)} + \|\cdot\|^2_{H^1(\mathsf{J};V')}$, which is sometimes used in place of $\|\cdot\|^2_{\mathcal{X}}$, yields an equivalent norm on $\mathcal{X}$. Only slightly more generally, we have the following result.

Lemma 3.2.8. *Let $\mathsf{J} = (0,\mathsf{T})$. Let Assumption 3.2.6 and Assumption 3.2.4 hold with $a_{\text{shift}} = 0$. Let $0 \le t_0, \tau \le \mathsf{T}$ and $\beta,\gamma \ge 0$ be arbitrary. Define for all $u \in C^\infty(\overline{\mathsf{J}};V)$*

$$\|u\|^2_{\mathcal{X},\beta,\gamma,\tau} := \int_{\mathsf{J}} \|u(t)\|^2_V dt + \int_{\mathsf{J}} \left\{\|\partial_t u(t)\|^2_{V'} + \beta\|u(t)\|^2_{V'}\right\} dt + \gamma\|u(\tau)\|^2_H,$$
$$\|u\|^2_{\mathcal{E},t_0} := \int_{\mathsf{J}} \langle \widehat{A}(t)u(t), u(t)\rangle dt + \int_{\mathsf{J}} \langle \widehat{A}(t)^{-1}P(t)u(t), P(t)u(t)\rangle dt + \|u(t_0)\|^2_H.$$

Then $\|\cdot\|_{\mathcal{X},\beta,\gamma,\tau} \sim \|\cdot\|_{\mathcal{E},t_0}$ are norms on $\mathcal{X}$ equivalent to $\|\cdot\|_{\mathcal{X}}$.

Proof. We first note that the integrals are well-defined by Proposition 3.2.7. By the continuity of the embedding $V \hookrightarrow V'$ and Lemma 2.7.4 the expression $\|\cdot\|_{\mathcal{X}} := \|\cdot\|_{\mathcal{X},0,0,\tau}$ is a norm, equivalent to $\|\cdot\|_{\mathcal{X},\beta,\gamma,\tau}$. Therefore it suffices to verify $\|\cdot\|_{\mathcal{X}} \sim \|\cdot\|_{\mathcal{E},t_0}$. Since $\|\cdot\|_{\mathcal{E},t_0} \lesssim \|\cdot\|_{\mathcal{X}}$ is straightforward, we concentrate on showing $\|\cdot\|_{\mathcal{X}} \lesssim \|\cdot\|_{\mathcal{E},t_0}$. Assume to the contrary that there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{\mathsf{J}};V)$ (with $V \hookrightarrow V'$ densely, this space is dense in $\mathcal{X}$ by [LM72, Volume 1, Chapter I, Theorem 2.1]) with $\|x_n\|_{\mathcal{E},t_0} \to 0$ as $n \to \infty$, yet $\|x_n\|_{\mathcal{X}} \ge 1$ for all $n \in \mathbb{N}$. Then, as $n \to \infty$, we have $x_n \to 0$ in $L^2(J;V)$, thus also $Ax_n \to 0$ in $L^2(J;V')$, as well as $\partial_t x_n + Ax_n = -Px_n + \widehat{A}x_n \to 0$ in $L^2(J;V')$. This implies $\partial_t x_n \to 0$ in $L^2(J;V')$ which is a contradiction to $\|x_n\|_{\mathcal{X}} \ge 1$. □

We will call a norm on $\mathcal{X}$ or $\mathcal{Y}$ defined in terms of the operator $A(t)$, such as $\|\cdot\|_{\mathcal{E},t_0}$ a **parabolic energy norm**. For consistency of notation we define the spaces

$$\mathcal{Y}_1 := L^2(\mathsf{J}; V) \quad \text{and} \quad \mathcal{Y}_2 := H, \tag{3.2.13}$$

thus $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$.

We now state the announced well-posedness results.

Theorem 3.2.9. *Let Assumption 3.2.4 hold. Then for all $(g, u^0) \in \mathcal{Y}'$ there exists a unique $u \in \mathcal{X}$ such that $\partial_t u(t) + A(t)u(t) = g(t)$, (a.e.) $t \in \mathsf{J}$ and $u(0) = u^0$ hold; the solution $u \in \mathcal{X}$ depends continuously on the data $(g, u^0) \in \mathcal{Y}'$.*

Proof. [DL92, Chapter XVIII, §3, Theorem 2]. □

For our purposes it is useful to formulate this result in terms of operators between Hilbert spaces. We recall first that the trace map $\mathcal{X} \ni u \mapsto u(0) \in H$ is continuous, and there holds

$$\|u(0)\|_H \leq \|u \mapsto u(0)\|_{\mathcal{L}(\mathcal{X},H)} \|u\|_{\mathcal{X}} \quad \text{for all} \quad u \in \mathcal{X}, \tag{3.2.14}$$

see [LM72, Chapter 1] or Remark 2.7.5. We define the space-time **parabolic operator** $\mathcal{B} : \mathcal{X} \to \mathcal{Y}'$ as the linear mapping given by

$$\langle \mathcal{B}u, v\rangle_{\mathcal{Y}'\times\mathcal{Y}} := \int_{\mathsf{J}} \langle (\partial_t + A(t))u(t), v_1(t)\rangle dt + \langle u(0), v_2\rangle, \quad \begin{cases} u \in \mathcal{X}, \\ v = (v_1, v_2) \in \mathcal{Y}. \end{cases} \tag{3.2.15}$$

Under Assumption 3.2.4, indeed $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$. In fact, $\mathcal{B} \in \mathrm{Iso}(\mathcal{X}, \mathcal{Y}')$:

Theorem 3.2.10. *For any $\mathcal{F} \in \mathcal{Y}'$ there exists a unique solution $u \in \mathcal{X}$ of the variational problem*

$$\langle \mathcal{B}u, v\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \mathcal{F}v \quad \forall v \in \mathcal{Y}. \tag{3.2.16}$$

Moreover, the solution map $\mathcal{F} \mapsto u$ is continuous.

Proof. See [SS09, Theorem 5.1]. □

Definition 3.2.11. Let $(g, u^0) \in \mathcal{Y}'$ be given, define $\mathcal{F} : \mathcal{Y} \to \mathbb{R}$ by

$$\mathcal{F}v := \langle u^0, v_2\rangle + \int_{\mathsf{J}} \langle g(t), v_1(t)\rangle dt \quad \forall v = (v_1, v_2) \in \mathcal{Y}. \tag{3.2.17}$$

We call $u \in \mathcal{X}$ a **generalized solution** of (3.1.1) if (3.2.16) holds.

By Theorem 3.2.10, a generalized solution of (3.2.16) exists and is unique. We remark that our terminology follows [Ama95, Section III.1.3]. Observe that in the equation $\mathcal{B}u = \mathcal{F} \in \mathcal{Y}'$, which is a restatement of (3.2.16) by reflexivity of Hilbert spaces, the initial condition $u(0) = u^0$ is enforced by means of a Lagrange multiplier $v_2 \in H$.

Remark 3.2.12. An alternative variational formulation is used in e.g. [BJ90; CS11]: integrating (3.1.1) against a test function $v_1 \in C^\infty(\overline{\mathsf{J}}; V)$ with $v_1(\mathsf{T}) = 0$ we obtain

$$\int_{\mathsf{J}} \langle u, -\partial_t v_1 + A' v_1\rangle dt + \underbrace{\langle u(\mathsf{T}), v_1(\mathsf{T})\rangle}_{=0} = \underbrace{\langle u^0, v_1(0)\rangle}_{=\langle u(0), v_1(0)\rangle} + \int_{\mathsf{J}} \langle g, v_1\rangle dt. \tag{3.2.18}$$

This leads to the definition of trial and test spaces as

$$\widetilde{\mathcal{X}} = L^2(\mathsf{J}; V) \quad \text{and} \quad \widetilde{\mathcal{Y}} = L^2(\mathsf{J}; V) \cap H^1_{0)}(\mathsf{J}; V') \tag{3.2.19}$$

where the subscript indicates vanishing trace at $t = \mathsf{T}$. In this formulation the initial value enters the formulation *naturally*; it is neither incorporated in the trial space, nor is it enforced by an additional equation via a Lagrange multiplier. A function $u \in \widetilde{\mathcal{X}}$ satisfying (3.2.18) for all admissible v_1 is called a **weak solution** in [Ama95, Section V.2.6], cf. [Rou05, Definition 8.2] (which is different from the "weak solutions" of [Bal77; Eva98]).

3.2.2 Semigroup theory

The semigroup approach is naturally suited to study abstract evolution equations in the constant generator case, but has been extended to the non-constant generator case. The pertinent notions here are given in the following definitions, followed by a result on existence of solutions to the abstract Cauchy problem (3.1.1).

Definition 3.2.13. Let $u^0 \in X$.

- Let $(A, g) \in C^0([0, \mathsf{T}]; \mathcal{L}(\mathcal{D}(A), X) \times X)$. A function
$$u \in C^1([0, \mathsf{T}]; X) \cap C^0([0, \mathsf{T}]; \mathcal{D}(A)) \tag{3.2.20}$$
is said to be a **strict solution** of (3.1.1) in the interval $[0, \mathsf{T}]$ if $\partial_t u(t) + A(t)u(t) = g(t)$ for all $t \in [0, \mathsf{T}]$ and $u(0) = u^0$.
- Let again $(A, g) \in C^0([0, \mathsf{T}]; \mathcal{L}(\mathcal{D}(A), X) \times X)$. A function $u \in C^0([0, \mathsf{T}]; X)$ is said to be a **strong solution** of (3.1.1) in the interval $[0, \mathsf{T}]$ if there exists
$$\{x_n\}_{n\in\mathbb{N}} \subset C^1([0, \mathsf{T}]; X) \cap C^0([0, \mathsf{T}]; \mathcal{D}(A)) \tag{3.2.21}$$
such that $x_n \to u$ and $\partial_t x_n + Ax_n \to f$ in $C^0([0, \mathsf{T}]; X)$ and $x_n(0) \to u^0$ in X as $n \to \infty$.
- Let now $(A, g) \in C^0((0, \mathsf{T}]; \mathcal{L}(\mathcal{D}(A), X) \times X)$. A function
$$u \in C^1((0, \mathsf{T}]; X) \cap C^0((0, \mathsf{T}]; \mathcal{D}(A)) \cap C^0([0, \mathsf{T}]; X) \tag{3.2.22}$$
is said to be a **classical solution** of (3.1.1) in the interval $[0, \mathsf{T}]$ if $\partial_t u(t) + A(t)u(t) = g(t)$ for all $t \in (0, \mathsf{T}]$ and $u(0) = u^0$.

Definition 3.2.14 (Following Definition 2.0.1 in [Lun95])**.** Let X be a Banach space. An operator $B : \mathcal{D}(B) \subseteq X \to X$ is called **sectorial** if there exist $a_{\text{shift}} \in \mathbb{R}$, $\frac{\pi}{2} < \theta < \pi$ and $M > 0$ such that

1. $\lambda \in \rho(B)$,
2. $\|(\lambda \operatorname{Id} - B)^{-1}\|_{\mathcal{L}(X)} |\lambda - a_{\text{shift}}| \leq M$,

hold for all $\lambda \in \{\lambda \in \mathbb{C} \setminus \{a_{\text{shift}}\} : |\arg(\lambda - a_{\text{shift}})| < \theta\}$.

The classical treatise [Paz83] does not contain a definition of a "sectorial operator", but characterizes a densely defined operator which is sectorial in the sense of Definition 3.2.14 with $a_{\text{shift}} = 0$ as the *infinitesimal generator of an analytic semigroup*, see [Paz83, Chapter 2, Theorem 5.2] and the foregoing remarks therein. In turn, such an operator B is necessarily closed and densely defined, see [Paz83, Chapter 1, Theorem 5.3]. In particular, a sectorial operator with $a_{\text{shift}} = 0$ is a sectorial operator in X in the sense of [EN06; Yag10]. Let us remark that Assumption 3.2.2 and Assumption 3.2.4 lead to a family of sectorial operators as in Assumption 3.2.15, see [Sch99, Proposition 2.3] or [Yag10, Chapter 2, Section 1.1], and Section 3.3.2.

A notion of solution to the abstract Cauchy problem (3.1.1) is obtained using the following assumption.

Assumption 3.2.15. We assume that $A(t) : \mathcal{D}(A) \to X$, $0 \leq t \leq \mathsf{T}$, is family of linear operators

1. with a common domain $\mathcal{D}(A) \subseteq X$,
2. such that $-A(t)$ is sectorial, $0 \leq t \leq \mathsf{T}$,
3. and $\mathcal{D}(A) \overset{d}{\hookrightarrow} X$, i.e., $A(t)$ is densely defined, $0 \leq t \leq \mathsf{T}$.

Definition 3.2.16 (Adapted from Definition 6.0.1 in [Lun95])**.** A family
$$\{G(t, s) : 0 \leq s \leq t \leq \mathsf{T}\} \subseteq \mathcal{L}(X) \tag{3.2.23}$$
is said to be a **parabolic evolution operator** for the problem (3.1.1) if

1. $G(t,s)G(s,r) = G(t,r)$, $G(s,s) = \mathrm{Id}$ for all $0 \le r \le s \le t \le \mathsf{T}$,

2. $G(t,s) \in \mathcal{L}(X, \mathcal{D}(A))$ for all $0 \le s < t \le \mathsf{T}$,

3. $(s, \mathsf{T}] \ni t \mapsto G(t,s)$ is differentiable with values in $\mathcal{L}(X)$, and

$$\partial_t G(t,s) = -A(t)G(t,s), \quad 0 \le s < t \le \mathsf{T}. \tag{3.2.24}$$

If $g \in L^1(\mathsf{J}; X)$ and $u^0 \in X$, the **mild solution of** (3.1.1) is defined by the variation-of-constants formula

$$u(t) = G(t,0)u^0 + \int_0^t G(t,s)g(s)ds, \quad 0 \le t \le \mathsf{T}. \tag{3.2.25}$$

Remark 3.2.17. A parabolic evolution operator in the sense of Definition 3.2.16:

1. is a parabolic evolution operator with regularity subspace $\mathcal{D}(A)$ in the sense of the definition given in [Ama95, Chapter II, Section 2].

2. is unique, if it exists [Ama95, Chapter II, Remark 2.1.2].

One has to impose some smoothness on the map $t \mapsto A(t)$ for the parabolic evolution operator $G(\cdot,\cdot)$, also called "fundamental solution" or "propagator", to exist. Its existence is desirable, as typically any solution, be it strict or classical, etc., satisfies the variation-of-constants formula (3.2.25), also called "Duhamel's formula", from which a number of properties of the solution (in a given sense) can be deduced [Lun95, Section 6], [Tan97, Section 6.10], [Yag10, Chapter 3, Section 6]; there exist, however, other representation formulas, see e.g. [AT87], the survey [Acq93], and references in those works. One of the technically easier possibilities for obtaining a parabolic evolution operator is to assume that $A(\cdot)$ is Hölder continuous (more general conditions are discussed in e.g. [DG84; But92; PS01; MT01]).

Assumption 3.2.18. There exists $0 < \alpha < 1$ such that $A \in C^\alpha([0,\mathsf{T}]; \mathcal{L}(\mathcal{D}(A), X))$.

Theorem 3.2.19. *Under Assumption 3.2.18 there exists a parabolic evolution operator $G(\cdot,\cdot)$. Let $u^0 \in X$. The function u defined by the variation-of-constants formula* (3.2.25)

1. *coincides with any strict solution of problem* (3.1.1) *if $g \in C^0([0,\mathsf{T}]; X)$ and $u^0 \in \mathcal{D}(A)$,*

2. *is the unique strong solution of problem* (3.1.1) *if $g \in C^0([0,\mathsf{T}]; X)$,*

3. *is, if such exists, the classical solution of problem* (3.1.1) *if $g \in C^0((0,\mathsf{T}]; X) \cap L^1(\mathsf{J}; X)$.*

Proof. See [Lun95, Definition 6.1.7] and [Lun95, Corollary 6.2.2, Corollary 6.2.3 and Corollary 6.2.4]. □

Properties of $G(\cdot,\cdot)$ can be found in [Lun95, Corollary 6.1.10 and Proposition 6.2.6], e.g. for every $t \in (0,\mathsf{T}]$ we have $G(t,\cdot) \in C^1([0,t); \mathcal{L}(X, \mathcal{D}(A)))$, and

$$\partial_s G(t,s)u = -G(t,s)A(s)u \quad \forall u \in \mathcal{D}(A), \quad 0 \le s < t \le \mathsf{T}. \tag{3.2.26}$$

In the case $\alpha = 0$, a parabolic evolution operator was constructed in [PS01], cf. Theorem 3.2.21.

3.2.3 Maximal regularity

'Maximal regularity' is studied in a framework which deemphasizes the temporal dimension, thus considering problem (3.1.1) as an operator equation of the form $\mathcal{A}u = (g, u^0)$ between suitable Banach spaces, where $\mathcal{A} = (\partial_t + A, u \mapsto u(0))$. This allows to avoid a loss of regularity, where the temporal derivative of the solution is less regular than the right-hand side, as observed in analytic semigroups [Ama95, Introduction to Chapter III], and to obtain the 'maximal regularity' that is possible in general for the given class of right hand sides.

Recall that $\mathcal{D}(A)$ denotes the common domain of the family of linear operators $A(t) : \mathcal{D}(A) \subseteq X \to X$, $t \in \mathsf{J}$. Assume that for (a.e.) $t \in \mathsf{J}$

1. $A(t)$ is a closed operator,
2. $A(t)$ is densely defined.

Then $\mathcal{D}(A)$ equipped with the norm $\|A(t)\cdot\|_X$ is a Banach space densely embedded in X for (a.e.) $t \in \mathsf{J}$. We assume that all these norms are uniformly equivalent to a norm denoted by $\|\cdot\|_{\mathcal{D}(A)}$, hence

$$\mathcal{D}(A) \overset{d}{\hookrightarrow} X. \tag{3.2.27}$$

If $A \in C^0(\overline{\mathsf{J}}; \mathcal{L}(\mathcal{D}(A), X))$ then this is necessarily so [PS01, p. 409]. For any $1 < p < \infty$ and $0 \leq \nu < 1/p'$, where $p' := p/(p-1)$ is the index dual to p, we define the spaces (cf. Section 2.7)

$$\mathcal{X}^p_\nu := W^{1,p}_\nu(\mathsf{J}; X) \cap L^p_\nu(\mathsf{J}; \mathcal{D}(A)) \tag{3.2.28}$$

with the norm $\|\cdot\|_{\mathcal{X}^p_\nu}$ given by

$$\|\cdot\|^p_{\mathcal{X}^p_\nu} := \|\cdot\|^p_{L^p_\nu(\mathsf{J};\mathcal{D}(A))} + \|\partial_t \cdot\|^p_{L^p_\nu(\mathsf{J};X)}. \tag{3.2.29}$$

Let $0 \leq \nu < 1/p'$. Recall from Lemma 2.7.7 that the trace $u(0) \in X_{\nu,p}$ is well-defined for functions $u \in \mathcal{X}^p_\nu$, where

$$X_{\nu,p} := (X, \mathcal{D}(A))_{1/p'-\nu,p}, \tag{3.2.30}$$

and the map $u \mapsto u(0)$ is continuous. The interpolation space $(X, \mathcal{D}(A))_{\theta,p}$ is also denoted by $D_A(\theta, p)$ in the literature. When $\mathcal{X}^p_\nu$ is defined w.r.t. a different interval, say J, we write $\mathcal{X}^p_\nu(J)$, etc. We further define the data spaces

$$\widetilde{\mathcal{Y}}^p_\nu := \widetilde{\mathcal{Y}}^p_{\nu,1} \times \widetilde{\mathcal{Y}}^p_{\nu,2} := L^p_\nu(\mathsf{J}; X) \times X_{\nu,p} \tag{3.2.31}$$

with the norm given by $\|\cdot\|^p_{\widetilde{\mathcal{Y}}^p_\nu} := \|\cdot\|^p_{\widetilde{\mathcal{Y}}^p_{\nu,1}} + \|\cdot\|^p_{\widetilde{\mathcal{Y}}^p_{\nu,2}} := \|\cdot\|^p_{L^p_\nu(\mathsf{J};X)} + \|\cdot\|^p_{X_{\nu,p}}$.

We remark that $\mathcal{X}^p_\nu$ and $\widetilde{\mathcal{Y}}^p_\nu$ depend on the (sometimes subjective) choice of $\mathcal{D}(A)$ and the norm on $\mathcal{D}(A)$ in applications.

The central notion of this section is the following.

Definition 3.2.20. Let $1 < p < \infty$ and $0 \leq \nu < 1/p'$. We say that A **has maximal L^p_ν regularity** (or L^p if $\nu = 0$), if there exists a constant $c_{\nu,p}(A) \geq 0$ such that for all $(g, u^0) \in \widetilde{\mathcal{Y}}^p_\nu$ there exists a unique $u \in \mathcal{X}^p_\nu$ satisfying

1. $\partial_t u + Au = g$ in $\widetilde{\mathcal{Y}}^p_{\nu,1}$,
2. $u(0) = u^0$ in $\widetilde{\mathcal{Y}}^p_{\nu,2}$,
3. $\|u\|_{\mathcal{X}^p_\nu} \leq c_{\nu,p}(A)\|(g, u^0)\|_{\widetilde{\mathcal{Y}}^p_\nu}$.

Maximal L^p_ν regularity can accommodate solutions with "rough" initial datum, i.e., belonging to the intermediate space $(X, \mathcal{D}(A))_{1/p'-\nu,p}$, with regularity in weighted Sobolev-Bochner spaces $\mathcal{X}^p_\nu$. In applications, the intermediate space usually captures a degree of differentiability, for instance in the case $X = L^2(D)$, $\mathcal{D}(A) = H^1_0(D) \cap H^2(D)$, $A(t) = -\Delta$ on a bounded convex domain $D \subset \mathbb{R}^d$, cf. Example 3.3.5 and Example 3.3.8. Note that Theorem 3.2.10 is a statement on maximal L^2 regularity of A for $X := V'$ and $\mathcal{D}(A) := V$. Another early maximal regularity result in the Hilbert space setting is [Sim64]. The following generalizations are more recent.

Theorem 3.2.21 (Non-constant generator case, $\nu = 0$)**.** *Assume*

1. $A \in C^0(\bar{\mathsf{J}}; \mathcal{L}(\mathcal{D}(A), X))$,
2. *the constant mapping* $\bar{A} : t \mapsto A(s)$ *has maximal* L^p *regularity for all* $s \in \bar{\mathsf{J}}$.

Then A *has maximal* L^p *regularity.*

Proof. See [PS01, Theorem 2.5] or [Ama04, Theorem 7.1, Remark 7.1]. □

Theorem 3.2.22 (Constant generator case, $\nu \geq 0$)**.** *Let* $1 < p < \infty$ *and* $0 \leq \nu < 1/p'$. *Assume that* $A(\cdot)$ *is constant. Then* A *has maximal* L^p *regularity if and only if it has maximal* L^p_ν *regularity.*

Proof. See [PS04], in particular Remark 3.3 therein. □

We are, however, interested in the non-constant generator case with $\nu \geq 0$. It may be derived from the local result for quasi-linear parabolic equations [KPW10, Theorem 2.1]. Instead, we show that among operators with maximal L^p regularity, the property of maximal L^p_ν regularity is stable under perturbations.

Proposition 3.2.23. *Let* $1 < p < \infty$ *and* $0 \leq \nu < 1/p'$. *Let* $A, \bar{A} \in L^\infty(\mathsf{J}; \mathcal{L}(\mathcal{D}(A), X))$. *Assume that* $\bar{A}$ *has maximal* L^p_ν *regularity. Assume further that*

$$\rho := c_{\nu,p}(\bar{A}) \|\bar{A} - A\|_{L^\infty(\mathsf{J}; \mathcal{L}(\mathcal{D}(A), X))} < 1. \tag{3.2.32}$$

If A *has maximal* L^p *regularity then it also has maximal* L^p_ν *regularity.*

Proof. Let $(g, u^0) \in \widetilde{\mathcal{Y}}^p_\nu$ be given. Assume that A has maximal L^p regularity. Then there exists a unique $u_\star \in \mathcal{X}^p_0$ satisfying

$$1. \quad \partial_t u_\star + A u_\star = t^\nu g, \qquad 2. \quad u_\star(0) = 0, \quad \text{and} \quad 3. \quad \|u_\star\|_{\mathcal{X}^p_0} \leq c_{0,p}(A) \|g\|_{L^p_\nu(\mathsf{J}; X)}.$$

Since the maps

$$\{u \in \mathcal{X}^p_0 : \|u(0)\|_X = 0\} \to \{\widetilde{u} \in \mathcal{X}^p_\nu : \|\widetilde{u}(0)\|_X = 0\}, \quad u \mapsto t^{-\nu} u \tag{3.2.33}$$

and

$$\{u \in \mathcal{X}^p_0 : \|u(0)\|_X = 0\} \to L^p_\nu(\mathsf{J}; X), \quad u \mapsto t^{-\nu-1} u \tag{3.2.34}$$

are continuous [PS04, Proposition 2.2], there exists a $C > 0$ independent of g and u^0 such that

$$\max\left\{\|t^{-\nu} u_\star\|_{\mathcal{X}^p_\nu}, \|t^{-\nu-1} u_\star\|_{L^p_\nu(\mathsf{J}; X)}\right\} \leq C \|g\|_{L^p_\nu(\mathsf{J}; X)}. \tag{3.2.35}$$

We claim that there exists a unique $\Delta u \in \mathcal{X}^p_\nu$ which solves

$$\partial_t \Delta u + \bar{A} \Delta u = \nu t^{-\nu-1} u_\star + (\bar{A} - A) \Delta u \in L^p_\nu(\mathsf{J}; X), \quad \Delta u(0) = u^0. \tag{3.2.36}$$

To see this, define the mapping $\Phi : \mathcal{X}^p_\nu \to \mathcal{X}^p_\nu$ by $\Phi(u) = \widetilde{u}$, the solution of

$$\partial_t \widetilde{u} + \bar{A} \widetilde{u} = \nu t^{-\nu-1} u_\star + (\bar{A} - A) u, \quad \widetilde{u}(0) = u^0. \tag{3.2.37}$$

Then Φ is well-defined, since $t^{-\nu-1}u_\star \in L^p_\nu(\mathsf{J};X)$ and $(\bar{A}-A)u \in L^p_\nu(\mathsf{J};X)$ for all $u \in \mathcal{X}^p_\nu$. Moreover, Φ is a strict contraction:

$$\|\Phi(u)-\Phi(v)\|_{\mathcal{X}^p_\nu} \le c_{\nu,p}(\bar{A})\|(\bar{A}-A)(u-v)\|_{L^p_\nu(\mathsf{J};X)} \le \rho\|u-v\|_{\mathcal{X}^p_\nu} \tag{3.2.38}$$

with $\rho < 1$. Therefore, the Banach fixed point theorem ensures the existence of a unique fixed point $\Delta u = \Phi(\Delta u)$, and

$$(1-\rho)\|\Delta u\|_{\mathcal{X}^p_\nu} \le c_{\nu,p}(\bar{A})\|(\nu t^{-\nu-1}u_\star, u^0)\|_{\widetilde{\mathcal{Y}}^p_\nu} \le C\|(g,u^0)\|_{\widetilde{\mathcal{Y}}^p_\nu} \tag{3.2.39}$$

holds with $C > 0$ independent of (g,u^0). By construction, $u := t^{-\nu}u_\star + \Delta u$ satisfies

$$1.\quad \partial_t u + Au = g, \qquad 2.\quad u(0) = u^0, \quad \text{and} \quad 3.\quad \|u\|_{\mathcal{X}^p_\mu} \le C\|(g,u^0)\|_{\widetilde{\mathcal{Y}}^p_\nu}, \tag{3.2.40}$$

with $C > 0$ independent of (g,u^0). This shows the claim. □

3.2.4 Self-dual Lagrangians

In this section we obtain an alternative variational formulation for the abstract Cauchy problem (3.1.1) in the Hilbert space setting and establish a connection to the generalized solutions in the sense of Definition 3.2.11. This formulation is *coercive* on $\mathcal{X}$ and *symmetric* on $L^2(\mathsf{J};H)$ (with domain $\mathcal{X} \hookrightarrow L^2(\mathsf{J};H)$), where the space $\mathcal{X}$ is defined in (3.2.46). Hence, given any finite-dimensional subspace of $\mathcal{X}$ to serve as a trial *and* test space, the resulting Galerkin system will be stable. This variational formulation derives from a general variational principle exposed in [Gho07], which is formulated in the language of convex analysis and, in particular, is not restricted to the linear, Hilbert space setting. For parabolic problems, this has been known as the Brézis-Ekeland principle [BE76], cf. the bibliographical remarks in [Rou05, Section 8.11]. This variational formulation reveals some of the structure of the parabolic evolution equation. Unfortunately (for numerical computations), it involves the inverse of the spatial operator A and in the simple case of the heat equation, $A = -\Delta$, the inverse A^{-1} is a non-local operator. Efficient algorithms based on this formulation using an approximate realization of the inverse A^{-1} may still be possible.

Let $V \hookrightarrow H \cong H' \hookrightarrow V'$ be a Gelfand triple of separable real Hilbert spaces with dense embeddings. Recall from (3.2.3)–(3.2.4) that $\widehat{A}$ and $\widetilde{A}$ denote the symmetric and the anti-symmetric part of A, resp., and $Pw := -\partial_t w - \widetilde{A}w$. We need the following basic notion from convex analysis [ET76].

Definition 3.2.24. The **Legendre-Fenchel dual** of a function $\psi : V \to \mathbb{R}\cup\{\infty\}$ is given by

$$\psi^\star : V' \to \mathbb{R}\cup\{\infty\}, \quad p \mapsto \psi^\star(p) := \sup_{\chi\in V}\{\langle p,\chi\rangle_{V'\times V} - \psi(\chi)\}. \tag{3.2.41}$$

Let Assumption 3.2.4 hold with $a_{\text{shift}} = 0$. Let $g \in L^2(\mathsf{J};V')$ and $u^0 \in H$ be given. Define for (a.e.) $t \in \mathsf{J}$ the **potential** functional

$$\psi(t;\chi) := \frac{1}{2}\langle \widehat{A}(t)\chi,\chi\rangle_{V'\times V} - \langle g(t),\chi\rangle_{V'\times V}, \quad \chi \in V, \tag{3.2.42}$$

which is a strictly convex function of χ. The Legendre-Fenchel dual of $\psi(t;\cdot)$ is computed as follows: for $p \in V'$ and (a.e.) $t \in \mathsf{J}$ we have

$$\begin{aligned}
\psi^\star(t;p) &= \sup_{\chi\in V}\{\langle p,\chi\rangle_{V'\times V} - \psi(t;\chi)\} && (3.2.43)\\
&= \sup_{\chi\in V}\{\langle p - (\frac{1}{2}\widehat{A}(t)\chi - g(t)),\chi\rangle_{V'\times V}\} && (3.2.44)\\
&= \langle p, \widehat{A}(t)^{-1}(g(t)+p)\rangle_{V'\times V} - \psi(t;\widehat{A}(t)^{-1}(g(t)+p)) && (3.2.45)
\end{aligned}$$

since the supremum is realized at $\chi = \widehat{A}(t)^{-1}(g(t)+p)$. We work with the space

$$\mathcal{X} = L^2(\mathsf{J};V) \cap H^1(\mathsf{J};V') \tag{3.2.46}$$

with the norm $\|\cdot\|_{\mathcal{X}}$ defined as in (3.2.11) by $\|w\|_{\mathcal{X}}^2 := \|w\|_{L^2(\mathsf{J};V)}^2 + \|\partial_t w\|_{L^2(\mathsf{J};V')}^2$. Motivated by [Gho07, Corollary 6.4], we define the functionals (**Lagrangians**)

$$L(t;\chi,p) := \psi(t;\chi) + \psi^\star(t;-\widetilde{A}(t)\chi + p), \quad \chi \in V, p \in V', \tag{3.2.47}$$

$$\ell(w) := \|u^0 - w(0)\|_H^2 + \int_{\mathsf{J}} \langle \partial_t w(t), w(t)\rangle_{V'\times V}\, dt, \quad w \in \mathcal{X}, \tag{3.2.48}$$

which are combined to the **action**

$$\mathcal{S}(w) := \int_{\mathsf{J}} L(t;w(t),-\partial_t w(t))dt + \ell(w), \quad w \in \mathcal{X}. \tag{3.2.49}$$

For completeness we cite from [Gho07, Corollary 6.4] the following result.

Theorem 3.2.25. *For any $w \in \mathcal{X}$*

1. $\mathcal{S}(w) \geq 0$,
2. $\mathcal{S}(w) = 0$ *if and only if* w *satisfies* (3.1.1) *for (a.e.)* $t \in \mathsf{J}$.

We now derive a symmetric variational formulation for the abstract Cauchy problem (3.1.1). First, for the (Fréchet) derivative of $\mathcal{S}$ we obtain by a direct computation

$$d\mathcal{S}(w)(v) = \int_{\mathsf{J}} \left\{ \langle \widehat{A}w, v\rangle + \langle \widehat{A}^{-1}Pw, Pv\rangle \right\} dt - \int_{\mathsf{J}} \langle g, v - \widehat{A}^{-1}Pv\rangle dt + \tag{3.2.50}$$

$$+ \int_{\mathsf{J}} \{\langle \partial_t w, v\rangle + \langle \partial_t v, w\rangle\}\, dt - \langle u^0 - w(0), v(0)\rangle. \tag{3.2.51}$$

By means of the integration-by-parts formula (2.7.8), the condition of stationarity, $d\mathcal{S}(w)(\cdot) \equiv 0$, is equivalent to

$$\langle \widehat{\mathcal{B}}w, v\rangle_{\mathcal{X}'\times\mathcal{X}} = \widehat{\mathcal{F}}v \quad \forall v \in \mathcal{X}, \tag{3.2.52}$$

where $\widehat{\mathcal{B}} : \mathcal{X} \to \mathcal{X}'$ and $\widehat{\mathcal{F}} : \mathcal{X} \to \mathbb{R}$ are defined by

$$\langle \widehat{\mathcal{B}}w, v\rangle_{\mathcal{X}'\times\mathcal{X}} := \int_{\mathsf{J}} \left\{ \langle \widehat{A}w, v\rangle + \langle \widehat{A}^{-1}Pw, Pv\rangle \right\} dt + \langle w(T), v(T)\rangle, \quad w, v \in \mathcal{X}, \tag{3.2.53}$$

$$\widehat{\mathcal{F}}v := \int_{\mathsf{J}} \left\{ \langle g, v\rangle - \langle g, \widehat{A}^{-1}Pv\rangle \right\} dt + \langle u^0, v(0)\rangle, \quad v \in \mathcal{X}. \tag{3.2.54}$$

Observe that $\langle \widehat{\mathcal{B}}\cdot, \cdot\rangle_{\mathcal{X}'\times\mathcal{X}}$ is a *symmetric* bilinear form on $\mathcal{X}\times\mathcal{X}$. The symmetric variational formulation (3.2.52) is indeed well-posed.

Proposition 3.2.26. $\widehat{\mathcal{B}} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$.

Proof. The operator $\widehat{\mathcal{B}} : \mathcal{X} \to \mathcal{X}'$ is obviously linear and continuous. It follows from Lemma 3.2.8 that the parabolic energy norm $\|\cdot\|_{\mathcal{E}}$, defined by

$$\|w\|_{\mathcal{E}}^2 := \int_{\mathsf{J}} \langle \widehat{A}w, w\rangle dt + \int_{\mathsf{J}} \langle \widehat{A}^{-1}Pw, Pw\rangle dt + \|w(T)\|_H^2, \quad w \in \mathcal{X}, \tag{3.2.55}$$

satisfies the norm equivalence $\|\cdot\|_{\mathcal{E}} \sim \|\cdot\|_{\mathcal{X}}$ on $\mathcal{X}$. Thus, $\langle \widehat{\mathcal{B}}\cdot, \cdot\rangle_{\mathcal{X}'\times\mathcal{X}} = \|\cdot\|_{\mathcal{E}}^2 \sim \|\cdot\|_{\mathcal{X}}^2$. The Lax-Milgram lemma (Theorem 4.1.2) shows the claim. □

We now provide a link to the abstract Cauchy problem (3.1.1).

Proposition 3.2.27. *If $u \in \mathcal{X}$ is a generalized solution of (3.1.1) (in the sense of Definition 3.2.11) then it satisfies the symmetric variational formulation (3.2.52).*

Proof. Let $u \in \mathcal{X}$ satisfy (3.1.1) in the sense of Definition 3.2.11. Owing to $-Pu + \widehat{A}u = g$ we then have

1. $\langle \widehat{A}^{-1}Pu, Pv\rangle + \langle g, \widehat{A}^{-1}Pv\rangle = \langle \widehat{A}^{-1}(Pu+g), Pv\rangle = \langle u, Pv\rangle$,
2. $\langle \widehat{A}u, v\rangle - \langle g, v\rangle = \langle Pu, v\rangle$.

Observing the anti-symmetry of $\widetilde{A}$ and the integration-by-parts formula (2.7.8),

$$\langle \widehat{\mathcal{B}}u - \widehat{\mathcal{F}}, v\rangle_{\mathcal{X}'\times\mathcal{X}} = \int_{\mathsf{J}} \{\langle u, Pv\rangle + \langle Pu, v\rangle\}\, dt + \langle u(T), v(T)\rangle - \langle u^0, v(0)\rangle \tag{3.2.56}$$

$$= -\int_{\mathsf{J}} \{\langle u, \partial_t v\rangle + \langle \partial_t u, v\rangle\}\, dt + \langle u(T), v(T)\rangle - \langle u^0, v(0)\rangle \tag{3.2.57}$$

$$= \langle u(0), v(0)\rangle - \langle u^0, v(0)\rangle = 0, \tag{3.2.58}$$

for any $v \in \mathcal{X}$, as claimed. □

3.3 Regularity

It is a known feature of parabolic evolution equations that the initial datum is promoted to spaces of higher smoothness (e.g. in terms of intermediate spaces between X and $\mathcal{D}(A)$) by the flow. This "smoothing effect" has been been exploited to obtain numerical methods with exponential accuracy in terms of the number of unknowns (degrees of freedom), e.g. in discontinuous Galerkin time-stepping methods [Sch99; SS00]. To quantify this, we will express the temporal regularity of solutions to the abstract Cauchy problem (3.1.1) in terms of its membership to countably normed weighted Bochner spaces $B^\ell_\beta(\mathsf{J}; X)$ introduced in Section 2.8. This is the subject of the present Section 3.3, first using semigroup theory (only for the constant generator case), then using the framework of maximal regularity (for the non-constant generator case).

The rationale for this approach is the following: just like analytic functions $(0,1) \to \mathbb{R}$ can be approximated by polynomials with exponentially decaying coefficients, functions in $B^\ell_\beta(\mathsf{J}; X)$, $\ell = 1, 2$, can be approximated in $H^{\ell-1}(\mathsf{J}; X)$ by piecewise polynomials $\mathsf{J} \to X$ with exponential accuracy in terms of the number of degrees of freedom. Thus, membership of the solution in a space like $B^1_\beta(\mathsf{J}; Y) \cap B^2_\beta(\mathsf{J}; X)$, where $Y \hookrightarrow X$, concisely explains the exponential accuracy of methods that are of *hp*-type in the temporal direction, such as the discontinuous Galerkin time-stepping method.

3.3.1 Regularity results based on semigroup theory

In this section we will restrict ourselves to the constant generator case of the abstract Cauchy problem (3.1.1) and postpone the non-constant generator case to Section 3.3.2. First, we recall some pointwise estimates on the temporal derivative of the solution to (3.1.1) given in [Sch99, Section 2.2]. These are based on the semigroup theory and the variation of constants formula (3.2.25). We find that the solution belongs to spaces of the type $B^\ell_\beta(\mathsf{J}; X)$. For the remainder of the section recall from Section 3.2.1 the definition of the Gelfand triple of real Hilbert spaces

$$V \hookrightarrow H \cong H' \hookrightarrow V'. \tag{3.3.1}$$

We assume that V is a separable Hilbert space and that the embeddings are dense. We proceed under the following assumption.

Assumption 3.3.1. $t \mapsto A(t)$ is a constant map, itself denoted by $A \in \mathcal{L}(V, V')$, and there exists $a_{\min} \in (0, \infty)$ such that $\langle A\chi, \chi\rangle_{V' \times V} \geq a_{\min}\|\chi\|_V^2$ for all $\chi \in V$.

Under Assumption 3.3.1 the operator $(-A)$ is sectorial in the sense of Definition 3.2.14 by means of the following lemma.

Lemma 3.3.2. *Let $B \in \mathcal{L}(V, V')$ satisfy the Gårding inequality*

$$\langle B\chi, \chi\rangle_{V' \times V} + a_{\text{shift}}\|\chi\|_H^2 \geq a_{\min}\|\chi\|_V^2 \quad \forall \chi \in V \tag{3.3.2}$$

for some $a_{\text{shift}} \in \mathbb{R}$ and $a_{\min} \in (0, \infty)$. Then the operator $(-B)$ is sectorial in the sense of Definition 3.2.14.

Proof. Assuming $a_{\text{shift}} = 0$, this is shown in [Sch99, Proposition 2.3], cf. [Yag10, Chapter 1, Theorem 2.1] and [Lun09, Lemma 4.31]. If $a_{\text{shift}} \neq 0$, just consider $(B + a_{\text{shift}}\,\mathrm{Id})$ instead. □

Since the embedding $\mathcal{D}(A) = V \hookrightarrow V' =: X$ is dense by assumption, the (constant) family $t \mapsto A$ satisfies all the hypotheses of Assumption 3.2.15 and Assumption 3.2.18. By Theorem 3.2.19 therefore there exists a parabolic evolution operator $G(\cdot, \cdot)$, see Definition 3.2.16. Moreover, it has the form $G(t, s) = G(t - s)$, $0 \leq s \leq t \leq \mathsf{T}$, and the variation of constants formula (3.2.25) for the mild solution reduces to

$$u(t) = G(t)u^0 + \int_0^t G(t - s)g(s)ds, \quad 0 \leq t \leq \mathsf{T}. \tag{3.3.3}$$

The derivatives of G at $t \in (0, \mathsf{T})$ may be computed from the integral representation via the resolvent of $G(t)$, see [Sch99, Section 2.2]. These may be used to estimate the derivatives of the mild solution (3.3.3) which results in the following.

Proposition 3.3.3. *Let $u^0 \in (V', V)_{\frac{1}{2}+\frac{s}{2},2}$ for $0 \leq s \leq 1$, with the convention $(V', V)_{1,2} := V$. Let $g \in L^2(\mathsf{J}; H)$ be sufficiently smooth (e.g. analytic with values in H). Then $u \in C^\infty(\mathsf{J}; V)$ and there exist $C > 0$ and $d > 0$ such that*

$$\|u^{(n)}(t)\|_V^2 \leq Ct^{-(2n+1)+s}d^{2n}(2n)! \leq Ct^{-(2n+1)+s}((2d)^n n!)^2, \quad n \in \mathbb{N}_0, \tag{3.3.4}$$

for any $n \in \mathbb{N}_0$ and $0 < t < \mathsf{T}$, where u is given by (3.3.3).

Proof. The first estimate is shown in [Sch99, Proposition 2.11]. The second inequality is due to

$$(2n)! \sim \sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n} \sim \frac{2^{2n}}{\sqrt{\pi n}}\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2 \sim \frac{4^n}{\sqrt{n}}(n!)^2, \quad n \in \mathbb{N}, \tag{3.3.5}$$

using the Stirling's approximation of $n!$. □

Corollary 3.3.4. *Under the assumptions of the previous proposition there exist $C > 0$ and $d > 0$ such that*

$$t^{2(\beta+k)} \max\{\|u^{(k+1)}(t)\|_V^2, \|u^{(k+2)}(t)\|_{V'}^2\} \leq C(d^k k!)^2 t^{-3+s+2\beta} \tag{3.3.6}$$

holds for all $k \in \mathbb{N}_0$ and $0 < t < \mathsf{T}$. Moreover, if $2\beta + s > 2$ then

$$u \in B^1_\beta(\mathsf{J}; V) \cap B^2_\beta(\mathsf{J}; V'). \tag{3.3.7}$$

Proof. From (3.3.6) the second claim follows by integration over J. Thus, only (3.3.6) is to be shown. To that end we use (3.3.4) with $n = k + 1$, $k \in \mathbb{N}_0$, to obtain the estimate for $\|u^{(k+1)}(t)\|_V$. The estimate for $\|u^{(k+2)}(t)\|_{V'}$ follows by differentiating the equation $\partial_t u(t) = -Au(t) + g(t)$ w.r.t. t once. □

The second statement of Corollary 3.3.4 may be reformulated as follows: if $s > 0$ then there exists $0 \leq \beta < 1$ such that $u \in B^1_\beta(\mathsf{J}; V) \cap B^2_\beta(\mathsf{J}; V')$.

Example 3.3.5. Let $D = (-1,1) \subset \mathbb{R}^1$. Set $H := L^2(D)$, $V := H_0^1(D)$ and $A := (-\partial_{xx}) \in \mathcal{L}(V, V')$. Let $u^0 \in H$ be the function

$$u^0(x) = \begin{cases} -x-1 & \text{for} \quad -1 < x < 0, \\ -x+1 & \text{for} \quad 0 < x < 1. \end{cases}$$

The Fourier series expansion shows

$$u^0 = \sum_{k\in\mathbb{N}} \frac{2}{k\pi} \sin(k\pi\cdot) \in \bigcap_{0<s<1/2} (H,V)_{s,2}.$$

From Corollary 3.3.4 we conclude that $u \in B^1_\beta(\mathsf{J}; V) \cap B^2_\beta(\mathsf{J}; V')$ for any $3/4 < \beta < 1$.

3.3.2 Higher order regularity from maximal regularity

In this section we obtain a regularity result for the solution of (3.1.1) based on the framework of maximal regularity for parabolic equations in weighted L^p spaces. The proof of Theorem 3.3.6 below is an adaptation of the unweighted case from [Prü02, Theorem 5.1] (cf. [Lun95, Section 8.3.2–3]), where the claims i) and iii) are shown; the idea to use the implicit function theorem in this context is attributed to [Ang90]. In addition, we verify that the solution to the abstract evolution equation (3.1.1) belongs to certain weighted Bochner spaces of vector valued functions defined in Section 2.8.

