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Optimism, Pessimism, and Realism in Economic Growth Theory

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Для моих родителей

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Abstract

This thesis analyzes non-standard issues in the context of economic growth, such as optimism, war, and globalization. First, the introductory chapter provides a motivation and an overview of the research.

In Chapter 2, we study the source and desirability of research bubbles. Research bubbles are instances when decision-makers overestimate the productivity of research, and thus, overinvest in it. We build an OLG model with endogenous growth where growth stems from the accumulation of knowledge. In out setting, the productivity of knowledge creation, which is research, is unknown. As a consequence of this, the labor market equilibrium and the outcome of the economy depend on the distribution and aggregation of beliefs. We find that under complete rationality of households, firms, and the government, research bubbles emerge. They appear due to the self-selection of optimistic agents into research and the subsequent aggregation of beliefs by the government. Research bubbles typically fail to implement the social optimum in a decentralized economy, but are welfare-improving overall. We discuss institutional arrangements that can prevent such bubbles from bursting and other mechanisms that move the economy closer to the social planner solution, such as wage contracts and debt financing.

In Chapter 3, we study the effects of disease and war on the accumulation of human and physical capital. In an OLG model with three generations, both types of capital are prone to destruction. Altruistic young adults decide about investments in schooling and reproducible capital. Schooling allows children to increase their human capital. Also, it depends on the human capital stock of young adults,

so that shocks to this stock have repercussions for future generations. We find that, for a wide rage of parameter values, an economy cannot escape the state of backwardness, and thus, finds itself in a poverty trap. However, altruism of young adults can rule out the fall into such a trap. Moreover, if altruism is strong enough, progress is the only steady state of the economy. We observe that there are steady states between the two extremes of progress and backwardness. Some of them are stationary. Finally, we demonstrate that the robustness of an economy to stochastic shocks crucially depends on its initial endowment.

In Chapter 4, we present a unified framework that incorporates institutions, the accumulation of human capital and international capital flows. The model that we use is built around two countries that compete for international capital. We can show that a small initial inequality in institutions can lead to substantial differences between countries in the long-run. The reason is that a small difference in institutions can lead to inflows of capital that set the accumulation of human capital in motion.

Finally, in Chapter 5, we turn to the topic of artificial intelligence, and discuss the impact of this technological development on growth. We provide an overview of selected articles and describe which considerations might be important for future research.

Zusammenfassung

Diese Arbeit diskutiert und analysiert relevante Themen im Bereich des ökonomischen Wachstums, und zwar Optimismus, Krieg und Globalisierung. Wir beginnen damit, in einem Einleitungskapitel, einen Überblick über unsere Forschung zu geben und die Forschungsfragen gleichzeitig zu begründen.

Danach, in Kapital 2, untersuchen wir die Ursache und Erwünschtheit von sogenannten "research bubbles". Research bubbles sind eine Art von Hochstimmungsphasen und sie entstehen in Situationen, wenn Entscheidungsträger die Produktivität von Forschung überschätzen und daher zu viel in sie investieren. Wir bauen ein *overlapping generations* (OLG) Modell mit endogenem Wachstum, in welchem Wachstum durch die Akkumulation von Wissen entsteht. In diesem Modell ist die Produktivität der Forschung, also des Prozesses der Schaffung von Wissen, unbekannt. Als Konsequenz hängt das Gleichgewicht des Arbeitsmarktes sowie der gesamten Wirtschaft von der Verteilung und Aggregation von Vorstellungen über die besagte Produktivität ab. Wir finden, dass, obwohl Haushalte, Firmen und der Staat sich rational verhalten, *research bubbles* auftreten. Sie entstehen wegen der Selbstselektion von Forschern in Forschung und wegen der anschließenden Aggregation von Vorstellungen durch den Staat. In einer dezentralen Ökonomie erzielen *research bubbles* das soziale Optimum für gewöhnlich nicht. Jedoch erhöhen sie grundsätzlich die Wohlfahrt. Wir diskutieren institutionelle Konstrukte, die das Platzen einer *research bubble* verhindern können, und weitere Mechanismen, die die Ökonomie näher an die Lösung des sozialen Planers bewegen, und zwar spezielle Lohnverträge und Schuldenfinanzierung.

In Kapitel 3 untersuchen wir die Effekte von Krankheit und Krieg auf die Akkumulation von Humankapital und in physisches Kapital. In einem OLG Modell mit drei Generationen unterliegen beide Formen von Kapital dem Risiko, zerstört zu werden. Altruistische junge Erwachsene entscheiden über Investitionen in Ausbildung und physischem Kapital. Ausbildung ermöglicht es Kindern, ihr menschliches Kapital auszubauen. Darüber hinaus, hängt ihr menschliches Kapital von dem ihrer Eltern ab, so dass negative Schocks gegen das elterliche Kapital langfristige Konsequenzen haben. Wir finden für ein breites Spektrum an Parameterwerten, dass eine Ökonomie der Armutsfalle nicht entkommen kann und sich somit im Zustand der Rückständigkeit befindet. Jedoch kann Altruismus den Abstieg in solch eine Falle verhindern. Des Weiteren kann der Zustand des Fortschritts, welcher das Gegenteil von Rückständigkeit ist, der einzige Wachstumspfad für die Ökonomie sein, falls der Altruismus der Eltern stark genug ist. We finden, dass es auch Wachstumspfade zwischen diesen beiden Extremen gibt, von denen einige stationär sind. Zum Schluss zeigen wir, dass die Robustheit einer Ökonomie gegenüber stochastischen Schocks sehr stark von der anfänglichen Ausstattung der Wirtschaft mit beiden Formen von Kapital abhängt.

In Kapitel 4 präsentieren wir ein Modell, das Institutionen, die Akkumulation von Humankapital und internationale Kapitalströme vereint. Dieses Modell beinhaltet zwei Länder, welche um internationales Kapital konkurrieren. Wir können zeigen, dass anfängliche kleine Unterschiede in der Qualität der Institutionen, auf lange Sicht, zu erheblichen Einkommensunterschieden zwischen Ländern führen können. Der Grund dafür ist, dass besagte kleine Unterschiede zum Zufluss von Kapital führen können, welcher dann die Akkumulation von Humankapital in Gang setzt.

