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BOUNDARY INTEGRAL EXTERIOR CALCULUS

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presented by

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Abstract. In this thesis, we ultimately develop first-kind boundary integral equations for boundary value problems involving the Hodge–Dirac and Hodge–Laplace operators associated with the de Rham Hilbert complex on compact Riemannian manifolds and in Euclidean space. We show that first-kind boundary integral operators associated with these boundary value problems posed on submanifolds with Lipschitz boundaries are Hodge–Dirac and Hodge–Laplace operators as well, but associated with trace de Rham complexes on the boundary whose spaces are equipped with non-local inner products defined through boundary potentials. The correspondence is to some extent structure-preserving in the sense that adding zero-order terms to these operators lead to the addition of zero-order terms in the trace de Rham complexes at the level of boundary integral operators. We put forth Boundary Integral Exterior Calculus (BIEC), a calculus of boundary potentials that significatively ease the derivation of boundary integral equations for (possibly perturbed) Hodge–Dirac, Hodge–Yukawa and possibly other boundary value problems. The ability to appeal to the powerful theory of Hilbert complexes greatly simplifies their analysis. This paves the way for the developement of Boundary Element Exterior Calculus (BEEC), where Galerkin discretizations of variational boundary integral equations could be studied in the language of differential forms.

Résumé. Dans cette thèse, nous développerons ultimement des équations intégrales de frontière du premier type pour des problèmes aux limites qui impliquent les opérateurs Hodge–Dirac et Hodge–Laplace associés à un complexe d'Hilbert (de Rham) sur des variétés riemanniennes compactes et dans l'espace euclidien. Nous montrerons que les opérateurs intégrals du premier type associés à ces problèmes limites posés sur des sous-variétés lipschitziennes sont aussi des opérateurs Hodge–Dirac et Hodge–Laplace, mais associés à des complexes de traces (de Rham) sur la frontière dont les espaces sont équipés de produits intérieurs non-locaux définis à l'aide de potentiels de frontière. La correspondance préserve dans une certaine mesure la structure, dans ce sens qu'ajouter un terme d'ordre zéro à ces opérateurs mène à l'ajout d'un terme d'ordre zéro dans le complexe de trace au niveau des opérateurs intégraux. On propose Boundary Integral Exterior Calculus (BIEC), un calculus de potentiels de frontière qui facilite significativement la dérivation d'équations intégrales de frontières pour des problèmes aux limites impliquant l'opérateur de Hodge–Dirac (possiblement perturbé), de Hodge–Yukawa et autres. La capacité de faire appel à la puissante théorie des complexes de Hilbert simplifie considérablement leur analyse. Cela ouvre la voie au développement de Boundary Element Exterior Calculus (BEEC), où les discrétisations de Galerkin des équations intégrales variationnelles de frontière pourraient être étudiées dans le langage des formes différentielles.

To Daria

"If you are receptive and humble, mathematics will lead you by the hand."

— Paul Dirac

Preface

This quotation from Paul Dirac is quite *à-propos*. When I started working with Ralf Hiptmair in 2018, our objective was broadly speaking to study boundary integral equations for non-standard differential operators. In the most general context, our distant goal was to characterize those formally self-adjoint linear operators acting on distributions that admit an (hopefully explicit) fundamental solution. The Hodge–Laplacian, the related Hodge–Helmholtz and Hodge–Yukawa operators, the (possibly perturbed) Hodge–Dirac operator and Friedrich operators, are all special cases belonging to that interesting class of problems. At the time, I imagined that our focus would be technical in nature, probably we would prove a number of stability results before establishing the convergence of boundary element methods for a given type of problems. This was an inaccurate forecast. I was naively underestimating the unpredictability of mathematical research.

In 2020, we had a new vision for the thesis after we made a surprising discovery: first-kind boundary integral operators for Hodge–Dirac operators arising from the de Rham complex are Hodge–Dirac operators themselves in trace de Rham complexes. It is sometimes difficult to draw a line between what can justifiably be called a *discovery* versus an *observation*. What best describes the creative drive behind the most exciting developments of this thesis is a combination of meticulousness, perceptiveness and luck. On that account, at least two things are worth mentioning. First, there are various ways in which boundary integral equations for the Hodge–Dirac operator can be written, especially in the fairly unfriendly setting of classical vector calculus used in Chapter 3, where we initially noticed the correspondence. Without intuition and awareness, we might not have uncovered that particular connection between the studied boundary value problems and their associated boundary integral equations. Our work on the Hodge–Dirac operator would then have been somewhat of a missed opportunity. Secondly, we dared to believe that this hinted at a deeper structure that was worth investigating. Thus, we embarked on a study of boundary integral equations for Hodge–Laplace and Hodge–Dirac operators on manifolds, which led to the boundary integral exterior calculus described in the last chapter bearing the same title as this thesis.

I believe that readers will agree that the few selected articles assembled in this thesis together make for a sound whole. It is divided in three parts. To each chapter corresponds a different article. The articles are ordered chronologically. In my opinion, this structure is optimal to convey how the research I've carried at ETH followed a coherent storyline. It is always easier in hindsight to look back at previous results and find where things could be improved or simplified. For instance, Chapter 5 generalizes and to some extent simplifies a large part of Chapter 3. The proof of the main result in Chapter 1 would also have been simpler using the theory developed at the end of the thesis—at least simpler to read. But until recently, we didn't necessarily have the tools to see it clearly. It's one thing to derive boundary integral equations, be it in the language of exterior calculus, it's another to develop a mature theory to better understand where they come from and how to work with them efficiently. So, in Part I, we solely work within the framework of classical vector analysis. Literature concerned with boundary integral equations in Euclidean space based on the later has a long history. It is rich, well-established and provided important results which helped us move forward. In particular, unbounded domains could be considered. While the presentation is technically heavy, our community is familiar with the operators involved. Hence, it may appeal to a wider audience. In Part II, we study traces for abstract Hilbert complexes and leverage the language of exterior calculus. The articles in the second part of this thesis can be regarded as the culmination of our recent research and (with Chapter 3) certainly the most original. In Appendix A, we introduce a boundary element Galerkin discretization for the first-kind boundary integral equations associated with Hodge–Dirac boundary value problems in three-dimensional Euclidean space. We briefly discuss stability and convergence before performing a simple numerical experiment to empirically confirm a theoretical result of Chapter 3.

Needless to say, Chapter 5 is interesting in its own right and might find theoretical applications at the continuous level. Be that as it may, it is noteworthy that I had first proposed the title *Boundary Element Exterior Calculus* (BEEC), a title which also conveys well the philosophy behind our investigations. I believe that we can create a BEEC that would be to Finite Element Exterior Calculus (FEEC) what the boundary element method (BEM) is to the finite element method (FEM). This thesis shows that such a program is possible using Boundary Integral Exterior Calculus (BIEC) as foundation at the continuous level. In Appendix A, even though we do not exploit exterior calculus, the ease with which we use the Hilbert complex framework to cover stability and convergence demonstrates the potential usefulness of this perspective to study related boudary integral equations in general. There, we rely on a discrete Poincaré inequality previously established by Ralf Hiptmair and Xavier Claeys for piecewise linear tangential surface vector fields with continuous tangential components across interelement edges [6], but the proof can be generalized without difficulty to discrete differential forms. To develop bounded projections from the spaces of the trace de Rham complex to conforming finite element spaces that commute with the exterior derivative (or finding a systematic way to avoid the need of using them in establishing improved convergence estimates *à la* FEEC) is an interesting open problem that may lead to fruitful research in the future.

Chapter 1. The contents of Chapter 1 were my first scientific contributions at ETH. Before writing this paper, I knew next to nothing about boundary integral equations. Therefore, the project served the dual purpose of contributing to ongoing research and at the same time learning about classical theory of boundary integral equations in Euclidean space before moving on to manifolds. Ralf Hiptmair suggested that we study transmission problems for Hodge–Helmholtz operators in late 2018. He had recently developed, together with Xavier Claeys, boundary integral equations for Hodge–Helmholtz boundary value problems posed in Lipschitz subdomains of three-dimensional Euclidean space [5]. My task was to couple the mixed variational formulation of the interior boundary value problem with the exterior scattering problem using the Dirichlet-to-Neumann map provided by the new boundary integral operators. The project proved very technical, partly due to the insufficient level of abstraction provided by the framework of vector analysis. We could already foresee the benefits of introducing exterior calculus to simplify the equations.

Chapter 2. It is well-known that the operators associated with common coupled domain-boundary variational formulations of acoustic and electromagnetic transmission problems are singular at certain frequencies. It was reasonable to expect that the issue would also arise for the coupled system of equations developed in Chapter 1. Since the matter belonged somewhat outside the scope of that chapter, our investigation of the topic eventually deserved its own paper. In Chapter 2, the main challenge is that working at an abstract level is needed to account for the peculiarities of the new operator: (1) a mixed formulation is used for the interior problem and (2) the boundary data lives in product spaces. By abstracting some of the main (functional analytic) features of first-kind boundary integral operators, the kernels of a *class* of operators associated with symmetrically coupled domain-boundary variational problems can be characterized explicitly in terms of the spectra of interior "Dirichlet" operators and "Neumann" traces of their eigenspaces. We conclude that the phenomenon is rooted in the formal structure of Calderon's identities. The desired result concerning the operator from Chapter 1 is obtained as a special case. A couple of results from classical theory are generalized. The research conducted in Chapter 2 is another example where mathematics has guided me by the hand. The difficulties that arise from the complicated structure of the coupled domain-boundary system of equations for the Hodge–Helmholtz operator were fortuitous in two ways. First, along with analysing the issue successfully for the transmission problem studied in Chapter 1, we found an approachable proof that covers at once a few coupled systems of interest in acoustics and electromagnetism. Secondly, it is by following the blueprint detailed in Chapter 2 that the boundary integral equations of Chapter 3 and Chapter 5 are established. In a way, the projects behind Chapter 1 and Chapter 2 gave us the right tools to tackle what comes in the rest of the thesis.

Chapter 3. Scientifically speaking, the article in this chapter is very well in continuity with the work of Xavier Claeys and Ralf Hiptmair on the Hodge–Helmholtz operator [5], but it is a turning point *mathematically*. Up to that point, the study of the Hodge–Helmholtz operator by our research group had been largely motivated by boundary value problems in electromagnetism. The Hodge–Helmholtz operator arise when imposing the Lorenz gauge to the potential formulation of Maxwell's equations in the frequency domain. In the limit as the frequency tends to zero, classical second-order electric and magnetic formulations break down [7,9], because the **curl curl** operator has an infinite dimensional kernel. Contrastingly, the low frequency limit of the Hodge–Helmholtz operator is the Hodge–Laplacian, which merely has a finite dimensional kernel whose dimension depends on the topology of the domain. The hope is then to find some clever regularization strategy for Hodge–Laplace boundary value problems that could lead to new ways of overcoming low frequency instabilities in electromagnetic simulation. This is a promising research direction, especially on "almost trivial" topologies where we face a one-dimensional kernel made of constants, such as on Lipschitz multi-screens which have relatively recently been a popular area of research [1,3,4]. Studying the Hodge–Dirac operator can be motivated in a similar way. The operator appears under a change of variables in the works of M. Taskinen, S. Vänskä and P. Ylä-Oijala [12–14] as Rainer Picard's extended Maxwell operator. It was originally assembled by Rainer Picard by combining the first-order Maxwell operator with the principal part of the equations of linear acoustics [10,11].

Roughly speaking, Chapter 3 harnesses the same tips and tricks as in [5]. Thanks to the previous chapters, I had a deep understanding of the new theory on first-kind boundary integral equations for the Hodge–Helmholtz operator and I could repeat its key developments for the Hodge–Dirac operator. Initially, my work had the same intentions—it simply concerned a different model. In this context, what makes Chapter 3 special is a surprising mathematical discovery: first-kind boundary integral operators for the Hodge–Dirac operator were Hodge–Dirac operators in trace de Rham complexes where the trace spaces are equipped with non-local inner products defined through boundary potentials. We found ourselves at a crossroads. The Hodge–Dirac operator was not to be seen exclusively as a candidate for the stable simulation of electromagnetic phenomena anymore, but as a possible highway to a deeper understanding of the structure of first-kind boundary integral operators associated with what came to call "Hodge–X" operators in general (Hodge–Dirac, Hodge–Laplace, Hodge–Yukawa, Hodge–Helmholtz, etc.).

The intuition that such a connection could be unveiled between the "Dirichlet" and "Neumann" boundary value problems involving these operators and the associated boundary integral equations was in the air. In light of our discovery, it was now clear that one of the first-kind boundary integral operator associated with the Hodge–Laplacian in [5] was indeed an Hodge–Laplacian in mixed-order form in a trace de Rham complex. However, the other integral operator featured a curious term involving unit normal vector-fields at the boundary that was difficult to identify.

Chapter 4. Our work with Dirk Pauly on traces for Hilbert complexes slightly breaks apart from the central theme of this thesis. The other chapters mainly concern the formulation and analysis of boundary integral equations. In Chapter 4, we focus on the trace complexes themselves. Before we would utilize the discovery made in Chapter 3, we wanted to better understand how surface operators spawn Fredholm Hilbert complexes. A mature literature is available concerning traces on the boundary of three-dimensional Lipschitz subdomains of Euclidean space. Notably, tangential traces are analyzed in the important work of A. Buffa, M. Costabel and D. Sheen [2], later generalized to differential forms by N. Weck [15]. D. Mitrea, M. Mitrea and M.-C. Shaw also published a comprehensive analysis of traces on Lipschitz subdomains of compact Riemannian manifolds in which properties of the trace de Rham complex are studied [8]. By adopting a new notion of trace operators for abstract Hilbert complexes, our contribution to the subject was to show that many of the properties established in [2], [8] and [15] for the trace de Rham complex are rooted in the general structure of Hilbert complexes. The trace spaces are introduced as annihilators/quotient spaces. The quotient space point of view is particularly relevant to boundary integral equations. By doing away with the concept of function space on the boundary, this alternative framework paves the way for the definition of traces on more complicated sets than Lipschitz boundaries. In that regard, it recently proved successful for the de Rham complex in [4], where boundary integral operators are defined on multi-screens. Beyond shedding new light on the origin of the duality between the classical traces, the main results of Chapter 4 are (1) that so-called stable "regular" decompositions are sufficient to generalize the classical trace theorems and (2) that if the lifting operators involved in those decompositions are compact, then the associated trace Hilbert complexes are Fredholm (they satisfy the compactness property). Evidently, since in the abstract setting there is no boundary, the analysis is detached from regularity considerations. Nevertheless, it is particularly interesting that the theory in Chapter 4 is built from the ground up using relatively elementary results of functional analysis, making it accessible to a very wide audience. In the future, more traces could be studied within this framework by applying the theory to other Hilbert complexes, such as to the elasticity complex.

Chapter 5. In this chapter, we generalize [5] and Chapter 3 to differential forms on compact manifolds without boundaries and in Euclidean space. First-kind boundary integral operators associated with Hodge–Dirac boundary value problems are shown to be Hodge–Dirac operators in trace de Rham complexes whose spaces are equipped with non-local inner products defined through boundary potentials. We also confirm our suspicion that the first kind boundary integral operators associated with Hodge–Laplace boundary value problems are Hodge-Laplace operators in mixed-order formulation in those same complexes. This discovery greatly simplifies their analysis, because we know from existing literature on abstract Hilbert complexes that the Hodge–Laplacian and the Hodge–Dirac operator are Fredholm operators of index zero. We found that the correspondence is structure-preserving to the extent that adding zero-order terms to the Hodge–Dirac and Hodge–Laplace operators lead to the addition of zero order terms in the trace de Rham complexes at the level of boundary integral operators. In Chapter 5, we study in particular Hodge–Yukawa operators and purely imaginary perturbations of the Hodge–Dirac operator.

As a byproduct of our investigations, Chapter 5 introduces a calculus of boundary potentials which leverage the language of differential forms to ease the derivation of boundary integral equations for Hodge-X operators in general. I call *atomic* the boundary potentials at the heart of this calculus, because every other boundary potential in this work is also obtained from them by applying the exterior derivative or the codifferential. Moreover, they are the elementary building blocks in the definitions of the non-local inner products that we equip on the spaces of the trace de Rham complexes. In a few words, the gist of Boundary Integral Exterior Calculus (BIEC) is the observation that a few commutation identities involving the traces, the exterior derivative, the codifferential and these atomic boundary potentials streamline the derivation of boundary integral equations related to the Hodge–Laplacian (which provides fundamental solutions) and allow expressing them in trace de Rham complexes.

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I would like to extend my thanks to my collaborators for the beautiful projects we've worked on during the past four years. The names of Dirk Pauly and Stefan Kurz appear in this thesis. It has been a pleasure to work together on the exciting ideas that led to the last chapters of this thesis. I also want to thank Matthias Kirchhart for our spontaneous collaboration, and Andy Wan as well, for many years of interesting mathematics. I look forward to discuss new exciting ideas and to continue solving mathematical problems in the future.

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BIEC: A Brief Overview

Applying the so-called tangential and normal traces $\mathsf{t} = \imath^*$ and $\mathsf{n} = \star^{-1}\,\imath^*\star$ to the de Rham complex in a Lipschitz subdomain Ω of a manifold $\mathcal{M}$, we obtain the trace de Rham complexes

$$... \xrightarrow{\mathrm{d}} H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma) \xrightarrow{\mathrm{d}} H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d},\Gamma) \xrightarrow{\mathrm{d}} ..., \tag{0.1a}$$

$$... \underset{\delta}{\longleftarrow} H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma) \underset{\delta}{\longleftarrow} H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\delta,\Gamma) \underset{\delta}{\longleftarrow} ...., \tag{0.1b}$$

on the boundary $\Gamma = \partial\Omega$, respectively. We equip these spaces with the non-local inner products

$$(u,v)_{-\frac{1}{2},\lambda,\mathsf{t}} = \langle \mathsf{t}\mathsf{S}(u), \overline{v}\rangle_{\Gamma}, \qquad u,v \in H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma), \tag{0.2a}$$

$$(w,z)_{-\frac{1}{2},\lambda,\mathsf{n}} = \langle \mathsf{n}\mathsf{D}(w), \overline{w}\rangle_{\Gamma}, \qquad w,z \in H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma), \tag{0.2b}$$

involving the boundary potentials

$$\mathsf{S}u(x) = \langle u, \mathsf{t}\,\mathcal{G}(x,\cdot)\rangle_{\Gamma}, \qquad \text{and} \qquad \mathsf{D}w(x) = \langle w, \mathsf{n}\mathcal{G}(x,\cdot)\rangle_{\Gamma}, \tag{0.3}$$

where $\mathcal{G}$ is a fundamental solution for the Hodge–Laplace or Yukawa operator

$$-\Delta + \lambda = \mathrm{d}\delta + \delta\mathrm{d} + \lambda, \qquad \lambda \geq 0. \tag{0.4}$$

The gist of the calculus of atomic boundary potentials presented in Chapter 5 is the observation that two commutation identities are available which greatly simplify the derivation of boundary integral equations for operators associated to the Hodge–Laplace/Yukawa operator acting on differential forms of order ℓ and to the (possibly perturbed) Hodge–Dirac operator

$$\mathfrak{D} + i\kappa = \mathbf{d} + \boldsymbol{\delta} + i\kappa, \qquad \kappa \in \mathbb{R}. \tag{0.5}$$

acting on the full graded algebra of differential forms:

1. The pullback commutes with exterior differentiation as

$$\mathsf{t}\circ\mathrm{d} = \mathrm{d}\circ\mathsf{t} \qquad \text{and} \qquad \mathsf{n}\circ\delta = -\delta\circ\mathsf{n}. \tag{0.6}$$

2. The boundary potentials commute with exterior differentiation as

$$\delta\mathsf{S}(v) = \mathsf{S}(\delta v) \qquad \text{and} \qquad \mathrm{d}\mathsf{D}(u) = -\mathsf{D}(\mathrm{d}u). \tag{0.7}$$

Since $\partial\Gamma = \emptyset$, integrating by parts then reveals that

$$\langle \mathsf{td}\mathsf{S}(u), \overline{v}\rangle_\Gamma = (\delta^* u, v)_{-\frac{1}{2},\lambda,\mathsf{t}}\,, \qquad \langle \mathsf{t}\delta\mathsf{S}(u), \overline{v}\rangle_\Gamma = (\delta u, v)_{-\frac{1}{2},\lambda,\mathsf{t}}\,, \tag{0.8a}$$

$$\langle \mathsf{nd}\mathsf{D}(w), \overline{z}\rangle_\Gamma = -(\mathrm{d}w, z)_{-\frac{1}{2},\lambda,\mathsf{n}}, \qquad \langle \mathsf{n}\delta\mathsf{D}(w), \overline{z}\rangle_\Gamma = -(\mathrm{d}^* w, z)_{-\frac{1}{2},\lambda,\mathsf{n}}, \tag{0.8b}$$

where d^* and δ^* are Hilbert space adjoint to the exterior derivative and codifferential under the non-local inner products (0.2a) and (0.2b). These identities are crucial, because they make it possible to recognize that *the first-kind boundary integral operators for Hodge–Dirac and Hodge–Laplace operators are Hodge–Dirac and Hodge–Laplace operators themselves in the trace de Rham complexes*.

For example, suppose that a full form $\boldsymbol{U} \in L^2\Lambda(\mathcal{M})$ is compactly supported and that there exists $\boldsymbol{F} \in L^2\Lambda(\mathcal{M})$ such that $\boldsymbol{F}|_\Omega = (\mathfrak{D} + i\kappa)\boldsymbol{U}|_\Omega$ and $\boldsymbol{F}|_{\Omega^+} = (\mathfrak{D} + i\kappa)\boldsymbol{U}|_{\Omega^+}$. Then, we will see in Chapter 5 that have the representation formula

$$\boldsymbol{U} = (\mathfrak{D} - i\kappa)\left(\mathbf{N}\boldsymbol{F} - \mathbf{S}_\lambda[\![\mathsf{n}\boldsymbol{U}]\!] + \mathbf{D}_\lambda[\![\mathsf{t}\boldsymbol{U}]\!]\right), \tag{0.9}$$

where $\mathbf{N}$ is the Newton operator given by the integral transformation involving $\mathcal{G}$ and $[\![\bullet]\!]$ denotes the jump of a trace across Γ, $\bullet = \mathsf{t}$ or n. Taking average traces $\{\bullet\}$ on both sides of (0.9) yields the boundary integral operators

$$\mathsf{V}[\mathfrak{D}] := \{\mathbf{t}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{S}, \qquad \mathsf{K}[\mathfrak{D}] := \{\mathbf{t}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{D}, \tag{0.10a}$$

$$\mathsf{A}[\mathfrak{D}] := \{\mathbf{n}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{S}, \qquad \mathsf{W}[\mathfrak{D}] := \{\mathbf{n}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{D}, \tag{0.10b}$$

that enter the variational formulations

$$\boldsymbol{h} \in H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}, \Gamma): \quad \langle \mathsf{V}[\mathfrak{D}]\boldsymbol{h}, \overline{\boldsymbol{w}}\rangle_\Gamma = \langle (\frac{1}{2}\mathrm{Id} + \mathsf{K}[\mathfrak{D}])\boldsymbol{g}, \overline{\boldsymbol{w}}\rangle_\Gamma, \quad \forall \boldsymbol{w} \in H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}, \Gamma), \tag{0.11a}$$

$$\boldsymbol{g} \in H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}, \Gamma): \quad \langle \mathsf{W}[\mathfrak{D}]\boldsymbol{g}, \overline{\boldsymbol{v}}\rangle_\Gamma = \langle (\frac{1}{2}\mathrm{Id} - \mathsf{A}[\mathfrak{D}])\boldsymbol{h}, \overline{\boldsymbol{v}}\rangle_\Gamma, \quad \forall \boldsymbol{v} \in H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}, \Gamma), \tag{0.11b}$$

associated with first-kind direct boundary integral equations. But from the jump relations

$$[\![\mathsf{t}]\!]\,\mathsf{S} = 0, \qquad [\![\mathrm{td}]\!]\,\mathsf{S} = 0, \qquad [\![\mathsf{t}\delta]\!]\,\mathsf{S} = 0, \tag{0.12a}$$

$$[\![\mathsf{n}]\!]\,\mathsf{S} = 0, \qquad [\![\mathrm{nd}]\!]\,\mathsf{S} = -\mathrm{Id}, \qquad [\![\mathsf{n}\delta]\!]\,\mathsf{S} = 0, \tag{0.12b}$$

$$[\![\mathsf{t}]\!]\,\mathsf{D} = 0, \qquad [\![\mathrm{td}]\!]\,\mathsf{D} = 0, \qquad [\![\mathsf{t}\delta]\!]\,\mathsf{D} = \mathrm{Id}, \tag{0.12c}$$

$$[\![\mathsf{n}]\!]\,\mathsf{D} = 0, \qquad [\![\mathrm{nd}]\!]\,\mathsf{D} = 0, \qquad [\![\mathsf{n}\delta]\!]\,\mathsf{D} = 0, \tag{0.12d}$$

that were used to establish those equations, we also have that the first-kind boundary integral operators can be evaluated as one-sided traces, i.e. $\mathsf{V}[\mathfrak{D}] = \mathsf{t}\;(\mathfrak{D} - i\kappa)\,\mathbf{S}_\lambda$ and $\mathsf{W}[\mathfrak{D}] = \mathbf{n}(\mathfrak{D} - i\kappa)\mathbf{D}$. Based on the previous identities, we can thus evaluate

$$\langle\, \mathsf{V}[\mathfrak{D}]\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle_\Gamma = (\boldsymbol{\delta h}, \boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}} + (\boldsymbol{h}, \boldsymbol{\delta w})_{-\frac{1}{2},\lambda,\mathsf{t}} - i\kappa(\boldsymbol{h}, \boldsymbol{\delta w})_{-\frac{1}{2},\lambda,\mathsf{t}}, \tag{0.13a}$$

$$\langle\, \mathsf{W}[\mathfrak{D}]\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle_\Gamma = -(\mathbf{d}\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}} - (\boldsymbol{g}, \mathbf{d}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{n}} - i\kappa(\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}}, \tag{0.13b}$$

from which we discover that

$$\mathsf{V}[\mathfrak{D}] = \boldsymbol{\delta} + \boldsymbol{\delta}^* - i\kappa \qquad \text{and} \qquad \mathsf{W}[\mathfrak{D}] = -(\mathbf{d} + \mathbf{d}^*) - i\kappa \tag{0.14}$$

(0.15)

on the boundary.

More involved calculations reveals that the first-kind boundary integral operators for the Hodge–Laplacian are also Hodge–Laplace operators on the boundary. This is true for both the strong and the mixed formulation

$$\mathfrak{M} = \begin{pmatrix} -\mathbf{Id} & \delta \\ \mathrm{d} & \delta\mathrm{d} + \lambda \end{pmatrix},$$

obtained by introducing an auxiliary unknown $V = \delta U$. Indeed, we will prove in Chapter 5 that these operators admit the representation formulas

$$U = \mathsf{N}F - \begin{pmatrix} \mathrm{d} & \mathbf{Id} \end{pmatrix} \begin{pmatrix} \mathsf{S}[\![\mathsf{n}U]\!] \\ \mathsf{S}[\![\mathsf{n}\mathrm{d}U]\!] \end{pmatrix} + \begin{pmatrix} \mathbf{Id} & \delta \end{pmatrix} \begin{pmatrix} \mathsf{D}[\![\mathsf{t}\delta U]\!] \\ \mathsf{D}[\![\mathsf{t}U]\!] \end{pmatrix}$$

and

$$\begin{pmatrix} V \\ U \end{pmatrix} = \begin{pmatrix} -\mathrm{d}\delta - \lambda\mathbf{Id} & \delta \\ \mathrm{d} & \mathbf{Id} \end{pmatrix} \left(\begin{pmatrix} \mathsf{N}H \\ \mathsf{N}F \end{pmatrix} - \begin{pmatrix} 0 \\ \mathsf{D}\,[\![\mathsf{t}V]\!] + \delta\mathsf{D}\,[\![\mathsf{t}U]\!] \end{pmatrix} + \begin{pmatrix} \mathsf{S}\,[\![\mathsf{n}U]\!] \\ \mathsf{S}\,[\![\mathsf{n}\mathrm{d}U]\!] \end{pmatrix} \right),$$

respectively, that eventually lead to the boundary integral operators

$$\mathsf{V}[\varDelta] = \mathsf{V}[\mathfrak{M}] = \begin{pmatrix} -\delta^*\delta - \lambda\mathbf{Id} & \delta \\ \delta^* & \mathbf{Id} \end{pmatrix} \tag{0.16a}$$

and

$$\mathsf{W}[\varDelta] = \mathsf{W}[\mathfrak{M}] = \begin{pmatrix} \mathbf{Id} & -\mathrm{d}^* \\ -\mathrm{d}_{\ell-1} & -\mathrm{d}^*\mathrm{d} - \lambda\mathbf{Id} \end{pmatrix} \tag{0.16b}$$

formulated in the trace de Rham complexes equipped with the non-local inner products.

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CHAPTER 1

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CHAPTER 2

E. Schulz And R. Hiptmair. *Spurious resonances in coupled domain-boundary variational formulations of transmission problems in electromagnetism and acoustics*, Computational Methods in Applied Mathematics, (2022), https://doi.org/doi:10.1515/cmam-2021-0197.

CHAPTER 3

E. Schulz And R. Hiptmair. *First-kind boundary integral equations for the Dirac operator in 3-dimensional Lipschitz domains*, SIAM J. Math. Anal., 54 (2022), pp. 616–6648,, https://doi.org/10.1137/20M1389224.

Chapter 1
Coupled Domain-Boundary Variational Formulations for Hodge–Helmholtz Operators

Erick Schulz and Ralf Hiptmair

Abstract We couple the mixed variational problem for the generalized Hodge-Helmholtz or Hodge-Laplace equation posed on a bounded 3D Lipschitz domain with the first-kind boundary integral equations arising from the latter when constant coefficients are assumed in the unbounded complement. Recently developed Calderón projectors for the relevant boundary integral operators are used to perform a symmetric coupling. We prove stability of the coupled problem away from resonant frequencies by establishing a generalized Gårding inequality (T-coercivity). The resulting system of equations describes the scattering of monochromatic electromagnetic waves at a bounded inhomogeneous isotropic body possibly having a “rough" surface. The low-frequency robustness of the potential formulation of Maxwell's equations makes this model a promising starting point for Galerkin discretization.

1.1 Introduction

Let $\Omega_s \subset \mathbb{R}^3$ be a bounded Lipschitz domain [38, Def. 2.1] representing a region of space occupied by a dielectric object, the scatterer, with spatially varying material properties. The scalar material coefficients are assumed to be bounded, i.e. $\mu, \epsilon \in L^\infty(\mathbb{R}^3)$. In a non-dissipative medium, the functions μ and ϵ are real-valued and uniformly positive. Dissipative effects are captured by allowing the coefficients to have non-negative imaginary parts [5, Sec. 1.1.3]. We follow [22] and suppose that

$$
\begin{aligned}
&0 < \mu_{\text{min}} \leq \mathfrak{Re}(\mu) \leq \mu_{\text{max}}, && 0 \leq \mathfrak{Im}(\mu),\\
&0 < \epsilon_{\text{min}} \leq \mathfrak{Re}(\epsilon) \leq \epsilon_{\text{max}}, && 0 \leq \mathfrak{Im}(\epsilon),\\
&0 \leq \mathfrak{Re}(\kappa^2), && 0 \leq \mathfrak{Im}(\kappa^2).
\end{aligned}
$$

We assume for simplicity that Ω_s has trivial cohomology, in other words that its first and second Betti numbers are zero [2, Sec. 4.4]. Qualitatively, this means that it doesn't feature handles nor interior voids: it is homeomorphic to a ball.

Remark 1.1 The hypothesis that the second Betti number is zero is only used to prove injectivity of the coupling problem for Hodge–Laplace operators. It can be dropped without any changes to the following development for couplings involving the Hodge–Helmholtz operator (*non-static* electromagnetic transmission problems). The hypothesis that the first Betti number is zero is used in Section 1.5 to guarantee the existence of a certain “scalar potential lifting" that greatly simplifies the Fredholm arguments.

Inside this possibly inhomogeneous isotropic physical body, the potential formulation of Maxwell's equations in frequency domain driven by a source current $\mathbf{J} \in \mathbf{L}^2(\Omega_s)$ with angular frequency $\omega > 0$ reads [12]

$$\mathbf{curl}\,(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{U}) + i\omega\epsilon(\mathbf{x})\nabla V - \omega^2\epsilon(\mathbf{x})\,\mathbf{U} = \mathbf{J}, \tag{1.1.1a}$$
$$\operatorname{div}\,(\epsilon(\mathbf{x})\mathbf{U}) + i\omega V = 0, \tag{1.1.1b}$$

where the Lorenz gauge (1.1.1b) relates the scalar potential V to the vector potential $\mathbf{U}$. Elimination of V using this relation leads to the Hodge–Helmholtz equation

$$\mathbf{curl}\,(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{U}) - \epsilon(\mathbf{x})\,\nabla\operatorname{div}\,(\epsilon(\mathbf{x})\mathbf{U}) - \omega^2\epsilon(\mathbf{x})\,\mathbf{U} = \mathbf{J}. \tag{1.1.2}$$

Away from the source current, in the unbounded region $\Omega' := \mathbb{R}^3 \backslash \overline{\Omega}_s$ outside the scatterer Ω_s, where we assume a homogeneous material with scalar constant permeability $\mu_0 > 0$ and dielectric permittivity $\epsilon_0 > 0$, equation (1.1.2) reduces to

$$-\Delta_\eta \mathbf{U} - \kappa^2\mathbf{U} := \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - \eta\,\nabla\operatorname{div}\mathbf{U} - \kappa^2\mathbf{U} = 0,$$

with constant coefficients $\eta = \mu_0\epsilon_0^2$ and $\kappa^2 = \mu_0\epsilon_0\omega^2$.

For given data $\mathbf{g}_R \in \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$, $g_n \in H^{-1/2}(\Gamma)$, $\zeta_D \in H^{1/2}(\Gamma)$ and $\boldsymbol{\zeta}_t \in \mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$ on the boundary $\Gamma = \partial\Omega_s$, we are interested in the following transmission problem, cf. [22, Sec. 2.1.2], [12]:

Volume equations

$$\mathbf{curl}\,(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{U}) - \epsilon(\mathbf{x})\,\nabla\operatorname{div}\,(\epsilon(\mathbf{x})\,\mathbf{U}) - \omega^2\epsilon(\mathbf{x})\,\mathbf{U} = \mathbf{J} \text{ in } \Omega_s, \tag{1.1.3a}$$
$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}^{\text{ext}} - \eta\,\nabla\operatorname{div}\mathbf{U}^{\text{ext}} - \kappa^2\,\mathbf{U}^{\text{ext}} = 0 \text{ in } \Omega', \tag{1.1.3b}$$

Transmission conditions

$$\gamma^-_{R,\mu}(\mathbf{U}) = \gamma^+_R\mathbf{U}^{\text{ext}} + \mathbf{g}_R, \quad \gamma^-_{n,\epsilon}(\mathbf{U}) = \gamma^+_n(\mathbf{U}^{\text{ext}}) + g_n \qquad \text{on } \Gamma, \tag{1.1.4a}$$
$$\gamma^-_{D,\epsilon}(\mathbf{U}) = \eta\,\gamma^+_D\mathbf{U}^{\text{ext}} + \zeta_D, \quad \gamma^-_t\mathbf{U} - \gamma^+_t\mathbf{U} = \boldsymbol{\zeta}_t \qquad \text{on } \Gamma. \tag{1.1.4b}$$

The traces $\gamma^\mp_\bullet$, $\bullet = R, D, n$, etc., on Γ from inside (superscript $-$) and outside (superscript $+$) Ω_s are defined for a smooth vector-field $\mathbf{U}$ by

$$\gamma^-_{R,\mu}(\mathbf{U}) := -\gamma^-_\tau(\mu^{-1}(\mathbf{x})\,\mathbf{curl}(\mathbf{U})), \qquad \gamma^+_R(\mathbf{U}^{\text{ext}}) := -\gamma^+_\tau(\mathbf{curl}(\mathbf{U}^{\text{ext}})),$$
$$\gamma^-_{D,\epsilon}(\mathbf{U}) := \gamma^-(\operatorname{div}(\epsilon(\mathbf{x})\,\mathbf{U})), \qquad \gamma^+_D(\mathbf{U}^{\text{ext}}) := \gamma^+(\operatorname{div}(\mathbf{U}^{\text{ext}})),$$
$$\gamma^-_{n,\epsilon}(\mathbf{U}) := \gamma^-_n(\epsilon(\mathbf{x})\,\mathbf{U}) \qquad \gamma^\pm_t(\mathbf{U}) := \mathbf{n} \times (\gamma^\pm_\tau(\mathbf{U})),$$

involving the classical traces

$$\gamma(\mathbf{U}) := \mathbf{U}\big|_{\Gamma}, \qquad \gamma_n(\mathbf{U}) := \gamma(\mathbf{U}) \cdot \mathbf{n}, \qquad \gamma_\tau(\mathbf{U}) := \gamma(\mathbf{U}) \times \mathbf{n},$$

where $\mathbf{n} \in \mathbf{L}^\infty(\Gamma)$ is the essentially bounded unit normal vector field on Γ directed toward the exterior of Ω_s [20, Thm. 3.1.6].

For positive frequencies $\omega > 0$, we supplement (1.1.4b) with the variants of the Silver-Müller's radiation condition imposed at infinity provided in [22]. In the static case where $\kappa = \omega = 0$, we seek a solution in an appropriate weighted Sobolev space that accounts for decay conditions [37, Sec. 2.5].

Remark 1.2 When derived from Maxwell's equations stated in terms of the magnetic and electric fields, the classical wave equation for an electric field $\mathbf{E}$ reads

$$\mathbf{curl}(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{E}) - \kappa^2\,\epsilon(\mathbf{x})\,\mathbf{E} = \mathbf{J}. \tag{1.1.5}$$

The regularizing term $\epsilon\nabla\mathrm{div}(\epsilon\mathbf{U})$ which appears in (1.1.3a), but not in (1.1.5), makes for a significant structural difference [22]. For suitable boundary conditions, the zero-order term $\omega^2\epsilon\mathbf{U}$ in (1.1.3) is a compact perturbation in the weak formulation of the Hodge-Helmholtz equation. Ergo, coercivity of the associated boundary value problem is preserved in the low frequency limit $\omega \to 0$. This is not the case for the "Maxwell operator" found on the left hand side of (1.1.5), whose associated scattering equation is characterized by an "incessant conversion" between electric and magnetic energies that play symmetric roles [11]. Functionally, the infinite dimensional kernel of the curl operator thwart compactness of the embedding $\mathbf{H}(\mathbf{curl}, \Omega_s) \hookrightarrow \mathbf{L}^2(\Omega_s)$. This is different from the weak variational formulation of the scalar Helmholtz equation $-\Delta u - \kappa^2 u = f$. In that model of acoustic scattering, potential energy turns out to be a compact perturbation of the kinetic energy due to Rellich's compact embedding $H^1(\Omega_s) \hookrightarrow L^2(\Omega_s)$.

Remark 1.3 It is stressed in [12] that from the rapid development in quantum optics emerged the need for electromagnetic models valid in both classical and quantum regimes. Robustness of the potential formulation of Maxwell's equations in the low frequency limit makes it a promising candidate for bridging physical scales.

Remark 1.4 The terminology used above is rooted in geometry. The equations (1.1.3a)-(1.1.3b) contain generalized instances of the Hodge–Helmholtz operator $-\Delta - \kappa^2\mathrm{Id} = \delta\mathrm{d} + \mathrm{d}\,\delta - \kappa^2\mathrm{Id}$ as it applies to differential 1-forms defined over 3D differentiable manifolds. When $\omega = \kappa = 0$, the left hand sides reduce to applications of the Hodge-Laplace operator. We refer to [26] and [24] for a thorough introduction to the formulation of Maxwell's equations in terms of differential/integral forms.

Remark 1.5 Boundary integral operators of the *second-kind* were extensively studied in the literature devoted to the Hodge-Laplace and Hodge-Helmholtz operators acting on differential forms over smooth manifolds (e.g. [33], [34], [37] and [32]). However, little attention was paid to the formulation of Hodge-Helmholtz/Laplace boundary value problems as *first-kind* boundary integral equations. Only recently, a boundary integral representation formula for Hodge-Helmholtz/Laplace equation in three-dimensional Lipschitz domains was derived in [15] which leads to boundary integral operators of the first-kind inducing bounded and coercive sesquilinear forms in the natural energy spaces for that equation. These innovative investigations are particularly relevant to the numerical analysis community. Operators admitting natural variational formulations in well-known energy trace spaces via duality are appealing for the development and numerical analysis of new Galerkin discretizations. For the case $\kappa^2 = 0$ of the Hodge–Laplace operator in 3D, a thorough *a priori* analysis of a Galerkin BEM was already proposed in [16] with additional experimental evidence.

1.1.1 Our contributions.

In the following, we couple the *mixed formulation* of the weak variational problem associated to (1.1.3a) with the first-kind boundary integral equation (BIE) arising from (1.1.3b) using these recently developed Calderón projectors for the Hodge–Helmholtz and Hodge–Laplace operators. The proof of the well-posedness of the coupled problem relies on T-coercivity (c.f. [14]) and is given in Subsection 1.5.2. It draws on and integrates several fundamental results of the theory of first-kind boundary integral operators on Lipschitz domains and of the mathematical analysis of Maxwell's equations:

- ▷ M. Costabel's symmetric coupling approach linking volume variational equations with BIEs [18],
- ▷ T-coercivity for electromagnetic variational problems via Hodge–type decompositions [15,25],
- ▷ mixed variational formulations of boundary value problems for Hodge–Laplace operators [3].

A crucial and surprising discovery is the perfect match of the interface terms naturally arising from the mixed variational formulation and from the first-kind BIE, see Section 1.3, and in particular (1.3.6), for details.

1.2 Preliminaries

Let $\Omega \in \{\Omega_s, \Omega'\}$. As usual, $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ denote the Hilbert spaces of square integrable scalar and vector-valued functions defined over Ω. We denote their inner products using round brackets, e.g. $(\cdot,\cdot)_\Omega$. Similarly, $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ refer to the corresponding Sobolev spaces. We write $C_0^\infty(\Omega)$ for the space of smooth compactly supported functions in Ω, but denote by $\mathscr{D}(\Omega)^3$ the analogous space of vector fields to simplify notation. The Banach spaces

$$\begin{aligned}
\mathbf{H}(\mathrm{div},\Omega) &:= \{\mathbf{U} \in L^2(\Omega) \,|\, \mathrm{div}(\mathbf{U}) \in L^2(\Omega)\},\\
\mathbf{H}(\epsilon;\mathrm{div},\Omega) &:= \{\mathbf{U} \in L^2(\Omega) \,|\, \epsilon(\mathbf{x})\,\mathbf{U} \in \mathbf{H}(\mathrm{div},\Omega)\},\\
\mathbf{H}(\mathbf{curl},\Omega) &:= \{\mathbf{U} \in L^2(\Omega) \,|\, \mathbf{curl}\,(\mathbf{U}) \in L^2(\Omega)\},\\
\mathbf{H}(\nabla\mathrm{div},\Omega) &:= \{\mathbf{U} \in \mathbf{H}(\mathrm{div},\Omega) \,|\, \mathrm{div}(\mathbf{U}) \in H^1(\Omega)\},\\
\mathbf{H}(\epsilon;\nabla\mathrm{div},\Omega_s) &:= \{\mathbf{U} \in \mathbf{L}^2(\Omega) \,|\, \epsilon(\mathbf{x})\,\mathbf{U} \in \mathbf{H}(\nabla\mathrm{div},\Omega)\},\\
\mathbf{H}(\mathbf{curl}^2,\Omega) &:= \{\mathbf{U} \in \mathbf{H}(\mathbf{curl},\Omega) \,|\, \mathbf{curl}(\mathbf{U}) \in \mathbf{H}(\mathbf{curl},\Omega)\},\\
\mathbf{H}(\mu^{-1};\mathbf{curl}^2,\Omega) &:= \{\mathbf{U} \in \mathbf{H}(\mathbf{curl},\Omega) \,|\, \mu^{-1}\,\mathbf{curl}(\mathbf{U}) \in \mathbf{H}(\mathbf{curl},\Omega)\},
\end{aligned}$$

equipped with the natural graph norms will be important. The variational space for the primal variational formulation of the classical and generalized Hodge–Helmholtz/Laplace operator is given by

$$\mathbf{X}(\Delta, \Omega) := \mathbf{H}(\mathbf{curl}^2, \Omega) \cap \mathbf{H}(\nabla \mathrm{div}, \Omega). \tag{1.2.1}$$

A subscript is used to identify spaces of locally integrable functions or vector fields, e.g. $U \in L^2_{\mathrm{loc}}(\Omega)$ if and only if ϕU is square-integrable for all $\phi \in C_0^\infty(\mathbb{R}^3)$. Dual spaces, e.g. $H_0^1(\Omega_s)' = H^{-1}(\Omega_s)$, and dual operators, e.g. $(\gamma^-)'$ are written with primes. We use an asterisk to indicate spaces of functions with zero mean, e.g. $H^1_*(\Omega_s)$, and let $\mathbf{mean} : H^1(\Omega_s) \to \mathbb{R}$ be the continuous operator defined by

$$\mathbf{mean}(P) := \frac{1}{|\Omega_s|} \int_{\Omega_s} P(\mathbf{x}) \mathrm{d}\mathbf{x}.$$

Since its range is finite dimensional, $\mathbf{mean}$ is a compact operator [30, Thm. 2.18]. The operator $Q_* : H^1(\Omega_s) \to H^1_*(\Omega_s)$ defined by $Q_* = \mathrm{Id} - \mathbf{mean}$ is a projection onto mean zero functions.

1.2.1 Trace spaces

Development of trace-related theory for Lipschitz domains and detailed definitions for the surface differential operators ∇_Γ, curl_Γ, $\mathbf{curl}_\Gamma$ and div_Γ can be found in [7], [8] and [10]. In this section, we define the product trace spaces required for a variational treatment of the Hodge–Laplace/Helmholtz operator. The traces are adapted to the system of equations at hand by accounting for the varying coefficients of (1.1.3a).

Based on the continuous and surjective extensions

$$\begin{aligned}
\gamma &: H^1(\Omega) \to H^{1/2}(\Gamma), && \text{[29, Thm. 4.2.1]}\\
\gamma_n &: \mathbf{H}(\mathrm{div}, \Omega) \to H^{-1/2}(\Gamma), && \text{[21, Thm. 2.5, Cor. 2.8]}\\
\gamma_\tau &: \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma), && \text{[10, Thm. 4.1]}\\
\gamma_t &: \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), && \text{[10, Thm. 4.1]}
\end{aligned}$$

the traces previously introduced can also be extended by continuity to the relevant Sobolev spaces. We denote the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ by $\langle \cdot, \cdot \rangle_\Gamma$, but use $\langle \cdot, \cdot \rangle_\tau$ for the duality pairing between the trace spaces $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [10, Eq. 36].

The duality pairings enter Green's formulas (+ for $\Omega = \Omega_s$)

$$\langle \gamma(P), \gamma_n(\mathbf{W}) \rangle_\Gamma = \pm \int_\Omega \mathrm{div}(\mathbf{W})\, P + \mathbf{W} \cdot \nabla P \,\mathrm{d}\mathbf{x}, \tag{1.2.2a}$$

$$\langle \gamma_t(\mathbf{V}), \gamma_\tau(\mathbf{U}) \rangle_\tau = \pm \int_\Omega \mathbf{U} \cdot \mathbf{curl}\,(\mathbf{V}) - \mathbf{curl}\,(\mathbf{U}) \cdot \mathbf{V} \mathrm{d}\mathbf{x}, \tag{1.2.2b}$$

$$\langle \gamma_t(\mathbf{V}), \gamma_R(\mathbf{E}) \rangle_\tau = \pm \int_\Omega \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} \cdot \mathbf{V} - \mathbf{curl}\,\mathbf{E} \cdot \mathbf{curl}\,\mathbf{V} \mathrm{d}\mathbf{x}, \tag{1.2.2c}$$

which hold for all $P \in H^1(\Omega)$, $\mathbf{W} \in \mathbf{H}(\mathrm{div}, \Omega)$, $\mathbf{U}, \mathbf{V} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{E} \in \mathbf{H}(\mathbf{curl}^2, \Omega)$.

As explained in [15, Sec. 3], a theory of differential equations for the Hodge–Helmholtz/Laplace problem in three dimensions entails partitioning our collection of traces into two *dual* pairs. Accordingly, we now introduce the continuous and surjective mappings

$$\begin{aligned}
&\mathcal{T}^-_{D,\epsilon} : \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega_s) \cap \mathbf{H}_{\text{loc}}(\epsilon; \nabla\text{div}, \Omega_s) \to \mathcal{H}_D(\Gamma),\\
&\mathcal{T}^+_{D} : \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega') \cap \mathbf{H}_{\text{loc}}(\nabla\text{div}, \Omega') \to \mathcal{H}_D(\Gamma),\\
&\mathcal{T}^-_{N,\mu} : \mathbf{H}_{\text{loc}}(\mu^{-1}; \mathbf{curl}^2, \Omega_s) \cap \mathbf{H}_{\text{loc}}(\epsilon; \text{div}, \Omega_s) \to \mathcal{H}_N(\Gamma),\\
&\mathcal{T}^+_{N} : \mathbf{H}_{\text{loc}}(\mathbf{curl}^2, \Omega') \cap \mathbf{H}_{\text{loc}}(\text{div}, \Omega') \to \mathcal{H}_N(\Gamma),
\end{aligned}$$

defined by

$$\mathcal{T}^-_{D,\epsilon}(\mathbf{U}) := \begin{pmatrix} \gamma_t^-(\mathbf{U}) \\ \gamma^-_{D,\epsilon}(\mathbf{U}) \end{pmatrix}, \qquad \mathcal{T}^-_{N,\mu}(\mathbf{U}) := \begin{pmatrix} \gamma_{R,\mu}(\mathbf{U}) \\ \gamma^-_{n,\epsilon}(\mathbf{U}) \end{pmatrix},$$

$$\mathcal{T}^+_{D}(\mathbf{U}) := \begin{pmatrix} \gamma_t^+(\mathbf{U}) \\ \gamma^+_{D,\eta}(\mathbf{U}) \end{pmatrix}, \qquad \mathcal{T}^+_{N}(\mathbf{U}) := \begin{pmatrix} \gamma_{R}(\mathbf{U}) \\ \gamma_{n}(\mathbf{U}) \end{pmatrix},$$

where

$$\begin{aligned}
\mathcal{H}_D &:= \mathbf{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \times H^{1/2}(\Gamma),\\
\mathcal{H}_N &:= \mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma).
\end{aligned}$$

They admit continuous right-inverses, i.e. lifting maps from the trace spaces into $\mathbf{X}(\Delta, \Omega)$ [15, Lem. 3.2].

In literature the pair of traces involved in $\mathcal{T}_N$ is labelled as *magnetic*, while the pair in $\mathcal{T}_D$ is referred to as *electric*—simply because one recovers the magnetic field by taking the curl of the potential $\mathbf{U}$. However, our choice of subscripts is motivated by the analogy between this pair of product traces and the classical Dirichlet and Neumann boundary conditions for second-order elliptic BVPs.

The trace spaces $\mathcal{H}_D$ and $\mathcal{H}_N$ are put in duality using the sum of the inherited component-wise duality parings. That is, for $\overrightarrow{\mathbf{p}} = (\mathbf{p}, q) \in \mathcal{H}_N$ and $\overrightarrow{\boldsymbol{\eta}} = (\boldsymbol{\eta}, \zeta) \in \mathcal{H}_D$, we define

$$\langle \overrightarrow{\mathbf{p}}, \overrightarrow{\boldsymbol{\eta}} \rangle := \langle \mathbf{p}, \boldsymbol{\eta} \rangle_\tau + \langle q, \zeta \rangle_\Gamma.$$

We indicate with curly brackets the average

$$\{\gamma_\bullet\} := \frac{1}{2}(\gamma_\bullet^+ + \gamma_\bullet^-)$$

of a trace and with square brackets its jump

$$[\gamma_\bullet] := \gamma_\bullet^- - \gamma_\bullet^+$$

over the interface Γ, $\bullet = R, D, t, \tau$, or n. Corresponding notation is used for the product traces.

Warning. *Notice the sign in the jump* $[\gamma] = \gamma^- - \gamma^+$*, which is often taken to be the opposite in literature.*

1.2.2 Boundary potentials

By exploiting the radiating fundamental solution

$$G_\nu(\mathbf{x}) := \exp\left(i\nu|\mathbf{x}|\right)/4\pi|\mathbf{x}|$$

for the scalar Helmholtz operator $-\Delta - \nu^2\mathrm{Id}$, it is shown in [15, Sec. 4.2] that a distributional solution $\mathbf{U} \in \mathbf{L}^2(\mathbb{R}^3)$ such that $\mathbf{U}|_{\Omega_s} \in \mathbf{X}(\Delta, \Omega_s)$ and $\mathbf{U}|_{\Omega'} \in \mathbf{X}_{\text{loc}}(\Delta, \Omega')$ of the homogeneous (scaled) Hodge–Helmholtz/Laplace equation (1.1.3b) with constant coefficients $\eta > 0$, $\kappa \geq 0$, stated in the whole of $\mathbb{R}^3$ with radiation conditions at infinity as considered in Section 1.1, affords a representation formula

$$\mathbf{U} = \mathcal{SL}_\kappa \cdot [\mathcal{T}_N(\mathbf{U})] + \mathcal{DL}_\kappa \cdot [\mathcal{T}_D(\mathbf{U})] \qquad \text{in } \mathbb{R}^3\backslash\Gamma. \tag{1.2.3}$$

Letting $\tilde{\kappa} = \kappa/\sqrt{n}$, the Hodge-Helmholtz single layer potential is explicitly given by

$$\mathcal{SL}_\kappa(\begin{pmatrix}\mathbf{p}\\ q\end{pmatrix}) = -\boldsymbol{\Psi}_\kappa(\mathbf{p}) - \nabla\tilde{\psi}_k(\mathrm{div}_\Gamma(\mathbf{p})) + \nabla\psi_{\tilde{\kappa}}(q), \tag{1.2.4}$$

where the Helmholtz scalar single-layer, vector single-layer and the regular potentials are written individually for $\mathbf{p} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $q \in H^{-1/2}(\Gamma)$ as

$$\psi_\nu(q)(\mathbf{x}) := \int_\Gamma q(\mathbf{y})G_\nu(\mathbf{x}-\mathbf{y})\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3\backslash\Gamma, \tag{1.2.5a}$$

$$\boldsymbol{\Psi}_\nu(\mathbf{p})(\mathbf{x}) := \int_\gamma \mathbf{p}(\mathbf{y})G_\nu(\mathbf{x}-\mathbf{y})\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3\backslash\Gamma, \tag{1.2.5b}$$

$$\tilde{\psi}_\kappa(q)(\mathbf{x}) := \int_\Gamma q(\mathbf{y})\frac{G_\kappa - G_{\tilde{\kappa}}}{\kappa^2}(\mathbf{x}-\mathbf{y})\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3\backslash\Gamma, \tag{1.2.5c}$$

respectively. The expression (1.2.4) is derived with (1.2.5a)-(1.2.5c) understood as duality pairings. However, if the essential supremum of $\mathbf{p}$, q and $\mathrm{div}_\Gamma(\mathbf{p})$ is bounded, then they can safely be computed as improper integrals [15, Rmk. 4.2]. These classical potentials satisfy

$$-\Delta\psi_{\tilde{\kappa}}(q) = \tilde{\kappa}^2\psi_{\tilde{\kappa}}(q), \tag{1.2.6a}$$

$$-\Delta\boldsymbol{\Psi}_\kappa(\mathbf{p}) = \kappa^2\boldsymbol{\Psi}_\kappa(\mathbf{p}), \tag{1.2.6b}$$

$$-\Delta\tilde{\psi}_\kappa(q) = \psi_\kappa(q) - \frac{1}{\eta}\psi_{\tilde{\kappa}}(q), \tag{1.2.6c}$$

and the identity [31, Lem. 2.3]

$$\mathrm{div}\,\boldsymbol{\Psi}_\nu(\mathbf{p}) = \psi_\nu\left(\mathrm{div}_\Gamma\mathbf{p}\right) \qquad \forall\,\mathbf{p} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma). \tag{1.2.7}$$

The mapping properties of ψ_ν, $\boldsymbol{\Psi}_\nu$, $\nabla\psi_{\tilde{\kappa}}$ and $\nabla\tilde{\psi}_\kappa$ are detailed in [15, Sec. 5].

Ultimately, we will resort to a Fredholm alternative argument to prove well-posedness of the coupled system. It is therefore evident that the compactness properties of the boundary integral operators introduced in the next Lemma will be extensively used both explicitly and implicitly—notably through exploiting the results found in [15, Sec. 6].

From [35, Lem. 3.9.8] and [11, Lem. 7], we know that for any $\nu \geq 0$, the following operators are compact:

$$\gamma^{\pm}(\psi_\nu - \psi_0) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \tag{1.2.8a}$$
$$\gamma_n^{\pm}(\nabla\psi_\nu - \nabla\psi_0) : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \tag{1.2.8b}$$
$$\gamma_t^{\pm}(\boldsymbol{\Psi}_\nu - \boldsymbol{\Psi}_0) : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \tag{1.2.8c}$$
$$\gamma_n^{\pm}\nabla\tilde{\psi}_\nu : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma). \tag{1.2.8d}$$

Compactness of the second boundary integral operator listed immediately entails compactness of

$$\nu^2\gamma_n^{\pm}\nabla\tilde{\psi}_\nu = \gamma_n^{\pm}(\nabla\psi_\nu - \nabla\psi_{\tilde{\nu}}) = \gamma_n^{\pm}(\nabla\psi_\nu - \nabla\psi_0) - (\gamma_n^{\pm}(\nabla\psi_{\tilde{\nu}} - \nabla\psi_0))$$

by linearity. While it seems that blow-up occurs in $\tilde{\psi}_\nu$ as $\nu \to 0$, $\nabla\tilde{\psi}_\nu$ happens to be an entire function of ν that vanishes at $\nu = 0$ [15, Sec. 4.1].

The Hodge–Helmholtz double layer potential is given for boundary data $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\xi \in H^{1/2}(\Gamma)$ by

$$\mathcal{DL}_\kappa(\begin{pmatrix}\boldsymbol{\eta}\\ \xi\end{pmatrix}) := \mathbf{curl}\,\boldsymbol{\Psi}_\kappa(\boldsymbol{\eta} \times \mathbf{n}) + \Upsilon_\kappa(\xi). \tag{1.2.9}$$

We recognize in (1.2.9) the (electric) Maxwell double layer potential (c.f. [25, Sec. 4], [11, Eq. 28]) and the normal vector single-layer potential

$$\Upsilon_\kappa(\xi) := \int_\Gamma \xi(\mathbf{y})\mathbf{G}_\kappa(\mathbf{x} - \mathbf{y})\mathbf{n}(\mathbf{y})\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3\backslash\Gamma,$$

in which appears the matrix-valued fundamental solution

$$\mathbf{G}_\kappa := G_\kappa \mathrm{Id} + \kappa^{-2}\nabla^2(G_\kappa - G_{\tilde{\kappa}})$$

satisfying $-\Delta_\eta \mathbf{G}_\kappa - \kappa^2\mathbf{G}_\kappa = \delta_0\,\mathrm{Id}$ exploited in [15] and detailed in [22, App. A]. This surface potential satisfies

$$-\Delta_\eta \Upsilon_\kappa(\xi) = \kappa^2\Upsilon_\kappa(\xi) \tag{1.2.10}$$

and the identity [15, Sec.5.4] $\mathbf{curl}\,\Upsilon_\kappa(\xi) = \mathbf{curl}\boldsymbol{\Psi}_\kappa(\xi\mathbf{n})$.

The mapping properties of the potentials $\mathbf{curl}\boldsymbol{\Psi}_\kappa(\cdot \times \mathbf{n})$ and Υ_κ are detailed in [15, Sec. 5].

1.2.3 Integral operators

In this section, we extend the analysis performed in [11,25] for the classical electric wave equation to the boundary integral operators arising from Hodge–Helmholtz and Hodge–Laplace problems.

The well-known Caldéron identities are obtained from (1.2.3) upon taking the classical compounded traces on both sides and utilizing the jump relations

$$[\mathcal{T}_D] \cdot \mathcal{DL}_\kappa(\vec{\boldsymbol{\eta}}) = \vec{\boldsymbol{\eta}}, \qquad [\mathcal{T}_N] \cdot \mathcal{DL}_\kappa(\vec{\boldsymbol{\eta}}) = 0, \qquad \vec{\boldsymbol{\eta}} \in \mathcal{H}_D, \tag{1.2.11a}$$
$$[\mathcal{T}_D] \cdot \mathcal{SL}_\kappa(\vec{\mathbf{p}}) = 0, \qquad [\mathcal{T}_N] \cdot \mathcal{SL}_\kappa(\vec{\mathbf{p}}) = \vec{\mathbf{p}}, \qquad \vec{\mathbf{p}} \in \mathcal{H}_N, \tag{1.2.11b}$$

given in [15, Thm. 5.1]. The operator forms of the interior and exterior Caldéron projectors defined on $\mathcal{H}_D \times \mathcal{H}_N$, which we denote $\mathbb{P}_\kappa^-$ and $\mathbb{P}_\kappa^+$ respectively, enter the Caldéron identites:

$$\underbrace{\begin{pmatrix} \{\mathcal{T}_D\}\cdot\mathcal{DL}_k + \frac{1}{2}\mathrm{Id} & \{\mathcal{T}_D\}\cdot\mathcal{SL}_k \\ \{\mathcal{T}_N\}\cdot\mathcal{DL}_k & \{\mathcal{T}_N\}\cdot\mathcal{SL}_k + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{=:\mathbb{P}_\kappa^-} \begin{pmatrix} \mathcal{T}_D^-\mathbf{U} \\ \mathcal{T}_N^-\mathbf{U} \end{pmatrix} = \begin{pmatrix} \mathcal{T}_D^-\mathbf{U} \\ \mathcal{T}_N^-\mathbf{U} \end{pmatrix}, \tag{1.2.12a}$$

$$\underbrace{\begin{pmatrix} -\{\mathcal{T}_D\}\cdot\mathcal{DL}_k + \frac{1}{2}\mathrm{Id} & -\{\mathcal{T}_D\}\cdot\mathcal{SL}_k \\ -\{\mathcal{T}_N\}\cdot\mathcal{DL}_k & -\{\mathcal{T}_N\}\cdot\mathcal{SL}_k + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{=:\mathbb{P}_\kappa^+} \begin{pmatrix} \mathcal{T}_D^+\mathbf{U}^{\mathrm{ext}} \\ \mathcal{T}_N^+\mathbf{U}^{\mathrm{ext}} \end{pmatrix} = \begin{pmatrix} \mathcal{T}_D^+\mathbf{U}^{\mathrm{ext}} \\ \mathcal{T}_N^+\mathbf{U}^{\mathrm{ext}} \end{pmatrix}. \tag{1.2.12b}$$

Note that $\mathbb{P}_\kappa^- + \mathbb{P}_\kappa^+ = \mathrm{Id}$ and that the range of $\mathbb{P}_\kappa^+$ coincides with the kernel of $\mathbb{P}_\kappa^-$ and vice-versa [11, Sec. 5]. As a consequence of the jump relations (1.2.11a)-(1.2.11b), the representation formula (1.2.3) and the existence of trace liftings, the pair of “magnetic" and “electric" traces $(\overrightarrow{\boldsymbol{\eta}}\ \ \overrightarrow{\mathbf{p}})^\top \in \mathcal{H}_D \times \mathcal{H}_N$ is valid interior or exterior Cauchy data, if and only if it lies in the kernel of $\mathbb{P}_\kappa^+$ or $\mathbb{P}_\kappa^-$ respectively (c.f. [38, Lem. 6.18], [11, Thm. 8] and [15, Prop. 5.2]).

Inspecting equations (1.2.12a)-(1.2.12b) reveals that the Caldéron projectors share a common structure. They can be written as

$$\mathbb{P}_\kappa^- = \frac{1}{2}\mathrm{Id} + \mathbb{A}_\kappa \qquad \text{and} \qquad \mathbb{P}_\kappa^+ = \frac{1}{2}\mathrm{Id} - \mathbb{A}_\kappa,$$

and where the Caldéron operator $\mathbb{A}_\kappa : \mathcal{H}_\mathcal{D} \times \mathcal{H}_\mathcal{N} \to \mathcal{H}_\mathcal{D} \times \mathcal{H}_\mathcal{N}$ is given by

$$\mathbb{A}_\kappa := \begin{pmatrix} \mathbb{A}_\kappa^{DD} & \mathbb{A}_\kappa^{ND} \\ \mathbb{A}_\kappa^{DN} & \mathbb{A}_\kappa^{NN} \end{pmatrix} := \begin{pmatrix} \{\mathcal{T}_D\}\cdot\mathcal{DL}_\kappa & \{\mathcal{T}_D\}\cdot\mathcal{SL}_\kappa \\ \{\mathcal{T}_N\}\cdot\mathcal{DL}_\kappa & \{\mathcal{T}_N\}\cdot\mathcal{SL}_\kappa \end{pmatrix}. \tag{1.2.13}$$

An analog of the operator matrix $\mathbb{A}_k$ was found convenient in the study of the boundary integral equations of electromagnetic scattering problems [11, Sec. 6]. It is known from [15] that the off-diagonal blocks $\mathbb{A}_\kappa^{DN}$ and $\mathbb{A}_\kappa^{ND}$ of $\mathbb{A}_\kappa$ independently satisfy generalized Gårding inequalities making them of Fredholm type with index 0. Injectivity holds when κ^2 lies outside a discrete set of “forbidden resonant frequencies” accumulating at infinity [15, Sec. 3]. More explanations will be given in Section 1.3. In the static case $\kappa = 0$, the dimensions of $\ker\left(\{\mathcal{T}_N\}\cdot\mathcal{SL}_0\right)$ and $\ker\left(\{\mathcal{T}_D\}\cdot\mathcal{DL}_0\right)$ agree with the zeroth and first Betti number of Γ, respectively [15, Sec. 7].

In the case of the classical electric wave equation, the boundary integral operators involved in the Caldéron projectors enjoy a hidden symmetry: there exists a compact linear operator

$$\mathbf{C}_\kappa : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

such that

$$\langle\{\gamma_R\}\boldsymbol{\Psi}_\kappa(\mathbf{p}), \boldsymbol{\eta}\rangle_\tau = \langle\mathbf{p}, \{\gamma_t\}\boldsymbol{\Psi}_\kappa\mathbf{curl}(\boldsymbol{\eta}\times\mathbf{n})\rangle_\tau + \langle\mathbf{C}_\kappa\mathbf{p}, \boldsymbol{\eta}\rangle_\tau \tag{1.2.14}$$

for all $\mathbf{p} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma, \Gamma)$, cf. [25, Lem. 5.4] and [11, Lem. 6].

We will extend this result to the integral operators defined for the scaled Hodge–Helmholtz equation to better characterize the structure of (1.2.13). The symmetry we are about to reveal in the diagonal blocks $\mathbb{A}_\kappa^{NN}$ and $\mathbb{A}_\kappa^{DD}$ of the Caldéron projectors will be crucial in the derivation of the main T-coercivity estimate of this work. It will be exploited for complete *cancellation*, *up to compact terms*, of the operators lying on the *off-diagonal* of the block operator matrix associated to the coupled variational system introduced in Section 1.3. The following lemmas are required.

Lemma 1.1 *There is a compact linear operator* $C_\kappa : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *such that*

$$\langle \{\gamma_n\} \nabla \psi_{\tilde{\kappa}}(q), \xi \rangle_\Gamma = -\langle q, \{\eta\, \gamma_D\} \Upsilon_\kappa(\xi) \rangle_\Gamma + \langle C_k q, \xi \rangle_\Gamma,$$

for all $q \in H^{-1/2}(\Gamma)$, $\xi \in H^{1/2}(\Gamma)$.

Proof. This proof utilizes a strategy found in [25, Lem. 5.4] and [9, Thm. 3.9]. Let $\rho > 0$ be such that B_ρ is an open ball containing $\overline{\Omega}_s$. We will indicate with a hat (e.g. $\widehat{\gamma}$) the traces taken over the boundary ∂B_ρ of that ball and use Green's formula to compare the following terms.

On the one hand, using the scalar Helmholtz equation (1.2.6a) and recalling that $\tilde{\kappa} = \kappa/\sqrt{\eta}$, we have

$$\begin{aligned} &\langle \eta\, \gamma_D^- \nabla \psi_{\tilde{\kappa}}(q),\, \gamma_n^- \Upsilon_\kappa(\xi) \rangle_\Gamma \\ &\qquad = \int_{\Omega_s} \eta \,\mathrm{div}\left(\nabla \psi_{\tilde{\kappa}}(q)\right) \mathrm{div} \Upsilon_k(\xi) + \eta \nabla \mathrm{div}\left(\nabla \psi_{\tilde{\kappa}}(q)\right) \cdot \Upsilon_\kappa(\xi) \mathrm{d}\mathbf{x} \\ &\qquad = -\int_{\Omega_s} \kappa^2 \psi_{\tilde{\kappa}}(q) \mathrm{div} \Upsilon_k(\xi) \mathrm{d}\mathbf{x} - \int_{\Omega_s} \kappa^2 \nabla \psi_{\tilde{\kappa}}(q) \cdot \Upsilon_\kappa(\xi) \mathrm{d}\mathbf{x}, \end{aligned} \tag{1.2.15}$$

and similarly,

$$\begin{aligned} \langle \eta\, \gamma_D^+ \nabla \psi_{\tilde{\kappa}}(q), \gamma_n^+ \Upsilon_\kappa(\xi) \rangle_\Gamma = \int_{\Omega' \cap B_\rho} &\kappa^2 \psi_{\tilde{\kappa}}(q)\, \mathrm{div} \Upsilon_k(\xi) + \nabla \psi_{\tilde{\kappa}}(q) \cdot \Upsilon_\kappa(\xi) \mathrm{d}\mathbf{x} \\ &+ \langle \eta\, \widehat{\gamma}_D^+ \nabla \psi_\kappa(q), \widehat{\gamma}_n^+ \Upsilon_\kappa(\xi) \rangle_{\partial B_\rho}. \end{aligned}$$

On the other hand, using (1.2.6a) together with the scaled Hodge–Helmholtz equation (1.2.10), we also have

$$\begin{aligned} &\langle\, \gamma_n^- \nabla \psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^- \Upsilon_\kappa(\xi) \rangle_\Gamma \\ &\quad = \int_{\Omega_s} \eta \,\mathrm{div}\left(\nabla \psi_{\tilde{\kappa}}(q)\right) \mathrm{div} \Upsilon_\kappa(\xi) \mathrm{d}\mathbf{x} + \int_{\Omega_s} \eta\, \nabla \psi_{\tilde{\kappa}}(q) \cdot \nabla \mathrm{div} \Upsilon_\kappa(\xi) \mathrm{d}\mathbf{x} \\ &\quad = -\int_{\Omega_s} \kappa^2 \psi_{\tilde{\kappa}}(q)\, \mathrm{div} \Upsilon_\kappa(\xi) \mathrm{d}\mathbf{x} + \int_{\Omega_s} \nabla \psi_{\tilde{\kappa}}(q) \cdot \mathbf{curl}\,\mathbf{curl}\, \Upsilon_\kappa(\xi) \mathrm{d}\mathbf{x} \\ &\qquad\qquad - \int_{\Omega_s} \kappa^2 \nabla \psi_{\tilde{\kappa}}(q) \cdot \Upsilon_\kappa(\xi) \mathrm{d}\mathbf{x}. \end{aligned} \tag{1.2.16}$$

Equations (1.2.15) and (1.2.16) together yield

$$\begin{aligned} \langle\, \gamma_n^- \nabla \psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^- \Upsilon_\kappa(\xi) \rangle_\Gamma = \langle \eta\, \gamma_D^- \nabla \psi_\kappa(q),\, \gamma_n^- \Upsilon_\kappa(\xi) \rangle_\Gamma& \\ + \int_{\Omega_s} \nabla \psi_{\tilde{\kappa}}(q) \cdot \mathbf{curl}\,\mathbf{curl}\, \Upsilon_\kappa(\xi) \mathrm{d}\mathbf{x}&. \end{aligned}$$

Similarly, the terms involving the exterior traces satisfy

$$\langle \gamma_n^+ \nabla \psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^+ \Upsilon_\kappa(\xi) \rangle_\Gamma = \langle \eta\, \gamma_D^+ \nabla \psi_\kappa(q), \gamma_n^+ \Upsilon_\kappa(\xi) \rangle_\Gamma$$

$$
\begin{aligned}
&- \langle \eta\, \widehat{\gamma}_D^+ \nabla\psi_\kappa(q), \widehat{\gamma}_n^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho} \\
&- \int_{\Omega' \cap B_\rho} \nabla\psi_{\tilde{\kappa}}(q) \cdot \mathbf{curl}\,\mathbf{curl}\,\Upsilon_\kappa(\xi)\mathrm{d}\mathbf{x} \\
&+ \langle \widehat{\gamma}_n^+ \nabla\psi_{\tilde{\kappa}}(q), \eta\, \widehat{\gamma}_D^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho}.
\end{aligned}
$$

From the first row of the jump properties [15, Sec. 5]

$$
[\gamma_D]\nabla\psi_{\tilde{\kappa}}(q) = 0, \qquad [\gamma_n]\Upsilon_\kappa(\xi) = 0, \tag{1.2.17a}
$$

$$
[\gamma_D]\Upsilon_\kappa(\xi) = \xi/\eta, \qquad [\gamma_n]\nabla\psi_{\tilde{\kappa}}(q) = q, \tag{1.2.17b}
$$

we obtain, by gathering the above results, integrating by parts again and using that $\mathbf{curl} \circ \nabla \equiv 0$,

$$
\begin{aligned}
&\langle \gamma_n^- \nabla\psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^- \Upsilon_\kappa(\xi)\rangle_\Gamma \\
&\quad = \langle \eta\, \gamma_D^+ \nabla\psi_\kappa(q), \gamma_n^+ \Upsilon_\kappa(\xi)\rangle_\Gamma + \int_{\Omega_s} \kappa^2 \nabla\psi_{\tilde{\kappa}}(q) \cdot \boldsymbol{\Psi}_\kappa(\xi\mathbf{n})\mathrm{d}\mathbf{x} \\
&\quad = \langle \gamma_n^+ \nabla\psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^+ \Upsilon_\kappa(\xi)\rangle_\Gamma + \int_{B_\rho} \nabla\psi_{\tilde{\kappa}}(q) \cdot \mathbf{curl}\,\mathbf{curl}\,\Upsilon_\kappa(\xi)\mathrm{d}\mathbf{x} \\
&\qquad + \langle \eta\, \widehat{\gamma}_D^+ \nabla\psi_\kappa(q), \widehat{\gamma}_n^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho} - \langle \widehat{\gamma}_n^+ \nabla\psi_{\tilde{\kappa}}(q), \eta\, \widehat{\gamma}_D^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho}. \\
&\quad = \langle \gamma_n^+ \nabla\psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^+ \Upsilon_\kappa(\xi)\rangle_\Gamma + \langle \gamma_t \nabla\psi_{\tilde{\kappa}}(q), \gamma_R \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho} \\
&\qquad + \langle \eta\, \widehat{\gamma}_D^+ \nabla\psi_\kappa(q), \widehat{\gamma}_n^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho} - \langle \widehat{\gamma}_n^+ \nabla\psi_{\tilde{\kappa}}(q), \eta\, \widehat{\gamma}_D^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho}.
\end{aligned} \tag{1.2.18}
$$

Fortunately, when restricted to domains away from Γ, the potentials are C^∞-smoothing. Hence, their evaluation on ∂B_ρ, the highlighted terms in (1.2.18), induce compact operators. This shows that for some compact operator $C_\kappa : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$,

$$
\langle \gamma_n^- \nabla\psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^- \Upsilon_\kappa(\xi)\rangle_\Gamma = \langle \gamma_n^+ \nabla\psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^+ \Upsilon_\kappa(\xi)\rangle_\Gamma + \langle C_\kappa q, \xi\rangle_\Gamma. \tag{1.2.19}
$$

The jump identities (1.2.17b) for the potentials yield formulas of the form $\{\gamma_\bullet\}K = \gamma_\bullet^\pm K \pm (1/2)\mathrm{Id}$, where $\bullet = n$, D and $K = \nabla\psi_{\tilde{\kappa}}$, Υ_κ accordingly. Substituting each one-sided trace involved in the two leftmost duality pairings of (1.2.19) for the integral operators using these equations completes the proof. □

Lemma 1.2 *For all* $\mathbf{p} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *and* $\xi \in H^{1/2}(\Gamma)$*, we have*

$$
\langle \mathbf{p}, \gamma_t^\pm \Upsilon_\kappa(\xi)\rangle_\tau = \langle \gamma_n^\pm \boldsymbol{\Psi}_\kappa(\mathbf{p}), \xi\rangle_\Gamma + \langle \gamma_n^\pm \nabla\tilde{\psi}_\kappa(\mathrm{div}_\Gamma(\mathbf{p})), \xi\rangle_\Gamma.
$$

Proof. In the following calculations, the boundary integrals are to be understood as duality pairings. Since $\mathbf{p} \in \mathbf{L}_t^2(\Gamma)$ is a tangent vector field lying in the image of γ_t, the tangential trace operator can safely be dropped in expanding these integrals using the definitions of Subsection 1.2.2. On the one hand, this leads to

$$
\langle \mathbf{p}, \gamma_t^\pm \Upsilon_\kappa(\xi)\rangle_\tau = \int_\Gamma \int_\Gamma \xi(\mathbf{y})\mathbf{p}(\mathbf{x}) \cdot (\mathbf{G}_\kappa(\mathbf{x}-\mathbf{y})\mathbf{n}(\mathbf{y}))\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})
$$

$$
\begin{aligned}
&= \int_\Gamma \int_\Gamma \xi(\mathbf{y}) G_\kappa(\mathbf{x}-\mathbf{y}) \mathbf{p}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \mathrm{d}\sigma(\mathbf{x}) \\
&\quad + \int_\Gamma \int_\Gamma \xi(\mathbf{y}) \mathbf{p}(\mathbf{x}) \cdot (\nabla^2 \tilde{G}_\kappa(\mathbf{x}-\mathbf{y}) \mathbf{n}(\mathbf{y})) \mathrm{d}\sigma(\mathbf{y}) \mathrm{d}\sigma(\mathbf{x}),
\end{aligned}
$$

where $\tilde{G}_\kappa := (G_\kappa - G_{\tilde{\kappa}})/\kappa^2$.

On the other hand, the same observation implies that $\langle \mathbf{p}, \nabla_\Gamma \gamma \mathbf{V}) \rangle_\tau = \langle \mathbf{p}, \gamma \nabla \mathbf{V}) \rangle_\tau$ for any $\mathbf{V} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$, and thus that

$$
\begin{aligned}
\langle \gamma_n^\pm \nabla \tilde{\psi}_\kappa(\mathrm{div}_\Gamma(\mathbf{p})), \xi \rangle_\Gamma &= \int_\gamma \int_\gamma \xi(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \nabla \tilde{G}_\kappa(\mathbf{y}-\mathbf{x}) \mathrm{div}_\Gamma(\mathbf{p}(\mathbf{x})) \mathrm{d}\sigma(\mathbf{y}) \mathrm{d}\sigma(\mathbf{x}) \\
&= -\int_\gamma \int_\gamma \xi(\mathbf{y}) \mathbf{p}(x) \nabla_{\mathbf{x}} (\mathbf{n}(\mathbf{y}) \cdot \nabla \tilde{G}_\kappa(\mathbf{y}-\mathbf{x})) \mathrm{d}\sigma(\mathbf{y}) \mathrm{d}\sigma(\mathbf{x}) \\
&= \int_\gamma \int_\gamma \xi(\mathbf{y}) \mathbf{p}(x) (\nabla^2 \tilde{G}_\kappa(\mathbf{x}-\mathbf{y}) \mathbf{n}(\mathbf{y})) \mathrm{d}\sigma(\mathbf{y}) \mathrm{d}\sigma(\mathbf{x}),
\end{aligned}
$$

where we have remembered that the tangential divergence defined in Subsection 1.2.1 was adjoint to the negative surface gradient. Recognizing the Helmholtz vector single-layer potential in the first expression on the right hand side concludes the proof. □

Proposition 1.1 *There exists a compact operator $\mathcal{C}_\kappa : \mathcal{H}_N \to \mathcal{H}_N$ such that*

$$
\langle \mathbb{A}_\kappa^{NN}(\vec{\mathbf{p}}), \vec{\boldsymbol{\eta}} \rangle = -\langle \vec{\mathbf{p}}, \mathbb{A}_\kappa^{DD}(\vec{\boldsymbol{\eta}}) \rangle + \langle \mathcal{C}_k \vec{\mathbf{p}}, \vec{\boldsymbol{\eta}} \rangle
$$

for all $\vec{\boldsymbol{\eta}} := (\boldsymbol{\eta},\ \xi)^\top \in \mathcal{H}_D$ and $\vec{\mathbf{p}} := (\mathbf{p},\ q)^\top \in \mathcal{H}_N$.

Proof. Recall that $\mathbb{A}_\kappa^{NN} = \{\mathcal{T}_N\} \cdot \mathcal{SL}_\kappa$ and $\mathbb{A}_\kappa^{DD} = \{\mathcal{T}_D\} \cdot \mathcal{DL}_\kappa$. Since $\mathbf{curl} \circ \nabla = 0$, $\langle \{\gamma_R\} \nabla \psi_{\tilde{k}}(q), \boldsymbol{\eta} \rangle_\tau = 0$ and $\langle \{\gamma_R\} \nabla \tilde{\psi}_k(\mathrm{div}_\Gamma(\mathbf{p})), \boldsymbol{\eta} \rangle_\tau = 0$; therefore,

$$
\begin{aligned}
\langle \{\mathcal{T}_N\} \cdot \mathcal{SL}_k(\vec{\mathbf{p}}), \vec{\boldsymbol{\eta}} \rangle = &\langle -\{\gamma_R\} \boldsymbol{\Psi}_\kappa(\mathbf{p}), \boldsymbol{\eta} \rangle_\tau + \langle \{\gamma_n\} \nabla \psi_{\tilde{\kappa}}(q), \xi \rangle_\Gamma \\
&- \langle \{\gamma_n\} \boldsymbol{\Psi}_\kappa(\mathbf{p}), \xi \rangle_\Gamma - \langle \{\gamma_n\} \nabla \tilde{\psi}_\kappa(\mathrm{div}_\Gamma(\mathbf{p})), \xi \rangle_\Gamma. \qquad (1.2.20)
\end{aligned}
$$

Since $\mathrm{div} \circ \mathbf{curl} = 0$, we also have $\{\gamma_D\}\, \mathbf{curl} \boldsymbol{\Psi}_{\boldsymbol{\kappa}} = 0$. Hence, we need to compare (1.2.20) with

$$
\begin{aligned}
\langle \vec{\mathbf{p}}, \{\mathcal{T}_D\} \cdot \mathcal{DL}_k(\vec{\boldsymbol{\eta}}) \rangle = &\langle \mathbf{p}, \{\gamma_t\} \mathbf{curl} \boldsymbol{\Psi}_k(\boldsymbol{\eta} \times \mathbf{n}) \rangle_\tau + \langle q, \{\eta\, \gamma_D\} \Upsilon_\kappa(\xi) \rangle_\Gamma \\
&+ \langle \mathbf{p}, \{\gamma_t\} \Upsilon_\kappa(\xi) \rangle_\tau.
\end{aligned}
$$

The desired result follows by combining the known symmetry result from (1.2.14) with Lemma 1.1 and Lemma 1.2. □

As consequence of Proposition 1.1, we have

$$
(\mathbb{P}_\kappa^+)^*_{11} \hat{=} (\mathbb{P}_\kappa^-)_{22},
$$

where $\hat{=}$ is used to indicate equality up to compact terms.

1.3 Coupled problem

In this section, we derive a variational formulation for the system (1.1.3a)-(1.1.4b) which couples a mixed variational formulation defined in the interior domain to a boundary integral equation of the first kind that arises in the exterior domain.

As proposed in [3], we introduce a new variable $P = -\mathrm{div}(\epsilon(\mathbf{x})\mathbf{U})$ into equation (1.1.3a) to dispense with trial spaces contained in $\mathbf{H}(\mathbf{curl}, \Omega_s) \cap \mathbf{H}(\mathrm{div}, \Omega_s)$. Applying Green's formula (1.2.2c) in Ω_s, we obtain

$$\begin{aligned}\int_{\Omega_s} \mu^{-1}\,\mathbf{curl}\,\mathbf{U}\cdot\mathbf{curl}\,\mathbf{V}\mathrm{d}\mathbf{x} + \int_{\Omega_s}\epsilon\,\nabla P\cdot\mathbf{V}\mathrm{d}\mathbf{x}&\\ -\omega^2\int_{\Omega_s}\epsilon\,\mathbf{U}\cdot\mathbf{V}\mathrm{d}\mathbf{x} + \langle\gamma^-_{R,\mu}\mathbf{U}, \gamma^-_t\mathbf{V}\rangle_\tau &= (\mathbf{J},\mathbf{V})_{\Omega_s},\\ \int_{\Omega_s} P\,Q\mathrm{d}\mathbf{x} - \int_{\Omega_s}\epsilon\,\mathbf{U}\cdot\nabla Q\mathrm{d}\mathbf{x} + \langle\gamma^-_{n,\epsilon}\mathbf{U}, \gamma^- Q\rangle_\Gamma &= 0\end{aligned} \tag{1.3.1}$$

for all $\mathbf{V} \in \mathbf{H}(\mathbf{curl}, \Omega_s)$, $Q \in H^1(\Omega_s)$. The volume integrals in these equations enter the interior bi-linear form

$$\begin{aligned}\mathfrak{B}_\kappa(\begin{pmatrix}\mathbf{U}\\ P\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\ Q\end{pmatrix}) := \int_{\Omega_s}\mu^{-1}\,\mathbf{curl}\,\mathbf{U}\cdot\mathbf{curl}\,\mathbf{V}\mathrm{d}\mathbf{x} + \int_{\Omega_s}\epsilon\,\nabla P\cdot\mathbf{V}\mathrm{d}\mathbf{x}&\\ + \int_{\Omega_s} P\,Q\mathrm{d}\mathbf{x} - \int_{\Omega_s}\epsilon\,\mathbf{U}\cdot\nabla Q\mathrm{d}\mathbf{x} - \omega^2\int_{\Omega_s}\epsilon\,\mathbf{U}\cdot\mathbf{V}\mathrm{d}\mathbf{x}&\end{aligned} \tag{1.3.2}$$

related to the one supplied for the Hodge-Laplace operator in [4, Sec. 3.2]. We aim to couple (1.3.2) with the BIEs replacing the PDEs in Ω'. We use the transmission conditions (1.1.4a)-(1.1.4b) to couple (1.3.1) to the variational equation

$$\mathfrak{B}_\kappa(\begin{pmatrix}\mathbf{U}\\ P\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\ Q\end{pmatrix}) + \left\langle \mathcal{T}^+_N(\mathbf{U}^{\mathrm{ext}}), \begin{pmatrix}\gamma^-_t\mathbf{V}\\ \gamma^- Q\end{pmatrix}\right\rangle = \mathscr{G}(\begin{pmatrix}\mathbf{V}\\ Q\end{pmatrix}),$$

which involves a functional

$$\mathscr{G}((\mathbf{V}\,Q)^\top) := (\mathbf{J},\mathbf{V})_{\Omega_s} - \langle(\mathbf{g}_R\,g_n)^\top, (\gamma^-_t\mathbf{V}\,\gamma^- Q)^\top\rangle$$

bounded over the test space. The exterior Calderón projector can be used to express the so-called Dirichlet-to-Neumann operator in two different ways.

1. Introducing the jump conditions into the *first exterior Calderón identity* given on the first line of (1.2.12b) along with a new unknown $\overrightarrow{\mathbf{p}} = \mathcal{T}^+_N(\mathbf{U}^{\mathrm{ext}})$ yields a variational system

$$\begin{aligned}\mathfrak{B}_\kappa(\begin{pmatrix}\mathbf{U}\\ P\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\ Q\end{pmatrix}) + \left\langle \overrightarrow{\mathbf{p}}, \begin{pmatrix}\gamma^-_t\mathbf{V}\\ \gamma^- Q\end{pmatrix}\right\rangle &= \mathscr{G}(\begin{pmatrix}\mathbf{V}\\ Q\end{pmatrix}),\\ \left\langle(\{\mathcal{T}_\mathcal{D}\}\cdot\mathcal{DL}_\kappa + \frac{1}{2}\mathrm{Id})\mathcal{T}^-_{D,\epsilon}(\mathbf{U}), \overrightarrow{\mathbf{a}}\right\rangle + \left\langle\{\mathcal{T}_D\}\cdot\mathcal{SL}_\kappa(\overrightarrow{\mathbf{p}}), \overrightarrow{\mathbf{a}}\right\rangle &= \mathscr{R}(\overrightarrow{\mathbf{a}}),\end{aligned} \tag{1.3.3}$$

for all $(\mathbf{V}\,Q)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$ and $\vec{\mathbf{a}} \in \mathcal{H}_N$, resembling the original Johnson-Nedélec coupling [6]. The new functional appearing on the right hand side of (1.3.3) is defined as

$$\mathscr{R}(\vec{\mathbf{a}}) := \langle(\{\mathcal{T}_D\} \cdot \mathcal{DL}_\kappa + \frac{1}{2}\mathrm{Id})(\boldsymbol{\zeta}_t, \zeta_D)^\top, \vec{\mathbf{a}}\rangle. \tag{1.3.4}$$

2. Following the exposition of Costabel in [18], we also retain the *second exterior Calderón identity* —in which we again introduce the jump conditions to eliminate the dependence on the exterior solution— and insert the resulting equation in (1.3.3) to obtain the symmetrically coupled problem. Again, the right hand side of our system of equations has to be modified to include a new bounded linear functional

$$\mathscr{F}(\vec{\mathbf{V}}) := \mathscr{G}(V) + \langle -\{\mathcal{T}_N\} \cdot \mathcal{DL}_\kappa(\boldsymbol{\zeta}_t, \zeta_D)^\top, (\gamma_t^- \mathbf{V}, \gamma^- Q)^\top\rangle. \tag{1.3.5}$$

We arrive at the following variational problem.

Find $\vec{\mathbf{U}} := (\mathbf{U},\ P)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$ and $\vec{\mathbf{p}} \in \mathcal{H}_N$ such that

$$\begin{aligned}\mathfrak{B}_\kappa(\vec{\mathbf{U}}, \vec{\mathbf{V}}) + \Big\langle(-\mathbb{A}_\kappa^{NN} + \frac{1}{2}\mathrm{Id})\vec{\mathbf{p}}, \begin{pmatrix}\gamma_t^-\mathbf{V}\\ \gamma^- Q\end{pmatrix}\Big\rangle &\\ + \Big\langle -\mathbb{A}_\kappa^{DN}\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix}, \begin{pmatrix}\gamma_t^-\mathbf{V}\\ \gamma^- Q\end{pmatrix}\Big\rangle &= \mathscr{F}(\vec{\mathbf{V}}) \\ \Big\langle(\mathbb{A}_\kappa^{DD} + \frac{1}{2}\mathrm{Id})\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix}, \vec{\mathbf{a}}\Big\rangle + \Big\langle\mathbb{A}_\kappa^{DD}(\vec{\mathbf{p}}), \vec{\mathbf{a}}\Big\rangle &= \mathscr{R}(\vec{\mathbf{a}}),\end{aligned} \tag{1.3.6}$$

for all $\vec{\mathbf{V}} := (\mathbf{V},\ Q)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$, $\vec{\mathbf{a}} \in \mathcal{H}_N$.

Remark 1.6 Part of the justification for using mixed formulations for problems involving the Hodge–Helmholtz/Laplace operator is the need to avoid trial spaces contained in $\mathbf{H}(\mathbf{curl}, \Omega_s) \cap \mathbf{H}(\mathrm{div}, \Omega_s)$, because the latter doesn't allow for viable discretizations using finite elements [4]. While from (1.3.3) the issue seems to reappear after using the Caldéron identities, the benefits of the introduced new unknown $P \in H^1(\Omega_s)$ in the mixed formulation conveniently carries over to the coupled system (1.3.6) upon substituting $-\gamma^-(P)$ in place of $\gamma_{D,\epsilon}(\mathbf{U})$ in $\mathcal{T}_{D,\epsilon}^-(\mathbf{U})$.

In the following proposition, we call *forbidden resonant frequencies* the interior "Dirichlet" (or electric) eigenvalues of the scaled Hodge-Laplace operator with constant coefficient $\eta = \mu_0\epsilon_0^2$. That is, κ^2 is a forbidden frequency if there exists a *non-trivial* solution $\mathbf{U} \neq 0$ in $\mathbf{X}(\Delta, \Omega)$ to

$$\begin{aligned}\Delta_\eta \mathbf{U} - \kappa^2\mathbf{U} &= 0, && \text{in } \Omega_s,\\ \mathcal{T}_D^-\mathbf{U} &= 0, && \text{on } \Gamma.\end{aligned}$$

We refer the reader to [15], where the spectrum of the scaled Hodge-Laplace operator is completely characterized. See for e.g. [36], [35], [13], [19] and [17] for an overview of the issue of spurious resonances in electromagnetic and acoustic scattering models based on integral equations.

Proposition 1.2 *Suppose that $\kappa^2 \in \mathbb{C}$ avoids forbidden resonant frequencies. By retaining an interior solution $U \in \mathbf{H}(\mathbf{curl}, \Omega_s)$ and producing $\mathbf{U}^{ext} \in \mathbf{X}_{\text{loc}}(\Delta, \Omega')$ using the representation formula (1.2.3) for the obtained Cauchy data $(\overrightarrow{\mathbf{p}}, \mathcal{T}^-_{D,\epsilon}U - (\boldsymbol{\zeta}_t,\ \zeta_D)^\top)$ with $\gamma^-_{D,\epsilon}(\mathbf{U}) = -\gamma^-(P)$, a solution to (1.3.6) solves the transmission system (1.1.3a)-(1.1.4b) in the sense of distribution.*

Proof. The proof follows the approach in [25, Lem. 6.1]. Since $\mathscr{D}(\Omega_s)^3 \times C_0^\infty(\Omega_s)$ is a subset of the volume test space, any solution to the problem (1.3.6) solves (1.1.3a) in Ω_s in the sense of distribution. It follows that (1.3.1) holds for all admissible $\overrightarrow{\mathbf{V}}$, which reduces (1.3.6) to the variational system

$$
\begin{aligned}
0 &= \left\langle (\mathbb{A}^{DD}_\kappa + \frac{1}{2}\mathrm{Id})\overrightarrow{\boldsymbol{\xi}}, \overrightarrow{\boldsymbol{\eta}} \right\rangle + \left\langle \{\mathbb{A}^{ND}_\kappa(\overrightarrow{\mathbf{p}}), \overrightarrow{\boldsymbol{\eta}} \right\rangle \\
0 &= -\left\langle \overrightarrow{\mathbf{q}}, \begin{pmatrix} \gamma_t^-\mathbf{V} \\ \gamma^- Q \end{pmatrix} \right\rangle + \left\langle (-\mathbb{A}^{NN}_\kappa + \frac{1}{2}\mathrm{Id})\overrightarrow{\mathbf{p}}, \begin{pmatrix} \gamma_t^-\mathbf{V} \\ \gamma^- Q \end{pmatrix} \right\rangle - \left\langle \mathbb{A}^{DN}_\kappa(\overrightarrow{\boldsymbol{\xi}}), \begin{pmatrix} \gamma_t^-\mathbf{V} \\ \gamma^- Q \end{pmatrix} \right\rangle
\end{aligned}
$$

where $\overrightarrow{\mathbf{q}} := \mathcal{T}^-_{N,\mu}(\mathbf{U}) - (\mathbf{g}_R,\ g_n)^\top$ and $\overrightarrow{\boldsymbol{\xi}} := \mathcal{T}^-_{D,\epsilon}(\mathbf{U}) - (\boldsymbol{\zeta}_t,\ \zeta_D)^\top$.

We recognize in the equivalent operator equation

$$
\underbrace{\begin{pmatrix} \mathbb{A}^{NN}_\kappa + \frac{1}{2}\mathrm{Id} & \mathbb{A}^{DN}_\kappa \\ \mathbb{A}^{ND}_\kappa & \mathbb{A}^{DD}_\kappa + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{\mathbb{P}^-_\kappa} \begin{pmatrix} \overrightarrow{\mathbf{p}} \\ \overrightarrow{\boldsymbol{\xi}} \end{pmatrix} = \begin{pmatrix} \overrightarrow{\mathbf{p}} - \overrightarrow{\mathbf{q}} \\ 0 \end{pmatrix} \tag{1.3.7}
$$

the interior Caldéron projector (1.2.12a) whose image is the space of valid Cauchy data for the homogeneous (scaled) Hodge–Laplace/Helmholtz interior equation with constant coefficient η. In particular, $\overrightarrow{\mathbf{p}} - \overrightarrow{\mathbf{q}} = \mathcal{T}^-_N(\tilde{\mathbf{U}})$ for some vector-field $\tilde{\mathbf{U}} \in \mathbf{X}(\Delta, \Omega_s)$ satisfying

$$
\begin{aligned}
\Delta_\eta \tilde{\mathbf{U}} - \kappa^2\tilde{\mathbf{U}} &= 0, && \text{in } \Omega_s \\
\mathcal{T}^-_D(\tilde{\mathbf{U}}) &= 0, && \text{on } \Gamma.
\end{aligned} \tag{1.3.8}
$$

If $\kappa^2 \neq 0$, we rely on the hypothesis that κ^2 doesn't belong to the set of forbidden resonant frequencies to guarantee injectivity of the above boundary value problem [15, Sec. 3] [22, Sec. 3]. Otherwise, the second Betti number of Ω_s being zero implies that zero is not a Dirichlet eigenvalue [2, Sec. 4.5.3]. We conclude that $\tilde{\mathbf{U}} = 0$ is the unique trivial solution to (1.3.8). Therefore, for the right hand side of (1.3.7) to exhibit valid Neumann data, it must be that $\overrightarrow{\mathbf{p}} = \overrightarrow{\mathbf{q}}$.

Now, the null space of the interior Caldéron projector $\mathbb{P}^-_\kappa$ coincides with valid Cauchy data for the exterior boundary value problem (1.1.3b) complemented with the radiation conditions at infinity introduced in Subsection 1.1. In particular $(\overrightarrow{\mathbf{p}},\ \overrightarrow{\boldsymbol{\xi}})^\top$ is valid Cauchy data for that exterior Hodge–Helmholtz or Hodge–Laplace problem and $\mathbf{U}^{\text{ext}} = \mathcal{SL}_\kappa(\overrightarrow{\mathbf{p}}) + \mathcal{DL}_\kappa(\overrightarrow{\boldsymbol{\xi}})$ solves (1.1.3b) and (1.1.4b) by construction. The fact that $\overrightarrow{\mathbf{p}} = \mathcal{T}^+_N(\mathbf{U}^{\text{ext}})$ solves (1.1.4a) is confirmed by the earlier observation that $\overrightarrow{\mathbf{p}} = \overrightarrow{\mathbf{q}}$. □

Corollary 1.1 *Suppose that $\kappa^2 \in \mathbb{C}$ avoids forbidden resonant frequencies. A solution pair $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{p}})$ to the coupled problem (1.3.6) is unique.*

Remark 1.7 We show in [36], where the kernel of the coupled problem is completely characterized, that when κ^2 happens to be a resonant frequency, the interior solution $\mathbf{U}$ remains unique. This is no longer true for $\overrightarrow{\mathbf{p}}$ however, which is in general only unique up to Neumann traces of interior Dirichlet eigenfunctions of Δ_η associated to the eigenvalue κ^2. Fortunately, this kernel vanishes under the exterior representation formula obtained from (1.2.3).

