Multigraph limit of the dense configuration model and the preferential attachment graph

Journal Article

Author(s):
Rath, B.; Szakacs, L.

MULTIGRAPH LIMIT OF THE DENSE CONFIGURATION MODEL AND THE PREFERENTIAL ATTACHMENT GRAPH

B. RÁTH[1,*] and L. SZAKÁCS[2]

[1]ETH Zürich, Department of Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland
e-mail: rathb@math.ethz.ch

[2]Eötvös Loránd University, Institute of Mathematics, Pázmány Péter sétány 1/C, 1117
Budapest, Hungary
e-mail: szakacsl@cs.elte.hu

Abstract. The configuration model is the most natural model to generate a random multigraph with a given degree sequence. We use the notion of dense graph limits to characterize the special form of limit objects of convergent sequences of configuration models. We apply these results to calculate the limit object corresponding to the dense preferential attachment graph and the edge reconnecting model. Our main tools in doing so are (1) the relation between the theory of graph limits and that of partially exchangeable random arrays (2) an explicit construction of our random graphs that uses urn models.

1. Introduction

The notion of dense graph limits was introduced in [10] and has been further developed over the years, see [9] for a recent survey. Heuristically, the theory of dense graph limits gives a compact way to characterize the statistics of a randomly chosen small subgraph of a large dense graph. In [5] the graph limits of various sequences of random dense graphs were calculated and in this paper we proceed with the investigation of this topic.

Our objects of study are multigraphs rather than simple graphs, i.e. we allow parallel and loop edges: this choice makes the definition of the limit objects of convergent multigraph sequences (*multigraphons*) slightly more complicated than the limit objects of simple graph sequences (graphons), but on the other hand the multigraph models defined below are easier to study than the corresponding simple graph models.

The simplest way to generate a random multigraph with a prescribed degree sequence is called the *configuration model:* we draw $d(v)$ stubs (half-edges) at each vertex v and then we uniformly choose one from the set of possible matchings of these stubs. In this paper we call such random multigraphs *edge stationary* (for reasons that will become clear later) and in Theorem 1 we characterize the special form of limiting multigraphons that arise as the limit of random dense edge stationary multigraph sequences. Rougly speaking, our theorem states that the number of edges connecting the vertices v and w has Poisson distribution with parameter proportional to $d(v)d(w)$.

We also investigate two random graph models which have different definitions but turn out to have the same distribution:

• The *edge reconnecting model* is a random multigraph evolving in time. Denote the multigraph at time T by $\mathcal{G}_n(T)$, where $T = 0, 1, 2, \ldots$ and $n = \big|V\big(\mathcal{G}_n(T)\big)\big|$ is the number of vertices. We denote by $m = \big|E\big(\mathcal{G}_n(T)\big)\big|$ the number of edges (the number of vertices and edges does not change over time). Given the multigraph $\mathcal{G}_n(T)$ we get $\mathcal{G}_n(T+1)$ by uniformly choosing an edge in $E\big(\mathcal{G}_n(T)\big)$, choosing one of the endpoints of that edge with a coin flip and reconnecting the edge to a new endpoint which is chosen using the rule of linear preferential attachment: a vertex v is chosen with probability $\frac{d(v)+\kappa}{2m+n\kappa}$, where $d(v)$ is the degree of vertex v in $\mathcal{G}_n(T)$ and $\kappa \in (0, +\infty)$ is a fixed parameter of the edge reconnecting model. We look at the unique stationary distribution of this multigraph-valued Markov chain which is a random multigraph on n vertices and m edges.

• In Section 3.4 of [5] a random multigraph called *preferential attachment graph with n nodes and m edges* (briefly $\mathrm{PAG}(n, m)$) is defined. We slightly generalize the definition to obtain $\mathrm{PAG}_\kappa(n, m)$ where $\kappa \in (0, +\infty)$ is a fixed parameter: let $V = \{v_1, \ldots, v_n\}$ be a set of vertices. We create a sequence $v_1^*, \ldots, v_{2m}^*$ with elements from V by starting with the empty sequence and appending random elements of V one by one. If the current length of the sequence is L then we choose the next element v_{L+1}^* to be equal to $v \in V$ with probability $\frac{d(v)+\kappa}{L+n\kappa}$, where $d(v)$ is the multiplicity of v in the sequence $v_1^*, \ldots, v_L^*$. Now we create the random multigraph $\mathrm{PAG}_\kappa(n, m)$ on the vertex set V by adding the edges of form $\{v_{2k-1}^*, v_{2k}^*\}$ for each $k = 1, \ldots, m$.

Lemma 2.1 states that the above described two random multigraphs have the same distribution. In Theorem 2 we give the limiting multigraphon of this random multigraph when $n \to \infty$ and $m \approx \frac{1}{2}\rho n^2$, where $\rho \in (0, +\infty)$ is a fixed parameter of the model called the *edge density.* Roughly speaking, the limiting multigraphon can be described as follows: it is edge stationary, and the rescaled degrees of vertices have Gamma distribution with parameters depending on κ and ρ.

The precise statements of these theorems along with the necessary notations can be found in Section 2. We end the Introduction with mentioning a few related results:

The configuration model is a random multigraph, but if we condition it to have no multiple and loop edges, then the resulting random simple graph is uniformly distributed given its degree sequence. In [6] the description of the limiting graphon of such sequences of simple dense graphs (and a continuous version of the Erdős–Gallai characterization of degree sequences) is given.

In [13] we give a characterization of the time evolution of the edge reconnecting model, viewed through the prism of the theory of multigraphons: roughly speaking, if we start the edge reconnecting model from an arbitrary initial multigraph, then we have to run our process for $n^2 \ll T$ steps until $\mathcal{G}_n(T)$ becomes "edge stationary" and run it for $n^3 \ll T$ steps until $\mathcal{G}_n(T)$ becomes "stationary".

2. Notation and results

Denote $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $[n] := \{1, \dots, n\}$ and $[k..n] := \{k, \dots, n\}$. If H_1 and H_2 are arbitrary sets, denote by $f : H_1 \hookrightarrow H_2$ a generic injective function from H_1 to H_2. Denote by $\mathcal{M}$ the set of undirected multigraphs (graphs with multiple and loop edges) and by $\mathcal{M}_n$ the set of multigraphs on n vertices. Let $G \in \mathcal{M}_n$. The adjacency matrix of a labeling of the multigraph G with $[n]$ is denoted by $\big(B(i,j) \big)_{i,j=1}^n$, where $B(i,j) \in \mathbb{N}_0$ is the number of edges connecting the vertices labeled by i and j. $B(i,j) = B(j,i)$ since the graph is undirected and $B(i,i)$ is twice the number of loop edges at vertex i (thus $B(i,i)$ is an even number).

Denote the set of adjacency matrices of multigraphs on n nodes by $\mathcal{A}_n$, thus

$$\mathcal{A}_n = \big\{ B \in \mathbb{N}_0^{n \times n} : B^T = B, \ \forall i \in [n] \ 2 \mid B(i,i) \big\}.$$

The degree of the vertex labeled by i in G with adjacency matrix $B \in \mathcal{A}_n$ is defined by $d(B,i) := \sum_{j=1}^n B(i,j)$, thus $d(B,i)$ is the number of stubs at i

(loop edges count twice). Let

$$m = m(G) = m(B) = \frac{1}{2} \sum_{i,j=1}^{n} B(i,j) = \frac{1}{2} \sum_{i=1}^{n} d(B,i)$$

denote the number of edges. Denote by $\mathcal{A}_n^m$ the set of adjacency matrices on n vertices with m edges.

An unlabeled multigraph is the equivalence class of labeled multigraphs where two labeled graphs are equivalent if one can be obtained by relabeling the other. Thus $\mathcal{M}$ is the set of these equivalence classes of labeled multigraphs, which are also called isomorphism types.

Suppose $F \in \mathcal{M}_k$, $G \in \mathcal{M}_n$ and denote by $A \in \mathcal{A}_k$ and $B \in \mathcal{A}_n$ the adjacency matrices of F and G. If $g : \mathcal{M} \to \mathbb{R}$ then we say that g is a multigraph parameter. Let $g(A) := g(F)$. Conversely, if $g : \bigcup_{k=1}^{\infty} \mathcal{A}_k \to \mathbb{R}$ is constant on isomorphism classes, then g defines a multigraph parameter.

2.1. Multigraphons and multigraph convergence. We define the *induced homomorphism density* of F into G by

(1)
$$t_=(F,G) := t_=(A,B) := \frac{1}{n^k} \sum_{\varphi : [k] \to [n]} \mathbb{1}\big[\forall i,j \in [k] : A(i,j) = B\big(\varphi(i),\varphi(j)\big)\big].$$

The notion of convergence of simple graph sequences and several equivalent characterizations of *graphons* (limit objects of convergent graph sequences) were given in [10]. In [8] a natural generalization of the theory of dense graph limits to multigraphs is given (see also [12] for similar results in a more general setting). We say that a sequence of multigraphs $(G_n)_{n=1}^{\infty}$ is convergent if for every $k \in \mathbb{N}$ and every multigraph $F \in \mathcal{M}_k$ the limit $g(F) = \lim_{n \to \infty} t_=(F, G_n)$ exists, moreover we have $\sum_{A \in \mathcal{A}_k} g(A) = 1$. The limit object of a convergent multigraph sequence is a measurable function $W : [0,1] \times [0,1] \times \mathbb{N}_0 \to [0,1]$ satisfying

(2) $$W(x,y,k) \equiv W(y,x,k), \quad \sum_{k=0}^{\infty} W(x,y,k) \equiv 1, \quad W(x,x,2k+1) \equiv 0.$$

Such functions are called *multigraphons*. For every multigraphon W and multigraph $F \in \mathcal{M}_k$ with adjacency matrix $A \in \mathcal{A}_k$ define

(3) $$t_=(F,W) := t_=(A,W) := \int_{[0,1]^k} \prod_{i \leqq j \leqq k} W\big(x_i, x_j, A(i,j)\big)\, dx_1\, dx_2 \ldots dx_k.$$

We say that $G_n \to W$ if for every $k \in \mathbb{N}$ and every $F \in \mathcal{M}_k$ we have $\lim_{n\to\infty} t_=(F, G_n) = t_=(F, W)$. Theorem 1 of [8] states that if a sequence of multigraphs $(G_n)_{n=1}^\infty$ is convergent then $G_n \to W$ for some multigraphon W and conversely, every multigraphon W arises this way. The limiting multigraphon of a convergent sequence is not unique, but if we define the equivalence relation $W_1 \cong W_2$ by $\forall F \in \mathcal{M} : t_=(F, W_1) = t_=(F, W_2)$ then obviously $G_n \to W_1$, $G_n \to W_2$ implies $W_1 \cong W_2$. For other characterisations of the equivalence relation $\cong$ for graphons, see [4].