Theorem 3.3.6. *Let $1 < p < \infty$ and $0 \le \nu < 1/p'$. Assume $u^0 \in X_{\nu,p}$. For some fixed $m \in \mathbb{N} \cup \{\infty\}$, let $A \in C^m(\overline{\mathsf{J}}; \mathcal{L}(\mathcal{D}(A), X))$ and $g \in C^m(\overline{\mathsf{J}}; X)$. Assume that A has maximal L^p_ν regularity. Assume $u \in \mathcal{X}^p_\nu$ satisfies $\partial_t u + Au = g$ in $L^p_\nu(\mathsf{J}; X)$ and $u(0) = u^0$ in $X_{\nu,p}$. Let $0 < \mathsf{T}' < \mathsf{T}$, set $\mathsf{J}' := (0, \mathsf{T}')$. Then*

i) $t^\ell \partial_t^\ell u \in \mathcal{X}^p_\nu(\mathsf{J}')$ for each $\ell = 0, \dots, m$,

ii) $u \in W^{0,p}_{\nu,m}(\mathsf{J}'; \mathcal{D}(A)) \cap W^{1,p}_{\nu,m}(\mathsf{J}'; X)$.

Assume further that $A : \mathsf{J} \to \mathcal{L}(\mathcal{D}(A), X)$ and $g : \mathsf{J} \to X$ are analytic. Then

iii) $u : (0, \mathsf{T}) \to \mathcal{D}(A)$ is analytic,

iv) for any $0 < \theta < 1$ we have $u \in B^{1,p}_{\nu+\theta}(\mathsf{J}'; (X, \mathcal{D}(A))_{\theta,p})$.

Proof. The proof is given in several steps.

Preparations. For all $\delta > 0$ define $\Lambda_\delta := (1-\delta, 1+\delta)$. For $0 < \epsilon < 1$ arbitrary, set $\mathsf{J}_\epsilon := \frac{1}{1+\epsilon}\mathsf{J}$. For $\lambda \in \Lambda_\epsilon$ and $\tau \in \mathsf{J}_\epsilon$ write $u_\lambda(\tau) := u(\lambda\tau)$, $A_\lambda(\tau) := A(\lambda\tau)$, $A'_\lambda(\tau) := [\partial_t A](\lambda\tau)$, etc. Consider now the map

$$H : \Lambda_\epsilon \times \mathcal{X}^p_\nu(\mathsf{J}_\epsilon) \to \widetilde{\mathcal{Y}}^p_\nu(\mathsf{J}_\epsilon), \quad H(\lambda, u) = (\partial_t u + \lambda A_\lambda u - \lambda g_\lambda, u(0) - u^0). \tag{3.3.8}$$

Since A and g are of class C^m by hypothesis, so is H. For the partial Fréchet derivatives we compute

$$D_\lambda H(\lambda, u) = (A_\lambda u - g_\lambda + \lambda\tau(A'_\lambda u - g'_\lambda), 0), \tag{3.3.9}$$

$$D_u H(\lambda, u)\widetilde{u} = (\partial_t \widetilde{u} + \lambda A_\lambda \widetilde{u}, \widetilde{u}(0)) \quad \forall \widetilde{u} \in \mathcal{X}^p_\nu(\mathsf{J}_\epsilon). \tag{3.3.10}$$

Let now $u \in \mathcal{X}^p_\nu(\mathsf{J})$ be as in the hypothesis. Then we have $H(\lambda, u_\lambda) = 0$ for $\lambda = 1$. By assumption, A has maximal L^p_ν regularity, and thus $D_u H(1, u_1) : \mathcal{X}^p_\nu(\mathsf{J}_\epsilon) \to \widetilde{\mathcal{Y}}^p_\nu(\mathsf{J}_\epsilon)$ is an isomorphism.

Proof of i). By the implicit function theorem (Theorem 2.6.7) there exists $0 < \delta \le \epsilon$ and $\phi \in C^m(\Lambda_\delta; \mathcal{X}^p_\nu(\mathsf{J}_\epsilon))$ such that $\phi(1) = u_1$ and $H(\lambda, \phi(\lambda)) = 0$ for each $\lambda \in \Lambda_\delta$. Observe that $\phi(\lambda) = u_\lambda$, hence the map $\lambda \mapsto u_\lambda$ is in $C^m(\Lambda_\delta; \mathcal{X}^p_\nu(\mathsf{J}_\epsilon))$. Given that $(\partial_\lambda^k u_\lambda)(\tau) = \tau^k(\partial_t^k u)(\lambda\tau)$, we may set $\lambda = 1$ to obtain $[t \mapsto t^k \partial_t^k u(t)] \in \mathcal{X}^p_\nu(\mathsf{J}_\epsilon)$ for each $k = 0, \dots, m$. Since $0 < \epsilon < 1$ may be arbitrary small, *i)* follows.

Proof of ii). By definition, we have $\mathcal{X}^p_\nu(\mathsf{J}') \hookrightarrow L^p_\nu(\mathsf{J}'; \mathcal{D}(A))$ with continuous embedding. Thus, using *i)* we have $\|\Phi_k \partial_t^k u\|_{L^p_\nu(\mathsf{J}'; \mathcal{D}(A))} < \infty$ for all $k = 0, \dots, m$, i.e., $u \in W^{0,p}_{\nu,m}(\mathsf{J}'; \mathcal{D}(A))$. In order to show

$u \in W^{1,p}_{\nu,m}(\mathsf{J}';X)$ we verify that $\|\Phi_k \partial_t^k \partial_t u\|_{L^p_\nu(\mathsf{J}';X)}$ is finite for all $k = 0, \dots, m$. Note that this holds for $k = 0$ by hypothesis. Observing that

$$\Phi_k \partial_t^k \partial_t u = \partial_t(\Phi_k \partial_t^k u) - k\Phi_{k-1}\partial_t^{k-1}\partial_t u, \quad k = 1, \dots, m, \tag{3.3.11}$$

the claim follows by induction over k using $\|\partial_t \Phi_k \partial_t^k u\|_{L^p_\nu(\mathsf{J}';X)} < \infty$, which is valid due to $\mathcal{X}^p_\nu(\mathsf{J}') \hookrightarrow W^{1,p}_\nu(\mathsf{J}';X)$ and *i)* for each $k = 0, \dots, m$.

Proof of iii). Assume now additionally that A and g are analytic on J. Then the map H, defined in (3.3.8), is analytic (with domain and range as in (3.3.8)), and therefore ϕ, given in the proof of *i)*, is also analytic by the implicit function theorem (Theorem 2.6.7). Let $I \subset\subset \mathsf{J}_\epsilon$ be an open interval with $\overline{I} \subset \mathsf{J}_\epsilon$. Given the embedding $W^{1,p}(I;\mathcal{D}(A)) \hookrightarrow C^0(I;\mathcal{D}(A))$, the function u can be shown to be analytic in a sufficiently small neighborhood of any $t \in I$ with values in $\mathcal{D}(A)$ using Theorem 2.6.6. The technical details are skipped here (cf. [Prü02, Proof of Theorem 5.1]). Since $I \subset\subset \mathsf{J}_\epsilon$ and $0 < \epsilon < 1$ were arbitrary, u is analytic on the whole open interval $\mathsf{J} = \bigcup_{0<\epsilon<1} \mathsf{J}_\epsilon$.

Before continuing with the proof of *iv)* we give a lemma.

Lemma 3.3.7. *Let $Y \hookrightarrow X$ be real Banach spaces, $\nu \in \mathbb{R}$, $1 \le p < \infty$. Let $\Lambda \subseteq \mathbb{R}$ be an open set, $J \subset (0,\infty)$ an interval. Let $\psi : \Lambda \to L^p_\nu(J;X) \cap L^p_{\nu+1}(J;Y)$ be such that both, $\psi : \Lambda \to L^p_\nu(J;X)$ and $\psi : \Lambda \to L^p_{\nu+1}(J;Y)$, are analytic. Then $\psi : \Lambda \to L^p_{\nu+\theta}(J;(X,Y)_{\theta,p})$ is analytic for any $0 < \theta < 1$.*

Proof. Let $\lambda \in \Lambda$. By Theorem 2.6.6 there exists a neighborhood $\widetilde{\Lambda} \subset \Lambda$ with $\lambda \in \widetilde{\Lambda}$ and constants $C, d \ge 0$ such that

$$\|\partial_\lambda^k \psi(\widetilde{\lambda})\|_{L^p_\nu(J;X)} \le C d^k k! \quad \text{and} \quad \|\partial_\lambda^k \psi(\widetilde{\lambda})\|_{L^p_{\nu+1}(J;Y)} \le C d^k k! \quad \forall k \in \mathbb{N}_0 \quad \forall \widetilde{\lambda} \in \widetilde{\Lambda}. \tag{3.3.12}$$

Now, for any $0 < \theta < 1$, the interpolation inequality (2.4.4) and the Hölder inequality with exponents $1/(1-\theta)$ and $1/\theta$ yield

$$\|u\|^p_{L^p_{\nu+\theta}(J;(X,Y)_{\theta,p})} = \int_J \|t^{\nu+\theta} u(t)\|^p_{(X,Y)_{\theta,p}} dt \tag{3.3.13}$$

$$\le c^p_{\theta,p} \int_J \{t^{\nu p}\|u(t)\|^p_X\}^{1-\theta}\{t^{(\nu+1)p}\|u(t)\|^p_Y\}^\theta dt \tag{3.3.14}$$

$$\le c^p_{\theta,p} \|u\|^{p(1-\theta)}_{L^p_\nu(J;X)} \|u\|^{p\theta}_{L^p_{\nu+1}(J;Y)} \tag{3.3.15}$$

for any $u \in L^p_\nu(J;X) \cap L^p_{\nu+1}(J;Y)$. Hence, we obtain $\|\partial_\lambda^k \psi(\widetilde{\lambda})\|_{L^p_{\nu+\theta}(J;(X,Y)_{\theta,p})} \le c_{\theta,p} C d^k k!$ for all $k \in \mathbb{N}_0$ and $\widetilde{\lambda} \in \widetilde{\Lambda}$. Since $\lambda \in \Lambda$ was arbitrary, the claim follows from Theorem 2.6.6. □

Proof of iv). We show that the map $\lambda \mapsto \psi(\lambda) := [\partial_t u](\lambda \cdot)$ is analytic on Λ_δ with values in $L^p_{\nu+1}(\mathsf{J}_\epsilon;\mathcal{D}(A))$ and in $L^p_\nu(\mathsf{J}_\epsilon;X)$. To that end recall that $\lambda \mapsto \phi(\lambda) \in \mathcal{X}^p_\nu(\mathsf{J}_\epsilon)$ is analytic on Λ_δ and that $[\partial_\lambda^k \phi(\lambda)](\tau) = \tau^k [\partial_t^k u](\lambda\tau)$ for all $\lambda \in \Lambda_\delta$, $\tau \in \mathsf{J}_\epsilon$. For $k = 1$ and $\lambda = 1$ this immediately implies that $\psi : \Lambda_\delta \to L^p_{\nu+1}(\mathsf{J}_\epsilon;\mathcal{D}(A))$ is analytic. It remains to check analyticity of ψ with values in $L^p_\nu(\mathsf{J}_\epsilon;X)$. Since $\lambda \mapsto \phi(\lambda) \in \mathcal{X}^p_\nu(\mathsf{J}_\epsilon)$ is analytic on Λ_δ, so is $\lambda \mapsto \partial_\tau[\phi(\lambda)] \in L^p_\nu(\mathsf{J}_\epsilon;X)$, hence, also $\lambda \mapsto \lambda^{-1}\partial_\tau[\phi(\lambda)] \in L^p_\nu(\mathsf{J}_\epsilon;X)$. But, $\partial_\tau[\phi(\lambda)](\tau) = \lambda[\partial_t u](\lambda\tau) = \lambda[\psi(\lambda)](\tau)$, $\tau \in \mathsf{J}_\epsilon$. Therefore, $\psi : \Lambda_\delta \to L^p_\nu(\mathsf{J}_\epsilon;X)$ is analytic. An application of the foregoing lemma and Proposition 2.8.1 shows *iv)*. □

Example 3.3.8. Returning to Example 3.3.5 with the time-constant generator $A = (-\partial_{xx})$ on $D = (-1,1)$ we have with $X := L^2(D)$, $V := H^1_0(D)$ and $\mathcal{D}(A) = V \cap H^2(D)$ that $u^0 \in (X,\mathcal{D}(A))_{1/2-\nu,2}$ for all $1/4 < \nu < 1/2$. With $\theta := 1/2$ and $\beta := \theta + \nu$, Theorem 3.3.6 implies $u \in B^1_\beta(\mathsf{J};V)$ for any $3/4 < \beta < 1$. Further, using $\partial_t u = -Au$ and $u \in H^1(\mathsf{J};V')$ it follows

$$u \in B^1_\beta(\mathsf{J};V) \;\Rightarrow\; Au \in B^1_\beta(\mathsf{J};V') \;\Rightarrow\; \partial_t u \in B^1_\beta(\mathsf{J};V') \tag{3.3.16}$$

$$\Rightarrow\; u \in B^1_\beta(\mathsf{J};V) \cap B^2_\beta(\mathsf{J};V') \tag{3.3.17}$$

for any $3/4 < \beta < 1$. This precisely recovers the conclusion of Example 3.3.5.

3.4 Semi-linear equations with small data

We use solution techniques for the abstract linear Cauchy problem (3.1.1) to study *semi-linear* evolution equations of the form

$$\begin{cases} \partial_t u(t) + (Au)(t) + (G(u))(t) = g(t), \quad \text{(a.e.) } t \in \mathsf{J}, \\ u(0) = u^0, \end{cases} \tag{3.4.1}$$

where the non-linear mapping G is assumed to satisfy suitable growth and/or Lipschitz conditions. We work under smallness assumptions on the input data. The notation and assumptions of Section 3.2.3 on maximal regularity are retained here.

Proposition 3.4.1. *Let $1 < p, p' < \infty$, $1/p + 1/p' = 1$, and $0 \le \nu < 1/p'$. Assume that A has maximal L^p_ν regularity (see Definition 3.2.20), let $0 < \alpha < 1/c_{\nu,p}(A)$. Assume that $G : \mathcal{X}^p_\nu \to \widetilde{\mathcal{Y}}^p_{\nu,1}$ satisfies $G(0) = 0$ and the local Lipschitz condition*

$$\|G(u) - G(w)\|_{\widetilde{\mathcal{Y}}^p_{\nu,1}} \le \eta(\max\{\|u\|_{\mathcal{X}^p_\nu}, \|w\|_{\mathcal{X}^p_\nu}\})\|u - w\|_{\mathcal{X}^p_\nu} \quad \forall u, w \in \mathcal{X}^p_\nu, \tag{3.4.2}$$

where $\eta \in C^0([0, \infty))$ with $\eta(0) = 0$.

Then there exists $\delta > 0$ such that for all input data $\mathcal{F} := (g, u^0) \in \widetilde{\mathcal{Y}}^p_\nu$ with $\|\mathcal{F}\|_{\widetilde{\mathcal{Y}}^p_\nu} \le \delta$ the problem (3.4.1) has a unique solution $u \in \mathcal{X}^p_\nu$.

Proof. Define the mappings $\mathcal{A}, \mathcal{G} : \mathcal{X}^p_\nu \to \widetilde{\mathcal{Y}}^p_\nu$ by $\mathcal{A}w := (\partial_t w + Aw, w(0))$ and $\mathcal{G}(w) := (G(w), 0)$. By assumption of maximal L^p_ν regularity on A, the linear operator $\mathcal{A}$ is an isomorphism and the norm of $\mathcal{A}^{-1}$ is bounded above by $c_{\nu,p}(A)$. Let $0 < \alpha < 1/c_{\nu,p}(A)$, and choose $\rho > 0$ such that

$$\|\mathcal{G}(u) - \mathcal{G}(w)\|_{\widetilde{\mathcal{Y}}^p_\nu} \le \alpha\|u - w\|_{\mathcal{X}^p_\nu} \quad \forall u, w \in B_\rho := \{w \in \mathcal{X}^p_\nu : \|w\|_{\mathcal{X}^p_\nu} \le \rho\}.$$

Being a closed subset of $\mathcal{X}^p_\nu$, the set B_ρ is complete. Assume that $\|\mathcal{F}\|_{\widetilde{\mathcal{Y}}^p_\nu} \le \delta := \rho(1/c_{\nu,p}(A) - \alpha)$.

Defining $\Phi : \mathcal{X}^p_\nu \to \mathcal{X}^p_\nu$, $w \mapsto \Phi(w) := \mathcal{A}^{-1}(\mathcal{F} - \mathcal{G}(w))$, equation (3.4.1) is equivalent to the fixed point equation $\Phi(u) = u$. Since for any $u, w \in B_\rho$ there holds

$$\|\Phi(w)\|_{\mathcal{X}^p_\nu} \le c_{\nu,p}(A)(\|\mathcal{F}\|_{\widetilde{\mathcal{Y}}^p_\nu} + \alpha\|w\|_{\mathcal{X}^p_\nu}) \le \rho$$

and

$$\|\Phi(u) - \Phi(w)\|_{\mathcal{X}^p_\nu} \le \alpha c_{\nu,p}(A)\|u - w\|_{\mathcal{X}^p_\nu},$$

the Banach fixed point theorem applied to $\Phi|_{B_\rho}$ shows that there exists a unique fixed point $u = \Phi(u) \in B_\rho$, hence of a unique solution to (3.4.1). □

4 Minimal residual FEM for operator equations

A variational solution strategy for well-posed operator equations, which deviates from the finite element method as presented in the standard text books of numerical analysis, is discussed. It is based on the minimization of the residual of the continuous operator equation on suitable conforming trial and test spaces, where the dimension of the test space is allowed to be larger than that of the trial space. This leads to a possibly rectangular, overdetermined linear system of equations when the equation is reformulated as a matrix vector equation for given bases on the trial and test spaces, rather than a square one. However, the freedom gained renders the inf-sup condition, which is crucial for the stable resolution of the problem, much easier to achieve for non-symmetric problems.

In this framework, it is natural to transport the norms of the continuous problem to the discrete problem. This has the following important consequences: the discrete solution satisfies a quasi-optimality error estimate analogous to Céa's lemma and, with slightly more effort, the discrete equations can be efficiently preconditioned, making the approach amenable to fully parallelizable iterative solvers. In the context of parabolic problems these features are particularly interesting, since they seem to be difficult to obtain with conventional methods.

To be more precise, let $\mathcal{B} : \mathcal{X} \to \mathcal{Y}'$ be a linear map between a real Hilbert space, $\mathcal{X}$, and the dual of another, $\mathcal{Y}$. For instance, $\mathcal{B}$ could be the linear parabolic operator defined in (3.2.15). For a given $\mathcal{F} \in \mathcal{Y}'$ we aim at approximating the solution $u \in \mathcal{X}$ of the linear equation $\mathcal{B}u = \mathcal{F}$. For reasons already sketched we rewrite this equation as the functional residual minimization problem

$$u = \arg\min_{w \in \mathcal{X}} \sup_{v \in \mathcal{Y} \setminus \{0\}} \frac{\langle \mathcal{B}w - \mathcal{F}, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|v\|_{\mathcal{Y}}}. \tag{4.0.1}$$

The **minimal residual Petrov-Galerkin** method is the following: let $\mathcal{X}_h \subseteq \mathcal{X}$ and $\mathcal{Y}_h \subseteq \mathcal{Y}$, $\mathcal{Y}_h \neq \{0\}$, be a pair of closed subspaces (not necessarily finite-dimensional), let $\|\cdot\|_{\mathcal{N}} \sim \|\cdot\|_{\mathcal{Y}}$ be an equivalent norm on $\mathcal{Y}$, and consider the (discrete) functional residual minimization problem

$$u_h := \arg\min_{w_h \in \mathcal{X}_h} \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B}w_h - \mathcal{F}, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|v_h\|_{\mathcal{N}}}. \tag{4.0.2}$$

This method relies on the validity of the so-called **discrete inf-sup condition**

$$\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) := \inf_{w_h \in \mathcal{X}_h \setminus \{0\}} \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B}w_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|w_h\|_{\mathcal{X}} \|v_h\|_{\mathcal{Y}}} > 0. \tag{4.0.3}$$

Such a pair of spaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ will be called stable for $\mathcal{B}$. Assuming this stability condition, in Section 4.1 we show that the discrete residual minimization problem is uniquely solvable and the minimizer is quasi-optimal in the discrete trial space $\mathcal{X}_h$ (i.e., the error $w_h \mapsto \|u - w_h\|_{\mathcal{X}}$ is the minimal possible up to a multiplicative constant proportional to $1/\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)$). In Section 4.2 we reduce the problem to a possibly overdetermined matrix vector equation, for which we also propose a generic preconditioner. An iterative solver for this linear system is formulated in Section 4.3. Given the significance of the discrete inf-sup condition, we elaborate in Section 4.4 on several characterizations of similar conditions, some of particular importance to parabolic problems; the notation introduced there is relevant to later sections.

Most results of Sections 4.1–4.4 have appeared in [And10; And12].

4.1 Solution of linear equations via residual minimization

Typically, the conforming finite element method proceeds as follows: given $\mathcal{B} \in \mathrm{Iso}(\mathcal{X}, \mathcal{Y}')$ and $\mathcal{F} \in \mathcal{Y}'$, we look for an approximate solution $u_h \in \mathcal{X}_h$ in a discrete trial space $\mathcal{X}_h \subseteq \mathcal{X}$ as the solution of the discrete variational problem

$$\text{find} \quad u_h \in \mathcal{X}_h : \quad \langle \mathcal{B}u_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}} = \mathcal{F}v_h \quad \forall v_h \in \mathcal{Y}_h, \tag{4.1.1}$$

where $\mathcal{Y}_h$ is a suitable discrete test space of the same dimension as $\mathcal{X}_h$. Choosing suitable bases on $\mathcal{X}_h$ and $\mathcal{Y}_h$ leads to a square system of linear algebraic equations $\mathbf{Bu} = \mathbf{f}$ for the coefficients $\mathbf{u}$ of u_h. Due to Theorem 4.1.1, a necessary (and, if $\dim \mathcal{X}_h = \dim \mathcal{Y}_h < \infty$, also sufficient) condition for the unique solvability of this system is the discrete inf-sup condition (4.0.3). Note that if the bilinear form $\langle \mathcal{B}\cdot, \cdot \rangle_{\mathcal{Y}' \times \mathcal{Y}}$ is continuous on $\mathcal{X} \times \mathcal{Y}$, then it is also continuous on $\mathcal{X}_h \times \mathcal{Y}_h$, but, in general, the same conclusion fails to hold for the inf-sup condition![1] This renders the construction of a suitable discrete test space $\mathcal{Y}_h$ difficult for non-symmetric $\mathcal{B}$ such as the parabolic operator (3.2.15). Hence, we abandon the requirement $\dim \mathcal{Y}_h = \dim \mathcal{X}_h$ in favor of $\dim \mathcal{Y}_h \geq \dim \mathcal{X}_h$. This generalization of the standard conforming finite element method to what we call **the minimal residual Petrov-Galerkin method** is the subject of Section 4.1.2.

4.1.1 Well-posed linear operator equations

The following well-known theorem due to [Nir55; Neč62; Bab71], see also [Bra07, Satz 3.6] or [EG04, Theorem 2.6], provides necessary and sufficient conditions on a linear continuous operator $\mathcal{B}$ for the well-posedness (= the solution mapping is continuous) of the linear operator equation $\mathcal{B}u = \mathcal{F}$.

Theorem 4.1.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be real Hilbert spaces. Let $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$. Then $\mathcal{B} \in \mathrm{Iso}(\mathcal{X}, \mathcal{Y}')$ if and only if the following two conditions are satisfied:*

$$\gamma_{\mathcal{B}} := \inf_{u \in \mathcal{X} \setminus \{0\}} \sup_{v \in \mathcal{Y} \setminus \{0\}} \frac{\langle \mathcal{B}u, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} > 0, \qquad \textit{(injectivity)} \tag{4.1.2}$$

$$\forall v \in \mathcal{Y} \setminus \{0\} : \quad \exists u \in \mathcal{X} : \quad \langle \mathcal{B}u, v \rangle_{\mathcal{Y}' \times \mathcal{Y}} \neq 0. \qquad \textit{(surjectivity)} \tag{4.1.3}$$

In that event we have the bound $\|\mathcal{B}^{-1}\|_{\mathcal{L}(\mathcal{Y}', \mathcal{X})} \leq \gamma_{\mathcal{B}}^{-1}$.

The following special case of Theorem 4.1.1 is known as the Lax-Milgram lemma.

Theorem 4.1.2. *Let $\mathcal{X}$ be a real Hilbert space. Let $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{X}')$ be $\mathcal{X}$-elliptic, i.e.,*

$$\exists \alpha_{\mathcal{B}} > 0 : \quad \langle \mathcal{B}u, u \rangle_{\mathcal{X}' \times \mathcal{X}} \geq \alpha_{\mathcal{B}} \|u\|_{\mathcal{X}}^2 \quad \forall u \in \mathcal{X}. \tag{4.1.4}$$

Then $\mathcal{B} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ and $\|\mathcal{B}^{-1}\|_{\mathcal{L}(\mathcal{X}', \mathcal{X})} \leq \alpha_{\mathcal{B}}^{-1}$.

The proof of the related results in [Nir55; Neč62; Bab71] used the implication "2. $\Rightarrow$ 1." of the following lemma, cited here from [EG04, Lemma A.36]. It will be used on several occasions, and we therefore detail the proof.

Lemma 4.1.3. *Let X and Y be Banach spaces and $A \in \mathcal{L}(X, Y)$. Equivalent are:*

1. *the image $A(X)$ of A is a closed subspace of Y,*
2. *there exists $\gamma_A > 0$ such that for all $y \in A(X)$ there exists $x \in X$ with $Ax = y$ and $\gamma_A \|x\|_X \leq \|y\|_Y$.*

[1] Consider $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$ and let $\mathcal{B}$ be the diagonal matrix $\mathrm{diag}(1, -1)$. Then $\mathcal{B}$ satisfies the inf-sup condition on $\mathcal{X} = \mathcal{Y}$ but not on the subspace spanned by $(1, 1)^\top$.

Proof. Assume that $R := A(X) \subseteq Y$ is a closed subspace, hence a Banach space. The Banach open mapping theorem implies that the surjective operator $A \in \mathcal{L}(X, R)$ maps the open unit ball $B_X \subseteq X$ onto an open set $A(B_X) \subset R$. Hence, there exists $\gamma_A > 0$ such that for any $y \in R$ we have $\widetilde{y} := \gamma_A y/\|y\|_Y \in A(B_X)$, which implies the existence of an element $\widetilde{x} \in B_X$ with $A\widetilde{x} = \widetilde{y}$. Now, $x := \widetilde{x}\|y\|_Y/\gamma_A$ satisfies $Ax = y$ and $\gamma_A\|x\|_X = \|\widetilde{x}\|_X\|y\|_Y \leq \|y\|_Y$.

Conversely, take $\gamma_A > 0$ as in the second hypothesis. To show that $R := A(X)$ is a closed subspace of Y, let $\{y_n\}_{n\in\mathbb{N}} \subset R$ be a (Cauchy) sequence converging to, say, $y \in Y$. By assumption, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $Ax_n = y_n$ and $\gamma_A\|x_n - x_m\|_X \leq \|y_m - y_n\|_X \to 0$ as $n, m \to \infty$. Hence, $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence with limit, say, $x \in X$. But, A being continuous implies $y \leftarrow y_n = Ax_n \to Ax$ as $n \to \infty$, hence $y = Ax \in R$. This shows that R is closed. □

The following standard theorem that we quote from [Bra07, Satz 3.7], cf. [Bab71, Theorem 2.2], addresses the well-posedness of the discrete variational problem (4.1.1). It should be contrasted with Theorem 4.1.9, where we construct a quasi-optimal approximate solution with the sharper constant $\gamma_h^{-1}\|\mathcal{B}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')}$ under slightly relaxed requirements as the minimizer of the functional residual.

Theorem 4.1.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be real Hilbert spaces, and $\mathcal{F} \in \mathcal{Y}'$. Let $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ satisfy the conditions of injectivity and surjectivity stated in Theorem 4.1.1. Let $\mathcal{X}_h \subseteq \mathcal{X}$ and $\mathcal{Y}_h \subseteq \mathcal{Y}$ be closed subspaces such that these two conditions still hold if $\mathcal{X}$ and $\mathcal{Y}$ are replaced by $\mathcal{X}_h$ and $\mathcal{Y}_h$, respectively (possibly with a different constant, say $\gamma_h > 0$). Then there exists a unique solution $u_h \in \mathcal{X}_h$ of the discrete variational problem (4.1.1). Moreover, there holds quasi-optimality estimate*

$$\|u - u_h\|_{\mathcal{X}} \leq \left(1 + \gamma_h^{-1}\|\mathcal{B}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')}\right) \inf_{w_h \in \mathcal{X}_h} \|u - w_h\|_{\mathcal{X}}. \tag{4.1.5}$$

4.1.2 Residual minimization

In this subsection we introduce the minimal residual Petrov-Galerkin solution for the linear operator equation $\mathcal{B}u = \mathcal{F}$ and show that it satisfies a quasi-optimality estimate. To that end, let $\mathcal{X}$ and $\mathcal{Y}$ be real Hilbert spaces. Let $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$ and $\mathcal{M} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ be s.p.d. operators. We set

$$c_{\mathcal{N}} := \inf_{v\in\mathcal{Y}\setminus\{0\}} \frac{\|v\|_{\mathcal{N}}}{\|v\|_{\mathcal{Y}}} \leq \sup_{v\in\mathcal{Y}\setminus\{0\}} \frac{\|v\|_{\mathcal{N}}}{\|v\|_{\mathcal{Y}}} =: C_{\mathcal{N}} \tag{4.1.6}$$

and

$$c_{\mathcal{M}} := \inf_{u\in\mathcal{X}\setminus\{0\}} \frac{\|u\|_{\mathcal{M}}}{\|u\|_{\mathcal{X}}} \leq \sup_{u\in\mathcal{X}\setminus\{0\}} \frac{\|u\|_{\mathcal{M}}}{\|u\|_{\mathcal{X}}} =: C_{\mathcal{M}}. \tag{4.1.7}$$

It follows that $c_{\mathcal{N}}\|\cdot\|_{\mathcal{Y}} \leq \|\cdot\|_{\mathcal{N}} \leq C_{\mathcal{N}}\|\cdot\|_{\mathcal{Y}}$ on $\mathcal{Y}$ and $c_{\mathcal{M}}\|\cdot\|_{\mathcal{X}} \leq \|\cdot\|_{\mathcal{M}} \leq C_{\mathcal{M}}\|\cdot\|_{\mathcal{X}}$ on $\mathcal{X}$ in the sense of equivalent norms. We give two examples of such operators.

Example 4.1.5. Let $\mathcal{M} : \mathcal{X} \to \mathcal{X}'$ be the Riesz map, defined by $\langle \mathcal{M}w, \widetilde{w}\rangle_{\mathcal{X}'\times\mathcal{X}} = \langle w, \widetilde{w}\rangle_{\mathcal{X}}$, $w, \widetilde{w} \in \mathcal{X}$. Then $c_{\mathcal{M}} = 1 = C_{\mathcal{M}}$ and $\|\cdot\|_{\mathcal{X}} \equiv \|\cdot\|_{\mathcal{M}}$.

Example 4.1.6. Let $D \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Set $\mathcal{X} := \mathcal{Y} := H_0^1(D)$. A possible choice for $\mathcal{M} = \mathcal{N}$ is the Laplace operator $\mathcal{M} := \mathcal{N} := -\Delta$.

Theorem 4.1.4 has already highlighted the crucial role of the discrete inf-sup condition, for which we now introduce some terminology.

Definition 4.1.7. A pair of subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ is called **non-trivial** if $\mathcal{X}_h \neq \{0\}$ and $\mathcal{Y}_h \neq \{0\}$.

Definition 4.1.8. Let $\mathcal{B} : \mathcal{X} \to \mathcal{Y}'$ and $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ be a pair of subspaces. We define

$$\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) := \inf_{u_h \in \mathcal{X}_h \setminus \{0\}} \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B} u_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|u_h\|_{\mathcal{X}} \|v_h\|_{\mathcal{Y}}} \tag{4.1.8}$$

if $\mathcal{X}_h \times \mathcal{Y}_h$ is non-trivial and $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) = 0$ otherwise. A given non-trivial pair $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ is said to **satisfy the discrete inf-sup condition for** $\mathcal{B}$ if we have $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$. A family of subspaces $\{\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}\}_h$ indexed by h is called **stable for** $\mathcal{B}$ if there exists $\gamma_0 > 0$ such that $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \geq \gamma_0 > 0$ for each h.

The following theorem is the basis for the minimal residual Petrov-Galerkin method. The quasi-optimality estimate obtained here is analogous to (4.1.5). The proof uses [XZ03, Lemma 5] to remove the classical "1+" from the discrete inf-sup constant.

Theorem 4.1.9. *Let*

1. *$\mathcal{X}$ and $\mathcal{Y}$ be real Hilbert spaces and $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$.*
2. *$\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ be a non-trivial pair of subspaces that satisfies the discrete inf-sup condition for $\mathcal{B}$, i.e., $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$ in (4.1.8).*
3. *$\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$ be an s.p.d. operator.*

Then for any $u \in \mathcal{X}$ there exists a unique $u_h \in \mathcal{X}_h$ which satisfies

$$\mathcal{R}_h(u_h) = \inf_{w_h \in \mathcal{X}_h} \mathcal{R}_h(w_h), \quad \mathcal{R}_h(w_h) := \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{|\langle \mathcal{B} w_h - \mathcal{B} u, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}|}{\|v_h\|_{\mathcal{N}}}. \tag{4.1.9}$$

Moreover, there holds the quasi-optimality estimate

$$\|u - u_h\|_{\mathcal{X}} \leq C_h \inf_{w_h \in \mathcal{X}_h} \|u - w_h\|_{\mathcal{X}} \tag{4.1.10}$$

with

$$C_h := \frac{C_{\mathcal{N}}}{c_{\mathcal{N}}} \frac{\|\mathcal{B}\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')}}{\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)} < \infty, \tag{4.1.11}$$

where $0 < c_{\mathcal{N}} \leq C_{\mathcal{N}} < \infty$ are given by (4.1.6).

Proof. For the proof we abbreviate $\gamma_h := \gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)$. Suppose, we have constructed a linear projector $P_h : \mathcal{X} \to \mathcal{X}_h$ such that for any $w \in \mathcal{X}$ the element $w_h := P_h w \in \mathcal{X}_h$ satisfies (4.1.9), and $\|P_h\|_{\mathcal{L}(\mathcal{X}, \mathcal{X})} \leq C_h$, where C_h is the constant in (4.1.10). Then (we can w.l.o.g. assume that $P_h \neq \mathrm{Id}_{\mathcal{X}}$ and $P_h \neq 0$), by [XZ03, Lemma 5] we have $\|\mathrm{Id}_{\mathcal{X}} - P_h\|_{\mathcal{L}(\mathcal{X}, \mathcal{X})} = \|P_h\|_{\mathcal{L}(\mathcal{X}, \mathcal{X})}$, and the quasi-optimality estimate (4.1.10) is due to

$$\|u - u_h\|_{\mathcal{X}} = \|(\mathrm{Id}_{\mathcal{X}} - P_h)(u - w_h)\|_{\mathcal{X}} \leq C_h \|u - w\|_{\mathcal{X}} \quad \forall w \in \mathcal{X}_h. \tag{4.1.12}$$

The remainder of the proof is devoted to the construction of this projector. Let $\langle \cdot, \cdot \rangle_{\mathcal{N}} := \langle \mathcal{N} \cdot, \cdot \rangle_{\mathcal{Y}' \times \mathcal{Y}}$ denote the scalar product on $\mathcal{Y}$ generated by $\mathcal{N}$. For each $w \in \mathcal{X}$ let $\widehat{w} \in \mathcal{Y}_h$ denote the unique element which satisfies

$$\langle \widehat{w}, v_h \rangle_{\mathcal{N}} = \langle \mathcal{B} w, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}} \quad \forall v_h \in \mathcal{Y}_h. \tag{4.1.13}$$

This element exists and is unique by the Riesz representation theorem on the Hilbert space $(\mathcal{Y}_h, \langle \cdot, \cdot \rangle_{\mathcal{N}})$, since $\mathcal{Y}_h \subseteq \mathcal{Y}$ is a closed subspace w.r.t. $\|\cdot\|_{\mathcal{Y}}$ by assumption, hence also w.r.t. $\|\cdot\|_{\mathcal{N}}$. It is obvious that the map $w \mapsto \widehat{w}$ is linear on $\mathcal{X}$. Choosing $v_h := \widehat{w}$ in the above formula we find $\|\widehat{w}\|_{\mathcal{N}} \leq c_{\mathcal{N}}^{-1} \|\mathcal{B}\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} \|w\|_{\mathcal{X}}$ for all $w \in \mathcal{X}$, and by definition of γ_h we have $\|\widehat{w_h}\|_{\mathcal{N}} \geq C_{\mathcal{N}}^{-1} \gamma_h \|w_h\|_{\mathcal{X}}$ for all $w_h \in \mathcal{X}_h$. Thus, the linear map $w_h \mapsto \widehat{w_h}$ is continuous and injective on $\mathcal{X}_h$. By Lemma 4.1.3, the image of $\mathcal{X}_h$ under this map, which is the subspace $\widehat{\mathcal{X}_h} := \{\widehat{w_h} : w_h \in \mathcal{X}_h\} \subseteq \mathcal{Y}_h$, is a closed (w.r.t. $\|\cdot\|_{\mathcal{N}}$) subspace of $\mathcal{Y}$.

Now, let $w \in \mathcal{X}$ be arbitrary. By closedness of the subspace $\widehat{\mathcal{X}_h} \subseteq \mathcal{Y}_h$ there exists exactly one $w_h \in \mathcal{X}_h$ with the unique corresponding $\widehat{w_h} \in \widehat{\mathcal{X}_h}$, which minimizes $\|\widehat{w_h} - \widehat{w}\|_{\mathcal{N}}$ on $\widehat{\mathcal{X}_h}$, i.e., $\widehat{w_h}$ is the $\mathcal{N}$-orthogonal projection of $\widehat{w}$ onto $\widehat{\mathcal{X}_h}$. Hence, it is the unique minimizer of

$$\|\widehat{w_h} - \widehat{w}\|_{\mathcal{N}} = \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \widehat{w_h} - \widehat{w}, v_h \rangle_{\mathcal{N}}}{\|v_h\|_{\mathcal{N}}} = \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{|\langle \mathcal{B}(w_h - w), v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}|}{\|v_h\|_{\mathcal{N}}}, \tag{4.1.14}$$

which is just the statement of (4.1.9). Let us call $P_h : \mathcal{X} \to \mathcal{X}_h$ the map which takes $w \in \mathcal{X}$ to $P_h w := w_h \in \mathcal{X}_h$ in this fashion. Since $P_h w$ is the composition $w \mapsto \widehat{w} \mapsto \widehat{w_h} \mapsto w_h$, it is linear and idempotent, i.e., $P_h^2 = P_h$.

To obtain a bound on the norm of P_h, we note that $\widehat{w_h} \in \widehat{\mathcal{X}_h}$ is alternatively characterized as the unique element in $\widehat{\mathcal{X}_h}$ with $\langle \widehat{w_h}, v_h \rangle_{\mathcal{N}} = \langle \widehat{w}, v_h \rangle_{\mathcal{N}}$ for all $v_h \in \widehat{\mathcal{X}_h} \subseteq \mathcal{Y}_h$. This implies,

$$C_{\mathcal{N}}^{-1} \gamma_h \|P_h w\|_{\mathcal{X}} \leq \sup_{v_h \in \widehat{\mathcal{X}_h} \setminus \{0\}} \frac{\langle \widehat{w_h}, v_h \rangle_{\mathcal{N}}}{\|v_h\|_{\mathcal{N}}} = \sup_{v_h \in \widehat{\mathcal{X}_h} \setminus \{0\}} \frac{\langle \widehat{w}, v_h \rangle_{\mathcal{N}}}{\|v_h\|_{\mathcal{N}}} \leq c_{\mathcal{N}}^{-1} \|\mathcal{B} w\|_{\mathcal{Y}'}, \tag{4.1.15}$$

where injectivity of $w_h \mapsto \widehat{w_h}$ on $\mathcal{X}_h$ was used in the first inequality. The continuity of $\mathcal{B}$ yields the desired bound. □

This theorem motivates the following definition.

Definition 4.1.10. Let $\mathcal{X}$ and $\mathcal{Y}$ be real Hilbert spaces. Let $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ and $\mathcal{F} \in \mathcal{Y}'$ be given. Let $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$ be an s.p.d. operator. Assume that $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ is a stable pair for $\mathcal{B}$. We then define the **minimal residual Petrov-Galerkin solution** of $\mathcal{B}u = \mathcal{F}$ (for the pair $\mathcal{X}_h \times \mathcal{Y}_h$ and the operator $\mathcal{N}$) as the minimizer of the functional residual,

$$u_h := \operatorname*{arg\,min}_{w_h \in \mathcal{X}_h} \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{|\langle \mathcal{B} w_h - \mathcal{F}, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}|}{\|v_h\|_{\mathcal{N}}}. \tag{4.1.16}$$

If there exists $u \in \mathcal{X}$ such that $\mathcal{B}u = \mathcal{F}$ then by Theorem 4.1.9 the minimal residual Petrov-Galerkin solution $u_h \in \mathcal{X}_h$ is unique and satisfies the quasi-optimality estimate (4.1.10). Moreover, the solution map is continuous by the following corollary to (4.1.15).

Theorem 4.1.11. *In addition to the hypotheses of Theorem 4.1.9, assume that $\mathcal{B} \in \mathrm{Iso}(\mathcal{X}, \mathcal{Y}')$. Then the solution map $\mathcal{F} \mapsto u_h$, given by (4.1.16), is continuous with continuity constant $\frac{1}{\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)} \frac{C_{\mathcal{N}}}{c_{\mathcal{N}}}$.*

4.2 Operator preconditioning

Operator preconditioning [Hip06] or *canonical preconditioning* [MW10] is a methodology for obtaining a well-posed system of linear equations from a Petrov-Galerkin or a finite element method discretization of a well-posed operator equation. We generalize this idea to problems of residual minimization (4.1.9). Hence, in this section we analyze the spectral properties of the (preconditioned) Petrov-Galerkin system matrix and describe how the residual minimization equation (4.1.16) can be formulated as an equivalent generalized linear least squares system. Most importantly, the discrete solution obtained from this least squares system coincides with the minimal residual Petrov-Galerkin solution (4.1.16), and in particular inherits the quasi-optimality bound (4.1.10).

Throughout this subsection, we assume that we are given

- real Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$,
- operators $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ and $\mathcal{F} \in \mathcal{Y}'$,
- s.p.d. operators $\mathcal{M} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ and $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$,

- a *fixed* nontrivial pair of closed subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ of dimensions $M := \dim \mathcal{X}_h \in \mathbb{N} \cup \{\infty\}$ and $N := \dim \mathcal{Y}_h \in \mathbb{N} \cup \{\infty\}$,
- bases $\Phi = \{\phi_m\}_{m\in[M]} \subset \mathcal{X}_h$ for $\mathcal{X}_h$ and $\Psi = \{\psi_n\}_{n\in[N]} \subset \mathcal{Y}_h$ for $\mathcal{Y}_h$.

We then define the matrices $\mathbf{N} \in \mathbb{R}^{N\times N}$, $\mathbf{B} \in \mathbb{R}^{N\times M}$ and $\mathbf{M} \in \mathbb{R}^{M\times M}$ by

$$\mathbf{N} := \langle \mathcal{N}\Psi, \Psi\rangle_{\mathcal{Y}'\times\mathcal{Y}}, \quad \mathbf{B} := \langle \mathcal{B}\Phi, \Psi\rangle_{\mathcal{Y}'\times\mathcal{Y}}, \quad \mathbf{M} := \langle \mathcal{M}\Phi, \Phi\rangle_{\mathcal{X}'\times\mathcal{X}}, \tag{4.2.1}$$

i.e., the components are $\mathbf{B}_{nm} = \langle \mathcal{B}\phi_m, \psi_n\rangle_{\mathcal{Y}'\times\mathcal{Y}}$, $n \in [N]$, $m \in [M]$, and similarly for $\mathbf{N}$ and $\mathbf{M}$. The vector $\mathbf{f} \in \mathbb{R}^N$ is the vector with the components $\mathbf{f}_n = \langle \mathcal{F}, \psi_n\rangle_{\mathcal{Y}'\times\mathcal{Y}}$, $n \in [N]$, for which we also write

$$\mathbf{f} = \langle \mathcal{F}, \Psi\rangle_{\mathcal{Y}'\times\mathcal{Y}}. \tag{4.2.2}$$

We call $\mathbf{B}$ the **system matrix** and $\mathbf{f}$ the **load vector**. Given vectors $\mathbf{u} \in \mathbb{R}^M$ and $\mathbf{v} \in \mathbb{R}^N$, we write

$$\mathbf{u}^\top\Phi := \sum_{m\in[M]} \mathbf{u}_m\phi_m \in \mathcal{X}_h \quad \text{and} \quad \mathbf{v}^\top\Psi := \sum_{n\in[N]} \mathbf{v}_n\psi_n \in \mathcal{Y}_h \tag{4.2.3}$$

if the respective sum converges in norm. The following observation requires no proof.

Observation 4.2.1. Given finitely supported vectors $\mathbf{u}, \widetilde{\mathbf{u}} \in \mathbb{R}^M$ and $\mathbf{v}, \widetilde{\mathbf{v}} \in \mathbb{R}^N$ set $u_h := \mathbf{u}^\top\Phi \in \mathcal{X}_h$ and $v_h := \mathbf{v}^\top\Psi \in \mathcal{Y}_h$, and similarly for $\widetilde{u_h}$ and $\widetilde{v_h}$. Then

1. $\langle \mathcal{B}u_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \mathbf{v}^\top\mathbf{B}\mathbf{u}$ and $\langle \mathcal{F}, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \mathbf{v}^\top\mathbf{f}$,
2. $\langle \mathcal{M}u_h, \widetilde{u_h}\rangle_{\mathcal{X}'\times\mathcal{X}} = \widetilde{\mathbf{u}}^\top\mathbf{M}\mathbf{u}$ and $\langle \mathcal{N}v_h, \widetilde{v_h}\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \widetilde{\mathbf{v}}^\top\mathbf{N}\mathbf{v}$.