1.4 Space decompositions

Using the classical Hodge decomposition, a general inf-sup condition for Hodge–Laplace problems posed on closed Hilbert complexes was derived in [4]. However, as orthogonality won't be important, we rather opt for the enhanced regularity of the regular decomposition proposed in [11] and [15]. There, a continuous projection $\mathsf{Z} : \mathbf{H}(\mathbf{curl}, \Omega_s) \to \mathbf{H}^1(\Omega_s)$ is defined such that $\ker(\mathsf{Z}) = \ker(\mathbf{curl}) \cap \mathbf{H}(\mathbf{curl}, \Omega_s)$ and $\mathbf{curl}(\mathsf{Z}(\mathbf{U})) = \mathbf{curl}(\mathbf{U})$. From Rellich's theorem, this operator is compact as a mapping $\mathsf{Z} : \mathbf{H}(\mathbf{curl}, \Omega_s) \to \mathbf{L}^2(\Omega_s)$. Therefore, a stable direct regular decomposition

$$\mathbf{H}(\mathbf{curl}, \Omega_s) = \mathbf{X}(\mathbf{curl}, \Omega_s) \oplus \mathbf{N}(\mathbf{curl}, \Omega_s). \tag{1.4.1}$$

is provided by defining the subspaces

$$\mathbf{X}(\mathbf{curl}, \Omega_s) := \mathsf{Z}(\mathbf{H}(\mathbf{curl}, \Omega_s)),$$
$$\mathbf{N}(\mathbf{curl}, \Omega_s) := \ker(\mathbf{curl}) \cap \mathbf{H}(\mathbf{curl}, \Omega_s).$$

A decomposition with similar properties can be designed for the space $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ with a projection operator $\mathsf{Z}^\Gamma : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_R^{1/2}(\Gamma)$ satisfying $\ker(\mathsf{Z}^\Gamma) = \ker(\mathrm{div}_\Gamma) \cap \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathrm{div}_\Gamma(\mathsf{Z}^\Gamma(\mathbf{p})) = \mathrm{div}_\Gamma(\mathbf{p})$.

As before, the extra regularity of the range, in this case provided by [25, Lem. 3.2], leads to compactness of the mapping $\mathsf{Z}^\Gamma : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_R^{-1/2}(\Gamma)$.

The subspaces

$$\mathbf{X}(\mathrm{div}_\Gamma, \Gamma) := \mathsf{Z}^\Gamma(\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)),$$
$$\mathbf{N}(\mathrm{div}_\Gamma, \Gamma) := \ker(\mathrm{div}_\Gamma) \cap \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma),$$

provide a stable direct regular decomposition

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \oplus \mathbf{N}(\mathrm{div}_\Gamma, \Gamma). \tag{1.4.2}$$

In the following, we may simplify notation by using $\mathbf{U}^\perp := \mathsf{Z}\mathbf{U}$, $\mathbf{p}^\perp := \mathsf{Z}^\Gamma\mathbf{p}$, $\mathbf{U}^0 := (\mathrm{Id} - \mathsf{Z})\mathbf{U}$ and $\mathbf{p}^0 := (\mathrm{Id} - \mathsf{Z}^\Gamma)\mathbf{p}$.

A very useful property of this pair of decompositions is stated is shown in [25, Lem. 8.1] and [25, Lem. 8.2]: The operators

$$(\gamma_t^-)' \circ (\{\gamma_R\}\boldsymbol{\Psi}_\kappa + \frac{1}{2}\mathrm{Id}) : \mathbf{N}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{N}(\mathbf{curl}, \Omega_s)', \tag{1.4.3a}$$

and

$$(\gamma_t^-)' \circ (\{\gamma_R\}\boldsymbol{\Psi}_\kappa + \frac{1}{2}\mathrm{Id}) : \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}(\mathbf{curl}, \Omega_s)' \tag{1.4.3b}$$

are compact.

Another benefit of this pair of regular decompositions will become explicit in the poof of Lemma 1.4 found in the next section.

It follows from [15, Lem. 6.4] that $\mathrm{div}_\Gamma : \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \to H_*^{-1/2}(\Gamma)$ is a continuous bijection. The bounded inverse theorem guarantees the existence of a continuous inverse $(\mathrm{div}_\Gamma)^\dagger : H_*^{-1/2}(\Gamma) \to \mathbf{X}(\mathrm{div}_\Gamma, \Gamma)$ such that

$$(\mathrm{div}_\Gamma)^\dagger \circ \mathrm{div}_\Gamma = \mathrm{Id}\Big|_{\mathbf{X}(\mathrm{div}_\Gamma,\Gamma)}, \qquad \mathrm{div}_\Gamma \circ (\mathrm{div}_\Gamma)^\dagger = \mathrm{Id}\Big|_{H_*^{-1/2}(\Gamma)}.$$

1.5 Well-posedness of the coupled variational problem

We use the direct decompositions introduced in Section 1.4 to prove that the bilinear form associated to the coupled system (1.2) of Section 1.3 satisfies a generalized Gårding inequality.

The coupled variational problem (1.3.6) translates into the operator equation

$$\mathbb{G}_\kappa \begin{pmatrix} \overrightarrow{\mathbf{U}} \\ \overrightarrow{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathscr{F} \\ \mathscr{R} \end{pmatrix} \in (\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega))' \times (\mathcal{H}_N)'.$$

Letting $\mathsf{B}_\kappa : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \to (\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s))'$ be the operator

$$\langle \mathsf{B}_\kappa(\overrightarrow{\mathbf{U}})\overrightarrow{\mathbf{V}} \rangle := \mathfrak{B}_\kappa(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}})$$

associated with the Hodge–Helmholtz/Laplace volume contribution to the system, the operator

$$\mathbb{G}_\kappa : (\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega)) \times \mathcal{H}_N \to (\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega))' \times (\mathcal{H}_N)'$$

can be represented by the block operator matrix

$$\mathbb{G}_\kappa = \left(\begin{array}{c|c} \mathsf{B}_\kappa - \begin{pmatrix} (\gamma_t^-)' \\ (\gamma^-)' \end{pmatrix} \cdot \mathbb{A}_\kappa^{DN} \cdot \begin{pmatrix} \gamma_t^- \\ -\gamma^- \end{pmatrix} & \begin{pmatrix} (\gamma_t^-)' \\ (\gamma^-)' \end{pmatrix} \cdot (\mathbb{P}_\kappa^+)_{22} \\ \hline (\mathbb{P}_\kappa^-)_{11} \cdot \begin{pmatrix} \gamma_t^- \\ -\gamma^- \end{pmatrix} & \mathbb{A}_\kappa^{ND} \end{array}\right),$$

shown here in “variational arrangement".

The symmetry revealed in Subsection 1.2.3 makes explicit much of the structure of the above operator. We have introduced colors to better highlight the contribution of each individual block in the following sections.

Our goal is to design an isomorphism $\mathbb{X}$ of the test space and resort to compact perturbations of $\mathbb{G}_\kappa \circ \mathbb{X}^{-1}$ to achieve an operator block structure with diagonal blocks that are elliptic over the splittings of Section 1.4 and off-diagonal blocks that fit a skew-symmetric pattern. Stability of the coupled system can then be obtained from the next theorem. An overline indicates component-wise complex conjugation.

Theorem 1.1 ([11, Thm. 4]) *If a bilinear form $a : V \times V \to \mathbb{C}$ on a reflexive Banach space V is T-coercive:*

$$\left|a(u, \mathbb{X}\overline{u}) + c(u, \overline{u})\right| \geq C\|u\|_V^2 \quad \forall u \in V, \tag{1.5.1}$$

with $C > 0$, $c : V \times V \to \mathbb{C}$ compact and $\mathbb{X} : V \to V$ an isomorphism of V, then the operator $A : V \to V'$ defined by $A : u \mapsto a(u, \cdot)$ is Fredholm with index 0.

The authors of [9] refer to (1.5.1) as "Generalized Gårding inequality", because

$$\left|a(u, \mathbb{X}\overline{u})\right| \geq C\|u\|_V^2 - \left|c(u, \overline{u})\right| \qquad \forall\, u \in V,$$

generalizes the classical Gårding inequality for a bilinear form b associated with uniformly elliptic operator of even order 2ℓ: $\exists\, C_2 \geq 0, C_1 > 0$ such that

$$b(u, u) \geq C_1\|u\|^2_{H^\ell(\Omega)} - C_2\|u\|_{L^2(\Omega)} \qquad \forall\, u \in H_0^\ell(\Omega).$$

Assuming that (1.5.1) holds with $\mathbb{X} = \mathrm{Id}$, a simple proof of the stability estimate $\|u\|_V \leq C\|f\|_{V'}$, obtained for the unique solution of the operator equation $Au = f$ when A is injective is given in [38, Thm. 3.15]. A proof of the general case can be deduced from [23]. T-coercivity theory is a reformulation of the Banach-Necas-Babuska theory. The former relies on the construction of explicit inf-sup operators at the discrete and continuous levels, whereas the later develops on an abstract inf-sup condition [14].

In deriving the following results, it will be convenient to denote $\overrightarrow{\mathbf{U}} := (\mathbf{U},\ P)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega)$ and $\overrightarrow{\mathbf{p}} := (\mathbf{p},\ q)^\top \in \mathcal{H}_N$. We indicate with a hat equality up to a compact perturbation (e.g. $\hat{=}$).

1.5.1 Space isomorphisms

In this section, we take up the challenge of finding a suitable isomorphism $\mathbb{X}$. We build it separately for the function spaces in Ω_s and on the boundary Γ. Crucial hints are offered by the construction of the sign-flip isomorphism for the classical electric wave equation in [11].

We start with devising an isomorphism Ξ of the volume function spaces and show that the upper-left diagonal block of $\mathbb{G}_\kappa$ satisfy a generalized Gårding inequality.

Under the assumption that the first Betti number of Ω_s is zero, there exists a bijective "scalar potential lifting" $\mathsf{S} : \mathbf{N}(\mathbf{curl}, \Omega_s) \to H^1_*(\Omega_s)$ satisfying $\nabla\mathsf{S}(\mathbf{U}) = \mathbf{U}$. The Poincaré-Friedrichs inequality guarantees that this map is continuous.

Notice that since it also follows from the Poincaré-Friedrichs inequality that $\nabla : H^1_*(\Omega_s) \to \mathbf{N}(\mathbf{curl}, \Omega_s)$ is injective, $\mathsf{S} \circ \nabla : H^1(\Omega_s) \to H^1_*(\Omega_s)$ is a bounded projection onto the space of Lebesgue measurable functions having zero mean. Its nullspace consists of the constant functions in Ω_s.

Proposition 1.3 *For any $\theta > 0$ and $\beta > 0$, the bounded linear operator $\Xi : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \to \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$ defined by*

$$\Xi(\overrightarrow{\mathbf{U}}) := \begin{pmatrix} \mathbf{U}^\perp - \mathbf{U}^0 + \beta \nabla P \\ -\theta(\mathsf{S}(\mathbf{U}^0) + \beta\, \mathbf{mean}(P)) \end{pmatrix}, \qquad \overrightarrow{\mathbf{U}} = (\mathbf{U},\ P)^\top,$$

has a continuous inverse. In other words, Ξ is an isomorphism of Banach spaces.

Proof. By showing that Ξ is a bijection, the theorem follows as a consequence of the bounded inverse theorem.

Let $(\mathbf{V}\ Q)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$. Since $\nabla Q \in \mathbf{N}(\mathbf{curl}, \Omega_s)$, we immediately have $\mathsf{Z}(\mathbf{V}^\perp - \theta^{-1}\nabla Q) = \mathbf{V}^\perp$ and $(\mathrm{Id} - \mathsf{Z})(\mathbf{V}^\perp - \theta^{-1}\nabla Q) = -\theta^{-1}\nabla Q$. Hence, relying on the resulting observation that

$$\nabla \mathsf{S}((\mathbf{V}^\perp - \theta^{-1}\nabla Q)^0) = -\theta^{-1}\nabla Q$$

and exploiting that $\mathbf{mean}(H^1_*(\Omega_s)) = \{0\}$, we have

$$\Xi(\begin{pmatrix} \mathbf{V}^\perp - \theta^{-1}\nabla Q \\ \beta^{-1}(\mathsf{S}(\mathbf{V}^0) - \theta^{-1}Q) \end{pmatrix}) = \begin{pmatrix} \mathbf{V} \\ \mathsf{S}(\nabla Q) + \mathbf{mean}(Q) \end{pmatrix}. \tag{1.5.2}$$

Since $H^1(\Omega_s)$ decomposes into the stable direct sum of $H^1_*(\Omega_s)$ and the space of constant functions in Ω_s, (1.5.2) shows that Ξ is surjective.

Now, suppose that $\Xi(\overrightarrow{\mathbf{V}}) = \Xi(\overrightarrow{\mathbf{U}})$. Then, we have

$$\mathbf{U}^0 - \mathbf{V}^0 = \nabla \mathsf{S}(\mathbf{U}^0 - \mathbf{V}^0) = \beta\, \nabla(\mathbf{mean}(Q - P)) = 0.$$

Since the considerations of Section 1.4 readily yield that $\mathbf{V}^\perp = \mathbf{U}^\perp$, we conclude that $\mathbf{V} = \mathbf{U}$. In turn, it follows that $\nabla P = \nabla Q$ and $\mathbf{mean}(P) = \mathbf{mean}(Q)$. Therefore, Ξ is injective. □

We now turn to the design of an isomorphism for the Neumann trace space $\mathcal{H}_N$ and prove that the lower-right block $\mathbb{A}_\kappa^{ND}$ of $\mathbb{G}_\kappa$ satisfies a generalized Gårding inequality.

Proposition 1.4 *For any $\tau > 0$ and $\lambda > 0$, the bounded linear operator $\Xi^\Gamma : \mathcal{H}_N \to \mathcal{H}_N$ defined by*

$$\Xi^\Gamma(\overrightarrow{\mathbf{p}}) := \begin{pmatrix} \mathbf{p}^\perp - \mathbf{p}^0 - \lambda(\mathrm{div}_\Gamma)^\dagger \mathsf{Q}_* q \\ -\tau(\mathrm{div}_\Gamma(\mathbf{p}) + \lambda\, \mathbf{mean}(q)) \end{pmatrix}, \qquad \overrightarrow{\mathbf{p}} = (\mathbf{p},\ q)^\top,$$

has a continuous inverse. In other words, Ξ^Γ is an isomorphism of Banach spaces.

Proof. We proceed as in proposition 1.3. Since $(\mathrm{div}_\Gamma)^\dagger \mathsf{Q}_* q \in \mathbf{X}(\mathrm{div}_\Gamma, \Gamma)$, we have $\mathsf{Z}^\Gamma(\Xi^\Gamma_1(\overrightarrow{\mathbf{p}})) = \mathbf{p}^\perp - (\mathrm{div}_\Gamma)^\dagger \mathsf{Q}_* q$. Using that $\mathbf{mean} \circ \mathrm{div}_\Gamma = 0$ and $(\mathrm{div}_\Gamma)^\dagger \mathrm{div}_\Gamma \mathbf{p} = \mathbf{p}^\perp$, we evaluate

$$\Xi^\Gamma(\begin{pmatrix} -\mathbf{p}^0 - \tau^{-1}(\mathrm{div}_\Gamma)^\dagger \mathsf{Q}_* q \\ \lambda^{-1}(-\mathrm{div}_\Gamma(\mathbf{p}) - \tau^{-1} q) \end{pmatrix}) = \begin{pmatrix} \mathbf{p}^0 + \mathbf{p}^\perp \\ \mathsf{Q}_* q + \mathbf{mean}(q) \end{pmatrix}.$$

This shows that Ξ^Γ is surjective.

Suppose that $X^\Gamma(\overrightarrow{\mathbf{p}}) = X^\Gamma(\overrightarrow{\mathbf{a}})$. It is immediate that $\mathbf{p}^0 = \mathbf{a}^0$. On the one hand, we obtain from $X^\Gamma_1(\overrightarrow{\mathbf{p}}) = X^\Gamma_1(\overrightarrow{\mathbf{a}})$ that

$$\mathbf{p}^\perp - \mathbf{a}^\perp = \lambda(\mathrm{div}_\Gamma)^\dagger(\mathsf{Q}_* q - \mathsf{Q}_* b). \tag{1.5.3}$$

On the other hand, $X_2^\Gamma(\overrightarrow{\mathbf{p}}) = X_2^\Gamma(\overrightarrow{\mathbf{a}})$ implies that

$$\mathrm{div}_\Gamma(\mathbf{p} - \mathbf{a}) = \lambda\,\mathbf{mean}(q - b). \tag{1.5.4}$$

Relying on the fact that $\mathrm{div}_\Gamma = \mathrm{div}_\Gamma \circ \mathsf{Z}^\Gamma$ again, combining (1.5.3) and (1.5.4) yields

$$\mathsf{Q}_* q + \mathbf{mean}(q) = \mathsf{Q}_* b + \mathbf{mean}(b).$$

Evidently, (1.5.3) then also guarantees that $\mathbf{p}^\perp = \mathbf{a}^\perp$. We can finally conclude that X^Γ is injective and thus the result follows from the bounded inverse theorem. □

In the following, we will write Ξ_1^Γ and Ξ_2^Γ for the components of the isomorphism of the trace space.

1.5.2 Main result

The main result of this work, stated in Theorem 1.2, asserts that the operator $\mathbb{G}_\kappa$ associated with the coupled system (1.3.6) is well-posed when κ^2 lies outside the discrete set of forbidden frequencies described in [15]. It relies on two propositions, whose proofs are postponed until the end of Section 1.5.

The first claims that the *block diagonal of* $\mathbb{G}_\kappa$ (as a sum of block operators) *is T-coercive*.

Proposition 1.5 *For any frequency $\omega \geq 0$, there exist a compact operator $\mathsf{K} : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N \to \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N$, a positive constant $C > 0$ and parameters $\theta > 0$ and $\tau > 0$, possibly depending on Ω_s, ϵ, μ, κ and ω, such that*

$$\mathfrak{Re}\left\langle \mathrm{diag}(\mathbb{G}_\kappa)\begin{pmatrix}\overrightarrow{\mathbf{U}}\\ \overrightarrow{\mathbf{p}}\end{pmatrix}, \begin{pmatrix}\Xi\,\overrightarrow{\overrightarrow{\mathbf{U}}}\\ \Xi^\Gamma\overrightarrow{\overrightarrow{\mathbf{p}}}\end{pmatrix}\right\rangle + \left\langle \mathsf{K}\begin{pmatrix}\overrightarrow{\mathbf{U}}\\ \overrightarrow{\mathbf{p}}\end{pmatrix}, \begin{pmatrix}\overrightarrow{\overrightarrow{\mathbf{U}}}\\ \overrightarrow{\overrightarrow{\mathbf{p}}}\end{pmatrix}\right\rangle \geq C(\|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_s)} + \|P\|^2_{H^1(\Omega_s)} + \|\overrightarrow{\mathbf{p}}\|^2_{\mathcal{H}_N})$$

for all $\overrightarrow{\mathbf{U}} := (\mathbf{U}\ \ P)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$ and $\overrightarrow{\mathbf{p}} \in \mathcal{H}_N$.

The proof of this proposition will rely on several steps: Lemma 1.3, Lemma 1.4 and Lemma 1.5.

The second proposition states that the off-diagonal blocks are compact operators. The proof of that fact relies on definitions and results that belong to the next technical section. It will materialize as the last piece of the puzzle that completes the proof of the T-coercivity of $\mathbb{G}_\kappa$.

Proposition 1.6 *For any frequency $\omega \geq 0$, there exists, for a suitable choice of τ, β, θ and λ, a continuous compact endomorphism K of the space $\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N$ such that*

$$\mathfrak{Re}\left\langle (\mathbb{G}_\kappa - \mathrm{diag}(\mathbb{G}_\kappa))\begin{pmatrix}\overrightarrow{\mathbf{U}}\\ \overrightarrow{\mathbf{p}}\end{pmatrix}, \begin{pmatrix}\Xi\,\overrightarrow{\overrightarrow{\mathbf{U}}}\\ \Xi^\Gamma\overrightarrow{\overrightarrow{\mathbf{p}}}\end{pmatrix}\right\rangle = \left\langle \mathsf{K}\begin{pmatrix}\overrightarrow{\mathbf{U}}\\ \overrightarrow{\mathbf{p}}\end{pmatrix}, \begin{pmatrix}\overrightarrow{\overrightarrow{\mathbf{U}}}\\ \overrightarrow{\overrightarrow{\mathbf{p}}}\end{pmatrix}\right\rangle. \tag{1.5.5}$$

The main result immediately follows from the two previous propositions.

Theorem 1.2 *For any $\omega \geq 0$, there exists an isomorphism $\mathbb{X}_\kappa$ of the trial space $\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N$, and compact operator $\mathbb{K} : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N \to (\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s))' \times \mathcal{H}'_N$ such that*

$$\mathfrak{Re} \left\langle (\mathbb{G}_\kappa + \mathbb{K}) \begin{pmatrix} \overrightarrow{\mathbf{U}} \\ \overrightarrow{\mathbf{p}} \end{pmatrix}, \mathbb{X} \begin{pmatrix} \overrightarrow{\overrightarrow{\mathbf{U}}} \\ \overrightarrow{\mathbf{p}} \end{pmatrix} \right\rangle \geq C(\|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl}, \Omega_s)} + \|P\|^2_{H^1(\Omega_s)} + \|\overrightarrow{\mathbf{p}}\|^2_{\mathcal{H}_N})$$

for some positive constant $C > 0$.

Proof. The proof will amount to the validation that the choices of parameters in the previous propositions 1.5 and 1.6 are compatible. □

The following corollary is immediate upon applying Theorem 1.1.

Corollary 1.2 *The system operator $\mathbb{G}_k : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N \to (\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s))' \times \mathcal{H}'_N$ associated with the variational problem (1.3.6) is Fredholm of index 0.*

Injectivity, guaranteed when κ^2 avoids resonant frequencies by corollary 1.1, yields well-posedness.

1.5.3 T-Coercivity of the diagonal blocks

Equipped with the isomorphism Ξ, let us now study coercivity of the bilinear form $\mathfrak{B}_\kappa$ defined in (1.3.2) and associated to the Hodge–Helmholtz/Laplace operator.

Lemma 1.3 *For any frequency $\omega \geq 0$ and parameter $\beta > 0$, there exist a positive constant $C > 0$ and a parameter $\theta > 0$, possibly depending on Ω_s, μ, ϵ and ω, and a compact bounded sesqui-linear form $\mathfrak{K}$ defined over $\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$, such that*

$$\mathfrak{Re}(\mathfrak{B}_\kappa(\overrightarrow{\mathbf{U}}, \Xi\,\overrightarrow{\overrightarrow{\mathbf{U}}}) - \mathfrak{K}(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{U}})) \geq C(\|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl}, \Omega_s)} + \|P\|^2_{H^1(\Omega_s)})$$

for all $\overrightarrow{\mathbf{U}} := (\mathbf{U},\ P)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$.

Proof. As $\mathbf{curl}(\mathbf{U}^0) = 0$, $\mathbf{curl}(\nabla P) = 0$, and $\nabla \circ \mathbf{mean} = 0$, we evaluate

$$\begin{aligned} &\mathfrak{B}_\kappa(\begin{pmatrix} \mathbf{U} \\ P \end{pmatrix}, \begin{pmatrix} \overline{\mathbf{U}}^\perp - \overline{\mathbf{U}}^0 + \beta \overline{\nabla P} \\ -\theta(\mathsf{S}(\overline{\mathbf{U}}^0) + \beta\, \mathbf{mean}(\overline{P})) \end{pmatrix}) \\ &= (\mu^{-1}\mathbf{curl}(\mathbf{U}^\perp), \mathbf{curl}(\mathbf{U}^\perp))_{\Omega_s} + (\epsilon \nabla P, \mathbf{U}^\perp)_{\Omega_s} - (\epsilon \nabla P, \mathbf{U}^0)_{\Omega_s} \\ &\quad + \beta(\epsilon \nabla P, \nabla P)_{\Omega_s} + \theta(\epsilon \mathbf{U}^\perp, \mathbf{U}^0)_{\Omega_s} + \theta(\epsilon \mathbf{U}^0, \mathbf{U}^0)_{\Omega_s} \end{aligned}$$

$$
\begin{aligned}
&- \omega^2(\epsilon \mathbf{U}^\perp, \mathbf{U}^\perp - \mathbf{U}^0 + \beta \nabla P)_{\Omega_s} - \omega^2(\epsilon \mathbf{U}^0, \mathbf{U}^\perp) + \omega^2(\epsilon \mathbf{U}^0, \mathbf{U}^0) \\
&- \beta\omega^2(\epsilon \mathbf{U}^0, \nabla P) - (P, \theta \mathsf{S}(\mathbf{U}^0))_{\Omega_s} - (P, \theta\beta\, \mathbf{mean}(P))_{\Omega_s}.
\end{aligned}
$$

Upon application of the Cauchy-Schwartz inequality, the bounded sesqui-linear form

$$
\begin{aligned}
\mathfrak{K}(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{U}}) := {}& (\epsilon \nabla P, \mathbf{U}^\perp)_{\Omega_s} - (P, \theta \mathsf{S}(\mathbf{U}^0))_{\Omega_s} + \theta(\epsilon \mathbf{U}^\perp, \mathbf{U}^0)_{\Omega_s} \\
&- \omega^2(\epsilon \mathbf{U}^0, \mathbf{U}^\perp)_{\Omega_s} - \omega^2(\epsilon \mathbf{U}^\perp, \mathbf{U}^\perp - \mathbf{U}^0 + \beta \nabla P)_{\Omega_s} \\
&- (P, \theta\beta\, \mathbf{mean}(P))_{\Omega_s}
\end{aligned}
$$

is shown to be compact by compactness of Z and the Rellich theorem. Using Young's inequality twice with $\delta > 0$, we estimate

$$
\begin{aligned}
&\mathfrak{Re}(\mathfrak{B}_\kappa(\overrightarrow{\mathbf{U}}, \Xi\, \overrightarrow{\mathbf{U}}) - \mathfrak{K}(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{U}})) \\
&\quad \geq \mu_{\text{max}}^{-1} \|\mathbf{curl}\, \mathbf{U}^\perp\|^2_{\Omega_s} + (\epsilon_{\text{min}}(\theta + \omega^2) - \delta\, \epsilon_{\text{max}}(1 + \beta\omega^2))\|\mathbf{U}^0\|^2_{\Omega_s} \\
&\qquad + \mathfrak{Re}(\epsilon_{\text{min}}\, \beta - \frac{1}{\delta}\epsilon_{\text{max}}\, (1 + \beta\omega^2))\|\nabla P\|^2_{\Omega_s}.
\end{aligned}
$$

The operator $\mathbf{curl} : \mathsf{Z}(\mathbf{H}(\mathbf{curl}, \Omega)) \to \mathbf{L}^2(\Omega_s)$ is a continuous injection, hence since its image is closed in $\mathbf{L}^2(\Omega_s)$, it is also bounded below. Hence, for any $\beta > 0$, choose $\delta > 0$ large enough, then $\theta > 0$ accordingly large, and the desired inequality follows. □

The complex inner products

$$
\begin{aligned}
(a, b)_{-1/2} &:= \int_\Gamma \int_\Gamma G_0(\mathbf{x} - \mathbf{y}) a(\mathbf{x})\, \overline{b(\mathbf{y})} \mathrm{d}\sigma(\mathbf{x}) \mathrm{d}\sigma(\mathbf{y}), \\
(\mathbf{a}, \mathbf{b})_{-1/2} &:= \int_\Gamma \int_\Gamma G_0(\mathbf{x} - \mathbf{y}) \mathbf{a}(\mathbf{x}) \cdot \overline{\mathbf{b}(\mathbf{y})} \mathrm{d}\sigma(\mathbf{x}) \mathrm{d}\sigma(\mathbf{y}),
\end{aligned}
$$

defined over $H^{-1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ respectively, are positive definite Hermitian forms and they induce equivalent norms on the trace spaces [9, Sec. 4.1]. Combined with the stability of the decomposition introduced in Section 1.4, this observation also allows us to conclude that

$$
\mathbf{a} \mapsto \|\mathbf{div}_\Gamma(\mathbf{a})\|_{-1/2} + \|(\mathrm{Id} - P^\Gamma)\, \mathbf{a}\|_{-1/2}
$$

also defines an equivalent norm in $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

Let us denote the two components of the isomorphism Ξ by

$$
\Xi_1(\overrightarrow{\mathbf{U}}) := \mathbf{U}^\perp - \mathbf{U}^0 + \nabla P, \quad \Xi_2(\overrightarrow{\mathbf{U}}) := -\theta(\mathsf{S}(\mathbf{U}^0) + \mathbf{mean}(P)).
$$

We now derive an estimate similar to the one found in Lemma 1.3 that completes the proof of the coercivity of the upper-left diagonal block of $\mathbb{G}_\kappa$.

Lemma 1.4 *For any frequency $\omega \geq 0$ and parameter $\beta > 0$, there exist a positive constant $C > 0$ and a parameter $\theta > 0$, possibly depending on Ω_s, μ, ϵ and κ, and a compact linear operator $\mathcal{K} : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \to \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$ such that*

$$\mathfrak{Re}\left(\left\langle -\mathbb{A}_\kappa^{DN}\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix},\begin{pmatrix}\gamma_t^-\Xi_1\overrightarrow{\mathbf{U}}\\ \gamma^-\Xi_2\overrightarrow{\mathbf{U}}\end{pmatrix}\right\rangle\right.$$
$$\left.+\left\langle \mathcal{K}\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix},\begin{pmatrix}\gamma_t^-\Xi_1\overrightarrow{\mathbf{U}}\\ \gamma^-\Xi_2\overrightarrow{\mathbf{U}}\end{pmatrix}\right\rangle\right)\geq C\|\begin{pmatrix}\gamma_t^-\mathbf{U}\\ \gamma^-(P)\end{pmatrix}\|^2_{\mathcal{H}_D(\Omega_s)}$$

for all $\overrightarrow{\mathbf{U}} := (\mathbf{U}\ P)^\top \in \mathbf{H}(\mathbf{curl},\Omega_s)\times H^1(\Omega_s)$.

Proof. The jump condition (1.2.11a) yield $\{\mathcal{T}_N\}\cdot\mathcal{DL}_\kappa = \mathcal{T}_N\cdot\mathcal{DL}_\kappa$. We deduce from [15, Sec. 6.4] that,

$$\begin{aligned}
&\left\langle -\mathcal{T}_N\cdot\mathcal{DL}_\kappa\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix},\begin{pmatrix}\gamma_t^-\Xi_1\overrightarrow{\mathbf{U}}\\ \gamma^-\Xi_2\overrightarrow{\mathbf{U}}\end{pmatrix}\right\rangle\\
&\hat{=}(\mathrm{div}_\Gamma(\mathbf{n}\times\gamma_t^-\mathbf{U})),\mathrm{div}_\Gamma(\mathbf{n}\times\gamma_t^-\Xi_1\overrightarrow{\mathbf{U}})))_{-1/2}\\
&\qquad-\kappa^2(\mathbf{n}\times\gamma_t^-\mathbf{U},\mathbf{n}\times\gamma_t^-\Xi_1\overrightarrow{\mathbf{U}})_{-1/2}+(\mathbf{n}\times\gamma_t^-\mathbf{U},\mathbf{curl}_\Gamma(\gamma^-\Xi_2\overrightarrow{\mathbf{U}}))_{-1/2}\\
&\qquad-(\mathbf{n}\times\gamma_t^-\Xi_1\overrightarrow{\mathbf{U}},\mathbf{curl}_\Gamma(\gamma^-(P)))_{-1/2}\\
&=(\mathrm{div}_\Gamma(\gamma_\tau^-\mathbf{U})),\mathrm{div}_\Gamma(\gamma_\tau^-\Xi_1\overrightarrow{\mathbf{U}})))_{-1/2}-\kappa^2(\gamma_\tau^-\mathbf{U},\gamma_\tau^-\Xi_1\overrightarrow{\mathbf{U}})_{-1/2}\\
&\qquad-(\gamma_\tau^-\mathbf{U},\mathbf{curl}_\Gamma(\gamma^-\Xi_2\overrightarrow{\mathbf{U}}))_{-1/2}+(\gamma_\tau^-\Xi_1\overrightarrow{\mathbf{U}},\mathbf{curl}_\Gamma(\gamma^-(P)))_{-1/2}
\end{aligned}\tag{1.5.6}$$

We consider each component of the isomorphim Ξ in turn. Since $\mathsf{Z}(\mathbf{U})\in\mathbf{H}^1(\Omega_s)$ [1, Lem. 3.5] and $\gamma_t\mathbf{H}^1(\Omega_s)$ is compactly embedded in $\mathbf{L}_t^2(\Gamma)$ [25, Lem. 3.2], the continous mapping $\gamma_\tau\circ\mathsf{Z}:\mathbf{H}(\mathbf{curl},\Omega_s)\to\mathbf{H}_R^{1/2}(\Omega_s)$ is compact. Therefore,

$$\begin{aligned}
\gamma_\tau^-\Xi_1(\overrightarrow{\mathbf{U}})&=\gamma_\tau^-\mathbf{U}^\perp-\gamma_\tau^-\mathbf{U}^0+\beta\gamma_\tau^-\nabla P\\
&\hat{=}\mathsf{Z}^\Gamma(\gamma_\tau^-\mathbf{U})-(\mathrm{Id}-\mathsf{Z}^\Gamma)\gamma_\tau^-\mathbf{U}+\beta\,\mathbf{curl}_\Gamma(\gamma^-P).
\end{aligned}\tag{1.5.7}$$

Let's introduce expression (1.5.7) in the various terms of (1.5.6) involving $\Xi_1(\overrightarrow{\mathbf{U}})$. We find that

$$\begin{aligned}
&\Big(\mathrm{div}_\Gamma\left(\gamma_\tau^-\mathbf{U}\right)\Big),\mathrm{div}_\Gamma(\gamma_\tau^-\Xi_1\overrightarrow{\mathbf{U}}))\Big)_{-1/2}\\
&\qquad\hat{=}(\mathrm{div}_\Gamma(\gamma_\tau\mathbf{U}),\mathrm{div}_\Gamma(\mathsf{Z}^\Gamma(\gamma_\tau^-\mathbf{U})))_{-1/2}\\
&\qquad\qquad-(\mathrm{div}_\Gamma(\gamma_\tau\mathbf{U}),\mathrm{div}_\Gamma((\mathrm{Id}-\mathsf{Z}^\Gamma)\gamma_\tau^-\mathbf{U}))_{-1/2}\\
&\qquad\qquad+\beta(\mathrm{div}_\Gamma(\gamma_\tau\mathbf{U}),\mathrm{div}_\Gamma(\mathbf{curl}_\Gamma(\gamma^-P)))_{-1/2}\\
&\qquad=(\mathrm{div}_\Gamma(\gamma_\tau^-\mathbf{U}),\mathrm{div}_\Gamma(\gamma_\tau^-\mathbf{U}))_{-1/2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\kappa^2(\gamma_\tau^-\mathbf{U},\gamma_\tau^-\Xi_1\overrightarrow{\mathbf{U}})_{-1/2}\hat{=}\ &\kappa^2((\mathrm{Id}-\mathsf{Z}^\Gamma)\gamma_\tau^-\mathbf{U},(\mathrm{Id}-\mathsf{Z}^\Gamma)\gamma_\tau^-\mathbf{U})_{-1/2}\\
&-\beta\kappa^2((\mathrm{Id}-\mathsf{Z}^\Gamma)\gamma_\tau^-\mathbf{U},\mathbf{curl}_\Gamma(\gamma^-P))_{-1/2}
\end{aligned}$$

and

$$(\gamma_\tau^- \Xi_1 \overrightarrow{\mathbf{U}}, \mathbf{curl}_\Gamma(\gamma^-(P)))_{-1/2} \hat{=} -((\mathrm{Id} - \mathsf{Z}^\Gamma)\gamma_\tau^- \mathbf{U}, \mathbf{curl}_\Gamma(\gamma^-(P)))_{-1/2} + \beta(\mathbf{curl}_\Gamma(\gamma^- P), \mathbf{curl}_\Gamma(\gamma^-(P)))_{-1/2}.$$

We now want to evaluate the terms involving $\Xi_2(\overrightarrow{\mathbf{U}})$. We introduce

$$\mathbf{curl}_\Gamma(\gamma^- \Xi_2 \overrightarrow{\mathbf{U}}) = -\theta \gamma_\tau^- \nabla(\mathsf{S}(\mathbf{U}^0) + \mathbf{mean}(P)) = -\theta(\mathrm{Id} - \mathsf{Z}^\Gamma)\gamma_\tau^- \mathbf{U},$$

in (1.5.6) to obtain

$$-(\gamma_\tau^- \mathbf{U}, \mathbf{curl}_\Gamma(\gamma^- \Xi_2 \overrightarrow{\mathbf{U}}))_{-1/2} = \theta((\mathrm{Id} - \mathsf{Z}^\Gamma)\gamma_\tau^- \mathbf{U}, (\mathrm{Id} - \mathsf{Z}^\Gamma)\gamma_\tau \mathbf{U})_{-1/2}$$

Using Young's inequality twice with $\delta > 0$,

$$\begin{aligned}
&\mathfrak{Re}\Big(\Big\langle -\{\mathcal{T}_N\} \cdot \mathcal{DL}_\kappa \begin{pmatrix} \gamma_t^- \mathbf{U} \\ -\gamma^-(P) \end{pmatrix}, \begin{pmatrix} \gamma_t^- \Xi_1 \overrightarrow{\mathbf{U}} \\ \gamma^- \Xi_2 \overrightarrow{\mathbf{U}} \end{pmatrix} \Big\rangle\Big) \\
&\hat{=} \|\mathrm{div}_\Gamma(\gamma_\tau^- \mathbf{U})\|_{-1/2}^2 + (\mathfrak{Re}(\kappa^2) + \theta)\|(\mathrm{Id} - \mathsf{Z}^\Gamma)\gamma_\tau^- \mathbf{U}\|_{-1/2}^2 \\
&\qquad + \beta\|\mathbf{curl}_\Gamma(\gamma^-(P))\|^2 - ((\mathrm{Id} - \mathsf{Z}^\Gamma)\gamma_\tau^- \mathbf{U}, \mathbf{curl}_\Gamma(\gamma^-(P)))_{-1/2} \\
&\qquad - \beta\,\mathfrak{Re}(\kappa^2)((\mathrm{Id} - \mathsf{Z}^\Gamma)\gamma_\tau^- \mathbf{U}, \mathbf{curl}_\Gamma(\gamma^- P))_{-1/2} \\
&\geq \|\mathrm{div}_\Gamma(\gamma_\tau^- \mathbf{U})\|_{-1/2}^2 + (\beta - \frac{1}{\delta}(1 + \beta\,\mathfrak{Re}(\kappa^2)))\|\mathbf{curl}_\Gamma(\gamma^-(P))\|^2 \\
&\qquad + (\mathfrak{Re}(\kappa^2) + \theta - \delta\,(1 + \beta\,\mathfrak{Re}(\kappa^2)))\|(\mathrm{Id} - \mathsf{Z}^\Gamma)\gamma_\tau^- \mathbf{U}\|_{-1/2}^2.
\end{aligned}$$

The operator $\mathbf{curl}_\Gamma : H^1_*(\Omega_s) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is a continuous injection [15, Lem. 6.4]. It is thus bounded below. Since the mean operator has finite rank, it is compact. Therefore, for any $\beta > 0$, choose $\delta > 0$ large enough, then $\theta > 0$ accordingly large, and the desired inequality follows by equivalence of norms. □

In the next lemma, we prove coercivity of the lower diagonal block of the coupling operator $\mathbb{G}_\kappa$.

Lemma 1.5 *For any frequency $\omega \geq 0$, there exist a compact linear operator $\mathcal{K} : \mathcal{H}_N \to \mathcal{H}_D$, a positive constants $C > 0$ and parameters $\tau > 0$ and $\lambda > 0$, possibly depending on Ω_s, μ, ϵ and κ, such that*

$$\mathfrak{Re}\Big(\Big\langle \mathbb{A}_\kappa^{ND}(\overrightarrow{\mathbf{p}}), \Xi^\Gamma \overrightarrow{\mathbf{p}} \Big\rangle + \Big\langle \mathcal{K}\,\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{p}} \Big\rangle\Big) \geq C\|\overrightarrow{\mathbf{p}}\|_{\mathcal{H}_N}^2$$

for all $\overrightarrow{\mathbf{p}} \in \mathcal{H}_N$. In particular, for $\mathfrak{Re}(k^2) \neq 0$, the inequality holds with $\tau = 1/\kappa^2$.

Proof. The jump conditions (1.2.11b) yield $\{\mathcal{T}_D\} \cdot \mathcal{SL}(\overrightarrow{\mathbf{p}}) = \mathcal{T}_D \cdot \mathcal{SL}(\overrightarrow{\mathbf{p}})$. We deduce from [15, Sec. 6.3] and the compact embedding of $\mathbf{X}(\mathrm{div}_\Gamma, \Gamma)$ into $\mathbf{H}_R^{-1/2}(\Gamma)$ that

$$\begin{aligned}
\Big\langle \mathcal{T}_D \cdot \mathcal{SL}(\overrightarrow{\mathbf{p}}), \Xi^\Gamma \overrightarrow{\mathbf{p}} \Big\rangle &\hat{=} -(\mathbf{p}^0, \Xi_1^\Gamma(\mathbf{p}))_{-1/2} - (q, \mathrm{div}_\Gamma(\Xi_1^\Gamma(\mathbf{p})))_{-1/2} \\
&\qquad - (\mathrm{div}_\Gamma(\mathbf{p}), \Xi_2^\Gamma \overrightarrow{\mathbf{p}})_{-1/2} - \kappa^2(q, \Xi_2^\Gamma(\overrightarrow{\mathbf{p}}))_{-1/2} \\
&\hat{=} (\mathbf{p}^0, \mathbf{p}^0)_{-1/2} - (q, \mathrm{div}_\Gamma(\mathbf{p}^\perp))_{-1/2} + \lambda(q, \mathsf{Q}_* q)_{-1/2} \\
&\qquad + \tau(\mathrm{div}_\Gamma(\mathbf{p}), \mathrm{div}_\Gamma(\mathbf{p}))_{-1/2} + \tau\kappa^2(q, \mathrm{div}_\Gamma(\mathbf{p}^\perp))_{-1/2}.
\end{aligned}$$

When $\mathfrak{Re}(\kappa^2) > 0$, setting $\tau = 1/\kappa^2$ immediately yields the existence of a compact linear operator $\mathcal{K} : \mathcal{H}_N \to \mathcal{H}_D$ such that

$$\left\langle \mathcal{T}_D \cdot \mathcal{SL}(\overrightarrow{\mathbf{p}}), \Xi^\Gamma \overrightarrow{\mathbf{p}} \right\rangle + \left\langle \mathcal{K}\overrightarrow{\mathbf{p}}, \Xi^\Gamma \overrightarrow{\mathbf{p}} \right\rangle \geq C(\|\mathrm{div}_\Gamma(\mathbf{p})\|^2_{-1/2} + \|\mathbf{p}^0\|^2_{-1/2} + \|\mathsf{Q}_* q\|^2_{-1/2}).$$

When $\kappa^2 = 0$, the same inequality is obtained for any $\lambda > 0$ by using Young's inequality as in the proof of Lemma 1.4 and choosing τ large enough. The claimed inequality follows by equivalence of norms. □

Equipped with the previous three lemmas, we are now ready to prove Proposition 1.5.

Proof of Proposition 1.5. For any parameters $\beta > 0$ and $\lambda > 0$, the choices of δ and θ in the proofs of Lemma 1.3 and Lemma 1.4 are not mutually exclusive. The choice of τ in Lemma 1.5 is independent of the choice of θ. □

1.5.4 Compactness of the off-diagonal blocks

Finally, The off-diagonal blocks remain to be considered. We will show that, up to compact perturbations, a suitable choice of parameters in the isomorphisms Ξ and Ξ^Γ of the test space leads to a skew-symmetric pattern in $\mathbb{G}_\kappa$. In other words, up to compact terms, the volume and boundary parts of the system decouples over the space decompositions introduced in Section 1.4.

Proof of Proposition 1.6. The isomorphisms Ξ and Ξ^Γ were designed so that favorable cancellations occur in evaluating the left hand side of (1.5.5).

From the jump properties (1.2.11b), we have $\{\mathcal{T}_N\}\mathcal{SL}_\kappa = \mathcal{T}_N^-\mathcal{SL}_\kappa - (1/2)\mathrm{Id}$. Therefore, as in (1.2.20), we evaluate

$$\begin{aligned}
&\left\langle (\mathbb{P}^+_\kappa)_{22}\,\overrightarrow{\mathbf{p}}, \begin{pmatrix} \gamma_t^- \Xi_1 \overrightarrow{\overline{\mathbf{U}}} \\ \gamma^- \Xi_2 \overrightarrow{\overline{\mathbf{U}}} \end{pmatrix} \right\rangle \\
&= \left\langle (-\{\mathcal{T}_N\} \cdot \mathcal{SL}_\kappa + \frac{1}{2}\mathrm{Id})\,\overrightarrow{\mathbf{p}}, \begin{pmatrix} \gamma_t^- \Xi_1 \overrightarrow{\overline{\mathbf{U}}} \\ \gamma^- \Xi_2 \overrightarrow{\overline{\mathbf{U}}} \end{pmatrix} \right\rangle \\
&= \left\langle -\mathcal{T}_N^- \cdot \mathcal{SL}_\kappa(\overrightarrow{\mathbf{p}}), \begin{pmatrix} \gamma_t^- \Xi_1 \overrightarrow{\overline{\mathbf{U}}} \\ \gamma^- \Xi_2 \overrightarrow{\overline{\mathbf{U}}} \end{pmatrix} \right\rangle + \left\langle \overrightarrow{\mathbf{p}}, \begin{pmatrix} \gamma_t^- \Xi_1 \overrightarrow{\overline{\mathbf{U}}} \\ \gamma^- \Xi_2 \overrightarrow{\overline{\mathbf{U}}} \end{pmatrix} \right\rangle \\
&= \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\mathbf{p}), \gamma_t^- \Xi_1 \overrightarrow{\overline{\mathbf{U}}} \rangle_\tau - \langle \gamma_n^- \nabla \psi_{\tilde{\kappa}}(q), \gamma^- \Xi_2 \overrightarrow{\overline{\mathbf{U}}} \rangle_\Gamma + \langle \gamma_n^- \boldsymbol{\Psi}_\kappa(\mathbf{p}), \gamma^- \Xi_2 \overrightarrow{\overline{\mathbf{U}}} \rangle_\Gamma \\
&\quad + \langle \gamma_n^- \nabla \tilde{\psi}_\kappa(\mathrm{div}_\Gamma \mathbf{p}), \gamma^- \Xi_2 \overrightarrow{\overline{\mathbf{U}}} \rangle_\Gamma + \langle \mathbf{p}, \gamma_t^- \Xi_1 \overrightarrow{\overline{\mathbf{U}}} \rangle_\tau + \langle q, \gamma^- \Xi_2 \overrightarrow{\overline{\mathbf{U}}} \rangle_\Gamma \\
&\hat{=} \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\mathbf{p}^0), \gamma_t \overline{\mathbf{U}}^\perp \rangle_\tau - \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\mathbf{p}^0), \gamma_t \overline{\mathbf{U}^0} \rangle_\tau + \beta \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\mathbf{p}^0), \gamma_t \nabla \overline{P} \rangle_\tau \\
&\quad + \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\mathbf{p}^\perp), \gamma_t \overline{\mathbf{U}}^\perp \rangle_\tau - \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\mathbf{p}^\perp), \gamma_t \overline{\mathbf{U}^0} \rangle_\tau \\
&\quad + \beta \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\mathbf{p}^\perp), \gamma_t \nabla \overline{P} \rangle_\tau + \theta \langle \gamma_n^- \nabla \psi_{\tilde{\kappa}}(q), \gamma^- \mathsf{S}(\overline{\mathbf{U}}^0) \rangle_\Gamma \\
&\quad - \theta \langle \gamma_n^- \boldsymbol{\Psi}_\kappa(\mathbf{p}), \gamma^- \mathsf{S}(\overline{\mathbf{U}}^0) \rangle_\Gamma - \langle \gamma_n^- \nabla \tilde{\psi}_\kappa(\mathrm{div}_\Gamma \mathbf{p}), \theta\, \gamma^- \mathsf{S}(\overline{\mathbf{U}}^0) \rangle_\Gamma
\end{aligned}$$

$$
\begin{aligned}
&+\langle \mathbf{p}^0, \gamma_t^- \overline{\mathbf{U}}^\perp\rangle_\tau + \langle \mathbf{p}^\perp, \gamma_t^- \overline{\mathbf{U}}^\perp\rangle_\tau - \langle \mathbf{p}^0, \gamma_t^- \overline{\mathbf{U}}^0\rangle_\tau - \langle \mathbf{p}^\perp, \gamma_t^- \overline{\mathbf{U}}^0\rangle_\tau \\
&+\beta \langle \mathbf{p}^0, \gamma_t^- \nabla \overline{P}\rangle_\tau + \beta \langle \mathbf{p}^\perp, \gamma_t^- \nabla \overline{P}\rangle_\tau - \theta \langle q, \gamma^- \mathsf{S}(\mathbf{U}^0)\rangle_\Gamma,
\end{aligned} \tag{1.5.8}
$$

where we have used that the finite rank of the mean operator implies compactness.

Similarly, using Proposition 1.1, we find

$$
\begin{aligned}
&\left\langle (\mathbb{P}_\kappa^-)_{11} \begin{pmatrix} \gamma_t^- \mathbf{U} \\ -\gamma^-(P) \end{pmatrix}, \Xi^\Gamma \overrightarrow{\overline{\mathbf{p}}} \right\rangle = \left\langle \begin{pmatrix} \gamma_t^- \mathbf{U} \\ -\gamma^-(P) \end{pmatrix}, (\mathbb{P}_\kappa^+)_{22} \Xi^\Gamma \overrightarrow{\overline{\mathbf{p}}} \right\rangle \\
&\hat{=} \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\overline{\mathbf{p}}^\perp), \gamma_t \mathbf{U}^0\rangle_\tau - \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\overline{\mathbf{p}}^0), \gamma_t \mathbf{U}^\perp\rangle_\tau \\
&\qquad - \lambda \langle \gamma_R^- \boldsymbol{\Psi}_\kappa((\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}), \gamma_t^- \mathbf{U}^0\rangle_\tau + \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\overline{\mathbf{p}}^0), \gamma_t \mathbf{U}^\perp\rangle_\tau \\
&\qquad - \langle \gamma_R^- \boldsymbol{\Psi}_\kappa(\overline{\mathbf{p}}^0), \gamma_t \mathbf{U}^\perp\rangle_\tau - \lambda \langle \gamma_R^- \boldsymbol{\Psi}_\kappa((\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}), \gamma_t^- \mathbf{U}^\perp\rangle_\tau \\
&\qquad - \tau \langle \gamma_n^- \nabla \psi_{\tilde{\kappa}}(\mathrm{div}_\Gamma \overline{\mathbf{p}}^\perp), \gamma^- P\rangle - \langle \gamma_n^- \boldsymbol{\Psi}_\kappa(\overline{\mathbf{p}}^\perp), \gamma^- P\rangle_\Gamma \quad \ldots
\end{aligned} \tag{1.5.9}
$$

$$
\begin{aligned}
\ldots \qquad &+ \langle \gamma_n^- \boldsymbol{\Psi}_\kappa(\overline{\mathbf{p}}^0), \gamma^- P\rangle_\Gamma + \lambda \langle \gamma_n^- \boldsymbol{\Psi}_\kappa((\mathrm{div})^\dagger Q_* \overline{q}), \gamma^- P\rangle_\Gamma \\
&- \langle \gamma_n^- \nabla \tilde{\psi}_\kappa(\mathrm{div}_\Gamma \overline{\mathbf{p}}^\perp), \gamma^- P\rangle_\Gamma + \lambda \langle \gamma_n^- \nabla \tilde{\psi}_\kappa(Q_* \overline{q}), \gamma^- P\rangle_\Gamma \\
&+ \langle \gamma_t^- \mathbf{U}^0, \overline{\mathbf{p}}^\perp\rangle_\tau + \langle \gamma_t^- \mathbf{U}^\perp, \overline{\mathbf{p}}^\perp\rangle_\tau - \langle \gamma_t^- \mathbf{U}^\perp, \overline{\mathbf{p}}^0\rangle_\tau \\
&- \langle \gamma_t^- \mathbf{U}^0, \overline{\mathbf{p}}^0\rangle_\tau - \lambda \langle \gamma_t^- \mathbf{U}^\perp, (\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}\rangle_\Gamma \\
&- \lambda \langle \gamma_t^- \mathbf{U}^0, (\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}\rangle + \tau \langle \gamma^- P, \mathrm{div}_\Gamma(\overline{\mathbf{p}}^\perp)\rangle_\Gamma.
\end{aligned} \tag{1.5.10}
$$

Many terms in these equations can be combined and asserted compact by (1.4.3a) and (1.4.3b). They are indicated in blue. When summing the real parts of (1.5.8) and (1.5.10), the terms in red cancel. Relying on (1.2.8a) to (1.2.8d), some terms amount to compact perturbations so that we may replace κ and $\tilde{\kappa}$ by zero in those instances. We have arrived at the following identity:

$$
\begin{aligned}
&\mathfrak{Re}\left(\left\langle (\mathbb{G}_\kappa - \mathrm{diag}(\mathbb{G}_\kappa)) \begin{pmatrix} \overrightarrow{\mathbf{U}} \\ \overrightarrow{\mathbf{p}} \end{pmatrix}, \begin{pmatrix} \Xi \overrightarrow{\overline{\mathbf{U}}} \\ \Xi^\Gamma \overrightarrow{\overline{\mathbf{p}}} \end{pmatrix} \right\rangle\right) \\
&\hat{=} \mathfrak{Re}\Bigg(\beta \langle \gamma_R^- \boldsymbol{\Psi}_0(\mathbf{p}^\perp), \gamma_t \nabla \overline{P}\rangle_\tau + \theta \langle \gamma_n^- \nabla \psi_0(q), \gamma^- \mathsf{S}(\overline{\mathbf{U}}^0)\rangle_\Gamma \\
&\qquad - \theta \langle \gamma_n^- \boldsymbol{\Psi}_0(\mathbf{p}), \gamma^- \mathsf{S}(\overline{\mathbf{U}}^0)\rangle_\Gamma + \beta \langle \mathbf{p}^\perp, \gamma_t^- \nabla \overline{P}\rangle_\tau - \theta \langle q, \gamma^- \mathsf{S}(\mathbf{U}^0)\rangle_\Gamma \\
&\qquad - \lambda \langle \gamma_R^- \boldsymbol{\Psi}_0((\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}), \gamma_t^- \mathbf{U}^0\rangle_\tau - \tau \langle \gamma_n^- \nabla \psi_0(\mathrm{div}_\Gamma \overline{\mathbf{p}}^\perp), \gamma^- P\rangle_\Gamma \\
&\qquad - \langle \gamma_n^- \boldsymbol{\Psi}_0(\overline{\mathbf{p}}^\perp), \gamma^- P\rangle_\Gamma + \langle \gamma_n^- \boldsymbol{\Psi}_0(\overline{\mathbf{p}}^0), \gamma^- P\rangle_\Gamma \\
&\qquad + \lambda \langle \gamma_n^- \boldsymbol{\Psi}_0((\mathrm{div})^\dagger Q_* \overline{q}), \gamma^- P\rangle_\Gamma \\
&\qquad - \lambda \langle \gamma_t^- \mathbf{U}^0, (\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}\rangle_\tau + \tau \langle \gamma^- P, \mathrm{div}_\Gamma(\overline{\mathbf{p}}^\perp)\rangle_\Gamma \Bigg).
\end{aligned}
$$

We claim that the terms colored in green are compact. Indeed, the integral identities of Subsection 1.2.1 together with equality (1.2.7) yield

$$\begin{aligned}
&\langle \gamma_n^- \boldsymbol{\Psi}_0(\mathbf{p}), \gamma^- \mathsf{S}(\overline{\mathbf{U}}^0)\rangle_\Gamma \\
&\qquad\qquad\qquad \leq (\|\psi_0(\mathrm{div}_\Gamma \mathbf{p})\|_{L^2(\Omega_s)} + \|\boldsymbol{\Psi}_0(\mathbf{p})\|_{\mathbf{L}^2(\Omega_s)})\|\overline{\mathbf{U}}^0\|_{\mathbf{L}^2(\Omega_s)}, \\
&\langle \gamma_n^- \boldsymbol{\Psi}_0(\overline{\mathbf{p}}), \gamma^- P\rangle_\Gamma \\
&\qquad\qquad\qquad \leq (\|\psi_0(\mathrm{div}_\Gamma \overline{\mathbf{p}})\|_{L^2(\Omega_s)} + \|\boldsymbol{\Psi}_0(\overline{\mathbf{p}})\|_{\mathbf{L}^2(\Omega_s)})\|P\|_{H^1(\Omega_s)} \\
&\langle \gamma_n^- \boldsymbol{\Psi}_0((\mathrm{div})^\dagger Q_* \overline{q}), \gamma^- P\rangle_\Gamma, \\
&\qquad\qquad\qquad \leq (\|\psi_0(Q_* q)\|_{L^2(\Omega_s)} + \|\boldsymbol{\Psi}_0(\mathrm{div}_\Gamma \overline{\mathbf{p}})\|_{\mathbf{L}^2(\Omega_s)})\|P\|_{H^1(\Omega_s)}.
\end{aligned}$$

Since $\psi_0 : H^{-1/2}(\Gamma) \to H^1(\Omega_s)$ and $\boldsymbol{\Psi}_0 : \mathbf{H}^{-1/2}(\Gamma) \to \mathbf{H}^1(\Omega_s)$ are continuous, compactness is guaranteed by Rellich's Theorem.

To go further, we need to settle for a choice of parameters in the volume and boundary isomorphisms. Choose τ to satisfy the requirements of Lemma 1.5, then set $\beta = \tau$. We are still free to let θ satisfy both Lemma 1.3 and Lemma 1.4, and then choose $\lambda = \theta$.

Under this choice of parameters, the terms in orange vanish, because we have $\langle \mathbf{p}^\perp, \gamma_t^- \nabla \overline{P}\rangle_\tau = \langle \mathbf{p}^\perp, \nabla_\Gamma \gamma^- \overline{P}\rangle_\tau = -\langle \mathrm{div}_\Gamma(\mathbf{p}^\perp), \gamma^- \overline{P}\rangle_\Gamma$, and similarly

$$\begin{aligned}
\langle \gamma_t^- \mathbf{U}^0, (\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}\rangle_\tau &= \langle \gamma_t^- \nabla \mathsf{S}(\mathbf{U}^0), (\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}\rangle_\tau \\
&= -\langle \gamma^- \mathsf{S}(\mathbf{U}^0), Q_* \overline{q}\rangle_\Gamma.
\end{aligned}$$

Finally, relying on (1.2.6a), (1.2.6b) and (1.2.7) once more, we observe that

$$\begin{aligned}
\langle \gamma_R^- \boldsymbol{\Psi}_0(\mathbf{p}^\perp), \gamma_t^- \nabla \overline{P}\rangle_\tau &= (\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Psi}_0(\mathbf{p}^\perp), \nabla P)_{\Omega_s} \\
&= (\nabla \psi_0(\mathrm{div}_\Gamma \mathbf{p}^\perp), \nabla P)_{\Omega_s} = \langle \gamma_n^- \nabla \psi_0(\mathrm{div}_\Gamma \mathbf{p}^\perp), \gamma^- \overline{P}\rangle_\Gamma.
\end{aligned}$$

A similar derivation shows that

$$\langle \gamma_n^- \nabla \psi_0(q), \gamma^- \mathsf{S}(\overline{\mathbf{U}}^0)\rangle_\Gamma \mathrel{\hat{=}} \langle \gamma_R^- \boldsymbol{\Psi}_0((\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}), \gamma_t^- \mathbf{U}^0\rangle_\tau.$$

We conclude that for such a choice of parameters,

$$\mathfrak{Re}\left(\left\langle (\mathbb{G}_\kappa - \mathrm{diag}(\mathbb{G}_\kappa)) \begin{pmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{p}} \end{pmatrix}, \begin{pmatrix} \Xi\, \overrightarrow{\overline{\mathbf{U}}} \\ \Xi^\Gamma \overrightarrow{\overline{\mathbf{p}}} \end{pmatrix} \right\rangle\right) \mathrel{\hat{=}} 0,$$

which concludes the proof of this proposition. □

1.6 Conclusion

In Section 1.3 we have proposed a system of equations coupling the *mixed formulation* of the variational form of the Hodge-Helmholtz and Hodge-Laplace equation with *first-kind* boundary integral equations. Well-posedness of the coupled problem was obtained using a T-coercivity argument demonstrating that the operator associated to the coupled variational problem was Fredholm of index 0. When $\kappa^2 \in \mathbb{C}$ avoids resonant frequencies, the operator's injectivity is guaranteed, and thus stability of the problem is obtained along with the existence and uniqueness of the solution. For such κ^2, Proposition 1.2 shows how solutions to the coupled variational problem are in one-to-one correspondence with solutions of the transmission system. In principle, the CFIE-type stabilization strategy applicable to transmission problems for the scalar Helmholtz operator [27] or the electric wave equation [28] could also be attempted here to get rid of the spurious resonances haunting the coupled problem (1.3.6), but such developments lie outside the scope of this work.

The symmetrically coupled system (1.3.6) offers a variational formulation of the transmission problem (1.1.3) in well-known energy spaces suited for discretization by finite and boundary elements. It is therefore a promising starting point for Galerkin discretization.

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Chapter 2
Spurious Resonances in Coupled Domain–Boundary Variational Formulations of Transmission Problems in Electromagnetism and Acoustics

Erick Schulz and Ralf Hiptmair

Abstract We develop a framework shedding light on common features of coupled variational formulations arising in electromagnetic scattering and acoustics. We show that spurious resonances haunting coupled domain-boundary formulations based on direct boundary integral equations of the first kind originate from the formal structure of their Calderón identities. Using this observation, the kernel of the coupled problem is characterized explicitly and we show that it completely vanishes under the exterior representation formula.

2.1 Introduction

Transmission problems in electromagnetism and acoustics model the following typical experiment. An incident wave penetrates an object and travels inside the possibly inhomogeneous medium. Concurrently, it also scatters at its surface and propagates in the outside homogeneous region to eventually decay at infinity. Simulation of the complete phenomenon entails coupling the interior and exterior problems. A vast literature is devoted to the design of such couplings for various physical situations. Notably, the described setting is considered in [17], [21], [8], [19] and [27].

On the one hand, domain based variational methods offer a familiar way of modeling wave propagation in materials whose properties vary in space. The texts [20], [18], [1] and [24] are thorough analyses for electromagnetism. Standard references such as [28] and [13] introduce the reader to the Helmholtz operator as it appears in acoustic scattering.

On the other hand, boundary integral equations are capable of describing the behavior of the waves in unbounded homogeneous regions, because they provide valid Cauchy data that can be fed to the representation formula. Their complete derivation and properties can be found in [28], [25], [22] and [23]. In the following, we consider in particular the *direct* boundary integral equations of the *first kind* detailed in [26], [7] and [11].

Even if a transmission problem involving a Helmholtz-like operator $\mathsf{P}-\lambda\mathrm{Id}$ has a unique solution at a given fixed frequency $\lambda \in \mathbb{C}$, the standard direct first-kind boundary integral equations obtained for the associated exterior problem is haunted by the existence of “spurious frequencies": the kernel of the Dirichlet-to-Neumann map supplied by the first exterior Calderón identity is spanned by the interior Dirichlet λ-eigenfunctions of P. Similarly, the related Neumann eigenspace corresponds to the kernel of the Neumann-to-Dirichlet map supplied by the second exterior Calderón identity. This issue was investigated for the electric field integral equation in [10]. Eigenvalues of the Laplacian were studied in [16] and [26] from the perspective of resonant frequencies.

Unsurprisingly, this deficiency of the boundary integral equations carries over to the coupled domain-boundary variational formulations. Its consequences for the symmetric approach to the coupling problem in the context of electromagnetism (classical $\mathbf{E}$–$\mathbf{H}$ formulation) and acoustics (Helmholtz equation) were stated without proof in [7] and [21], respectively. The development presented below is inspired by the analysis carried out in [19, Lem. 6.1] and [27, Prop. 3.1] for electromagnetism, where equivalence between domain-boundary couplings and associated transmission problems is established based on ideas from [29, Sec. 4.3] for acoustics.

This article is motivated by our impression that the occurrence and nature of spurious resonances is presumably "well-known in the community", but that it is difficult to locate a systematic analysis and rigorous results in literature. We thus give in this essay a unified treatment of a few symmetric domain-boundary variational formulations for the time-harmonic solutions of transmission problems in electromagnetism and acoustics under a common framework. Particular problems are discussed in Section 2.5. Costabel's symmetric approach introduced in [14] is generalized to allow for the mixed formulation of the interior problem. The lack of uniqueness due to resonant frequencies is shown to result from the formal structure of the Calderón identities. The phenomenon is thus shared by all three couplings under consideration. The kernel of the abstract coupled problem is fully characterized in Section 2.4.

We point out that, from a theoretical point of view, the post-processing required to recover the scattered waves in the exterior region restores uniqueness of the solutions. Indeed, the kernel of the Dirichlet-to-Neumann map vanishes under the representation formula. In practice however, the poor conditioning of the linear systems of boundary integral equations near so-called resonant frequencies leads to severe impact of round-off errors in computations and to slow convergence of iterative solvers, but while these so-called "spurious resonant frequencies" generally cause instabilities after discretization that enforce the use of regularization strategies, their mere existence is harmless to the physical validity of the domain-boundary coupling models. This explains why classical coercive symmetric couplings remain nonetheless valuable pilot formulations for Galerkin discretization. We refer to [26] for an introduction to a classical approach originally suggested by Brakhage and Werner [3] to regularize the *indirect* BIEs for the Helmholtz operator. We also point out that a CFIE-type stabilization procedure for the Helmholtz transmission problem is studied in [21], where a symmetric coupling stable for all positive frequencies is obtained. However, stabilization techniques are not the focus of this paper.

2.2 Formal framework

2.2.1 Notation and conventions

Let $\Omega^- \subset \mathbb{R}^3$ be a bounded simply connected domain with Lipschitz boundary $\Gamma := \partial\Omega^-$. We think of Ω^- as a bounded volume occupied by an inhomogeneous object with a possibly "rough" surface. Usually, Ω^- is assumed to be a curvilinear polyhedron. Throughout this work, we use Ω generically to denote either Ω^- or $\Omega^+ := \mathbb{R}^3 \backslash \overline{\Omega^-}$. Physically, Ω^+ often represents an unbounded homogeneous air region around Ω^-. We let $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(\Gamma)$ denote, respectively, spaces of square-integrable functions over Ω and Γ. Whenever it is possible, we use bold letters to differentiate vector-valued quantities from scalars. Capitals are often used to denote fields defined over a volume, while small characters usually refer to functions on Γ. The space of smooth fields compactly supported in Ω is written $\mathscr{D}(\Omega)$. The subscript 'loc' is used to extend a given function space V to the larger space V_{loc} comprising all functions u such that $u\psi \in V$ for all $\psi \in \mathscr{D}(\Omega)$. A prime will be used to indicate the dual of a space, e.g. V'. Duality parings are written with angular brackets, e.g. $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, but we also often allow ourselves to substitute integrals for these angular brackets when we want to emphasize $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(\Gamma)$ as pivot spaces or highlight the analogy between the identities introduced in this formal framework and Green's classical formulas.

We call *weak differential operator matrices* the various linear operators that can be represented by a matrix of partial derivatives. We understand their arrangement in a weak sense. If no particular structure is recognized, then we must accept to define them on the Sobolev space $H^1(\Omega)$. However, in the models we consider in this work, the partial derivatives often sum up to form divergence and curl operators respectively defined on

$$\mathbf{H}(\text{div}, \Omega) := \{\mathbf{U} \in \mathbf{L}^2(\Omega) \,|\, \text{div}\,\mathbf{U} \in \mathbf{L}^2(\Omega)\}, \tag{2.2.1a}$$

$$\mathbf{H}(\mathbf{curl}, \Omega) := \{\mathbf{U} \in \mathbf{L}^2(\Omega) \,|\, \mathbf{curl}\,\mathbf{U} \in \mathbf{L}^2(\Omega)\}. \tag{2.2.1b}$$

The following Green's formulas can be extended to these spaces:

$$\pm \int_{\Omega^\mp} \text{div}(\mathbf{U})\, P + \mathbf{U} \cdot \nabla P \, \mathrm{d}\mathbf{x} = \int_\Gamma P(\mathbf{U} \cdot \mathbf{n}) \mathrm{d}\sigma, \tag{2.2.2a}$$

$$\pm \int_{\Omega^\mp} \mathbf{U} \cdot \mathbf{curl}\,(\mathbf{V}) - \mathbf{curl}\,(\mathbf{U}) \cdot \mathbf{V} \mathrm{d}\mathbf{x} = \int_\Gamma \mathbf{V} \cdot (\mathbf{U} \times \mathbf{n}) \mathrm{d}\sigma. \tag{2.2.2b}$$

Here and in the remainder of the paper, $\mathbf{n}(\mathbf{x})$ stands for the unit normal boundary vector field oriented outward from Ω^-. The same notation is kept throughout this section.

2.2.2 Boundary value problems

We consider a formally self-adjoint linear weak differential operator matrix

$$\mathsf{P} : \mathbf{X}_{\text{loc}}(\Omega) := \mathbf{X}_{\text{loc}}(\mathsf{P}, \Omega) \to \mathbf{L}^2_{\text{loc}}(\Omega). \tag{2.2.3}$$

In accordance with this definition, we assume that $\mathbf{X}_{\text{loc}}(\Omega) \subset \mathbf{L}^2_{\text{loc}}(\Omega)$. Ultimately, our goal is to develop variational transmission equations in which exterior problems of the form

$$(\mathsf{P} - \lambda\, \text{Id}) U^{\text{ext}} = 0 \quad \text{in } \Omega^+ \tag{2.2.4}$$

for some fixed $\lambda \in \mathbb{C}$ are formulated using BIEs. What we have in mind for P is a range of important operators. A few examples that arise in the study of acoustic and electromagnetic scattering will be presented in Section 2.5.

The first step in the formulation of BVPs for P is to establish a definition of boundary data. In the following, square brackets indicate the jump $[\mathsf{T}_\bullet] := \mathsf{T}^-_\bullet - \mathsf{T}^+_\bullet$ of a trace, specified by $\bullet = D$ or N, over the boundary Γ. Let $\mathbf{X}^D_{\text{loc}}(\Omega)$ and $\mathbf{X}^N_{\text{loc}}(\Omega)$ be two subspaces of $\mathbf{L}^2(\Omega)$ such that

$$\mathbf{X}_{\text{loc}}(\Omega) \subset \mathbf{X}^{D}_{\text{loc}}(\Omega) \cap \mathbf{X}^{N}_{\text{loc}}(\Omega)\,.$$

The next assumption is motivated by [23, Sec. 3 & 4], [7, Sec. 2] and [11, Sec. 3.1], among others.

Assumption I (Existence of traces and Green's second formula) There exist two non-trivial Hilbert trace spaces of distributions $\mathbf{H}_N$ and $\mathbf{H}_D$ supported on Γ that are dual under a pairing $\langle\langle\cdot,\cdot\rangle\rangle_\Gamma$, together with continuous and *surjective* linear operators

$$\mathsf{T}_D^{\mp} : \mathbf{X}^{D}_{loc}(\Omega) \to \mathbf{H}_D, \qquad \mathsf{T}_N^{\mp} : \mathbf{X}^{N}_{loc}(\Omega) \to \mathbf{H}_N, \tag{2.2.5}$$

admitting continuous right inverses and satisfying Green's second formula

$$\int_{\Omega^\mp} \mathsf{P}U\cdot V - U\cdot \mathsf{P}V\,\mathrm{d}\mathbf{x} = \pm\langle\langle \mathsf{T}_N^{\mp}U, \mathsf{T}_D^{\mp}V\rangle\rangle_\Gamma \mp \langle\langle \mathsf{T}_N^{\mp}V, \mathsf{T}_D^{\mp}U_\Gamma\rangle\rangle_\Gamma, \qquad \forall\, U,V\in \mathbf{X}(\Omega^\mp). \tag{2.2.6}$$

We suppose that $\mathscr{D}(\Omega^\mp)\subset \ker(\mathsf{T}_D^{\mp})\cap\ker(\mathsf{T}_N^{\mp})$. Moreover, we take for granted that $\left[\mathsf{T}_D(\phi)\right] = \left[\mathsf{T}_N(\phi)\right] = 0$ whenever ϕ is smooth in a neighborhood of Γ.

The archetypes behind these operators are the Dirichlet and Neumann traces, but all the traces occurring in the examples presented in this paper also satisfy Assumption I.

Remark 2.1 Roughly speaking, the hypothesis that $\mathscr{D}(\Omega)\subset\ker(\mathsf{T}_D^{\mp})\cap\ker(\mathsf{T}_N^{\mp})$ simply asks for the traces of functions vanishing on the boundary to vanish.

Given boundary data $g\in\mathbf{H}_D$ and $\eta\in\mathbf{H}_N$, we use the traces supplied in Assumption I to impose boundary conditions in the statement of interior and exterior BVPs:

$$\begin{cases} \mathsf{P}U - \lambda U = 0, & \text{in } \Omega^\mp \\ \mathsf{T}_D^{\mp}U = g, & \text{on } \Gamma, \\ \text{radiation conditions at } \infty, & \text{if } \Omega = \Omega^+ \end{cases} \qquad (\textit{abstract}\text{ Dirichlet BVPs for } \mathsf{P}) \tag{2.2.7a}$$

$$\begin{cases} \mathsf{P}U - \lambda U = 0, & \text{in } \Omega^\mp \\ \mathsf{T}_N^{\mp}U = \eta, & \text{on } \Gamma, \\ \text{radiation conditions at } \infty, & \text{if } \Omega = \Omega^+. \end{cases} \qquad (\textit{abstract}\text{ Neumann BVPs for } \mathsf{P}) \tag{2.2.7b}$$

Assumption II (Uniqueness for exterior BVPs) The solutions to the *exterior* (abstract) Dirichlet and Neumann BVPs (2.2.7a) and (2.2.7b) posed on $\mathbf{X}\left(\Omega^+\right)$ are unique.

See [23, Thm. 9.11], [13, Thm. 6.10], [17] and [11, Cor. 3.9].

2.2.3 Transmission problems

Let P be defined on $\mathbf{X}(\mathsf{P}, \Omega^{\mp}) \subset \mathbf{X}_{\mathsf{P}}^{D}(\Omega^{\mp}) \cap \mathbf{X}_{\mathsf{P}}^{N}(\Omega^{\mp})$ such that it satisfies assumptions I and II for continuous and surjective traces $\mathsf{T}_{\mathsf{P},D}^{\mp} : \mathbf{X}_{\mathsf{P}}^{D}(\Omega^{\mp}) \to \mathbf{H}_{D}(\Gamma)$ and $\mathsf{T}_{\mathsf{P},N}^{\mp} : \mathbf{X}_{\mathsf{P}}^{N}(\Omega^{\mp}) \to \mathbf{H}_{N}(\Gamma)$. Further suppose that L is a linear differential operator defined on $\mathbf{X}(\mathsf{L}, \Omega^{-}) \subset \mathbf{X}_{\mathsf{L}}^{D}(\Omega^{\mp}) \cap \mathbf{X}_{\mathsf{L}}^{N}(\Omega^{\mp})$ that satisfies Assumption I for the traces $\mathsf{T}_{\mathsf{L},D}^{-} : \mathbf{X}_{\mathsf{L}}^{D}(\Omega^{\mp}) \to \mathbf{H}_{D}(\Gamma)$ and $\mathsf{T}_{\mathsf{L},N}^{-} : \mathbf{X}_{\mathsf{L}}^{N}(\Omega^{\mp}) \to \mathbf{H}_{D}(\Gamma)$. Notice that the trace spaces associated with the two operators are required to correspond.

We are interested in well-posed transmission problems of the form: given a source term $f \in \mathbf{L}^{2}(\Omega^{-})$ and boundary data $(g, \eta) \in \mathbf{H}_{D} \times \mathbf{H}_{N}$, find $(U, U^{\text{ext}}) \in \mathbf{X}(\mathsf{L}, \Omega^{-}) \times \mathbf{X}(\mathsf{P}, \Omega^{+})$ satisfying

$$\begin{cases} \mathsf{L}U = f, & \text{in } \Omega^{-} \\ \mathsf{P}U^{\text{ext}} - \lambda U^{\text{ext}} = 0, & \text{in } \Omega^{+} \\ \mathsf{T}_{\mathsf{L},D}^{-} U = \mathsf{T}_{\mathsf{P},D}^{+} U^{\text{ext}} + g, & \text{on } \Gamma, \\ \mathsf{T}_{\mathsf{L},N}^{-} U = \mathsf{T}_{\mathsf{P},N}^{+} U^{\text{ext}} + \eta, & \text{on } \Gamma, \\ \text{radiation conditions at } \infty, \end{cases} \qquad (\textit{abstract} \text{ transmission problem}) \tag{2.2.8}$$

cf. [21, Eq. 2], [19, Eq. 1.1], [27, Eq. 3-4], [17, Sec. 2] and related literature.

The operator L models propagation of waves inside the object Ω^{-}. The later phenomenon can be described using different formulations, thus we emphasize that vector-valued functions $U \in \mathbf{X}(\mathsf{L}, \Omega^{-})$ need *not* have the same number of entries as vector-valued functions $U^{\text{ext}} \in \mathbf{X}_{\text{loc}}(\mathsf{P}, \Omega^{+})$. In other words, the number of unknowns in the interior problem may differ from the number of unknowns in the exterior problem. For instance, this occurs with mixed formulations, in which auxiliary variables increase the dimensionality of the system of equations. Nevertheless, the transmission problem (2.2.8) covers the important and common case where $\mathsf{L} = \mathsf{P} - \lambda \mathrm{Id}$. Intuitively, it is a good heuristic to think of L as "the operator P in which the spacial coefficients may vary in space". See Figure 2.1.

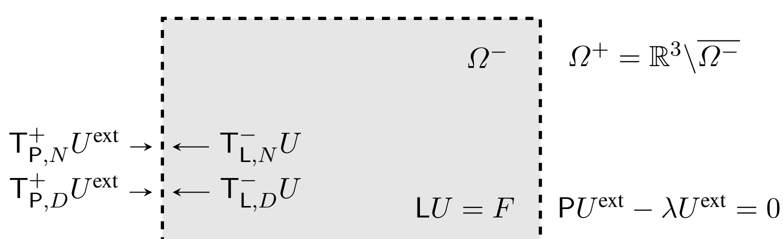

Fig. 2.1 Depiction of the abstract transmission problem (2.2.8). The shaded region represents a volume occupied by a scattering object.

We refer to [15], [24], [7], [11] and [27, Sec. 3] for the next assumption.

Assumption III (Green's first formula) There exist a non-trivial subspace $\mathbf{V}(\Omega^-) \subset \mathbf{X}^D_{\mathsf{L}}(\Omega^-)$ and a continuous bilinear form $\boldsymbol{\Phi}$ on $\mathbf{V}(\Omega^-) \times \mathbf{V}(\Omega^-)$ such that

$$\int_{\Omega^-} \mathsf{L}U \cdot V \mathrm{d}\mathbf{x} = \boldsymbol{\Phi}(U,V) + \langle\!\langle \mathsf{T}^-_{\mathsf{L},N} U, \mathsf{T}^-_{\mathsf{L},D} V \rangle\!\rangle_\Gamma \qquad \forall U \in \mathbf{X}\left(\mathsf{L}, \Omega^-\right), V \in \mathbf{V}(\Omega^-). \tag{2.2.9}$$

Assumption III states that for $g \in \mathbf{H}_D$, the problem

$$U \in \mathbf{V}(\Omega^-) \cap \{\mathsf{T}^-_{\mathsf{L},D} U = g\} : \qquad \boldsymbol{\Phi}(U,V) = 0 \qquad \forall V \in \mathbf{V}(\Omega^-) \cap \ker(\mathsf{T}^-_{\mathsf{L},D}) \tag{2.2.10}$$

is a weak variational formulation for the interior Dirichlet problem

$$\begin{cases} \mathsf{L}U = 0 & \text{in } \Omega^-, \\ \mathsf{T}^-_{\mathsf{L},D} U = g & \text{on } \Gamma. \end{cases} \qquad (\textit{abstract} \text{ interior Dirichlet BVP for } \mathsf{L}) \tag{2.2.11}$$

By testing with $V \in \mathscr{D}(\Omega^-)$, we immediately find that a solution $U \in \mathbf{V}(\Omega^-)$ of (2.2.10) solves $\mathsf{L}U = 0$ in the sense of distributions. Therefore, it also solves (2.2.11) in $\mathbf{L}^2(\Omega)$ if it is regular enough. It is necessary and sufficient that $U \in \mathbf{X}\left(\mathsf{L}, \Omega^-\right)$. It is thus reasonable to assume the following regularity result.

Assumption IV (Regularity) A distribution $U \in \mathbf{V}(\Omega^-)$ which solves $\mathsf{L}U = 0$ in the sense of distributions also belongs to $\mathbf{X}(\mathsf{L}, \Omega^-)$.

This assumption is modeled on the examples below. For e.g., in the simple case where $U \in H^1(\Omega^-)$ is a weak solution of the interior variational problem associated with the scalar Helmholtz operator, then it follows that $\nabla U \in \mathbf{H}(\mathrm{div}, \Omega)$, i.e. $U \in H(\Delta, \Omega)$.

Assumption V (Uniqueness) The transmission problem (2.2.8) is uniquely solvable.

In the following sections, we use $\mathsf{T}_D := \mathsf{T}_{\mathsf{P},D}$ and $\mathsf{T}_N := \mathsf{T}_{\mathsf{P},N}$ to ease notation.

2.2.4 Representation by boundary potentials

Given a formally self-adjoint weak differential operator matrix L and a locally integrable source term F, we say that $\mathsf{L}U = F$ holds in Ω *in the sense of distributions* if

$$\langle\!\langle \mathsf{L}U, V \rangle\!\rangle := \int_\Omega U \cdot \mathsf{L}V \mathrm{d}\mathbf{x} = \int_\Omega F \cdot V \mathrm{d}\mathbf{x}, \qquad \forall V \in \mathscr{D}(\Omega). \tag{2.2.12}$$

From this point of view, we have $U, F \in \mathscr{D}(\Omega)'$ and the action of L is extended by the left hand side of (2.2.12) to be also defined on the space of distributions. That is to say, the solution U is interpreted as a bounded linear functional over the space of test functions.

Let $U \in \mathbf{L}^2_{\text{loc}}(\mathbb{R}^3)$ be such that $U|_{\Omega^-} \in \mathbf{X}\left(\Omega^-\right)$ and $U|_{\Omega^+} \in \mathbf{X}_{\text{loc}}\left(\mathsf{P}, \Omega^+\right)$ with $(\mathsf{P}-\lambda \mathrm{Id})U|_{\Omega^\mp} = 0$, where the restrictions are to be understood in the sense of distributions. Using Green's second formula (2.2.6) both in Ω^- and Ω^+ as in [11, Sec. 4.2], we obtain under Assumption I that

$$\langle \mathsf{P}U - \lambda U, \psi \rangle = \langle\!\langle [\mathsf{T}_D U], \mathsf{T}_N^- \psi \rangle\!\rangle_\Gamma - \langle\!\langle \mathsf{T}_D^- \psi, [\mathsf{T}_N U] \rangle\!\rangle_\Gamma \tag{2.2.13}$$

for all smooth compactly supported fields ψ defined over $\mathbb{R}^3$. Therefore, in the sense of distributions,

$$\mathsf{P}U - \lambda U = (\mathsf{T}_N^-)^* [\mathsf{T}_D u] - (\mathsf{T}_D^-)^* [\mathsf{T}_N u], \tag{2.2.14}$$

where the mappings $(\mathsf{T}_N^-)^*$ and $(\mathsf{T}_D^-)^*$ are adjoint to T_N^- and T_D^-, respectively—to be compared with [23, Thm. 6.10], [15, Eq. 3.8], [11, Eq. 38].

Let δ_0 be the Dirac distribution centered at 0, i.e. $\langle \delta_0, V \rangle = V(0)$ for all $V \in \mathscr{D}(\mathbb{R}^3)$. Recall that convolution by a matrix-valued function $\mathbf{M}$ defined on $\mathbb{R}^3 \backslash \{0\}$ is given by

$$\mathbf{M} \star U = \int_{\mathbb{R}^3} \mathbf{M}(\mathbf{x} - \mathbf{y}) U(\mathbf{y}) \mathrm{d}\mathbf{x}. \tag{2.2.15}$$

Compare the following assumption with [26, Sec. 1.1.3], [28, Chap. 5], [23, Chap. 6], [7, Sec. 4] and [11, 4.1].

Assumption VI (Fundamental solution) There exists a smooth (possibly matrix-valued) complex Green tensor $\mathbf{G}_\lambda$ defined over $\mathbb{R}^3 \backslash \{0\}$ satisfying

$$(\mathsf{P} - \lambda\, Id)\mathbf{G}_\lambda = \delta_0\, Id \tag{2.2.16}$$

as a distribution and the radiation conditions at infinity stated in (2.2.7a) and (2.2.7b).

Convolution with $\mathbf{G}_\lambda$ on both sides of (2.2.14) using (2.2.16) yields the representation formula

$$U = \mathcal{SL}_\lambda([\mathsf{T}_N U]) + \mathcal{DL}_\lambda([\mathsf{T}_D U]), \tag{2.2.17}$$

where we have defined for all $g \in \mathbf{H}_D$ and $\eta \in \mathbf{H}_N$ the layer potentials

$$\mathcal{SL}_\lambda(g) := -\mathbf{G}_\lambda \star ((\mathsf{T}_D^-)^* g), \qquad \mathcal{DL}_\lambda(\eta) := \mathbf{G}_\lambda \star ((\mathsf{T}_N^-)^* \eta). \tag{2.2.18}$$

2.2.5 Boundary integral operators

Boundary integral equations for the BVPs (2.2.7a) and (2.2.7b) are obtained by establishing the famous Caldéron identities.

The following continuity result and jump relations can be found for all examples to be covered below in [21, Eq. 5], [28, Chap. 6], [7, Thm. 7] and [11, Thm. 5.1] (beware of the sign in the definition of the jump across Γ, which may differ from one reference to another).

Assumption VII (Jump identities) We assume that the boundary potentials are continuous as mappings

$$\mathcal{SL}_\lambda : \mathbf{H}_N \to \mathbf{X}_{loc}(\Omega), \qquad \mathcal{DL}_\lambda : \mathbf{H}_D \to \mathbf{X}_{loc}(\Omega), \tag{2.2.19}$$

and suppose that they satisfy the jump relations

$$[\mathsf{T}_D]\,\mathcal{DL}_\lambda = \mathit{Id}, \qquad [\mathsf{T}_D]\,\mathcal{DL}_\lambda = 0, \tag{2.2.20a}$$
$$[\mathsf{T}_D]\,\mathcal{SL}_\lambda = 0, \qquad [\mathsf{T}_N]\,\mathcal{SL}_\lambda = \mathit{Id}. \tag{2.2.20b}$$

Applying averaged traces $\{\mathsf{T}_\bullet\} := 1/2(\mathsf{T}^+_\bullet + \mathsf{T}^-_\bullet)$ specified with $\bullet = D$ and N to $\mathcal{SL}_\lambda$ and $\mathcal{DL}_\lambda$ yields four continuous boundary integral operators:

$$\mathcal{V}_\lambda := \{\mathsf{T}_D\}\,\mathcal{SL}_\lambda : \mathbf{H}_N \to \mathbf{H}_D, \qquad \mathcal{K}^\dagger_\lambda := \{\mathsf{T}_N\}\,\mathcal{SL}_\lambda : \mathbf{H}_N \to \mathbf{H}_N, \tag{2.2.21a}$$
$$\mathcal{K}_\lambda := \{\mathsf{T}_D\}\,\mathcal{DL}_\lambda : \mathbf{H}_D \to \mathbf{H}_D, \qquad \mathcal{W}_\lambda := \{\mathsf{T}_N\}\,\mathcal{DL}_\lambda : \mathbf{H}_D \to \mathbf{H}_N. \tag{2.2.21b}$$

Taking the traces on both sides of the representation formula (2.2.17) and using the jump relations of Assumption VII, we obtain the interior and exterior Caldéron identities

$$\mathsf{P}U - \lambda U \text{ in } \Omega^- \quad \Longrightarrow \quad \underbrace{\begin{pmatrix} \mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id} & \mathcal{V}_\lambda \\ \mathcal{W}_\lambda & \mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{=:\mathbb{P}^-_\lambda} \begin{pmatrix} \mathsf{T}^-_D U \\ \mathsf{T}^-_N U \end{pmatrix} = \begin{pmatrix} \mathsf{T}^-_D U \\ \mathsf{T}^-_N U \end{pmatrix}, \tag{2.2.22a}$$

$$\mathsf{P}U - \lambda U \text{ in } \Omega^+ \quad \Longrightarrow \quad \underbrace{\begin{pmatrix} -\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id} & -\mathcal{V}_\lambda \\ -\mathcal{W}_\lambda & -\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{=:\mathbb{P}^+_\lambda} \begin{pmatrix} \mathsf{T}^+_D U \\ \mathsf{T}^+_N U \end{pmatrix} = \begin{pmatrix} \mathsf{T}^+_D U \\ \mathsf{T}^+_N U \end{pmatrix}, \tag{2.2.22b}$$

respectively, cf. [28, Sec. 6.6.], [7, Sec. 5], [11, Sec. 5], [27, Sec. 2.3], [26, Sec. 3.6], [21, Sec. 4]. Note that $\mathbb{P}^+_\lambda + \mathbb{P}^-_\lambda = \mathrm{Id}$ so that the range of $\mathbb{P}^+_\lambda$ coincides with the kernel of $\mathbb{P}^-_\lambda$ and vice-versa. The next theorem is a consequence of the existence of continuous right inverses for the traces stated in Assumption I. It promotes the Caldéron projectors to a pivotal role in domain–boundary formulations of transmission problems.

A pair of boundary functions $(g, \eta) \in \mathbf{H}_D \times \mathbf{H}_N$ is said to be valid interior/exterior Cauchy data if there exists a distribution $U^\mp \in \mathbf{X}_{\mathrm{loc}}\left(\Omega^\mp\right)$ solving the Dirichlet and Neumann BVPs (2.2.7a) and (2.2.7b) in $\Omega^\mp$ such that $\mathsf{T}^\mp_{\mathsf{P},D} U^\mp = g$ and $\mathsf{T}^\mp_{\mathsf{P},N} U^\mp = \eta$. We refer to [7, Thm. 8], [29, Thm. 3.7] and [26, Prop. 3.6.2] for the proof of the next result (cf. [28, Lem. 6.18], [11, Prop. 5.2] and [27, Sec. 2.3]).

Lemma 2.1 *A pair $(g, \eta) \in \mathbf{H}_D \times \mathbf{H}_N$ is valid interior or exterior Cauchy data if and only if it lies in the kernel of $\mathbb{P}^+_\lambda$ or $\mathbb{P}^-_\lambda$, respectively.*

2.2.6 Boundary integral equations

The rows of the exterior Caldéron identities give rise to the following two direct variational BIEs of the first kind for the exterior Dirichlet (2.2.7a) and Neumann (2.2.7b) problems respectively:

$$\xi \in \mathbf{H}_N(\Gamma): \qquad \int_\Gamma \mathcal{V}_\lambda \xi \cdot \zeta \mathrm{d}\sigma = -\int_\Gamma (\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id}) g \cdot \zeta \mathrm{d}\sigma \qquad \forall \zeta \in \mathbf{H}_N(\Gamma), \tag{2.2.23a}$$

$$\xi \in \mathbf{H}_D(\Gamma): \qquad \int_\Gamma \mathcal{W}_\lambda \xi \cdot \zeta \mathrm{d}\sigma = -\int_\Gamma (\mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id}) \eta \cdot \zeta \mathrm{d}\sigma \qquad \forall \zeta \in \mathbf{H}_D(\Gamma). \tag{2.2.23b}$$

2.3 Coupled domain–boundary variational formulations

The idea behind the so-called symmetric approach to marrying domain and boundary variational formulations (originally developed in [14] for problems involving linear strongly elliptic differential operators) is to introduce a particularly clever choice of Dirichlet-to-Neumann map

$$\mathsf{DtN} : \mathbf{H}_D \to \mathbf{H}_N \tag{2.3.1}$$

into Green's first formula—the validity of which, following M. Costabel, we have required in Assumption III. Notice that both rows of the exterior Caldéron projection $\mathbb{P}_\lambda^+$ realize a Dirichlet-to-Neumann map [26, Sec. 3.7], [21, Sec. 4]:

$$\mathsf{DtN}_1 := -\mathcal{V}_\lambda^{-1}(\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id}) : \mathbf{H}_D \to \mathbf{H}_N, \qquad \mathsf{DtN}_2 := -(\mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id})^{-1}\mathcal{W}_\lambda : \mathbf{H}_D \to \mathbf{H}_N.$$

M. Costabel's insight was to combine both rows into the expression

$$\mathsf{DtN} := -\mathcal{W}_\lambda + (-\mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id})\mathsf{DtN}_1.$$

Introducing the transmission conditions into (2.2.9), the transmission problem (2.2.8) can be cast into the operator equation

$$\boldsymbol{\Phi} U + (\mathsf{T}_{\mathsf{L},D}^-)^*(\mathsf{DtN} \circ \mathsf{T}_{\mathsf{L},D}^- U) = \mathbf{r.h.s.},$$

where $(\mathsf{T}_{\mathsf{L},D}^-)^* : \mathbf{H}_N = \mathbf{H}_D' \to \mathbf{X}^D(\mathsf{L}, \Omega^-)'$ denotes the adjoint of $\mathsf{T}_{\mathsf{L},D}^-$. Here, $\mathbf{H}_D'$ and $\mathbf{H}_N$ were identified using the duality pairing from Assumption I. Indeed, the details read

$$\begin{aligned}
\mathsf{T}_{\mathsf{L},N}^- U &= \mathsf{T}_{\mathsf{P},N}^+ U^{\mathrm{ext}} + \eta \\
&= -\mathcal{W}_\lambda \mathsf{T}_{\mathsf{P},D}^+ U^{\mathrm{ext}} + (-\mathcal{K}^\dagger + \frac{1}{2}\mathrm{Id})\mathsf{T}_{\mathsf{P},N}^+ U^{\mathrm{ext}} + \eta \\
&= -\mathcal{W}_\lambda \mathsf{T}_{\mathsf{P},D}^+ U^{\mathrm{ext}} + (-\mathcal{K}^\dagger + \frac{1}{2}\mathrm{Id})\mathsf{DtN}_1(\mathsf{T}_{\mathsf{P},D}^+ U^{\mathrm{ext}}) + \eta \\
&= \mathsf{DtN}(\mathsf{T}_{\mathsf{L},D}^- U) + \eta - \mathsf{DtN}(g).
\end{aligned} \tag{2.3.2}$$

Of course, we dispense with the explicit inverse of $\mathcal{V}_\lambda$ by introducing an auxiliary unknown

$$\xi := \mathsf{T}^+_{\mathsf{P},N} U^{\text{ext}} = \mathsf{DtN}_1 \mathsf{T}^+_{\mathsf{P},D} U^{\text{ext}} \in \mathbf{H}_N \tag{2.3.3}$$

in (2.3.2), seeking instead a solution pair $(U, \xi) \in \mathbf{V}(\Omega) \times \mathbf{H}_N$ to the variational problem

$$\begin{aligned} \boldsymbol{\Phi}(U,V) + \langle\!\langle(-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id})\xi, \mathsf{T}_{\mathsf{L},D}V\rangle\!\rangle + \langle\!\langle -\mathcal{W}_\lambda \mathsf{T}^-_{\mathsf{L},D}U, \mathsf{T}_{\mathsf{L},D}V\rangle\!\rangle &= \mathsf{R}_{\mathbf{V}}(v), \\ \langle\!\langle(\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id})\mathsf{T}^-_{\mathsf{L},D}U, \zeta\rangle\!\rangle + \langle\!\langle \mathcal{V}_\lambda \,\xi, \zeta\rangle\!\rangle &= \mathsf{R}_{\mathsf{T}}(\zeta), \end{aligned} \tag{2.3.4}$$

for all $(V, \zeta) \in \mathbf{V}(\Omega) \times \mathbf{H}_N$.

A few terms were moved to the continuous functionals on the right hand sides. In particular,

$$\mathsf{R}_{\mathbf{V}}(V) := \int_{\Omega^-} f \cdot V \,\mathrm{d}\mathbf{x} - \langle\!\langle \eta, \mathsf{T}_{\mathsf{L},D}V\rangle\!\rangle - \langle\!\langle \mathcal{W}_\lambda g, \mathsf{T}_{\mathsf{L},D}V\rangle\!\rangle, \tag{2.3.5a}$$

$$\mathsf{R}_{\mathsf{T}}(\zeta) := \langle\!\langle(\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id})g, \zeta\rangle\!\rangle. \tag{2.3.5b}$$

An alternative (arguably simpler) derivation based on the idea of modifying the Johnson-Nedélec coupling is presented in [2] and [27]. In this work, to choice to obtain (2.3.4) based on (2.3.2) is motivated by our intention to share the details behind the derivation available in [7, Sec. 10].

2.4 Resonant frequencies

We call *Dirichlet or Neumann resonant frequency* any eigenvalue in the Dirichlet or Neumann spectrum

$$\Lambda_D(\mathsf{P}, \Omega^-) := \{\lambda \in \mathbb{C} |\, \exists U \in \mathbf{X}\left(\mathsf{P}, \Omega^-\right), 0 \neq U \text{ solving (2.2.7a) in } \Omega^- \text{ with } g = 0\},$$
$$\Lambda_N(\mathsf{P}, \Omega^-) := \{\lambda \in \mathbb{C} |\, \exists U \in \mathbf{X}\left(\mathsf{P}, \Omega^-\right), 0 \neq U \text{ solving (2.2.7b) in } \Omega^- \text{ with } \eta = 0\},$$

respectively. Given a frequency $\lambda \in \Lambda_D$ or Λ_N, we denote the λ-eigenspaces by

$$E^\lambda_D(\mathsf{P}, \Omega^-) := \{U \in \mathbf{X}\left(\mathsf{P}, \Omega^-\right) |\, U \text{ solving (2.2.7a) in } \Omega^- \text{ with } g = 0\},$$
$$E^\lambda_N(\mathsf{P}, \Omega^-) := \{U \in \mathbf{X}\left(\mathsf{P}, \Omega^-\right) |\, U \text{ solving (2.2.7a) in } \Omega^- \text{ with } \eta = 0\},$$

respectively.