For a multigraphon W and $x \in [0, 1]$ define the *average degree* of W at x and the *edge density* of W by

$$(4) \qquad D(W, x) := \int_0^1 \sum_{k=0}^\infty k \cdot W(x, y, k) \, dy,$$

$$(5) \qquad \rho(W) := \int_0^1 \int_0^1 \sum_{k=0}^\infty k \cdot W(x, y, k) \, dy \, dx.$$

If $\rho(W) < +\infty$ then $D(W, x) < +\infty$ for Lebesgue-almost all x.

Given a multigraphon W define the *degree distribution function* of W by

$$(6) \qquad F_W(z) = \int_0^1 \mathbb{1}\big[D(W, x) \leqq z\big] \, dx, \qquad z \geqq 0.$$

Indeed, $F_W(\cdot)$ is a probability distribution function on $[0, \infty)$, i.e. it is non-negative, right continuous, increasing and satisfies $\lim_{z\to\infty} F_W(z) = 1$. It is easy to see that $\rho(W) = \int_0^\infty z \, dF_W(z)$. Denote

$$(7) \qquad F_W^{-1}(u) := \min\big\{z : F_W(z) \geqq u\big\}, \qquad u \in (0, 1).$$

2.2. Random multigraphs and random adjacency matrices. Denote a random element of $\mathcal{A}_n$ by $\mathbf{X}_n$. We may associate a random multigraph $\mathcal{G}_n$ to $\mathbf{X}_n$ by taking the isomorphism class of $\mathbf{X}_n$.

We say that a sequence of random multigraphs $(\mathcal{G}_n)_{n=1}^\infty$ converges in probability to a multigraphon W (or briefly write $\mathcal{G}_n \xrightarrow{\mathrm{p}} W$) if for every multigraph F we have $t_=(F, \mathcal{G}_n) \xrightarrow{\mathrm{p}} t_=(F, W)$, i.e.

$$(8) \qquad \forall F \in \mathcal{M} \ \forall \varepsilon > 0 : \lim_{n\to\infty} \mathbf{P}\big(\big|t_=(F, \mathcal{G}_n) - t_=(F, W)\big| > \varepsilon\big) = 0.$$

We say that $\mathbf{X}_n \xrightarrow{\mathrm{p}} W$ if $\mathcal{G}_n \xrightarrow{\mathrm{p}} W$ holds for the associated random multigraphs.

Note that the definitions of the edge reconnecting model and the PAG_κ (see Section 1) in fact naturally give rise to a random labeled graph, i.e.

a random element $\mathbf{X}_n$ of $\mathcal{A}_n$. The edge reconnecting Markov chain is easily seen to be irreducible and aperiodic on the state space $\mathcal{A}_n^m$, thus the stationary distribution is indeed unique.

We say that the distribution of $\mathbf{X}_n$ is *edge stationary* if the conditional distribution of $\mathbf{X}_n$ given the degree sequence $\big(d(\mathbf{X}_n, i) \big)_{i=1}^n$ is the same as that of the configuration model (see Section 1) with the same degree sequence.

Recall the formulas defining the Poisson and Gamma distributions:

$$
(9) \qquad\qquad \mathbf{p}(k, \lambda) := e^{-\lambda} \frac{\lambda^k}{k!}
$$

$$
(10) \qquad\qquad \mathbf{g}(x, \alpha, \beta) := x^{\alpha-1} \frac{\beta^\alpha e^{-\beta x}}{\Gamma(\alpha)} \mathbb{1}[x > 0].
$$

We say that a nonnegative integer-valued random variable X has Poisson distribution with parameter λ (or briefly denote $X \sim \mathrm{POI}(\lambda)$) if $\mathbf{P}(X = k) = \mathbf{p}(k, \lambda)$ for all $k \in \mathbb{N}$. We say that a nonnegative real-valued random variable Z has gamma distribution with parameters α and β (or briefly denote $Z \sim \mathrm{Gamma}(\alpha, \beta)$) if $\mathbf{P}(Z \leqq z) = \int_0^z \mathbf{g}(x, \alpha, \beta)\, dx$.

For a real-valued nonnegative random variable X define

$$
\mathbf{E}(X; m) := \mathbf{E}\big(X \cdot \mathbb{1}[X \geqq m] \big).
$$

2.3. Statements of main results. First we state our theorem characterizing the form of multigraph limits of edge stationary multigraph sequences:

THEOREM 1. *Let W denote a multigraphon with $\rho(W) < +\infty$. If $\mathbf{X}_n$ is an $\mathcal{A}_n$-valued edge stationary random variable for all $n \in \mathbb{N}$, $\mathbf{X}_n \xrightarrow{\mathrm{P}} W$ for some multigraphon W, and the sequence $(\mathbf{X}_n)_{n=1}^\infty$ satisfies*

$$
(11) \qquad\qquad \lim_{m \to \infty} \sup_{n \in \mathbb{N}} \frac{1}{\binom{n}{2}} \sum_{i < j \leqq n} \mathbf{E}\big(X_n(i, j); m \big) = 0
$$

$$
(12) \qquad\qquad \lim_{m \to \infty} \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^n \mathbf{E}\big(X_n(i, i); m \big) = 0,
$$

then the limiting multigraphon W can be rewritten in the form $\hat{W} \cong W$ where

$$
(13) \qquad \hat{W}(x,y,k) \overset{(7),(9)}{:=}
\begin{cases}
\mathbf{p}\left(k, \dfrac{F_W^{-1}(x)F_W^{-1}(y)}{\rho(W)}\right) & \text{if } x \neq y, \\[4mm]
\mathbb{1}[2 \mid k] \cdot \mathbf{p}\left(\dfrac{k}{2}, \dfrac{F_W^{-1}(x)F_W^{-1}(y)}{2\rho(W)}\right) & \text{if } x = y.
\end{cases}
$$

Now we state our results describing the multigraph limit of the $\mathrm{PAG}_\kappa(n,m)$ and the stationary distribution of the edge reconnecting model.

For an adjacency matrix $B \in \mathcal{A}_n$ denote by $m'(B) = \sum_{i=1}^n \sum_{j=1}^{i-1} B(i,j)$ the number of non-loop edges of the corresponding graph.

LEMMA 2.1. *The unique stationary distribution of the edge reconnecting model with linear preferential attachment parameter κ and state space $\mathcal{A}_n^m$ has the same distribution as $\mathrm{PAG}_\kappa(n,m)$. If $\mathbf{X}_n$ has this distribution then for all $B \in \mathcal{A}_n^m$*

(14)

$$
\mathbf{P}(\mathbf{X}_n = B) = \frac{\prod_{i=1}^n \prod_{j=1}^{d(B,i)}(\kappa + j - 1)}{\prod_{j=1}^{2m}(\kappa n + j - 1)} \frac{m!\, 2^{m'(B)}}{\left(\prod_{i=1}^n \prod_{j=1}^{i-1} B(i,j)!\right)\left(\prod_{i=1}^n \frac{B(i,i)}{2}!\right)}.
$$

At the end of Section 3.4 of [5] the following theorem is stated:

Let $\mathrm{SPAG}\,(n,m)$ denote the simple graph obtained from $\mathrm{PAG}\,(n,m)$ by deleting loops and keeping only one copy of the parallel edges. Then

$$
(15) \qquad \mathrm{SPAG}\left(n, \frac{n^2}{2} \cdot \left(\rho + o(1)\right)\right) \overset{\mathrm{p}}{\longrightarrow} W_s,
$$

$$
W_s(x,y) := 1 - \exp\left(-\rho \ln(x)\ln(y)\right),
$$

where (analogously to (8)) the symbol $\overset{\mathrm{p}}{\longrightarrow}$ denotes convergence in probability of a sequence of random simple graphs to a (simple) graphon.

It is easy to see that (15) is a corollary of the following theorem:

THEOREM 2. *Let us fix $\kappa, \rho \in (0, +\infty)$. If $\mathbf{X}_n$ is a random element of $\mathcal{A}_n^{m(n)}$ with distribution (14) for $n = 1, 2, \ldots,$ moreover the asymptotic edge density is*

$$
\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho,
$$

then $\mathbf{X}_n \overset{\mathrm{p}}{\longrightarrow} W$ *where*

$$
(16) \qquad W(x, y, k) = \begin{cases} \mathbf{p}\left(k, \dfrac{F^{-1}(x)F^{-1}(y)}{\rho}\right) & \text{if } x \neq y \\[4mm] \mathbb{1}[2|k] \cdot \mathbf{p}\left(\dfrac{k}{2}, \dfrac{F^{-1}(x)F^{-1}(y)}{2\rho}\right) & \text{if } x = y \end{cases}
$$

and F^{-1} *is the inverse function of* $F(z) = \int_0^z \mathbf{g}\big(y, \kappa, \frac{\kappa}{\rho}\big)\, dy$, *see* (10).

Note the similarity of the multigraphons appearing in (13) and (16): as we will see later, this is a consequence of the fact that the distribution of $\mathrm{PAG}_\kappa(n, m)$ is edge stationary.

The proofs of the above stated theorems rely on the following ideas:

• We relate our random graph models to urn models with multiple colors (e.g. the well-known Pólya urn model): the number of balls is $2m$ and they are colored with n possible colors. Each ball corresponds to a stub, each color corresponds to a labeled vertex and the edge set of the multigraph depends on the positions of balls in the urn.

• We make use of the underlying symmetries of the distributions of our random graphs by relating the theory of graph limits to the theory of partially exchangeable arrays of random variables, a connection first observed in [7].

