Delay Equations with Rapidly Oscillating Stable Periodic Solutions

Journal Article

Author(s):
Stoffer, Daniel

Publication date:
2008

Permanent link:
https://doi.org/10.3929/ethz-b-000005024

Originally published in:
Journal of dynamics and differential equations 20(1), https://doi.org/10.1007/s10884-006-9068-4

Journal of Dynamics and Differential Equations, Vol. 20, No. 1, March 2008 (© 2007)
DOI: 10.1007/s10884-006-9068-4

Delay Equations with Rapidly Oscillating Stable Periodic Solutions

Daniel Stoffer

Received June 29, 2006

We prove analytically that there exist delay equations admitting rapidly oscillating stable periodic solutions. Previous results were obtained with the aid of computers, only for particular feedback functions. Our proofs work for stiff equations with several classes of feedback functions. Moreover, we prove that for negative feedback there exists a class of feedback functions such that the larger the stiffness parameter is, the more stable rapidly oscillating periodic solutions there are. There are stable periodic solutions with arbitrarily many zeros per unit time interval if the stiffness parameter is chosen sufficiently large.

KEY WORDS: Delay differential equations; rapidly oscillating solutions; stable periodic solutions.

1. INTRODUCTION

In this paper, we consider the delay equation

$$\frac{1}{\mu}\dot{x} = -x + f(x(t-1)), \tag{1}$$

where f models either positive feedback, i.e. $\operatorname{sign} f(x) = \operatorname{sign} x$, or negative feedback, i.e. $\operatorname{sign} f(x) = -\operatorname{sign} x$.

If f is monotone the dynamics of (1) is relatively well understood. For positive feedback Krisztin, Walther and Wu [5] show that under weak assumptions on f there is a compact invariant set for the flow of (1). This invariant set becomes a global attractor provided some additional restrictions are imposed on f and the damping constant μ, cf. Krisztin and Walther [4]. It was also proved that stable periodic motion cannot occur,

Department of Mathematics, ETH-Zürich, CH-8092 Zürich, Switzerland. E-mail: stoffer@math.ethz.ch

cf. [5]. For negative feedback Mallet-Paret and Walther [6] show that if f is monotone there do not exist rapidly oscillating stable periodic solutions.

If f is not monotone the behaviour of (1) is more intricate. It was very surprising when Ivanov and Losson [3] showed that for a specific choice of μ and a negative feedback function f the delay equation (1) admits a rapidly oscillating stable periodic solution. Later, Schulze–Halberg [9] found an example of a positive feedback function f for which there is a rapidly oscillating stable periodic solution. Aschwanden, Schulze-Halberg and Stoffer [1] simplified the computer-assisted proofs of Ivanov and Losson [3] and Schulze-Halberg [9] and they presented a new type of rapidly oscillating stable periodic solutions motivated by numerical computations in [9].

In this paper, we prove *analytically* the existence of rapidly oscillating stable periodic solutions for classes of functions f which in the case of negative feedback include the particular function taken by Ivanov and Losson. Our proofs work for stiff equations, i.e. we state that there exists μ_0 such that for all $\mu > \mu_0$ Eq. (1) admits a rapidly oscillating stable periodic solution. Our proof does not guarantee that μ_0 is smaller than the particular value of μ chosen by Ivanov and Losson. For positive feedback we give conditions on f which include the function considered by Aschwanden *et al.* [1].

Let us introduce some notions used in this paper. A solution of Eq. (1) is a continuous function $x\colon[-1,\infty)\to\mathbb{R}$, differentiable for positive arguments and satisfying (1) for $t>0$. Let $\mathcal{C}$ be the space of continuous functions defined for $[-1,0]$. $\mathcal{C}$ is endowed with the maximum norm. For any $\varphi\in\mathcal{C}$ there is a unique solution x with initial condition $\varphi = x|_{[-1,0]}$. It may be computed recursively using the variation-of-constant formula

$$x(t+1) = x(t_0+1)e^{-\mu(t-t_0)} + \mu\int_0^{t-t_0} e^{-\mu(t-t_0-s)} f(x(t_0+s))\,ds\,. \qquad (2)$$

For a solution x and $t\geqslant 0$ define $x_t\in\mathcal{C}$ by $x_t\colon s\mapsto x(t+s),\, s\in[-1,0]$.

We give an outline of the paper. In Section 2, we consider Eq. (1) with positive feedback functions f of the kind as in Fig. 1. We consider the case $a>1>b>0$ considered in [1]. We give conditions on a and b (which we believe also to be necessary) under which Eq. (1) admits a rapidly oscillating stable periodic solution for all sufficiently large μ.

In Section 3, we consider Eq. (1) with negative feedback functions f as shown in Fig. 5 with $a>b>1$ and $d>c>1$. We give conditions on the parameters a, b, c, d of f implying the existence of a rapidly oscillating stable periodic solution if μ is sufficiently large.

In Section 4, we investigate a new type of rapidly oscillating solutions for odd negative feedback functions f with $a>1>b>0$. In this situation, Eq. (1) admits periodic solutions with a particularly simple structure.

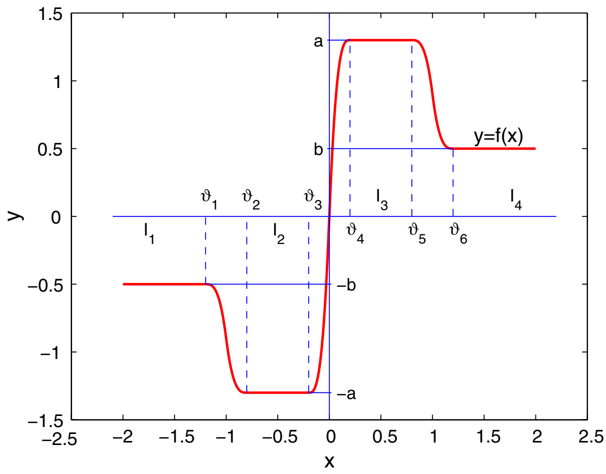

Figure 1. A typical function $f \in F_\varepsilon^+$.

For odd $n \in \mathbb{N}$ there are explicitely known numbers $\mu_n(a)$ such that for all $\mu > \mu_n(a)$ Eq. (1) admits rapidly oscillating stable periodic solutions of period $\tau \in [2/n, 2/(n-1)]$. This states the existence of stable periodic solutions with an arbitrary large number of oscillations per unit time interval if the stiffness parameter μ is sufficiently large. We believe that our conditions on the parameters μ, a and b are sharp.

In the final Section 5, we make some comments on the results obtained and report on some numerical investigations. This leads to some conjectures concerning the case of (positive or negative) feedback functions f with $a > 1 > b > 0$.

2. DELAY EQUATIONS WITH POSITIVE FEEDBACK

In this section, we consider delay equations with positive feedback functions f, as depicted in Fig. 1. More precisely, define $\vartheta_{1,2} := -1 \mp \varepsilon$, $\vartheta_{3,4} := \mp\varepsilon$, $\vartheta_{5,6} := 1 \mp \varepsilon$ for a small positive constant ε and let a, b be constants satisfying $a > b > 0$. Let F_ε^+ denote the set of all functions $f \in C_b^1$ (differentiable functions with bounded derivative) satisfying $|f(x)| \leqslant a$ for $x \in \mathbb{R}$ and

$$f(x) = \begin{cases} -b, & x \in (-\infty, \vartheta_1) =: I_1, \\ -a, & x \in (\vartheta_2, \vartheta_3) \quad =: I_2, \\ a, & x \in (\vartheta_4, \vartheta_5) \quad =: I_3, \\ b, & x \in (\vartheta_6, \infty) \quad =: I_4. \end{cases}$$

We denote the set of these functions by F_ε^+ because they model positive feedback. In this section, we restrict our considerations to the case $a > 1 >$

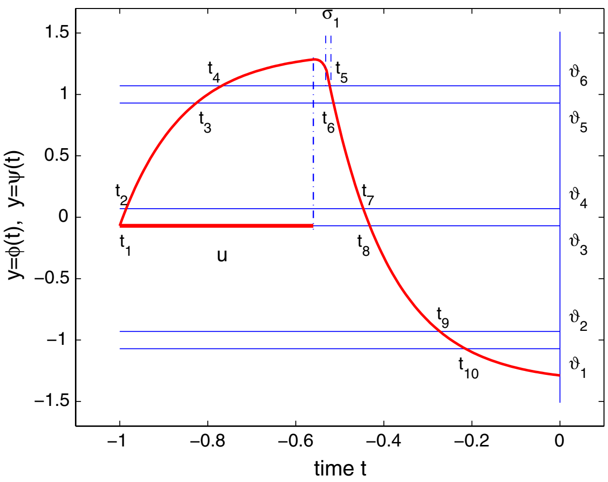

Figure 2. An initial function $\varphi \in \Phi$.

$b > 0$. If $x(t) \in I_j$ for $t \in [t_0, t_1]$ then $f(x(t))$ is constant and by the variation-of-constant formula (2) it follows that

$$x(t+1) = c + (x(t_0+1) - c)e^{-\mu(t-t_0)}$$

holds for $c = f(x(t)) \in \{-a, -b, b, a\}$. We say that x is *of exponential type with limit c in the interval* $[t_0+1, t_1+1]$. Let $T_1 := \frac{1}{\mu} \log \frac{a+\varepsilon}{a-\varepsilon}$ be the time needed for a function of exponential type with limit a to increase from $\vartheta_3 = -\varepsilon$ to $\vartheta_4 = \varepsilon$. We introduce the following set of initial conditions, cf. Fig. 2

$\Phi := \{\varphi \in \mathcal{C} |$there is $u \in (0, 1)$ such that

- $\varphi(-1) = \vartheta_3$,
- φ is of exponential type with limit a in $[-1, -1+u]$,
- $\varphi(-1+u) > \vartheta_6$,
- For $t \in [0, T_1]$: $\varphi(-1+u+t) = \varphi(-1+u)e^{-\mu t} + \mu \int_0^t e^{-\mu(t-s)} f(-a + (\varepsilon + a)e^{-\mu s})\, ds > \vartheta_6$,
- φ is of exponential type with limit $-a$ in $[-1+u+T_1, 0]$,
- $\varphi(0) < \vartheta_1\}$.

We define the map $S: \Phi \to \mathbb{R}$, $\varphi \mapsto S(\varphi) := u$. By definition of Φ the map S is injective. Hence, the solution x of (1) with initial function $\varphi \in \Phi$ is uniquely determined by the single real number u. For such a solution let $t_1 = -1 < t_2 < t_3 < \ldots$ be the consecutive times for which $x(t_i) \in \{\vartheta_1, \ldots, \vartheta_6\}$, define the numbers $T_i := t_{i+1} - t_i$ and let $x_i := x(t_i + 1)$, $i = 1, 2, \ldots$ In the next lemma we state that the set Φ may serve as Poincaré section.

Lemma 1. *Let the constants a, b satisfy* $a>1$, $b>0$, $b<2/a-a$ *and let* $I=[\frac{1}{4},\frac{3}{4}]$. *Then there is* $\mu_0>0$ *and* $\varepsilon_0>0$ *such that for* $\mu>\mu_0$, $\varepsilon\in(0,\varepsilon_0)$ *and* $f\in F_\varepsilon^+$ *the following holds:*

1. For every $u\in I$ *there is* $\varphi\in\Phi$ *with* $S(\varphi)=u$.

For any solution $x(t)$ *with initial function* φ *satisfying* $S(\varphi)\in I$ *the following holds:*

2. $x(t)<\vartheta_1$ *for* $t\in[0,t_2+1]$.
3. $x(t)$ *is of exponential type with limit a for* $t\in[t_2+1,t_3+1]$.
4. $\vartheta_4<x(t)<\vartheta_5$ *for* $t\in[t_3+1,t_8+1]$.
5. $x(t)$ *is of exponential type with limit* $-a$ *for* $t\in[t_8+1,t_9+1]$.
6. $\vartheta_2<x(t)<\vartheta_3$ *for* $t\in[t_9+1,t_{14}+1]$.
7. $x(t)$ *is of exponential type with limit a for* $t\in[t_{14}+1,t_{15}+1]$.
8. $\vartheta_6<x(t)$ *for* $t\in[t_{15}+1,t_{16}+1]$.
9. $x(t)$ *is of exponential type with limit* $-a$ *for* $t\in[t_{16}+1,\tau]$ *for* $\tau:=t_{17}+1$.
10. $x_\tau\in\Phi$.

Remark

1. Figure 3 shows a sketch of a solution as described in Lemma 1.
2. Let the assumptions of Lemma 1 be satisfied and μ and ε be chosen according to Lemma 1. Then we define the set $\Phi_0:=S^{-1}(I)$.

Proof: Proof of Assertion 1. The graph of the initial function φ consists of three arcs. The first one exists for a time interval of length u, starts with $\varphi(-1)=\vartheta_3=-\varepsilon$, is of exponential type with limit a and thus lies entierly below $x=a$. The second arc has a time interval of length T_1 and by definition of Φ also lies below $x=a$. The third arc is of exponential type with limit $-a$, has a time interval of length $1-u-T_1\geqslant\frac{1}{4}-T_1$ and begins with an x-value below a. Therefore it is sufficient to require that a function of exponential type with limit $-a$ decays from a to $\vartheta_1=-1-\varepsilon$ in a time interval of length $\leqslant\frac{1}{4}-T_1$. This leads to the condition $\mu\geqslant4\log\frac{2a(a+\varepsilon)}{(a-1-\varepsilon)(a-\varepsilon)}$. Note that this condition also guarantees that the first arc satisfies the condition $\varphi(-1+u)>\vartheta_6$.

For the proof of Assertions 2. – 10. we make some preliminary remarks. For the solution x with initial function $\varphi\in\Phi$ remember the definition of t_i, T_i, x_i introduced just before Lemma 1.