2.4.1 Kernels of first-kind direct boundary integral equations

The nontrivial eigenfunctions in $E^\lambda_D(\mathsf{P}, \Omega^-)$ and $E^\lambda_N(\mathsf{P}, \Omega^-)$ foil uniqueness of solutions of the boundary integral problems (2.2.23a) and (2.2.23b). The next lemmas completely characterize the kernels of the operators $\mathcal{V}_\lambda$ and $\mathcal{W}_\lambda$.

Lemma 2.2 $\ker(\mathcal{V}_\lambda) = \mathsf{T}^-_{\mathsf{P},N}(E^\lambda_D(\mathsf{P},\Omega^-))$

Proof. $(\supset)$ Suppose that $\lambda \in \Lambda_D$ and let $0 \neq U \in E^\lambda_D(\mathsf{P},\Omega^-)$. By Lemma 2.1, the valid Cauchy data $(0, \mathsf{T}^-_{\mathsf{P},N}U) \in \mathbf{H}_D \times \mathbf{H}_N$ for the interior problem is in the kernel of the exterior Caldéron projection. The first row of the matrix equation

$$\begin{aligned}\begin{pmatrix} -\mathcal{V}_\lambda \mathsf{T}^-_{\mathsf{P},N}U \\ (-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id})\mathsf{T}^-_{\mathsf{P},N}U \end{pmatrix} &= \begin{pmatrix} -\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id} & -\mathcal{V}_\lambda \\ -\mathcal{W}_\lambda & -\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 \\ \mathsf{T}^-_{\mathsf{P},N}U \end{pmatrix} \\ &= \mathbb{P}^+_\lambda \begin{pmatrix} 0 \\ \mathsf{T}^-_{\mathsf{P},N}U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\end{aligned} \tag{2.4.1}$$

implies that $\mathsf{T}^-_{\mathsf{P},N}U \in \ker(\mathcal{V}_\lambda)$.

$(\subset)$ If $\xi \in \mathbf{H}_N$ is such that $\mathcal{V}_\lambda \xi = 0$, then

$$\mathbb{P}^+_\lambda \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ (-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id})\xi \end{pmatrix}.$$

Lemma 2.1 then guarantees that $(0, (-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id})\xi)^\top$ is valid Cauchy data for the exterior boundary value problem (2.2.7a) in Ω^+. By Assumption II, the unique solution to the exterior Dirichlet boundary value problem (2.2.7a) with $g = 0$ is trivial, so it must be that $\mathcal{K}^\dagger_\lambda \xi = \frac{1}{2}\xi$. Therefore, we find that

$$\mathbb{P}^-_\lambda \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} \mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id} & \mathcal{V}_\lambda \\ \mathcal{W}_\lambda & \mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ \xi \end{pmatrix}.$$

We conclude relying on Lemma 2.1 again that there exists $0 \neq U \in E^\lambda_D(\mathsf{P},\Omega^-)$ with $\mathsf{T}^-_{\mathsf{P},N}U = \xi$. □

Because of the formal symmetry in the structure of the Caldéron identities, we also conclude from the above proof that the kernel of $\mathcal{W}_\lambda$ is spanned by the Dirichlet traces of the interior Neumann eigenfunctions of P.

Lemma 2.3 $\ker(\mathcal{W}_\lambda) = \mathsf{T}^-_{\mathsf{P},D}(E^\lambda_N(\mathsf{P},\Omega^-))$

The operators on the right-hand sides of the Dirichlet and Neumann variational boundary integral equations (2.2.23a) and (2.2.23b) display similar properties.

Lemma 2.4 $\ker(-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id}) = \mathsf{T}^-_{\mathsf{P},N}(E^\lambda_D(\mathsf{P},\Omega^-))$

Proof. $(\subset)$ Suppose that $\lambda \in \Lambda_D$ and let $U \in E^\lambda_D(\mathsf{P},\Omega^-)$. Using Theorem 1, the valid Cauchy data $(0, \mathsf{T}^-_{\mathsf{P},N}U) \in \mathbf{H}_D \times \mathbf{H}_N$ belongs to the kernel of $\mathbb{P}^+_\lambda$. We read from (2.4.1) that $\mathsf{T}^-_{\mathsf{P},N}U \in \ker(-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id})$.

$(\supset)$ If $(-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id})\xi = 0$, then similarly as in the proof of Lemma 2.2,

$$\mathbb{P}^+_\lambda \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} -\mathcal{V}_\lambda \xi \\ 0 \end{pmatrix},$$

which by Lemma 2.1 shows that $(\mathcal{V}_\lambda \xi, 0)$ is valid Cauchy data for the exterior boundary value problem. By Assumption II, the unique solution to (2.2.7b) in Ω^+ with $\eta = 0$ is trivial, so it must be that $\mathcal{V}_\lambda \xi = 0$. The conclusion follows from Lemma 2.2. □

The following result shouldn't come as a surprise now.

Lemma 2.5 $\ker(-\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id}) = \mathsf{T}^-_{\mathsf{P},D}(E^\lambda_N(\mathsf{P}, \Omega^-))$

Corollary 2.1 *A solution of the Dirichlet variational boundary integral equations* (2.2.23a) *is unique if and only if* $\lambda \notin \Lambda_D$.

Corollary 2.2 *A solution to the Neumann variational boundary integral equations* (2.2.23b) *is unique if and only if* $\lambda \notin \Lambda_N$.

2.4.2 Kernel of the domain-boundary coupled variational formulation

At this point, we are well equipped to study the kernel of the operator

$$\mathscr{P} := \begin{pmatrix} \boldsymbol{\Phi} - (\mathsf{T}^-_{\mathsf{L},D})^* \mathcal{W}_\lambda \circ \mathsf{T}^-_{\mathsf{L},D} & (\mathsf{T}^-_{\mathsf{L},D})^*(-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id}) \\ (\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id})\mathsf{T}^-_{\mathsf{L},D} & \mathcal{V}_\lambda \end{pmatrix}$$

arising from the variational problem (2.3.4).

Proposition 2.1 *The following are equivalent.*

1. $(U, \xi) \in \mathbf{V}(\Omega^-) \times \mathbf{H}_N$ *is in the kernel of* $\mathscr{P}$.
2. *The pair* $(U, \xi) \in \mathbf{V}(\Omega^-) \times \mathbf{H}_N$ *is such that*
   - $\mathsf{L}U = 0$ *in the sense of distributions,*
   - $(\mathsf{T}^-_{\mathsf{L},N}U - \xi) \in \mathsf{T}^-_{\mathsf{P},N}E^\lambda_D(\mathsf{P}, \Omega^-)$,
   - $(\mathsf{T}^-_{\mathsf{L},D}U, \mathsf{T}^-_{\mathsf{L},N}U)$ *is valid Cauchy data in* Ω^+.

Proof. $(1 \Rightarrow 2)$ Suppose that $(U, \xi) \in \mathbf{V}(\Omega^-) \times \mathbf{H}_N$ is such that for all $V \in \mathbf{V}(\Omega^-)$,

$$\boldsymbol{\Phi}(U, V) + \langle\!\langle(-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id})\xi, \mathsf{T}_{\mathsf{L},D}V\rangle\!\rangle + \langle\!\langle -\mathcal{W}_\lambda \mathsf{T}^-_{\mathsf{L},D}U, \mathsf{T}_{\mathsf{L},D}V\rangle\!\rangle = 0,$$
$$\langle\!\langle(\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id})\mathsf{T}^-_{\mathsf{L},D}U, \zeta\rangle\!\rangle + \langle\!\langle \mathcal{V}_\lambda\, \xi, \zeta\rangle\!\rangle = 0.$$

There are three elements that we need to check.

Testing with $V \in \mathscr{D}(\Omega)$, we immediately find that $\mathsf{L}U = 0$ holds in the sense of distributions ✓.

Therefore, we can rely on Assumption IV and use the generalized version (2.2.9) of Green's first formula to obtain

$$\langle\!\langle(-\mathcal{K}^\dagger_\lambda + \frac{1}{2}\mathrm{Id})\xi, \mathsf{T}^-_{\mathsf{L},D}V\rangle\!\rangle + \langle\!\langle -\mathcal{W}_\lambda \mathsf{T}^-_{\mathsf{L},D}U, \mathsf{T}^-_{\mathsf{L},D}V\rangle\!\rangle = \langle\!\langle \mathsf{T}^-_{\mathsf{L},N}U, \mathsf{T}^-_{\mathsf{L},D}V\rangle\!\rangle, \quad (2.4.2a)$$
$$\langle\!\langle(\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id})\mathsf{T}^-_{\mathsf{L},D}U, \zeta\rangle\!\rangle + \langle\!\langle \mathcal{V}_\lambda\, \xi, \zeta\rangle\!\rangle = 0. \quad (2.4.2b)$$

This allows us to evaluate

$$\mathbb{P}_\lambda^+ \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \xi \end{pmatrix} = \begin{pmatrix} -\mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id} & -\mathcal{V}_\lambda \\ -\mathcal{W}_\lambda & -\mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \xi \end{pmatrix} = \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix},$$

where the last equality was obtained by subtracting $\mathsf{T}_{\mathsf{L},D}^- U$ on both sides of (2.4.2b). Since the range of the exterior Calderón projector coincides with the kernel of its interior counterpart, Lemma 2.1 implies that the pair $(\mathsf{T}_{\mathsf{L},D}^- U, \mathsf{T}_{\mathsf{L},N}^- U) \in \mathbf{H}_D \times \mathbf{H}_N$ is valid exterior Cauchy data for (2.2.7a) ✓.

Moreover, from (2.4.2a) and (2.4.2b), we know that

$$\mathcal{V}_\lambda \mathsf{T}_{\mathsf{L},D} U = -\mathcal{V}_\lambda \xi, \qquad \text{and} \qquad \mathcal{W}_\lambda \mathsf{T}_{\mathsf{L},D} U = (-\mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id})\xi - \mathsf{T}_{\mathsf{L},N} U.$$

Hence,

$$\begin{aligned} 0 = \mathbb{P}_\lambda^- \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix} &= \begin{pmatrix} \mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id} & \mathcal{V}_\lambda \\ \mathcal{W}_\lambda & \mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{V}_\lambda (\mathsf{T}_{\mathsf{L},N}^- U - \xi) \\ (-\mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id})(\xi - \mathsf{T}_{\mathsf{L},N}^- U) \end{pmatrix} \end{aligned} \tag{2.4.3}$$

and conclude from Lemma 2.2 that $\mathsf{T}_{\mathsf{L},N}^- U - \xi \in \mathsf{T}_{\mathsf{P},N}^- E_D^\lambda(\mathsf{P}, \Omega^-)$ ✓.

$(2 \Rightarrow 1)$ Since $\mathsf{T}_{\mathsf{L},N}^- U - \xi$ is the interior Neumann trace of a Dirichlet λ-eigenfunction of P, it follows from Lemma 2.2 and Lemma 2.4 that

$$\mathcal{V}_\lambda \mathsf{T}_N^- U = \mathcal{V}_\lambda \xi,$$

and

$$(-\mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id})\mathsf{T}_N^- U = (-\mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id})\xi.$$

Moreover, because $\mathsf{L}U = 0$ in the sense of distributions, then Assumption IV guarantees that $U \in \mathbf{X}(\mathsf{L}, \Omega^-)$. We can thus integrate by parts using Assumption III to verify that $\boldsymbol{\Phi}(U, V) = \langle\!\langle -\mathsf{T}_{\mathsf{L},N}^- U, \mathsf{T}_{\mathsf{L},D}^- V \rangle\!\rangle_\Gamma$ for all $V \in \mathbf{V}(\Omega^-)$.

Therefore,

$$\begin{aligned} \langle\!\langle \mathscr{P} \begin{pmatrix} U \\ \xi \end{pmatrix}, \begin{pmatrix} V \\ \zeta \end{pmatrix} \rangle\!\rangle &= \left\langle \begin{pmatrix} -\mathcal{W}_\lambda & -\mathcal{K}_\lambda^\dagger + \frac{1}{2}\mathrm{Id} \\ \mathcal{K}_\lambda + \frac{1}{2}\mathrm{Id} & \mathcal{V}_\lambda \end{pmatrix} \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix} - \begin{pmatrix} \mathsf{T}_{\mathsf{L},N}^- U \\ 0 \end{pmatrix}, \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- V \\ \zeta \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \mathbb{P}_\lambda^- \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix}, \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- V \\ \zeta \end{pmatrix} \right\rangle \end{aligned}$$

vanishes for all $(V, \zeta) \in \mathbf{V}(\Omega) \times \mathbf{H}_N$, since valid exterior Cauchy data for $\mathsf{P} - \lambda\mathrm{Id}$ lies in the kernel of the interior Caldéron projector. This shows $(U, \xi) \in \ker(\mathscr{P})$. □

The previous characterization is technical, but it tells us a lot more than what is immediately apparent. It leads to the following main result.

Theorem 2.1 *The interior function $U \in \mathbf{V}(\Omega^-)$ of a solution pair $(U, \xi) \in \mathbf{V}(\Omega^-) \times \mathbf{H}_N$ solving the coupled variational problem (2.3.4) is always unique. If $\lambda \notin \Lambda_D$, then the boundary data ξ is also unique. It is otherwise only unique up to adding a boundary function lying in $\mathsf{T}^-_{\mathsf{P},N}(E^\lambda_D(\mathsf{P}, \Omega^-))$. In other words,*

$$\ker(\mathscr{P}) = \{0\} \times \mathsf{T}^-_{\mathsf{P},N} E^\lambda_D(\mathsf{P}, \Omega^-). \tag{2.4.4}$$

Proof. $(\subset)$ Suppose that the pair $(U, \xi) \in \mathbf{V}(\Omega^-) \times \mathbf{H}_N$ is in the kernel of $\mathscr{P}$. By Proposition 2.1, $U \in \mathbf{X}(\mathsf{L}, \Omega^-)$ and it solves $\mathsf{L}U = 0$ in the sense of distributions. The lemma also guarantees that the boundary field $(\mathsf{T}^-_{\mathsf{L},D}U, \mathsf{T}^-_{\mathsf{L},N}U)$ is valid Cauchy data for the Dirichlet BVP (2.2.7a) in Ω^+. Thus, $\exists\, U^{\text{ext}} \in \mathbf{X}(\mathsf{P}, \Omega^+)$ with $\mathsf{P}U - \lambda U = 0$ satisfying $\mathsf{T}^-_{\mathsf{L},D}U = \mathsf{T}^+_{\mathsf{P},D}U^{\text{ext}}$ and $\mathsf{T}^-_{\mathsf{L},N}U = \mathsf{T}^+_{\mathsf{P},N}U^{\text{ext}}$.

The pair (U, U^{ext}) solves the transmission problem (2.2.8) with $g = 0$ and $\eta = 0$. Therefore, by Assumption V, it can only be the trivial solution.

In particular, $U = 0$. Going back to Proposition 2.1 with this new information, we are left with the assertion (using $(1) \implies (2)$) that $\xi \in \mathsf{T}^-_{\mathsf{P},N} E^\lambda_D(\mathsf{P}, \Omega^-)$.

$(\supset)$ It follows immediately from $((2) \implies (1))$ in Proposition 2.1 that $(0, \xi) \in \ker(\mathscr{P})$ for all $\xi \in \mathsf{T}^-_{\mathsf{P},N} E^\lambda_D(\mathsf{P}, \Omega^-)$, because $(0, 0)$ is the valid Cauchy data associated with the trivial solution. □

2.4.3 Recovery of field solution in Ω^+

In practice, one is less interested by the solution pair $(U,\ \xi)$ of (2.3.4) than by the actual simulation $(U,\ U^{\text{ext}})$ solving the transmission problem (2.2.8). To recover the exterior function U^{ext}, we use the exterior representation formula

$$U^{\text{ext}} = -\mathcal{SL}_\lambda(\mathsf{T}^+_{\mathsf{P},N}U) - \mathcal{DL}_\lambda(\mathsf{T}^+_{\mathsf{P},D}U), \tag{2.4.5}$$

obtained from (2.2.17). This step was called *post-processing* in the introduction.

It goes as follows. The right hand side of (2.4.5) defines an operator $\mathfrak{R} : \mathbf{H}_D \times \mathbf{H}_N \to \mathbf{X}(\mathsf{P}, \Omega^+)$ by

$$\mathfrak{R}\begin{pmatrix} h \\ \zeta \end{pmatrix} = -\mathcal{SL}_\lambda(\zeta) - \mathcal{DL}_\lambda(h)$$

Therefore, given a solution pair $(U,\ \xi)$ solving (2.3.4), one retrieves the value of the scattered wave at a location $\mathbf{x} \in \Omega^+$ in the exterior region by computing

$$U^{\text{ext}}(\mathbf{x}) = \mathfrak{R}\begin{pmatrix} \mathcal{T}_{\mathsf{L},D}U - g \\ \xi \end{pmatrix}(\mathbf{x}). \tag{2.4.6}$$

Because (2.4.4) was established in Theorem 2.1, we need to verify the following.

Proposition 2.2 $\{0\} \times \mathsf{T}^-_{\mathsf{P},N} E^\lambda_D(\mathsf{P}, \Omega^-) \subset \ker(\mathfrak{R})$

Proof. Let $\xi \in \mathsf{T}^-_{\mathsf{P},N} E^\lambda_D(\mathsf{P}, \Omega^-)$. Using the jump identities of Assumption VII, we notice that

$$\begin{aligned} \mathsf{T}^+_{\mathsf{P},D}\, \mathfrak{R} \begin{pmatrix} 0 \\ \xi \end{pmatrix} &= -\{\mathsf{T}_{\mathsf{P},D}\}\, \mathcal{SL}_\lambda(\xi) \\ &= -\mathcal{V}_\lambda(\xi) \end{aligned}$$

vanishes by Lemma 2.2. We conclude that $\mathfrak{R} \begin{pmatrix} 0 \\ \xi \end{pmatrix}$ solves (2.2.7a) in Ω^+ with $g = 0$. By assumption II, this can only occur for $\mathfrak{R}(0\ \xi)^\top = 0$ in $\mathbf{L}^2(\Omega^+)$. □

Since $\mathfrak{R}$ is linear, this confirms uniqueness of the pair (U, U^{ext}), and along with it validity of the coupled problem (2.3.4) as a physical model for electromagnetic and acoustic transmission problems.

2.5 Examples

We now survey three concrete examples of transmission problems where the above assumptions are met.

2.5.1 Acoustics in frequency domain

The simplest examples of BVPs satisfying these hypotheses are obtained from elliptic operators acting on scalar real-valued functions, of which the Laplacian

$$\mathsf{P} := -\Delta = -\operatorname{div} \circ \nabla = -\sum_{i=1}^{3} \partial_i^2$$

is the most famous one. It acts on a suitably scaled pressure amplitude U in the scalar Helmholtz equation

$$\mathsf{L}U := -\operatorname{div}(\nabla U) - \kappa^2 r(\mathbf{x}) U = 0 \tag{2.5.1}$$

that models the propagation of plane time harmonic sound waves with real positive wave number $\kappa > 0$. While the bounded refractive index $r(\mathbf{x})$ may vary inside the inhomogeneous body Ω^-, it is a constant $r_0 \in \mathbb{R}$ in the unbounded air region Ω^+, leading to an exterior problem involving $\mathsf{P} - \lambda$ with $\lambda = \kappa^2 r_0$. BIEs offer the most flexible way of tackling the exterior problem, but a domain formulation is best suited to deal with the interior inhomogeneity. Because of its simplicity, acoustic scattering thus presents itself as a canonical example to illustrate the relevance of coupled domain–boundary variational formulations.

In this framework, the domain of the Laplace operator is easily seen to be

$$\mathbf{X}_{\text{loc}}(\Omega) := H_{\text{loc}}(\Delta, \Omega) = \{U \in H^1_{\text{loc}}(\Omega) |\ \nabla U \in \mathbf{H}_{\text{loc}}(\operatorname{div}, \Omega)\}. \tag{2.5.2}$$

Boundary value problems are stated using the classical Dirichlet and Neumann traces

$$\gamma^{\mp}U\,(x) = \lim_{\Omega^{\mp}\ni y\to x} U(y),$$
$$\gamma_n^{\mp}U\,(x) = -\lim_{\Omega^{\mp}\ni y\to x} \mathbf{n}(x)\cdot\nabla U\,(y),$$

which enter Green's identity (2.2.2a). These traces are well-defined on smooth scalar fields and can be extended continuously to Sobolev spaces:

$$\mathsf{T}^{\mp}_{\mathsf{P},D} := \gamma^{\mp} : H^1_{\text{loc}}(\Omega^{\mp}) \to H^{1/2}(\Gamma) =: \mathbf{H}_D, \tag{2.5.3a}$$
$$\mathsf{T}^{\mp}_{\mathsf{P},N} := \gamma_n^{\mp} : \mathbf{H}_{\text{loc}}(\text{div}, \Omega^{\mp}) \to H^{-1/2}(\Gamma) =: \mathbf{H}_N. \tag{2.5.3b}$$

The classical symmetric coupling for (2.5.1) derived in [21] fits the abstract framework of the previous sections with $\mathsf{T}^-_{\mathsf{L},D} = \mathsf{T}^-_{\mathsf{P},D}$, $\mathsf{T}^-_{\mathsf{L},N} = \mathsf{T}^-_{\mathsf{P},N}$ and

$$\boldsymbol{\Phi}(U,V) := \int_{\Omega} \nabla U\cdot\nabla V - \kappa^2 r(\mathbf{x})\mathrm{d}\mathbf{x}$$

defined on $\mathbf{V}(\Omega^-)\times\mathbf{V}(\Omega^-)$ where $\mathbf{V}(\Omega^-) := H^1(\Omega^-)$.

In this case, the bilinear form on the left-hand side of (2.3.4) is $H^1(\Omega^-)\times H^{-1/2}(\Gamma)$-coercive [21, Lem. 5.1].

2.5.2 E–H electromagnetism

As explained in the introduction, we also consider the non-elliptic linear operators arising in the simulation of electromagnetic scattering phenomena. A prominent example is the $\mathbf{curl}\,\mathbf{curl}$ operator

$$\mathbf{E} \mapsto \mathbf{curl}(\,\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{E}) \tag{2.5.4}$$

occurring in the frequency domain formulation of the electric wave equation

$$\mathsf{L}\mathbf{E} := \mathbf{curl}(\,\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{E}) - \omega^2\epsilon(\mathbf{x})\mathbf{E} = 0, \tag{2.5.5}$$

in which $\epsilon(\mathbf{x})$ and $\mu(\mathbf{x})$ are material properties known, respectively, as the dielectric and permeability tensors. Again, these quantities are assumed constant outside the scatterer, i.e. $\mu(\mathbf{x}) = \mu_0$ and $\epsilon(\mathbf{x}) = \epsilon_0$ in Ω^+. This is the most standard time-harmonic model for the propagation of an electromagnetic wave with angular frequency ω. As opposed to the Helmholtz equation of acoustic scattering, the unknown is a vector-valued function.

We note that the $\mathbf{curl}\,\mathbf{curl}$ operator can be represented by the operator matrix

$$\mathsf{P} := \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}^2 \tag{2.5.6}$$

involved in the the exterior problem for $\mathsf{P}-\lambda$, where $\lambda := \omega^2\mu_0\epsilon_0$. Its domain of definition is

$$\mathbf{X}_{\text{loc}}(\Omega) := \mathbf{H}_{\text{loc}}(\mathbf{curl}^2,\Omega) := \{\mathbf{E}\in\mathbf{H}_{\text{loc}}(\mathbf{curl},\Omega)\,|\,\mathbf{curl}(\mathbf{E})\in\mathbf{H}_{\text{loc}}(\mathbf{curl},\Omega)\}.$$

Well-posed boundary value problems are established for the electric wave equations by continuously extending the tangential traces

$$\mathsf{T}^{\mp}_{\mathsf{P},D} := \gamma_t^{\mp}\,\mathbf{E}(x) := \mathbf{n}(x) \times \gamma_\tau(\mathbf{E}(\mathbf{x})), \tag{2.5.7a}$$
$$\mathsf{T}^{\mp}_{\mathsf{P},N} := \gamma_R^{\mp}\,\mathbf{E}(x) := -\gamma_\tau^{\mp}\mathbf{curl}\,\mathbf{E}(x) \tag{2.5.7b}$$

to mappings

$$\mathsf{T}^{\mp}_{\mathsf{P},D} : \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^{\mp}) \to \mathbf{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) =: \mathbf{H}_D, \tag{2.5.8}$$
$$\mathsf{T}^{\mp}_{\mathsf{P},L} : \mathbf{H}_{\text{loc}}(\mathbf{curl}^2, \Omega^{\mp}) \to \mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma) =: \mathbf{H}_N. \tag{2.5.9}$$

Note that $\gamma_\tau^{\mp}\mathbf{E} := \mathbf{E} \times \mathbf{n}$ enters Green's identity (2.2.2b). The "magnetic trace" $\gamma_R^{\mp}\,\mathbf{E}$ plays a role akin to the Neumann trace. The relatively recent development of tangential traces theory for Lipschitz domains can be found in [4], [5] and [6]. A symmetric domain-boundary variational coupling for (2.5.5) fitting the framework of this article is performed in [19]. There, the bilinear form

$$\boldsymbol{\Phi}(\mathbf{E}, \mathbf{V}) := \int_\Omega \mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{E} \cdot \mathbf{curl}\,\mathbf{V} - \omega^2\epsilon(\mathbf{x})\mathrm{d}\mathbf{x}$$

enters Green's first formula together with the traces

$$\mathsf{T}^{-}_{\mathsf{L},D} := \mathsf{T}^{-}_{\mathsf{P},D},$$
$$\mathsf{T}^{-}_{\mathsf{L},N} := \gamma_R^{-}\,(\mu^{-1}(\mathbf{x})\,\mathbf{E}(x)).$$

As proved in [19], the bilinear form underlying the coupled variational problem (2.3.4) in this case satisfies a generalized Gårding inequality (T-coercivity) in $\mathbf{H}(\mathbf{curl}, \Omega^-) \times \mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)$.

2.5.3 A-ϕ electromagnetism

Equation (2.5.5) is obtained upon combining the dynamical equations

$$\mathbf{curl}\,\mathbf{E} = -i\omega\mu(x)\mathbf{H}, \qquad \mathbf{curl}\,\mathbf{H} = i\omega\epsilon(\mathbf{x})\mathbf{E},$$

that are part of the **E**–**H** formulation of Maxwell's equations. When the magnetic and electric fields are expressed in terms of the vector and scalar electromagnetic potentials, which satisfy $\mathbf{H} = \mu^{-1}(x)\,\mathbf{curl}\,\mathbf{A}$ and $\mathbf{E} = -\partial_t\mathbf{A} - \nabla\phi$, these two equations instead combine to form

$$\mathbf{curl}(\,\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{A}) + i\omega\epsilon(\mathbf{x})\nabla\phi - \omega^2\epsilon(\mathbf{x})\mathbf{A} = 0.$$

Elimination of ϕ using the Lorenz gauge

$$\text{div}(\epsilon(\mathbf{x})\mathbf{A}) + i\omega\phi = 0 \tag{2.5.10}$$

leads to the Hodge-Helmholtz equation

$$\mathbf{curl}(\,\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{A}) - \epsilon(\mathbf{x})\nabla\,\text{div}(\epsilon(\mathbf{x})\mathbf{A}) - \omega^2\epsilon(\mathbf{x})\mathbf{A} = 0. \tag{2.5.11}$$

Remark 2.2 The link between electromagnetism and geometry through the Hodge-Laplace operator is the subject of a vast literature. Because (2.5.11) is robust in the low-frequency limit $\omega \to 0$, its extension to inhomogeneous materials through the generalized Lorenz gauge (2.5.10) has resurfaced relatively recently as an interesting alternative to the standard electric wave equation for the simulation of some contemporary physical experiments in quantum optics [9].

When the material properties $\epsilon(\mathbf{x}) = \epsilon_0$ and $\mu(\mathbf{x}) = \mu_0$ are assumed constant, equation (2.5.11) reduces to

$$\mathsf{P} := \mathbf{curl}\,\mathbf{curl}\,\mathbf{A} - \eta\,\nabla\,\mathrm{div}\,\mathbf{A} - \kappa^2\mathbf{A} = 0, \tag{2.5.12}$$

where $\eta = \mu_0\epsilon_0^2$ and $\kappa^2 = \mu_0\epsilon_0\omega^2$. The domain of the so-called Hodge–Helmholtz operator on the left hand side is the intersection space $\mathbf{X}(\Omega^\mp) := \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega^\mp) \cap \mathbf{H}_{\mathrm{loc}}(\nabla\mathrm{div}, \Omega^\mp)$, where

$$\mathbf{H}_{\mathrm{loc}}(\nabla\mathrm{div}, \Omega) := \{\mathbf{U} \in \mathbf{H}_{\mathrm{loc}}(\mathrm{div}, \Omega)\,|\,\mathrm{div}\,\mathbf{U} \in H^1_{\mathrm{loc}}(\Omega)\}.$$

A pair of suitable traces for the formulation of boundary value problems is given by [11, 12]

$$\mathsf{T}^\mp_{\mathsf{P},N}\mathbf{A}(\mathbf{x}) := \mathcal{T}^\mp_{\mathrm{mg}}\,\mathbf{A}(\mathbf{x}) = \begin{pmatrix} \gamma^\mp_R\,\mathbf{A}(\mathbf{x}) \\ \gamma^\mp_n\,\mathbf{A}(\mathbf{x}) \end{pmatrix}, \tag{2.5.13a}$$

$$\mathsf{T}^\mp_{\mathsf{P},D}\mathbf{A}(\mathbf{x}) := \mathcal{T}^\mp_{\mathrm{el}}\,\mathbf{A}(\mathbf{x}) = \begin{pmatrix} \gamma^\mp_t\,\mathbf{A}(\mathbf{x}) \\ \eta\,\gamma^\mp\mathrm{div}\mathbf{A}(\mathbf{x}) \end{pmatrix}. \tag{2.5.13b}$$

Notice that their ranges are product trace spaces. This is partly due to the fact that the $\mathbf{A}$–ϕ potential formulation of Maxwell's equations initially introduced two unknowns in the wave equation. Going back to the Lorenz gauge (2.5.10), we see in the context of transmission problems that the second component of the "electric trace" $\mathcal{T}^\mp_{\mathrm{el}}$ is in hiding a continuity condition for the scalar potential. Once again, it is the "magnetic trace" $\mathcal{T}^\mp_{\mathrm{mg}}$ that resembles the Neumann trace. We refer to [9] for more details.

The natural trial and test subspaces of $\mathbf{H}(\mathrm{div}, \Omega^-) \cap \mathbf{H}(\mathbf{curl}, \Omega^-)$ readily obtained upon establishing domain based variational formulations for (2.5.12) using (2.2.2a) and (2.2.2b) are unfortunately not viable for discretization by finite elements [18, Sec. 6.2]. This is the reason why in [27] the mixed formulation

$$\begin{aligned} \mathbf{curl}\,(\,\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{A}) + \epsilon(\mathbf{x})\nabla P - \omega^2\epsilon(\mathbf{x})\mathbf{A} = \mathbf{F}, \\ -\,\mathrm{div}\,(\epsilon(\mathbf{x})\mathbf{A}) - P = 0, \end{aligned} \tag{2.5.14}$$

is considered. The operator matrix

$$\mathsf{L} := \begin{pmatrix} \mathbf{curl} \circ \mu^{-1}(\mathbf{x})\mathbf{curl} - \omega^2\epsilon(\mathbf{x}) & \epsilon(\mathbf{x})\nabla \\ -\,\mathrm{div}(\epsilon(\mathbf{x})\cdot) & -\,\mathrm{Id} \end{pmatrix}$$

is well defined over $\mathbf{X}(\mathsf{L}, \Omega^-) := \mathbf{H}(\mathbf{curl}^2, \Omega^-) \times H^1(\Omega^-)$ and therefore more convenient to model the interior problem. This subtlety justifies generalizing Green's first formula in Assumption III, because integration by parts yields

$$\int_{\Omega^-} \mathsf{L}\begin{pmatrix}\mathbf{A}\\P\end{pmatrix} \cdot \begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}\mathrm{d}\mathbf{x} = \boldsymbol{\Phi}_\kappa(\begin{pmatrix}\mathbf{A}\\P\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}) + \Big\langle \mathsf{T}^-_{\mathsf{L},N}\begin{pmatrix}\mathbf{A}\\P\end{pmatrix}, \mathsf{T}^-_{\mathsf{L},D}\begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}\Big\rangle, \tag{2.5.15}$$

where the bilinear form defined on $\mathbf{V}(\Omega^-) \times \mathbf{V}(\Omega^-)$ with $\mathbf{V}(\Omega^-) := \mathbf{H}(\mathbf{curl}, \Omega) \times H^1(\Omega^-)$ is given by

$$\boldsymbol{\Phi}_\kappa(\begin{pmatrix}\mathbf{A}\\ P\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\ Q\end{pmatrix}) := \int_{\Omega_s} \mu^{-1}\,\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\mathbf{V}\mathrm{d}\mathbf{x} + \int_{\Omega_s} \epsilon\,\nabla P\cdot\mathbf{V}\mathrm{d}\mathbf{x} - \int_{\Omega_s} P\,Q\mathrm{d}\mathbf{x} + \int_{\Omega_s} \mathbf{A}\cdot\epsilon\nabla Q\mathrm{d}\mathbf{x} - \omega^2\int_{\Omega_s} \epsilon\,\mathbf{A}\cdot\mathbf{V}\mathrm{d}\mathbf{x}$$

and the traces are

$$\mathsf{T}^-_{\mathsf{L},N}\begin{pmatrix}\mathbf{A}\\ P\end{pmatrix} = \begin{pmatrix}\gamma_R^-(\mu^{-1}(\mathbf{x})\mathbf{A}(\mathbf{x}))\\ \gamma_n^-\,(\epsilon(\mathbf{x})\mathbf{A}(\mathbf{x}))\end{pmatrix},$$

$$\mathsf{T}^-_{\mathsf{L},D}\begin{pmatrix}\mathbf{V}\\ Q\end{pmatrix} = \begin{pmatrix}\gamma_t^-\mathbf{V}(\mathbf{x})\\ -\gamma^-\,Q\end{pmatrix}.$$

For this formulation, T-coercivity in $\mathbf{H}(\mathbf{curl}, \Omega^-) \times H^1(\Omega^-) \times \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma) \times H^{-1/2}(\Gamma)$ of the bilinear form in (2.3.4) is established in [27, Thm. 5.6].

2.6 Conclusion

We have abstracted the common characteristics of the three particular problems presented in Section 2.5. As a consequence, Costabel's original symmetric coupling was generalized to allow for a larger class of operators. The issues raised by spurious resonant frequencies were found to be rooted in the formal structure detailed by the framework of Section 2.2. In Section 2.4, the consequences of their existence were investigated. In doing so, the kernels of the operators entering the problems (2.2.23a), (2.2.23b) and (2.3.4) were completely characterized. It was also shown that the Neumann eigenfunctions which thwart the uniqueness of solutions for the coupled problem vanish under the exterior representation formula, thus showing that the complete field solution U remains unique despite the existence of spurious resonance frequencies. The symmetric approach to symmetric domain-boundary coupling therefore remains a valuable starting point for Galerkin discretization.

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Chapter 3
First-Kind Boundary Integral Equations for the Dirac Operator in 3-Dimensional Lipschitz Domains

Erick Schulz and Ralf Hiptmair

Abstract We develop novel first-kind boundary integral equations for Euclidean Dirac operators in 3D Lipschitz domains. They comprise square-integrable potentials and involve only weakly singular kernels. Generalized Gårding inequalities are derived and we establish that the obtained boundary integral operators are Fredholm of index zero. Their finite dimensional nullspaces are characterized and we show that their dimensions are equal to the number of topological invariants of the domain's boundary, in other words, to the sum of its Betti numbers. This is explained by the fundamental discovery that the associated bilinear forms agree with those induced by the 2D Dirac operators for surface de Rham Hilbert complexes whose underlying inner-products are the non-local inner products defined through the classical single-layer boundary integral operators for the Laplacian. Decay conditions for well-posedness in natural energy spaces of the Dirac system in unbounded exterior domains are also presented.

3.1 Introduction

We develop first-kind boundary integral equations for the Hodge-Dirac operator in 3-dimensional Euclidean space

$$\mathsf{D} := \mathbf{d} + \boldsymbol{\delta} : \mathbf{H}(\mathbf{d}, \Omega^{\mp}) \cap \mathbf{H}(\boldsymbol{\delta}, \Omega^{\mp}) \to L^2(\Omega^{\mp})^8, \tag{3.1.1}$$

involving the exterior derivative and codifferential

$$\mathbf{d} := \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{0}^\top & 0 \\ \nabla & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0} \\ \mathbf{0} & \mathbf{curl} & \mathbf{0}_{3\times 3} & \mathbf{0} \\ 0 & \mathbf{0}^\top & \operatorname{div} & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\delta} := \begin{pmatrix} 0 & -\operatorname{div} & \mathbf{0}^\top & 0 \\ \mathbf{0} & \mathbf{0}_{3\times 3} & \mathbf{curl} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & -\nabla \\ 0 & \mathbf{0}^\top & \mathbf{0}^\top & 0 \end{pmatrix}. \tag{3.1.2}$$

We are concerned with the partial differential equations $\mathsf{D}\overrightarrow{\mathbf{U}} = \overrightarrow{\mathbf{F}}$, which in components $\overrightarrow{\mathbf{U}} = (U_0, \mathbf{U}_1, \mathbf{U}_2, U_3)^\top$ and $\overrightarrow{\mathbf{F}} = (F_0, \mathbf{F}_1, \mathbf{F}_2, F_3)^\top$ read

$$\mathcal{N} \quad \begin{aligned} -\operatorname{div} \mathbf{U}_1 &= F_0, \\ \nabla U_0 + \mathbf{curl}\, \mathbf{U}_2 &= \mathbf{F}_1, \\ -\nabla U_3 + \mathbf{curl}\, \mathbf{U}_1 &= \mathbf{F}_2, \\ \operatorname{div} \mathbf{U}_2 &= F_3. \end{aligned} \tag{3.1.3}$$

We will consider both interior and exterior boundary value problems, and assume that (3.1.3) is either posed on a bounded domain Ω^- having a Lipschitz boundary $\Gamma := \partial\Omega^-$, or on the unbounded complement $\Omega^+ := \mathbb{R}^3\backslash\overline{\Omega^-}$. In the latter case, suitable decay conditions at infinity will be needed. Throughout, $\Omega \in \{\Omega^-, \Omega^+\}$.

3.1.1 Related work

Current work discussing Dirac operators from the point of view of Hodge theory offers solutions to boundary value problems for (3.1.3) and related eigenvalue problems based on domain variational formulations [13, 26].

The operator matrix in (3.1.1) appears under a change of variables in the works of M. Taskinen, S. Vänskä and P. Ylä-Oijala [41–43] as R. Picard's extended Maxwell operator. It was originally assembled by R. Picard by combining the first-order Maxwell operator with the principal part of the equations of linear acoustics [25, 34, 35]. In [41–43], Helmholtz-like boundary value problems for Picard's operator are studied with a focus on *second-kind* boundary integral equations.

Eigenvalue problems related to acoustic and electromagnetic scattering, that is transmission problems for the so-called perturbed Dirac operator, have also guided the study of second-kind boundary integral equations in the literature of harmonic and hypercomplex analysis. Important contributions were made in that direction by E. Marmolejo-Olea, I. Mitrea, M. Mitrea, Q. Shi [28], A. Axelsson, A. Rosén and J. Helsing [4, 20, 36]. There, the Dirac operator enters larger systems of equations that encompass or correspond to Maxwell's equations [20, 28]. An extensive body of work, created by these authors together with R. Grognard and J. Hogan [5], S. Keith [6], A. McIntosh and S. Monniaux [30,31], is devoted to the harmonic analysis of Dirac operators in L_p spaces [7,29].

3.1.2 Our contributions

In this work, we derive novel *first-kind* boundary integral equations for the Dirac equation $\mathsf{D}\overrightarrow{\mathbf{U}} = 0$ with suitable boundary and decay conditions. Two boundary integral operators are obtained and shown to satisfy generalized Gårding inequalities, making them Fredholm of index 0. Their finite dimensional nullspaces are characterized in Section 3.7, where we show that their dimension equals the number of topological invariants of the boundary—counted as the sum of its Betti numbers. Indeed, the integral representations of their associated bilinear forms turn out to be related to the variational formulations of the surface Dirac operators introduced in Section 3.8. Recognizing these surface operators will simultaneously reveal how the boundary integral operators introduced in Section 3.5, which are related to two different sets of boundary conditions, arise as "rotated" versions of one another. The exterior representation formula of Lemma 3.11 and the condition at infinity identified in (3.4.66) eventually lead, together with the coercivity results of Section 3.6, to well-posedness of Euclidean Dirac exterior boundary value problems in natural energy spaces in the complement of the finite dimensional nullspaces.

The new integral formulas display desirable properties: the surface potentials are square-integrable and the kernels of the bilinear forms associated with the boundary integral operators are merely weakly singular, i.e. they are bounded by $|x-y|^{-\alpha}$, $\alpha < 2$, cf. [24, Sec. 2.4]. Nevertheless, we want to emphasize that the main result is the discovery that they relate to the Hodge–Dirac operators of surface de Rham Hilbert complexes equipped with the non-local inner products defined as the bilinear forms associated with the classical single-layer potential for the Laplacian. As a consequence, we already know a lot about these first-kind boundary integral operators for the Dirac operator. Moreover, this relationship suggests that they are related to the first-kind boundary integral operators for the Hodge–Laplacian.

For the sake of readability, we adopt the framework of classical vector analysis rather than exterior calculus. It is in this framework that the structural relationship between the following development and the standard theory for second-order elliptic operators seemed most explicit.

In summary, our main contributions are:

- ▷ We derive representation formulas for the Dirac equation posed on domains having a Lipschitz boundary by following the approach pioneered by M. Costabel [16]. The novelty here is to follow and extend the elegant strategy used in [14]—there used to find a representation formula for Hodge–Laplace and Helmholtz operators—that leads to potentials having simple explicit expressions. By adapting the arguments in the now classical monographs by W. McLean [32, Chap. 7] and A. Sauter and C. Schwab [37, Chap. 3], we also establish an exterior representation formula. We will observe that the development of this theory is possible due to the strong structural similarity between integration by parts for the first-order Dirac operator and Green's second formula for second-order elliptic operators.
- ▷ A sneak peek at the potentials presented in (3.4.39) and (3.4.42) will already convince the reader that the approach we have adopted leads to simple formulas for the *square-integrable* potentials involved in the representation formula. Some terms are recognizable from [14, 15], while others occur in well-known theory for elliptic second-order operators. The simplicity that comes with the calculation procedure provided by Lemma 3.6 allows for a straightforward analysis of their mapping and jump properties.
- ▷ Given the previous items, it is not surprising that decay conditions at infinity for exterior boundary value problems posed on the unbounded domain Ω^+ can be easily established by adapting the approach for second-order elliptic operators presented in [32, Chap. 7].
- ▷ The crux of our calculations are the formulas (3.5.12) and (3.5.13) for the bilinear forms associated with the obtained weakly-singular first-kind boundary integral operators. We provide generalized Gårding inequalities for the two operators and characterize their null-spaces.
- ▷ Our main discovery is presented in Section 3.8, where we expose the relationship between these boundary integral operators and surface Dirac operators in an Hilbert complex framework.

3.2 Function spaces and traces

As usual, $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ denote the Hilbert spaces of complex square-integrable scalar and vector-valued functions defined over Ω. We denote their inner products using round brackets, e.g. $(\cdot,\cdot)$. The spaces $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ refer to the corresponding Sobolev spaces. The notation $C^\infty(\Omega)$ is used for smooth functions. The subscript in $C_0^\infty(\Omega)$ further specifies that these smooth functions have compact support in Ω. $C^\infty(\overline{\Omega})$ is defined as the space of uniformly continuous functions over Ω that have uniformly continuous derivatives of all order. A subscript is used to identify spaces of locally integrable functions/vector fields, e.g. $U \in L^2_{\text{loc}}(\Omega)$ if and only if φU is square-integrable for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. We denote with an asterisk the spaces of functions with zero mean, e.g. $H^1_*(\Omega)$.

In general, given an operator L acting on square-integrable fields in the sense of distributions, we equip

$$\mathbf{H}(\mathsf{L},\Omega) := \{\mathbf{U} \in (\mathbf{L}^2(\Omega))^\bullet \,\big|\, \mathsf{L}\mathbf{U} \in (\mathbf{L}^2(\Omega))^\dagger\} \tag{3.2.1}$$

with the natural graph norm, where $\bullet = 8$ or 3 and $\dagger = 8$, 3 or 1. Important specimens are

$$\mathbf{H}(\mathrm{div},\Omega) := \left\{\mathbf{U} \in (L^2(\Omega))^3 \,|\, \mathrm{div}\,\mathbf{U} \in L^2(\Omega)\right\}, \tag{3.2.2}$$

$$\mathbf{H}(\mathbf{curl},\Omega) := \left\{\mathbf{U} \in (L^2(\Omega))^3 \,|\, \mathbf{curl}\,\mathbf{U} \in (L^2(\Omega))^3\right\}. \tag{3.2.3}$$

Of course, in all of the above definitions, Ω can be replaced by $\mathbb{R}^3$, or any other domain. We understand restrictions in the sense of distributions when working with domains having disconnected components. For example, in line with the above notation we mean in particular

$$\mathbf{H}(\mathsf{D},\mathbb{R}^3\backslash\Gamma) := \mathbf{H}(\mathsf{D},\Omega) \times \mathbf{H}(\mathsf{D},\mathbb{R}^3\backslash\Omega) \subset (L^2(\mathbb{R}^3))^8. \tag{3.2.4}$$

We use a prime superscript to denote dual spaces, for instance $C_0^\infty(\Omega)'$ is the space of distributions in Ω. Angular brackets indicate duality pairings, e.g. $\langle\cdot,\cdot\rangle_\Omega$ or $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\Gamma$. The former will be used for domain-based quantities in Ω, while the latter will pair spaces on Γ.

Trace-related theory for Lipschitz domains can be found in [8, 9, 11] and [19, 32], where it is established that the traces

$$\gamma W := \mathbf{W}\big|_\Gamma, \qquad \forall\, W \in C^\infty(\overline{\Omega}), \tag{3.2.5a}$$

$$\gamma_n \mathbf{W} := \gamma\mathbf{W}\cdot\mathbf{n}, \qquad \forall\, \mathbf{W} \in \mathbf{C}^\infty(\overline{\Omega}), \tag{3.2.5b}$$

$$\gamma_\tau \mathbf{W} := \gamma\mathbf{W}\times\mathbf{n}, \qquad \forall\, \mathbf{W} \in \mathbf{C}^\infty(\overline{\Omega}), \tag{3.2.5c}$$

$$\gamma_t \mathbf{W} := \mathbf{n}\times(\gamma_\tau\mathbf{W}), \qquad \forall\, \mathbf{W} \in \mathbf{C}^\infty(\overline{\Omega}), \tag{3.2.5d}$$

extend to continuous and surjective linear operators

$$\gamma : H^1(\Omega) \to H^{1/2}(\Gamma), \qquad \text{[22, Thm. 4.2.1]} \tag{3.2.6a}$$

$$\gamma_n : \mathbf{H}(\mathrm{div},\Omega) \to H^{-1/2}(\Gamma), \qquad \text{[19, Thm. 2.5, Cor. 2.8]} \tag{3.2.6b}$$

$$\gamma_\tau : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma), \qquad \text{[11, Thm. 4.1]} \tag{3.2.6c}$$

$$\gamma_t : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma), \qquad \text{[11, Thm. 4.1]} \tag{3.2.6d}$$

with nullspaces

$$H^1_0(\Omega) := \overline{C_0^\infty(\Omega)}^{H^1(\Omega)} = \ker\gamma, \qquad \text{[32, Thm 3.40]} \tag{3.2.7}$$

$$\mathbf{H}_0(\mathrm{div},\Omega) := \overline{C_0^\infty(\Omega)^3}^{\mathbf{H}(\mathrm{div},\Omega)} = \ker\gamma_n, \qquad \text{[33, Thm. 3.25]} \tag{3.2.8}$$

$$\mathbf{H}_0(\mathbf{curl},\Omega) := \overline{C_0^\infty(\Omega)^3}^{\mathbf{H}(\mathbf{curl},\Omega)} = \ker\gamma_\tau = \ker\gamma_t. \qquad \text{[33, Thm. 3.33]} \tag{3.2.9}$$

Here, $\mathbf{n} \in \mathbf{L}^\infty(\Gamma)$ is the essentially bounded unit normal vector field on Γ directed toward the exterior of Ω^-. Detailed definitions can be found in [8, 9, 11] together with a study of the involved surface differential operators. Short practical summaries are also provided in [12, 14, 23, 38].

Similarly as for the Hodge–Laplace operator [14,15,38,39], a theory of boundary value problems for the Hodge–Dirac problem in three dimensions entails partitioning our collection of traces into two "dual" pairs. Accordingly, we assemble the traces into

$$\gamma_{\mathsf{T}}(\overrightarrow{\mathbf{U}}) := \begin{pmatrix} \gamma(U_0) \\ \gamma_t(\mathbf{U}_1) \\ \gamma_n(\mathbf{U}_2) \end{pmatrix} \qquad \text{and} \qquad \gamma_{\mathsf{R}}(\overrightarrow{\mathbf{U}}) := \begin{pmatrix} \gamma_n(\mathbf{U}_1) \\ \gamma_\tau(\mathbf{U}2) \\ \gamma(U_3) \end{pmatrix}. \tag{3.2.10}$$

Warning. *We want to highlight that in spite of the notation, γ_{T} and γ_{R} are **not** defined as in [14], [15] and related work.*

The trace spaces

$$\mathcal{H}_{\mathsf{T}} := H^{1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma), \tag{3.2.11a}$$

$$\mathcal{H}_{\mathsf{R}} := H^{-1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{1/2}(\Gamma), \tag{3.2.11b}$$

are dual to each other with respect to the $L^2(\Gamma)$ duality pairing (c.f. [11, Lem. 5.6]). In this sense, we can identify

$$\mathcal{H}'_{\mathsf{T}} = \mathcal{H}_{\mathsf{R}} \qquad \text{and} \qquad \mathcal{H}'_{\mathsf{R}} = \mathcal{H}_{\mathsf{T}}. \tag{3.2.12}$$

Naturally, the traces can also be taken from the exterior domain. The extensions (3.2.6) will be tagged with a minus subscript (only when required to avoid confusion), e.g. γ^-, to distinguish them from the extensions obtained from (3.2.5) by replacing Ω with $\Omega^+ := \mathbb{R}^3 \backslash \overline{\Omega}$, which we will label with a plus superscript, e.g. γ^+.

Lemma 3.1 (See [14, Lem. 6.4]) *The linear mappings*

$$\gamma^{\pm}_{\mathsf{T}} : \mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \Omega^{\pm}) \to \mathcal{H}_{\mathsf{T}}, \qquad \gamma^{\pm}_{\mathsf{R}} : \mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \Omega^{\pm}) \to \mathcal{H}_{\mathsf{R}}, \tag{3.2.13}$$

defined by (3.2.10) *are continuous and surjective. There exist continuous lifting maps $\mathcal{E}_{\mathsf{T}} : \mathcal{H}_{\mathsf{T}} \to \mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \mathbb{R}^3 \backslash \Gamma)$ and $\mathcal{E}_{\mathsf{R}} : \mathcal{H}_{\mathsf{R}} \to \mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \mathbb{R}^3 \backslash \Gamma)$ such that $\gamma_{\mathsf{T}} \circ \mathcal{E}_{\mathsf{T}} = \mathrm{Id}$ and $\gamma_{\mathsf{R}} \circ \mathcal{E}_{\mathsf{R}} = \mathrm{Id}$.*

Lemma 3.2 (See [14, Lem. 6.4]) *The surface divergence extends to a continuous surjection $\mathrm{div}_\Gamma : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to H_*^{-1/2}(\Gamma)$, while $\mathbf{curl}_\Gamma : H_*^{1/2} \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is a bounded injection with closed range such that $\mathbf{curl}_\Gamma \xi = \nabla_\Gamma \xi \times \mathbf{n}$ for all $\xi \in H^{1/2}(\Gamma)$. These operators satisfy $\mathrm{div}_\Gamma \circ \mathbf{curl}_\Gamma = 0$.*

Lemma 3.3 *For all $\overrightarrow{\mathbf{U}} \in \mathbf{H}(\mathbf{d}, \Omega^{\mp})$ and $\overrightarrow{\mathbf{V}} \in \mathbf{H}(\boldsymbol{\delta}, \Omega^{\mp})$,*

$$\int_{\Omega^{\mp}} \mathbf{d}\overrightarrow{\mathbf{U}} \cdot \overrightarrow{\mathbf{V}} \mathrm{d}\mathbf{x} = \int_{\Omega^{\mp}} \overrightarrow{\mathbf{U}} \cdot \boldsymbol{\delta}\overrightarrow{\mathbf{V}} \mathrm{d}\mathbf{x} \pm \langle\!\langle \gamma_{\mathsf{T}} \overrightarrow{\mathbf{U}}, \gamma_{\mathsf{R}} \overrightarrow{\mathbf{V}} \rangle\!\rangle_\Gamma. \tag{3.2.14}$$

Proof. We integrate by parts using Green's identities to obtain

$$\begin{aligned} \int_{\Omega^{\mp}} \mathbf{dU} \cdot \mathbf{V} \mathrm{d}\mathbf{x} &= \int_{\Omega^{\mp}} \nabla U_0 \cdot \mathbf{V}_1 \mathrm{d}\mathbf{x} + \int_{\Omega^{\mp}} \mathbf{curl}\, \mathbf{U}_1 \cdot \mathbf{V}_2 \mathrm{d}\mathbf{x} + \int_{\Omega^{\mp}} (\mathrm{div}\, \mathbf{U}_2) V_3 \mathrm{d}\mathbf{x} \\ &= -\int_{\Omega^{\mp}} \mathbf{U}_0\, (\mathrm{div}\, \mathbf{V}_1) \mathrm{d}\mathbf{x} + \int_{\Omega^{\mp}} \mathbf{U}_1 \cdot \mathbf{curl}\, \mathbf{V}_2 \mathrm{d}\mathbf{x} - \int_{\Omega^{\mp}} \mathbf{U}_2 \cdot \nabla V_3 \mathrm{d}\mathbf{x} \end{aligned}$$

$$+\langle\gamma U_0,\gamma_n\mathbf{V}_1\rangle_\Gamma+\langle\gamma_t\mathbf{U}_1,\gamma_\tau\mathbf{V}_2\rangle_\Gamma+\langle\gamma_n\mathbf{U}_2,\gamma V_3\rangle_\Gamma$$
$$=\int_{\Omega^\mp}\mathbf{U}\cdot\boldsymbol{\delta}\mathbf{V}\mathrm{d}\mathbf{x}+\langle\!\langle\gamma_{\mathsf{T}}\mathbf{U},\gamma_{\mathsf{R}}\mathbf{V}\rangle\!\rangle_\Gamma.$$

□

Corollary 3.1 (Green's formula for Dirac operator) *For all* $\vec{\mathbf{U}},\vec{\mathbf{V}}\in\mathbf{H}(\mathsf{D},\Omega^\mp)$*, we have*

$$\int_{\Omega^\mp}\mathsf{D}\vec{\mathbf{U}}\cdot\vec{\mathbf{V}}\mathrm{d}\mathbf{x}=\int_{\Omega^\mp}\vec{\mathbf{U}}\cdot\mathsf{D}\vec{\mathbf{V}}\mathrm{d}\mathbf{x}\pm\langle\!\langle\gamma_{\mathsf{T}}\vec{\mathbf{U}},\gamma_{\mathsf{R}}\vec{\mathbf{V}}\rangle\!\rangle_\Gamma\mp\langle\!\langle\gamma_{\mathsf{T}}\vec{\mathbf{V}},\gamma_{\mathsf{R}}\vec{\mathbf{U}}\rangle\!\rangle_\Gamma. \tag{3.2.15}$$

Remark 3.1 It is remarkable that despite the fact that D is a first-order operator, eq. (3.2.15) nevertheless resembles Green's classical second formula for the Laplacian. This induces profound structural similarities between the representation formula, potentials and boundary integral equations for the Dirac operator established in the next sections and the already well-known theory for second-order elliptic operators. As emphasized in [39], a formula such as eq. (3.2.15) paves the way for harnessing powerful established techniques.

We will indicate with curly brackets the average $\{\gamma_\bullet\}:=\frac{1}{2}(\gamma_\bullet^++\gamma_\bullet^-)$ of a trace and with square brackets its jump $[\gamma_\bullet]:=\gamma_\bullet^--\gamma_\bullet^+$ over the interface Γ.

Warning. *Notice the sign in the jump* $[\gamma]=\gamma^--\gamma^+$*, which is often taken to be the opposite in the literature!*

3.3 Boundary value problems

In light of Lemma 3.1 and the duality in (3.2.12), the integration by parts formula (3.2.15) points towards two types of boundary conditions. Consider the boundary value problems of finding $\vec{\mathbf{U}}\in\mathbf{H}(\mathsf{D},\Omega)$ satisfying

$$\begin{cases}\mathsf{D}\vec{\mathbf{U}}&=\vec{\mathbf{0}}, \quad \text{in } \Omega,\\ \gamma_{\mathsf{T}}\vec{\mathbf{U}}&=\vec{\mathbf{b}}, \quad \text{on } \Gamma,\end{cases}\qquad \vec{\mathbf{b}}\in\mathcal{H}_{\mathsf{T}}, \tag{T}$$

or

$$\begin{cases}\mathsf{D}\vec{\mathbf{U}}&=\vec{\mathbf{0}}, \quad \text{in } \Omega,\\ \gamma_{\mathsf{R}}\vec{\mathbf{U}}&=\vec{\mathbf{a}}, \quad \text{on } \Gamma,\end{cases}\qquad \vec{\mathbf{a}}\in\mathcal{H}_{\mathsf{R}}. \tag{R}$$

For $\Omega=\Omega^+$, also impose the decay condition that $\vec{\mathbf{U}}(\mathbf{x})\to 0$ uniformly as $\mathbf{x}\to\infty$, cf. Lemma 3.12. In the following sections, development related to problem (T) will be colored in blue, while red will be used for (R).

When Ω is bounded, the self-adjoint Dirac operator behind (R) is

$$\mathsf{D}_{\mathsf{R}}^{\Omega}=\mathbf{d}+\mathbf{d}^*, \tag{3.3.1}$$

where $\mathbf{d}:L^2(\Omega)^8\to L^2(\Omega)^8$ is the closed densely defined Fredholm-nilpotent linear operator associated with the L^2 de Rham cochain complex [1,26]

$$H^1(\Omega) \xrightarrow{\nabla} \mathbf{H}(\mathbf{curl},\Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega), \tag{3.3.2}$$

cf. [1, Chap. 3-4], [26, Sec. 2]. The Hilbert space adjoint $\mathbf{d}^*$ is the nilpotent operator associated with the dual chain complex [1, Sec. 4.3, Thm. 6.5]

$$L^2_*(\Omega) \xleftarrow[-\mathrm{div}]{} \mathbf{H}_0(\mathrm{div},\Omega) \xleftarrow[\mathbf{curl}]{} \mathbf{H}_0(\mathbf{curl},\Omega) \xleftarrow[-\nabla]{} \mathbf{H}^1_0(\Omega). \tag{3.3.3}$$

The mapping properties of D_R and its domain are detailed in Figure 3.1.

Similarly, the self-adjoint operator

$$\mathsf{D}^\Omega_\mathsf{T} := \boldsymbol{\delta} + \boldsymbol{\delta}^* \tag{3.3.4}$$

behind (T) arises from the dual perspective, where we view the codifferential operator

$$\boldsymbol{\delta} : L^2(\Omega)^8 \to L^2(\Omega)^8$$

as the nilpotent operator associated with the Hilbert chain complex

$$L^2(\Omega) \xleftarrow[-\mathrm{div}]{} \mathbf{H}(\mathrm{div},\Omega) \xleftarrow[\mathbf{curl}]{} \mathbf{H}(\mathbf{curl},\Omega) \xleftarrow[-\nabla]{} \mathbf{H}^1(\Omega). \tag{3.3.5}$$

The adjoint $\boldsymbol{\delta}^*$ is spawned by the chain complex

$$H^1_0(\Omega) \xrightarrow{\nabla} \mathbf{H}_0(\mathbf{curl},\Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}_0(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2_*(\Omega). \tag{3.3.6}$$

See Figure 3.2 for the explicit mapping properties of $\mathsf{D}^\Omega_\mathsf{T}$ and its domain of definition.

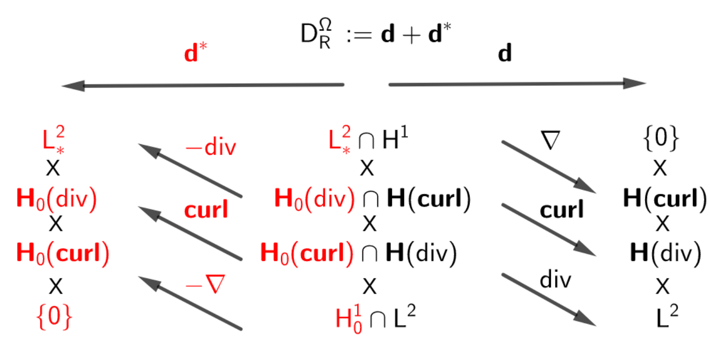

Fig. 3.1 This diagram shows the mapping properties of the exterior derivatives and their Hilbert space adjoints corresponding to the functional analytic setting of [26] for problem (R) in Ω^-. In the figure, the operators on the left-hand side are to be understood as the adjoint operators $-\mathrm{div} = \nabla^*$, $\mathbf{curl} = \mathbf{curl}^*$ and $-\nabla = \mathrm{div}^*$.

So unlike second-order operators, the Hodge–Dirac operator admits two distinct fundamental symmetric bilinear forms

$$\mathsf{a}_{\boldsymbol{\delta}}(\vec{\mathbf{U}},\vec{\mathbf{V}}) = \int_\Omega \boldsymbol{\delta}\vec{\mathbf{U}}\cdot\vec{\mathbf{V}} + \vec{\mathbf{U}}\cdot\boldsymbol{\delta}\vec{\mathbf{V}}\,\mathrm{d}\mathbf{x}, \qquad \vec{\mathbf{U}},\vec{\mathbf{V}} \in \mathbf{H}(\boldsymbol{\delta},\Omega), \tag{3.3.7a}$$

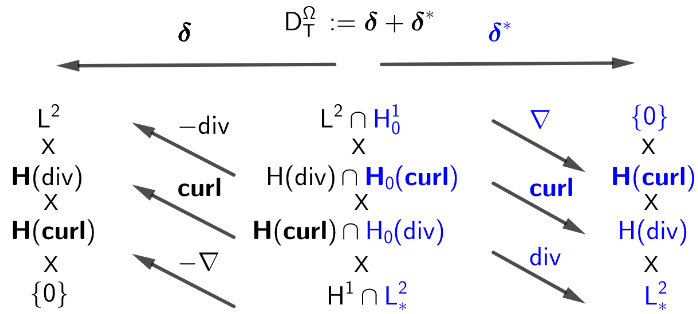

Fig. 3.2 This diagram shows the mapping properties of the codifferentials and their Hilbert space adjoints corresponding to the functional analytic setting of [26] for problem (T) in Ω^-. In the figure, the operators on the left-hand side are to be understood as the adjoint operators $\nabla = -\mathrm{div}^*$, $\mathbf{curl} = \mathbf{curl}^*$ and $\mathrm{div} = -\nabla$.

$$\mathsf{a}_{\mathbf{d}}(\vec{\mathbf{U}}, \vec{\mathbf{V}}) = \int_\Omega \mathbf{d}\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} + \vec{\mathbf{U}} \cdot \mathbf{d}\vec{\mathbf{V}} \mathrm{d}\mathbf{x}, \qquad \vec{\mathbf{U}}, \vec{\mathbf{V}} \in \mathbf{H}(\mathbf{d}, \Omega), \tag{3.3.7b}$$

that rest on an equal footing. They readily appear upon integrating by parts with Lemma 3.3 and they are involved in the first-order analogs of Green's identities

$$\int_{\Omega^\mp} \mathsf{D}\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} = \mathsf{a}_{\delta}(\vec{\mathbf{U}}, \vec{\mathbf{V}}) \pm \langle\!\langle \gamma_\mathsf{T}\vec{\mathbf{U}}, \gamma_\mathsf{R}\vec{\mathbf{V}} \rangle\!\rangle_\Gamma, \tag{3.3.8a}$$

$$\int_{\Omega^\mp} \mathsf{D}\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} = \mathsf{a}_{\mathbf{d}}(\vec{\mathbf{U}}, \vec{\mathbf{V}}) \mp \langle\!\langle \gamma_\mathsf{T}\vec{\mathbf{V}}, \gamma_\mathsf{R}\vec{\mathbf{U}} \rangle\!\rangle_\Gamma, \tag{3.3.8b}$$

which hold for all $\vec{\mathbf{U}}, \vec{\mathbf{V}} \in \mathbf{H}(\mathsf{D}, \Omega)$.

These identities lead to the variational problems:

$$\vec{\mathbf{U}} \in \mathbf{H}(\delta, \Omega) : \qquad \mathsf{a}_{\delta}(\vec{\mathbf{U}}, \vec{\mathbf{V}}) = -\langle\!\langle \vec{\mathbf{b}}, \gamma_\mathsf{R}\vec{\mathbf{V}} \rangle\!\rangle_\Gamma, \qquad \forall \vec{\mathbf{V}} \in \mathbf{H}(\delta, \Omega), \tag{VT}$$

and

$$\vec{\mathbf{U}} \in \mathbf{H}(\mathbf{d}, \Omega) : \qquad \mathsf{a}_{\mathbf{d}}(\vec{\mathbf{U}}, \vec{\mathbf{V}}) = \langle\!\langle \vec{\mathbf{a}}, \gamma_\mathsf{T}\vec{\mathbf{V}} \rangle\!\rangle_\Gamma, \qquad \forall \vec{\mathbf{V}} \in \mathbf{H}(\mathbf{d}, \Omega). \tag{VR}$$

3.3.1 Compatibility conditions

Either from Green's second formula for the Dirac operator (3.2.15) or the variational problems themselves, we see that the boundary values $\vec{\mathbf{b}} \in \mathcal{H}_\mathsf{T}$ and $\vec{\mathbf{a}} \in \mathcal{H}_\mathsf{R}$ must fulfill compatibility conditions. For the problems to admit solutions, we require that

$$\langle\!\langle \vec{\mathbf{b}}, \gamma_\mathsf{R}\vec{\mathbf{V}} \rangle\!\rangle_\Gamma = \vec{\mathbf{0}}, \qquad \forall \vec{\mathbf{V}} \in \mathfrak{H}_\mathsf{T}, \tag{CCT}$$

and

$$\langle\!\langle \vec{\mathbf{a}}, \gamma_\mathsf{T}\vec{\mathbf{V}} \rangle\!\rangle_\Gamma = \vec{\mathbf{0}}, \qquad \forall \vec{\mathbf{V}} \in \mathfrak{H}_\mathsf{R}, \tag{CCR}$$

where

$$\mathfrak{H}_{\mathsf{T}}(\Omega) := \left\{ \overrightarrow{\mathbf{V}} \in \mathbf{H}(\mathsf{D}, \Omega) : \mathsf{D}\overrightarrow{\mathbf{V}} = 0, \gamma_{\mathsf{T}}\overrightarrow{\mathbf{V}} = \overrightarrow{\mathbf{0}} \right\} \tag{3.3.9a}$$

and

$$\mathfrak{H}_{\mathsf{R}}(\Omega) := \{ \overrightarrow{\mathbf{V}} \in \mathbf{H}(\mathsf{D}, \Omega) : \mathsf{D}\overrightarrow{\mathbf{V}} = 0, \gamma_{\mathsf{R}}\overrightarrow{\mathbf{V}} = \overrightarrow{\mathbf{0}} \} \tag{3.3.9b}$$

are spaces of harmonic vector-fields. We refer to [1–3] and [26] for explanations on how these spaces exactly correspond to the nullspaces of the Hodge-Laplacian with natural and essential boundary conditions.

The fact that there are two distinct bilinear forms in the expressions (VT) and (VR) is one of the appealing use of the dual perspective involving the codifferential $\boldsymbol{\delta}$. It points to the symmetry presented in Remark 3.7 below, and it highlights the necessity of imposing compatibility conditions on the data. For example, we could alternatively formulate (T) as the variational problem

$$\overrightarrow{\mathbf{U}} \in \mathbf{H}(\mathbf{d}, \Omega) \quad \text{with} \quad \gamma_{\mathsf{T}}\overrightarrow{\mathbf{U}} = \overrightarrow{\mathbf{b}} : \qquad \mathsf{a}_{\mathbf{d}}(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}}) = 0, \qquad \forall\, \overrightarrow{\mathbf{V}} \in \mathbf{H}_0(\mathbf{d}, \Omega), \tag{3.3.10}$$

where $\mathbf{H}_0(\mathbf{d}, \Omega) = H^1_0(\Omega) \times \mathbf{H}_0(\mathbf{curl}, \Omega) \times \mathbf{H}_0(\mathrm{div}, \Omega) \times L^2(\Omega)$. But according to (3.2.15) the condition (CCT) must remain, and it now appears less obviously so when the type of boundary condition is *essential*. Anyway, in a formulation such as (3.3.10), one proceeds with a lifting of the boundary data and is left with the solvability of the problem

$$\overrightarrow{\mathbf{U}}_0 \in \mathbf{H}_0(\mathbf{d}, \Omega) : \qquad \mathsf{a}_{\mathbf{d}}(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}}) = -\mathsf{a}_{\mathbf{d}}(\mathcal{E}_{\mathsf{T}}\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{V}}), \qquad \forall\, \overrightarrow{\mathbf{V}} \in \mathbf{H}_0(\mathbf{d}, \Omega). \tag{3.3.11}$$

So the question of compatibly cannot be avoided: integrating by parts with the right-hand side evaluated at a nullspace element in $\mathfrak{H}_{\mathsf{T}}$ using (3.3.8b) leads to (CCT). We discuss in greater details the reason why the two boundary conditions can be formulated both as *natural* and *essential* in Remark 3.7.

3.3.2 Well-posedness

Since the bilinear form $\mathsf{a}_{\boldsymbol{\delta}}$ is associated with the self-adjoint operator D_{T} obtained from the chain complex (3.3.6) and $\mathsf{a}_{\mathbf{d}}$ to the self-adjoint operator D_{R} spawned by the cochain complex (3.3.2), they fit the framework of [26, Sec. 2]. The abstract inf-sup inequality supplied in [26, Thm. 6] applies to both bilinear forms and leads to well-posedness of the mixed variational problems:

$$\begin{aligned} \mathsf{a}_{\boldsymbol{\delta}}(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}}) + (\overrightarrow{\mathbf{P}}, \overrightarrow{\mathbf{V}})_{\Omega} &= -\langle\!\langle \overrightarrow{\mathbf{b}}, \gamma_{\mathsf{R}}\overrightarrow{\mathbf{V}} \rangle\!\rangle_{\Gamma} \qquad && \forall\, \overrightarrow{\mathbf{V}} \in \mathbf{H}(\boldsymbol{\delta}, \Omega^-), \\ (\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{W}})_{\Omega} &= 0 && \forall\, \overrightarrow{\mathbf{W}} \in \ker \mathsf{D}^{\Omega}_{\mathsf{T}}, \end{aligned} \tag{MVT}$$

and

$$\begin{aligned} \mathsf{a}_{\mathbf{d}}(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}}) + (\overrightarrow{\mathbf{Q}}, \overrightarrow{\mathbf{V}})_{\Omega} &= \langle\!\langle \overrightarrow{\mathbf{a}}, \gamma_{\mathsf{T}}\overrightarrow{\mathbf{V}} \rangle\!\rangle_{\Gamma} \qquad && \forall\, \overrightarrow{\mathbf{V}} \in \mathbf{H}(\mathbf{d}, \Omega^-), \\ (\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{W}})_{\Omega} &= 0 && \forall\, \overrightarrow{\mathbf{W}} \in \ker \mathsf{D}^{\Omega}_{\mathsf{R}}, \end{aligned} \tag{MVR}$$

for unknown pairs $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{P}}) \in \mathbf{H}(\boldsymbol{\delta}, \Omega^-) \times \ker \mathsf{D}_\mathsf{T}$ and $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{Q}}) \in \mathbf{H}(\mathbf{d}, \Omega^-) \times \ker \mathsf{D}_\mathsf{R}$.

Consistency of the right-hand side in (VT) exactly corresponds to requiring that (CCT) holds for the given data $\overrightarrow{\mathbf{b}} \in \mathcal{H}_\mathsf{T}$, while (CCR) similarly guarantees consistency of the right-hand side in (VR). We conclude that if the compatibility conditions are satisfied, solutions to (VT) and (VR) in Ω^- are unique up to contributions of harmonic vector-fields in $\ker \mathsf{D}_\mathsf{T}$ and $\ker \mathsf{D}_\mathsf{R}$. Moreover, they continuously depend on the boundary data.

3.4 Representation formulas

We derive interior and exterior representation formulas for solutions of the Dirac equation. It is expressed through known boundary potentials, whose jump properties across Γ are elaborated.

3.4.1 Fundamental solution

Convolution of a vector field $\overrightarrow{\mathbf{U}} : \mathbb{R}^3 \to \mathbb{R}^8$ by a matrix-valued function $\mathsf{K} : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}^{8,8}$ possibly having a singularity at the $\mathbf{0} \in \mathbb{R}^3$ is defined, if the limit exists, as the Cauchy principal value

$$(\mathsf{K} * \overrightarrow{\mathbf{U}})(\mathbf{x}) := \lim_{\epsilon \to 0} \int_{\mathbb{R}^3 \backslash B_\epsilon(\mathbf{x})} \mathsf{K}(\mathbf{x} - \mathbf{y}) \overrightarrow{\mathbf{U}}(\mathbf{y}) \mathrm{d}\mathbf{y} \in \mathbb{R}^8, \tag{3.4.1}$$

where $B_\epsilon(\mathbf{x}) \subset \mathbb{R}^3$ is a ball of radius ϵ centered at $\mathbf{x}$.

Let $G : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}$ be given by $G(\mathbf{z}) := (4\pi|\mathbf{z}|)^{-1}$, and set

$$\mathsf{G}(\mathbf{z}) := G(\mathbf{z}) \mathsf{I}_8 \in \mathbb{R}^{8,8}, \qquad \mathbf{z} \neq \mathbf{0}, \tag{3.4.2}$$

where I_8 is the identity matrix on $\mathbb{R}^8$. Then, define $\Phi : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}^{8,8}$ by applying the Dirac operator to the columns of G as

$$\Phi(\mathbf{z}) := \begin{pmatrix} 0 & -(\nabla G)^\top(\mathbf{z}) & \mathbf{0}^\top & 0 \\ (\nabla G)(\mathbf{z}) & \mathbf{0}_{3\times 3} & \mathsf{A}_{3\times 3}(\mathbf{z}) & \mathbf{0} \\ \mathbf{0} & \mathsf{A}_{3\times 3}(\mathbf{z}) & \mathbf{0}_{3\times 3} & -(\nabla G)(\mathbf{z}) \\ 0 & \mathbf{0}^\top & (\nabla G)^\top(\mathbf{z}) & 0 \end{pmatrix} \in \mathbb{R}^{8\times 8}, \; \mathbf{z} \neq \mathbf{0},$$

where the anti-symmetric blocks

$$\mathsf{A}_{3\times 3}(\mathbf{z}) := \begin{pmatrix} 0 & -(\partial_3 G)(\mathbf{z}) & (\partial_2 G)(\mathbf{z}) \\ (\partial_3 G)(\mathbf{z}) & 0 & -(\partial_1 G)(\mathbf{z}) \\ -(\partial_2 G)(\mathbf{z}) & (\partial_1 G)(\mathbf{z}) & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \qquad \mathbf{z} \neq \mathbf{0}, \tag{3.4.3}$$

are associated with the curl operator.

Lemma 3.4 *For* $\mathbf{z} \neq \mathbf{0}$*,*

$$\Phi(-\mathbf{z}) = -\Phi(\mathbf{z}) \qquad \textit{and} \qquad \Phi(\mathbf{z}) \overrightarrow{\mathbf{U}} \cdot \overrightarrow{\mathbf{V}} = -\overrightarrow{\mathbf{U}} \cdot \Phi(\mathbf{z}) \overrightarrow{\mathbf{V}} \tag{3.4.4}$$

for all $\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}} \in \mathbb{R}^8$.

Proof. Let $\mathsf{s} : \mathbb{R}^3 \to \mathbb{R}^3$ be the sign flip operation $\mathsf{s}(\mathbf{z}) = -\mathbf{z}$. For the fist identity, we simply rely on the fact that $G(\mathbf{x}) = G(|\mathbf{x}|)$ to verify that for any $\overrightarrow{\mathbf{U}} \in \mathbb{R}^8$,

$$\Phi(-\mathbf{z})\overrightarrow{\mathbf{U}} = \mathsf{D}(\mathsf{G}\overrightarrow{\mathbf{U}})\Big|_{\mathsf{s}(\mathbf{z})} = -\mathsf{D}_{\mathbf{x}}(\mathsf{G}(\mathsf{s}(\mathbf{x}))\overrightarrow{\mathbf{U}})\Big|_{\mathbf{x}=\mathbf{z}}$$
$$= -\mathsf{D}_{\mathbf{x}}(G(\mathsf{s}(\mathbf{x}))\overrightarrow{\mathbf{U}})\Big|_{\mathbf{x}=\mathbf{z}} = -\mathsf{D}_{\mathbf{x}}(\mathsf{G}(\mathbf{x})\overrightarrow{\mathbf{U}})\Big|_{\mathbf{x}=\mathbf{z}} = -\Phi(\mathbf{z})\overrightarrow{\mathbf{U}}. \quad (3.4.5)$$

The second identity is clear by definition. □

This lemma allows to extend the domain of the Newton-type potential

$$\mathsf{N} : C_0^\infty(\mathbb{R}^3)^8 \to C^\infty(\mathbb{R}^3)^8$$
$$\overrightarrow{\mathbf{U}} \mapsto \Phi * \overrightarrow{\mathbf{U}}$$

to distributions.

Lemma 3.5 *For all* $\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}} \in C_0^\infty(\mathbb{R}^3)^8$,

$$\left(\mathsf{N}\,\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}}\right) = \left(\overrightarrow{\mathbf{U}}, \mathsf{N}\,\overrightarrow{\mathbf{V}}\right). \quad (3.4.6)$$

Proof. Using Lemma 3.4, we can change the order of integration using Fubini's theorem and evaluate

$$\left(\mathsf{N}\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}}\right) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi(\mathbf{x}-\mathbf{y})\overrightarrow{\mathbf{U}}(\mathbf{y}) \cdot \overrightarrow{\mathbf{V}}(\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \quad (3.4.7)$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \overrightarrow{\mathbf{U}}(\mathbf{y}) \cdot \Phi(\mathbf{y}-\mathbf{x})\overrightarrow{\mathbf{V}}(\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \quad (3.4.8)$$
$$= \int_{\mathbb{R}^3} \overrightarrow{\mathbf{U}}(\mathbf{y}) \cdot \int_{\mathbb{R}^3} \Phi(\mathbf{y}-\mathbf{x})\overrightarrow{\mathbf{V}}(\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \quad (3.4.9)$$
$$= \left(\overrightarrow{\mathbf{U}}, \mathsf{N}\,\overrightarrow{\mathbf{V}}\right). \quad (3.4.10)$$

□

Remark 3.2 Lemma 3.5 reflects the fact that the Dirac operator is symmetric as an unbounded operator on $(L^2(\mathbb{R}^3))^8$.

The extension

$$\mathsf{N} : (C^\infty(\mathbb{R}^3)^8)' \to (C_0^\infty(\mathbb{R}^3)^8)' \quad (3.4.11)$$

is obtained as in [37, Sec. 3.1.1] via dual mapping by defining the action of the distribution $\mathsf{N}\overrightarrow{\mathbf{U}} \in (C_0^\infty(\mathbb{R}^3)^8)'$ on $\overrightarrow{\mathbf{V}} \in C_0^\infty(\mathbb{R}^3)^8$ as

$$\langle \mathsf{N}\,\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}}\rangle := \langle \overrightarrow{\mathbf{U}}, \mathsf{N}\,\overrightarrow{\mathbf{V}}\rangle. \quad (3.4.12)$$

Proposition 3.1 (Fundamental solution) *For all compactly supported distributions* $\overrightarrow{\mathbf{U}} \in (C^\infty(\mathbb{R}^3)^8)'$,

$$\mathsf{N}\,\mathsf{D}\,\overrightarrow{\mathbf{U}} = \overrightarrow{\mathbf{U}} = \mathsf{D}\,\mathsf{N}\,\overrightarrow{\mathbf{U}} \quad (3.4.13)$$

holds on $C_0^\infty(\mathbb{R}^3)^8$.

Proof. We first show that for $\overrightarrow{\mathbf{U}} \in (C^\infty(\mathbb{R}^3)^8)'$,

$$\langle \mathsf{N}\,\mathsf{D}\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}}\rangle = \langle \overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}}\rangle \tag{3.4.14}$$

for all $\overrightarrow{\mathbf{V}} \in C_0^\infty(\mathbb{R}^3)^8$.

The argument is inspired by the proof of [18, Thm.1]. Let $\mathbf{e}_i \in \mathbb{R}^3$ be the vector with 1 at the i-th entry and zeros elsewhere, $i = 1, 2, 3$. Since

$$\mathsf{N}\overrightarrow{\mathbf{V}} = \int_{\mathbb{R}^3} \Phi(\mathbf{x}-\mathbf{y})\overrightarrow{\mathbf{V}}(\mathbf{y})\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^3} \Phi(\mathbf{y})\overrightarrow{\mathbf{V}}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}, \tag{3.4.15}$$

we have

$$\frac{\mathsf{N}\overrightarrow{\mathbf{V}}(\mathbf{x}+h\mathbf{e}_i) - \mathsf{N}\overrightarrow{\mathbf{V}}(\mathbf{x})}{h} = \int_{\mathbb{R}^3} \Phi(\mathbf{y})\frac{\overrightarrow{\mathbf{V}}(\mathbf{x}+h\mathbf{e}_i-\mathbf{y}) - \overrightarrow{\mathbf{V}}(\mathbf{x}-\mathbf{y})}{h}\mathrm{d}\mathbf{y}. \tag{3.4.16}$$

Hence,

$$\mathsf{D}_{\mathbf{x}}\mathsf{N}\overrightarrow{\mathbf{V}}(\mathbf{x}) = \int_{\mathbb{R}^3} \Phi(\mathbf{y})\mathsf{D}\overrightarrow{\mathbf{V}}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}, \tag{3.4.17}$$

because the assumption that $\overrightarrow{\mathbf{V}}$ is smooth and compactly supported guarantees that

$$\frac{\overrightarrow{\mathbf{V}}(\mathbf{x}+h\mathbf{e}_i-\mathbf{y}) - \overrightarrow{\mathbf{V}}(\mathbf{x}-\mathbf{y})}{h} \to \frac{\partial}{\partial \mathbf{x}_i}\overrightarrow{\mathbf{V}}(\mathbf{x}-\mathbf{y}) \tag{3.4.18}$$

uniformly for $h \to 0$. The main idea is to isolate Φ's singularity at the origin by splitting the right hand side of eq. (3.4.17) into two integrals as

$$\mathsf{D}_{\mathbf{x}}\mathsf{N}\overrightarrow{\mathbf{V}}(\mathbf{x}) = \underbrace{\int_{B_\epsilon(\mathbf{0})} \Phi(\mathbf{y})\mathsf{D}\overrightarrow{\mathbf{V}}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}}_{I_\epsilon} + \underbrace{\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \Phi(\mathbf{y})\mathsf{D}\overrightarrow{\mathbf{V}}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}}_{J_\epsilon} \tag{3.4.19}$$

whose limits as $\epsilon \to 0$ we can control.

The main difficulty is that we cannot readily mimic the standard proof commonly given for the Poisson equation, because the integration by parts formula supplied for the product of two vectors by eq. (3.2.15) is not applicable to the matrix–vector multiplication involved in the integrands of eq. (3.4.19). The analysis of

$$\Phi(\mathbf{y})\mathsf{D}\overrightarrow{\mathbf{V}}(\mathbf{x}-\mathbf{y}) = \begin{pmatrix} -\nabla G(\mathbf{y})\cdot\nabla V_0(\mathbf{x}-\mathbf{y}) - \nabla G(\mathbf{y})\cdot\mathbf{curl}\,\mathbf{V}_1(\mathbf{x}-\mathbf{y}) \\ -\mathrm{div}\,\mathbf{V}_1(\mathbf{x}-\mathbf{y})\nabla G(\mathbf{y}) - \nabla G(\mathbf{y})\times\nabla V_3(\mathbf{x}-\mathbf{y}) + \nabla G(\mathbf{y})\times\mathbf{curl}\,\mathbf{V}_1(\mathbf{x}-\mathbf{y}) \\ -\nabla G(\mathbf{y})\times\nabla V_0(\mathbf{x}-\mathbf{y}) + \nabla G(\mathbf{y})\times\mathbf{curl}\,\mathbf{V}_2(\mathbf{x}-\mathbf{y}) - \mathrm{div}\,\mathbf{V}_2(\mathbf{x}-\mathbf{y})\nabla G(\mathbf{y}) \\ -\nabla G(\mathbf{y})\cdot\nabla V_3(\mathbf{x}-\mathbf{y}) + \nabla G(\mathbf{y})\cdot\mathbf{curl}\,\mathbf{V}_2(\mathbf{x}-\mathbf{y}) \end{pmatrix} \tag{3.4.20}$$

is carried out component-wise.

There are five different types of terms whose limit need to be investigated. Let $\mathbf{V} \in (C_0^\infty(\mathbb{R}^3))^3$ and $V \in C_0^\infty(\mathbb{R}^3)$ be arbitrary fields. To ease the reading, we write $V_{\mathbf{x}}(\mathbf{y}) := V(\mathbf{x}-\mathbf{y})$ and

$\mathbf{V_x}(\mathbf{y}) := \mathbf{V}(\mathbf{x}-\mathbf{y})$. We denote by $\mathbf{n}_\epsilon$ the unit normal vector field pointing towards the interior of $B_\epsilon(\mathbf{0})$.

Integrating by parts using that $\Delta G = 0$ in $\mathbb{R}^3\backslash(\mathbf{0})$ and $\mathbf{curl} \circ \nabla \equiv \mathbf{0}$, we find that

$$\begin{aligned}\int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \cdot \nabla V(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} &= \int_{\partial B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \cdot \mathbf{n}_\epsilon(\mathbf{y}) V(\mathbf{x}-\mathbf{y})\mathrm{d}\sigma(\mathbf{y}) \\ &= \frac{1}{4\pi}\int_{\partial B_\epsilon(\mathbf{0})} \frac{V(\mathbf{x}-\mathbf{y})}{|\mathbf{y}|^3}(-\mathbf{y}\cdot\frac{\mathbf{y}}{|\mathbf{y}|})\mathrm{d}\sigma(\mathbf{y}) = -\frac{1}{4\pi\epsilon^2}\int_{\partial B_\epsilon(\mathbf{0})} V(\mathbf{x}-\mathbf{y})\mathrm{d}\sigma(\mathbf{y}) \\ &= -\fint_{\partial B_\epsilon(\mathbf{x})} V(\mathbf{y})\mathrm{d}\sigma(\mathbf{y}) \xrightarrow[\epsilon\to 0]{} -V(\mathbf{x}) \end{aligned} \quad (3.4.21)$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \cdot \mathbf{curl}\,\mathbf{V}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} \\ &\quad = -\int_{\partial B_\epsilon(\mathbf{0})} (\nabla G(\mathbf{y}) \times \mathbf{n}_\epsilon(\mathbf{y})) \cdot \mathbf{V}(\mathbf{x}-\mathbf{y})\mathrm{d}\sigma(\mathbf{y}) \\ &\quad = -\frac{1}{4\pi\epsilon^4}\int_{\partial B_\epsilon(\mathbf{0})} (\mathbf{y}\times\mathbf{y}) \cdot \mathbf{V}(\mathbf{x}-\mathbf{y})\mathrm{d}\sigma(\mathbf{y}) = 0. \end{aligned} \quad (3.4.22)$$

Similarly, integrating by parts component-wise yields

$$\begin{aligned}&\int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \times \nabla V_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} \begin{pmatrix} \partial_2 G(\mathbf{y})\partial_3 V_{\mathbf{x}}(\mathbf{y}) - \partial_3 G(\mathbf{y})\partial_2 V_{\mathbf{x}}(\mathbf{y}) \\ \partial_3 G(\mathbf{y})\partial_1 V_{\mathbf{x}}(\mathbf{y}) - \partial_1 G(\mathbf{y})\partial_3 V_{\mathbf{x}}(\mathbf{y}) \\ \partial_1 G(\mathbf{y})\partial_2 V_{\mathbf{x}}(\mathbf{y}) - \partial_2 G(\mathbf{y})\partial_1 V_{\mathbf{x}}(\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} \begin{pmatrix} G(\mathbf{y})\partial_2\partial_3 V_{\mathbf{x}}(\mathbf{y}) - G(\mathbf{y})\partial_3\partial_2 V_{\mathbf{x}}(\mathbf{y}) \\ G(\mathbf{y})\partial_3\partial_1 V_{\mathbf{x}}(\mathbf{y}) - G(\mathbf{y})\partial_1\partial_3 V_{\mathbf{x}}(\mathbf{y}) \\ G(\mathbf{y})\partial_1\partial_2 V_{\mathbf{x}}(\mathbf{y}) - G(\mathbf{y})\partial_2\partial_1 V_{\mathbf{x}}(\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y} \\ &+ \int_{\partial B_\epsilon(\mathbf{0})} \begin{pmatrix} -(\mathbf{n}_\epsilon)_2(\mathbf{y})G(\mathbf{y})\partial_3 V_{\mathbf{x}}(\mathbf{y}) + (\mathbf{n}_\epsilon)_3(\mathbf{y})G(\mathbf{y})\partial_2 V_{\mathbf{x}}(\mathbf{y}) \\ -(\mathbf{n}_\epsilon)_2(\mathbf{y})G(\mathbf{y})\partial_1 V_{\mathbf{x}}(\mathbf{y}) + (\mathbf{n}_\epsilon)_1(\mathbf{y})G(\mathbf{y})\partial_3 V_{\mathbf{x}}(\mathbf{y}) \\ -(\mathbf{n}_\epsilon)_1(\mathbf{y})G(\mathbf{y})\partial_2 V_{\mathbf{x}}(\mathbf{y}) + (\mathbf{n}_\epsilon)_2(\mathbf{y})G(\mathbf{y})\partial_1 V_{\mathbf{x}}(\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y}. \end{aligned} \quad (3.4.23)$$

Since V is smooth everywhere in $\mathbb{R}^3$, partial derivatives commute and the volume integral vanishes, leading to

$$\int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \times \nabla V_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{y} = -\int_{\partial B_\epsilon(\mathbf{0})} G(\mathbf{y})\mathbf{n}_\epsilon(\mathbf{y}) \times \nabla V_{\mathbf{x}}(\mathbf{y})\mathrm{d}\sigma(\mathbf{y}). \quad (3.4.24)$$

This integral vanishes under the limit $\epsilon \to 0$, because

$$\sup_{\mathbf{x}\in\mathbb{R}^3} \left| \int_{\partial B_\epsilon(\mathbf{0})} G(\mathbf{y})\mathbf{n}_\epsilon(\mathbf{y}) \times \nabla V(\mathbf{x}-\mathbf{y})\mathrm{d}\sigma(\mathbf{y}) \right|$$

$$\leq \|\nabla V\|_\infty \int_{\partial B_\epsilon(\mathbf{0})} \big|G(\mathbf{y})\big| \,\mathrm{d}\sigma(\mathbf{y}) = \mathcal{O}(\epsilon). \quad (3.4.25)$$

Moving on to the next term, one eventually obtains from similar calculations that

$$\int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \times \mathbf{curl}\,\mathbf{V}_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} G(\mathbf{y})\mathbf{curl}\,\mathbf{curl}\,\mathbf{V}_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{y} + \int_{\partial B_\epsilon(\mathbf{0})} G(\mathbf{y})(\mathbf{curl}\,\mathbf{V}_{\mathbf{x}}(\mathbf{y}) \times \mathbf{n}_\epsilon(\mathbf{y}))\mathrm{d}\sigma(\mathbf{y}). \quad (3.4.26)$$

Since $\|\mathbf{curl}\,\mathbf{V}\|_\infty < \infty$, the boundary integral on the right hand side vanishes under the limit by repeating the argument of eq. (3.4.25). Finally, commuting partial derivatives after integrating by parts also yields

$$\int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} \mathrm{div}\,\mathbf{V}(\mathbf{x}-\mathbf{y})\nabla G(\mathbf{y})\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} G(\mathbf{y})\nabla \mathrm{div}\mathbf{V}(\mathbf{x}-\mathbf{y}) - \int_{\partial B_\epsilon(\mathbf{0})} G(\mathbf{y})\mathrm{div}\mathbf{V}(\mathbf{x}-\mathbf{y})\mathbf{n}_\epsilon(\mathbf{y})\mathrm{d}\sigma(\mathbf{y}) \quad (3.4.27)$$

Putting the two previous calculations together, we find that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \times \mathbf{curl}\,\mathbf{V}(\mathbf{x}-\mathbf{y}) - \mathrm{div}\,\mathbf{V}(\mathbf{x}-\mathbf{y})\nabla G(\mathbf{y})\mathrm{d}\mathbf{y} = -\lim_{\epsilon\to 0}\int_{\mathbb{R}^3\backslash B_\epsilon(\mathbf{0})} G(\mathbf{y})\boldsymbol{\Delta}\mathbf{V}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} = \mathbf{V}(\mathbf{x}), \quad (3.4.28)$$

where we recognized the vector (Hodge-) Laplace operator $-\boldsymbol{\Delta} \equiv \mathbf{curl}\,\mathbf{curl} - \nabla\,\mathrm{div}$.

We have found that $J_\epsilon \longrightarrow \overrightarrow{\mathbf{V}}(\mathbf{x})$ as $\epsilon \to 0$. Meanwhile,

$$\|I_\epsilon\|_\infty \leq \|\mathsf{D}\overrightarrow{\mathbf{V}}\|_\infty \int_{B_\epsilon(\mathbf{0})} \|\Phi\|_\infty \mathrm{d}\mathbf{y} = \mathcal{O}\Big(\int_{B_\epsilon(\mathbf{0})} \|\nabla G\|_\infty \mathrm{d}\mathbf{y}\Big) = \mathcal{O}(\epsilon). \quad (3.4.29)$$

The calculations for $\overrightarrow{\mathbf{U}} = \mathsf{D}\,\mathsf{N}\,\overrightarrow{\mathbf{U}}$ follow similarly starting from (3.4.17). □

In light of Proposition 3.1, we say that the kernel Φ of N is a fundamental solution for the Dirac operator.

3.4.2 Surface potentials

Adopting the perspective on first-kind boundary integral operators from [16], [32], [37] and [14]—in the later works for the study of second-order elliptic operators—for the first-order Dirac operator, we define the surface potentials

$$\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) := \mathsf{N}(\gamma'_{\mathsf{T}}\overrightarrow{\mathbf{a}}), \qquad \forall\,\overrightarrow{\mathbf{a}} = (\mathsf{a}_0, \mathbf{a}_1, a_2) \in \mathcal{H}_{\mathsf{R}}, \quad (3.4.30)$$

$$\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) := -\mathsf{N}(\gamma'_{\mathsf{R}}\overrightarrow{\mathbf{b}}), \qquad \forall\,\overrightarrow{\mathbf{b}} = (b_0, \mathbf{b}_1, \mathsf{b}_2) \in \mathcal{H}_{\mathsf{T}}, \quad (3.4.31)$$

where the mappings $\gamma_{\mathsf{T}}' : \mathcal{H}_{\mathsf{R}} = \mathcal{H}_{\mathsf{T}}' \to \mathbf{H}_{\text{loc}}(\mathsf{D}, \mathbb{R}^3 \backslash \overline{\Omega})'$ and $\gamma_{\mathsf{R}}' : \mathcal{H}_{\mathsf{T}} = \mathcal{H}_{\mathsf{R}}' \to \mathbf{H}_{\text{loc}}(\mathsf{D}, \mathbb{R}^3 \backslash \overline{\Omega})'$ are adjoint to the trace operators γ_{T} and γ_{R} defined in (3.2.10).

It will be convenient to denote by $\Phi_{\mathbf{x}}$ the map $\mathbf{y} \mapsto \Phi(\mathbf{x}-\mathbf{y})$. Let $\overrightarrow{\mathbf{E}}_j \in \mathbb{R}^8$ denote the constant vector with 1 at the j-th entry and zeros elsewhere, $j = 1, ..., 8$. Similarly for $\mathbf{E}_k \in \mathbb{R}^3$, $k = 1, 2, 3$.

Adapting the calculations found in [14, Sec. 4.2], we will establish integral representation formulas for these potentials by splitting the pairings into their components.

Lemma 3.6 *Given* $\overrightarrow{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$ *and* $\overrightarrow{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}}$*, it holds for* $\mathbf{x} \in \Omega \backslash \Gamma$ *that*

$$\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_j = -\langle\!\langle \overrightarrow{\mathbf{a}}\,,\, \gamma_{\mathsf{T}}^{-}(\Phi_{\mathbf{x}}\, \overrightarrow{\mathbf{E}}_j) \rangle\!\rangle_{\Gamma}, \tag{3.4.32a}$$

$$\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_j = \langle\!\langle \overrightarrow{\mathbf{b}}\,,\, \gamma_{\mathsf{R}}^{-}(\Phi_{\mathbf{x}}\, \overrightarrow{\mathbf{E}}_j) \rangle\!\rangle_{\Gamma}. \tag{3.4.32b}$$

Proof. Let $V \in C_0^\infty(\mathbb{R}^3)$ and suppose that $\overrightarrow{\mathbf{a}}$ is the trace of a smooth 8-dimensional vector-field. Using Fubini's theorem, Lemma 3.5 and the fact that Φ is smooth away from the origin,

$$\langle \mathsf{N}(\gamma_{\mathsf{T}}'\overrightarrow{\mathbf{a}}),\, V\overrightarrow{\mathbf{E}}_j \rangle_{\mathbb{R}^3} = \langle\!\langle \overrightarrow{\mathbf{a}}, \gamma_{\mathsf{T}} \mathsf{N}(V\overrightarrow{\mathbf{E}}_j) \rangle\!\rangle_{\Gamma} \tag{3.4.33}$$

$$= \int_\Gamma \overrightarrow{\mathbf{a}}(\mathbf{y}) \cdot \gamma_{\mathsf{T}} \int_{\mathbb{R}^3} \Phi(\mathbf{y}-\mathbf{x}) V(\mathbf{x}) \overrightarrow{\mathbf{E}}_j(\mathbf{x}) \mathrm{d}\mathbf{x} \mathrm{d}\sigma(\mathbf{y}) \tag{3.4.34}$$

$$\overset{(*)}{=} -\int_{\mathbb{R}^3} V(\mathbf{x}) (\int_\Gamma \overrightarrow{\mathbf{a}}(\mathbf{y}) \cdot \gamma_{\mathsf{T}} \Phi(\mathbf{x}-\mathbf{y}) \overrightarrow{\mathbf{E}}_j \mathrm{d}\sigma(\mathbf{y})) \mathrm{d}\mathbf{x}, \tag{3.4.35}$$

where the sign was obtained in $(*)$ thanks to Lemma 3.4. The integrals on the right-hand side of (3.4.35) can be extended to duality pairings by a standard density argument exploiting Lemma 3.1.