Let $\widetilde{\Phi}$ denote the **analysis operator**, defined by $\widetilde{\Phi}(\mathbf{u}^\top\Phi) := \mathbf{u}$ for any $\mathbf{u} \in \mathbb{R}^M$ with $\#\operatorname{supp}\mathbf{u} < \infty$. Since Φ is a basis for $\mathcal{X}_h$, the subspace of $\mathbf{u}^\top\Phi$ with $\mathbf{u} \in \mathbb{R}^M$ finitely supported is dense in $\mathcal{X}_h$. Therefore, $\widetilde{\Phi}$ extends uniquely by continuity to an isometric isomorphism $\widetilde{\Phi} : (\mathcal{X}_h, \|\cdot\|_{\mathcal{M}}) \to \ell^2_{\mathbf{M}}$ and the identities in Observation 4.2.1 extend to all $\mathbf{u}, \widetilde{\mathbf{u}} \in \ell^2_{\mathbf{M}}$ and $\mathbf{v}, \widetilde{\mathbf{v}} \in \ell^2_{\mathbf{N}}$. Immediate consequences are:

- $\mathbf{M}$ is s.p.d. on $\ell^2_{\mathbf{M}}$ and $\mathbf{N}$ is s.p.d. on $\ell^2_{\mathbf{N}}$,
- $\mathbf{f} \in \ell^2_{\mathbf{N}^{-1}}$ and $\mathbf{B} \in \mathcal{L}(\ell^2_{\mathbf{M}}, \ell^2_{\mathbf{N}^{-1}})$,
- if $\mathcal{B} \in \operatorname{Iso}(\mathcal{X}, \mathcal{Y})$ then $\mathbf{B} \in \operatorname{Iso}(\ell^2_{\mathbf{M}}, \ell^2_{\mathbf{N}^{-1}})$,
- for all $\mathbf{u} \in \ell^2_{\mathbf{M}}$ and $\mathbf{v} \in \ell^2_{\mathbf{N}}$, the products $\mathbf{v}^\top\mathbf{B}\mathbf{u}$ and $\mathbf{v}^\top\mathbf{f}$ are finite,
- there holds

$$\frac{\langle \mathcal{B}u_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}}}{\|u_h\|_{\mathcal{M}}\|v_h\|_{\mathcal{N}}} = \frac{\mathbf{v}^\top\mathbf{B}\mathbf{u}}{\|\mathbf{u}\|_{\mathbf{M}}\|\mathbf{v}\|_{\mathbf{N}}} \quad \text{for all} \quad \begin{cases} u_h = \mathbf{u}^\top\Phi \in \mathcal{X}_h \setminus \{0\}, \\ v_h = \mathbf{v}^\top\Psi \in \mathcal{Y}_h \setminus \{0\}. \end{cases} \tag{4.2.4}$$

The identity (4.2.4) is the essence of the operator preconditioning methodology.

Proposition 4.2.2. *Assume that $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$ holds in (4.1.8). Then $\mathbf{B}$ is injective.*

Proof. Let $\mathbf{u} \in \ell^2_{\mathbf{M}}$ be arbitrary, $\mathbf{u} \neq 0$. Then $u_h := \mathbf{u}^\top\Phi \in \mathcal{X}_h$ is non-zero. Since $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \neq 0$ by assumption, there exists $v_h = \mathbf{v}^\top\Psi \in \mathcal{Y}_h \neq 0$ such that $\langle \mathcal{B}u_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}} \neq 0$, and hence also $\mathbf{v}^\top\mathbf{B}\mathbf{u} \neq 0$, i.e., $\mathbf{B}\mathbf{u} \neq 0$. This shows that $\mathbf{B}$ is injective. □

Since $\mathbf{N}$ is s.p.d., there exists a non-singular square matrix $\mathbf{R}$ such that $\mathbf{R}^\top\mathbf{R} = \mathbf{N}$. This may be the Cholesky factor or the matrix square root [GV96, Section 4.2.10]. We write $\mathbf{N}^{1/2} := \mathbf{R}$, $\mathbf{N}^{\top/2} := \mathbf{R}^\top$, $\mathbf{N}^{-1/2} := \mathbf{R}^{-1}$ and $\mathbf{N}^{-\top/2} := (\mathbf{R}^{-1})^\top$. Similar notation is adopted for $\mathbf{M}$. Now, we obtain bounds on the singular values of the preconditioned system matrix $\mathbf{N}^{-\top/2}\mathbf{B}\mathbf{M}^{-1/2}$. To simplify the argument, we assume that $\mathcal{X}_h$ and $\mathcal{Y}_h$ are finite-dimensional.

Proposition 4.2.3. *Assume that $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ is a non-trivial pair of finite-dimensional subspaces that satisfy the discrete inf-sup condition, i.e., $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$ holds in (4.1.8). Then:*

1. *The matrix $\mathbf{B}^\top \mathbf{N}^{-1} \mathbf{B}$ is s.p.d.*
2. *Every singular value σ of the matrix $\widetilde{\mathbf{B}} := \mathbf{N}^{-\top/2} \mathbf{B} \mathbf{M}^{-1/2}$ satisfies the bounds*

$$\frac{\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)}{C_{\mathcal{N}} C_{\mathcal{M}}} \leq \sigma \leq \frac{\|\mathcal{B}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')}}{c_{\mathcal{N}} c_{\mathcal{M}}}. \tag{4.2.5}$$

3. *The matrix $\widetilde{\mathbf{B}}^\top \widetilde{\mathbf{B}}$ is s.p.d. Its condition number $\kappa_2(\widetilde{\mathbf{B}}^\top \widetilde{\mathbf{B}})$ is bounded by*

$$\sqrt{\kappa_2(\widetilde{\mathbf{B}}^\top \widetilde{\mathbf{B}})} \leq \frac{C_{\mathcal{N}} C_{\mathcal{M}}}{c_{\mathcal{N}} c_{\mathcal{M}}} \frac{\|\mathcal{B}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')}}{\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)}. \tag{4.2.6}$$

Here, $0 < c_{\mathcal{N}} \leq C_{\mathcal{N}} < \infty$ and $0 < c_{\mathcal{M}} \leq C_{\mathcal{M}} < \infty$ are as in (4.1.6)–(4.1.7).

Proof. By Proposition 4.2.2, $\mathbf{B} \in \mathbb{R}^{N \times M}$ is injective, and we necessarily have $N \geq M$. Moreover, $\mathbf{M}$ and $\mathbf{N}$ being s.p.d. matrices, the matrices $\mathbf{B}^\top \mathbf{N}^{-1} \mathbf{B} \in \mathbb{R}^{N \times N}$ and $\widetilde{\mathbf{B}}^\top \widetilde{\mathbf{B}} \in \mathbb{R}^{M \times M}$ are s.p.d. The condition number of $\widetilde{\mathbf{B}}^\top \widetilde{\mathbf{B}}$ is therefore the ratio of the largest to the smallest singular value of $\widetilde{\mathbf{B}}$, hence (4.2.6) follows from (4.2.5).

To prove (4.2.5), let $0 \leq \sigma_{\min}(\widetilde{\mathbf{B}}) \leq \sigma_{\max}(\widetilde{\mathbf{B}})$ denote the minimal and the maximal singular value of $\widetilde{\mathbf{B}}$, respectively. From (4.2.4) we obtain

$$\frac{\widetilde{\mathbf{v}}^\top \widetilde{\mathbf{B}} \widetilde{\mathbf{u}}}{\|\widetilde{\mathbf{u}}\|_{\ell^2_M} \|\widetilde{\mathbf{v}}\|_{\ell^2_N}} = \frac{\mathbf{v}^\top \mathbf{B} \mathbf{u}}{\|\mathbf{u}\|_{\ell^2_{\mathbf{M}}} \|\mathbf{v}\|_{\ell^2_{\mathbf{N}}}} = \frac{\langle \mathcal{B} u_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|u_h\|_{\mathcal{M}} \|v_h\|_{\mathcal{N}}}$$

for all $u_h = \mathbf{u}^\top \Phi \in \mathcal{X}_h \setminus \{0\}$ and $v_h = \mathbf{v}^\top \Psi \in \mathcal{Y}_h \setminus \{0\}$ with $\widetilde{\mathbf{u}} = \mathbf{M}^{1/2} \mathbf{u}$ and $\widetilde{\mathbf{v}} = \mathbf{N}^{1/2} \mathbf{v}$. Inserting this into the variational characterization of singular values,

$$\sigma_{\min}(\widetilde{\mathbf{B}}) = \inf_{\widetilde{\mathbf{u}} \in \mathbb{R}^M \setminus \{0\}} \sup_{\widetilde{\mathbf{v}} \in \mathbb{R}^N \setminus \{0\}} \frac{\widetilde{\mathbf{v}}^\top \widetilde{\mathbf{B}} \widetilde{\mathbf{u}}}{\|\widetilde{\mathbf{u}}\|_{\ell^2_M} \|\widetilde{\mathbf{v}}\|_{\ell^2_N}}, \tag{4.2.7}$$

$$\sigma_{\max}(\widetilde{\mathbf{B}}) = \sup_{\widetilde{\mathbf{u}} \in \mathbb{R}^M \setminus \{0\}} \sup_{\widetilde{\mathbf{v}} \in \mathbb{R}^N \setminus \{0\}} \frac{\widetilde{\mathbf{v}}^\top \widetilde{\mathbf{B}} \widetilde{\mathbf{u}}}{\|\widetilde{\mathbf{u}}\|_{\ell^2_M} \|\widetilde{\mathbf{v}}\|_{\ell^2_N}}, \tag{4.2.8}$$

the claim (4.2.5) is immediate from the definition of $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)$ and (4.1.6)–(4.1.7). □

Discrete variational problems of the type discussed in Theorem 4.1.4 readily lead to a linear algebraic equation $\mathbf{B}\mathbf{u} = \mathbf{f}$. Since we have $\mathbf{B} \in \mathbb{R}^{N \times M}$ with possibly $N \neq M$ or $M = N = \infty$, we discuss the linear algebraic system corresponding to the discrete residual minimization (4.1.16). The following proposition provides several options.

Proposition 4.2.5. *Let $u_h = \mathbf{u}^\top \Phi \in \mathcal{X}_h$. Equivalent are:*

i) The vector $u_h \in \mathcal{X}_h$ is the minimal residual Petrov-Galerkin solution (4.1.16),

ii) The vector $\mathbf{u} \in \ell^2_{\mathbf{M}}$ minimizes the discrete algebraic residual:

$$\|\mathbf{B}\mathbf{u} - \mathbf{f}\|_{\mathbf{N}^{-1}} = \inf_{\mathbf{w} \in \ell^2_{\mathbf{M}}} \|\mathbf{B}\mathbf{w} - \mathbf{f}\|_{\mathbf{N}^{-1}}, \tag{4.2.9}$$

iii) The vector $\mathbf{u} \in \ell^2_{\mathbf{M}}$ solves the generalized Gauss normal equations:

$$\mathbf{B}^\top \mathbf{N}^{-1} \mathbf{B} \mathbf{u} = \mathbf{B}^\top \mathbf{N}^{-1} \mathbf{f}, \tag{4.2.10}$$

iv) The vector $\widetilde{\mathbf{u}} \in \ell^2_M$ solves the preconditioned generalized Gauss normal equations:

$$\widetilde{\mathbf{B}}^\top \widetilde{\mathbf{B}} \widetilde{\mathbf{u}} = \widetilde{\mathbf{B}}^\top \widetilde{\mathbf{f}}, \tag{4.2.11}$$

where $\widetilde{\mathbf{B}} := \mathbf{N}^{-\top/2}\mathbf{B}\mathbf{M}^{-1/2}$, $\widetilde{\mathbf{u}} := \mathbf{M}^{1/2}\mathbf{u}$, $\widetilde{\mathbf{f}} := \mathbf{N}^{-\top/2}\mathbf{f}$.

Proof. The equivalence i) $\Leftrightarrow$ ii) is due to the identity

$$\sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B} w_h - \mathcal{F}, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|v_h\|_{\mathcal{N}}} = \sup_{\mathbf{v} \in \ell^2_{\mathbf{N}} \setminus \{0\}} \frac{\mathbf{v}^\top (\mathbf{B}\mathbf{w} - \mathbf{f})}{\|\mathbf{v}\|_{\mathbf{N}}} = \|\mathbf{B}\mathbf{w} - \mathbf{f}\|_{\mathbf{N}^{-1}} \tag{4.2.12}$$

for all $w_h = \mathbf{w}^\top \Phi \in \mathcal{X}_h$.

To obtain ii) $\Leftrightarrow$ iii) note that for all $\mathbf{w} \in \ell^2_{\mathbf{M}}$ the map

$$\mathbb{R} \to \mathbb{R}, \quad \varepsilon \mapsto \|\mathbf{B}(\mathbf{u} + \varepsilon \mathbf{w}) - \mathbf{f}\|^2_{\mathbf{N}^{-1}} \tag{4.2.13}$$

is differentiable and convex, and the derivative vanishes at $\varepsilon = 0$ for all $\mathbf{w} \in \ell^2_{\mathbf{M}}$ if and only if $\mathbf{u}$ satisfies (4.2.10). Hence, ii) $\Rightarrow$ iii). Since the map $\ell^2_{\mathbf{M}} \to \mathbb{R}$, $\mathbf{u} \mapsto \|\mathbf{B}\mathbf{u} - \mathbf{f}\|^2_{\mathbf{N}^{-1}}$ is strictly convex, it possesses at most one (local) minimum. This shows iii) $\Rightarrow$ ii).

The equivalence iii) $\Leftrightarrow$ iv) is clear. □

Corollary 4.2.6. *Let $u \in \mathcal{X}$ and $\mathcal{F} := \mathcal{B}u \in \mathcal{Y}'$. Let the discrete inf-sup condition $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$ hold. Let $\mathbf{u} \in \ell^2_{\mathbf{M}}$ be a solution of either equation, (4.2.9), (4.2.10) or (4.2.11). Then $u_h := \mathbf{u}^\top \Phi \in \mathcal{X}_h$ is the unique minimal residual Petrov-Galerkin solution satisfying (4.1.16).*

Proof. By Proposition 4.2.5, the element $u_h := \mathbf{u}^\top \Phi \in \mathcal{X}_h$ satisfies (4.1.9). Since $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$, Theorem 4.1.9 shows that such a $u_h \in \mathcal{X}_h$ is unique, and hence uniquely solves (4.1.16). □

4.3 Iterative solution

Proposition 4.2.5 and Theorem 4.1.9 can be summarized by saying that the optimization problem

$$\text{find} \quad \mathbf{u} \in \ell^2_{\mathbf{M}} \quad \text{s.t.} \quad \|\mathbf{B}\mathbf{u} - \mathbf{f}\|_{\mathbf{N}^{-1}} \xrightarrow{!} \min \tag{4.3.1}$$

has a unique solution if the discrete inf-sup condition $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$ is satisfied. The solution is given by the corresponding Gauss normal equations (4.2.10) or, in preconditioned form, by (4.2.10). Efficient iterative Krylov subspace methods for such problems are known [Cho06]. One option is the LSQR algorithm due to Paige and Saunders [PS82] applied to the normal equations. We formulate it in such a way that only the the preconditioners $\mathbf{M}^{-1}$ and $\mathbf{N}^{-1}$ need to be applied, but not the square roots, cf. [Ben99]. We refer to [CPT09] for the discussion of stopping criteria for the LSQR algorithm.

Algorithm 4.3.1 (Generalized least squares)**.** For $\mathbf{B} \in \mathbb{R}^{N \times M}$, $N, M \in \mathbb{N}_0$, $M \le N < \infty$, of full rank, $\mathbf{M} \in \mathbb{R}^{M \times M}$ and $\mathbf{N} \in \mathbb{R}^{N \times N}$ s.p.d., $\mathbf{f} \in \mathbb{R}^N$, compute an approximate solution $\mathbf{u}_{i^\star} \approx \mathbf{u} \in \mathbb{R}^M$ to

$$\mathbf{B}^\top \mathbf{N}^{-1} \mathbf{B} \mathbf{u} = \mathbf{B}^\top \mathbf{N}^{-1} \mathbf{f} \tag{4.3.2}$$

using $\mathbf{M}$ as a preconditioner.

1. a) $(\mathbf{v}_1, \widehat{\mathbf{v}}_1, \beta_1) := \textsc{Normalize}(\mathbf{f}, \mathbf{N})$
   b) $(\mathbf{w}_1, \widehat{\mathbf{w}}_1, \alpha_1) := \textsc{Normalize}(\mathbf{B}^\top \mathbf{v}_1, \mathbf{M})$
   c) $\mathbf{d}_1 := \mathbf{w}_1$, $\mathbf{u}_0 := \mathbf{0}$, $\bar{\phi}_1 = \beta_1$, $\bar{\rho}_1 = \alpha_1$
2. For $i = 1, 2, \ldots, i^\star$ do the following steps (until convergence)

a) $(\mathbf{v}_{i+1}, \widehat{\mathbf{v}}_{i+1}, \beta_{i+1}) := \textsc{Normalize}(\mathbf{B}\mathbf{w}_i - \alpha_i \widehat{\mathbf{v}}_i, \mathbf{N})$

b) $(\mathbf{w}_{i+1}, \widehat{\mathbf{w}}_{i+1}, \alpha_{i+1}) := \textsc{Normalize}(\mathbf{B}^\top \mathbf{v}_{i+1} - \beta_{i+1} \widehat{\mathbf{w}}_i, \mathbf{M})$

c) $\rho_i := \sqrt{\bar{\rho}_i^2 + \beta_{i+1}^2}, \qquad c_i := \bar{\rho}_i / \rho_i, \qquad s_i := \beta_{i+1}/\rho_i$

d) $\theta_{i+1} := s_i \alpha_{i+1}, \qquad \bar{\rho}_{i+1} := -c_i \alpha_{i+1}, \qquad \phi_i := c_i \bar{\phi}_i, \qquad \bar{\phi}_{i+1} := s_i \bar{\phi}_i$

e) $\mathbf{u}_i := \mathbf{u}_{i-1} + (\phi_i/\rho_i)\mathbf{d}_i, \qquad \mathbf{d}_{i+1} := \mathbf{w}_{i+1} - (\theta_{i+1}/\rho_i)\mathbf{d}_i$

$\textsc{Normalize} : \mathbb{R}^K \times \mathbb{R}^{K\times K} \ni (\mathbf{s}, \mathbf{S}) \mapsto (\mathbf{z}, \widehat{\mathbf{z}}, z) \in \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}$, with $\mathbf{S}$ s.p.d.

1. Solve $\mathbf{S}\mathbf{s}^\star = \mathbf{s}$ for $\mathbf{s}^\star$. Set $z := \sqrt{\mathbf{s}^\top \mathbf{s}^\star}$ and $(\mathbf{z}, \widehat{\mathbf{z}}) := (z^{-1}\mathbf{s}^\star, z^{-1}\mathbf{s})$

4.4 On the inf-sup condition

We will formulate and verify results concerning the stability of pairs of subspaces using the following notation.

Definition 4.4.1. Let X and Y be normed real vector spaces. Let $\langle \cdot, \cdot \rangle_{X\times Y} : X \times Y \to \mathbb{R}$ be a map. For any pair of subspaces $U \times V \subseteq X \times Y$ we define

$$\mathcal{K}_{X\times Y}(U, V) := \inf_{u \in U\setminus\{0\}} \sup_{v \in V\setminus\{0\}} \frac{\langle u, v\rangle_{X\times Y}}{\|u\|_X \|v\|_Y} \tag{4.4.1}$$

if $U \times V$ is a non-trivial pair, and $\mathcal{K}_{X\times Y}(U, V) := 0$ if otherwise.

A frequent situation will be $X = Y$ or $Y = X'$, as the following examples illustrate.

Example 4.4.2. If X is a Banach space and $Y = X'$ is its dual, then $\langle \cdot, \cdot \rangle_{X\times Y}$ will mean the duality pairing $\langle \cdot, \cdot \rangle_{X\times X'}$. In this case we have $0 \le \mathcal{K}_{X\times Y}(U, V) \le 1$ for any pair of subspaces $U \times V \subseteq X \times Y$.

Example 4.4.3. Let $X = Y$ be a Hilbert space. In this case $\langle \cdot, \cdot \rangle_{X\times Y}$ will mean the scalar product on $X = Y$. Owing to the Cauchy-Schwarz inequality we have $\mathcal{K}_{Y\times Y}(V, V) = 1$ for any subspace $\{0\} \neq V \subseteq X$.

Example 4.4.4. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces, and $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$. Set $\langle \cdot, \cdot \rangle_{\mathcal{X}\times\mathcal{Y}} := \langle \mathcal{B}\cdot, \cdot \rangle_{\mathcal{Y}'\times\mathcal{Y}}$ on $\mathcal{X} \times \mathcal{Y}$. Then, tautologically, $\gamma_{\mathcal{B}}(\cdot, \cdot) = \mathcal{K}_{\mathcal{X}\times\mathcal{Y}}(\cdot, \cdot)$.

4.4.1 General properties

Several useful characterizations for the quantity $\mathcal{K}_{X\times Y}(U, V)$ are stated and discussed. The proofs are given at the end of this subsection.

Proposition 4.4.5. *Let $X = Y$ be a Hilbert space. Let $\{0\} \neq U \subseteq X$ and $W \subseteq X$ be closed subspaces. Let further $W^+ \subseteq X$ be any closed subspace such that $U \subseteq W \oplus W^+$ (direct sum) and $W \perp_X W^+$ (orthogonal subspaces in X). Let $Q : X \to W$ be the X-orthogonal projector onto W. Then for any $0 \le \kappa \le 1$ the following are equivalent:*

$$\mathcal{K}_{X\times X}(U, W) \ge \kappa, \tag{4.4.2}$$

$$\sup_{w \in W\setminus\{0\}} \frac{\langle u, w\rangle_X}{\|w\|_X} \ge \kappa \|u\|_X \qquad \forall u \in U, \tag{4.4.3}$$

$$\|Qu\|_X \ge \kappa \|u\|_X \qquad \forall u \in U, \tag{4.4.4}$$

$$\|u - Qu\|_X \le \sqrt{1-\kappa^2}\|u\|_X \qquad \forall u \in U, \tag{4.4.5}$$

$$\langle u, w^+\rangle_X \le \sqrt{1-\kappa^2}\|u\|_X \|w^+\|_X \qquad \forall u \in U, w^+ \in W^+. \tag{4.4.6}$$

The inequality (4.4.5) may be interpreted as a statement on the quality of the approximation of elements in U by elements in W. If U and V are finite-dimensional then $\mathcal{K}_{X\times X}(U,V) = \cos\theta$, where $0 \le \theta \le \pi/2$ is the **largest principal angle** between the subspaces U and V, cf. [GV96, Section 12.4.3]. An inequality of the form (4.4.6) is called a **strengthened Cauchy-Schwarz inequality**.

If $U_1, U_2, W \subseteq X$ are subspaces, then not necessarily $\mathcal{K}_{X\times X}(U,W) > 0$ for $U = U_1 + U_2$, even if this holds for $U = U_1$ and $U = U_2$. Indeed, consider $U_1 := \mathbb{R}\times\{0\}$, $U_2 := \{0\}\times\mathbb{R}$ and $W := \{(r,r) : r \in \mathbb{R}\}$ as subsets of $X := \mathbb{R}^2$. However, we may state the following as a consequence of Proposition 4.4.5.

Corollary 4.4.6. *Let $U_1, U_2, W \subseteq X$ be closed subspaces such that either $U_1 \subseteq W$ or $U_2 \subseteq W$. Then*

$$\mathcal{K}_{X\times X}(U_1 + U_2, W) \ge \min\{\mathcal{K}_{X\times X}(U_1, W), \mathcal{K}_{X\times X}(U_2, W)\}. \tag{4.4.7}$$

We are going to generalize some of the equivalences in Proposition 4.4.5. Consider two Hilbert spaces X and Y and a continuous bilinear form $\langle\cdot,\cdot\rangle_{X\times Y} : X\times Y \to \mathbb{R}$. Let $U\times V \subseteq X\times Y$ be a pair of subspaces. By the Riesz representation theorem on the Hilbert space $(V, \|\cdot\|_Y)$ there exists a unique $\Gamma \in \mathcal{L}(X,V)$ such that

$$\langle \Gamma x, v\rangle_Y = \langle x, v\rangle_{X\times Y} \quad \forall (x,v) \in X\times V. \tag{4.4.8}$$

In the following we will simply write

$$\langle \Gamma\cdot,\cdot\rangle_Y := \langle\cdot,\cdot\rangle_{X\times Y} \quad \text{on} \quad X\times V \tag{4.4.9}$$

to define $\Gamma \in \mathcal{L}(X,V)$.

Proposition 4.4.7. *Let X and Y be Hilbert spaces, and $U\times V \subseteq X\times Y$ a pair of closed subspaces. Let $\langle\cdot,\cdot\rangle_{X\times Y}$ be a continuous bilinear form on $X\times Y$ and define $\Gamma \in \mathcal{L}(X,V)$ by $\langle\Gamma\cdot,\cdot\rangle_Y := \langle\cdot,\cdot\rangle_{X\times Y}$ on $X\times V$. Then for any $\kappa \ge 0$ the following are equivalent:*

i) $\|\Gamma u\|_Y \ge \kappa\|u\|_X$ *for all* $u \in U$,

ii) $\mathcal{K}_{X\times Y}(U,V) \ge \kappa$,

iii) $\mathcal{K}_{X\times Y}(U,\widetilde{V}) \ge \kappa\,\mathcal{K}_{Y\times Y}(V,\widetilde{V})$ *for any closed subspace* $\widetilde{V} \subseteq Y$,

iv) there exists $\Xi \in \mathcal{L}(\Gamma(U),X)$ *with* $\|\Xi\|_{\mathcal{L}(\Gamma(U),X)} \le \kappa^{-1}$ *such that* $\Xi\circ\Gamma = \mathrm{Id}_U$.

The equivalence i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iv) is related to the statement of [Bre74, Theorem 0.1]. By setting $\kappa := \mathcal{K}_{X\times Y}(U,V)$, and exchanging the roles of $\widetilde{V}$ and V, we obtain an important consequence from Proposition 4.4.7, ii) $\Rightarrow$ iii): for arbitrary closed subspaces $U \subseteq X$ and $V, \widetilde{V} \subseteq Y$ we have

$$\mathcal{K}_{X\times Y}(U,V) \ge \mathcal{K}_{X\times Y}(U,\widetilde{V})\mathcal{K}_{Y\times Y}(\widetilde{V},V). \tag{4.4.10}$$

This estimate means that, in order to obtain $\mathcal{K}_{X\times Y}(U,V) > 0$, we may first identify an "optimal" space $\widetilde{V}$ for which $\mathcal{K}_{X\times Y}(U,\widetilde{V})$ is positive, and then pass to a more "practical" space V which approximates $\widetilde{V}$ well, see Proposition 4.4.5 for several characterizations of the latter event, Section 4.4.3 for further discussion, and Theorem 5.2.18 for an application.

In a Gelfand triple $V \hookrightarrow H \cong H' \hookrightarrow V'$ we have the following characterization for $\mathcal{K}_{V'\times V}(U,U)$.

Proposition 4.4.8 ([And12])**.** *Let $V \hookrightarrow H \cong H' \hookrightarrow V'$ be a Gelfand triple of Hilbert spaces. Let Q be the H-orthogonal projector onto a closed subspace $U \subseteq V$. Then, for any $\kappa > 0$, t.f.a.e.:*

a) $\mathcal{K}_{V'\times V}(U,U) \ge \kappa$,

b) $\|Qv\|_V \le \kappa^{-1}\|v\|_V$ *for all* $v \in V$.

Remark 4.4.9. The inequality $\|Qu\|_V \le C\|u\|_V$, i.e., the stability of the H-orthogonal projector onto U in V, has been investigated for finite element spaces, see [Car02] and references therein. Note that any *finite*-dimensional non-trivial subspace $U \subset V$ satisfies $\mathcal{K}_{V'\times V}(U,U) > 0$.

We now give the proofs for the statements of this subsection.

Proof of Proposition 4.4.5. We will use several times the fact that $\langle Qu, u\rangle_X = \|Qu\|_X^2$ for all $u \in U$. Let now $u \in U$ be arbitrary.

- The equivalence (4.4.2) $\Leftrightarrow$ (4.4.3) is clear from the definition.
- (4.4.3) $\Leftrightarrow$ (4.4.4): using $\langle Qu, w\rangle_X = \langle u, w\rangle_X$ for all $w \in W$ we have

$$\|Qu\|_X = \sup_{w\in W\setminus\{0\}} \frac{\langle Qu, w\rangle_X}{\|w\|_X} = \sup_{w\in W\setminus\{0\}} \frac{\langle u, w\rangle_X}{\|w\|_X}.$$

- (4.4.4) $\Rightarrow$ (4.4.5): $\|u - Qu\|_X^2 = \|u\|_X^2 - \|Qu\|_X^2 \le (1-\kappa^2)\|u\|_X^2$.
- (4.4.4) $\Leftarrow$ (4.4.5): $\kappa^2\|u\|_X^2 \le \|u\|_X^2 - \|u - Qu\|_X^2 = \|Qu\|_X^2$.
- (4.4.5) $\Rightarrow$ (4.4.6): $\langle u, w^+\rangle_X \le \|u - Qu\|_X\|w^+\|_X \le \sqrt{1-\kappa^2}\|u\|_X\|w^+\|_X$ for all $w^+ \in W^+$.
- (4.4.5) $\Leftarrow$ (4.4.6): setting $w^+ := u - Qu$, we have $\|w^+\|_X^2 = \langle u, w^+\rangle_X \le \sqrt{1-\kappa^2}\|u\|_X\|w^+\|_X$.

The proof is complete. □

Proof of Corollary 4.4.6. Assume w.l.o.g. that $U_2 \subseteq W$. Set $\kappa_1 := \mathcal{K}_{X\times X}(U_1, W)$. Define $P : X \to U_1 \cap W^{\perp_X}$ and $Q : X \to W$ as the X-orthogonal surjective projectors. For any $u = u_1 + u_2$ with $(u_1, u_2) \in U_1 \times U_2$ we have

$$\|u - Qu\|_X^2 = \|Pu_1 - QPu_1\|_X^2 \le (1-\kappa_1^2)\|Pu_1\|_X^2 \tag{4.4.11}$$

and we obtain (4.4.5) using $\|Pu_1\|_X = \|Pu\|_X \le \|u\|_X$. Since $\mathcal{K}_{X\times X}(U_2, W) = 1$, the implication (4.4.5) $\Rightarrow$ (4.4.2) shows the claim. □

Proof of Proposition 4.4.7. Let $\kappa > 0$. We show i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) and i) $\Leftrightarrow$ iv). Let $u \in U\setminus\{0\}$ be arbitrary.

- i) $\Leftrightarrow$ ii): is seen from

$$\mathcal{K}_{X\times Y}(U, V) = \inf_{u\in U\setminus\{0\}} \sup_{v\in V\setminus\{0\}} \frac{\langle \Gamma u, v\rangle_Y}{\|u\|_X\|v\|_Y} = \inf_{u\in U\setminus\{0\}} \frac{\|\Gamma u\|_Y}{\|u\|_X}. \tag{4.4.12}$$

- ii) $\Rightarrow$ iii): observing the valid implication ii) $\Rightarrow$ i), we have for any subspace $\widetilde{V} \subset Y$

$$\mathcal{K}_{X\times Y}(U, \widetilde{V}) = \inf_{u\in U\setminus\{0\}} \left\{ \frac{\|\Gamma u\|_Y}{\|u\|_X} \sup_{\widetilde{v}\in\widetilde{V}\setminus\{0\}} \frac{\langle u, \widetilde{v}\rangle_{X\times Y}}{\|\Gamma u\|_Y\|\widetilde{v}\|_Y} \right\} \tag{4.4.13}$$

$$\ge \kappa \inf_{u\in U\setminus\{0\}} \sup_{\widetilde{v}\in\widetilde{V}\setminus\{0\}} \frac{\langle \Gamma u, \widetilde{v}\rangle_Y}{\|\Gamma u\|_Y\|\widetilde{v}\|_Y} \ge \kappa\mathcal{K}_{Y\times Y}(V, \widetilde{V}). \tag{4.4.14}$$

- iii) $\Leftarrow$ ii): is immediate with $\widetilde{V} := V$.
- i) $\Rightarrow$ iv): First, Γ is injective on U due to i). By Lemma 4.1.3, the image $R := \Gamma(U)$ is a closed subspace of Y, hence a Banach space. Hence, $\Gamma|_U \in \mathcal{L}(U, R)$ is a bijection. The Banach open mapping theorem implies $\Gamma|_U \in \mathrm{Iso}(U, R)$, i.e., $\Xi := \Gamma|_U^{-1} \in \mathcal{L}(R, U)$. Using i) we find $\|\Xi(\Gamma u)\|_X = \|u\|_X \le \kappa^{-1}\|\Gamma u\|_Y$, and $u \in U$ being arbitrary implies $\|\Xi r\|_X \le \kappa^{-1}\|r\|_Y$ for all $r \in R = \Gamma(U)$, hence $\|\Xi\|_{\mathcal{L}(R,U)} \le \kappa^{-1}$.
- i) $\Leftarrow$ iv): using $\Xi \circ \Gamma = \mathrm{Id}_U$ and $\|\Xi\|_{\mathcal{L}(\Gamma(U),X)} \le \kappa^{-1}$ we obtain $\kappa\|u\|_X \le \kappa\|\Xi\|_{\mathcal{L}(\Gamma(U),X)}\|\Gamma u\|_Y \le \|\Gamma u\|_Y$.

This finishes the proof. □

Proof of Proposition 4.4.8. To check a) $\Rightarrow$ b), let $\langle \Gamma\cdot,\cdot\rangle_V := \langle\cdot,\cdot\rangle_{V'\times V}$ on $V'\times U$. The operator $\Gamma|_U : U \to U$ has a closed range by Proposition 4.4.7, ii) $\Rightarrow$ iv). Moreover, it is surjective, since the mapping $v \mapsto \langle \Gamma u, Qv\rangle_V = \langle u, Qv\rangle_{V'\times V} = \langle u, v\rangle_{V'\times V}$ does not vanish on V regardless of $u \in U\setminus\{0\}$. Thus, for all $v \in V\setminus\{0\}$ there exists $u \in U \subset V'$, $u \neq 0$, such that $\Gamma u = Qv$, and

$$\kappa\|u\|_{V'}\|Qv\|_V \le \|\Gamma u\|_V\|Qv\|_V = \langle \Gamma u, Qv\rangle_V = \langle u, v\rangle_{V'\times V} \le \|u\|_{V'}\|v\|_V,$$

where the implication ii) $\Rightarrow$ i) of Proposition 4.4.7 was used. Canceling $\|u\|_{V'}$ shows b).

Conversely, assume b). Let $u \in U\setminus\{0\}$ be arbitrary. Then, using $\langle u, v\rangle_{V'\times V} = \langle u, v\rangle_H = \langle u, Qv\rangle_H = \langle u, Qv\rangle_{V'\times V}$, $v \in V$, we obtain

$$\kappa\|u\|_{V'} = \sup_{v\in V\setminus\{0\}} \frac{\langle u, v\rangle_{V'\times V}}{\kappa^{-1}\|v\|_V} = \sup_{v\in V, Qv\neq 0} \frac{\langle u, Qv\rangle_{V'\times V}}{\kappa^{-1}\|v\|_V} \le \sup_{v\in U\setminus\{0\}} \frac{\langle u, v\rangle_{V'\times V}}{\|v\|_V},$$

where $\kappa^{-1}\|v\|_V \ge \|Qv\|_V$ was used in the last step. Dividing on both sides by $\|u\|_{V'}$ and taking the infimum over $u \in U\setminus\{0\}$ shows $\kappa \le \mathcal{K}_{V'\times V}(U, U)$, which is a). □

4.4.2 Case of sums and tensor products of subspaces

Some further properties of the quantity $\mathcal{K}_{X\times Y}(\cdot,\cdot)$ in the case that X and Y are Hilbert spaces, themselves composed of several Hilbert spaces are reproduced here from [And12]. These are motivated by applications to parabolic problems: there, we consider spaces such as $X = [L^2(0,1)\otimes V]\cap[H^1(0,1)\otimes V']$ and $Y = [L^2(0,1)\otimes V]\times H$, where V and H are Hilbert spaces.

Lemma 4.4.10. *Let X and Y be Hilbert spaces. Let $U_i \subseteq X$, $i \in \mathbb{N}$, and $V \subseteq Y$ be closed subspaces. Let $\langle\cdot,\cdot\rangle_{X\times Y}$ be a continuous bilinear form, set $\kappa_i := \mathcal{K}_{X\times Y}(U_i, V)$, $i \in \mathbb{N}$. Define $\Gamma : X \to V$ by $\langle\Gamma\cdot,\cdot\rangle_Y := \langle\cdot,\cdot\rangle_{X\times Y}$ on $X\times V$. If*

$$\langle u_i, u_j\rangle_X = 0 = \langle \Gamma u_i, \Gamma u_j\rangle_Y \quad \forall (u_i, u_j) \in U_i\times U_j \quad \textit{with} \quad i \neq j \tag{4.4.15}$$

then

$$\mathcal{K}_{X\times Y}(U, V) \ge \inf_{i\in\mathbb{N}} \kappa_i \quad \textit{for} \quad U := \sum_{i=0}^{L} U_i, \quad L \in \mathbb{N}_0. \tag{4.4.16}$$

Proof. Set $\kappa := \inf_{i\in\mathbb{N}} \kappa_i$, let $L \in \mathbb{N}_0$. For any $u = \sum_{i=0}^{L} u_i$, where $u_i \in U_i$, Proposition 4.4.7, ii) $\Rightarrow$ i), implies

$$\|\Gamma u\|_Y^2 = \sum_{i=0}^{L} \|\Gamma u_i\|_Y^2 \ge \sum_{i=0}^{L} \kappa_i^2\|u_i\|_X^2 \ge \kappa^2 \sum_{i=0}^{L} \|u_i\|_X^2 = \kappa^2\|u\|_X^2.$$

Proposition 4.4.7, i) $\Rightarrow$ ii), shows $\mathcal{K}_{X\times Y}(U, V) \ge \kappa$, as claimed. □

If X_i and Y_i are Hilbert spaces, and $\langle\cdot,\cdot\rangle_{X_i\times Y_i} : X_i\times Y_i \to \mathbb{R}$ are continuous bilinear maps, then we define $\langle\cdot,\cdot\rangle_{X\times Y} : X\times Y \to \mathbb{R}$ on $X := X_1\otimes X_2$, $Y := Y_1\otimes Y_2$ as the unique continuous bilinear map which satisfies

$$\langle x_1\otimes x_2, y_1\otimes y_2\rangle_{X\times Y} = \langle x_1, y_1\rangle_{X_1\times Y_1}\langle x_2, y_2\rangle_{X_2\times Y_2} \tag{4.4.17}$$

for all $(x_i, y_i) \in X_i\times Y_i$, $i = 1, 2$. This is the setting for the following lemma.

Lemma 4.4.11. *Let X_i and Y_i be Hilbert spaces and $\langle\cdot,\cdot\rangle_{X_i\times Y_i}$ a continuous bilinear form on $X_i\times Y_i$, $i = 1, 2$. Let $U_i \subseteq X_i$ and $V_i \subseteq Y_i$, $i = 1, 2$, be closed subspaces. Set $X := X_1\otimes X_2$ and $Y = Y_1\otimes Y_2$, as well as $U := U_1\otimes U_2$ and $V = V_1\otimes V_2$. Then*

$$\mathcal{K}_{X\times Y}(U, V) \ge \mathcal{K}_{X_1\times Y_1}(U_1, V_1)\mathcal{K}_{X_2\times Y_2}(U_2, V_2). \tag{4.4.18}$$

Proof. We set $\kappa_i := \mathcal{K}_{X_i \times Y_i}(U_i, V_i)$, $i = 1, 2$, and proceed with the nontrivial case $\kappa_1 \kappa_2 > 0$. Define $\Gamma_i : X_i \to V_i$, $i = 1, 2$, by $\langle \Gamma_i \cdot, \cdot \rangle_Y := \langle \cdot, \cdot \rangle_{X_i \times Y_i}$ on $X_i \times V_i$. Set $X := X_1 \otimes X_2$, $Y := Y_1 \otimes Y_2$ and $\Gamma := \Gamma_1 \otimes \Gamma_2 : X \to Y$. We now use Proposition 4.4.7, i) $\Rightarrow$ iv): the range $R_i := \Gamma_i U_i$ is closed in Y_i, and there exists $\Xi_i \in \mathcal{L}(R_i, X_i)$ with $\Xi_i \circ \Gamma_i = \mathrm{Id}_{U_i}$ and $\|\Xi_i\|_{\mathcal{L}(R_i, X_i)} \leq \kappa_i^{-1}$. Hence, $R := R_1 \otimes R_2$ is closed in $Y_1 \otimes Y_2$ and the operator $\Xi := \Xi_1 \otimes \Xi_2 : R \to U := U_1 \otimes U_2$ satisfies $\Xi \circ \Gamma = \mathrm{Id}_U$ and $\|\Xi\|_{\mathcal{L}(R,X)} \leq \kappa_1^{-1} \kappa_2^{-1}$. Since $R = \Gamma U$ and Γ is precisely the operator given by $\langle \Gamma \cdot, \cdot \rangle_Y = \langle \cdot, \cdot \rangle_{X \times Y}$, the claim follows from Proposition 4.4.7, iv) $\Rightarrow$ i). □

The following proposition combines the preparations of this section and specifically targets the construction of stable pairs of subspaces for linear parabolic evolution equations (cf. Proposition 5.2.14).

Proposition 4.4.12. *Let F and V be Hilbert spaces, set $\mathbb{X} := F \otimes V'$ and $\mathbb{Y} := F \otimes V$. Let*

$$F_k^1 \subseteq F_{k+1}^1 \subseteq F \quad \text{and} \quad F_k^2 \subseteq F_{k+1}^2 \subseteq F, \quad k \in \mathbb{N}_0, \tag{4.4.19}$$

and

$$U_\ell \subseteq U_{\ell+1} \subseteq V' \quad \text{and} \quad V_\ell \subseteq V_{\ell+1} \subseteq V, \quad \ell \in \mathbb{N}_0, \tag{4.4.20}$$

be families of nontrivial nested closed subspaces. Set

$$\tau := \inf_{k \in \mathbb{N}_0} \mathcal{K}_{F \times F}(F_k^1, F_k^2) \quad \text{and} \quad \eta := \inf_{\ell \in \mathbb{N}_0} \mathcal{K}_{V' \times V}(U_\ell, V_\ell). \tag{4.4.21}$$

Let $L \in \mathbb{N}_0$ be arbitrary, fixed. Define the pair of subspaces $\mathbb{U} \times \mathbb{V} \subseteq \mathbb{X} \times \mathbb{Y}$ as

$$\mathbb{U} := \bigcup_{0 \leq k+\ell \leq L} F_k^1 \otimes U_\ell \quad \text{and} \quad \mathbb{V} := \bigcup_{0 \leq k+\ell \leq L} F_k^2 \otimes V_\ell, \tag{4.4.22}$$

where k and ℓ range in $\mathbb{N}_0$. Then

$$\mathcal{K}_{\mathbb{X} \times \mathbb{Y}}(\mathbb{U}, \mathbb{V}) \geq \tau \eta. \tag{4.4.23}$$

Proof. First, for the auxiliary subspace

$$\widetilde{\mathbb{V}} := \bigcup_{0 \leq k+\ell \leq L} F_k^1 \otimes V_\ell \subseteq \mathbb{Y} \tag{4.4.24}$$

we show

i) $\widetilde{\eta} := \mathcal{K}_{\mathbb{X} \times \mathbb{Y}}(\mathbb{U}, \widetilde{\mathbb{V}}) \geq \eta$,

ii) $\widetilde{\tau} := \mathcal{K}_{\mathbb{Y} \times \mathbb{Y}}(\widetilde{\mathbb{V}}, \mathbb{V}) \geq \tau$.

Then, the claim follows immediately from (4.4.10). Proof of i)–ii):

i) Define the closed subspaces

$$G_0^1 := F_0^1, \quad G_k^1 := F_k^1 \cap \left(F_{k-1}^1\right)^{\perp_F} \quad \forall k \in \mathbb{N}. \tag{4.4.25}$$

Using the nestedness $U_\ell \subseteq U_{\ell+1}$ and $V_\ell \subseteq V_{\ell+1}$, there holds

$$\mathbb{U} = \sum_{k=0}^{L} G_k^1 \otimes U_{L-k} \quad \text{and} \quad \widetilde{\mathbb{V}} = \sum_{k=0}^{L} G_k^1 \otimes V_{L-k}. \tag{4.4.26}$$

Lemma 4.4.11 now ensures

$$\mathcal{K}_{\mathbb{X} \times \mathbb{Y}}(G_k^1 \otimes U_\ell, G_k^1 \otimes V_\ell) \geq \underbrace{\mathcal{K}_{F \times F}(G_k^1, G_k^1)}_{=1} \mathcal{K}_{V' \times V}(U_\ell, V_\ell) \geq \eta$$

for all $k, \ell \in \mathbb{N}_0$. In order to extend this to $\mathbb{U}$ and $\mathbb{V}$ we use Lemma 4.4.10: observe that $G_k^1 \perp_F G_{k'}^1$ when $k \neq k'$, therefore also

$$[G_k^1 \otimes U_\ell] \perp_{\mathbb{X}} [G_{k'}^1 \otimes U_{\ell'}] \quad \text{and} \quad [G_k^1 \otimes V_\ell] \perp_{\mathbb{Y}} [G_{k'}^1 \otimes V_{\ell'}] \tag{4.4.27}$$

for all non-negative integers $k \neq k'$, ℓ and ℓ'. Define $\Gamma \in \mathcal{L}(\mathbb{X}, \widetilde{\mathbb{V}})$ by $\langle \Gamma \cdot, \cdot \rangle_{\mathbb{Y}} = \langle \cdot, \cdot \rangle_{\mathbb{X} \times \mathbb{Y}}$ on $\mathbb{X} \times \widetilde{\mathbb{V}}$. Since $\langle u, \widetilde{v} \rangle_{\mathbb{X} \times \mathbb{Y}} = 0$ for all $u \in G_k^1 \otimes U_\ell$ and $\widetilde{v} \in G_{k'}^1 \otimes V_{\ell'}$ with nonnegative integers $k \neq k'$, ℓ and ℓ', there holds $\Gamma(G_k^1 \otimes U_\ell) \subseteq G_k^1 \otimes V_\ell$ for all $k, \ell \in \mathbb{N}_0$.

ii) Define closed subspaces $W_0 := V_0$ and $W_\ell := V_\ell \cap V_{\ell-1}^{\perp_V}$, $\ell \in \mathbb{N}$. The property ii) follows in the same fashion as i) by recognizing that

$$\widetilde{\mathbb{V}} = \sum_{\ell=0}^{L} F_{L-\ell}^1 \otimes W_\ell \quad \text{and} \quad \mathbb{V} = \sum_{\ell=0}^{L} F_{L-\ell}^2 \otimes W_\ell, \tag{4.4.28}$$

and that both sums are, in fact, orthogonal in $\mathbb{Y}$.