We first compute $x_1=\varphi(0)$ for $\varphi\in\Phi_0$. We have

$$\varphi(-1+u)=a+(\vartheta_3-a)\,e^{-\mu u}=a+\delta_{-1}$$

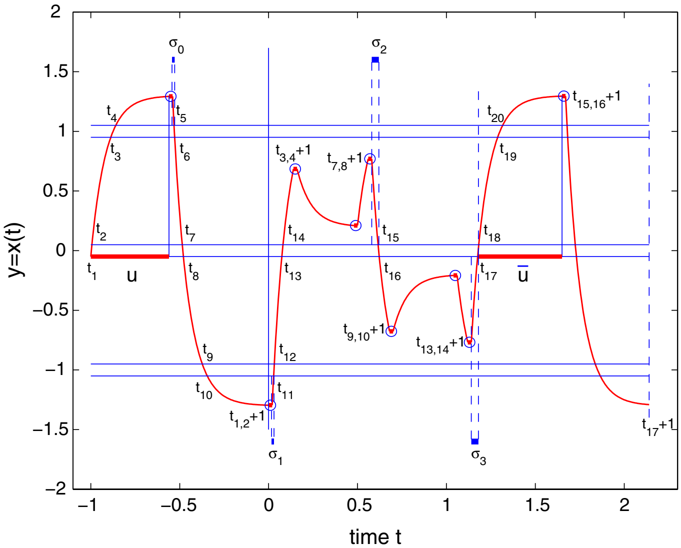

Figure 3. Sketch of a solution as described in Lemma 1. Note that the bold points within the small circles actually represent very small arcs of the solution, the t–coordinate of the starting point and the end point being denoted by $t_{1,2}+1$, $t_{3,4}+1$,... Note that $x_i = x(t_i+1)$, $i=1,\dots,17$.

with $\delta_{-1} = -(a+\varepsilon)\,e^{-\mu u}$. Then by definition of $\varphi \in \Phi$ we get

$$\begin{aligned}\varphi(-1+u+T_1) &= (a+\delta_{-1})\frac{a-\varepsilon}{a+\varepsilon} \\ &\quad + \mu \int_0^{T_1} e^{-\mu(T_1-s)} f(-a+(\varepsilon+a)\,e^{-\mu s})\,ds \qquad (3) \\ &= a + K_0 + \delta_0\,.\end{aligned}$$

with $K_0 = -2a\varepsilon/(a+\varepsilon) + \mu \int_0^{T_1} e^{-\mu(T_1-s)} f(-a+(\varepsilon+a)\,e^{-\mu s})\,ds = O(\varepsilon)$ being independent of u and with $\delta_0 = \frac{a-\varepsilon}{a+\varepsilon}\delta_{-1} = -(a-\varepsilon)\,e^{-\mu u}$ (as to K_0 we used $T_1 = \frac{1}{\mu}O(\varepsilon)$).

Here and in the following we consider $x_1, x_2, \dots, \delta_1, \delta_2, \dots$ as functions of u with parameters ε and μ. We use the small-o notation in the following very specific sense:

Writing $g(u) = o$ means: $g = O(e^{-\mu u} + e^{-\mu(1-u)})$ and $D_u g = O(\mu(e^{-\mu u} + e^{-\mu(1-u)}))$ for $u \in [\frac{1}{4}, \frac{3}{4}]$, $\varepsilon \to 0$ and $\mu \to \infty$.

Since we shall need derivatives with respect to u we emphasize that every time we use the small–o notation one has to make sure that the estimated expression has a derivative with respect to u of size $O(\mu(e^{-\mu u} + e^{-\mu(1-u)}))$.

Since φ is of exponential type with limit $-a$ in $[-1+u+T_1, 0]$ we obtain by (6), (2), (3) and the definition of T_1

$$\begin{aligned} x_1 &= -a + (\varphi(-1+u+T_1)+a)\, e^{-\mu(1-u-T_1)} \\ &= -a + (2a+K_0+o)\, \frac{a+\varepsilon}{a-\varepsilon}\, e^{-\mu(1-u)} \end{aligned}$$

or

$$x_1 = -a + \delta_1, \quad \delta_1 = (2a + L_1 + o)\, e^{-\mu(1-u)}\,. \tag{4}$$

Note that in (4) the o-symbol represents a multiple of $\delta_0 = -(a-\varepsilon)e^{-\mu u}$ and therefore has derivative with respect to u of order $O(\mu(e^{-\mu u} + e^{-\mu(1-u)}))$.

The numbers x_i may be determined recursively. By the variation-of-constant formula (2) we have

$$x_{i+1} = x_i e^{-\mu T_i} + \mu \int_0^{T_i} e^{-\mu(T_i - s)} f(x(t_i+s))\, ds\,. \tag{5}$$

If i is even then $f(x(t_i+s))$ is constant for $s \in [0, T_i]$, say $f(x(t_i+s)) = k_i \in \{-a, -b, b, a\}$. Then $x(t)$ is of exponential type with limit k_i in $[t_i+1, t_{i+1}+1]$ and (5) simplifies to

$$x_{i+1} = k_i + (x_i - k_i)\, e^{-\mu T_i}, \quad (i \text{ even})\,. \tag{6}$$

We show that the functions $x_i = x_i(u)$ have a specific structure.

Claim A. *For $i = 1, 2, \dots, 17$ there are constants c_i, K_i, d_i, L_i such that*

$$x_i = c_i + K_i + \delta_i$$

with

$$\delta_i = (d_i + L_i + o)\, e^{-\mu u} \quad \text{or} \quad (d_i + L_i + o)\, e^{-\mu(1-u)}$$

and where

$$\begin{aligned} &c_i \text{ and } d_i \text{ are independent of } u, \varepsilon \text{ and } \mu, \\ &K_i = C(\varepsilon) \quad \text{and } L_i = C(\varepsilon). \end{aligned}$$

The notation $g = C(\varepsilon)$ means: g is independent of u and satisfies $g = O(\varepsilon)$ for $\varepsilon \to 0$, $\mu \to \infty$.

Below we shall recursively compute the constants c_i, d_i. To simplify the computations we already remark that whenever the solution crosses one of the small intervals $[\vartheta_1, \vartheta_2]$, $[\vartheta_3, \vartheta_4]$ or $[\vartheta_5, \vartheta_6]$ it is of exponential type with limit a or $-a$, cf. Fig. 3, as will be confirmed in the course of the subsequent computations. It follows that if i is odd, then

$$T_i = \frac{1}{\mu} \log \frac{a+\vartheta_{k+1}}{a+\vartheta_k}, \quad e^{-\mu T_i} = 1 + C(\varepsilon) \ (i \text{ odd}) \tag{7}$$

for $k=1$, 3 or 5. We thus easily get

Claim B. *If Claim A holds for all $j \leqslant i$ and if i is odd then Claim A also holds for $i+1$ with $c_{i+1} = c_i$ and $\delta_{i+1} = (1+C(\varepsilon))\delta_i$.*

Proof of Claim B. From (5) we get

$$x_{i+1} = (c_i + K_i + \delta_i)(1+C(\varepsilon)) + \int_0^{T_i} \mu\, e^{-\mu(T_i - s)} f(x(t_i+s))\, ds\,. \tag{8}$$

Assume by induction that the function $x(t_i+s)$ is of exponential type with limit a or $-a$ and is starting at some ϑ_j. It follows that the integral Int_i in (8) does not depend on u and since $T_i = \frac{1}{\mu}C(\varepsilon)$ we have $\mathrm{Int}_i = C(\varepsilon)$. Splitting $x_{i+1} = (c_{i+1} + K_{i+1}) + \delta_{i+1}$ we get $c_{i+1} + K_{i+1} = (c_i + K_i)(1+C(\varepsilon)) + \mathrm{Int}_i$, $\delta_{i+1} = (1+C(\varepsilon))\delta_i$. From $\delta_{i+1} = (1+C(\varepsilon))\delta_i$ we also get that $D_u\delta_{i+1} = D_u(d_{i+1} + L_{i+1} + o) = D_u(1+C(\varepsilon))(d_i + L_i) + o = O(\mu(e^{-\mu u} + e^{-\mu(1-u)}))$ and Claim B follows at once.

In the following, we successively prove the assertions of Lemma 1 together with the corresponding part of Claim A.

Proof of Assertion 2. We compute the constants c_i, d_i for odd i using (6). From (4) and Claim B we already know

$$c_1 = c_2 = -a, \quad \delta_1 = (2a + C(\varepsilon) + o)\, e^{-\mu(1-u)}, \quad \delta_2 = (2a + C(\varepsilon) + o)\, e^{-\mu(1-u)}\,. \tag{9}$$

Hence $x(t) = -a + O(\varepsilon) + o$ holds for $t \in [0, t_2+1]$. If ε is sufficiently small and μ is sufficiently large then obviously statement 2 of Lemma 1 holds.

Proof of Assertion 3. With

$$e^{-\mu T_2} = \frac{a-\vartheta_5}{a-\vartheta_4} = \frac{a-1+\varepsilon}{a-\varepsilon} = \frac{a-1}{a} + C(\varepsilon) \tag{10}$$

we find with (6)

$$x_3 = a + (-2a + K_2 + \delta_2)\Big(\frac{a-1}{a} + C(\varepsilon)\Big)$$

leading to

$$c_3 = c_4 = 2 - a, \quad \delta_3 = \Big(\frac{a-1}{a} + C(\varepsilon)\Big)\delta_1, \quad \delta_4 = \Big(\frac{a-1}{a} + C(\varepsilon)\Big)\delta_1\,.$$

Note that K_3, K_4 are of the required form although we do not compute them explicitely. Since by assumption $a \in (1, \sqrt{2})$ note that $x_3, x_4 \in [\vartheta_4, \vartheta_5]$ if ε is sufficiently small and if μ is sufficiently large. Hence $t_2 + 1 < t_{11} < t_{12} < t_{13} < t_{14} < t_3 + 1$ and the solution x crosses the intervals $[\vartheta_1, \vartheta_2]$, $[\vartheta_3, \vartheta_4]$ where x is of exponential type with limit a, cf. Fig. 3. The formulas for δ_3, δ_4 immediately imply the estimates for the derivative with respect to u. This proves statement 3 of Lemma 1.

Proof of Assertion 4. Note that the solution arc corresponding to $t \in [t_3 + 1, t_4 + 1]$ is covered by Claim B, $i = 3$. Next we discuss the time interval $[t_4 + 1, t_6 + 1]$. We define the numbers

$$\sigma_0 := \frac{1}{\mu} \log \frac{a + \varphi(-1 + u + T_1)}{a + \vartheta_6}, \quad \sigma_1 := \frac{1}{\mu} \log \frac{a - x_2}{a - \vartheta_1}. \tag{11}$$

σ_0 is the time needed for x to decrease from $\varphi(-1 + u + T_1)$ to ϑ_6 and σ_1 is the time in which x increases from x_2 to ϑ_1, cf. Fig. 3. We now obtain

$$T_4 = (u - T_1 - T_2 - T_3) + T_1 + \sigma_0 = -T_2 - T_3 + \sigma_0 + u \tag{12}$$

and therefore, see (10), (7), (3)

$$e^{-\mu T_4} = \left(\frac{a}{a-1} + C(\varepsilon)\right)(1 + C(\varepsilon))\left(\frac{a+1}{2a} + C(\varepsilon) + o\right)e^{-\mu u}.$$

It is easily verified that the derivative of the third bracket ( ) is of order $O(\mu(e^{-\mu u} + e^{-\mu(1-u)}))$. Equation (6) yields

$$x_5 = b + (c_4 - b + K_4 + \delta_4)\left(\frac{a+1}{2(a-1)} + C(\varepsilon) + o\right)e^{-\mu u}$$

and hence

$$c_5 = c_6 = b, \quad \delta_5 = \left(\frac{(2-a-b)(a+1)}{2(a-1)} + C(\varepsilon) + o\right)e^{-\mu u}, \quad \delta_6 = (1 + C(\varepsilon))\delta_5.$$

The derivative D_u of the term o in the expression for δ_5 is of order $O(\mu(e^{-\mu u} + e^{-\mu(1-u)}))$. Note that the solution remains in I_3. We discuss the time interval $[t_6 + 1, t_8 + 1]$. We have

$$e^{-\mu T_6} = \frac{a + \varepsilon}{a + 1 - \varepsilon} = \frac{a}{a+1} + C(\varepsilon),$$

$$x_7 = a + (b - a + K_6 + \delta_6)\left(\frac{a}{a+1} + C(\varepsilon)\right)$$

leading to

$$c_7 = c_8 = a\,\frac{b+1}{a+1}, \quad \delta_7 = \Big(\frac{a}{a+1} + C(\varepsilon)\Big)\delta_5, \quad \delta_8 = (1 + C(\varepsilon))\delta_7\,. \tag{13}$$

The inequality $c_7 < 1$ is equivalent to $a(b+1) < a+1$ or $ab < 1$ which is satisfied since by assumption $ab < 2 - a^2 < 1$. We conclude that all the numbers x_4, x_5, x_6, x_7, x_8 lie in the interval $[\vartheta_4, \vartheta_5]$ if ε and $1/\mu$ are sufficiently small. Taking into account that the arcs of exponential type are monotone this proves assertion 4 of Lemma 1.

Proof of Assertion 5. Noting that $e^{-\mu T_8} = e^{-\mu T_2}$ yields with (10), (13)

$$x_9 = -a + \Big(a\,\frac{b+1}{a+1} + a + K_8 + \delta_8\Big)\Big(\frac{a-1}{a} + C(\varepsilon)\Big)$$

leading to

$$c_9 = c_{10} = -\frac{2 - ab + b}{a+1}, \quad \delta_9 = \Big(\frac{a-1}{a+1} + C(\varepsilon)\Big)\delta_5, \quad \delta_{10} = (1 + C(\varepsilon))\delta_9\,.$$

Assertion 5 of Lemma 1 holds.