Similar calculations can be carried out for $\mathcal{L}_{\mathsf{R}}$. □

In particular,

$$\Phi_{\mathbf{x}}(\mathbf{y})\, \overrightarrow{\mathbf{E}}_1 = \begin{pmatrix} 0 \\ \nabla G(\mathbf{x}-\mathbf{y}) \\ \mathbf{0} \\ 0 \end{pmatrix}, \qquad \Phi_{\mathbf{x}}(\mathbf{y})\, \overrightarrow{\mathbf{E}}_8 = \begin{pmatrix} 0 \\ \mathbf{0} \\ -\nabla G(\mathbf{x}-\mathbf{y}) \\ 0 \end{pmatrix}, \tag{3.4.36}$$

$$\Phi_{\mathbf{x}}(\mathbf{y})\, \overrightarrow{\mathbf{E}}_i = \begin{pmatrix} -\frac{\partial}{\partial \mathbf{z}_{\mu(i)}} G(\mathbf{z}) \\ \mathbf{0} \\ \nabla G(\mathbf{z}) \times \mathbf{E}_{\mu(i)} \\ 0 \end{pmatrix} \Bigg|_{\mathbf{z}=\mathbf{x}-\mathbf{y}}, \quad \Phi_{\mathbf{x}}(\mathbf{y})\, \overrightarrow{\mathbf{E}}_k = \begin{pmatrix} 0 \\ \nabla G(\mathbf{z}) \times \mathbf{E}_{\nu(k)} \\ \mathbf{0} \\ \frac{\partial}{\partial \mathbf{z}_{\nu(k)}} G(\mathbf{z}) \end{pmatrix} \Bigg|_{\mathbf{z}=\mathbf{x}-\mathbf{y}}, \tag{3.4.37}$$

for $i = 2, 3, 4$, $k = 5, 6, 7$, $\mu(i) = i - 1$ and $\nu(k) = k - 4$.

Therefore, we can evaluate

$$\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_1 = -\int_\Gamma \mathbf{a}_1(\mathbf{y}) \cdot \nabla G(\mathbf{x}-\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \tag{3.4.38a}$$

$$\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_i = \int_\Gamma \mathsf{a}_0(\mathbf{y})\, \partial_{\mu(i)} G(\mathbf{x}-\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \tag{3.4.38b}$$

$$
\begin{aligned}
&\quad - \int_\Gamma a_2(\mathbf{y}) \left(\nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{E}_{\mu(i)}\right) \cdot \mathbf{n}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \\
&= \partial_{\mu(i)} \int_\Gamma \mathsf{a}_0(\mathbf{y}) G(\mathbf{x}-\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \\
&\quad + \mathbf{E}_{\mu(i)} \cdot \int_\Gamma a_2(\mathbf{y}) \nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{n}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_k &= - \int_\Gamma \mathbf{a}_1(\mathbf{y}) \cdot (\nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{E}_{\nu(k)}) \mathrm{d}\sigma(\mathbf{y}) && \text{(3.4.38c)} \\
&= \mathbf{E}_{\nu(k)} \cdot \int_\Gamma \nabla_{\mathbf{y}} G(\mathbf{x}-\mathbf{y}) \times \mathbf{a}_1(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \\
\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_8 &= \int_\Gamma a_2(\mathbf{y}) \nabla_{\mathbf{y}} G_{\mathbf{x}}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}), && \text{(3.4.38d)}
\end{aligned}
$$

where we have used the fact that $\mathbf{a}_1 \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ was “tangential" to safely drop the trace γ_t everywhere. Similarly as in the proof of Lemma 3.6, all these integrals should be understood as duality pairings and the following explicit representations do not only hold in the sense of distributions, but also pointwise on $\mathbb{R}^3 \backslash \Gamma$.

We collect the above entries to obtain

$$
\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) = \begin{pmatrix} -\,\mathrm{div}\, \mathbf{\Psi}(\mathbf{a}_1) \\ \nabla \psi(\mathsf{a}_0) + \mathbf{curl}\, \mathbf{\Upsilon}(a_2) \\ \mathbf{curl}\, \mathbf{\Psi}(\mathbf{a}_1) \\ \mathrm{div}\, \mathbf{\Upsilon}(a_2) \end{pmatrix}, \quad \text{pointwise on } \mathbb{R}^3 \backslash \Gamma, \qquad \text{(3.4.39)}
$$

where we respectively recognize in

$$
\begin{aligned}
\psi(q)(\mathbf{x}) &:= \int_\Gamma q(\mathbf{y}) G(\mathbf{x}-\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}), && \mathbf{x} \in \mathbb{R}^3 \backslash \Gamma, && \text{(3.4.40a)} \\
\boldsymbol{\Psi}(\mathbf{p})(\mathbf{x}) &:= \int_\gamma \mathbf{p}(\mathbf{y}) G(\mathbf{x}-\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}), && \mathbf{x} \in \mathbb{R}^3 \backslash \Gamma, && \text{(3.4.40b)} \\
\mathbf{\Upsilon}(q)(\mathbf{x}) &:= \int_\Gamma q(\mathbf{y}) G(\mathbf{x}-\mathbf{y}) \mathbf{n}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) && \mathbf{x} \in \mathbb{R}^3 \backslash \Gamma, && \text{(3.4.40c)}
\end{aligned}
$$

the well-known single layer, vector single layer and normal vector single layer potentials. They notably enter eq. (3.4.39) in the expression for the classical double layer potential $\mathrm{div}\, \mathbf{\Upsilon}(q)$ and for the Maxwell double layer potential $\mathbf{curl} \mathbf{\Psi}(\mathbf{p})$ as they arise in acoustic and electromagnetic scattering respectively.

Similarly, for $i = 2, 3, 4$ and $k = 5, 6, 7$,

$$
\begin{aligned}
\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_1 &= \int_\Gamma b_0(\mathbf{y}) \nabla G(\mathbf{x}-\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) && \text{(3.4.41a)} \\
\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_i &= \int_\Gamma \mathbf{b}_1(\mathbf{y}) \cdot (\nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{E}_{\mu(i)}) \times \mathbf{n}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) && \text{(3.4.41b)}
\end{aligned}
$$

$$= \int_\Gamma (\nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{E}_{\mu(i)}) \cdot \mathbf{n}(\mathbf{y}) \times \mathbf{b}_1(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y})$$
$$= \mathbf{E}_{\mu(i)} \cdot \int_\Gamma (\mathbf{n}(\mathbf{y}) \times \mathbf{b}_1(\mathbf{y})) \times \nabla G(\mathbf{x}-\mathbf{y}) \mathrm{d}\sigma(\mathbf{y})$$

$$\begin{aligned} \mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_k &= \int_\Gamma b_0(\mathbf{y})(\nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{E}_{\nu(k)}) \cdot \mathbf{n}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \\ &\quad + \int_\Gamma \mathsf{b}_2(\mathbf{y}) \partial_j G(\mathbf{x}-\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \\ &= \mathbf{E}_{\nu(k)} \cdot \int_\Gamma b_0(\mathbf{y}) \mathbf{n}(\mathbf{y}) \times \nabla G(\mathbf{x}-\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \\ &\quad + \int_\Gamma \mathsf{b}_2(\mathbf{y}) \partial_j G(\mathbf{x}-\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \end{aligned} \tag{3.4.41c}$$

$$\begin{aligned} \mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}})(\mathbf{x}) \cdot \overrightarrow{\mathbf{E}}_8 &= -\int_\Gamma \mathbf{b}_1(\mathbf{y}) \cdot \nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{n}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \\ &= -\int_\Gamma \nabla G(\mathbf{x}-\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \times \mathbf{b}_1(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \end{aligned} \tag{3.4.41d}$$

so that we have

$$\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) = \begin{pmatrix} \mathrm{div}\boldsymbol{\Upsilon}(b_0) \\ \mathbf{curl}\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n}) \\ -\mathbf{curl}\boldsymbol{\Upsilon}(b_0) + \nabla\psi(\mathsf{b}_2) \\ \mathrm{div}\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n}) \end{pmatrix}, \quad \text{pointwise on } \mathbb{R}^3 \backslash \Gamma. \tag{3.4.42}$$

3.4.3 Mapping properties of the surface potentials

Fortunately, we already know a lot about each potential entering eq. (3.4.39) and eq. (3.4.42).

Lemma 3.7 *The potentials* $\mathcal{L}_{\mathsf{T}} : \mathcal{H}_{\mathsf{R}} \to \mathbf{H}(\mathsf{D}, \mathbb{R}^3 \backslash \Gamma)$ *and* $\mathcal{L}_{\mathsf{R}} : \mathcal{H}_{\mathsf{T}} \to \mathbf{H}(\mathsf{D}, \mathbb{R}^3 \backslash \Gamma)$ *explicitly given by eq.* (3.4.39) *and eq.* (3.4.42) *are continuous.*

Proof. Recall that if $\mathbf{b}_1 \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, then $\mathbf{n} \times \mathbf{b}_1 \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. So the proof simply boils down to extracting from the discussion of Section 5 in [14] the mapping properties

$$\nabla\psi : H^{-1/2}(\Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \mathbb{R}^3 \backslash \Gamma) \cap \mathbf{H}_{\mathrm{loc}}(\nabla \mathrm{div}, \mathbb{R}^3 \backslash \Gamma), \tag{3.4.43a}$$
$$\mathrm{div}\boldsymbol{\Upsilon} : H^{1/2} \to H^1_{\mathrm{loc}}(\Delta, \mathbb{R}^3 \backslash \Gamma), \tag{3.4.43b}$$
$$\mathbf{curl}\,\boldsymbol{\Upsilon} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3 \backslash \Gamma), \tag{3.4.43c}$$
$$\mathrm{div}\,\boldsymbol{\Psi} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to H^1_{\mathrm{loc}}(\mathbb{R}^3 \backslash \Gamma), \tag{3.4.43d}$$
$$\mathbf{curl}\,\boldsymbol{\Psi} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3 \backslash \Gamma). \tag{3.4.43e}$$

Since div $\circ$ $\mathbf{curl} \equiv 0$, we have in particular

$$\mathbf{curl}\,\boldsymbol{\Upsilon} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3 \backslash \Gamma) \cap \mathbf{H}_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^3 \backslash \Gamma), \tag{3.4.44a}$$

$$\mathbf{curl}\,\boldsymbol{\Psi} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3 \backslash \Gamma) \cap \mathbf{H}_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^3 \backslash \Gamma). \tag{3.4.44b}$$

Now, for $\mathbf{z} \neq \mathbf{0}$, the kernels of the two surface potentials decay as

$$\|\nabla G(\mathbf{z})\| \lesssim \|\mathbf{z}\|^{-2},$$

thus are not only square-integrable locally, but in fact belong to $(L^2(\mathbb{R}^3 \backslash \Gamma))^8$. □

The next lemma shows that the surface potentials solve the homogeneous Dirac equation.

Lemma 3.8 *For all* $\overrightarrow{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}}$ *and* $\overrightarrow{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$*, it holds on* $\mathbb{R}^3 \backslash \Gamma$ *that*

$$\mathsf{D}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) \equiv \overrightarrow{\mathbf{0}}, \tag{3.4.45a}$$

$$\mathsf{D}\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) \equiv \overrightarrow{\mathbf{0}}. \tag{3.4.45b}$$

Proof. The well-known vector and scalar potentials of (3.4.40) are harmonic. Hence, since div $\circ$ $\mathbf{curl} \equiv 0$ and $\mathbf{curl} \circ \nabla \equiv 0$, we directly evaluate

$$\begin{aligned} \mathsf{D}\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) &= \begin{pmatrix} -\mathrm{div}\,\nabla\psi(\mathsf{a}_0) - \mathrm{div}\,\mathbf{curl}\boldsymbol{\Upsilon}(a_2) \\ -\nabla\mathrm{div}\,\boldsymbol{\Psi}(\mathbf{a}_1) + \mathbf{curl}\,\mathbf{curl}\boldsymbol{\Psi}(\mathbf{a}_1) \\ \mathbf{curl}\,\nabla\psi(\mathsf{a}_0) + \mathbf{curl}\,\mathbf{curl}\boldsymbol{\Upsilon}(a_2) - \nabla\mathrm{div}\boldsymbol{\Upsilon}(a_2) \\ \mathrm{div}\,\mathbf{curl}\boldsymbol{\Psi}(\mathbf{a}_1) \end{pmatrix} \\ &= \begin{pmatrix} -\Delta\psi(\mathsf{a}_0) \\ -\nabla\mathrm{div}\,\boldsymbol{\Psi}(\mathbf{a}_1) + \mathbf{curl}\,\mathbf{curl}\boldsymbol{\Psi}(\mathbf{a}_1) \\ -\nabla\,\mathrm{div}\,\boldsymbol{\Upsilon}(a_2) + \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Upsilon}(a_2) \\ 0 \end{pmatrix} = \overrightarrow{\mathbf{0}}. \end{aligned} \tag{3.4.46}$$

A similar calculation holds for $\mathsf{D}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}})$. □

Remark 3.3 Lemma 3.8 was proved using the explicit representations (3.4.39) and (3.4.42). The technique revealed some structure behind the two boundary potentials. However, notice that adapting the argument found in the proof of [37, Thm. 3.1.6], the desired result could also be obtained by observing that

$$\gamma'_{\mathsf{T}} : \mathcal{H}_{\mathsf{R}} \to (\mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \mathbb{R}^3 \backslash \Gamma))' \subset (C^\infty(\mathbb{R}^3 \backslash \Gamma)^8)', \tag{3.4.47}$$

together with Proposition 3.1, guarantees the equality $\mathsf{D}\mathcal{L}_{\mathsf{T}}\overrightarrow{\mathbf{a}} = \gamma'_{\mathsf{T}}\overrightarrow{\mathbf{a}}$ as continuous linear functionals on $C_0^\infty(\mathbb{R}^3 \backslash \Gamma)$.

Remark 3.4 It is a nice and unusual property for the potentials to belong to $(L^2(\Omega^+))^8$ as opposed to being only locally square-integrable. We see from Lemma 3.6 that this is a consequence of two ingredients: the stronger singularity of the Dirac fundamental solution, combined with the absence of differential operators acting on the relevant traces.

Lemma 3.9 (Jump relations) *For all* $\overrightarrow{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$ *and* $\overrightarrow{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}}$,

$$[\gamma_{\mathsf{T}}]\,\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) = \overrightarrow{\mathbf{0}}, \qquad [\gamma_{\mathsf{R}}]\,\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) = \mathrm{Id}, \tag{3.4.48}$$

$$[\gamma_{\mathsf{T}}]\,\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) = \mathrm{Id}, \qquad [\gamma_{\mathsf{R}}]\,\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) = \overrightarrow{\mathbf{0}}. \tag{3.4.49}$$

Proof. For the most part, the following jump relations can be inferred from known theory. We carefully evaluate

$$[\gamma_{\mathsf{T}}]\,\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) = \begin{pmatrix} -[\gamma]\,\mathrm{div}\,\boldsymbol{\Psi}(\mathbf{a}_1) \\ [\gamma_t]\,\nabla\psi(\mathsf{a}_0) + [\gamma_t]\,\mathbf{curl}\,\boldsymbol{\Upsilon}(a_2) \\ [\gamma_n]\,\mathbf{curl}\,\boldsymbol{\Psi}(\mathbf{a}_1) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \end{pmatrix}, \tag{3.4.50a}$$

$$[\gamma_{\mathsf{R}}]\,\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) = \begin{pmatrix} [\gamma_n]\,\nabla\psi(\mathsf{a}_0) + [\gamma_n]\,\mathbf{curl}\,\boldsymbol{\Upsilon}(a_2) \\ [\gamma_\tau]\,\mathbf{curl}\,\boldsymbol{\Psi}(\mathbf{a}_1) \\ [\gamma]\,\mathrm{div}\,\boldsymbol{\Upsilon}(a_2) \end{pmatrix} = \begin{pmatrix} \mathsf{a}_0 \\ \mathbf{a}_1 \\ a_2 \end{pmatrix}, \tag{3.4.50b}$$

$$[\gamma_{\mathsf{T}}]\,\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) = \begin{pmatrix} [\gamma]\,\mathrm{div}\,\boldsymbol{\Upsilon}(b_0) \\ [\gamma_t]\,\mathbf{curl}\,\boldsymbol{\Psi}(\mathbf{b}_1\times\mathbf{n}) \\ -[\gamma_n]\,\mathbf{curl}\,\boldsymbol{\Upsilon}(b_0) + [\gamma_n]\,\nabla\psi(\mathsf{b}_2) \end{pmatrix} = \begin{pmatrix} b_0 \\ \mathbf{b}_1 \\ \mathsf{b}_2 \end{pmatrix}, \tag{3.4.50c}$$

$$[\gamma_{\mathsf{R}}]\,\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) = \begin{pmatrix} [\gamma_n]\,\mathbf{curl}\,\boldsymbol{\Psi}(\mathbf{b}_1\times\mathbf{n}) \\ -[\gamma_\tau]\,\mathbf{curl}\,\boldsymbol{\Upsilon}(b_0) + [\gamma_\tau]\,\nabla\psi(\mathsf{b}_2) \\ [\gamma]\,\mathrm{div}\,\varPsi(\mathbf{b}_1\times\mathbf{n}) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \end{pmatrix}. \tag{3.4.50d}$$

The individual terms appearing in the above calculations can be found in [14, Sec. 5] and [21, Sec. 4], possibly up to tangential rotation by $90°$. Some terms slightly differ. In both eq. (3.4.50b) and eq. (3.4.50c), we are particularly concerned with the normal jump of $\mathbf{curl}\,\boldsymbol{\Upsilon}$ across $\varGamma$. Fortunately, we know that the restriction of $\boldsymbol{\Upsilon}$ to $H^{1/2}(\varGamma)$ is a continuous map with codomain $\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \varOmega)$. Its image is therefore regular enough for the identity

$$[\gamma_n]\,\mathbf{curl}\,\boldsymbol{\Upsilon}(q) = \mathrm{div}_{\varGamma}\big([\gamma_\tau]\,\boldsymbol{\Upsilon}(q)\big) = 0$$

to hold for all $q \in H^{1/2}(\varGamma)$ [12, Eq. 8]. □

Remark 3.5 The formal structure of these jump relations is the same as that of the jump identities for the potentials associated with other operators such as

- ◇ scalar second-order strongly elliptic operators [32, 37],
- ◇ second-order Maxwell wave operators [10, 12],
- ◇ Hodge–Laplace and Hodge–Helmholtz operators [14, 15].

3.4.4 Representation by surface potentials

Following McLean in [32, Chap. 7], we mimic the approach introduced by Costabel and Dauge [16, 17]. Corollary 3.1 plays the role of Green's second identity. We begin with the case where a solution of the Dirac equation defines a compactly supported distribution. This covers for instance interior problems and yields a representation formula in $\varOmega^-$. However, a condition on the behavior of solutions at infinity will be needed for $\varOmega^+$.

Proposition 3.2 (Interior representation formula) *If* $\overrightarrow{\mathbf{U}} \in \mathbf{H}(\mathsf{D}, \mathbb{R}^3 \backslash \varGamma)$ *is compactly supported and* $\overrightarrow{\mathbf{F}} \in (L^2(\mathbb{R}^3))^8$ *is such that* $\overrightarrow{\mathbf{F}}\big|_{\varOmega} := (\mathsf{D}\mathbf{U})\big|_{\varOmega}$ *and* $\overrightarrow{\mathbf{F}}\big|_{\varOmega^+} := (\mathsf{D}\mathbf{U})\big|_{\varOmega^+}$*. Then*

$$\overrightarrow{\mathbf{U}}(\mathbf{x}) = \varPhi * \overrightarrow{\mathbf{F}}(\mathbf{x}) + \mathcal{L}_{\mathsf{T}}\left[\gamma_{\mathsf{R}}\overrightarrow{\mathbf{U}}\right](\mathbf{x}) + \mathcal{L}_{\mathsf{R}}\left[\gamma_{\mathsf{T}}\overrightarrow{\mathbf{U}}\right](\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^3 \backslash \varGamma. \tag{3.4.51}$$

Proof. According to eq. (3.2.15),

$$\begin{aligned}
\langle \mathsf{D}\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{V}} \rangle_{\mathbb{R}^3} &\overset{(*)}{=} \int_{\varOmega} \overrightarrow{\mathbf{U}} \cdot \mathsf{D}\overrightarrow{\mathbf{V}} \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3 \backslash \overline{\varOmega}} \overrightarrow{\mathbf{U}} \cdot \mathsf{D}\overrightarrow{\mathbf{V}} \mathrm{d}\mathbf{x} \\
&= \int_{\varOmega} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{V}} \mathrm{d}\mathbf{x} + \langle\!\langle \gamma_{\mathsf{R}}^- \overrightarrow{\mathbf{U}}, \gamma_{\mathsf{T}}^- \overrightarrow{\mathbf{V}} \rangle\!\rangle_{\varGamma} - \langle\!\langle \gamma_{\mathsf{T}}^- \overrightarrow{\mathbf{U}}, \gamma_{\mathsf{R}}^- \overrightarrow{\mathbf{V}} \rangle\!\rangle_{\varGamma} \\
&\quad + \int_{\mathbb{R}^3 \backslash \overline{\varOmega}} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{V}} \mathrm{d}\mathbf{x} - \langle\!\langle \gamma_{\mathsf{R}}^+ \overrightarrow{\mathbf{U}}, \gamma_{\mathsf{T}}^+ \overrightarrow{\mathbf{V}} \rangle\!\rangle_{\varGamma} + \langle\!\langle \gamma_{\mathsf{T}}^+ \overrightarrow{\mathbf{U}}, \gamma_{\mathsf{R}}^+ \overrightarrow{\mathbf{V}} \rangle\!\rangle_{\varGamma} \\
&= \int_{\mathbb{R}^3} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{V}} \mathrm{d}\mathbf{x} + \langle\!\langle \left[\gamma_{\mathsf{R}} \overrightarrow{\mathbf{U}}\right], \gamma_{\mathsf{T}} \overrightarrow{\mathbf{V}} \rangle\!\rangle_{\varGamma} - \langle\!\langle \left[\gamma_{\mathsf{T}} \overrightarrow{\mathbf{U}}\right], \gamma_{\mathsf{R}} \overrightarrow{\mathbf{V}} \rangle\!\rangle_{\varGamma}
\end{aligned} \tag{3.4.52}$$

for all $\overrightarrow{\mathbf{V}} \in (C_0^\infty(\mathbb{R}^3))^8$. The regularity assumptions on $\overrightarrow{\mathbf{U}}$ guarantee that the traces are well-defined. We have used the fact that $\overrightarrow{\mathbf{V}}$ is smooth across the boundary to obtain the last equality, because smoothness guarantees that $\gamma_{\mathsf{T}}^- \overrightarrow{\mathbf{V}} = \gamma_{\mathsf{T}}^+ \overrightarrow{\mathbf{V}}$ and $\gamma_{\mathsf{R}}^- \overrightarrow{\mathbf{V}} = \gamma_{\mathsf{R}}^+ \overrightarrow{\mathbf{V}}$. Therefore, in the sense of distributions, we have

$$\mathsf{D}\overrightarrow{\mathbf{U}} = \mathbf{F} + (\gamma_{\mathsf{T}}^-)' \left[\gamma_{\mathsf{R}}^- \overrightarrow{\mathbf{U}}\right] - (\gamma_{\mathsf{R}}^-)' \left[\gamma_{\mathsf{T}}^- \overrightarrow{\mathbf{U}}\right]. \tag{3.4.53}$$

Since $\overrightarrow{\mathbf{U}}$ is assumed to have compact support, it can interpreted as a continuous linear functional on $C^\infty(\mathbb{R}^3)^8$ and convolution with $\varPhi$ using Proposition 3.1 shows that the identity is valid when interpreted in the sense distributions. Lemma 3.7 confirms that the equality holds in $(L^2(\mathbb{R}^3))^8$. □

In the following, we will work over the domains defined as the interior B_ρ and exterior B_ρ^+ of an open ball of radius ρ. Therefore, we must introduce the traces γ_{T}^ρ and γ_{R}^ρ that extend the operators defined in (3.2.5) where $\varGamma$ is replaced by the boundary ∂B_ρ of the open ball. The surface potentials $\mathcal{L}_{\mathsf{R}}^\rho$ and $\mathcal{L}_{\mathsf{T}}^\rho$ are defined accordingly with respect to these trace mappings. Similarly, a dagger † will refer to any given Lipschitz domain $\varOmega_\dagger \subset \mathbb{R}^3$. The following development parallels that of [32, Sec. 7].

Lemma 3.10 *For* $\overrightarrow{\mathbf{U}} \in (C_0^\infty(\varOmega^+)^8)'$ *such that* $\mathsf{D}\overrightarrow{\mathbf{U}}$ *has compact support in* $\varOmega^+$*, there exists a* *unique* *vector field* $\mathsf{M}\overrightarrow{\mathbf{U}} \in (C^\infty(\mathbb{R}^3))^8$ *such that*

$$\mathsf{M}\overrightarrow{\mathbf{U}}(\mathbf{x}) = \mathcal{L}_{\mathsf{T}}^\dagger(\gamma_{\mathsf{R}}^\dagger \overrightarrow{\mathbf{U}})(\mathbf{x}) + \mathcal{L}_{\mathsf{R}}^\dagger(\gamma_{\mathsf{T}}^\dagger \overrightarrow{\mathbf{U}})(\mathbf{x}) \tag{3.4.54}$$

for all $\mathbf{x}$ *inside* *any* *bounded Lipschitz domain* $\varOmega_\dagger$ *such that*

$$\overline{\varOmega} \cup \mathrm{supp}(\mathsf{D}\overrightarrow{\mathbf{U}}) \Subset \varOmega_\dagger. \tag{3.4.55}$$

Remark 3.6 It is key in the statement of Lemma 3.10 that the vector field $\mathsf{M}\overrightarrow{\mathbf{U}}$ is independent of $\varOmega_\dagger$.

Proof. Under the above hypotheses, $\overrightarrow{\mathbf{U}}$ is harmonic in $\Omega^+\backslash\mathrm{supp}(\mathsf{D}\overrightarrow{\mathbf{U}})$, because $\mathsf{D}\overrightarrow{\mathbf{U}} = \overrightarrow{\mathbf{0}}$ implies that $\overrightarrow{\boldsymbol{\Delta}}\overrightarrow{\mathbf{U}} = \mathsf{D}^2\overrightarrow{\mathbf{U}} = \overrightarrow{\mathbf{0}}$. Standard elliptic regularity theory [32, Thm. 6.4] further tells us that $\overrightarrow{\mathbf{U}}$ is a regular distribution whose components are smooth in that domain. Therefore, we can *define* $\mathsf{M}\overrightarrow{\mathbf{U}}$ in B_{ρ_1} as in the right hand side of (3.4.54) by

$$\mathsf{M}\overrightarrow{\mathbf{U}}(\mathbf{x}) := \mathcal{L}_{\mathsf{T}}^{\rho_1}(\gamma_{\mathsf{R}}^{\rho_1}\overrightarrow{\mathbf{U}})(\mathbf{x}) + \mathcal{L}_{\mathsf{R}}^{\rho_1}(\gamma_{\mathsf{T}}^{\rho_1}\overrightarrow{\mathbf{U}})(\mathbf{x}), \tag{3.4.56}$$

where the radius ρ_1 is large enough that $\overline{\Omega}\cup\mathrm{supp}(\mathsf{D}\overrightarrow{\mathbf{U}}) \Subset B_{\rho_1}$.

Applying eq. (3.2.15) inside $B_{\rho_2}\backslash\overline{B}_{\rho_1}$ with $\rho_1 < \rho_2$ eventually shows that this definition is independent of the radius. Indeed, for any $\mathbf{x}\in B_{\rho_1}$, $\Phi_{\mathbf{x}}$ is a smooth matrix in $\mathbb{R}^3\backslash\overline{B_{\rho_1}}$, and, thus, $\mathrm{supp}(\mathsf{D}\overrightarrow{\mathbf{U}}) \Subset B_{\rho_1}$ guarantees for $i = 1, ..., 8$ that

$$\begin{aligned}
0 = \int_{B_{\rho_2}\backslash\overline{B}_{\rho_1}} &\Phi(\mathbf{x}-\mathbf{y})\mathsf{D}\overrightarrow{\mathbf{U}}(\mathbf{y})\mathrm{d}\mathbf{y}\cdot\overrightarrow{\mathbf{E}}_i = \int_{B_{\rho_2}\backslash\overline{B}_{\rho_1}} \Phi_{i,:}(\mathbf{x}-\mathbf{y})\mathsf{D}\overrightarrow{\mathbf{U}}(\mathbf{y})\mathrm{d}\mathbf{y} \\
&\overset{(*)}{=} -\int_{B_{\rho_2}\backslash\overline{B}_{\rho_1}} \Phi_{:,i}(\mathbf{x}-\mathbf{y})\cdot\mathsf{D}\overrightarrow{\mathbf{U}}(\mathbf{y})\mathrm{d}\mathbf{y} = -\int_{B_{\rho_2}\backslash\overline{B}_{\rho_1}} \mathsf{D}_{\mathbf{y}}\Phi_{:,i}(\mathbf{x}-\mathbf{y})\cdot\overrightarrow{\mathbf{U}}(\mathbf{y})\mathrm{d}\mathbf{y} \\
&\qquad - \langle\!\langle \gamma_{\mathsf{T}}^{\rho_2}\overrightarrow{\mathbf{U}}, \gamma_{\mathsf{R}}^{\rho_2}\Phi_{:,i}(\mathbf{x}-\cdot)\rangle\!\rangle_\Gamma - \langle\!\langle \gamma_{\mathsf{T}}^{\rho_2}\Phi_{:,i}(\mathbf{x}-\cdot), \gamma_{\mathsf{R}}^{\rho_2}\overrightarrow{\mathbf{U}}\rangle\!\rangle_\Gamma \\
&\qquad + \langle\!\langle \gamma_{\mathsf{T}}^{\rho_1}\overrightarrow{\mathbf{U}}, \gamma_{\mathsf{R}}^{\rho_1}\Phi_{:,i}(\mathbf{x}-\cdot)\rangle\!\rangle_\Gamma + \langle\!\langle \gamma_{\mathsf{T}}^{\rho_1}\Phi_{:,i}(\mathbf{x}-\cdot), \gamma_{\mathsf{R}}^{\rho_1}\overrightarrow{\mathbf{U}}\rangle\!\rangle_\Gamma,
\end{aligned} \tag{3.4.57}$$

where $\Phi_{i,:}$ corresponds to the i-th row of Φ, $\Phi_{:,j}$ to its j-th column, and Lemma 3.4 was used to obtain $(*)$.

On the one hand, for $\mathbf{x}\neq\mathbf{y}$,

$$\begin{aligned}
\mathsf{D}_{\mathbf{y}}\Phi_{:,i}(\mathbf{x}-\mathbf{y})\cdot\overrightarrow{\mathbf{U}}(\mathbf{y}) &= \mathsf{D}_{\mathbf{x}}(\Phi_{\mathbf{x}}(\mathbf{y})\overrightarrow{\mathbf{U}}(\mathbf{y}))\cdot\overrightarrow{\mathbf{E}}_i \\
&= \mathsf{D}_{\mathbf{x}}\mathsf{D}_{\mathbf{x}}(\mathsf{G}(\mathbf{x}-\mathbf{y})\overrightarrow{\mathbf{U}}(\mathbf{y}))\cdot\overrightarrow{\mathbf{E}}_i = (-\Delta_{\mathbf{x}}G(\mathbf{x}-\mathbf{y}))\overrightarrow{\mathbf{U}}(\mathbf{y})\cdot\overrightarrow{\mathbf{E}}_i = 0.
\end{aligned} \tag{3.4.58}$$

On the other hand,

$$\langle\!\langle \gamma_{\mathsf{T}}^{\rho_2}\overrightarrow{\mathbf{U}}, \gamma_{\mathsf{R}}^{\rho_2}\Phi_{:,i}(\mathbf{x}-\cdot)\rangle\!\rangle = -\langle\!\langle \gamma_{\mathsf{T}}^{\rho_2}\overrightarrow{\mathbf{U}}, \gamma_{\mathsf{R}}^{\rho_2}(\Phi_{\mathbf{x}}\overrightarrow{\mathbf{E}}_i)\rangle\!\rangle = -\mathcal{L}_{\mathsf{R}}^{\rho_2}(\gamma_{\mathsf{T}}^{\rho_2}\overrightarrow{\mathbf{U}})(\mathbf{x})\cdot\overrightarrow{\mathbf{E}}_j \tag{3.4.59}$$

by Lemma 3.6, and similarly for the remaining boundary terms. These two pieces of information together prove the validity of the independence claim.

In fact, the same argument can be repeated in $B_{\rho_1}\backslash\overline{\Omega_\dagger}$ to confirm that (3.4.54) holds independently of the chosen Lipschitz domain satisfying the hypotheses.

Smoothness of $\mathsf{M}\overrightarrow{\mathbf{U}}$ is inherited from the smoothness of the integrands. □

Lemma 3.11 *Let* $\overrightarrow{\mathbf{F}} \in (L^2(\Omega^+))^8$ *be compactly supported and suppose that* $\overrightarrow{\mathbf{U}} \in (C_0^\infty(\Omega^+)^8)'$ *satisfies* $\mathsf{D}\overrightarrow{\mathbf{U}} = \overrightarrow{\mathbf{F}}$ *on* Ω^+*. If the restriction of* $\overrightarrow{\mathbf{U}}$ *to* $\Omega^+\cap B_\rho$ *belongs to* $\mathbf{H}(\mathsf{D}, \Omega^+\cap B_\rho)$ *for some* ρ *large enough that* $\Omega\cup\Gamma\Subset B_\rho$ *and* $\mathrm{supp}\,\overrightarrow{\mathbf{F}} \Subset \Omega^+\cap B_\rho$*, then*

$$\overrightarrow{\mathbf{U}} = \Phi * \overrightarrow{\mathbf{F}} - \mathcal{L}_{\mathsf{T}}\gamma_{\mathsf{R}}^+\overrightarrow{\mathbf{U}} - \mathcal{L}_{\mathsf{R}}\gamma_{\mathsf{T}}^+\overrightarrow{\mathbf{U}} + \mathsf{M}\overrightarrow{\mathbf{U}} \tag{3.4.60}$$

holds in $\mathbf{H}(\mathsf{D}, \Omega^+)$.

Proof. Upon applying Proposition 3.2 to the distribution

$$\vec{\mathbf{U}}_0 := \begin{cases} \vec{\mathbf{0}}, & \text{in } \Omega, \\ \vec{\mathbf{U}}, & \text{in } \Omega^+ \cap B_\rho, \\ \vec{\mathbf{0}}, & \text{in } \mathbb{R}^3 \backslash \overline{B_\rho}, \end{cases} \tag{3.4.61}$$

that is compactly supported and belongs to $\mathbf{H}_{\text{loc}}(\mathsf{D}, \mathbb{R}^3 \backslash (\Gamma \cup \partial B_\rho))$, we obtain

$$\vec{\mathbf{U}}_0 = \Phi * \vec{\mathbf{F}} - \mathcal{L}_{\mathsf{T}}(\gamma^+_{\mathsf{R}} \vec{\mathbf{U}}) - \mathcal{L}_{\mathsf{R}}(\gamma^+_{\mathsf{T}} \vec{\mathbf{U}}) + \mathcal{L}^\rho_{\mathsf{T}}(\gamma^\rho_{\mathsf{R}} \vec{\mathbf{U}}) + \mathcal{L}^\rho_{\mathsf{R}}(\gamma^\rho_{\mathsf{T}} \vec{\mathbf{U}}) \tag{3.4.62}$$

as a functional on $(C_0^\infty(\mathbb{R}^3))^8$. Since B_ρ satisfies the hypotheses imposed on $\Omega_\dagger$ in the statement of Lemma 3.10, we recognize that

$$\mathcal{L}^\rho_{\mathsf{T}}(\gamma^\rho_{\mathsf{R}} \vec{\mathbf{U}})(\mathbf{x}) + \mathcal{L}^\rho_{\mathsf{R}}(\gamma^\rho_{\mathsf{T}} \vec{\mathbf{U}})(\mathbf{x}) = \mathsf{M}\vec{\mathbf{U}}(\mathbf{x}) \tag{3.4.63}$$

for all $\mathbf{x} \in B_\rho$. Hence,

$$\vec{\mathbf{U}} = \Phi * \vec{\mathbf{F}} - \mathcal{L}_{\mathsf{T}} \gamma^+_{\mathsf{R}} \vec{\mathbf{U}} - \mathcal{L}_{\mathsf{R}} \gamma^+_{\mathsf{T}} \vec{\mathbf{U}} + \mathsf{M}\vec{\mathbf{U}} \qquad \text{in } \Omega^+ \cap B_\rho. \tag{3.4.64}$$

As in Lemma 3.10, it follows from supp $\vec{\mathbf{F}} \subset B_\rho$ that $\vec{\mathbf{U}}$ is harmonic in $\mathbb{R}^3 \backslash B_\rho$, and thus smooth everywhere outside the ball B_ρ by well-known elliptic regularity theory [32, Thm. 6.4]. Hence, the hypothesis that $\vec{\mathbf{U}} \in \mathbf{H}(\mathsf{D}, \Omega^+ \cap B_\rho)$ for at least one ball B_ρ satisfying the hypotheses in fact guarantees that it belongs to that space independently of the radius satisfying those same requirements. Therefore, (3.11) holds in the whole of Ω^+. Based on Lemma 3.10, the mapping properties of the potentials established in Lemma 3.7 and Proposition 3.1, we conclude that the equality (3.4.64) holds in fact not only in $\mathbf{H}_{\text{loc}}(\mathsf{D}, \Omega^+)$, but in $\mathbf{H}(\mathsf{D}, \Omega^+)$—which is the desired result. □

Lemma 3.12 *Under the hypotheses of Lemma 3.11,*

$$\mathsf{M}\vec{\mathbf{U}} = \vec{\mathbf{0}} \tag{3.4.65}$$

if and only if

$$\|\vec{\mathbf{U}}(\mathbf{z})\| \to 0 \textit{ uniformly as } \mathbf{z} \to \infty. \tag{3.4.66}$$

Proof. The condition (3.4.66) is well-defined, because as in Lemma 3.11, there exists a radius ρ_1 large enough that the vector-field $\vec{\mathbf{U}}$ is smooth outside B_{ρ_1}. For the same reason, the traces of $\vec{\mathbf{U}}$ appearing in the following inequalities are smooth boundary fields.

Recall that for $\mathbf{z} \neq \mathbf{0}$,

$$\|\nabla G(\mathbf{z})\| \lesssim \|\mathbf{z}\|^{-2}. \tag{3.4.67}$$

Therefore, it is easily seen from (3.4.39) and (3.4.42) that if $\rho_2 > \rho_1$,

$$\|\mathcal{L}^{\rho_2}_\bullet(\gamma^{\rho_2}_\bullet \vec{\mathbf{U}})(\mathbf{x})\| \lesssim \rho_2^{-2} \| \int_{\partial B_{\rho_2}} \gamma_\bullet \vec{\mathbf{U}}(\mathbf{y}) d\sigma(\mathbf{y})\| \lesssim \max_{\mathbf{y} \in \partial B_{\rho_2}} \|\vec{\mathbf{U}}(\mathbf{y})\| \tag{3.4.68}$$

for all $\mathbf{x} \in B_{\rho_1}$, $\bullet = \mathsf{T}$ or R. Notice that the left hand side of (3.4.68) is well-defined, because as in Lemma 3.10, Lemma 3.8 and $\mathsf{D}^2 = -\Delta$ guarantee that away from the boundary ∂B_{ρ_2}, the potentials are smooth harmonic vector fields. No differential operator appears in the definition of the trace mappings γ_{R} and γ_{T}. The independence of $\mathsf{M}\overrightarrow{\mathbf{U}}$ from its domain of definition thus directly yields one implication of the lemma upon taking $\rho_2 \to \infty$.

The converse follows from the exterior representation formula (3.4.60) with $\mathsf{M}\overrightarrow{\mathbf{U}} = \overrightarrow{\mathbf{0}}$ and an analysis exploiting (3.4.67) that leads to an inequality similar to (3.4.68). However, this time the potentials are computed as integrals (duality pairings) on the fixed boundary Γ and an inverse square decay is inherited from the decay of the fundamental solution. □

Proposition 3.3 (Exterior representation formula) *If* $\overrightarrow{\mathbf{U}} \in \mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \Omega^+)$ *is such that* $\overrightarrow{\mathbf{U}}(\mathbf{z}) \to 0$ *as* $\mathbf{z} \to \infty$ *and* $\overrightarrow{\mathbf{F}} := \mathsf{D}\mathbf{U}$ *is compactly supported. Then*

$$\overrightarrow{\mathbf{U}}(\mathbf{x}) = \Phi * \overrightarrow{\mathbf{F}}(\mathbf{x}) - \mathcal{L}_{\mathsf{T}}\gamma_{\mathsf{R}}^{+}\overrightarrow{\mathbf{U}}(\mathbf{x}) - \mathcal{L}_{\mathsf{R}}^{+}\gamma_{\mathsf{T}}^{+}\overrightarrow{\mathbf{U}}(\mathbf{x}), \qquad \mathbf{x} \in \Omega^+. \tag{3.4.69}$$

3.5 Boundary integral equations

Boundary integral equations are obtained by taking the traces γ_{R} and γ_{T} on both sides of the representation formulas (3.4.51) and (3.4.69). The operator form of the interior and exterior Calderón projectors defined on $\mathcal{H}_{\mathsf{R}} \times \mathcal{H}_{\mathsf{T}}$, which we denote P^- and P^+ respectively, enter the Calderón identities

$$\underbrace{\begin{pmatrix} \{\gamma_{\mathsf{R}}\}\,\mathcal{L}_{\mathsf{T}} + \frac{1}{2}\mathrm{Id} & \{\gamma_{\mathsf{R}}\}\,\mathcal{L}_{\mathsf{R}} \\ \{\gamma_{\mathsf{T}}\}\,\mathcal{L}_{\mathsf{T}} & \{\gamma_{\mathsf{T}}\}\,\mathcal{L}_{\mathsf{R}} + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{\mathsf{P}^-} \begin{pmatrix} \gamma_{\mathsf{R}}^{-}(\mathbf{U}) \\ \gamma_{\mathsf{T}}^{-}(\mathbf{U}) \end{pmatrix} = \begin{pmatrix} \gamma_{\mathsf{R}}^{-}(\mathbf{U}) \\ \gamma_{\mathsf{T}}^{-}(\mathbf{U}) \end{pmatrix}, \tag{3.5.1}$$

$$\underbrace{\begin{pmatrix} -\{\gamma_{\mathsf{R}}\}\,\mathcal{L}_{\mathsf{T}} + \frac{1}{2}\mathrm{Id} & -\{\gamma_{\mathsf{R}}\}\,\mathcal{L}_{\mathsf{R}} \\ -\{\gamma_{\mathsf{T}}\}\,\mathcal{L}_{\mathsf{T}} & -\{\gamma_{\mathsf{T}}\}\,\mathcal{L}_{\mathsf{R}} + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{\mathsf{P}^+} \begin{pmatrix} \gamma_{\mathsf{R}}^{+}(\mathbf{U}) \\ \gamma_{\mathsf{T}}^{+}(\mathbf{U}) \end{pmatrix} = \begin{pmatrix} \gamma_{\mathsf{R}}^{+}(\mathbf{U}) \\ \gamma_{\mathsf{T}}^{+}(\mathbf{U}) \end{pmatrix}. \tag{3.5.2}$$

For example, extend a solution $\overrightarrow{\mathbf{U}} \in \mathbf{H}(\mathsf{D}, \Omega)$ of the homogeneous Dirac equation in Ω^- to the whole of $\mathbb{R}^3$ by zero. Using Proposition 3.2,

$$\overrightarrow{\mathbf{U}}(\mathbf{x}) = \mathcal{L}_{\mathsf{T}}\gamma_{\mathsf{R}}^{-}\overrightarrow{\mathbf{U}}(\mathbf{x}) + \mathcal{L}_{\mathsf{R}}\gamma_{\mathsf{T}}^{-}\overrightarrow{\mathbf{U}}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^3 \backslash \Gamma. \tag{3.5.3}$$

Then, applying γ_{R}^{-} on both sides of the equation yields

$$\gamma_{\mathsf{R}}^{-}\overrightarrow{\mathbf{U}}(\mathbf{x}) = \gamma_{\mathsf{R}}^{-}\mathcal{L}_{\mathsf{T}}\gamma_{\mathsf{R}}^{-}\overrightarrow{\mathbf{U}}(\mathbf{x}) + \gamma_{\mathsf{R}}^{-}\mathcal{L}_{\mathsf{R}}\gamma_{\mathsf{T}}^{-}\overrightarrow{\mathbf{U}}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma. \tag{3.5.4}$$

It is a simple calculation to verify that the jump identities of Lemma 3.9 implies

$$\{\gamma_{\mathsf{T}}\}\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) = \gamma_{\mathsf{T}}^{-}\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}), \qquad \{\gamma_{\mathsf{R}}\}\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) = \gamma_{\mathsf{R}}^{-}\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) - \frac{1}{2}\overrightarrow{\mathbf{a}}, \tag{3.5.5}$$

$$\{\gamma_{\mathsf{T}}\}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) = \gamma_{\mathsf{T}}^{-}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) - \frac{1}{2}\overrightarrow{\mathbf{b}}, \qquad \{\gamma_{\mathsf{R}}\}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) = \gamma_{\mathsf{R}}^{-}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}). \tag{3.5.6}$$

Substituting the interior traces for the averages using these relations leads to the top row of (3.5.1). The other identities are obtained similarly.

A classical argument, cf. [40, lem. 6.18], shows that P^- and P^+ are indeed projectors, i.e. $(\mathsf{P}^{\mp})^2 = \mathsf{P}^{\mp}$. The proof, which for the homogeneous Dirac equation is essentially based on Lemma 3.8, also shows as a byproduct, cf. [44, Thm. 3.7], that the images of P^- and P^+ are spaces of valid interior and exterior Cauchy data, respectively. In fact, as observed in [12, Sec. 5], we have $\mathsf{P}^- + \mathsf{P}^+ = \mathrm{Id}$. So the range of P^- coincides with the nullspace of P^+ and vice versa. Therefore, we find the important property that $(\vec{a}, \vec{\mathbf{b}}) \in \mathcal{H}_{\mathsf{R}} \times \mathcal{H}_{\mathsf{T}}$ is valid interior or exterior Cauchy data if and only if it lies in the nullspace of P^+ or P^-, respectively.

The two direct boundary integral equations of the first-kind related to (R) and (T) then read as follows. Given $\gamma_{\mathsf{R}}\vec{\mathbf{U}} = \vec{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$, the task is to determine the unknown $\vec{\mathbf{b}} = \gamma_{\mathsf{T}}\vec{\mathbf{U}} \in \mathcal{H}_{\mathsf{T}}$ by solving

$$\gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}}) = \frac{1}{2}\vec{\mathbf{a}} - \{\gamma_{\mathsf{R}}\}\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}). \tag{BR}$$

If $\vec{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}}$ is known instead, then we solve

$$\gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}) = \frac{1}{2}\vec{\mathbf{b}} - \{\gamma_{\mathsf{T}}\}\mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}}) \tag{BT}$$

for the unknown $\vec{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$.

Remark 3.7 (Duality and symmetry) Let us revisit the boundary value problems of Section 3.3. We wish to highlight that (T) and (R) are really the same problem in hiding. For example, we can always relabel the components of an unknown vector-field $\vec{\mathbf{U}} \in \mathbf{H}(\mathsf{D}, \Omega)$ to

$$V_0 := U_3, \qquad \mathbf{V}_1 := -\mathbf{U}_2 \qquad \mathbf{V}_2 := -\mathbf{U}_1 \qquad \text{and} \qquad V_3 := V_0, \tag{3.5.7}$$

and set

$$a_0 := -b_2 \qquad \mathbf{a}_1 := \mathbf{n} \times \mathbf{b}_1 \qquad \text{and} \qquad a_3 = b_0. \tag{3.5.8}$$

This turns a problem (T) for $\vec{\mathbf{U}}$ into a problem (R) for $\vec{\mathbf{V}} \in \mathbf{H}(\mathsf{D}, \Omega)$.

Since both a solution $\vec{\mathbf{U}}$ of (T) and a solution $\vec{\mathbf{V}}$ of (R) can be written using the representation formula (3.4.51), we expect (3.5.8) to define an isomorphism $\mathsf{T} : \mathcal{H}_{\mathsf{T}} \to \mathcal{H}_{\mathsf{R}}$ that also turns one of the boundary integral equation into the other. And indeed, one can verify that

$$\{\gamma_{\mathsf{R}}\}\mathcal{L}_{\mathsf{T}}\left(\mathsf{T}\,\vec{\mathbf{b}}\right) = \mathsf{T}\,\gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{R}}\left(\vec{\mathbf{b}}\right) \qquad \text{and} \qquad \{\gamma_{\mathsf{T}}\}\mathcal{L}_{\mathsf{T}}\left(\mathsf{T}\,\vec{\mathbf{b}}\right) = \mathsf{T}\,\{\gamma_{\mathsf{R}}\}\mathcal{L}_{\mathsf{R}}\left(\vec{\mathbf{b}}\right).$$

Hence, (BT) can be equivalently formulated as a problem (BR) with unknown "$\mathsf{T}^{-1}\vec{\mathbf{a}}$" and given data $\mathsf{T}\,\vec{\mathbf{b}}$.

Let us take a closer look at the bilinear forms naturally associated with the continuous first-kind boundary integral operators

$$\gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}} : \mathcal{H}_{\mathsf{R}} \to \mathcal{H}_{\mathsf{T}}, \tag{3.5.9}$$
$$\gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}} : \mathcal{H}_{\mathsf{T}} \to \mathcal{H}_{\mathsf{R}}, \tag{3.5.10}$$

that map trace spaces to their dual spaces.

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{c}}$ be trial and test boundary vector fields lying in $\mathcal{H}_{\mathsf{R}}$, and similarly for $\vec{\mathbf{b}}$ and $\vec{\mathbf{d}}$ in $\mathcal{H}_{\mathsf{T}}$. Catching up with the calculations of Subsection 3.4.2, we want to derive convenient integral formulas for

$$\langle\!\langle \overrightarrow{\mathbf{c}}, \gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}}\left(\overrightarrow{\mathbf{a}}\right)\rangle\!\rangle = -\langle \mathsf{c}_0, \gamma \operatorname{div}\boldsymbol{\Psi}(\mathbf{a}_1)\rangle_\Gamma + \langle \mathbf{c}_1, \gamma_t \nabla\psi(\mathsf{a}_0)\rangle_\tau \\ + \langle \mathbf{c}_1, \gamma_t \operatorname{\mathbf{curl}}\boldsymbol{\Upsilon}(a_2)\rangle_\tau + \langle c_2, \gamma_n \operatorname{\mathbf{curl}}\boldsymbol{\Psi}(\boldsymbol{a}_1)\rangle_\Gamma$$

and

$$\langle\!\langle \overrightarrow{\mathbf{d}}, \gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}})\rangle\!\rangle = \langle d_0, \gamma_n \operatorname{\mathbf{curl}}\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n})\rangle_\Gamma - \langle \mathbf{d}_1, \gamma_\tau \operatorname{\mathbf{curl}}\boldsymbol{\Upsilon}(b_0)\rangle_\tau \\ + \langle \mathbf{d}_1, \gamma_\tau \nabla\psi(\mathsf{b}_2)\rangle_\tau + \langle \mathsf{d}_2, \gamma \operatorname{div}\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n})\rangle_\Gamma.$$

In the course of our derivation, we will often rely implicitly on the fact that $\mathbf{a}_1$ and $\mathbf{b}_1$ are tangential vector fields.

Using the fact that $\operatorname{div}\boldsymbol{\Psi}(\mathbf{a}_1) = \psi\,(\operatorname{div}_\Gamma \mathbf{a}_1)$ and $\operatorname{div}\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n}) = \psi\,(\operatorname{curl}_\Gamma \mathbf{b}_1)$ [27, Lem. 2.3], we immediately find that

$$\langle \mathsf{c}_0, \gamma \operatorname{div}\boldsymbol{\Psi}(\mathbf{a}_1)\rangle_\Gamma = \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \mathsf{c}_0(\mathbf{x}) \operatorname{div}_\Gamma \mathbf{a}_1(\mathbf{y}) \mathrm{d}\sigma(\mathbf{x})\mathrm{d}\sigma(\mathbf{y})$$

and

$$\langle \mathsf{d}_2, \gamma \operatorname{div}\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n})\rangle_\Gamma = \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \mathsf{d}_2(\mathbf{x}) \operatorname{curl}_\Gamma \mathbf{b}_1(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}). \tag{3.5.11}$$

We know from [14, Sec. 6.4] that

$$\begin{aligned}\langle \mathbf{d}_1, \gamma_\tau \operatorname{\mathbf{curl}}\boldsymbol{\Upsilon}(b_0)\rangle_\tau \\ &= -\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \big(\mathbf{n}(\mathbf{x}) \times \mathbf{d}_1(\mathbf{x})\big) \cdot \big(\mathbf{n}(\mathbf{y}) \times \nabla_\Gamma b_0(\mathbf{y})\big)\, \mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}) \\ &= \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \big(\mathbf{n}(\mathbf{x}) \times \mathbf{d}_1(\mathbf{x})\big) \cdot \operatorname{\mathbf{curl}}_\Gamma b_0(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}).\end{aligned}$$

Adapting the arguments, we also obtain

$$\begin{aligned}\langle \mathbf{c}_1, \gamma_t \operatorname{\mathbf{curl}}\boldsymbol{\Upsilon}(a_2)\rangle_\tau &= \langle \mathbf{c}_1 \times \mathbf{n}, \gamma_\tau \operatorname{\mathbf{curl}}\boldsymbol{\Upsilon}(\mathsf{a}_0)\rangle_\tau \\ &= \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \big(\mathbf{n}(\mathbf{x}) \times (\mathbf{c}_1(\mathbf{x}) \times \mathbf{n}(\mathbf{x}))\big) \cdot \operatorname{\mathbf{curl}}_\Gamma a_2(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}) \\ &= \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \mathbf{c}_1(\mathbf{x}) \cdot \operatorname{\mathbf{curl}}_\Gamma a_2(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}).\end{aligned}$$

Again, from [14, Sec. 6.4], we can similarly extract

$$\begin{aligned}\langle c_2, \gamma_n \operatorname{\mathbf{curl}}\boldsymbol{\Psi}(\mathbf{a}_1)\rangle_\Gamma &= -\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \mathbf{a}_1(\mathbf{y}) \cdot \big(\mathbf{n}(\mathbf{x}) \times \nabla_\Gamma c_2(\mathbf{x})\big)\, \mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}) \\ &= \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \mathbf{a}_1(\mathbf{y}) \cdot \operatorname{\mathbf{curl}}_\Gamma c_2(\mathbf{x}) \mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})\end{aligned}$$

and

$$\langle d_0, \gamma_n \operatorname{\mathbf{curl}}\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n})\rangle_\Gamma = -\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, (\mathbf{n}(\mathbf{y}) \times \mathbf{b}_1(\mathbf{y})) \cdot \operatorname{\mathbf{curl}}_\Gamma d_0(\mathbf{x}) \mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})$$

Finally, it follows almost directly by definition that

$$\langle \mathbf{c}_1, \gamma_t \nabla\psi(\mathsf{a}_0)\rangle_\tau = -\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\mathsf{a}_0(\mathbf{y})\,\mathrm{div}_\Gamma\,\mathbf{c}_1(\mathbf{x})\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}),$$

and

$$\langle \mathbf{d}_1, \gamma_\tau \nabla\psi(\mathsf{b}_2)\rangle_\tau = \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\mathsf{b}_2(\mathbf{y})\,\mathrm{curl}_\Gamma\,\mathbf{d}_1(\mathbf{x})\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}).$$

Putting everything together yields the symmetric bilinear forms

$$\begin{aligned}\langle\!\langle \overrightarrow{\mathbf{c}}, \gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}}\left(\overrightarrow{\mathbf{a}}\right)\rangle\!\rangle = &-\int_\Gamma\int_\Gamma G(\mathbf{x}-\mathbf{y})\,\mathsf{c}_0(\mathbf{x})\,\mathrm{div}_\Gamma\mathbf{a}_1(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})\mathrm{d}\sigma(\mathbf{y})\\ &-\int_\Gamma\int_\Gamma G(\mathbf{x}-\mathbf{y})\,\mathsf{a}_0(\mathbf{y})\,\mathrm{div}_\Gamma\,\mathbf{c}_1(\mathbf{x})\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})\\ &+\int_\Gamma\int_\Gamma G(\mathbf{x}-\mathbf{y})\,\mathbf{c}_1(\mathbf{x})\cdot\mathbf{curl}_\Gamma\,a_2(\mathbf{y})\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})\\ &+\int_\Gamma\int_\Gamma G(\mathbf{x}-\mathbf{y})\,\mathbf{a}_1(\mathbf{y})\cdot\mathbf{curl}_\Gamma\,c_2(\mathbf{x})\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}),\end{aligned} \tag{3.5.12}$$

$$\begin{aligned}\langle\!\langle \overrightarrow{\mathbf{d}}, \gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}})\rangle\!\rangle = &-\int_\Gamma\int_\Gamma G(\mathbf{x}-\mathbf{y})\,\big(\mathbf{n}(\mathbf{y})\times\mathbf{b}_1(\mathbf{y})\big)\cdot\mathbf{curl}_\Gamma\,d_0(\mathbf{x})\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})\\ &-\int_\Gamma\int_\Gamma G(\mathbf{x}-\mathbf{y})\,\big(\mathbf{n}(\mathbf{x})\times\mathbf{d}_1(\mathbf{x})\big)\cdot\mathbf{curl}_\Gamma\,b_0(\mathbf{y})\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})\\ &+\int_\Gamma\int_\Gamma G(\mathbf{x}-\mathbf{y})\,\mathsf{b}_2(\mathbf{y})\,\mathrm{curl}_\Gamma\,\mathbf{d}_1(\mathbf{x})\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})\\ &+\int_\Gamma\int_\Gamma G(\mathbf{x}-\mathbf{y})\,\mathsf{d}_2(\mathbf{x})\,\mathrm{curl}_\Gamma\mathbf{b}_1(\mathbf{y})\mathrm{d}\sigma(\mathbf{y})\mathrm{d}\sigma(\mathbf{x}).\end{aligned} \tag{3.5.13}$$

The above integrals must be understood as duality pairings.

Remark 3.8 Let us highlight here, as we have announced in the introduction, that in the sense of [24, Chap. 2.5], these double integrals feature only weakly singular kernels!

The non-local inner products

$$(u, v)_{-1/2} := \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})u(\mathbf{x})\,v(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})\mathrm{d}\sigma(\mathbf{y}), \tag{3.5.14a}$$

$$(\mathbf{u}, \mathbf{v})_{-1/2,\mathsf{T}} := \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{y})\mathrm{d}\sigma(\mathbf{x})\mathrm{d}\sigma(\mathbf{y}), \tag{3.5.14b}$$

$$(\mathbf{u}, \mathbf{v})_{-1/2,\mathsf{R}} := \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})(\mathbf{n}(\mathbf{x})\times\mathbf{u}(\mathbf{x}))\cdot(\mathbf{n}(\mathbf{y})\times\mathbf{v}(\mathbf{y}))\mathrm{d}\sigma(\mathbf{x})\mathrm{d}\sigma(\mathbf{y}), \tag{3.5.14c}$$

respectively defined over $H^{-1/2}(\Gamma)$, $\mathbf{H}_{\mathsf{T}}^{-1/2}(\Gamma) := (\mathbf{H}_{\mathsf{T}}^{1/2}(\Gamma))'$ and $\mathbf{H}_{\mathsf{R}}^{-1/2}(\Gamma) := (\mathbf{H}_{\mathsf{R}}^{1/2}(\Gamma))'$, where

$$\mathbf{H}_{\mathsf{T}}^{1/2}(\Gamma) := \gamma_t(\mathbf{H}^1(\Omega)) \qquad \text{and} \qquad \mathbf{H}_{\mathsf{R}}^{1/2}(\Gamma) := \gamma_\tau(\mathbf{H}^1(\Omega)), \tag{3.5.15}$$

are positive definite Hermitian forms, and induce equivalent norms on the trace spaces [10, Sec. 4.1]. In the following, we will concern ourselves with the coercivity and geometric structure of the bilinear forms

$$\begin{aligned} \mathsf{b}_{\mathsf{T}}(\vec{\mathbf{a}}, \vec{\mathbf{c}}) &:= \langle\!\langle \gamma_{\mathsf{T}} \mathcal{L}_{\mathsf{T}} \left(\vec{\mathbf{a}}\right), \vec{\mathbf{c}} \rangle\!\rangle \\ &= (-\mathrm{div}_\Gamma\, \mathbf{a}_1, \mathsf{c}_0)_{-1/2} + (\mathsf{a}_0, -\mathrm{div}_\Gamma\, \mathbf{c}_1)_{-1/2} \\ &\quad + (\mathbf{curl}_\Gamma\, a_2, \mathbf{c}_1)_{-1/2,\mathsf{T}} + (\mathbf{a}_1, \mathbf{curl}_\Gamma c_2)_{-1/2,\mathsf{T}} \end{aligned} \tag{3.5.16}$$

and

$$\begin{aligned} \mathsf{b}_{\mathsf{R}}(\vec{\mathbf{b}}, \vec{\mathbf{d}}) &:= \langle\!\langle \gamma_{\mathsf{R}} \mathcal{L}_{\mathsf{R}} \left(\vec{\mathbf{b}}\right), \vec{\mathbf{d}} \rangle\!\rangle \\ &= (\mathbf{b}_1, \nabla_\Gamma\, d_0)_{-1/2,\mathsf{R}} + (\nabla_\Gamma\, b_0, \mathbf{d}_1)_{-1/2,\mathsf{R}} \\ &\quad + (\mathsf{b}_2, \mathrm{curl}_\Gamma \mathbf{d}_1)_{-1/2} + (\mathrm{curl}_\Gamma \mathbf{b}_1, \mathsf{d}_2)_{-1/2}\,. \end{aligned} \tag{3.5.17}$$

3.6 T-coercivity

Based on the space decomposition introduced by the next lemma, we design isomorphisms $\mathcal{H}_{\mathsf{R}} \to \mathcal{H}_{\mathsf{R}}$ and $\mathcal{H}_{\mathsf{T}} \to \mathcal{H}_{\mathsf{T}}$ that are instrumental for obtaining the desired generalized Gårding inequalities for b_{T} and b_{R}.

Lemma 3.13 (See [21, Sec. 7] and [12, Lem. 2]) *There exists a continuous projection* $\mathsf{Z}^\Gamma : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathsf{R}}^{1/2}(\Gamma)$ *with*

$$\ker(\mathsf{Z}^\Gamma) = \ker(\mathrm{div}_\Gamma) \cap \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \tag{3.6.1}$$

and satisfying

$$\mathrm{div}_\Gamma(\mathsf{Z}^\Gamma(\mathbf{v})) = \mathrm{div}_\Gamma(\mathbf{v}). \tag{3.6.2}$$

The closed subspaces $\mathbf{X}(\mathrm{div}_\Gamma, \Gamma) := \mathsf{Z}^\Gamma(\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))$ and $\mathbf{N}(\mathrm{div}_\Gamma, \Gamma) := \ker(\mathrm{div}_\Gamma) \cap \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ provide a stable direct regular decomposition

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \oplus \mathbf{N}(\mathrm{div}_\Gamma, \Gamma). \tag{3.6.3}$$

Hence, it follows from (3.6.2) that

$$\mathbf{v} \mapsto \|\mathrm{div}_\Gamma(\mathbf{v})\|_{-1/2} + \|(\mathrm{Id} - \mathsf{Z}^\Gamma)\,\mathbf{v}\|_{-1/2} \tag{3.6.4}$$

also defines an equivalent norm in $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

Note that since, by Rellich's embedding theorem, $\mathbf{H}^{1/2}_{\mathsf{R}}(\Gamma)$ compactly embeds in the space $\mathbf{L}^2_t(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2(\Gamma) \,|\, \mathbf{u} \cdot \mathbf{n} \equiv 0\}$ of square-integrable tangential vector-fields, this is also the case for $\mathbf{X}(\mathrm{div}_\Gamma, \Gamma)$.

From Lemma 3.2, $\mathrm{div}_\Gamma : \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \to H^{-1/2}_*(\Gamma)$ is a continuous bijection, thus the bounded inverse theorem guarantees the existence of a continuous inverse $(\mathrm{div}_\Gamma)^\dagger : H^{-1/2}_*(\Gamma) \to \mathbf{X}(\mathrm{div}_\Gamma, \Gamma)$ such that

$$(\mathrm{div}_\Gamma)^\dagger \circ \mathrm{div}_\Gamma = \mathrm{Id}\Big|_{\mathbf{X}(\mathrm{div}_\Gamma, \Gamma)}, \qquad \mathrm{div}_\Gamma \circ (\mathrm{div}_\Gamma)^\dagger = \mathrm{Id}\Big|_{H^{-1/2}_*(\Gamma)}.$$

The existence of an operator $\mathbf{curl}^\dagger_\Gamma : \mathbf{N}(\mathrm{div}_\Gamma, \Gamma) \to H^{1/2}_*(\Gamma)$ satisfying $\mathbf{curl}^\dagger_\Gamma \circ \mathbf{curl}_\Gamma = \mathrm{Id}$ and $\mathbf{curl}_\Gamma \circ \mathbf{curl}^\dagger_\Gamma =$ '$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-orthogonal projection onto (surface) divergence-free vector-fields' also follows by Lemma 3.2.

In the following, we will denote by Q_* both the projection $H^{1/2}(\Gamma) \to H^{1/2}_*(\Gamma)$ onto mean zero functions and the projection $H^{-1/2}(\Gamma) \to H^{-1/2}_*(\Gamma)$ onto the space of annihilators of the characteristic function.

Lemma 3.14 *The bounded linear operator*

$$\Xi : H^{-1/2}_*(\Gamma) \times \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{1/2}_*(\Gamma) \to H^{-1/2}_*(\Gamma) \times \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{1/2}_*(\Gamma)$$

defined by

$$\Xi \begin{pmatrix} \mathsf{a}_0 \\ \mathbf{a}_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\mathrm{div}_\Gamma\, \mathbf{a}_1 \\ -(\mathrm{div}_\Gamma)^\dagger(Q_*\mathsf{a}_0) + \mathbf{curl}_\Gamma(Q_* a_2) \\ (\mathbf{curl}_\Gamma)^\dagger\left((\mathrm{Id} - \mathsf{Z}^\Gamma)\mathbf{a}_1\right) \end{pmatrix}$$

is a continuous involution. In particular, Ξ is an isomorphism of Banach spaces.

Proof. We directly evaluate

$$\begin{aligned}
\Xi^2 \begin{pmatrix} \mathsf{a}_0 \\ \mathbf{a}_1 \\ a_2 \end{pmatrix} &= \Xi \begin{pmatrix} -\mathrm{div}_\Gamma\, \mathbf{a}_1 \\ -(\mathrm{div}_\Gamma)^\dagger(Q_*\mathsf{a}_0) + \mathbf{curl}_\Gamma(Q_* a_2) \\ (\mathbf{curl}_\Gamma)^\dagger\left((\mathrm{Id} - \mathsf{Z}^\Gamma)\mathbf{a}_1\right) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{N} & \begin{matrix} \mathrm{div}_\Gamma((\mathrm{div}_\Gamma)^\dagger(Q_*\mathsf{a}_0)) - \mathrm{div}_\Gamma(\mathbf{curl}_\Gamma(Q_* a_2)) \\ (\mathrm{div}_\Gamma)^\dagger(Q_*(\mathrm{div}_\Gamma\,\mathbf{a}_1)) + \mathbf{curl}_\Gamma(Q_*(\mathbf{curl}_\Gamma)^\dagger((\mathrm{Id}-\mathsf{Z}^\Gamma)\mathbf{a}_1)) \\ -(\mathbf{curl}_\Gamma)^\dagger((\mathrm{Id}-\mathsf{Z}^\Gamma)((\mathrm{div}_\Gamma)^\dagger(Q_*\mathsf{a}_0))) + (\mathbf{curl}_\Gamma)^\dagger((\mathrm{Id}-\mathsf{Z}^\Gamma)(\mathbf{curl}_\Gamma(Q_* a_2))) \end{matrix} \end{pmatrix} \\
&= \begin{pmatrix} Q_*\mathsf{a}_0 \\ \mathsf{Z}^\Gamma\mathbf{a}_1 + (\mathrm{Id} - \mathsf{Z}^\Gamma)\mathbf{a}_1 \\ Q_* a_2 \end{pmatrix} = \begin{pmatrix} \mathsf{a}_0 \\ \mathbf{a}_1 \\ a_2 \end{pmatrix}.
\end{aligned}$$

□

Proposition 3.4 *There exists a constant $C > 0$ and a compact bilinear form $\mathsf{c} : \mathcal{H}_\mathsf{R} \times \mathcal{H}_\mathsf{R} \to \mathbb{R}$ such that*

$$\left|\langle\!\langle \Xi\,\vec{\mathbf{a}}, \gamma_\mathsf{T}\mathcal{L}_\mathsf{T}\left(\vec{\mathbf{a}}\right)\rangle\!\rangle_\times + \mathsf{c}(\vec{\mathbf{a}}, \vec{\mathbf{a}})\right| \geq C\|\vec{\mathbf{a}}\|^2_{\mathcal{H}_\mathsf{R}} \qquad \forall \vec{\mathbf{a}} \in \mathcal{H}_\mathsf{R}. \tag{3.6.5}$$

Proof. The operator $\mathbf{curl}_\Gamma : H^1_*(\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma)$ is a continuous injection with closed range, it is thus bounded below. Since the mean operator has finite rank, it is compact. Moreover, $(\mathrm{div}_\Gamma)^\dagger(H^{-1/2}_*(\Gamma)) \subset \mathbf{H}^{1/2}_{\mathsf{R}}(\Gamma)$ is compactly embedded in $\mathbf{L}^2_t(\Gamma)$. Hence, the proof ultimately follows from

$$\begin{aligned}\langle\!\langle \Xi\, \overrightarrow{\mathbf{a}}, \gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}}\left(\overrightarrow{\mathbf{a}}\right)\rangle\!\rangle_\times \hat{=} &\,(\mathrm{div}_\Gamma\, \mathbf{a}_1, \mathrm{div}_\Gamma\, \mathbf{a}_1)_{-1/2} + (a_2, Q_* a_2)_{-1/2} \\ &+ ((\mathrm{div}_\Gamma)^\dagger Q_* a_2, \mathbf{curl}_\Gamma\, \mathsf{a}_0)_{-1/2} + (\mathbf{curl}_\Gamma\, Q_* \mathsf{a}_0, \mathbf{curl}_\Gamma\, \mathsf{a}_0)_{-1/2} \\ &+ (\mathbf{a}_1, (\mathrm{Id} - \mathsf{Z}^\Gamma)\mathbf{a}_1)_{-1/2}\end{aligned}$$

and the opening observations of this section. □

Since $\mathrm{curl}_\Gamma(\mathbf{d}) = \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{d})$ for all $\mathbf{d} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, tinkering with the signs and introducing rotations in the definition of Ξ easily leads to an analogous generalized Gårding inequality for $\gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}}$.

Corollary 3.2 *The boundary integral operators* $\gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}} : \mathcal{H}_{\mathsf{R}} \to \mathcal{H}_{\mathsf{T}}$ *and* $\gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}} : \mathcal{H}_{\mathsf{T}} \to \mathcal{H}_{\mathsf{R}}$ *are Fredholm of index* 0*.*

3.7 Kernels

We conclude from Corollary 3.2 that the nullspaces of $\gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}}$ and $\gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}}$ are finite dimensional. In this section, we proceed similarly as in [14, Sec. 7.1] and [15, Sec. 3] to characterize them explicitly.

Suppose that $\overrightarrow{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$ is such that $\gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathbf{a}}) = 0$.

- Since $\mathrm{div}_\Gamma\, \mathbf{a}_1 \in H^{-1/2}(\Gamma)$, we can test the bilinear form of Equation (3.5.12) with $\mathsf{c}_0 = \mathrm{div}_\Gamma\, \mathbf{a}_1$, $\mathbf{c}_1 = 0$ and $c_2 = 0$ to find that $\mathrm{div}_\Gamma\, \mathbf{a}_1 = 0$.
- Testing with $\mathsf{c}_0 = 0$ and $\mathbf{c}_1 = 0$ shows that $(\mathbf{a}_1, \mathbf{curl}_\Gamma\, v)_{-1/2} = 0\ \forall\, v \in H^{1/2}(\Gamma)$.
- Because $\mathrm{div}_\Gamma \circ \mathbf{curl}_\Gamma = 0$, we can choose $c_2 = 0$, $\mathsf{c}_0 = 0$ and $\mathbf{c}_1 = \mathbf{curl}_\Gamma a_2$ to conclude that $\mathbf{curl}_\Gamma a_2 = 0$.
- We are left with $(\mathsf{a}_0, \mathrm{div}_\Gamma \mathbf{v})_{-1/2} = 0\ \forall\, \mathbf{v} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

In $H^{1/2}(\Gamma)$, $\ker(\mathbf{curl}_\Gamma) = \ker(\nabla_\Gamma)$ is the space of functions $\mathsf{c}(\Gamma)$ that are constant over connected components of Γ. Defining $\mathbf{\Psi}_t := \gamma_t \mathbf{\Psi}$, we have found that

$$\ker(\gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}}) = \left\{ \overrightarrow{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}} \;\middle|\; a_0 \in \mathsf{c}(\Gamma),\ \mathrm{curl}_\Gamma \mathbf{\Psi}_t(\mathbf{a}_1) = 0,\ \mathrm{div}_\Gamma \mathbf{a}_1 = 0,\ \nabla_\Gamma \psi(a_0') = 0 \right\}. \quad (3.7.1)$$

Now, suppose that $\overrightarrow{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}}$ is such that $\gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathbf{b}}) = 0$.

- As $\mathrm{curl}_\Gamma(\mathbf{b}_1) \in H^{-1/2}(\Gamma)$, we may test Equation (3.5.13) with $\mathsf{d}_2 = \mathrm{curl}_\Gamma\, \mathbf{b}_1$, $\mathbf{d}_1 = 0$ and $d_0 = 0$ to find that $\mathrm{curl}_\Gamma\, \mathbf{b}_1 = 0$.

- Testing with $\mathsf{d}_2 = 0$ and $\mathbf{d}_1 = 0$, we find that $(\mathbf{n} \times \mathbf{b}_1, \mathbf{curl}_\Gamma\, v)_{-1/2} = 0$ for all $v \in H^{1/2}(\Gamma)$.
- Since $\operatorname{curl}_\Gamma \circ \nabla_\Gamma = 0$, we can choose $d_0 = 0$, $\mathsf{d}_2 = 0$ and $\mathbf{d}_1 = \nabla_\Gamma b_0$ to conclude that $\mathbf{curl}_\Gamma\, b_0 = 0$.
- Finally, it follows that $(\mathsf{b}_2, \operatorname{curl}_\Gamma \mathbf{v})_{-1/2} = 0$ for all $\mathbf{v} \in \mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$.

Notice that since $\nabla_\Gamma(v)$ is tangential for all $v \in H^{1/2}(\Gamma)$,

$$(\mathbf{n} \times \mathbf{b}_1, \mathbf{curl}_\Gamma\, v)_{-1/2} = (\mathbf{n} \times \mathbf{b}_1, \nabla_\Gamma v \times \mathbf{n})_{-1/2} = \langle \mathbf{n} \times \boldsymbol{\Psi}(\mathbf{n} \times \mathbf{b}_1), \nabla_\Gamma v \rangle$$

for all $v \in H^{1/2}(\Gamma)$. Therefore, we let $\boldsymbol{\Psi}_\tau(\cdot) := -\gamma_\tau \boldsymbol{\Psi}(\mathbf{n} \times \cdot)$ and conclude that

$$\ker(\gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}}) = \left\{ \overrightarrow{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}} \,\middle|\, b_0 \in \mathsf{c}(\Gamma), \operatorname{curl}_\Gamma \mathbf{b}_1 = 0, \operatorname{div}_\Gamma \boldsymbol{\Psi}_\tau(\mathbf{b}_1) = 0, \mathbf{curl}_\Gamma \psi(b_0') = 0 \right\}. \quad (3.7.2)$$

Equation (3.7.1) and Equation (3.7.2) together with the mapping properties of the scalar and vector single layer potentials allow us to determine as in [14, Sec. 7.2] and [15, Lem. 2, Lem. 6] that the dimension of these nullspaces relate to the Betti numbers of Γ.

Proposition 3.5 *The dimensions of* $\ker(\gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}})$ *and* $\ker(\gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}})$ *are finite and equal to the sum of the Betti numbers* $\beta_0(\Gamma) + \beta_1(\Gamma) + \beta_2(\Gamma)$.

Remark 3.9 The zeroth Betti number $\beta_0(\Gamma)$ indicates the number of connected components of Γ. The first Betti number $\beta_1(\Gamma)$ amounts to the number of equivalence classes of non-bounding cycles in Γ. For the second Betti number, it holds that $\beta_2(\Gamma) = \beta_2(\Omega^+) + \beta_2(\Omega^-)$, which sums the number of holes in Ω^+ and Ω^-, respectively.

3.8 Surface Dirac operators

In this section, we reveal the geometric structure behind the formulas of the bilinear forms b_{R} and b_{T} established in Section 3.5. They turn out to be associated with the 2D surface Dirac operators induced by the chain and cochain Hilbert complexes

$$H^{-1/2}(\Gamma) \xrightarrow{\nabla_\Gamma} \mathbf{H}_{\mathsf{T}}^{-1/2}(\Gamma) \xrightarrow{\operatorname{curl}_\Gamma} H^{-1/2}(\Gamma) \quad (3.8.1)$$

and

$$H^{-1/2}(\Gamma) \xleftarrow[-\operatorname{div}_\Gamma]{} \mathbf{H}_{\mathsf{R}}^{-1/2}(\Gamma) \xleftarrow[\mathbf{curl}_\Gamma]{} H^{-1/2}(\Gamma), \quad (3.8.2)$$

equipped with the non-local inner products (3.5.14a), (3.5.14b) and (3.5.14c). Their associated domain complexes

$$H^{1/2}(\Gamma) \xrightarrow{\nabla_\Gamma} \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \xrightarrow{\mathrm{curl}_\Gamma} H^{-1/2}(\Gamma) \tag{3.8.3}$$

and

$$H^{-1/2}(\Gamma) \xleftarrow[-\mathrm{div}_\Gamma]{} \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \xleftarrow[\mathbf{curl}_\Gamma]{} H^{1/2}(\Gamma), \tag{3.8.4}$$

are equipped with the natural graph inner products.

Remark 3.10 Notice that (3.8.3) and (3.8.4) are dual to each other with respect to the duality pairing on the boundary introduced in Section 3.2.

The Hilbert space adjoint $\mathbf{d}_\Gamma^*$ and $\boldsymbol{\delta}_\Gamma^*$ of the nilpotent operators

$$\mathbf{d}_\Gamma : \mathcal{H}_\mathsf{T} \to \mathcal{H}_\mathsf{T}, \tag{3.8.5}$$
$$\boldsymbol{\delta}_\Gamma : \mathcal{H}_\mathsf{R} \to \mathcal{H}_\mathsf{R}, \tag{3.8.6}$$

represented by the block operator matrices

$$\mathbf{d}_\Gamma := \begin{pmatrix} 0 & \mathbf{0}^\top & 0 \\ \nabla_\Gamma & \mathbf{0}_{3\times 3} & \mathbf{0} \\ 0 & \mathrm{curl}_\Gamma & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\delta}_\Gamma := \begin{pmatrix} 0 & -\mathrm{div}_\Gamma & 0 \\ \mathbf{0} & \mathbf{0}_{3\times 3} & \mathbf{curl}_\Gamma \\ 0 & \mathbf{0}^\top & 0 \end{pmatrix}$$

are *non-local* operators.

In terms of variational formulations, the bilinear forms associated with the surface Dirac operators

$$\mathsf{D}_\mathsf{R}^\Gamma := \mathbf{d}_\Gamma + \mathbf{d}_\Gamma^* \tag{3.8.7}$$
$$\mathsf{D}_\mathsf{T}^\Gamma := \boldsymbol{\delta}_\Gamma + \boldsymbol{\delta}_\Gamma^* \tag{3.8.8}$$

are precisely b_R and b_T defined in (3.5.17) and (3.5.16), previously associated to the boundary integral operators $\gamma_\mathsf{R}\mathcal{L}_\mathsf{R}$ and $\gamma_\mathsf{T}\mathcal{L}_\mathsf{T}$:

$$\begin{aligned}(\mathsf{D}_\mathsf{R}^\Gamma \vec{\mathbf{b}}, \vec{\mathbf{d}})_{\mathcal{H}_\mathsf{T}} &= (\mathbf{d}_\Gamma \vec{\mathbf{b}}, \vec{\mathbf{d}})_{\mathcal{H}_\mathsf{T}} + (\vec{\mathbf{b}}, \mathbf{d}_\Gamma \vec{\mathbf{d}})_{\mathcal{H}_\mathsf{T}} \\ &= (\nabla_\Gamma b_0, \mathbf{d}_1)_{-1/2,\mathsf{R}} + (\mathrm{curl}_\Gamma \mathbf{b}_1, \mathsf{d}_2)_{-1/2} \\ &\qquad + (\mathbf{b}_1, \nabla_\Gamma d_0)_{-1/2,\mathsf{R}} + (\mathsf{b}_2, \mathrm{curl}_\Gamma \mathbf{d}_1)_{-1/2} \\ &= \mathsf{b}_\mathsf{R}(\vec{\mathbf{b}}, \vec{\mathbf{d}}),\end{aligned} \tag{3.8.9}$$

and similarly

$$\begin{aligned}(\mathsf{D}_\mathsf{T}^\Gamma \vec{\mathbf{a}}, \vec{\mathbf{c}})_{\mathcal{H}_\mathsf{R}} &= (\boldsymbol{\delta}_\Gamma \vec{\mathbf{a}}, \vec{\mathbf{c}})_{\mathcal{H}_\mathsf{R}} + (\vec{\mathbf{a}}, \boldsymbol{\delta}_\Gamma \vec{\mathbf{c}})_{\mathcal{H}_\mathsf{R}} \\ &= (-\mathrm{div}_\Gamma \mathbf{a}_1, \mathsf{c}_0)_{-1/2} + (\mathsf{a}_0, -\mathrm{div}_\Gamma \mathbf{c}_1)_{-1/2} \\ &\qquad + (\mathbf{curl}_\Gamma a_2, \mathbf{c}_1)_{-1/2,\mathsf{T}} + (\mathbf{a}_1, \mathbf{curl}_\Gamma c_2)_{-1/2,\mathsf{T}} \\ &= \mathsf{b}_\mathsf{T}(\vec{\mathbf{a}}, \vec{\mathbf{c}}).\end{aligned} \tag{3.8.10}$$

First-kind boundary integral operators spawned by the (volume) Dirac operators in 3D Euclidean space thus coincide with (surface) Dirac operators on 2D boundaries: boundary value problems related to $\mathsf{D}_{\mathsf{R}}^{\Omega} = \mathbf{d} + \mathbf{d}^*$ in Ω can be formulated as problems for $\mathsf{D}_{\mathsf{R}}^{\Gamma} = \mathbf{d}_{\Gamma} + \mathbf{d}_{\Gamma}^*$ in Γ, and similarly problems for $\mathsf{D}_{\mathsf{T}}^{\Omega} = \boldsymbol{\delta} + \boldsymbol{\delta}^*$ in Ω correspond to problems for $\mathsf{D}_{\mathsf{T}}^{\Gamma} = \boldsymbol{\delta}_{\Gamma} + \boldsymbol{\delta}_{\Gamma}^*$ in Γ.

This explains why the dimension of the nullspaces of first-kind boundary integral operators is the sum of the dimensions of the standard spaces of surface harmonic scalar and vector fields.

3.9 Solvability

Thanks to the duality between the trace spaces, (BT) and (BR) can be reformulated into the variational problems:

$$\vec{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}} : \qquad \mathsf{b}_{\mathsf{T}}(\vec{\mathbf{a}}, \vec{\mathbf{c}}) = \ell_{\mathsf{T}}(\vec{\mathbf{c}}), \qquad \forall\, \vec{\mathbf{c}} \in \mathcal{H}_{\mathsf{R}}, \tag{BVT}$$

and

$$\vec{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}} : \qquad \mathsf{b}_{\mathsf{R}}(\vec{\mathbf{b}}, \vec{\mathbf{d}}) = \ell_{\mathsf{R}}(\vec{\mathbf{d}}), \qquad \forall\, \vec{\mathbf{d}} \in \mathcal{H}_{\mathsf{T}}, \tag{BVR}$$

with right-hand side functionals

$$\ell_{\mathsf{T}}(\vec{\mathbf{c}}) = \langle\!\langle \frac{1}{2}\vec{\mathbf{b}} - \{\gamma_{\mathsf{T}}\}\mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}}), \vec{\mathbf{c}} \rangle\!\rangle_{\Gamma} \tag{3.9.1}$$

and

$$\ell_{\mathsf{R}}(\vec{\mathbf{d}}) = \langle\!\langle \frac{1}{2}\vec{\mathbf{a}} - \{\gamma_{\mathsf{R}}\}\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}), \vec{\mathbf{d}} \rangle\!\rangle_{\Gamma}. \tag{3.9.2}$$

As explained in Remark 3.7, it is sufficient when it comes to well-posedness to restrict our considerations to only one of the two boundary integral equations stated in Section 3.5. The following result makes explicit the condition under which a solution to (BVR) exists.

Proposition 3.6 *If the boundary data $\vec{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$ satisfies the compatibility condition* (CCR)*, then the right-hand side $\ell \in \mathcal{H}'_{\mathsf{T}}$ of* (BVR) *is consistent in the sense that*

$$\ell_{\mathsf{R}}(\vec{\mathbf{d}}) = 0, \qquad \forall\, \vec{\mathbf{d}} \in \ker \mathcal{B}_{\mathsf{T}}. \tag{3.9.3}$$

Proof. Following the strategy found in the proofs of [15, Lem. 4] and [15, Lem. 8], we use (3.4.39) to directly evaluate

$$\begin{aligned}
\ell_{\mathsf{R}}(\vec{\mathbf{d}}) &= \langle\!\langle \frac{1}{2}\vec{\mathbf{a}} - \{\gamma_{\mathsf{R}}\}\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}), \vec{\mathbf{d}} \rangle\!\rangle_{\Gamma} \\
&= \langle\!\langle \frac{1}{2}\vec{\mathbf{a}}, \vec{\mathbf{d}} \rangle\!\rangle_{\Gamma} - \langle \{\gamma_n\}\, \mathbf{curl}\Upsilon(a_2), d_0 \rangle_{\Gamma} + \langle \mathsf{K}'(a_0), d_0 \rangle_{\Gamma} \\
&\qquad + \langle \mathbf{a}_1, \mathsf{C}(\mathbf{b}_1) \rangle_{\Gamma} - \langle \mathsf{K}(a_2), \mathsf{b}_2 \rangle_{\Gamma} \\
&= \langle (\frac{1}{2}\mathrm{Id} - \mathsf{K}')a_0, d_0 \rangle_{\Gamma} + \langle \{\gamma_n\}\, \mathbf{curl}\Upsilon(a_2), d_0 \rangle_{\Gamma}
\end{aligned}$$

$$+\langle \mathbf{a}_1, (\frac{1}{2}\mathrm{Id}+\mathsf{C})\mathbf{b}_1\rangle_\Gamma + \langle(\frac{1}{2}\mathrm{Id}+\mathsf{K})a_2, \mathsf{b}_2\rangle_\Gamma,$$

where we recognize the "Maxwell double layer boundary integral operator" C, and the double layer boundary integral operator K for the Laplacian.

Locally constant functions are trivially harmonic. They can thus be written using the classical representation formula for the scalar Laplacian in which the Neumann trace vanishes to yield $d_0 = \gamma\,(\frac{1}{2}\mathrm{Id}-\mathsf{K})d_0$. Since K is dual to K', the first term on the right-hand side vanishes because of the compatibility condition (CCR).

The second term also evaluates to zero. On the one hand, $\ker \mathbf{curl}_\Gamma = \ker \nabla_\Gamma$. On the other hand, $\gamma_n\,\mathbf{curl} = \mathrm{curl}_\Gamma\,\gamma_t$ in $\mathbf{H}(\mathbf{curl}, \Omega)$, and curl_Γ is dual to $\mathbf{curl}_\Gamma$.

The third and fourth terms are shown to vanish in [15, Lem.4] and [15, Lem.3] with similar arguments. □

In the framwork of Section 3.8, a standard result is the Poincaré inequality: $\exists C > 0$, only depending on Γ, such that [1,26]

$$\|\overrightarrow{\mathbf{b}}\|_{\mathcal{H}_\mathsf{R}} \leq C\|\mathbf{d}_\Gamma \overrightarrow{\mathbf{b}}\|_{\mathcal{H}_\mathsf{R}}, \qquad \forall\,\overrightarrow{\mathbf{b}} \in \mathfrak{K}, \tag{3.9.4}$$

where $\mathfrak{K} := (\ker \mathbf{d}_\Gamma)^\perp \cap \mathrm{dom}(\mathbf{d}_\Gamma)$ and orthogonality is taken in the non-local inner products introduced in Section 3.5. From the complex (3.8.3),

$$\mathrm{dom}(\mathbf{d}_\Gamma) = H^{1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma), \tag{3.9.5}$$

and thus

$$\mathfrak{K} = \mathfrak{K}_0 \times \mathfrak{K}_1 \times \mathfrak{K}_2 \in \mathcal{H}_\mathsf{T} \tag{3.9.6}$$

with

$$\mathfrak{K}_0 := \ker \nabla_\Gamma, \qquad \mathfrak{K}_1 := \ker \mathrm{curl}_\Gamma \cap \left(\nabla_\Gamma H^{\frac{1}{2}}(\Gamma)\right)^\perp, \qquad \mathfrak{K}_2 := \left(\mathrm{curl}_\Gamma\,\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)\right)^\perp.$$

It is routine to verify from (3.7.2) that $\mathfrak{K} = \ker \mathsf{b}_\mathsf{R}$. Hence, due to the inf-sup inequality supplied in [26, Thm. 2.4], the problem of finding $\overrightarrow{\mathbf{b}} \in \mathcal{H}_\mathsf{T}$ and $\overrightarrow{\mathbf{p}} \in \mathfrak{K}$ such that

$$\begin{aligned} \mathsf{b}_\mathsf{R}\left(\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{d}}\right) + \langle\!\langle \overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{d}} \rangle\!\rangle_\Gamma = \ell_\mathsf{R}(\overrightarrow{\mathbf{d}}) \qquad & \forall\,\overrightarrow{\mathbf{d}} \in \mathcal{H}_\mathsf{T}, \\ \langle\!\langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{g}} \rangle\!\rangle_\Gamma = 0 \qquad & \forall\,\overrightarrow{\mathbf{g}} \in \ker \mathsf{b}_\mathsf{R} \end{aligned} \tag{MBVR}$$

is well-posed.

Similarly, the problem of solving

$$\begin{aligned} \mathsf{b}_\mathsf{T}\left(\overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{c}}\right) + \langle\!\langle \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{c}} \rangle\!\rangle_\Gamma = \ell_\mathsf{T}(\overrightarrow{\mathbf{c}}), \qquad & \forall\,\overrightarrow{\mathbf{c}} \in \mathcal{H}_\mathsf{T}, \\ \langle\!\langle \overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{g}} \rangle\!\rangle_\Gamma = 0, \qquad & \forall\,\overrightarrow{\mathbf{g}} \in \ker \mathsf{b}_\mathsf{T} \end{aligned} \tag{MBVT}$$

for the unknown pair $(\overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{q}}) \in \mathcal{H}_\mathsf{R} \times \ker \mathsf{b}_\mathsf{T}$ is well-posed.

Theorem 3.1 *The mixed variational problems* (MBVR) *has a unique solution* $\overrightarrow{\mathrm{b}} \in \mathcal{H}_{\mathsf{T}}$ *such that* $\overrightarrow{\mathrm{b}} \perp \ker \mathsf{b}_{\mathsf{R}}$. *Moreover,*

$$\|\overrightarrow{\mathrm{b}}\|_{-1/2} + \|\overrightarrow{\mathrm{p}}\|_{-1/2} \lesssim \|\frac{1}{2}\overrightarrow{\mathrm{a}} - \{\gamma_{\mathsf{R}}\}\mathcal{L}_{\mathsf{T}}(\overrightarrow{\mathrm{a}})\|_{\mathcal{H}_{\mathsf{R}}}, \tag{3.9.7}$$

where the constant depends only on the constant in the Poincaré inequality (3.9.4). *If* $\overrightarrow{\mathrm{a}}$ *satisfies* (CCR)*, then this result extends to the variational problem* (BVR) *and* (3.9.7) *holds with* $\overrightarrow{\mathrm{p}} = 0$.

Similarly, the mixed variational problems (MBVT) *has a unique solution* $\overrightarrow{\mathrm{a}} \in \mathcal{H}_{\mathsf{R}}$ *such that* $\overrightarrow{\mathrm{a}} \perp \ker \mathsf{b}_{\mathsf{T}}$. *Moreover,*

$$\|\overrightarrow{\mathrm{a}}\|_{-1/2} + \|\overrightarrow{\mathrm{q}}\|_{-1/2} \lesssim \|\frac{1}{2}\overrightarrow{\mathrm{b}} - \{\gamma_{\mathsf{T}}\}\mathcal{L}_{\mathsf{R}}(\overrightarrow{\mathrm{b}})\|_{\mathcal{H}_{\mathsf{T}}}, \tag{3.9.8}$$

where the constant depends only on the constant in the Poincaré inequality for $\boldsymbol{\delta}_{\Gamma}$. *If* $\overrightarrow{\mathrm{b}}$ *satisfies* (CCT)*, then this result extends to the variational problem* (BVT) *and* (3.9.8) *holds with* $\overrightarrow{\mathrm{q}} = 0$.

3.10 Conclusion

First-kind boundary integral equations are appealing to the numerical analysis community because they lead to variational problems posed in natural “energy" trace spaces that are generally well-suited for Galerkin discretization. Therefore, on the one hand, the new equations pave the way for development of new Galerkin boundary element methods. On the other hand, our results simultaneously open a new perspective towards the recent developments in boundary integral equations for Hodge-Laplace problems. As it stands, the rich theories of Hilbert complexes and nilpotent operators not only support our observations with the help of already established abstract inf-sup conditions, but in fact also supply the framework and analysis tools needed to relate the studied non-standard surface Dirac operators to the mixed variational formulations associated with the first-kind boundary integral operators for the Hodge-Laplacian. In fact, this insight already led us to observe that the variational formulation [15, Eq. 25] is associated with the Laplace-Beltrami of the Hilbert complex (3.8.1). We note that [15, Eq. 34] also appears to be related to higher-order differential forms on surfaces. The significant observation that our integral operators arise as “non-standard” surface Dirac operators associated to trace Hilbert complexes suggests a new analysis of Hodge-Dirac and Hodge-Laplace related first-kind boundary integral equations which has yet to be explored.

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Part II
Compact Manifolds and Abstract Hilbert Complexes

The manuscripts in Part II are publicly available online. They are submitted for publication and awaiting peer-review.

CHAPTER 4

R. Hiptmair, D. Pauly and E. Schulz. *Traces for Hilbert complexes*, Seminar for Applied Mathematics, Report no. 2022-07, (2022), https://www.sam.math.ethz.ch/sam_reports/reports_final/reports2022/2022-07.pdf.

CHAPTER 5

E. Schulz, R. Hiptmair and S. Kurz. *Boundary Integral Exterior Calculus*, Seminar for Applied Mathematics, Report no. 2022-36, (2022), https://www.sam.math.ethz.ch/sam_reports/reports_final/reports2022/2022-36.pdf.

Chapter 4
Traces for Hilbert Complexes

Erick Schulz, Dirk Pauly and Ralf Hiptmair

Abstract We study a new notion of trace operators and trace spaces for abstract Hilbert complexes. We introduce trace spaces as quotient spaces/annihilators. We characterize the kernels and images of the related trace operators and discuss duality relationships between trace spaces. We elaborate that many properties of the classical boundary traces associated with the Euclidean de Rham complex on bounded Lipschitz domains are rooted in the general structure of Hilbert complexes. We arrive at abstract trace Hilbert complexes that can be formulated using quotient spaces/annihilators. We show that, if a Hilbert complex admits stable "regular decompositions" with compact lifting operators, then the associated trace Hilbert complex is Fredholm. Incarnations of abstract concepts and results in the concrete case of the de Rham complex in three-dimensional Euclidean space will be discussed throughout.

4.1 Introduction

4.1.1 Starting point: the de Rham complex

In vector-analytic notation, the L^2 de Rham complex in a bounded domain $\Omega \subset \mathbb{R}^3$ reads[1]

$$\mathbb{R} \xhookrightarrow{\imath_{\mathbb{R}}} L^2(\Omega) \xrightarrow{\mathbf{grad}} L^2(\Omega) \xrightarrow{\mathbf{curl}} L^2(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{\pi_{\{0\}}} \{0\}. \tag{4.1.1}$$

It involves unbounded first-order differential operators inducing the domain Hilbert complex

$$\mathbb{R} \xhookrightarrow{\imath_{\mathbb{R}}} H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl},\Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{\pi_{\{0\}}} \{0\}, \tag{4.1.2}$$

where customary notation for Sobolev spaces equipped with graph inner products was adopted[2]. Taking the closure of compactly supported functions in these Sobolev spaces and tagging the resulting closed subspaces with '$\circ$' on top, we obtain a subcomplex

[1] Throughout, we use special arrows to indicate properties of mappings: '$\twoheadrightarrow$' for surjectivity, '$\hookrightarrow$' for injectivity and '$\dashrightarrow$' for isometry.

[2] For instance, the spaces $H^1(\Omega)$, $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{H}(\mathrm{div},\Omega)$ are discussed in [22]. They are equipped with the obvious graph norms making the operators involved in the domain Hilbert complex trivially bounded. In the Euclidean setting, we distinguish vector quantities from scalars by using a bold font.

$$\{0\} \xhookrightarrow{\imath} \mathring{H}^1(\Omega) \xrightarrow{\mathbf{grad}} \mathring{\mathbf{H}}(\mathbf{curl},\Omega) \xrightarrow{\mathbf{curl}} \mathring{\mathbf{H}}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2_*(\Omega) \xrightarrow{0} \{0\}, \tag{4.1.3}$$

giving rise to the following structure:

$$\begin{array}{ccccccc} H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \mathbf{H}(\mathbf{curl},\Omega) & \xrightarrow{\mathbf{curl}} & \mathbf{H}(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \cup & & \cup & & \cup & & \cup \\ \mathring{H}^1(\Omega) & \xrightarrow{\mathbf{grad}} & \mathring{\mathbf{H}}(\mathbf{curl},\Omega) & \xrightarrow{\mathbf{curl}} & \mathring{\mathbf{H}}(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega). \end{array} \tag{4.1.4}$$

4.1.2 The de Rham complex and trace operators

The focus of this work is on trace operators. For the de Rham complex above, those are usually introduced as linear mappings of functions in Ω to functions on $\Gamma = \partial\Omega$. The classical traces are obtained by extending the restriction operators[3]

$$\gamma u := u\big|_\Gamma \qquad \text{(pointwise trace)}, \tag{4.1.5a}$$
$$\gamma_t \mathbf{u} := \mathbf{n} \times (\mathbf{u}\big|_\Gamma \times \mathbf{n}) \qquad \text{(pointwise tangential component trace)}, \tag{4.1.5b}$$
$$\gamma_n \mathbf{u} := \mathbf{u}\big|_\Gamma \cdot \mathbf{n} \qquad \text{(pointwise normal component trace)}, \tag{4.1.5c}$$

to continuous and surjective mappings

$$\gamma : H^1(\Omega) \to H^{1/2}(\Gamma) \qquad \text{[26, Thm. 4.2.1]}, \tag{4.1.6a}$$
$$\gamma_t : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \qquad \text{[15, Thm. 4.1]}, \tag{4.1.6b}$$
$$\gamma_n : \mathbf{H}(\mathrm{div},\Omega) \to H^{-1/2}(\Gamma) \qquad \text{[22, Thm. 2.5, Cor. 2.8]}. \tag{4.1.6c}$$

from the Sobolev spaces involved in the domain de Rham complex to so-called trace spaces whose characterization is the main assertion of the standard trace theorems for a Lipschitz domain Ω.