This completes the proof. □

The type of subspaces $\mathbb{U}$ and $\mathbb{V}$ discussed in the preceding proposition is referred to as **sparse tensor product** subspaces. As a particular, but important, example of a pair of subspaces $\mathbb{U} \times \mathbb{V} \subseteq \mathbb{X} \times \mathbb{Y}$ we single out the so-called **full tensor product** subspaces

$$\mathbb{U} := F_k^1 \otimes U_\ell \quad \text{and} \quad \mathbb{V} := F_k^2 \otimes V_\ell, \tag{4.4.29}$$

where $(k, \ell) \in \mathbb{N}_0 \times \mathbb{N}_0$ is any fixed pair. Under the assumptions of Proposition 4.4.12, these satisfy $\mathcal{K}_{\mathbb{X} \times \mathbb{Y}}(\mathbb{U}, \mathbb{V}) \geq \tau\eta$, by renaming the subspaces, if necessary, and setting $L = 0$.

4.4.3 Robustness of stability

In this section we comment on the sensitivity of the constant $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)$ in (4.1.8) w.r.t. the test space $\mathcal{Y}_h$. This question plays a central role in some recent related discretization methods cited here.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$. Suppose we are given a stable pair $\mathcal{X}_h \times \widetilde{\mathcal{Y}_h} \subseteq \mathcal{X} \times \mathcal{Y}$ of closed subspaces for $\mathcal{B}$, i.e., $\gamma_{\mathcal{B}}(\mathcal{X}_h, \widetilde{\mathcal{Y}_h}) > 0$. Let $\mathcal{Y}_h \subseteq \mathcal{Y}$ be another closed subspace. If $\mathcal{Y}_h$ and $\widetilde{\mathcal{Y}_h}$ are "close" we may expect that the pair $\mathcal{X}_h \times \mathcal{Y}_h$ is still stable for $\mathcal{B}$. To quantify this, assume that there exists a (not necessarily linear) mapping $P : \widetilde{\mathcal{Y}_h} \to \mathcal{Y}_h$ such that for some fixed $\delta \geq 0$ there holds

$$\|v_h - P(v_h)\|_{\mathcal{Y}} \leq \delta \|v_h\|_{\mathcal{Y}} \quad \forall v_h \in \widetilde{\mathcal{Y}_h}. \tag{4.4.30}$$

Observation 4.4.13. It holds

$$\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \geq \frac{\gamma_{\mathcal{B}}(\mathcal{X}_h, \widetilde{\mathcal{Y}_h}) - \delta \|\mathcal{B}\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')}}{1 + \delta}. \tag{4.4.31}$$

Proof. Let $\varepsilon > 0$ and $u_h \in \mathcal{X}_h$ be arbitrary. Then there exists $v_h \in \widetilde{\mathcal{Y}_h}$ such that $\langle \mathcal{B}u_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}} \geq (\gamma_{\mathcal{B}}(\mathcal{X}_h, \widetilde{\mathcal{Y}_h}) - \varepsilon) \|u_h\|_{\mathcal{X}} \|v_h\|_{\mathcal{Y}}$. With this, and (4.4.30), we obtain

$$\begin{aligned} \langle \mathcal{B}u_h, P(v_h) \rangle_{\mathcal{Y}' \times \mathcal{Y}} &= \langle \mathcal{B}u_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}} - \langle \mathcal{B}u_h, v_h - P(v_h) \rangle_{\mathcal{Y}' \times \mathcal{Y}} && (4.4.32) \\ &\geq \Big((\gamma_{\mathcal{B}}(\mathcal{X}_h, \widetilde{\mathcal{Y}_h}) - \varepsilon) - \delta \|\mathcal{B}\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} \Big) \|u_h\|_{\mathcal{X}} \|v_h\|_{\mathcal{Y}}. && (4.4.33) \end{aligned}$$

Estimating $\|v_h\|_{\mathcal{Y}}$ from below using $\|P(v_h)\|_{\mathcal{Y}} \leq \|v_h\|_{\mathcal{Y}} + \|P(v_h) - v_h\|_{\mathcal{Y}} \leq (1 + \delta)\|v_h\|_{\mathcal{Y}}$ shows the claim. □

Thus, the property $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$ is robust under certain perturbations of the test space. This observation suggests that we first identify a test space $\widetilde{\mathcal{Y}_h} \subseteq \mathcal{Y}$ for which $\gamma_{\mathcal{B}}(\mathcal{X}_h, \widetilde{\mathcal{Y}_h}) > 0$ holds, and then pass to a possibly more "practical" test space $\mathcal{Y}_h \subseteq \mathcal{Y}$. However, the conclusion seems suboptimal in the sense that it requires δ to be small enough to guarantee $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$. By choosing suitable norms on $\mathcal{X}$ and $\mathcal{Y}$ involving the operator $\mathcal{B}$ itself we may assume $\gamma_{\mathcal{B}}(\mathcal{X}, \mathcal{Y}) = 1 = \|\mathcal{B}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')}$. Then any $0 \le \delta < 1$ can be chosen, and Observation 4.4.13 coincides with [Dah+11, Lemma 4.1], if the dimensions of $\mathcal{X}_h$, $\mathcal{Y}_h$ and $\widetilde{\mathcal{Y}_h}$ are all finite and equal.

A different route is to apply (4.4.10) with $\langle \cdot, \cdot \rangle_{\mathcal{X}\times\mathcal{Y}} := \langle \mathcal{B}\cdot, \cdot \rangle_{\mathcal{Y}'\times\mathcal{Y}}$. Indeed, for any subspace $\widetilde{\mathcal{Y}_h} \subseteq \mathcal{Y}$ we have

$$\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \ge \mathcal{K}_{\mathcal{Y}\times\mathcal{Y}}(\widetilde{\mathcal{Y}_h}, \mathcal{Y}_h)\gamma_{\mathcal{B}}(\mathcal{X}_h, \widetilde{\mathcal{Y}_h}). \tag{4.4.34}$$

A discrete test space $\widetilde{\mathcal{Y}_h}$ which maximizes $\gamma_{\mathcal{B}}(\mathcal{X}_h, \widetilde{\mathcal{Y}_h})$ and has the same dimension as $\mathcal{X}_h$ can always be found (cf. the proof of Theorem 4.1.9) for which, in fact, $\gamma_{\mathcal{B}}(\mathcal{X}_h, \widetilde{\mathcal{Y}_h}) \ge \gamma_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})$. Such a discrete test space $\widetilde{\mathcal{Y}_h}$ was termed the space of "optimal test functions" in [DG11]. A discrete test space $\mathcal{Y}_h$ of the same (finite) dimension as $\widetilde{\mathcal{Y}_h}$ was said to be δ-proximal in [Dah+11], provided $\inf_{v\in\mathcal{Y}_h} \|\widetilde{v} - v\|_{\mathcal{Y}} \le \delta\|\widetilde{v}\|_{\mathcal{Y}}$ for all $\widetilde{v} \in \widetilde{\mathcal{Y}_h}$. But, due to (4.4.5) $\Leftrightarrow$ (4.4.2), the latter condition is equivalent to $\mathcal{K}_{\mathcal{Y}\times\mathcal{Y}}(\widetilde{\mathcal{Y}_h}, \mathcal{Y}_h) \ge \sqrt{1-\delta^2}$. Hence, the constant $\frac{1-\delta}{1+\delta}$ in [Dah+11, Lemma 4.1] may be improved to $\sqrt{1-\delta^2}$.

Corollary 4.4.14. *Let $\mathcal{X}_h \times \widetilde{\mathcal{Y}_h} \subseteq \mathcal{X} \times \mathcal{Y}$ be a nontrivial pair of closed subspaces. Let $\mathcal{Y}_h \subseteq \mathcal{Y}$ be a closed subspace such that (4.4.30) holds for some $P : \widetilde{\mathcal{Y}_h} \to \mathcal{Y}_h$ and $0 \le \delta \le 1$. Then*

$$\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \ge \sqrt{1-\delta^2}\,\gamma_{\mathcal{B}}(\mathcal{X}_h, \widetilde{\mathcal{Y}_h}). \tag{4.4.35}$$

4.5 Extension to semi-linear equations with small data

Let us consider a generalization of minimal residual Petrov-Galerkin discretizations for the linear problem $\mathcal{B}u = \mathcal{F}$ with $\mathcal{B} \in \mathrm{Iso}(\mathcal{X}, \mathcal{Y}')$, to the non-linear problem

$$\mathcal{B}u + \mathcal{G}(u) = \mathcal{F} \tag{4.5.1}$$

where $\mathcal{G} : \mathcal{X} \to \mathcal{Y}'$ is a non-linear mapping satisfying certain Lipschitz conditions specified below. We think of $\mathcal{G}$ as being of lower order w.r.t. $\mathcal{B}$. To discuss the key arguments, let us for the moment assume, as in Section 4.2, that we are given a fixed pair of closed subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ with $M = \dim \mathcal{X}_h$ and $N = \dim \mathcal{Y}_h$ and bases $\Phi \subset \mathcal{X}_h$, resp. $\Psi \subset \mathcal{Y}_h$, as well as s.p.d. operators $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$ and $\mathcal{M} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$. Recall further from (4.2.1) the definitions of the matrices $\mathbf{N}$, $\mathbf{B}$ and $\mathbf{M}$. For simplicity of notation we assume here that $M \le N < \infty$. Define the mapping $\mathbf{G} : \mathbb{R}^M \to \mathbb{R}^N$ by

$$\mathbf{G}(\mathbf{w}) := \langle \mathcal{G}(w), \Psi \rangle_{\mathcal{Y}'\times\mathcal{Y}} \tag{4.5.2}$$

where $w = \mathbf{w}^\top \Phi$. We propose the following fixed point iteration for the approximate solution of the non-linear equation (4.5.1):

1. Take an initial guess $\mathbf{u}_0 \in \mathbb{R}^M$.
2. For $i = 0, 1, 2, \ldots$, solve

$$\|\mathbf{B}\mathbf{u}_{i+1} - [\mathbf{f} - \mathbf{G}(\mathbf{u}_i)]\|_{\mathbf{N}^{-1}} \overset{!}{\to} \min \tag{4.5.3}$$

for $\mathbf{u}_{i+1}$ using one of the alternatives given in Proposition 4.2.5 with $[\mathbf{f} - \mathbf{G}(\mathbf{u}_i)]$ in place of $\mathbf{f}$.

In this section we show by adapting the proof of Proposition 3.4.1 that, for sufficiently small data $\mathcal{F}$, this iteration converges linearly, and the convergence rate may be estimated in terms of the discrete inf-sup constant $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)$ defined in (4.1.8). Moreover, the so obtained discrete solution is quasi-optimal. To

that end recall that for any $u \in \mathcal{X}$, the minimal residual Petrov-Galerkin solution to $\mathcal{B}u = \mathcal{F}$ with $\mathcal{F} := \mathcal{B}u$ is defined by

$$u_h := \arg\min_{w_h \in \mathcal{X}_h} \mathcal{R}(w_h), \quad \mathcal{R}(w_h) := \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B}w_h - \mathcal{B}u, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|v_h\|_{\mathcal{N}}}. \tag{4.5.4}$$

In the proof of Theorem 4.1.9 we showed that the mapping $P_h : u \mapsto u_h$ is a linear continuous projection with norm bounded by $C_h := \frac{C_{\mathcal{N}}}{c_{\mathcal{N}}} \frac{\|\mathcal{B}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')}}{\gamma_{\mathcal{B}}(\mathcal{X}_h,\mathcal{Y}_h)}$. This will play a crucial role in the proof of convergence of the iterates $\mathbf{u}^i$.

At the core of the proofs is the following fixed point argument. Assume that we are given $\alpha > 0$ and $r > 0$ such that

$$\|\mathcal{G}(w) - \mathcal{G}(\widetilde{w})\|_{\mathcal{Y}'} \leq \alpha \|w - \widetilde{w}\|_{\mathcal{X}} \quad \forall w, \widetilde{w} \in B_r := \{w \in \mathcal{X} : \|w\|_{\mathcal{X}} \leq r\}.$$

For arbitrary $w, \widetilde{w} \in B_r$ let $u, \widetilde{u} \in \mathcal{X}$ be the unique solutions of

$$\mathcal{B}u = \mathcal{F} - \mathcal{G}(w) \quad \text{and} \quad \mathcal{B}\widetilde{u} = \mathcal{F} - \mathcal{G}(\widetilde{w}). \tag{4.5.5}$$

By the Lipschitz assumption on $\mathcal{G}$ we have $\|u - \widetilde{u}\|_{\mathcal{X}} \leq \frac{\alpha}{\gamma_{\mathcal{B}}(\mathcal{X},\mathcal{Y})} \|w - \widetilde{w}\|_{\mathcal{X}}$. The corresponding minimal residual Petrov-Galerkin solutions $u_h, \widetilde{u_h} \in \mathcal{X}_h$ then satisfy

$$\|u_h - \widetilde{u_h}\|_{\mathcal{X}} = \|P_h(u - \widetilde{u})\|_{\mathcal{X}} \leq C_h \|u - \widetilde{u}\|_{\mathcal{X}} \leq \frac{\alpha C_h}{\gamma_{\mathcal{B}}(\mathcal{X},\mathcal{Y})} \|w - \widetilde{w}\|_{\mathcal{X}} \tag{4.5.6}$$

and

$$\|u_h\|_{\mathcal{X}} \leq C_h \|u\|_{\mathcal{X}} \leq \frac{C_h}{\gamma_{\mathcal{B}}(\mathcal{X},\mathcal{Y})} \left(\|\mathcal{F}\|_{\mathcal{Y}'} + \alpha \|w\|_{\mathcal{X}}\right) \leq \frac{C_h}{\gamma_{\mathcal{B}}(\mathcal{X},\mathcal{Y})} \left(\|\mathcal{F}\|_{\mathcal{Y}'} + \alpha r\right). \tag{4.5.7}$$

In order to be able to apply the Banach fixed point theorem to the mapping $w \mapsto u_h$ on B_r, we need to assume $\frac{\alpha C_h}{\gamma_{\mathcal{B}}(\mathcal{X},\mathcal{Y})} < 1$ and $\frac{C_h}{\gamma_{\mathcal{B}}(\mathcal{X},\mathcal{Y})} \left(\|\mathcal{F}\|_{\mathcal{Y}'} + \alpha r\right) \leq r$. These two conditions are satisfied if

$$\alpha < \frac{\gamma_{\mathcal{B}}(\mathcal{X},\mathcal{Y})}{C_h} \quad \text{and} \quad \|\mathcal{F}\|_{\mathcal{Y}'} \leq \delta := r \left(\frac{\gamma_{\mathcal{B}}(\mathcal{X},\mathcal{Y})}{C_h} - \alpha \right). \tag{4.5.8}$$

Now we assume

1. A family of pairs of closed subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ indexed by $h > 0$ that is stable for $\mathcal{B}$, i.e.,

$$\gamma_0 := \inf_{h>0} \gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0.$$

2. A fixed mapping $\mathcal{G} : \mathcal{X} \to \mathcal{Y}'$ satisfying $G(0) = 0$ and the local Lipschitz condition

$$\|G(w) - G(\widetilde{w})\|_{\mathcal{Y}'} \leq \eta(\max\{\|w\|_{\mathcal{X}}, \|\widetilde{w}\|_{\mathcal{X}}\}) \|w - \widetilde{w}\|_{\mathcal{X}} \quad \forall w, \widetilde{w} \in \mathcal{X}, \tag{4.5.9}$$

   where $\eta \in C^0([0,\infty))$ with $\eta(0) = 0$.

3. A s.p.d. operator $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$.

For any $\mathcal{F} \in \mathcal{Y}'$ and $h > 0$ we define the mapping

$$\Phi^h_{\mathcal{F}} : \mathcal{X} \to \mathcal{X}_h, \quad w \mapsto w_h := \arg\min_{\widetilde{w_h} \in \mathcal{X}_h} \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B}\widetilde{w_h} + \mathcal{G}(w) - \mathcal{F}, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|v_h\|_{\mathcal{N}}}. \tag{4.5.10}$$

The discrete algebraic iteration proposed at the beginning of this section is, of course, the algebraic equivalent of the iteration $u_{h,i+1} := \Phi^h_{\mathcal{F}}(u_{h,i})$, $i \in \mathbb{N}_0$, $u_{h,0} \in \mathcal{X}_h$.

With the above assumptions and notation we obtain the following proposition.

Proposition 4.5.1. *There exists $r > 0$ such that for any $\alpha > 0$ and $\mathcal{F} \in \mathcal{Y}'$ with*

1. $0 < \alpha < \gamma_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})/C_0$, *where* $C_0 := \frac{C_{\mathcal{N}}}{c_{\mathcal{N}}} \frac{\|\mathcal{B}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')}}{\gamma_0}$,
2. $\|\mathcal{F}\|_{\mathcal{Y}'} \leq \delta := r\,(\gamma_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})/C_0 - \alpha)$,

the mapping $\Phi^h_{\mathcal{F}}$ is a strict contraction on the complete set $B_r \cap \mathcal{X}_h$ where its Lipschitz constant L satisfies

$$L \leq \rho := \frac{\alpha C_0}{\gamma_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})} < 1. \tag{4.5.11}$$

The constant $\rho < 1$ is, in particular, independent of h and $\mathcal{F}$.

Proof. With the above arguments, the proof is a straightforward modification of the proof of Proposition 3.4.1. □

This proposition allows us to construct discrete solutions for small data:

Corollary 4.5.2. *With any $\mathcal{F}$ as in the previous proposition, there exists a unique sufficiently small solution $u_h \in \mathcal{X}_h$ to the discrete fixed point equation $\Phi^h_{\mathcal{F}}(u_h) = u_h$. It is obtained as the limit in $\mathcal{X}_h$ of the fixed point iteration $u_{h,i} := \Phi^h_{\mathcal{F}}(u_{h,i})$, $i \in \mathbb{N}_0$, with the initial guess $u_{h,0} := 0$. The iterates satisfy*

$$\|u_h - u_{h,i}\|_{\mathcal{X}} \leq \rho^i \|u_h\|_{\mathcal{X}} \quad \forall i \in \mathbb{N}_0. \tag{4.5.12}$$

More generally, for any two such $\mathcal{F}, \widetilde{\mathcal{F}} \in \mathcal{Y}'$ and any initial guesses $u_{h,0}, \widetilde{u_{h,0}} \in B_r \cap \mathcal{X}_h$, the corresponding iterates $u_{h,i} := [\Phi^h_{\mathcal{F}}]^i(u_{h,0})$ and $\widetilde{u_{h,i}} := [\Phi^h_{\widetilde{\mathcal{F}}}]^i(\widetilde{u_{h,0}})$ satisfy, for all $i \in \mathbb{N}_0$,

$$\|u_{h,i} - \widetilde{u_{h,i}}\|_{\mathcal{X}} \leq \frac{a}{1-\rho} + \rho^i \left(\|u_{h,0} - \widetilde{u_{h,0}}\|_{\mathcal{X}} - \frac{a}{1-\rho} \right) \quad \text{where} \quad a := \frac{1}{\gamma_0} \frac{C_{\mathcal{N}}}{c_{\mathcal{N}}} \|\mathcal{F} - \widetilde{\mathcal{F}}\|_{\mathcal{Y}'}.$$

Proof of corollary. Let us prove the second, more general, statement. It follows by induction over i from

$$\|\Phi^h_{\mathcal{F}}(w_h) - \Phi^h_{\widetilde{\mathcal{F}}}(\widetilde{w_h})\|_{\mathcal{X}} \leq a + \rho \|w_h - \widetilde{w_h}\|_{\mathcal{X}} \quad \forall w_h, \widetilde{w_h} \in B_r \cap \mathcal{X}_h \tag{4.5.13}$$

where $a = \frac{1}{\gamma_0} \frac{C_{\mathcal{N}}}{c_{\mathcal{N}}} \|\mathcal{F} - \widetilde{\mathcal{F}}\|_{\mathcal{Y}'}$, and hence we only show the latter estimate: for arbitrary $w_h, \widetilde{w_h} \in B_r \cap \mathcal{X}_h$ we have

$$\|\Phi^h_{\mathcal{F}}(w_h) - \Phi^h_{\widetilde{\mathcal{F}}}(\widetilde{w_h})\|_{\mathcal{X}} \leq \|\Phi^h_{\mathcal{F}}(w_h) - \Phi^h_{\widetilde{\mathcal{F}}}(w_h)\|_{\mathcal{X}} + \|\Phi^h_{\widetilde{\mathcal{F}}}(w_h) - \Phi^h_{\widetilde{\mathcal{F}}}(\widetilde{w_h})\|_{\mathcal{X}}. \tag{4.5.14}$$

The claimed estimate follows by applying the continuity of the (linear) minimal residual Petrov-Galerkin solution mapping (Theorem 4.1.11) to the first term and the contraction property of $\Phi^h_{\widetilde{\mathcal{F}}}$ to the second. □

Thus, the discrete solution u_h to the non-linear problem (4.5.1), which we define as the limit of the fixed point iteration $\Phi^h_{\mathcal{F}}$ with zero initial guess, depends Lipschitz continuously on $\mathcal{F}$ with Lipschitz constant $\leq \frac{1}{1-\rho} \frac{1}{\gamma_0} \frac{C_{\mathcal{N}}}{c_{\mathcal{N}}}$. This will be used in the next step to show that u_h indeed approximates the exact solution u.

Theorem 4.5.3. *Let $r > 0$, $\alpha > 0$ and $\delta > 0$ be as in Proposition 4.5.1. Take $\mathcal{F} \in \mathcal{Y}'$ with $\|\mathcal{F}\|_{\mathcal{Y}'} < \delta$. Let $u \in \mathcal{X}$ be the unique sufficiently small solution to (4.5.1). Assume that*

$$\limsup_{h \searrow 0} \inf_{w_h \in \mathcal{X}_h} \|u - w_h\| = 0. \tag{4.5.15}$$

For all $h > 0$ let u_h be limit of the fixed point iteration $\Phi^h_{\mathcal{F}}$ with zero initial guess. Then there exist $C > 0$ and $h_0 > 0$ such that the quasi-optimality estimate

$$\|u - u_h\|_{\mathcal{X}} \leq C \inf_{w_h \in \mathcal{X}_h} \|u - w_h\|_{\mathcal{X}} \quad \text{for all} \quad 0 < h \leq h_0 \tag{4.5.16}$$

holds.

Proof. Let $\mathcal{F} \in \mathcal{Y}'$ satisfy $\|\mathcal{F}\|_{\mathcal{Y}'} < \delta$. Fix $\epsilon_0 > 0$ such that for any $w \in \mathcal{X}$ the implication

$$\|u - w\|_{\mathcal{X}} \leq \epsilon_0 \quad \Longrightarrow \quad \|\widetilde{\mathcal{F}}\|_{\mathcal{Y}'} \leq \delta \quad \text{for} \quad \widetilde{\mathcal{F}} := \mathcal{F} + \mathcal{B}(w - u) + [\mathcal{G}(w) - \mathcal{G}(u)] \tag{4.5.17}$$

is valid. Let $h_0 > 0$ be such that $\inf_{w_h \in \mathcal{X}_h} \|u - w_h\| \leq \epsilon_0$ for all $0 < h \leq h_0$. For any $0 < h \leq h_0$ take $w_h \in \mathcal{X}_h$ with $\|u - w_h\|_{\mathcal{X}} \leq \epsilon_0$. Setting $\mathcal{F}_h := \mathcal{F} + \mathcal{B}(w_h - u) + [\mathcal{G}(w_h) - \mathcal{G}(u)]$, the vector w_h is the unique solution to $\mathcal{B}w_h + \mathcal{G}(w_h) = \mathcal{F}_h$ in the minimal residual sense (i.e., $w_h = \lim_{i\to\infty}[\Phi^h_{\mathcal{F}_h}]^i(0)$), the residual being, in fact, zero. Therefore, by the Lipschitz continuous dependence of u_h on $\mathcal{F}$ we have

$$\|u_h - w_h\|_{\mathcal{X}} \lesssim \|\mathcal{F} - \mathcal{F}_h\|_{\mathcal{Y}'} \lesssim \|u - w_h\|_{\mathcal{X}}, \tag{4.5.18}$$

where the implied constants are independent of h and $\mathcal{F}$. Estimating $\|u - u_h\|_{\mathcal{X}} \leq \|u - w_h\|_{\mathcal{X}} + \|u_h - w_h\|_{\mathcal{X}}$ and taking the infimum over w_h that satisfy $\|u - w_h\|_{\mathcal{X}} \leq \epsilon_0$ yields the desired quasi-optimality estimate. □

5 Stability of space-time Petrov-Galerkin discretizations

For the parabolic operator (3.2.15) we develop and discuss criteria for stability of families of trial and test spaces, in particular of space-time sparse tensor product type. This is one of the main contributions of the thesis. We proceed as follows. First, to motivate the subsequent development, we briefly review selected numerical methods for the solution of parabolic evolution equations in Section 5.1 (a thorough survey, if possible at all, is beyond the scope of this work). In a series of examples in Section 5.2.1 we highlight some aspects of stability for certain space-time trial and test spaces and use explicit techniques to show their stability. In Section 5.2.2 we give stability results for abstract trial and test spaces w.r.t. certain subspace-dependent norms. The implications are discussed for some concrete types of space-time trial and test spaces in Section 5.2.3, where we first show that continuous Galerkin time-stepping schemes are not stable, in general, unless a CFL condition is satisfied. This motivates the subsequent construction of stable space-time trial and test spaces of space-time sparse tensor product type. With this, we obtain an **a priori stable**, fully **parallelizable**, space-time **compressive** Petrov-Galerkin discretization scheme for parabolic evolution equations.

5.1 Discretization schemes for parabolic equations

In this section we give a very brief overview of selected discretization schemes for parabolic equation with focus on stability, parallelism, and space-time compressivity. We consider the linear abstract evolution equation in the setting of Section 3.2.1

$$\partial_t u(t) + Au(t) = g(t), \quad t \in \mathsf{J}, \quad u(0) = u^0, \tag{5.1.1}$$

in a Gelfand triple of real separable Hilbert spaces $V \hookrightarrow H \cong H' \hookrightarrow V'$ with initial datum $u^0 \in H$. To avoid unnecessary generality we assume for this overview, if not indicated otherwise, that $g \in C^0(\mathsf{J}; V)$ and $A \in \mathcal{L}(V, V')$ is a constant-in-time and V-elliptic (i.e., $\exists \alpha > 0$: $\langle A\nu, \nu\rangle_{V' \times V} \geq \alpha \|\nu\|_V^2 \ \forall \nu \in V$).

5.1.1 Time-stepping methods

The traditional numerical recipes for the solution of parabolic equations collect under the name *time-stepping* or *time marching*. These are schemes that approximately compute the solution iteratively on successive temporal subintervals, i.e., purely "upwind" in the positive temporal direction, and consequently are inherently difficult to parallelize to full scalability (some attempts are mentioned below). Time-stepping algorithms for parabolic (partial integro-differential) evolution equations may be roughly grouped into two categories. In *Rothe's method*, semi-discretization in time leads to a sequence of stationary (elliptic) problems. Vice versa, in the *method of lines*, semi-discretization in space reduces the problem to a system of coupled ordinary differential equations. Besides their importance in numerical computations, both were used to show existence of solutions to parabolic problems, in particular non-linear ones, see e.g. [Rou05, Section 8.2] and [DL92, Chapter XVIII, §2-3].

Elementary time-stepping procedures for the (semi-)discretization in time comprise single step, e.g. Runge-Kutta, and multi-step methods, e.g. backward differentiation formulae. These collocate the approximate solution at a number of discrete time points on successive subintervals in time, and may be *explicit* or *implicit*. Explicit methods are usually computationally fast (per time-step) but are associated

with a restriction, called the CFL condition, on the time-step size to ensure the convergence of the method. This restriction may be severe for *stiff* evolution equations, such as many parabolic evolution equations (after semi-discretization in space). Implicit methods are required to remove this restriction, usually at the cost of the solution of a large linear or non-linear system of equations at each time step. In this context, the model ordinary differential equation

$$\dot{y}(t) = \lambda y(t), \quad t > 0, \tag{5.1.2}$$

for any $y(0) \in \mathbb{R}$, and any $\lambda \in \mathbb{C}$ with large negative real part, is of interest. A method is called **A-stable** if it yields a bounded solution to this model problem for any λ in the left complex plane, an example is the Crank-Nicolson (**CN**) time-stepping method. We will see in Section 5.2.3, however, that the CN method is not stable as a space-time method in the sense of this thesis, in general, unless indeed a CFL condition is observed (otherwise, the CN method is known to be "too energy conservative" for *parabolic* evolution equations).

Contrary to collocation methods, the so-called *continuous Galerkin* (**cG**) and *discontinuous Galerkin* (**dG**) implicit time-stepping methods, to our knowledge originally developed for ordinary differential equations in [Hul72b; Hul72a], seek the approximate solution in a particular subspace of, say, $L^2(\mathsf{J}; V)$, by constructing a suitable projector on successive subintervals in time. We shall briefly discuss the lowest order cG, for its interpretation as a space-time method and for its relation with the CN scheme. Assume that we are given a family of nested finite-dimensional subspaces

$$V_\ell \subseteq V_{\ell+1} \subseteq V, \quad \ell \in \mathbb{N}_0, \tag{5.1.3}$$

such that $\bigcup_{\ell \in \mathbb{N}_0} V_\ell$ is dense in V. For instance, V_ℓ could comprise the piecewise polynomial continuous functions on an ℓ-times uniformly refined simplicial mesh. Suppose further we have at hand a sequence of linear continuous operators $P_\ell : V' \to V_\ell \subseteq V$, $\ell \in \mathbb{N}_0$, say defined by $\langle P_\ell \cdot, \cdot \rangle_{V' \times V} = \langle \cdot, \cdot \rangle_{V' \times V}$ on $V' \times V_\ell$. The adjoint $P'_\ell : V_\ell \to V$ thereof is then the (unique linear continuous) operator that satisfies $\langle \cdot, P'_\ell \cdot \rangle_{V' \times V} = \langle P_\ell \cdot, \cdot \rangle_{V' \times V}$ on $V' \times V_\ell$. Now we set $A_\ell := P_\ell A P'_\ell : V_\ell \to V_\ell$ and constrain the original evolution equation to the finite-dimensional subspace V_ℓ. More precisely, we look for the solution $u_\ell : \mathsf{J} \to V_\ell$ of the evolution equation $\partial_t u_\ell(t) + A_\ell u_\ell(t) = g(t)$, $t \in \mathsf{J}$, with the initial condition $u_\ell(0) = P_\ell u^0$. We set $g = 0$ for simplicity. Choosing a basis on V_ℓ, this leads to a linear system of ordinary differential equations which may be solved approximately by means of any time-stepping method of choice.

Let us constrain the set of candidate solutions. Let $E_k \subset H^1(\mathsf{J})$ be the space of continuous piecewise affine functions on a uniform partition $\{0 = t_0 \le t_r \le t_{R_k} = \mathsf{T}\}_{r=0}^{R_k}$ of J into, say, 2^{k+1} subintervals, and set $F_k := \partial_t E_k$, i.e., as the space of piecewise constant functions on the same partition. For given $k, \ell \in \mathbb{N}_0$, we now look for an approximate solution $u_{k,\ell} \in E_k \otimes V_\ell$. To that end, we require $u_{k,\ell}$ to satisfy (we suppress the dependence on t of the integrands)

$$\int_{\mathsf{J}} \langle \partial_t u_{k,\ell} + A_\ell u_{k,\ell}, v \rangle_{V' \times V} dt = 0 \qquad \forall v \in F_k \otimes V_\ell, \tag{5.1.4}$$

$$\langle u_{k,\ell}(0), \chi_\ell \rangle_{V' \times V} = \langle u^0, \chi_\ell \rangle_{V' \times V} \qquad \forall \chi_\ell \in V_\ell. \tag{5.1.5}$$

Formally, this is a space-time variational formulation. However, any test function $v \in F_k \otimes V_\ell$ is piecewise constant in time, and (5.1.4) is equivalent to

$$\int_{t_{r-1}}^{t_r} \langle \partial_t u_{k,\ell} + A_\ell u_{k,\ell}, \chi_\ell \rangle_{V' \times V} dt = 0 \quad \forall \chi_\ell \in V_\ell \quad \forall r = 1, \dots, R_k. \tag{5.1.6}$$

Discretizing the temporal integral (5.1.6) by means of the trapezoidal rule, *which is exact in this case*, we precisely obtain the CN scheme for the computation of $u_{k,\ell}(t_r) \in V_\ell$, given $u_{k,\ell}(t_{r-1}) \in V_\ell$, $r = 1, 2, \dots$. The correspondence still holds for piecewise affine (on the given mesh) functions $t \mapsto g(t)$.

Exponential integrators [HO10] are time-stepping methods that derive from various explicit representation formulas of the solutions to (5.1.1), e.g. such as the variation of constants formula (3.2.25) with $G(t, s) \approx e^{-A_{[s,t]}(t-s)}$ where $A_{[s,t]}$ is a suitable approximation of $A(\cdot)$ on the interval $[s, t]$, or such as the

Magnus expansion [GOT06; IN99; Mag54]. Consequently, in the constant generator case (with a suitable right hand side), the equation is, in principle, solved exactly. For a practical method, usually Krylov subspace iterative methods for the approximate computation of the matrix exponential are suggested [HO10]. In line with the theory of semigroups sketched in Section 3.2.2, the cited publications assume different types of Lipschitz or Hölder regularity on the family $A(\cdot)$, e.g. [GOT06, Hypothesis 2]. In this respect, numerical methods that we obtain based on the space-time variational theory of Section 3.2.1 (or Section 3.2.4) are more general, at least in the Hilbert space setting. Other methods based on the integral representation formula were presented in [SST00; GHK05].

Details on time-stepping methods may be found in [HNW93; HW96; Tho06; HO10].

5.1.2 Space-time discretization methods

Let us reconsider the cG formulation (5.1.4). Let $\Theta_k = \{\theta_r^k : r = 0, 1, \ldots, R_k\}$ denote the basis for E_k consisting of the standard piecewise affine nodal interpolants that satisfy $\theta_r^k(t_{r'}) = \delta_{r,r'}$. Expanding $u_{k,\ell}$ we obtain $u_{k,\ell} = \sum_{r=0}^{R_k} \theta_r^k \otimes u_{k,\ell}^{(r)}$ with coefficients $u_{k,\ell}^{(r)} \in V_\ell$. We set $\Delta t := t_r - t_{r-1}$ for the time step size, which we assume to be the same for each r. The sequence of equations (5.1.6) may now be written as a single *block diagonally implicit* linear system of equations

$$\left((\mathrm{Id} - S) \otimes \mathrm{Id} + \frac{1}{2}\Delta t(S + SS') \otimes A_\ell\right)(u_{k,\ell}^{(0)}, u_{k,\ell}^{(1)}, \ldots, u_{k,\ell}^{(R_k)})^\top = (u_{k,\ell}^{(0)}, 0, 0, \ldots)^\top \tag{5.1.7}$$

where S is the right shift operator $(u^{(0)}, u^{(1)}, \ldots)^\top \mapsto (0, u^{(0)}, u^{(1)}, \ldots)^\top$, with its adjoint left shift operator S', and $\otimes$ denotes the Kronecker product. The iterative solution of the above system simultaneously for the full vector of coefficients can be parallelized. This approach (in various related formulations) has therefore attracted some attention in the literature, we mention the so-called waveform relaxation method [VH95; JV96; Hal08], dating back in its essentials at least to [Pic93], as well as the low rank tensor approximation ansatz [DKO11]. The "parareal" method [LMT01; GV07] is an iterative method that corrects the solution all temporal subintervals simultaneously. Analysis and numerics for a parallelizable space-time multigrid method were presented in [HV95], cf. the references therein.

None of the numerical approaches mentioned above exploits the essential fact that the parabolic evolution equation is a well-posed operator equation in space-time Banach spaces, as discussed in Section 3. This is a crucial difference to [SS09] (as well as this work), where the applicability of adaptive wavelet methods in space-time to parabolic evolution equations was shown. This approach is parallelizable, space-time compressive, and is of optimal complexity, i.e., the work is proportional to the minimal number of degrees of freedom that is needed to represent the solution up to the given accuracy in the chosen Riesz basis. However, the necessary construction of space-time tensor product wavelet bases that can be rescaled to be Riesz bases in certain spaces is intricate, cf. [Sta11; CS11; CS12a]. By means of a reduction to a boundary integral equation suggested in [Cos90], space-time compressive algorithms for the heat equation (with constant coefficients) based on sparse tensor product subspaces were constructed in [CS12b].

A parallelizable and space-time compressive approach using simpler hierarchical tensor bases of wavelet type on a sparse grid in space-time and a heuristic space-time adaptive algorithm were previously presented in [GOV06; GO07]. There, however, the question of stability and well-posedness was not answered satisfactorily; indeed, we will show in Section 5.2.3 that the Crank-Nicolson method is not a stable space-time method, in general, and therefore, the best approximation rates derived in [GO07] may not be achieved even for the exact solution to the discrete system.

5.2 Stability of space-time discretizations: main results

Throughout this section we work in the setting of Section 3.2.1. We assume we are given a Gelfand triple of separable real Hilbert spaces $V \hookrightarrow H \cong H' \hookrightarrow V'$, where the embeddings are dense; the "pivot" space H is identified with its dual H' via the scalar product $\langle \cdot, \cdot \rangle_H$ on H; the duality pairing $\langle \cdot, \cdot \rangle$ on

$V \times V'$ coincides with the unique continuous extension of $\langle \cdot, \cdot \rangle_H : V \times V \to \mathbb{R}$. The spaces $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{X} = L^2(\mathsf{J}; V) \cap H^1(\mathsf{J}; V')$ and $\mathcal{Y} = L^2(\mathsf{J}; V) \times H$ with norms given by (3.2.11) and (3.2.12). Here, $\mathsf{J} = (0, \mathsf{T})$ is a non-trivial bounded temporal interval. For a family of operators $a(t; \cdot, \cdot)$, (a.e.) $t \in \mathsf{J}$, on $V \times V$ that satisfy Assumption 3.2.4, the parabolic operator $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ is defined as in (3.2.15) by

$$\langle \mathcal{B}u, v \rangle_{\mathcal{Y}' \times \mathcal{Y}} := \int_{\mathsf{J}} \langle (\partial_t + A(t))u(t), v_1(t) \rangle dt + \langle u(0), v_2 \rangle, \quad (u, v) \in \mathcal{X} \times \mathcal{Y}, \tag{5.2.1}$$

where $\langle A(t) \cdot, \cdot \rangle := a(t; \cdot, \cdot)$ on $V \times V$ for (a.e.) $t \in \mathsf{J}$.

In view of Theorem 4.1.9, in order to construct a sequence of approximations $\{u_h\}_{h>0} \subset \mathcal{X}$ to u, we look for a sequence of non-trivial pairs of (finite-dimensional) subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$, increasing as $h \searrow 0$, such that $\cup_{h>0} \mathcal{X}_h$ is dense in $\mathcal{X}$ and $\inf_{h>0} \gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$ in (4.1.8). Then, the quasi-optimality estimate (4.1.10) implies $u_h \to u$ in $\mathcal{X}$, as $h \searrow 0$. The present section is therefore concerned with the identification of conditions on the subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ such that the discrete inf-sup constant $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)$ in (4.1.8) can be bounded away from zero in terms of these conditions.

5.2.1 Examples

In a series of examples we study the discrete inf-sup condition for the parabolic operator $\mathcal{B} : \mathcal{X} \to \mathcal{Y}'$ defined by (3.2.15). Some explicit techniques for obtaining a lower bound on the discrete inf-sup constant are indicated. We set, unless specified otherwise, $\mathsf{T} := 1$, $D \subset \mathbb{R}^d$ a bounded domain with a Lipschitz boundary, $V := H_0^1(D)$, $H := L^2(D)$, and let $A \equiv -\Delta : V \to V'$ denote the Laplace operator. The space V is equipped with the energy norm, such that $A : V \to V'$ is an isometry. We argue that polynomials and trigonometric polynomials may be used in the temporal direction to define stable pairs of subspaces, provided we are given a (finite-dimensional) subspace $U \subset V$ that itself satisfies the stability condition $\mathcal{K}_{V' \times V}(U, U) > 0$. A pathological example is also given. See also Example 5.2.11 and Example 5.2.12.

Example 5.2.1. Let $D := (-\pi/2, \pi/2) \subset \mathbb{R}^1$. The one-dimensional subspaces $\mathcal{X}_h := \operatorname{span}\{1 \otimes \cos\}$ and $\mathcal{Y}_h := \{0\} \times \operatorname{span}\{\cos\}$ then satisfy $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$.

Example 5.2.2. For a fixed $k \in \mathbb{N}_0$ consider the space of trigonometric polynomials

$$E_k := \operatorname{span}\{\sin_j(t) := \sin(j\omega t), \cos_j(t) := \cos(j\omega t) : j = 0, \dots, k\},$$

where $\omega := 2\pi/\mathsf{T}$. Then $\{\sin_j, \cos_j : j = 0, \dots, k\}$ is an orthogonal basis for E_k in $L^2(\mathsf{J})$. Let $U \subset V$ be a non-trivial finite-dimensional subspace. Set $\mathcal{X}_h := E_k \otimes U \subset \mathcal{X}$ and $\mathcal{Y}_h = \mathcal{Y}_{h,1} \times \mathcal{Y}_{h,2} := [E_k \otimes U] \times U \subset \mathcal{Y}$. By construction, functions in $\mathcal{X}_h$ are time-periodic. We show that $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \geq \mathcal{K}_{V' \times V}(U, U)$.

Take any $u_h \in \mathcal{X}_h$. We can expand u_h into the Fourier series $u_h = \sum_{j=0}^k (\sin_j \otimes u_j^s + \cos_j \otimes u_j^c)$, where $u_j^s, u_j^c \in U$ for each $j = 0, \dots, k$. The Fourier series coefficients of $\widetilde{u_h} := \partial_t u_h + A u_h$ are then

$$\widetilde{u}_j^s := -j\omega u_j^c + A u_j^s \quad \text{and} \quad \widetilde{u}_j^c := j\omega u_j^s + A u_j^c.$$

If we defined $v_1 := \sum_{j=0}^k (\sin_j \otimes v_j^s + \cos_j \otimes v_j^c)$ via the Fourier series coefficients

$$v_j^s := -j\omega A^{-1} u_j^c + u_j^s \quad \text{and} \quad v_j^c := j\omega A^{-1} u_j^s + u_j^c,$$

and set $v_2 := u_h(0) \in \mathcal{Y}_{h,2}$, we would obtain for $v := (v_1, v_2)$ the (optimal) result

$$\langle \mathcal{B}u_h, v \rangle_{\mathcal{Y}' \times \mathcal{Y}} = \|u_h\|_{\mathcal{X}}^2 + \|u_h(\mathsf{T})\|_H^2 \quad \text{and} \quad \langle \mathcal{B}u_h, v \rangle_{\mathcal{Y}' \times \mathcal{Y}} = \|v\|_{\mathcal{Y}}^2. \tag{5.2.2}$$

However, $v = (v_1, v_2)$ is not necessarily in $\mathcal{Y}_h$ because of the first component. To project it to $\mathcal{Y}_h$ we first define $w_j^s \in U$ and $w_j^c \in U$ by

$$\langle A w_j^\alpha, \chi \rangle_{V' \times V} = \langle u_j^\alpha, \chi \rangle_{V' \times V} \quad \forall \chi \in U, \quad \alpha = s, c. \tag{5.2.3}$$

Note that $w_j^\alpha \in U$ approximates $A^{-1}u_j^\alpha$. This definition implies

$$\|Aw_j^\alpha\|_{V'} \geq \sup_{\chi\in U\setminus\{0\}} \frac{\langle Aw_j^\alpha, \chi\rangle_{V'\times V}}{\|\chi\|_V} = \sup_{\chi\in U\setminus\{0\}} \frac{\langle u_j^\alpha, \chi\rangle_{V'\times V}}{\|\chi\|_V} \geq \kappa\|u_j^\alpha\|_{V'} \tag{5.2.4}$$

for $\alpha = s, c$, where we abbreviate $\kappa := \mathcal{K}_{V'\times V}(U, U)$. Consequently,

$$\langle u_j^\alpha, w_j^\alpha\rangle_{V'\times V} = \langle Aw_j^\alpha, w_j^\alpha\rangle_{V'\times V} = \|w_j^\alpha\|_V^2 \geq \|Aw_j^\alpha\|_{V'}^2 \geq \kappa^2\|u_j^\alpha\|_{V'}^2, \tag{5.2.5}$$

for $\alpha = s, c$. The fact that $A : V \to V'$ is an isometry was invoked in some of the inequalities, which still hold up to constants if A is merely an isomorphism. Now, the function $v_{h,1} := \sum_{j=0}^k(\sin_j \otimes v_j^s + \cos_j \otimes v_j^s)$ with the Fourier series coefficients

$$v_j^s := -j\omega w_j^c + u_j^s \quad \text{and} \quad v_j^c := +j\omega w_j^s + u_j^c \tag{5.2.6}$$

is in $\mathcal{Y}_{h,1}$, and so we have $v_h := (v_{h,1}, v_{h,2}) \in \mathcal{Y}_h$ for $v_{h,2} := u_h(0) \in \mathcal{Y}_{h,2}$. Employing the Fourier series expansions of u_h and v_h, and using mutual orthogonality of $\sin_j$ and $\cos_j$, we find

$$\langle \mathcal{B}u_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \sum_{j=0}^k \left\{ \|\sin_j\|_{L^2(\mathsf{J})}^2 \langle \widetilde{u}_j^s, v_j^s\rangle_{V'\times V} + \|\cos_j\|_{L^2(\mathsf{J})}^2 \langle \widetilde{u}_j^c, v_j^c\rangle_{V'\times V} \right\} + \langle u_h(0), v_{h,2}\rangle_H. \tag{5.2.7}$$

Using the definition of w_j^α, two types of estimates follow for each term in the sum $\sum\{\ldots\}$.