Letztlich, wenden wir uns in Kapitel 5 dem Thema der künstlichen Intelligenz zu und diskutieren die Auswirkungen dieses Phänomens auf wirtschaftliches Wachstum. Im Gegensatz zu den vorangehenden Kapital, verschaffen wir einen Überblick über ausgewählte Aufsätze und beschreiben, welche Überlegungen in zukünftige Forschung zu diesem Thema einfließen könnten.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The importance of economic growth is indisputable. One central indicator of economic growth—the level of gross domestic product per capita (or simply GDP per capita)—is positively correlated with consumption per capita, life expectancy, and personal happiness, as measured by the happiness index.[1][2] GDP per capita may not be an indicator of all economically relevant factors, as it ignores, for instance, environmental and societal issues. However, its level and growth rate are at the focus of political and academic attention.

Since the Solow-Swan-model, i.e. the neoclassical workhorse model, economists have scrutinized the growth process, identified the driving factors of growth, and partially rejected some of its core-assumptions. Neither do households always save a constant fraction of their income, nor is technological progress completely exogenous, i.e. independent of the economic environment and of agents' actions. Instead, households react to incentives and adjust their investment accordingly. Also, technological progress is not a total black box anymore. It stems from the actions of households and firms. On the one hand, households increase their human

[1]This chapter is single-authored.

[2]The happiness index stems from the World Happiness Report, which defines itself as follows: "The World Happiness Report is a landmark survey of the state of global happiness that ranks 156 countries by how happy their citizens perceive themselves to be. This year's World Happiness Report [2019] focuses on happiness and the community: how happiness has evolved over the past dozen years, with a focus on the technologies, social norms, conflicts and government policies that have driven those changes." (Source: https://worldhappiness.report/ed/2019/; accessed on 19.03.20)

capital and perform basic research, creating new knowledge and patents. On the other hand, firms strive to become monopolists, and thus buy patents and turn knowledge into marketable products. They do this through the process of applied research.

While in the last decades our understanding of the growth process has increased tremendously[3], a variety of important observations still awaits thorough examination. We address several of those issues in this study. The first observation is that growth models usually assume rational and well-informed agents who know the return on their investments in advance, or, at least, have an idea of the distribution of these returns. Yet, the particular field of basic research is especially prone to uncertainty. The return to basic research only accrues with a substantial delay and is notoriously difficult to measure, due to the heterogeneity of patent data and the difficulty to distinguish basic from applied research. Because of this uncertainty, society has frequently overestimated the value of new technologies. Hence, we construct a growth model in the style of Romer (1990), and assign an important role to the beliefs about the impact of research. In Chapter 2, we allow beliefs to be either right or wrong, and show how the outcome of an economy can depend on their distribution. Overoptimism emerges, resulting from the aggregation of beliefs, and it crucially determines the state of the economy.

We build an overlapping generations model (OLG model) with endogenous growth where growth stems from the accumulation of knowledge. Knowledge is created through research, which is conducted by households. They decide whether to spend their time in the research sector or in the production sector, and have heterogeneous beliefs about the productivity of research. This productivity is unknown to

[3]Seminal studies that contributed to our understanding are, of course, Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992). There is also more recent work that sheds light on the topic of economic growth, such as Schmassmann (2018).

the agents, as well as to the government which forms the demand for researchers. Thus, the government relies on the beliefs of agents to form an estimate for the productivity of research. However, the beliefs which are communicated to the government are biased. We find that agents self-select into research, meaning that rather optimistic households decide to work in research. The consequence is an overly optimistic estimate of the productivity. This phenomenon is a "research bubble". It arises in face of completely rational behavior.

We discuss the desirability of such a research bubble when research has dynamic externalities. Such externalities imply that the value of research today is twofold. First, research today generates more knowledge tomorrow, and thus, more output tomorrow. Second, more knowledge tomorrow simplifies research tomorrow, leading to even more research the day after tomorrow. If such externalities are ignored by the government, which decides how much research should be conducted, research bubbles can improve the economy's outcome. Under such a bubble, the government overestimates the effect of research on output tomorrow, and hence, invests more in research. This effect partially compensates the government's ignorance about the impact of research today on output the day after tomorrow.

The second observation is that households do not live for infinitely many periods. The OLG model has become the standard approach to modeling this fact. However, a household's lifetime is not only finite, but also random, as it faces sickness and war. These two calamities are highly important for a growth model, as they affect the accumulation of human capital and physical capital. Incorporating sickness and war is particularly important for discussing growth in developing countries, where the risk of premature death is especially high. In Chapter 3, we discuss the impact of destruction rates for human capital and physical capital on the growth opportunities of a country.

In an OLG model with three generations—children, young adults and the old—output is produced through physical capital and human capital. Both are supplied by young adults and children. While human capital is accumulated through savings, human capital must be produced. Children are born with a fixed stock of human capital which can be increased through schooling and their parents' human capital. Under such conditions, the untimely death of young adults has two negative implications. First, their human capital cannot be employed in production, leading to less consumption for all generations. Second, they cannot contribute to the education of their children. This effect has consequences that play out over time, as less educated children will raise children that are even less educated, and so on. In our model, we include social norms which govern how much each generation receives of total income, and consider altruism of parents to their children's education.

We find conditions for a path along which a country grows at a constant rate. This is not guaranteed, in the economy that we model. Then, we show that a country can find itself on one of two extreme paths. Along the first, it grows at a positive rate—the state of progress. Along the second, it does not grow, but ends in what we call a "poverty trap". In such a poverty trap, investment is only able to offset the destruction of capital, but not to accumulate capital. We also study whether there are paths that lie between the extremes, and demonstrate that sufficiently severe stochastic shocks can move an economy from the state of progress into a poverty trap.

Third, agents in a specific country do not take their decisions in an economic vacuum. Countries are integrated in the global economy, competing for scarce resources. Over the last decades, we observe increasing inequality between countries. In addition, flows of capital from poorer to richer countries are triggered.

Such flows are called "uphill" capital flows, and we ask whether they can have an impact on international inequality. Kose et al. (2009) suggest the following: Inflows of capital do not impact the growth rate of a country directly. Instead, they interact with other relevant factors of growth. In Chapter 4, we show how the interaction of institutions, the accumulation of human capital, and the inflows of capital can lead to long-term economic growth.