The classical trace spaces can be defined based on the vector-valued rotated surface gradient $\mathbf{curl}_\Gamma$ and the scalar-valued surface rotation curl_Γ as

$$H^{1/2}(\Gamma) := \left\{ \boldsymbol{\phi} \in H^{-1/2}(\Gamma) \mid \mathbf{curl}_\Gamma\, \boldsymbol{\phi} \in \mathbf{H}_t^{-1/2}(\Gamma) \right\}, \tag{4.1.7a}$$
$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) := \left\{ \boldsymbol{\phi} \in \mathbf{H}_t^{-1/2}(\Gamma) \mid \mathrm{curl}_\Gamma\, \boldsymbol{\phi} \in H^{-1/2}(\Gamma) \right\}, \tag{4.1.7b}$$

where $\mathbf{H}_t^{-1/2}(\Gamma)$ is defined as the dual of the range of the tangential trace applied to $\mathbf{H}^1(\Omega)$. The mathematical theory of the pointwise trace γ is well established, cf. [29, Chap. 3]. That for the normal component trace γ_n is carefully developed in [22, Chap. 1]. Regarding the tangential trace γ_t in (4.1.6b) and the trace space (4.1.7b), we recommend the comprehensive and profound analysis of [15], based on the earlier works [1, 13, 14].

These important results were generalized to arbitrary dimensions by Weck in [43] using the framework of differential forms, where pullback by the boundary's inclusion map provides a

[3] We denote by $\mathbf{n} \in \mathbf{L}^\infty(\Gamma)$ the exterior unit normal vector-field on the boundary Γ.

unified description and generalization of the traces (4.1.6). A similar characterization of the range of the boundary restriction operator for Lipschitz subdomains of compact manifolds is given in [30], where a boundary de Rham complex involving surface operators is also studied.

One may wonder whether the structures shining through in (4.1.7a) and (4.1.7b) hint at a more general pattern governing the structure of trace spaces. Thus, in this article, we are going to elaborate this structure in the abstract framework of Hilbert complexes, of which the de Rham complex is the best-known representative. Since there is no notion of "boundary" in that abstract framework, we have to detach the concept of a trace space from the idea of a function space on a boundary. This can be accomplished by adopting a quotient-space view of traces.

Let us sketch this idea for the Euclidean de Rham complex. Since the kernels of the classical trace operators (4.1.6a)-(4.1.6c) are[4]

$$\mathcal{N}(\gamma) = \mathring{H}^1(\Omega) := \overline{C_0^\infty(\Omega)}^{H^1(\Omega)} \qquad [29, \text{Thm. } 3.40], \tag{4.1.8a}$$

$$\mathcal{N}(\gamma_t) = \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) := \overline{C_0^\infty(\Omega)^3}^{\mathbf{H}(\mathbf{curl},\Omega)} \qquad [31, \text{Thm. } 3.33], \tag{4.1.8b}$$

$$\mathcal{N}(\gamma_n) = \mathring{\mathbf{H}}(\mathrm{div}, \Omega) := \overline{C_0^\infty(\Omega)^3}^{\mathbf{H}(\mathrm{div},\Omega)} \qquad [31, \text{Thm. } 3.25], \tag{4.1.8c}$$

we immediately conclude that these trace operators induce isomorphisms between the classical trace spaces and the quotient spaces:

$$H^1(\Omega)/\mathring{H}^1(\Omega) \cong H^{1/2}(\Gamma), \tag{4.1.9a}$$

$$\mathbf{H}(\mathbf{curl}, \Omega)/\mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \cong \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \tag{4.1.9b}$$

$$\mathbf{H}(\mathrm{div}, \Omega)/\mathring{\mathbf{H}}(\mathrm{div}, \Omega) \cong H^{-1/2}(\Gamma). \tag{4.1.9c}$$

This paves the way for an alternative characterization of trace spaces independent of the notion of "function space on Γ". We remark that the quotient space approach to the definition of trace spaces has also proved successful for the de Rham complex in order to define traces on sets more complicated than boundaries of Lipschitz domains [17, 18].

Classical theory of trace spaces for $H^1(\Omega)$, $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$ also addresses duality between trace spaces:

- The $L^2(\Gamma)$ inner product induces a duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$; cf. [26, Chap. 4.2] and [29, Chap. 3].
- The skew-symmetric pairing[5]

$$\langle \mathbf{u}, \mathbf{v} \rangle_\times := \int_\Gamma (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v} \, \mathrm{d}\sigma \tag{4.1.10}$$

  can be extended from $\mathbf{L}^2(\Gamma) \times \mathbf{L}^2(\Gamma)$ to $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, allowing the identification of $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ with its own dual space; cf. [15, 16, 31].

The possibility to put trace spaces for the 3D de Rham complex into duality seems to follow general rules:

[4] We write $\mathcal{N}(\mathsf{T})$ and $\mathcal{R}(\mathsf{T})$ for the kernel/nullspace and range/image space, respectively, of a linear operator T.
[5] We denote by σ the surface measure on the boundary.

$$
\begin{array}{ccccc}
H^1(\Omega) & \xrightarrow{\ \mathbf{grad}\ } & \mathbf{H}(\mathbf{curl},\Omega) & \xrightarrow{\ \mathrm{curl}\ } & \mathbf{H}(\mathrm{div},\Omega) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma_t} & & \downarrow{\scriptstyle \gamma_n} \\
\mathbf{H}^{1/2}(\Gamma) & & \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \circlearrowleft \ L^2\text{-self duality} & & H^{-1/2}(\Gamma) \\
 & \underbrace{\hphantom{xxxxxxxxxxxxxxxx}}_{} & L^2\text{-duality} & \underbrace{\hphantom{xxxxxxxxxxxxxxxx}}_{} &
\end{array}
\tag{4.1.11}
$$

4.1.3 Goals, outline, and main results

There are obvious parallels in the definitions of the different trace spaces and their duality relations. One may wonder if this kind of resemblance between the trace spaces arise only for the de Rham complex or whether it is already manifest in a more basic/general setting, of which the de Rham complex is just a prominent specimen. That setting is the framework of *Hilbert complexes*[6], first introduced in [12]. Therefore, the guiding question behind this work is:

> *To what extent can results about traces for the de Rham domain complex be transferred to abstract Hilbert complexes?*

Of course, abstract Hilbert complexes know neither domains nor boundaries. Therefore, as already mentioned above, we cannot expect to arrive at a characterization of trace spaces as function spaces on a boundary. Yet, a theory based on the quotient space view of trace spaces is feasible. Its development will be pursued in Section 4.3. There, we first propose trace operators induced by "generalized integration by parts formulas" and mapping into dual spaces, and then generalize (4.1.9) to a quotient-space understanding of trace spaces.

Next, in Section 4.4, we shed light on duality relationships between trace spaces and find that the observation made in (4.1.11) is a generic pattern; see Theorem 4.3. This even holds in a setting simpler than Hilbert complexes. "Minimal Hilbert complexes" will only enter the stage in Section 4.5 in order to define so-called "surface operators", which are abstract counterparts of the classical surface differential operators such as $\mathbf{grad}_\Gamma$ and $\mathbf{curl}_\Gamma$. The full structure of Hilbert complexes is exploited starting from Section 4.6. Augmenting it by assumptions about the existence of so-called stable regular decompositions (Assumptions B and C), we obtain characterizations of traces spaces, in Theorem 4.6 and Theorem 4.7, which reveal that the definitions (4.1.7a) and (4.1.7b) of classical trace spaces reflect a more general pattern. This paves the way for the key insight expressed in Theorem 4.9 that trace spaces and surface operators are the building blocks of what we call a trace Hilbert complex, a full-fledged Hilbert complex of unbounded, densely defined, and closed operators.

Parallel to its development, we will apply our new abstract theory to the de Rham complex in three-dimensional Euclidean space. We hope that this will motivate some of the assumptions

[6] For the functional analytic foundations, we refer to parts of the FA-ToolBox from [35, Sec. 2], which is a compilation of useful functional analysis results that grew from its use in previous works, cf. [33, Sec. 4.1], [34, Sec. 2], [36, Sec. 2.1], [37, Sec. 2.1], [38, 2.2], [35, Sec. 2] and [32, App. 3]. We find the introduction in [6, Chap. 4] to be an accessible resource for readers unacquainted with Hilbert complexes, because it reviews in detail the material more concisely presented in [9, Sec. 3], cf. [7, Sec. 2] and [12].

made on the abstract spaces. The discussion will take the form of an ongoing specialization of the definitions and results, set apart from the main line of reasoning.

3D de Rham setting I: Traces and integration by parts

The key trace operators and trace spaces associated with the Euclidean de Rham complex in three space dimensions have already been introduced in (4.1.5) and (4.1.6). We just want to add the well-known fact that the trace operators (4.1.6a)-(4.1.6c) have a close link with Green's formulas

$$\langle \gamma u, \gamma_n \mathbf{v}\rangle_\Gamma = \int_\Omega \mathbf{grad}\, u \cdot v + u\,\mathrm{div}(\mathbf{v})\,\mathrm{d}\mathbf{x} \qquad \forall u \in H^1(\Omega), \mathbf{v} \in \mathbf{H}(\mathrm{div},\Omega), \tag{4.1.12a}$$

$$\langle \gamma_t \mathbf{u}, \gamma_t \mathbf{v}\rangle_\times = \int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\mathrm{d}\mathbf{x} \qquad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}(\mathbf{curl},\Omega). \tag{4.1.12b}$$

On the left, we denoted the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ by $\langle\cdot,\cdot\rangle_\Gamma$, but wrote $\langle\cdot,\cdot\rangle_\times$ for the skew-symmetric self-duality pairing on $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$, cf. [15, Lem. 5.6].

Finally, we stress that we could have demonstrated the specialization of our results also in the setting of general exterior calculus, but refrained from it in the interest of readability.

List of symbols

A_k	$\hat{=}$ closed densely defined unbounded operators	4.2.2, (4.2.5a)
A_k^*	$\hat{=}$ Hilbert space adjoint of A_k	4.2.2, (4.2.5b)
$\mathring{\mathsf{A}}_k$	$\hat{=}$ closed densely defined unbounded operator $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k$	4.2.3, (4.2.8a)
$\mathsf{A}_k^\top$	$\hat{=}$ Hilbert space adjoint of $\mathring{\mathsf{A}}_k$	4.2.3, (4.2.8b)
$\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}$	$\hat{=}$ Riesz isomorphism $\mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{D}(\mathsf{A}_k^\top)'$	4.3.3, (4.2)
T_k^t	$\hat{=}$ primal Hilbert trace $\mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathsf{A}_k^\top)'$	4.3.1, (4.3.3)
T_k^n	$\hat{=}$ dual Hilbert trace $\mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{D}(\mathsf{A}_k)'$	4.4.1, (4.4.2)
$\mathcal{T}(\mathsf{A}_k)$	$\hat{=}$ quotient space $\mathcal{D}(\mathsf{A}_k)/\mathcal{D}(\mathring{\mathsf{A}}_k)$	4.3.2, (4.3.23)
$\mathcal{T}(\mathsf{A}_k^\top)$	$\hat{=}$ quotient space $\mathcal{D}(\mathsf{A}_k^\top)/\mathcal{D}(\mathsf{A}_k^*)$	4.4.1, (4.4.8)
I_k^t	$\hat{=}$ isometric isomorphism $\mathcal{D}(\mathsf{A}_k) \to \mathcal{R}(\mathsf{T}_k^t)$	4.3.2, (4.3.38)
I_k^n	$\hat{=}$ isometric isomorphism $\mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{R}(\mathsf{T}_k^n)$	4.4.1, (4.4.19)
$\langle\langle\cdot,\cdot\rangle\rangle_k$	$\hat{=}$ duality pairing	4.4.2, (4.4.24b)
K_k	$\hat{=}$ isometric isomorphism induced by $\langle\cdot,\cdot\rangle_k$	4.4.2, (4.4.26)
P_k^t	$\hat{=}$ orthogonal projection $\mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}})^\perp$	4.3.1, (4.3.28)
P_k^n	$\hat{=}$ orthogonal projection $\mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{D}(\mathsf{A}_k^*)^\perp$	4.4.1, (4.4.12)
$\boldsymbol{\pi}_k^t$	$\hat{=}$ canonical quotient map $\mathcal{D}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_k)$	4.3.1, (4.3.28)
$\boldsymbol{\pi}_k^n$	$\hat{=}$ canonical quotient map $\mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{T}(\mathsf{A}_k^\top)$	4.3.1, (4.4.12)
$\mathbf{W}_k^+$	$\hat{=}$ dense inclusion $\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathsf{A}_k)$ and/or $\mathcal{D}(\mathsf{A}_{k-1}^\top)$	4.6.1, (4.6.1)

$\mathbf{W}_k^-$	$\hat{=}$ dual space $(\mathbf{W}_k^+)'$	4.6.1, (4.6.7)
$\mathring{\mathbf{W}}_k^{n,+}$	$\hat{=}$ intersection space $\mathcal{D}(\mathsf{A}_{k-1}^*) \cap \mathbf{W}_k^+ = \mathcal{N}(\mathsf{T}_{k-1}^n) \cap \mathbf{W}_k^+$	4.6.3, (4.6.32)
$\mathring{\mathbf{W}}_k^{t,+}$	$\hat{=}$ intersection space $\mathcal{D}(\mathring{\mathsf{A}}_k) \cap \mathbf{W}_k^+ = \mathcal{N}(\mathsf{T}_k^t) \cap \mathbf{W}_k^+$	4.6.3, (4.6.32)
$\mathbf{T}_k^{n,+}$	$\hat{=}$ quotient space $\mathbf{W}_k^+/\mathring{\mathbf{W}}^{n,+}$	4.6.4, (4.6.41b)
$\mathbf{T}_k^{t,+}$	$\hat{=}$ quotient space $\mathbf{W}_k^+/\mathring{\mathbf{W}}^{t,+}$	4.6.4, (4.6.41a)
$\mathbf{T}_k^{n,-}$	$\hat{=}$ dual space $(\mathbf{T}_k^{n,+})'$	4.6.4, (4.6.41b)
$\mathbf{T}_k^{t,-}$	$\hat{=}$ dual space $(\mathbf{T}_k^{t,+})'$	4.6.4, (4.6.41a)
D_k^t	$\hat{=}$ surface operator $(\mathsf{A}_{k+1}^\top)' : \mathcal{D}(\mathsf{A}_k^\top)' \to \mathcal{D}(\mathsf{A}_{k+1}^\top)'$	4.5.1, (4.5.4a)
D_k^n	$\hat{=}$ surface operator $\mathsf{A}_{k-1}' : \mathcal{D}(\mathsf{A}_k)' \to \mathcal{D}(\mathsf{A}_{k-1})'$	4.5.1, (4.5.4b)
S_k^t	$\hat{=}$ surface operator $\mathsf{A}_k : \mathcal{T}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_{k+1})$	4.5.2, (4.5.22)
S_k^t	$\hat{=}$ surface operator $\mathsf{A}_k^\top : \mathcal{T}(\mathsf{A}_k^\top) \to \mathcal{T}(\mathsf{A}_{k-1}^\top)$	4.5.2, (4.5.22)
$\hat{\mathsf{S}}_k^t$	$\hat{=}$ surface operator $\mathsf{A}_k : \mathbf{T}_{k+1}^{t,+} \to \mathcal{T}(\mathsf{A}_{k+1})$	4.6.4 (4.6.44)
$\hat{\mathsf{S}}_k^n$	$\hat{=}$ surface operator $\mathsf{A}_k^\top : \mathbf{T}_{k+1}^{n,+} \to \mathcal{T}(\mathsf{A}_{k-1}^\top)$	4.6.4, (4.6.44)
$\hat{\mathsf{D}}_k^t$	$\hat{=}$ surface operator $(\hat{\mathsf{S}}_{k+1}^n)' : \mathcal{T}(\mathsf{A}_k^\top)' \to \mathbf{T}_{k+2}^{n,-}$	4.6.4, (4.6.46)
$\hat{\mathsf{D}}_k^n$	$\hat{=}$ surface operator $(\hat{\mathsf{S}}_k^t)' : \mathcal{T}(\mathsf{A}_{k+1})' \to \mathbf{T}_k^{t,-}$	4.6.4, (4.6.46)

4.2 Hilbert Complexes

4.2.1 Operators on Hilbert spaces

In this article, both *bounded* and *unbounded* linear operators take center stage[7]. We distinguish them using the following notation. Let $\mathbf{X}$ and $\mathbf{Y}$ be two Hilbert spaces equipped with the inner products $(\cdot,\cdot)_{\mathbf{X}}$ and $(\cdot,\cdot)_{\mathbf{Y}}$, respectively. We will consistently write $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ to indicate that A is regarded as an *unbounded* linear operator from $\mathbf{X}$ to $\mathbf{Y}$ with domain $\mathcal{D}(\mathsf{A})$, whereas we mean by $\mathsf{A} : \mathbf{X} \to \mathbf{Y}$ that A is viewed as a *bounded* operator from $\mathbf{X}$ to $\mathbf{Y}$ defined on the whole space $\mathbf{X}$.

Recall that the difference between $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ and $\mathsf{A} : \mathcal{D}(\mathsf{A}) \to \mathbf{Y}$ comes from whether the topology of the subspace $\mathcal{D}(\mathsf{A}) \subset \mathbf{X}$ is given by the norm of $\mathbf{X}$ or the graph norm induced by the inner product $(\mathbf{x}_1, \mathbf{x}_2)_{\mathcal{D}(\mathsf{A})} := (\mathbf{x}_1, \mathbf{x}_2)_{\mathbf{X}} + (\mathsf{A}\mathbf{x}_1, \mathsf{A}\mathbf{x}_2)_{\mathbf{Y}}\ \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}(\mathsf{A})$.

An unbounded operator $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ is said to be *closed* if and only if its domain $\mathcal{D}(\mathsf{A})$ is a Hilbert space when endowed with the graph norm, cf. [6, Prop. 3.1]. It is *densely defined* if $\mathcal{D}(A)$ is a dense subset of $\mathbf{X}$. The kernel and range of A, whether it is bounded or not, will be denoted $\mathcal{N}(\mathsf{A})$ and $\mathcal{R}(\mathsf{A})$, respectively.

Topological dual spaces will be tagged with prime, e.g. $\mathbf{X}'$. We use angle brackets for duality pairings, e.g. $\langle \phi, \mathbf{x} \rangle_{\mathbf{X}'}$, $\phi \in \mathbf{X}'$, $\mathbf{x} \in \mathbf{X}$. Accordingly, the operator dual to a *bounded* linear operator $\mathsf{A} : \mathbf{X} \to \mathbf{Y}$ is a bounded operator $\mathsf{A}' : \mathbf{Y}' \to \mathbf{X}'$.

The Hilbert space adjoint of $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ is written $\mathsf{A}^* : \mathcal{D}(\mathsf{A}^*) \subset \mathbf{Y} \to \mathbf{X}$. Recall that it is the unbounded linear operator satisfying

$$(\mathsf{A}^*\mathbf{y}, \mathbf{x})_{\mathbf{X}} = (\mathbf{y}, \mathsf{A}\mathbf{x})_{\mathbf{Y}} \quad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}^*), \forall \mathbf{x} \in \mathcal{D}(\mathsf{A}), \tag{4.2.1}$$

[7] Standard references concerning bounded and unbounded linear operators are [27, Chap. 3] and [44, Chap. 7]. We also particularly recommend [6, Chap. 3], [11, Chap. 1-6] and [40, Chap. 6-8].

whose domain $\mathcal{D}(\mathsf{A}^*)$ consists of all $\mathbf{y} \in \mathbf{Y}$ for which the linear functional $\mathcal{D}(\mathsf{A}) \to \mathbb{R}$ defined by $\mathbf{x} \mapsto (\mathbf{y}, \mathsf{A}\mathbf{x})_{\mathbf{Y}}$ is continuous in the $\mathbf{X}$ norm, i.e. for every $\mathbf{y} \in \mathcal{D}(\mathsf{A}^*)$, $\exists C_{\mathbf{y}} > 0$ such that $|(\mathbf{y}, \mathsf{A}\mathbf{x})_{\mathbf{Y}}| \leq C_{\mathbf{y}} \|\mathbf{x}\|_{\mathbf{X}}$, $\forall \mathbf{x} \in \mathcal{D}(\mathsf{A})$. If A is closed and densely defined, then A^* is also closed and densely defined [6, Prop. 3.3]—in which case $\mathsf{A}^{**} = \mathsf{A}$.

We write $\mathring{\mathsf{A}} \subset \mathsf{A}$ and say that an unbounded linear operator $\mathsf{A} : \mathcal{D}(\mathsf{A}) \subset \mathbf{X} \to \mathbf{Y}$ is an extension of another unbounded linear operator $\mathring{\mathsf{A}} : \mathcal{D}(\mathring{\mathsf{A}}) \subset \mathbf{X} \to \mathbf{Y}$ when $\mathcal{D}(\mathring{\mathsf{A}}) \subset \mathcal{D}(\mathsf{A})$ and $\mathsf{A}\mathbf{x}_\circ = \mathring{\mathsf{A}}\mathbf{x}_\circ$ for all $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}})$.

3D de Rham setting II: Differential operators

We refer to [6, Chap. 3] for the following mappings properties. The linear differential operators

$$\mathbf{grad} : H^1(\Omega) \subset L^2(\Omega) \to \mathbf{L}^2, \tag{4.2.2a}$$
$$\mathbf{curl} : \mathbf{H}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2 \to \mathbf{L}^2, \tag{4.2.2b}$$
$$\mathrm{div} : \mathbf{H}(\mathrm{div}, \Omega) \subset \mathbf{L}^2 \to L^2(\Omega), \tag{4.2.2c}$$

are densely defined and closed unbounded linear operators. They are extensions of

$$\mathring{\mathbf{grad}} : \mathring{H}^1(\Omega) \subset L^2(\Omega) \to \mathbf{L}^2, \tag{4.2.3a}$$
$$\mathring{\mathbf{curl}} : \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2 \to \mathbf{L}^2, \tag{4.2.3b}$$
$$\mathring{\mathrm{div}} : \mathring{\mathbf{H}}(\mathrm{div}, \Omega) \subset \mathbf{L}^2 \to L^2(\Omega). \tag{4.2.3c}$$

The L^2 Hilbert space adjoints of (4.2.2a)-(4.2.2c) are

$$\mathbf{grad}^* = -\mathring{\mathrm{div}} : \mathring{\mathbf{H}}(\mathrm{div}, \Omega) \subset \mathbf{L}^2 \to L^2(\Omega), \tag{4.2.4a}$$
$$\mathbf{curl}^* = \mathring{\mathbf{curl}} : \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2 \to \mathbf{L}^2, \tag{4.2.4b}$$
$$\mathrm{div}^* = -\mathring{\mathbf{grad}} : \mathring{H}^1(\Omega) \subset L^2(\Omega) \to \mathbf{L}^2, \tag{4.2.4c}$$

respectively. Then, the adjoint operators of (4.2.3a)-(4.2.3c) are obtained using the fact that $\mathsf{A}^{**} = \mathsf{A}$ for all densely defined and closed unbounded linear operators between Hilbert spaces.

By abuse of notation, we generally write $\mathbf{grad} = \mathring{\mathbf{grad}}$, $\mathbf{curl} = \mathring{\mathbf{curl}}$ and $\mathrm{div} = \mathring{\mathrm{div}}$.

4.2.2 Definition

A *Hilbert complex* is a sequence of Hilbert spaces $\mathbf{W}_k$, $k \in \mathbb{Z}$, together with a sequence of closed and densely defined unbounded linear operators $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1}$ such that $\mathcal{R}(\mathsf{A}_k) \subset \mathcal{N}(\mathsf{A}_{k+1})$, i.e. $\mathsf{A}_{k+1} \circ \mathsf{A}_k \equiv 0$ for all $k \in \mathbb{Z}$. It can be written as

$$\cdots \xrightarrow{\mathsf{A}_{k-2}} \mathcal{D}(\mathsf{A}_{k-1}) \subset \mathbf{W}_{k-1} \xrightarrow{\mathsf{A}_{k-1}} \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k \xrightarrow{\mathsf{A}_k} \mathcal{D}(\mathsf{A}_{k+1}) \subset \mathbf{W}_{k+1} \xrightarrow{\mathsf{A}_{k+1}} \cdots, \tag{4.2.5a}$$

cf. [6, Def. 4.1]. The associated sequence of adjoint operators spawns the so-called dual Hilbert complex

$$\cdots \xleftarrow[\mathsf{A}^*_{k-2}]{} \mathcal{D}(A^*_{k-2}) \subset \mathbf{W}_{k-1} \xleftarrow[\mathsf{A}^*_{k-1}]{} \mathcal{D}(A^*_{k-1}) \subset \mathbf{W}_k \xleftarrow[\mathsf{A}^*_{k}]{} \mathcal{D}(A^*_{k}) \subset \mathbf{W}_{k+1} \xleftarrow[\mathsf{A}^*_{k+1}]{} \cdots, \tag{4.2.5b}$$

which by (4.2.1) is itself a Hilbert complex, because $\mathsf{A}^*_{k-1} \circ \mathsf{A}^*_k \equiv 0$ for all $k \in \mathbb{Z}$. "Finite" Hilbert complexes can be embedded into (4.2.5a) by setting $\mathbf{W}_k = \{0\}$ for all $k \notin \{0, 1, ..., N\}$.

Notice that since $\mathcal{R}(\mathsf{A}_k) \subset \mathcal{D}(\mathsf{A}_{k+1})$ and $\mathcal{R}(\mathsf{A}^*_{k+1}) \subset \mathcal{D}(\mathsf{A}^*_k)$, the sequences of *bounded* operators $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \to \mathbf{W}_{k+1}$ and $\mathsf{A}^*_k : \mathcal{D}(\mathsf{A}^*_k) \to \mathbf{W}_k$ also induce Hilbert complexes themselves:

$$\cdots \xrightarrow{\mathsf{A}_{k-2}} \mathcal{D}(\mathsf{A}_{k-1}) \xrightarrow{\mathsf{A}_{k-1}} \mathcal{D}(\mathsf{A}_k) \xrightarrow{\mathsf{A}_{k}} \mathcal{D}(\mathsf{A}_{k+1}) \xrightarrow{\mathsf{A}_{k+1}} \cdots, \tag{4.2.6a}$$

$$\cdots \xleftarrow[\mathsf{A}^*_{k-2}]{} \mathcal{D}(\mathsf{A}^*_{k-2}) \xleftarrow[\mathsf{A}^*_{k-1}]{} \mathcal{D}(\mathsf{A}^*_{k-1}) \xleftarrow[\mathsf{A}^*_{k}]{} \mathcal{D}(\mathsf{A}^*_{k}) \xleftarrow[\mathsf{A}^*_{k+1}]{} \cdots. \tag{4.2.6b}$$

These are examples of *bounded* Hilbert complexes in which every operator is continuous. We refer to (4.2.6a) and (4.2.6b) as the *domain complexes* of (4.2.5a) and (4.2.5b).

If the range $\mathcal{R}(\mathsf{A}_k)$ is a closed subset of $\mathbf{W}_{k+1}$ for all k, we say that the Hilbert complex (4.2.5a) is *closed*. If this is the case, then $\mathcal{R}(\mathsf{A}^*_k)$ is also closed in $\mathbf{W}_k$ by the closed range theorem [6, Thm. 3.7], making the dual complex (4.2.5b) a closed Hilbert complex too. Furthermore, (4.2.5a) is said to be *Fredholm* if the codimension of $\mathcal{R}(\mathsf{A}_k)$ is finite in $\mathcal{N}(\mathsf{A}_{k+1})$—in which case it is also closed by [6, Thm. 3.8]. Equivalently, a Hilbert complex is Fredholm if the quotient spaces $\mathcal{N}(\mathsf{A}_{k+1})/\mathcal{R}(\mathsf{A}_k)$ and $\mathcal{N}(\mathsf{A}^*_k)/\mathcal{R}(\mathsf{A}^*_{k+1})$ are finite dimensional, in other words, if the cohomology spaces of (4.2.5a) and (4.2.5b) have finite dimension. It is a sufficient condition for a Hilbert complex to be Fredholm to satisfy the *compactness property*, that is, the embedding $\mathcal{D}(\mathsf{A}_k) \cap \mathcal{D}(\mathsf{A}^*_{k-1}) \hookrightarrow \mathbf{W}_k$ is compact for all $k \in \mathbb{Z}$.

3D de Rham setting III: The L^2 de Rham complex in $\mathbb{R}^3$

The L^2 de Rham complex (4.1.1) is a standard example of a Hilbert complex, where $\mathsf{A}_k \equiv 0$ and $\mathbf{W}_k = \{0\}$ is set for $k \in \mathbb{Z} \setminus \{0, 1, 2, 3\}$. Its dual complex is represented by the sequence

$$\{0\} \xleftarrow[0]{} L^2(\Omega) \xleftarrow[-\mathrm{div}]{} \mathring{\mathbf{H}}(\mathrm{div}, \Omega) \xleftarrow[\mathbf{curl}]{} \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \xleftarrow[-\mathbf{grad}]{} \mathring{H}^1(\Omega) \xleftarrow[\imath]{} \{0\}, \tag{4.2.7}$$

cf. [6, Sec. 3.4] and [6, Sec. 4.3], and its embedding into our abstract framework is summarized in the following table:

k	$\mathbf{W}_k$	A_k	$\mathcal{D}(\mathsf{A}_k)$	A^*_k	$\mathcal{D}(\mathsf{A}^*_k)$	$\mathcal{D}(\mathsf{A}_k) \cap \mathcal{D}(\mathsf{A}^*_{k-1})$
0	$L^2(\Omega)$	$\mathbf{grad}$	$H^1(\Omega)$	$-\mathrm{div}$	$\mathring{\mathbf{H}}(\mathrm{div}, \Omega)$	$H^1(\Omega)$
1	$\mathbf{L}^2(\Omega)$	$\mathbf{curl}$	$\mathbf{H}(\mathbf{curl}, \Omega)$	$\mathbf{curl}$	$\mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$	$\mathbf{H}(\mathbf{curl}, \Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega)$
2	$\mathbf{L}^2(\Omega)$	div	$\mathbf{H}(\mathrm{div}, \Omega)$	$-\mathbf{grad}$	$\mathring{H}^1(\Omega)$	$\mathbf{H}(\mathrm{div}, \Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$
3	$L^2(\Omega)$	0	$L^2(\Omega)$	$\mathbf{Id}$	$\{0\}$	$\mathring{H}^1(\Omega)$

The de Rham complex satisfies the compactness property, and thus it is Fredholm. Indeed, recall that Rellich's compact embedding theorem states that the inclusion of $H^1(\Omega)$ and $\mathring{H}^1(\Omega)$ in $L^2(\Omega)$ is compact. We refer to [39] for a proof that $\mathbf{H}(\mathbf{curl},\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div},\Omega)$ and $\mathbf{H}(\mathrm{div},\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl},\Omega)$ are compactly embedded in $\mathbf{L}^2(\Omega)$.

4.2.3 Basic setting

Now, let a Hilbert complex as in (4.2.5a) be given and suppose that the unbounded linear operators of a second Hilbert complex

$$\cdots \xrightarrow{\mathring{\mathsf{A}}_{k-2}} \mathcal{D}(\mathring{\mathsf{A}}_{k-1}) \subset \mathbf{W}_{k-1} \xrightarrow{\mathring{\mathsf{A}}_{k-1}} \mathcal{D}(\mathring{\mathsf{A}}_{k}) \subset \mathbf{W}_{k} \xrightarrow{\mathring{\mathsf{A}}_{k}} \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \subset \mathbf{W}_{k+1} \xrightarrow{\mathring{\mathsf{A}}_{k+1}} \cdots \tag{4.2.8a}$$

are such that $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k$, i.e. $\mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathcal{D}(\mathsf{A}_k)$ and $\mathsf{A}_k|_{\mathcal{D}(\mathring{\mathsf{A}}_k)} = \mathring{\mathsf{A}}_k$. In other words, for all $k \in \mathbb{Z}$, A_k is an extension of $\mathring{\mathsf{A}}_k$. It is easy to verify that the adjoint operators $\mathsf{A}_k^\top := \mathring{\mathsf{A}}_k^* : \mathcal{D}(\mathring{\mathsf{A}}_k^*) \subset \mathbf{W}_{k+1} \to \mathbf{W}_k$ involved in the dual complex

$$\cdots \xleftarrow[\mathsf{A}_{k-2}^\top]{} \mathcal{D}(\mathsf{A}_{k-2}^\top) \subset \mathbf{W}_{k-1} \xleftarrow[\mathsf{A}_{k-1}^\top]{} \mathcal{D}(\mathsf{A}_{k-1}^\top) \subset \mathbf{W}_{k} \xleftarrow[\mathsf{A}_{k}^\top]{} \mathcal{D}(\mathsf{A}_{k}^\top) \subset \mathbf{W}_{k+1} \xleftarrow[\mathsf{A}_{k+1}^\top]{} \cdots \tag{4.2.8b}$$

are such that $\mathsf{A}_k^* \subset \mathsf{A}_k^\top$. In particular, the bounded domain complexes

$$\cdots \xrightarrow{\mathring{\mathsf{A}}_{k-2}} \mathcal{D}(\mathring{\mathsf{A}}_{k-1}) \xrightarrow{\mathring{\mathsf{A}}_{k-1}} \mathcal{D}(\mathring{\mathsf{A}}_{k}) \xrightarrow{\mathring{\mathsf{A}}_{k}} \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \xrightarrow{\mathring{\mathsf{A}}_{k+1}} \cdots, \tag{4.2.9a}$$

$$\cdots \xleftarrow[\mathsf{A}_{k-2}^*]{} \mathcal{D}(\mathsf{A}_{k-2}^*) \xleftarrow[\mathsf{A}_{k-1}^*]{} \mathcal{D}(\mathsf{A}_{k-1}^*) \xleftarrow[\mathsf{A}_{k}^*]{} \mathcal{D}(\mathsf{A}_{k}^*) \xleftarrow[\mathsf{A}_{k+1}^*]{} \cdots, \tag{4.2.9b}$$

are examples of Hilbert *subcomplexes* of the domain Hilbert complexes (4.2.6a) and (4.2.6b).

For reference, this basic setting is summarized in the following assumption.

Assumption A For all $k \in \mathbb{Z}$ let $\mathbf{W}_k$ be real Hilbert spaces, and suppose that

$$\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1} \qquad \text{and} \qquad \mathring{\mathsf{A}}_k : \mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1}$$

are densely defined and closed unbounded linear operators such that $\mathcal{R}(\mathsf{A}_k) \subset \mathcal{N}(\mathsf{A}_{k+1})$, $\mathcal{R}(\mathring{\mathsf{A}}_k) \subset \mathcal{N}(\mathring{\mathsf{A}}_{k+1})$, and A_k is an extension of $\mathring{\mathsf{A}}_k$, i.e. $\mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathcal{D}(\mathsf{A}_k)$ and $\mathsf{A}_k \mathbf{x}_\circ = \mathring{\mathsf{A}}_k \mathbf{x}_\circ$ for all $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$.

3D de Rham setting IV: Boundary conditions

The Hilbert complex

$$\{0\} \xhookrightarrow{\imath} H_0^1(\Omega) \xrightarrow{\mathbf{grad}} \mathring{\mathbf{H}}(\mathbf{curl},\Omega) \xrightarrow{\mathbf{curl}} \mathring{\mathbf{H}}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\} \tag{4.2.10a}$$

fulfills the hypothesis on (4.2.8a) for the L^2 de Rham complex (4.1.1). Owing to (4.2.4a)-(4.2.4c), its dual complex is written

$$\{0\} \xleftarrow[0]{} L^2(\Omega) \xleftarrow[-\mathrm{div}]{} \mathbf{H}(\mathrm{div},\Omega) \xleftarrow[\mathbf{curl}]{} \mathbf{H}(\mathbf{curl},\Omega) \xleftarrow[-\mathbf{grad}]{} H^1 \xleftarrow[\imath]{} \{0\}. \tag{4.2.10b}$$

Summing up, the various operators and spaces have the following incarnations for the de Rham complex in three-dimensional Euclidean space:

k	$\mathbf{W}_k$	$\mathring{\mathsf{A}}_k$	$\mathcal{D}(\mathring{\mathsf{A}}_k)$	$\mathsf{A}_k^\top$	$\mathcal{D}(\mathsf{A}_k^\top)$	$\mathcal{D}(\mathring{\mathsf{A}}_k) \cap \mathcal{D}(\mathsf{A}_{k-1}^\top)$
0	$L^2(\Omega)$	$\mathbf{grad}$	$\mathring{H}^1(\Omega)$	$-\mathrm{div}$	$\mathbf{H}(\mathrm{div},\Omega)$	$\mathring{H}^1(\Omega)$
1	$\mathbf{L}^2(\Omega)$	$\mathbf{curl}$	$\mathring{\mathbf{H}}(\mathbf{curl},\Omega)$	$\mathbf{curl}$	$\mathbf{H}(\mathbf{curl},\Omega)$	$\mathring{\mathbf{H}}(\mathbf{curl},\Omega) \cap \mathbf{H}(\mathrm{div},\Omega)$
2	$\mathbf{L}^2(\Omega)$	div	$\mathring{\mathbf{H}}(\mathrm{div},\Omega)$	$-\mathbf{grad}$	$H^1(\Omega)$	$\mathring{\mathbf{H}}(\mathrm{div},\Omega) \cap \mathbf{H}(\mathbf{curl},\Omega)$
3	$L^2(\Omega)$	0	$L^2(\Omega)$	$\mathbf{Id}$	$\{0\}$	$H^1(\Omega)$

4.3 Trace Operators

The following sections lay the foundations of a general quotient-based abstract theory for traces in Hilbert spaces. To that end, we do not require the full structure of Hilbert complexes, but it suffices to focus on the following snippet of the Hilbert complexes (4.2.5a) and (4.2.8a):

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\mathsf{A}_{k-2}} & \mathcal{D}(\mathsf{A}_{k-1}) \subset \mathbf{W}_{k-1} & \xrightarrow{\mathsf{A}_{k-1}} & \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k & \xrightarrow{\mathsf{A}_k} & \mathcal{D}(\mathsf{A}_{k+1}) \subset \mathbf{W}_{k+1} & \xrightarrow{\mathsf{A}_{k+1}} & \cdots, \\
 & & \cup & & \cup & & \cup & & \\
\cdots & \xrightarrow{\mathring{\mathsf{A}}_{k-2}} & \mathcal{D}(\mathring{\mathsf{A}}_{k-1}) \subset \mathbf{W}_{k-1} & \xrightarrow{\mathring{\mathsf{A}}_{k-1}} & \mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathbf{W}_k & \xrightarrow{\mathring{\mathsf{A}}_k} & \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \subset \mathbf{W}_{k+1} & \xrightarrow{\mathring{\mathsf{A}}_{k+1}} & \cdots.
\end{array}$$

In the sequel, we fix $k \in \mathbb{Z}$ and take for granted Assumption A.

4.3.1 Hilbert traces

Using the shorthand $\mathsf{A}_k^\top := \mathring{\mathsf{A}}_k^* : \mathcal{D}(\mathsf{A}_k^\top) \subset \mathbf{W}_{k+1} \to \mathbf{W}_k$, it follows from the estimate

$$\begin{aligned}
|(\mathsf{A}_k\mathbf{x},\mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x},\mathsf{A}_k^\top\mathbf{y})_{\mathbf{W}_k}| &\le \|\mathsf{A}_k\mathbf{x}\|_{\mathbf{W}_{k+1}}\|\mathbf{y}\|_{\mathbf{W}_{k+1}} + \|\mathbf{x}\|_{\mathbf{W}_k}\|\mathsf{A}_k^\top\mathbf{y}\|_{\mathbf{W}_k} \\
&\le \|\mathbf{x}\|_{\mathcal{D}(\mathsf{A}_k)}\|\mathbf{y}\|_{\mathcal{D}(\mathsf{A}_k^\top)}
\end{aligned} \tag{4.3.1}$$

that the following definition of a particular notion of a trace makes sense.

Definition 4.1 In the setting of Assumption A, the bounded linear operator

$$\mathsf{T}_k^t : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathsf{A}_k^\top)' \tag{4.3.2}$$

defined for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ and $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$ by

$$\langle \mathsf{T}_k^t \mathbf{x}, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k^\top)'} := (\mathsf{A}_k \mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_k} \tag{4.3.3}$$

is called the *(primal) Hilbert trace* associated with the pair of operators A_k and $\mathring{\mathsf{A}}_k$.

It also follows from (4.3.1) that

$$\|\mathsf{T}_k^t\| = 1, \tag{4.3.4}$$

where $\|\cdot\|$ is the operator norm.

We point out that defining a trace operator as a mapping into a dual space has precedents in the theory of Friedrichs operators, has been pursued in [21, Sect. 2.2] and [20, Sect. 56.3.2], and is also discussed in [3–5]. In these works, the authors have dubbed “boundary operators” what we have decided to call “Hilbert traces”.

Let us motivate the above notion of trace with classical examples.

3D de Rham setting V: Hilbert traces

Applying Definition 4.1 in the 3D de Rham setting II, we obtain the Hilbert traces

$$\mathsf{T}_0^t = \mathsf{T}_{\mathbf{grad}}^t : H^1(\Omega) \to \mathbf{H}(\mathrm{div}, \Omega)', \tag{4.3.5a}$$
$$\mathsf{T}_1^t = \mathsf{T}_{\mathbf{curl}}^t : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}(\mathbf{curl}, \Omega)', \tag{4.3.5b}$$
$$\mathsf{T}_2^t = \mathsf{T}_{\mathrm{div}}^t : \mathbf{H}(\mathrm{div}, \Omega) \to H^1(\Omega)', \tag{4.3.5c}$$

defined by

$$\langle \mathsf{T}_{\mathbf{grad}}^t v, \mathbf{u} \rangle_{\mathbf{H}(\mathrm{div},\Omega)'} := (\mathbf{grad} v, \mathbf{u})_{\mathbf{L}^2} + (v, \mathrm{div}\, \mathbf{u})_{L^2(\Omega)}, \tag{4.3.6a}$$
$$\langle \mathsf{T}_{\mathbf{curl}}^t \mathbf{z}, \mathbf{w} \rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} := (\mathbf{curl}\, \mathbf{z}, \mathbf{w})_{\mathbf{L}^2} - (\mathbf{z}, \mathbf{curl}\, \mathbf{w})_{\mathbf{L}^2}, \tag{4.3.6b}$$
$$\langle \mathsf{T}_{\mathrm{div}}^t \mathbf{u}, v \rangle_{H^1(\Omega)'} := (\mathrm{div}\, \mathbf{u}, v)_{\mathbf{L}^2} + (\mathbf{u}, \mathbf{grad}\, v)_{\mathbf{L}^2}, \tag{4.3.6c}$$

for all $v \in H^1$, $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega)$ and $\mathbf{z}, \mathbf{w} \in \mathbf{H}(\mathbf{curl}, \Omega)$.

We recognize on the right hand sides of (4.3.6a)-(4.3.6c) the continuous bilinear forms occurring in Green’s formulas (4.1.12a) and (4.1.12b). Introducing the operators

$$\gamma_n' : H^{1/2}(\Gamma) \to \mathbf{H}(\mathrm{div}, \Omega)', \tag{4.3.7}$$
$$\gamma_t' : \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}(\mathbf{curl}, \Omega)', \tag{4.3.8}$$
$$\gamma' : H^{-1/2} \to H^1(\Omega)', \tag{4.3.9}$$

dual to the classical traces, where we have identified $H^{-1/2}(\Gamma)$ with $(H^{1/2}(\Gamma))'$ through the $L^2(\Gamma)$-pairing on the boundary and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ with its own dual through the skew-symmetric pairing defined in (4.1.10), we obtain

$$\mathsf{T}^t_{\mathbf{grad}} = \gamma_n' \circ \gamma, \qquad \mathsf{T}^t_{\mathbf{curl}} = \gamma_t' \circ \gamma_t, \qquad \mathsf{T}^t_{\mathrm{div}} = \gamma' \circ \gamma_n. \tag{4.3.10}$$

Observe that

$$(\mathsf{T}^t_{\mathbf{grad}})' = \mathsf{T}^t_{\mathrm{div}}. \tag{4.3.11}$$

The appeal of definitions (4.3.6a)-(4.3.6c) is that they do not explicitly depend on Γ. In fact, notice that they are well-defined for general bounded open sets Ω without any assumption on the regularity of their boundary $\Gamma := \partial\Omega$.

Proposition 4.1 *Under Assumption A,*

$$\mathcal{N}(\mathsf{T}^t_k) = \mathcal{D}(\mathring{\mathsf{A}}_k). \tag{4.3.12}$$

Proof. On the one hand, for any $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$, it follows from $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k$ and (4.2.1) that

$$\begin{aligned}\langle \mathsf{T}^t_k \mathbf{x}_\circ, \mathbf{y}\rangle_{\mathcal{D}(\mathsf{A}_k^\top)'} &= (\mathsf{A}_k \mathbf{x}_\circ, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}_\circ, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_k} = (\mathring{\mathsf{A}}_k \mathbf{x}_\circ, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}_\circ, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_k} \\ &= (\mathbf{x}_\circ, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}_\circ, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_k} = 0\end{aligned} \tag{4.3.13}$$

for all $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$. This shows that $\mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathcal{N}(\mathsf{T}^t_k)$.

On the other hand, if $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ is such that $\mathbf{x} \in \mathcal{N}(\mathsf{T}^t_k)$, then

$$0 = \langle \mathsf{T}^t_k \mathbf{x}, \mathbf{y}\rangle_{\mathcal{D}(\mathsf{A}_k^\top)'} = (\mathsf{A}_k \mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_k} \quad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top). \tag{4.3.14}$$

If we set $C_{\mathbf{x}} := \|\mathbf{x}\|_{\mathcal{D}(\mathsf{A}_k)}$, we see that

$$|(\mathbf{x}, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_k}| = |(\mathsf{A}_k \mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}}| \leq \|\mathsf{A}_k \mathbf{x}\|_{\mathbf{W}_{k+1}} \|\mathbf{y}\|_{\mathbf{W}_{k+1}} \leq C_{\mathbf{x}} \|\mathbf{y}\|_{\mathbf{W}_{k+1}} \quad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top).$$

As explained in Subsection 4.2.2, this means that $\mathbf{x} \in \mathcal{D}((\mathsf{A}_k^\top)^*) = \mathcal{D}(\mathring{\mathsf{A}}_k^{**}) = \mathcal{D}(\mathring{\mathsf{A}}_k)$. □

3D de Rham setting VI: Kernels of classical Hilbert traces

Comparing Proposition 4.1 with (4.1.8a)-(4.1.8c), we verify that

$$\mathcal{N}(\mathsf{T}^t_{\mathbf{grad}}) = \mathcal{N}(\gamma), \qquad \mathcal{N}(\mathsf{T}^t_{\mathbf{curl}}) = \mathcal{N}(\gamma_t), \qquad \mathcal{N}(\mathsf{T}^t_{\mathrm{div}}) = \mathcal{N}(\gamma_n). \tag{4.3.15}$$

Remark 4.3.1. *Intuitively, we think of a trace operator as a means of imposing "boundary conditions". The idea behind Definition 4.1 is to impose these boundary conditions on the operator itself, which is a common strategy in the analysis of variational problems and related operator equations. In this work, A_k is the operator of interest. We regard $\mathring{\mathsf{A}}_k$ as the operator on which boundary conditions are imposed. From that perspective, the operator $\mathsf{A}_k^\top$ does not feature boundary conditions. The right hand side of (4.3.3) plays a role akin to the bilinear form involved in classical integration by parts formulas.*

4.3.2 Trace spaces

Recall that by hypothesis, $\mathcal{D}(\mathsf{A}_k^*) \subset \mathcal{D}(\mathsf{A}_k^\top)$. The next proposition involves the annihilator of $\mathcal{D}(\mathsf{A}_k^*)$ in $\mathcal{D}(\mathsf{A}_k^\top)'$:

$$\mathcal{D}(\mathsf{A}_k^*)^\circ := \left\{ \phi \in \mathcal{D}(\mathsf{A}_k^\top)' \mid \langle \phi, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k^\top)'} = 0 \ \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_k^*) \right\} \subset \mathcal{D}(\mathsf{A}_k^\top)'. \tag{4.3.16}$$

Proposition 4.2 *Under Assumption A, we find for the ranges of the Hilbert traces*

$$\mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ. \tag{4.3.17}$$

Proof. Suppose that $\phi \in \mathcal{D}(\mathsf{A}_k^*)^\circ$ and let $\mathbf{w} \in \mathcal{D}(\mathsf{A}_k^\top)$ be its Riesz representative in $\mathcal{D}(\mathsf{A}_k^\top)$, that is

$$\langle \phi, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k^\top)'} = (\mathbf{w}, \mathbf{y})_{\mathcal{D}(\mathsf{A}_k^\top)} \qquad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top). \tag{4.3.18}$$

We claim that $\mathbf{x} := -\mathsf{A}_k^\top \mathbf{w} \in \mathcal{D}(\mathsf{A}_k)$. Indeed, (4.3.18) implies that for all $\mathbf{y}_* \in \mathcal{D}(\mathsf{A}_k^*)$, we have

$$0 = (\mathbf{w}, \mathbf{y}_*)_{\mathcal{D}(\mathsf{A}_k^\top)} = (\mathbf{w}, \mathbf{y}_*)_{\mathbf{W}_{k+1}} + (\mathsf{A}_k^\top \mathbf{w}, \mathsf{A}_k^\top \mathbf{y}_*)_{\mathbf{W}_k} = (\mathbf{w}, \mathbf{y}_*)_{\mathbf{W}_{k+1}} + (\mathsf{A}_k^\top \mathbf{w}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_k}. \tag{4.3.19}$$

This means $(\mathbf{w}, \mathbf{y}_*)_{\mathbf{W}_{k+1}} = (\mathbf{x}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_k}$. Therefore, if we set $C_\mathbf{x} := \|\mathbf{w}\|_{\mathbf{W}_{k+1}}$, we find the estimate

$$|(\mathbf{x}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_k}| = |(\mathbf{w}, \mathbf{y}_*)_{\mathbf{W}_{k+1}}| \leq \|\mathbf{w}\|_{\mathbf{W}_{k+1}} \|\mathbf{y}_*\|_{\mathbf{W}_{k+1}} = C_\mathbf{x} \|\mathbf{y}_*\|_{\mathbf{W}_{k+1}} \quad \forall \mathbf{y}_* \in \mathcal{D}(\mathsf{A}_k^*), \tag{4.3.20}$$

which as explained in Subsection 4.2.1 implies that $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k^{**}) = \mathcal{D}(\mathsf{A}_k)$.

In particular, according to (4.2.1), it also follows from (4.3.19) that $\mathsf{A}_k \mathbf{x} = \mathbf{w}$. Hence, the inclusion $\mathcal{R}(\mathsf{T}_k^t) \supset \mathcal{D}(\mathsf{A}_k^*)^\circ$ is verified by observing that for all $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$,

$$\begin{aligned} \langle \mathsf{T}_k^t \mathbf{x}, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k^\top)'} &= (\mathsf{A}_k \mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_k} = (\mathbf{w}, \mathbf{y})_{\mathbf{W}_{k+1}} + (\mathsf{A}_k^\top \mathbf{w}, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_k} \\ &= (\mathbf{w}, \mathbf{y})_{\mathcal{D}(\mathsf{A}_k^\top)} = \langle \phi, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k^\top)'}, \end{aligned} \tag{4.3.21}$$

i.e. $\mathsf{T}_k^t \mathbf{x} = \phi$.

To show that $\mathcal{R}(\mathsf{T}_k^t) \subset \mathcal{D}(\mathsf{A}_k^*)^\circ$, let $\phi = \mathsf{T}_k^t \mathbf{x}$ for some $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$. Then, since $\mathsf{A}_k^* \subset \mathsf{A}_k^\top$, we obtain by (4.2.1) that for all $\mathbf{y}_* \in \mathcal{D}(\mathsf{A}_k^*)$

$$\langle \phi, \mathbf{y}_* \rangle_{\mathcal{D}(\mathsf{A}_k^\top)'} = (\mathsf{A}_k \mathbf{x}, \mathbf{y}_*)_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathsf{A}_k^\top \mathbf{y}_*)_{\mathbf{W}_k} = (\mathbf{x}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathsf{A}_k^* \mathbf{y}_*)_{\mathbf{W}_k} = 0, \tag{4.3.22}$$

i.e. $\phi \in \mathcal{D}(\mathsf{A}_k^*)^\circ$. □

Since $\mathcal{D}(\mathring{\mathsf{A}}_k)$ is a Hilbert subspace of $\mathcal{D}(\mathsf{A}_k)$, it is closed and we can proceed with the next definition.

Definition 4.2 In the setting of Definition 4.1, we call *trace spaces* the quotient spaces

$$\mathcal{T}(\mathsf{A}_k) := \mathcal{D}(\mathsf{A}_k) / \mathcal{D}(\mathring{\mathsf{A}}_k), \tag{4.3.23}$$

equipped with the quotient norm

$$\|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)} := \inf_{\mathring{\mathbf{z}} \in \mathcal{D}(\mathring{\mathsf{A}}_k)} \|\mathbf{x} - \mathring{\mathbf{z}}\|_{\mathcal{D}(\mathsf{A}_k)} \quad \forall \mathbf{x} \in \mathcal{D}(\mathsf{A}_k). \tag{4.3.24}$$

Remark 4.3.2. *Notice that due to Proposition 4.1,*

$$\mathcal{T}(\mathsf{A}_k) = \mathcal{D}(\mathsf{A}_k)/\mathcal{N}(\mathsf{T}_k^t). \tag{4.3.25}$$

In Definition 4.2, the equivalence class in $\mathcal{T}(\mathsf{A}_k)$ of $\mathbf{x} \in D(\mathsf{A}_k)$ is denoted $[\mathbf{x}] = \{\mathbf{x} + \mathring{\mathbf{z}} \,|\, \mathring{\mathbf{z}} \in \mathcal{D}(\mathring{\mathsf{A}}_k)\}$. Write $\boldsymbol{\pi}_{\mathsf{A}_k}^t : \mathcal{D}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_k)$ for the canonical projection (also frequently called quotient map), i.e. $\boldsymbol{\pi}_{\mathsf{A}_k}^t(\mathbf{x}) = [\mathbf{x}]$. It is an application of a classical theorem of functional analysis that there exists a bounded orthogonal projection $\mathsf{P}_k^t : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ onto the complement space

$$\mathcal{D}(\mathring{\mathsf{A}}_k)^\perp := \left\{ \mathbf{x} \in \mathcal{D}(\mathsf{A}_k) \mid (\mathbf{x}, \mathring{\mathbf{z}})_{\mathcal{D}(\mathsf{A}_k)} = 0 \ \forall \mathring{\mathbf{z}} \in \mathcal{D}(\mathring{\mathsf{A}}_k) \right\} \subset \mathcal{D}(\mathsf{A}_k) \tag{4.3.26}$$

such that

$$\|\mathsf{P}_k^t \mathbf{x}\|_{\mathcal{D}(\mathsf{A})} = \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)} \qquad \forall \mathbf{x} \in \mathcal{D}(\mathsf{A}_k), \tag{4.3.27}$$

cf. [44, Chap. 3.1] and [11, Chap. 5]. Write $\imath_k^t : \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp \hookrightarrow \mathcal{D}(\mathsf{A}_k)$ for canonical inclusion maps. Since $\mathcal{N}(\mathsf{P}_k^t) = \mathcal{D}(\mathring{\mathsf{A}}_k)$ by (4.3.27) , the bounded linear map $\mathsf{G}_k^t : \mathcal{T}(\mathsf{A}_k) \to \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ defined by $\mathsf{G}_k^t[\mathbf{x}] := \mathsf{P}_k^t \mathbf{x}$ and involved in the commutative diagram

$$\begin{array}{ccc} \mathcal{D}(\mathsf{A}_k) & \xrightarrow{\ \mathsf{P}_k^t\ } & \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp \\ & {\scriptstyle \boldsymbol{\pi}_k^t} \searrow \quad \nearrow {\scriptstyle \mathsf{G}_k^t} & \\ & \mathcal{D}(\mathsf{A}_k)/\mathcal{N}(\mathsf{P}_k^t) = \mathcal{T}(\mathsf{A}_k) & \end{array} \tag{4.3.28}$$

as provided by the first isomorphism theorem for modules is a well-defined isometric isomorphism, cf. [19, Chap. 10.2, Thm. 4]. Since $\mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ is closed [44, Chap. 3.1, Thm. 1], it is a Hilbert space, and therefore so is $\mathcal{T}(\mathsf{A}_k)$. The quotient norm is induced by the inner product

$$([\mathbf{x}], [\mathbf{z}])_{\mathcal{T}(\mathsf{A}_k)} := (\mathsf{P}_k^t \mathbf{x}, \mathsf{P}_k^t \mathbf{z})_{\mathcal{D}(\mathsf{A}_k)} \qquad \forall [\mathbf{x}], [\mathbf{z}] \in \mathcal{T}(\mathsf{A}_k). \tag{4.3.29}$$

Remark 4.3.3. *Notice that* $\mathcal{N}(\mathsf{P}_k^t) = \mathcal{D}(\mathring{\mathsf{A}}_k) = \mathcal{N}(\mathsf{T}_k^t)$.

That the projection P_k^t is orthogonal means that $(\mathbf{x} - \mathsf{P}_k^t \mathbf{x}, \mathbf{z}_\perp)_{\mathcal{D}(\mathsf{A}_k)} = 0$ for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ and $\mathbf{z}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$. In other words, $(\mathrm{Id} - \mathsf{P}_k^t)\mathbf{x} \in \mathcal{D}(\mathring{\mathsf{A}}_k)$ for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$. Hence, the simple observation that $\mathrm{Id} = \mathsf{P}_k^t + (\mathrm{Id} - \mathsf{P}_k^t)$ shows that any element $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ can be decomposed as

$$\mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_\circ \tag{4.3.30}$$

where $\mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ and $\mathbf{x}_\circ \in D(\mathring{\mathsf{A}}_k)$. It is easy to see that the decomposition (4.3.30) is unique.

3D de Rham setting VII: Trace spaces

In the 3D de Rham setting V, applying Definition 4.2 leads to

$$\mathcal{T}(\mathsf{A}_0) = \mathcal{T}(\mathbf{grad}) = H^1(\Omega)/\mathring{H}^1(\Omega), \tag{4.3.31a}$$

$$\mathcal{T}(\mathsf{A}_1) = \mathcal{T}(\mathbf{curl}) = \mathbf{H}(\mathbf{curl}, \Omega)/\mathring{\mathbf{H}}(\mathbf{curl}, \Omega), \tag{4.3.31b}$$

$$\mathcal{T}(\mathsf{A}_2) = \mathcal{T}(\mathrm{div}) \quad = \mathbf{H}(\mathrm{div}, \Omega)/\mathring{\mathbf{H}}(\mathrm{div}, \Omega). \tag{4.3.31c}$$

Based on Example 3.3, the linear mappings

$$\mathsf{X}_{\mathbf{grad}} : H^1(\Omega)/\mathring{H}^1(\Omega) \to H^{1/2}(\Gamma), \tag{4.3.32a}$$

$$\mathsf{X}_{\mathbf{curl}} : \mathbf{H}(\mathbf{curl}, \Omega)/\mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \tag{4.3.32b}$$

$$\mathsf{X}_{\mathrm{div}} : \mathbf{H}(\mathrm{div}, \Omega)/\mathring{\mathbf{H}}(\mathrm{div}, \Omega) \to H^{-1/2}(\Gamma) \tag{4.3.32c}$$

defined by

$$\mathsf{X}_{\mathbf{grad}}[u] = \gamma u \qquad \forall u \in H^1(\Omega), \tag{4.3.33a}$$

$$\mathsf{X}_{\mathbf{curl}}[\mathbf{u}] = \gamma_t \mathbf{u} \qquad \forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega), \tag{4.3.33b}$$

$$\mathsf{X}_{\mathrm{div}}[\mathbf{v}] = \gamma_n \mathbf{v} \qquad \forall \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega), \tag{4.3.33c}$$

are the Hilbert space isomorphisms induced by the canonical projections involved in the following commutative diagrams:

$$\begin{array}{ccc} H^1(\Omega) & \overset{\gamma}{\twoheadrightarrow} & H^{1/2}(\Gamma) \\ \boldsymbol{\pi}^t_{\mathbf{grad}} \downarrow & \swarrow \mathsf{X}_{\mathbf{grad}} & \\ \mathcal{T}(\mathbf{grad}) & & \end{array} \qquad \begin{array}{ccc} \mathbf{H}(\mathbf{curl}, \Omega) & \overset{\gamma_t}{\twoheadrightarrow} & \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \\ \boldsymbol{\pi}^t_{\mathbf{curl}} \downarrow & \swarrow \mathsf{X}_{\mathbf{curl}} & \\ \mathcal{T}(\mathbf{curl}) & & \end{array} \qquad \begin{array}{ccc} \mathbf{H}(\mathbf{curl}, \Omega) & \overset{\gamma_n}{\twoheadrightarrow} & H^{-1/2}(\Gamma) \\ \boldsymbol{\pi}^t_{\mathrm{div}} \downarrow & \swarrow \mathsf{X}_{\mathbf{curl}} & \\ \mathcal{T}(\mathrm{div}) & & \end{array}$$

The trace spaces $H^{1/2}(\Gamma)$, $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $H^{-1/2}(\Gamma)$ can therefore be identified with the quotient spaces $\mathcal{T}(\mathbf{grad})$, $\mathcal{T}(\mathbf{curl})$ and $\mathcal{T}(\mathrm{div})$, respectively, as we have already observed in (4.1.9). Under these identifications, the bounded inverse theorem guarantees that the quotient spaces are equipped with equivalent norms. Moreover, due to the Lipschitz regularity of Γ and Sobolev extension theorems, the definitions of $\mathcal{T}(\mathbf{grad})$, $\mathcal{T}(\mathbf{curl})$ and $\mathcal{T}(\mathrm{div})$ are intrinsic, in the sense that the quotient spaces $H^1(\mathbb{R}^3\backslash\overline{\Omega})/\mathring{H}^1(\mathbb{R}^3\backslash\overline{\Omega})$, $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3\backslash\overline{\Omega})/\mathring{\mathbf{H}}(\mathbf{curl}, \mathbb{R}^3\backslash\overline{\Omega})$ and $\mathbf{H}(\mathrm{div}, \mathbb{R}^3\backslash\overline{\Omega})/\mathring{\mathbf{H}}(\mathrm{div}, \mathbb{R}^3\backslash\overline{\Omega})$ are also Hilbert spaces with equivalent norms [17].

Lemma 4.1 *Under Assumption A, if $\mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$, then $\mathsf{A}_k\mathbf{x}_\perp \in \mathcal{D}(\mathsf{A}_k^\top)$ and*

$$(\mathsf{A}_k^\top \mathsf{A}_k + \mathrm{Id})\, \mathbf{x}_\perp = 0. \tag{4.3.34}$$

Proof. Suppose that $\mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$. Since $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k$, we have by definition that

$$\begin{aligned} 0 &= (\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathcal{D}(\mathsf{A}_k)} = (\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathbf{W}_k} + (\mathsf{A}_k\mathbf{x}_\perp, \mathsf{A}_k\mathbf{z}_\circ)_{\mathbf{W}_{k+1}} \\ &= (\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathbf{W}_k} + (\mathsf{A}_k\mathbf{x}_\perp, \mathring{\mathsf{A}}_k\mathbf{z}_\circ)_{\mathbf{W}_{k+1}} \end{aligned} \tag{4.3.35}$$

for all $\mathbf{z}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$, which means

$$(\mathsf{A}_k\mathbf{x}_\perp, \mathring{\mathsf{A}}_k\mathbf{z}_\circ)_{\mathbf{W}_k} = -(\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathbf{W}_k} \qquad \forall \mathbf{z}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k). \tag{4.3.36}$$

So by setting $C_{\mathbf{x}_\perp} := \|\mathbf{x}_\perp\|_{\mathbf{W}_k}$, we conclude from the estimate

$$|(\mathsf{A}_k \mathbf{x}_\perp, \mathring{\mathsf{A}}_k \mathbf{z}_\circ)_{\mathbf{W}_{k+1}}| = |(\mathbf{x}_\perp, \mathbf{z}_\circ)_{\mathbf{W}_k}| \leq \|\mathbf{x}_\perp\|_{\mathbf{W}_k} \|\mathbf{z}_\circ\|_{\mathbf{W}_k} = C_{\mathbf{x}_\perp} \|\mathbf{z}_\circ\|_{\mathbf{W}_k} \qquad \forall \mathbf{z}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k),$$

that $\mathsf{A}_k \mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k^*) = \mathcal{D}(\mathsf{A}_k^\top)$. Then as in (4.2.1), the identity (4.3.34) follows from (4.3.36). ☐

Corollary 4.3.4. *Under Assumption A, the linear map* $\mathsf{A}_k : \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp \to \mathcal{D}(\mathsf{A}_k^\top)$ *is an isometry.*

Proof. Suppose that $\mathbf{x}_\perp \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$. Then, by Lemma 4.1,

$$\|\mathsf{A}_k \mathbf{x}_\perp\|^2_{\mathcal{D}(\mathsf{A}_k^\top)} = \|\mathsf{A}_k \mathbf{x}_\perp\|^2_{\mathbf{W}_{k+1}} + \|\mathsf{A}_k^\top \mathsf{A}_k \mathbf{x}_\perp\|^2_{\mathbf{W}_k} = \|\mathsf{A}_k \mathbf{x}_\perp\|^2_{\mathbf{W}_{k+1}} + \|\mathbf{x}_\perp\|^2_{\mathbf{W}_k} = \|\mathbf{x}_\perp\|^2_{\mathcal{D}(\mathsf{A}_k)}. \tag{4.3.37}$$

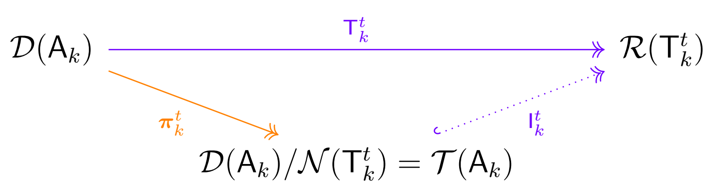
☐

Theorem 4.1 *Under Assumption A, the linear map*

$$\mathsf{I}_k^t : \begin{cases} \mathcal{T}(\mathsf{A}_k) \to \mathcal{R}(\mathsf{T}_k^t) \\ [\mathbf{x}] \mapsto \mathsf{T}_k^t \mathbf{x} \end{cases} \tag{4.3.38}$$

is a well-defined isometric isomorphism.

Proof. Since $\mathcal{D}(\mathring{\mathsf{A}}_k) = \mathcal{N}(\mathsf{T}_k^t)$ by Proposition 4.1, notice that $\mathsf{I}_k^t : \mathcal{T}(\mathsf{A}_k) \to \mathcal{R}(\mathsf{A}_k)$ is simply the well-defined induced isomorphism of modules involved in the commutative diagram



provided by the first isomorphism theorem [19, Chap. 10.2, Thm. 4]. It only remains to show that it is an isometry.

Let $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$. By Proposition 4.1,

$$\|\, \mathsf{I}_k^t[\mathbf{x}]\, \|_{\mathcal{D}(\mathsf{A}_k^\top)'} = \|\mathsf{T}_k^t \mathbf{x}\|_{\mathcal{D}(\mathsf{A}_k^\top)'} = \|\mathsf{T}_k^t(\mathbf{x}_\perp + \mathbf{x}_\circ)\|_{\mathcal{D}(\mathsf{A}_k^\top)'} = \|\mathsf{T}_k^t \mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k^\top)'}. \tag{4.3.39}$$

Using that $\mathsf{A}_k^\top \mathsf{A}_k \mathbf{x}_\perp = -\mathbf{x}_\perp$ by Lemma 4.1, we can choose $\mathbf{y} = \mathsf{A}_k \mathbf{x}_\perp \in \mathcal{D}(\mathsf{A}_k^\top)$ to obtain

$$\begin{aligned} \|\mathsf{T}_k^t \mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k^\top)'} &= \sup_{0 \neq \mathbf{y} \in D(\mathsf{A}_k^\top)} \frac{|\langle \mathsf{T}_k^t \mathbf{x}_\perp, \mathbf{y}\rangle|}{\|\mathbf{y}\|_{\mathcal{D}(\mathsf{A}_k^\top)}} \geq \frac{|\langle \mathsf{T}_k^t \mathbf{x}_\perp, \mathsf{A}_k \mathbf{x}_\perp\rangle|}{\|\mathsf{A}_k \mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k^\top)}} \\ &= \frac{|(\mathsf{A}_k \mathbf{x}_\perp, \mathsf{A}_k \mathbf{x}_\perp)_{\mathbf{W}_{k+1}} - (\mathbf{x}_\perp, \mathsf{A}_k^\top \mathsf{A}_k \mathbf{x}_\perp)_{\mathbf{W}_k}|}{\|\mathsf{A}_k \mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k^\top)}} = \frac{\|\mathbf{x}_\perp\|^2_{\mathcal{D}(\mathsf{A}_k)}}{\|\mathsf{A}_k \mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k^\top)}}. \end{aligned} \tag{4.3.40}$$

Recalling that $\|\mathsf{A}_k \mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k^\top)} = \|\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k)}$ by Corollary 4.3.4, we arrive at the inequality

$$\|\mathsf{T}_k^t \mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k^\top)'} \geq \frac{\|\mathbf{x}_\perp\|^2_{\mathcal{D}(\mathsf{A}_k)}}{\|\mathbf{x}_\perp\|_{D(\mathsf{A}_k)}} = \|\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k)}. \tag{4.3.41}$$

Therefore, on the one hand, $\| \mathsf{I}_k^t[\mathbf{x}] \|_{\mathcal{D}(\mathsf{A}_k^\top)'} \geq \|\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k)} = \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)}$ by (4.3.27).
On the other hand, inserting (4.3.4) in (4.3.39) leads to the estimate

$$\| \mathsf{I}_k^t[\mathbf{x}] \|_{\mathcal{D}(\mathsf{A}_k^\top)'} = \|\mathsf{T}_k^t\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k^\top)} \leq \|\mathsf{T}_k^t\|\|\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k)} = \|\mathbf{x}_\perp\|_{\mathcal{D}(\mathsf{A}_k)} = \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)}, \tag{4.3.42}$$

which concludes the proof. □

It is natural to think of a trace operator as a bounded linear operator from a domain to a trace space. Therefore, based on the identification provided by Theorem 4.1, we introduce the following perspective: in the setting of Definition 4.1, we call *quotient trace* the canonical projection

$$\boldsymbol{\pi}_k^t : \begin{cases} \mathcal{D}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_k) \\ \quad \mathbf{x} \mapsto [\mathbf{x}] \end{cases}. \tag{4.3.43}$$

Notice that because I_k^t is an isomorphism, it follows from $\mathsf{I}_k^t(\mathsf{I}_k^t)^{-1}\mathsf{T}_k^t\mathbf{x} = \mathsf{T}_k^t\mathbf{x} = \mathsf{I}_k^t[\mathbf{x}]$ that

$$\boldsymbol{\pi}_k^t\mathbf{x} = (\mathsf{I}_k^t)^{-1}\mathsf{T}_k^t\mathbf{x}. \tag{4.3.44}$$

4.3.3 Riesz representatives

Let $\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)} : \mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{D}(\mathsf{A}_k^\top)'$ be the Riesz isomorphism defined by $\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}\mathbf{y} = (\mathbf{y}, \cdot)_{\mathcal{D}(\mathsf{A}_k^\top)}$ for all $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$, cf. [11, Thm. 5.5]. Notice that in the first part of the proof of Proposition 4.2, we have shown that the following result holds with $\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi} \in \mathcal{D}(\mathsf{A}_k)$.

Lemma 4.2 *Under Assumption A, if $\boldsymbol{\phi} \in \mathcal{D}(\mathsf{A}_k^*)^\circ$, then $\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi} \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$ with*

$$(\mathsf{A}_k\mathsf{A}_k^\top + \mathrm{Id})\,\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi} = 0 \qquad \textit{and} \qquad \mathsf{T}_k^t(\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi}) = -\boldsymbol{\phi}. \tag{4.3.45}$$

Proof. It only remains to show that in particular $\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi} \in \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp$. Recall that $\mathsf{A}_k^\top := \mathring{\mathsf{A}}_k^*$. Since $\mathsf{A}_k^* \subset \mathsf{A}_k^\top$, we find, using $(\mathsf{A}_k\mathsf{A}_k^\top + \mathrm{Id})\,\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi} = 0$, that for all $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$,

$$\begin{aligned}(\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi}, \mathbf{x}_\circ)_{\mathcal{D}(\mathsf{A}_k)} &= (\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi}, \mathbf{x}_\circ)_{\mathbf{W}_k} + (\mathsf{A}_k\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi}, \mathsf{A}_k\mathbf{x}_\circ)_{\mathbf{W}_{k+1}} \\ &= (\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi}, \mathring{\mathsf{A}}_k\mathbf{x}_\circ)_{\mathbf{W}_{k+1}} - (\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi}, \mathring{\mathsf{A}}_k\mathbf{x}_\circ)_{\mathbf{W}_{k+1}} = 0.\end{aligned} \tag{4.3.46}$$

□

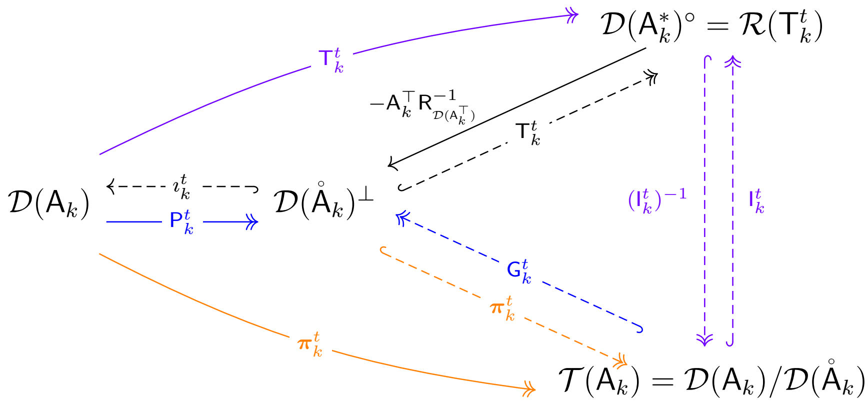
Applying $(\mathsf{I}_k^t)^{-1}$ on both sides of the second identity in Lemma 4.2, we find using (4.3.44) a slightly more explicit expression of the inverse $(\mathsf{I}_k^t)^{-1}$.

Lemma 4.3 *Under Assumption A, we have*

$$(\mathsf{I}_k^t)^{-1}\boldsymbol{\phi} = -\boldsymbol{\pi}_{\mathsf{A}_k}^t(\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi}) \qquad \forall\boldsymbol{\phi} \in D(\mathsf{A}_k^*)^\circ = \mathcal{R}(\mathsf{T}_k^t). \tag{4.3.47}$$

Remark 4.3.5. *The operators $\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to \mathcal{D}(\mathring{\mathsf{A}}_k)^\perp \subset \mathcal{D}(\mathsf{A}_k)$ could be called $\mathcal{D}(\mathsf{A}_k)$-harmonic extension operators.*

In summary, we have shown so far in Section 4.3 that the following diagram is commutative:



4.4 Duality

In this section, we maintain the setting of Assumption A, and we focus on the following snippet of the dual Hilbert complex (cf. Sections 4.2.2 and 4.2.3):

$$\begin{array}{ccccccccc}
\cdots \xleftarrow[\mathsf{A}_{k-2}^\top]{} & \mathcal{D}(\mathsf{A}_{k-2}^\top) \subset \mathbf{W}_{k-1} & \xleftarrow[\mathsf{A}_{k-1}^\top]{} & \mathcal{D}(\mathsf{A}_{k-1}^\top) \subset \mathbf{W}_{k} & \xleftarrow[\mathsf{A}_{k}^\top]{} & \mathcal{D}(\mathsf{A}_{k}^\top) \subset \mathbf{W}_{k+1} & \xleftarrow[\mathsf{A}_{k+1}^\top]{} \cdots \\
& \cup & & \cup & & \cup & \\
\cdots \xleftarrow[\mathsf{A}_{k-2}^*]{} & \mathcal{D}(A_{k-2}^*) \subset \mathbf{W}_{k-1} & \xleftarrow[\mathsf{A}_{k-1}^*]{} & \mathcal{D}(A_{k-1}^*) \subset \mathbf{W}_{k} & \xleftarrow[\mathsf{A}_{k}^*]{} & \mathcal{D}(A_{k}^*) \subset \mathbf{W}_{k+1} & \xleftarrow[\mathsf{A}_{k+1}^*]{} \cdots
\end{array}$$

Recall the simple though important observation that because $(\mathsf{A}_k^\top)^* = \mathring{\mathsf{A}}_k^{**} = \mathring{\mathsf{A}}_k$, then we have $\mathring{\mathsf{A}}_k \subset \mathsf{A}_k \iff \mathsf{A}_k^* \subset \mathsf{A}_k^\top$. Given two operators $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1}$ and $\mathring{\mathsf{A}}_k : \mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathbf{W}_k \to \mathbf{W}_{k+1}$ satisfying Assumption A, the Hilbert space adjoints $\mathsf{A}_k^\top : \mathcal{D}(\mathsf{A}_k^\top) \subset \mathbf{W}_{k+1} \to \mathbf{W}_k$ and $\mathsf{A}_k^* : \mathcal{D}(\mathsf{A}_k^*) \subset \mathbf{W}_{k+1} \to \mathbf{W}_k$ thus also satisfy Assumption A, but with the roles of $\mathbf{W}_k$ and $\mathbf{W}_{k+1}$ swapped. Indeed, both $\mathsf{A}_k^\top$ and A_k^* are densely defined and closed unbounded linear operators between the Hilbert spaces and $\mathsf{A}_k^\top$ is an extension of A_k^*, i.e. $\mathcal{D}(\mathsf{A}_k^*) \subset \mathcal{D}(\mathsf{A}_k^\top)$ and $\mathsf{A}_k^* \mathbf{y}_* = \mathsf{A}_k^\top \mathbf{y}_*$ for all $\mathbf{y}_* \in \mathcal{D}(\mathsf{A}_k^*)$.

In Subsection 4.4.1, the *dual Hilbert trace* T_k^n will be nothing more than the primal Hilbert trace from Definition 4.1 but associated with the pair of operators $\mathsf{A}_k^\top$ and A_k^*. Nevertheless, we state its properties for completeness and to set up notation, because it will be used for the important duality results of Subsection 4.4.2.

4.4.1 Dual traces

As before, it follows from (4.3.1) that the following operator is well-defined.

Definition 4.3 Under Assumption A, we call *dual Hilbert trace* the bounded operator

$$\mathsf{T}_k^n : \mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{D}(\mathsf{A}_k)', \tag{4.4.1}$$

defined for all $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$ and $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ by

$$\langle \mathsf{T}_k^n \mathbf{y}, \mathbf{x} \rangle_{\mathcal{D}(\mathsf{A}_k)'} := (\mathsf{A}_k^\top \mathbf{y}, \mathbf{x})_{\mathbf{W}_k} - (\mathbf{y}, \mathsf{A}_k \mathbf{x})_{\mathbf{W}_{k+1}}. \tag{4.4.2}$$

As in (4.3.4), we have $\|\mathsf{T}_k^n\| = 1$, where $\|\cdot\|$ is the operator norm. Note that for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ and $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$,

$$\langle \mathsf{T}_k^t \mathbf{x}, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k)'} = -\langle \mathbf{x}, \mathsf{T}_k^n \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k^\top)'}. \tag{4.4.3}$$

In other words

$$(\mathsf{T}_k^t)' = -\mathsf{T}_k^n \qquad \text{and} \qquad (\mathsf{T}_k^n)' = -\mathsf{T}_k^t. \tag{4.4.4}$$

The results of Section 4.3 can be mirrored by interchanging the roles of A_k and $\mathsf{A}_k^\top$ (and the roles of $\mathring{\mathsf{A}}_k$ and A_k^* accordingly). We translate a few of them without proof.

Proposition 4.3 (cf. Proposition 4.1) *Under Assumption A, we have*

$$\mathcal{N}(\mathsf{T}_k^n) = \mathcal{D}(\mathsf{A}_k^*). \tag{4.4.5}$$

The next proposition involves the annihilator of $\mathcal{D}(\mathring{\mathsf{A}}_k)$ in $\mathcal{D}(\mathsf{A}_k)'$:

$$\mathcal{D}(\mathring{\mathsf{A}}_k)^\circ := \{\, \phi \in \mathcal{D}(\mathsf{A}_k)' \mid \langle \phi, \mathbf{x}_\circ \rangle = 0,\ \forall \mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k) \,\}. \tag{4.4.6}$$

Proposition 4.4 (cf. Proposition 4.2) *Under Assumption A, we have*

$$\mathcal{R}(\mathsf{T}_k^n) = \mathcal{D}(\mathring{\mathsf{A}}_k)^\circ. \tag{4.4.7}$$

Definition 4.4 (cf. Definition 4.2) We call *dual trace spaces* the quotient spaces

$$\mathcal{T}(\mathsf{A}_k^\top) := \mathcal{D}(\mathsf{A}_k^\top)/\mathcal{D}(\mathsf{A}_k^*), \tag{4.4.8}$$

equipped with the quotient norm

$$\|[\mathbf{y}]\|_{\mathcal{T}(\mathsf{A}_k^\top)} := \inf_{\mathbf{z}_* \in \mathcal{D}(\mathsf{A}_k^*)} \|\mathbf{y} - \mathbf{z}_*\|_{\mathcal{D}(\mathsf{A}_k^\top)} \qquad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top). \tag{4.4.9}$$

Remark 4.4.1. *Just as in Remark 4.3.2, notice that due to Proposition 4.3,*

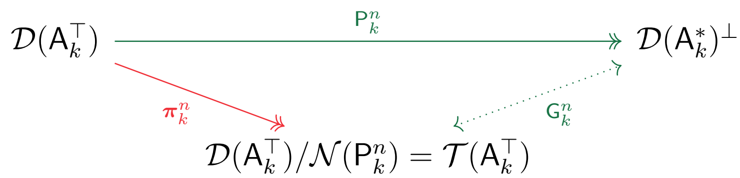
$$\mathcal{T}(\mathsf{A}_k^\top) = \mathcal{D}(\mathsf{A}_k^\top)/\mathcal{N}(\mathsf{T}_{\mathsf{A}_k}^n). \tag{4.4.10}$$

In (4.4.9), we used square brackets to denote the equivalence class in $\mathcal{T}(\mathsf{A}_k^\top)$ of $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$, i.e. $[\mathbf{y}] = \{\mathbf{y} + \mathbf{z}_* \,|\, \mathbf{z}_* \in \mathcal{D}(\mathsf{A}_k^*)\}$. We will write $\boldsymbol{\pi}_k^n : D(\mathsf{A}_k^\top) \to \mathcal{T}(\mathsf{A}_k^\top)$ for the associated canonical projection (quotient map), i.e. $\boldsymbol{\pi}_k^n(\mathbf{y}) = [\mathbf{y}]$. Then, as previously detailed in Subsection 4.3.2, there exists a bounded orthogonal projection $\mathsf{P}_k^n : \mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{D}(\mathsf{A}_k^*)^\perp$ onto the complement space

$$\mathcal{D}(\mathsf{A}_k^*)^\perp := \left\{ \mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top) \mid (\mathbf{y}, \mathbf{z}_*)_{\mathcal{D}(\mathsf{A}_k^\top)} = 0,\ \forall \mathbf{z}_* \in \mathcal{D}(\mathsf{A}_k^*) \right\} \tag{4.4.11}$$

satisfying $\|\mathsf{P}_k^n \mathbf{y}\|_{\mathcal{D}(\mathsf{A}_k^\top)} = \|[\mathbf{y}]\|_{\mathcal{T}(\mathsf{A}_k^\top)}$ for all $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$. We denote by $\imath_k^n : \mathcal{D}(\mathsf{A}_k^*)^\perp \hookrightarrow \mathcal{D}(\mathsf{A}_k^\top)$ the canonical inclusion maps.

The induced operator $\mathsf{G}_k^n : \mathcal{T}(\mathsf{A}_k^\top) \to \mathcal{D}(\mathsf{A}_k^*)^\perp$ involved in the commutative diagram



(4.4.12)

is an isometric isomorphism. Accordingly, any $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$ can be uniquely decomposed as

$$\mathbf{y} = \mathsf{P}_k^n \mathbf{y} + \mathbf{y}_*, \qquad \mathbf{y}_* := (\mathrm{Id} - \mathsf{P}_k^n)\mathbf{y} \in \mathcal{N}(\mathsf{P}_k^n) = \mathcal{D}(\mathsf{A}_k^*). \tag{4.4.13}$$

3D de Rham setting VIII: Classical dual traces

Using (4.4.4), we find for the de Rham complex that

$$\mathsf{T}_{\mathbf{grad}}^n = -\gamma' \circ \gamma_n, \qquad \mathsf{T}_{\mathbf{curl}}^n = \gamma_t' \circ \gamma_t, \qquad \mathsf{T}_{\mathrm{div}}^n = -\gamma_n' \circ \gamma. \tag{4.4.14}$$

Recalling (4.1.8a) to (4.1.8c), we see from the table of the 3D de Rham setting IV that based on Proposition 4.3,

$$\mathcal{N}(\mathsf{T}_{\mathbf{grad}}^n) = \mathcal{N}(\gamma_n), \qquad \mathcal{N}(\mathsf{T}_{\mathbf{curl}}^n) = \mathcal{N}(\gamma_t), \qquad \mathcal{N}(\mathsf{T}_{\mathrm{div}}^n) = \mathcal{N}(\gamma). \tag{4.4.15}$$

The trace spaces provided by Definition 4.4 in this setting are

$$\mathcal{T}(\mathbf{grad}^\top) = \mathcal{T}(\mathrm{div}) = \mathbf{H}(\mathrm{div}, \Omega)/\mathring{\mathbf{H}}(\mathrm{div}, \Omega), \tag{4.4.16a}$$
$$\mathcal{T}(\mathbf{curl}^\top) = \mathcal{T}(\mathbf{curl}) = \mathbf{H}(\mathbf{curl}, \Omega)/\mathring{\mathbf{H}}(\mathbf{curl}, \Omega), \tag{4.4.16b}$$
$$\mathcal{T}(\mathrm{div}^\top) = \mathcal{T}(\mathbf{grad}) = H^1(\Omega)/\mathring{H}^1(\Omega). \tag{4.4.16c}$$

Notice that from (4.3.11), we also

$$(\mathsf{T}_{\mathrm{div}}^t)' = \mathsf{T}_{\mathbf{grad}}^t = -\mathsf{T}_{\mathrm{div}}^n = -(\mathsf{T}_{\mathbf{grad}}^n)' \quad \text{and} \quad (\mathsf{T}_{\mathbf{grad}}^t)' = \mathsf{T}_{\mathrm{div}}^t = -\mathsf{T}_{\mathbf{grad}}^n = -(\mathsf{T}_{\mathrm{div}}^n)'. \tag{4.4.17}$$

Moreover, we see that the skew-symmetry behind (4.1.10) is rooted in the fact that the identity $\mathsf{A}_1 = \mathbf{curl} = \mathsf{A}_1^\top$ leads to skew-symmetry of the pairing

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{curl}\,\mathbf{x}, \mathbf{y})_{\mathbf{L}^2(\Omega)} - (\mathbf{x}, \mathbf{curl}\,\mathbf{y})_{\mathbf{L}^2(\Omega)}. \tag{4.4.18}$$

This is reflected in the observation that $(\gamma_t' \circ \gamma_t)' = (\mathsf{T}_1^t)' = -\mathsf{T}_1^n = -\gamma_t' \circ \gamma_t$, which indeed occurs when duality is taken with respect to the skew-symmetric pairing (4.1.10).

Theorem 4.2 (cf. Theorem 4.1) *Under Assumption A, the linear map*

$$\mathsf{I}_k^n : \begin{cases} \mathcal{T}(\mathsf{A}_k^\top) \to \mathcal{R}(\mathsf{T}_k^n) \\ [\mathbf{y}] \mapsto \mathsf{T}_k^n \mathbf{y} \end{cases} \tag{4.4.19}$$

is a well-defined isometric isomorphism.

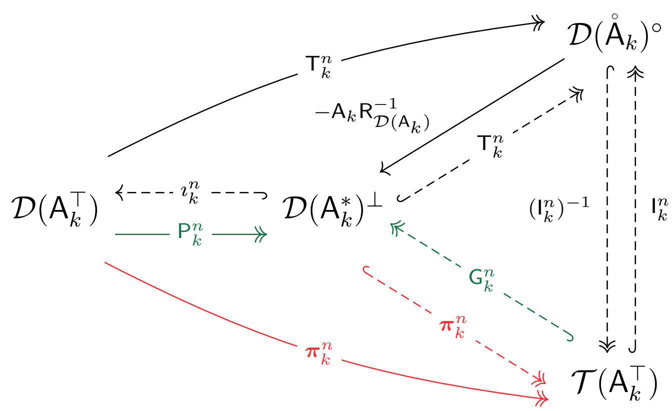
We call *dual quotient trace* the canonical projection (cf. (4.3.43))

$$\boldsymbol{\pi}_k^n : \begin{cases} \mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{T}(\mathsf{A}_k^\top) \\ \mathbf{y} \mapsto [\mathbf{y}] \end{cases}. \tag{4.4.20}$$

Similarly as before, notice that (cf. (4.3.44))

$$\boldsymbol{\pi}_k^n \mathbf{y} = (\mathsf{I}_k^n)^{-1} \mathsf{T}_k^n \mathbf{y}, \tag{4.4.21}$$

and the following diagram is commutative:



4.4.2 Duality of trace spaces

In this section, we show that the trace spaces $\mathcal{T}(\mathsf{A}_k)$ and $\mathcal{T}(\mathsf{A}_k^\top)$ can be put in duality through an isometry. In fact, this follows immediately from a classical result in functional analysis. Indeed, according to [41, Thm. 4.9], we have the isometric isomorphisms

$$\mathcal{D}(\mathsf{A}_k^*)^\circ \cong \left(\mathcal{D}(\mathsf{A}_k^\top)/\mathcal{D}(\mathsf{A}_k^*)\right)' \qquad \text{and} \qquad \mathcal{D}(\mathring{\mathsf{A}}_k)^\circ \cong \left(\mathcal{D}(\mathsf{A}_k)/\mathcal{D}(\mathring{\mathsf{A}}_k)\right)'. \tag{4.4.22}$$

Combining these results with propositions 4.2 and 4.4, along with theorems 4.1 and 4.2,

$$\mathcal{T}(\mathsf{A}_k) \cong \mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ \cong \left(\mathcal{D}(\mathsf{A}_k^\top)/\mathcal{D}(\mathsf{A}_k^*)\right)' = (\mathcal{T}(\mathsf{A}_k^\top))', \tag{4.4.23a}$$

$$\mathcal{T}(\mathsf{A}_k^\top) \cong \mathcal{R}(\mathsf{T}_k^n) = \mathcal{D}(\mathring{\mathsf{A}}_k)^\circ \cong \left(\mathcal{D}(\mathsf{A}_k)/\mathcal{D}(\mathring{\mathsf{A}}_k)\right)' = (\mathcal{T}(\mathsf{A}_k))'. \tag{4.4.23b}$$

Nevertheless, we provide a detailed proof below, not only for convenience and completeness, but also because the exercise is illuminating. We proceed with the definition of a continuous bilinear form on $\mathcal{T}(\mathsf{A}_k) \times \mathcal{T}(\mathsf{A}_k^\top)$ and prove that the associated induced linear operator is an isometry. This pairing will be at the heart of sections 4.7.2 and 4.7, where it will be used to prove that Hilbert complexes affording so-called compact regular decompositions spawn Fredholm trace Hilbert complexes.

Lemma 4.4 *Under Assumption A, the bilinear form*

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_k : \mathcal{T}(\mathsf{A}_k) \times \mathcal{T}(\mathsf{A}_k^\top) \to \mathbb{R}, \tag{4.4.24a}$$

defined by

$$\langle\!\langle [\mathbf{x}], [\mathbf{y}] \rangle\!\rangle_k := (\mathsf{A}_k\mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathsf{A}_k^\top \mathbf{y})_{\mathbf{W}_k} \quad \forall [\mathbf{x}] \in \mathcal{T}(\mathsf{A}_k), \forall [\mathbf{y}] \in \mathcal{T}(\mathsf{A}_k^\top), \tag{4.4.24b}$$

is well-defined and continuous with norm ≤ 1.

Proof. Since $\langle\!\langle [\mathbf{u}], [\mathbf{v}] \rangle\!\rangle_k = \langle \mathsf{T}_k^t \mathbf{x}, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k)'}$, it is well-defined thanks to Proposition 4.1 and Proposition 4.2. By the same propositions, the orthogonal decompositions (4.3.30) and (4.4.13) yield the estimate

$$\begin{aligned}
|\langle \mathsf{T}_k^t \mathbf{x}, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k)'}| &= |\langle \mathsf{T}_k^t \mathsf{P}_k^t \mathbf{x}, \mathsf{P}_k^n \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k)'}| \\
&= |(\mathsf{A}_k \mathsf{P}_k^t \mathbf{x}, \mathsf{P}_k^n \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathsf{P}_k^t \mathbf{x}, \mathsf{A}_k^\top \mathsf{P}_k^n \mathbf{y})_{\mathbf{W}_k}| \\
&\leq \|\mathsf{A}_k \mathsf{P}_k^t \mathbf{x}\|_{\mathbf{W}_{k+1}} \|\mathsf{P}_k^n \mathbf{y}\|_{\mathbf{W}_{k+1}} + \|\mathsf{P}_k^t \mathbf{x}\|_{\mathbf{W}_k} \|\mathsf{A}_k^\top \mathsf{P}_k^n \mathbf{y}\|_{\mathbf{W}_k} \\
&\leq \|\mathsf{P}_k^t \mathbf{x}\|_{\mathcal{D}(\mathsf{A}_k)} \|\mathsf{P}_k^n \mathbf{y}\|_{\mathcal{D}(\mathsf{A}_k^\top)} \\
&= \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)} \|[\mathbf{y}]\|_{\mathcal{T}(\mathsf{A}_k^\top)},
\end{aligned} \tag{4.4.25}$$

showing that the bilinear form is continuous with norm ≤ 1. □

The next result shows in particular that $\mathcal{T}(\mathsf{A}_k)$ and $\mathcal{T}(\mathsf{A}_k^\top)$ can be put in duality through the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_k$.

Theorem 4.3 *Under Assumption A, the bounded linear operator*

$$\mathsf{K}_k : \begin{cases} \mathcal{T}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_k^\top)' \\ \quad [\mathbf{x}] \mapsto \langle\!\langle [\mathbf{x}], \cdot \rangle\!\rangle_k \end{cases} \tag{4.4.26}$$

induced by the bilinear form defined in Lemma 4.4 is an isometric isomorphism.

Proof. The key to the proof is that (4.4.24b) permits us to appeal to Theorem 4.1.

Notice that since $\mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ$, it follows from the orthogonal decomposition (4.4.13) that K_k is the pullback by G_k^n of I_k^t, i.e. $\mathsf{K}_k[\mathbf{x}]([\mathbf{y}]) = \mathsf{I}_k^t[\mathbf{x}](\mathsf{G}_k^n[\mathbf{y}])$. We first show that it is an isomorphism.

If $\mathsf{K}_k[\mathbf{x}] = \mathsf{K}_k[\mathbf{z}]$, then since G_k^n is an isomorphism onto $\mathcal{D}(\mathsf{A}_k^*)^\perp$, it then follows from Proposition 4.2 and decomposition (4.4.13) that $\mathsf{I}_k^t[\mathbf{x}](\mathbf{y}) = \mathsf{I}_k^t[\mathbf{z}](\mathbf{y})$ for all $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$. But I_k^t is also an isomorphism, so $\mathsf{I}_k^t[\mathbf{x}] = \mathsf{I}_k^t[\mathbf{z}]$ implies that $\mathbf{x} = \mathbf{z}$ and we conclude that K_k is injective.

Suppose that $\phi \in \mathcal{T}(\mathsf{A}_k^\top)'$. Then the pullback of ϕ by the canonical quotient map $\boldsymbol{\pi}_k^n : \mathcal{D}(\mathsf{A}_k^\top) \to \mathcal{T}(\mathsf{A}_k^\top)$ is a bounded linear functional on $\mathcal{D}(\mathsf{A}_k^\top)$, i.e. $\phi \circ \boldsymbol{\pi}_k^n \in \mathcal{D}(\mathsf{A}_k^\top)'$. Indeed, this simply holds because

$$|\phi(\boldsymbol{\pi}_k^n\mathbf{y})| = \|\phi\|\|\boldsymbol{\pi}_k^n\mathbf{y}\|_{\mathcal{T}(\mathsf{A}_k^\top)} \leq \|\phi\|\|\boldsymbol{\pi}_k^n\|\|\mathbf{y}\|_{\mathcal{D}(\mathsf{A}_k^\top)} \qquad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top). \tag{4.4.27}$$

Moreover, since $\mathcal{N}(\boldsymbol{\pi}_k^n) = \mathcal{D}(\mathsf{A}_k^*)$, we find in particular that $\phi \circ \boldsymbol{\pi}_k^n \in \mathcal{D}(\mathsf{A}_k^*)^\circ = \mathcal{R}(\mathsf{T}_k^t)$. But I_k^t is an isomorphism onto $\mathcal{R}(\mathsf{T}_k^t)$, so there exists $[\mathbf{x}] \in \mathcal{T}(\mathsf{A}_k)$ such that $\mathsf{I}_k^t[\mathbf{x}] = \phi \circ \boldsymbol{\pi}_k^n$. Evaluating

$$\mathsf{K}_k[\mathbf{x}] = \mathsf{I}_k^t[\mathbf{x}] \circ \mathsf{G}_k^n = \phi \circ \boldsymbol{\pi}_k^n \circ \mathsf{G}_k^n = \phi \tag{4.4.28}$$

shows that K_k is surjective.