The rest of this paper is organized as follows. In Section 3 we introduce the notion of random, vertex exchangeable, infinite adjacency matrices as well as W-random multigraphons and deduce some useful results relating the convergence of these objects to graph limits. In Section 4 we relate the notion of edge stationarity to the ball exchangeability of the corresponding urn models and prove the convergence results stated above.

3. Vertex exchangeable arrays

In this section we introduce random infinite arrays $\mathbf{X} = \big(X(i, j)\big)_{i,j=1}^{\infty}$ that arise as the adjacency matrices of random infinite labeled multigraphs and give probabilistic meaning to the homomorphism densities $t_=(F, W)$ by introducing W-random infinite multigraphs $\mathbf{X}_W$. We also introduce the notion of the average degree $D(\mathbf{X}, i)$ of a vertex i in an infinite, dense, vertex exchangeable multigraph.

In Subsection 3.1 we give a useful alternative characterisation of $\mathcal{G}_n \overset{\mathrm{p}}{\longrightarrow} W$ using exchangeable arrays and prove that under certain technical conditions the average degrees of $\mathcal{G}_n$ converge in distribution to the average degrees $D(\mathbf{X}_W, i)$ of the limiting W-random infinite array.

Let $\mathcal{A}_{\mathbb{N}}$ denote the set of adjacency matrices $\big(A(i,j)\big)_{i,j=1}^{\infty}$ of countable multigraphs:

$$\mathcal{A}_{\mathbb{N}} = \big\{ A \in \mathbb{N}_0^{\mathbb{N}\times\mathbb{N}} : \forall i,j \in \mathbb{N} \; A(i,j) \equiv A(j,i), \; \forall i \in \mathbb{N} \; 2 \mid A(i,i) \big\}.$$

Consider the probability space $(\mathcal{A}_{\mathbb{N}}, \mathcal{F}, \mathbf{P})$ where $\mathcal{F}$ is the coarsest sigma-algebra with respect to which $A(i,j)$ is measurable for all i,j and $\mathbf{P}$ is a probability measure on the measurable space $(\mathcal{A}_{\mathbb{N}}, \mathcal{F})$. We are going to denote the infinite random array with distribution $\mathbf{P}$ by $\mathbf{X} = \big(X(i,j)\big)_{i,j=1}^{\infty}$. We use the standard notation $\mathbf{X} \sim \mathbf{Y}$ if $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed (i.e. their distribution $\mathbf{P}$ is identical on $(\mathcal{A}_{\mathbb{N}}, \mathcal{F})$).

If $\mathbf{X}$ is a random element of $\mathcal{A}_{\mathbb{N}}$, let $\mathbf{X}^{[k]}$ be the random element of $\mathcal{A}_k$ defined by $\mathbf{X}^{[k]} := \big(X(i,j)\big)_{i,j=1}^{k}$.

DEFINITION 3.1 (W-random infinite multigraphons). Let $(U_i)_{i=1}^{\infty}$ be independent random variables uniformly distributed in $[0,1]$. Given a multigraphon W define the random countable adjacency matrix $\mathbf{X}_W = \big(X_W(i,j)\big)_{i,j=1}^{\infty}$ as follows: Given the background variables $(U_i)_{i=1}^{\infty}$ the random variables $\big(X_W(i,j)\big)_{i \leqq j \in \mathbb{N}}$ are conditionally independent and

$$\mathbf{P}\big(X_W(i,j) = m \mid (U_i)_{i=1}^{\infty}\big) = W(U_i, U_j, m),$$

that is if $A \in \mathcal{A}_k$ then we have

$$(17) \qquad \mathbf{P}\big(\mathbf{X}_W^{[k]} = A \mid (U_i)_{i=1}^{\infty}\big) := \prod_{i \leqq j \leqq k} W\big(U_i, U_j, A(i,j)\big).$$

In plain words: if $i \neq j$ and $U_i = x$, $U_j = y$ then the number of multiple edges between the vertices labeled by i and j in $\mathbf{X}_W$ has distribution $\big(W(x,y,k)\big)_{k=1}^{\infty}$ and the number of loop edges at vertex i has distribution $\big(W(x,x,2k)\big)_{k=1}^{\infty}$ (these are indeed proper probability distributions by (2)).

For every multigraphon W and multigraph $F \in \mathcal{M}_k$ with adjacency matrix $A \in \mathcal{A}_k$ we have

$$(18) \qquad t_{=}(F,W) \overset{(3),\,(17)}{=} \mathbf{P}\big(\mathbf{X}_W^{[k]} = A\big).$$

Recalling (4) and (5) we have

$$(19) \qquad D(W,x) = \mathbf{E}(X_W(1,2) \mid U_1 = x), \qquad \rho(W) = \mathbf{E}\big(X_W(1,2)\big).$$

If $\rho(W) < +\infty$ then $D(W,U_1) < +\infty$ almost surely.

We say that a random infinite array $\mathbf{X} = \big(X(i,j)\big)_{i,j=1}^{\infty}$ is *vertex exchangeable* if

$$(20) \qquad \big(X\big(\tau(i), \tau(j)\big)\big)_{i,j=1}^{\infty} \sim \big(X(i,j)\big)_{i,j=1}^{\infty}$$

for all finitely supported permutations $\tau : \mathbb{N} \to \mathbb{N}$. We call $\mathbf{X} = \big(X(i,j) \big)_{i,j=1}^{\infty}$ *dissociated* if for all $m, n \in \mathbb{N}$ the $\mathcal{A}_n$-valued random variable $\big(X(i,j) \big)_{i,j=1}^{n}$ is independent of the $\mathcal{A}_m$-valued random variable $\big(X(i,j) \big)_{i,j=n+1}^{n+m}$.

In our case an infinite exchangeable array can be thought of as the adjacency matrix of a random multigraph with vertex set $\mathbb{N}$: the adjacency matrix of this random infinite multigraph is vertex exchangeable if and only if the distribution of the random graph is invariant under the relabeling of the vertices and dissociated if and only if subgraphs spanned by disjoint vertex sets are independent.

It follows from Definition 3.1 that $\mathbf{X}_W$ is vertex exchangeable and dissociated and by Aldous' representation theorem (see Theorem 1.4, Proposition 3.3 and Theorem 5.1 in [1]), the converse holds: a random element $\mathbf{X}$ of $\mathcal{A}_{\mathbb{N}}$ is vertex exchangeable and dissociated if and only if $\mathbf{X} \sim \mathbf{X}_W$ for some multigraphon W. Although the notion of the W-random graph (see Definition 3.1) is already present in [10], the connection of Aldous' representation theorem with the theory of graph limits was first observed in [7]. See also Theorem 3.1, Theorem 3.2, Proposition 3.4 of [11]. For a self-contained proof of this representation theorem for multigraphons, see Theorem 1 and Theorem 2 in [8].

For a vertex exchangeable infinite array $\mathbf{X}$ satisfying $\mathbf{E}\big(X(1,2) \big) < +\infty$ define the average degree of $\mathbf{X}$ at vertex i by

$$(21) \qquad D(\mathbf{X}, i) := \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} X(i,j).$$

The sum $\frac{1}{n} \sum_{j=1}^{n} X(i,j)$ indeed almost surely converges to a random variable as $n \to \infty$ by de Finetti's theorem (see Section 2.1 of [2]) and the conditional strong law of large numbers. From (4), Definition 3.1 and (19) we get

$$(22) \qquad D(\mathbf{X}_W, i) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} X_W(i,j) \overset{\text{a.s.}}{=} D(W, U_i).$$

3.1. Convergence of exchangeable arrays. In this subsection we state and prove two lemmas: in Lemma 3.1 we relate convergence of dense random multigraphs to convergence of the probability measures of the corresponding random arrays, and in Lemma 3.2 we give sufficient conditions under which convergence of dense random multigraphs imply convergence of the degree distribution of these graphs.

We say that a sequence of random infinite arrays $(\mathbf{X}_n)_{n=1}^{\infty}$ *converges in distribution* to a random infinite array $\mathbf{X}$ (or briefly denote $\mathbf{X}_n \overset{\text{d}}{\longrightarrow} \mathbf{X}$) if

$\mathbf{X}_n^{[k]}$ converges in distribution to $\mathbf{X}^{[k]}$ for all $k \in \mathbb{N}$, i.e.

$$(23) \qquad \forall\, k \in \mathbb{N}, \ A \in \mathcal{A}_n : \ \lim_{n \to \infty} \mathbf{P}\big(A = \mathbf{X}_n^{[k]}\big) = \mathbf{P}\big(A = \mathbf{X}^{[k]}\big).$$

If $\mathbf{X}_n$ is vertex exchangeable for all n, then $\mathbf{X}$ is also vertex exchangeable.

Let $\mathbf{X}_n$ denote a random element of $\mathcal{A}_n$. We say that the distribution $\mathbf{X}_n$ is *vertex exchangeable* if for all permutations $\tau : [n] \to [n]$ and $B \in \mathcal{A}_n$

$$(24) \qquad \mathbf{P}\big(\forall\, i,j \in [n] : B(i,j) = X_n(i,j)\big)$$

$$= \mathbf{P}\big(\forall\, i,j \in [n] : B(i,j) = X_n\big(\tau(i),\tau(j)\big)\big),$$

that is $\big(X(i,j)\big)_{i,j=1}^{n} \sim \big(X\big(\tau(i),\tau(j)\big)\big)_{i,j=1}^{n}$ holds.

If $\mathbf{X}_n$ is a random element of $\mathcal{A}_n$ then $\mathbf{X}_n^{[k]} = \big(X_n(i,j)\big)_{i,j=1}^{k}$ is well-defined for $k \leqq n$, thus we might define $\mathbf{X}_n \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}$ (where $\mathbf{X}$ is a random element of $\mathcal{A}_{\mathbb{N}}$) by (23). It is easy to show that if $\mathbf{X}_n$ is vertex exchangeable for each $n \in \mathbb{N}$ then $\mathbf{X}$ inherits this property.

Also note that by (18) we have $\mathbf{X}_n \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}_W$ if and only if for all $k \in \mathbb{N}$ and for all $A \in \mathcal{A}_k$ we have $\lim_{n \to \infty} \mathbf{P}\big(\mathbf{X}_n^{[k]} = A\big) = t_=(A,W)$.