We model a world with two countries, with capital flowing between them. There is no global financial market where countries can borrow as much as they want at a constant rate. Instead, the rate of return is determined by the countries' institutions. The levels of institutions determine which country experiences inflows, and which country experiences outflows of capital. If the change of institutions and the inflow of capital increase the income of agents sufficiently, agents might begin to invest in human capital. Hereby, human capital is the driving force of growth.

We find that a small initial difference in institutions can translate into substantial differences in growth in the long-run. A country that extends its institutions can find itself on a growth path with a positive rate, while another country remains with the same output. We study conditions under which such a scenario occurs and provide a simulation that reflects our theoretical findings.

Of course, many other issues are relevant in the context of economic growth, and it is impossible to discuss them all in one thesis. However, we would like to mention another phenomenon that emerged quite recently. This phenomenon is artificial intelligence (AI). Its importance is drastically increasing, thanks partly to the abundant data which is generated by social media. Hence, we study how AI can contribute to economic growth. This question is particularly relevant, as economic growth currently seems to slow down.

We review the literature on AI in Chapter 5.1. After providing the key insights from existing studies, we shortly discuss possible avenues for future research.

Chapter 2

Research bubbles

Abstract

We develop a model to rationalize and examine so-called “research bubbles”, i.e. research activities based on overoptimistic beliefs about the impact of this research on the economy.[1] Research bubbles occur when researchers self-select into research activities and the government aggregates the assessment of active researchers on the way advances in research may spur innovation and growth. In an overlapping generations framework, we study the occurrence of research bubbles and show that they tend to be welfare-improving. Particular forms can even implement the socially optimal solution. However, research bubbles can collapse, and we discuss institutional devices and the role of debt financing that ensure the sustainability of such bubbles. Finally, we demonstrate that research bubbles emerge in various extensions of our baseline model.

2.1 Introduction

Motivation

The world spends huge amounts of money on basic research and science in general.

[1]This chapter is joint work with Prof. Dr. Hans Gersbach.

In 2014, basic research accounted for nearly one-third (27.8 percent) of total R&D expenditures in OECD countries (OECD (2016)), with total R&D equaling 2.4 percent of GDP. Basic research may lead to innovations that result in technological progress and thus long-term material benefits. Historical examples are advances in biology and medicine—from X-rays to DNA sequencing—and the introduction of running water and sewer systems (see Gordon (2012) for many examples).

Nevertheless, the success of effort expended on basic research is highly uncertain, and the value of research is often difficult to assess for the generation making the investment. From 1991 to 1995, for example, only 12 percent of all university patents were ready for commercial use once they were licensed, and whether manufacturing would be feasible was known for only 8 percent (Jensen and Thursby (2001)). Estimating the value of research is also difficult. HERG-OHE-Rand (2008) find a return of 9 percent for medical research on physical health but state that "[...] rates of return need to be treated with extreme caution. Most aspects of the methods unavoidably involve considerable uncertainties." Due to this uncertainty, there are several examples where society has overestimated the value of newly discovered technologies. Perez (2009) documents several examples of investment in new technologies, namely canal building in England, starting in 1771, railway development in Great Britain, starting in 1829, and the establishment of internet-related companies in the US, starting in 1971. With hindsight, these instances displayed a concentration of investments, divorced from actual technology needs in the real economy. Typically, these projects were fueled by great optimism about potential real-world application. But, after an initial surge, investments often slowed down or collapsed altogether.

There are also more recent examples of such occurrences, like the Apollo Program (Gisler and Sornette (2009)) and the Human Genome Project (Gisler et al.

(2011)). Also, the Google Lunar Xprize (XPRIZE Foundation (2016)) or current concentrated expenditures focused on projects such as the European Flagships seem to involve outstanding optimism. "*Flagships are visionary, science-driven, large-scale research initiatives addressing grand Scientific and Technological (S&T) challenges. They are long-term initiatives bringing together excellent research teams across various disciplines, sharing a unifying goal and an ambitious research roadmap on how to achieve it*".[2] Currently, one focus is on Graphene, a single, thin layer of graphite, that is considered the world's strongest and most conductive material, another the explanation of the human brain.

According to the literature above, these examples, while seeming very different at first glance, share three main features:

- Large basic research investments are involved which may collapse at some point in time.
- The projects are fueled by great optimism and enthusiasm about the scientific and economic benefits of the project, while a more realistic assessment would lead to more cautious calculations.
- Typically, the outcomes are disappointing compared to the initial expectations. However, over time, various types of benefits are generated. The Apollo Program, for example, led to improvements in the production of microprocessors and to greater memory capacity for computers, from which other industries have greatly benefited (Mezzucato (2014)).

We call occurrences that fulfill these criteria "research bubbles[3]". As they seem to

[2] For more information see http://ec.europa.eu/programmes/horizon2020/en/h2020-section/fet-flagships (accessed on 18.09.2017).

[3] Research bubbles are, of course, quite distinct from the well observed bubbles in the financial sector. Instead, they can be understood as a subset of "social bubbles", which are defined in Gisler and Sornette (2009) and Gisler et al. (2011), occurring in the realm of public research. For a recent survey of the asset bubble literature, see Scherbina and Schlusche (2014).

be a pervasive feature in the discovery of knowledge, questions about the causes and the desirability of such bubbles arise. This is the focus of this chapter.

One might suggest that such bubbles are the result of mere irrationality and since agents overinvest, can only be detrimental to welfare. However, we suggest that research bubbles are generated by the self-selection of researchers into research activities and result from rational decisions on the part of governments as to whether to embark on such adventures on the basis of the assessments by the researchers involved.[4] Moreover, while such bubbles may lead to disappointment and may not benefit the current generation, they tend to be desirable from a long-term perspective, taking the welfare of future generations into account. However, they may also be excessive, even from a long-run perspective.

In classic innovation-driven growth theory and its extensions, research bubbles do not figure at all (Aghion and Howitt (1992), Grossman and Helpman (1991), Romer (1990)). Cozzi (2007) is an exception, presenting a model that allows for self-fulfilling prophecies. In our model, research bubbles are not the result of multiple equilibria, but arise from the government's aggregation of heterogeneous assessment by researchers who self-select into research activities. A related study is Olivier (2000) where financial bubbles increase firms' value and enables them to attract more researchers. A positive effect on growth is the result. However, we focus on instances when the allocation of researchers is the cause of a research bubble, and not the consequence of a financial bubble.