Proof of Assertion 6. We show $c_9 \in [\vartheta_2, \vartheta_3]$. Note that $0 < 2 - ab + b < 2$ and $a + 1 > 2$ imply $c_9 \in (-1, 0)$. Hence $x_9, x_{10} \in [\vartheta_2, \vartheta_3]$ if ε and $1/\mu$ are sufficiently small. In addition we conclude $t_8 + 1 < t_{15} < t_{16} < t_9 + 1$. Note that also here the solution arc crossing $[\vartheta_3, \vartheta_4]$ is of exponential type with limit $-a$. We continue with

$$\begin{aligned} T_{10} &= (1 - u - T_1 - \sigma_0 - (T_5 + \cdots + T_9)) + T_1 + \sigma_1 \\ &= -(T_5 + \cdots + T_9) + (\sigma_1 - \sigma_0) + (1 - u) \end{aligned} \tag{14}$$

implying

$$e^{-\mu T_{10}} = \Big(\frac{a+1}{a-1} + C(\varepsilon)\Big)\Big(1 + C(\varepsilon) + o\Big)\,e^{-\mu(1-u)}$$

and compute

$$x_{11} = -b + (c_{10} + b + K_{10} + \delta_{10})\Big(\frac{a+1}{a-1} + C(\varepsilon) + o\Big)e^{-\mu(1-u)}$$

leading to

$$c_{11} = c_{12} = -b, \quad \delta_{11} = -\Big(\frac{2(1-ab)}{a-1} + C(\varepsilon) + o\Big)\,e^{-\mu(1-u)}, \quad \delta_{12} = (1 + C(\varepsilon))\delta_{11}\,.$$

Note that the o in the expression for $e^{-\mu T_{10}}$ and x_{11} only involve δ_0 and δ_2 which have small derivatives as required. We conclude that the solution remains in the interval $(\vartheta_2, \vartheta_3)$ if ε and $1/\mu$ are sufficiently small. Similarly as for x_7 we get

$$e^{-\mu T_{12}} = e^{-\mu T_6} = \frac{a}{a+1} + C(\varepsilon)\,,$$

$$x_{13} = -a + (-b + a + K_{12} + \delta_{12})\Big(\frac{a}{a+1} + C(\varepsilon)\Big)$$

implying

$$c_{13} = c_{14} = -a\,\frac{b+1}{a+1}, \quad \delta_{13} = \Big(\frac{a}{a+1} + C(\varepsilon)\Big)\delta_{11}, \quad \delta_{14} = (1 + C(\varepsilon))\delta_{13}\,. \tag{15}$$

Since $c_{13} = -c_7$ we have $c_{13} > -1$ and we conclude that the solution $x(t)$ remains in the interval $(\vartheta_2, \vartheta_3)$ for $t \in [t_9 + 1, t_{14} + 1]$. This verifies Assertion 6.

Proof of Assertion 7 and 8. In order to determine T_{14} and T_{16} and $\overline{u}$ we need the numbers

$$\sigma_2 := \frac{1}{\mu}\log\frac{a + x_8}{a + \vartheta_4}, \quad \sigma_3 := \frac{1}{\mu}\log\frac{a - x_{14}}{a - \vartheta_3}\,. \tag{16}$$

σ_2 is the time needed for x to decrease from x_8 to ϑ_4 and σ_3 is the time in which x increases from x_{14} to ϑ_3, cf. Fig. 3. We remark that for $i = 0, 1, 2, 3$ one has $e^{\mu\sigma_i} =$ const $+ C(\varepsilon) + o$ where all o–terms have small derivative. We find, cf. Figure 3

$$T_{14} = (T_2 - \sigma_1 - T_{11} - T_{12} - T_{13}) + T_3 + T_4 + T_5 + T_6 + T_7 + \sigma_2$$

and with (12) and

$$T_{11} = T_5, \quad T_{12} = T_6, \quad T_{13} = T_7 \tag{17}$$

we get

$$T_{14} = u + \sigma_0 - \sigma_1 + \sigma_2\,. \tag{18}$$

We thus get

$$e^{-\mu T_{14}} = (\text{const } + C(\varepsilon) + o)\,e^{-\mu u}$$

and

$$x_{15} = a + (c_{14} - a + K_{14} + \delta_{14})(\text{const } + C(\varepsilon) + o)\,e^{-\mu u}$$

leading to

$$c_{15} = c_{16} = a, \quad \delta_{15} = (d_{15} + C(\varepsilon) + o)\,e^{-\mu u}, \quad \delta_{16} = (1 + C(\varepsilon))\delta_{15}$$

for some constant d_{15}. All o–terms denote expressions with derivatives with respect to u of order $O(\mu(e^{-\mu u}+e^{-\mu(1-u)}))$. We find $x_{15}, x_{16} > \vartheta_6$ if ε and $1/\mu$ are sufficiently small. We conclude $t_{14}+1 < t_{17} < t_{18} < t_{19} < t_{20} < t_{15}+1$. This implies that the solution arc crossing the intervals $[\vartheta_3, \vartheta_4]$ and $[\vartheta_5, \vartheta_6]$ is of exponential type with limit a. Assertions 7 and 8 hold, cf. Fig. 3. Note that if f is odd then $K_2 = -K_0$, $K_{14} = -K_8$ holds.

Proof of Assertion 9 and 10. We set $\tau := t_{17}+1$, $\overline{u} := (t_{15}+1) - t_{17}$ and verify that $\overline{\varphi} := x_\tau \in \Phi$ with $S(\overline{\varphi}) = \overline{u}$.

- $\overline{\varphi}(-1) = x(t_{17}) = \vartheta_3$.
- $x(t)$ being of exponential type with limit a in $[t_{17}, t_{15}+1]$ we conclude that $\overline{\varphi}$ is of exponential type with limit a in $[-1, -1+\overline{u}]$. With (18) we get

$$
\begin{aligned}
\overline{u} &= (t_{15}+1) - t_{17} = (t_{15}-t_{14}) - (t_{17} - (t_{14}+1)) = T_{14} - \sigma_3, \\
\overline{u} &= u + \sigma_0 - \sigma_1 + \sigma_2 - \sigma_3 \,. \qquad (19)
\end{aligned}
$$

- $\overline{\varphi}(-1+\overline{u}) = x_{15} > \vartheta_6$ holds if ε and $1/\mu$ are sufficiently small.
- By (5) we get $\overline{\varphi}(-1+\overline{u}+t) = \overline{\varphi}(-1+\overline{u})e^{-\mu t} + \int_0^t e^{-\mu(t-s)} f(-a + (\varepsilon+a)e^{-\mu s})\, ds$ for $t \in [0, T_1]$ since $T_{15} = T_1$ and since x is of exponential type with limit $-a$ in $[t_{15}, t_{16}]$.
- $\overline{\varphi}(t) > \vartheta_6$ for $t \in [-1+\overline{u}, -1+\overline{u}+T_1]$ if ε and $1/\mu$ are sufficiently small.
- $\overline{\varphi}$ is of exponential type with limit $-a$ in $[-1+u+T_1, 0]$ since x is of exponential type with limit $-a$ in $[t_{16}+1, t_{17}+1]$. This verifies assertion 9.
- $\overline{\varphi}(0) = x_{17}$. It remains to compute x_{17}. We have $T_{16} = (T_8 - \sigma_2 - T_{15}) + T_9 + T_{10} + T_{11} + T_{12} + T_{13} + \sigma_3$ and with the formula (14) for T_{10} and the relation (17) we get $T_{16} = 1 - u - \sigma_0 + \sigma_1 - \sigma_2 + \sigma_3 - T_{15}$. Using that T_{15} and the σ_i are of order $O(1/\mu)$ we get $T_{16} = 1 - u + O(1/\mu)$. We thus obtain

$$
x_{17} = -a + (x_{16}+a)e^{-\mu T_{16}} = -a + o \,.
$$

Thus $\overline{\varphi}(0) = x_{17} < \vartheta_1$ holds if ε and $1/\mu$ are sufficiently small. We conclude that $\overline{\varphi} \in \Phi$.

This completes the proof of Lemma 1. □

Lemma 1 states conditions on a, b, μ and ε such that for every solution x of the delay Eq. (1) with initial condition $\varphi \in \Phi_0$ there is a return time τ such that $P: \varphi \mapsto \overline{\varphi} := x_\tau \in \Phi$. This Poincaré map P induces a map $\Pi := S \circ P \circ S^{-1}$: $I \to \mathbb{R}$, $u \mapsto \overline{u} := \Pi(u) := S(\overline{\varphi})$. We call this map the *reduced Poincaré map*.

Lemma 2. *Let the constants a, b satisfy $a>1$, $b>0$, $b<2/a-a$ and let $I=[\frac{1}{4},\frac{3}{4}]$. Then there is $\mu_0>0$ such that for $\mu>\mu_0$ there is $\varepsilon=\varepsilon(\mu)$ such that for $f\in F_\varepsilon^+$ the reduced Poincaré map Π admits a unique fixed point $u^*\in I$. Moreover, the fixed point u^* is attractive.*

Remark

3. If in Lemma 2 f is odd then ε does not depend on μ: Under the given assumptions there is $\varepsilon>0$ and μ_0 such that for every $f\in F_\varepsilon^+\cap\{f\mid f(-x)=-f(x)\}$ and $\mu>\mu_0$ the assertion of Lemma 2 holds: Π admits a unique attractive fixed point $u^*\in I$. This follows from the fact that in the proof below $K_0=-K_2$, $K_8=-K_{14}$ holds if f is odd. ◁

Proof. Equation (19) together with (11) and (16) gives a formula for the map $\Pi\colon u\mapsto\bar{u}$. We find with (3), (9), (13), (15), $\vartheta_6=-\vartheta_1$, $\vartheta_4=-\vartheta_3$

$$
\begin{aligned}
e^{\mu(\bar{u}-u)} &= e^{\mu(\sigma_0-\sigma_1+\sigma_2-\sigma_3)} \qquad (20)\\
&= \frac{a+\varphi(-1+u+T_1)}{a+\vartheta_6}\,\frac{a-\vartheta_1}{a-x_2}\,\frac{a+x_8}{a+\vartheta_4}\,\frac{a-\vartheta_3}{a-x_{14}}\\
&= \frac{2a+K_0+\delta_0}{2a-K_2-\delta_2}\,\frac{a+b+2+\frac{a+1}{a}K_8+(1+C(\varepsilon))\delta_5}{a+b+2-\frac{a+1}{a}K_{14}-(1+C(\varepsilon))\delta_{11}}\\
&= \frac{1+C(\varepsilon)+(\frac{1}{2a}+C(\varepsilon)+o)\delta_0+(\frac{1}{a+b+2}+C(\varepsilon)+o)\delta_5}{1-(\frac{1}{2a}+C(\varepsilon)+o)\delta_2-(\frac{1}{a+b+2}+C(\varepsilon)+o)\delta_{11}}\\
&= 1+C(\varepsilon)+\big[(\tfrac{1}{2a}+C(\varepsilon)+o)\delta_0+(\tfrac{1}{a+b+2}+C(\varepsilon)+o)\delta_5\big]\\
&\quad+\big[(\tfrac{1}{2a}+C(\varepsilon)+o)\delta_2+(\tfrac{1}{a+b+2}+C(\varepsilon)+o)\delta_{11}\big]. \qquad (21)
\end{aligned}
$$

Inserting the expressions for δ_0 and δ_5 we find that the expression in the first bracket [ ] in (21) is equal to

$$
\frac{1}{2}\Big[\frac{(2-a-b)(a+1)}{(a+b+2)(a-1)}-1+C(\varepsilon)+o\Big]e^{-\mu u}=\Big[\frac{2-a^2-ab}{(a+b+2)(a-1)}+C(\varepsilon)+o\Big]e^{-\mu u}
$$

and that the expression in the second bracket [ ] is equal to

$$
\Big[\frac{-2(1-ab)}{(a+b+2)(a-1)}+1+C(\varepsilon)+o\Big]e^{-\mu(1-u)}=\Big[-\frac{4-a-a^2-b(3a-1)}{(a+b+2)(a-1)}+C(\varepsilon)+o\Big]e^{-\mu(1-u)}.
$$

We thus have

$$
e^{\mu(\bar{u}-u)}=1+C(\varepsilon)+(p_0+C(\varepsilon)+o)e^{-\mu u}-(p_1+C(\varepsilon)+o)e^{-\mu(1-u)} \qquad (22)
$$

with

$$p_0 = \frac{2-a^2-ab}{(a+b+2)(a-1)}, \quad p_1 = \frac{4-a-a^2-b(3a-1)}{(a+b+2)(a-1)}.$$

Note that if f is odd then the first $C(\varepsilon)$ in (22) vanishes. This follows from the fact that for symmetric f one has $K_2 = -K_0$, $K_{14} = -K_8$.

We show that p_0 and p_1 are positive. $p_0 > 0$ follows immediately from the assumption $b < \frac{2}{a} - a$. p_1 is positive if

$$b < \frac{4-a-a^2}{3a-1}.$$

It is therefore sufficient to show that

$$\frac{2}{a} - a < \frac{4-a-a^2}{3a-1}.$$

This is equivalent to each of the following inequalities (we use $a > 1$)

$$\begin{aligned} -3a^3 + a^2 + 6a - 2 &< 4a - a^2 - a^3, \\ 0 &< 2(a^3 - a^2 - a + 1), \\ 0 &< (a-1)^2(a+1). \end{aligned}$$

This yields that p_1 is positive.

We show that $\Pi : u \mapsto \bar{u}$ is a contraction in I if ε and μ are chosen appropriately. From (22) we obtain for $u \in I$

$$\Pi(u) = u + \frac{1}{\mu} \log\left(1 + C(\varepsilon) + [p_0 + C(\varepsilon) + o]e^{-\mu u} - [p_1 + C(\varepsilon) + o]e^{-\mu(1-u)}\right) \tag{23}$$

implying

$$\Pi'(u) = 1 - [p_0 + C(\varepsilon) + o]e^{-\mu u} - [p_1 + C(\varepsilon) + o]e^{-\mu(1-u)}) < 1$$

for ε and $1/\mu$ sufficiently small. Since I is compact we conclude that Π is uniformly contracting in I with a contraction factor smaller than 1. To show $\Pi(I) \subset I$ it is sufficient to verify $\Pi(1/4) > 1/4$, $\Pi(3/4) < 3/4$. For given sufficiently small μ it is possible to find $\varepsilon > 0$ such that equation (23) yields

$$\Pi\left(\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{\mu}\log\Big(1 + C(\varepsilon) + [p_0 + C(\varepsilon) + o$$
$$-(p_1 + C(\varepsilon) + o)e^{-\mu/2}]e^{-\mu/4}\Big) > \frac{1}{4},$$
$$\Pi\left(\frac{3}{4}\right) = \frac{3}{4} + \frac{1}{\mu}\log\Big(1 + C(\varepsilon) - [p_1 + C(\varepsilon) + o$$
$$-(p_0 + C(\varepsilon) + o)e^{-\mu/2}]e^{-\mu/4}\Big) < \frac{3}{4}.$$

Note that if the first $C(\varepsilon)$ in both equations vanish then ε may be chosen independently of μ. By the contraction mapping principle the map Π admits a unique fixed point $u^* \in I$. This completes the proof of Lemma 2. $\perp$

We now define the initial function $\varphi^* \in \Phi$ by setting $\varphi^* := S^{-1}(u^*)$. Since $\Pi(u^*) = u^*$ it follows that $P(\varphi^*) = \varphi^*$ and hence φ^* generates a periodic solution x^* of (1). We now show that x^* is an attractive, periodic solution in the space $\mathcal{C}$.