We now prove that K_k is an isometry. Using similar arguments as above, we estimate

$$\|\mathsf{K}_k[\mathbf{x}]\| = \sup_{\substack{[\mathbf{y}]\in\mathcal{T}(\mathsf{A}_k^\top),\\ \|[\mathbf{y}]\|_{\mathcal{T}(\mathsf{A}_k^\top)}=1}} |\mathsf{K}_k[\mathbf{x}]([\mathbf{y}])| = \sup_{\substack{\mathbf{y}_\perp\in\mathcal{D}(\mathsf{A}_k^*)^\perp,\\ \|\mathbf{y}_\perp\|_{\mathcal{D}(\mathsf{A}_k^\top)}=1}} |\mathsf{I}_k^t[\mathbf{x}](\mathbf{y}_\perp)| = \|\mathsf{I}_k^t[\mathbf{x}]\| = \|[\mathbf{x}]\|_{\mathcal{T}(\mathsf{A}_k)}. \tag{4.4.29}$$

□

We have arrived at an integration by parts formula involving the traces from Subsection 4.3.1 and Subsection 4.4.1: for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ and $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top)$,

$$(\mathsf{A}_k\mathbf{x}, \mathbf{y})_{\mathbf{W}_{k+1}} - (\mathbf{x}, \mathsf{A}_k^\top\mathbf{y})_{\mathbf{W}_k} = \langle\!\langle \boldsymbol{\pi}_k^t\mathbf{x}, \boldsymbol{\pi}_k^n\mathbf{y}\rangle\!\rangle_{\mathsf{A}_k}. \tag{4.4.30}$$

Theorem 4.3, in combination with (4.1.12a) and (4.1.12b), reveals the abstract version of the duality observed for the de Rham complex in Section 4.1.

4.5 Operators on Trace Spaces

Starting from this section, we start exploiting more of the structure of Hilbert complexes by introducing the *minimal* Hilbert complex setting required to define what we will call *surface operators*. We “zoom in” on short snippets of (4.2.5a) and (4.2.8a) of the form

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\mathsf{A}_{k-1}} & \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k & \xrightarrow{\mathsf{A}_k} & \mathcal{D}(\mathsf{A}_{k+1}) \subset \mathbf{W}_{k+1} & \xrightarrow{\mathsf{A}_{k+1}} & \mathcal{D}(\mathsf{A}_{k+2}) \subset \mathbf{W}_{k+2} & \xrightarrow{\mathsf{A}_{k+2}} \cdots \\
& \cup & & \cup & & \cup & \\
\cdots \xrightarrow{\mathring{\mathsf{A}}_{k-1}} & \mathcal{D}(\mathring{\mathsf{A}}_k) \subset \mathbf{W}_k & \xrightarrow{\mathring{\mathsf{A}}_k} & \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \subset \mathbf{W}_{k+1} & \xrightarrow{\mathring{\mathsf{A}}_{k+1}} & \mathcal{D}(\mathring{\mathsf{A}}_{k+2}) \subset \mathbf{W}_{k+2} & \xrightarrow{\mathring{\mathsf{A}}_{k+2}} \cdots
\end{array} \tag{4.5.1}$$

We may call the highlighted sequences “minimal Hilbert complexes”. The index k should be considered arbitrary but fixed in this section.

3D de Rham setting IX: Minimal Hilbert complexes

Based on the 3D de Rham setting III and IV, we obtain two minimal complexes such as (4.5.1). For $k = 0$, we have

$$\begin{aligned} H^1(\Omega) \subset L^2(\Omega) &\xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2 \xrightarrow{\mathbf{curl}} \mathbf{L}^2, \\ \mathring{H}^1(\Omega) \subset L^2(\Omega) &\xrightarrow{\mathbf{grad}} \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2 \xrightarrow{\mathbf{curl}} \mathbf{L}^2. \end{aligned} \tag{4.5.2}$$

For $k = 1$, we get

$$\begin{aligned} \mathbf{H}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2 &\xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div}, \Omega) \subset \mathbf{L}^2 \xrightarrow{\mathrm{div}} \subset \mathbf{L}^2, \\ \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \subset \mathbf{L}^2 &\xrightarrow{\mathbf{curl}} \mathring{\mathbf{H}}(\mathrm{div}, \Omega) \subset \mathbf{L}^2 \xrightarrow{\mathrm{div}} \mathbf{L}^2. \end{aligned} \tag{4.5.3}$$

4.5.1 Surface operators in domains

Notice that due to the complex property, we have in particular that $\mathcal{R}(\mathsf{A}_k) \subset \mathcal{D}(\mathsf{A}_{k+1})$ and $\mathcal{R}(\mathsf{A}_{k+1}^\top) \subset \mathcal{D}(\mathsf{A}_k^\top)$. The following key operators are thus well-defined.

Definition 4.5 We call *surface operators* the bounded linear maps

$$\mathsf{D}_k^t := (\mathsf{A}_{k+1}^\top)' : \mathcal{D}(\mathsf{A}_k^\top)' \to \mathcal{D}(\mathsf{A}_{k+1}^\top)', \tag{4.5.4a}$$
$$\mathsf{D}_{k+1}^n := \mathsf{A}_k' : \mathcal{D}(\mathsf{A}_{k+1})' \to \mathcal{D}(\mathsf{A}_k)', \tag{4.5.4b}$$

dual to $\mathsf{A}_{k+1}^\top : \mathcal{D}(\mathsf{A}_{k+1}^\top) \to \mathcal{D}(\mathsf{A}_k^\top)$ and $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathsf{A}_{k+1})$, respectively. Equivalently,

$$\langle \mathsf{D}_k^t \boldsymbol{\phi}, \mathbf{z} \rangle_{\mathcal{D}(\mathsf{A}_{k+1}^\top)'} = \langle \boldsymbol{\phi}, \mathsf{A}_{k+1}^\top \mathbf{z} \rangle_{\mathcal{D}(\mathsf{A}_k^\top)'}, \quad \forall \boldsymbol{\phi} \in \mathcal{D}(\mathsf{A}_k^\top)', \forall \mathbf{z} \in \mathcal{D}(\mathsf{A}_{k+1}^\top) \subset \mathbf{W}_{k+2}, \tag{4.5.5a}$$
$$\langle \mathsf{D}_{k+1}^n \boldsymbol{\psi}, \mathbf{y} \rangle_{\mathcal{D}(\mathsf{A}_k)'} = \langle \boldsymbol{\psi}, \mathsf{A}_k \mathbf{x} \rangle_{\mathcal{D}(\mathsf{A}_{k+1})'}, \quad \forall \boldsymbol{\psi} \in \mathcal{D}(\mathsf{A}_{k+1})', \forall \mathbf{x} \in \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k. \tag{4.5.5b}$$

Remark 4.5.1. *Recall the distinction made in Subsection 4.2.1 between the notation for bounded and unbounded linear operators. We point out that in Definition 4.5, the operators* $\mathsf{A}_{k+1}^\top : \mathcal{D}(\mathsf{A}_{k+1}^\top) \to \mathcal{D}(\mathsf{A}_k^\top)$ *and* $\mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathsf{A}_{k+1})$ *are* bounded.

Remark 4.5.2. *The name 'surface operators' was chosen by analogy with standard surface operators on the boundary of a domain, despite the fact that there is no boundary involved in the above definition. The relation between Definition 4.5 and standard surface operators is made more explicit in the 3D de Rham settings X and XI.*

3D de Rham setting X: Surface operators in domains

In the 3D de Rham setting IX, we find the surface operators

$$\mathsf{D}_0^t := \mathbf{curl}' : \mathbf{H}(\mathrm{div}, \Omega)' \to \mathbf{H}(\mathbf{curl}, \Omega)', \tag{4.5.6a}$$

$$\mathsf{D}_1^t := (-\mathbf{grad})' : \mathbf{H}(\mathbf{curl}, \Omega)' \to \widetilde{H}^{-1}(\Omega), \tag{4.5.6b}$$

dual to the *bounded* operators

$$\mathbf{curl} : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}(\mathrm{div}, \Omega) \quad \text{and} \quad -\mathbf{grad} : H^1(\Omega) \to \mathbf{H}(\mathbf{curl}, \Omega), \tag{4.5.7}$$

where we have written $\widetilde{H}^{-1}(\Omega) := H^1(\Omega)'$. That is,

$$\begin{aligned} \langle \mathsf{D}_0^t \boldsymbol{\phi}, \mathbf{v} \rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} &= \langle \boldsymbol{\phi}, \mathbf{curl}\, \mathbf{v} \rangle_{\mathbf{H}(\mathrm{div},\Omega)'}, && \forall \boldsymbol{\phi} \in \mathbf{H}(\mathrm{div}, \Omega)', \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega), \\ \langle \mathsf{D}_1^t \boldsymbol{\phi}, \mathbf{u} \rangle_{\widetilde{H}^{-1}(\Omega)} &= \langle \boldsymbol{\psi}, -\mathbf{grad} u \rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} && \forall \boldsymbol{\phi} \in \mathbf{H}(\mathbf{curl}, \Omega)', \forall u \in H^1(\Omega). \end{aligned}$$

In the adjoint perspective, the bounded linear operators

$$\mathsf{D}_1^n := \mathbf{grad}' : \mathbf{H}(\mathbf{curl}, \Omega)' \to \widetilde{H}^{-1}(\Omega) \tag{4.5.8a}$$

$$\mathsf{D}_2^n := \mathbf{curl}' : \mathbf{H}(\mathrm{div}, \Omega)' \to \mathbf{H}(\mathbf{curl}, \Omega)' \tag{4.5.8b}$$

are dual to the bounded linear operators

$$\mathbf{grad} : H^1(\Omega) \to \mathbf{H}(\mathbf{curl}, \Omega) \quad \text{and} \quad \mathbf{curl} : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}(\mathrm{div}, \Omega). \tag{4.5.9}$$

That is,

$$\begin{aligned} \langle \mathsf{D}_1^n \boldsymbol{\phi}, u \rangle_{\widetilde{H}^{-1}(\Omega)} &= \langle \boldsymbol{\phi}, \mathbf{grad} u \rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} && \forall \boldsymbol{\phi} \in \mathbf{H}(\mathbf{curl}, \Omega)', \forall u \in H^1(\Omega), \\ \langle \mathsf{D}_2^n \boldsymbol{\psi}, \mathbf{v} \rangle_{\mathbf{H}(\mathbf{curl},\Omega)'} &= \langle \boldsymbol{\psi}, \mathbf{curl}\, \mathbf{v} \rangle_{\mathbf{H}(\mathrm{div},\Omega)'} && \forall \boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}, \Omega)', \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega). \end{aligned}$$

Since

$$\mathcal{R}(\mathsf{A}_k) \subset \mathcal{D}(\mathsf{A}_{k+1}) = \mathcal{D}(\mathsf{T}_{k+1}^t), \qquad \mathcal{R}(\mathsf{T}_k^t) \subset \mathcal{D}(\mathsf{A}_k^\top)' = \mathcal{D}(\mathsf{D}_k^t), \tag{4.5.11a}$$

$$\mathcal{R}(\mathsf{A}_{k+1}^\top) \subset \mathcal{D}(\mathsf{A}_k^\top) = \mathcal{D}(\mathsf{T}_k^n), \qquad \mathcal{R}(\mathsf{T}_{k+1}^n) \subset \mathcal{D}(\mathsf{A}_{k+1})' = \mathcal{D}(\mathsf{D}_{k+1}^n), \tag{4.5.11b}$$

the linear operators

$$\mathsf{D}_k^t \circ \mathsf{T}_k^t : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathsf{A}_{k+1}^\top)', \qquad \mathsf{T}_{k+1}^t \circ \mathsf{A}_k : \mathcal{D}(\mathsf{A}_k) \to \mathcal{D}(\mathsf{A}_{k+1}^\top)', \tag{4.5.12a}$$

$$\mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n : \mathcal{D}(\mathsf{A}_{k+1}^\top) \to \mathcal{D}(\mathsf{A}_k)', \qquad \mathsf{T}_k^n \circ \mathsf{A}_{k+1}^\top : \mathcal{D}(\mathsf{A}_{k+1}^\top) \to \mathcal{D}(\mathsf{A}_k)', \tag{4.5.12b}$$

are also well-defined and bounded.

Lemma 4.5 *Assumption A implies the following commuting relations:*

$$-\mathsf{D}_k^t \circ \mathsf{T}_k^t = \mathsf{T}_{k+1}^t \circ \mathsf{A}_k \quad \textit{and} \quad -\mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n = \mathsf{T}_k^n \circ \mathsf{A}_{k+1}^\top. \tag{4.5.13}$$

Proof. By symmetry, we need to verify only one relation. Recall that because of the complex property $\mathsf{A}_{k+1} \circ \mathsf{A}_k = 0$, we also have $\mathsf{A}_k^\top \circ \mathsf{A}_{k+1}^\top = 0$. Therefore, for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k$ and $\mathbf{z} \in \mathcal{D}(\mathsf{A}_{k+1}^\top) \subset \mathbf{W}_{k+2}$, we have on the one hand that

$$\begin{aligned}\langle \mathsf{D}_k^t \mathsf{T}_k^t \mathbf{x}, \mathbf{z}\rangle_{\mathcal{D}(\mathsf{A}_{k+1}^\top)'} = \langle \mathsf{T}_k^t \mathbf{x}, \mathsf{A}_{k+1}^\top \mathbf{z}\rangle_{\mathcal{D}(\mathsf{A}_k^\top)'} &= (\mathsf{A}_k \mathbf{x}, \mathsf{A}_{k+1}^\top \mathbf{z})_{\mathbf{W}_{k+1}} - (\mathbf{u}, \mathsf{A}_k^\top \mathsf{A}_{k+1}^\top \mathbf{z})_{\mathbf{W}_k} \\ &= (\mathsf{A}_k \mathbf{x}, \mathsf{A}_{k+1}^\top \mathbf{z})_{\mathbf{W}_{k+1}}.\end{aligned} \tag{4.5.14}$$

On the other hand, we also evaluate

$$\begin{aligned}\langle \mathsf{T}_{k+1}^t \mathsf{A}_k \mathbf{x}, \mathbf{z}\rangle_{\mathcal{D}(\mathsf{A}_{k+1}^\top)'} &= (\mathsf{A}_{k+1}\mathsf{A}_k \mathbf{x}, \mathbf{z})_{\mathbf{W}_{k+2}} - (\mathsf{A}_k \mathbf{x}, \mathsf{A}_{k+1}^\top \mathbf{z})_{\mathbf{W}_{k+1}} \\ &= -(\mathsf{A}_k \mathbf{x}, \mathsf{A}_{k+1}^\top \mathbf{z})_{\mathbf{W}_{k+1}}.\end{aligned} \tag{4.5.15}$$

□

Remark 4.5.3. *Consistent with* (4.4.4), $(\mathsf{D}_k^t \circ \mathsf{T}_k^t)' = \mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n$ *and* $\mathsf{D}_k^t \circ \mathsf{T}_k^t = (\mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n)'$.

Lemma 4.5 states that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{D}(\mathsf{A}_k) & \xrightarrow{\mathsf{A}_k} & \mathcal{D}(\mathsf{A}_{k+1}) \\ \downarrow{\scriptstyle \mathsf{T}_k^t} & & \downarrow{\scriptstyle \mathsf{T}_{k+1}^t} \\ \mathcal{R}(\mathsf{T}_k^t) & \xrightarrow{-\mathsf{D}_k^t} & \mathcal{R}(\mathsf{T}_{k+1}^t) \end{array} \qquad \begin{array}{ccc} \mathcal{D}(\mathsf{A}_{k+1}^\top) & \xrightarrow{\mathsf{A}_{k+1}^\top} & \mathcal{D}(\mathsf{A}_k^\top) \\ \downarrow{\scriptstyle \mathsf{T}_{k+1}^n} & & \downarrow{\scriptstyle \mathsf{T}_k^n} \\ \mathcal{R}(\mathsf{T}_{k+1}^n) & \xrightarrow{-\mathsf{D}_{k+1}^n} & \mathcal{R}(\mathsf{T}_k^n) \end{array} \tag{4.5.16}$$

An important consequence of this result is that

$$\mathsf{D}_k^t(\mathcal{R}(\mathsf{T}_k^t)) \subset \mathcal{R}(\mathsf{T}_{k+1}^t) = \mathcal{D}(A_{k+1}^*)^\circ, \tag{4.5.17}$$

an observation that is key to the introduction of trace Hilbert complexes in later sections.

3D de Rham setting XI: Commutative relations

In the 3D de Rham setting, it follows from (4.4.17) that the four relations obtained from Lemma 4.5 boil down to the single identity

$$\mathbf{grad}' \gamma_t' \circ \gamma_t = \gamma' \circ \gamma_n \mathbf{curl}. \tag{4.5.18}$$

In particular, (4.5.18) states that for all $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $v \in H^1(\Omega)$,

$$\int_\Gamma v\, \mathbf{n} \cdot \mathbf{curl}\, \mathbf{u}\, \mathrm{d}\sigma = \int_\Gamma \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) \cdot (\mathbf{grad}\, v \times \mathbf{n})\, \mathrm{d}\sigma. \tag{4.5.19}$$

Recall that $\mathbf{n} \cdot \mathbf{curl} = \mathrm{curl}_\Gamma \circ \gamma_t$ on $\mathbf{H}(\mathbf{curl}, \Omega)$, while the $L^2(\Gamma)$-dual operator $\mathrm{curl}_\Gamma = \mathbf{curl}_\Gamma'$ is such that $\mathbf{grad} \cdot \times \mathbf{n} = \mathbf{curl}_\Gamma \circ \gamma$ on $H^1(\Omega)$. Therefore, (4.5.19) expresses that

$$\int_\Gamma u\, \mathrm{curl}_\Gamma \mathbf{u}\, \mathrm{d}\sigma = \int_\Gamma \mathbf{curl}_\Gamma \mathbf{u} \cdot u\, \mathrm{d}\sigma \quad \forall u \in H^{1/2}(\Gamma),\ \mathbf{u} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma). \tag{4.5.20}$$

We conclude that the duality between the surface operators and their surface vector calculus counterparts in classical trace spaces is indeed captured by the duality in Subsection 4.4.2 and Lemma 4.5.

We point out that if one works with the $\mathbf{L}^2(\Gamma)$-pairing instead of the skew-symmetric pairing (4.1.10) from the start, then the two isometrically isomorphic perspectives of tangential and "rotated" tangential traces from [15] are also captured by the abstract theory. Indeed, by introducing the trace $\gamma_\tau : \cdot \mapsto \cdot \times \mathbf{n}$, one obtains $\mathsf{T}^t_{\mathbf{curl}} = \gamma_t' \circ \gamma_\tau$ and $\mathsf{T}^n_{\mathbf{curl}} = -\gamma_\tau' \circ \gamma_\tau$, which also satisfy (4.4.4). With these definitions, Lemma 4.5 leads to two identities corresponding to (4.5.20) and

$$\int_\Gamma v \operatorname{div}_\Gamma \mathbf{v} \, \mathrm{d}\sigma = -\int_\Gamma \mathbf{grad}_\Gamma v \cdot \mathbf{v} \mathrm{d}\sigma \quad \forall v \in H^{1/2}(\Gamma),\ \mathbf{v} \in \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma), \tag{4.5.21}$$

which is a "rotated" version of (4.5.20), where $\gamma_n \mathbf{curl} = \operatorname{div}_\Gamma \gamma_\tau$ on $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ is defined by analogy with (4.1.7b).

4.5.2 Surface operators in quotient spaces

Let us investigate the properties of the linear operators between trace spaces induced by the surface operators defined in Subsection 4.5.1.

Definition 4.6 We call *quotient surface operators* the bounded linear maps

$$\mathsf{S}^t_k : \begin{cases} \mathcal{T}(\mathsf{A}_k) \to \mathcal{T}(\mathsf{A}_{k+1}) \\ \quad [\mathbf{x}] \mapsto \boldsymbol{\pi}^t_{k+1} \mathsf{A}_k \mathbf{x} \end{cases} \quad \text{and} \quad \mathsf{S}^n_{k+1} : \begin{cases} \mathcal{T}(\mathsf{A}^\top_{k+1}) \to \mathcal{T}(\mathsf{A}^\top_k) \\ \quad [\mathbf{z}] \mapsto \boldsymbol{\pi}^n_k \mathsf{A}^\top_{k+1} \mathbf{z} \end{cases}. \tag{4.5.22}$$

We verify that S^t_k is well-defined. The analogous result holds for S^n_{k+1} by duality. Suppose that $\mathbf{x}_\circ \in \mathcal{D}(\mathring{\mathsf{A}}_k)$. By the complex property, we evaluate

$$\begin{aligned} \langle\!\langle \boldsymbol{\pi}^t_{k+1} \mathsf{A}_k \mathbf{x}_\circ, [\mathbf{z}] \rangle\!\rangle_{\mathsf{A}_{k+1}} &= (\mathsf{A}_{k+1} \mathsf{A}_k \mathbf{x}_\circ, \mathbf{z})_{\mathbf{W}_{k+2}} - (\mathsf{A}_k \mathbf{x}_\circ, \mathsf{A}^\top_{k+1} \mathbf{z})_{\mathbf{W}_{k+1}} \\ &= -(\mathring{\mathsf{A}}_k \mathbf{x}_\circ, \mathring{\mathsf{A}}^*_{k+1} \mathbf{z})_{\mathbf{W}_{k+1}} \\ &= -(\mathring{\mathsf{A}}_{k+1} \mathring{\mathsf{A}}_k \mathbf{x}_\circ, \mathbf{z})_{\mathbf{W}_{k+1}} = 0 \end{aligned} \tag{4.5.23}$$

for all $\mathbf{z} \in \mathcal{D}(\mathsf{A}^\top_{k+1}) \subset \mathbf{W}_{k+2}$. By Subsection 4.4.2, we conclude that $\boldsymbol{\pi}^t_{k+1} \mathsf{A}_k \mathring{\mathbf{x}} = 0$.

From the above, we also find that for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k) \subset \mathbf{W}_k$ and $\mathbf{z} \in \mathcal{D}(\mathsf{A}^\top_{k+1}) \subset \mathbf{W}_{k+2}$,

$$\langle\!\langle \mathsf{S}^t_k \circ \boldsymbol{\pi}^t_k \mathbf{x}, \boldsymbol{\pi}^n_{k+1} \mathbf{z} \rangle\!\rangle_{k+1} = -(\mathsf{A}_k \mathbf{x}, \mathsf{A}^\top_{k+1} \mathbf{z})_{\mathbf{W}_{k+1}} = -\langle\!\langle \boldsymbol{\pi}^t_k \mathbf{x}, \mathsf{S}^n_{k+1} \circ \boldsymbol{\pi}^n_{k+1} \mathbf{z} \rangle\!\rangle_k. \tag{4.5.24}$$

We can view the identity

$$\langle\!\langle \mathsf{S}^t_k [\mathbf{x}], [\mathbf{z}] \rangle\!\rangle_{k+1} = -\langle\!\langle [\mathbf{x}], \mathsf{S}^n_{k+1} [\mathbf{z}] \rangle\!\rangle_k \quad \forall [\mathbf{x}] \in \mathcal{T}(\mathsf{A}_k), \forall [\mathbf{z}] \in \mathcal{T}(\mathsf{A}^\top_{k+1}), \tag{4.5.25}$$

as an integration by parts formula in (quotient) trace spaces.

Recalling Subsection 4.4.2, we can rewrite (4.5.25) as

$$\mathsf{K}_{k+1} \circ \mathsf{S}^t_k = -(\mathsf{S}^n_{k+1})' \circ \mathsf{K}_k, \tag{4.5.26}$$

which gives rise to the commutative diagram

$$\begin{array}{ccc} \mathcal{T}(\mathsf{A}_k^\top)' & \xrightarrow{-(\mathsf{S}_{k+1}^n)'} & \mathcal{T}(\mathsf{A}_{k+1}^\top)' \\ \mathsf{K}_k \Big\Uparrow & & \Big\Uparrow \mathsf{K}_{k+1} \\ \mathcal{T}(\mathsf{A}_k) & \xrightarrow{\mathsf{S}_k^t} & \mathcal{T}(\mathsf{A}_{k+1}) \end{array} \tag{4.5.27}$$

We end this section by putting the results of the subsections 4.5.1 and 4.5.2 together into a single diagram. On the one hand, for all $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$ and $\mathbf{z} \in \mathcal{D}(\mathsf{A}_{k+1}^\top)$, we find from the proof of Lemma 4.5 that

$$\begin{aligned} \langle \mathsf{D}_k^t \circ \mathsf{T}_k^t\,\mathbf{x}, \mathbf{z}\rangle_{\mathcal{D}(\mathsf{A}_{k+1}^\top)'} &= \langle \mathsf{D}_{k+1}^n \circ \mathsf{T}_{k+1}^n \mathbf{z}, \mathbf{x}\rangle_{\mathcal{D}(\mathsf{A}_k)'} \\ &= \langle\!\langle \boldsymbol{\pi}_k^t\,\mathbf{x}, \mathsf{S}_{k+1}^n \circ \boldsymbol{\pi}_{k+1}^n\,\mathbf{z}\rangle\!\rangle_k \\ &= -\langle\!\langle \mathsf{S}_k^t \circ \boldsymbol{\pi}_k^t\,\mathbf{x}, \boldsymbol{\pi}_{k+1}^n\,\mathbf{z}\rangle\!\rangle_{k+1}. \end{aligned} \tag{4.5.28}$$

On the other hand, we have by definition

$$\mathsf{S}_k^t \boldsymbol{\pi}_k^t\,\mathbf{x} = \boldsymbol{\pi}_{k+1}^t \mathsf{A}_k \mathbf{x} \qquad \text{and} \qquad \mathsf{S}_{k+1}^n \boldsymbol{\pi}_{k+1}^n\,\mathbf{z} = \boldsymbol{\pi}_k^n \mathsf{A}_{k+1}^\top \mathbf{z}. \tag{4.5.29}$$

Also recall (4.3.44) and (4.4.21). In summary, the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{T}(\mathsf{A}_k) & \xrightarrow{\mathsf{S}_k^t} & \mathcal{T}(\mathsf{A}_{k+1}) \\ \boldsymbol{\pi}_k^t \Big\Uparrow & & \Big\Uparrow \boldsymbol{\pi}_{k+1}^t \\ \mathcal{D}(\mathsf{A}_k) & \xrightarrow{\mathsf{A}_k} & \mathcal{D}(\mathsf{A}_{k+1}) \\ \mathsf{T}_k^t \Big\Downarrow & & \Big\Downarrow \mathsf{T}_{k+1}^t \\ \mathcal{R}(\mathsf{T}_k^t) & \xrightarrow{-\mathsf{D}_k^t} & \mathcal{R}(\mathsf{T}_{k+1}^t) \end{array} \quad (\mathsf{I}_k^t,\ \mathsf{I}_{k+1}^t) \qquad \begin{array}{ccc} \mathcal{T}(\mathsf{A}_{k+1}^\top) & \xrightarrow{\mathsf{S}_{k+1}^n} & \mathcal{T}(\mathsf{A}_k^\top) \\ \boldsymbol{\pi}_{k+1}^n \Big\Uparrow & & \Big\Uparrow \boldsymbol{\pi}_k^n \\ \mathcal{D}(\mathsf{A}_{k+1}^\top) & \xrightarrow{\mathsf{A}_{k+1}^\top} & \mathcal{D}(\mathsf{A}_k^\top) \\ \mathsf{T}_{k+1}^n \Big\Downarrow & & \Big\Downarrow \mathsf{T}_k^n \\ \mathcal{R}(\mathsf{T}_{k+1}^n) & \xrightarrow{-\mathsf{D}_{k+1}^n} & \mathcal{R}(\mathsf{T}_k^n) \end{array} \quad (\mathsf{I}_{k+1}^n,\ \mathsf{I}_k^n) \tag{4.5.30}$$

4.6 Trace spaces: Characterization by Regular Subspaces

4.6.1 Bounded regular decompositions

In this section, we augment Assumption A. We first detail results in the setting of Definition 4.1 for primal Hilbert traces, then formulate their analogs in the dual setting of Definition 4.3. By symmetry, the primal and dual settings are evidently two faces of the same coin. From an abstract point of view, they are identical. Nevertheless, the dual setting is presented for convenience. The two settings are covered independently to avoid loosing sight of the core considerations.

4.6.1.1 Primal decomposition

Now, we aim at a more detailed characterization of the space $\mathcal{D}(\mathsf{A}_k^\top)'$. Recall that by the complex property, $\mathcal{R}(\mathsf{A}_{k+1}^\top) \subset \mathcal{D}(\mathsf{A}_k^\top)$.

We refer to [35, Def. 2.12] for the next assumption, which introduces additional structure.

Assumption B For all $k \in \mathbb{Z}$, Assumption A holds along with the following hypotheses:

I The Hilbert spaces $\mathbf{W}_k^+ \subset \mathbf{W}_k$ are such that the inclusion maps spawn continuous and dense embeddings

$$\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathsf{A}_{k-1}^\top). \tag{4.6.1}$$

II There exist bounded linear operators

$$\mathsf{L}_{k+1}^t : \mathcal{D}(\mathsf{A}_k^\top) \to \mathbf{W}_{k+1}^+ \qquad \text{and} \qquad \mathsf{V}_{k+1}^t : \mathcal{D}(\mathsf{A}_k^\top) \to \mathbf{W}_{k+2}^+ \tag{4.6.2}$$

such that

$$\mathbf{y} = (\mathsf{L}_{k+1}^t + \mathsf{A}_{k+1}^\top \mathsf{V}_{k+1}^t)\,\mathbf{y} \qquad\qquad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_k^\top). \tag{4.6.3}$$

III The Hilbert spaces

$$\mathbf{W}_{k+2}^+(\mathsf{A}_{k+1}^\top) := \left\{ \mathbf{z} \in \mathbf{W}_{k+2}^+ \mid \mathsf{A}_{k+1}^\top \mathbf{z} \in \mathbf{W}_{k+1}^+ \right\}, \tag{4.6.4}$$

equipped with the graph inner product defined for all $\mathbf{z}_1, \mathbf{z}_z \in \mathbf{W}_{k+2}^+(\mathsf{A}_{k+1}^\top)$ by

$$(\mathbf{z}_1, \mathbf{z}_2)_{\mathbf{W}_{k+2}^+(\mathsf{A}_{k+1}^\top)} := (\mathbf{z}_1, \mathbf{z}_2)_{\mathbf{W}_{k+2}^+} + (\mathsf{A}_{k+1}^\top \mathbf{z}_1, \mathsf{A}_{k+1}^\top \mathbf{z}_2)_{\mathbf{W}_{k+1}^+}, \tag{4.6.5}$$

are such that the inclusions $\mathbf{W}_{k+2}^+ \subset \mathbf{W}_{k+2}$ induce continuous and dense embeddings

$$\mathbf{W}_{k+2}^+(\mathsf{A}_{k+1}^\top) \hookrightarrow \mathcal{D}(\mathsf{A}_{k-1}^\top). \tag{4.6.6}$$

We adopt a shorter notation for the dual spaces:

$$\mathbf{W}_k^- := (\mathbf{W}_k^+)', \quad k \in \mathbb{Z}. \tag{4.6.7}$$

Remark 4.6.1. *In Hypothesis* **II**, (4.6.3) *is a stable regular decomposition of the form*

$$\mathcal{D}(\mathsf{A}_k^\top) = \mathbf{W}_{k+1}^+ + \mathsf{A}_{k+1}^\top \mathbf{W}_{k+2}^+, \quad k \in \mathbb{Z}. \tag{4.6.8}$$

By stable, we mean that the lifting and potential operators in (4.6.2) *are bounded. We call it regular due to Hypothesis* **I**, *based on which we can imagine the* $\mathbf{W}_k^+$*s as subspaces of "extra regularity".*

Remark 4.6.2. *The decomposition in* (4.6.3)/(4.6.8) *need not be direct.*

Remark 4.6.3. *Assumption* B *is stated for all* $k \in \mathbb{Z}$*. Strictly speaking, in the setting of a minimal complex with* $k \in \mathbb{Z}$ *fixed, to which we adhere in this section, only one stable regular decomposition (the one written in* (4.6.3) *and involving the regular spaces* $\mathbf{W}_{k+1}^+$ *and* $\mathbf{W}_{k+2}^+$*) is necessary for the characterization of* $\mathcal{D}(\mathsf{A}_k^\top)'$ *and* $\mathcal{R}(\mathsf{T}_k^t)$*.*

Lemma 4.6 *Under Assumption B, the surface operator* $\mathsf{D}_k^t : \mathcal{D}(\mathsf{A}_k^\top)' \to \mathcal{D}(\mathsf{A}_{k+1}^\top)'$ *defined in* (4.5.4a) *can be extended to a continuous mapping*

$$\mathsf{D}_k^t : \begin{cases} \mathbf{W}_{k+1}^- \to \mathbf{W}_{k+2}^+(\mathsf{A}_{k+1}^\top)' \\ \quad \phi \mapsto \langle \phi, \mathsf{A}_{k+1}^\top \cdot \rangle_{\mathbf{W}_{k+1}^-} \end{cases}, \tag{4.6.9}$$

still designated by the same notation.

Proof. For all $\phi \in \mathbf{W}_{k+1}^-$, it follows by definition that $\forall \mathbf{z} \in \mathbf{W}_{k+2}^+(\mathsf{A}_{k+1}^\top)$,

$$|\langle \phi, \mathsf{A}_{k+1}^\top \mathbf{z} \rangle_{\mathbf{W}_{k+1}^-}| \leq \|\phi\|_{\mathbf{W}_{k+1}^-} \|\mathsf{A}_{k+1}^\top \mathbf{z}\|_{\mathbf{W}_{k+1}^+} \leq \|\phi\|_{\mathbf{W}_{k+1}^-} \|\mathbf{z}\|_{\mathbf{W}_{k+2}^+(\mathsf{A}_{k+1}^\top)}. \tag{4.6.10}$$

□

4.6.1.2 Dual decomposition

We may also adopt the adjoint perspective. It goes without saying that the development is completely symmetric to Subsection 4.6.1.1. We present it for completeness.

Assumption C *(cf. Assumption B)* For all $k \in \mathbb{Z}$, beside Assumption A we stipulate the following:

I The Hilbert spaces $\mathbf{W}_k^+ \subset \mathbf{W}_k$ are such that the inclusion maps spawn continuous and dense embeddings

$$\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathsf{A}_k). \tag{4.6.11}$$

II There exist bounded operators

$$\mathsf{L}_{k+1}^n : \mathcal{D}(\mathsf{A}_{k+1}) \to \mathbf{W}_{k+1}^+ \qquad \text{and} \qquad \mathsf{V}_{k+1}^n : \mathcal{D}(\mathsf{A}_{k+1}) \to \mathbf{W}_k^+ \tag{4.6.12}$$

such that

$$\mathbf{y} = (\mathsf{L}_{k+1}^n + \mathsf{A}_k \mathsf{V}_{k+1}^n)\,\mathbf{y} \qquad\qquad \forall \mathbf{y} \in \mathcal{D}(\mathsf{A}_{k+1}). \tag{4.6.13}$$

III The Hilbert spaces

$$\mathbf{W}_k^+(\mathsf{A}_k^\top) := \left\{ \mathbf{x} \in \mathbf{W}_k^+ \mid \mathsf{A}_k \mathbf{x} \in \mathbf{W}_{k+1}^+ \right\}, \tag{4.6.14}$$

equipped with the graph inner product defined for all $\mathbf{x}_1, \mathbf{x}_z \in \mathbf{W}_k^+(\mathsf{A}_k)$ by

$$(\mathbf{x}_1, \mathbf{x}_2)_{\mathbf{W}_k^+(\mathsf{A}_k)} := (\mathbf{x}_1, \mathbf{x}_2)_{\mathbf{W}_k^+} + (\mathsf{A}_k \mathbf{x}_1, \mathsf{A}_{k+1}^\top \mathbf{x}_2)_{\mathbf{W}_{k+1}^+}, \tag{4.6.15}$$

are such that the inclusions $\mathbf{W}_k^+ \subset \mathbf{W}_k$ induce continuous and dense embeddings

$$\mathbf{W}_k^+(\mathsf{A}_k) \hookrightarrow \mathcal{D}(\mathsf{A}_k). \tag{4.6.16}$$

Lemma 4.7 *Under Assumption C, the surface operator* D^n_{k+1} *can be extended to a continuous mapping*

$$\mathsf{D}^n_k : \begin{cases} \mathbf{W}^-_{k+1} \to \mathbf{W}^+_k(\mathsf{A}_k)' \\ \quad \boldsymbol{\psi} \mapsto \langle \boldsymbol{\psi}, \mathsf{A}_k \cdot \rangle_{\mathbf{W}^-_{k+1}} \end{cases} . \tag{4.6.17}$$

Proof. Parallel to the proof of Lemma 4.6, it follows by definition that given $\boldsymbol{\psi} \in \mathbf{W}^-_{k+1}$,

$$|\langle \boldsymbol{\psi}, \mathsf{A}_k \mathbf{x} \rangle_{\mathbf{W}^-_{k+1}}| \leq \|\boldsymbol{\psi}\|_{\mathbf{W}^-_{k+1}} \|\mathsf{A}_k \mathbf{x}\|_{\mathbf{W}^+_{k+1}} \leq \|\boldsymbol{\psi}\|_{\mathbf{W}^-_{k+1}} \|\mathbf{x}\|_{\mathbf{W}^+_k(\mathsf{A}_k)} \qquad \forall \mathbf{x} \in \mathbf{W}^+_k(\mathsf{A}_k). \tag{4.6.18}$$

□

It is not excluded that both assumptions B and C hold, in which case the inclusion

$$\mathbf{W}^+_{k+1} \hookrightarrow \mathcal{D}(\mathsf{A}^\top_k) \cap \mathcal{D}(\mathsf{A}_{k+1}) \tag{4.6.19}$$

is assumed to be a dense embedding.

3D de Rham setting XII: Stable regular decompositions

There is some freedom in choosing the spaces $\mathbf{W}^+_k$, $k \in \mathbb{Z}$. For the de Rham complex though, there are obvious candidates satisfying (4.6.19) that also satisfy both assumptions B and C: functions in the Sobolev space $H^1(\Omega)$ and vector-fields with components in $H^1(\Omega)$, which by Rellich's lemma are compactly embedded in the spaces $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively.

k	0	1	2	3
$\mathbf{W}_k$	$L^2(\Omega)$	$\mathbf{L}^2$	$\mathbf{L}^2$	$L^2(\Omega)$
$\mathbf{W}^+_k$	$H^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$H^1(\Omega)$
$\mathcal{D}(\mathsf{A}_k)$	$H^1(\Omega)$	$\mathbf{H}(\mathbf{curl}, \Omega)$	$\mathbf{H}(\mathrm{div}, \Omega)$	$L^2(\Omega)$
$\mathcal{D}(\mathsf{A}^\top_k)$	$\mathbf{H}(\mathrm{div}, \Omega)$	$\mathbf{H}(\mathbf{curl}, \Omega)$	$H^1(\Omega)$	$\{0\}$

It is well-known (cf. [24, Sec.2], [23, Lem. 2.4] and [25, Sec. 3]) that the graph spaces $\mathcal{D}(\mathsf{A}_k)$ and $\mathcal{D}(\mathsf{A}^\top_k)$ given in the above table admit the stable decompositions

$$\mathcal{D}(\mathsf{A}_2) = \mathcal{D}(\mathsf{A}^\top_0) = \mathbf{H}(\mathrm{div}, \Omega) = \mathbf{H}^1(\Omega) + \mathbf{curl}\,\mathbf{H}^1(\Omega), \tag{4.6.20a}$$

$$\mathcal{D}(\mathsf{A}_1) = \mathcal{D}(\mathsf{A}^\top_1) = \mathbf{H}(\mathbf{curl}, \Omega) = \mathbf{H}^1(\Omega) + \mathbf{grad}\,H^1(\Omega) \tag{4.6.20b}$$

These satisfy assumptions B and C. Moreover, you may recall that

$$\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega) \tag{4.6.21}$$

is a dense embedding [2, Prop. 2.3].

4.6.2 Characterization of dual spaces

In light of Lemma 4.6, the Hilbert space

$$\mathbf{W}^-_{k+1}(\mathsf{D}^t_k) := \left\{ \phi \in \mathbf{W}^-_{k+1} \mid \mathsf{D}^t_k \phi \in \mathbf{W}^-_{k+2} \right\}, \tag{4.6.22}$$

equipped with the graph norm $\|\cdot\|^2_{\mathbf{W}^-_{k+1}(\mathsf{D}^t_k)} := \|\cdot\|^2_{\mathbf{W}^-_{k+1}} + \|\mathsf{D}^t_k \cdot\|^2_{\mathbf{W}^-_{k+2}}$, is well-defined under Assumption B. In this setting, observe that, if $\phi \in \mathbf{W}^-_{k+1}(\mathsf{D}^t_k)$, then based on the decomposition (4.6.3), the evaluation

$$\phi(\mathbf{y}) = \phi(\mathsf{L}^t_{k+1}\mathbf{y}) + \phi(\mathsf{A}^\top_{k+1}\mathsf{V}^t_{k+1}\mathbf{y}) = \phi(\mathsf{L}^t_{k+1}\mathbf{y}) + \mathsf{D}^t_k\phi(\mathsf{V}^t_{k+1}\mathbf{y}) \tag{4.6.23}$$

is well-defined for all $\mathbf{y} \in \mathcal{D}(\mathsf{A}^\top_k)$ thanks to the hypothesis that guarantees $\mathcal{R}(\mathsf{L}^t_{k+1}) \subset \mathbf{W}^+_{k+1}$ and $\mathcal{R}(\mathsf{V}^t_{k+1}) \subset \mathbf{W}^+_{k+2}$.

Theorem 4.4 *Assumption B guarantees the following isomorphism of normed vector spaces,*

$$\mathcal{D}(\mathsf{A}^\top_k)' \cong \mathbf{W}^-_{k+1}(\mathsf{D}^t_k). \tag{4.6.24}$$

Proof. Due to (4.6.1) from Hypothesis **I** of Assumption B, the restriction of functionals $\mathcal{D}(\mathsf{A}^\top_{k+1})' \hookrightarrow \mathbf{W}^-_{k+2}$ is a continuous embedding, so the inclusion $\mathcal{D}(\mathsf{A}^\top_k)' \subset \mathbf{W}^-_{k+1}(\mathsf{D}^t_k)$ is immediate from Definition 4.5.

Moreover, for all $\phi \in \mathbf{W}^-_{k+1}(\mathsf{D}^t_k)$, we estimate using (4.6.23) that

$$\begin{aligned} |\phi(\mathbf{y})| &\le \|\phi\|_{\mathbf{W}^-_{k+1}} \|\mathsf{L}^t_{k+1}\mathbf{y}\|_{\mathbf{W}^+_{k+1}} + \|\mathsf{D}^t_k\phi\|_{\mathbf{W}^-_{k+2}} \|\mathsf{V}^t_{k+1}\mathbf{y}\|_{\mathbf{W}^+_{k+2}} \\ &\le C(\|\phi\|_{\mathbf{W}^-_{k+1}} + \|\mathsf{D}^t_k\phi\|_{\mathbf{W}^-_{k+2}})\|\mathbf{y}\|_{\mathcal{D}(\mathsf{A}^\top_k)} \end{aligned} \tag{4.6.25}$$

for all $\mathbf{y} \in \mathcal{D}(\mathsf{A}^\top_k)$, where $C > 0$ is a constant of continuity related to the boundedness of the potential and lifting operators in hypothesis **II** of Assumption B. We conclude that

$$\mathbf{W}^-_{k+1}(\mathsf{D}^t_k) \subset \mathcal{D}(\mathsf{A}^\top_k)'. \tag{4.6.26}$$

Notice that it also follows from (4.6.25) that

$$\|\phi\|_{\mathcal{D}(\mathsf{A}^\top_k)'} = \sup_{0 \neq \mathbf{y} \in \mathcal{D}(\mathsf{A}^\top_k)} \frac{|\phi(\mathbf{y})|}{\|\mathbf{y}\|_{\mathcal{D}(\mathsf{A}^\top_k)}} \le C(\|\phi\|_{\mathbf{W}^-_{k+1}} + \|\mathsf{D}^t_k\phi\|_{\mathbf{W}^-_{k+2}}) = C\|\phi\|_{\mathbf{W}^-_{k+1}(\mathsf{D}^t_k)} \tag{4.6.27}$$

for all $\phi \in \mathbf{W}^-_{k+1}(\mathsf{D}^t_k)$. In other words, the identity map is continuous as a mapping

$$\mathbf{W}^-_{k+1}(\mathsf{D}^t_k) \hookrightarrow \mathcal{D}(\mathsf{A}^\top_k)'. \tag{4.6.28}$$

Appealing to the bounded inverse theorem verifies the equivalence of norms. □

Similarly, under Assumption C, Lemma 4.7 ensures that the Hilbert space

$$\mathbf{W}^-_{k+1}(\mathsf{D}^n_{k+1}) := \left\{ \psi \in \mathbf{W}^-_{k+1} \mid \mathsf{D}^n_{k+1}\psi \in \mathbf{W}^-_k \right\}, \tag{4.6.29}$$

equipped with the graph norm $\|\cdot\|_{\mathbf{W}^-_{k+1}(\mathsf{D}^n_{k+1})} := \|\cdot\|_{\mathbf{W}^-_{k+1}} + \|\mathsf{D}^n_{k+1}\cdot\|_{\mathbf{W}^-_k}$, is well-defined. We obtain the following analogous result.

Theorem 4.5 (cf. Theorem 4.4) *Under Assumption C, we conclude the isomorphism of normed vector spaces*

$$\mathcal{D}(\mathsf{A}_k)' \cong \mathbf{W}^-_k(\mathsf{D}^n_k). \tag{4.6.30}$$

3D de Rham setting XIII: Characterization of dual spaces

Now, we specialize the theoretical results of Subsection 4.6.2 to the 3D de Rham setting using the table in example XII. We obtain the following characterization of the dual spaces:

$$\mathbf{H}(\mathbf{curl}, \Omega)' = \mathcal{D}(\mathsf{A}_1)' = \mathcal{D}(\mathsf{A}_1^\top)' \cong \left\{ \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \mid \mathbf{grad}'\, \boldsymbol{\phi} \in \widetilde{H}^{-1}(\Omega) \right\}, \quad (4.6.31a)$$

$$\mathbf{H}(\mathrm{div}, \Omega)' = \mathcal{D}(\mathsf{A}_2)' = \mathcal{D}(\mathsf{A}_0^\top)' \cong \left\{ \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \mid \mathbf{curl}'\, \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \right\}. \quad (4.6.31b)$$

Note that these characterizations are interesting in their own right. They do not depend on the theory of traces developed in the previous sections. The take-home message from the de Rham settings XII and XIII is that via the decompositions (4.6.20a) and (4.6.20b), the dual spaces of $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$ can be characterized using more regular spaces such as $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$.

4.6.3 Characterization of trace spaces

We have almost reached characterizations of the ranges of the Hilbert traces $\mathcal{R}(\mathsf{T}_k^t)$ and $\mathcal{R}(\mathsf{T}_k^n)$ in terms of the spaces of "extra regularity" provided by Assumptions B and C. To achieve these new characterizations, we introduce the following spaces for all $k \in \mathbb{Z}$:

$$\mathring{\mathbf{W}}_k^{n,+} := \mathbf{W}_k^+ \cap \mathcal{D}(\mathsf{A}_{k-1}^*), \qquad \text{and} \qquad \mathring{\mathbf{W}}_k^{t,+} := \mathbf{W}_k^+ \cap \mathcal{D}(\mathring{\mathsf{A}}_k). \quad (4.6.32)$$

Notice that by propositions 4.3 and 4.1, we have

$$\mathring{\mathbf{W}}_k^{n,+} = \mathbf{W}_k^+ \cap \mathcal{N}(\mathsf{T}_{k-1}^n), \qquad \text{and} \qquad \mathring{\mathbf{W}}_k^{t,+} = \mathbf{W}_k^+ \cap \mathcal{N}(\mathsf{T}_k^t), \quad (4.6.33)$$

respectively.

Assumption D Suppose that Assumption B holds. For all $k \in \mathbb{Z}$, we make the hypothesis that the inclusion map $\mathbf{W}_k^+ \subset \mathcal{D}(\mathsf{A}_{k-1}^\top)$ spawns a *continuous* and *dense* embedding

$$\mathring{\mathbf{W}}_k^{n,+} \hookrightarrow \mathcal{D}(\mathsf{A}_{k-1}^*). \quad (4.6.34)$$

The next result involves the annihilator

$$(\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ := \left\{ \boldsymbol{\phi} \in \mathbf{W}_{k+1}^- \mid \langle \boldsymbol{\phi}, \mathbf{y} \rangle_{\mathbf{W}_{k+1}^-} = 0,\ \forall \mathbf{y} \in \mathring{\mathbf{W}}_{k+1}^{n,+} \right\}. \quad (4.6.35)$$

Theorem 4.6 *Taking for granted Assumption D we obtain the characterization*

$$\mathcal{R}(\mathsf{T}_k^t) = \mathbf{W}_{k+1}^-(\mathsf{D}_k^t) \cap (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ = \left\{ \boldsymbol{\psi} \in (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ \mid \mathsf{D}_k^t \boldsymbol{\psi} \in (\mathring{\mathbf{W}}_{k+2}^{n,+})^\circ \right\}, \tag{4.6.36}$$

in the sense of equality of functionals in $\mathbf{W}_{k+1}^-$ *and with equivalent norms.*

Proof. We already know by Proposition 4.2 that $\mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ$. To verify the equality on the right, recall that $\mathsf{D}_k^t(\mathcal{R}(\mathsf{T}_k^t)) \subset \mathcal{R}(\mathsf{T}_{k+1}^t) = \mathcal{D}(\mathsf{A}_{k+1}^*)^\circ$.

"$\subset$": On the one hand, since $\mathcal{D}(\mathsf{A}_k^*)^\circ \subset \mathcal{D}(\mathsf{A}_k^\top)'$, it follows immediately from Theorem 4.4 and (4.6.34) that $\mathcal{R}(\mathsf{T}_k^t) \subset \mathbf{W}_{k+1}^-(\mathsf{D}_k^t)$. Moreover, as $\mathring{\mathbf{W}}_{k+1}^{n,+} \subset \mathcal{D}(\mathsf{A}_k^*)$, any functional in the annihilator of $\mathcal{D}(\mathsf{A}_k^*)$ will, in particular, vanish on $\mathring{\mathbf{W}}_{k+1}^{n,+}$, which implies $\mathcal{D}(\mathsf{A}_k^*)^\circ \subset (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ$.

Thanks to the continuous embedding of Assumption B***I*** and (4.5.5a) from the definition of the operator D_k^t, we find for every $\boldsymbol{\varphi} \in \mathcal{D}(\mathsf{A}_k^\top)'$:

$$\begin{aligned}\|\boldsymbol{\varphi}\|_{\mathbf{W}_{k+1}^-} + \|\mathsf{D}_k^t\boldsymbol{\varphi}\|_{\mathbf{W}_{k+2}^-} &= \sup_{0\neq\mathbf{w}\in\mathbf{W}_{k+1}^+} \frac{|\boldsymbol{\varphi}(\mathbf{w})|}{\|\mathbf{w}\|_{\mathbf{W}_{k+1}^+}} + \sup_{0\neq\mathbf{w}\in\mathbf{W}_{k+2}^+} \frac{|\boldsymbol{\varphi}(\mathsf{A}_{k+1}^\top\mathbf{w})|}{\|\mathbf{w}\|_{\mathbf{W}_{k+2}^+}} \\ &\leq C \sup_{0\neq\mathbf{w}\in\mathcal{D}(\mathsf{A}_k^\top)} \frac{|\boldsymbol{\varphi}(\mathbf{w})|}{\|\mathbf{w}\|_{\mathcal{D}(\mathsf{A}_k^\top)}} + \sup_{0\neq\mathbf{w}\in\mathcal{D}(\mathsf{A}_{k+1}^\top)} \frac{|\boldsymbol{\varphi}(\mathsf{A}_{k+1}^\top\mathbf{w})|}{\|\mathbf{w}\|_{\mathcal{D}(\mathsf{A}_{k+1}^\top)}} \leq 2C\|\boldsymbol{\varphi}\|_{\mathcal{D}(\mathsf{A}_k^\top)'},\end{aligned}$$

for some constant $C > 0$ independent of $\boldsymbol{\varphi}$.

"$\supset$": On the other hand, it also follows by Theorem 4.4 that any $\boldsymbol{\phi} \in \mathbf{W}_{k+1}^-(\mathsf{D}_k^t) \cap (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ$ is a continuous functional in $\mathcal{D}(\mathsf{A}_k^\top)'$ vanishing on $\mathring{\mathbf{W}}_{k+1}^{n,+}$. By Assumption D $\mathring{\mathbf{W}}_{k+1}^{n,+}$ is densely embedded in $\mathcal{D}(\mathsf{A}_k^*)$. Thus, $\boldsymbol{\phi}$ must also vanish on $\mathcal{D}(\mathsf{A}_k^*)$ by continuity. We conclude that the inclusion $\mathbf{W}_{k+1}^-(\mathsf{D}_k^t) \cap (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ \subset \mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ$ holds.

Finally, the estimate (4.6.25) gives us

$$\|\boldsymbol{\phi}\|_{\mathcal{D}(\mathsf{A}_k^\top)'} \leq C(\|\boldsymbol{\phi}\|_{\mathbf{W}_{k+1}^-} + \|\mathsf{D}_k^t\boldsymbol{\phi}\|_{\mathbf{W}_{k+2}^-})$$

with $C > 0$ independent of $\boldsymbol{\phi}$. □

Of course, there is a symmetric statement on the dual side.

Assumption E *(cf. Assumption D)* Suppose that Assumption C holds. For all $k \in \mathbb{Z}$, we make the hypothesis that the inclusion map $\mathbf{W}_k^+ \subset \mathcal{D}(\mathsf{A}_k)$ spawns a continuous and dense embedding

$$\mathring{\mathbf{W}}_k^{t,+} \hookrightarrow \mathcal{D}(\mathring{\mathsf{A}}_k). \tag{4.6.37}$$

Theorem 4.7 (cf. Theorem 4.6) *Under Assumption E we have equality in* $\mathbf{W}^-_{k+1}$ *with equivalent norms,*

$$\mathcal{R}(\mathsf{T}^n_{k+1}) = \mathbf{W}^-_{k+1}(\mathsf{D}^n_{k+1}) \cap (\mathring{\mathbf{W}}^{t,+}_{k+1})^\circ = \left\{ \boldsymbol{\psi} \in (\mathring{\mathbf{W}}^{t,+}_{k+1})^\circ \mid \mathsf{D}^n_{k+1}\boldsymbol{\psi} \in (\mathring{\mathbf{W}}^{t,+}_{k})^\circ \right\}. \quad (4.6.38)$$

3D de Rham setting XIV: Characterization of trace spaces

We specialize the theoretical results of Subsection 4.6.3 to the 3D de Rham setting.

k	0	1	2	3
$\mathbf{W}_k$	$L^2(\Omega)$	$\mathbf{L}^2$	$\mathbf{L}^2$	$L^2(\Omega)$
$\mathbf{W}^+_k$	$H^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$H^1(\Omega)$
$\mathring{\mathbf{W}}^{t,+}_k$	$\mathring{H}^1(\Omega)$	$\mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$	$\mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega)$	$\mathring{H}^1(\Omega)$
$\mathring{\mathbf{W}}^{n,+}_k$	$\mathring{H}^1(\Omega)$	$\mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega)$	$\mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$	$\mathring{H}^1(\Omega)$

Loosely speaking, theorems 4.6 and 4.7 state that the range of the Hilbert trace is a subspace of functionals in the dual of a regular space $\mathbf{W}^+_k$ whose image under the corresponding surface operator also lies in the dual of $\mathbf{W}^+_{k+1}$. Linear functionals in that subspace vanish on a dense subset of the dual trace's kernel:

$$\mathcal{R}(\mathsf{T}^n_{\mathbf{curl}}) = \left\{ \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)^\circ | \mathbf{grad}'\, \boldsymbol{\phi} \in \widetilde{H}^{-1}(\Omega) \cap \mathring{H}^1(\Omega)^\circ \right\}, \quad (4.6.39a)$$

$$\mathcal{R}(\mathsf{T}^n_{\mathrm{div}}) = \left\{ \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega)^\circ | \mathbf{curl}'\, \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)^\circ \right\}, \quad (4.6.39b)$$

with

$$\mathcal{R}(\mathsf{T}^t_{\mathbf{curl}}) = \mathcal{R}(\mathsf{T}^n_{\mathbf{curl}}) \quad (4.6.39c)$$

$$\mathcal{R}(\mathsf{T}^t_{\mathbf{grad}}) = \mathcal{R}(\mathsf{T}^n_{\mathrm{div}}) \quad (4.6.39d)$$

The identities (4.6.39c) and (4.6.39d) are expected, because we already know from previous sections that

$$\mathcal{R}(\mathsf{T}^n_{\mathbf{curl}}) = \mathcal{D}(\mathring{\mathsf{A}}_1)^\circ = \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)' = \mathcal{D}(\mathsf{A}^*_1)^\circ = \mathcal{R}(\mathsf{T}^t_{\mathbf{curl}}), \quad (4.6.40a)$$

$$\mathcal{R}(\mathsf{T}^n_{\mathrm{div}}) = \mathcal{D}(\mathring{\mathsf{A}}_2)^\circ = \mathring{\mathbf{H}}(\mathrm{div}, \Omega)' = \mathcal{D}(\mathsf{A}^*_1) = \mathcal{R}(\mathsf{T}^t_{\mathbf{grad}}). \quad (4.6.40b)$$

Before we compare these characterizations with (4.1.7a) and (4.1.7b), we want to reformulate them in terms of quotient spaces in the next section.

4.6.4 Characterization of trace spaces in quotient spaces

We can reformulate the characterizations of Subsection 4.6.3 in terms of quotient spaces. To proceed, let us set

$$\mathbf{T}_k^{t,+} := \mathbf{W}_k^+ / \mathring{\mathbf{W}}_k^{t,+}, \qquad \mathbf{T}_k^{t,-} := \left(\mathbf{T}_k^{t,+}\right)', \tag{4.6.41a}$$

$$\mathbf{T}_k^{n,+} := \mathbf{W}_k^+ / \mathring{\mathbf{W}}_k^{n,+}, \qquad \mathbf{T}_k^{n,-} := \left(\mathbf{T}_k^{n,+}\right)'. \tag{4.6.41b}$$

Under Assumption D (resp. E), it follows by definition of the space $\mathring{\mathbf{W}}_k^{n,+}$ (resp. $\mathring{\mathbf{W}}_k^{t,+}$) that the dense embedding $\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathsf{A}_{k-1}^\top)$ (resp. $\mathbf{W}_k^+ \hookrightarrow \mathcal{D}(\mathsf{A}_k)$) induces a well-defined and dense embedding

$$\begin{cases} \mathbf{T}_k^{n,+} \hookrightarrow \mathcal{T}(\mathsf{A}_{k-1}^\top) \\ \quad [\mathbf{x}] \mapsto \boldsymbol{\pi}_{k-1}^n \mathbf{x} \end{cases} \qquad \left(\text{resp.} \qquad \begin{cases} \mathbf{T}_k^{t,+} \hookrightarrow \mathcal{T}(\mathsf{A}_k) \\ \quad [\mathbf{x}] \mapsto \boldsymbol{\pi}_k^t \mathbf{x} \end{cases} \right) \tag{4.6.42}$$

on the quotient spaces. Accordingly, the associated restriction of functionals

$$\begin{cases} \mathcal{T}(\mathsf{A}_{k-1}^\top)' \hookrightarrow \mathbf{T}_k^{n,-} \\ \qquad \psi \mapsto \{ [\mathbf{x}] \mapsto \psi(\boldsymbol{\pi}_{k-1}^n \mathbf{x}) \} \end{cases} \quad \left(\text{resp.} \quad \begin{cases} \mathcal{T}(\mathsf{A}_k)' \hookrightarrow \mathbf{T}_k^{t,-} \\ \qquad \phi \mapsto \{ [\mathbf{x}] \mapsto \phi(\boldsymbol{\pi}_k^t \mathbf{x}) \} \end{cases} \right) \tag{4.6.43}$$

is also well-defined and gives rise to dense embeddings.

In the next lemma, we make explicit the mappings induced on the quotient spaces by restricting the operators $\mathsf{A}_{k-1}^\top$ and A_k to $\mathbf{W}_k^+$. Those are the restrictions of the surface operators S_{k-1}^n and S_k^t to $\mathbf{T}_k^{n,+}$ and $\mathbf{T}_k^{t,+}$, respectively; cf. Definition 4.6.

Lemma 4.8 *Assumptions D and E imply that the mappings*

$$\hat{\mathsf{S}}_{k+1}^n : \begin{cases} \mathbf{T}_{k+2}^{n,+} \to \mathcal{T}(\mathsf{A}_k^\top) \\ \quad [\mathbf{z}] \mapsto \boldsymbol{\pi}_k^n \mathsf{A}_{k+1}^\top \mathbf{z} \end{cases} \qquad \textit{and} \qquad \hat{\mathsf{S}}_k^t : \begin{cases} \mathbf{T}_k^{t,+} \to \mathcal{T}(\mathsf{A}_{k+1}) \\ \quad [\mathbf{x}] \mapsto \boldsymbol{\pi}_{k+1}^t \mathsf{A}_k \mathbf{x} \end{cases}, \tag{4.6.44}$$

respectively, are well-defined and continuous.

Proof. Consider the mapping on the left. We know from the complex property for $\mathsf{A}_k^\top$ in Assumption A that $\mathsf{A}_{k+1}^\top \mathbf{z} \in \mathcal{D}(\mathsf{A}_k^\top)$ for all $\mathbf{z} \in \mathbf{W}_{k+1}^+$. We only need to verify that $\mathsf{A}_{k+1}^\top \mathbf{z}_\circ \in \mathcal{D}(\mathsf{A}_k^*)$ for all $\mathbf{z}_\circ \in \mathring{\mathbf{W}}_{k+2}^{n,+} = \mathbf{W}_{k+2}^+ \cap \mathcal{D}(\mathsf{A}_{k+1}^*)$, but this immediately follows from the complex property for A_{k+1}^*, also provided by Assumption A. The proof is similar for $\hat{\mathsf{S}}_k^t$. □

Using the same strategy as in lemmas 4.6 and 4.7, the mappings

$$\hat{\mathsf{D}}_k^t := (\hat{\mathsf{S}}_{k+1}^n)' : \mathcal{T}(\mathsf{A}_k^\top)' \to \mathbf{T}_{k+2}^{n,-} \qquad \text{and} \qquad \hat{\mathsf{D}}_k^n := (\hat{\mathsf{S}}_k^t)' : \mathcal{T}(\mathsf{A}_{k+1})' \to \mathbf{T}_k^{t,-}, \tag{4.6.45}$$

defined as the bounded operators dual to $\hat{\mathsf{S}}_{k+1}^n$ and $\hat{\mathsf{S}}_k^t$, can be extended, using (4.6.43), to the continuous mappings

$$\hat{\mathsf{D}}_k^t : \mathbf{T}_{k+1}^{n,-} \to \mathbf{T}_{k+2}^{n,+}(\hat{\mathsf{S}}_{k+1}^n)' \qquad \text{and} \qquad \hat{\mathsf{D}}_k^n : \mathbf{T}_{k+1}^{t,-} \to \mathbf{T}_k^{t,+}(\hat{\mathsf{S}}_k^t)', \tag{4.6.46}$$

involving the dual spaces of the Hilbert spaces

$$\mathbf{T}^{n,+}_{k+2}(\hat{\mathsf{S}}^n_{k+1}) := \left\{ [\mathbf{z}] \in \mathbf{T}^{n,+}_{k+2} \mid \hat{\mathsf{S}}^n_{k+1}[\mathbf{z}] \in \mathbf{T}^{n,+}_{k+1} \right\}, \tag{4.6.47a}$$

$$\mathbf{T}^{t,+}_{k}(\hat{\mathsf{S}}^t_{k}) := \left\{ [\mathbf{x}] \in \mathbf{T}^{t,+}_{k} \mid \hat{\mathsf{S}}^t_{k}[\mathbf{x}] \in \mathbf{T}^{t,+}_{k+1} \right\}, \tag{4.6.47b}$$

equipped with the natural graph inner products.

With the operators (4.6.46), we can reformulate theorems 4.6 and 4.7 using the isometric isomorphisms

$$(\mathbf{W}^+_k/\mathring{\mathbf{W}}^{t,+}_k)' \cong (\mathring{\mathbf{W}}^{t,+}_k)^\circ \qquad \text{and} \qquad \mathbf{W}^+_k/\mathring{\mathbf{W}}^{n,+}_k \cong (\mathring{\mathbf{W}}^{n,+}_k)^\circ \tag{4.6.48}$$

provided by [41, Thm. 4.9].

Theorem 4.8 *Under assumptions D and E we have the isomorphisms of Hilbert spaces*

$$\mathcal{R}(\mathsf{T}^t_k) \cong \left\{ \boldsymbol{\phi} \in \mathbf{T}^{n,-}_{k+1} \mid \hat{\mathsf{D}}^t_k \boldsymbol{\phi} \in \mathbf{T}^{n,-}_{k+2} \right\}, \tag{4.6.49a}$$

$$\mathcal{R}(\mathsf{T}^n_k) \cong \left\{ \boldsymbol{\phi} \in \mathbf{T}^{t,-}_{k} \mid \hat{\mathsf{D}}^n_k \boldsymbol{\phi} \in \mathbf{T}^{t,-}_{k-1} \right\}, \tag{4.6.49b}$$

respectively.

3D de Rham setting XV: Characterization of trace spaces by quotient spaces

Recall from (4.1.8b) and (4.1.8c) that $\mathcal{N}(\gamma_t) = \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)$ and $\mathcal{N}(\gamma_n) = \mathring{\mathbf{H}}(\mathrm{div}, \Omega)$. So let us denote the spaces of $\mathbf{H}^1$-regular vector fields with vanishing tangential and normal traces by

$$\mathbf{H}^1_t(\Omega) := \mathcal{N}(\gamma_t|_{\mathbf{H}^1(\Omega)}) = \mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega) \tag{4.6.50a}$$

$$\mathbf{H}^1_n(\Omega) := \mathcal{N}(\gamma_n|_{\mathbf{H}^1(\Omega)}) = \mathbf{H}^1(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega), \tag{4.6.50b}$$

respectively.

k	0	1	2	3
$\mathbf{W}_k$	$L^2(\Omega)$	$\mathbf{L}^2$	$\mathbf{L}^2$	$L^2(\Omega)$
$\mathbf{W}^+_k$	$H^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)$	$H^1(\Omega)$
$\mathbf{T}^{t,+}_k$	$H^1(\Omega)/\mathring{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)$	$\mathbf{H}^1(\Omega)/\mathbf{H}^1_n(\Omega)$	$H^1(\Omega)/\mathring{H}^1(\Omega)$
$\mathbf{T}^{n,+}_k$	$H^1(\Omega)/\mathring{H}^1(\Omega)$	$\mathbf{H}^1(\Omega)/\mathbf{H}^1_n(\Omega)$	$\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)$	$H^1(\Omega)/\mathring{H}^1(\Omega)$

Reformulating (4.6.39a) and (4.6.39b), we obtain

$$\mathcal{R}(\mathsf{T}^n_{\mathbf{curl}}) \cong \left\{ \boldsymbol{\phi} \in \left(\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)\right)' \mid \mathbf{grad}'\,\boldsymbol{\phi} \in \left(H^1(\Omega)/\mathring{H}^1(\Omega)\right)' \right\}, \tag{4.6.51a}$$

$$\mathcal{R}(\mathsf{T}^n_{\mathrm{div}}) \cong \left\{ \boldsymbol{\phi} \in \left(\mathbf{H}^1(\Omega)/\mathbf{H}^1_n(\Omega)\right)' \mid \mathbf{curl}'\,\boldsymbol{\phi} \in \left(\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)\right)' \right\}, \tag{4.6.51b}$$

still with

$$\mathcal{R}(\mathsf{T}^t_{\mathbf{curl}}) = \mathcal{R}(\mathsf{T}^n_{\mathbf{curl}}), \tag{4.6.51c}$$
$$\mathcal{R}(\mathsf{T}^t_{\mathbf{grad}}) = \mathcal{R}(\mathsf{T}^n_{\mathrm{div}}). \tag{4.6.51d}$$

These characterizations are to be compared with

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = \left\{ \boldsymbol{\phi} \in \mathbf{H}_t^{-1/2}(\Gamma) \mid \mathrm{curl}_\Gamma\,\boldsymbol{\phi} \in H^{-1/2}(\Gamma) \right\} = \mathcal{R}(\gamma_t), \tag{4.6.52a}$$
$$H^{1/2}(\Gamma) = \left\{ \boldsymbol{\phi} \in H^{-1/2}(\Gamma) \mid \mathbf{curl}_\Gamma\,\boldsymbol{\phi} \in \mathbf{H}_t^{-1/2}(\Gamma) \right\} = \mathcal{R}(\gamma), \tag{4.6.52b}$$

where as before the two spaces

$$H^{-1/2}(\Gamma) = \left(H^{1/2}(\Gamma)\right)' = \left(\gamma H^1(\Omega)\right)' \tag{4.6.53a}$$
$$\mathbf{H}_t^{-1/2}(\Gamma) = \left(\mathbf{H}_t^{1/2}(\Gamma)\right)' = \left(\gamma_t \mathbf{H}^1(\Omega)\right)' \tag{4.6.53b}$$

are dual to the more regular spaces $\gamma\, H^1(\Omega)$ and $\gamma_t\, \mathbf{H}^1(\Omega)$, respectively.

In the classical trace spaces, the quotient spaces involved in (4.6.51a) and (4.6.51b) are featured implicitly, because as previously stated in (4.6.50a) and (4.6.50b), $\mathbf{H}^1_t(\Omega)$ and $\mathbf{H}^1_n(\Omega)$ are kernels which vanish under application of the traces. In fact, since $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ and $\gamma_t : \mathbf{H}^1(\Omega) \to \mathbf{H}_t^{1/2}(\Gamma)$ are surjective, it follows from (4.6.50a) and (4.6.50b) that the same argument as in the 3D de Rham setting VII shows that the traces induce the isomorphisms

$$\mathbf{H}_t^{1/2}(\Gamma) \cong \mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega) \qquad \text{and} \qquad H^{1/2}(\Gamma) \cong H^1(\Omega)/\mathring{H}^1(\Omega), \tag{4.6.54}$$

which in turn imply isomorphisms between the dual spaces.

We would like to draw the reader's attention to the fact that it is an annihilator related to the kernel of the dual trace that is used to characterize the range of the primal trace and vice-versa. This is in agreement with the characterizations provided in [15], where the range of γ_t is characterized using the dual space $(\gamma_\tau \mathbf{H}^1(\Omega))'$, involving the rotated tangential trace γ_τ discussed in the 3D de Rham setting XI. As in [15], recall that if the skew-symmetric pairing (4.1.10) is replaced with the $\mathbf{L}^2(\Gamma)$-pairing, the dual trace $\mathsf{T}^n_{\mathbf{curl}}$, corresponding with the rotated tangential trace (roughly speaking), arises in the abstract setting of Subsection 4.4.1 as dual to $\mathsf{T}^t_{\mathbf{curl}}$, which corresponds to γ_t.

Finally, notice that the surface operators curl_Γ and $\mathbf{curl}_\Gamma$ are dual to the domain operators on which the relevant traces are applied, which is in line with (4.6.51a) and (4.6.51b), i.e. (cf. [15])

$$\mathrm{curl}_\Gamma \circ \gamma = (\gamma_t \circ \nabla)' \qquad \text{and} \qquad \mathbf{curl}_\Gamma \circ \gamma_t = (\gamma_n \circ \mathbf{curl})'. \tag{4.6.55}$$

4.7 Trace Hilbert Complexes

From now on, we make use of the full setting of Hilbert complexes as presented in Subsection 4.2.2. Both Assumptions D and E are not required for the mere characterization of the trace Hilbert complexes in Subsection 4.7.1: each one of these hypotheses suffices for the corresponding characterization. However, we do rely on *both* decompositions for the upcoming compactness result in Subsection 4.7.2, where we must take (4.6.19) for granted.

4.7.1 Complexes of quotient spaces

It is easy to verify that $\mathsf{D}^t_{k+1} \circ \mathsf{D}^t_k = 0$, $\mathsf{D}^n_k \circ \mathsf{D}^n_{k+1} = 0$, $\mathsf{S}^t_{k+1} \circ \mathsf{S}^t_k = 0$ and $\mathsf{S}^n_k \circ \mathsf{S}^n_{k+1} = 0$. Therefore, we have already seen from (4.5.30) that Hilbert complexes give rise to Hilbert complexes in trace spaces. The bounded complexes

$$\cdots \xrightarrow{\mathsf{D}^t_k} \mathcal{R}(\mathsf{T}^t_k) \xrightarrow{\mathsf{D}^t_k} \mathcal{R}(\mathsf{T}^t_{k+1}) \xrightarrow{\mathsf{D}^t_{k+1}} \mathcal{R}(\mathsf{T}^t_{k+2}) \xrightarrow{\mathsf{D}_{k+2}} \cdots, \tag{4.7.1a}$$

and

$$\cdots \xleftarrow[\mathsf{D}_k]{} \mathcal{R}(\mathsf{T}^n_k) \xleftarrow[\mathsf{D}^n_{k+1}]{} \mathcal{R}(\mathsf{T}^n_{k+1}) \xleftarrow[\mathsf{D}^n_{k+2}]{} \mathcal{R}(\mathsf{T}^n_{k+2}) \xleftarrow[\mathsf{D}_{k+3}]{} \cdots, \tag{4.7.1b}$$

are isometrically isomorphic to the bounded complexes of quotient spaces

$$\cdots \xrightarrow{\mathsf{S}^t_k} \mathcal{T}(\mathsf{A}_k) \xrightarrow{\mathsf{S}^t_k} \mathcal{T}(\mathsf{A}_{k+1}) \xrightarrow{\mathsf{S}^t_{k+1}} \mathcal{T}(\mathsf{A}_{k+2}) \xrightarrow{\mathsf{S}^t_{k+2}} \cdots, \tag{4.7.2a}$$

and

$$\cdots \xleftarrow[\mathsf{S}^n_k]{} \mathcal{T}(\mathsf{A}^\top_k) \xleftarrow[\mathsf{S}^n_{k+1}]{} \mathcal{T}(\mathsf{A}^\top_{k+1}) \xleftarrow[\mathsf{S}^n_{k+2}]{} \mathcal{T}(\mathsf{A}^\top_{k+2}) \xleftarrow[\mathsf{S}^n_{k+3}]{} \cdots. \tag{4.7.2b}$$

While the bounded domain complexes are interesting in their own right, the rich structure of Hilbert complexes reveals itself when closed densely defined unbounded operators are introduced. As stated in [6, Chap. 4], the complex produced by the latter contains more information than the associated domain complexes. It turns out that the characterizations provided in Section 4.6 shed more light on the structure of (4.7.1a)-(4.7.2b). The next theorem provides a first characterization of what we call *trace Hilbert complexes*.

Theorem 4.9 *Under assumptions D and E respectively, the sequences of unbounded operators*

$$\cdots \xrightarrow{\mathsf{D}^t_{k-1}} \mathcal{R}(\mathsf{T}^t_k) \subset (\mathring{\mathbf{W}}^{n,+}_{k+1})^\circ \xrightarrow{\mathsf{D}^t_k} \mathcal{R}(\mathsf{T}^t_{k+1}) \subset (\mathring{\mathbf{W}}^{n,+}_{k+2})^\circ \xrightarrow{\mathsf{D}^t_{k+2}} \cdots \tag{4.7.3}$$

and

$$\cdots \xleftarrow[\mathsf{D}^n_k]{} \mathcal{R}(\mathsf{T}^n_k) \subset (\mathring{\mathbf{W}}^{t,+}_k)^\circ \xleftarrow[\mathsf{D}^n_{k+1}]{} \mathcal{R}(\mathsf{T}^n_{k+1}) \subset (\mathring{\mathbf{W}}^{t,+}_{k+1})^\circ \xleftarrow[\mathsf{D}^n_{k+2}]{} \cdots \tag{4.7.4}$$

are Hilbert complexes as defined in Subsection 4.2.2.

Proof. By symmetry, it is sufficient to verify the claim for (4.7.3). In light for (4.7.1a) and Theorem 4.6, we simply need to show that $\mathsf{D}_k^t : \mathcal{R}(\mathsf{T}_k^t) \subset (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ \to (\mathring{\mathbf{W}}_{k+2}^{n,+})^\circ$ is a densely defined and closed unbounded linear operator. In fact, since $\mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ \subset \mathcal{D}(\mathsf{A}_k^\top)'$ is a Hilbert space by Proposition 4.2, we already know that such an operator must be closed, and we only need to confirm that $\mathcal{R}(\mathsf{T}_k^t)$ is dense in $(\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ$.

We need two key mappings:

- Recall that since $\mathbf{W}_{k+1}^+$ is a Hilbert space and Hilbert spaces are reflexive (cf. [41, Sec. 4.5], [11, Thm. 5.5]), the map

$$\boldsymbol{\rho} : \begin{cases} \mathbf{W}_{k+1}^+ \longrightarrow (\mathbf{W}_{k+1}^-)' \\ \quad \mathbf{y} \;\mapsto \begin{cases} \mathbf{W}_{k+1}^- \to \mathbb{R} \\ \quad \boldsymbol{\phi} \mapsto \boldsymbol{\rho}\mathbf{y}(\boldsymbol{\phi}) = \boldsymbol{\phi}(\mathbf{y}) \end{cases} \end{cases} \tag{4.7.5}$$

  is an isometric isomorphism. Substituting $\boldsymbol{\rho}^{-1}(\widetilde{\boldsymbol{\phi}})$ for $\mathbf{y}$ in the definition $(\boldsymbol{\rho}\mathbf{y})(\boldsymbol{\phi}) = \boldsymbol{\phi}(\mathbf{y})$, we find a useful formula involving the inverse:

$$\widetilde{\boldsymbol{\psi}}(\boldsymbol{\phi}) = \boldsymbol{\phi}(\boldsymbol{\rho}^{-1}\widetilde{\boldsymbol{\psi}}) \tag{4.7.6}$$

  for all $\boldsymbol{\phi} \in \mathbf{W}_{k+1}^-$ and $\widetilde{\boldsymbol{\psi}} \in (\mathbf{W}_{k+1}^-)'$.
- Since the inclusion $\mathbf{W}_{k+1}^+ \hookrightarrow \mathcal{D}(\mathsf{A}_k^\top)$ is continuous and dense by Assumption B, the *restriction* of functionals $\mathsf{J} : \mathcal{D}(\mathsf{A}_k^\top)' \to \mathbf{W}_{k+1}^-$ is also a continuous and dense embedding. In particular, because $\mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}^*)^\circ$ by Proposition 4.2 and $\mathring{\mathbf{W}}_{k+1}^{n,+} \subset \mathcal{D}(\mathsf{A}_k^*)$ by definition, it satisfies the important property that $\mathsf{J}(\mathcal{R}(\mathsf{T}_k^t)) \subset (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ$.

To prove density, we show that an arbitrary functional $\widetilde{\boldsymbol{\phi}}_\circ \in ((\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ)'$ such that $\widetilde{\boldsymbol{\phi}}_\circ(\mathsf{J}\boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi} \in \mathcal{R}(\mathsf{T}_k^t)$ vanish in $((\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ)'$. We proceed in three short steps.

1) First, we use the Hahn–Banach theorem to extend $\widetilde{\boldsymbol{\phi}}_\circ$ to a functional $\widetilde{\boldsymbol{\phi}} \in (\mathbf{W}_{k+1}^-)'$. By definition,

$$\widetilde{\boldsymbol{\phi}}(\mathsf{J}\boldsymbol{\xi}) = 0 \qquad\qquad \forall \boldsymbol{\xi} \in \mathcal{R}(\mathsf{T}_k^t). \tag{4.7.7}$$

2) Secondly, we set $\mathbf{y} := \boldsymbol{\rho}^{-1}\widetilde{\boldsymbol{\phi}} \in \mathbf{W}_{k+1}^+ \subset \mathcal{D}(\mathsf{A}_k^\top)$. Based on (4.7.6), it follows from (4.7.7) that

$$\boldsymbol{\xi}(\mathbf{y}) = \mathsf{J}\boldsymbol{\xi}(\mathbf{y}) = \mathsf{J}\boldsymbol{\xi}(\boldsymbol{\rho}^{-1}\widetilde{\boldsymbol{\phi}}) = \widetilde{\boldsymbol{\phi}}(\mathsf{J}\boldsymbol{\xi}) = 0 \qquad \forall \boldsymbol{\xi} \in \mathcal{R}(\mathsf{T}_k^t) = \mathcal{D}(\mathsf{A}_k^*)^\circ. \tag{4.7.8}$$

In particular, we obtain from (4.7.8) that $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^*)$. Thus, under the choice made in (4.6.32), $\mathbf{y} \in \mathcal{D}(\mathsf{A}_k^*) \cap \mathbf{W}_{k+1}^+ = \mathring{\mathbf{W}}_k^{n,+}$.

3) Finally, the previous step implies that

$$\widetilde{\boldsymbol{\phi}}(\boldsymbol{\phi}_\circ) = \boldsymbol{\rho}\mathbf{y}(\boldsymbol{\phi}_\circ) = \boldsymbol{\phi}_\circ(\mathbf{y}) = 0 \qquad\qquad \forall \boldsymbol{\phi}_\circ \in (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ. \tag{4.7.9}$$

Therefore, $\widetilde{\boldsymbol{\phi}}_\circ = \widetilde{\boldsymbol{\phi}}\big|_{(\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ} = 0$, which concludes the proof. □

Now, rewriting the trace Hilbert complexes (4.7.3) and (4.7.4) in terms of the isometrically isomorphic characterizations given in Theorem 4.8, we obtain the Hilbert complexes

$$\cdots \xrightarrow{\hat{\mathsf{D}}^t_{k-1}} \mathbf{T}^{n,-}_{k+1}(\hat{\mathsf{D}}^t_k) \subset \mathbf{T}^{n,-}_{k+1} \xrightarrow{\hat{\mathsf{D}}^t_k} \mathbf{T}^{n,-}_{k+2}(\hat{\mathsf{D}}^t_{k+1}) \subset \mathbf{T}^{n,-}_{k+2} \xrightarrow{\hat{\mathsf{D}}^t_{k+1}} \cdots \tag{4.7.10a}$$

and

$$\cdots \xleftarrow[\hat{\mathsf{D}}^n_k]{} \mathbf{T}^{t,-}_{k}(\hat{\mathsf{D}}^n_k) \subset \mathbf{T}^{t,-}_{k} \xleftarrow[\hat{\mathsf{D}}^n_{k+1}]{} \mathbf{T}^{t,-}_{k+1}(\hat{\mathsf{D}}^n_{k+1}) \subset \mathbf{T}^{t,-}_{k+1} \xleftarrow[\hat{\mathsf{D}}^n_{k+2}]{} \cdots . \tag{4.7.10b}$$

4.7.2 Compactness property

It is well-known that compact embeddings of the regular spaces $\mathbf{W}^+_k \subset \mathbf{W}_k$ in the stable decompositions (4.6.3) and (4.6.13) lead to the Hilbert complexes (4.2.5a) and (4.2.8b) being Fredholm. For convenience, we review this result in the next lemma.

Assumption F Suppose that the dense inclusions $\imath^+_k : \mathbf{W}^+_k \hookrightarrow \mathbf{W}_k$ are compact for all $k \in \mathbb{Z}$.

Lemma 4.9 *Under Assumption F, Assumptions B and C guarantee compactness of the inclusions*

$$\mathcal{D}(\mathsf{A}^\top_k) \cap \mathcal{D}(\mathring{\mathsf{A}}_{k+1}) \hookrightarrow \mathbf{W}_{k+1} \qquad \textit{and} \qquad \mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}^*_k) \hookrightarrow \mathbf{W}_{k+1}, \tag{4.7.11}$$

respectively.

Proof. By symmetry, it is sufficient to prove that, under Assumption F, it follows from Assumption C that the dense inclusion $\mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}^*_k) \hookrightarrow \mathbf{W}_k$ is a compact operator. In particular, let $(\mathbf{y}_\ell)_{\ell\in\mathbb{Z}} \subset \mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}^*_k)$ be an arbitrary sequence that is bounded in $\mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}^*_k)$. We only need to show that there exists a subsequence $(\mathbf{y}_{\ell_\rho})_{\rho\in\mathbb{Z}}$ that is Cauchy in $\mathbf{W}_k$.

By Assumption C, for all $\ell \in \mathbb{Z}$, there exist $\mathbf{p}^+_\ell \in \mathbf{W}^+_{k+1}$ and $\mathbf{x}^+_\ell \in \mathbf{W}^+_k$ such that

$$\mathbf{y}_\ell = \mathbf{p}^+_\ell + \mathsf{A}_k \mathbf{x}^+_\ell \qquad \Big(\text{in particular, } \mathbf{p}^+_\ell := \mathsf{L}^n_{k+1}\mathbf{y}_\ell \text{ and } \mathbf{x}^+_\ell := \mathsf{V}^n_{k+1}\mathbf{y}_\ell\Big). \tag{4.7.12}$$

The norm in $\mathcal{D}(\mathsf{A}_{k+1}) \cap \mathcal{D}(\mathsf{A}^*_k)$ is stronger than the norm in $\mathcal{D}(\mathsf{A}_{k+1})$, so since the decomposition is stable by hypothesis **II** from Assumption C, the sequences $(\mathbf{p}^+_\ell)_\ell$ and $(\mathbf{x}^+_\ell)_\ell$ are bounded in $\mathbf{W}^+_{k+1}$ and $\mathbf{W}^+_k$, respectively. Under Assumption F, we can thus find subsequences $(\mathbf{p}^+_{\ell_\rho})_\rho$ and $(\mathbf{x}^+_{\ell_\rho})_\rho$ that are Cauchy in $\mathbf{W}_{k+1}$ and $\mathbf{W}_k$, respectively. Evaluating

$$\begin{aligned}\|\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_m}\|^2_{\mathbf{W}_{k+1}} &= \left(\mathbf{p}^+_{\ell_n} - \mathbf{p}^+_{\ell_n}, \mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)_{\mathbf{W}_{k+1}} + \left(\mathsf{A}_k\left(\mathbf{x}^+_{\ell_n} - \mathbf{x}^+_{\ell_n}\right), \mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)_{\mathbf{W}_{k+1}} \\ &\leq \|\mathbf{p}^+_{\ell_n} - \mathbf{p}^+_{\ell_n}\|_{\mathbf{W}_{k+1}} \|\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\|_{\mathbf{W}_{k+1}} + \left(\mathbf{x}^+_{\ell_n} - \mathbf{x}^+_{\ell_n}, \mathsf{A}^*_k\left(\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)\right)_{\mathbf{W}_{k+1}}\end{aligned}$$

$$\leq \underbrace{\|\mathbf{p}_{\ell_n}^{+} - \mathbf{p}_{\ell_n}^{+}\|_{\mathbf{W}_{k+1}}}_{\to 0 \text{ as } n,m\to 0} \|\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\|_{\mathbf{W}_{k+1}} + \underbrace{\|\mathbf{x}_{\ell_n}^{+} - \mathbf{x}_{\ell_n}^{+}\|_{\mathbf{W}_k}}_{\to 0 \text{ as } n,m\to 0} \|\mathsf{A}_k^{*}\left(\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)\|_{\mathbf{W}_{k+1}},$$

we arrive at the conclusion once noticing that $\|\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\|_{\mathbf{W}_{k+1}}$ and $\|\mathsf{A}_k^{*}\left(\mathbf{y}_{\ell_n} - \mathbf{y}_{\ell_n}\right)\|_{\mathbf{W}_{k+1}}$ are also bounded by hypothesis. □

In other words, under Assumption F, the stable decompositions of Subsection 4.6.1 imply complex properties, which as stated in Subsection 4.2.2, guarantee that the associated Hilbert complexes are Fredholm. The goal of this section is to show that this carries over to the trace spaces. Ultimately, this is because what is essential for Lemma 4.9 to go through is not compactness of the spaces, but rather that the potential and lifting operators are compact operators.

In order to obtain the complex properties for the trace Hilbert complexes, we find it most convenient to work with the characterizations provided in Theorem 4.9, because it allows us to harness the theory developed in Subsection 4.3.3. By symmetry, we may focus on (4.7.3).

For any $\mathbf{x} \in \mathcal{D}(\mathsf{A}_k)$, it follows from Assumption C and the commuting relations of Lemma 4.5 that

$$\mathsf{T}_k^t\mathbf{x} = \mathsf{T}_k^t\mathsf{L}_k^n\mathbf{x} + \mathsf{T}_k^t\mathsf{A}_{k-1}\mathsf{V}_k^n\mathbf{x} = \mathsf{T}_k^t\mathsf{L}_k^n\mathbf{x} - \mathsf{D}_{k-1}^t\mathsf{T}_{k-1}^t\mathsf{V}_k^n\mathbf{x}. \tag{4.7.13}$$

Recall from Lemma 4.2 that the $\mathcal{D}(\mathsf{A}_k)$-harmonic extension operators $-\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to \mathcal{D}(\mathsf{A}_k)$ satisfy $\mathsf{T}_k^t(-\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi}) = \boldsymbol{\phi}$ for all $\boldsymbol{\phi} \in \mathcal{R}(\mathsf{T}_k^t)$. Inserting this identity in (4.7.13) yields the decomposition

$$\boldsymbol{\phi} = -\mathsf{T}_k^t\mathsf{L}_k^n\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi} + \mathsf{D}_{k-1}^t\mathsf{T}_{k-1}^t\mathsf{V}_k^n\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1}\boldsymbol{\phi} \tag{4.7.14}$$

for all $\boldsymbol{\phi} \in \mathcal{R}(\mathsf{T}_k^t)$.

Compare (4.7.14) with the regular decompositions provided in (4.6.3) and (4.6.13). In (4.7.14), the bounded maps

$$-\mathsf{T}_k^t\mathsf{L}_k^n\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to \mathsf{T}_k^t(\mathbf{W}_k^+) \subset \mathcal{R}(\mathsf{T}_k^t) \tag{4.7.15}$$

and

$$\mathsf{T}_{k-1}^t\mathsf{V}_k^n\mathsf{A}_k^\top\mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to \mathsf{T}_{k-1}^t(\mathbf{W}_{k-1}^+) \subset \mathcal{R}(\mathsf{T}_{k-1}^t) \tag{4.7.16}$$

play the roles of lifting and potential operators. Compactness of these operators as mappings $\mathcal{R}(\mathsf{T}_k^t) \to (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ$ and $\mathcal{R}(\mathsf{T}_k^t) \to (\mathring{\mathbf{W}}_{k}^{n,+})^\circ$ follows upon observing that under Assumption F, the map

$$\mathsf{T}_k^t : \mathbf{W}_k^+ \to (\mathring{\mathbf{W}}_k^{n,+})^\circ \tag{4.7.17}$$

is a compact operator, because the product of two bounded linear operators between normed spaces is compact if any one of the operand is [28, Thm. 2.16]. To confirm that (4.7.17) is compact, it is sufficient to recall from Definition 4.1 that it is the operator associated with the compact bilinear form (cf. [42, Chap. 3])

$$\begin{cases} \mathbf{W}_k^+ \times \mathbf{W}_{k+1}^+ \to \mathbb{R} \\ \qquad\qquad (\mathbf{x}, \mathbf{y}) \mapsto (\mathsf{A}_k\mathbf{x}, \imath_{k+1}^+\mathbf{y})_{\mathbf{W}_{k+1}} - (\imath_k^+\mathbf{x}, \mathsf{A}_k^\top\mathbf{y})_{\mathbf{W}_k} \end{cases} \tag{4.7.18}$$

where we have introduced for clarity the compact inclusions supplied by Assumption F.

In the next theorem, the unbounded linear operators

$$(\mathsf{D}_k^t)^* : \mathcal{D}\left((\mathsf{D}_k^t)^*\right) \subset (\mathring{\mathbf{W}}_{k+2}^{n,+})^\circ \to (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ, \tag{4.7.19a}$$

$$(\mathsf{D}_k^n)^* : \mathcal{D}\left((\mathsf{D}_k^n)^*\right) \subset (\mathring{\mathbf{W}}_{k-1}^{t,+})^\circ \to (\mathring{\mathbf{W}}_{k}^{t,+})^\circ, \tag{4.7.19b}$$

are the Hilbert space adjoints of the closed densely defined unbounded operators

$$\mathsf{D}_k^t : \mathcal{R}(\mathsf{T}_k^t) \subset (\mathring{\mathbf{W}}_{k+1}^{n,+})^\circ \to (\mathring{\mathbf{W}}_{k+2}^{n,+})^\circ \quad \text{and} \quad \mathsf{D}_k^n : \mathcal{R}(\mathsf{T}_k^n) \subset (\mathring{\mathbf{W}}_{k}^{t,+})^\circ \to (\mathring{\mathbf{W}}_{k-1}^{t,+})^\circ, \tag{4.7.20}$$

respectively.

Theorem 4.10 *Under assumptions D, E and F, the inclusions*

$$\mathcal{R}(\mathsf{T}_k^t) \cap \mathcal{D}\left((\mathsf{D}_{k-1}^t)^*\right) \hookrightarrow (\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ \quad \textit{and} \quad \mathcal{R}(\mathsf{T}_k^n) \cap \mathcal{D}\left((\mathsf{D}_{k+1}^n)^*\right) \hookrightarrow (\mathring{\mathbf{W}}_{k}^{n,+})^\circ \tag{4.7.21}$$

are compact.

Proof. We follow the arguments in the proof of Lemma 4.9. Let $(\phi_\ell)_{\ell\in\mathbb{Z}} \subset \mathcal{R}(\mathsf{T}_k^t) \cap \mathcal{D}\left((\mathsf{D}_k^t)^*\right)$ be a bounded sequence in $\mathcal{R}(\mathsf{T}_k^t) \cap \mathcal{D}\left((\mathsf{D}_k^t)^*\right)$.

The goal is to find a subsequence $(\phi_{\ell_\rho})_{\rho\in\mathbb{Z}}$ that is Cauchy in $(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ$. Similarly to (4.7.12), we use the stable decomposition in trace spaces (4.7.14):

$$\phi_\ell = \xi_\ell^+ + \mathsf{D}_{k-1}^t \zeta_\ell^+ \tag{4.7.22}$$

for all $\ell \in \mathbb{Z}$, where $\xi_\ell^+ := -\mathsf{T}_k^t \mathsf{L}_k^n \mathsf{A}_k^\top \mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1} \phi_\ell$ and $\zeta_\ell := \mathsf{T}_{k-1}^t \mathsf{V}_k^n \mathsf{A}_k^\top \mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1} \phi_\ell$. Since the norm in $\mathcal{R}(\mathsf{T}_k^t) \cap \mathcal{D}\left((\mathsf{D}_k^t)^*\right)$ is stronger than the norm in $\mathcal{R}(\mathsf{T}_k^t)$, the sequence $(\phi_\ell)_{\ell\in\mathbb{Z}}$ is bounded in the norm of $\mathcal{R}(\mathsf{T}_k^t)$. Hence, by compactness of the operators $-\mathsf{T}_k^t \mathsf{L}_k^n \mathsf{A}_k^\top \mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to (\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ$ and $\mathsf{T}_{k-1}^t \mathsf{V}_k^n \mathsf{A}_k^\top \mathsf{R}_{\mathcal{D}(\mathsf{A}_k^\top)}^{-1} : \mathcal{R}(\mathsf{T}_k^t) \to (\mathring{\mathbf{W}}_{k}^{t,+})^\circ$, there exist subsequences $(\xi_{\ell_\rho}^+)_{\rho\in\mathbb{Z}}$ and $(\zeta_{\ell_\rho}^+)_{\rho\in\mathbb{Z}}$ that are Cauchy in $(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ$ and $(\mathring{\mathbf{W}}_{k}^{t,+})^\circ$, respectively.

Now, we verify that $(\phi_{\ell_\rho})_{\rho\in\mathbb{Z}}$ is indeed Cauchy in $(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ$. We evaluate directly

$$\begin{aligned}
&\|\phi_{\ell_n} - \phi_{\ell_n}\|_{(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ} \\
&\quad = \left(\xi_{\ell_n} - \xi_{\ell_n}, \phi_{\ell_n} - \phi_{\ell_n}\right)_{(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ} + \left(\mathsf{D}_{k-1}^t\left(\zeta_{\ell_n} - \zeta_{\ell_n}\right), \phi_{\ell_n} - \phi_{\ell_n}\right)_{(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ} \\
&\quad \leq \|\xi_{\ell_n} - \xi_{\ell_n}\|_{(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ} \|\phi_{\ell_n} - \phi_{\ell_n}\|_{(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ} + \left(\zeta_{\ell_n} - \zeta_{\ell_n}, (\mathsf{D}_{k-1}^t)^*\left(\phi_{\ell_n} - \phi_{\ell_n}\right)\right)_{(\mathring{\mathbf{W}}_{k}^{t,+})^\circ},
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
\|\phi_{\ell_n} - \phi_{\ell_n}\|_{(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ} \leq{}& \underbrace{\|\xi_{\ell_n} - \xi_{\ell_n}\|_{(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ}}_{\to 0 \text{ as } m,n\to 0} \|\phi_{\ell_n} - \phi_{\ell_n}\|_{(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ} \\
&+ \underbrace{\|\xi_{\ell_n} - \xi_{\ell_n}\|_{(\mathring{\mathbf{W}}_{k}^{t,+})^\circ}}_{\to 0 \text{ as } m,n\to 0} \|(\mathsf{D}_{k-1}^t)^*\left(\phi_{\ell_n} - \phi_{\ell_n}\right)\|_{(\mathring{\mathbf{W}}_{k}^{t,+})^\circ}.
\end{aligned}$$

The desired result thus follows because $\|\phi_{\ell_n} - \phi_{\ell_n}\|_{(\mathring{\mathbf{W}}_{k+1}^{t,+})^\circ}$ and $\|(\mathsf{D}_{k-1}^t)^*\left(\phi_{\ell_n} - \phi_{\ell_n}\right)\|_{(\mathbf{W}_{k}^{t,+})^\circ}$ are bounded by hypothesis. □

Corollary 4.7.1. *Under assumptions D, E and F, the trace Hilbert complexes introduced in Theorem 4.9 are Fredholm.*

It is particularly interesting that while only one decomposition was sufficient to obtain Lemma 4.9, we needed *both* decompositions (assumptions B and C) to achieve a proof of the compactness property for the trace Hilbert complex: one for the space characterization and the other for the decomposition formula itself. The question whether it is *necessary* to have both remains open.

3D de Rham setting XVI: Trace de Rham complexes

Trace Hilbert complexes for the de Rham complex in 3D arise from the results of XV:

$$\begin{array}{ccc}
\{0\} & & \{0\} \\
\Big\downarrow \imath & & \Big\downarrow \imath \\
\mathcal{D}(\mathbf{curl}') \subset \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathrm{div}, \Omega)^{\circ} & & \mathcal{D}(\mathbf{curl}') \subset \Big(\mathbf{H}^1(\Omega)/\mathbf{H}^1_n(\Omega)\Big)' \\
\Big\downarrow \mathbf{curl}' & & \Big\downarrow \mathbf{curl}' \\
\mathcal{D}(\mathbf{grad}') \subset \widetilde{\mathbf{H}}^{-1}(\Omega) \cap \mathring{\mathbf{H}}(\mathbf{curl}, \Omega)^{\circ} & & \mathcal{D}(\mathbf{grad}') \subset \Big(\mathbf{H}^1(\Omega)/\mathbf{H}^1_t(\Omega)\Big)' \\
\Big\downarrow \mathbf{grad}' & & \Big\downarrow \mathbf{grad}' \\
\widetilde{H}^{-1}(\Omega) \cap \mathring{H}^1(\Omega)^{\circ} & & \Big(H^1(\Omega)/\mathring{H}^1(\Omega)\Big)' \\
\Big\downarrow 0 & & \Big\downarrow 0 \\
\{0\} & & \{0\}
\end{array} \tag{4.7.23}$$

In light of the de Rham setting XV, they correspond to

$$\{0\} \overset{\imath}{\hookrightarrow} H^{1/2}(\Gamma) \subset H^{-1/2}(\Gamma) \xrightarrow{\mathbf{curl}_\Gamma} \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \subset \mathbf{H}_t^{-1/2} \xrightarrow{\mathrm{curl}_\Gamma} H^{-1/2}(\Gamma) \xrightarrow{0} \{0\} \tag{4.7.24}$$

or its rotated version.