Approach and results

More specifically, we develop a framework that rationalizes research bubbles in an

[4]The optimism bias in our study, which will be substantiated in the following sections is an aggregate phenomenon. Our definition differs from the standard explanation in psychology and behavioral economics, where individuals overestimate the likelihood of positive events and underestimate the likelihood of negative ones.

overlapping generation model with endogenous growth. Here, growth stems from the accumulation of knowledge, a production factor created in the basic research sector (henceforth simply research sector). Conducting research today leads to more knowledge tomorrow. It requires labor input, and the amount of labor in the economy is a finite resource. It can be either employed in the research sector, to increase output tomorrow, or in the productive sector, to produce output today. Hence, employing labor in research means forfeiting output today for more output tomorrow, so that conducting research represents a trade-off between output today and output tomorrow. We assume that the demand for research labor is formed by a decentralized myopic government. We use the term "myopic" to indicate that the government has a shorter horizon than a social planner.

In a first simple model without bubbles, we demonstrate that the government fails to internalize the dynamic externality of research, leading to too little research activity over and against the social optimum. We find that the decentralized outcome can be improved both by lengthening the decision-maker's horizon, and by an overestimation of the short-term impact of research, i.e. a research bubble.

In a second, more complex model, we introduce bubbles that derive from rational behavior of households and the government. By allowing for heterogeneous beliefs about productivity among agents in the research sector, we focus on the way agents self-select into the research sector. Those with higher beliefs, i.e. more optimistic agents, will want to work in the research sector, while more pessimistic agents will choose the productive sector. The government does not know how productive research will be. It relies on the assessment of agents in the research sector for its estimate of research productivity, which, in turn, is the basis for its demand for researchers. As optimistic agents self-select into the research sector, the government overestimates productivity, and a research bubble arises.

When governments form average assessments of the technology potential from research by listening to researchers, the emerging research bubble will typically fail to reach the socially optimal level of research investments. But other aggregation methods can produce research bubbles that generate socially optimal research activities.

We further examine how research bubbles may burst and how such collapses can be avoided through institutional remedies such as establishing constitutional rules or giving optimistic researchers a big say in basic research investment. An alternative route is debt financing, where the amount of debt that the government can borrow on international capital markets depends on the amount of research conducted in the economy. Finally, we recast the occurrence of research bubbles in variant models in which research success also depends on research effort decisions. Models with alternative welfare functions of the government are also discussed.

Structure

The rest of the chapter is organized as follows: In the next section, we introduce the baseline model describing our research economy. Section 2.3 presents the same model, this time with research bubbles, and Section 2.4 studies the implementation of the social optimum in a decentralized economy. Section 2.5 discusses the potential and the drawbacks of the decentralized solution. In Section 2.6, we present possible extensions of the model. Section 2.7 concludes.

2.2 A research economy

Let us turn first to our baseline model without research bubbles. We use an OLG model where endogenous growth results from an increasing stock of knowledge.

Knowledge is created by basic research and research is conducted in the public research sector, which competes with the production sector for skilled labor.

2.2.1 Households

Households live for two periods and at any point in time, two generations coexist. An agent is labeled "young" in the first period and "old" in the second period. Each generation is represented by a single household and possesses one unit of time supplied inelastically in the market for labor. Hence, total labor endowment $\overline{L}$ is normalized to 1. There is one physical commodity that can be either used for consumption or as capital for production, while capital is fully depreciated in each period. Consumption is the only source of utility. The life-time utility of a household born in period t is

$$U_t = \log\left(c_t^1\right) + \beta \log\left(c_{t+1}^2\right), \tag{2.1}$$

where c_t^1 denotes consumption of the physical good in the first period, c_{t+1}^2 in the second, and the parameter β stands for the discount factor, with $0 < \beta < 1$. When young, the agent makes decisions on saving and on how much time to allocate to work in the research sector, and/or the productive sector. When old, the household only consumes its savings. The variables $L_t^{S,R}$ and $L_t^{S,P}$ stand for the time the agent supplies to the research sector and the productive sector, respectively. Furthermore, s_t stands for savings, w_t^P for the wage in the productive sector, w_t^R for the wage in the research sector, and r_{t+1} for the gross interest rate. We assume that in order to finance research, wage income in the productive sector is taxed at rate τ_t . Hence, when young and old, consumption for an agent born in t are

$$c_t^1 = w_t^R L_t^{S,R} + (1-\tau_t) w_t^P L_t^{S,P} - s_t, \tag{2.2}$$

$$c_{t+1}^2 = s_t r_{t+1}, \text{ and} \tag{2.3}$$

$$L_t^{S,R} + L_t^{S,P} = 1. \tag{2.4}$$

We plug in these definitions and maximize utility with respect to $L_t^{S,R}$ and s_t to obtain

$$L_t^{S,P} = \begin{cases} 0, & \text{if } w_t^R > w_t^P(1-\tau_t), \\ \text{arbitrary}, & \text{if } w_t^R = w_t^P(1-\tau_t), \\ 1, & \text{if } w_t^R < w_t^P(1-\tau_t), \end{cases}$$

and

$$s_t = \frac{\beta}{1+\beta}\left(w_t^R L_t^{S,R} + (1-\tau_t) w_t^P \left(1 - L_t^{S,R}\right)\right).$$

Throughout the chapter, we focus on constellations in which both sectors are active, which in this section requires that $w_t^R = w_t^P(1-\tau_t)$. We denote this wage by w_t and obtain that savings are a constant share of income due to logarithmic utility.

The tax rate balances the budget and fulfills the following condition:

$$w_t^R L_t^R = \tau_t w_t^P (1 - L_t^R),$$

where L_t^R is the eventual market equilibrium. Hence, by the required equality of net wages, we have

$$(1-\tau_t)w_t^P L_t^R = \tau_t w_t^P (1-L_t^R) \Rightarrow \tau_t = L_t^R, \tag{2.5}$$

implying that the tax rate on wage income from productive activity is equal to the share of labor in the research sector.