Theorem 3. *Let the constants* a, b *satisfy* $a > 1$, $b > 0$ *and* $b < 2/a - a$. *Then there is* $\mu_0 > 0$ *such that for* $\mu > \mu_0$ *there is* $\varepsilon = \varepsilon(\mu)$ *such that for* $f \in F_\varepsilon^+$ *the following holds: The delay equation* (1) *admits a rapidly oscillating orbitally asymptotically stable periodic solution* x^*. *Moreover, for every* $\varphi \in \Phi_0$ *there is* $\delta > 0$ *such that every* $\psi \in \mathcal{C}$ *with* $|\psi - \varphi| < \delta$ *generates a solution tending orbitally to* x^*.

Remark

4. We again point out that if f is odd then in Theorem 3 ε does not depend on μ: Under the given assumptions there is $\varepsilon > 0$ and μ_0 such that for every $f \in F_\varepsilon^+ \cap \{f \mid f(-x) = -f(x)\}$ and $\mu > \mu_0$ the assertion of Theorem 3 holds.
5. If f is odd then the first term $C(x)$ in (23) vanishes. It follows that for odd f the estimate $u^* = \frac{1}{2} + O(\frac{1}{\mu})$ holds. $\triangleleft$

Proof. From Lemma 2 we already know that there is a periodic solution x^* with initial function φ^*. From the proof of Lemma 2 we also know that the reduced Poincaré map Π is a contraction in I. It follows that x^* orbitally attracts every solution with initial condition $\varphi \in \Phi_0$. Assume $|\psi - \varphi| < \delta$ for a small constant δ still to be determined. We denote the solution with initial condition ψ by $\hat{x}$, the one with initial condition $\varphi \in \Phi_0$ by x. We define $\hat{\vartheta}_{1,2} := \vartheta_{1,2} \mp \delta$, $\hat{\vartheta}_{3,4} := \vartheta_{3,4} \mp \delta$, $\hat{\vartheta}_{5,6} := \vartheta_{5,6} \mp \delta$. Let $\hat{t}_1 = -1 < \hat{t}_2 < \cdots < \hat{t}_{10}$ be the consecutive times such that $\varphi(\hat{t}_i) = x(\hat{t}_i) \in \{\hat{\vartheta}_1, \ldots, \hat{\vartheta}_6\}$, cf. Figure 4. By construction we have $\hat{t}_i = t_i + O(\delta)$,

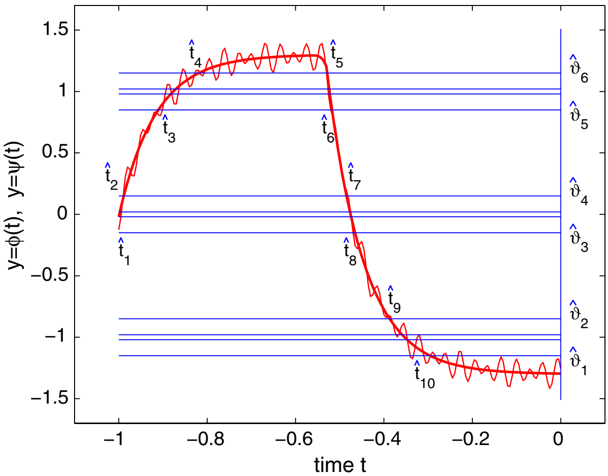

Figure 4. The functions φ and ψ.

$i=2,3,\ldots,10$, and for $t\in[\hat{t}_i,\hat{t}_{i+1}]$, $i=2,4,6,8$, the identity $f(\hat{x}(t))=f(\psi(t))=f(\varphi(t))=$ const holds, cf. Fig. 3. We conclude $\hat{x}(\hat{t}_i+1)=x(t_i+1)+O(\delta)$, $i=1,2,\ldots,10$ and that $\hat{x}$ is of exponential type in the intervals $[\hat{t}_i+1,\hat{t}_{i+1}+1]$, $i=2,4,6,8$. For $i=11,12,\ldots$ let $\hat{t}_i$ be the consecutive positive times such that $\hat{x}(\hat{t}_i)\in\{\vartheta_1,\ldots,\vartheta_6\}$ and set $\hat{x}_i:=\hat{x}(\hat{t}_i+1)$, $i=1,2,\ldots,$. We also find $\hat{x}_i=x_i+O(\delta)$ for $i=11,12,\ldots,17$. Since $\hat{x}_2<\vartheta_1$ and $\hat{x}_3\in(\vartheta_4,\vartheta_5)$ if δ is small enough we conclude $\hat{t}_2+1<\hat{t}_{11}<\hat{t}_{12}<\hat{t}_{13}<\hat{t}_{14}<\hat{t}_3+1$. For small δ we also find $\hat{x}_i\in(\vartheta_4,\vartheta_5)$ for $i=4,5,\ldots,8$ and $\hat{x}_i\in(\vartheta_2,\vartheta_3)$ for $i=9,10,\ldots,14$, etc., quite similarly as for x_i. Analogously as in the proof of Lemma 1 we get that $\overline{\psi}:=\hat{x}_{\hat{\tau}}\in\Phi$ for $\hat{\tau}=\hat{t}_{17}+1$. Since $S(\overline{\varphi})$ is in the interior of I and since $|S(\overline{\varphi})-S(\overline{\psi})|=O(\delta)$ we conclude that $S(\overline{\psi})\in I$ if δ is chosen sufficiently small. This completes the proof of Theorem 3. □

3. DELAY EQUATIONS WITH NEGATIVE FEEDBACK

In this section we consider the delay equation (1) with negative feedback function f. Since we also want to consider the feedback function used by Ivanov and Losson [3] we also include asymmetric functions in the definition of $\hat{F}_\varepsilon^-$. Let $\vartheta_i, i=1,\ldots,6$ be defined as in Section 2 and let $a>b>0>-c>-d$. We denote by $\hat{F}_\varepsilon^-$ the set of all functions $f\in C_b^1$ satisfying $f(x)\in[-d,a]$ for $x\in\mathbb{R}$ and

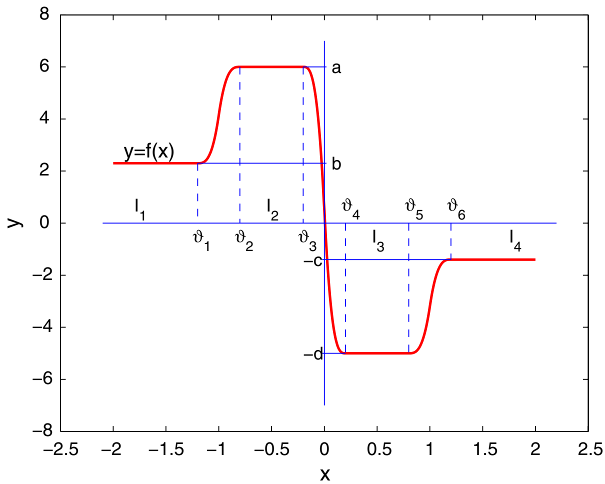

Figure 5. A function $f \in \hat{F}_\varepsilon^-$ modelling negative feedback.

$$f(x) = \begin{cases} b, & x \in (-\infty, \vartheta_1) = I_1, \\ a, & x \in (\vartheta_2, \vartheta_3) \quad = I_2, \\ -c, & x \in (\vartheta_4, \vartheta_5) \quad = I_3, \\ -d, & x \in (\vartheta_6, \infty) \quad = I_4. \end{cases}$$

In this section, we restrict the considerations to the case $a > b > 1$ and $d > c > 1$, as in the particular case treated in [3]

We first review the considerations of Section 2. The key computations are contained in the proofs of Lemma 1 and 2, where the numbers c_i and d_i were computed, c.f. Claims A and B. Remember that c_i and d_i determine the leading terms in the asymptotic expansion of $x_i = x_i(u)$ for $\mu \to \infty$. For the proof it is essential that the numbers c_i and d_i do not depend on ε. These numbers may also be computed for the limiting equation obtained by setting $\varepsilon = 0$. This simplifies the computations considerably. Since we already know how the x_i depend on ε we compute the c_i, d_i for $\varepsilon = 0$ and infer the results for ε small. A solution of Eq. (1) for $f \in \hat{F}_0^-$ is a continuous function $x: [-1, \infty) \mapsto \mathbb{R}$ which is piecewise of exponential type for $t > 0$ and satisfies (1) in each open interval of exponential type.

For the type of solutions considered in this section we need a new set of inital functions. Let $T_6 := \frac{1}{\mu} \log \frac{c+1}{c}$ be the time needed for a function of exponential type with limit $-c$ to decrease from 1 to 0. For the following definition see also Fig. 6.

$\Phi := \{\varphi \in \mathcal{C} |$ there are (small) numbers $\kappa > 0$, $\sigma > 0$ and $t_1 = -1 < t_2 < \cdots < t_8 < -T_6 - \kappa$ such that

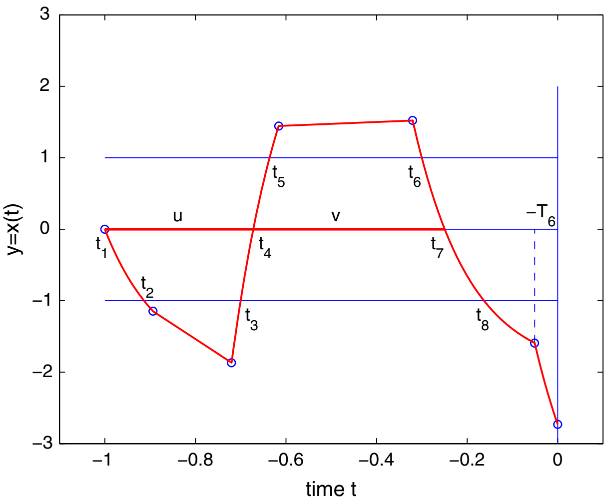

Figure 6. An initial function $\varphi \in \Phi$ for negative feedback.

- $\varphi(t_1) = 0$,
- φ is of exponential type with limit $-c$ in $[t_1, t_2+\kappa]$,
- $\varphi(t_2) = -1$,
- $\varphi(t) \leqslant -1-\sigma$ for $t \in [t_2+\kappa, t_3-\kappa]$,
- φ is of exponential type with limit a in $[t_3-\kappa, t_5+\kappa]$,
- $\varphi(t_3) = -1, \varphi(t_4) = 0, \varphi(t_5) = 1$,
- $\varphi(t) \geqslant 1+\sigma$ for $t \in [t_5+\kappa, t_6-\kappa]$,
- φ is of exponential type with limit $-c$ in $[t_6-\kappa, -T_6]$,
- $\varphi(t_6) = 1, \varphi(t_7) = 0, \varphi(t_8) = -1$,
- φ is of exponential type with limit $-d$ in $[-T_6, 0]\}$.

Similarly as in Section 2, we define a map $S\colon \Phi \to \mathbb{R}^2$, $\varphi \mapsto S(\varphi) := (u, v) := (t_4 - t_1, t_7 - t_4)$. It is easy to see that for a function $\varphi \in \Phi$ the expression $f(\varphi(t))$ only depends on u, v, t. More precisely, $f(\varphi(t)) = f(\psi(t))$ holds if $S(\varphi) = S(\psi)$. It follows that if $\varphi \in \Phi$ then for $t > 0$ the solution x of (1) is uniquely determined by $(u, v) = S(\varphi)$. For such solutions let $t_1 = -1 < t_2 < t_3 < \ldots$ be the consecutive times such that $x(t_i) \in \{-1, 0, 1\}$. Note that for $i = 1, \ldots, 8$ the numbers t_i coincide with the t_i defined before. Moreover, define the numbers $T_i := t_{i+1} - t_i$ and let $x_i := x(t_i + 1)$, $i = 1, 2, \ldots$ Note that the definition of the numbers t_i slightly differs from the one in Section 2 since we have put $\varepsilon = 0$. In the next lemma we state that the set Φ still may serve as Poincaré section, c.f. Fig. 7.

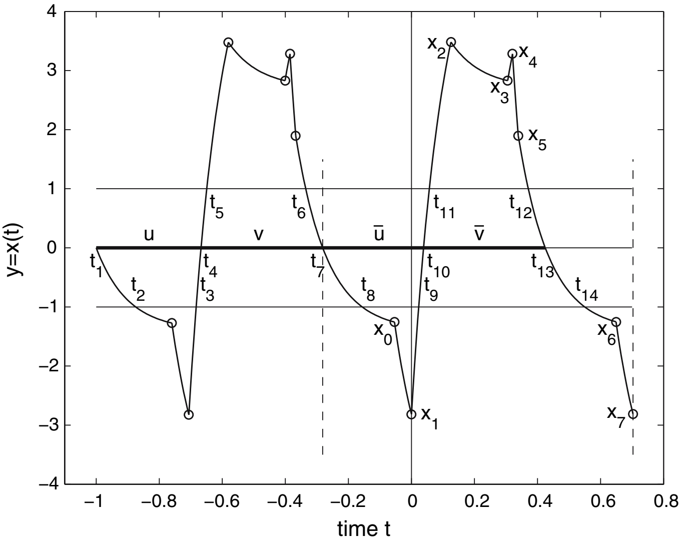

Figure 7. A solution of (1) with negative feedback.

Lemma 4. *Let the constants a, b, c, d satisfy* $a > b > 1 > -1 > -c > -d$, $a(c+1) > c(c^2+1) + d(c-1)$, $a(a-1)b > 2a + (a+1)d$ *and let* $I := [\frac{1}{6}, \frac{5}{12}] \times [\frac{1}{6}, \frac{5}{12}]$.