LEMMA 3.1. *Let* $\mathbf{X}_n = \big(X_n(i,j)\big)_{i,j=1}^{n}$ *be a random, vertex exchangeable element of* $\mathcal{A}_n$ *for all* $n \in \mathbb{N}$. *The following statements are equivalent:*

(a) $\mathbf{X}_n \overset{\mathrm{p}}{\longrightarrow} W$, *that is* $\forall\, k \ \forall\, A \in \mathcal{A}_k : t_=(A, \mathbf{X}_n) \overset{\mathrm{p}}{\longrightarrow} t_=(A,W)$,

(b) $\mathbf{X}_n \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}_W$, *that is* $\forall\, k \ \forall\, A \in \mathcal{A}_k : \lim_{n \to \infty} \mathbf{P}\big(\mathbf{X}_n^{[k]} = A\big) = t_=(A,W)$.

PROOF. We are going to use the fact $\lim_{n \to \infty} \frac{n \cdot (n-1) \dots (n-k+1)}{n^k} = 1$ many times in this proof.

We first prove (a) $\implies$ (b):

(25)

$$\lim_{n \to \infty} \mathbf{P}\big(\mathbf{X}_n^{[k]} = A\big) \overset{(24)}{=} \lim_{n \to \infty} \frac{(n-k)!}{n!} \sum_{\varphi : [k] \hookrightarrow [n]} \mathbf{P}\Big(\big(X_n\big(\varphi(i),\varphi(j)\big)\big)_{i,j=1}^{k} = A\Big)$$

$$= \lim_{n \to \infty} \frac{1}{n^k} \sum_{\varphi : [k] \to [n]} \mathbf{P}\Big(\big(X_n\big(\varphi(i),\varphi(j)\big)\big)_{i,j=1}^{k} = A\Big)$$

$$\overset{(1)}{=} \lim_{n \to \infty} \mathbf{E}\big(t_=(A, \mathbf{X}_n)\big) \overset{(a)}{=} t_=(A,W).$$

Now we prove (b) $\implies$ (a): The idea of this proof comes from Lemma 2.4 of [10].

From (b) we get $\mathbf{E}\big(t_=(A, \mathbf{X}_n)\big) \to t_=(A, W)$ for all A by the argument used in (25). In order to have $t_=(A, \mathbf{X}_n) \xrightarrow{\mathrm{p}} t_=(A, W)$ we only need to show

$$\lim_{n\to\infty} \mathbf{D}^2\big(t_=(A, \mathbf{X}_n)\big) = \lim_{n\to\infty} \mathbf{E}\big(t_=(A, \mathbf{X}_n)^2\big) - t_=(A, W)^2 = 0.$$

This follows by the computation

$$\lim_{n\to\infty} \mathbf{E}\big(t_=(A, \mathbf{X}_n)^2\big)$$

$$\stackrel{(1)}{=} \lim_{n\to\infty} \frac{1}{n^{2k}} \sum_{\varphi\colon [2k]\to[n]} \mathbf{P}\Big(A = \big(X_n(\varphi(i), \varphi(j))\big)_{i,j=1}^k,$$

$$A = \big(X_n(\varphi(i), \varphi(j))\big)_{i,j=k+1}^{2k}\Big)$$

$$= \lim_{n\to\infty} \frac{(n-2k)!}{n!} \sum_{\varphi\colon [2k]\hookrightarrow[n]} \mathbf{P}\Big(A = \big(X_n(\varphi(i), \varphi(j))\big)_{i,j=1}^k,$$

$$A = \big(X_n(\varphi(i), \varphi(j))\big)_{i,j=k+1}^{2k}\Big)$$

$$\stackrel{(24)}{=} \lim_{n\to\infty} \mathbf{P}\big(A = \big(X_n(i,j)\big)_{i,j=1}^k,\ A = \big(X_n(i,j)\big)_{i,j=k+1}^{2k}\big)$$

$$\stackrel{(b)}{=} \mathbf{P}\big(A = \big(X_W(i,j)\big)_{i,j=1}^k,\ A = \big(X_W(i,j)\big)_{i,j=k+1}^{2k}\big) \stackrel{(*)}{=} t_=(A, W)^2.$$

In the equation $(*)$ we used the fact that $\mathbf{X}_W$ is dissociated and (18). $\square$

Recall that for a real-valued nonnegative random variable X we denote $\mathbf{E}(X; m) := \mathbf{E}\big(X \cdot \mathbb{1}[X \geqq m]\big)$. A sequence of real-valued nonnegative random variables $(X_n)_{n=1}^\infty$ is uniformly integrable (see Ch. 13 of [15]) if

$$\lim_{m\to\infty} \max_n \mathbf{E}(X_n; m) = 0.$$

Now we state and prove a lemma which gives sufficient conditions under which $\tilde{\mathbf{X}}_n \xrightarrow{\mathrm{d}} \mathbf{X}$ implies $\frac{1}{n}d\big(\tilde{\mathbf{X}}_n, i\big) \xrightarrow{\mathrm{d}} D(\mathbf{X}, i)$. Note that some extra conditions are indeed needed, because it might happen that very few pairs of vertices of $\tilde{\mathbf{X}}_n$ with a huge number of parallel edges between them remain invisible if we only sample small subgraphs of $\tilde{\mathbf{X}}_n$, but still cause a sigificant distortion in the distribution of the degrees of vertices in $\tilde{\mathbf{X}}_n$. This phenomenon is related to the fact that weak convergence of a sequence of random variables $X_n \xrightarrow{\mathrm{d}} X$ does not necessarily imply the convergence of

the means of X_n to that of X: the uniform integrability of $(X_n)_{n=1}^\infty$ is a sufficient (and essentially necessary) condition that rules out pathological behavior.

LEMMA 3.2. (i) *If* $(\mathbf{X}_n)_{n=1}^\infty$ *is a sequence of infinite vertex exchangeable arrays, the sequence* $\big(X_n(1,2)\big)_{n=1}^\infty$ *is uniformly integrable and* $\mathbf{X}_n \xrightarrow{\mathrm{d}} \mathbf{X}$, *then for all* $k \in \mathbb{N}$ *we have*

$$(26) \qquad \big(\mathbf{X}_n^{[k]}, \big(D(\mathbf{X}_n, i)\big)_{i=1}^k\big) \xrightarrow{\mathrm{d}} \big(\mathbf{X}^{[k]}, \big(D(\mathbf{X}, i)\big)_{i=1}^k\big).$$

(ii) *If* $\tilde{\mathbf{X}}_n$ *is a random, vertex exchangeable element of* $\mathcal{A}_n$ *for each* $n \in \mathbb{N}$, $\tilde{\mathbf{X}}_n \xrightarrow{\mathrm{d}} \mathbf{X}$ *holds for some infinite vertex exchangeable array* $\mathbf{X}$ *and the sequences* $\big(\tilde{X}_n(1,1)\big)_{n=1}^\infty$ *and* $\big(\tilde{X}_n(1,2)\big)_{n=1}^\infty$ *are uniformly integrable, then for all* $k \in \mathbb{N}$

$$(27) \qquad \left(\tilde{\mathbf{X}}_n^{[k]}, \left(\frac{1}{n}d\big(\tilde{\mathbf{X}}_n, i\big)\right)_{i=1}^k\right) \xrightarrow{\mathrm{d}} \big(\mathbf{X}^{[k]}, \big(D(\mathbf{X}, i)\big)_{i=1}^k\big).$$

PROOF. (i) We first prove that (26) holds if we further assume $\mathbf{P}\big(X_n(i,j) \leqq m\big) \equiv 1$ for some $m \in \mathbb{N}$. By the method of moments we only need to show that for all $\mu_{i,j} \in \mathbb{N}_0$, $1 \leqq i \leqq j \leqq k$ and $\nu_i \in \mathbb{N}_0$, $1 \leqq i \leqq k$ we have

$$(28) \qquad \lim_{n \to \infty} \mathbf{E}\bigg(\prod_{i \leqq j \leqq k} X_n(i,j)^{\mu_{i,j}} \cdot \prod_{i=1}^k D(\mathbf{X}_n, i)^{\nu_i}\bigg)$$

$$= \mathbf{E}\bigg(\prod_{i \leqq j \leqq k} X(i,j)^{\mu_{i,j}} \cdot \prod_{i=1}^k D(\mathbf{X}, i)^{\nu_i}\bigg).$$

For every $i \in [k]$ choose $J(i) \subseteqq \mathbb{N}$ such that for all i we have $\big|J(i)\big| = \nu_i$ and $J(i) \cap [k] = \emptyset$, moreover for all $i \neq i'$ we have $J(i) \cap J(i') = \emptyset$. In order to prove (28) we first show that if $\mathbf{P}\big(X(i,j) \leqq m\big) \equiv 1$ for some $m \in \mathbb{N}$ then

$$(29) \qquad \mathbf{E}\bigg(\prod_{i \leqq j \leqq k} X(i,j)^{\mu_{i,j}} \cdot \prod_{i=1}^k D(\mathbf{X}, i)^{\nu_i}\bigg)$$

$$= \mathbf{E}\bigg(\prod_{i \leqq j \leqq k} X(i,j)^{\mu_{i,j}} \cdot \prod_{i=1}^k \prod_{j \in J(i)} X(i,j)\bigg).$$

Denote $\nu = \sum_{i=1}^{k} \nu_i$ and $\underline{\nu} := \{(i, l) : i \in [k], \quad l \in [\nu_i]\}$ and $\mathbf{X}^{[k],\mu} := \prod_{i \leq j \leq k} X(i, j)^{\mu_{i,j}}$. Using (21) and dominated convergence, the left-hand side of (29) is equal to

$$\lim_{n \to \infty} \mathbf{E}\left(\mathbf{X}^{[k],\mu} \prod_{i=1}^{k} \left(\frac{1}{n} \sum_{j=1}^{n} X(i, j) \right)^{\nu_i} \right)$$

$$= \lim_{n \to \infty} \frac{1}{n^\nu} \sum_{j: \underline{\nu} \to [n]} \mathbf{E}\left(\mathbf{X}^{[k],\mu} \prod_{i=1}^{k} \prod_{l=1}^{\nu_i} X\big(i, j(i, l)\big) \right)$$

$$= \lim_{n \to \infty} \frac{1}{n^\nu} \sum_{j: \underline{\nu} \hookrightarrow [k..n]} \mathbf{E}\left(\mathbf{X}^{[k],\mu} \prod_{i=1}^{k} \prod_{l=1}^{\nu_i} X\big(i, j(i, l)\big) \right)$$

$$\overset{(20)}{=} \lim_{n \to \infty} \frac{1}{n^\nu} \sum_{j: \underline{\nu} \hookrightarrow [k..n]} \mathbf{E}\left(\mathbf{X}^{[k],\mu} \prod_{i=1}^{k} \prod_{j' \in J(i)} X(i, j') \right).$$

Now the right-hand side of the above equation is easily shown to be equal to the right-hand side of (29).