2.2.2 Productive sector

A single firm produces output using knowledge B_t, capital K_t and labor $L_t^{D,P}$, with D, P indicating demand in the productive sector. The production function takes the form

$$Y_t = (L_t^{D,P} B_t)^{1-\alpha} K_t^{\alpha}.$$

Labor is supplied by the young household, knowledge is created in the research sector, as is described below, and capital is created from the household's savings. As capital depreciates fully within one generation, $s_t = K_{t+1}$ is the equilibrium condition. Also, this implies that r_{t+1} is the net and gross interest rate. Knowledge will be useful in production,[5] and while the output of basic research can be used free of charge, the other two production factors are rented by the firm and compensated by wage w_t^P and interest rate r_t. Hence, the profit of the firm reads

$$\Pi_t = Y_t - w_t^P L_t^{D,P} - r_t K_t.$$

[5]Since basic research output is of no immediate commercial use, there is typically time lag between basic research and its use in production. Estimates of this time lag range between 6 and 20 years on average (see Adams (1990)).

The firm takes all prices as given, so that optimal behavior is described by the standard demand functions. Since both labor and capital are supplied inelastically, we obtain the standard equilibrium condition for wages and the interest rate:

$$w_t^P = (1-\alpha)\frac{Y_t}{L_t^{D,P}} \text{ and } r_t = \alpha\frac{Y_t}{K_t}.$$

2.2.3 Research sector

We assume that the research sector is run by a government. It employs labor $L_t^{D,R}$ to create new knowledge on the basis of the existing knowledge stock. From the government's perspective, the knowledge production function is

$$B_{t+1} = (1 + \theta \cdot L_t^{D,R})B_t.$$

The function depends on labor demand in research $L_t^{D,R}$ and on the productivity parameter θ.

Conducting research inhibits a fundamental trade-off: Increasing knowledge and output tomorrow means forfeiting output today, as labor has to be reallocated from the productive sector to the research sector. Thus, it is the government's task to decide how much labor should be employed in the research sector.

The economy allows for balanced growth paths. A steady state is characterized by Proposition 1.

Proposition 1 *A steady state of the economy for a given constant share of labor invested in research in each period $\hat{L}^R$ is uniquely characterized by a constant labor input $\hat{L}^P$ in the productive sector and a constant return to capital $\hat{r}$. Consumption of young and old agents, c_t^1 and c_t^2, output Y_t, capital K_t and the knowledge stock B_t all grow at a constant rate $\hat{g} = \theta\hat{L}^R$.*

The proof of Proposition 1 can be found in the Appendix for Chapter 2.

2.2.4 Decentralized solution

First, we look at a planner who is only concerned with the current generation. He has preferences over output today and discounted output tomorrow, where both depend on labor input in research. We call this the "government solution"

$$\max_{L_t^{D,R}} \log(Y_t) + \beta \log(Y_{t+1}), \quad \text{or}$$
$$\max_{L_t^{D,R}} \log\left(((1 - L_t^{D,R})B_t)^{1-\alpha} K_t^{\alpha}\right) + \beta \log\left(((1 - L_{t+1}^{D,R})B_{t+1})^{1-\alpha} K_{t+1}^{\alpha}\right),$$

where we assume that the government has the same logarithmic utility function and discount factor β as the household. Maximizing with respect to $L_t^{D,R}$ yields

$$\frac{1}{1 - L_t^{D,R}} = \frac{\beta\theta}{1 + \theta L_t^{D,R}} \quad \text{for} \quad 0 \leq L_t^{D,R} < 1. \tag{2.6}$$

The left hand side of (2.6) is the marginal product of labor and reflects the marginal cost of one more unit of labor in research. The right hand side is the discounted marginal product of research today on output tomorrow via an increase in the knowledge stock.

Expression (2.6) only contains the contemporary value of $L_t^{D,R}$ but not $L_{t+1}^{D,R}$, so that the government's solution is static and takes the form

$$L^{D,R} = \frac{1}{1+\beta}\left(\beta - \frac{1}{\theta}\right) \quad \forall t, \tag{2.7}$$

where $L^{D,R}$ only depends on the parameters β and θ. A condition for research to occur in the decentralized economy is $L^{D,R} > 0 \leftrightarrow \beta > 1/\theta$. As we have assumed that β is smaller than 1, the condition can be reduced to $\theta > 1$. A more impatient government with a lower discount factor β will invest less in research.

We can be certain that we have found a utility maximum, as the second derivative of the objective function is simply

$$-(1 - L_t^{D,R})^{-2} - \beta\theta^2(1 + \theta L_t^{D,R})^{-2} < 0 \ \ \forall L_t^{D,R} \ \in (0, 1).$$

The static nature of the government's solution results from two structural assumptions: Logarithmic utility and a Cobb-Douglas production function. The logarithmic utility causes Y_t and Y_{t+1} to appear in the denominator of the derivative, and the production function causes them to appear in the numerator, so that they cancel out.

To ensure that the household supplies the demanded share of labor, the government sets the wage in the research sector

$$w_t^R = (1 - \tau_t)\frac{(1 - \alpha)Y_t}{1 - L_t^R} = (1 - \tau_t)w_t^P = w_t, \tag{2.8}$$

in each period. By doing this, the government can implement its demand as an equilibrium, so that $L^{D,R}$ stands for the equilibrium value of labor in research L^R.

Definition 1 *We define an equilibrium of the economy as the paths of w_t, r_t, Y_t, K_t and B_t, given a sequence $\{L_t^R\}_{t=0}^{\infty}$ that fulfill the following conditions: The household maximizes utility, the firm maximizes profits, the government maximizes its own utility, and the market for capital clears as do the good and the labor market, i.e. Equation* (2.8) *holds.*