Then there is $\mu_0 > 0$ *such that for* $\mu > \mu_0$ *the following holds.*

1. *For every* $(u, v) \in I$ *there is* $\varphi \in \Phi$ *with* $S(\varphi) = (u, v)$. *If* $f \in \hat{F}_0^-$ *then any solution* $x(t)$ *with initial condition* φ *satisfying* $S(\varphi) \in I$ *has the following properties:*
2. $x(t)$ *is of exponential type with limit a for* $t \in [0, t_2 + 1]$.
3. $x(t_2 + 1) > 1$.
4. $x(t)$ *is of exponential type with limit b for* $t \in [t_2 + 1, t_3 + 1]$.
5. $x(t)$ *is of exponential type with limit a for* $t \in [t_3 + 1, t_4 + 1]$.
6. $x(t)$ *is of exponential type with limit* $-d$ *for* $t \in [t_4 + 1, t_5 + 1]$.
7. $x(t_5 + 1) > 1$.
8. $x(t)$ *is of exponential type with limit* $-c$ *for* $t \in [t_5 + 1, t_6 + 1]$.
9. $x(t_6 + 1) < -1$.
10. $x(t)$ *is of exponential type with limit* $-d$ *for* $t \in [t_6 + 1, t_7 + 1]$.
11. $x_\tau \in \Phi$ *for* $\tau := t_7 + 1$.

Proof.

1. The larger μ is, the closer are the numbers t_1, t_2, the numbers t_3, t_4, t_5 and the numbers t_6, t_7, t_8 and the existence of some $\varphi \in \Phi$ is guaranteed

for sufficiently large μ (e.g. it is sufficient to choose $\mu \geqslant 12 \max\{\log(c/(c-1)), \log(a/(a-1))\}$, this guarantees that $t_2 < t_3, t_5 < t_6, t_8 < -T_6$).

2.–3. We first compute $x_1 = \varphi(0)$. Since by definition of Φ we have $\varphi(-1+u+v) = 0$ and φ is of exponential type with limit $-c$ in $[-1+u+v, -T_6]$ we get $x_0 := \varphi(-T_6) = -c + c\, e^{-\mu(1-u-v-T_6)} = -c+(c+1)\, e^{-\mu(1-u-v)}$. The function φ being of exponential type with limit $-d$ in $[-T_6, 0]$ we get

$$\begin{aligned} x_1 &= \varphi(0) = -d + (x_0 + d)\, e^{-\mu T_6} \\ &= -d + (-c + d + (c+1)\, e^{-\mu(1-u-v)}) \frac{c}{c+1} \\ &= -\frac{c^2+d}{c+1} + \delta_1 \quad \text{with} \quad \delta_1 = c\, e^{-\mu(1-u-v)}. \end{aligned}$$

Since for $\varepsilon = 0$ the feedback function f is piecewise constant the formula (6) holds for all $i = 1, 2, \ldots$ We thus get with $e^{-\mu T_1} = (c-1)/c$

$$\begin{aligned} x_2 &= a + (x_1 - a)\, e^{-\mu T_1} \\ &= \frac{a}{c} - \frac{(c^2+d)(c-1)}{c(c+1)} + \delta_2 \quad \text{with} \quad \delta_2 = \frac{c-1}{c}\, \delta_1. \end{aligned}$$

All the numbers $\delta_1, \delta_2, \ldots$ are exponentially small with respect to μ. Thus, $x_2 > 1$ holds if the condition

$$\frac{a}{c} - \frac{(c^2+d)(c-1)}{c(c+1)} > 1$$

is satisfied and if μ is sufficiently large. This condition is equivalent to the assumption $a(c+1) > c(c^2+1) + d(c-1)$.

4.–7. We proceed with

$$\begin{aligned} x_3 &= b + (x_2 - b)\, e^{-\mu T_2} \\ &= b + \delta_3 \quad \text{with} \quad \delta_3 = (x_2 - b) \frac{c}{c-1} \frac{a+1}{a} e^{-\mu u}, \\ x_4 &= a + (x_3 - a) \frac{a}{a+1} \\ &= a \frac{b+1}{a+1} + \delta_4 \quad \text{with} \quad \delta_4 = \frac{a}{a+1}\, \delta_3, \\ x_5 &= -d + (x_4 + d) \frac{a-1}{a} \\ &= (a-1) \frac{b+1}{a+1} - \frac{d}{a} + \delta_5 \quad \text{with} \quad \delta_5 = \frac{a-1}{a+1}\, \delta_3. \end{aligned}$$

We conclude that Assertion 7 holds for μ large enough if

$$(a-1) \frac{b+1}{a+1} - \frac{d}{a} > 1\,.$$

This is equivalent to the assumption $a(a-1)b > 2a + (a+1)d$.

8.–11. We continue with

$$\begin{aligned} x_6 &= -c + (x_5 + c)\, e^{-\mu T_5} \\ &= -c + \delta_6 \quad \text{with} \quad \delta_6 = (x_5 + c)\, \frac{a}{a-1}\, \frac{c+1}{c}\, e^{-\mu v} \end{aligned}$$

and similarly as for x_1 we get

$$x_7 = -d + (x_6 + d)\, e^{-\mu T_6} = -\frac{c^2+d}{c+1} + \delta_7 \quad \text{with} \quad \delta_7 = \frac{c}{c+1}\, \delta_6\, .$$

It follows that $x_\tau \in \Phi$ for $\tau = t_7 + 1$. □

We conclude from Lemma 4 that the hyperplane $H := \{\varphi \in \mathcal{C} \mid \varphi(-1) = 0\} \supset \Phi$ may serve as Poincaré section. For every solution x of the delay equation (1) with initial condition $\varphi \in \Phi_0$ there is a Poincaré return time τ such that $P\colon \varphi \mapsto \overline{\varphi} := x_\tau \in \Phi \subset H$. Also in this case we may reduce P to a finite dimensional map. For any $w = (u, v) \in I$ there is $\varphi \in \Phi$ with $S(\varphi) = (u, v)$. We have already mentioned that for $t > 0$ the solution x with initial function φ only depends on (u, v). It follows that $\overline{\varphi} := P(\varphi)$ is uniquely determined by $(u, v) \in I$. We thus may define the so called *reduced Poincaré map* Π by $\Pi\colon I \to \mathbb{R}^2$, $w = (u, v) \mapsto \overline{w} = (\overline{u}, \overline{v}) := \Pi(u, v) := S(\overline{\varphi})$.

Lemma 5. *Let the constants a, b, c, d satisfy $a > b > 1 > -1 > -c > -d$, $a(c+1) > c(c^2+1) + d(c-1)$, $a(a-1)b > 2a + (a+1)d$,*

$$\frac{a - bc}{c(c-1)} > \frac{c^2+d}{c(c+1)} + \frac{ac-d}{a(a-1)} + \frac{b+1}{a+1} \tag{24}$$

and let $I := [\frac{1}{6}, \frac{5}{12}] \times [\frac{1}{6}, \frac{5}{12}]$.

Then there is $\mu_0 > 0$ such that for $\mu > \mu_0$ and $f \in \hat{F}_0^-$ the reduced Poincaré map Π admits a unique fixed point $w^* = (u^*, v^*) \in I$. Moreover, w^* is attractive.

Proof. With the same notation as in the proof of Lemma 4 we get

$$\begin{aligned} \overline{u} &= 1 - u - v + \frac{1}{\mu} \log\Big(\frac{a - x_1}{a}\Big), \qquad (25) \\ \overline{v} &= t_{13} - t_{10} = (t_4 + 1) - t_{10} + (t_5 - t_4) + (t_{13} - t_5 - 1) \\ &= u - \frac{1}{\mu} \log\Big(\frac{a - x_1}{a}\Big) + \frac{1}{\mu} \log\Big(\frac{a}{a-1}\Big) + \frac{1}{\mu} \log\Big(\frac{c + x_5}{c}\Big). \end{aligned}$$

For the the derivatives of Π we get

$$D_u\,\Pi = \begin{pmatrix} -1+\frac{1}{\mu}D_u\log(a-x_1) \\ 1-\frac{1}{\mu}D_v\log(a-x_1)+\frac{1}{\mu}D_u\log(c+x_5) \end{pmatrix},$$
$$D_v\,\Pi = \begin{pmatrix} -1+\frac{1}{\mu}D_v\log(a-x_1) \\ -\frac{1}{\mu}D_v\log(a-x_1)+\frac{1}{\mu}D_v\log(c+x_5) \end{pmatrix}$$

with

$$\frac{1}{\mu}D_u\log(a-x_1) = \frac{1}{\mu}D_v\log(a-x_1) = -\frac{c}{a-x_1}\,e^{-\mu(1-u-v)} =: -\rho,$$
$$\frac{1}{\mu}D_u\log(c+x_5) = \left(-\frac{1}{c+x_5}\,\frac{a-1}{a+1}\,(x_2-b)\,\frac{c}{c-1}\,\frac{a+1}{a}+o\right)e^{-\mu u} =: -\delta,$$
$$\frac{1}{\mu}D_v\log(c+x_5) = o^2$$

using the o-notation to denote terms of order $O(e^{-\mu u}+e^{-\mu(1-u-v)})$. We therefore have

$$\Pi(w) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} w + \Gamma(w)$$

with $\Gamma(w)=O(1/\mu)$ and $D\Gamma(w)=O(e^{-\mu u}+e^{-\mu(1-u-v)})$ for $w\in I$. It easily follows that Π has a unique fixed point $w^*=(1/3,1/3)+O(1/\mu)$ in I. It remains to show that w^* is attractive. It is sufficient to show that the eigenvalues of the Jacobian $D\Pi$ at the fixed point have magnitude smaller than 1. We already know

$$D\Pi = \begin{pmatrix} -1-\rho & -1-\rho \\ 1+\rho-\delta & \rho+o^2 \end{pmatrix}$$

with ρ and δ introduced above. The characteristic polynomial of $D\Pi$ is $P_{D\Pi}(\lambda)=\lambda^2+(1+o^2)\lambda+(1+\rho-\delta+o^2)$. The zeros of $P_{D\Pi}$ have magnitude smaller than 1 iff $\delta-\rho>0$. We have

$$\delta-\rho = \frac{(a-1)c(x_2-b)+o}{a(c-1)(c+x_5)}\,e^{-\mu u^*} - \frac{c}{a-x_1}\,e^{-\mu(1-u^*-v^*)}.$$

According to (25) one has $e^{-\mu(1-u^*-v^*)}=\frac{a-x_1}{a}\,e^{-\mu u^*}$. This yields

$$\delta-\rho = \left[\frac{(a-1)(x_2-b)}{(c-1)(c+x_5)}-1\right]\frac{c}{a}\,e^{-\mu u^*}.$$

We thus have $\delta-\rho>0$ if

$$\frac{x_2-b}{c-1} > \frac{c+x_5}{a-1}$$

or

$$\frac{a}{c(c-1)} - \frac{c^2+d}{c(c+1)} - \frac{b}{c-1} > \frac{c}{a-1} + \frac{b+1}{a+1} - \frac{d}{a(a-1)} + o\,.$$

We conclude that for sufficiently large μ the fixed point is attractive if the condition (24) holds. □

As we have mentioned in the beginning of this section the results may be tranfered to the case with small $\varepsilon > 0$.

Theorem 6. *Let the constants a, b, c, d with* $a > b > 1 > -1 > -c > -d$ *satisfy the conditions*

$$a(c+1) > c(c^2+1) + d(c-1), \tag{26}$$

$$a(a-1)b > 2a + (a+1)d, \tag{27}$$

$$\frac{a-bc}{c(c-1)} > \frac{c^2+d}{c(c+1)} + \frac{ac-d}{a(a-1)} + \frac{b+1}{a+1}\,. \tag{28}$$

Then there is $\mu_0 > 0$ *such that for* $\mu > \mu_0$ *there is* $\varepsilon = \varepsilon(\mu)$ *such that for* $f \in \hat{F}_\varepsilon^-$ *the following holds: the delay equation* (1) *admits an orbitally asymptotically stable periodic solution* x^* *(of the type as depicted in Fig. 7).*

Proof. Let $\varphi \in \Phi$ be such that $S(\varphi) = (u^*, v^*)$. Then set $\varphi^* := P(\varphi)$. The function φ^* generates a periodic solution x^*. Similarly as in the proof of Theorem 3 one can show that if $\psi \in \mathcal{C}$ is sufficiently close to φ^* then $P(\psi) \in \Phi$ holds with $P(\psi)$ arbitrarily close to φ^*. By Lemma 5 such initial functions generate solutions tending orbitally to x^*. □

4. VERY RAPID OSCILLATIONS

In this section, we consider Eq. (1) with an odd negative feedback function f. For simplicity we only consider the case $\varepsilon = 0$ and thus take $f \in F_0^- := \hat{F}_0^- \cap \{f \mid f \text{ is odd}\}$. We leave it to the reader to transfer the results to the case with small $\varepsilon > 0$. As in Section 2, we choose a and b to satisfy the conditions

$$a > 1 > b > 0, \quad b < \frac{2}{a} - a\,.$$

These conditions lead to periodic solutions of a particularly simple structure. The periodic solution restricted to one period consists of only four arcs of exponential types.

We first report on numerical experiments and show plots for $a = 1.2$ and $b = 0.3$. Figure 8 shows a plot of a stable periodic solution with 7 “humps” per unit time interval (more precisely: per time interval of length

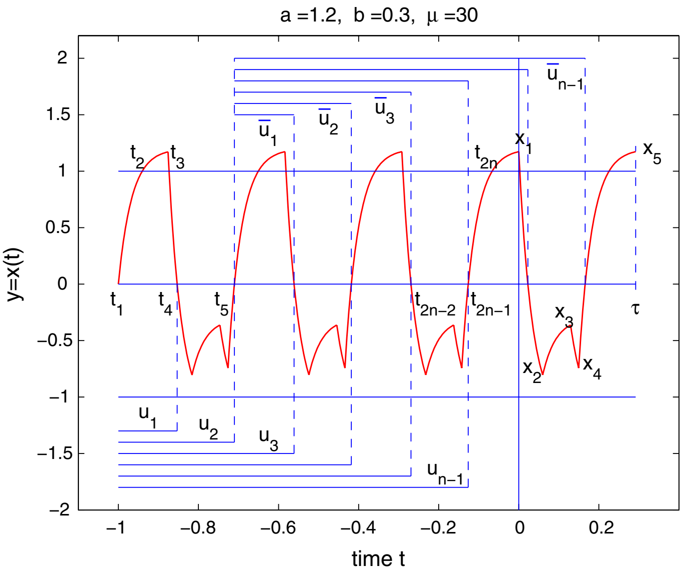

Figure 8. A solution with $n = 7$ humps.