Having established (29), our assumptions $\mathbf{X}_n \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}$ and $\mathbf{P}\big(X_n(i, j) \leq m\big) \equiv 1$ imply the equality (28): if we rewrite both the left and the right hand side of (28) in the form corresponding to the right hand side of (29), then we only need to check that the expected value of a polynomial function of finitely many values of $\mathbf{X}_n$ converge, and this follows from the definition of $\mathbf{X}_n \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}$ (for details on $\overset{\mathrm{d}}{\longrightarrow}$, see [3]).

Having established (26) under the condition $\mathbf{P}\big(X_n(i, j) \leq m\big) \equiv 1$ we now prove (26) without assuming this condition. If we define the truncated array $X^m(i, j) := \min\{X(i, j), m\}$, then for each $m \in \mathbb{N}$ we have $\mathbf{X}_n^m \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}^m$ from which

$$(30) \qquad \left(\mathbf{X}_n^{m,[k]}, \big(D(\mathbf{X}_n^m, i)\big)_{i=1}^{k} \right) \overset{\mathrm{d}}{\longrightarrow} \left(\mathbf{X}^{m,[k]}, \big(D(\mathbf{X}^m, i)\big)_{i=1}^{k} \right)$$

follows by the previous argument. By uniform integrability for every $\varepsilon > 0$ there is an m such that for all n we have

$$(31) \quad \mathbf{E}\big(D(\mathbf{X}_n, i) - D(\mathbf{X}_n^m, i)\big) \overset{(19)}{=} \mathbf{E}\big(X(1, 2) - \min\{X(1, 2), m\}\big) \leq \varepsilon.$$

It follows from Fatou's lemma that $\mathbf{E}\big(D(\mathbf{X}, i) - D(\mathbf{X}^m, i)\big) \leq \varepsilon$ also holds.

In order to prove (26) we only need to check

$$\lim_{n \to \infty} \mathbf{E}\Big(f\big(\mathbf{X}_n^{[k]}, \big(D(\mathbf{X}_n, i) \big)_{i=1}^{k} \big) \Big) = \mathbf{E}\Big(f\big(\mathbf{X}^{[k]}, \big(D(\mathbf{X}, i) \big)_{i=1}^{k} \big) \Big)$$

for any bounded and continuous $f : \mathcal{A}_k \times [0, +\infty)^k \to \mathbb{R}$. This can be easily proved using (30), (31) and the $\varepsilon/3$-argument (see Ch. 1.5 of [14]). This finishes the proof of (i).

(ii) For each $n \in \mathbb{N}$ let $(\eta_i^n)_{i=1}^{\infty}$ be i.i.d. and uniformly distributed on $[n]$. Define the infinite array $\mathbf{X}_n$ by $X_n(i,j) := \hat{X}_n(\eta_i^n, \eta_j^n)$. Now $\mathbf{X}_n$ is vertex exchangeable and using the vertex exchangeability of $\tilde{\mathbf{X}}_n$ we get

$$\mathbf{E}\big(X_n(1,2); m \big) = \left(1 - \frac{1}{n} \right) \mathbf{E}\big(\tilde{X}_n(1,2); m \big) + \frac{1}{n} \mathbf{E}\big(\tilde{X}_n(1,1); m \big),$$

and if we combine this with the assumptions of (ii) we get that $\big(X_n(1,2) \big)_{n=1}^{\infty}$ is uniformly integrable.

Note that by (21) and the law of large numbers $D(\mathbf{X}_n, i) = \frac{1}{n} d\big(\tilde{\mathbf{X}}_n, \eta_i^n \big)$. Using the vertex exchangeability of $\tilde{\mathbf{X}}_n$ the following two $(\mathcal{A}_k, \mathbb{R}_+^k)$-valued random variables have the same distribution:

• $\big(\mathbf{X}_n^{[k]}, \big(D(\mathbf{X}_n, i) \big)_{i=1}^{k} \big)$ under the condition $\big| \{\eta_1^n, \ldots, \eta_k^n\} \big| = k$,
• $\big(\tilde{\mathbf{X}}_n^{[k]}, \big(\frac{1}{n} d\big(\tilde{\mathbf{X}}_n, i \big) \big)_{i=1}^{k} \big)$.

Let us call this fact $(*)$.

$\mathbf{X}_n \xrightarrow{\mathrm{d}} \mathbf{X}$ easily follows from $\tilde{\mathbf{X}}_n \xrightarrow{\mathrm{d}} \mathbf{X}$, $(*)$ and

$$\tag{32} \lim_{n \to \infty} \mathbf{P}\big(\big| \{\eta_1^n, \ldots, \eta_k^n\} \big| = k \big) = 1,$$

so we can apply (i) to obtain (26). Now using $(*)$ and (32) again we obtain (27). □

4. Random urn configurations and edge stationarity

In this section we define a way of constructing random adjacency matrices using random urn configurations (the basic idea comes from Section 3.4 of [5]). This construction relates edge stationary random adjacency matrices to ball exchangeable urn models and gives an easy proof of Lemma 2.1 using the fact that the distribution of the $\mathrm{PAG}_\kappa(n, m)$ and that of the stationary state of the edge reconnecting model both arise from the Pólya urn model via our construction.

In Subsection 4.1 we prove Theorem 1 and Theorem 2 using this machinery.

Let $n, m \in \mathbb{N}$. A random urn configuration with $2m$ balls of n different colors is a probability distribution on $[n]^{[2m]}$, that is a random function $\Psi : [2m] \to [n]$. If $l \in [2m]$ we say that the l'th ball has color $\Psi(l)$. Let $d(\Psi, i) := \sum_{l=1}^{2m} \mathbb{1}\big[\Psi(l) = i\big]$ for $i \in [n]$ denote the multiplicity of color i in Ψ.

We say that a random urn configuration Ψ is *ball exchangeable* if for all permutations $\tau : [2m] \to [2m]$ we have

$$\big(\Psi(l)\big)_{l=1}^{2m} \sim \big(\Psi\big(\tau(l)\big)\big)_{l=1}^{2m}.$$

Ψ is ball exchangeable if and only if the following property holds: conditioned on the value of the *type vector* $\big(d(\Psi, i)\big)_{i=1}^{n}$, the distribution of Ψ is uniform on the elements of $[n]^{[2m]}$ with this particular type vector, more precisely if $\psi \in [n]^{[2m]}$ then

$$\mathbf{P}(\Psi = \psi) = \frac{\mathbf{P}\big(\big(d(\Psi, i)\big)_{i=1}^{n} = \big(d(\psi, i)\big)_{i=1}^{n}\big)}{\Big(\frac{(2m)!}{\prod_{i=1}^{n} d(\psi, i)!}\Big)}.$$

We say that Ψ is *color exchangeable* if for all permutations $\tau : [n] \to [n]$ we have

$$\big(\Psi(l)\big)_{l=1}^{2m} \sim \big(\tau\big(\Psi(l)\big)\big)_{l=1}^{2m}.$$

To a random urn configuration Ψ we assign a random element $\mathbf{X}$ of $\mathcal{A}_n^m$ by defining

(33)

$$X(i,j) := \sum_{e=1}^{m} \mathbb{1}\big[\Psi(2e-1) = i, \Psi(2e) = j\big] + \mathbb{1}\big[\Psi(2e-1) = j, \Psi(2e) = i\big]$$

for all $i, j \in [n]$. In plain words: the colors of the balls correspond to the labels of the vertices and if for any $1 \leqq e \leqq m$ we see a ball of color i at position $2e - 1$ and a ball of color j at position $2e$ then we draw an edge between the vertices i and j in the corresponding labeled multigraph (and if $i = j$ then we draw a loop edge at vertex i).

With the definition (33) we have $\mathbf{P}\big(d(\mathbf{X}, i) = d(\Psi, i)\big) = 1$. It is easy to see that all probability measures on $\mathcal{A}_n^m$ arise this way.

If Ψ is color exchangeable then $\mathbf{X}$ is vertex exchangeable. All vertex exchangeable probability measures on $\mathcal{A}_n^m$ arise this way.

If Ψ is ball exchangeable then for all $B \in \mathcal{A}_n^m$ we have

(34)

$$\mathbf{P}(\mathbf{X} = B) = \frac{\mathbf{P}\big(\big(d(\mathbf{X}, i)\big)_{i=1}^{n} = \big(d(B, i)\big)_{i=1}^{n}\big)}{\Big(\frac{(2m)!}{\prod_{i=1}^{n} d(B, i)!}\Big)} \frac{m! 2^{m'(B)}}{\big(\prod_{i<j} B(i,j)!\big)\big(\prod_{i=1}^{n} \frac{B(i,i)}{2}!\big)}$$

where $m'(B)$ denotes the number of non-loop edges. The first term in (34) is $\mathbf{P}(\Psi = \psi)$ for some ψ that produces B via (33), the second term is the number of elements of $[n]^{[2m]}$ that produce B via (33).

Recalling the definition of the configuration model (see Section 1) we can see that if we generate $\mathbf{X}$ using (33) from a ball exchangeable urn configuration Ψ with a given degree sequence $(d_i)_{i=1}^n$ then we in fact uniformly choose one from the the set of possible matchings of the stubs where the vertex $i \in [n]$ has d_i stubs. Thus (34) holds for a random element $\mathbf{X}$ of $\mathcal{A}_n^m$ if and only if the distribution of $\mathbf{X}$ is edge stationary. It is easy to see that all edge-stationary probability distributions on $\mathcal{A}_n^m$ arise from ball exchangeable distributions on $[n]^{[2m]}$ via (33).