2.2.5 Social planner solution

Next we turn to the social optimum for the economy. The maximization problem of the social planner reads

$$
\begin{aligned}
\max_{\{c_t^1, c_{t+1}^2, L_t^R, B_{t+1}, K_{t+1}\}_{t=0}^{\infty}} W &= \sum_{t=0}^{\infty} \beta_s^t \left(\log(c_t^1) + \beta \log(c_{t+1}^2)\right) \\
\text{s.t. } ((1 - L_t^R)B_t)^{1-\alpha} K_t^{\alpha} &= c_t^1 + c_t^2 + K_{t+1}, \\
B_{t+1} &= (1 + \theta L_t^R) B_t,
\end{aligned}
$$

where $\beta_s \in (0, 1)$ is the social planner's discount factor and W denotes social welfare. We define λ_t as the Lagrange Multiplier on the budget constraint and μ_t as the multiplier on the knowledge-production function, and obtain the following first order conditions:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t^1} &= \beta_s^t \left(\frac{1}{c_t^1} + \lambda_t \right) = 0, \\
\frac{\partial \mathcal{L}}{\partial c_{t+1,}^2} &= \beta_s^t \left(\frac{\beta}{c_{t+1}^2} + \beta_s \lambda_{t+1} \right) = 0, \\
\frac{\partial \mathcal{L}}{\partial K_{t+1}} &= \lambda_t - \beta_s \lambda_{t+1} \alpha \frac{Y_{t+1}}{K_{t+1}} = 0, \\
\frac{\partial \mathcal{L}}{\partial L_t^R} &= \lambda_t (1 - \alpha) \frac{Y_t}{1 - L_t^R} + \mu_t \theta B_t = 0, \\
\frac{\partial \mathcal{L}}{\partial B_{t+1}} &= -\mu_t + \beta_s (-\lambda_{t+1} (1 - \alpha) \frac{Y_{t+1}}{B_{t+1}} + \mu_{t+1} (1 + \theta L_{t+1}^R)) = 0,
\end{aligned}
$$

where $\mathcal{L}$ is the Lagrange Function. The first three conditions are common, but the last two deserve attention. To understand them, we interpret λ_t as the change in the life-time utility of an agent born in t if one more unit of output Y_t were available. Analogously, we see μ_t as the change in the life-time utility of such an agent if one more unit of knowledge B_{t+1} were available tomorrow. With this, the derivative with respect to L_t^R implies that the marginal loss from allocating one more unit of labor to research, which is λ_t times the marginal product of labor, must be equal to the benefit which results from having θB_t more units of knowledge tomorrow. In the derivative of the Lagrange Function with respect to B_{t+1}, μ_t stands for the welfare loss associated with creating one more unit of B_{t+1}. The loss is equal to the discounted sum of two different benefits. First, more knowledge tomorrow will increase production by the marginal product $(1-\alpha)Y_{t+1}/B_{t+1}$. Second, having more knowledge tomorrow will reduce the necessity to conduct research tomorrow and hence yields the benefit $\mu_{t+1}(1+\theta L^R_{t+1})$.

From the five first-order conditions we obtain two dynamic equations. The first is the common Euler equation, and the second describes the dynamic allocation of labor in research:

$$\frac{1}{c_t^1} = \frac{\beta_s \alpha Y_{t+1}}{K_{t+1} c_{t+1}^2} \quad \text{and} \tag{2.9}$$

$$\frac{Y_t}{1-L_t^R} = \frac{K_{t+1}}{\alpha Y_{t+1}} \left(\frac{\theta Y_{t+1}}{1+\theta L_t^R} + \frac{Y_{t+1}}{1-L_{t+1}^R} \frac{1+\theta L_{t+1}^R}{1+\theta L_t^R} \right). \tag{2.10}$$

Together with the budget-constraint and the knowledge-production function, they describe the model. If we assume $\beta_s = \beta$, we obtain the following steady state condition:

$$\frac{1}{1-L^R} = \beta \left(\frac{\theta}{1+\theta L^R} + \frac{1}{1-L^R} \right) \quad \text{and thus}$$

$$L^R = L^{O,R} = \beta - \frac{1-\beta}{\theta}. \tag{2.11}$$

For the derivation of the equation, see the Appendix for Chapter 2. We denote the steady state value of labor input in research by $L^{O,R}$ and compare it to the government's demand for research $L^{D,R} = \frac{1}{1+\beta}\left(\beta - \frac{1}{\theta}\right)$. We find three differences. First, when θ goes to infinity, the social optimum converges to β in the limit, while the government solution converges to $\beta/(1+\beta) < \beta$. Second, when θ increases, the social planner will increase his demand less than the government:

$$\frac{\partial L^{O,R}}{\partial \theta} = \frac{1-\beta}{\theta^2} < \frac{1}{(1+\beta)\theta^2} = \frac{\partial L^{D,R}}{\partial \theta},$$

which holds if and only if
$(1-\beta)(1+\beta) < 1 \Leftrightarrow \beta^2 > 0$. Third, the social planner always employs more labor in research than the government:

$$L^{O,R} = \beta - \frac{1-\beta}{\theta} > \frac{1}{1+\beta}\left(\beta - \frac{1}{\theta}\right) = L^{D,R} \quad \text{and thus} \quad 0 < \beta^2(\theta+1),$$

which holds because we are assuming that $\beta, \theta > 0$.

To explain this result, we compare (2.6) and (2.10). We rewrite (2.10) as

$$\frac{1}{1-L_t^R} = \frac{K_{t+1}}{\alpha Y_{t+1}} \frac{Y_{t+1}}{Y_t} \left(\frac{\theta}{1+\theta L_t^R} + \frac{1+\theta L_{t+1}^R}{(1-L_{t+1}^R)(1+\theta L_t^R)} \right)$$

and observe that the government and the social planner discount different benefits, using different discount factors. The government only takes into consideration the immediate benefit from research that arises in the next period and uses the constant β to discount it. The social planner internalizes the additional intergenerational effects and uses $\frac{K_{t+1}}{\alpha Y_{t+1}} \frac{Y_{t+1}}{Y_t}$ to scale the benefits occurring in the future. Note that this factor is the product of two components: first, the inverse of the marginal product of capital, which under complete depreciation is the economy's discount factor, and second, 1 plus the growth rate of output.

The social planner's awareness of the long-term benefits of research explains why the socially optimal steady state can be greater than the decentralized solution. It also explains why the social planner solution is less sensitive to changes in θ.