1. First, letting $\langle\cdot,\cdot\rangle$ denote the duality pairing on $V' \times V$ for brevity,

$$\begin{aligned}
\langle \widetilde{u}_j^s, v_j^s\rangle &= \langle -j\omega u_j^c + Au_j^s, -j\omega w_j^c + u_j^s\rangle && (5.2.8)\\
&= |j\omega|^2\langle u_j^c, w_j^c\rangle - j\omega\left(\langle u_j^c, u_j^s\rangle + \langle Au_j^s, w_j^c\rangle\right) + \langle Au_j^s, u_j^s\rangle && (5.2.9)\\
&= |j\omega|^2\langle Aw_j^c, w_j^c\rangle - j\omega\left(\langle Aw_j^c, u_j^s\rangle + \langle Au_j^s, w_j^c\rangle\right) + \langle Au_j^s, u_j^s\rangle && (5.2.10)\\
&= \langle Av_j^s, v_j^s\rangle = \|v_j^s\|_V^2. && (5.2.11)
\end{aligned}$$

   Similarly, $\langle \widetilde{u}_j^c, v_j^c\rangle = \|v_j^c\|_V^2$. Hence, $\langle \mathcal{B}u_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \|v_h\|_{\mathcal{Y}}^2$.

2. Second,

$$\begin{aligned}
\langle \widetilde{u}_j^s, v_j^s\rangle &= |j\omega|^2\langle u_j^c, w_j^c\rangle - j\omega\left(\langle u_j^c, u_j^s\rangle + \langle Au_j^s, w_j^c\rangle\right) + \langle Au_j^s, u_j^s\rangle && (5.2.12)\\
&\geq |j\omega|^2\kappa^2\|u_j^c\|_{V'}^2 - 2j\omega\langle u_j^c, u_j^s\rangle + \|u_j^s\|_V^2. && (5.2.13)
\end{aligned}$$

   Together with a similar computation for $\langle \widetilde{u}_j^c, v_j^c\rangle$, this leads to

$$\begin{aligned}
\langle \mathcal{B}u_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}} &\geq \kappa^2\|\partial_t u_h\|_{L^2(\mathsf{J};V')}^2 + 2\int_{\mathsf{J}}\langle \partial_t u_h, u_h\rangle dt + \|u_h\|_{L^2(\mathsf{J};V)}^2 + \|u_h(0)\|_H^2\\
&= \kappa^2\|\partial_t u_h\|_{L^2(\mathsf{J};V')}^2 + \|u_h\|_{L^2(\mathsf{J};V)}^2 + \|u_h(\mathsf{T})\|_H^2.
\end{aligned}$$

   Owing to $\kappa \leq 1$, we have $\langle \mathcal{B}u_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}} \geq \kappa^2\left(\|u_h\|_{\mathcal{X}}^2 + \|u_h(\mathsf{T})\|_H^2\right) \geq \kappa^2\|u_h\|_{\mathcal{X}}^2$.

In summary, we obtain $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \geq \mathcal{K}_{V'\times V}(U, U)$. From the proof it is also clear that, due to orthogonality of the Fourier modes, we could choose a different subspace $U_j \subset V$ for each mode j, and κ then has to be replaced by the infimal $\mathcal{K}_{V'\times V}(U_j, U_j)$ over those subspaces U_j.

Example 5.2.3. For a fixed $k \in \mathbb{N}$ let $E_k := \operatorname{span}\{t^j : j = 0, \ldots, k\}$ be the space of polynomials on J of degree at most k. Let $\{0\} \neq U \subset V$ be a finite-dimensional subspace. Set $\mathcal{X}_k := E_k \otimes U$ and $\mathcal{Y}_k := [E_k \otimes U] \times U$. Then $\gamma_{\mathcal{B}}(\mathcal{X}_k, \mathcal{Y}_k) \geq \mathcal{K}_{V'\times V}(U, U)$.

To see this, we first consider a function $\varphi \in V$ which is an eigenfunction of A such that $A\varphi = \lambda^2\varphi$ for some $\lambda > 0$, with $\|\varphi\|_H = 1$. Note that $\|\varphi\|_V = \lambda$ and $\|\varphi\|_{V'} = \lambda^{-1}$. Let $p \in E$ be any polynomial and

set $w_k := p \otimes \varphi$. Set $q := \lambda^{-2}(p' + \lambda^2 p)$, and define $v_{k,1} := q \otimes \varphi \in \mathcal{Y}_1$ and $v_{k,2} := p(0)\varphi \in \mathcal{Y}_2$. Then

$$\langle \mathcal{B} w_k, v_k \rangle_{\mathcal{Y}' \times \mathcal{Y}} = \int_J \langle \partial_t w_k + A u_k, v_{k,1} \rangle_{V' \times V} dt + \langle w_k(0), v_{k,2} \rangle_H \tag{5.2.14}$$

$$= \int_{\mathsf{J}} (p' + \lambda^2 p)(\lambda^{-2} p' + p) dt + |p(0)|^2 \tag{5.2.15}$$

$$= \lambda^{-2} \|p'\|^2_{L^2(\mathsf{J})} + \lambda^2 \|p\|^2_{L^2(\mathsf{J})} + |p(\mathsf{T})|^2 \geq \|u_k\|^2_{\mathcal{X}}, \tag{5.2.16}$$

where the last inequality is due to $\|\varphi\|^2_V = \lambda^2$ and $\|\varphi\|^2_{V'} = \lambda^{-2}$. On the other hand,

$$\langle \mathcal{B} u_k, v_k \rangle_{\mathcal{Y}' \times \mathcal{Y}} = \lambda^2 \int_{\mathsf{J}} |q|^2 dt + |p(0)|^2 = \|q\|^2_{L^2(\mathsf{J})} \|\varphi\|^2_V + |p(0)|^2 = \|v_k\|^2_{\mathcal{Y}}. \tag{5.2.17}$$

These estimates combine to $\langle \mathcal{B} u_k, v_k \rangle_{\mathcal{Y}' \times \mathcal{Y}} \geq \|u_k\|_{\mathcal{X}} \|v_k\|_{\mathcal{Y}}$.

For a more general function $u_k \in \mathcal{X}_k$ one can show similarly to [BJ89, Section B] (by expanding $u_k(t)$ into the discrete eigenbasis of A, i.e., those pairs $(\varphi, \lambda^2) \in U \times (0, \infty)$ which satisfy $\langle A\varphi, \cdot \rangle_{V' \times V} = \lambda^2 \langle \varphi, \cdot \rangle_H$ on U) that there exists $v_k \in \mathcal{Y}_k$ with

$$\langle \mathcal{B} u_k, v_k \rangle_{\mathcal{Y}' \times \mathcal{Y}} \geq \mathcal{K}_{V' \times V}(U, U) \|u_k\|_{\mathcal{X}} \|v_k\|_{\mathcal{Y}}. \tag{5.2.18}$$

The details are omitted here, since this claim follows from the more general approach of Section 5.2.2, see Example 5.2.16.

Example 5.2.4. The space E_k in the previous example may be replaced by the span of $e^{-\lambda_j t}$, $j = 0, \ldots, k$, for any $\lambda \in \ell^\infty(\mathbb{N}_0)$.

The following is a pathological example: although both components under the integral $\int_{\mathsf{J}} \cdots dt$ contribute non-trivially, they cancel out. Thus, the discrete inf-sup condition fails to hold.

Example 5.2.5. Set $D := (-\pi/2, \pi/2) \subset \mathbb{R}^1$. Let $e(t) = 2t$ and $f(t) = 5 - 9t$. Then $\int_{\mathsf{J}} e' f dt = 1$ and $\int_{\mathsf{J}} e f dt = -1$. Set $u_h := e \otimes \cos$ and $v_{h,1} := f \otimes \cos$, take any $v_{h,2} \in H$. Then

$$\langle \mathcal{B} u_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}} = \int_{\mathsf{J}} \{ \underbrace{\langle e' \cos, f \cos \rangle}_{2e'f} + \underbrace{\langle -e\Delta \cos, f \cos \rangle}_{2ef} \} dt + \underbrace{\langle e(0) \cos, v_{h,2} \rangle}_{0} = 0.$$

It follows that $\langle \mathcal{B} u_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}} = 0$ for all $u_h \in \mathcal{X}_h := \operatorname{span}\{e \otimes \cos\}$ and $v_h \in \mathcal{Y}_h := \operatorname{span}\{f \otimes \cos\} \times H$. It is clear that the "optimal" function f is $\widetilde{f} = e' + e$, and no test function is required for the initial datum. However, the resulting "optimal test space" $\widetilde{\mathcal{Y}_h} := \operatorname{span}\{\widetilde{f} \otimes \cos\} \times \{0\}$ is precisely orthogonal in $\mathcal{Y}$ to our pathological test space $\mathcal{Y}_h$.

5.2.2 Stability for subspace-dependent norms

This section contains the core statements of the thesis: introducing certain subspace-dependent norms on the trial space $\mathcal{X}_h$ we obtain lower bounds on the stability constant of the parabolic operator $\mathcal{B}$ for abstract families of pairs of subspaces $\mathcal{X}_h \times \mathcal{Y}_h$ w.r.t. those norms. In the following two subsections we derive two slightly different results, using two slightly different techniques, which differ in the way the anti-symmetric part of the generator A is handled. The first generalizes the main result of [And12] to non-symmetric generators $A(\cdot)$, the second adopts the proof of [SS09, Theorem 5.1] to the discrete setting. The introduction of subspace-dependent norms was motivated by [UP11], where stability for the subspace-dependent norms was shown for the Crank-Nicolson method for the heat equation with reference to [SS09, Theorem 5.1]. However, we regard stability for the subspace-dependent norms as an intermediate abstract step: the stability bounds that we obtain are in terms of the quantities defined later in (5.2.26) and (5.2.48), which need to be bounded from below in a subsequent step for *particular* instances of $\mathcal{X}_h \times \mathcal{Y}_h$, see Section 5.2.3.

A. Estimates for the parabolic energy norms

Let Assumption 3.2.4 hold with $a_{\text{shift}} = 0$. Recall the notation $\widehat{A} = \frac{1}{2}(A + A')$ for the symmetric part, and $\widetilde{A} = \frac{1}{2}(A - A')$ for the anti-symmetric part of A. We shall work with the scalar products and the corresponding induced norms

$$\langle v_1, \widetilde{v_1}\rangle_+ := \int_{\mathsf{J}} \langle \widehat{A} v_1, \widetilde{v_1}\rangle dt, \qquad \|v_1\|_+^2 := \langle v_1, v_1\rangle_+, \qquad v_1, \widetilde{v_1} \in \mathcal{Y}_1, \tag{5.2.19}$$

$$\langle z, \widetilde{z}\rangle_- := \int_{\mathsf{J}} \langle \widehat{A}^{-1} z, \widetilde{z}\rangle dt, \qquad \|z\|_-^2 := \langle z, z\rangle_-, \qquad z, \widetilde{z} \in \mathcal{Y}_1', \tag{5.2.20}$$

and

$$\langle\!\langle v, \widetilde{v}\rangle\!\rangle := \langle v_1, \widetilde{v_1}\rangle_+ + \langle v_2, \widetilde{v_2}\rangle_H, \qquad |\!|\!| v |\!|\!|^2 := \langle\!\langle v, v\rangle\!\rangle, \qquad v, \widetilde{v} \in \mathcal{Y}. \tag{5.2.21}$$

Assume that we are given a family of pairs of non-trivial closed subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ indexed by $h > 0$. For each h define the operator

$$P^h : \mathcal{X} \to \mathcal{Y}_h, \quad w \mapsto P^h w = (P_1^h w, P_2^h w) \in \mathcal{Y}_h \subseteq \mathcal{Y}_1 \times \mathcal{Y}_2 \tag{5.2.22}$$

by

$$\langle\!\langle P^h w, v_h\rangle\!\rangle := \langle\!\langle (w, w(0)), v_h\rangle\!\rangle \quad \forall (w, v_h) \in \mathcal{X} \times \mathcal{Y}_h. \tag{5.2.23}$$

Introducing the abbreviation

$$G^h w := \partial_t w + \widetilde{A} P_1^h w, \quad w \in \mathcal{X}, \tag{5.2.24}$$

we define the mapping $|\!|\!|\cdot|\!|\!|_h : \mathcal{X} \to \mathbb{R}$ by

$$|\!|\!| w |\!|\!|_h^2 := \|G^h w\|_-^2 + |\!|\!| P^h w |\!|\!|^2 + 2 \int_{\mathsf{J}} \langle \partial_t w, w\rangle dt, \quad w \in \mathcal{X}. \tag{5.2.25}$$

Whether this defines a norm on $\mathcal{X}_h$ depends on the structure of the subspaces $\mathcal{X}_h \times \mathcal{Y}_h$, and we will either assume or prove this for each occasion below. We note that setting $G^h w := \partial_t w + \widetilde{A} w$ would not affect the statement of the following theorem (Theorem 5.2.6). Finally, we define $\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h) \geq 0$ by

$$\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h) := \inf_{w_h \in \mathcal{X}_h} \sup_{v_h \in \mathcal{Y}_h} \frac{\int_{\mathsf{J}} \langle \partial_t w_h + \widetilde{A} w_h, v_{h,1}\rangle dt}{\|G^h w_h\|_- \|v_{h,1}\|_+}, \tag{5.2.26}$$

where the infimum and the supremum are taken w.r.t. all elements such that the denominator is non-zero.

Theorem 5.2.6. *Let Assumption 3.2.4 hold with $a_{\text{shift}} = 0$. With the above definitions assume further that **a)** $|\!|\!|\cdot|\!|\!|_h$ is a norm on $\mathcal{X}_h$, and **b)** the operator P_1^h satisfies*

$$\int_{\mathsf{J}} \langle \partial_t w_h, w_h\rangle dt = \int_{\mathsf{J}} \langle \partial_t w_h + \widetilde{A} w_h, P_1^h w_h\rangle dt \quad \forall w_h \in \mathcal{X}_h. \tag{5.2.27}$$

Then

$$\inf_{w_h \in \mathcal{X}_h \setminus \{0\}} \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B} w_h, v_h\rangle_{\mathcal{Y}' \times \mathcal{Y}}}{|\!|\!| w_h |\!|\!|_h |\!|\!| v_h |\!|\!|} \geq \min\{\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h), 1\}. \tag{5.2.28}$$

Proof. Fix $h > 0$ and $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$, and let $w_h \in \mathcal{X}_h$ be arbitrary. Define $\Gamma^h \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_h)$ by

$$\langle\!\langle \Gamma^h \cdot, \cdot \rangle\!\rangle := \langle \mathcal{B} \cdot, \cdot \rangle \quad \text{on} \quad \mathcal{X} \times \mathcal{Y}_h, \tag{5.2.29}$$

i.e., for each $w \in \mathcal{X}$, the element $\Gamma^h w \in \mathcal{Y}_h$ is the Riesz representative of the linear continuous map $\langle \mathcal{B}w, \cdot \rangle$ on the Hilbert space $(\mathcal{Y}_h, \langle\!\langle \cdot, \cdot \rangle\!\rangle)$. Using (5.2.29) for the pair $(w_h, P^h w_h) \in \mathcal{X}_h \times \mathcal{Y}_h$, the hypothesis (5.2.27) and the definition (5.2.22)–(5.2.23) of P^h, we obtain

$$\begin{aligned}
\langle\!\langle \Gamma^h w_h, P^h w_h \rangle\!\rangle &= \langle \mathcal{B} w_h, P^h w_h \rangle_{\mathcal{Y}' \times \mathcal{Y}} && (5.2.30)\\
&= \int_{\mathsf{J}} \langle \partial_t w_h + A w_h, P_1^h w_h \rangle dt + \langle w_h(0), P_2^h w_h \rangle_H && (5.2.31)\\
&= \int_{\mathsf{J}} \langle \partial_t w_h, w_h \rangle dt + |\!|\!| P^h w_h |\!|\!|^2. && (5.2.32)
\end{aligned}$$

This and the properties of the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ imply

$$\begin{aligned}
|\!|\!| \Gamma^h w_h |\!|\!|^2 - |\!|\!| \Gamma^h w_h - P^h w_h |\!|\!|^2 &= 2 \langle\!\langle \Gamma^h w_h, P^h w_h \rangle\!\rangle - |\!|\!| P^h w_h |\!|\!|^2 && (5.2.33)\\
&= |\!|\!| P_1^h w_h |\!|\!|^2 + 2 \int_{\mathsf{J}} \langle \partial_t w_h, w_h \rangle dt. && (5.2.34)
\end{aligned}$$

By definition, $\Gamma^h w_h - P^h w_h \in \mathcal{Y}_h$, hence, using (5.2.29), the definition (5.2.22)–(5.2.23) of P^h, and the definition (5.2.26) of $\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h)$, we can estimate

$$\begin{aligned}
|\!|\!| \Gamma^h w_h - P^h w_h |\!|\!| &= \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle\!\langle \Gamma^h w_h - P^h w_h, v_h \rangle\!\rangle}{|\!|\!| v_h |\!|\!|} && (5.2.35)\\
&= \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B} w_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}} - \langle\!\langle P^h w_h, v_h \rangle\!\rangle}{|\!|\!| v_h |\!|\!|} && (5.2.36)\\
&= \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\int_{\mathsf{J}} \langle \partial_t w_h + \widetilde{A} w_h, v_{h,1} \rangle dt}{\| v_{h,1} \|} && (5.2.37)\\
&\geq \mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h) \| G^h w_h \|_{-}. && (5.2.38)
\end{aligned}$$

This, combined with the previous identity, shows

$$|\!|\!| \Gamma^h w_h |\!|\!|^2 \geq [\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h)]^2 \| G^h w_h \|_{-}^2 + |\!|\!| P_1^h w_h |\!|\!|^2 + 2 \int_{\mathsf{J}} \langle \partial_t w_h, w_h \rangle dt. \tag{5.2.39}$$

We conclude that $|\!|\!| \Gamma^h w_h |\!|\!| \geq \min\{\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h), 1\} |\!|\!| w_h |\!|\!|_h$ for any $w_h \in \mathcal{X}_h$, and from

$$\sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B} w_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{|\!|\!| w_h |\!|\!|_h |\!|\!| v_h |\!|\!|} = \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle\!\langle \Gamma^h w_h, v_h \rangle\!\rangle}{|\!|\!| w_h |\!|\!|_h |\!|\!| v_h |\!|\!|} = \frac{|\!|\!| \Gamma^h w_h |\!|\!|}{|\!|\!| w_h |\!|\!|_h} \quad \forall w_h \in \mathcal{X}_h \setminus \{0\},$$

the claim (5.2.28) follows. □

By the aforegoing theorem, we obtain a lower bound on the discrete inf-sup constant $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h)$ for a family of subspaces $\mathcal{X}_h \times \mathcal{Y}_h$ if we can verify the norm equivalence $|\!|\!| \cdot |\!|\!|_h \sim \| \cdot \|_{\mathcal{X}}$ on $\mathcal{X}_h$, and bound from below the quantity $\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h)$, defined in (5.2.26). We will see later for subspaces of space-time tensor product type that $|\!|\!| \cdot |\!|\!|_h \sim \| \cdot \|_{\mathcal{X}}$ is essentially a condition on the temporal discretization, while the requirement $\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h) > 0$ describes a certain stability of the spatial discretization.

Corollary 5.2.7. *Let Assumption 3.2.4 hold with $a_{\text{shift}} = 0$. With the above definitions assume further that the projector P_1^h satisfies (5.2.27), and that for each $h > 0$ there are constants $0 < d_h \leq D_h < \infty$ such that*

$$d_h \| w_h \|_{\mathcal{X}} \leq |\!|\!| w_h |\!|\!|_h \leq D_h \| w_h \|_{\mathcal{X}} \quad \forall w_h \in \mathcal{X}_h. \tag{5.2.40}$$

Then there exists $\gamma_0 > 0$, independent of $\mathcal{X}_h \times \mathcal{Y}_h$, such that

$$\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \geq \gamma_0 d_h \min\{\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h), 1\} \quad \forall h > 0. \tag{5.2.41}$$

B. Alternative estimates

In this subsection, we obtain in Theorem 5.2.9 below a stability bound which is slightly less sharp than that of Theorem 5.2.6. However, the assumptions will be easier to check (cf. Corollary 5.2.13).

Assume again that we are given a family of pairs of non-trivial closed subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ indexed by $h > 0$, such that $\mathcal{Y}_h$ has the form $\mathcal{Y}_h = \mathcal{Y}_{h,1} \times \mathcal{Y}_{h,2}$. We replace the definition of the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ by the continuous (not necessarily symmetric) bilinear form

$$\langle\!\langle v, \widetilde{v} \rangle\!\rangle := \int_{\mathsf{J}} \langle A v_1, \widetilde{v_1} \rangle dt + \langle v_2, \widetilde{v_2} \rangle_H, \quad v, \widetilde{v} \in \mathcal{Y}. \tag{5.2.42}$$

Let Assumption 3.2.4 hold with $a_{\text{shift}} = 0$. The operator $P^h : \mathcal{X} \to \mathcal{Y}_h$, $w \mapsto P^h w = (P_1^h w, P_2^h w)$ is defined analogously to (5.2.22)–(5.2.23) (well-defined by Proposition 3.2.7), namely

$$\langle\!\langle P^h w, v_h \rangle\!\rangle := \langle\!\langle (w, w(0)), v_h \rangle\!\rangle \quad \forall (w, v_h) \in \mathcal{X} \times \mathcal{Y}_h. \tag{5.2.43}$$

We define the norm $\|\cdot\|_h : \mathcal{X} \to \mathbb{R}$ by

$$\|w\|_h^2 := \|\partial_t w\|_{L^2(\mathsf{J};V')}^2 + \|P_1^h w\|_{L^2(\mathsf{J};V)}^2 + \|w(\mathsf{T})\|_H^2, \quad w \in \mathcal{X}. \tag{5.2.44}$$

Note, this norm may not be equivalent to $\|\cdot\|_{\mathcal{X}}$. The following lemma motivates the term $\|w(\mathsf{T})\|_H$ in the definition of $\|\cdot\|_h$.

Lemma 5.2.8. *For any $w \in \mathcal{X}$ there holds the implication*

$$\int_{\mathsf{J}} \langle \partial_t w, w \rangle dt = \int_{\mathsf{J}} \langle \partial_t w, P_1^h w \rangle dt \quad \Rightarrow \quad \|w(0)\|_H \le \|w\|_h. \tag{5.2.45}$$

Proof. Assume $w \in \mathcal{X}$ satisfies the premise of the implication. Then the integration-by-parts formula (2.7.8), the continuity of the duality pairing, and the inequality $|2ab| \le a^2 + b^2$ yield

$$\|w(0)\|_H^2 \le \|w(\mathsf{T})\|_H^2 + \left| \int_{\mathsf{J}} 2 \langle \partial_t w, w \rangle dt \right| \tag{5.2.46}$$

$$\le \|w(\mathsf{T})\|_H^2 + \int_{\mathsf{J}} \left\{ \|\partial_t w\|_{V'}^2 + \|P_1^h w\|_V^2 \right\} dt = \|w\|_h^2, \tag{5.2.47}$$

hence the assertion. □

Finally, let us recall the definition (4.4.1), adapted to the present situation,

$$\mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}(\partial_t \mathcal{X}_h, \mathcal{Y}_{h,1}) = \inf_{z \in \partial_t \mathcal{X}_h \setminus \{0\}} \sup_{v_{h,1} \in \mathcal{Y}_{h,1} \setminus \{0\}} \frac{\int_{\mathsf{J}} \langle z, v_{h,1} \rangle dt}{\|z\|_{\mathcal{Y}_1'} \|v_{h,1}\|_{\mathcal{Y}_1}}. \tag{5.2.48}$$

The following stability theorem is analogous to Theorem 5.2.6.

Theorem 5.2.9. *Let Assumption 3.2.4 hold with $a_{\text{shift}} = 0$. With the above definitions assume further that **a)** the inclusion $\{w_h(0) : w_h \in \mathcal{X}_h\} \subseteq \mathcal{Y}_{h,2}$ is valid, **b)** the operator $P_1^h \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_{h,1})$, defined by (5.2.43), satisfies*

$$\int_{\mathsf{J}} \langle \partial_t w_h, w_h \rangle dt = \int_{\mathsf{J}} \langle \partial_t w_h, P_1^h w_h \rangle dt \quad \forall w_h \in \mathcal{X}_h. \tag{5.2.49}$$

Then

$$\inf_{w_h \in \mathcal{X}_h} \sup_{v_h \in \mathcal{Y}_h} \frac{\langle \mathcal{B} w_h, v_h \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|w_h\|_h \|v_h\|_{\mathcal{Y}}} \ge \gamma_1 \gamma_2, \tag{5.2.50}$$

where the infimum and the supremum are taken over all elements such that the denominator is non-zero, and

$$\gamma_1 := \min\{a_{\min}, 1\} / \sqrt{2 \max\{1, a_{\max}^2\} + 1}, \tag{5.2.51}$$

$$\gamma_2 := \min\{a_{\min} a_{\max}^{-2} [\mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}(\partial_t \mathcal{X}_h, \mathcal{Y}_{h,1})]^2, a_{\min}, 1\}. \tag{5.2.52}$$

Proof. Consider one particular pair of subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$. Define $\Gamma^h \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_h)$ by

$$\langle\!\langle v_h, \Gamma^h w \rangle\!\rangle := \int_{\mathsf{J}} \langle \partial_t w, v_{h,1} \rangle dt + \langle\!\langle v_h, P^h w \rangle\!\rangle, \quad (w, v_h) \in \mathcal{X} \times \mathcal{Y}_h. \tag{5.2.53}$$

Let Γ_1^h and Γ_2^h be such that $(\Gamma_1^h w, \Gamma_2^h w) = \Gamma^h w$ for all $w \in \mathcal{X}$. The linear operator Γ^h is indeed continuous, since (5.2.43) implies

$$\min\{a_{\min}, 1\} \|\Gamma^h w_h\|_{\mathcal{Y}}^2 \leq \langle\!\langle \Gamma^h w_h, \Gamma^h w_h \rangle\!\rangle$$

and (5.2.53) implies

$$\begin{aligned} \langle\!\langle \Gamma^h w_h, \Gamma^h w_h \rangle\!\rangle &\leq \sqrt{2} \max\{1, a_{\max}\} \|w_h\|_h \|\Gamma_1^h w_h\|_{L^2(\mathsf{J};V)} + \|P_2^h w_h\|_H \|\Gamma_2^h w_h\|_H \\ &\leq \sqrt{2 \max\{1, a_{\max}^2\} + 1} \|w_h\|_h \|\Gamma^h w_h\|_{\mathcal{Y}}, \end{aligned}$$

where, in addition, the fact $\|P_2^h w\|_H \leq \|w(0)\|_H$, $w \in \mathcal{X}$, and Lemma 5.2.8 were used. Hence,

$$\gamma_1 \|\Gamma^h w_h\|_{\mathcal{Y}} \leq \|w_h\|_h \quad \forall w_h \in \mathcal{X}_h \tag{5.2.54}$$

for γ_1 defined in (5.2.51).

Further, by definition of the operator Γ^h, of the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, and of P^h, for any $w \in \mathcal{X}$ we have

$$\begin{aligned} \langle \mathcal{B} w, \Gamma^h w \rangle_{\mathcal{Y}' \times \mathcal{Y}} &= \int_{\mathsf{J}} \langle \partial_t w, \Gamma_1^h w \rangle dt + \langle\!\langle (w, w(0)), \Gamma^h w \rangle\!\rangle \\ &= \langle\!\langle \Gamma^h w, \Gamma^h w \rangle\!\rangle - \langle\!\langle \Gamma^h w, P^h w \rangle\!\rangle + \langle\!\langle P^h w, \Gamma^h w \rangle\!\rangle \\ &= \langle\!\langle \Gamma^h w - P^h w, \Gamma^h w - P^h w \rangle\!\rangle + 2 \langle\!\langle P^h w, \Gamma^h w \rangle\!\rangle - \langle\!\langle P^h w, P^h w \rangle\!\rangle \\ &= \underbrace{\langle\!\langle (\Gamma^h - P^h) w, (\Gamma^h - P^h) w \rangle\!\rangle}_{T_1} + \underbrace{\langle\!\langle P^h w, P^h w \rangle\!\rangle + 2 \int_{\mathsf{J}} \langle \partial_t w, P_1^h w \rangle dt}_{T_2}, \end{aligned}$$

and we can estimate the terms from below if $w \in \mathcal{X}_h$ as follows. For first term T_1, the identity (5.2.53) yields (note that $\Gamma_2^h w = P_2^h w$)

$$\|A'(\Gamma_1^h - P_1^h) w_h)\|_{\mathcal{Y}_1'} \geq \sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle\!\langle v_h, (\Gamma^h - P^h) w_h \rangle\!\rangle}{\|v_h\|_{\mathcal{Y}}} \tag{5.2.55}$$

$$= \sup_{v_{h,1} \in \mathcal{Y}_{h,1} \setminus \{0\}} \frac{\int_{\mathsf{J}} \langle \partial_t w_h, v_{h,1} \rangle dt}{\|v_{h,1}\|_{\mathcal{Y}_1}} \geq \mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}(\partial_t \mathcal{X}_h, \mathcal{Y}_{h,1}) \|\partial_t w_h\|_{\mathcal{Y}_1'}, \tag{5.2.56}$$

which we use to estimate

$$T_1 \geq a_{\min} \|(\Gamma_1^h - P_1^h) w_h\|_{\mathcal{Y}_1}^2 \geq \frac{a_{\min}}{a_{\max}^2} \|A'(\Gamma_1^h - P_1^h) w_h\|_{\mathcal{Y}_1'}^2 \geq \frac{a_{\min}}{a_{\max}^2} [\mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}(\partial_t \mathcal{X}_h, \mathcal{Y}_{h,1})]^2 \|\partial_t w_h\|_{\mathcal{Y}_1'}^2.$$

For the second term T_2, we use the hypothesis (5.2.49), the integration-by-parts formula (2.7.8), and the hypothesis $w_h(0) \in \mathcal{Y}_{h,2}$, to obtain

$$\begin{aligned} T_2 &= \langle\!\langle P^h w_h, P^h w_h \rangle\!\rangle + 2 \int_{\mathsf{J}} \langle \partial_t w_h, w_h \rangle dt \\ &= \int_{\mathsf{J}} \langle A P_1^h w_h, P_1^h w_h \rangle dt + \|w_h(\mathsf{T})\|_H^2 + \underbrace{\|P_2^h w_h\|_H^2 - \|w_h(0)\|_H^2}_{=0} \\ &\geq \min\{a_{\min}, 1\} \left(\|P_1^h w_h\|_{L^2(\mathsf{J};V)}^2 + \|w_h(\mathsf{T})\|_H^2 \right). \end{aligned}$$

Thus, $T_1 + T_2 \geq \gamma_2 \|w_h\|_h^2$ for γ_2 defined in (5.2.52).

This last observation and continuity (5.2.54) of Γ^h yield

$$\langle \mathcal{B} w_h, \Gamma^h w_h \rangle_{\mathcal{Y}' \times \mathcal{Y}} = T_1 + T_2 \geq \gamma_2 \|w_h\|_h^2 \geq \gamma_1 \gamma_2 \|w_h\|_h \|\Gamma^h w_h\|_{\mathcal{Y}} \quad \forall w_h \in \mathcal{X}_h.$$

In particular, $\|w_h\|_h \neq 0$ implies $\Gamma^h w_h \neq 0$. Thus, dividing by $\|w_h\|_h \|\Gamma^h w_h\|_{\mathcal{Y}}$ and taking the infimum over $w_h \in \mathcal{X}_h$ with $\|w_h\|_h \neq 0$ shows the claim. □

Corollary 5.2.10. *Let Assumption 3.2.4 hold with $a_{\text{shift}} = 0$. With the above definitions assume further that* ***a)*** $\{w_h(0) : w_h \in \mathcal{X}_h\} \subseteq \mathcal{Y}_{h,2}$, ***b)*** *the projector P_1^h satisfies (5.2.49),* ***c)*** *for each $h > 0$ there are constants $0 < d_h \leq D_h < \infty$ such that*

$$d_h \|w_h\|_{\mathcal{X}} \leq \|w_h\|_h \leq D_h \|w_h\|_{\mathcal{X}} \quad \forall w_h \in \mathcal{X}_h. \tag{5.2.57}$$

Then there exists $\gamma_0 > 0$, independent of $\mathcal{X}_h \times \mathcal{Y}_h$, such that

$$\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \geq \gamma_0 d_h \min\{[\mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}(\partial_t \mathcal{X}_h, \mathcal{Y}_{h,1})]^2, 1\} \quad \forall h > 0. \tag{5.2.58}$$

5.2.3 Applications: conditionally and unconditionally stable pairs

Using the stability results of the previous subsection w.r.t. subspace-dependent norms we discuss particular constructions of pairs of subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ for which **stability** w.r.t. the natural space-time norms holds

A) **conditionally**: the family of continuous Galerkin time-stepping schemes (e.g. Crank-Nicolson) is shown not to be stable, in general, unless a CFL condition is observed;

B) **unconditionally**: families of space-time sparse tensor product trial and test spaces "of inclusion type" are constructed and shown to be stable (irrespective of the mesh-width in the temporal direction);

C) **unconditionally**: the stabilized Crank-Nicolson scheme.

A. Continuous Galerkin time-stepping: CFL condition

Continuous Galerkin (henceforth, **cG**) time-stepping schemes, see [SW10] and references therein, may be interpreted as Petrov-Galerkin schemes for the space-time variational formulation (3.2.16) with space-time trial and test spaces that are piecewise polynomial in time, time-continuous in the trial space and time-discontinuous in the test space.

Using the discrete inf-sup estimate (5.2.28) for subspace-dependent norms, we show that stability in the natural space-time norms $\inf_{h>0} \gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) > 0$ is coupled to a *CFL condition* (5.2.65). We will show this in the simpler situation that Assumption 3.2.4 holds with $a_{\text{shift}} = 0$ and that $t \mapsto A(t)$ is constant, equal to a self-adjoint operator (i.e., $\widetilde{A} \equiv 0$). Let us assume this for the remainder of the subsection. The CFL condition is of the well-known form, e.g. for the heat equation in one dimension with a fixed spatial discretization it reads $\Delta t / \Delta x^2 \leq C < \infty$, where Δt is the maximal time step and $1/\Delta x^2$ is (of the order of) the maximal eigenvalue of the discretized spatial operator. This can be expected from the fact that the space-time norm $\mathcal{X}$ includes the first temporal derivative.

To set up the notation, for a temporal mesh $\mathcal{T} = \{0 =: t_0 < t_1 < \ldots < t_N := \mathsf{T}\} \subset [0, \mathsf{T}]$ and a vector of polynomial degrees $\mathbf{p} = (p_1, p_2, \ldots, p_N) \in \mathbb{N}_0^N$ we introduce the spline spaces

$$S^{r,\mathbf{p}}(\mathcal{T}) := \{f \in H^r(\mathsf{J}) : f|_{(t_{n-1}, t_n)} \in \mathbb{P}^{p_n}, n = 1, \ldots, N\} \tag{5.2.59}$$

of global Sobolev smoothness $r \in \mathbb{N}_0$, where $\mathbb{P}^d$ denotes polynomials (real-valued, of one real variable) of degree $d \in \mathbb{N}_0$. If $\mathbf{p} = (p, p, \ldots, p)$, we may write $S^{r,p} := S^{r,\mathbf{p}}$, and $\mathbf{p} + 1$ will denote the vector $(\mathbf{p}_1 + 1, \ldots, \mathbf{p}_N + 1)$. The maximal time step size of the temporal mesh $\mathcal{T}$ is denoted by

$$\max \Delta \mathcal{T} := \max_{n=1,\ldots,N} |t_n - t_{n-1}|. \tag{5.2.60}$$

Suppose now that we are given families of **a)** closed non-trivial subspaces $V_h \subseteq V$, **b)** temporal meshes $\mathcal{T}_h \subset [0, \mathsf{T}]$, and **c)** vectors of polynomial degrees $\mathbf{p}_h \in \mathbb{N}_0^{\#\mathcal{T}_h - 1}$, indexed by $h > 0$. We then define the continuous Galerkin subspaces

$$\mathcal{X}_h := S^{1,\mathbf{p}_h+1}(\mathcal{T}_h) \otimes V_h \subset \mathcal{X} \quad \text{and} \quad \mathcal{Y}_h := [S^{0,\mathbf{p}_h}(\mathcal{T}_h) \otimes V_h] \times V_h \subset \mathcal{Y}. \tag{5.2.61}$$

By counting the degrees of freedom one finds that $\dim \mathcal{X}_h = \dim \mathcal{Y}_h$, whenever either is finite. We define the CFL number (finite if $\dim V_h < \infty$)

$$\mathrm{CFL}_h := \max \Delta\mathcal{T}_h \sup_{\chi_h \in V_h \setminus \{0\}} \frac{\|\chi_h\|_V}{\|\chi_h\|_{V'}}, \quad h > 0. \tag{5.2.62}$$

Since $A(\cdot) = A(\cdot)'$ is constant, the operator P^h, given by (5.2.22)–(5.2.23) for the cG subspaces (5.2.61) is easily seen to satisfy (5.2.27). Here, we focus on the norm equivalence $|\!|\!|\cdot|\!|\!|_h \sim \|\cdot\|_{\mathcal{X}}$, see (5.2.25) for the definition of $|\!|\!|\cdot|\!|\!|_h$. We will show that there exists a constant $C > 0$ independent of the parameters appearing in the definition (5.2.61) of the cG subspaces $\mathcal{X}_h \times \mathcal{Y}_h$ such that

$$\|w_h\|_{\mathcal{X}} \le C \max\{1, \mathrm{CFL}_h\} |\!|\!|w_h|\!|\!|_h \quad \forall w_h \in \mathcal{X}_h \quad \forall h > 0. \tag{5.2.63}$$

Then, Corollary 5.2.7 implies that there exists $\widetilde{\gamma}_0 > 0$ such that

$$\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \ge \widetilde{\gamma}_0 \min\{1, \mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h)\} \min\{1, \mathrm{CFL}_h^{-1}\} \quad \forall h > 0, \tag{5.2.64}$$

and consequently, under the assumptions of this subsection, and assuming that $\inf_{h>0} \mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h) > 0$ holds in (5.2.26), a sufficient condition for the stability of cG schemes is the CFL condition

$$\sup_{h>0} \mathrm{CFL}_h < \infty. \tag{5.2.65}$$

Proof of (5.2.63). Let $w_h \in \mathcal{X}_h$ be arbitrary. Fix one subinterval, say $I := (t_{n-1}, t_n)$ of length $k := |t_n - t_{n-1}|$. On this subinterval, we have $w_h|_I = \sum_{j=0}^{p+1} L_j^I \otimes \chi_j^h$ with some $\chi_j^h \in V_h$ and $p \in \mathbb{N}_0$, where

$$L_j^I(t) := L_j(2k^{-1}(t - t_{n-1}) - 1), \quad t \in I, \tag{5.2.66}$$

is the Legendre polynomial L_i on the reference interval $(-1, 1)$ of degree $j \in \mathbb{N}_0$ transported to I. The normalization is such that $\int_{-1}^1 L_i(s)L_j(s)ds = \delta_{ij}$, and $L_j(1) = +\sqrt{j + 1/2}$. Thus, by definition of P^h we have $(P_1^h w_h)|_I = \sum_{j=0}^{p} L_j^I \otimes \chi_j^h$. The identity $\int_{-1}^1 L'_{p+1}(s)L_p(s)ds = \sqrt{4(p+1)^2 - 1}$ implies that $L'_{p+1} = \sqrt{4(p+1)^2 - 1} L_p + \alpha L_{p-1} + \cdots$, which we use to estimate

$$\begin{aligned}
\|\partial_t w_h\|^2_{L^2(I;V')} &= 2k^{-1} \left\| \sum_{j=1}^{p+1} L'_j \otimes \chi_j^h \right\|^2_{L^2((-1,1);V')} \\
&= 2k^{-1} \left\| \sqrt{4(p+1)^2 - 1} L_p \otimes \chi_{p+1}^h + \left\{ \begin{array}{c} \text{terms in} \\ \mathbb{P}^{p-1} \otimes V' \end{array} \right\} \right\|^2_{L^2((-1,1);V')} \\
&\ge 2k^{-1}(4(p+1)^2 - 1)\|\chi_{p+1}^h\|^2_{V'} \ge 6k^{-1}\|\chi_{p+1}^h\|^2_{V'}.
\end{aligned}$$

Using the mutual L^2 orthogonality of the Legendre polynomials, and the last estimate, we compute

$$\|w_h\|^2_{L^2(I;V)} = \frac{k}{2}\|\chi_{p+1}^h\|^2_V + \|P_1^h w_h\|^2_{L^2(I;V)} \tag{5.2.67}$$

$$\le \frac{k^2}{12} \frac{\|\chi_{p+1}^h\|^2_V}{\|\chi_{p+1}^h\|^2_{V'}} \|\partial_t w_h\|^2_{L^2(I;V')} + \|P_1^h w_h\|^2_{L^2(I;V)} \tag{5.2.68}$$

$$\le \frac{1}{12}\mathrm{CFL}_h^2 \|\partial_t w_h\|^2_{L^2(I;V')} + \|P_1^h w_h\|^2_{L^2(I;V)}. \tag{5.2.69}$$

Adding $\|\partial_t w_h\|^2_{L^2(I;V')}$ on both sides, the claim (5.2.63) follows by summation over all subintervals. □

The following two examples study numerically and analytically the inf-sup condition for the cG method of lowest order (Crank-Nicolson) on an equidistant temporal mesh. We observe that the CFL condition (5.2.65) can not be removed, in general (it could, however, be refined).

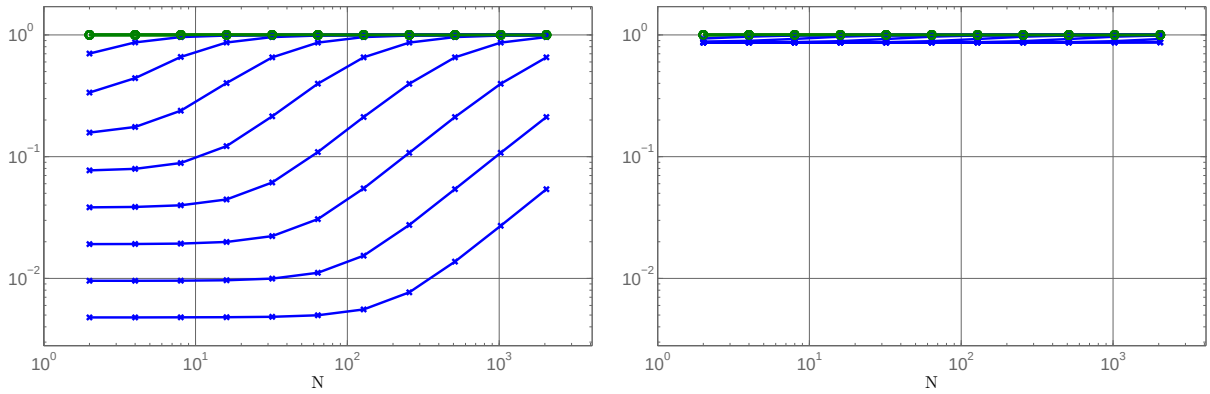

Figure 5.2.1: Stability of the Crank-Nicolson scheme. **Left**, from top to bottom: discrete continuity constant $\Gamma_{\lambda,h} \approx 1$ and discrete inf-sup constant $\gamma_{\lambda,h}$ for $\lambda = 2^1, \dots, 2^8$, as a function of the number of time steps $N = h^{-1}\mathsf{T}$. **Right**: the same, for an over-refined test space with $2N$ time steps.