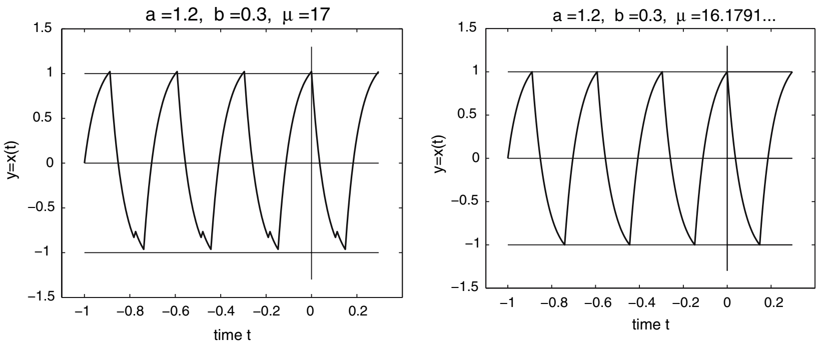

Figure 9. Periodic solutions for μ close to μ_7 and for $\mu = \mu_7$.

$l \in [1 + \frac{1}{2\mu} \log 2, 1 + \frac{1}{\mu} \log 2]$) for $\mu = 30$. We call a *hump* the restriction of the solution between two consecutive zeros. We distiguish two types of humps. We call a hump $x(\cdot)|_I$ of type A if $\max_{t \in I} |x(t)| > 1$ and of type B if $\max_{t \in I} |x(t)| \leqslant 1$, respectively. The solution of Fig. 8 has consecutive humps of type $A, B, A, B, \ldots$

We gradually decrease μ. For $\mu = 17$ the stable periodic solution is shown in the first plot of Fig. 9. One can observe that the maximum of $|x(t)|$ is only slightly larger than 1 for a hump of type A and only slightly smaller than 1 for a hump of type B. As μ approaches some limit value μ_7 the shape of the two types of humps look more and more the same, and the stable periodic solution approaches the function shown in the second plot of Fig. 9. This limiting function consists of arcs of exponential type with limit a increasing from -1 to 1 and of arcs of exponential type with limit $-a$ decreasing from 1 to -1. It is not difficult to determine μ_7 such that if the first hump begins at $t=-1$ then the 7th hump attains its maximum at $t=0$. In this section, we prove that if

$$\mu > \mu_n(a) := (n-1)\log\frac{a+1}{a-1} + \log\frac{a}{a-1} \tag{29}$$

for n odd then there exists a stable periodic solution with n humps per time interval of length $1+O(1/\mu)$. We believe that condition (29) is sharp, i.e. that for $\mu \leqslant \mu_n(a)$ there is no stable periodic solution with n humps per time interval of length $1+O(1/\mu)$.

Our proof is organized as follows. In Section 4.1, we first prove the existence of rapidly oscillating periodic solutions with an arbitrarily high number of oscillations per unit time interval. Then in Section 4.2, we show that the constructed periodic solutions are stable.

4.1. Existence of Rapidly Oscillating Periodic Solutions

We construct rapidly oscillating periodic solutions as follows. We first consider a slowly oscillating periodic solution of (1) with humps of type $A, B, A, B, \ldots$ By rescaling the time we obtain rapidly oscillating periodic solutions of arbitrarily high frequency.

Lemma 7. *If $t \mapsto \hat{p}(t)$ is a $\hat{\tau}$–periodic solution of $(1)_{\hat{\mu}}$ then for $m \in \mathbb{N}$ the function $t \mapsto p(t) := \hat{p}((1+m\hat{\tau})t)$ is a τ–periodic solution of $(1)_\mu$ with*

$$\tau = \frac{\hat{\tau}}{1+m\hat{\tau}}, \qquad \mu = \hat{\mu}(1+m\hat{\tau}).$$

Proof. We verify that p is a solution of $(1)_\mu$. We get

$$\begin{aligned}\frac{1}{\mu}\,\dot{p}(t) &= \frac{1}{\hat{\mu}(1+m\hat{\tau})}\,\frac{d}{dt}\,\hat{p}((1+m\hat{\tau})t)\\ &= \frac{1}{\hat{\mu}}\,\dot{\hat{p}}((1+m\hat{\tau})t)\\ &= -\hat{p}((1+m\hat{\tau})t) + f(\hat{p}((1+m\hat{\tau})t-1)).\end{aligned}$$

Using the $\hat{\tau}$–periodicity of $\hat{p}$ we get

$$\hat{p}((1+m\hat{\tau})t-1) = \hat{p}((1+m\hat{\tau})t-1-m\hat{\tau}) = \hat{p}((1+m\hat{\tau})(t-1)) = p(t-1)$$

and we therefore have

$$\frac{1}{\mu}\,\dot{p}(t) = -p(t) + f(p(t))\,.$$

□

We next compute our slowly oscillating periodic solution.

Lemma 8. *Let $f \in F_0^-$ with $0 < b < 1 < a$, $b < \frac{2}{a} - a$ and let $\hat{\mu} > \mu_1(a) := \log \frac{a}{a-1}$. Then, up to time shifts, Eq. $(1)_{\hat{\mu}}$ admits a unique slowly oscillating periodic solution $\hat{p}$ with $\min_t \hat{p}(t) > -1$ and $\max_t \hat{p}(t) > 1$. The period $\hat{\tau}(\hat{\mu})$ of $\hat{p}$ tends to $\frac{2}{\mu_1(a)} \log \frac{a+1}{a-1}$ as $\hat{\mu}$ tends to $\mu_1(a)$.*

Proof. Let $\hat{p}$ be the solution of $(1)_{\hat{\mu}}$ with initial function $\hat{\varphi}(t) = a - ae^{-\hat{\mu}(t+1)}$, $t \in [-1, 0]$. To show that $\hat{p}$ has the desired properties we compute the solution, cf. Fig. 10.

As in Section 3, we introduce the numbers $t_1 = -1 < t_2 < \ldots$ of consequtive times such that $\hat{p}(t_i) \in \{-1, 0, 1\}$, the numbers $T_i := t_{i+1} - t_i$ and $x_i^* := \hat{p}(t_i + 1)$. We have

$$x_1^* = \hat{p}(0) = a - ae^{-\hat{\mu}} > a - ae^{-\mu_1(a)} = 1$$

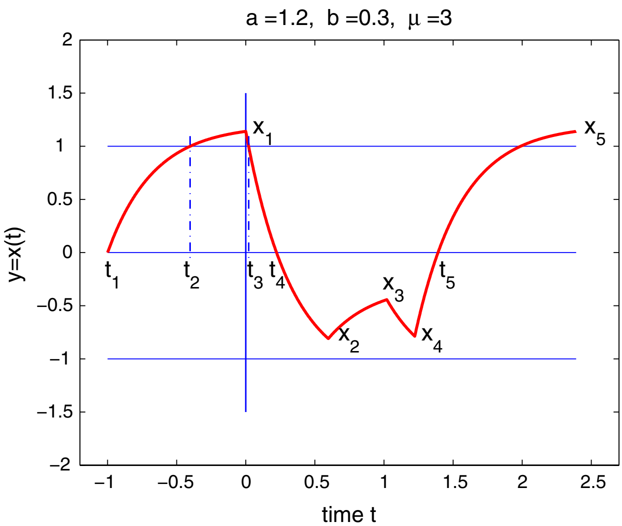

Figure 10. The slowly oscillating periodic solution.

and $\lim_{\hat{\mu}\to\mu_1(a)} x_1^* = 1$. We further get with $e^{-\hat{\mu}T_1} = (a-1)/a$

$$\begin{aligned} x_2^* &= \hat{p}(t_2+1) = -a + (x_1^*+a)e^{-\hat{\mu}T_1} \\ &= -(2-a) - (a-1)e^{-\hat{\mu}} \in \big(-1/a, -(2-a)\big) \end{aligned}$$

and $\lim_{\hat{\mu}\to\mu_1(a)} x_2^* = -1/a$. We continue with

$$x_3^* = -b + (x_2^*+b)e^{-\hat{\mu}T_2}\,.$$

Since $x_2^* < -b$ the solution $\hat{p}$ is monotonically increasing in $[t_2+1, t_3+1]$ and since $\lim_{\hat{\mu}\to\mu_1(a)} T_2 = 0$ we have $x_3^* \in \big(-1/a, -b\big)$ with $\lim_{\hat{\mu}\to\mu_1(a)} x_3^* = -1/a$. We compute

$$x_4^* = -a + (x_3^*+a)\frac{a}{a+1} \in \Big(-1, -a\,\frac{b+1}{a+1}\Big)$$

and $\lim_{\hat{\mu}\to\mu_1(a)} x_4^* = -1$. After t_4+1 the solution $\hat{p}$ is of exponential type with limit a. It admits the next zero at t_5 and is of exponential type with limit a until t_5+1. It follows that $\hat{p}$ is periodic with period

$$\hat{\tau} = t_5 + 1 = 1 + \frac{1}{\hat{\mu}}\,\log\frac{a+x_1^*}{a} + 1 + \frac{1}{\hat{\mu}}\,\log\frac{a-x_4^*}{a}\,. \tag{30}$$

For $\mu \to \mu_1(a)$ we therefore have

$$\lim_{\hat{\mu}\to\mu_1(a)} \hat{\tau}(\hat{\mu}) = 2\Big(1 + \frac{1}{\mu_1(a)}\,\log\frac{a+1}{a}\Big) = \frac{2}{\mu_1(a)}\,\log\frac{a+1}{a-1}\,.$$

□

We are ready to prove the existence of our rapidly oscillating periodic solutions.

Lemma 9. *Let $f \in F_0^-$ with $0 < b < 1 < a$, $b < \frac{2}{a} - a$, for $m \in \mathbb{N}$ let $n = 2m+1$ and let $\mu > \mu_n(a) := (n-1)\log\frac{a+1}{a-1} + \log\frac{a}{a-1}$. Then there is $\hat{\mu} > \mu_1(a)$ such that $\mu = \hat{\mu}(1+m\hat{\tau})$ where $\hat{\tau}$ is the period of the slowly oscillating periodic solution $\hat{p}$ constructed in Lemma 8. The function $p_n(t) := \hat{p}((1+m\hat{\tau})t)$ is a rapidly oscillating periodic solution of period $\tau_n = \hat{\tau}/(1+m\hat{\tau}) \in [\frac{2}{n}, \frac{2}{n-1}]$.*

Remark

6. A rapidly oscillating periodic solution as described in Lemma 9 is ploted in Fig. 8 for $m = 3$. ◃

Proof. Consider the function

$$h\colon \hat{\mu} \mapsto h(\hat{\mu}) = \hat{\mu}(1 + m\hat{\tau}(\hat{\mu})), \quad \hat{\mu} > \mu_1(a)\,.$$

Since by Lemma 8 we know that $\lim_{\hat{\mu}\to\mu_1(a)} \hat{\tau}(\hat{\mu}) = \frac{2}{\mu_1(a)} \log \frac{a+1}{a-1}$ it follows that

$$\lim_{\hat{\mu}\to\mu_1(a)} h(\hat{\mu}) = 2m \log \frac{a+1}{a-1} + \log \frac{a}{a-1} = \mu_n(a) < \mu\,.$$

Since $\hat{\tau}(\hat{\mu}) > 2$ by (30) one has $h(\hat{\mu}) > \hat{\mu}(1+2m) > \mu$ if $\hat{\mu}$ is sufficiently large. We conclude that there is $\hat{\mu}$ such that $h(\hat{\mu}) = \hat{\mu}(1+m\hat{\tau}) = \mu$ holds. Now Lemma 7 asserts that p_n is a rapidly oscillating periodic solution of $(1)_\mu$ of period $\tau_n = \frac{2}{2m+2/\tau} \in \big(2/(2m+1), 2/(2m)\big)$. □

4.2. Stability of Rapidly Oscillating Periodic Solutions

For odd $n = 2m+1$ we define a set Φ_n of initial functions appropriate to introduce a reduced Poincaré map Π_n. Note that Φ_n contains the rapidly oscillating periodic solutions p_n constructed in Section 4.1.

$\Phi_n := \{\varphi \in \mathcal{C} |$there are numbers $t_1 = -1 < t_2 < \cdots < t_{2n} < 0$ such that

- φ is of exponential type with limit a in $[t_1, t_2]$,
- $\varphi(t_1) = 0, \quad \varphi(t_2) = 1,$
- $\varphi(t) \geqslant 1 \quad$ for $t \in (t_2, t_3)$,
- φ is of exponential type with limit $-a$ in $[t_3, t_4]$,
- $\varphi(t_3) = 1, \quad \varphi(t_4) = 0,$
- $-1 < \varphi(t) \leqslant 0 \quad$ for $t \in [t_4, t_5]$,
- φ is of exponential type with limit a in $[t_5, t_6]$,
- $\varphi(t_5) = 0, \quad \varphi(t_6) = 1,$
- $\ldots$
- $-1 \leqslant \varphi(t) \leqslant 0 \quad$ for $t \in [t_{2n-2}, t_{2n-1}]$,
- φ is of exponential type with limit a in $[t_{2n-1}, 0]$,
- $\varphi(t_{2n-1}) = 0, \quad \varphi(t_{2n}) = 1$.