The government only enjoys the benefits of a higher θ and increased productivity in the next period. To capitalize on the increased productivity, the government strongly raises labor input in research. The social planner, by contrast, is aware that the benefits of a higher θ extend beyond the next period. Therefore he has smaller incentives to increase labor input today

We prove that the difference between the social planner and the government solution arise only because of two differing decision horizons. For this, we show that the government solution converges to the socially optimal solution with a rising decision horizon. A government that is aware that research today has an impact on all future generations faces the following problem of maximizing the sum of all future output over investment today: $\max_{L_t^{D,R}} \sum_{s=t}^{\infty} \beta^{s-t} \log(Y_s)$, which due to logarithmic utility can simply be written as

$$\max_{L_t^{D,R}} \log(1 - L_t^{D,R}) + \sum_{s=t+1}^{\infty} \beta^{s-t} \log(1 + \theta L_t^{D,R}) \text{ or}$$
$$\max_{L_t^{D,R}} \log(1 - L_t^{D,R}) + \frac{\beta}{1-\beta} \log(1 + \theta L_t^{D,R}),$$

which yields

$$(1 + \theta L_t^{D,R})(1 - \beta) = \beta\theta(1 - L_t^{D,R}) \quad \text{and thus} \quad L^{D,R} = \beta - \frac{1-\beta}{\theta}.$$

This expression precisely yields the social optimum. Thus, we obtain the following proposition:

Proposition 2

For the decentralized government and the social planner, the steady state levels of labor in research are given by (2.7) *and* (2.11). *The social optimum implies more research than the decentralized solution and has a higher upper bound but is less* *sensitive to changes in productivity.*

2.2.6 Implementing the socially optimal solution via research bubbles

The preceding analysis reveals that decentralized basic research investments can yield lower social welfare. We note that a more optimistic government view, i.e. the assumption that θ is higher than it actually is, would increase welfare. However, it would do so at the expense of generation t's utility. In addition, if all governments had a more optimistic view, i.e. if their assumption, $\widetilde{\theta}$, is greater than the true value of θ, social welfare would be higher at the expense of the first few generations. In particular, if the first generations could finance part of their research expenditures by issuing debt, a combination of research bubbles and public debt could implement the socially optimal solution and make everybody better off compared to the decentralized solution. We define the following:

Definition 2 *The economy exhibits a research bubble in a particular time frame* $(0, T]$ *for some* $T \in \mathbb{R}$ *if the governments assume* $\widetilde{\theta} > \theta$ *when they decide on investment in basic research.*

In the following, we show that sufficient optimism in the decentralized economy can implement the social optimum. Assume that the government does not know the true value of θ but beliefs $\widetilde{\theta}$ to be the true productivity. To achieve the socially optimal outcome, this $\widetilde{\theta}$ must fulfill

$$\beta - \frac{1-\beta}{\theta} = \frac{1}{1+\beta}\left(\beta - \frac{1}{\widetilde{\theta}}\right) \quad \text{and thus} \quad \widetilde{\theta} = \frac{\theta}{1-\beta^2(1+\theta)}. \tag{2.12}$$

The implementation of the social optimum hinges on the relation between β and θ.

If $1/\beta^2-1 > \theta$, then implementation is possible, otherwise it is not. This restriction arises, because the decentralized solution has a lower upper bound than the social planner solution, i.e. $\frac{\beta}{1+\beta} < \beta$.

Proposition 3 *If there is optimism in the decentralized economy and the government beliefs $\widetilde{\theta}$, given by* (2.12)*, to be the true productivity, the economy will achieve the social optimum if* $1/\beta^2 - 1 > \theta$.

In the next section, we derive a microfoundation for optimistic beliefs and suggest that they are a natural outcome of decisions on basic research. Moreover, we suggest an institutional arrangement that can implement the socially optimal solution.

2.3 Research with heterogeneous beliefs

We next explore whether research bubbles arise naturally in scenarios where the government does not know the true parameter θ. Also, we ask whether there are institutional arrangements that support welfare-enhancing research bubbles.

We substitute the single household in each generation by a continuum of infinitely many households of measure 1. A subset of these agents holds beliefs about the parameter θ. The beliefs are heterogeneous and the government has to make an estimate for θ based on the given beliefs.

2.3.1 Households

The economy is populated by infinitely many agents represented by the interval $[0, 1]$ and of mass 1. All agents possess one unit of time. Hence, the overall labor endowment in the economy is 1, as before. A share $\overline{L}_B$ of all agents is able to work in the research sector and these agents hold beliefs about θ. The set of agents with the capacity to work in the research sector is $\mathcal{L}_B$. We denote agent $i \in \mathcal{L}_B$'s belief about θ as θ_i and allow it to lie in $[\theta_l, \theta_h]$, with $\theta_l < \theta < \theta_h$. Belief types are uniformly distributed in $[\theta_l, \theta_h]$, with density $\frac{1}{\theta_h - \theta_l}$, where the latter follows from the assumption that households have mass 1.

The belief determines the sector in which an agent will want to work. If an agent works in the productive sector, he earns a wage and consumes. If the agent works in the research sector, additional considerations matter, since research has a strong non-pecuniary utility component. We assume that a researcher derives utility from research achievements and thus from knowledge creation, e.g. through intrinsic means—satisfaction about achievements— or extrinsic means—such as status and prestige. More specifically, utility derived from research depends on how efficient the agent beliefs his research to be, θ_i, so that the utility function of a scientist reads

$$U_{t,i}^R = \log(w_t^R - s_{t,i}^R) + (1+\beta)\log(1 - \theta_l + \theta_i) + \beta \log(r_{t+1} s_{t,i}^R),$$
$$\text{with } \; i \in \mathcal{L}_B,$$

where we calibrate utility in such a way that the least optimistic agent, i.e. the one who holds the belief $\theta_i = \theta_l$, receives no additional utility from working in research. With this calibration we ensure that no agents receive negative utility from working in research and hence require compensation. Also, we scale this utility by $1+\beta$, as this simplifies later derivations. The utility of a worker is given by

$$U_{t,i}^P = \log((1-\tau_t)w_t^P - s_{t,i}^P) + \beta \log(r_{t+1} s_{t,i}^P).$$

Since an individual has no impact on prices and aggregate variables, an agent takes r_{t+1} as given and maximizes utility with respect to savings $s_{t,i}$. The solutions of the worker problem the researcher problem are given by the expressions

$$s^P_{t,i} = \frac{\beta(1-\tau_t)w^P_t}{1+\beta} \text{ and } s^R_{t,i} = \frac{\beta w^R_t}{1+\beta}. \tag{2.13}$$

Plugging these results back into the utility function, we obtain

$$U^R_{t,i} = \log\left(\frac{w^R_t}{1+\beta}\right) + (1+\beta)\log(1-\theta_l+\theta_i) + \beta\log\left(\frac{\beta r_{t+1}w^R_t}{1+\beta}\right) \text{ and}$$
$$U^P_{t,i} = \log\left(\frac{(1-\tau_t)w^P_t}{1+\beta}\right) + \beta\log\left(\frac{\beta r_{t+1}(1-\tau_t)w^P_t}{1+\beta}\right).$$

Setting both utilities equal yields Proposition 4.