Example 5.2.11. On $\mathsf{J} = (0, \mathsf{T}) = (0, 2)$ consider the scalar ODE $u' + \lambda^2 u = g$, $u(0) = u^0 \in \mathbb{R}$, with the corresponding bilinear form

$$\langle \mathcal{B}_\lambda w, v\rangle_{\mathcal{Y}'\times\mathcal{Y}} := \int_{\mathsf{J}} (w' + \lambda^2 w) v_1 dt + w(0) v_2, \quad \begin{cases} w \in \mathcal{X} := H^1(\mathsf{J}), \\ v \in \mathcal{Y} := L^2(\mathsf{J}) \times \mathbb{R}. \end{cases} \tag{5.2.70}$$

The spaces $\mathcal{X}$ and $\mathcal{Y}$ are endowed with the norms

$$\|w\|^2_{\mathcal{X},\lambda} = \lambda^{-2}\|w'\|^2_{L^2(\mathsf{J})} + \lambda^2\|w\|^2_{L^2(\mathsf{J})} + |w(\mathsf{T})|^2, \quad \|v\|^2_{\mathcal{Y},\lambda} = \lambda^2\|v_1\|^2_{L^2(\mathsf{J})} + |v_2|^2.$$

The motivation for these definitions is sketched in Example 5.2.3, cf. [BJ89; BJ90]. We compute numerically the discrete continuity constant $\Gamma_{\lambda,h}$ and the discrete inf-sup constant $\gamma_{\lambda,h}$,

$$\Gamma_{\lambda,h} := \sup_{w_h \in \mathcal{X}_h\setminus\{0\}} \sup_{v_h \in \mathcal{Y}_h\setminus\{0\}} \frac{\langle \mathcal{B}_\lambda w_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}}}{\|w_h\|_{\mathcal{X},\lambda}\|v_h\|_{\mathcal{Y},\lambda}}, \quad \gamma_{\lambda,h} := \inf_{w_h \in \mathcal{X}_h\setminus\{0\}} \sup_{v_h \in \mathcal{Y}_h\setminus\{0\}} \frac{\langle \mathcal{B}_\lambda w_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}}}{\|w_h\|_{\mathcal{X},\lambda}\|v_h\|_{\mathcal{Y},\lambda}},$$

for $\mathcal{X}_h \times \mathcal{Y}_h \subset \mathcal{X} \times \mathcal{Y}$ given by the cG scheme of lowest order (Crank-Nicolson) based on an equidistant temporal mesh $\mathcal{T}_h$ with $1 + h^{-1}\mathsf{T} \le \#\mathcal{T}_h < 2 + h^{-1}\mathsf{T}$ for $0 < h \le \mathsf{T}$. The results for a range of h and λ are shown in Figure 5.2.1 (left). We observe the following behavior:

$$\forall \lambda > 0: \quad \gamma_{\lambda,h} \nearrow \Gamma_{\lambda,h} \approx 1 \quad \text{as} \quad h \searrow 0, \tag{5.2.71}$$

and

$$\gamma_{\lambda,h} \sim \min\{1, \max\{\lambda^{-1}, \mathrm{CFL}^{-1}_{\lambda,h}\}\} \quad \forall \lambda \ge 1 \quad \forall h > 0, \tag{5.2.72}$$

where $\mathrm{CFL}_{\lambda,h} := h\lambda^2$. Thus, the discrete variational problem

$$\text{find} \quad u_h \in \mathcal{X}_h: \quad \langle \mathcal{B}_\lambda u_h, v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \langle (g, u^0), v_h\rangle_{\mathcal{Y}'\times\mathcal{Y}} \quad \forall v_h \in \mathcal{Y}_h \tag{5.2.73}$$

becomes well-conditioned for the above choice of norms as $h \searrow 0$, cf. (5.2.64). However, the pair $\mathcal{X}_h \times \mathcal{Y}_h$ of continuous Galerkin trial and test spaces is *not stable uniformly in* λ (cf. Example 5.2.12). This behavior is in sharp contrast to what can be observed in Figure 5.2.1 (right) when we refine the test space, i.e., for the pair $\mathcal{X}_h \times \mathcal{Y}_{h/2}$. This family is unconditionally stable for $\mathcal{B}_\lambda$, i.e., uniformly in λ and h (cf. Proposition 5.2.20).

Example 5.2.12. We show that the behavior of (5.2.63) and (5.2.72) w.r.t. CFL_h cannot be improved, in general. The hidden positive constants in the following statements are understood to be independent

of $h > 0$. Consider a sequence of equidistant temporal meshes $\mathcal{T}_h$ with $\#\mathcal{T}_h \sim h^{-1}$ for the lowest order discontinuous Galerkin scheme with $\mathbf{p}_h = 0 \in \mathbb{N}_0^{\mathcal{T}_h - 1}$. Let $A = -\Delta$ be the Laplace operator $H_0^1(D) =: V \hookrightarrow V'$ on a bounded domain/interval D, with compact inverse as a mapping $H' \cong H \to H := L^2(D)$. Assume that $\varphi_h \in V_h$ satisfies $\|\varphi_h\|_H = 1$ and $\langle A\varphi_h, \chi_h\rangle = \lambda_h^2 \langle \varphi_h, \chi_h\rangle$ for all $\chi_h \in V_h$, with $\lambda_h \sim h^{-1}$. Assume further $\kappa_h := \mathcal{K}_{V' \times V}(V_h, V_h) \gtrsim 1$; this implies $\|\varphi_h\|_{V'} \leq \kappa_h^{-1} \sup_{\chi_h \in V_h \setminus \{0\}} \frac{\langle \varphi_h, \chi_h\rangle}{\|\chi_h\|_V} \leq \kappa_h^{-1} \lambda_h^{-2} \|\varphi_h\|_V$. Consequently, $\mathrm{CFL}_h \gtrsim h\lambda_h^2 \gtrsim h^{-1}$. Let $w_h \in \mathcal{X}_h$ be of the form $w_h = e_h \otimes \varphi_h$ with $e_h \in H^1(\mathsf{J})$, $e_h(0) = 0$, to be specified. We have $\|w_h\|_{\mathcal{X}}^2 \gtrsim \lambda_h^{-2} \|e_h'\|_{L^2(\mathsf{J})}^2 + \lambda_h^2 \|e_h\|_{L^2(\mathsf{J})}^2$. Then, using an analogous construction for e_h as in the counter-example [BJ90, pp. 353–354] (namely, with time reversed such that $e_h(0) = 0$), and with an analogous proof, we find

$$\sup_{v_h \in \mathcal{Y}_h \setminus \{0\}} \frac{\langle \mathcal{B} w_h, v_h\rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|v_h\|_{\mathcal{Y}}} = \sup_{f_h \in S^{0,\mathbf{p}_h}(\mathcal{T}_h) \setminus \{0\}} \frac{\int_{\mathsf{J}} (e_h' + \lambda_h^2 e_h) f_h dt}{\lambda_h \|f_h\|_{L^2(\mathsf{J})}} \lesssim h \|w_h\|_{\mathcal{X}},$$

and hence, $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \lesssim h \lesssim \mathrm{CFL}_h^{-1}$ for all $h > 0$.

B. Stable space-time sparse tensor product trial and test spaces

Let us focus on the particular situation that $\mathcal{X}_h \times \{w_h(0) : w_h \in \mathcal{X}_h\} \subseteq \mathcal{Y}_h = \mathcal{Y}_{h,1} \times \mathcal{Y}_{h,2}$. Note that this is not the case for the continuous Galerkin time-stepping schemes discussed above. Then

1. $P^h : \mathcal{X} \to \mathcal{Y}_h$, defined in (5.2.22)–(5.2.23), coincides on $\mathcal{X}_h$ with the mapping $w_h \mapsto (w_h, w_h(0))$;
2. consequently, P_1^h satisfies (5.2.27) by anti-symmetry of $\widetilde{A}$;
3. the norm equivalence (5.2.40) holds with constants that are independent of the structure of $\mathcal{X}_h \times \mathcal{Y}_h$ by the integration-by-parts formula (2.7.8) and Lemma 3.2.8.

With this, we may combine Corollary 5.2.7 and Corollary 5.2.10 as follows.

Corollary 5.2.13. *Let Assumption 3.2.4 hold with $a_{\mathrm{shift}} = 0$. Then a constant $\gamma_0 > 0$ exists such that for any pair of non-trivial closed subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$, having the structure*

$$\mathcal{X}_h \times \{w_h(0) : w_h \in \mathcal{X}_h\} \subseteq \mathcal{Y}_h = \mathcal{Y}_{h,1} \times \mathcal{Y}_{h,2}, \tag{5.2.74}$$

the estimates

1) $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \geq \gamma_0 \min\{\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h), 1\}$,

2) $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \geq \gamma_0 \min\{[\mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}(\partial_t \mathcal{X}_h, \mathcal{Y}_{h,1})]^2, 1\}$

hold.

The definitions of $\mathcal{K}_1^h$ and $\mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}$ can be found in (5.2.26) and (5.2.48). Now, the task of proving stability of a family $\mathcal{X}_h \times \mathcal{Y}_h$, $h > 0$, reduces to showing either of:

1) $\inf_{h>0} \mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h) > 0$,
2) $\inf_{h>0} \mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}(\partial_t \mathcal{X}_h, \mathcal{Y}_{h,1}) > 0$.

Observe that $\mathcal{K}_1^h(\mathcal{X}_h, \mathcal{Y}_h) \sim \mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}(\partial_t \mathcal{X}_h, \mathcal{Y}_{h,1})$ with constants that are independent of the pair $\mathcal{X}_h \times \mathcal{Y}_h$, if the anti-symmetric part of A vanishes, i.e., if $\widetilde{A} = 0$. Let us therefore discuss a particular construction of the pairs of trial and test spaces $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$ for which

$$\inf_{h>0} \mathcal{K}_{\mathcal{Y}_1' \times \mathcal{Y}_1}(\partial_t \mathcal{X}_h, \mathcal{Y}_{h,1}) > 0 \tag{5.2.75}$$

can be shown. These will be of tensor product structure based on suitable "temporal" subspaces of $L^2(\mathsf{J})$ and "spatial" subspaces of V', that themselves are required to satisfy a stability condition. The following result is an immediate consequence of Proposition 4.4.12.

Proposition 5.2.14. *Let*

a) $E_k \subseteq E_{k+1} \subseteq H^1(\mathsf{J})$ *and* $F_k \subseteq F_{k+1} \subseteq L^2(\mathsf{J})$, $k \in \mathbb{N}_0$,

b) $U_\ell \subseteq U_{\ell+1} \subseteq V'$, *and* $V_\ell \subseteq V_{\ell+1} \subseteq V$, $\ell \in \mathbb{N}_0$,

be nested sequences of non-trivial closed subspaces. Set

a) $\tau := \inf_{k\in\mathbb{N}_0} \mathcal{K}_{L^2(\mathsf{J})\times L^2(\mathsf{J})}(\partial_t E_k, F_k)$,

b) $\eta := \inf_{\ell\in\mathbb{N}_0} \mathcal{K}_{V'\times V}(U_\ell, V_\ell)$.

For $L \in \mathbb{N}_0$ *define the subspaces* $\mathcal{X}_L \subseteq \mathcal{X}$ *and* $\mathcal{Y}_{L,1} \subseteq \mathcal{Y}_1$ *by*

$$\mathcal{X}_L := \bigcup_{0\le k+\ell\le L} E_k \otimes U_\ell, \quad \mathcal{Y}_{L,1} := \bigcup_{0\le k+\ell\le L} [F_k \otimes V_\ell], \tag{5.2.76}$$

where k, ℓ *range over* $\mathbb{N}_0$. *Then* $\mathcal{K}_{\mathcal{Y}_1'\times\mathcal{Y}_1}(\partial_t \mathcal{X}_L, \mathcal{Y}_{L,1}) \ge \eta\tau$.

Remark 5.2.15. The pairs of subspaces $\mathcal{X}_L \times \mathcal{Y}_L \subseteq \mathcal{X} \times \mathcal{Y}$ with $\mathcal{Y}_L := \mathcal{Y}_{L,1} \times \mathcal{Y}_{L,2}$ given by

$$\mathcal{X}_L := \bigcup_{0\le k+\ell\le L} E_k \otimes U_\ell, \quad \mathcal{Y}_{L,1} := \bigcup_{0\le k+\ell\le L} [F_k \otimes V_\ell], \quad \mathcal{Y}_{L,2} := V_L, \tag{5.2.77}$$

will be referred to as

- the sparse tensor product subspaces.

By relabeling the subspaces, for any fixed L, the conclusion of Proposition 5.2.14 also holds for

- the full tensor product subspaces,

$$\mathcal{X}_L := E_L \otimes U_L, \quad \mathcal{Y}_{L,1} := F_L \otimes V_L, \quad \mathcal{Y}_{L,2} := V_L. \tag{5.2.78}$$

- the sparse tensor product subspaces with anisotropy,

$$\mathcal{X}_L^\rho := \bigcup_{0\le k+\rho\ell\le L} E_k \otimes U_\ell, \quad \mathcal{Y}_{L,1}^\rho := \bigcup_{0\le k+\rho\ell\le L} F_k \otimes V_\ell, \quad \mathcal{Y}_{L,2}^\rho := \bigcup_{\rho\ell\le L} V_\ell, \tag{5.2.79}$$

  where $\rho \in [0,\infty]$ determines the anisotropy in the "refinement" with the convention "$\infty \cdot 0 = 0$". In the case $\rho = 0$, all V_ℓ are replaced by the closure of $\bigcup_{\ell\in\mathbb{N}_0} V_\ell$ in V, similarly for U_ℓ in V'.

- the subspaces $\mathcal{X}_{L_1}^{\rho_1} + \mathcal{X}_{L_2}^{\rho_2}$ and $\mathcal{Y}_{L_1}^{\rho_1} + \mathcal{Y}_{L_2}^{\rho_2}$ where $\rho_1, \rho_2 \in [0,\infty]$ and $L_1, L_2 \in \mathbb{N}_0$.

In order to ensure the inclusion property (5.2.74) *and* $\tau > 0$, it is natural to simply set

$$F_k := E_k + \partial_t E_k, \tag{5.2.80}$$

where $\partial_t E_k := \{e' : e \in E_k\}$. Then, in fact, $\mathcal{K}_{L^2(\mathsf{J})\times L^2(\mathsf{J})}(E_k', F_k) \ge \tau = 1$ for all $k \in \mathbb{N}_0$.

Example 5.2.16. A class of finite-dimensional temporal subspaces $F_k = E_k$ for which $\tau > 0$ holds are subspaces that are closed under derivatives. Some examples, cf. Section 5.2.1, are

- polynomials $E_k := \operatorname{span}\{t^j : j = 0, \dots, k\}$,
- trigonometric polynomials $E_k := \operatorname{span}\{\sin(j\omega t), \cos(j\omega t) : j = 0, \dots, k\}$, where $\omega \in \mathbb{R}$,
- exponentials $E_k := \operatorname{span}\{\exp(\lambda_j t) : j = 0, \dots, k\}$ where $\{\lambda_j\}_{j\in\mathbb{N}_0} \subset \mathbb{R}$.

Setting $F_k := E_k$, in each case we indeed have $F_k = E_k + \partial_t E_k$. Using Corollary 5.2.13, i), and Proposition 5.2.14 we recover the observation $\gamma_{\mathcal{B}}(\mathcal{X}_h, \mathcal{Y}_h) \ge \mathcal{K}_{V'\times V}(U, U)$ obtained by explicit computations in Example 5.2.3 and Example 5.2.2. Further types of temporal subspaces E_k and F_k that are admissible in the sense of Proposition 5.2.14 with $\tau > 0$ are subject of Section 7.

More generally, we suggest the following definition.

Definition 5.2.17 (Admissible pairs of temporal subspaces)**.** A pair of families of closed temporal subspaces $E_k \subseteq H^1(\mathsf{J})$ and $F_k \subseteq L^2(\mathsf{J})$, $k \in \mathbb{N}_0$, is called **admissible** if it has the properties

1. E_0 and F_0 are non-trivial,
2. of nestedness, $E_k \subseteq E_{k+1}$ and $F_k \subseteq F_{k+1}$, $k \in \mathbb{N}_0$,
3. of stability, $\inf_{k\in\mathbb{N}_0} \mathcal{K}_{L^2(\mathsf{J})\times L^2(\mathsf{J})}(E_k + \partial_t E_k, F_k) > 0$.

Such admissible pairs together with a stability condition on the spatial subspaces lead to **stable sparse space-time discrete trial and test spaces**:

Theorem 5.2.18 (Stable sparse space-time subspaces)**.** *Let $E_k \subset H^1(\mathsf{J})$, $F_k \subseteq L^2(\mathsf{J})$, $k \in \mathbb{N}_0$, be an admissible pair of families of closed subspaces in the sense of Definition 5.2.17. Let $U_\ell \subseteq V_\ell \subseteq V$, $\ell \in \mathbb{N}_0$, be closed subspaces that satisfy the conditions of Proposition 5.2.14. For all $L \in \mathbb{N}_0$ define $\mathcal{X}_L \times \mathcal{Y}_L \subseteq \mathcal{X} \times \mathcal{Y}$ as any of the pairs of Remark 5.2.15. Then*

$$\inf_{L\in\mathbb{N}_0} \gamma_{\mathcal{B}}(\mathcal{X}_L, \mathcal{Y}_L) > 0. \tag{5.2.81}$$

Proof. We consider w.l.o.g. the sparse tensor product case ($\rho = 1$). Set $\widetilde{F_k} := E_k + \partial_t E_k$, $k \in \mathbb{N}_0$, and define the auxiliary subspaces $\widetilde{\mathcal{Y}_L} := \sum_{0\le k+\ell\le L}[\widetilde{F_k} \otimes V_\ell] \times V_\ell$. Then $\inf_{L\in\mathbb{N}_0} \gamma_{\mathcal{B}}(\mathcal{X}_L, \widetilde{\mathcal{Y}_L}) > 0$ by Proposition 5.2.14 and Corollary 5.2.13. In view of (4.4.34), it remains to show $\inf_{L\in\mathbb{N}_0} \mathcal{K}_{\mathcal{Y}\times\mathcal{Y}}(\widetilde{\mathcal{Y}_L}, \mathcal{Y}_L) > 0$. This final argument is similar to the first step in the proof of Proposition 4.4.12, and is therefore omitted. □

Remark 5.2.19. Assume that either $E_k \subseteq F_k$ or $\partial_t E_k \subseteq F_k$. Then, Corollary 4.4.6 implies

$$\mathcal{K}_{L^2(\mathsf{J})\times L^2(\mathsf{J})}(E_k + \partial_t E_k, F_k) \ge \min\{\mathcal{K}_{(\dots)}(E_k, F_k), \mathcal{K}_{(\dots)}(\partial_t E_k, F_k)\}. \tag{5.2.82}$$

C. Stabilized Crank-Nicolson

The computations in Example 5.2.11 suggested that the lowest order continuous Galerkin subspaces (Crank-Nicolson) become stable if the test space is h-refined in the temporal direction. Using the results of the foregoing subsection this can be shown to be indeed the case. Note, however, that the dimension of the discrete test space is then larger than that of the trial space; therefore, the resulting discretization, does not have an interpretation of a time-stepping scheme anymore, and the discrete solution has to be computed as the minimizer of the non-local-in-time residual functional (see Proposition 4.2.5).

Let $\mathcal{T}_k \subset [0, \mathsf{T}]$, $k \in \mathbb{N}_0$, be a sequence of temporal meshes (with $\#\mathcal{T}_k \sim 2^k$, say) and for each $k \in \mathbb{N}_0$ let $\mathcal{T}_k^\star \subset [0, \mathsf{T}]$ be the uniformly refined version of $\mathcal{T}_k$, i.e., $\#\mathcal{T}_k^\star = 2\#\mathcal{T}_k - 1$ and every $t \in \mathcal{T}_k^\star$ is either in $\mathcal{T}_k$ or is exactly in between two neighboring nodes in $\mathcal{T}_k$. Let $V_\ell \subseteq V$, $\ell \in \mathbb{N}_0$, be a family of finite-dimensional subspaces. Set $E_k := S^{1,1}(\mathcal{T}_k)$ and $F_k := S^{0,0}(\mathcal{T}_k^\star)$, where $S^{r,p}$ is the spline space defined in (5.2.59). Consider the "stabilized Crank-Nicolson" subspaces

$$\mathcal{X}_L := E_L \otimes V_L \subset \mathcal{X}, \quad \mathcal{Y}_L := [F_L \otimes V_L] \times V_L \subset \mathcal{Y}, \quad L \in \mathbb{N}_0. \tag{5.2.83}$$

From Theorem 5.2.18 we obtain the following.

Proposition 5.2.20. *Assume* $\inf_{\ell>0} \mathcal{K}_{V'\times V}(V_\ell, V_\ell) > 0$. *Then* $\inf_{L>0} \gamma_{\mathcal{B}}(\mathcal{X}_L, \mathcal{Y}_L) > 0$.

Proof. Set $\widetilde{F_k} := E_k + \partial_t E_k$, note that $\dim \widetilde{F_k} = \dim F_k$. Now, in order to apply Theorem 5.2.18 we need to verify $\inf_{k\in\mathbb{N}_0} \mathcal{K}_{L^2(\mathsf{J})\times L^2(\mathsf{J})}(\widetilde{F_k}, F_k) \geq \tau > 0$ for some $\tau > 0$. This follows using the equivalence (4.4.2) $\Leftrightarrow$ (4.4.5) from

$$\inf_{f\in S^{0,0}(\mathcal{T}_k^\star)} \|e - f\|^2_{L^2(\mathsf{J})} \leq (1-\tau^2)\|e\|^2_{L^2(\mathsf{J})} \quad \forall e \in S^{0,1}(\mathcal{T}_k) \quad \forall k \in \mathbb{N}_0, \tag{5.2.84}$$

with $\tau = \sqrt{3/4}$, which we will shown now. Let $k \in \mathbb{N}_0$ and $e \in S^{0,1}(\mathcal{T}_k)$ (i.e., piecewise linear with discontinuous allowed) be arbitrary, fixed. As in the proof of (5.2.63), we can write $e = \sum_{n=1}^N (e_n^0 L_0^{I_n} + e_n^1 L_1^{I_n})$ where $L_j^{I_n}$ is the j-th Legendre polynomial on the reference interval $I = (-1,1)$ transported to the n-th subinterval $I_n = (t_{n-1}, t_n)$ of the temporal mesh $\mathcal{T}_k$. Since the piecewise constant functions are contained in $S^{0,0}(\mathcal{T}_k^\star)$, and using mutual orthogonality of $L_0^{I_n}$ and $L_1^{I_n}$ in $L^2(\mathsf{J})$, we may w.l.o.g. assume that $e_n^0 = 0$. Further, by mapping each I_n to the reference interval $I = (-1,1)$, the problem reduces to approximating L_1 on I by a function of the form $f = \beta\chi_{(-1,0)} + \alpha\chi_{(0,1)}$. Since $L_1(\pm 1) = \pm\sqrt{3/2}$, that best approximation of L_1 is given by $f = -\alpha\chi_{(-1,0)} + \alpha\chi_{(0,1)}$ with $\alpha = \sqrt{3/8}$. This yields $\|L_1 - f\|^2_{L^2(I)} = 1/4 = (1-\tau^2)$ and hence (5.2.84) with $\tau = \sqrt{3/4}$, as claimed. □

Having shown $\inf_{k\in\mathbb{N}_0} \mathcal{K}_{L^2(\mathsf{J})\times L^2(\mathsf{J})}(E_k + \partial_t E_k, F_k) \geq \tau > 0$ in the above proof, the sparse tensor product "stabilized Crank-Nicolson" subspaces of the form (5.2.77) with F_k being the piecewise constant functions on a refined temporal mesh as discussed in this subsection are also stable according to Theorem 5.2.18.

Sparse tensor product Crank-Nicolson subspaces could also be defined (as was done in [GO07]), but we do not expect these to be stable, in general, cf. Example 5.2.11 and Example 5.2.12.

6 Parabolic BPX preconditioning

In Section 4.2 we have described the numerical procedure for obtaining the minimal residual Petrov-Galerkin solution (Definition 4.1.10) to the operator equation $\mathcal{B}u = \mathcal{F}$ w.r.t. stable pairs $\mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{X} \times \mathcal{Y}$. There, we assumed the existence of norm-inducing s.p.d. operators $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$ and $\mathcal{M} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$. This is not restrictive, the canonical example being the Riesz operators defined by $\langle \mathcal{N}\cdot, \cdot\rangle_{\mathcal{Y}' \times \mathcal{Y}} := \langle \cdot, \cdot\rangle_{\mathcal{Y}}$ and $\langle \mathcal{M}\cdot, \cdot\rangle_{\mathcal{X}' \times \mathcal{X}} := \langle \cdot, \cdot\rangle_{\mathcal{X}}$. When used as preconditioners with the finite element method, however, the respective matrices should be "easily" invertible, cf. Algorithm 4.3.1, and this is not necessarily the case for this canonical choice. This chapter is therefore devoted to the construction of such operators for the space-time variational formulation of the abstract parabolic evolution equation considered in Chapter 3, namely for $\mathcal{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ defined in (3.2.15). We will not specifically address the operator $\widehat{\mathcal{B}} \in \mathcal{L}(\mathcal{X}, \mathcal{X}')$ defined in (3.2.53), which would be a rather straightforward adaptation. Two different constructions are given. In Section 6.1 we employ Riesz bases, and adapt arguments from [SS09]. Section 6.2 contains the second main contribution of this thesis: a multilevel **parabolic BPX preconditioner** for the space-time variational formulations of the parabolic evolution equation. It is derived from the classical BPX preconditioner [BPX90; BY93] for elliptic problems; the idea may be adapted to other problems posed in anisotropic Sobolev spaces.

The setting will be essentially that of Section 3.2.1: we will work in a Gelfand triple of *separable* real Hilbert spaces

$$V \overset{d}{\hookrightarrow} H \cong H' \overset{d}{\hookrightarrow} V', \tag{6.0.1}$$

where the embeddings are *dense*; the "pivot" space H is identified with its dual H' via the scalar product $\langle \cdot, \cdot\rangle_H$ on H; the duality pairing $\langle \cdot, \cdot\rangle$ on $V \times V'$ is then, in fact, the unique continuous extension of $\langle \cdot, \cdot\rangle_H : V \times V \to \mathbb{R}$. Recall further the spaces $\mathcal{X}$ and $\mathcal{Y}$, that for the purpose of this chapter are better understood as tensor product spaces

$$\mathcal{X} = [L^2(\mathsf{J}) \otimes V] \cap [H^1(\mathsf{J}) \otimes V'] \quad \text{and} \quad \mathcal{Y} = [L^2(\mathsf{J}) \otimes V] \times H, \tag{6.0.2}$$

where $\mathsf{J} = (0, \mathsf{T})$ is a non-trivial bounded temporal interval.

6.1 Wavelet preconditioning

The variational formulation (3.2.16) of the abstract parabolic evolution equation (3.1.1) was motivated by [SS09]. There, the operator equation $\mathcal{B}u = \mathcal{F}$ was transformed to an equivalent well-posed bi-infinite matrix-vector equation $\widetilde{\mathbf{B}}\widetilde{\mathbf{u}} = \widetilde{\mathbf{f}}$ with $\widetilde{\mathbf{B}} \in \mathrm{Iso}(\ell^2(\mathbb{N}), \ell^2(\mathbb{N}))$ by means of Riesz bases on $\mathcal{X}$ and $\mathcal{Y}$. A basis $\{x_i : i \in I\}$ for a separable Hilbert space X is called a Riesz basis, if the synthesis operator $\ell^2(I) \ni c \mapsto \sum_{i \in I} c_i x_i \in X$ is an isomorphism. Riesz bases for Sobolev (and Besov) spaces on domains are typically of wavelet type [Dah97; DK92]. In this section we assume the point of view of operator preconditioning discussed in Section 4.2, and construct norm-inducing operators $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$ and $\mathcal{M} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ using such Riesz bases. The tensor product structure of $\mathcal{X}$ and $\mathcal{Y}$ allows to construct Riesz bases on those spaces from suitable Riesz bases on $L^2(\mathsf{J})$ and H.

6.1.1 Norm-inducing operator on the test space

Let $\Theta = \{\theta_\lambda : \lambda \in \mathcal{I}^\Theta\} \subset H^1(\mathsf{J})$ and $\Sigma = \{\sigma_\mu : \mu \in \mathcal{I}^\Sigma\} \subset V$ be bases. We assume that Θ is a Riesz basis for $L^2(\mathsf{J})$, and that Σ is a Riesz basis for H which may be rescaled to a Riesz basis for V. Note that on the test space side, Riesz bases for $H^1(\mathsf{J})$ and V' are not required (cf. the following subsection). For $\lambda \in \mathcal{I}^\Theta$ and $\mu \in \mathcal{I}^\Sigma$ set[1] $c_{\lambda\mu} := \|\theta_\lambda\|_{L^2(\mathsf{J})}\|\sigma_\mu\|_V$. Define

$$\Psi_1 := \{c_{\lambda\mu}^{-1}\theta_\lambda \otimes \sigma_\mu : (\lambda,\mu) \in \mathcal{I}^\Theta \times \mathcal{I}^\Sigma\} \quad \text{and} \quad \Psi_2 := \Sigma. \tag{6.1.1}$$

Then $\Psi := (\Psi_1 \times \{0\}) \cup (\{0\} \times \Psi_2)$ is a Riesz basis for $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$. Define the symmetric operator $\mathcal{N} \in \mathcal{L}(\mathcal{Y},\mathcal{Y}')$ by $\langle \mathcal{N}\psi, \psi'\rangle_{\mathcal{Y}'\times\mathcal{Y}} = c_\psi \delta_{\psi\psi'}$ for $\psi, \psi' \in \Psi$, where $c_\psi > 0$ are arbitrary constants of order one. Then, for all $v = \Psi^\top \mathbf{v}$ with (finitely supported) $\mathbf{v} \in \ell^2(\mathbb{N})$ we have $\langle \mathcal{N}v, v\rangle_{\mathcal{Y}'\times\mathcal{Y}} \sim \|\mathbf{v}\|^2_{\ell^2(\mathbb{N})} \sim \|v\|^2_{\mathcal{Y}}$, which implies $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y},\mathcal{Y}')$.

6.1.2 Norm-inducing operator on the trial space

Let $\Theta \subset H^1(\mathsf{J})$ and $\Sigma \subset V$ be bases with the properties stated in the previous subsection (but possibly different ones). In addition, we require now that Θ may be rescaled to a Riesz basis on $H^1(\mathsf{J})$, and that Σ may be rescaled to a Riesz basis on V'. We remark, that the latter property is necessarily true (cf. Lemma 6.2.3) if Σ is an *orthonormal* basis on H which may be rescaled to a Riesz basis on V. Such Riesz bases of wavelet type were constructed using the notion of an intertwining multiresolution analysis in one and two dimensions in [DGH96; DGH99; DGH00; Goo03]. Set[1]

$$c_{\lambda\mu} := \|\theta_\lambda \otimes \sigma_\mu\|_{\mathcal{X}} = \sqrt{\|\theta_\lambda\|^2_{L^2(\mathsf{J})}\|\sigma_\mu\|^2_V + \|\theta_\lambda\|^2_{H^1(\mathsf{J})}\|\sigma_\mu\|^2_{V'}}.$$

The collection $\Phi := \{c_{\lambda\mu}^{-1}\theta_\lambda \otimes \sigma_\mu : (\lambda,\mu) \in \mathcal{I}^\Theta \times \mathcal{I}^\Sigma\}$ is a Riesz basis for $\mathcal{X}$ [SS09, Section 6]. Define the symmetric operator $\mathcal{M} \in \mathcal{L}(\mathcal{X},\mathcal{X}')$ by $\langle \mathcal{M}\phi, \phi'\rangle_{\mathcal{X}'\times\mathcal{X}} = c_\phi \delta_{\phi\phi'}$ for $\phi, \phi' \in \Phi$, where $c_\phi > 0$ are arbitrary constants of order one. Then, for all $w = \Phi^\top \mathbf{w}$ with (finitely supported) $\mathbf{w} \in \ell^2(\mathbb{N})$ we have $\langle \mathcal{M}w, w\rangle_{\mathcal{X}'\times\mathcal{X}} \sim \|\mathbf{w}\|^2_{\ell^2(\mathbb{N})} \sim \|w\|^2_{\mathcal{X}}$, hence $\mathcal{M} \in \mathrm{Iso}(\mathcal{X},\mathcal{X}')$.

6.2 Parabolic BPX preconditioner

The construction of suitable wavelet bases required in Section 6.2.1 and especially in Section 6.2.2 is rather intricate, especially on non-trivial domains. In this section we therefore develop a space-time **parabolic BPX** multilevel preconditioner. It is based on the BPX preconditioner well-known for its optimality for certain elliptic problems [BPX90; BY93].

Let $\{0\} = E_0 \subseteq E_1 \subseteq \ldots \subseteq H^1(\mathsf{J})$ and $\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V$ be sequences of (closed) nested finite-dimensional subspaces. Assume that $\bigcup_{k\in\mathbb{N}} E_k$ is dense in $L^2(\mathsf{J})$ and $\bigcup_{\ell\in\mathbb{N}} V_\ell$ is dense in H. Let $P_k^\Delta : L^2(\mathsf{J}) \to E_k$, $k \in \mathbb{N}$, be projectors such that $E_k = E_{k-1} \oplus P_k^\Delta L^2(\mathsf{J})$; similarly, let $Q_\ell^\Delta : H \to V_\ell$, $\ell \in \mathbb{N}$, be projectors such that $V_\ell = V_{\ell-1} \oplus Q_\ell^\Delta H$. Clearly, $\sum_{k\in\mathbb{N}} P_k^\Delta = \mathrm{Id}_{L^2(\mathsf{J})}$ and $\sum_{\ell\in\mathbb{N}} Q_\ell^\Delta = \mathrm{Id}_H$ (pointwise convergence). Note that these projectors are not necessarily surjective or orthogonal for the respective inner products. Further, we assume that P_k^Δ and Q_ℓ^Δ generate **stable subspace decompositions**, i.e., there exist constants $d_{L^2(\mathsf{J})}, D_{L^2(\mathsf{J})} \in (0,\infty)$ and $d_H, D_H \in (0,\infty)$ such that the norm equivalences

$$d_{L^2(\mathsf{J})}\|f\|^2_{L^2(\mathsf{J})} \le \sum_{k\in\mathbb{N}} \|P_k^\Delta f\|^2_{L^2(\mathsf{J})} \le D_{L^2(\mathsf{J})}\|f\|^2_{L^2(\mathsf{J})} \quad \forall f \in L^2(\mathsf{J}) \tag{6.2.1}$$

and

$$d_H\|\chi\|^2_H \le \sum_{\ell\in\mathbb{N}} \|Q_\ell^\Delta \chi\|^2_H \le D_H\|\chi\|^2_H \quad \forall \chi \in H \tag{6.2.2}$$

hold. For the construction of the parabolic BPX operators we will work under the following hypotheses.

[1] One may drop the term $\|\theta_\lambda\|_{L^2(\mathsf{J})}$ in this definition, since $\|\theta_\lambda\|_{L^2(\mathsf{J})} \sim 1$ by assumption.

Assumption 6.2.1. There exist constants $d_{H^1(\mathsf{J})}, D_{H^1(\mathsf{J})} \in (0,\infty)$ and a monotone sequence $\{p_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $p_k \nearrow \infty$ as $k \to \infty$, such that

$$d_{H^1(\mathsf{J})}\|f\|^2_{H^1(\mathsf{J})} \le \sum_{k\in\mathbb{N}} p_k^2 \|P_k^\Delta f\|^2_{L^2(\mathsf{J})} \le D_{H^1(\mathsf{J})}\|f\|^2_{H^1(\mathsf{J})} \quad \forall f \in H^1(\mathsf{J}). \tag{6.2.3}$$

Assumption 6.2.2. There exist constants $d_V, D_V \in (0,\infty)$ and a monotone sequence $\{q_\ell\}_{\ell\in\mathbb{N}} \subset (0,\infty)$ with $q_\ell \nearrow \infty$ as $\ell \to \infty$, such that

$$d_V\|\chi\|^2_V \le \sum_{\ell\in\mathbb{N}} q_\ell^2 \|Q_\ell^\Delta \chi\|^2_H \le D_V\|\chi\|^2_V \quad \forall \chi \in V. \tag{6.2.4}$$

Norm equivalences of the form (6.2.3) and (6.2.4) are known in the finite element literature, e.g. in the simplest case of continuous piecewise linear finite elements on regular quasi-uniform triangulations, for proof and discussion see [BY93] and references therein. Assumption 6.2.2 allows to obtain multilevel norm equivalences on the dual V' of V.

Lemma 6.2.3. *Let Assumption 6.2.2 hold for a sequence of H-orthogonal projectors Q_ℓ^Δ on H, that are, moreover, mutually orthogonal. Then*

$$d_{V'}\|\chi\|^2_{V'} \le \sum_{\ell\in\mathbb{N}} q_\ell^{-2} \|Q_\ell^\Delta \chi\|^2_H \le D_{V'}\|\chi\|^2_{V'} \quad \forall \chi \in V \tag{6.2.5}$$

for $d_{V'} := D_V^{-1}$ and $D_{V'} := d_V^{-1}$.

Proof. As in [Osw98, Lemma 1]. □

Remark 6.2.4. The notion of stable subspace decompositions generalizes that of a Riesz basis used in Section 6.1 in the following sense. Assume that $P_k^\Delta L^2(\mathsf{J})$ is one-dimensional for each k. Then $P_k^\Delta = \langle \cdot, \widetilde{f}_k\rangle_{L^2(\mathsf{J})} f_k$ for some non-zero $\widetilde{f}_k, f_k \in L^2(\mathsf{J})$ with $\|f_k\|_{L^2(\mathsf{J})} = 1$. Thus, for each $f \in L^2(\mathsf{J})$ we have **a)** $f = \sum_{k\in\mathbb{N}} P_k^\Delta f = \sum_{k\in\mathbb{N}} c_k(f) f_k$ with $c_k(f) := \langle f, \widetilde{f}_k\rangle_{L^2(\mathsf{J})}$ and **b)** $\|P_k^\Delta f\|_{L^2(\mathsf{J})} = |c_k(f)|$. This with (6.2.1) shows the Riesz basis property in $L^2(\mathsf{J})$ for the set $\{f_k\}_{k\in\mathbb{N}}$. The converse is obviously also true.

6.2.1 Norm-inducing operator on the test space

For the construction of an s.p.d. operator $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$ we require that the multilevel norm equivalences (6.2.1) in $L^2(\mathsf{J})$ and (6.2.2) in H, and Assumption 6.2.2 on the multilevel norm equivalence in V hold. Here, P_k^Δ and Q_ℓ^Δ are not necessarily orthogonal projectors for the respective inner products.

An s.p.d. operator $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$ is defined follows.

1. Define the linear operator $\mathcal{N}_1 : \mathcal{D}(\mathcal{N}_1) := \bigcup_{k,\ell\in\mathbb{N}} E_k \otimes V_\ell \subset \mathcal{Y}_1 \to \mathcal{Y}_1'$ by

$$\langle \mathcal{N}_1 v_1, \widetilde{v}_1\rangle_{\mathcal{Y}_1'\times\mathcal{Y}_1} := \sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}} q_\ell^2 \langle (P_k^\Delta \otimes Q_\ell^\Delta) v_1, (P_k^\Delta \otimes Q_\ell^\Delta)\widetilde{v}_1\rangle_{L^2(\mathsf{J})\otimes H}, \tag{6.2.6}$$

   for $v_1 \in \mathcal{D}(\mathcal{N}_1)$ and $\widetilde{v}_1 \in \mathcal{Y}_1$. Note that the sum is finite for each $v_1 \in \mathcal{D}(\mathcal{N}_1)$. Then, for any $v_1 \in \mathcal{D}(\mathcal{N}_1)$,

$$\begin{aligned}
\langle \mathcal{N}_1 v_1, v_1\rangle_{\mathcal{Y}_1'\times\mathcal{Y}_1} &= \sum_{\ell\in\mathbb{N}} q_\ell^2 \sum_{k\in\mathbb{N}} \|(P_k^\Delta \otimes \mathrm{Id})(\mathrm{Id}\otimes Q_\ell^\Delta) v_1\|^2_{L^2(\mathsf{J})\otimes H} && (6.2.7)\\
&\le D_{L^2(\mathsf{J})} \sum_{\ell\in\mathbb{N}} q_\ell^2 \|(\mathrm{Id}\otimes Q_\ell^\Delta) v_1\|^2_{L^2(\mathsf{J})\otimes H} && (6.2.8)\\
&\le D_{L^2(\mathsf{J})} D_V \|v_1\|^2_{L^2(\mathsf{J})\otimes V}, && (6.2.9)
\end{aligned}$$

and the Cauchy-Schwarz inequality on the series implies

$$\langle \mathcal{N}_1 v_1, \widetilde{v_1}\rangle_{\mathcal{Y}_1'\times\mathcal{Y}_1} \le \sqrt{\langle \mathcal{N}_1 v_1, v_1\rangle_{\mathcal{Y}_1'\times\mathcal{Y}_1}}\sqrt{\langle \mathcal{N}_1 \widetilde{v_1}, \widetilde{v_1}\rangle_{\mathcal{Y}_1'\times\mathcal{Y}_1}} \tag{6.2.10}$$

$$\le D_{L^2(\mathsf{J})} D_V \|v_1\|_{L^2(\mathsf{J})\otimes V}\|\widetilde{v_1}\|_{L^2(\mathsf{J})\otimes V} \tag{6.2.11}$$

for $v_1, \widetilde{v_1} \in \mathcal{D}(\mathcal{N}_1)$. Here, we have used

$$\sum_{k\in\mathbb{N}} \|(P_k^\Delta \otimes \mathrm{Id})v_1\|^2_{L^2(\mathsf{J})\otimes H} \le D_{L^2(\mathsf{J})}\|v_1\|^2_{L^2(\mathsf{J})\otimes H}, \quad v_1 \in \mathcal{Y}_1 \tag{6.2.12}$$

which is obtained by expanding $v_1 = \sum_{i\in\mathbb{N}} f_i \otimes \chi_i$ with an H-orthonormal basis $\{\chi_i\}_{i\in\mathbb{N}}$ and using (6.2.1), as well as similar inequalities. Since $\mathcal{D}(\mathcal{N}_1)$ is dense in $\mathcal{Y}_1$ and also in $\mathcal{Y}_1' = L^2(\mathsf{J}) \otimes V'$, the operator $\mathcal{N}_1$ extends by continuity to an operator (again denoted by) $\mathcal{N}_1 \in \mathcal{L}(\mathcal{Y}_1, \mathcal{Y}_1')$, and is obviously symmetric.[2] Similarly, for $v_1 \in \mathcal{D}(\mathcal{N}_1)$, we obtain

$$\langle \mathcal{N}_1 v_1, v_1\rangle_{\mathcal{Y}_1'\times\mathcal{Y}_1} \ge d_{L^2(\mathsf{J})} d_V \|v_1\|^2_{L^2(\mathsf{J})\otimes V}, \tag{6.2.13}$$

which implies that $\mathcal{N}_1$ is s.p.d., and from the Lax-Milgram Lemma (Theorem 4.1.2) that $\mathcal{N}_1 \in \mathrm{Iso}(\mathcal{Y}_1, \mathcal{Y}_1')$, the norm of its inverse being bounded above by $(d_{L^2(\mathsf{J})} d_V)^{-1}$.

2. For $\chi, \widetilde{\chi} \in \bigcup_{\ell\in\mathbb{N}} V_\ell$ set

$$\langle \mathcal{N}_2 \chi, \widetilde{\chi}\rangle_H := \sum_{\ell\in\mathbb{N}} \langle Q_\ell^\Delta \chi, Q_\ell^\Delta \widetilde{\chi}\rangle_H. \tag{6.2.14}$$

Then

$$d_H \|\chi\|_H^2 \le \langle \mathcal{N}_2 \chi, \chi\rangle_H \le D_H \|\chi\|_H^2 \quad \forall \chi \in \bigcup_{\ell\in\mathbb{N}} V_\ell \tag{6.2.15}$$

holds, and $\mathcal{N}_2$ extends by continuity to an s.p.d. operator $\mathcal{N}_2 \in \mathrm{Iso}(H, H)$ with the same bounds on all of H. Note that a different set of projectors $\{Q_\ell^\Delta\}_{\ell\in\mathbb{N}}$ from that for $\mathcal{N}_1$ could be used here.

3. Define $\mathcal{N} v := (\mathcal{N}_1 v_1, \mathcal{N}_2 v_2)$, $v \in \mathcal{Y}$, Then, $\mathcal{N} \in \mathrm{Iso}(\mathcal{Y}, \mathcal{Y}')$, and $\mathcal{N}$ is s.p.d.

6.2.2 Norm-inducing operator on the trial space

Here we assume the validity of the multilevel norm equivalences (6.2.1) for $L^2(\mathsf{J})$ and (6.2.2) for H, as well of Assumption 6.2.2 for $H^1(\mathsf{J})$ and Assumption 6.2.1 for V. Further, assume that Q_ℓ^Δ are H-orthogonal, which implies the dual multilevel norm equivalence (6.2.5) for V'.

Define

$$\mathcal{M} : \mathcal{D}(\mathcal{M}) := \bigcup_{k,\ell\in\mathbb{N}} E_k \otimes V_\ell \subset \mathcal{X} \to \mathcal{X}'$$

by

$$\langle \mathcal{M} w, \widetilde{w}\rangle_{\mathcal{X}'\times\mathcal{X}} := \sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}} \left(q_\ell^2 + p_k^2 q_\ell^{-2}\right) \langle (P_k^\Delta \otimes Q_\ell^\Delta) w, (P_k^\Delta \otimes Q_\ell^\Delta)\widetilde{w}\rangle_{L^2(\mathsf{J})\otimes H}, \tag{6.2.16}$$

where $w \in \mathcal{D}(\mathcal{M})$, $\widetilde{w} \in \mathcal{X}$. Note that the sum is finite for each $w \in \mathcal{D}(\mathcal{M})$. As in the previous subsection, we obtain

$$\langle \mathcal{M} w, w\rangle_{\mathcal{X}'\times\mathcal{X}} \le D_{L^2(\mathsf{J})} D_V \|w\|^2_{L^2(\mathsf{J})\otimes V} + D_{H^1(\mathsf{J})} D_{V'} \|w\|^2_{H^1(\mathsf{J})\otimes V'} \tag{6.2.17}$$

and

$$\langle \mathcal{M} w, w\rangle_{\mathcal{X}'\times\mathcal{X}} \ge d_{L^2(\mathsf{J})} d_V \|w\|^2_{L^2(\mathsf{J})\otimes V} + d_{H^1(\mathsf{J})} d_{V'} \|w\|^2_{H^1(\mathsf{J})\otimes V'} \tag{6.2.18}$$

for any $w \in \mathcal{D}(\mathcal{M})$, where $D_{V'} = d_V^{-1}$ and $d_{V'} = D_V^{-1}$. This implies that $\mathcal{M}$ can be extended by continuity to an s.p.d. operator (again denoted by) $\mathcal{M} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$.

[2] More precisely, we extend the bounded real valued bilinear form $\langle \mathcal{N}_1 \cdot, \cdot\rangle_{\mathcal{Y}_1'\times\mathcal{Y}_1}$ on $\mathcal{D}(\mathcal{N}_1) \times \mathcal{D}(\mathcal{N}_1)$ to $\mathcal{Y}_1 \times \mathcal{Y}_1$ and define the extension $\mathcal{N}_1 \in \mathcal{L}(\mathcal{Y}_1, \mathcal{Y}_1')$ as the corresponding linear continuous operator.