Now we define two different dynamics on random urn configurations:

• *The Pólya urn model:* Fix $\kappa \in (0, +\infty)$. Let Ψ_L be a random element of $[n]^{[L]}$. Given Ψ_L we generate a random element of $[n]^{[L+1]}$ which we denote by Ψ_{L+1} in the following way: let $\Psi_{L+1}(l) := \Psi_L(l)$ for all $l \in [L]$ and

$$\forall\, i \in [n] : \mathbf{P}\big(\Psi_{L+1}(L+1) = i \mid \Psi_L\big) = \frac{d(\Psi_L, i) + \kappa}{L + n\kappa}.$$

• *The ball replacement model:* Fix $\kappa \in (0, +\infty)$. Let Ψ_T be a random element of $[n]^{[2m]}$. Given Ψ_T we generate a random element of $[n]^{[2m]}$ which we denote by Ψ_{T+1} in the following way: let ξ_T denote a uniformly chosen element of $[2m]$. For all $l \in [2m] \setminus \xi_T$ let $\Psi_{T+1}(l) := \Psi_T(l)$ and

$$(35) \qquad \forall\, i \in [n] : \mathbf{P}\big(\Psi_{T+1}(\xi_T) = i \mid \Psi_T, \xi_T\big) = \frac{d(\Psi_T, i) + \kappa}{2m + n\kappa}.$$

It is well-known that if we start with an empty urn Ψ_0 and repeatedly apply the Pólya urn scheme to get Ψ_L for $L = 1, 2, \ldots, 2m$, then the distribution of Ψ_{2m} is of the following form:

$$(36) \qquad \forall\, \psi \in [n]^{[2m]} : \mathbf{P}(\Psi_{2m} = \psi) = \frac{\prod_{i=1}^n \prod_{j=1}^{d(\psi,i)}(\kappa + j - 1)}{\prod_{j=1}^{2m}(\kappa n + j - 1)}.$$

Thus the distribution of Ψ_{2m} is ball and color exchangeable. The $\mathrm{PAG}_\kappa(n, m)$ (defined in Section 1) is in fact the random multigraph obtained as the image of the random urn configuration (36) under the mapping (33).

The ball replacement model is an $[n]^{[2m]}$-valued Markov chain, which is irreducible and aperiodic with unique stationary distribution (36): if we delete the ξ_T'th ball from Ψ_{2m}, then by ball exchangeability the distribution of the resulting $[n]^{[2m-1]}$-valued random variable is the same as deleting the $2m$'th ball: Pólya-Ψ_{2m-1}. Thus replacing the removed ξ_T'th ball with a new

one according to (35) we get a $[n]^{[2m]}$-valued random variable with Pólya-Ψ_{2m} distribution again by ball exchangeability.

Now consider the ball replacement Markov chain Ψ_T, $T = 0, 1, \dots$ with Ψ_0 being an arbitrary $[n]^{[2m]}$-valued random variable. If we use the mapping (33) to create $\mathbf{X}(T)$ from Ψ_T, then it is easily seen that the resulting $\mathcal{A}_n^m$-valued stochastic process $\mathbf{X}(T)$, $T = 0, 1, \dots$ evolves according to the rules of the edge reconnecting Markov chain defined in Section 1. Some consequences of this fact:

• If the distribution of Ψ_0 is ball exchangeable then Ψ_T is also ball exchangeable for all T, thus if $\mathbf{X}(0)$ is edge stationary then so is $\mathbf{X}(T)$ for all T (hence the name "edge stationarity").

• The distribution (36) is stationary for the ball replacement model, thus the image of this distribution under the mapping (33) is the unique stationary distribution of the edge reconnecting model. Lemma 2.1 follows from (36) and (34).

4.1. Limits of edge stationary multigraph sequences. The key result of this subsection is Lemma 4.1 which can be roughly summarized as follows: in a large dense edge stationary random multigraph the number of edges connecting the vertices v and w has Poisson distribution with parameter proportional to $d(v)d(w)$. Given Lemma 4.1 the proof of Theorem 1 is straightforward and the proof of Theorem 2 reduces to a limit theorem which states that the rescaled number of balls with color $1, 2, \dots, k$ in the Pólya urn model converge in distribution to i.i.d. random variables with Gamma distribution.

LEMMA 4.1. *Let $F : [0, +\infty) \to [0, 1]$ denote the cumulative distribution function of a nonnegative random variable Z. Let $F^{-1}(u) := \min \big\{ x : F(x) \geqq u \big\}$. Let $Z_1, Z_2, \dots$ be i.i.d. random variables with $Z_i \sim Z \sim F^{-1}(U_i)$ (where U_i are uniform on $[0, 1]$).*

If $\mathbf{X}_n$ is an $\mathcal{A}_n$-valued random variable for $n = 1, 2, \dots$, moreover the distribution of $\mathbf{X}_n$ is vertex exchangeable and edge stationary, and

$$(37) \qquad \frac{2m(\mathbf{X}_n)}{n^2} \xrightarrow{\text{p}} \rho, \qquad n \to \infty,$$

where $0 < \rho < +\infty$ is positive real parameter, moreover for all $k \in \mathbb{N}$ we have

$$(38) \qquad \left(\frac{1}{n} d(\mathbf{X}_n, i) \right)_{i=1}^{k} \xrightarrow{\text{d}} (Z_i)_{i=1}^{k}, \qquad n \to \infty$$

then $\mathbf{X}_n \xrightarrow{\mathrm{p}} W$ *where*

$$(39) \qquad W(x,y,k) = \begin{cases} \mathbf{p}\left(k, \dfrac{F^{-1}(x)F^{-1}(y)}{\rho}\right) & \text{if } x \neq y, \\[3ex] \mathbb{1}[2|k] \cdot \mathbf{p}\left(\dfrac{k}{2}, \dfrac{F^{-1}(x)F^{-1}(y)}{2\rho}\right) & \text{if } x = y. \end{cases}$$

PROOF. The infinite random array $\mathbf{X}_W$ (see Definition 3.1) can be alternatively defined in the following way: Let $\big(X_W(i,j)\big)_{i \leqq j}$ be conditionally independent given $(Z_i)_{i \in \mathbb{N}}$ with conditional distribution $X_W(i,j) \sim \mathrm{POI}\big(\frac{Z_i Z_j}{\rho}\big)$ if $i < j$ and $\frac{X_W(i,i)}{2} \sim \mathrm{POI}\big(\frac{Z_i Z_j}{2\rho}\big)$.

If $A \in \mathcal{A}_k$ let A^* denote the following modified matrix: $A^*(i,j) := A(i,j)$ if $i \neq j$ but $A^*(i,i) := \frac{A(i,i)}{2}$. Thus $A^*(i,i)$ is the number of loop edges at vertex i.

Let $m_{[k]} := \frac{1}{2} \sum_{i,j} A(i,j)$. Define

$$\mathbf{p}\big(A, (z_i)_{i=1}^k, \rho\big)$$

$$:= \exp\left(\frac{-1}{2\rho}\left(\sum_{i=1}^k z_i\right)^2\right) \cdot \prod_{i \leqq j} \frac{1}{A^*(i,j)!} \cdot \prod_{i=1}^k (z_i)^{d(A,i)} \cdot \rho^{-m_{[k]}} \cdot 2^{-\sum_{i=1}^k A^*(i,i)}.$$

By (17) and (39) we have

$$(40) \qquad \mathbf{P}\big(\mathbf{X}_W^{[k]} = A \mid (Z_i)_{i=1}^k\big) = \prod_{i=1}^k \prod_{j=i}^k \mathbf{p}\left(A^*(i,j), \frac{Z_i \cdot Z_j}{\rho \cdot \big(1 + \mathbb{1}[i=j]\big)}\right)$$

$$= \mathbf{p}\big(A, (Z_i)_{i=1}^k, \rho\big).$$

By Lemma 3.1 we only need to show that we have

$$(41) \qquad \forall k \in \mathbb{N},\ \forall A \in \mathcal{A}_k : \ \lim_{n \to \infty} \mathbf{P}\big(\mathbf{X}_n^{[k]} = A\big) = \mathbf{P}\big(\mathbf{X}_W^{[k]} = A\big)$$

in order to prove $\mathbf{X}_n \xrightarrow{\mathrm{p}} W$.

Let $(d_i)_{i=1}^n$ denote an arbitrary degree sequence with $m = \frac{1}{2} \sum_{i=1}^n d_i$ and denote by

$$(42) \qquad\qquad\qquad z_i := \frac{d_i}{n}, \quad \rho_n := \frac{2m}{n^2}.$$

Fix $\varepsilon > 0$ and $A \in \mathcal{A}_k$. We are going to prove that if

$$\text{(43)} \qquad \varepsilon \leqq \rho_n \leqq \varepsilon^{-1}, \qquad \forall\, i \in [k] :\ z_i \leqq \varepsilon^{-1}$$

then

$$\text{(44)} \qquad \mathbf{P}\left(\mathbf{X}_n^{[k]} = A \,\middle|\, \big(d(\mathbf{X}_n, i)\big)_{i=1}^k = (d_i)_{i=1}^k,\ \frac{2m(\mathbf{X}_n)}{n^2} = \rho_n \right)$$

$$\text{(45)} \qquad = \mathbf{p}\big(A, (z_i)_{i=1}^k, \rho_n \big) + \operatorname{Err}(n, A, \varepsilon)$$

with $\lim_{n\to\infty} \operatorname{Err}(n, A, \varepsilon) = 0$. We adapt the convention that the value of $\operatorname{Err}(n, A, \varepsilon)$ might change from line to line.

First we assume that $(44) = (45)$ holds under the condition (43), and deduce (41) from it. Define the events B_n^ε and B^ε by

$$B_n^\varepsilon := \left\{ \varepsilon \leqq \frac{2m(\mathbf{X}_n)}{n^2} \leqq \varepsilon^{-1},\ \forall\, i \in [k] :\ \frac{1}{n} d(\mathbf{X}_n, i) \leqq \varepsilon^{-1} \right\}$$

$$B^\varepsilon := \left\{ \varepsilon \leqq \rho \leqq \varepsilon^{-1},\ \forall\, i \in [k] :\ Z_i \leqq \varepsilon^{-1} \right\}.$$

Using the Portmanteau theorem and (37), (38) we obtain the inequality $\limsup_{n\to\infty} \mathbf{P}(B_n^\varepsilon) \leqq \mathbf{P}(B^\varepsilon)$.