For $\varphi \in \Phi_n$ we define $S(\varphi) := u = (u_1, u_2, \ldots, u_{n-1})$ with $u_j = t_{2j+2} - t_1$ for odd j and $u_j = t_{2j+1} - t_1$ for even j, $j = 1, \ldots, n-1 = 2m$, cf. Fig. 8. Also here we find that $f(\varphi(t))$ only depends on t and $u = S(\varphi)$ for $\varphi \in \Phi_n$. It follows that if $\varphi \in \Phi_n$ then for $t > 0$ the solution x of (1) is uniquely determined by $u = S(\varphi)$. For a solution x with initial function $\varphi \in \Phi_n$ let $t_1 = -1 < t_2 < t_3 < \ldots$ be the consecutive times such that $x(t_i) \in \{-1, 0, 1\}$. Also here the set $H := \{\varphi \in \mathcal{C} | \varphi(-1) = 0\}$ may serve as Poincaré section. We denote the initial function of the periodic solution p_n defined in Lemma 9 by $\varphi_n^* := p_n(\cdot)|_{[-1,0]}$ and the corresponding u–vector by $u_n^* := S(\varphi_n^*)$.

Lemma 10. *Let* $f \in F_0^-$ *with* $0 < b < 1 < a$, $b < \frac{2}{a} - a$, *for* $m \in \mathbb{N}$ *let* $n = 2m+1$ *and let* $\mu > \mu_n(a)$. *Then there is a neighbourhood* $\Phi_n^0 \subset \Phi_n$ *of*

$\varphi_n^* = p_n(\cdot)|_{[-1,0]}$ *such that for any solution* $x: t \mapsto x(t)$ *with initial* $\varphi \in \Phi_n^0$ *the inclusion* $x_\tau \in \Phi_n$ *holds for* $\tau := t_5 + 1$.

Proof. In order to prove the assertion we compute the solution x up to $\tau := t_5 + 1$. We first compute $x_1 = \varphi(0)$. Since by definition of Φ_n we have $\varphi(-1 + u_{n-1}) = 0$ and φ is of exponential type with limit a in $[-1 + u_{n-1}, 0]$ we get

$$x_1 = a - a\, e^{-\mu(1-u_{n-1})}.$$

We have $e^{-\mu T_1} = (a-1)/a$ and thus get

$$\begin{aligned} x_2 &= -a + (x_1 + a)\,\frac{a-1}{a} \\ &= -(2-a) - (a-1)\, e^{-\mu(1-u_{n-1})}. \end{aligned}$$

If the neighbourhood Φ_n^0 of φ_n^* is taken sufficiently small then x_2 is so close to $x_2^* \in (-1, 0)$ such that $x_2 \in (-1, 0)$. Remember that $x_2^*, x_3^*, \dots$ are the numbers computed in the proof of Lemma 8. We proceed with

$$\begin{aligned} x_3 &= -b + (x_2 + b)\, e^{-\mu T_2} \\ &= -b - (-b - x_2)\frac{a+1}{a-1} e^{-\mu u_1} \\ &= -b - \frac{(a+1)(2-a-b)}{a-1} e^{-\mu u_1} - (a+1)\, e^{-\mu(1-u_{n-1})} e^{-\mu u_1}, \\ x_4 &= -a + (x_3 + a)\,\frac{a}{a+1}, \\ x_4 &= -a\frac{b+1}{a+1} - \frac{a(2-a-b)}{a-1} e^{-\mu u_1} - a\, e^{-\mu(1-u_{n-1})} e^{-\mu u_1}. \end{aligned} \tag{31}$$

We conclude that if Φ_n^0 is taken sufficiently small then x_3, x_4 and x_5 are close enough to x_3^*, x_4^* and x_5^*, respectively, to guarantee $x_3, x_4 \in (-1, 0)$ and $x_5 > 1$. It follows that $x_\tau \in \Phi_n$ for $\tau = t_5 + 1$. □

We conclude from Lemma 10 that the hyperplane $H := \{\varphi \in \mathcal{C} \mid \varphi(-1) = 0\} \supset \Phi_n$ may serve as Poincaré section. For every solution x of the delay equation (1) with initial condition $\varphi \in \Phi_n^0$ there is a Poincaré return time τ such that $P: \varphi \mapsto \overline{\varphi} := x_\tau \in \Phi_n \subset H$. Since $x_\tau \in \Phi_n$ we may reduce P to a finite dimensional map. For any $u \in U := S(\Phi_n^0)$ there is $\varphi \in \Phi_n^0$ with $S(\varphi) = u$. We have already mentioned that for $t > 0$ the solution x with initial function φ only depends on u. It follows that $\overline{\varphi} := P(\varphi)$ is uniquely determined by $u \in U$. We thus may define the *reduced Poincaré map* Π_n by $\Pi_n: U \to \mathbb{R}^{n-1}$, $u \mapsto \overline{u} := \Pi_n(u) := S(\overline{\varphi})$. Simple considerations, cf. Fig. 8, lead to the following formulas for the map $\Pi_n: u \mapsto \overline{u}$

$$\overline{u}_j = u_{j+2} - u_2, \quad j = 1, \dots, n-3, \tag{32}$$
$$\overline{u}_{n-2} = 1 - u_2 + \sigma_1, \tag{33}$$
$$\overline{u}_{n-1} = 1 - u_2 + u_1 + \sigma_2, \tag{34}$$

where

$$\sigma_1 = \frac{1}{\mu} \log \frac{a + x_1}{a}, \quad \sigma_2 = \frac{1}{\mu} \log \frac{a - x_4}{a} \tag{35}$$

are the time needed for the solution x to decrease from x_1 to 0 and the time needed for the solution x to increase from x_4 to 0, respectively. We compute the Jacobian of the reduced Poincaré map Π_n. We get (written here for $n = 7$)

$$D\Pi_n = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & -A \\ 1-B & -1 & 0 & 0 & 0 & C \end{pmatrix}, \tag{36}$$

where we used the following abreviations (and we already introduce D for later use):

$$\begin{aligned} A &= -D_{u_{n-1}}\sigma_1 = \frac{a\, e^{-\mu(1-u_{n-1})}}{a + x_1}, \\ B &= -D_{u_1}\sigma_2 = \frac{\frac{2-a-b}{a-1}\, e^{-\mu u_1} + e^{-\mu(1-u_{n-1})}\, e^{-\mu u_1}}{\frac{a-x_4}{a}}, \\ C &= D_{u_{n-1}}\sigma_2 = \frac{e^{-\mu(1-u_{n-1})}\, e^{-\mu u_1}}{\frac{a-x_4}{a}}, \\ D &= B - A + AB\,. \end{aligned} \tag{37}$$

The aim is to show that the spectral radius of the Jacobian $D\Pi_n$, evaluated at the fixed point u_n^*, is smaller than 1. We consider the characteristic polynomial

$$P_{D\Pi_n}(\lambda) = \det \begin{pmatrix} -\lambda & -1 & 1 & 0 & 0 & 0 \\ 0 & -1-\lambda & 0 & 1 & 0 & 0 \\ 0 & -1 & -\lambda & 0 & 1 & 0 \\ 0 & -1 & 0 & -\lambda & 0 & 1 \\ 0 & -1 & 0 & 0 & -\lambda & -A \\ 1-B & -1 & 0 & 0 & 0 & -\lambda + C \end{pmatrix}$$

of $D\Pi_n$. In order to simplify the matrix we subtract the λ–fold of the $(2(m-j)+1)$th and the $(2(m-j)+2)$th column from the $(2(m-j)-1)$th and the $2(m-j)$th column first for $j=1$, then for $j=2$ and so on up to $j=m-1$. This eliminates all λ-s in the first $2m-2$ rows. We get

$$P_{D\Pi_n}(\lambda) = \det \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -\lambda^3 & -1-A\lambda^2 & -\lambda^2 & -A\lambda & -\lambda & -A \\ 1-B & -1-\lambda^3+C\lambda^2 & 0 & -\lambda^2+C\lambda & 0 & -\lambda+C \end{pmatrix}.$$

We add the 3rd, the 4th, ..., the $(n-1)$st column to the second. Then the determinant reduces to the determinant of the 2×2-matrix consisting of the first two elements in the last two rows. We thus have

$$\begin{aligned} P_{D\Pi_n}(\lambda) &= \det \begin{pmatrix} -\lambda^m & -(1+A)(\lambda^{m-1}+\cdots+1) \\ 1-B & -\lambda^m-(1+C)(\lambda^{m-1}+\cdots+1) \end{pmatrix} \\ &= \lambda^{2m} + (1+C)\,\lambda^m\,\frac{\lambda^m-1}{\lambda-1} + (1+A)(1-B)\,\frac{(\lambda^m-1)}{\lambda-1}. \end{aligned}$$

Multiplying $P_{D\Pi_n}$ by $\lambda-1$ and using $(1+A)(1-B)=1-D$ by definition (37) yields

$$(\lambda-1)\,P_{D\Pi_n}(\lambda) = \lambda^{2m+1} - \left[C\lambda^{2m} + (D-C)\lambda^m + (1-D)\right]. \tag{38}$$

Lemma 11. *If a polynomial P is given by*

$$(\lambda-1)\,P(\lambda) = \lambda^{2m+1} - \left[C\lambda^{2m} + (D-C)\lambda^m + (1-D)\right]$$

with $0<C<D<1$ then all zeros of P lie in the interior of the unit circle.

Proof. We show that $P(\lambda)\neq 0$ for any λ with $|\lambda|\geqslant 1$. We distinguish several cases.

1. $|\lambda|=r>1$.
   We have

$$\begin{aligned} |(\lambda-1)\,P(\lambda)| &\geqslant r^{2m+1} - \left[Cr^{2m} + (D-C)r^m + (1-D)\right] \\ &\geqslant r^{2m+1} - \left[C + (D-C) + (1-D)\right]r^{2m} = r^{2m}(r-1) > 0\,. \end{aligned}$$

2. $|\lambda|=1$, $\lambda^m\neq 1$.
   We have

$$\begin{aligned} |(\lambda-1)\,P(\lambda)| &\geqslant 1 - \left|C\lambda^{2m} + (D-C)\lambda^m + 1 - D\right| \\ &> 1 - \big(C + (D-C) + 1 - D\big) = 0. \end{aligned}$$

3. $\lambda^m = 1,\ \lambda \neq 1$.
   We have
   $$(\lambda - 1)\,P(\lambda) = \lambda^{2m+1} - \big[C + (D - C) + 1 - D\big] = (\lambda^m)^2\lambda - 1 = \lambda - 1 \neq 0.$$
4. $\lambda = 1$.
   For $\lambda \neq 1$ we have
   $$P(\lambda) = \frac{\lambda^{2m+1} - 1}{\lambda - 1} - (D + C\lambda^m)\frac{\lambda^m - 1}{\lambda - 1}\,.$$
   Taking the limit $\lambda \to 1$ leads to
   $$P(1) = 2m + 1 - (D + C)m > 2m + 1 - 2m = 1\,.$$

□

We now prove

Lemma 12. *Let $f \in F_0^-$ with $0 < b < 1 < a$, $b < \frac{2}{a} - a$, for $m \in \mathbb{N}$ let $n = 2m + 1$ and let $\mu > \mu_n(a)$. Let u_n^* be a fixed point of the reduced Poincaré map Π_n.*

Then all zeros of the characteristic polynomial of the Jacobian $D\Pi_n(u_n^*)$ lie in the interior of the unit circle.

Proof. By Lemma 11 it is sufficient to show that the constants C, D given in (37) satisfy $0 < C < D < 1$. Note that by definition 37 the constants A, B, C are positive. We first prove $C < D$. By (32) we find that at the fixed point $u = u_n^*$ the following equations hold

$$\begin{aligned} u_{2j+1} &= ju_2 + u_1, \quad j = 0, 1, \ldots, m - 1,\\ u_{2j} &= ju_2, \quad j = 1, 2, \ldots, m\,. \end{aligned}$$

Now (33) implies $(m - 1)u_2 + u_1 = 1 - u_2 + \sigma_1$ or $u_1 = 1 - mu_2 + \sigma_1 = 1 - u_{n-1} + \sigma_1$. Multiplying this equation by μ and taking exponentials yields by definition (35) of σ_1

$$e^{\mu u_1} = \frac{e^{\mu(1 - u_{n-1})}(a + x_1)}{a}\,.$$

It follows from the definition of A that $A = e^{-\mu u_1}$ holds at the fixed point u_n^*. By definition (37) the following inequalities are equivalent

$$\begin{aligned} D &> C,\\ B - C - A + AB &> 0,\\ \frac{\frac{2-a-b}{a-1}e^{-\mu u_1}}{\frac{a-x4}{a}} - A + AB &> 0. \end{aligned}$$

Using that $e^{-\mu u_1} = A$, deviding by A and multiplying by $\frac{a-x4}{a}$ we get

$$\frac{2-a-b}{a-1} - \frac{a-x_4}{a} + \frac{2-a-b}{a-1} e^{-\mu u_1} + e^{-\mu(1-u_{n-1})} e^{-\mu u_1} > 0.$$

By the definition (31) of x_4 this is equivalent to

$$\frac{2-a-b}{a-1} - \frac{a+b+2}{a+1} > 0.$$

Multiplying by $(a+1)(a-1)$ and deviding by 2 we get the equivalent inequality $\frac{2}{a} - a - b > 0$ which is satisfied by assumption.

It remains to prove $D < 1$. This is equivalent to the following inequalities

$$\begin{aligned} B - A + AB &< 1, \\ B(A+1) &< A+1, \\ B &< 1, \\ \frac{2-a-b}{a-1} e^{-\mu u_1} + e^{-\mu(1-u_{n-1})} e^{-\mu u_1} &< \frac{a-x_4}{a}. \end{aligned}$$

Using the definition (31) of x_4 this inequality is equivalent to

$$0 < \frac{a+b+2}{a+1},$$

which is obviously satisfied. □

We thus have proved the following result.

Lemma 13. *Let $f \in F_0^-$ with $0 < b < 1 < a$, $b < \frac{2}{a} - a$, for $m \in \mathbb{N}$ let $n = 2m+1$ and let $\mu > \mu_n(a)$. Then the reduced Poincaré map Π_n admits an orbitally attractive fixed point u_n^*.*

Similarly as at the end of Sections 2 and 3 one may infer the result to

Theorem 14. *Let $f \in F_0^-$ with $0 < b < 1 < a$, $b < \frac{2}{a} - a$, let n be an odd integer and let $\mu > \mu_n(a)$.*

Then the delay equation $(1)_\mu$ admits an orbitally asymptotically stable periodic solution p_n of period $\tau_n \in [\frac{2}{n}, \frac{2}{n-1}]$.