6.2.3 Explicit representation of the inverse

In the framework of operator preconditioning in Section 4.2 we have obtained optimal preconditioners $\mathbf{N}$ and $\mathbf{M}$ for the discrete algebraic residual minimization problem (4.2.9) from norm inducing s.p.d. operators $\mathcal{N} \in \mathcal{L}(\mathcal{X}, \mathcal{X}')$ and $\mathcal{M} \in \mathcal{L}(\mathcal{X}, \mathcal{X}')$. Recall that, given a fixed pair of finite-dimensional subspaces $\mathcal{X}_h \times \mathcal{Y}_h \subset \mathcal{X} \times \mathcal{Y}$, with bases $\Phi \subset \mathcal{X}_h$ and $\Psi \subset \mathcal{Y}_h$, these were defined as $\mathbf{M} := \langle \mathcal{M}\Phi, \Phi\rangle_{\mathcal{X}'\times\mathcal{X}}$ and $\mathbf{N} := \langle \mathcal{N}\Psi, \Psi\rangle_{\mathcal{Y}'\times\mathcal{Y}}$. However, the iterative solution of the discrete algebraic residual minimization problem as described in Section 4.3 requires (the possibly approximate action of) the inverses $\mathbf{N}^{-1}$ and $\mathbf{M}^{-1}$. This is easy for the operator $\mathcal{N}$ and $\mathcal{M}$ defined in terms of Riesz bases as done in Section 6.1 which gives rise to diagonal matrices $\mathbf{N}$ and $\mathbf{M}$. For the parabolic BPX preconditioner we can construct $\mathbf{M}^{-1}$ and $\mathbf{N}^{-1}$ from the inverses $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$ if the projectors P_k^Δ and Q_ℓ^Δ are orthogonal projectors for the respective inner products. This will be used in the numerical experiments in Section 8.3. We shall elaborate on this for the parabolic BPX operator $\mathcal{M}$ defined in (6.2.16) in more detail here. First, we explicitly write the of inverse $\mathcal{M}$.

Proposition 6.2.5. *Assume that P_k^Δ and Q_ℓ^Δ are orthogonal projectors in $L^2(\mathsf{J})$ and H, respectively, such that the multilevel norm equivalences (6.2.1) in $L^2(\mathsf{J})$, (6.2.2) in H, (6.2.3) in $H^1(\mathsf{J})$, (6.2.4) in V, and consequently also (6.2.5) in V', hold. Let $\mathcal{M} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ be defined by (6.2.16). Define $\mathcal{M}_+$ and $\mathcal{M}_-$ on $\mathcal{D}(\mathcal{M}) := \bigcup_{k,\ell\in\mathbb{N}} E_k \otimes V_\ell$ by*

$$\mathcal{M}_\pm = \sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}} \left(q_\ell^2 + p_k^2 q_\ell^{-2}\right)^{\pm 1} (P_k^\Delta \otimes Q_\ell^\Delta). \tag{6.2.19}$$

Then $\mathcal{M}_+ = \mathcal{M}$ and $\mathcal{M}_- = \mathcal{M}^{-1}$ on $\mathcal{D}(\mathcal{M})$.

Proof. It is clear that the inverse of $\mathcal{M}_+$ on $\mathcal{D}(\mathcal{M})$ is given by $\mathcal{M}_-$. Hence, only the identity $\mathcal{M}_+ = \mathcal{M}$ is to be shown. Let us abbreviate $m_{k\ell} := q_\ell^2 + p_k^2 q_\ell^{-2}$. For any $w, \widetilde{w} \in \mathcal{D}(\mathcal{M})$ we have

$$\begin{aligned}
\langle \mathcal{M}_+ w, \widetilde{w}\rangle_{\mathcal{X}'\times\mathcal{X}} &= \sum_{k',\ell'\in\mathbb{N}} \langle \mathcal{M}_+ w, (P_{k'}^\Delta \otimes Q_{\ell'}^\Delta)\widetilde{w}\rangle_{\mathcal{X}'\times\mathcal{X}} && (6.2.20)\\
&= \sum_{k,\ell,k',\ell'\in\mathbb{N}} m_{k,\ell}\langle (P_k^\Delta \otimes Q_\ell^\Delta)w, (P_{k'}^\Delta \otimes Q_{\ell'}^\Delta)\widetilde{w}\rangle_{L^2(\mathsf{J})\otimes H} && (6.2.21)\\
&= \sum_{k,\ell\in\mathbb{N}} m_{k,\ell}\langle (P_k^\Delta \otimes Q_\ell^\Delta)w, (P_k^\Delta \otimes Q_\ell^\Delta)\widetilde{w}\rangle_{\mathcal{X}'\times\mathcal{X}} = \langle \mathcal{M} w, \widetilde{w}\rangle_{\mathcal{X}'\times\mathcal{X}}, && (6.2.22)
\end{aligned}$$

where the mutual orthogonality property of the projectors P_k^Δ and Q_ℓ^Δ, namely

$$\langle (P_k^\Delta \otimes Q_\ell^\Delta)w, (P_{k'}^\Delta \otimes Q_{\ell'}^\Delta)\widetilde{w}\rangle_{L^2(\mathsf{J})\otimes H} = \delta_{kk'}\delta_{\ell\ell'},$$

was used. The claim follows by density of $\mathcal{D}(\mathcal{M})$ in $\mathcal{X}$. □

Let us define the matrices $\mathbf{M}_+$, $\mathbf{M}_-$ and $\mathbf{M}_0$ by

$$\mathbf{M}_\pm := \langle \mathcal{M}_\pm \Phi, \Phi\rangle_{L^2(\mathsf{J})\otimes H} \quad \text{and} \quad \mathbf{M}_0 := \langle \Phi, \Phi\rangle_{L^2(\mathsf{J})\otimes H}. \tag{6.2.23}$$

Note that $\mathbf{M} = \mathbf{M}_+$. The inverse of $\mathbf{M} = \mathbf{M}_+$ is obtained from $\mathbf{M}_-$ as follows.

Proposition 6.2.6. *The matrices $\mathbf{M}_\pm$ and $\mathbf{M}_0$ are related by $\mathbf{M}_+^{-1} = \mathbf{M}_0^{-1}\mathbf{M}_-\mathbf{M}_0^{-1}$.*

Proof. We show that $\mathbf{M}_0 = \mathbf{M}_-\mathbf{M}_0^{-1}\mathbf{M}_+$. Indeed, we have

$$\begin{aligned}
\widetilde{\mathbf{w}}^\top \mathbf{M}_0 \mathbf{w} &= \langle \mathcal{M}_+^{-1}\widetilde{w}, \mathcal{M}_+ w\rangle_{L^2(\mathsf{J})\otimes H} = \langle \mathcal{M}_-\widetilde{w}, \mathcal{M}_+ w\rangle_{L^2(\mathsf{J})\otimes H}\\
&= (\mathbf{M}_0^{-1}\mathbf{M}_-\widetilde{\mathbf{w}})^\top \mathbf{M}_0 (\mathbf{M}_0^{-1}\mathbf{M}_+\mathbf{w}) = \widetilde{\mathbf{w}}^\top \mathbf{M}_-\mathbf{M}_0^{-1}\mathbf{M}_+\mathbf{w}
\end{aligned}$$

for all $\widetilde{w} = \widetilde{\mathbf{w}}^\top\Phi$ and $w = \mathbf{w}^\top\Phi$. □

7 Temporal discretization

In the chapters 3, 4 and 5 we have discussed the operator formulation $\mathcal{B}u = \mathcal{F}$ and the minimal residual Petrov-Galerkin solution strategy of the abstract parabolic evolution equation for a stable pair $\mathcal{X}_L \times \mathcal{Y}_L \subseteq \mathcal{X} \times \mathcal{Y}$ of discrete trial and test spaces. Among others, we have obtained the quasi-optimality estimate

$$\|u - u_L\|_{\mathcal{X}} \leq C \inf_{w_L \in \mathcal{X}_L} \|u - w_L\|_{\mathcal{X}} \tag{7.0.1}$$

of the approximate solution $u_L \in \mathcal{X}_L$, see Theorem 4.1.9. Recall that $\mathcal{X} = L^2(\mathsf{J}; V) \cap H^1(\mathsf{J}; V')$, where V is a separable Hilbert space. From the quasi-optimality estimate, convergence rates for the discrete solution u_L may be obtained if the exact solution u is assumed to belong to a certain smoothness class, and the discrete trial spaces $\mathcal{X}_L$ are adapted to that class.

Assume, for instance, that $u \in H^\tau_\beta(\mathsf{J}; V) \cap L^2(\mathsf{J}; W)$ for some $0 \leq \beta < 1$ and $\tau \geq 0$, where $W \hookrightarrow V$ is a continuously embedded Hilbert space. Let $V_\ell \subseteq W$, $\ell \in \mathbb{N}_0$, be a sequence of finite-dimensional subspaces with projectors $Q_\ell : V \to V_\ell$ with the approximation property $\|\mathrm{Id} - Q_\ell\|_{\mathcal{L}(W,V)} \lesssim (\dim V_\ell)^{-\sigma}$, $\ell \in \mathbb{N}_0$, for some $\sigma \geq 0$. We can construct a sequence of finite-dimensional subspaces $E_k \subset H^1(\mathsf{J})$, e.g. based on the local wavelet refinement towards $t = 0$ as discussed in Section 7.2.3 below, with associated $L^2(\mathsf{J})$-stable projectors $P_k : L^2(\mathsf{J}) \to E_k$, such that $\|\mathrm{Id} - P_k\|_{\mathcal{L}(H^\tau_\beta(\mathsf{J}), L^2(\mathsf{J}))} \lesssim (\dim E_k)^{-\tau}$, $k \in \mathbb{N}_0$. Owing to the identity $(\mathrm{Id} \otimes \mathrm{Id}) - (P_k \otimes Q_\ell) = ((\mathrm{Id} - P_k) \otimes \mathrm{Id}) + (P_k \otimes (\mathrm{Id} - Q_\ell))$ on $L^2(\mathsf{J}) \otimes V$, and the $L^2(\mathsf{J})$-stability of the projectors P_k, the approximation estimates combine to

$$\|u - (P_k \otimes Q_\ell)u\|_{L^2(\mathsf{J};V)} \lesssim (\dim E_k)^{-\tau} \|u\|_{H^\tau_\beta(\mathsf{J};V)} + (\dim V_\ell)^{-\sigma} \|u\|_{L^2(\mathsf{J};W)} \quad \forall k, \ell \in \mathbb{N}_0. \tag{7.0.2}$$

Consider the full tensor product discrete trial space $\mathcal{X}_L := E_L \otimes V_L \subseteq \mathcal{X}$. In order to equilibrate the error let us couple $(\dim E_L)^{-\tau} \sim (\dim V_L)^{-\sigma}$ by relabeling the subspaces if necessary. Since the total dimension is $\dim \mathcal{X}_L = \dim E_L \times \dim V_L$, we obtain the approximation rate estimate

$$\|u - (P_L \otimes Q_L)u\|_{L^2(\mathsf{J};V)} \lesssim (\dim \mathcal{X}_L)^{-r} \|u\|_{H^\tau_\beta(\mathsf{J};V) \cap L^2(\mathsf{J};W)}, \quad r = \frac{1}{\frac{1}{\tau} + \frac{1}{\sigma}} \quad \forall L \in \mathbb{N}_0. \tag{7.0.3}$$

A similar estimate may be obtained for the approximation in $H^1(\mathsf{J}; V')$. A rather detailed discussion of such approximation rates for the sparse tensor product case can be found in [SS09, Section 7] and [GO07]. In [Hac81], space-time error estimates in anisotropic Sobolev spaces for some time-stepping methods were derived.

Thus, given a regularity class, suitable families of subspaces $E_k \subseteq H^1(\mathsf{J})$ are of interest. In addition, in order to use E_k in the minimal residual space-time Petrov-Galerkin framework, another family of subspaces F_k is required that renders the pairs E_k, F_k admissible in the sense of Definition 5.2.17. Having established some notation in Section 7.1, we therefore discuss in Section 7.2 several smoothness classes and suitable approximation subspaces E_k that replace the estimate $\|\mathrm{Id} - P_k\|_{\mathcal{L}(H^\tau_\beta(\mathsf{J}), L^2(\mathsf{J}))} \lesssim (\dim E_k)^{-\tau}$ in the above argument, and continue by giving examples of corresponding admissible subspaces F_k in Section 7.3.

7.1 Notation

Throughout this chapter, $\mathsf{J} := (0, \mathsf{T})$ denotes a finite temporal interval, $0 < \mathsf{T} < \infty$. A **temporal mesh** $\mathcal{T}$ is a finite set of **nodes** with $\mathcal{T} = \{0 =: t_0 < t_1 < \ldots < t_N := \mathsf{T}\} \subset [0, \mathsf{T}]$. For a temporal mesh and a

vector of polynomial degrees $\mathbf{p} = (p_1, p_2, \dots, p_N) \in \mathbb{N}_0^N$ we introduce the spline spaces

$$S^{r,\mathbf{p}}(\mathcal{T}) := \{f \in H^r(\mathsf{J}) : f|_{(t_{n-1},t_n)} \in \mathbb{P}^{p_n}, n = 1, \dots, N\} \tag{7.1.1}$$

of global Sobolev smoothness $r \in \mathbb{N}_0$, where $\mathbb{P}^p$ denotes polynomials (real-valued, of one real variable) of degree $p \in \mathbb{N}_0$. If $\mathbf{p} = (p, p, \dots, p)$ then we may write $S^{r,p}$ in place of $S^{r,\mathbf{p}}$. For $k \in \mathbb{N}_0$, the addition $\mathbf{p} + k$ is defined component-wise. For a Banach space X, the space $S^{r,\mathbf{p}}(\mathcal{T}; X)$ of X-valued splines is defined analogously.

Given any temporal mesh $\mathcal{T} = \{t_n\}_{n=0}^N$ we set $\mathcal{T}_{\star 0} := \mathcal{T}$, and for $j \in \mathbb{N}$ define $\mathcal{T}_{\star j}$ as the **uniform subdivision** of $\mathcal{T}_{\star(j-1)}$, i.e., $\#\mathcal{T}_{\star j} = 2\#\mathcal{T}_{\star(j-1)} - 1$ and each $t \in \mathcal{T}_{\star j}$ is either in $\mathcal{T}_{\star(j-1)}$, or is in between two neighboring nodes in $\mathcal{T}_{\star(j-1)}$. Given $\xi \geq 1$, we define the **algebraically graded version** of $\mathcal{T}$ with **grading exponent** ξ by $\mathcal{T}^\xi := \{t_n^\xi \mathsf{T}^{1-\xi}\}_{n=0}^N$.

7.2 Approximation of functions in countably normed spaces

In Section 3.3 we have seen that solutions to parabolic evolution equations tend to be smooth but have singular support $\{t = 0\}$. We have quantified this type of regularity in time as membership to (countably normed) weighted Bochner spaces. Here we briefly discuss how it can be exploited in numerical approximations, first using graded and geometric meshes, then using local wavelet refinement.

7.2.1 Graded temporal meshes

Graded meshes may be used to capture a singularities in the setting of h-refinement for functions in $H^\ell_{\beta,k}$ (see Section 2.8 for the definition of those spaces), recovering the optimal convergence rates of piecewise polynomial approximation. Let us first recall the following well-known result.

Theorem 7.2.1. *Let X be a separable Hilbert space. Then there exists $C > 0$ such that for any integers $\ell, p \geq 1$ with $p \geq \ell - 1$, for any bounded interval $(a, b) \subset \mathbb{R}$ and for any $f \in H^\ell((a, b); X)$ there exists $q \in \mathbb{P}^p$ satisfying $q(a) = f(a)$ and $q(b) = f(b)$ and*

$$\|f - q\|_{H^m((a,b);X)} \leq C\,(b-a)^{\ell-m}\,|f|_{H^\ell((a,b);X)}, \quad m = 0, 1. \tag{7.2.1}$$

The following theorem is known, but seems to be hard to find in the literature in the present form (cf. e.g. [Sch99, Proposition 3.7]), and we therefore give the proof. In the statement of the theorem, $S^{m,p}$ may be replaced by $S^{1,p}$.

Theorem 7.2.2. *Let $\mathcal{T} \subset [0, \mathsf{T}]$ be an equidistant temporal mesh, let X be a separable Hilbert space. Let further*

- $0 \leq \beta < 1$,
- $m \in \{0, 1\}$, $\ell \in \mathbb{N}$, *with* $\ell \geq m + 1$, *and* $k \in \mathbb{N}_0$,
- $p \in \mathbb{N}$ *with* $p \geq k + \ell - 1$,
- $\xi \geq \frac{k+\ell-m}{\ell-m-\beta}$.

Then there exists $C > 0$ such that for any $f \in H^\ell_{\beta,k}(\mathsf{J}; X)$ there holds the approximation estimate

$$\inf_{g \in S^{m,p}(\mathcal{T}_j;X)} \|f - g\|_{H^m(\mathsf{J};X)} \leq C(\#\mathcal{T}_j)^{-(k+\ell-m)}\|f\|_{H^\ell_{\beta,k}(\mathsf{J};X)} \quad \forall j \in \mathbb{N}_0, \tag{7.2.2}$$

where $\mathcal{T}_j := [\mathcal{T}_{\star j}]^\xi$ is the graded version with grading exponent ξ of the j-th uniform refinement of $\mathcal{T}$.

Proof. It suffices to give the proof for $X = \mathbb{R}$ (otherwise the proof applies to the coefficients w.r.t. an orthonormal basis of X). The proof is based on the three facts:

1. $|f|^2_{H^{k+\ell}(a,b)} \le a^{-2(k+\beta)}|f|^2_{H^\ell_{\beta,k}(a,b)}$ for any $f \in H^\ell_{\beta,k}(a,b)$ and $0 < a < b < \mathsf{T}$.

2. Given the interval $(0,s)$, $s > 0$, there exists $C > 0$ such that for any $f \in H^\ell_{\beta,m}(0,s)$ there exists $q \in \mathbb{P}^m$ with $\|f-q\|^2_{H^m(0,s)} \le Cs^{2(\ell-m-\beta)}|f|^2_{H^\ell_{\beta,0}(0,s)}$, and, in addition, $q(s) = f(s)$ if $m = 1$. The inequality can be proven using [Sch98, (4.3.11)].

3. Setting $h(t) := t^\xi \mathsf{T}^{1-\xi}$, The function $t \mapsto h(t)^{\sigma-\theta}h'(t)^\theta = \mathsf{T}^{(1-\xi)\sigma}\xi^\theta t^{\sigma\xi-\theta}$ is continuous and therefore bounded on $[0,\mathsf{T}]$ for any $\sigma,\theta \ge 0$ with $\sigma\xi \ge \theta$.

Let $j \in \mathbb{N}_0$ be fixed. Write $\mathcal{T}_{\star j} = \{t_n : n = 0,\dots,N\}$ where $0 \le t_{n-1} < t_n \le \mathsf{T}$, $n = 1,\dots,N$. Then $\mathcal{T}_j = h(\mathcal{T}_{\star j})$ for $h(t) := t^\xi \mathsf{T}^{1-\xi}$, and the elements of $\mathcal{T}_j$ are $s_n := h(t_n)$, $n = 0,\dots,N$. By Theorem 7.2.1, fact 1. and fact 2., there exists $g \in S^{m,p}(\mathcal{T}_j;X)$ subject to the condition $g(s_n) = f(s_n)$ for all $s_n \in \mathcal{T}_j$ if $m = 1$, such that the error $\|f-g\|^2_{H^m(s_0,s_N)}$ can be estimated interval-wise by

$$\|f-g\|^2_{H^m(s_0,s_1)} \le Cs_1^{2(\ell-m-\beta)}|f|^2_{H^\ell_{\beta,0}(s_0,s_1)} \tag{7.2.3}$$

and

$$\|f-g\|^2_{H^m(s_{n-1},s_n)} \le Cs_{n-1}^{-2(k+\beta)}(s_n-s_{n-1})^{2(k+\ell-m)}|f|^2_{H^\ell_{\beta,k}(s_{n-1},s_n)}, \quad n = 2,\dots,N. \tag{7.2.4}$$

The error on the first interval is further estimated using $s_1 = t_1^\xi \mathsf{T}^{1-\xi} \le CN^{-\xi}$ and the hypothesis on ξ. The collective error on the remaining intervals is estimated using $s_n - s_{n-1} = N^{-1}\mathsf{T}h'(\widetilde{t}_n)$ for some $\widetilde{t}_n \in [t_{n-1},t_n]$, monotonicity of h', and n,j-uniform boundedness of the ratio s_n/s_{n-1}, by the n-independent bound

$$s_{n-1}^{-2(k+\beta)}(s_n-s_{n-1})^{2(k+\ell-m)} \le CN^{-\theta}h(t_n)^{\sigma-\theta}h'(t_n)^\theta \le CN^{-\theta}\max_{t\in[0,\mathsf{T}]} h(t)^{\sigma-\theta}h'(t)^\theta, \tag{7.2.5}$$

where $\theta := 2(k+\ell-m)$ and $\sigma := -2(k+\beta)+\theta$. By hypothesis, we have $\xi \ge \theta/\sigma$. These estimates and assertion 3. yield the claim. □

7.2.2 Analyticity away from zero: geometric meshes

Using *hp*-refinement, exponential convergence can be obtained for a piecewise polynomial approximation of functions in the countably normed spaces B^ℓ_β (see Section 2.8 for definitions).

Theorem 7.2.3. *Let $\ell \in \{1,2\}$, $0 \le \beta < 1$ and $f \in B^\ell_\beta(\mathsf{J};X)$, where X is a separable Hilbert space. Let $\mathcal{T}_N$, $N \in \mathbb{N}$, be a sequence of temporal meshes given by $\mathcal{T}_N = \{0\} \cup \{\sigma^{N-n}\mathsf{T}\}_{n=1}^N$, where $0 < \sigma < 1$ is arbitrary. Then there exist constants $C > 0$, $\alpha \ge 0$ and $b > 0$ such that*

$$\inf_{g\in S^{\ell,\mathbf{p}}(\mathcal{T}_N)} \|f-g\|_{H^{\ell-1}(\mathsf{J};X)} \le Ce^{-bN} \quad \forall N \in \mathbb{N}_0 \tag{7.2.6}$$

for any degree vector $\mathbf{p} \in \mathbb{N}_0^N$ satisfying the slope conditions

$$\mathbf{p}_1 \ge \ell-1 \quad \text{and} \quad \mathbf{p}_n \ge \max\{\ell,\alpha n\}, \quad n = 2,\dots,N. \tag{7.2.7}$$

If the components of $\mathbf{p}$ are minimal subject to this restriction then $N = \mathcal{O}(\sqrt{\dim S^{\ell,\mathbf{p}}(\mathcal{T}_N)})$ as $N \to \infty$.

Proof. The statement of the claim is a minor modification of [Sch98, Theorem 3.36]. □

7.2.3 Local wavelet refinement

In [Nit04; Nit05; DS10], local wavelet refinement was used to resolve singularities of functions in the weighted Sobolev spaces $H^\ell_\beta(0,1)$, see definitions in Section 2.8 (similar ideas appeared in e.g. [PS96; BN99]). This type of refinement may be applied to functions $u \in H^\ell_\beta(\mathsf{J};X)$ as well. This is of interest for the space-time discretizations and preconditioning discussed in Section 5.2.3 and Chapter 6: the requirements on the temporal wavelet basis made in Section 6.1 on wavelet preconditioning are not as stringent as that on the spatial wavelet basis; therefore, it is reasonable to combine the wavelet preconditioning in time with the multilevel subspace decomposition in space as discussed in Section 6.2, i.e., the projectors P_k in Section 6.2 may be defined in terms of wavelet Riesz bases; this leads to *block-diagonal* preconditioners $\mathbf{M}$ and $\mathbf{N}$, where each block is of the size of a spatial problem.

On the bounded interval $\mathsf{J} = (0,\mathsf{T})$ assume a dyadic wavelet collection $\Theta := \{\theta_\lambda : \lambda \in \mathcal{I}^\Theta\} \subset H^{\bar s}(\mathsf{J})$ such that

1. Θ is a Riesz basis for $L^2(\mathsf{J})$,
2. $\{2^{-|\lambda|}\theta_\lambda : \lambda \in \mathcal{I}^\Theta\}$ is a Riesz basis for $H^1(\mathsf{J})$, where $|\lambda| \in \mathbb{N}_0$ denotes the level of λ or θ_λ.

Let $\mathbb{N} \ni \bar s > 1$. For the dual basis $\widetilde\Theta := \{\widetilde\theta_\lambda : \lambda \in \mathcal{I}^\Theta\}$ of Θ (i.e., $\langle \widetilde\theta_\lambda, \theta_{\lambda'}\rangle_{L^2(\mathsf{J})} = \delta_{\lambda\lambda'}$) we assume for all $|\lambda| > 0$, and for some $\widetilde\omega_\lambda \subset \overline{\mathsf{J}}$ that contain the support of $\widetilde\theta_\lambda$, that

1. $|\langle \widetilde\theta_\lambda, f\rangle_{L^2(\widetilde\omega_\lambda)}| \lesssim 2^{-|\lambda|\bar s}\|f\|_{H^{\bar s}(\widetilde\omega_\lambda)}$ for all $f \in H^{\bar s}(\widetilde\omega_\lambda)$,
2. $\operatorname{diam}\widetilde\omega_\lambda \sim 2^{-|\lambda|}$,
3. $\sup_{j,k\in\mathbb{N}_0} \#\{|\lambda| = j : [k2^{-j},(k+1)2^{-j}] \cap \widetilde\omega_\lambda \neq \emptyset\} < \infty$.

Orthogonal an biorthogonal wavelets of this type can be found in, among others, [CDV93; DKU99]. For $f \in L^2(\mathsf{J})$ and $\lambda \in \mathcal{I}^\Theta$ we write $f_\lambda := \langle \widetilde\theta_\lambda, f\rangle_{L^2(\mathsf{J})}$. For any $0 \le \eta < 1$ and $L \in \mathbb{N}_0$ define the index sets

$$\mathcal{I}^\Theta_{L,\eta} := \{\lambda \in \mathcal{I}^\Theta : (1-\eta)|\lambda| \le L \quad \text{and} \quad \widetilde\omega_\lambda \cap (0, 2^{\eta^{-1}(L-|\lambda|)}\mathsf{T}) \neq \emptyset\} \tag{7.2.8}$$

and the projector $P^\Theta_{L,\eta}$ on $L^2(\mathsf{J})$ by $P^\Theta_{L,\eta} : f \mapsto \sum_{\lambda\in\mathcal{I}^\Theta_{L,\eta}} f_\lambda\theta_\lambda$. We will refer to η as the **wavelet grading factor**. The wavelets involved in $P^\Theta_{L,\eta}$ that are close to the left boundary have support size on the order of $2^{-|\lambda|} \sim 2^{-L/(1-\eta)}$. This is of the same order as the local mesh width of an algebraically graded version of a uniform mesh with 2^L nodes with grading exponent $\xi = 1/(1-\eta)$. Moreover, for any fixed $\eta \in [0,1)$, the estimate $\#\mathcal{I}^\Theta_{L,\eta} \sim 2^L$ holds for $L \in \mathbb{N}_0$. The projectors $P^\Theta_{L,\eta}$ resolve the singularities of functions $H^{\bar s}_\beta(\mathsf{J})$ while maintaining the same order of the number of degrees of freedom that is necessary for non-singular functions in $H^{\bar s}(\mathsf{J})$.

Theorem 7.2.4 (Adapted from Proposition 2.5 and Theorem 2.6 in [DS10])**.** *Let $s \in \{0,1\}$ and $\beta \in [0,\bar s - s)$, and take $\eta \in [\beta/(\bar s - s), 1)$ with $\eta \ge 1 - \frac{1}{2(\bar s - s)}$. Then*

$$\|f - P^\Theta_{L,\eta} f\|_{H^s(\mathsf{J})} \lesssim (\#I^\Theta_{L,\eta})^{-r}\|f\|_{H^{\bar s}_\beta(\mathsf{J})}, \quad r := \bar s - s \quad \forall f \in H^{\bar s}_\beta(\mathsf{J}) \quad \forall L \in \mathbb{N}_0. \tag{7.2.9}$$

The approximation estimate (7.2.9) remains valid for functions with values in a separable Hilbert space X if we replace $H^s(\mathsf{J})$ by $H^s(\mathsf{J};X)$, etc.

Assuming weighted norm equivalences in weighted Sobolev spaces, such as were shown in [BSS04], the approximation inequality (7.2.9) follows if we merely require $\eta \in [\beta/(\bar s - s), 1)$. To see that, define for $s \in [0,\bar s]$, where now $\bar s \ge 1$, the expression

$$|||f|||^2_{H^s_\beta(\mathsf{J})} := \sum_{j\in\mathbb{N}_0} 2^{2sj} \sum_{|\lambda|=j} (\widetilde t_\lambda)^{2\beta}|f_\lambda|^2, \tag{7.2.10}$$

where $\widetilde t_\lambda$ is an inner point of $\widetilde\omega_\lambda$ with $\widetilde t_\lambda \gtrsim \operatorname{diam}\widetilde\omega_\lambda \sim 2^{-|\lambda|}$ uniformly in $\lambda \in \mathcal{I}^\Theta$.

Proposition 7.2.5. *Assume the weighted norm equivalences*

1. $\|f\|_{H^s(\mathsf{J})} \lesssim |\!|\!| f |\!|\!|_{H^s_0(\mathsf{J})} < \infty$ *for all* $f \in L^2(\mathsf{J})$ *with* $|\!|\!| f |\!|\!|_{H^s_0(\mathsf{J})} < \infty$, *where* $s = 0, 1$,
2. $|\!|\!| f |\!|\!|_{H^s_\beta(\mathsf{J})} \lesssim \|f\|_{H^s_\beta(\mathsf{J})}$ *for all* $f \in H^s_\beta(\mathsf{J})$ *and* $s \in [0, \bar{s}]$.

Let $s \in \{0, 1\}$. *Then the approximation inequality* (7.2.9) *holds for each fixed* $\eta \in [\beta/(\bar{s} - s), 1)$.

Proof. Let $s \in \{0, 1\}$ and $\eta \in [\beta/(\bar{s} - s), 1)$. In the proof, all implied constants are independent of the essential parameters L, λ and f. We start with the estimate

$$|\!|\!| f - P^\Theta_{L,\eta} f |\!|\!|^2_{H^s_0(\mathsf{J})} \le \underbrace{\sum_{j \ge L} 2^{2sj} \sum_{|\lambda|=j} |f_\lambda|^2 1_{(\widetilde{t}_\lambda)^\eta > 2^{L-j}\mathsf{T}^\eta}}_{T_1} + \underbrace{\sum_{j \ge L/(1-\eta)} 2^{2sj} \sum_{|\lambda|=j} |f_\lambda|^2}_{T_2} \tag{7.2.11}$$

and show $T_1, T_2 \lesssim 2^{-2(\bar{s}-s)L} |\!|\!| f |\!|\!|^2_{H^{\bar{s}}_\beta(\mathsf{J})}$ for T_1 and T_2 separately:

i) The implication $t^\eta > 2^{L-j}\mathsf{T}^\eta \Rightarrow 1 \lesssim 2^{2(j-L)(\bar{s}-s)} t^{2\beta}$ that holds for any $t \in \mathsf{J}$, $j \in \mathbb{N}_0$, shows

$$T_1 \lesssim \sum_{j \ge L} 2^{2sj} 2^{2(j-L)(\bar{s}-s)} \sum_{|\lambda|=j} (\widetilde{t}_\lambda)^{2\beta} |f_\lambda|^2.$$

Cancelling 2^{2sj} results in $T_1 \lesssim 2^{-2(\bar{s}-s)L} |\!|\!| f |\!|\!|^2_{H^{\bar{s}}_\beta(\mathsf{J})}$.

ii) The choice of $\widetilde{t}_\lambda$ is such that $1 \lesssim 2^{2|\lambda|\beta} (\widetilde{t}_\lambda)^{2\beta}$ holds. Thus, T_2 is bounded by

$$T_2 \lesssim \sum_{j \ge L/(1-\eta)} \underbrace{2^{-2L(\bar{s}-(s+\beta))/(1-\eta)} 2^{2j(\bar{s}-(s+\beta))}}_{1 \le} 2^{2js} 2^{2j\beta} \sum_{|\lambda|=j} (\widetilde{t}_\lambda)^{2\beta} |f_\lambda|^2.$$

Using $(\bar{s} - (s + \beta))/(1 - \eta) \ge (\bar{s} - s)$ and cancelling $2^{2j(s+\beta)}$ shows $T_2 \lesssim 2^{-2(\bar{s}-s)L} |\!|\!| f |\!|\!|^2_{H^{\bar{s}}_\beta(\mathsf{J})}$.

Since $\#\mathcal{I}^\Theta_{L,\eta} \sim 2^L$, the norm equivalences assumed in the hypothesis yield the claim. □

Example 7.2.6. We estimate numerically the rate $r \ge 0$ in the approximation inequality (7.2.9) for $f_\alpha(t) := t^\alpha$ on $\mathsf{J} = (0, \mathsf{T}) = (0, 2)$ for some fractional positive exponents α. Note that $f_\alpha \in H^{\bar{s}}_\beta(\mathsf{J})$ if and only if the exponent satisfies $\alpha > \bar{s} - \beta - 1/2$. For the wavelet collection Θ we adapt the piecewise linear B-spline wavelets, shown in [HS06] to form Riesz bases for a scale of Sobolev spaces $H^s(\mathbb{R})$, to the interval J by anti-symmetric reflection at the boundary. The index set $\mathcal{I}^\Theta$ is given by

$$\mathcal{I}^\Theta = \{(0, 0), (1, 0), (2, 0)\} \cup \bigcup_{j \in \mathbb{N}} \{(k, j) : k \text{ is an odd integer with } 1 \le k \le 2^{j+1}\}. \tag{7.2.12}$$

For each $\lambda = (k, j) \in \mathcal{I}^\Theta$, the function $\theta_\lambda \in H^1(\mathsf{J})$ is the piecewise affine function w.r.t. the mesh $\mathcal{T}_j = \{t^j_k := k2^{-(j+1)}\mathsf{T}\}_{k=0}^{2^{j+1}}$ which for $j \ge 1$ attains the values

$$(\theta_\lambda(t^j_{k-1}), \theta_\lambda(t^j_k), \theta_\lambda(t^j_{k+1})) = \begin{cases} (0, -\frac{3}{2}, \frac{1}{2}) & \text{if} \quad t^j_{k-1} = 0, \\ (\frac{1}{2}, -\frac{3}{2}, 0) & \text{if} \quad t^j_{k+1} = \mathsf{T}, \\ (\frac{1}{2}, -1, \frac{1}{2}) & \text{else,} \end{cases} \tag{7.2.13}$$

for each inner node $t^j_k \in \mathcal{T}_j$ and zero at all other nodes. For $\lambda = (k, 0) \in \mathcal{I}^\Theta$, the function θ_λ is the nodal interpolant with $\theta_{(k,0)}(t^0_{k'}) = \delta_{kk'}$. While it may be possible to obtain dual wavelets with compact support [Lem97], we follow [Nit04, Section 2.2] and use the simpler definition (than (7.2.8)) for the locally refined index sets,

$$\mathcal{I}^\Theta_{L,\eta} := \{(k, j) \in \mathcal{I}^\Theta : (k2^{-j})^\eta \le 2^{L-j}\} \tag{7.2.14}$$

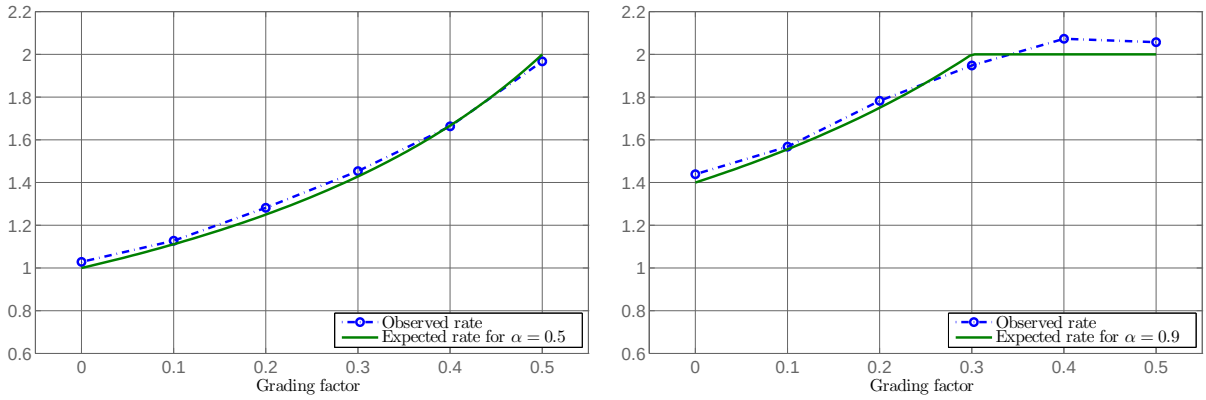

Figure 7.2.1: Observed and expected (computed as $r = \min\{2, (\alpha + 1/2)/(1 - \eta)\}$) approximation rates of local wavelet refinement at $t = 0$ as a function of the wavelet grading factor η for $f_\alpha(t) = t^\alpha$ with $\alpha = 0.5$ (**left**) and $\alpha = 0.9$ (**right**), see Example 7.2.6.

with the associated biorthogonal projector $P^{\Theta}_{L,\eta}$, where $0 \le \eta < 1$ and $L \in \mathbb{N}_0$. In Figure 7.2.1 we plot the observed approximation rates r in $L^2(\mathsf{J})$ (i.e., in (7.2.9) we have $s = 0$) for wavelet grading factors $\eta = 0, 0.1, \dots, 0.5$ and $f_\alpha(t) = t^\alpha$ with $\alpha = 0.5, 0.9$. We compute the expected rate as follows: *one* of the requirements of Theorem (7.2.4) is $\eta \ge \beta/\bar{s}$, let us therefore constrain $\beta = \eta\bar{s}$; the maximal smoothness $\bar{s}$ is then limited by $\alpha > \bar{s} - \beta - 1/2 \ge \bar{s}(1 - \eta) - 1/2$ which gives $\bar{s} \le (\alpha + 1/2)/(1 - \eta)$; since we use piecewise linear trial functions, the expected rate r is bounded by $r \le \max\{2, (\alpha + 1/2)/(1 - \eta)\}$, which is indeed very close to the observed behavior.

7.3 Admissible temporal subspaces

This section is devoted to the construction of examples of temporal subspaces $E_k \subseteq H^1(\mathsf{J})$ and $F_k \subseteq L^2(\mathsf{J})$ that are admissible in the sense of Definition 5.2.17. According to Theorem 5.2.18, such subspaces give rise to stable (sparse) space-time pairs of subspaces $\mathcal{X}_L \times \mathcal{Y}_L \subset \mathcal{X} \times \mathcal{Y}$. To check admissibility of temporal subspaces $E_k \subseteq H^1(\mathsf{J})$ and $F_k \subseteq L^2(\mathsf{J})$, $k \in \mathbb{N}_0$, in the sense of Definition 5.2.17, we need to find $\tau > 0$ such that

$$\mathcal{K}(E_k + \partial_t E_k, F_k) \ge \tau > 0 \quad \forall k \in \mathbb{N}_0, \tag{7.3.1}$$

where we abbreviate $\partial_t E := \{\partial_t e : e \in E\}$, and

$$\mathcal{K}(E, F) := \mathcal{K}_{L^2(\mathsf{J}) \times L^2(\mathsf{J})}(E, F) := \inf_{e \in E \setminus \{0\}} \sup_{f \in F \setminus \{0\}} \frac{\langle e, f \rangle_{L^2(\mathsf{J})}}{\|e\|_{L^2(\mathsf{J})} \|f\|_{L^2(\mathsf{J})}}. \tag{7.3.2}$$

Recall also from Remark 5.2.19 that in order to obtain $\mathcal{K}(E + \partial_t E, F) \ge \tau$, it suffices to check

$$\mathcal{K}(\partial_t E, F) \ge \tau \quad \text{if} \quad E \subseteq F, \qquad \text{or} \qquad \mathcal{K}(E, F) \ge \tau \quad \text{if} \quad \partial_t E \subseteq F. \tag{7.3.3}$$

7.3.1 Graded and geometric meshes

Let us recall the generic construction of admissible pairs in which $F_k = E_k + \partial_t E_k$. We will be more specific in the subsequent paragraphs.

Proposition 7.3.1. *Let $\mathcal{T}_k \subseteq \mathcal{T}_{k+1} \subset \bar{\mathsf{J}}$, $k \in \mathbb{N}_0$, be a nested sequence of temporal meshes. Let $\mathbf{p}_k \in \mathbb{N}_0^{\#\mathcal{T}_k - 1}$ be a sequence of vectors of polynomial degrees that is non-decreasing in every given component. Define the continuous spline spaces $E_k := S^{1,\mathbf{p}_k+1}(\mathcal{T}_k) \subset H^1(\mathsf{J})$ and the spaces of (discontinuous) piecewise polynomials $F_k := S^{0,\mathbf{p}_k+1}(\mathcal{T}_k) \subset L^2(\mathsf{J})$. Then the pairs E_k, F_k are admissible in the sense of Definition 5.2.17.*

Proof. The identity $F_k = E_k + \partial_t E_k$ is valid and implies $\mathcal{K}(E_k, F_k) = 1$ for each $k \in \mathbb{N}_0$. The nestedness property $E_k \subseteq E_{k+1}$ and $F_k \subseteq F_{k+1}$ is also clear by nestedness of the temporal meshes and the monotonicity property of $\mathbf{p}_k$. Thus, E_k and F_k are admissible in the sense of Definition 5.2.17. □

For this generic setup where F_k admit discontinuous functions, we can easily construct $L^2(\mathsf{J})$-orthogonal bases for F_k, namely as scaled Legendre polynomials transported to the subintervals of the temporal mesh $\mathcal{T}_k$. We point out that this may be very useful for the construction and the implementation of the norm-inducing operator $\mathcal{N}$ on the test space.

Graded meshes

Let $\mathcal{T}_0 \subset \bar{\mathsf{J}}$ be an initial temporal mesh, let $\xi \geq 1$ be a grading exponent. Define the sequence of meshes $\mathcal{T}_k$, $k \in \mathbb{N}$, by $\mathcal{T}_k := [\mathcal{T}_{\star k}]^\xi$ = the graded version with grading exponent ξ of the k-th uniform refinement of $\mathcal{T}_0$. Fix $p \in \mathbb{N}_0$, and define $\mathbf{p}_k := p$ for each $k \in \mathbb{N}_0$. Now, Proposition 7.3.1 applies in this setup.

For the lowest order case $p_k = 0$, several alternatives for the test space

i) $F_k := S^{0,1}(\mathcal{T}_k)$ as described above

are possible, e.g.

ii) the space $F_k := S^{0,0}(\mathcal{T}_{k+1})$ of piecewise constant functions on the mesh that is the graded version of the $(k+1)$-st uniform refinement of $\mathcal{T}_0$. A variant of this is $F_k := S^{0,0}([\mathcal{T}_k]_{\star 1})$, i.e., the space of piecewise constant functions on a uniformly refined version of the k-th temporal mesh. In both cases we have $\partial_t E_k \subseteq F_k$. This was essentially discussed in Section 5.2.3, **C**.

iii) the continuous piecewise linear functions $F_k := S^{1,1}([\mathcal{T}_k]_{\star 1})$ on a uniformly refined version of the k-th temporal mesh $[\mathcal{T}_k]_{\star 1}$; or similarly, the case $F_k := S^{1,1}(\mathcal{T}_{k+1})$. In both cases we have $E_k \subseteq F_k$ and $\inf_{k \in \mathbb{N}_0} \mathcal{K}(\partial_t E_k, F_k) > 0$ can be shown by adapting [And12, Proof of Proposition 6.1], which implies admissibility of E_k, F_k by (7.3.3). These will be used in the numerical experiments in Section 8.2.

Geometric meshes

Let $\mathcal{T}_k$, $k \in \mathbb{N}_0$, be a sequence of temporal meshes given by $\mathcal{T}_k = \{0\} \cup \{\sigma^{k+1-n}\mathsf{T}\}_{n=1}^{k+1}$, where $0 < \sigma < 1$. Let $\mathbf{p}_k \in \mathbb{N}_0^{\#\mathcal{T}_k - 1}$ be a sequence of polynomial degrees, non-decreasing in every given component. This setup is valid for Proposition 7.3.1 to apply. Thus, also hp-refinement towards the initial time is admissible in the construction of stable (sparse) space-time discrete trial and test subspaces. In order to apply the approximation result Theorem 7.2.3, additional slope conditions akin to (7.2.7) will be necessary.

7.3.2 Wavelet bases

Let $\Theta \subset L^2(\mathsf{J})$ be a wavelet basis as in Section 7.2.3, which we will use in the construction of the temporal subspaces E_k on the trial side. In order to be able to resolve singularities at the initial time by means of the approximation estimate (7.2.9), we define $E_k := P^{\Theta}_{k,\eta} L^2(\mathsf{J})$, where $0 \leq \eta < 1$ is a given wavelet grading factor. In other words, E_k contains the functions spanned by the wavelets Θ with active coefficients from the index set $\mathcal{I}^{\Theta}_{k,\eta}$ (defined in (7.2.8)), which includes indices of wavelets up to level k, but also of some higher levels close to the boundary $t = 0$ of J.

Assuming that Θ are piecewise polynomials, an admissible choice for F_k in the sense of Definition 5.2.17 is any nested sequence of subspaces $F_k \supseteq E_k + \partial_t E_k$ (not necessarily wavelet based) that are piecewise polynomial w.r.t. the set of singular supports $\bigcup_{\lambda \in \mathcal{I}^{\Theta}_{k,\eta}} \operatorname{sing\,supp} \theta_\lambda$ and of sufficiently high piecewise-polynomial degree. Since usually $\operatorname{sing\,supp} \theta_\lambda \lesssim 1$, the dimension of F_k is bounded by a constant multiple of the dimension of $E_k + \partial_t E_k$.

Let us briefly discuss the possibility of defining F_k in terms of another wavelet collection, $\Xi \subset L^2(\mathsf{J})$, which is assumed to have the same properties as Θ, except that the Riesz basis property in $H^1(\mathsf{J})$ is not required. Define the projector $P^{\Xi}_{k,\eta}$ analogously to $P^{\Theta}_{k,\eta}$, and set $F_k := P^{\Xi}_{k+\Delta k,\eta} L^2(\mathsf{J})$, where $\Delta k \in \mathbb{N}_0$ is a fixed number of extra levels. We are now interested in showing admissibility of the subspaces E_k, F_k in the sense of Definition 5.2.17, in particular we need to check the condition $\inf_{k\in\mathbb{N}} \mathcal{K}(E_k + \partial_t E_k, F_k) > 0$, provided Δk is chosen large enough. Here, we only mention that for Θ being the wavelet collection as in Example 7.2.6 and Ξ being the Haar wavelets on the interval J we have observed numerically that such a Δk exists for any $\eta \in [0, 1)$, and leave the validity of the statement in a general setting as a conjecture.