(46)

$$\left| \mathbf{P}\big(\mathbf{X}_n^{[k]} = A \big) - \mathbf{P}\big(\mathbf{X}_W^{[k]} = A \big) \right| \overset{(40)}{=} \left| \mathbf{P}\big(\mathbf{X}_n^{[k]} = A \big) - \mathbf{E}\big(\mathbf{p}\big(A, (Z_i)_{i=1}^k, \rho \big) \big) \right|$$

(47)

$$\leqq \left| \mathbf{E}\left(\mathbf{p}\left(A, \left(\frac{1}{n} d(\mathbf{X}_n, i) \right)_{i=1}^k, \frac{2m(\mathbf{X}_n)}{n^2} \right); B_n \right) - \mathbf{E}\big(\mathbf{p}\big(A, (Z_i)_{i=1}^k, \rho \big) \big) \right|$$

$$\text{(48)} \qquad + \operatorname{Err}(\varepsilon, A, n) + \big(1 - \mathbf{P}(B_n^\varepsilon) \big).$$

By (37), (38), $\lim_{n\to\infty} \operatorname{Err}(n, A, \varepsilon) = 0$ and the fact that $\mathbf{p}\big(A, (\cdot)_{i=1}^k, \cdot \big)$ is a bounded continuous function on the domain (43) we obtain

$$\limsup_{n\to\infty} (47) \leqq 1 - \mathbf{P}(B^\varepsilon) \quad \text{and} \quad \limsup_{n\to\infty} (48) \leqq 1 - \mathbf{P}(B^\varepsilon).$$

Now $\mathbf{P}(B^\varepsilon) \to 1$ as $\varepsilon \to 0$, from which (41) and the statement of the lemma follows under the assumption that (43) implies $(44) = (45)$. $\quad\square$

PROOF OF (43) $\implies$ (44) = (45). We are using random urn configurations to generate $\mathbf{X}_n$. Let Ψ_n denote the ball and color exchangeable $[n]^{[2m]}$-valued random variable with $\big(d(\Psi, i) \big)_{i=1}^n = (d_i)_{i=1}^n$, thus Ψ_n is uniformly distributed on the set of urn configurations with this type vector. $\mathbf{X}_n$ can be generated from Ψ_n via (33). To determine the distribution of $\mathbf{X}_n^{[k]}$ we only need to know the positions of the balls of color $i \in [k]$. We paint the rest of the balls "grey". Let

$$ m_{[k]} := \frac{1}{2} \sum_{i,j} A(i,j), \quad d_{[k]} := \sum_{i=1}^k d_i, \quad m_g := m - d_{[k]} + m_{[k]}. $$

Thus m_g denotes the number of edges of the multigraph spanned by grey vertices.

In order to prove (44) = (45) we first give an explicit formula for (44). The number of grey balls is $2m - d_{[k]}$. The number of all urn configurations with type vector $\big(d_1, \ldots, d_k, 2m - d_{[k]} \big)$ is

$$ (49) \qquad \frac{(2m)!}{\big(\prod_{i=1}^k d_i! \big) \cdot \big(2m - d_{[k]} \big)!} $$

The number of urn configurations with type vector $\big(d_1, \ldots, d_k, 2m - d_{[k]} \big)$ for which $\mathbf{X}_n^{[k]} = A$ is

$$ (50) \qquad \frac{m! \cdot 2^{m - m_g - \sum_{i=1}^k A^*(i,i)}}{\big(\prod_{i \leqq j} A^*(i,j)! \big) \cdot \big(\prod_{i=1}^k \big(d_i - d(A,i) \big)! \big) \cdot m_g!}. $$

Thus $(44) = \frac{(50)}{(49)}$. Our aim is to prove $\frac{(50)}{(49)} = (45)$: after dividing both sides of this equality by $\prod_{i \leq j} \frac{1}{A^*(i,j)!} \cdot 2^{-\sum_i A^*(i,i)}$ we only need to prove

$$ (51) \qquad \frac{m! \cdot \big(\prod_{i=1}^k d_i! \big) \cdot \big(2m - d_{[k]} \big)! \cdot 2^{m - m_g}}{\prod_{i=1}^k \big(d_i - d(A,i) \big)! \cdot m_g! \cdot (2m)!} $$

$$ (52) \qquad = \exp\left(\frac{-1}{2\rho_n} \Big(\sum_{i=1}^k z_i \Big)^2 \right) \cdot \prod_{i=1}^k (z_i)^{d(A,i)} \cdot \rho_n^{-m_{[k]}} + \mathrm{Err}\,(n, A, \varepsilon). $$

Now we rewrite (51):

$$ (51) = \left(\prod_{i=1}^k \prod_{l=1}^{d(A,i)} \big(d_i - d(A,i) + l \big) \right) $$

$$\cdot \left(\prod_{l=1}^{m-m_g} (m_g + l) \right) \cdot \frac{2^{m-m_g}}{\prod_{l=1}^{d_{[k]}} \left(2m - d_{[k]} + l \right)}$$

$$(53) \qquad = \left(\prod_{l=1}^{m-m_g} \frac{2m_g + 2l}{2m - d_{[k]} + l} \right) \cdot \frac{\prod_{i=1}^{k} \prod_{l=1}^{d(A,i)} \left(d_i - d(A,i) + l \right)}{\prod_{l=1}^{m_{[k]}} \left(2m - m_{[k]} + l \right)}.$$

We approximate various terms that appear on the right hand side of (53) using our assumptions (43):

$$(54) \qquad \prod_{l=1}^{m-m_g} \frac{2m_g + 2l}{2m - d_{[k]} + l} = \left(\prod_{l=1}^{d_{[k]}} \frac{2m - 2l}{2m - l} \right) \cdot \left(1 + \frac{1}{n} \operatorname{Err}(A, \varepsilon) \right)$$

$$(55) \qquad \prod_{l=1}^{m_{[k]}} \left(2m - m_{[k]} + l \right) = (2m)^{m_{[k]}} \cdot \left(1 + \frac{1}{n^2} \operatorname{Err}(A, \varepsilon) \right)$$

where $0 \leqq \left| \operatorname{Err}(A, \varepsilon) \right| < +\infty$ is independent of n.

Let $d^* = \min \left\{ d_i : i \in [k], \ d(A,i) > 0 \right\}$. We consider two cases separately: If $d^* \leqq n^{1/2}$ then using (43) it is easy to see that $(53) \leqq \operatorname{Err}(A, \varepsilon) n^{-1/2}$ and also $\mathbf{p}\left(A, (z_i)_{i=1}^{k}, \rho_n \right) \leqq \operatorname{Err}(A, \varepsilon) n^{-1/2}$, so $(51) = (52)$ holds when $d^* \leqq n^{1/2}$. If $d^* > n^{1/2}$ then we have

$$(56) \qquad \prod_{i=1}^{k} \prod_{l=1}^{d(A,i)} (d_i - d(A,i) + l) = \left(\prod_{i=1}^{k} (d_i)^{d(A,i)} \right) \left(1 + \frac{1}{\sqrt{n}} \operatorname{Err}(A, \varepsilon) \right).$$

Putting (54), (55) and (56) together we get

$$(53) = \left(\prod_{l=1}^{d_{[k]}} \frac{2m - 2l}{2m - l} \right) \cdot \frac{\prod_{i=1}^{k} (d_i)^{d(A,i)}}{(2m)^{m_{[k]}}} \cdot \left(1 + \operatorname{Err}(n, A, \varepsilon) \right)$$

$$\stackrel{(42)}{=} \left(\prod_{l=1}^{d_{[k]}} \frac{1 - \frac{2l}{n^2 \rho_n}}{1 - \frac{l}{n^2 \rho_n}} \right) \cdot \frac{\prod_{i=1}^{k} (n \cdot z_i)^{d(A,i)}}{(n^2 \rho_n)^{m_{[k]}}} \cdot \left(1 + \operatorname{Err}(n, A, \varepsilon) \right)$$

$$\stackrel{(43)}{=} \exp \left(\frac{-1}{2\rho_n} \left(\sum_{i=1}^{k} z_i \right)^2 \right) \cdot \prod_{i=1}^{k} (z_i)^{d(A,i)} \cdot \rho_n^{-m_{[k]}} + \operatorname{Err}(n, A, \varepsilon).$$

This completes the proof of $(51) = (52)$. $\quad \square$

PROOF OF THEOREM 1. Given $\mathbf{X}_n$ for every $n \in \mathbb{N}$ let us define the vertex exchangeable random adjacency matrix $\tilde{\mathbf{X}}_n$ in the following way: let π_n denote a uniformly distributed permutation $\pi_n : [n] \hookrightarrow [n]$, independent from $\mathbf{X}_n$. Let

$$(57) \qquad \big(\tilde{X}_n(i,j) \big)_{i,j=1}^n := \big(X_n \big(\pi_n(i), \pi_n(j) \big) \big)_{i,j=1}^n.$$

Then $\tilde{\mathbf{X}}_n$ is indeed vertex exchangeable, moreover $\mathbf{P}\big(t_=(F, \tilde{\mathbf{X}}_n) = t_=(F, \mathbf{X}_n) \big)$ $= 1$ for every $F \in \mathcal{M}$, so $\mathbf{X}_n \xrightarrow{\mathrm{P}} W$ is equivalent to $\tilde{\mathbf{X}}_n \xrightarrow{\mathrm{P}} W$, which is in turn equivalent to $\tilde{\mathbf{X}}_n \xrightarrow{\mathrm{d}} \mathbf{X}_W$ by Lemma 3.1. By (57) the conditions (11) and (12) are equivalent to the uniform integrability of $\big(\tilde{X}_n(1,2) \big)_{n=1}^\infty$ and $\big(\tilde{X}_n(1,1) \big)_{n=1}^\infty$, respectively, thus we can apply Lemma 3.2(ii) to deduce that for all $k \in \mathbb{N}$

$$(58) \qquad \left(\frac{1}{n} d\big(\tilde{\mathbf{X}}_n, i \big) \right)_{i=1}^k \xrightarrow{\mathrm{d}} \big(D(\mathbf{X}_W, i) \big)_{i=1}^k.$$

Note that by (6), Definition 3.1 and (22) we have that $\big(D(\mathbf{X}_W, i) \big)_{i=1}^k$ are i.i.d. with probability distribution function $F_W(z)$.