5. FINAL REMARKS

In this last section, we discuss the obtained results and we report on numerical experiments. We proved analytically that delay equations with positive or negative feedback may admit orbitally asymptotically stable, rapidly oscillating periodic solutions. In the case of negative feedback this was first shown by Ivanov and Losson [3]. Their proof was computer-assisted. They showed, that for a particular feedback function there exists a stable, rapidly oscillating periodic solution. In the setting of Section 3 their feedback function corresponds to the parameter values $a = 5.41935...$, $b = 2.32258...$, $c = 1.49032...$ and $d = 5.22580...$ It is easily verified that the assumptions of Theorem 6 are satisfied for these parameter values. It hence follows that in the particular case of the feedback function of Ivanov and Losson stable, rapidly oscillating periodic solutions exist for *all* sufficiently large μ. In contrast, the paper by Ivanov and Losson covers the case $\mu = 7.75$. Moreover, we have an explanation why the eigenvalues of the Jacobian of the Poincaré return map have modulus very close to 1. Our computations show that the modulus of both eigenvalues are exponentially close to 1 as $\mu \to \infty$.

We report that in addition we did the computations for symmetric positive feedback functions in the case $a > b > 1$ considered by Schulze–Halberg [9]. We found that under the conditions

$$\begin{aligned} 0 &< 2 - b(a-1), \\ 0 &< a^3(b+1) - a^2(6b^2 - 3b - 1) - a(3b^3 + 5b^2 - 10b + 4) + b^3 + 5b^2 + 4b - 8 \end{aligned}$$

there are stable, rapidly oscillating periodic solutions of the type described by Schulze–Halberg for *all* sufficiently large μ. In this case, too, we were able to explain why one of the two eigenvalues is so close to 1.

Assume that for negative feedback $f \in F_\varepsilon^-$ is symmetric with $a > b > 1$. Now reconsider the hypotheses of Theorem 6. The first two conditions (26), (27) which guarantee the existence of a periodic solution reduce to

$$\begin{aligned} a &> \frac{b(b^2+1)}{2}, \\ (a-1)(b-1) &> 4. \end{aligned}$$

The third condition (28) for stability reads

$$a^2 - ab(b^2+1) + 2b^2 - 1 > 0.$$

In Fig. 11, we show for which parameter values the three conditions are satisfied.

Surprisingly, numerical experiments indicate that no Hopf bifurcation occurs when transitions from the stable to the unstable region are considered. For given a and large fixed μ, e.g. the following seems to hold. There is b_0 such that if b is slightly smaller than b_0 then there is an attractive fixed point w of Π with a neighbourhood U independent of b in its domain of attraction. If b is slightly larger than b_0 then the fixed point of Π becomes repelling and every trajectory starting in $U - \{w\}$ leaves U.

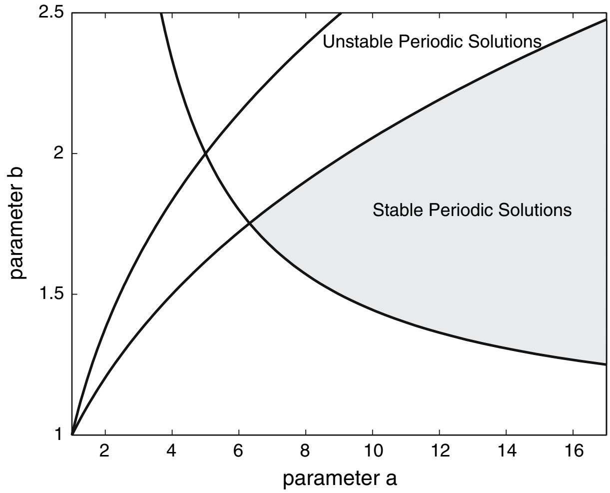

Figure 11. Domain for stable, rapidly oscillating, periodic solutions for negative feedback function and $a > b > 1$.

There do not only exist the rapidly oscillating stable periodic solutions discussed in Section 4. We assume $f \in F_0^-$ being symmetric. Let us consider solutions with n humps in a time interval of length $1 + O(1/\mu)$ and let us therefore assume $\mu > \mu_n(a)$. For the initial function φ an arbitrary sequence of n humps of types A and B may be chosen. As can easily be seen the following holds as long as the width of the humps are close enough to $1/n$ and μ is sufficiently large. If the kth hump is a hump of type A then the $(k+n)$th hump is a hump of type B and vice versa. In other words: a hump of type A at time t generates a hump of type B at time $t + 1 + O(1/\mu)$. We considered many such initial functions for $n = 3, 5, 7, 9$ and computed the corresponding solutions. Surprisingly, we observed that all these solutions tend to a stable periodic solution of the corresponding type. Let $\Gamma = \gamma_1\gamma_2 \ldots \gamma_n$ be a block of length n of symbols $\gamma_k \in \{A, B\}$. Let φ be an initial function consisting of humps of type as described by Γ. The solution x with initial function φ then has humps of types $\Gamma\overline{\Gamma}\Gamma\overline{\Gamma}\ldots$ where $\overline{\Gamma} = \overline{\gamma}_1\overline{\gamma}_2 \ldots \overline{\gamma}_n$ with $\overline{A} = B$ and $\overline{B} = A$. If x is a periodic solution, then we say that it is of type Γ. All these considerations lead us to the following

Conjecture 1. *Let $a > 1 > b > 0$ satisfy $b < 2/b - a$, let $f \in F_0^-$, and let $n \geqslant 3$ be an odd integer.*

1. *If $\mu > \mu_n(a)$ then for any $\Gamma \in \{A, B\}^n$ there exists a rapidly oscillating stable periodic solution of type Γ.*

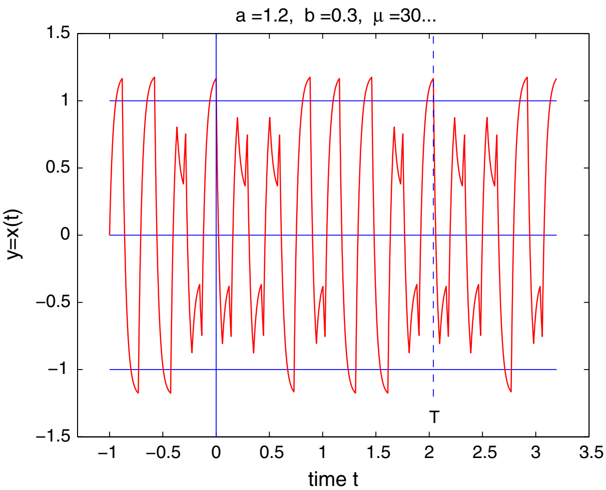

Figure 12. A rapidly oscillating periodic solution of type $AAAABBA$ with period T.

2. *If* $\mu \leqslant \mu_n(a)$ *then equation* (1) *does not admit a rapidly oscillating stable periodic solution with n humps per time interval of length* $1 + O(1/\mu)$.

To illustrate Conjecture 1 we have plotted a periodic solution with 7 humps of type $\Gamma = AAAABBA$ in Fig. 12.

Similar considerations can be done for Eq. (1) with positive feedback function $f \in F_0^+$. For positive feedback functions rapidly oscillating stable periodic solutions consist of an even number of humps in a time interval of length $1 + O(1/\mu)$. In Fig. 13, we have plotted a periodic solution with 6 humps of type $AAABBA$. We conjecture

Conjecture 2. *Let* $a > 1 > b > 0$ *satisfy* $b < 2/b - a$, *let* $f \in F_0^+$, *and let* $n \geqslant 2$ *be an even integer.*

1. *If* $\mu > \mu_n(a)$ *then for any* $\Gamma \in \{A, B\}^n$ *there exists a rapidly oscillating stable periodic solution of type* Γ.
2. *If* $\mu \leqslant \mu_n(a)$ *then Eq.* (1) *does not admit a rapidly oscillating stable periodic solution with n humps per time interval of length* $1 + O(1/\mu)$.

Let us discuss the number of stable periodic solutions with n humps per time interval of length $1 + O(1/\mu)$. Consider, e.g. the block $\Gamma = AAABA$. Conjecture 1 states that for $f \in F_0^-$, $\mu > \mu_5$ Eq. (1) admits a stable periodic solution with humps of types $AAABA\ BBBAB\ AAABA\ B \ldots$ (this

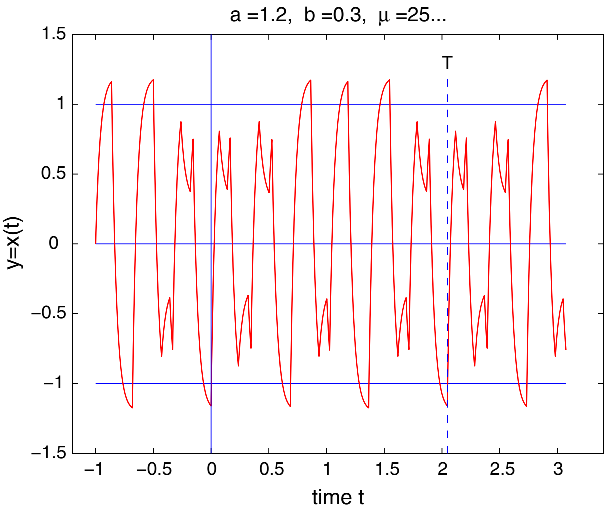

Figure 13. A rapidly oscillating periodic solution of type $AAABBA$ for positive feedback.

sequence of symbols is called a *binary n–stage shift register sequence*, cf. Golomb [2]). Now consider phase shifts of this periodic solution of shift lengths of 1, 2, 3,... humps. The resulting stable periodic solutions are of type $AABAB$, $ABABB$, $BABBB, \dots, BAAAB$. We call two blocks Γ and Γ' of length n *equivalent* if Γ' occurs in the sequence $\Gamma\overline{\Gamma}\Gamma$. Let $a(n)$ denote the number of essentially different blocks (the number of equivalence classes) of length n. For negative feedback (n odd) we found $a(3)=2$, $a(5)=4$, $a(7)=10$, $a(9)=30$ and for positive feedback (n even) $a(2)=1$, $a(4)=2$, $a(6)=6$, $a(8)=16$. In Sloane and Plouffe [10] we found the following formula for $a(n)$

$$a(n)=\sum_{d\,\text{odd},\ d|n}\frac{\varphi(d)2^{n/d}}{2n}, \tag{39}$$

where φ denotes the Euler function. With the methods of Colomb [2] it is not difficult to prove (39).

Note that with x also $-x$ is a solution of (1) since f is assumed to be an odd function. We show that if x is a periodic solution of type Γ then it is not possible that $-x$ is a time shift of the solution x. The proof is by contradiction. Without loss of generality let us assume that the first hump of x is positive. The symbol sequence of x is $\Gamma\overline{\Gamma}\Gamma\overline{\Gamma}\cdots=\gamma_1\gamma_2\gamma_3\dots$ with

$$\gamma_{j+n}=\overline{\gamma}_j\,. \tag{40}$$

Assuming that $-x(t) = x(t+\tau)$ implies that there is an odd integer p such that

$$\gamma_{j+p} = \gamma_j \,. \tag{41}$$

Let $k \in \mathbb{N}$ be the smallest natural number such that $n|kp$, hence $kp = ln$ for some $l \in \mathbb{N}$. Clearly, k and l are relatively prime. Assume l to be even. Then k is odd and $kp \neq ln$ since kp is odd and ln is even. Thus l has to be odd and (40) implies $\gamma_{ln+1} = \overline{\gamma}_1$. On the other hand (41) implies $\gamma_{kp+1} = \gamma_1$ which is a contradiction to $kp = ln$. This implies that for every type Γ there are precisely 2 periodic solutions of type Γ, one starting with a positive hump, the other with a negative hump.

We thus have the following

Result 14. *If Conjectures 1 and 2 are true then for $a > 1 > b > 0$ with $b < 2/b - a$ and for $\mu > \mu_n(a)$ the following holds. If n is even and $f \in F_0^+$ or if n is odd and $f \in F_0^-$ then, up to phase shifts, there are $2a(n)$ different rapidly oscillating stable periodic solutions with n humps per time interval of length $1 + O(1/\mu)$.*

We are very pleased to report that in her Diploma Thesis M. Rupflin [7] was able to show that also for positive feedback functions $f \in F_0^+$ there are stable periodic solutions with arbitrary high frequency. Her proof works if μ is taken sufficiently large. See also Rupflin [8] for related results concerning rapidly oscillating heteroclinic connections of periodic solutions.

REFERENCES

1. Aschwanden, A., Schulze–Halberg, A., and Stoffer, D. (2006). Stable periodic solutions for delay equations. *Disc. Cont. Dyn. Sys.* **14**(4), 721–736.
2. Golomb, S. W., with portions co-authored by Welch, L. R., Goldstein, R. M., and Hales, A. W. (1967). *Shift Register Sequences*, Holden–Day, San Francisco.
3. Ivanov, A. F., and Losson, J. (1999). Stable rapidly oscillating solutions in delay differential equations with negative feedback. *Differential Integral Equations* **12**, 811–832.
4. Krisztin, T., and Walther, H.-O. (2001). Unique periodic orbits for delayed positive feedback and the global attractor. *J. Dynam. Differential Equations* **13**(1), 1–57.
5. Krisztin, T., Walther, H.-O., and Wu, J. (1999). *Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback*, Fields Institute Monographs, 11. American Mathematical Society, Providence, RI.
6. Mallet-Paret, J., and Walther, H. O. (1994). *Rapid Oscillations are Rare in Scalar Systems Governed by Monotone Negative Feedback with a Time Delay*, Preprint.
7. Rupflin, M. (2006). *Existenz schnellschwingender periodischer Lösungen bei steifen Differentialgleichungen mit Verzögerung*, Diploma Thesis, ETH Zurich, Switzerland.
8. Rupflin, M. Heteroclinic connections of solutions of delay differential equations, preprint.
9. Schulze-Halberg, A. (2003). Orbital asymptotisch stabile periodische Lösungen von delay–Gleichungen mit positiver Rückkopplung. *Mitt. Math. Sem. Giessen* **252**, 1–106.
10. Sloane, N. J. A., and Plouffe, S. (1995). *The Encyclopedia of Integer Sequences*, Academic, San Diego, CA. See also *The online encyclopedia of integer sequences*, http://www.research.att.com/~ njas/sequences/.