8 Example: semi-linear heat equation

If we don't have a clue, we can still do numerics.
a workshop speaker

Well, this looks like a nice cartoon...
same workshop, next speaker

The chapter discusses one particular application of the minimal residual Petrov-Galerkin framework to the solution of semi-linear parabolic evolution equations (further examples may be found in [And12; AT12]). The model equation is introduced in Section 8.1. Several core statements of the thesis are verified in Section 8.2: we measure mesh-independent contraction factors in the fixed point iteration for both, the full tensor, and the sparse tensor product space-time discrete trial and test spaces. Moreover, optimal convergence rates in terms of the total number of degrees of freedom are observed. In Section 8.3, the stability of the discrete trial and test spaces and the optimality of the parabolic BPX preconditioner is confirmed numerically.

8.1 Problem setup

Let $D \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary. We use the standard notation for the Sobolev spaces $L^p(D)$, $H^1_0(D)$ for functions in $H^1(D)$ with vanishing trace on ∂D, and $H^{-1}(D)$ for the dual of $H^1_0(D)$. The norm on $H^1_0(D)$ is the $H^1(D)$ semi-norm. Let further $\mathsf{J} = (0, \mathsf{T})$, $\mathsf{T} > 0$, be a bounded temporal interval.

Consider the semi-linear parabolic partial differential equation

$$\partial_t u - \operatorname{div}(q \operatorname{grad} u) + \alpha u^m = g \quad \text{in} \quad \mathsf{J} \times D, \tag{8.1.1}$$

$$u = u^0 \quad \text{in} \quad \{0\} \times D, \tag{8.1.2}$$

$$u = 0 \quad \text{in} \quad \mathsf{J} \times \partial D, \tag{8.1.3}$$

where $m \in \mathbb{N}$, $\alpha \geq 0$, the source g satisfies $g \in L^2(\mathsf{J}; H^{-1}(D))$, and $u^0 \in L^2(D)$ is the initial datum. We assume that the exponent $m \in \mathbb{N}$ is such that the embedding $H^1_0(D) \hookrightarrow L^{m+1}(D)$ is valid. This is guaranteed by the Sobolev embedding theorem e.g. for any finite $m \in \mathbb{N}$ in $d = 1$ or $d = 2$ space dimensions and for $1 \leq m \leq 5$ in $d = 3$ space dimensions. Another example of a (non-smooth) non-linearity, that is of interest in obstacle problems, would be $\max\{0, u - \varphi\}$ for a non-negative measurable function $\varphi : \mathsf{J} \times D \to \mathbb{R}$.

For the space- and time-dependent diffusion coefficient q we assume $q \in L^\infty(\mathsf{J} \times D)$. This type of low regularity is not admissible in most standard numerical methods for parabolic evolution equations. In order to apply the abstract framework of Section 3.2.1, set $V := H^1_0(D)$ and $H := L^2(D)$, and recall the definition of the space-time trial and test spaces $\mathcal{X}$ and $\mathcal{Y}$, which now read

$$\mathcal{X} = L^2(\mathsf{J}; H^1_0(D)) \cap H^1(\mathsf{J}; H^{-1}(D)) \quad \text{and} \quad \mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 = L^2(\mathsf{J}; H^1_0(D)) \times L^2(D). \tag{8.1.4}$$

We define the family of operators $a(t; \cdot, \cdot) : H^1_0(D) \times H^1_0(D) \to \mathbb{R}$ for (a.e.) $t \in \mathsf{J}$ by

$$a(t; \chi, \widetilde{\chi}) := \langle q(t, \cdot) \operatorname{grad} \chi, \operatorname{grad} \widetilde{\chi} \rangle_{L^2(D)}, \quad \chi, \widetilde{\chi} \in H^1_0(D). \tag{8.1.5}$$

For the well-posedness of the parabolic equation we further need that the diffusion coefficient is strictly positive,

$$\operatorname*{ess\,inf}_{(t,x)\in \mathsf{J}\times D} q(t,x) > 0. \tag{8.1.6}$$

Lemma 8.1.1. *Assuming strict positivity (8.1.6) of the diffusion coefficient $q \in L^\infty(\mathsf{J}\times D)$, the family of operators $a(t;\cdot,\cdot)$, $t\in\mathsf{J}$, defined in 8.1.5, satisfies Assumption 3.2.4.*

Proof. We need to show measurability of the mapping $t \mapsto \langle q(t,\cdot)\operatorname{grad}\chi, \operatorname{grad}\widetilde{\chi}\rangle_{L^2(D)}$ for arbitrary fixed $\chi,\widetilde{\chi}\in H^1_0(D)$. This follows from Corollary 2.7.10: for $\chi,\widetilde{\chi}\in H^1_0(D)$, the function $\varphi := \operatorname{grad}\chi\cdot\operatorname{grad}\widetilde{\chi}$ is in $L^1(D)$, hence the mapping $t\mapsto \int_D q(t,x)\varphi(x)dx$ is measurable. Boundedness and Gårding inequality with $a_{\text{shift}} = 0$ are then clear. □

Remark 8.1.2 (See Example 1.42 in [Rou05], Example 5.0.10 in [Fat99])**.** Note that $q\in L^\infty(\mathsf{J}\times D)$ does not imply $q\in L^\infty(\mathsf{J};L^\infty(D))$. Indeed, consider $q(t,x) := 2+\operatorname{sign}(x-t)$ on $\mathsf{J}\times D = (0,1)\times(0,1)$. Then $q\in L^\infty(\mathsf{J}\times D)$, but the mapping $t\mapsto q(t,\cdot)$ has a non-separable range in $L^\infty(D)$. Thus (see Section 2.2), the mapping $t\mapsto q(t,\cdot)$ is not Bochner measurable with values in $L^\infty(D)$. A direct proof of Lemma 8.1.1 without resorting to the general statement Theorem 2.7.9 may be found in [Fat05, Proof of Lemma 4.4.1].

Define now the operator $\mathcal{B}\in\mathcal{L}(\mathcal{X},\mathcal{Y}')$ as in (3.2.15) by

$$\langle \mathcal{B}w, v\rangle_{\mathcal{Y}'\times\mathcal{Y}} := \int_{\mathsf{J}} \langle (\partial_t + A(t))w(t), v_1(t)\rangle dt + \langle w(0), v_2\rangle \quad \forall (w,v)\in\mathcal{X}\times\mathcal{Y}, \tag{8.1.7}$$

the load functional $\mathcal{F}\in\mathcal{Y}'$ as in (3.2.17) by

$$\mathcal{F}v := \langle u^0, v_2\rangle + \int_{\mathsf{J}} \langle g(t), v_1(t)\rangle dt \quad \forall v = (v_1,v_2)\in\mathcal{Y}, \tag{8.1.8}$$

and the non-linearity $\mathcal{G} : u\mapsto \alpha u^m$. We thus aim at approximately solving the semi-linear problem (8.1.1)–(8.1.3) in the variational form

$$\text{find}\quad u\in\mathcal{X}: \quad \langle \mathcal{B}u + \mathcal{G}(u), v\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \mathcal{F}v \quad \forall v\in\mathcal{Y}. \tag{8.1.9}$$

The minimal residual discretization method of Section 4.5 for abstract semi-linear equations becomes applicable once we have shown that the mapping $\mathcal{G}:\mathcal{X}\to\mathcal{Y}'$ is locally Lipschitz continuous, i.e., Lipschitz continuous on bounded sets. This is the subject of the following lemma (which uses arguments from [HS11, Lemma 2.1]). See [RS96, Section 5.3.2] for a more general study of the mapping $u\mapsto u^m$ in the context of Nemytskij operators.

Lemma 8.1.3. *Let $m\in\mathbb{N}$ be such that the embedding $H^1_0(D)\hookrightarrow L^{m+1}(D)$ holds. Let $\alpha>0$. Then the mapping $\mathcal{G}: w\mapsto \alpha w^m$ is locally Lipschitz continuous as a mapping $\mathcal{G}:\mathcal{Y}\to\mathcal{Y}'$.*

Proof. Take $u,w,v\in\mathcal{Y}$ arbitrary. Then

$$\begin{aligned}
|\langle \mathcal{G}(u)-\mathcal{G}(w), v\rangle_{\mathcal{Y}'\times\mathcal{Y}}| &= \alpha\left|\langle u^m - w^m, v\rangle_{L^2(\mathsf{J};L^2(D))}\right| && (8.1.10)\\
&\le \alpha\sum_{k=0}^{m-1}\left|\langle (u-w)v, u^{m-1-k}w^k\rangle_{L^2(\mathsf{J};L^2(D))}\right| && (8.1.11)\\
&\le \alpha C\|(u-w)v\|_{L^2(\mathsf{J};L^{(m+1)/2}(D))} && (8.1.12)\\
&\le \alpha C\|u-w\|_{L^2(\mathsf{J};L^{m+1}(D))}\|v\|_{L^2(\mathsf{J};L^{m+1}(D))} && (8.1.13)
\end{aligned}$$

where

$$C = \sum_{k=0}^{m-1}\|u^{m-1-k}w^k\|_{L^2(\mathsf{J};L^{(m+1)/(m-1)}(D))} \le \sum_{k=0}^{m-1}\|u\|^{m-1-k}_{L^2(\mathsf{J};L^{m+1}(D))}\|w\|^k_{L^2(\mathsf{J};L^{m+1}(D))}. \tag{8.1.14}$$

Given that $H^1_0(D)\hookrightarrow L^{m+1}(D)$, this shows local Lipschitz continuity of $\mathcal{G}$ as claimed. □

8.2 Convergence rates

For a smooth problem, we compare the approximation rates of the sparse space-time tensor product discrete trial and test spaces against the full space-time tensor product ones (see Remark 5.2.15). For a discussion of the advantages of the space-time sparse tensor product spaces see [GO07]. Further, we empirically verify the mesh-independence of the contraction factor of the fixed point iteration discussed in Section 4.5.

To completely define the problem (8.1.1)–(8.1.3), we set

- $D = (-1, 1)$ for the spatial domain, $\mathsf{J} = (0, \mathsf{T}) = (0, 2)$ for the temporal domain,
- $m = 3$ for the exponent of the non-linearity, and $\alpha = 10$,
- $q = 1$ for the diffusion coefficient,
- $g(t, x) = \sin(\pi t/2)^2 \cos(\cos(\pi t/2) + x)$ for the source term, see Figure 8.2.1
- $u^0 = 0$ for the initial datum.

The source term g and the corresponding solution u are shown in Figure 8.2.1.

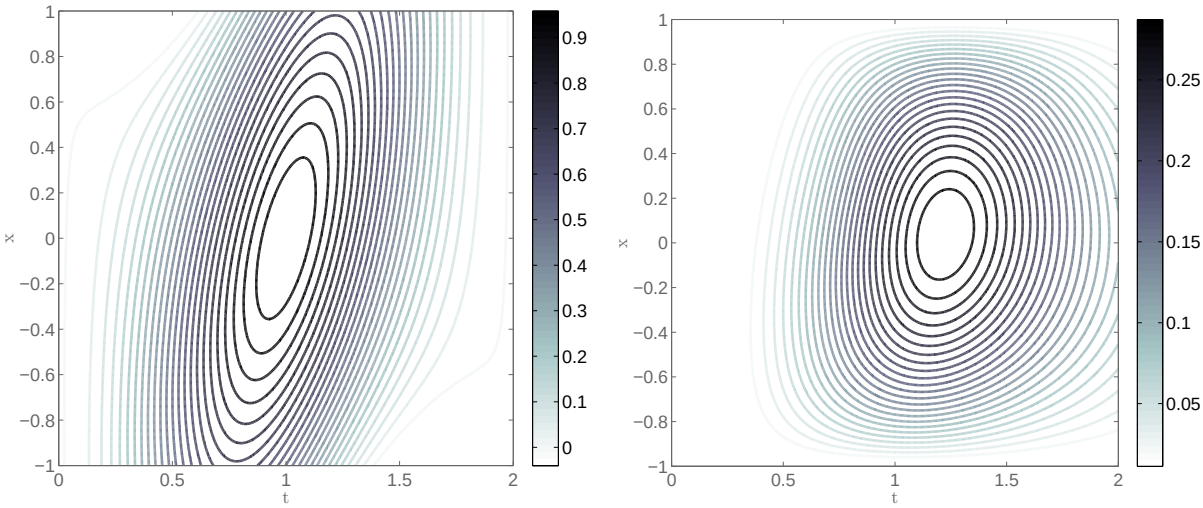

Figure 8.2.1: The source term g (**left**) and the solution u (**right**) for the setup of Section 8.2.

The subspaces $V_\ell \subset H^1_0(D)$, $\ell \in \mathbb{N}_0$, consist of continuous piecewise linear functions on an equidistant mesh with $2^{\ell+1} - 1$ inner nodes. Similarly, on the trial side, the subspaces $E_k \subset H^1(\mathsf{J})$, $k \in \mathbb{N}_0$, contain the continuous piecewise linear functions on an equidistant mesh with $2^{k+1} + 1$ nodes (counting the boundary nodes). On the test side, the subspaces $F_k \subset L^2(\mathsf{J})$ are defined by $F_k := E_{k+1}$. The nestedness properties $V_\ell \subseteq V_{\ell+1}$, $E_k \subseteq E_{k+1}$, and $E_k \subseteq F_k \subseteq F_{k+1}$ are obvious.

Let $Q_\ell : L^2(D) \to V_\ell$ and $R_\ell : H^1_0(D) \to V_\ell$ denote the $L^2(D)$-orthogonal and $H^1_0(D)$-orthogonal surjective projectors. We recall (e.g. from [BS08, Theorem 4.4.20 and Theorem 4.5.11] and quasi-optimality of R_ℓ) the direct estimate

$$\exists C_{\mathrm{dir}} > 0 : \quad \|\chi - R_\ell \chi\|_{L^2(D)} \le C_{\mathrm{dir}} 2^{-\ell} \|\chi\|_{H^1_0(D)} \quad \forall \chi \in H^1_0(D) \quad \forall \ell \in \mathbb{N}_0, \tag{8.2.1}$$

and the inverse estimate

$$\exists C_{\mathrm{inv}} > 0 : \quad \|\chi_\ell\|_{H^1_0(D)} \le C_{\mathrm{inv}} 2^{\ell} \|\chi_\ell\|_{L^2(D)} \quad \forall \chi_\ell \in V_\ell \quad \forall \ell \in \mathbb{N}_0. \tag{8.2.2}$$

These estimates imply stability of the $L^2(D)$-orthogonal projector Q_ℓ in $H^1_0(D)$ uniformly in $\ell \in \mathbb{N}_0$:

Lemma 8.2.1. *There exists $C > 0$ such that $\|Q_\ell \chi\|_{H^1_0(D)} \le C \|\chi\|_{H^1_0(D)}$ for all $\ell \in \mathbb{N}_0$, $\chi \in H^1_0(D)$.*

Proof. The direct (8.2.1) and the inverse (8.2.2) estimates imply for any $\chi \in H_0^1(D)$

$$\|Q_\ell(\chi - R_\ell\chi)\|_{H_0^1(D)} \le C_{\mathrm{inv}} 2^\ell \|Q_\ell(\chi - R_\ell\chi)\|_{L^2(D)} \le C_{\mathrm{inv}} 2^\ell \|\chi - R_\ell\chi\|_{L^2(D)} \le C_{\mathrm{inv}} C_{\mathrm{dir}} \|\chi\|_{H_0^1(D)},$$

and therefore,

$$\|Q_\ell\chi\|_{H_0^1(D)} \le \|Q_\ell R_\ell\chi\|_{H_0^1(D)} + \|Q_\ell(\chi - R_\ell\chi)\|_{H_0^1(D)} \le (1 + C_{\mathrm{inv}} C_{\mathrm{dir}})\|\chi\|_{H_0^1(D)},$$

which is the claim for $C := 1 + C_{\mathrm{inv}} C_{\mathrm{dir}}$. □

The following proposition shows suitability of V_ℓ and E_k for the space-time Petrov-Galerkin discretization. In particular, the subspaces E_k and F_k are admissible in the sense of Definition 5.2.17, cf. Remark 5.2.19.

Proposition 8.2.2. *There exist $\eta > 0$ and $\tau > 0$ for which*

$$\mathcal{K}_{V' \times V}(V_\ell, V_\ell) = \inf_{\widetilde{\chi} \in V_\ell \setminus \{0\}} \sup_{\chi \in V_\ell \setminus \{0\}} \frac{\langle \widetilde{\chi}, \chi \rangle}{\|\widetilde{\chi}\|_{V'} \|\chi\|_V} \ge \eta > 0 \quad \forall \ell \in \mathbb{N}_0, \tag{8.2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $V' \times V$, and

$$\mathcal{K}_{L^2(\mathsf{J}) \times L^2(\mathsf{J})}(\partial_t E_k, F_k) = \inf_{e' \in \partial_t E_k \setminus \{0\}} \sup_{f \in F_k \setminus \{0\}} \frac{\langle e', f \rangle_{L^2(\mathsf{J})}}{\|e'\|_{L^2(\mathsf{J})} \|f\|_{L^2(\mathsf{J})}} \ge \tau > 0 \quad \forall k \in \mathbb{N}_0, \tag{8.2.4}$$

hold.

Proof. The first claim (8.2.3) follows from Lemma 8.2.1 and Proposition 4.4.8. The second claim (8.2.4) may be shown explicitly, see [And12, Proposition 6.2]. □

For each $L \in \mathbb{N}_0$, the **space-time full tensor product**

$$\mathcal{X}_L := E_L \otimes V_L \subset \mathcal{X} \quad \text{and} \quad \mathcal{Y}_L := [F_L \otimes V_L] \times V_L \subset \mathcal{Y}, \tag{8.2.5}$$

and the **space-time sparse tensor product**

$$\widehat{\mathcal{X}_L} := \bigcup_{0 \le k+\ell \le L} E_k \otimes V_\ell \subset \mathcal{X} \quad \text{and} \quad \widehat{\mathcal{Y}_L} := \bigcup_{0 \le k+\ell \le L} [F_k \otimes V_\ell] \times V_\ell \subset \mathcal{Y}, \tag{8.2.6}$$

trial and test spaces are now defined.

Proposition 8.2.3. *There exists $\gamma > 0$ such that*

$$\gamma_{\mathcal{B}}(\mathcal{X}_L, \mathcal{Y}_L) \ge \gamma > 0 \quad \textit{and} \quad \gamma_{\mathcal{B}}(\widehat{\mathcal{X}_L}, \widehat{\mathcal{Y}_L}) \ge \gamma > 0 \quad \forall L \in \mathbb{N}_0. \tag{8.2.7}$$

Proof. The claim follows form Proposition 8.2.2 and Theorem 5.2.18. □

For the norm-inducing operators $\mathcal{M}$ and $\mathcal{N}$ we use the Riesz basis strategy described in Section 6.1 with the piecewise linear wavelets given in Example 7.2.6 in the temporal, as well as in the spatial direction. Concerning the operator $\mathcal{M}$, we point out that these wavelets cannot be rescaled to a Riesz basis of $H^{-1}(D)$; however, in low dimension and on reasonably low refinement levels, as is the case here, this does not significantly affect the computations. We use this operator $\mathcal{M}$ to estimate the norms in $\mathcal{X}$.

For the solution of the semi-linear equation, we consider the fixed point iteration

$$\Phi_L : \mathcal{X}_L \to \mathcal{X}_L, \quad w \mapsto w_L := \operatorname*{arg\,min}_{\widetilde{w_L} \in \mathcal{X}_L} \sup_{v_L \in \mathcal{Y}_L \setminus \{0\}} \frac{\langle \mathcal{B}\widetilde{w_L} + \mathcal{G}(w) - \mathcal{F}, v_L \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|v_L\|_{\mathcal{N}}} \tag{8.2.8}$$

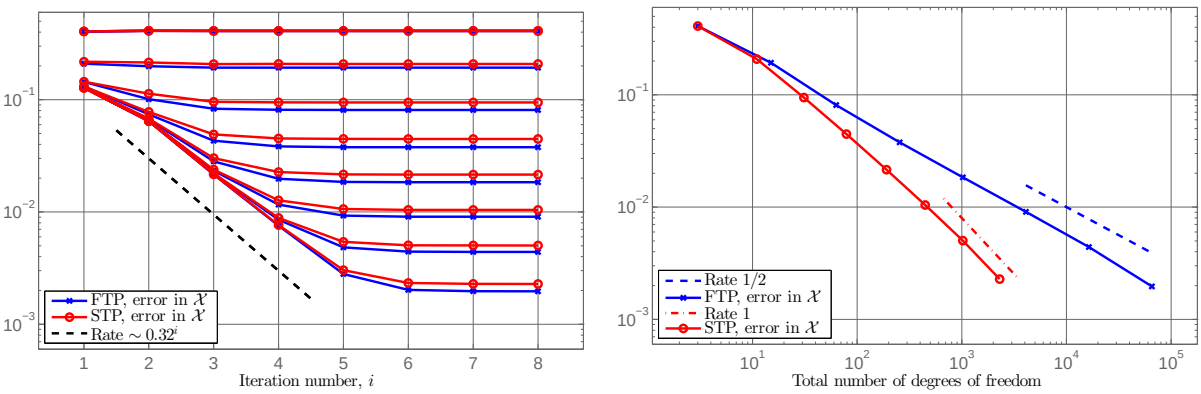

Figure 8.2.2: Estimated space-time error in $\mathcal{X}$ of the discrete fixed point iterations $[\Phi_L]^i(0)$ (FTP) and $[\widehat{\Phi}_L]^i(0)$ (STP), see Section 8.2 for definitions. In all cases, $[\widehat{\Phi}_L]^i(0)$ with $L = 8$, $i = 10$, is used as the reference solution. **Left**: error for $L = 0, 1, \ldots, 7$ (top to bottom), as a function of the iteration number i. **Right**: error for fixed $i = 10$, as a function of the total number of degrees of freedom corresponding to the discretization levels $L = 0, 1, \ldots, 7$.

and $\widehat{\Phi}_L : \mathcal{X} \to \widehat{\mathcal{X}_L}$, defined analogously with $\mathcal{X}_L \times \mathcal{Y}_L$ replaced by $\widehat{\mathcal{X}_L} \times \widehat{\mathcal{Y}_L}$. As was discussed in Section 4.5, owing to stability of the trial and test subspaces (Proposition 8.2.3), each of these mappings is strictly contractive on a ball around zero, provided the norm of $\mathcal{F}$ is sufficiently small (this seems to be the case in our example). Piecewise linear functions in the discrete trial and test space allow an exact evaluation of the non-linear term $\langle \mathcal{G}(w), v_L \rangle_{\mathcal{Y}' \times \mathcal{Y}}$ using a composite quadrature of sufficiently high order.

To estimate the errors we have used the finest solution obtained as the reference solution, namely $[\Phi_L]^i(0) \in \mathcal{X}_L$ with $L = 8$ and $i = 10$. The generalized least squares system in the preconditioned form (4.2.11) was solved in each iteration using MATLAB's function `lsqr` with relative residual tolerance of 10^{-6}. The `lsqr` solver always exited with success flag 0 within a few dozen or a few hundred steps depending on the discretization level L; this dependence on L in the number of iterations is accounted for by the fact that more iterations are required to approach possible consistency in the overdetermined equation $\mathbf{Bw} = \mathbf{f}$ for larger L, rather than by an increase of the condition number of the preconditioned system matrix.

In Figure 8.2.2 we document

- the convergence history of the fixed point iterations Φ_L and $\widehat{\Phi}_L$ for $L = 0, 1, \ldots, 7$, each with initial value zero. In all cases, a constant error reduction factor of about 10^{-1} per two iterations ($\rho \approx 0.32$ per iteration), *independently of the discretization level* L in the range where the discretization error is comparatively small, is observed.
- the convergence of the discrete solutions $[\Phi_L]^i(0) \in \mathcal{X}_L$ and $[\widehat{\Phi}_L]^i(0) \in \widehat{\mathcal{X}_L}$ with fixed $i = 10$, as the discretization level L is increased. We observe the convergence rate of *one half* in the space-time full tensor product case $\mathcal{X}_L$, and *the doubled rate of one* in the space-time sparse tensor product case $\widehat{\mathcal{X}_L}$, in terms of the total number of degrees of freedom ($\dim \mathcal{X}_L$ and $\dim \widehat{\mathcal{X}_L}$, respectively).

These observations are in complete agreement with the theory.

8.3 Parabolic BPX preconditioner

In this section we verify numerically the optimality of the parabolic BPX preconditioner discussed in Section 6.2 for the setup of the previous subsection. We only consider the full tensor product discrete trial and test spaces.

Recall from the previous subsection the definition of the nested subspaces of piecewise linear continuous splines $V_\ell \subset V_{\ell+1} \subset H_0^1(D)$ on $D = (-1,1)$ and $E_k \subset F_k := E_{k+1} \subset H^1(\mathsf{J})$ on $\mathsf{J} = (0,\mathsf{T}) = (0,2)$, and the resulting full tensor product $\mathcal{X}_L \times \mathcal{Y}_L \subset \mathcal{X} \times \mathcal{Y}$ pairs of discrete trial and test subspaces (8.2.5)–(8.2.6) for each discretization level $L \in \mathbb{N}_0$.

Let $Q_\ell : L^2(D) \to V_\ell$, $\ell \in \mathbb{N}_0$, denote the $L^2(D)$-orthogonal projector onto V_ℓ, and set $Q_\ell^\Delta := Q_\ell - Q_{\ell-1}$ with the convention $Q_{-1} \equiv 0$. Similarly, define $P_k^\Delta := P_k - P_{k-1}$, $k \in \mathbb{N}_0$, where $P_k : L^2(D) \to E_k$ is the $L^2(D)$-orthogonal projector onto E_k, and $P_{-1} \equiv 0$. By mutual orthogonality of the projectors, it is clear that the multilevel norm equivalences (6.2.1) in $L^2(\mathsf{J})$ and (6.2.2) in $H := L^2(D)$ hold. Further, in [Xu92, Proposition 8.6] it was shown that the direct and inverse estimates (8.2.1)–(8.2.2) imply the multilevel norm equivalence (6.2.4) in $V := H_0^1(D)$ for Q_ℓ^Δ with $q_\ell := 2^\ell$, and the multilevel norm equivalence (6.2.3) in $H^1(\mathsf{J})$ for P_k^Δ with $p_k := 2^k$ follows analogously. The "parabolic BPX" s.p.d. isomorphisms $\mathcal{M} \in \mathcal{L}(\mathcal{X},\mathcal{X}')$ and $\mathcal{N} \in \mathcal{L}(\mathcal{Y},\mathcal{Y}')$ are now defined as described in Section 6.2.

For each discretization level $L \in \mathbb{N}_0$, let $\Phi_L \subset \mathcal{X}_L$ and $\Psi_L \subset \mathcal{Y}_L$ denote the bases consisting of tensor products of the usual univariate hat functions. The system matrix and the parabolic BPX preconditioners are defined as

$$\mathbf{N}_L := \langle \mathcal{N}\Psi_L, \Psi_L \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \quad \mathbf{B}_L := \langle \mathcal{B}\Phi_L, \Psi_L \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \quad \mathbf{M}_L := \langle \mathcal{M}\Phi_L, \Phi_L \rangle_{\mathcal{X}' \times \mathcal{X}}. \tag{8.3.1}$$

To verify that the minimal and the maximal singular value of the preconditioned system matrix $\widetilde{\mathbf{B}}_L := \mathbf{N}_L^{-\top/2}\mathbf{B}_L\mathbf{M}_L^{-1/2}$ are of order one for all discretization levels L, we approximately compute the square roots of the extremal eigenvalues of the symmetric matrix $\widetilde{\mathbf{B}}_L^\top\widetilde{\mathbf{B}}_L = \mathbf{M}_L^{-\top/2}\mathbf{B}_L^\top\mathbf{N}_L^{-1}\mathbf{B}_L\mathbf{M}_L^{-1/2}$ for a range of values of L using the power iteration and the inverse power iteration. In the inverse power iteration, the conjugate gradient method is employed in each iteration to solve the linear system. Note that the matrix $\mathbf{B}_L$ is of dimension $N \times M$ with $N = (2^{L+1}-1)[(2^{L+2}+1)+1]$ and $M = (2^{L+1}-1)(2^{L+1}+1)$.

The computed maximal and minimal singular values of $\widetilde{\mathbf{B}}_L$ are shown in Figure 8.3.1. We observe that the maximal singular value is of order 10 with a slight (apparently preasymptotic) increase, while the minimal singular value is of order 1 for $L = 0,1,\dots,9$. This confirms that the parabolic BPX preconditioner is indeed very effective in this case. For comparison, the extremal singular values of $\widetilde{\mathbf{B}}_L$ if the test space is replaced by $\widetilde{\mathcal{Y}_L} := \mathcal{X}_L \times V_L$ are plotted, which show that the pair $\mathcal{X}_L \times \widetilde{\mathcal{Y}_L}$ is not stable uniformly in L.

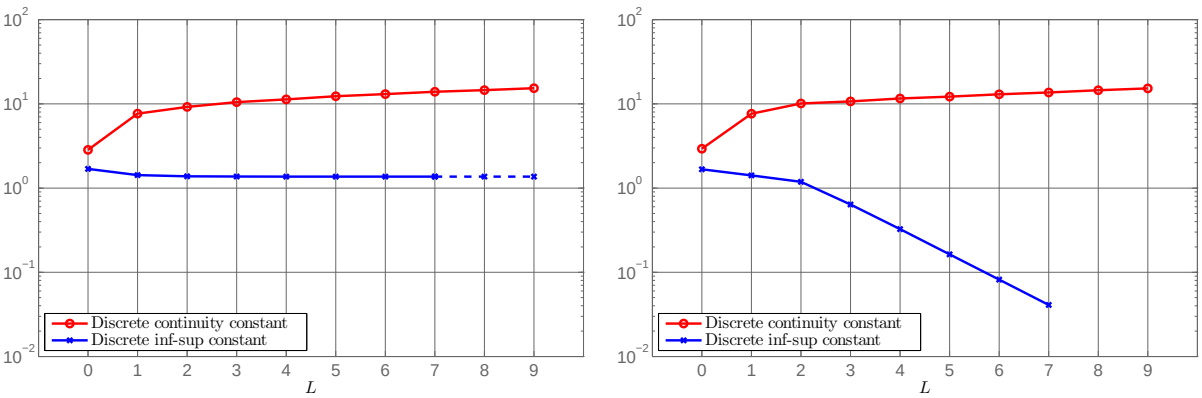

Figure 8.3.1: **Left**: max. and min. singular values of the system matrix $\mathbf{N}_L^{-\top/2}\mathbf{B}_L\mathbf{M}_L^{-1/2}$ preconditioned with the parabolic BPX preconditioner for the heat operator $(\partial_t - \Delta)$ on $D \times \mathsf{J} = (-1,1) \times (0,2)$ with hom. Dirichlet boundary conditions with $(2^{L+1}-1) \times (2^{L+1}+1)$, resp. $(2^{L+1}-1) \times [(2^{L+2}+1)+1]$, tensor products of univariate hat functions spanning the discrete trial space, resp. test space (dashed line is extrapolated data). **Right**: The same for the smaller test space of dimension $(2^{L+1}-1) \times [(2^{L+1}+1)+1]$. See Section 8.3 for details.

9 Conclusions and outlook

9.1 Conclusions

Motivated by the problem of constructing stable discrete space-time Petrov-Galerkin subspaces for a space-time variational formulation of abstract linear parabolic evolution equations in Hilbert spaces, we derived a minimal residual Petrov-Galerkin discretization framework for linear operator equations in Chapter 4. It is a generalization of the finite element method as found in standard text books on numerical analysis in that it allows discrete test spaces that are of larger dimension than that of the discrete trial spaces. This makes the discrete inf-sup condition easier to achieve for non-symmetric problems, leading to stable Petrov-Galerkin discretization schemes. Relations to "optimal test functions" [DG11] and "stability by adaptivity" [Dah+11] can be established, cf. Section 4.4.3; for parabolic problems we described in Chapters 5 and 7 ways for the a priori construction of space-time discrete test spaces that approximate the "optimal test functions" sufficiently well, and the minimal residual Petrov-Galerkin discretization framework disposes of the need to actually compute those. This leads to a stable family of discrete projectors in the natural solution space, which, among others, implies quasi-optimality of the discrete approximate solution. We have shown theoretically and verified numerically that sparse tensor product space-time trial and test spaces may be used, resulting in space-time compressive and fully parallelizable solution algorithms.

The space-time minimal residual Petrov-Galerkin discretization of parabolic evolution equations leads to a large, but possibly sparse, linear system of equations. To efficiently precondition this linear system we have developed a parabolic BPX preconditioner that is based on the BPX preconditioner well-known for its optimality for elliptic problems. The parabolic BPX preconditioner was shown theoretically (Section 6.2) and numerically (Section 8.3) to be optimal, and hence to provide a viable tool for the preconditioning of space-time Petrov-Galerkin discretizations of parabolic evolution equations, also beyond the ones discussed here.

9.2 Outlook

9.2.1 Further applications

Further possible applications of stable space-time Petrov-Galerkin discretizations of parabolic evolution equations are sketched. The detailed description is beyond the scope of this thesis and is the subject of ongoing or future work.

Diffusion on a high-dimensional cube and adaptive tensor methods

Consider the equation modeling diffusion on the cube $D = (-1,1)^d$ of moderate dimension (we think of, say, $d = 10$)

$$\partial_t u - \operatorname{div}(q \operatorname{grad} u) = g \quad \text{in} \quad \mathsf{J} \times D, \tag{9.2.1}$$

$$u = u^0 \quad \text{in} \quad \{0\} \times D, \tag{9.2.2}$$

$$u = 0 \quad \text{in} \quad \mathsf{J} \times \partial D, \tag{9.2.3}$$

where $g : \mathsf{J} \times D \to \mathbb{R}$, and $u^0 : D \to \mathbb{R}$ are given functions and $q \in \mathbb{R}^{d\times d}$ is a fixed symmetric positive definite matrix describing the material property of permeability. The diffusion equation posed in a high-dimensional domain arises naturally in several applications, e.g. financial engineering. Even for a moderate dimension d, a stable space-time minimal residual Petrov-Galerkin discretization w.r.t. tensor product trial and test subspaces similarly to Section 8.2 leads to a large linear least squares system which, however, exhibits tensor product structure. Therefore, iterative methods in an adaptive low rank tensor format are natural solution candidates for this system. This is exposed in more detail in [AT12].

Parametric parabolic evolution equations

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. Let $D \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary. Assume that $q : \mathsf{J} \times \Omega \times D \to \mathbb{R}$ is measurable (w.r.t. the natural product measure) with

$$0 < a_{\min} := \operatorname*{ess\,inf}_{\mathsf{J}\times\Omega\times D} q \leq \operatorname*{ess\,sup}_{\mathsf{J}\times\Omega\times D} q =: a_{\max} < \infty. \tag{9.2.4}$$

Set $V := L^2(\Omega, d\mathbb{P}; H^1_0(D))$. Let $a(t; \cdot, \cdot) : V \times V \to \mathbb{R}$ be given by

$$a(t; \nu, \widetilde{\nu}) = \int_\Omega \int_D q(t, \omega, \xi) \operatorname{grad} \nu(\omega, \xi) \cdot \operatorname{grad} \widetilde{\nu}(\omega, \xi) d\xi d\mathbb{P}(\omega) \tag{9.2.5}$$

for all $\nu, \widetilde{\nu} \in V$, (a.e.) $t \in \mathsf{J}$. Then Assumption 3.2.4 holds with $a_{\text{shift}} = 0$. Indeed, recognizing that $\operatorname{grad} \nu \cdot \operatorname{grad} \widetilde{\nu} \in L^1(\Omega \times D)$ for all $\nu, \widetilde{\nu} \in V$, this follows from Corollary 2.7.10 applied to the σ-finite measure space $(\Omega \times D, \Sigma \otimes \mathcal{B}(D), \mathbb{P} \otimes |\cdot|)$, where $\mathcal{B}(D)$ is the Borel σ-algebra (generated by open subsets) of D, and $\mathbb{P} \otimes |\cdot|$ is the product measure. Thus, parabolic evolution equations with parametric input data may be studied using the methods of this thesis. We interpret the parabolic evolution equation (3.1.1) corresponding to the operator defined in (9.2.5) as describing the process of heat conduction (or particle diffusion) in a physical body D, subject to a parametric or stochastic heat conduction coefficient modeling lack of knowledge of material properties. Various aspects of this model are studied in [NT09; HS10; GAS12].

Optimal control with a parabolic PDE constraint

Consider the linear-quadratic optimization problem without additional control or state constraints

$$\mathcal{J}_\epsilon(\mathbf{y}, \mathbf{u}) := \frac{1}{2}\|\mathbf{C}\mathbf{y} - \mathbf{z}^\star\|^2_{\mathbf{Z}} + \frac{\epsilon}{2}\|\mathbf{u}\|^2_{\mathbf{U}} \overset{!}{\to} \min \quad \text{s.t.} \quad \mathbf{A}\mathbf{y} = \mathbf{b} + \widetilde{\mathbf{E}}\mathbf{u} \tag{9.2.6}$$

with

- the **state variable** $\mathbf{y} \in \ell^2_{\mathbf{M}}$ with $\mathbf{M} \in \mathbb{R}^{m\times m}$ s.p.d., where $m \in \mathbb{N} \cup \{\infty\}$,
- the **control variable** $\mathbf{u} \in \ell^2_{\mathbf{U}}$ with $\mathbf{U}$ s.p.d.,
- the **target state** $\mathbf{z}^\star \in \ell^2_{\mathbf{Z}}$ and the **observable** $\mathbf{C} \in \mathcal{L}(\ell^2_{\mathbf{M}}, \ell^2_{\mathbf{Z}})$ with $\mathbf{Z}$ s.p.semi-d.,
- the (Tikhonov) **regularization parameter** $\epsilon > 0$,
- $\mathbf{A} \in \operatorname{Iso}(\ell^2_{\mathbf{M}}, (\ell^2_{\mathbf{M}})')$ and $\mathbf{b} \in (\ell^2_{\mathbf{M}})'$,
- $\widetilde{\mathbf{E}} \in \mathcal{L}(\ell^2_{\mathbf{U}}, (\ell^2_{\mathbf{M}})')$ injective.

For $\mathbf{A}$ and $\mathbf{b}$ we have in mind the symmetrized parabolic operator $\mathbf{A} = \mathbf{B}^\top \mathbf{N}^{-1}\mathbf{B}$ as discussed in Section 4.2 and correspondingly, the load vector $\mathbf{b} = \mathbf{B}^\top \mathbf{N}^{-1}\mathbf{f}$. The matrix $\mathbf{M}$ is then the Galerkin discretization of any norm inducing operator $\mathcal{M}$ as discussed in Chapter 6.

We note that the dual of $\ell^2_{\mathbf{M}}$ is $(\ell^2_{\mathbf{M}})' = \ell^2_{\mathbf{M}^{-1}}$, with the duality pairing given by the Euclidean scalar product. Because of the important role of the parameter ϵ it is convenient to work with $\epsilon\|\cdot\|^2_{\mathbf{U}}$ rather than the equivalent $\|\cdot\|^2_{\epsilon\mathbf{U}}$. If $\mathbf{u}$ is viewed as an unknown parameter, the optimization problem (9.2.6), oftentimes with $\|\mathbf{u}\|$ replaced by $\|\mathbf{u} - \mathbf{u}^{\text{prior}}\|$, is called a **parameter identification** problem.

Define the block matrices $\mathbf{Q}_\epsilon$ and $\mathbf{D}_\epsilon$ by

$$\mathbf{Q}_\epsilon := \begin{pmatrix} \mathbf{C}^\top\mathbf{Z}\mathbf{C} & \mathbf{0} & \mathbf{A}^\top \\ \mathbf{0} & \epsilon\mathbf{U} & -\widetilde{\mathbf{E}}^\top \\ \mathbf{A} & -\widetilde{\mathbf{E}} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{D}_\epsilon = \begin{pmatrix} \epsilon\mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \epsilon\mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{\epsilon}\mathbf{M} \end{pmatrix}. \tag{9.2.7}$$

Proposition 9.2.1. *There exist $0 < \sigma_0 \le \sigma_1$, monotonically dependent on the norms*

- $\|\widetilde{\mathbf{E}}\|$ *of* $\widetilde{\mathbf{E}}$ *in* $\mathcal{L}\left(\ell^2_{\mathbf{U}}, (\ell^2_{\mathbf{M}})'\right)$,
- $\|\mathbf{A}\|$ *of* $\mathbf{A}$ *in* $\mathcal{L}\left(\ell^2_{\mathbf{M}}, (\ell^2_{\mathbf{M}})'\right)$, *and* $\|\mathbf{A}^{-1}\|$ *of* $\mathbf{A}^{-1}$ *in* $\mathcal{L}\left((\ell^2_{\mathbf{M}})', \ell^2_{\mathbf{M}}\right)$,

but independent of $\epsilon > 0$, such that

$$\sigma_0\|\mathbf{s}\|^2_{\mathbf{D}_\epsilon} \le \left|\mathbf{s}^\top\mathbf{Q}_\epsilon\mathbf{s}\right| \le \sigma_1 \max\{\epsilon^{-1}\|\mathbf{C}^\top\mathbf{Z}\mathbf{C}\|, 1\}\|\mathbf{s}\|^2_{\mathbf{D}_\epsilon} \quad \forall \mathbf{s} \in \ell^2_{\mathbf{D}_\epsilon}, \tag{9.2.8}$$

where $\|\mathbf{C}^\top\mathbf{Z}\mathbf{C}\|$ is the norm of $\mathbf{C}^\top\mathbf{Z}\mathbf{C}$ in $\mathcal{L}\left(\ell^2_{\mathbf{M}}, (\ell^2_{\mathbf{M}})'\right)$.

This proposition can be proved along the same lines as [NM10, Proof Theorem 4.4]. There, the ϵ^{-1} term is removed from (9.2.8) at the expense of a preconditioner $\mathbf{D}_\epsilon$ that is more difficult to invert by adding $\mathbf{C}^\top\mathbf{Z}\mathbf{C}$ to the left-upper block of $\mathbf{D}_\epsilon$. The following standard proposition characterizes the unique solution to the linear-quadratic optimization problem (9.2.6).

Proposition 9.2.2. *Let $\epsilon > 0$. Let $\mathbf{s} = (\mathbf{y}, \mathbf{u}, \widetilde{\mathbf{p}}) \in \ell^2_{\mathbf{D}_\epsilon}$. Equivalent are*

1. *The triple $(\mathbf{y}, \mathbf{u}, \widetilde{\mathbf{p}})$ is a stationary point of the Lagrangian*

$$\mathcal{L}_\epsilon(\mathbf{y}, \mathbf{u}, \widetilde{\mathbf{p}}) := \mathcal{J}_\epsilon(\mathbf{y}, \mathbf{u}) + \widetilde{\mathbf{p}}^\top\left(\mathbf{A}\mathbf{y} - \mathbf{b} - \widetilde{\mathbf{E}}\mathbf{u}\right). \tag{9.2.9}$$

2. *There hold the first order optimality conditions*

$$\underbrace{\mathbf{A}^\top\widetilde{\mathbf{p}} = -\mathbf{C}^\top\mathbf{Z}(\mathbf{C}\mathbf{y} - \mathbf{z}^\star)}_{\text{costate/adjoint equation}}, \quad \underbrace{\widetilde{\mathbf{E}}^\top\widetilde{\mathbf{p}} = \epsilon\mathbf{U}\mathbf{u}}_{\text{design equation}} \quad \text{and} \quad \underbrace{\mathbf{A}\mathbf{y} = \mathbf{b} + \widetilde{\mathbf{E}}\mathbf{u}}_{\text{primal system}}. \tag{9.2.10}$$

3. *The pair $(\mathbf{y}, \mathbf{u})$ is a minimizer of the problem (9.2.6), while $\widetilde{\mathbf{p}}$ satisfies the above costate equation.*
4. *It holds $\mathbf{Q}_\epsilon\mathbf{s} = (\mathbf{C}^\top\mathbf{Z}\mathbf{z}^\star, \mathbf{0}, \mathbf{b})$.*
5. *It holds $\widehat{\mathbf{Q}_\epsilon}\mathbf{u} = \widehat{\mathbf{s}}$ where*

$$\begin{aligned} \widehat{\mathbf{Q}_\epsilon} &:= \widetilde{\mathbf{E}}^\top\mathbf{A}^{-\top}\mathbf{C}^\top\mathbf{Z}\mathbf{C}\mathbf{A}^{-1}\widetilde{\mathbf{E}} + \epsilon\mathbf{U} \quad \text{s.p.d.} \\ \text{and} \quad \widehat{\mathbf{s}} &:= \widetilde{\mathbf{E}}^\top\mathbf{A}^{-\top}\mathbf{C}^\top\mathbf{Z}\left(\mathbf{z}^\star - \mathbf{C}\mathbf{A}^{-1}\mathbf{b}\right), \end{aligned}$$

while $\mathbf{y}$ and $\widetilde{\mathbf{p}}$ are given by the primal system and the costate equation, respectively.

In [GK11], the adaptive wavelet method was shown to apply to the equation $\widehat{\mathbf{Q}_\epsilon}\mathbf{u} = \text{RHS}$ for the case that the constraint is a parabolic evolution equation using the results of [SS09]. As an alternative, the bound (9.2.8) implies that the MinRes method [SS11] with the preconditioner $\mathbf{D}_\epsilon$ is naturally suited for an "all-at-once" iterative solution of the optimality equation $\mathbf{Q}_\epsilon\mathbf{s} = \text{RHS}$, where $\mathbf{A} = \mathbf{B}^\top\mathbf{N}^{-1}\mathbf{B}$ is the "symmetrized" discretized parabolic operator (see Section 4.2) and $\mathbf{M}$ is e.g. the parabolic BPX preconditioner (see Section 6.2.2).

9.2.2 Some open questions

A challenge of major importance is the construction and the analysis of optimal space-time adaptive Petrov-Galerkin algorithms for parabolic evolution equations that are not based on the adaptive wavelet method.

Further, stable space-time discretizations of parabolic evolution equations in mixed or saddle point form are relevant for the computation of viscous fluid flow.

Finally, it would be interesting to find conditions on the discretization parameters under which the discrete maximum principle holds for the stable space-time Petrov-Galerkin discretizations, e.g. for the sparse and the full tensor product subspaces described in Section 8.2.

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