Since by Rellich's lemma the embeddings $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ are compact, the de Rham complexes in (4.1.4) satisfy Assumption F with the regular decompositions presented in the de Rham setting XII. Therefore, the associated trace de Rham complexes are Fredholm. As a consequence, their cohomology spaces are finite-dimensional.

4.8 Conclusion

As we have demonstrated in the present article, it takes only a pair of Hilbert complexes linked by the sub-complex relationship of their domain complexes to recover essential aspects of the structures inherent in the trace operators and trace spaces for the de Rham complex. Relying on notions of trace spaces as dual spaces or quotient spaces, we could establish detailed characterizations merely assuming the existence of stable regular decompositions induced by bounded lifting operators. These developments culminated in the discovery of associated trace Hilbert complexes, which are Fredholm under the mild additional assumption that the lifting operators are compact.

Hilbert complexes have recently moved into the focus of applied mathematicians, since they underlie a host of PDE-based mathematical models in areas as diverse as linear elasticity, gravity, and fluid dynamics. The related complexes are known as the elasticity complex, [8, Sect. 11] and [38], conformal complex, or Stokes complex [10, Sect. 4.4]. These and many more complexes [36, 37] arise from the de Rham complex through the powerful Bernstein-Gelfand-Gelfand (BGG) construction, as has been shown in [10]. Most likely, many more Hilbert complexes relevant for mathematical modeling still await discovery.

This backdrop lends relevance to our present work. Once the Hilbert complex structure is established, trace operators and trace spaces become available, which can serve as stepping stones towards the study of boundary value problems and the development of integral representations.

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Chapter 5
Boundary Integral Exterior Calculus

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Abstract We develop first-kind boundary integral equations for Hodge–Dirac and Hodge–Laplace operators associated with de Rham Hilbert complexes on compact Riemannian manifolds and in Euclidean space. We show that from a variational perspective, the first-kind boundary integral operators associated with Hodge–Dirac and Hodge–Laplace boundary value problems posed on submanifolds with Lipschitz boundaries are Hodge–Dirac and Hodge–Laplace operators as well, this time spawned by trace de Rham Hilbert complexes on the boundary whose spaces are equipped with non-local inner products defined through boundary potentials. The correspondence is to some extent structure-preserving in the sense that zero-order perturbations are also formally reproduced at the level of integral equations.

5.1 Introduction

Let $\mathcal{M}$ be either a smooth orientable compact Riemannian N-dimensional manifold without boundary or Euclidean space $\mathbb{R}^N$, cf. [1, Chap. 3 and 7] and [19, Chap. 6]. Assume that $\Omega = \Omega^- \subset \mathcal{M}$ is a submanifold of the same dimension with a compatibly oriented Lipschitz boundary $\Gamma := \partial\Omega$, cf. [23, Chap. 3], [24, sect. 2], [27, App. 1], [38, sect. 1] and [40, sect. 1]. Define $\Omega^+ = \mathcal{M}\backslash \mathrm{Int}\,\Omega$ and write $\imath_\Gamma^\mp : \Gamma \to \Omega^\mp$ for the inclusion maps. If $\mathcal{M} = \mathbb{R}^N$, we suppose for simplicity that Ω is bounded to avoid explicitly handling the necessary complications introduced by the need for decay or radiation conditions at infinity, cf. [11, sect. 3.3], [14], [23, Chap. 7] and [33, sect. 4.4].

5.1.1 Overview

Our goal is to understand the structural properties of *first-kind* boundary integral operators (BIOs) associated with boundary value problems (BVPs) for the Hodge–Dirac and Hodge–Laplace operators

$$\mathfrak{D} = \mathbf{d} + \boldsymbol{\delta} \qquad \text{and} \qquad -\Delta_\ell = \mathrm{d}_{\ell-1}\delta_\ell + \delta_{\ell+1}\mathrm{d}_\ell, \qquad 0 \le \ell \le N,$$

cf. [11, 13, 23, 31] and [33]. We will find that the obtained first-kind BIOs are Hodge–Dirac and Hodge–Laplace operators themselves, but associated with trace de Rham complexes whose spaces are equipped with non-local inner products defined through boundary potentials. This discovery reveals the importance of the trace de Rham complex to study related BVPs in general.

An unavoidable difficulty arises at the outset of our program. In Euclidean space, both the Hodge–Dirac operator and the Hodge–Laplacian admit two-sided inverses in the sense of distributions when suitable decay conditions are imposed. However, there are topological obstructions on compact manifolds that may prevent the existence of fundamental solutions. To recover their existence, we regularize the Hodge–Dirac and Hodge–Laplace operators by adding zero-order terms when they have non-trivial kernels on $\mathcal{M}$. Our main intent being to display the structure-preserving power of first-kind boundary integral equations (BIEs), it will be sufficient motivation to focus on the simplest type of perturbation.

In that regard, the simplest option is to work with modified Hodge–Dirac and Hodge–Yukawa operators of the form

$$\mathfrak{D} + i\kappa, \qquad \kappa \in \mathbb{R}\backslash\{0\}, \qquad\qquad \text{and} \qquad\qquad -\Delta_\ell + \lambda, \qquad \lambda > 0,$$

which are related by the identity

$$(\mathfrak{D} - i\kappa)\,(\mathfrak{D} + i\kappa) = -\boldsymbol{\Delta} + \kappa^2. \tag{5.1.1}$$

5.1.2 Related work

We draw on the previous article by Schulz and Hiptmair, presented in Chapter 3, in which the correspondence between domain and boundary Hodge–Dirac operators was initially discovered [33]. Inspired by [11] and [12], where first-kind boundary integral equations for Hodge–Helmholtz operators were studied, only three-dimensional Euclidean space $\mathcal{M} = \mathbb{R}^3$ is studied in [33]. The investigation was solely based on classical vector calculus. The idea was to emphasize that although the Hodge–Dirac operator is only first-order, there is a close formal relationship between our arguments and the well-known theory of first-kind boundary integral equations for second-order elliptic operators in Euclidean space. Our goal now is to generalize these results to arbitrary dimensions by translating [11] and [33] into the language of differential forms. In doing so, the theory naturally extends to Riemannian manifolds and hidden structures behind the integral equations are revealed.

We owe to a rich literature on boundary integral equations formulated in the framework of Grassmann algebras. Most notably, D. Mitrea, I. Mitrea, M. Mitrea and Taylor extensively studied *second-kind* boundary integral equations related to the Hodge–Laplacian on compact manifolds [25, 27]. Auchmann and Kurz also used exterior algebra to study boundary integral equations for Maxwell-type problems [20].

The important results of D. Mitrea, M. Mitrea, Shaw [26] and Weck [40] on the existence and properties of surjective trace operators for the relevant spaces of differential forms allow the development of boundary integral exterior calculus on boundaries of mere Lipschitz regularity. Abstract trace complexes are also studied in [18], where an alternative proof than that given in [26] is provided for the compactness property of the trace de Rham complex.

5.1.3 Exterior Calculus

Subspaces of the space of differential forms of order ℓ on $\mathcal{M}$ characterized by coefficient-based regularity properties will be denoted by $L^\infty\Lambda^\ell(\mathcal{M})$, $L^2\Lambda^\ell(\mathcal{M})$, $H^s\Lambda^\ell(\mathcal{M})$ and so forth, cf. [26] and [40]. Similar notation is used for submanifolds. Following [16, Chap. 3], we will shorthand $\mathcal{E}^\ell(\mathcal{M}) = C^\infty\Lambda^\ell(\mathcal{M})$ and $\mathcal{D}^\ell(\mathcal{M}) = C_0^\infty\Lambda^\ell(\mathcal{M})$ for spaces of test functions. Their topological duals will be written $\mathcal{D}'_\ell(\mathcal{M})$ and $\mathcal{E}'_\ell(\mathcal{M})$. Primes always refer to dual spaces or dual maps, e.g. $H^{-1}\Lambda^\ell(\Omega) = (H_0^1\Lambda^\ell(\Omega))'$. We write in a bold font, e.g. $\boldsymbol{U} = (U_\ell)_\ell$, the elements of full Grassman algebras such as $L^2\Lambda(\mathcal{M}) = \oplus_\ell L^2\Lambda^\ell(\mathcal{M})$. For convenience, we let $\ell \in \mathbb{Z}$ run over all integers, but identify forms of rank $\ell < 0$ and $\ell > n$ with zero.

The Hodge star $\star_\ell : L^2\Lambda^\ell(\mathcal{M}) \to L^2\Lambda^{N-\ell}(\mathcal{M})$ is induced by the Riemannian metric on $\mathcal{M}$. The symmetric pairing

$$\langle U_\ell, V_\ell\rangle_\Omega = \int_\Omega U_\ell \wedge \star_\ell V_\ell, \qquad \forall U_\ell, V_\ell \in L^2\Lambda^\ell(\Omega), \tag{5.1.2}$$

is distinguished from the Hermitian inner product $(U_\ell, V_\ell)_\Omega = \langle U_\ell, \overline{V_\ell}\rangle_\Omega$, where the overline indicates complex conjugation of the coefficients.

The codifferential $\delta_{\ell+1} = (-1)^{\ell+1} \star_\ell^{-1} \mathrm{d}_{N-\ell-1} \star_{\ell+1}$ is formally adjoint to the exterior derivative. We adopt the view that $\mathrm{d}_\ell : L^2\Lambda^\ell(\Omega) \to L^2\Lambda^{\ell+1}(\Omega)$ and $\delta_{\ell+1} : L^2\Lambda^{\ell+1}(\Omega) \to L^2\Lambda^\ell(\Omega)$ are the closed densely defined unbounded linear operators giving rise to the Fredholm Hilbert cochain and chain complexes

$$\dots \xrightarrow{\mathrm{d}_{\ell-1}} H\Lambda^\ell(\mathrm{d},\Omega) \xrightarrow{\mathrm{d}_\ell} H\Lambda^{\ell+1}(\mathrm{d},\Omega) \xrightarrow{\mathrm{d}_{\ell+1}} \dots \tag{5.1.3a}$$

and

$$\dots \xleftarrow[\delta_{\ell-1}]{} H\Lambda^{\ell-1}(\delta,\Omega) \xleftarrow[\delta_\ell]{} H\Lambda^\ell(\delta,\Omega) \xleftarrow[\delta_{\ell+1}]{} \dots \tag{5.1.3b}$$

satisfying the compactness property, cf. [2, Chap. 4 and 6], [4], [24] and [30].

The corresponding diffuse Fredholm–nilpotent operators $\mathbf{d} : L^2\Lambda(\Omega) \to L^2\Lambda(\Omega)$ and $\boldsymbol{\delta} : L^2\Lambda(\Omega) \to L^2\Lambda(\Omega)$ are formally adjoint under the Hermitian inner product $(\boldsymbol{U}, \boldsymbol{V})_\Omega = \langle \boldsymbol{U}, \overline{\boldsymbol{V}}\rangle_\Omega$ defined through the symmetric pairing

$$\langle \boldsymbol{U}, \boldsymbol{V}\rangle_\Omega = \sum_\ell \langle U_\ell, V_\ell\rangle_\Omega, \qquad \forall \boldsymbol{U}, \boldsymbol{V} \in L^2\Lambda(\Omega),$$

cf. [21, sect. 3 and 5] and [5, sect. 3]. As operator matrices acting on vectors of differential forms of the form $\boldsymbol{U} = (U_0, ..., U_N)^\top$, the *full* exterior derivative and codifferential read

$$\mathbf{d} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \mathrm{d}_0 & 0 & 0 & \dots & 0 \\ 0 & \mathrm{d}_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \mathrm{d}_{N-1} & 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\delta} = \begin{pmatrix} 0 & \delta_1 & 0 & \dots & 0 \\ 0 & 0 & \delta_2 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \delta_N \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.1.4}$$

Similarly, the full Hodge star is represented by

$$\star = \begin{pmatrix} & & & & \star_0 \\ & \mathbf{0} & & \star_1 & \\ & & \iddots & & \\ & \star_{N-1} & & \mathbf{0} & \\ \star_N & & & & \end{pmatrix}. \tag{5.1.5}$$

We find it convenient to write the duality pairings that extend symmetric L^2-type pairings of the form (5.1.2) using double angular brackets, e.g. $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{M}}$.

For later use, we define the Sobolev spaces of ℓ-forms

$$\begin{aligned} H\Lambda^{\ell}(\mathrm{d}\delta, \Omega) &= \left\{ U_\ell \in H\Lambda^{\ell}(\delta, \Omega) \;\middle|\; \delta_\ell U_\ell \in H\Lambda^{\ell-1}(\mathrm{d}, \Omega) \right\}, \\ H\Lambda^{\ell}(\delta\mathrm{d}, \Omega) &= \left\{ U_\ell \in H\Lambda^{\ell}(\mathrm{d}, \Omega) \;\middle|\; \mathrm{d}_\ell U_\ell \in H\Lambda^{\ell+1}(\delta, \Omega) \right\}, \\ H\Lambda^{\ell}(\Delta, \Omega) &= H\Lambda^{\ell}(\mathrm{d}\delta, \Omega) \cap H\Lambda^{\ell}(\delta\mathrm{d}, \Omega), \\ X\Lambda^{\ell}(\Omega) &= H\Lambda^{\ell}(\mathrm{d}, \Omega) \cap H\Lambda^{\ell}(\delta, \Omega), \end{aligned}$$

equipped with graph inner products.

Because $\oplus_\ell X\Lambda^\ell(\Omega)$ will be the domain of the Hodge–Dirac operator, we introduce the notation

$$H\Lambda(\mathfrak{D}, \Omega) = H\Lambda(\mathbf{d}, \Omega) \cap H\Lambda(\boldsymbol{\delta}, \Omega)$$

for that space of full forms.

5.1.4 Trace spaces

We will impose boundary conditions via trace operators. We briefly review their definition and mapping properties, cf. [9, 18, 26, 40]. In accordance with standard practice, we repurpose the notation from Subsection 5.1.3 for operators on the boundary, but point out that the indices must account for the change in dimension when passing to a submanifold. In particular, notice that the Hodge star associated with the induced metric on the boundary is a continuous mapping $\star_\ell : L^2\Lambda^\ell(\Gamma) \to L^2\Lambda^{N-\ell-1}(\Gamma)$, cf. [26, 40].

5.1.4.1 Traces of differential forms

Relevant traces for $H_{\mathrm{loc}}\Lambda^\ell(\mathrm{d}, \Omega^\mp)$ and $H_{\mathrm{loc}}\Lambda^\ell(\delta, \Omega^\mp)$ are obtained by extending the pullback and "rotated" pullback of differential forms, also called tangential and normal traces. They are defined for all smooth forms $U_\ell \in \mathcal{D}^\ell(\mathcal{M})$ by

$$\mathsf{t}_\ell^\mp U_\ell = \imath_\mp^* U_\ell \qquad \text{and} \qquad \mathsf{n}_\ell^\mp U_\ell = \star_{\ell-1}^{-1} \imath_\mp^* \star_\ell U_\ell. \tag{5.1.6}$$

Adopting the notation of [20], we define the dual spaces

$$H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\Gamma) := (H_\parallel^{\frac{1}{2}}\Lambda^\ell(\Gamma))' \qquad \text{and} \qquad H_\perp^{-\frac{1}{2}}\Lambda^\ell(\Gamma) := (H_\perp^{\frac{1}{2}}\Lambda^\ell(\Gamma))',$$

where the regular trace spaces are given by

$$H_{\|}^{\frac{1}{2}}\Lambda^{\ell}(\Gamma) := \mathsf{t}_{\ell}^{\mp} H^{1}\Lambda^{\ell}(\Omega^{\mp}) \qquad \text{and} \qquad H_{\perp}^{\frac{1}{2}}\Lambda^{\ell}(\Gamma) = \mathsf{n}_{\ell}^{\mp} H^{1}\Lambda^{\ell+1}(\Omega^{\mp}).$$

They generalize the well-known space of Dirichlet traces $H^{\frac{1}{2}}\Lambda^{0}(\Gamma)$.

On the boundary, we view the exterior derivative and the codifferential as the closed densely defined unbounded linear operators $\mathrm{d}_{\ell} : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell+1}(\Gamma)$ and $\delta_{\ell} : H_{\|}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma) \longrightarrow H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma)$ giving rise to the Fredholm Hilbert complexes

$$\ldots \xrightarrow{\mathrm{d}_{\ell-1}} H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma) \xrightarrow{\mathrm{d}_{\ell}} H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d},\Gamma) \xrightarrow{\mathrm{d}_{\ell+1}} \ldots \tag{5.1.7a}$$

and

$$\ldots \xleftarrow[\delta_{\ell-1}]{} H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma) \xleftarrow[\delta_{\ell}]{} H_{\|}^{-\frac{1}{2}}\Lambda^{\ell}(\delta,\Gamma) \xleftarrow[\delta_{\ell+1}]{} \ldots \tag{5.1.7b}$$

associated with the domain complexes (5.1.3a) and (5.1.3b), cf. Chapter 4 and [18, 26, 40].

It is the content of the trace theorems studied in Chapter 4 and [9, 18, 26, 40] that the operators

$$\mathsf{t}_{\ell}^{\mp} : H^{1}\Lambda_{\mathrm{loc}}^{\ell}(\mathcal{M}) \longrightarrow H_{\|}^{\frac{1}{2}}\Lambda^{\ell}(\Gamma) \qquad \text{and} \qquad \mathsf{n}_{\ell}^{\mp} : H^{1}\Lambda_{\mathrm{loc}}^{\ell}(\mathcal{M}) \longrightarrow H_{\perp}^{\frac{1}{2}}\Lambda^{\ell-1}(\Gamma) \tag{5.1.8}$$

extend to continuous and *surjective* mappings

$$\begin{aligned} \mathsf{t}_{\ell}^{\mp} &: H_{\mathrm{loc}}\Lambda^{\ell}(\mathrm{d},\Omega^{\mp}) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d},\Gamma) \\ \mathsf{n}_{\ell}^{\mp} &: H_{\mathrm{loc}}\Lambda^{\ell}(\delta,\Omega^{\mp}) \longrightarrow H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma) \end{aligned} \tag{5.1.9}$$

such that the integration by parts formula

$$\langle \mathrm{d}_{\ell}\, U_{\ell}, V_{\ell+1}\rangle_{\Omega^{\mp}} = \langle U_{\ell}, \delta_{\ell+1}\, V_{\ell+1}\rangle_{\Omega^{\mp}} \pm \langle\!\langle \mathsf{t}_{\ell}^{\mp} U_{\ell}, \mathsf{n}_{\ell+1}^{\mp} V_{\ell+1}\rangle\!\rangle_{\Gamma} \tag{5.1.10}$$

holds for all $U_{\ell} \in H\Lambda^{\ell}(\mathrm{d},\Omega^{\mp})$ and $V_{\ell+1} \in H\Lambda^{\ell+1}(\delta,\Omega^{\mp})$.

On the right-hand side of (5.1.10), the duality pairing on the boundary extends the $L^{2}\Lambda^{\ell}(\Gamma)$-pairing. That is, it puts $H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d},\Gamma)$ in duality with $H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma)$ using $L^{2}\Lambda^{\ell}(\Gamma)$ as a pivot space.

In a similar notation to [2, Thm. 6.5],

$$\begin{aligned} \mathring{H}\Lambda^{\ell}(\mathrm{d},\Omega) &= \overline{\mathcal{D}(\Omega)}^{H\Lambda^{\ell}(\mathrm{d},\Omega)} = \ker \mathsf{t}_{\ell} \cap H\Lambda^{\ell}(\mathrm{d},\Omega) \\ \mathring{H}\Lambda^{\ell}(\delta,\Omega) &= \overline{\mathcal{D}(\Omega)}^{H\Lambda^{\ell}(\delta,\Omega)} = \ker \mathsf{n}_{\ell} \cap H\Lambda^{\ell}(\delta,\Omega). \end{aligned}$$

Despite Γ being merely Lipschitz regular, the usual commutative relations

$$\mathsf{t}_{\ell}^{\mp} \circ \mathrm{d}_{\ell} = \mathrm{d}_{\ell} \circ \mathsf{t}_{\ell}^{\mp} \qquad \text{and} \qquad \mathsf{n}_{\ell-1}^{\mp} \circ \delta_{\ell} = -\delta_{\ell-1} \circ \mathsf{n}_{\ell}^{\mp}, \tag{5.1.11}$$

also hold at the level of trace spaces. In particular, the second identity can be obtained from the first:

$$\mathsf{n}_{\ell-1}\delta_{\ell} = \star_{\ell-2}^{-1}\imath^{*} \star_{\ell-1} \left((-1)^{\ell} \star_{\ell-1}^{-1} \mathrm{d}_{N-\ell}\star_{\ell}\right)$$

$$= -(-1)^{\ell-1} \star_{\ell-2} \mathrm{d}_{N-\ell}\, \imath^* \star_\ell$$
$$= -\left((-1)^{\ell-1} \star_{\ell-2} \mathrm{d}_{N-\ell} \star_{\ell-1}\right) \star_{\ell-1}^{-1} \imath^* \star_\ell = -\delta_{\ell-1} \circ \mathsf{n}_\ell^{\mp}.$$

We use a bold font to denote traces acting on the full algebra of forms, i.e.

$$\mathbf{t}^{\mp}\boldsymbol{U} = \boldsymbol{\imath}_{\mp}^{*}\boldsymbol{U} \qquad \text{and} \qquad \mathbf{n}^{\mp}\boldsymbol{V} = \star^{-1}\, \mathbf{t}_{\mp} \star \boldsymbol{V}. \tag{5.1.12}$$

Then, applying the integration by parts formula (5.1.10) order-wise yields

$$\langle \mathbf{d}\, \boldsymbol{U}, \boldsymbol{V} \rangle_{\Omega^{\mp}} = \langle \boldsymbol{U}, \boldsymbol{\delta}\, \boldsymbol{V} \rangle_{\Omega^{\mp}} \pm \langle\!\langle \mathbf{t}^{\mp}\boldsymbol{U}, \mathbf{n}^{\mp}\boldsymbol{V} \rangle\!\rangle_{\Gamma} \tag{5.1.13}$$

for all $\boldsymbol{U} \in H\Lambda(\mathbf{d}, \Omega)$ and $\boldsymbol{V} \in H\Lambda(\boldsymbol{\delta}, \Omega)$.

5.1.4.2 Lifting maps

The purpose of this section is twofold. Firstly, it is immediate by surjectivity that the traces in (5.1.9) admit continuous right inverses into $H\Lambda^\ell(\mathrm{d}, \Omega)$ and $H\Lambda^\ell(\delta, \Omega)$, respectively. We want to show in particular that these right inverses can be designed to lift the boundary data into the more regular space $H\Lambda^\ell(\Delta, \Omega)$. Secondly, we also build lifting maps for the continuous traces

$$\mathsf{t}_{\ell-1}^{\mp} \circ \delta_\ell : H\Lambda^\ell(\mathrm{d}\delta, \Omega) \to H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d}, \Gamma),$$
$$\mathsf{n}_{\ell+1}^{\mp} \circ \mathrm{d}_\ell : H\Lambda^\ell(\delta\mathrm{d}, \Omega) \to H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\delta, \Gamma),$$

that will be used to impose boundary conditions for the Hodge–Laplacian.

With the next two lemmas, we generalize to differential forms the results of [11, Sec. 2.5].

Lemma 5.1 *There exist continuous operators* $\mathcal{E}_\ell^{\mathsf{t}} : H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}, \Gamma) \to H\Lambda^\ell(\Delta, \Omega)$ *and* $\mathcal{E}_\ell^{\mathsf{n}} : H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta, \Gamma) \to H\Lambda^\ell(\Delta, \Omega)$ *such that*

$$\mathsf{t}_\ell\, \mathcal{E}_\ell^{\mathsf{t}}\, g_\ell = g_\ell \qquad \textit{and} \qquad \mathsf{n}_\ell\, \mathcal{E}_\ell^{\mathsf{n}}\, h_{\ell-1} = h_{\ell-1}$$

for all $g_\ell \in H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}, \Gamma)$ *and* $h_{\ell-1} \in H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta, \Gamma)$.

Proof. Given $g_\ell \in H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}, \Gamma)$, let $\mathcal{E}_\ell^{\mathsf{t}}(g_\ell)$ be the unique element in $H\Lambda(\mathrm{d}, \Omega)$ defined by

$$\mathcal{E}_\ell^{\mathsf{t}}(g_\ell) = \arg\min_{\substack{V_\ell \in H\Lambda^\ell(\mathrm{d},\Omega), \\ \mathsf{t}_\ell V_\ell = g_\ell}} \|V_\ell\|_{H\Lambda^\ell(\mathrm{d},\Omega)}.$$

This minimization problem is equivalent to satisfying the Euler equations

$$\langle \mathrm{d}_\ell\, \mathcal{E}_\ell^{\mathsf{t}}(g_\ell), \mathrm{d}_\ell\, V_\ell \rangle_\Omega + \langle \mathcal{E}_\ell^{\mathsf{t}}(g_\ell), V_\ell \rangle_\Omega = 0$$

for all $V_\ell \in \overset{\circ}{H}\Lambda^\ell(\mathrm{d}, \Omega)$. Testing with suitable choices of test functions shows that $\mathcal{E}_\ell^{\mathsf{t}} : H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}, \Gamma) \to H\Lambda^\ell(\mathrm{d}, \Omega)$ is a continuous operator satisfying the equations

$$\delta_{\ell+1}\mathrm{d}_\ell\,\mathcal{E}^{\mathsf{t}}_\ell g_\ell + \mathcal{E}^{\mathsf{t}}_\ell g_\ell = 0 \qquad \text{in } \Omega,, \tag{5.1.14a}$$
$$\mathsf{t}_\ell \mathcal{E}^{\mathsf{t}}_\ell g_\ell = g_\ell \qquad \text{on } \Gamma. \tag{5.1.14b}$$

In particular, (5.1.14a) not only reveals that $\mathcal{E}^{\mathsf{t}}_\ell g_\ell \in H\Lambda^\ell(\delta\mathrm{d},\Omega)$, but also that $\delta_\ell\,\mathcal{E}^{\mathsf{t}}_\ell g_\ell = 0$ in Ω. We conclude that $\mathcal{E}^{\mathsf{t}}_\ell$ exhibits the claimed regularity and (5.1.14b) confirms that the defined map is a right-inverse for the tangential trace.

The map $\mathcal{E}^{\mathsf{n}}_\ell$ can be defined similarly using a minimization problem involving the normal trace. □

Lemma 5.2 *There exist continuous operators* $\mathcal{R}^{\mathsf{t}}_\ell : H_\perp^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ *and* $\mathcal{R}^{\mathsf{n}}_\ell : H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ *such that*

$$\mathsf{t}_{\ell-1}\delta_\ell\mathcal{R}^{\mathsf{t}}_\ell\, g_{\ell-1} = g_{\ell-1} \qquad \textit{and} \qquad \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathcal{R}^{\mathsf{n}}_\ell\, h_\ell = h_\ell$$

for all $g_{\ell-1} \in H_\perp^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma)$ *and* $h_\ell \in H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma)$.

Proof. Given boundary data $h_\ell \in H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma)$, we define $\mathcal{R}^{\mathsf{n}}_\ell(h_\ell)$ as the unique element of $H\Lambda^\ell(\mathrm{d},\Omega)$ such that

$$\langle \mathrm{d}_\ell\,\mathcal{R}^{\mathsf{n}}_\ell(h_\ell), \mathrm{d}_\ell\,V_\ell\rangle_\Omega + \langle \mathcal{R}^{\mathsf{t}}_\ell(h_\ell), V_\ell\rangle_\Omega = \langle\!\langle h_\ell, \mathsf{t}_\ell V_\ell\rangle\!\rangle_\Gamma$$

for all $V_\ell \in H\Lambda^\ell(\mathrm{d},\Omega)$. Lax-Milgram lemma guarantees that $\mathcal{R}^{\mathsf{n}}_\ell$ is well-defined and continuous as a map $\mathcal{R}^{\mathsf{n}}_\ell : H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma) \to H\Lambda^\ell(\mathrm{d},\Omega)$.

Routine verification using suitable test functions and the integration by parts formula (5.1.10) shows that it satisfies the equations

$$\delta_{\ell+1}\mathrm{d}_\ell\,\mathcal{R}^{\mathsf{n}}_\ell(h_\ell) + \mathcal{R}^{\mathsf{n}}_\ell(h_\ell) = 0 \qquad \text{in } \Omega, \tag{5.1.15a}$$
$$\mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathcal{R}^{\mathsf{n}}_\ell\, h_\ell = h_\ell \qquad \text{on } \Gamma. \tag{5.1.15b}$$

Similarly as in the proof of Lemma 5.1, we obtain from (5.1.15a) that $\mathcal{R}^{\mathsf{n}}_\ell(h_\ell) \in H\Lambda^\ell(\delta\mathrm{d},\Omega)$ and $\delta_\ell\mathcal{R}^{\mathsf{t}}_\ell h_\ell = 0$ in Ω, i.e. $\mathcal{R}^{\mathsf{t}}_\ell h_\ell \in H\Lambda^\ell(\Delta,\Omega)$. Then, (5.1.15b) confirms that $\mathcal{R}^{\mathsf{t}}_\ell$ is a right-inverse for the trace $\mathsf{n}_{\ell+1}\mathrm{d}_\ell$.

The analogous result for $\mathsf{t}_{\ell-1}\delta_\ell$ is obtained similarly by defining $\mathcal{R}^{\mathsf{t}}_\ell(h_\ell)$ using the graph inner product on $H\Lambda^\ell(\delta,\Omega)$. □

Before moving on, we want to verify that the lifting operators from Lemma 5.1 and Lemma 5.2 can be used to construct right-inverses for the compound traces of the Hodge–Laplacian.

Recalling Subsection 5.1.4, the traces defined for all $U_\ell \in \mathcal{D}^\ell(\mathcal{M})$ by

$$\mathsf{T}^{\mathsf{t}}_\Delta\, U_\ell = \begin{pmatrix} \mathsf{t}_{\ell-1}\delta_\ell U_\ell \\ \mathsf{t}_\ell U_\ell \end{pmatrix} \qquad \text{and} \qquad \mathsf{T}^{\mathsf{n}}_\Delta\, U_\ell = \begin{pmatrix} \mathsf{n}_\ell U_\ell \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell \end{pmatrix} \tag{5.1.16}$$

are continuous as mappings

$$\mathsf{T}^{\mathsf{t}}_\Delta : H\Lambda^\ell(\mathrm{d}\delta,\Omega) \cap H\Lambda^\ell(\mathrm{d},\Omega) \longrightarrow H^{\mathsf{t}}_\Delta(\Gamma),$$
$$\mathsf{T}^{\mathsf{n}}_\Delta : H\Lambda^\ell(\delta,\Omega) \cap H\Lambda^\ell(\delta\mathrm{d},\Omega) \longrightarrow H^{\mathsf{n}}_\Delta(\Gamma),$$

where the product of trace spaces are given by

$$H_\Delta^{\mathsf{t}}(\Gamma) = H_\perp^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma) \times H_\perp^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d},\Gamma),$$
$$H_\Delta^{\mathsf{n}}(\Gamma) = H_\parallel^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma) \times H_\parallel^{-\frac{1}{2}}\Lambda^{\ell}(\delta,\Gamma).$$

We want to show that their restriction to $H\Lambda^\ell(\Delta,\Omega)$ are surjective and admit continuous lifting operators.

Lemma 5.3 *There exist continuous operators $\mathcal{L}_\ell^{\mathsf{t}} : H_\Delta^{\mathsf{t}}(\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ and $\mathcal{L}_\ell^{\mathsf{n}} : H_\Delta^{\mathsf{n}}(\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ such that*

$$\mathsf{T}_\Delta^{\mathsf{t}}\,\mathcal{L}^{\mathsf{t}}\begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} = \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} \qquad \text{and} \qquad \mathsf{T}_\Delta^{\mathsf{n}}\,\mathcal{L}_\ell^{\mathsf{n}}\begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix}$$

for all $(g_{\ell-1}, g_\ell)^\top \in H_\Delta^{\mathsf{t}}(\Gamma)$ and $(h_{\ell-1}, h_\ell)^\top \in H_\Delta^{\mathsf{n}}(\Gamma)$.

Proof. We prove the result for $\mathsf{T}_\Delta^{\mathsf{n}}$. The proof is similar for $\mathsf{T}_\Delta^{\mathsf{t}}$. The trick is to define the lifting for all $(h_{\ell-1}, h_\ell)^\top \in H_\Delta^{\mathsf{n}}(\Gamma)$ by

$$\mathcal{L}_\ell^{\mathsf{n}}\begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} = \mathcal{E}_\ell^{\mathsf{n}} h_{\ell-1} + \mathcal{R}_\ell^{\mathsf{n}} h_\ell - \mathrm{d}_{\ell-1}\mathcal{R}_{\ell-1}^{\mathsf{n}}(\mathsf{n}_\ell \mathcal{R}_\ell^{\mathsf{n}} h_\ell).$$

The first two terms are immediately seen to belong in $H\Lambda^\ell(\Delta,\Omega)$ thanks to the mapping results of Lemma 5.1 and Lemma 5.2. To confirm that the third term also displays the same regularity, we dig deeper into the proof of Lemma 5.2 and simply recall (5.1.15a).

By construction, $\mathrm{d}_\ell \circ \mathcal{E}_\ell^{\mathsf{n}} = 0$. Indeed, the analogous result for the tangential trace in the proof of Lemma 5.1 was that $\delta_\ell \circ \mathcal{E}^{\mathsf{t}} = 0$. Hence, using the established properties of the lifting operators and the fact that $\mathrm{d}^2 = 0$, we compute

$$\begin{aligned}
\mathsf{T}_\Delta^{\mathsf{n}}\mathcal{L}_\ell^{\mathsf{n}}\begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} &= \begin{pmatrix} \mathsf{n}_\ell\mathcal{E}_\ell^{\mathsf{n}} h_{\ell-1} + \mathsf{n}_\ell\mathcal{R}_\ell^{\mathsf{n}} h_\ell - \mathsf{n}_\ell \mathrm{d}_{\ell-1}\mathcal{R}_{\ell-1}^{\mathsf{n}}(\mathsf{n}_\ell\mathcal{R}_\ell^{\mathsf{n}} h_\ell) \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathcal{E}_\ell^{\mathsf{n}} h_{\ell-1} + \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathcal{R}_\ell^{\mathsf{n}} h_\ell - \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathrm{d}_{\ell-1}\mathcal{R}_{\ell-1}^{\mathsf{n}}(\mathsf{n}_\ell\mathcal{R}_\ell^{\mathsf{n}} h_\ell) \end{pmatrix} \\
&= \begin{pmatrix} h_{\ell-1} + \mathsf{n}_\ell\mathcal{R}_\ell^{\mathsf{n}} h_\ell - \mathsf{n}_\ell\mathcal{R}_\ell^{\mathsf{n}} h_\ell \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathcal{R}_\ell^{\mathsf{n}} h_\ell \end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix},
\end{aligned}$$

which shows that $\mathcal{L}_\ell^{\mathsf{n}}$ is a right-inverse for T_Δ^{n}. □

5.2 Boundary value problems

In this section, we formulate the BVPs of interest in this article. We begin with the Hodge–Dirac operator before moving on to the Hodge–Laplacian. In Subsection 5.2.1.1, readers might notice that because the Hodge star operators

$$\star_\ell : X\Lambda^\ell(\Omega) \to X\Lambda^{N-\ell}(\Omega) \qquad \text{and} \qquad \star_\ell : H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma) \to H_\parallel^{-\frac{1}{2}}\Lambda^{N-1-\ell}(\delta,\Gamma)$$

are isometric isomorphisms [40, Lem. 5], the two boundary value problems stated for the Hodge–Dirac operator are, at the abstract level of Hilbert complexes, equivalent in terms of solvability. One corresponds to the Hodge–Dirac operator associated with the cochain complex (5.1.3a), while the other corresponds to the Hodge–Dirac operator associated with the chain complex (5.1.3b). Each of these problems can be turned into the other, cf. [11, Rmk. 3.3] and [33, Rmk. 5.1]. A similar observation can be made for the two BVPs involving the Hodge–Laplacian that will be presented in Subsection 5.2.2.1. The reason we insist on formulating each of them explicitly and independently is to highlight the formal difference in the expressions of the self-adjoint operators behind them. It turns out that it is those expressions that we will recognize in the formulas of the associated first-kind BIOs.

5.2.1 Hodge–Dirac operators

We take $H\Lambda(\mathfrak{D},\Omega)$ to be the domain of the Hodge–Dirac operator

$$\mathfrak{D} = \boldsymbol{\delta} + \mathbf{d} : H\Lambda(\mathfrak{D},\Omega) \to L^2\Lambda(\Omega) \tag{5.2.1}$$

on which we want to impose boundary conditions.

5.2.1.1 BVPs for Hodge–Dirac operators

In light of Subsection 5.1.4, the duality between the trace spaces $H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma)$ and $H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma)$ involved in the integration by parts formula (5.1.13) points towards two types of boundary conditions. For $\kappa \in \mathbb{R}$, we consider the BVPs

$$\boldsymbol{U} \in H\Lambda(\mathfrak{D},\Omega) : \begin{cases} (\mathfrak{D} + i\kappa)\,\boldsymbol{U} = \mathbf{0} & \text{in } \Omega \\ \mathbf{t}\,\boldsymbol{U} = \boldsymbol{g} & \text{on } \partial\Omega \end{cases}, \qquad \boldsymbol{g} \in H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma), \tag{5.2.2a}$$

and

$$\boldsymbol{U} \in H\Lambda(\mathfrak{D},\Omega) : \begin{cases} (\mathfrak{D} + i\kappa)\,\boldsymbol{U} = \mathbf{0} & \text{in } \Omega \\ \mathbf{n}\,\boldsymbol{U} = \boldsymbol{h} & \text{on } \partial\Omega \end{cases}, \qquad \boldsymbol{h} \in H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma). \tag{5.2.2b}$$

The self-adjoint operators underlying (5.2.2a) and (5.2.2b) are

$$\mathfrak{D}_{\mathbf{t}} = \boldsymbol{\delta} + \boldsymbol{\delta}^* : H\Lambda(\boldsymbol{\delta},\Omega) \cap \mathring{H}\Lambda(\mathbf{d},\Omega) \to L^2\Lambda(\Omega), \tag{5.2.3a}$$

$$\mathfrak{D}_{\mathbf{n}} = \mathbf{d} + \mathbf{d}^* : \mathring{H}\Lambda(\boldsymbol{\delta},\Omega) \cap H\Lambda(\mathbf{d},\Omega) \to L^2\Lambda(\Omega), \tag{5.2.3b}$$

respectively. These are the Hodge–Dirac operators associated with the nilpotent operators $\boldsymbol{\delta}$ and $\mathbf{d}$ arising from the Hilbert complexes (5.1.3a) and (5.1.3b), respectively; cf. [21, Sec. 2], [33, Sec. 3].

We notice that the null-spaces

$$\mathfrak{H}_{\mathbf{t}} = \ker(\mathfrak{D}_{\mathbf{t}}) = \left\{\boldsymbol{U} \in H\Lambda(\mathfrak{D},\Omega) : \mathbf{d}\boldsymbol{U} = 0,\ \boldsymbol{\delta}\boldsymbol{U} = 0,\ \mathbf{t}\boldsymbol{U} = 0\right\}, \tag{5.2.4a}$$

$$\mathfrak{H}_{\mathbf{n}} = \ker(\mathfrak{D}_{\mathbf{n}}) = \left\{ \boldsymbol{U} \in H\Lambda(\mathfrak{D}, \Omega) : \mathbf{d}\boldsymbol{U} = 0,\, \boldsymbol{\delta}\boldsymbol{U} = 0,\, \mathbf{n}\boldsymbol{U} = 0 \right\}, \tag{5.2.4b}$$

are direct sums of harmonic spaces of all orders, cf. [2], [21, sect. 2], [33, sect. 3] and [38, Prop. 5.1]. In particular, when $\mathcal{M} = \mathbb{R}^N$ and $\kappa = 0$, the Hodge–Dirac operators have non-trivial finite dimensional kernels. Nevertheless, the BVPs (5.2.2a) and (5.2.2b) are well-posed on the orthogonal complements of $\ker(\mathfrak{D}_{\mathbf{t}})$ and $\ker(\mathfrak{D}_{\mathbf{n}})$ if we impose the following compatibility conditions on the boundary data:

$$\langle \boldsymbol{g}, \mathbf{n}\boldsymbol{V} \rangle_{\Gamma} = 0 \qquad \forall\, \boldsymbol{V} \in \ker(\mathfrak{D}_{\mathbf{t}}), \tag{5.2.5a}$$
$$\langle \boldsymbol{h}, \mathbf{t}\boldsymbol{V} \rangle_{\Gamma} = 0 \qquad \forall\, \boldsymbol{V} \in \ker(\mathfrak{D}_{\mathbf{n}}), \tag{5.2.5b}$$

respectively, cf. [21] and [33].

Otherwise, if $\kappa \neq 0$, recall that the inclusion map spawns compact embeddings $\mathrm{dom}(\mathfrak{D}_{\mathbf{t}}) \hookrightarrow L^2\Lambda(\Omega)$ and $\mathrm{dom}(\mathfrak{D}_{\mathbf{n}}) \hookrightarrow L^2\Lambda(\Omega)$, and so $\mathfrak{D}_{\mathbf{t}} + i\kappa$ and $\mathfrak{D}_{\mathbf{n}} + i\kappa$ are Fredholm operators of index zero [18, Lem. 7.2], [30, Lem. 4.1]. Because Lemma 5.1 offers continuous lifting maps from the trace spaces to the domain of the Hodge–Dirac operator $H\Lambda(\mathfrak{D}, \Omega)$, well-posedness of the boundary value problems (5.2.2a) and (5.2.2b) thus follow by injectivity, which is evidently guaranteed because the zero-order perturbations are purely imaginary.

5.2.1.2 Variational formulations for Hodge–Dirac BVPs

As discussed in [33, Sec. 3], a key feature of the Hodge–Dirac operator is that it admits the two distinct fundamental symmetric bilinear forms

$$\mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U}, \boldsymbol{V}) := \langle \boldsymbol{\delta}\boldsymbol{U}, \boldsymbol{V} \rangle_{\Omega} + \langle \boldsymbol{U}, \boldsymbol{\delta}\boldsymbol{V} \rangle_{\Omega}, \qquad \forall \boldsymbol{U}, \boldsymbol{V} \in H\Lambda(\boldsymbol{\delta}, \Omega) \tag{5.2.6a}$$
$$\mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \boldsymbol{V}) := \langle \mathbf{d}\boldsymbol{U}, \boldsymbol{V} \rangle_{\Omega} + \langle \boldsymbol{U}, \mathbf{d}\boldsymbol{V} \rangle_{\Omega}, \qquad \forall \boldsymbol{U}, \boldsymbol{V} \in H\Lambda(\mathbf{d}, \Omega), \tag{5.2.6b}$$

that rest on an equal footing. They arise in first-order analogs of Green's identities

$$\langle \mathfrak{D}\boldsymbol{U}, \boldsymbol{V} \rangle_{\Omega} = \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U}, \boldsymbol{V}) + \langle\!\langle \mathbf{t}\boldsymbol{U}, \mathbf{n}\boldsymbol{V} \rangle\!\rangle_{\Gamma}, \tag{5.2.7a}$$
$$\langle \mathfrak{D}\boldsymbol{U}, \boldsymbol{V} \rangle_{\Omega} = \mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \boldsymbol{V}) - \langle\!\langle \mathbf{n}\boldsymbol{U}, \mathbf{t}\boldsymbol{V} \rangle\!\rangle_{\Gamma}, \tag{5.2.7b}$$

which hold for all $\boldsymbol{U}, \boldsymbol{V} \in H\Lambda(\mathfrak{D}, \Omega)$. They lead to two variational problems associated with (5.2.2a) and (5.2.2b), respectively:

$$\boldsymbol{U} \in H\Lambda(\boldsymbol{\delta}, \Omega) : \quad \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U}, \boldsymbol{V}) + i\kappa \langle \boldsymbol{U}, \boldsymbol{V} \rangle_{\Omega} = -\langle\!\langle \boldsymbol{g}, \mathbf{n}\boldsymbol{V} \rangle\!\rangle_{\Gamma}, \quad \forall \boldsymbol{V} \in H\Lambda(\boldsymbol{\delta}, \Omega), \tag{5.2.8a}$$
$$\boldsymbol{U} \in H\Lambda(\mathrm{d}, \Omega) : \quad \mathcal{A}_{\mathrm{d}}(\boldsymbol{U}, \boldsymbol{V}) + i\kappa \langle \boldsymbol{U}, \boldsymbol{V} \rangle_{\Omega} = \langle\!\langle \boldsymbol{h}, \mathbf{t}\boldsymbol{V} \rangle\!\rangle_{\Gamma}, \quad \forall \boldsymbol{V} \in H\Lambda(\mathrm{d}, \Omega). \tag{5.2.8b}$$

It is a simple exercise in integration by parts to verify using suitable test functions that the variational problems (5.2.8a) and (5.2.8b) are equivalent with the strong formulations (5.2.2a) and (5.2.2b), respectively. Nevertheless, the analysis of the Hodge–Dirac operator is not as common as that of the Hodge–Laplacian. Because of the importance of inf-sup inequalities for the analysis of Galerkin discretization, we thus find meaningful to also cover solvability of the variational problems directly without assuming prior knowledge of the Hodge–Dirac operator's properties and take the opportunity to point at important references.

If $\kappa = 0$, solvability of the variational problems (5.2.8a) and (5.2.8b) is covered by the theory for the abstract Hodge–Dirac operator provided in [21, Sec. 2]. Specifically, the bilinear forms associated with each of the variational problems

$$\begin{aligned} \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U},\boldsymbol{V}) + \langle \boldsymbol{P},\boldsymbol{V}\rangle_{\Omega} &= -\langle\!\langle \boldsymbol{g}, \mathsf{n}\boldsymbol{V}\rangle\!\rangle_{\Gamma}, && \forall \boldsymbol{V} \in H\Lambda(\boldsymbol{\delta},\Omega), \\ \langle \boldsymbol{U},\boldsymbol{W}\rangle_{\Omega} &= 0 && \forall \boldsymbol{W} \in \mathfrak{H}_{\mathsf{t}}, \end{aligned} \tag{5.2.9a}$$

and

$$\begin{aligned} \mathcal{A}_{\mathbf{d}}(\boldsymbol{U},\boldsymbol{V}) + \langle \boldsymbol{Q},\boldsymbol{V}\rangle_{\Omega} &= \langle\!\langle \boldsymbol{h}, \mathsf{n}\boldsymbol{V}\rangle\!\rangle_{\Gamma}, && \forall \boldsymbol{V} \in H\Lambda(\mathbf{d},\Omega), \\ \langle \boldsymbol{U},\boldsymbol{W}\rangle_{\Omega} &= 0 && \forall \boldsymbol{W} \in \mathfrak{H}_{\mathsf{n}}, \end{aligned} \tag{5.2.9b}$$

satisfy inf-sup inequalities [21, Thm. 6]. The compatibility conditions (5.2.5b) and (5.2.5a) ensure compatibility of the right-hand sides and thus well-posedness.

We now show that when $\kappa \neq 0$, generalized Gårding inequalities hold for the bilinear forms associated with the variational problems (5.2.8a) and (5.2.8b), cf. [10, Thm. 4], [7, Chap. 11.4].

Lemma 5.4 *Let $\kappa \neq 0$. The bilinear forms associated with the variational problems (5.2.8a) and (5.2.8b) are T-coercive. In other words, there exist positive constants $C_{\mathsf{t}}, C_{\mathsf{n}} > 0$, isomorphisms $\Xi_{\mathsf{t}} : H\Lambda(\boldsymbol{\delta},\Omega) \to H\Lambda(\boldsymbol{\delta},\Omega)$ and $\Xi_{\mathsf{n}} : H\Lambda(\mathbf{d},\Omega) \to H\Lambda(\mathbf{d},\Omega)$, and compact operators $\mathsf{K}_{\mathsf{t}} : H\Lambda(\boldsymbol{\delta},\Omega) \to H\Lambda(\boldsymbol{\delta},\Omega)$ and $\mathsf{K}_{\mathsf{n}} : H\Lambda(\mathbf{d},\Omega) \to H\Lambda(\mathbf{d},\Omega)$, such that*

$$\|\boldsymbol{U}\|^2_{H\Lambda(\mathrm{d},\Omega)} \leq C_{\mathsf{n}} \left| \mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \overline{\Xi_{\mathsf{n}}\boldsymbol{U}}) + i\kappa\langle \boldsymbol{U}, \overline{\Xi_{\mathsf{n}}\boldsymbol{U}}\rangle_{\Omega} + \langle \mathsf{K}_{\mathsf{n}}\boldsymbol{U},\boldsymbol{U}\rangle_{\Omega} \right| \tag{5.2.10a}$$

$$\|\boldsymbol{V}\|^2_{H\Lambda(\delta,\Omega)} \leq C_{\mathsf{n}} \left| \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{V}, \overline{\Xi_{\mathsf{t}}\boldsymbol{V}}) + i\kappa\langle \boldsymbol{U}, \overline{\Xi_{\mathsf{t}}\boldsymbol{V}}\rangle_{\Omega} + \langle \mathsf{K}_{\mathsf{t}}\boldsymbol{V},\boldsymbol{V}\rangle_{\Omega} \right| \tag{5.2.10b}$$

for all $\boldsymbol{U} \in H\Lambda(\mathbf{d},\Omega)$ and $\boldsymbol{V} \in H\Lambda(\boldsymbol{\delta},\Omega)$.

Proof. By duality, it is sufficient to focus on (5.2.10a). The isomorphism $\Xi_{\mathbf{n}}$ is designed based on the $L^2\Lambda(\Omega)$-orthogonal Hodge decomposition

$$H\Lambda(\mathbf{d},\Omega) = \mathfrak{B} \oplus \mathfrak{H}_{\mathsf{n}} \oplus \mathfrak{Z}^{\perp}, \tag{5.2.11}$$

where $\mathfrak{B} = \mathrm{range}(\mathbf{d})$ and $\mathfrak{Z} = \ker(\mathbf{d})$. The intent is to exploit that the identity map spawns compact embeddings $\mathfrak{Z}^{\perp} \hookrightarrow L^2\Lambda(\Omega)$ and $\mathfrak{H}_{\mathsf{n}} \hookrightarrow L^2\Lambda(\Omega)$, cf. [3], [4], [29, Sec. 2], [30]. According to (5.2.11), any element $\boldsymbol{U} \in H\Lambda(\mathbf{d},\Omega)$ can be uniquely written as $\boldsymbol{U} = \boldsymbol{U}_{\mathfrak{B}} + \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} + \boldsymbol{U}_{\mathfrak{Z}^{\perp}}$.

Recall that $\mathbf{d}_{\mathfrak{Z}^{\perp}} = \mathbf{d}\big|_{\mathfrak{Z}^{\perp}} : \mathfrak{Z}^{\perp} \to \mathfrak{B}$ is a bounded isomorphism, because $\mathbf{d}$ has closed range (Fredholm property). Therefore, it has a continuous inverse $\mathbf{d}^{-1}_{\mathfrak{Z}^{\perp}} : \mathfrak{B} \to \mathfrak{Z}^{\perp}$. We define $\Xi_{\mathsf{n}} : H\Lambda(\mathrm{d},\Omega) \to H\Lambda(\mathrm{d},\Omega)$ by

$$\Xi_{\mathsf{n}}\boldsymbol{U} = \alpha\, \mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}} - i\kappa\alpha\, \boldsymbol{U}_{\mathfrak{B}} + \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} + \mathbf{d}^{-1}_{\mathfrak{Z}^{\perp}}\boldsymbol{U}_{\mathfrak{B}}, \tag{5.2.12}$$

where $0 < \alpha < 1/\kappa^2$. It is easy to see that Ξ_{n} is bounded.

We claim that Ξ_{n} is injective. Indeed, if we suppose that $\Xi_{\mathsf{n}}\boldsymbol{U} = 0$, then by orthogonality $\|\boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}}\| = \|\mathbf{d}^{-1}_{\mathfrak{Z}^{\perp}}\boldsymbol{U}_{\mathfrak{B}}\| = 0$, and thus $\boldsymbol{U}_{\mathfrak{B}} = \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} = 0$. We are left with the identity $0 = \mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}} = \mathbf{d}_{\mathfrak{Z}^{\perp}}\boldsymbol{U}_{\mathfrak{Z}^{\perp}}$, from which once again $\boldsymbol{U}_{\mathfrak{Z}^{\perp}} = 0$.

To see that Ξ_{n} is surjective, we simply verify that

$$\begin{aligned}
&\Xi_{\mathsf{n}}\left(\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}}+\boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}}+\alpha^{-1}\mathbf{d}_{\mathfrak{Z}^{\perp}}^{-1}\boldsymbol{U}_{\mathfrak{B}}+i\kappa\boldsymbol{U}_{\mathfrak{Z}^{\perp}}\right)\\
&\qquad=\alpha\,\mathbf{d}\left(\alpha^{-1}\mathbf{d}_{\mathfrak{Z}^{\perp}}^{-1}\boldsymbol{U}_{\mathfrak{B}}+i\kappa\boldsymbol{U}_{\mathfrak{Z}^{\perp}}\right)-i\kappa\alpha\,\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}}+\boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}}+\mathbf{d}^{-1}\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}}\\
&\qquad=\mathbf{d}\mathbf{d}_{\mathfrak{Z}^{\perp}}^{-1}\boldsymbol{U}_{\mathfrak{B}}+\boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}}+\boldsymbol{U}_{\mathfrak{Z}^{\perp}}=\boldsymbol{U}_{\mathfrak{B}}+\boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}}+\boldsymbol{U}_{\mathfrak{Z}^{\perp}}=\boldsymbol{U}.
\end{aligned}$$

Now, let us indicate by a hat inequalities and identities that hold *up to compact perturbation*, e.g. $\hat{=}$ and $\hat{\geq}$. Due to orthogonality, we find that

$$\langle \mathbf{d}\boldsymbol{U},\overline{\Xi_{\mathsf{n}}\boldsymbol{U}}\rangle_{\Omega}=\alpha\,\|\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}}\|^{2}+i\kappa\alpha\left(\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}},\boldsymbol{U}_{\mathfrak{B}}\right)_{\Omega}, \tag{5.2.13a}$$

$$\langle \boldsymbol{U}_{\mathfrak{B}},\mathbf{d}\overline{\Xi_{\mathsf{n}}\boldsymbol{U}}\rangle_{\Omega}=\|\boldsymbol{U}_{\mathfrak{B}}\|^{2}, \tag{5.2.13b}$$

$$i\kappa\langle \boldsymbol{U},\overline{\Xi_{n}\boldsymbol{U}}\rangle_{\Omega}\mathrel{\hat{=}}-\kappa^{2}\alpha\|\boldsymbol{U}_{\mathfrak{B}}\|^{2}+i\kappa\alpha\left(\boldsymbol{U}_{\mathfrak{B}},\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}}\right)_{\Omega}, \tag{5.2.13c}$$

where compact terms involving $\boldsymbol{U}_{\mathfrak{Z}^{\perp}}$ and $\boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}}$ were dropped. Summing the contributions of (5.2.13a) to (5.2.13c), we obtain

$$\mathcal{A}_{\mathbf{d}}(\boldsymbol{U},\Xi_{\mathsf{n}}\overline{\boldsymbol{U}})+i\kappa\langle \boldsymbol{U},\Xi_{\mathsf{n}}\overline{\boldsymbol{U}}\rangle_{\Omega}\mathrel{\hat{=}}\alpha\,\|\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}}\|^{2}+(1-\kappa^{2}\alpha)\|\boldsymbol{U}_{\mathfrak{B}}\|^{2}+i\kappa\alpha(\nu+\overline{\nu}),$$

where $\nu=(\boldsymbol{U}_{\mathfrak{B}},\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}})_{\Omega}$. Since the initial choice of parameter α guarantees that $1-\kappa^{2}\alpha>0$ and the last term is purely imaginary, we conclude that

$$\left|\,\mathcal{A}_{\mathbf{d}}(\boldsymbol{U},\Xi_{\mathsf{n}}\overline{\boldsymbol{U}})+i\kappa\langle \boldsymbol{U},\Xi_{\mathsf{n}}\overline{\boldsymbol{U}}\rangle_{\Omega}\,\right|\ \hat{\geq}\ C\left(\ \|\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^{\perp}}\|^{2}+\|\boldsymbol{U}_{\mathfrak{B}}\|^{2}\ \right)$$

for $C=\min\{\alpha,1-\kappa^{2}\alpha\}$, which concludes the proof. □

Corollary 5.1 *The variational problems* (5.2.8a) *and* (5.2.8b) *are well-posed.*

Proof. Based on Lemma 5.4, the operators associated with the variational problems (5.2.8a) and (5.2.8b) are Fredholm of index 0. We thus only need to show that they are injective. We focus on (5.2.8b).

Suppose that $\boldsymbol{U}\in H\Lambda(\mathrm{d},\Omega)$ is such that

$$\mathcal{A}_{\mathrm{d}}(\boldsymbol{U},\boldsymbol{V})+i\kappa\langle \boldsymbol{U},\boldsymbol{V}\rangle_{\Omega}=0$$

for all $\boldsymbol{V}\in H\Lambda(\mathrm{d},\Omega)$. Testing with $\boldsymbol{V}=\overline{\boldsymbol{U}}$, we find that

$$i\kappa\|\boldsymbol{U}\|_{\Omega}^{2}+\omega+\overline{\omega}=0,$$

where $\omega=(\mathbf{d}\boldsymbol{U},\boldsymbol{U})_{\Omega}$. As in the proof of Lemma 5.4, $\omega+\overline{\omega}$ is a real number, so $\|\boldsymbol{U}\|_{\Omega}^{2}=0$, from which we conclude that $\boldsymbol{U}=0$. □

The ability to introduce two distinct bilinear forms associated with (5.2.2a) and (5.2.2b) for the Hodge–Dirac operator is crucially rooted in the fact that both the cochain and chain perspective of the de Rham complex can be adopted in formulating BVPs for the Hodge–Dirac operator. Notably, it points to the symmetry between the BVPs (5.2.2a) and (5.2.2b) as discussed in the introduction of Subsection 5.2.1 and emphasizes for $\kappa=0$ the necessity of imposing the compatibility conditions (5.2.5a) and (5.2.5b) on the boundary data. For example, we could alternatively formulate (5.2.2a) as the variational problem of finding a full form $\boldsymbol{U}\in H\Lambda(\mathbf{d},\Omega)$ with $\mathbf{t}\boldsymbol{U}=\mathrm{g}$ such that

$$\mathcal{A}_{\mathrm{d}}(\boldsymbol{U},\boldsymbol{V}) + i\kappa\langle \boldsymbol{U},\boldsymbol{V}\rangle_{\Omega} = 0, \qquad \forall \boldsymbol{V} \in \overset{\circ}{H}\Lambda^{\ell}(\mathrm{d},\Omega). \tag{5.2.14}$$

Recall that in a formulation such as (5.2.14), we lift the boundary data and solve

$$\boldsymbol{W} \in \overset{\circ}{H}\Lambda(\mathbf{d},\Omega): \qquad \mathcal{A}_{\mathrm{d}}(\boldsymbol{W},\boldsymbol{V}) + i\kappa\langle \boldsymbol{W},\boldsymbol{V}\rangle_{\Omega} = \mathsf{F}_{\boldsymbol{g}}(\boldsymbol{V}), \qquad \forall \boldsymbol{V} \in \overset{\circ}{H}\Lambda(\mathbf{d},\Omega),$$

where $\mathsf{F}_{\boldsymbol{g}}(\boldsymbol{V}) = -\mathcal{A}_{\mathrm{d}}(\boldsymbol{\mathcal{E}}^{\mathsf{t}}\boldsymbol{g},\boldsymbol{V}) - i\kappa\langle \boldsymbol{\mathcal{E}}^{\mathsf{t}}\boldsymbol{g},\boldsymbol{V}\rangle_{\Omega}$. This is the mainstream perspective adopted in the literature of finite element exterior calculus. We depart from this standard because, as opposed to $\mathfrak{D}_{\mathsf{t}}$ and $\mathfrak{D}_{\mathsf{n}}$ in (5.2.3a) and (5.2.3b), the self-adjoint operator behind (5.2.14) is

$$\overset{\circ}{\mathfrak{D}}_{\mathbf{t}} = \overset{\circ}{\mathbf{d}} + \overset{\circ}{\mathbf{d}}{}^{*},$$

where

$$\overset{\circ}{\mathrm{d}} : \overset{\circ}{H}\Lambda(\mathbf{d},\Omega) \to \overset{\circ}{H}\Lambda(\mathbf{d},\Omega)$$

is obtained by restricting the exterior derivative to the kernel of the tangential trace. For the goal of this article, this approach is inconvenient because it *modifies the exterior derivative* such that it is no longer the one which enters the definition of the Hodge–Dirac operator introduced in (5.2.1) that leads to the BVPs (5.2.2a) and (5.2.2b). It is the *maximal* Hodge–Dirac operator $\mathfrak{D} = \boldsymbol{\delta} + \mathbf{d} : H\Lambda(\mathfrak{D},\Omega) \to L^2\Lambda(\Omega)$ involving the exterior derivative $\mathbf{d} : H\Lambda(\mathrm{d},\Omega) \to H\Lambda(\mathrm{d},\Omega)$ that appears in the representation formula given in Subsection 5.4.1.1 from which BIEs are derived. Indeed, the BVPs (5.2.2a) and (5.2.2b) lead to four variational problems: to each one of the two BVPs is associated both a variational problem featuring *natural* boundary conditions (such as in (5.2.8a)) and a variational problem with *essential* boundary conditions imposed on the domain of the operator (such as in (5.2.14)). As we will see, it is the structure of the variational problems with natural boundary conditions— and accordingly the expressions of the self-adjoint operators (5.2.3a) and (5.2.3b)—that is reproduced at the level of the trace de Rham complex in the first-kind BIOs.

5.2.2 Hodge–Laplace operators

We now turn to the Hodge–Laplacian and zero-order perturbations involving a non-negative constant $\lambda \geq 0$ (non-negative when $\mathcal{M} = \mathbb{R}^N$ and strictly positive when $\mathcal{M}$ is a compact manifold). We will be interested in both *strong* and *mixed* formulations of the operator. While equivalent from the point of view of solvability, the formal distinction in their structure is important in revealing the connection we seek with first-kind BIOs. It is a straightforward exercise in integration by parts to show that all the formulations presented below are indeed equivalent. Since well-posedness of BVPs for the Hodge–Laplacian has been extensively studied and is very well-known, we omit the details and refer to standard references such as [2, Chap. 4].

5.2.2.1 BVPs for the Hodge–Laplacian

Our starting point is the strong formulation

$$-\Delta_{\ell} + \lambda : H\Lambda^{\ell}(\Delta,\Omega) \to L^2\Lambda^{\ell}(\Omega). \tag{5.2.15}$$

Suitable boundary conditions for this operator can be imposed using the surjective compound traces introduced in Subsection 5.1.4.2. Recall that the traces

$$\mathsf{T}^{\mathsf{t}}_{\Delta} U_\ell = \begin{pmatrix} \mathsf{t}_{\ell-1}\delta_\ell U_\ell \\ \mathsf{t}_\ell U_\ell \end{pmatrix} \qquad \text{and} \qquad \mathsf{T}^{\mathsf{n}}_{\Delta} U_\ell = \begin{pmatrix} \mathsf{n}_\ell U_\ell \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell \end{pmatrix} \tag{5.2.16}$$

are continuous and surjective as mappings

$$\begin{aligned} \mathsf{T}^{\mathsf{t}}_{\Delta} &: H\Lambda^\ell(\mathrm{d}\delta, \Omega) \cap H\Lambda^\ell(\mathrm{d}, \Omega) \longrightarrow H^{\mathsf{t}}_{\Delta}(\Gamma), \\ \mathsf{T}^{\mathsf{n}}_{\Delta} &: H\Lambda^\ell(\delta, \Omega) \cap H\Lambda^\ell(\delta\mathrm{d}, \Omega) \longrightarrow H^{\mathsf{n}}_{\Delta}(\Gamma), \end{aligned}$$

where the product of trace spaces are given by

$$\begin{aligned} H^{\mathsf{t}}_{\Delta}(\Gamma) &= H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}, \Gamma) \times H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d}, \Gamma), \\ H^{\mathsf{n}}_{\Delta}(\Gamma) &= H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta, \Gamma) \times H_{\parallel}^{-\frac{1}{2}}\Lambda^\ell(\delta, \Gamma). \end{aligned}$$

The significance of these traces for the Hodge–Laplacian has long been recognized in related literature. They are covered extensively in [25, Sec.1.1] and [27, Chap. 5]. Imposing boundary conditions using these traces was shown in [35, Sec. 1.6] to render the Hodge–Laplacian elliptic in the sense of Sapiro–Lopatinski. In [11, 12], [15, Sec. 1.c], [17] and [33, 34], these traces are seen to appear naturally in variational problems from identities obtained using integration by parts. In particular, our derivation of a representation formula will use the fact that they give rise to a generalization of Green's second formula to differential forms.

For $0 \leq \ell \leq N$, we consider the BVPs

$$U_\ell \in H\Lambda^\ell(\Delta, \Omega) : \begin{cases} (-\Delta_\ell + \lambda)\, U_\ell = 0 & \text{in } \Omega \\ \mathsf{T}^{\mathsf{t}}_{\Delta} U_\ell = \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \qquad \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} \in H^{\mathsf{t}}_{\Delta}(\Gamma), \tag{5.2.17a}$$

and

$$U_\ell \in H\Lambda^\ell(\Delta, \Omega) : \begin{cases} (-\Delta_\ell + \lambda)\, U_\ell = 0 & \text{in } \Omega \\ \mathsf{T}^{\mathsf{n}}_{\Delta} U_\ell = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \qquad \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} \in H^{\mathsf{n}}_{\Delta}(\Gamma). \tag{5.2.17b}$$

We will derive BIEs for the BVPs (5.2.17a) and (5.2.17b) using the strong formulation of the Hodge–Laplacian in Subsection 5.4.2.2. However, as mentioned in the closing discussing of Subsection 5.1.1, if a Hodge–Laplace operator is to appear in the trace de Rham complex, it has to be in mixed form, because the boundary data lies in *product spaces*. With this guiding principle, we now introduce mixed formulations for (5.2.17a) and (5.2.17b). We will later recognize their structure in the first-kind BIOs.

Introducing an auxiliary variable $U_{\ell-1} = \delta_\ell U_\ell \in H\Lambda^{\ell-1}(\mathrm{d}, \Omega)$, we obtain the *mixed-order* formulation

$$\delta_\ell U_\ell - U_{\ell-1} = 0,$$

$$\delta_{\ell+1} \mathrm{d}_\ell U_\ell + \mathrm{d}_{\ell-1} U_{\ell-1} + \lambda U_\ell = 0.$$

More succinctly,

$$\mathfrak{M} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the perturbed Hodge–Laplacian in mixed form

$$\mathfrak{M} : \mathrm{dom}(\mathfrak{M}) = H\Lambda^{\ell-1}(\mathrm{d}, \Omega) \times \Big(H\Lambda^\ell(\delta\mathrm{d}, \Omega) \cap H\Lambda^\ell(\delta, \Omega) \Big) \to L^2\Lambda^{\ell-1}(\Gamma) \times L^2\Lambda^\ell(\Gamma)$$

can be represented by the operator matrix

$$\mathfrak{M} = \begin{pmatrix} -\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \delta_{\ell+1}\mathrm{d}_\ell + \lambda \end{pmatrix}. \tag{5.2.18}$$

By substituting the auxiliary variable $U_{\ell-1}$ in the traces (5.2.16) for Hodge–Laplace operators in strong formulation, we obtain the pair of traces

$$\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} \mathsf{t}_{\ell-1} U_{\ell-1} \\ \mathsf{t}_\ell U_\ell \end{pmatrix} \qquad \text{and} \qquad \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} \mathsf{n}_\ell U_\ell \\ \mathsf{n}_{\ell+1} \mathrm{d}_\ell U_\ell \end{pmatrix}, \tag{5.2.19}$$

which are continuous as mappings

$$\begin{aligned} &\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}} : H\Lambda^{\ell-1}(\mathrm{d}, \Omega) \times H\Lambda^\ell(\mathrm{d}, \Omega) \longrightarrow H^{\mathsf{t}}_{\mathfrak{M}}(\Gamma) = H^{\mathsf{t}}_{\Delta}(\Gamma), \\ &\mathsf{T}^{\mathsf{n}}_{\mathfrak{M}} : L^2\Lambda^{\ell-1}(\Omega) \times \Big(H\Lambda^\ell(\delta, \Omega) \cap H\Lambda^\ell(\delta\mathrm{d}, \Omega) \Big) \longrightarrow H^{\mathsf{n}}_{\mathfrak{M}}(\Gamma) = H^{\mathsf{n}}_{\Delta}(\Gamma). \end{aligned}$$

The associated boundary value problems read, respectively:

$$\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} \in \mathrm{dom}(\mathfrak{M}) : \begin{cases} \mathfrak{M} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega \\ \mathsf{T}^{\mathsf{t}}_{\mathfrak{M}} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \quad \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} \in H^{\mathsf{t}}_{\Delta}(\Gamma), \tag{5.2.20a}$$

$$\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} \in \mathrm{dom}(\mathfrak{M}) : \begin{cases} \mathfrak{M} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega \\ \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \quad \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} \in H^{\mathsf{n}}_{\Delta}(\Gamma). \tag{5.2.20b}$$

Starting from the BVPs (5.2.17a) and (5.2.17b) in strong form, we could have alternatively opted for an auxiliary variable $U_{\ell+1} = \mathrm{d}_\ell U_\ell \in H\Lambda^\ell(\delta, \Omega)$ to obtain the equivalent mixed-order formulation

$$\delta_{\ell+1} U_{\ell+1} + \mathrm{d}_{\ell-1} \delta_\ell U_\ell + \lambda U_\ell = 0,$$

$$\mathrm{d}_\ell U_\ell - U_{\ell+1} = 0,$$

or in short,

$$\mathfrak{R}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where this time the perturbed Hodge–Laplacian in mixed form is a continuous map

$$\mathfrak{R} : \mathrm{dom}(\mathfrak{R}) = H\Lambda^{\ell+1}(\delta, \Omega) \times H\Lambda^{\ell}(\mathrm{d}\delta, \Omega) \cap H\Lambda^{\ell}(\mathrm{d}, \Omega) \to L^2\Lambda^{\ell}(\Gamma) \times L^2\Lambda^{\ell+1}(\Gamma),$$

whose operator matrix representation reads

$$\mathfrak{R} = \begin{pmatrix} \mathrm{d}_{\ell+1}\delta_\ell + \lambda & \delta_{\ell+1} \\ \mathrm{d}_\ell & -\mathrm{Id} \end{pmatrix}. \tag{5.2.21}$$

We then reach instead the BVPs

$$\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} \in \mathrm{dom}(\mathfrak{R}) : \begin{cases} \mathfrak{R}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega \\ \mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \qquad \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} \in H^{\mathsf{t}}_{\mathfrak{R}}(\Gamma), \tag{5.2.22a}$$

$$\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} \in \mathrm{dom}(\mathfrak{R}) : \begin{cases} \mathfrak{R}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega \\ \mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \qquad \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} \in H^{\mathsf{n}}_{\mathfrak{R}}(\Gamma), \tag{5.2.22b}$$

involving the continuous and surjective traces

$$\mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} \mathsf{t}_{\ell-1}\delta_\ell U_\ell \\ \mathsf{t}_\ell U_\ell \end{pmatrix} \qquad \text{and} \qquad \mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} \mathsf{n}_\ell U_\ell \\ \mathsf{n}_{\ell+1} U_{\ell+1} \end{pmatrix}. \tag{5.2.23}$$

5.2.3 Variational formulations for Hodge–Laplace BVPs

In line with our goal, it is sufficient for our purposes to present only two variational problems equivalent to the BVPs (5.2.17a), (5.2.17b), (5.2.20a), (5.2.20b), (5.2.22a) and (5.2.22b). We focus on those two because they are the only variational formulations obtained by integration by parts sharing the two following characteristics:

- They are in mixed formulation. Therefore, they involve product spaces of differential forms of order $\ell - 1$ and ℓ, or ℓ and $\ell + 1$, analogous to the structure of the trace spaces $\mathsf{T}^{\mathsf{t}}_{\Delta}$ and $\mathsf{T}^{\mathsf{n}}_{\Delta}$ for the Hodge–Laplacian.

- The boundary conditions are *natural*, so that no restriction is needed on the domain of the featured exterior derivative or codifferential.

Integrating by parts using (5.1.10), we find the analog of Green's first identities for Hodge–Laplace operators in mixed formulation:

$$\left(\mathfrak{M}\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\begin{pmatrix}V_{\ell-1}\\V_\ell\end{pmatrix}\right)_\Omega = \mathcal{B}_{\mathrm{d}}\left(\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\begin{pmatrix}V_{\ell-1}\\V_\ell\end{pmatrix}\right) - \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\begin{pmatrix}\overline{V}_{\ell-1}\\\overline{V}_\ell\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma, \tag{5.2.24a}$$

$$\left(\mathfrak{R}\begin{pmatrix}U_{\ell}\\U_{\ell+1}\end{pmatrix},\begin{pmatrix}V_{\ell}\\V_{\ell+1}\end{pmatrix}\right)_\Omega = \mathcal{B}_{\delta}\left(\begin{pmatrix}U_{\ell}\\U_{\ell+1}\end{pmatrix},\begin{pmatrix}V_{\ell}\\V_{\ell+1}\end{pmatrix}\right) + \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}\begin{pmatrix}U_{\ell}\\U_{\ell+1}\end{pmatrix},\mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}\begin{pmatrix}\overline{V}_{\ell}\\\overline{V}_{\ell+1}\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma, \tag{5.2.24b}$$

where the fundamental bilinear forms associated with $\mathfrak{M}$ and $\mathfrak{R}$ are

$$\begin{aligned}\mathcal{B}_{\mathrm{d}}\left(\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\begin{pmatrix}V_{\ell-1}\\V_\ell\end{pmatrix}\right) &= (\mathrm{d}_\ell U_\ell,\mathrm{d}_\ell V_\ell)_\Omega + \lambda\,(U_\ell,V_\ell)_\Omega + (\mathrm{d}_{\ell-1}U_{\ell-1},V_\ell)_\Omega\\ &\quad + (U_\ell,\mathrm{d}_\ell V_{\ell-1})_\Omega - (U_{\ell-1},V_{\ell-1})_\Omega\,,\end{aligned} \tag{5.2.25}$$

$$\begin{aligned}\mathcal{B}_{\delta}\left(\begin{pmatrix}U_{\ell}\\U_{\ell+1}\end{pmatrix},\begin{pmatrix}V_{\ell}\\V_{\ell+1}\end{pmatrix}\right) &= (\delta_\ell U_\ell,\delta_\ell V_\ell)_\Omega + \lambda\,(U_\ell,V_\ell)_\Omega + (\delta_{\ell+1}U_{\ell+1},V_\ell)_\Omega\\ &\quad + (U_\ell,\delta_{\ell+1} V_{\ell+1})_\Omega - (U_{\ell+1},V_{\ell+1})_\Omega\,.\end{aligned} \tag{5.2.26}$$

They lead to two variational problems. In the first, we suppose that the boundary data $(h_{\ell-1},h_\ell)^\top \in T^{\mathsf{n}}_{\mathfrak{M}}(\Gamma)$ is given and we seek $(U_{\ell-1},U_\ell)^\top \in H\Lambda^{\ell-1}(\mathrm{d},\Omega)\times H\Lambda^{\ell}(\mathrm{d},\Omega)$ such that

$$\mathcal{B}_{\mathrm{d}}\left(\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\begin{pmatrix}V_{\ell-1}\\V_\ell\end{pmatrix}\right) = \left\langle\!\!\left\langle \begin{pmatrix}h_{\ell-1}\\h_\ell\end{pmatrix},\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\begin{pmatrix}\overline{V}_{\ell-1}\\\overline{V}_\ell\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma \tag{5.2.27a}$$

for all $(V_{\ell-1},V_\ell)^\top \in H\Lambda^{\ell-1}(\mathrm{d},\Omega)\times H\Lambda^{\ell}(\mathrm{d},\Omega)$. In the second, $(g_\ell,g_{\ell+1})^\top \in H^{\mathsf{t}}_{\mathfrak{R}}(\Gamma)$ is given and we seek $(U_\ell,U_{\ell+1}) \in H\Lambda^{\ell}(\delta,\Omega)\times H\Lambda^{\ell+1}(\delta,\Omega)$ such that

$$\mathcal{B}_{\delta}\left(\begin{pmatrix}U_{\ell}\\U_{\ell+1}\end{pmatrix},\begin{pmatrix}V_{\ell}\\V_{\ell+1}\end{pmatrix}\right) = \left\langle\!\!\left\langle \begin{pmatrix}g_{\ell-1}\\g_\ell\end{pmatrix},\mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}\begin{pmatrix}\overline{V}_{\ell}\\\overline{V}_{\ell+1}\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma \tag{5.2.27b}$$

for all $(V_\ell, V_{\ell+1})^\top \in H\Lambda^\ell(\delta,\Omega) \times H\Lambda^{\ell+1}(\delta,\Omega)$.

The variational problem (5.2.27a) is an equivalent reformulation of the problems (5.2.20b) and (5.2.17b) where the trace data $\mathsf{T}^{\mathsf{n}}_{\Delta} U_\ell$ is known, while (5.2.27b) is a variational formulation for (5.2.22a) and (5.2.17a) where $\mathsf{T}^{\mathsf{t}}_{\Delta} U_\ell$ is known.

The self-adjoint operators behind these BVPs and associated with the bilinear forms in the analogs of Green's first formulas are

$$\mathfrak{M}_{\mathsf{n}} = \begin{pmatrix} -\mathrm{Id} & \mathrm{d}^*_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{d}^*_\ell \mathrm{d}_\ell + \lambda \end{pmatrix} : H\Lambda^{\ell-1}(\mathrm{d},\Omega) \times \left(\mathring{H}\Lambda^\ell(\delta,\Omega) \cap \mathring{H}\Lambda^\ell(\delta\mathrm{d},\Omega) \right) \to L^2\Lambda(\Omega), \tag{5.2.28a}$$

$$\mathfrak{R}_{\mathsf{t}} = \begin{pmatrix} \delta^*_\ell \delta_\ell + \lambda & \delta_{\ell+1} \\ \delta^*_{\ell+1} & -\mathrm{Id} \end{pmatrix} : \left(\mathring{H}\Lambda^\ell(\mathrm{d},\Omega) \cap \mathring{H}\Lambda^\ell(\mathrm{d}\delta,\Omega) \right) \times H\Lambda^{\ell+1}(\delta,\Omega) \to L^2\Lambda(\Omega), \tag{5.2.28b}$$

where similarly as for (5.2.3a) and (5.2.3b), the traces vanish on

$$\mathring{H}\Lambda^\ell(\delta\mathrm{d},\Omega) = H\Lambda^\ell(\delta\mathrm{d},\Omega) \cap \ker \mathsf{n}_{\ell+1}\mathrm{d}_\ell,$$
$$\mathring{H}\Lambda^\ell(\mathrm{d}\delta,\Omega) = H\Lambda^\ell(\mathrm{d}\delta,\Omega) \cap \ker \mathsf{t}_{\ell-1}\delta_\ell.$$

5.3 Calculus of boundary potentials

Our main tool in deriving BIEs for Hodge–Dirac and Hodge–Laplace operators is a *calculus of atomic boundary potentials*. We call *atomic* the two boundary potentials defined in this section, because all the other layer potentials in this work are obtained from them by differentiation. We intend to convey that the commutation identities and jump relations involving the exterior derivative and codifferential are valuable instruments that greatly simplify derivations. In that sense, viewing these potentials as elementary building blocks unlocks the power of exterior calculus as a framework for calculations. Moreover, these atomic potentials are the crucial components in the definitions of the non-local inner products on the spaces of the trace de Rham complex where the claimed correspondence between the operators entering the BVPs of Section 5.2 and the first-kind BIOs studied in Section 5.4 is revealed.

5.3.1 Newtonian potential

For $\mathcal{M} = \mathbb{R}^N$, a fundamental solution Φ^λ_ℓ for the scalar differential operator $-\Delta_\ell + \lambda$ satisfying suitable decay conditions at infinity exists for all $\lambda \geq 0$, cf. [22, Eq. 4.1], [23, Chap. 8 and 9]. Denote by $\mathcal{I}_\ell$ the identity double form of bi-degree (ℓ,ℓ) on $\mathbb{R}^N \times \mathbb{R}^N$ and for $x \neq y$ let

$$\mathcal{G}^\lambda_\ell(x,y) = \Phi_\lambda\left(|x-y|\right) \mathcal{I}_\ell(x,y) \tag{5.3.1}$$

be the singular kernel of the symmetric integral transformation

$$\mathsf{N}^\lambda_\ell U_\ell(x) = \lim_{\epsilon\to 0} \langle \mathcal{G}^\lambda_\ell(x,\cdot), U_\ell(\cdot) \rangle_{\mathbb{R}^N \setminus B_\epsilon(x)}, \qquad U_\ell \in \mathcal{D}^\ell(\mathbb{R}^N), \tag{5.3.2}$$

where $B_\epsilon(x)$ is the N-dimensional ball of radius $\epsilon > 0$ centered at x. The extension $\mathsf{N}_\ell^\lambda : \mathcal{E}'_\ell(\mathbb{R}^N) \to \mathcal{D}'_\ell(\mathbb{R}^N)$ to distributions via the dual mapping is a two-sided inverse of $-\Delta_\ell + \lambda$ in the sense of distributions, cf. [16, Chap. 12 and 16], [20, sect. 2.2 and 2.3], [23, Chap. 6], [31, Chap. 3] and [39, sect. 3].

On a compact boundaryless manifold, not only the Hodge–Laplacian is not invertible, we must also be wary of its non-trivial eigenspaces. Thankfully though, $-\Delta+\lambda : H^1\Lambda^\ell(\mathcal{M}) \to H^{-1}\Lambda^\ell(\mathcal{M})$ is invertible for at least all $\lambda > 0$, in which case the Schwarz kernel of its continuous inverse is available, cf. [25, Chap. 3] and [27].

To keep our exposition simple, we thus settle for imposing on $\lambda \geq 0$ the condition that $\lambda > 0$ whenever $\mathcal{M}$ is compact.

Assumption A. *If $\mathcal{M} = \mathbb{R}^N$, we allow $\kappa \in \mathbb{R}$ and $\lambda \geq 0$, but we impose $\kappa > 0$ and $\lambda > 0$ when $\mathcal{M}$ is a compact manifold without boundary.*

Under Assumption A, we can always assume that a Newtonian potential

$$\mathsf{N}_\ell^\lambda : H^{-1}_{\text{comp}}\Lambda^\ell(\mathcal{M}) = H^{-1}\Lambda^\ell(\mathcal{M}) \cap \mathcal{E}'_\ell \to H^1_{\text{loc}}\Lambda^\ell(\mathcal{M}) \cap H_{\text{loc}}\Lambda^\ell(\Delta, \mathcal{M}) \tag{5.3.3}$$

for the Hodge–Yukawa operator exists whose integrable kernel satisfies

$$\mathrm{d}_{\ell,x}\, \mathcal{G}_\ell^\lambda(x,y) = \delta_{\ell+1,y}\, \mathcal{G}_{\ell+1}^\lambda(x,y) \qquad \text{and} \qquad \delta_{\ell,x}\, \mathcal{G}_\ell^\lambda(x,y) = \mathrm{d}_{\ell-1,y}\, \mathcal{G}_{\ell-1}^\lambda(x,y) \tag{5.3.4}$$

for $x \neq y$, cf. [20, Lem. 3] and [25, eq. 3.1.44]. Moreover,

$$\star_{\ell,y} \star_{\ell,x} \mathcal{G}_\ell^\lambda = \star_{\ell,x} \star_{\ell,y} \mathcal{G}_\ell^\lambda = \mathcal{G}_{N-\ell}^\lambda, \tag{5.3.5}$$

cf. [25, 3.1.23] and [20, Lem. 1]. At the level of the full Grassman algebra of differential forms, the identities in (5.3.4) translate for $\boldsymbol{\mathcal{G}}_\lambda = (\mathcal{G}_\ell^\lambda)_\ell$ to

$$\mathbf{d}_x\, \boldsymbol{\mathcal{G}}_\lambda = \boldsymbol{\delta}_y\, \boldsymbol{\mathcal{G}}_\lambda \qquad \text{and} \qquad \boldsymbol{\delta}_x\, \boldsymbol{\mathcal{G}}_\lambda = \mathbf{d}_y\, \boldsymbol{\mathcal{G}}_\lambda, \tag{5.3.6}$$

while property (5.3.5) implies that

$$\star_y \star_x \boldsymbol{\mathcal{G}}_\lambda = \star_x \star_y \boldsymbol{\mathcal{G}}_\lambda = \boldsymbol{\mathcal{G}}_\lambda.$$

5.3.2 Atomic boundary potentials

Consider the bounded operators

$$\mathsf{t}'_\ell : H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma) \to H^{-1}_{\text{comp}}\Lambda^\ell(\mathcal{M}) \qquad \text{and} \qquad \mathsf{n}'_\ell : H_\perp^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma) \to H^{-1}_{\text{comp}}\Lambda^\ell(\mathcal{M}),$$

dual to the trace mappings in (5.1.8). As previously stated, the *atomic* boundary potentials

$$\begin{aligned} \mathsf{S}_\lambda^\ell &= \mathsf{N}_\ell^\lambda\, \mathsf{t}'_\ell : H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma) \longrightarrow H^1_{\text{loc}}\Lambda^\ell(\mathcal{M}), \\ \mathsf{D}_\lambda^\ell &= \mathsf{N}_\ell^\lambda\, \mathsf{n}'_{\ell-1} : H_\perp^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma) \longrightarrow H^1_{\text{loc}}\Lambda^\ell(\mathcal{M}), \end{aligned} \tag{5.3.7}$$

take center stage throughout this article, cf. [11], [13] and [33].

If $u_\ell \in L^1\Lambda^\ell(\Gamma)$, it follows by symmetry of the fundamental solution that for $x \notin \Gamma$, they admit the integral representations

$$\mathsf{S}_\ell^\lambda u_\ell(x) = \langle u_\ell, \mathsf{t}_\ell\, \mathcal{G}_\ell^\lambda(x,\cdot)\rangle_\Gamma \qquad \text{and} \qquad \mathsf{D}_\ell^\lambda u_{\ell-1}(x) = \langle u_{\ell-1}, \mathsf{n}_\ell \mathcal{G}_\ell^\lambda(x,\cdot)\rangle_\Gamma, \tag{5.3.8}$$

cf. [23, Thm. 6.10] and [31, Thm. 3.1.6].

Since $\star_\ell$ is an isometry, we observe using (5.3.5) that for $x \notin \Gamma$,

$$\begin{aligned}\star_{\ell+1,x}^{-1}\langle \star_{\ell,y}\, u_\ell(y), \mathsf{t}_{N-\ell-1,y}\, \mathcal{G}_{N-\ell-1}^\lambda(x,y)\rangle_\Gamma &= \langle\, u_\ell(y), \star_{\ell,y}^{-1}\mathsf{t}_{N-\ell-1,y} \star_{\ell+1,y} \mathcal{G}_{\ell+1}^\lambda(x,y)\rangle_\Gamma \\ &= \langle\, u_\ell(y), \mathsf{n}_{\ell+1,y}\mathcal{G}_{\ell+1}^\lambda(x,y)\rangle_\Gamma.\end{aligned}$$

Therefore, a density argument eventually shows that

$$\star_\ell^{-1}\, \mathsf{S}_{N-\ell}^\lambda \star_{\ell-1} = \mathsf{D}_\ell^\lambda \qquad \text{and} \qquad \star^{-1}\, \mathbf{S}_\lambda \star = \mathbf{D}_\lambda, \tag{5.3.9}$$

where a bold font denote the boundary potentials $\mathbf{S}_\lambda = (\mathsf{S}_\ell^\lambda)_\ell$ and $\mathbf{D}_\lambda = (\mathsf{D}_\ell^\lambda)_\ell$ acting on the full algebra of differential forms.

Lemma 5.5 *For all $v_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma)$ and $u_\ell \in H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma)$,*

$$\delta_\ell\, \mathsf{S}_\ell^\lambda(v_\ell) = \mathsf{S}_{\ell-1}^\lambda(\delta_\ell\, v_\ell) \qquad \text{and} \qquad \mathrm{d}_{\ell+1}\, \mathsf{D}_{\ell+1}^\lambda(u_\ell) = -\mathsf{D}_{\ell+2}^\lambda(\mathrm{d}_\ell\, u_\ell).$$

Proof. We refer to [20, Lem. 3] and [25, Eq. 3.2.41] for the first identity. The second then follows as a consequence of (5.3.9), but can also be verified directly using (5.3.4) and integration by parts as follows.

Let $u_\ell \in L^\infty\Lambda^\ell(\Gamma) \cap H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma)$ be the tangential trace of a smooth ℓ-form on $\mathcal{M}$. Then, for $x \notin \Gamma$, we can evaluate directly using the integral representation of the boundary potential that

$$\begin{aligned}\mathrm{d}_{\ell+1}\mathsf{D}_{\ell+1}^\lambda u_\ell(x) &= \int_\Gamma u_\ell(y) \wedge_y i_y^* \star_{\ell+1,y} \mathrm{d}_{\ell+1,x}\, \mathcal{G}_{\ell+1,\ell+1}^\lambda(x,y)\, \mathrm{d}y \\ &= \int_\Gamma u_\ell(y) \wedge_y i_y^* \star_{\ell+1,y} \delta_{\ell+2,y}\, \mathcal{G}_{\ell+2,\ell+2}^\lambda(x,y)\, \mathrm{d}y \qquad (5.3.10) \\ &= (-1)^{\ell+2}\int_\Gamma u_\ell(y) \wedge_y \mathrm{d}_{n-\ell-2,y}\, i_y^* \star_{\ell+2,y} \mathcal{G}_{\ell+2,\ell+2}^\lambda(x,y)\, \mathrm{d}y \qquad (5.3.11) \\ &= -(-1)^\ell(-1)^{\ell+2}\int_\Gamma \mathrm{d}_{\ell,y}\, u_\ell(y) \wedge_y i_y^* \star_{\ell+2,y} G_{\ell+2,\ell+2}^\lambda(x,y)\, \mathrm{d}y \qquad (5.3.12) \\ &= -\langle \mathrm{d}_\ell\, u_\ell, \mathsf{n}_{\ell+2}\mathcal{G}_{\ell+2,\ell+2}^\lambda\rangle_\Gamma,\end{aligned}$$

where (5.3.10) is obtained using (5.3.4), (5.3.11) holds because the exterior derivative commutes with pullbacks, and (5.3.12) follows by integration by parts. □

Corollary 5.2 *For all $\boldsymbol{v} \in H_{\|}^{-\frac{1}{2}}\Lambda(\delta,\Gamma)$ and $\boldsymbol{u} \in H_\perp^{-\frac{1}{2}}\Lambda(\mathrm{d},\Gamma)$,*

$$\boldsymbol{\delta}\mathbf{S}_\lambda(\boldsymbol{v}) = \mathbf{S}_\lambda(\boldsymbol{\delta v}) \qquad \text{and} \qquad \mathbf{d}\mathbf{D}_\lambda(\boldsymbol{u}) = -\mathbf{D}_\lambda(\mathbf{d}\boldsymbol{u}).$$

Lemma 5.6 *The boundary potentials restrict to continuous mappings*

$$\mathsf{S}^{\ell}_{\lambda} : H_{\|}^{-\frac{1}{2}}\Lambda^{\ell}(\delta, \Gamma) \longrightarrow H^{1}_{\mathrm{loc}}\Lambda^{\ell}(\mathcal{M}) \cap H\Lambda^{\ell}(-\Delta, \Omega),$$
$$\mathsf{D}^{\ell+1}_{\lambda} : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d}, \Gamma) \longrightarrow H^{1}_{\mathrm{loc}}\Lambda^{\ell+1}(\mathcal{M}) \cap H\Lambda^{\ell+1}(-\Delta, \Omega),$$

satisfying, in the sense of distributions,

$$(-\Delta_{\ell} + \lambda)\,\mathsf{S}^{\lambda}_{\ell}(u_{\ell}) = 0, \qquad \textit{and} \qquad (-\Delta_{\ell+1} + \lambda)\,\mathsf{D}^{\lambda}_{\ell+1}(w_{\ell}) = 0, \tag{5.3.13}$$

for all $u_{\ell} \in H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta, \Gamma)$ *and* $w_{\ell} \in H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d}, \Gamma)$.

Proof. For the first identity in (5.3.13), we refer to [25, Eq. 3.2.5] and [20, Lem. 3 (ii)]. The second is obtained as a corollary using (5.3.9), because the Hodge star commutes with the Hodge–Laplacian [25, Lem. 2.8]. □

Denote the jump of a trace across Γ by $[\![\bullet]\!] = \bullet^{+} - \bullet^{-}$, where $\bullet = \mathsf{t}$ or n.

Lemma 5.7 *We have the jump relations*

$$[\![\mathsf{t}_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{t}_{\ell+1}\mathrm{d}_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{t}_{\ell-1}\delta_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \tag{5.3.14a}$$
$$[\![\mathsf{n}_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{n}_{\ell+1}\mathrm{d}_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = -\mathrm{Id}, \qquad [\![\mathsf{n}_{\ell-1}\delta_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \tag{5.3.14b}$$
$$[\![\mathsf{t}_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{t}_{\ell+1}\mathrm{d}_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{t}_{\ell-1}\delta_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = \mathrm{Id}, \tag{5.3.14c}$$
$$[\![\mathsf{n}_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{n}_{\ell+1}\mathrm{d}_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{n}_{\ell-1}\delta_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0. \tag{5.3.14d}$$

Proof. We appeal to continuity and [20, Lem. 10], which already gives us

$$[\![\mathsf{t}_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{t}_{\ell+1}\mathrm{d}_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{t}_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0,$$
$$[\![\mathsf{n}_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{n}_{\ell+1}\mathrm{d}_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = -\mathrm{Id}, \qquad [\![\mathsf{n}_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0.$$

Based on these jump identities, the commutative relations for the trace operators in (5.1.11) and for the boundary potentials given in Lemma 5.5 immediately yield

$$[\![\mathsf{t}_{\ell-1}\delta_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{t}_{\ell+1}\mathrm{d}_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{n}_{\ell-1}\delta_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0,$$
$$[\![\mathsf{n}_{\ell-1}\delta_{\ell}]\!]\,\mathsf{S}^{\lambda}_{\ell} = 0, \qquad [\![\mathsf{n}_{\ell+1}\mathrm{d}_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = 0.$$

It only remains to verify that $[\![\mathsf{t}_{\ell-1}\delta_{\ell}]\!]\,\mathsf{D}^{\lambda}_{\ell} = \mathrm{Id}$. Using (5.3.9), we expand the definition of the codifferential to get

$$\begin{aligned} \mathsf{t}_{\ell-1}\delta_{\ell}\,\mathsf{D}^{\lambda}_{\ell} &= (-1)^{\ell}\mathsf{t}_{\ell-1}\star^{-1}_{\ell-1}\mathrm{d}_{N-\ell}\star_{\ell}\star^{-1}_{\ell}\mathsf{S}^{\lambda}_{N-\ell}\star_{\ell-1} \\ &= (-1)^{\ell}\mathsf{t}_{\ell-1}\star^{-1}_{\ell-1}\mathrm{d}_{N-\ell}\,\mathsf{S}^{\lambda}_{N-\ell}\star_{\ell-1}. \end{aligned}$$

It is a tedious but straightforward calculation to invert the domain and boundary Hodge star operators to further conclude that

$$\mathsf{t}_{\ell-1}\delta_{\ell}\,\mathsf{D}^{\lambda}_{\ell} = (-1)^{\ell}(-1)^{\ell+1}\mathsf{t}_{\ell-1}\star_{N-\ell+1}\mathrm{d}_{N-\ell}\,\mathsf{S}^{\lambda}_{N-\ell}\star^{-1}_{N-\ell} = \mathsf{t}_{\ell-1}\Phi,$$

where we have recognized the double layer potential

$$\Phi = -\star_{N-\ell+1}\mathrm{d}_{N-\ell}\,\mathsf{S}^{\lambda}_{N-\ell}\star^{-1}_{N-\ell}$$

studied in [20] and for which we know from [20, Lem. 10] that $[\![\mathsf{t}_{\ell-1}]\!]\,\Phi = \mathrm{Id}$. □

5.3.3 Trace de Rham complex with non-local inner products

The trace de Rham complexes introduced in (5.1.7a) and (5.1.7b) are the canvas on which the theory in this article is drawn. Ultimately, it is by formulating the first-kind boundary integral equations for the Hodge–Dirac and Hodge–Laplace operators as variational problems in the trace de Rham complex that their structure is revealed with the most clarity.

We generalize to differential forms of any degree and to arbitrary dimension the theory presented in [33, Sec. 8]. It should be compared with [18] and [26].

The key observation is that the continuous sesquilinear forms

$$(u_\ell, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{t}} = \langle\langle u_\ell, \mathsf{t}_\ell \mathsf{S}^\lambda_\ell(\overline{v}_\ell)\rangle\rangle_\Gamma, \qquad u_\ell, v_\ell \in H^{-\frac{1}{2}}_{\|}\Lambda^\ell(\Gamma), \tag{5.3.15a}$$

$$(w_\ell, z_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} = \langle\langle w_\ell, \mathsf{n}_{\ell+1} \mathsf{D}^\lambda_\ell(\overline{z}_\ell)\rangle\rangle_\Gamma, \qquad w_\ell, z_\ell \in H^{-\frac{1}{2}}_{\perp}\Lambda^\ell(\Gamma), \tag{5.3.15b}$$

define non-local inner products on the spaces $H^{-\frac{1}{2}}_{\|}\Lambda^\ell(\Gamma)$ and $H^{-\frac{1}{2}}_{\perp}\Lambda^\ell(\Gamma)$.

In the following results, it is a convenient notation to write

$$\|U_\ell\|^2_{\lambda,X\Lambda^\ell(\mathcal{M})} = \lambda\,\|U_\ell\|^2_{\mathcal{M}} + \|\mathrm{d}_\ell U_\ell\|^2_{\mathcal{M}} + \|\delta_\ell U_\ell\|^2_{\mathcal{M}}, \qquad \forall U_\ell \in X\Lambda^\ell(\mathcal{M}),$$

where we allow $\lambda = 0$. Evidently, $\mathcal{M}$ can be replaced by $\Omega^\mp$.