Now we are going to prove that $\frac{2m(\tilde{\mathbf{X}}_n)}{n^2} \xrightarrow{\mathrm{P}} \rho(W)$. In order to do so we define the truncated adjacency matrix $\tilde{\mathbf{X}}_n^l$ by $\tilde{X}_n^l(i,j) := \min\big\{ \tilde{X}_n(i,j), l \big\}$ and the truncated multigraphon W^l which satisfies

$$\big(X_{W^l}(i,j) \big)_{i,j=1}^\infty \sim \big(\min\big\{ X_W(i,j), l \big\} \big)_{i,j=1}^\infty.$$

Now we show that if we fix $l \in \mathbb{N}$ then $\frac{2m(\tilde{\mathbf{X}}_n^l)}{n^2} \xrightarrow{\mathrm{P}} \rho(W^l)$. The equations marked by $(*)$ below are true by exchangeability:

$$\lim_{n\to\infty} \mathbf{E}\left(\frac{1}{n} \sum_{i=1}^n \frac{1}{n} d(\tilde{\mathbf{X}}_n^l, i) \right) \overset{(*)}{=} \lim_{n\to\infty} \mathbf{E}\left(\frac{1}{n} d(\tilde{\mathbf{X}}_n^l, 1) \right)$$

$$\overset{(58)}{=} \mathbf{E}\big(D(\mathbf{X}_{W^l}, 1) \big) \overset{(19)}{=} \rho(W^l).$$

$$\lim_{n\to\infty} \mathbf{D}^2\left(\frac{1}{n} \sum_{i=1}^n \frac{1}{n} d\big(\tilde{\mathbf{X}}_n^l, i \big) \right) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^n \mathbf{Cov}\left(\frac{1}{n} d(\tilde{\mathbf{X}}_n^l, i), \frac{1}{n} d(\tilde{\mathbf{X}}_n^l, j) \right)$$

$$\overset{(*)}{=} \lim_{n\to\infty} \left(\frac{1}{n} \mathbf{D}^2\left(\frac{1}{n} d\big(\tilde{\mathbf{X}}_n^l, 1 \big) \right) \right.$$

$$\left. + \frac{n-1}{n} \mathbf{Cov}\left(\frac{1}{n} d(\tilde{\mathbf{X}}_n^l, 1), \frac{1}{n} d(\tilde{\mathbf{X}}_n^l, 2) \right) \right) \overset{(58)}{=} 0.$$

Having established $\forall\, l : \frac{2m(\tilde{\mathbf{X}}_n^l)}{n^2} \overset{\text{p}}{\longrightarrow} \rho(W^l)$, the relation $\frac{2m(\tilde{\mathbf{X}}_n)}{n^2} \overset{\text{p}}{\longrightarrow} \rho(W)$ follows from

$$\lim_{l\to\infty} \rho(W^l) = \rho(W),$$

$$\forall\, \varepsilon > 0 : \ \limsup_{l\to\infty\,n\in\mathbb{N}} \mathbf{P}\left(\left|\frac{2m(\tilde{\mathbf{X}}_n)}{n^2} - \frac{2m(\tilde{\mathbf{X}}_n^l)}{n^2}\right| \geqq \varepsilon\right) \overset{(11),\,(12)}{=} 0,$$

and the $\varepsilon/3$-argument. So conditions (37) and (38) are satisfied, thus we can apply Lemma 4.1 to show that $\tilde{\mathbf{X}}_n \overset{\text{p}}{\longrightarrow} \hat{W}$, where $\hat{W}$ is of the form (13). $\square$

PROOF OF THEOREM 2. The distribution (14) arises from the Pólya Ψ_{2m}^n urn model (36) with $2m$ balls and n colors via (33). The distribution (36) is ball and color exchangeable, so $\mathbf{X}_n$ is vertex exchangeable and edge stationary. If we want to prove Theorem 2 then by Lemma 4.1 we only need to show that (38) holds for all $k \in \mathbb{N}$ where $(Z_i)_{i\in\mathbb{N}}$ are i.i.d. with density function $\mathbf{g}(x,\kappa,\frac{\kappa}{\rho})$ (see (10)). We may use the method of moments to prove convergence in distribution, since the Gamma distribution is uniquely determined by its moments. Thus we need to show that if $\nu_1,\dots,\nu_k \in \mathbb{N}$ then

$$\lim_{n\to\infty} \mathbf{E}\left(\prod_{i=1}^{k}\left(\frac{1}{n}d\big(\Psi_{2m(n)}^n, i\big)\right)^{\nu_i}\right) = \mathbf{E}\left(\prod_{i=1}^{k} Z_i^{\nu_i}\right)$$

$$= \prod_{i=1}^{k}\left(\frac{\rho}{\kappa}\right)^{\nu_i} \cdot \prod_{j=1}^{\nu_i}(\kappa + j - 1).$$

Fix k and ν_i, $i \in [k]$. Let $\nu = \sum_{i=1}^{k}\nu_i$ and denote by ψ a particular element of $[k]^{[\nu]}$ with type vector $(\nu_1,\dots,\nu_k)$. By the construction of the Pólya-Ψ_{2m}^n distribution we have

$$\mathbf{P}\big(\forall\, l \in [\nu] : \Psi_{2m}^n(l) = \psi(l)\big) = \frac{\prod_{i=1}^{k}\prod_{j=1}^{\nu_i}(\kappa + j - 1)}{\prod_{j=1}^{\nu}(\kappa n + j - 1)} = \mathcal{O}(n^{-\nu})$$

Denote $\underline{\nu} := \big\{(i,j) : i \in [k], j \in [\nu_i]\big\}$. The number of functions $f : \underline{\nu} \to [2m]$ with $\big|\mathcal{R}(f)\big| = N$ is $\mathcal{O}\big(\big(2m(n)\big)^N\big) = \mathcal{O}(n^{2N})$ if $1 \leqq N \leqq \nu$.

$$\lim_{n\to\infty} \mathbf{E}\left(\prod_{i=1}^{k}\left(\frac{1}{n}d\big(\Psi_{2m(n)}^n, i\big)\right)^{\nu_i}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^\nu} \sum_{f:\underline{\nu}\to[2m]} \mathbf{P}\big(\forall\,(i,j)\in\underline{\nu}:\ \Psi^n_{2m(n)}\big(f(i,j)\big)=i\big)$$

$$= \lim_{n\to\infty} \frac{1}{n^\nu} \sum_{f:\underline{\nu}\hookrightarrow[2m]} \mathbf{P}\big(\forall\,(i,j)\in\underline{\nu}:\ \Psi^n_{2m(n)}\big(f(i,j)\big)=i\big)$$

$$+ \lim_{n\to\infty} \frac{1}{n^\nu} \sum_{N=1}^{\nu-1} \mathcal{O}\big(n^{2N}\big)\mathcal{O}\big(n^{-N}\big)$$

$$\overset{(*)}{=} \lim_{n\to\infty} \frac{\prod_{k=1}^{\nu}\big(2m(n)-k+1\big)}{n^\nu}\mathbf{P}\big(\forall\,l\in[\nu]:\ \Psi^n_{2m}(l)=\psi(l)\big)$$

$$\overset{(37)}{=} \prod_{i=1}^{k}\left(\frac{\rho}{\kappa}\right)^{\nu_i}\cdot\prod_{j=1}^{\nu_i}(\kappa+j-1).$$

The equation $(*)$ holds true by ball exchangeability. $\square$

References

[1] D. J. Aldous, Representations for partially exchangeable arrays of random variables, *J. Multivar. Anal.*, **11** (1981), 581–598.

[2] D. J. Aldous, More uses of exchangeability: representations of complex random structures, arXiv: 0909.4339v2, to appear in *Probability and Mathematical Genetics: Papers in Honour of Sir John Kingman*, Cambridge University Press (2010).

[3] P. Billingsley, *Convergence of Probability Measures*, 2nd ed., John Wiley & Sons, Inc. (New York, 1999).

[4] C. Borgs, J. Chayes and L. Lovász, Moments of two-variable functions and the uniqueness of graph limits, *Geom. Funct. Anal.*, **19** (2010), 1597–1619.

[5] C. Borgs, J. Chayes, L. Lovász, V. Sós and K. Vesztergombi, Limits of randomly grown graph sequences, *Eur. J. Combin.*, **32** (2011), 985–999.

[6] S. Chatterjee, P. Diaconis and A. Sly, Random graphs with a given degree sequence, *Ann. Appl. Probab.*, **21** (2011), 1400–1435.

[7] P. Diaconis and S. Janson, Graph limits and exchangeable random graphs, *Rend. Mat. Appl.*, **28** (2008), 33–61.

[8] I. Kolossváry and B. Ráth, Multigraph limits and exchangeability, *Acta Math. Hungar.*, **130** (2011), 1–34.

[9] L. Lovász, Very large graphs, in: *Current Developments in Mathematics 2008*, International Press (Somerville, MA, 2009), pp. 67–128.

[10] L. Lovász and B. Szegedy, Limits of dense graph sequences, *J. Combin. Theory Ser. B*, **96** (2006), 933–957.

[11] L. Lovász and B. Szegedy, Random graphons and weak positivstellensatz for graphs, arXiv: 0902.1327v1, to appear in *Journal of Graph Theory* (2011).

[12] L. Lovász and B. Szegedy, Limits of compact decorated graphs, arXiv: 1010.5155v1 (2010).

[13] B. Ráth, Time evolution of dense multigraph limits under edge-conservative preferential attachment dynamics (accepted for publication in *Random Structures and Algorithms*), arXiv: 0912.3904v3 (2010).

[14] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, vol. 1: Functional Analysis*, Gulf Professional Publishing (1980).

[15] D. Williams, *Probability with Martingales*, Cambridge University Press (Cambridge, 1991).