Lemma 5.8 *For all $h_\ell \in H^{-\frac{1}{2}}_{\|}\Lambda^\ell(\Gamma)$ and $g_\ell \in H^{-\frac{1}{2}}_{\perp}\Lambda^\ell(\Gamma)$, we have*

$$\|h_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{t}} = \langle\langle \overline{h}_\ell, \mathsf{t}_\ell \mathsf{S}^\lambda_\ell(h_\ell)\rangle\rangle_\Gamma = \|\mathsf{S}^\lambda_\ell h_\ell\|^2_{\lambda,X\Lambda^\ell(\mathcal{M})}, \tag{5.3.16a}$$

$$\|g_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{n}} = \langle\langle \overline{g}_\ell, \mathsf{n}_\ell \mathsf{S}^\lambda_\ell(g_\ell)\rangle\rangle_\Gamma = \|\mathsf{D}^\lambda_\ell g_\ell\|^2_{\lambda,X\Lambda^\ell(\mathcal{M})}. \tag{5.3.16b}$$

Proof. For convenience, let us shorthand $\Psi = \mathsf{S}^\lambda_\ell h_\ell$. Integrating by parts the first term on the right-hand side of

$$\|\Psi\|^2_{\lambda,X\Lambda^\ell(\mathcal{M})} = \|\Psi\|^2_{\lambda,X\Lambda^\ell(\lambda,\Omega^-)} + \|\Psi\|^2_{\lambda,X\Lambda^\ell(\lambda,\Omega^+)},$$

we obtain

$$\begin{aligned}
\|\Psi\|^2_{X\Lambda^\ell(\lambda,\Omega^-)} &= (\mathrm{d}_\ell\Psi, \mathrm{d}_\ell\Psi)_{\Omega^-} + (\delta_\ell\Psi, \delta_\ell\Psi)_{\Omega^-} + \lambda\|\Psi\|^2_{\Omega^-} \\
&= (-\Delta_\ell\Psi, \Psi)_{\Omega^-} + \langle\langle \mathsf{n}^-_{\ell+1}\mathrm{d}_\ell\Psi, \mathsf{t}^-_\ell\overline{\Psi}\rangle\rangle_\Gamma - \langle\langle \mathsf{t}^-_{\ell-1}\delta_\ell\Psi, \mathsf{n}^-_\ell\overline{\Psi}\rangle\rangle_\Gamma + \lambda\|\Psi\|^2_{\Omega^-} \\
&= \langle\langle \mathsf{n}^-_{\ell+1}\mathrm{d}_\ell\Psi, \mathsf{t}^-_\ell\overline{\Psi}\rangle\rangle_\Gamma - \langle\langle \mathsf{t}^-_{\ell-1}\delta_\ell\Psi, \mathsf{n}^-_\ell\overline{\Psi}\rangle\rangle_\Gamma,
\end{aligned}$$

where we have used the fact that Ψ satisfies the equation $-\Delta\Psi = -\lambda\Psi$ in Ω^-, i.e. $(-\Delta_\ell\Psi, \Psi)_{\Omega^-} = -\lambda(\Psi, \Psi)_{\Omega^-} = -\lambda\|\Psi\|^2_{\Omega^-}$. We find similarly in Ω^+ that

$$\|\Psi\|^2_{\lambda,X\Lambda^\ell(\Omega^+)} = -\langle\langle \mathsf{n}^+_{\ell+1}\mathrm{d}_\ell\Psi, \mathsf{t}^+_\ell\overline{\Psi}\rangle\rangle_\Gamma + \langle\langle \mathsf{t}^+_{\ell-1}\delta_\ell\Psi, \mathsf{n}^+_\ell\overline{\Psi}\rangle\rangle_\Gamma.$$

Summing these contributions and using the jump relations from Lemma 5.7 yields

$$\|\Psi\|^2_{\lambda,X\Lambda^\ell(\mathcal{M})} = \langle\langle -[\![\, \mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\Psi, \mathsf{t}\overline{\Psi}\rangle\rangle_\Gamma = \langle\langle \overline{h}_\ell, \mathsf{t}\mathsf{S}^\lambda_\ell u_\ell\rangle\rangle_\Gamma = \|h_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{t}}.$$

□

The next result generalizes [8, Thm. 4] to arbitrary dimensions. We indicate inequalities that hold up to a positive constant multiple depending on Ω and λ using $\lesssim$.

Theorem 5.1 *Under Assumption A, we have*

$$\|h_\ell\|^2_{H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim (h_\ell, h_\ell)_{-\frac{1}{2},\lambda,\mathsf{t}}\,, \qquad \forall h_\ell \in H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\Gamma),$$

$$\|g_\ell\|^2_{H_\perp^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim (g_\ell, g_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}}\,, \qquad \forall w_\ell \in H_\perp^{-\frac{1}{2}}\Lambda^\ell(\Gamma).$$

Proof. We focus on the first inequality. The second can be obtained using analogous arguments.

In the first step, we use two ingredients:

- Recall from Subsection 5.1.4 that the tangential trace $\mathsf{t}_\ell : H^1\Lambda^\ell(\Omega) \to H_\parallel^{\frac{1}{2}}\Lambda^\ell(\Gamma)$ is a *surjective* operator admitting a bounded right-inverse

$$\mathsf{t}^\dagger : H_\parallel^{\frac{1}{2}}\Lambda^\ell(\Gamma) \to H^1\Lambda^\ell(\Omega),$$

  i.e $\|\mathsf{t}^\dagger g_\ell\|_{H^1\Lambda^\ell(\Omega)} \lesssim \|g_\ell\|_{H_\parallel^{\frac{1}{2}}\Lambda^\ell}(\Gamma)$ and $\mathsf{t}_\ell \circ \mathsf{t}^\dagger g_\ell = g_\ell$ for all $g_\ell \in H_\parallel^{\frac{1}{2}}\Lambda^\ell(\Gamma)$.
- According to [26, prop. 3.1], there exists a continuous *extension* operator

$$\mathsf{E} : H^1\Lambda^\ell(\Omega) \to H^1\Lambda^\ell(\mathcal{M})$$

  such that $(\mathsf{E}\,U_\ell)\big|_\Omega = U_\ell$ for all $U_\ell \in H^1\Lambda(\Omega)$.

Given $h_\ell \in H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$, we can introduce these operators in the definition of $H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$ to obtain the estimate

$$\begin{aligned}
\|h_\ell\|^2_{H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} &= \sup_{g_\ell \in H_\parallel^{\frac{1}{2}}\Lambda^\ell(\Gamma)} \frac{\left|\langle\!\langle h_\ell, g_\ell\rangle\!\rangle_\Gamma\right|}{\|g_\ell\|_{H_\parallel^{\frac{1}{2}}\Lambda^\ell(\Gamma)}} \lesssim \sup_{g_\ell \in H_\parallel^{\frac{1}{2}}\Lambda^\ell(\Gamma)} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell\mathsf{t}^\dagger g_\ell\rangle\!\rangle_\Gamma\right|}{\|\mathsf{t}^\dagger g_\ell\|_{H^1\Lambda^\ell(\Omega)}} \\
&\leq \sup_{W_\ell \in H^1\Lambda^\ell(\Omega)} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell W_\ell\rangle\!\rangle_\Gamma\right|}{\|W_\ell\|_{H^1\Lambda^\ell(\Omega)}} \lesssim \sup_{W_\ell \in H^1\Lambda^\ell(\Omega)} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell \mathsf{E} W_\ell\rangle\!\rangle_\Gamma\right|}{\|\mathsf{E}W_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}} \\
&\lesssim \sup_{W_\ell \in H^1\Lambda^\ell(\mathcal{M})} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell W_\ell\rangle\!\rangle_\Gamma\right|}{\|W_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}}.
\end{aligned} \tag{5.3.17}$$

In the second step, we recognize in the numerator the definition of the atomic boundary potential. Recall that $\mathsf{S}_\ell^\lambda = \mathsf{N}_\ell^\lambda \circ \mathsf{t}'_\ell = (-\Delta_\ell + \lambda\mathrm{Id})^{-1} \circ \mathsf{t}'_\ell$. In other words, $\mathsf{S}_\ell^\lambda h_\ell$ satisfies the variational equation

$$\left(\mathrm{d}_\ell \mathsf{S}_\ell^\lambda g_\ell, \mathrm{d}_\ell V_\ell\right)_\Omega + \left(\delta_\ell \mathsf{S}_\ell^\lambda g_\ell, \delta_\ell V_\ell\right)_\Omega + \lambda\left(\mathsf{S}_\ell^\lambda g_\ell, V_\ell\right)_\Omega = \langle\!\langle \mathsf{t}'_\ell h_\ell, V_\ell\rangle\!\rangle_\Gamma = \langle\!\langle h_\ell, \mathsf{t}_\ell V_\ell\rangle\!\rangle_\Gamma \tag{5.3.18}$$

for all $V_\ell \in H^1\Lambda^\ell(\Omega)$. Hence, we arrive at the identity

$$\left|\langle\langle h_\ell, \mathsf{t}_\ell V_\ell \rangle\rangle_\Gamma\right| = \left|\left(\mathsf{S}_\ell^\ell h_\ell, V_\ell\right)_{\lambda, X\Lambda^\ell(\mathcal{M})}\right| \tag{5.3.19}$$

and plug it into (5.3.17) to obtain

$$\|h_\ell\|^2_{H^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \sup_{V_\ell \in H^1\Lambda^\ell(\mathcal{M})} \frac{\left|\left(\mathsf{S}_\ell^\ell h_\ell, V_\ell\right)_{\lambda, X\Lambda^\ell(\mathcal{M})}\right|}{\|V_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}}. \tag{5.3.20}$$

In the third step, we simply appeal to a Gaffney inequality (the easy direction), which states that

$$\|V_\ell\|_{H^1\Lambda^\ell(\mathcal{M})} \sim \|V_\ell\|_{\lambda, X\Lambda^\ell(\mathcal{M})} \tag{5.3.21}$$

for all $V_\ell \in H^1\Lambda^\ell(\Omega)$, cf. [36], [28, Thm. 7.2.6], [6]. Going back to (5.3.20) and applying the Cauchy-Schwartz inequality yields

$$\|h_\ell\|^2_{H^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \sup_{V_\ell \in X\Lambda^\ell(\mathcal{M})} \frac{\left|\left(\mathsf{S}_\ell^\ell h_\ell, V_\ell\right)_{\lambda, X\Lambda^\ell(\mathcal{M})}\right|}{\|V_\ell\|_{X\Lambda^\ell(\lambda, \mathcal{M})}} = \|\mathsf{S}_\ell^\lambda h_\ell\|^2_{\lambda, X\Lambda^\ell(\mathcal{M})}. \tag{5.3.22}$$

Finally, using Lemma 5.8, we arrive at

$$\|h_\ell\|^2_{H^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \|h_\ell\|^2_{-\frac{1}{2}, \lambda, \mathsf{t}}.$$

□

5.4 Boundary integral equations

In this section, we exploit the results of Section 5.3 to derive boundary integral equations for the BVPs of Section 5.2. We follow the same recipe for every operator. The approach has a long history. Standard references for scalar-valued BVPs in Euclidean space are [23, 31] and [37]. We also particularly recommend [13]. We refer to [10] for classical electromagnetism, where the perspective is also prominently adopted. As mentioned in the introduction—and directly relevant to this work—the abstract procedure was used to derive BIEs for the Hodge–Dirac and Hodge–Helmholtz operators using vector calculus in [12] and [33] under the hypothesis that $\mathcal{M} = \mathbb{R}^3$. An overview similar to what follows is given in [34].

Let $\mathfrak{L}$ stand for any one of the operators introduced in Section 5.2: $\mathfrak{D} + i\kappa$, $-\Delta_\ell + \lambda$, $\mathfrak{M}$ or $\mathfrak{R}$.

1. We confirm that the operator satisfies an identity of the form

$$\langle \mathfrak{L}U, V\rangle_{\Omega^\mp} = \langle U, \mathfrak{L}V\rangle_{\Omega^\mp} \pm \langle\langle \mathsf{T}^{\mathsf{t}}_{\mathfrak{L}} U, \mathsf{T}^{\mathsf{n}}_{\mathfrak{L}} V\rangle\rangle_\Gamma \mp \langle\langle \mathsf{T}^{\mathsf{t}}_{\mathfrak{L}} V, \mathsf{T}^{\mathsf{n}}_{\mathfrak{L}} U\rangle\rangle_\Gamma \tag{5.4.1}$$

   resembling Green's second formula and identify a "Newton potential operator" $\mathsf{N}[\mathfrak{N}]$, i.e. an inverse of $\mathfrak{L}$ in the sense of distributions. Together, these two ingredients enable us to find a representation formula of the form

$$U = \mathsf{N}[\mathfrak{L}]U - \mathsf{SL}[\mathfrak{L}]\left(\llbracket \mathsf{T}^{\mathsf{n}}_{\mathfrak{L}} U\rrbracket\right) + \mathsf{DL}[\mathfrak{L}]\left(\llbracket \mathsf{T}^{\mathsf{t}}_{\mathfrak{L}} U\rrbracket\right), \tag{5.4.2}$$

   where $\mathsf{SL}[\mathfrak{L}]$ and $\mathsf{DL}[\mathfrak{L}]$ are potential operators playing roles analogous to the single and double layer potentials in the classical theory of BIEs for scalar Laplace problems or electromagnetic scattering.

2. We apply average traces $\{\mathsf{T}^{\bullet}_{\mathfrak{L}}\} = (\mathsf{T}^{\bullet,-}_{\mathfrak{L}} + \mathsf{T}^{\bullet,+}_{\mathfrak{L}})/2$ to the obtained boundary potentials $\mathsf{SL}[\mathfrak{L}]$ and $\mathsf{DL}[\mathfrak{L}]$ to define four BIOs:

$$\begin{aligned} \mathsf{V}[\mathfrak{L}] &= \{\mathsf{T}^{\mathfrak{t}}_{\mathfrak{L}}\}\mathsf{SL}[\mathfrak{L}] : H^{\mathfrak{n}}_{\mathfrak{L}}(\Gamma) \longrightarrow H^{\mathfrak{t}}_{\mathfrak{L}}(\Gamma), \\ \mathsf{K}[\mathfrak{L}] &= \{\mathsf{T}^{\mathfrak{t}}_{\mathfrak{L}}\}\mathsf{DL}[\mathfrak{L}] : H^{\mathfrak{t}}_{\mathfrak{L}}(\Gamma) \longrightarrow H^{\mathfrak{t}}_{\mathfrak{L}}(\Gamma), \\ \mathsf{A}[\mathfrak{L}] &= \{\mathsf{T}^{\mathfrak{n}}_{\mathfrak{L}}\}\mathsf{SL}[\mathfrak{L}] : H^{\mathfrak{n}}_{\mathfrak{L}}(\Gamma) \longrightarrow H^{\mathfrak{n}}_{\mathfrak{L}}(\Gamma), \\ \mathsf{W}[\mathfrak{L}] &= \{\mathsf{T}^{\mathfrak{n}}_{\mathfrak{L}}\}\mathsf{DL}[\mathfrak{L}] : H^{\mathfrak{t}}_{\mathfrak{L}}(\Gamma) \longrightarrow H^{\mathfrak{n}}_{\mathfrak{L}}(\Gamma). \end{aligned}$$

3. We verify that the jump relations

$$\begin{aligned} [\![\mathsf{T}^{\mathfrak{t}}_{\mathfrak{L}}]\!]\mathsf{SL}[\mathfrak{L}] &= 0, & \qquad [\![\mathsf{T}^{\mathfrak{t}}_{\mathfrak{L}}]\!]\mathsf{DL}[\mathfrak{L}] &= \mathrm{Id}, \\ [\![\mathsf{T}^{\mathfrak{n}}_{\mathfrak{L}}]\!]\mathsf{SL}[\mathfrak{L}] &= -\mathrm{Id}, & \qquad [\![\mathsf{T}^{\mathfrak{n}}_{\mathfrak{L}}]\!]\mathsf{DL}[\mathfrak{L}] &= 0, \end{aligned}$$

hold in the trace spaces $T^{\mathfrak{t}}_{\mathfrak{L}}$ and $T^{\mathfrak{n}}_{\mathfrak{L}}$. Applying average traces on both sides of the representation formula and appealing to these jump relations lead to a Calderón operator

$$\mathsf{C}[\mathfrak{L}] = \begin{pmatrix} \frac{1}{2}\mathrm{Id} + \mathsf{K}[\mathfrak{L}] & -\mathsf{V}[\mathfrak{L}] \\ -\mathsf{W}[\mathfrak{L}] & \frac{1}{2}\mathrm{Id} - \mathsf{A}[\mathfrak{L}] \end{pmatrix} \tag{5.4.3}$$

whose kernel fully characterizes the space of valid Cauchy data. In other words, boundary data $(g,h)^\top \in T^{\mathfrak{t}}_{\mathfrak{L}}(\Gamma) \times T^{\mathfrak{n}}_{\mathfrak{L}}(\Gamma)$ satisfies $\mathsf{C}[\mathfrak{L}](g,h)^\top = 0$ if and only if there exists $U \in \mathrm{dom}(\mathfrak{L})$ such that $\mathfrak{L}U = 0$ with $\mathsf{T}^{\mathfrak{t}}_{\mathfrak{L}}U = g$ and $\mathsf{T}^{\mathfrak{t}}_{\mathfrak{L}}U = h$.

4. We extract two first-kind BIEs:

$$h \in T^{\mathfrak{n}}_{\mathfrak{L}}(\Gamma) : \qquad \mathsf{V}[\mathfrak{L}]h = (\frac{1}{2}\mathrm{Id} + \mathsf{K}[\mathfrak{L}])g, \qquad g \in T^{\mathfrak{t}}_{\mathfrak{L}}(\Gamma), \tag{5.4.4a}$$

$$g \in H^{\mathfrak{t}}_{\mathfrak{L}}(\Gamma) : \qquad \mathsf{W}[\mathfrak{L}]g = (\frac{1}{2}\mathrm{Id} - \mathsf{A}[\mathfrak{L}])h, \qquad h \in H^{\mathfrak{n}}_{\mathfrak{L}}(\Gamma). \tag{5.4.4b}$$

It suffices to take duality pairing on both sides of these equations to obtain the equivalent variational problems

$$\langle\!\langle \mathsf{V}[\mathfrak{L}]h, \overline{w}\rangle\!\rangle_\Gamma = \langle\!\langle (\frac{1}{2}\mathrm{Id} + \mathsf{K}[\mathfrak{L}])g, \overline{w}\rangle\!\rangle_\Gamma, \qquad \forall w \in T^{\mathfrak{n}}_{\mathfrak{L}}(\Gamma), \tag{5.4.5a}$$

$$\langle\!\langle \mathsf{W}[\mathfrak{L}]g, \overline{v}\rangle\!\rangle_\Gamma = \langle\!\langle (\frac{1}{2}\mathrm{Id} - \mathsf{A}[\mathfrak{L}])h, \overline{v}\rangle\!\rangle_\Gamma, \qquad \forall v \in T^{\mathfrak{t}}_{\mathfrak{L}}(\Gamma). \tag{5.4.5b}$$

We will show that the first-kind BIOs operators $\mathsf{V}[\mathfrak{L}]$ and $\mathsf{W}[\mathfrak{L}]$ associated with the bilinear forms on the left-hand side of these variational problems are Hodge–Dirac and Hodge–Laplace operators in variational form in the trace de Rham complex with the non-local inner products presented in Subsection 5.3.3.

Remark 5.1 The signs in (5.4.3), and thus accordingly in (5.4.4a) and (5.4.4b), were chosen as per convention to mimic the well-known theory for the scalar Laplacian. This choice is somewhat arbitrary and the equations can be altered to avoid some sign flips that occur in the next sections. We restrain ourselves from doing so as we do not believe there is much to gain so far by departing from classical sign conventions.

5.4.1 BIEs for Hodge–Dirac BVPs

In this section, the abstract theory is instantiated according to the following table.

$\mathfrak{L}$	$\mathrm{dom}(\mathfrak{L})$	$\mathsf{T}^{\mathfrak{t}}_{\mathfrak{L}}$	$\mathsf{T}^{\mathfrak{n}}_{\mathfrak{L}}$	$H^{\mathfrak{t}}_{\mathfrak{L}}(\Gamma)$	$H^{\mathfrak{n}}_{\mathfrak{L}}(\Gamma)$
$\mathfrak{D}+i\kappa$	$H\Lambda(\mathfrak{D},\Omega)$	$\mathbf{t}$	$\mathbf{n}$	$H_{\perp}^{-\frac{1}{2}}\Lambda(\mathrm{d},\Gamma)$	$H_{\parallel}^{-\frac{1}{2}}\Lambda(\delta,\Gamma)$

Integration by parts reveals that the Hodge–Dirac operator satisfies an identity such as (5.4.1):

$$\langle \mathfrak{D}\boldsymbol{U},\boldsymbol{V}\rangle_{\Omega^{\mp}} = \langle \boldsymbol{U},\mathfrak{D}\boldsymbol{V}\rangle_{\Omega^{\mp}} \pm \langle\!\langle \mathbf{t}\,\boldsymbol{U},\mathbf{n}\,\boldsymbol{V}\rangle\!\rangle_{\Gamma} \mp \langle\!\langle \mathbf{t}\,\boldsymbol{V},\mathbf{n}\,\boldsymbol{U}\rangle\!\rangle_{\Gamma} \tag{5.4.6}$$

for all $\boldsymbol{U},\boldsymbol{V}\in H\Lambda(\mathfrak{D},\Omega)$

Remark 5.2 It is remarkable that an identity resembling Green's second formula is available *despite* the operator being only first-order. Evidently, this is due to its symmetric structure. The Hodge–Dirac operator is a sum of two operators that are formally adjoint to each other. Thanks to that, a representation by boundary potentials can be derived using the approach promoted by Costabel for second order elliptic operators, cf. [11, sect. 4.2], [13], [33, sect. 4.4] and [34, Sec. 2.4].

5.4.1.1 Representation formula for Hodge–Dirac operators

This section generalizes [33, Sec. 4]. It follows immediately from (5.3.6) that

$$\mathfrak{D}_x\boldsymbol{\mathcal{G}}_{\lambda} = \mathfrak{D}_y\boldsymbol{\mathcal{G}}_{\lambda}.$$

Integrating by parts after using the commutative relations (5.3.6) eventually verifies that

$$\mathbf{N}_{\lambda}\,\mathfrak{D} = \mathfrak{D}\,\mathbf{N}_{\lambda} \tag{5.4.7}$$

in the sense of distributions. Going back to (5.1.1) with $\lambda=\kappa^2$, we find that

$$(\mathfrak{D}-i\kappa)\,\mathbf{N}_{\lambda}\,(\mathfrak{D}+i\kappa) = (-\boldsymbol{\Delta}+\kappa^2)\mathbf{N}_{\lambda} = \mathrm{Id}.$$

In other words,

$$\mathsf{N}[\mathfrak{D}] = (\mathfrak{D}-i\kappa)\,\mathbf{N}_{\lambda} = \mathbf{N}_{\lambda}\,(\mathfrak{D}-i\kappa)$$

is a fundamental solution for the perturbed Hodge–Dirac operator $(\mathfrak{D}+i\kappa)$.

Proposition 5.1 *If $\boldsymbol{U}\in L^2\Lambda(\mathcal{M})$ is compactly supported and there exists $\boldsymbol{F}\in L^2\Lambda(\mathcal{M})$ such that $\boldsymbol{F}|_{\Omega}=(\mathfrak{D}+i\kappa)\boldsymbol{U}|_{\Omega}$ and $\boldsymbol{F}|_{\Omega^+}=(\mathfrak{D}+i\kappa)\boldsymbol{U}|_{\Omega^+}$, then*

$$\boldsymbol{U} = (\mathfrak{D}-i\kappa)\left(\mathbf{N}_{\lambda}\boldsymbol{F} - \mathbf{S}_{\lambda}[\![\mathsf{n}\boldsymbol{U}]\!] + \mathbf{D}_{\lambda}[\![\mathsf{t}\boldsymbol{U}]\!]\right). \tag{5.4.8}$$

Proof. According to (5.4.6), we have

$$\begin{aligned}\langle\!\langle (\mathfrak{D}+i\kappa)\,\boldsymbol{U},\boldsymbol{V}\rangle\!\rangle_{\mathcal{M}} &= \langle \boldsymbol{U},(\mathfrak{D}+i\kappa)\boldsymbol{V}\rangle_{\Omega} + \langle \boldsymbol{U},(\mathfrak{D}+i\kappa)\boldsymbol{V}\rangle_{\Omega^+}\\ &= \langle \boldsymbol{F},\boldsymbol{V}\rangle_{\Omega} + \langle\!\langle \mathbf{t}^-\boldsymbol{V},\mathbf{n}^-\boldsymbol{U}\rangle\!\rangle_{\Gamma} - \langle\!\langle \mathbf{t}^-\boldsymbol{U},\mathbf{n}^-\boldsymbol{V}\rangle\!\rangle_{\Gamma}\\ &\quad + \langle \boldsymbol{F},\boldsymbol{V}\rangle_{\Omega^+} - \langle\!\langle \mathbf{t}^+\boldsymbol{V},\mathbf{n}^+\boldsymbol{U}\rangle\!\rangle_{\Gamma} + \langle\!\langle \mathbf{t}^+\boldsymbol{U},\mathbf{n}^+\boldsymbol{V}\rangle\!\rangle_{\Gamma}\\ &= \langle \boldsymbol{F},\boldsymbol{V}\rangle_{\mathcal{M}} - \langle\!\langle \mathbf{t}\,\boldsymbol{V},[\![\mathbf{n}]\!]\,\boldsymbol{U}\rangle\!\rangle_{\Gamma} + \langle\!\langle [\![\mathbf{t}]\!]\boldsymbol{U},\mathbf{n}\boldsymbol{V}\rangle\!\rangle_{\Gamma}\end{aligned}$$

for all $\boldsymbol{V}\in\mathcal{D}(\mathcal{M})$. The regularity assumption on $\boldsymbol{U}$ guarantees that the traces are well-defined. We have used the fact that $\boldsymbol{V}$ is smooth across the boundary to obtain the last equality, because smoothness guarantees that $\mathbf{t}^+\boldsymbol{V}=\mathbf{t}^-\boldsymbol{V}$ and $\mathbf{n}^+\boldsymbol{V}=\mathbf{n}^-\boldsymbol{V}$, i.e. the jumps vanish on Γ. Hence, in the sense of distributions,

$$(\mathfrak{D}+i\kappa)\,\boldsymbol{U} = \boldsymbol{F} - \mathbf{t}'\,[\![\mathbf{n}\,\boldsymbol{U}]\!] + \mathbf{n}'\,[\![\mathbf{t}\,\boldsymbol{U}]\!].$$

Since $\boldsymbol{U}$ has compact support, it can be interpreted as a continuous linear functional on $\boldsymbol{\mathcal{E}}(\mathcal{M})$. With the definitions of the atomic boundary potentials from (5.3.7) at hand, applying the Newton potential operator $\mathbf{N}_{\boldsymbol{\lambda}}$ on both sides of this equation yields

$$\begin{aligned}\mathbf{N}_{\boldsymbol{\lambda}}(\mathfrak{D}+i\kappa)\,\boldsymbol{U} &= \mathbf{N}_{\boldsymbol{\lambda}}\boldsymbol{F} - \mathbf{N}_{\boldsymbol{\lambda}}\mathbf{t}'\,[\![\mathbf{n}\,\boldsymbol{U}]\!] + \mathbf{N}_{\boldsymbol{\lambda}}\mathbf{n}'\,[\![\mathbf{t}\,\boldsymbol{U}]\!]\\ &= \mathbf{N}_{\boldsymbol{\lambda}}\boldsymbol{F} - \mathbf{S}_{\lambda}\,[\![\mathbf{n}\,\boldsymbol{U}]\!] + \mathbf{D}_{\lambda}\,[\![\mathbf{t}\,\boldsymbol{U}]\!].\end{aligned}\tag{5.4.9}$$

Since $(\mathfrak{D}+i\kappa)\,\boldsymbol{U}$ is square-integrable, the mapping properties of the Newton potential provided in eq. (5.3.3) ensure that the left-hand side of this identity lies in the domain of the Hodge–Dirac operator, since it is in fact component-wise weakly differentiable. Moreover, that the images of the atomic boundary potentials belong to $\boldsymbol{H}^1_{\mathrm{loc}}\Lambda(\mathcal{M})$ was the result of Lemma 5.6. Therefore, we can apply $\mathfrak{D}-i\kappa$ on both sides of (5.4.9) and use the commutation relation (5.4.7) to reach (5.4.8). □

We are tempted to call *single* and *double* layer potentials for the Hodge–Dirac operator the boundary potentials

$$\mathsf{SL}[\mathfrak{D}] = (\mathfrak{D}-i\kappa)\,\mathbf{S}_{\lambda} : H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma) \longrightarrow H\Lambda(\mathfrak{D},\Omega),\tag{5.4.10a}$$

$$\mathsf{DL}[\mathfrak{D}] = (\mathfrak{D}-i\kappa)\,\mathbf{D}_{\lambda} : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d},\Gamma) \longrightarrow H\Lambda(\mathfrak{D},\Omega),\tag{5.4.10b}$$

respectively. However, while this nomenclature is a convenient way to highlight the similarities between our development and the classical theory of boundary integral equations for second-order elliptic operators, we stress that it may also be misleading. Both traces in (5.1.6) rest on an equal footing in that none involves a differential operator. We saw in (5.3.9) that the two boundary potentials are not only isometrically isomorphic, but also symmetric in the sense of Hodge duality.

It follows immediately from Lemma 5.7 that these boundary potentials satisfy the abstract jump relations stated above. For example,

$$[\![\mathsf{T}^{\mathsf{n}}_{\mathfrak{D}}]\!]\mathsf{SL}[\mathfrak{D}] = [\![\mathbf{n}]\!]\,(\mathfrak{D}-i\kappa)\,\mathbf{S}_{\lambda} = [\![\mathbf{n}\mathrm{d}]\!]\mathbf{S}_{\lambda} + [\![\mathbf{n}\boldsymbol{\delta}]\!]\mathbf{S}_{\lambda} - i\kappa[\![\mathbf{n}]\!]\mathbf{S}_{\lambda} = -\mathrm{Id}.$$

The other relations are computed similarly.

5.4.1.2 BIOs for Hodge–Dirac operators

Since the boundary potentials may jump across Γ, we resort as per convention to the average traces $\{\bullet\} = (\bullet^- + \bullet^+)/2$, where $\bullet = \mathsf{t}$ or n. In particular, we let

$$\begin{aligned}
\mathsf{V}[\mathfrak{D}] &= \{\mathsf{t}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{S}_\lambda : H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}, \Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}, \Gamma),\\
\mathsf{K}[\mathfrak{D}] &= \{\mathsf{t}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{D}_\lambda : H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}, \Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}, \Gamma),\\
\mathsf{A}[\mathfrak{D}] &= \{\mathsf{n}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{S}_\lambda : H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}, \Gamma) \longrightarrow H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}, \Gamma),\\
\mathsf{W}[\mathfrak{D}] &= \{\mathsf{n}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{D}_\lambda : H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}, \Gamma) \longrightarrow H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}, \Gamma).
\end{aligned}$$

As a consequence of the jump relations, these boundary integral operators enter a Calderón operator $\mathsf{C}[\mathfrak{D}]$ such as (5.4.3) whose kernel fully characterizes the space of valid Cauchy data. This last property is a consequence of three ingredients: the jump relations, the representation formula and the lifting maps from Subsection 5.1.4.2.

From the jump relations, $\mathsf{V}[\mathfrak{D}] = \mathsf{t}\,(\mathfrak{D} - i\kappa)\,\mathbf{S}_\lambda$, i.e. the average of the traces is equal to taking a single-sided trace. Using Corollary 5.2, it follows by integration by parts and (5.1.11) that

$$\begin{aligned}
\langle\!\langle\, \mathsf{V}[\mathfrak{D}]\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma &= \langle\!\langle\, \mathsf{t}\boldsymbol{\delta}\mathbf{S}_\lambda\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma + \langle\!\langle\, \mathsf{t}\mathbf{d}\mathbf{S}_\lambda\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma - i\kappa\langle\!\langle \mathsf{t}\mathbf{S}_\lambda\boldsymbol{h}, \overline{\boldsymbol{w}}\rangle\!\rangle_\Gamma\\
&= \langle\!\langle\, \mathsf{t}\mathbf{S}_\lambda\boldsymbol{\delta}\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma + \langle\!\langle\, \mathsf{t}\mathbf{S}_\lambda\boldsymbol{h}, \boldsymbol{\delta}\overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma - i\kappa\langle\!\langle \mathsf{t}\mathbf{S}_\lambda\boldsymbol{h}, \overline{\boldsymbol{w}}\rangle\!\rangle_\Gamma\\
&= (\boldsymbol{\delta}\boldsymbol{h}, \boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}} + (\boldsymbol{h}, \boldsymbol{\delta}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}} - i\kappa(\boldsymbol{h}, \boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}}
\end{aligned} \tag{5.4.11}$$

for all $\boldsymbol{h}, \boldsymbol{w} \in H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}, \Gamma)$.

Similarly, we can also compute

$$\begin{aligned}
\langle\!\langle\, \mathsf{W}[\mathfrak{D}]\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma &= \langle\!\langle\, \mathsf{n}\mathbf{d}\mathbf{D}_\lambda\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma + \langle\!\langle\, \mathsf{n}\boldsymbol{\delta}\mathbf{D}_\lambda\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma - i\kappa\langle\!\langle \mathsf{n}\mathbf{D}_\lambda\boldsymbol{g}, \overline{\boldsymbol{v}}\rangle\!\rangle_\Gamma\\
&= -\langle\!\langle\, \mathsf{n}\mathbf{D}_\lambda\mathbf{d}\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma - \langle\!\langle\, \mathsf{n}\mathbf{D}_\lambda\boldsymbol{g}, \mathbf{d}\overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma - i\kappa\langle\!\langle \mathsf{n}\mathbf{D}_\lambda\boldsymbol{g}, \overline{\boldsymbol{v}}\rangle\!\rangle_\Gamma\\
&= -(\mathbf{d}\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}} - (\boldsymbol{g}, \mathbf{d}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{n}} - i\kappa(\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}}
\end{aligned} \tag{5.4.12}$$

for all $\boldsymbol{g}, \boldsymbol{v} \in H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}, \Gamma)$.

We urge the reader to compare these bilinear forms on the boundary with the bilinear forms $\mathcal{A}_{\boldsymbol{\delta}}$ and $\mathcal{A}_{\mathbf{d}}$ that appear in the variational problems (5.2.8a) and (5.2.8b) for the Hodge–Dirac operator in the domain Ω.

We conclude from (5.4.11) and (5.4.12) that the first-kind boundary integral operators $\mathsf{V}[\mathfrak{D}]$ and $\mathsf{W}[\mathfrak{D}]$ associated with the direct first-kind boundary integral equations (5.4.4a) and (5.4.4b) are zero-order perturbations of Hodge–Dirac operators in the trace de Rham complexes of Subsection 5.3.3. More precisely,

$$\mathsf{V}[\mathfrak{D}] = \boldsymbol{\delta} + \boldsymbol{\delta}^* - i\kappa, \tag{5.4.13a}$$
$$\mathsf{W}[\mathfrak{D}] = -(\mathbf{d} + \mathbf{d}^*) - i\kappa, \tag{5.4.13b}$$

where the closed densely defined unbounded operators

$$\boldsymbol{\delta}^* : H_{\parallel}^{-\frac{1}{2}}\Lambda(\Gamma) \longrightarrow H_{\parallel}^{-\frac{1}{2}}\Lambda(\Gamma),$$
$$\mathbf{d}^* : H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma),$$

are the Hilbert space adjoint of the closed densely defined unbounded operators

$$\boldsymbol{\delta} : H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma) \subset H_{\parallel}^{-\frac{1}{2}}\Lambda(\Gamma) \longrightarrow H_{\parallel}^{-\frac{1}{2}}\Lambda(\Gamma),$$
$$\mathbf{d} : H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma) \subset H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma),$$

introduced in Subsection 5.1.4, but where the spaces $H_{\parallel}^{-\frac{1}{2}}\Lambda(\Gamma)$ and $H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma)$ are equipped with the non-local inner products defined in Subsection 5.3.3.

As in Subsection 5.2.1, it follows immediately from the abstract theory for the Hodge–Dirac operator in Hilbert complexes that $\mathsf{V}[\mathfrak{D}]$ and $\mathsf{W}[\mathfrak{D}]$ are invertible for $\kappa \neq 0$. They are Fredholm operators of index zero when $\kappa = 0$, in which case the dimension of their finite dimensional kernel is the sum of the Betti numbers of the boundary Γ.

The expressions (5.4.13a) and (5.4.13b) should be compared with the self-adjoint operators (5.2.3a) and (5.2.3b).

unknown	$\mathbf{n}\boldsymbol{U}$
boundary data	$\mathbf{t}\boldsymbol{U}$
self-adjoint op. in Ω	$\mathfrak{D}_{\mathbf{t}} + i\kappa = \boldsymbol{\delta} + \boldsymbol{\delta}^* + i\kappa$
first-kind BIO	$\mathsf{V}[\mathfrak{D}] = \boldsymbol{\delta} + \boldsymbol{\delta}^* - i\kappa$
bilinear form on Γ	$\langle\!\langle\, \mathsf{V}[\mathfrak{D}]\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma = (\boldsymbol{\delta}\boldsymbol{h}, \boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}} + (\boldsymbol{h}, \boldsymbol{\delta}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}}$ $-i\kappa(\boldsymbol{h}, \boldsymbol{\delta}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}}$
bilinear form in Ω	$\mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U},\boldsymbol{V}) + i\kappa(\boldsymbol{U},\boldsymbol{V})_\Omega = (\boldsymbol{\delta}\boldsymbol{U},\boldsymbol{V})_\Omega + (\boldsymbol{U},\boldsymbol{\delta}\boldsymbol{V})_\Omega$ $+i\kappa\,(\boldsymbol{U},\boldsymbol{V})_\Omega$

Fig. 5.1 Table of relations for the BVPs (5.2.2a) and (5.2.8a).

unknown	$\mathbf{t}\,\boldsymbol{U}$
boundary data	$\mathbf{n}\,\boldsymbol{U}$
self-adjoint op. in Ω	$\mathfrak{D}_{\mathbf{n}} = \mathbf{d} + \mathbf{d}^{*} + i\kappa$
first-kind BIO	$\mathsf{W}[\mathfrak{D}] = -\mathbf{d} - \mathbf{d}^{*} - i\kappa$
bilinear form on Γ	$\langle\!\langle\, \mathsf{W}[\mathfrak{D}]\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_{\Gamma} = -(\mathbf{d}\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}} - (\boldsymbol{g}, \mathbf{d}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{n}}$ $-i\kappa(\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}}$
bilinear form in Ω	$\mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \boldsymbol{V}) + i\kappa(\boldsymbol{U}, \boldsymbol{V})_{\Omega} = (\mathbf{d}\boldsymbol{U}, \boldsymbol{V})_{\Omega} + (\boldsymbol{U}, \mathbf{d}\boldsymbol{V})_{\Omega}$ $+i\kappa\,(\boldsymbol{U}, \boldsymbol{V})_{\Omega}$

Fig. 5.2 Table of relations for the BVPs (5.2.2b) and (5.2.8b).

5.4.2 BIEs for Hodge–Laplace BVPs

In Subsection 5.2.2, the notation was kept consistent with the abstract overview given at the beginning of Section 5.4, so we can jump straight into calculations. It is routine to verify that Green's second formulas such as (5.4.1) hold for the Hodge–Laplacian in both strong and mixed formulations, cf. [11, 12, 32, 34]. For the mixed formulations, this can be seen directly from the fact the bilinear forms in Subsection 5.2.3 are symmetric.

Remark 5.3 It is worth noting that the strong form of the Hodge–Laplace operator *fails to admit* an identity akin to Green's *first* formula:

$$\langle -\Delta_{\ell} U_{\ell}, V_{\ell} \rangle_{\Omega} \neq \mathcal{B}_{\Delta}(U_{\ell}, V_{\ell}) \pm \langle\!\langle \mathsf{T}^{\mathsf{n}}_{\Delta} U_{\ell}, \mathsf{T}^{\mathsf{t}}_{\Delta} U_{\ell} \rangle\!\rangle_{\Gamma},$$

where $\mathcal{B}_{\Delta}(U_{\ell}, V_{\ell}) = \langle \mathrm{d}_{\ell} U_{\ell}, \mathrm{d}_{\ell} V_{\ell} \rangle_{\Omega} + \langle \delta_{\ell} U_{\ell}, \delta_{\ell} V_{\ell} \rangle_{\Omega}$ is the fundamental bilinear form associated with $-\Delta_{\ell}$. When it comes to the use of BIEs in scattering and transmission problems, this can be a serious drawback. The perfect match between the boundary term that arises in domain variational problems for the mixed Hodge–Laplacian and the product space of traces on which first-kind BIEs are defined was crucial in [32] to establish variational formulations which coupled the two.

5.4.2.1 Representation formula for the strong Hodge–Laplacian

We have already seen in Subsection 5.3.1 that a Newton operator $\mathsf{N}[\Delta] = \mathsf{N}^{\lambda}_{\ell}$ is available for the Hodge–Laplacian in strong second-order formulation.

Proposition 5.2 *If $U_\ell \in L^2\Lambda^\ell(\mathcal{M})$ is compactly supported and there exists $F_\ell \in L^2\Lambda^\ell(\mathcal{M})$ such that $F_\ell|_\Omega = (-\Delta_\ell + \lambda)U_\ell|_\Omega$ and $F_\ell|_{\Omega^+} = (-\Delta_\ell + \lambda)U_\ell|_{\Omega^+}$, then*

$$U_\ell = \mathsf{N}_\ell^\lambda F_\ell - \begin{pmatrix} \mathrm{d}_{\ell-1}\ \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{S}_{\ell-1}^\lambda [\![\mathsf{n}_\ell U_\ell]\!] \\ \mathsf{S}_\ell^\lambda [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell]\!] \end{pmatrix} + \begin{pmatrix} \mathrm{Id}\ \delta_{\ell+1} \end{pmatrix} \begin{pmatrix} \mathsf{D}_\ell^\lambda [\![\mathsf{t}_{\ell-1}\delta_\ell U_\ell]\!] \\ \mathsf{D}_{\ell+1}^\lambda [\![\mathsf{t}_\ell U_\ell]\!] \end{pmatrix}.$$

Proof. Details of the argument are similar to those in the proof of (5.1), so we proceed faster through the derivation. From Green's second formula,

$$\begin{aligned}
\langle\!\langle(-\Delta_\ell + \lambda\mathrm{Id})U_\ell, V_\ell\rangle\!\rangle_\Omega &= \langle U_\ell, (-\Delta_\ell + \lambda\mathrm{Id})V_\ell\rangle_{\Omega^-} + \langle U_\ell, (-\Delta_\ell + \lambda\mathrm{Id})V_\ell\rangle_{\Omega^+} \\
&= \langle F_\ell, V_\ell\rangle_{\Omega^-} - \langle\!\langle \mathsf{T}^{\mathsf{t},-}_{\Delta_\ell} U_\ell, \mathsf{T}^{\mathsf{n},-}_{\Delta_\ell} V_\ell\rangle\!\rangle_\Gamma + \langle\!\langle \mathsf{T}^{\mathsf{n},-}_{\Delta_\ell} U_\ell, \mathsf{T}^{\mathsf{t},-}_{\Delta_\ell} V_\ell\rangle\!\rangle_\Gamma \\
&\quad + \langle F_\ell, V_\ell\rangle_{\Omega^+} + \langle\!\langle \mathsf{T}^{\mathsf{t},+}_{\Delta_\ell} U_\ell, \mathsf{T}^{\mathsf{n},+}_{\Delta_\ell} V_\ell\rangle\!\rangle_\Gamma - \langle\!\langle \mathsf{T}^{\mathsf{n},+}_{\Delta_\ell} U_\ell, \mathsf{T}^{\mathsf{t},+}_{\Delta_\ell} V_\ell\rangle\!\rangle_\Gamma \\
&= \langle F_\ell, V_\ell\rangle_{\mathcal{M}} + \langle\!\langle [\![\mathsf{T}^{\mathsf{t}}_{\Delta_\ell} U_\ell]\!], \mathsf{T}^{\mathsf{n}}_{\Delta_\ell} V_\ell\rangle\!\rangle_\Gamma - \langle\!\langle [\![\mathsf{T}^{\mathsf{n}}_{\Delta_\ell} U_\ell]\!], \mathsf{T}^{\mathsf{t}}_{\Delta_\ell} V_\ell\rangle\!\rangle_\Gamma
\end{aligned}$$

for all $V_\ell \in \mathcal{D}^\ell(\mathcal{M})$. Hence, in the sense of distributions, we have

$$U_\ell = \mathsf{N}_\ell^\lambda(-\Delta_\ell + \lambda\mathrm{Id})U_\ell = \mathsf{N}_\ell^\lambda F_\ell + \mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{n}}_{\Delta_\ell}\right)' [\![\mathsf{T}^{\mathsf{t}}_{\Delta_\ell} U_\ell]\!] - \mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{t}}_{\Delta_\ell}\right)' [\![\mathsf{T}^{\mathsf{n}}_{\Delta_\ell} U_\ell]\!].$$

Explicitly, we appeal to the integral representations provided in (5.3.8) to evaluate

$$\begin{aligned}
\mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{t}}_{\Delta_\ell}\right)' (h_\ell,\ h_{\ell-1})^\top &= \langle\!\langle h_{\ell-1}(y), \mathsf{t}_{\ell-1,y}\delta_{\ell,y}\,\mathcal{G}_\ell^\lambda(x,y)\rangle\!\rangle_\Gamma + \langle\!\langle h_\ell, \mathsf{t}_\ell \mathcal{G}_\ell^\lambda\rangle\!\rangle_\Gamma \\
&= \mathrm{d}_{\ell-1,x}\langle\!\langle h_{\ell-1}, \mathsf{t}_{\ell-1}\mathcal{G}_{\ell-1}^\lambda\rangle\!\rangle_\Gamma + \langle\!\langle h_\ell, \mathsf{t}_\ell \mathcal{G}_\ell^\lambda\rangle\!\rangle_\Gamma \\
&= \mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + \mathsf{S}_\ell^\lambda(h_\ell),
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{n}}_{\Delta_\ell}\right)' (g_\ell,\ g_{\ell-1})^\top &= \langle\!\langle g_{\ell-1}, \mathsf{n}_\ell \mathcal{G}_\ell^\lambda\rangle\!\rangle_\Gamma + \langle\!\langle g_\ell(y), \mathsf{n}_{\ell+1,y}\mathrm{d}_{\ell,y}\mathcal{G}_\ell^\lambda(x,y)\rangle\!\rangle_\Gamma \\
&= \langle\!\langle g_{\ell-1}, \mathsf{n}_\ell \mathcal{G}_\ell^\lambda\rangle\!\rangle_\Gamma + \delta_{\ell+1,x}\langle\!\langle g_\ell, \mathsf{n}_{\ell+1}\mathcal{G}_{\ell+1}^\lambda\rangle\!\rangle_\Gamma \\
&= \mathsf{D}_\ell^\lambda(g_{\ell-1}) + \delta_{\ell+1}\mathsf{D}_{\ell+1}^\lambda(g_\ell),
\end{aligned}$$

where we have used the identities stated in (5.3.4) to proceed.

We have arrived at the representation formula

$$U_\ell = \mathsf{N}_\ell^\lambda F_\ell - \mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda [\![\mathsf{n}_\ell U_\ell]\!] - \mathsf{S}_\ell^\lambda [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell]\!] + \mathsf{D}_\ell^\lambda [\![\mathsf{t}_{\ell-1}\delta_\ell U_\ell]\!] + \delta_{\ell+1}\mathsf{D}_{\ell+1}^\lambda [\![\mathsf{t}_\ell U_\ell]\!].$$

□

In the representation formula of Proposition 5.2, the boundary potentials

$$\mathsf{SL}[\Delta] : T^{\mathsf{n}}_\Delta(\Gamma) \to H\Lambda^\ell(\Delta, \Omega) \qquad \text{and} \qquad \mathsf{DL}[\Delta] : T^{\mathsf{t}}_\Delta(\Gamma) \to H\Lambda^\ell(\Delta, \Omega)$$

defined by

$$\begin{aligned}\mathsf{SL}[\Delta](h_{\ell-1},\ h_\ell)^\top &= \mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}(h_{\ell-1}) + \mathsf{S}^\lambda_\ell(h_\ell),\\ \mathsf{DL}[\Delta](g_{\ell-1},\ g_\ell)^\top &= \mathsf{D}^\lambda_\ell(g_{\ell-1}) + \delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1}(g_\ell),\end{aligned}$$

play the roles of single and double layer for the Hodge–Laplacian in strong form.

Once again, the jump relations for these potentials are obtained from those of the atomic potentials stated in Lemma 5.7. However, unlike for the Hodge–Dirac operator, for which the calculations were direct, we now need to appeal to Lemma 5.6. For example,

$$[\![\mathsf{T}^{\mathsf{n}}_\Delta]\!]\mathsf{SL}[\Delta](h_{\ell-1},\ h_\ell)^\top = \begin{pmatrix} [\![\mathsf{n}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}(h_{\ell-1}) + [\![\mathsf{n}_\ell]\!]\mathsf{S}^\lambda_\ell(h_\ell) \\ [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}(h_{\ell-1}) + [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\mathsf{S}^\lambda_\ell(h_\ell)\end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix}$$

simply follows from Lemma 5.7 because $\mathrm{d}^2 = 0$, but we must evaluate in

$$[\![\mathsf{T}^{\mathsf{t}}_\Delta]\!]\mathsf{SL}[\Delta](h_{\ell-1},\ h_\ell)^\top = \begin{pmatrix} [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}(h_{\ell-1}) + [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathsf{S}^\lambda_\ell(h_\ell) \\ [\![\mathsf{t}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}(h_{\ell-1}) + [\![\mathsf{t}_\ell]\!]\mathsf{S}^\lambda_\ell(h_\ell)\end{pmatrix}$$

the jump $[\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}$, which we haven't encountered before. To show that it vanishes, we use the fact that the atomic potential satisfies the equation in the interior and exterior domains to compute

$$[\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1} = -\mathrm{d}_{\ell-2}[\![\mathsf{t}_{\ell-1}\delta_{\ell-1}]\!]\mathsf{S}^\lambda_{\ell-1} - \lambda[\![\mathsf{t}_{\ell-1}]\!]\mathsf{S}^\lambda_{\ell-1} = 0.$$

The other jump relations are obtained similarly.

5.4.2.2 BIOs for the strong formulation of the Hodge–Laplacian

We want to find explicit expressions for the first-kind BIOs

$$\begin{aligned}\mathsf{V}[\Delta] &= \{\mathsf{T}^{\mathsf{t}}_\Delta\}\mathsf{SL}[\Delta] : H^{\mathsf{n}}_\Delta(\Gamma) \longrightarrow H^{\mathsf{t}}_\Delta(\Gamma),\\ \mathsf{W}[\Delta] &= \{\mathsf{T}^{\mathsf{n}}_\Delta\}\mathsf{DL}[\Delta] : H^{\mathsf{t}}_\Delta(\Gamma) \longrightarrow H^{\mathsf{n}}_\Delta(\Gamma).\end{aligned}$$

Once again, we work under the duality pairings on the left of the variational problems (5.4.5a) and (5.4.5b), which allows us to combine the "integration by parts trick" with the commutative relations of Lemma 5.5. Starting with $\mathsf{V}[\Delta]$, we evaluate using Lemma 5.5 (terms in green and blue), Lemma 5.6 (terms in blue) and (5.1.11) (terms in red) that

$$\mathsf{T}^{\mathsf{t}}_\Delta\mathsf{SL}[\Delta](h_{\ell-1},\ h_\ell)^\top = \begin{pmatrix} \mathsf{t}_{\ell-1}\delta_\ell\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell-1}\delta_\ell\mathsf{S}^\lambda_\ell h_\ell \\ \mathsf{t}_\ell\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}h_{\ell-1} + \mathsf{t}_\ell\mathsf{S}^\lambda_\ell h_\ell \end{pmatrix}$$

$$= \begin{pmatrix} -\mathrm{d}_{\ell-2}\mathsf{t}_{\ell-2}\mathsf{S}^{\lambda}_{\ell-2}\delta_{\ell-1}h_{\ell-1} - \lambda\mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}\delta_\ell h_\ell \\ \mathrm{d}_\ell\mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1} + \mathsf{t}_\ell\mathsf{S}^{\lambda}_\ell h_\ell \end{pmatrix},$$

from which we can further obtain

$$\begin{aligned} &\langle\!\langle \mathsf{T}^{\mathsf{t}}_{\Delta}\mathsf{SL}[\Delta](h_{\ell-1},\ h_\ell)^\top, (\overline{w}_{\ell-1},\ \overline{w}_\ell)^\top\rangle\!\rangle_\Gamma \\ &\quad = -\langle\!\langle \mathsf{t}_{\ell-2}\mathsf{S}^{\lambda}_{\ell-2}\delta_{\ell-1}h_{\ell-1}, \delta_{\ell-1}\overline{w}_{\ell-1}\rangle\!\rangle_\Gamma - \lambda\langle\!\langle \mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1}, \overline{w}_{\ell-1}\rangle\!\rangle_\Gamma \\ &\qquad + \langle\!\langle \mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}\delta_\ell h_\ell, \overline{w}_{\ell-1}\rangle\!\rangle_\Gamma + \langle\!\langle \mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1}, \delta_\ell\overline{w}_\ell\rangle\!\rangle_\Gamma \\ &\qquad + \langle\!\langle \mathsf{t}_\ell\mathsf{S}^{\lambda}_\ell h_\ell, \overline{w}_\ell\rangle\!\rangle_\Gamma \\ &\quad = -\left(\delta_{\ell-1}h_{\ell-1}, \delta_{\ell-1}\overline{w}_{\ell-1}\right)_{-\frac12,\lambda,\mathsf{t}} - \lambda\left(h_{\ell-1}, w_{\ell-1}\right)_{-\frac12,\lambda,\mathsf{t}} \\ &\qquad + \left(\delta_\ell h_\ell, w_{\ell-1}\right)_{-\frac12,\lambda,\mathsf{t}} + \left(h_{\ell-1}, \delta_\ell w_\ell\right)_{-\frac12,\lambda,\mathsf{t}} \\ &\qquad + \left(h_\ell, w_\ell\right)_{-\frac12,\lambda,\mathsf{t}} \end{aligned} \tag{5.4.14}$$

using integration by parts.

Similarly for $\mathsf{W}[\Delta]$, evaluating

$$\begin{aligned} &\mathsf{T}^{\mathsf{n}}_{\Delta_\ell}\mathsf{DL}^{\lambda}_\ell[\Delta](g_{\ell-1},\ g_\ell)^\top \\ &\quad = \begin{pmatrix} \mathsf{n}_\ell\mathsf{D}^{\lambda}_\ell g_{\ell-1} + \mathsf{n}_\ell\delta_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_\ell \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathsf{D}^{\lambda}_\ell g_{\ell-1} + \mathsf{n}_{\ell+1}\mathrm{d}_\ell\delta_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_\ell \end{pmatrix} \\ &\quad = \begin{pmatrix} \mathsf{n}_\ell\mathsf{D}^{\lambda}_\ell g_{\ell-1} - \delta_\ell\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_\ell \\ -\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}\mathrm{d}_{\ell-1}g_{\ell-1} - \mathsf{n}_{\ell+1}\delta_{\ell+2}\mathrm{d}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_\ell - \lambda\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_\ell \end{pmatrix} \\ &\quad = \begin{pmatrix} \mathsf{n}_\ell\mathsf{D}^{\lambda}_\ell g_{\ell-1} - \delta_\ell\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_\ell \\ -\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}\mathrm{d}_{\ell-1}g_{\ell-1} - \delta_{\ell+1}\mathsf{n}_{\ell+2}\mathsf{D}^{\lambda}_{\ell+2}\mathrm{d}_\ell g_\ell - \lambda\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_\ell \end{pmatrix} \end{aligned}$$

eventually leads to

$$\begin{aligned} &\langle\!\langle \mathsf{T}^{\mathsf{n}}_{\Delta_\ell}\mathsf{DL}^{\lambda}_\ell[\Delta](h_{\ell-1},\ h_\ell)^\top, (\overline{v}_{\ell-1},\ \overline{v}_\ell)^\top\rangle\!\rangle_\Gamma \\ &\quad = \left(g_{\ell-1}, v_{\ell-1}\right)_{-\frac12,\lambda,\mathsf{n}} - \left(g_\ell, \mathrm{d}_{\ell-1}v_{\ell-1}\right)_{-\frac12,\lambda,\mathsf{n}} \\ &\qquad - \left(\mathrm{d}_{\ell-1}g_{\ell-1}, v_\ell\right)_{-\frac12,\lambda,\mathsf{n}} - \left(\mathrm{d}_\ell g_\ell, \mathrm{d}_\ell v_\ell\right)_{-\frac12,\lambda,\mathsf{n}} \\ &\qquad - \lambda\left(g_\ell, v_\ell\right)_{-\frac12,\lambda,\mathsf{n}}. \end{aligned} \tag{5.4.15}$$

We conclude from (5.4.14) and (5.4.15) that the first-kind boundary integral operators $\mathsf{V}[\Delta]$ and $\mathsf{W}[\Delta]$ associated with the direct first-kind boundary integral equations (5.4.4a) and (5.4.4b) are zero-order perturbations of Hodge–Laplace operators in the trace de Rham complexes equipped with the non-local inner products introduced in Subsection 5.3.3. More precisely, in the same sense as (5.4.13a) and (5.4.13b), we have

$$\mathsf{V}[\Delta] = \begin{pmatrix} -\delta_{\ell-1}^* \delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \delta_\ell^* & \mathrm{Id} \end{pmatrix}, \tag{5.4.16a}$$

$$\mathsf{W}[\Delta] = \begin{pmatrix} \mathrm{Id} & -\mathrm{d}_\ell^* \\ -\mathrm{d}_{\ell-1} & -\mathrm{d}_\ell^* \mathrm{d}_\ell - \lambda \mathrm{Id} \end{pmatrix}. \tag{5.4.16b}$$

The results of the abstract theory of Hodge–Laplace operators in Hilbert complexes is therefore available to analyze the BIOS. When $\lambda > 0$, $\mathsf{V}[\Delta]$ and $\mathsf{W}[\Delta]$ are invertible. They are Fredholm operators of index zero when $\lambda = 0$, in which case the dimension of their finite dimensional kernel is the same as the Betti number of corresponding order on the boundary.

The expressions (5.4.16a) and (5.4.16b) should be compared with the self-adjoint operators (5.2.28a) and (5.2.28b), while the bilinear forms (5.4.14) and (5.4.15) should be compared with the bilinear forms (5.2.25) and (5.2.26).

5.4.2.3 Representation formula for the mixed-order Hodge–Laplacian

Similarly as for the Hodge–Dirac operator, we can build a fundamental solution for the mixed-order Hodge–Laplacian using the one available for the Hodge–Laplacian in strong formulation. Notice that

$$\begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \mathfrak{M} = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} -\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \delta_{\ell+1}\mathrm{d}_\ell + \lambda\, \mathrm{Id} \end{pmatrix}$$
$$= \begin{pmatrix} -\Delta_\ell + \lambda \mathrm{Id} & 0 \\ 0 & -\Delta_\ell + \lambda \mathrm{Id} \end{pmatrix}.$$

Moreover, integrating by parts after using the commutative relations (5.3.4) verifies that the commutation property

$$\begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix} = \begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix} \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix}$$

holds in the sense of distributions. We conclude that

$$\mathsf{N}[\mathfrak{M}] = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix}$$

is a fundamental solution for the Hodge–Laplacian $\mathfrak{M}$ in mixed formulation.

A similar fundamental solution can be designed for $\mathfrak{R}$, but since the following development is mirrored for the mixed formulation involving $\mathfrak{R}$, we will focus our attention on $\mathfrak{M}$.

Proposition 5.3 *If* $(U_{\ell-1}, U_\ell)^\top \in L^2\Lambda^{\ell-1}(\mathcal{M}) \times L^2\Lambda^\ell(\mathcal{M})$ *is compactly supported and there exists* $(F_{\ell-1}, F_\ell)^\top \in L^2\Lambda^{\ell-1}(\mathcal{M}) \times L^2\Lambda^\ell(\mathcal{M})$ *such that* $(F_{\ell-1}, F_\ell)^\top|_\Omega = \mathfrak{M}(U_{\ell-1}, U_\ell)^\top|_\Omega$ *and* $(F_{\ell-1}, F_\ell)^\top|_{\Omega^+} = \mathfrak{M}(U_{\ell-1}, U_\ell)^\top|_{\Omega^+}$*, then*

$$\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \left(\begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} F_{\ell-1} \\ \mathsf{N}^\lambda_\ell F_\ell \end{pmatrix} \right.$$
$$\left. - \begin{pmatrix} 0 \\ \mathsf{D}^\lambda_\ell [\![\mathsf{t}_{\ell-1} U_{\ell-1}]\!] + \delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1} [\![\mathsf{t}_\ell U_\ell]\!] \end{pmatrix} + \begin{pmatrix} \mathsf{S}^\lambda_{\ell-1} [\![\mathsf{n}_\ell U_\ell]\!] \\ \mathsf{S}^\lambda_\ell [\![\mathsf{n}_{\ell+1} \mathrm{d}_\ell U_\ell]\!] \end{pmatrix} \right)$$

Proof. As before, it follows from Green's second formula that

$$\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \left(\begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} F_{\ell-1} \\ \mathsf{N}^\lambda_\ell F_\ell \end{pmatrix} \right.$$
$$\left. - \begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix} (\mathsf{T}^{\mathsf{n}}_{\mathfrak{M}})' [\![\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}} (U_{\ell-1}, U_\ell)^\top]\!] \right.$$

$$+\begin{pmatrix} \mathsf{N}^{\lambda}_{\ell-1} & 0 \\ 0 & \mathsf{N}^{\lambda}_{\ell} \end{pmatrix} \left(\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\right)' [\![\mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}(U_{\ell-1}, U_\ell)^\top]\!]\Bigg).$$

Explicitly, we evaluate

$$\begin{pmatrix} \mathsf{N}^{\lambda}_{\ell-1} & 0 \\ 0 & \mathsf{N}^{\lambda}_{\ell} \end{pmatrix} \left(\mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}\right)' \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ \langle\!\langle g_{\ell-1}, \mathsf{n}_\ell \mathcal{G}^{\lambda}_{\ell} \rangle\!\rangle_{\Gamma} + \langle\!\langle g_\ell, \mathsf{n}_{\ell+1} \mathrm{d}_\ell \mathcal{G}^{\lambda}_{\ell} \rangle\!\rangle_{\Gamma} \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \mathsf{D}^{\lambda}_{\ell} g_{\ell-1} + \delta_{\ell+1} \mathsf{D}^{\lambda}_{\ell+1} g_\ell \end{pmatrix}$$

and

$$\begin{pmatrix} \mathsf{N}^{\lambda}_{\ell-1} & 0 \\ 0 & \mathsf{N}^{\lambda}_{\ell} \end{pmatrix} \left(\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\right)' \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} = \begin{pmatrix} \langle\!\langle h_{\ell-1}, \mathsf{t}_{\ell-1} \mathcal{G}^{\lambda}_{\ell-1} \rangle\!\rangle_{\Gamma} \\ \langle\!\langle h_\ell, \mathsf{t}_\ell \mathcal{G}^{\lambda}_{\ell} \rangle\!\rangle_{\Gamma} \end{pmatrix} = \begin{pmatrix} \mathsf{S}^{\lambda}_{\ell-1} h_{\ell-1} \\ \mathsf{S}^{\lambda}_{\ell} h_\ell \end{pmatrix}.$$

□

In the representation formula of Proposition 5.3, the potentials

$$\mathsf{SL}[\mathfrak{M}] : H^{\mathsf{n}}_{\mathfrak{M}}(\Gamma) \to \mathrm{dom}(\mathfrak{M}) \qquad \text{and} \qquad \mathsf{DL}[\mathfrak{M}] : H^{\mathsf{t}}_{\mathfrak{M}}(\Gamma) \to \mathrm{dom}(\mathfrak{M})$$

defined by

$$\mathsf{SL}[\mathfrak{M}] \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{S}^{\lambda}_{\ell-1} h_{\ell-1} \\ \mathsf{S}^{\lambda}_{\ell} h_\ell \end{pmatrix},$$
$$\mathsf{DL}^{\lambda}_{\ell}[\mathfrak{M}] \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 \\ \mathsf{D}^{\lambda}_{\ell} g_{\ell-1} + \delta_{\ell+1} \mathsf{D}^{\lambda}_{\ell+1} g_\ell \end{pmatrix}$$
$$= \begin{pmatrix} \delta_\ell \mathsf{D}^{\lambda}_{\ell} g_{\ell-1} \\ \mathsf{D}^{\lambda}_{\ell} g_{\ell-1} + \delta_{\ell+1} \mathsf{D}^{\lambda}_{\ell+1} g_\ell \end{pmatrix},$$

play the roles of single and double layer potentials for the Hodge–Laplace operator in mixed form $\mathfrak{M}$.

Jump relations for these boundary potentials are obtained using the same techniques as in the previous sections.

5.4.2.4 Boundary integral operators for the mixed Hodge–Laplacian

We now derive explicit expressions for the first-kind BIOs

$$\begin{aligned}\mathsf{V}[\mathfrak{M}] &= \{\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\}\mathsf{SL}[\mathfrak{M}] : H^{\mathsf{n}}_{\mathfrak{M}}(\Gamma) \longrightarrow H^{\mathsf{t}}_{\mathfrak{M}}(\Gamma),\\ \mathsf{W}[\mathfrak{M}] &= \{\mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}\}\mathsf{DL}[\mathfrak{M}] : H^{\mathsf{t}}_{\mathfrak{M}}(\Gamma) \longrightarrow H^{\mathsf{n}}_{\mathfrak{M}}(\Gamma).\end{aligned}$$

After evaluating

$$\begin{aligned}&\{\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\}\mathsf{SL}[\mathfrak{M}](h_{\ell-1},h_\ell)^\top\\ &\quad= \begin{pmatrix} -\mathrm{d}_{\ell-2}\mathsf{t}_{\ell-2}\mathsf{S}_{\ell-2}\delta_{\ell-1}h_{\ell-1} - \lambda\mathsf{t}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}\delta_\ell h_\ell \\ \mathrm{d}_{\ell-1}\mathsf{t}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}h_{\ell-1} + \mathsf{t}_\ell\mathsf{S}^\lambda_\ell h_\ell \end{pmatrix},\end{aligned}$$

we find that

$$\begin{aligned}&\langle\!\langle \mathsf{V}[\mathfrak{M}](h_{\ell-1},h_\ell)^\top, (\overline{w}_{\ell-1},\overline{w}_\ell)^\top\rangle\!\rangle_\Gamma\\ &\qquad= -\left(\delta_{\ell-1}h_{\ell-1},\delta_{\ell-1}\overline{w}_{\ell-1}\right)_{-\frac12,\lambda,\mathsf{t}} - \lambda\left(h_{\ell-1},w_{\ell-1}\right)_{-\frac12,\lambda,\mathsf{t}}\\ &\qquad\quad+\left(\delta_\ell h_\ell, w_{\ell-1}\right)_{-\frac12,\lambda,\mathsf{t}} + \left(h_{\ell-1},\delta_\ell w_\ell\right)_{-\frac12,\lambda,\mathsf{t}}\\ &\qquad\quad+\left(h_\ell,w_\ell\right)_{-\frac12,\lambda,\mathsf{t}}.\end{aligned} \tag{5.4.17}$$

Similarly, evaluating

$$\begin{aligned}\mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}\mathsf{DL}^\lambda_\ell[\mathfrak{M}](g_{\ell-1},g_\ell)^\top &= \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}\begin{pmatrix}\delta_\ell\mathsf{D}^\lambda_\ell g_{\ell-1}\\ \mathsf{D}^\lambda_\ell g_{\ell-1} + \delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1}g_\ell\end{pmatrix}\\ &= \begin{pmatrix}\mathsf{n}_\ell\mathsf{D}^\lambda_\ell g_{\ell-1} + \mathsf{n}_\ell\delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1}g_\ell\\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathsf{D}^\lambda_\ell g_{\ell-1} + \mathsf{n}_{\ell+1}\mathrm{d}_\ell\delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1}g_\ell\end{pmatrix}\\ &= \begin{pmatrix}\mathsf{n}_\ell\mathsf{D}^\lambda_\ell g_{\ell-1} - \delta_\ell\mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1}g_\ell\\ -\mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1}\mathrm{d}_{\ell-1}g_{\ell-1} + \mathsf{n}_{\ell+1}\delta_{\ell+2}\mathsf{D}^\lambda_{\ell+2}\mathrm{d}_\ell g_\ell - \lambda\mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1}g_\ell\end{pmatrix}\\ &= \begin{pmatrix}\mathsf{n}_\ell\mathsf{D}^\lambda_\ell g_{\ell-1} - \delta_\ell\mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1}g_\ell\\ -\mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1}\mathrm{d}_{\ell-1}g_{\ell-1} - \delta_{\ell+1}\mathsf{n}_{\ell+2}\mathsf{D}^\lambda_{\ell+2}\mathrm{d}_\ell g_\ell - \lambda\mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1}g_\ell\end{pmatrix}\end{aligned}$$

leads to

$$
\begin{aligned}
&\langle\!\langle \mathsf{W}[\mathfrak{M}](g_{\ell-1}, g_\ell)^\top, (\overline{v}_{\ell-1}, \overline{v}_\ell)^\top \rangle\!\rangle_\Gamma \\
&\qquad = (g_{\ell-1}, v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}} - (g_\ell, \mathrm{d}_{\ell-1} v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}} \\
&\qquad - (\mathrm{d}_{\ell-1} g_{\ell-1}, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} - (\mathrm{d}_\ell g_\ell, \mathrm{d}_\ell v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} \\
&\qquad - \lambda\,(g_\ell, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}}\,.
\end{aligned} \tag{5.4.18}
$$

We conclude from (5.4.17) and (5.4.18) that the first-kind boundary integral operators $\mathsf{V}[\mathfrak{M}]$ and $\mathsf{W}[\mathfrak{M}]$ associated with the direct first-kind boundary integral equations (5.4.4a) and (5.4.4b) are zero-order perturbations of Hodge–Laplace operators in the trace de Rham complexes equipped with the non-local inner products introduced in Subsection 5.3.3. More precisely, in the same sense as (5.4.13a), (5.4.13b), (5.4.16a) and (5.4.16b) we have

$$
\mathsf{V}[\mathfrak{M}] = \begin{pmatrix} -\delta^*_{\ell-1}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \delta^*_\ell & \mathrm{Id} \end{pmatrix}, \tag{5.4.19a}
$$

$$
\mathsf{W}[\mathfrak{M}] = \begin{pmatrix} \mathrm{Id} & -\mathrm{d}^*_\ell \\ -\mathrm{d}_{\ell-1} & -\mathrm{d}^*_\ell\mathrm{d}_\ell - \lambda\mathrm{Id} \end{pmatrix}. \tag{5.4.19b}
$$

The results of the abstract theory of Hodge–Laplace operators in Hilbert complexes is therefore available to analyze the BIOS. When $\lambda > 0$, $\mathsf{V}[\Delta]$ and $\mathsf{W}[\Delta]$ are invertible. They are Fredholm operators of index zero when $\lambda = 0$, in which case the dimension of their finite dimensional kernel is the same as the Betti number of corresponding order on the boundary.

Importantly, we have unveiled that

$$
\begin{aligned}
\mathsf{V}[\Delta] &= \mathsf{V}[\mathfrak{W}], \\
\mathsf{W}[\Delta] &= \mathsf{W}[\mathfrak{W}].
\end{aligned}
$$

Notice that in the mixed formulation, the tangential trace was relieved of differential operators, but these were account for in the factor

$$
\begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix}
$$

appearing in the fundamental solution, which should be compared with (5.4.19a) and the mixed Hodge–Laplacian $\mathfrak{N}$.

unknown	$\mathsf{T}^{\mathsf{n}}_{\Delta}, \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}, \mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}$
boundary data	$\mathsf{T}^{\mathsf{t}}_{\Delta}, \mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}, \mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}$
self-adjoint operator in Ω	$\mathfrak{R}_{\mathsf{t}} = \begin{pmatrix} \delta_\ell^* \delta_\ell + \lambda & \delta_{\ell+1} \\ \delta_{\ell+1}^* & -\mathrm{Id} \end{pmatrix}$
first-kind BIOs	$\mathsf{V}[\mathfrak{M}] = \mathsf{V}[\Delta] = \begin{pmatrix} -\delta_{\ell-1}^* \delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \delta_\ell^* & \mathrm{Id} \end{pmatrix}$
bilinear form on Γ	$\langle\!\langle \mathsf{V}[\mathfrak{M}](h_{\ell-1}, h_\ell)^\top, (\overline{w}_{\ell-1}, \overline{w}_\ell)^\top \rangle\!\rangle$ $= -(\delta_{\ell-1} h_{\ell-1}, \delta_{\ell-1} \overline{w}_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{t}} - \lambda (h_{\ell-1}, w_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{t}}$ $+ (\delta_\ell h_\ell, w_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{t}} + (h_{\ell-1}, \delta_\ell w_\ell)_{-\frac{1}{2},\lambda,\mathsf{t}}$ $+ (h_\ell, w_\ell)_{-\frac{1}{2},\lambda,\mathsf{t}}$
bilinear form in Ω	$\mathcal{B}_\delta \left((U_\ell, U_{\ell+1})^\top, (V_\ell, V_{\ell+1})^\top \right)$ $= (\delta_\ell U_\ell, \delta_\ell V_\ell)_\Omega + \lambda (U_\ell, V_\ell)_\Omega + (\delta_{\ell+1} U_{\ell+1}, V_\ell)_\Omega$ $+ (U_\ell, \delta_{\ell+1} V_{\ell+1})_\Omega - (U_{\ell+1}, V_{\ell+1})_\Omega$

Fig. 5.3 Table of relations for the BVPs (5.2.17a), (5.2.20a), (5.2.22a) and (5.2.27b).

5.5 Conclusion

We have seen in Subsection 5.4.2 that while the correspondence between the BVPs for the Hodge–Dirac operator in Ω and its first-kind BIOs on Γ displays a striking simplicity and elegance, the correspondence claimed in the introduction for the Hodge–Laplacian hid that the first-kind BIOs on Γ turn out to be Hodge–Laplace operators in *mixed* formulation. However, far from undermining the relevance of the connections revealed by boundary integral exterior calculus, this interesting complication sheds new light on the structure of the so-called “compound” traces for the Hodge–Laplacian, which appear naturally from integration by parts. In mixed formulation, the Hodge–Laplacian in Ω can be represented by an operator matrix acting on a product space. The associated BIOs are then also Hodge–Laplace operators in mixed formulation acting on products of trace spaces. But it is clear that such a correspondence cannot materialize for the second-order strong formulation of the Hodge–Laplacian: we cannot expect to obtain an Hodge–Laplacian on the boundary in strong formulation, because such an operator only acts on forms of a given order. In fact, if the BIOs associated with the strong formulation are to be Hodge–Laplace operators at all, then they *must* be in mixed formulation, because they operate on boundary data that lives in

unknown	$\mathsf{T}^{\mathsf{t}}_{\Delta}, \mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}, \mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}$
boundary data	$\mathsf{T}^{\mathsf{n}}_{\Delta}, \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}, \mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}$
self-adjoint operator in Ω	$\mathfrak{M}_{\mathsf{n}} = \begin{pmatrix} -\mathrm{Id} & \mathrm{d}_\ell^* \\ \mathrm{d}_{\ell-1} & \mathrm{d}_\ell^*\mathrm{d}_\ell + \lambda \end{pmatrix}$
first-kind BIO	$\mathsf{W}[\mathfrak{M}] = \mathsf{W}[\Delta] = \begin{pmatrix} \mathrm{Id} & -\mathrm{d}_\ell^* \\ -\mathrm{d}_{\ell-1} & -\mathrm{d}_\ell^*\mathrm{d}_\ell - \lambda\mathrm{Id} \end{pmatrix}$
bilinear form on Γ	$\langle\langle \mathsf{W}[\mathfrak{M}](g_{\ell-1}, g_\ell)^\top, (\overline{v}_{\ell-1}, \overline{v}_\ell)^\top \rangle\rangle_\Gamma$ $= (g_{\ell-1}, v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}} - (g_\ell, \mathrm{d}_{\ell-1} v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}}$ $- (\mathrm{d}_{\ell-1} g_{\ell-1}, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} - (\mathrm{d}_\ell g_\ell, \mathrm{d}_\ell v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}}$ $-\lambda\,(g_\ell, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}}$
bilinear form in Ω	$\mathcal{B}_{\mathrm{d}}\left((U_{\ell-1}, U_\ell)^\top, (V_{\ell-1}, V_\ell)^\top\right)$ $= (\mathrm{d}_\ell U_\ell, \mathrm{d}_\ell V_\ell)_\Omega + \lambda\,(U_\ell, V_\ell)_\Omega + (\mathrm{d}_{\ell-1} U_{\ell-1}, V_\ell)_\Omega$ $+ (U_\ell, \mathrm{d}_\ell V_{\ell-1})_\Omega - (U_{\ell-1}, V_{\ell-1})_\Omega$

Fig. 5.4 Table of relations for the BVPs (5.2.17b), (5.2.20b), (5.2.22b) and (5.2.27a).

product spaces. It turns out that first-kind BIOs for the Hodge–Laplacian in strong form and the first-kind BIOs for the Hodge–Laplacian in mixed form *are the same*! The difference in meaning of the solutions of the BIEs is accounted for on the right hand sides.

As a by product of our study, an exterior calculus of boundary potentials was described that eases calculations. Recognizing the structure of the BIOs as operators in trace de Rham complexes also enables us to harness a rich and powerful literature on Hilbert complexes for their analysis.

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Appendix A
Boundary Element Galerkin Discretization for Hodge–Dirac in 3D

A.1 Boundary element method

We will assume for simplicity that Ω is a Lipschitz polyhedron, so that no variational crime is committed when triangulating its surface in the creation of a piecewise flat, shape-regular mesh Γ_h. We follow [5, Sec. 4.1] in our choice of piecewise polynomial boundary element spaces. They are defined in Figure A.1.

$S^{1,0}(\Gamma_h)$	continuous piecewise linear scalar functions
$S^{0,-1}(\Gamma_h)$	piecewise constant scalar functions
$\boldsymbol{\mathcal{E}}^0(\Gamma_h)$	piecewise linear tangential surface vector fields with continuous tangential components across interelement edges

Fig. A.1 Choice of boundary element spaces for the Galerkin discretization of Appendix A.1.

These boundary element spaces are involved in the discrete surface de Rham complex

$$0 \longrightarrow S^{1,0}(\Gamma_h) \xrightarrow{\nabla_\Gamma} \boldsymbol{\mathcal{E}}^0(\Gamma_h) \xrightarrow{\mathrm{curl}_\Gamma} S^{0,-1}(\Gamma_h) \longrightarrow 0. \tag{A.1}$$

Following Section 3.8, we view

$$\mathcal{H}^h_{\mathsf{T}} := S^{1,0}(\Gamma_h) \times \boldsymbol{\mathcal{E}}^0(\Gamma_h) \times S^{0,-1}(\Gamma_h) \tag{A.2}$$

as a Hilbert space equipped with the non-local inner products (3.5.14a) and (3.5.14b). Accordingly, orthogonal complements indicated by $\perp$ are taken within $\mathcal{H}^h_{\mathsf{T}}$ throughout Appendix A.1. We write $\mathcal{V}^h_{\mathsf{T}}$ to speak of the same space of functions, but equipped with the graph inner products associated with the closed, *everywhere defined*, Fredholm-nilpotent linear operator

$$\mathbf{d}^h_{\mathsf{T}} := \mathbf{d}_{\mathsf{T}}\big|_{\mathcal{V}^h_{\mathsf{T}}} : \mathcal{V}^h_{\mathsf{T}} \to \mathcal{V}^h_{\mathsf{T}}. \tag{A.3}$$

Let $\mathfrak{K} = \ker \mathfrak{D}$. Similarly as in [12, Sec. 6], it can be verified using a collection of suitably chosen test functions, that the kernel of the bilinear form

$$\mathcal{B}_{\mathsf{T}}^{h}(\overrightarrow{\mathbf{b}}_h, \overrightarrow{\mathbf{d}}_h) := \mathcal{B}_{\mathsf{T}}\big|_{\mathcal{H}_{\mathsf{T}}^h \times \mathcal{H}_{\mathsf{T}}^h}(\overrightarrow{\mathbf{b}}_h, \overrightarrow{\mathbf{d}}_h) = (\mathbf{d}_{\mathsf{T}}^h \overrightarrow{\mathbf{b}}_h, \overrightarrow{\mathbf{d}}_h)_{-1/2,\mathsf{T}} + (\overrightarrow{\mathbf{b}}_h, \mathbf{d}_{\mathsf{T}}^h \overrightarrow{\mathbf{d}}_h)_{-1/2,\mathsf{T}} \tag{A.4}$$

is $\mathfrak{X}_h = \mathfrak{X}_0^h \times \mathfrak{X}_1^h \times \mathfrak{X}_2^h$, involving the discrete harmonic spaces

$$\mathfrak{X}_0^h = \mathfrak{K}_0, \qquad \mathfrak{X}_1^h = (\nabla_\Gamma S^{1,0}(\Gamma_h))^\perp \cap \ker \mathrm{curl}_\Gamma \qquad \text{and} \qquad \mathfrak{X}_2^h = (\mathrm{curl}_\Gamma \boldsymbol{\mathcal{E}}^0(\Gamma_h))^\perp \tag{A.5}$$

of the complex (A.1).

As pointed out in [5, Sec. 4], [1, Chap. 5] and [8, Sec. 3], notice that in general $\mathfrak{X}^h \not\subset \mathfrak{K}$, because in the discrete setting, orthogonality is taken over smaller spaces, i.e. $\nabla_\Gamma S^{1,0}(\Gamma_h) \subsetneq \nabla_\Gamma H^{\frac{1}{2}}(\Gamma)$ and $\mathrm{curl}_\Gamma \boldsymbol{\mathcal{E}}^0(\Gamma_h) \subsetneq \mathrm{curl}_\Gamma \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$. As a consequence, even though

$$S^{1,0}(\Gamma_h) \subset V^{\frac{1}{2}}(\Gamma), \qquad S^{0,-1}(\Gamma_h) \subset V^{-\frac{1}{2}}(\Gamma) \qquad \text{and} \qquad \boldsymbol{\mathcal{E}}^0(\Gamma_h) \subset \mathbf{V}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma), \tag{A.6}$$

the discrete mixed variational problem of finding a pair $(\mathbf{b}_h, \overrightarrow{\mathbf{p}}_h) \in \mathcal{V}_{\mathsf{T}}^h \times \mathfrak{X}_h$ satisfying

$$\begin{aligned} \mathcal{B}_{\mathsf{T}}^h\left(\overrightarrow{\mathbf{b}}_h, \overrightarrow{\mathbf{d}}_h\right) + \left(\overrightarrow{\mathbf{p}}_h, \overrightarrow{\mathbf{d}}_h\right)_{-1/2,\mathsf{T}} &= \ell_{\mathsf{R}}(\mathbf{d}_h), \qquad \forall\, \overrightarrow{\mathbf{d}}_h \in \mathcal{V}_{\mathsf{T}}^h, \\ \left(\overrightarrow{\mathbf{b}}_h, \overrightarrow{\mathbf{q}}_h\right)_{-1/2,\mathsf{T}} &= 0, \qquad \forall\, \overrightarrow{\mathbf{q}}_h \in \mathfrak{X}_h, \end{aligned} \tag{MBVR$_h$}$$

is *not* a conforming Galerkin discretization for (MBVR). Alternatively, while it is true that

$$\overrightarrow{\mathbf{b}}_h \in \mathcal{V}_{\mathsf{T}}^h : \qquad \mathcal{B}_{\mathsf{T}}(\overrightarrow{\mathbf{b}}_h, \overrightarrow{\mathbf{d}} h) = \ell(\overrightarrow{\mathbf{d}}_h), \qquad \forall\, \overrightarrow{\mathbf{d}}_h \in \mathcal{V}_{\mathsf{T}}^h, \tag{BVR$_h$}$$

is conforming for (BVR), it generally comes at the price of an inconsistent right-hand side.

Nevertheless, it is an important result of algebraic topology that the dimension of $\mathfrak{X}_h$ remains equal to the dimension of $\mathfrak{K}$ independently h. In particular, the dimensions $\mathfrak{X}_0^h = \beta_0$, $\mathfrak{X}_1^h = \beta_1$ and $\mathfrak{X}_2^h = \beta_2$ are the Betti numbers of the surface Γ, despite discretization [13, Chap. 4].

A.1.1 h-uniform stability

In this setting, a double inf-sup condition for the bilinear form on the left-hand side of (MBVR$_h$) is freely granted by the abstract theory [8, Thm. 2.4]: $\exists\, \gamma_h > 0$ such that for all non-vanishing $\mathbf{b}_h \in \mathcal{H}_{\mathsf{T}}^h$ and $\overrightarrow{\mathbf{p}}_h \in \mathfrak{X}_h$, one can find $\mathbf{d}_h \in \mathcal{V}_{\mathsf{T}}^h$ and $\overrightarrow{\mathbf{q}}_h \in \mathfrak{X}_h$ such that

$$\mathcal{B}_{\mathsf{T}}^h(\overrightarrow{\mathbf{b}}_h, \overrightarrow{\mathbf{d}}_h) + (\overrightarrow{\mathbf{p}}_h, \overrightarrow{\mathbf{d}}_h)_{\boldsymbol{\mathcal{V}}_{\mathsf{T}}} + (\overrightarrow{\mathbf{b}}_h, \overrightarrow{\mathbf{q}}_h)_{-1/2,\mathsf{T}} \geq \gamma_h(\|\overrightarrow{\mathbf{b}}_h\|_{\mathcal{H}_{\mathsf{T}}} + \|\overrightarrow{\mathbf{p}}_h\|_{\boldsymbol{\mathcal{V}}_{\mathsf{T}}}) + (\|\overrightarrow{\mathbf{d}}_h\|_{\mathcal{H}_{\mathsf{T}}} + \|\overrightarrow{\mathbf{q}}_h\|_{\boldsymbol{\mathcal{V}}_{\mathsf{T}}}), \tag{A.7}$$

where γ_h only depends on the parameter h of the mesh through the constant in the Poincaré inequality associated with the discrete complex (A.1), cf. [8, Sec. 3], [1, Chap. 5].

Therefore, it is sufficient for a proof of h-uniform stability, to show that there exists a constant $C > 0$, *depending only on Γ and the shape-regularity of Γ_h*, such that

$$\|\overrightarrow{\mathbf{b}}_h\|_{-1/2,\mathsf{T}} \leq C\, \|\mathbf{d}_{\mathsf{T}} \overrightarrow{\mathbf{b}}_h\|_{-1/2,\mathsf{T}}, \qquad \forall\, \overrightarrow{\mathbf{b}}_h \in \left(\ker \mathbf{d}_{\mathsf{T}}^h\right)^\perp, \tag{A.8}$$

because this guarantees that $\gamma_h = \gamma$ is independent of the discretization.

In finite element exterior calculus literature, a great deal of work is devoted to the design of bounded commuting projections [4, 6, 10, 11], which is a general approach to this problem. P. Leopardi and A. Stern rely on their existence in the discretization of their abstract theory for the Hodge-Dirac operator [8]. In the current setting, we are only aware of the existence of an interpolation operator $\Pi_h : \mathrm{dom}(\Pi_h) \to \boldsymbol{\mathcal{E}}^0(\Gamma_h)$ satisfying the commuting diagram property

$$\mathrm{curl}_\Gamma \circ \Pi_h \mathbf{u} = \mathsf{Q}_h \circ \mathrm{curl}_\Gamma \mathbf{u}, \qquad \mathbf{u} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \cap \mathrm{dom}(\Pi_h), \tag{A.9}$$

where Q_h is an $L^2(\Gamma)$-orthogonal projection onto a suitable space of piecewise polynomial discontinuous functions on the boundary [3, Sec.8], and it fails to be bounded $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$.

Fortunately, (A.7) also supplies a discrete inf-sup condition for the bilinear form $\mathcal{B}_\mathsf{T}^h$ of (BVR_h) in the orthogonal complement $\mathfrak{X}_h^\perp$, and this result can evidently also be obtained by approaching the problem from the point of view of Babuska-Brezzi theory.

This perspective is fruitful, because it points to the argumentation in the recent work of X. Claeys and R. Hiptmair, in which they use (A.9) to prove that

$$\|\boldsymbol{w}_h\|_{-\frac{1}{2},\mathsf{T}} \lesssim \|\mathrm{curl}_\Gamma \boldsymbol{w}_h\|_{-\frac{1}{2}}, \qquad \forall\, \mathbf{b}_h \in (\ker \mathrm{curl}_\Gamma\big|_{\boldsymbol{\mathcal{E}}^0(\Gamma_h)})^\perp, \tag{A.10}$$

with a constant that only depends on Γ and the shape-regularity of Γ_h. The inequality directly follows from [5, Lem. 11] and the injectivity of curl_Γ in the orthogonal complement of its kernel.

This proves (A.8) directly, because it simply remains to establish an analogous inequality for the surface gradient, and this is a particular case of the Poicaré inequality that holds in the infinite dimensional case! Recall from (A.5) that $\mathfrak{X}_0^h = \mathfrak{K}_0$ and the constant involved in (3.9.4) only depends on Γ.

Lemma A.1 *The constant $\gamma > 0$ entering (A.7) depends only on Γ and the shape-regularity of Γ_h.*

Proposition A.1 *Both the bilinear form associated with the left-hand side of (MBVR_h) and the restriction of $\mathcal{B}_\mathsf{T}^h$ to the orthogonal complement $\mathfrak{X}_h^\perp$ of its kernel are h-uniformly stable.*

A.1.2 Gap-based a priori Galerkin error estimate

According to Proposition A.1, the Galerkin discretization proposed in (MBVR_h) is h-uniformly well-posed, but as we've previously mentioned, it isn't conforming for (MBVR).

By the same token, we emphasize that since the consistency result of Proposition 3.6 does not carry to the discrete setting, existence of a solution to (BVR_h) is not guaranteed in general. Yes, the bilinear form $\mathcal{B}_\mathsf{T}^h$ is h-uniformly stable in the orthogonal complement of its kernel, and so the restriction of (BVR) to $\mathfrak{X}_h^\perp$ possesses a unique solution. But again, since $\mathfrak{X}_h^\perp \not\subset \mathfrak{K}^\perp$, the Galerkin formulation obtained upon restriction is not conforming either.

This calls for a *gap*-based a priori estimate based on a variant of the second Strang lemma, cf. [2, Chap. 3], [5, App. A], [1, Chap. 5]. Recall that the gap between two subspaces $V, W \subset X$ of a normed space X is defined by [7, Chap. 4]

$$\mathrm{gap}\,(V, W) := \sup_{v \in V\setminus\{0\}} \frac{1}{\|v\|_X} \inf_{w \in W} \|v - w\|_X. \tag{A.11}$$

Theorem A.1 *Let* $(\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{p}}) \in \mathcal{V}_{\mathsf{T}} \times \mathfrak{K}$ *be the unique solution to* (MBVR). *If* $(\mathbf{b}_h, \overrightarrow{\mathbf{p}}_h) \in \mathcal{V}^h_{\mathsf{T}} \times \mathfrak{X}_h$ *is the unique solution to* (MBVR$_h$), *then*

$$\|\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{b}}_h\|_{\mathcal{V}_{\mathsf{T}}} + \|\overrightarrow{\mathbf{p}} - \overrightarrow{\mathbf{p}}_h\|_{\mathcal{W}_{\mathsf{T}}} \lesssim \inf_{\overrightarrow{\mathbf{d}}_h \in \mathcal{V}^h_{\mathsf{T}}} \|\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{d}}_h\| + \inf_{\overrightarrow{\mathbf{q}}_h \in \mathfrak{X}_h} \|\overrightarrow{\mathbf{p}} - \overrightarrow{\mathbf{q}}_h\|_{\mathcal{W}_{\mathsf{T}}} + \mathrm{gap}(\mathfrak{X}_h, \mathfrak{K}), \quad \text{(A.12)}$$

with a constant depending only on Γ, *the shape regularity of* Γ_h *and the norm of* $\overrightarrow{\mathbf{b}}$. *Moreover, if* $\overrightarrow{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$ *satisfies* (CCR) *and* $\overrightarrow{\mathbf{b}}$ *is the unique solution to* (BVR) *such that* $\mathbf{b} \perp \mathfrak{K}$, *then the unique solution* $\overrightarrow{\mathbf{b}}_h$ *of the restriction of* (BVR$_h$) *to* $\mathfrak{X}_h^{\perp}$ *satisfies* (A.12) *with* $\overrightarrow{\mathbf{p}} = \overrightarrow{\mathbf{q}}_h = 0$.

The takeaway is that Galerkin solutions of (MBVR$_h$) and of the restriction of (BVR$_h$) to $\mathfrak{X}_h^{\perp}$ are quasi-optimal up to the discrepancy between the harmonic spaces $\mathfrak{K}$ and $\mathfrak{X}_h$. These both have the same dimension, c.f. [13, Chap. 4], [9, Sec. 4] and [5]—but exhibit a non-zero gap. The rate of convergence of $\mathrm{gap}(\mathfrak{X}_h, \mathfrak{K}) \to 0$ as $h \to 0$ depends on the regularity of the harmonic vector-fields in $\mathfrak{K}$ and the quality of local mesh refinement near corners [5, Rmk. 7].

A.2 Numerical experiments

In this section, we supply empirical evidence of the theory's correctness by performing the following numerical experiment: we compute the dimension of the Galerkin matrices for $\mathcal{B}^h_{\mathsf{T}}$ assembled over compact surfaces and verify that it is equal to the sum of their Betti numbers.

The numerical experiments are conducted using the open software GYPSILAB (version 0.61) developed at Institut Polytechnique de Paris by Matthieu Aussal, as well as standard MATLAB libraries.

A.2.1 Computing Betti numbers

Let $\{s^1_1, ..., s^1_n\}$, $\{\mathbf{e}^0_1, ..., \mathbf{e}^0_m\}$ and $\{s^0_1, ..., s^0_q\}$ be bases for $S^{1,0}(\Gamma_h)$, $S^{0,-1}(\Gamma_h)$ and $\boldsymbol{\mathcal{E}}^0(\Gamma_h)$, respectively. Then, $\dim(\mathcal{V}^h_{\mathsf{T}}) = N$ with $N = n + m + q$. We have produced the Galerkin matrix

$$B_{N\times N} = \begin{pmatrix} \mathbf{0}_{n\times n} & A_{n\times m} & 0 \\ (A^{\top})_{m\times n} & \mathbf{0}_{m\times m} & B_{m\times q} \\ \mathbf{0}_{q\times n} & (B^{\top})_{q\times m} & \mathbf{0}_{q\times q} \end{pmatrix} \quad \text{(A.1)}$$

of $\mathcal{B}^h_{\mathsf{T}}$, where

$$A_{ij} := (\mathbf{e}^0_i, \nabla_{\Gamma} s^1_j)_{-\frac{1}{2},\mathsf{R}} \qquad \text{and} \qquad B_{ij} := (s^0_i, \mathrm{curl}_{\Gamma} \mathbf{e}^0_j)_{-\frac{1}{2}}, \quad \text{(A.2)}$$

over different surfaces: the sphere, the torus and disjoint unions of those. An example of such a union mesh is provided in Figure A.2. The dimensions of the kernels were simply computed as the size of the null-space matrix provided by MATLAB's funtion `null`. In every cases, the correct dimension was obtained, independently of h: 2 for the sphere, 4 for the union of two spheres, 4 for the torus, 6 for the union of a sphere and a torus, etc.

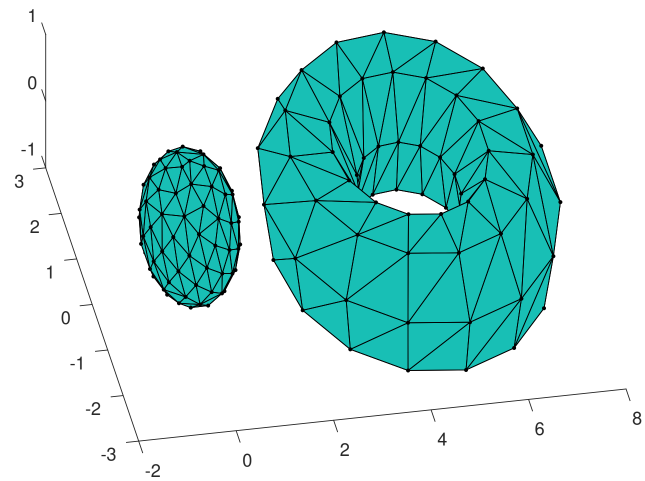

Fig. A.2 The discrete surface Γ_h is a coarse triangular meshing of the disjoint union of a sphere and a torus. The sum of the Betti numbers $\beta_0 + \beta_1 + \beta_2$ of the sphere is $1 + 0 + 1 = 2$, while it is $1 + 2 + 1 = 4$ for the torus. We assert numerically that $\ker \mathcal{B}_{\mathrm{T}}^h = 6$, the sum of the Betti numbers Γ_h.

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Appendix B
Generalized Lorenz Gauge in Vector Potential Electromagnetics

It was brought to my attention by Stefan Kurz that from a physical point of view, the gauge proposed in (1.1.1b) of Chapter 1 does not respect the units of the physical objects it claims to model. This appendix serves to resolve the issue. The change in the equations entering the transmission problem stated in (1.1.3) are inconsequential to the development of Chapter 1, whose setting is general enough to cover the corrected model. Nevertheless, it is meaningful to briefly comment on them for completeness.

In accordance with the SI base units, we write 's' for seconds, 'm' for meters, 'A' for ampere and 'V" for volts. We indicate in square brackets when we are calculating with units. Recall that

$$[\mathbf{U}] = \frac{\mathsf{Vs}}{\mathsf{m}}, \qquad [V] = \mathsf{V}, \qquad [\epsilon] = \frac{\mathsf{sA}}{\mathsf{mV}}, \qquad [\omega] = \frac{1}{s}, \qquad \text{and} \qquad [\mu] = \frac{\mathsf{sV}}{\mathsf{mA}}. \tag{B.1}$$

Following W.C. Chew [1, Sec. 2], we replace (1.1.1b) by the generalized Lorenz gauge

$$\operatorname{div}(\epsilon\mathbf{U}) + i\omega\chi V = 0, \tag{B.2}$$

in which the parameter χ has the appropriate units. From (B.1), we must have

$$[\chi] = [\operatorname{div}(\epsilon\mathbf{U})]/[\omega V] = \frac{s[\epsilon]\mathsf{A}[\mu]}{m\mathsf{V}} = [\epsilon]^2[\mu], \tag{B.3}$$

were we have used that $[\mathbf{U}] = \mathsf{A}[\mu]$.

Eliminating the scalar potential in (1.1.1a) using (B.2) leads to the equation

$$\mathbf{curl}\,(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{U}) - \epsilon(\mathbf{x})\,\nabla\chi^{-1}(\mathbf{x})\operatorname{div}(\epsilon(\mathbf{x})\mathbf{U}) - \omega^2\epsilon(\mathbf{x})\,\mathbf{U} = \mathbf{J}. \tag{B.4}$$

It is straightforward to verify that if (B.3) holds, then the units on the left-hand side of (B.4) respect the units $[\mathbf{J}] = \mathsf{m}^{-2}\mathsf{A}$ of the source current.

A natural choice is $\chi = \epsilon^2\mu$. Then, in the exterior domain, where the material properties are constant, (B.4) reduces to

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - \nabla\operatorname{div}\mathbf{U} - \mu_0\omega^2\mathbf{U} = 0. \tag{B.5}$$

Notice that (B.5) is even simpler than generalized Hoge–Helmholtz operator studied in Chapter 1 which involved η. In the interior domain, χ is absorbed in the bilinear form $\mathfrak{B}_\kappa$.

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