Proposition 4

The critical value for researcher i's belief is

$$\theta_{crit,t} = \frac{(1-\tau_t)w^P_t}{w^R_t} - (1-\theta_l). \tag{2.14}$$

Hence, every household with a belief θ_i above this value $\theta_{crit,t}$ will choose to work in the research sector. Every household with a belief below $\theta_{crit,t}$ will choose the productive sector. The agent with $\theta_i = \theta_{crit,t}$ is indifferent and, by assumption, will choose the research sector.[6]

We find that the critical belief is a linear function of the wage ratio. The greater the wage in the productive sector, the greater an agent's belief must be for him to choose the research sector. Given some wage ratio, an agent will choose the research sector if his belief θ_i lies between $\theta_{crit,t}$ and θ_h, so that labor supply is

[6] For a more detailed derivation, see the Appendix for Chapter 2.

given by

$$L_t^{S,R} = \overline{L}_B \frac{\theta_h - \theta_{crit,t}}{\theta_h - \theta_l}, \tag{2.15}$$

i.e. the product of the share of agents able to work in the research sector $\overline{L}_B$ and those who choose to do so $\frac{\theta_h - \theta_{crit,t}}{\theta_h - \theta_l}$.

2.3.2 Assessment of research productivity

Unlike before, the government does *not* know the parameter θ and has to form an estimate. Hereby, the researchers' beliefs are the only available source of information, and we assume that researchers truthfully signal their belief to the government. Equipped with this set of beliefs, the government then makes the following estimate:

$$\widetilde{\theta}_t = \eta\theta_h + (1-\eta)\theta_{crit,t}. \tag{2.16}$$

The parameter η $(0 < \eta < 1)$ is the weight that the government places on the most optimistic researcher belief, and $1-\eta$ is the weight placed on the most pessimistic counterpart. At this stage we do not specify how η is eventually determined. In Section 5.3 we explore different institutional arrangements leading to particular values of η.

Two remarks are in order. First, the expressed range of beliefs $[\theta_{crit,t}, \theta_h]$ is itself more optimistic than the range of beliefs $[\theta_l, \theta_h]$ in the entire population of researchers.

Second, at this stage we assume that researchers reveal their true beliefs to the government. In section 5.2 we discuss whether researchers do indeed have incentives to reveal their true beliefs.

2.3.3 The government's problem

The government relies on the following estimated production function to derive research labor demand:

$$\widetilde{B}_{t+1} = (1 + \widetilde{\theta}_t \cdot L_t^{D,R})B_t.$$

This demand differs from the previous one in two ways. First, it indicates the amount of knowledge that the government believes to be available tomorrow, $\widetilde{B}_{t+1}$. Second, the parameter θ is replaced by the government's estimate $\widetilde{\theta}_t$. To ease notational complexity, $L_t^{D,R}$ again stands for the government's demand for research labor, but now for the case with *heterogeneous* beliefs. The maximization problem for the government now reads

$$\max_{L_t^{D,R}} \log\left(((1 - L_t^{D,R})B_t)^{1-\alpha}K_t^{\alpha}\right) + \beta \log\left(((1 - L_{t+1}^{D,R})B_t(1 + \widetilde{\theta}_t L_t^{D,R}))^{1-\alpha}K_{t+1}^{\alpha}\right).$$

Maximizing with respect to $L_t^{D,R}$ yields

$$\frac{1}{1 - L_t^{D,R}} = \frac{\beta\widetilde{\theta}_t}{1 + \widetilde{\theta}_t L_t^{D,R}}, \quad \text{with} \tag{2.17}$$

$\widetilde{\theta}_t$ given by (2.16).

Equation (2.17) is analogous to Equation (2.6), but θ is now replaced by the estimate $\widetilde{\theta}_t$. However, Equation (2.17) alone does not enable us to determine L_t^R. It depends on $\widetilde{\theta}_t$, which in turn, depends on the labor supply to the research sector. As labor is no longer supplied inelastically, it is necessary to determine labor demand and supply for research labor simultaneously. We do this by examining the labor market equilibrium in the next subsection.

2.3.4 Labor market equilibrium

To determine the labor market equilibrium, we solve (2.17) for $L_t^{D,R}$

$$L_t^{D,R} = \frac{1}{1+\beta}\left(\beta - \frac{1}{\widetilde{\theta}_t}\right). \tag{2.18}$$

Using Definition (2.16) with $\eta = 1/2$ yields

$$L_t^{D,R} = \frac{1}{1+\beta}\left(\beta - \frac{2}{(\theta_h + \theta_{crit,t})}\right).$$

Note that unlike before, $L_t^{D,R}$ is not a fixed value but a strictly concave function of the variable $\theta_{crit,t}$. So is the labor supply from Equation (2.15). Thus, we have two equations, labor supply and demand in two variables, research labor L_t^R, and the critical belief $\theta_{crit,t}$. This means that labor demand and the critical belief are interdependent. Labor supply depends on $\theta_{crit,t}$ because every agent with a belief higher than $\theta_{crit,t}$ supplies his labor to the research sector. Hence, the supply falls linearly with the critical value. Labor demand depends on $\theta_{crit,t}$ because the critical belief determines $\widetilde{\theta}_t$. Demand is an increasing function in $\theta_{crit,t}$: The more optimistic the statement by researchers about the productivity of research, the greater is, of course, the government's demand.

In Figure 2.1 we plot supply and demand as functions of $\theta_{crit,t}$ for the purpose of illustration with the values $\beta = 0.85$, $\theta_h = 2$ and $\theta_l = 1$. Labor supply by households is shown by the linear falling function and labor demand of the government by the increasing one. Labor supply reaches its maximal value of $\overline{L}_B$ when the critical belief takes the smallest possible value. The supply decreases smoothly

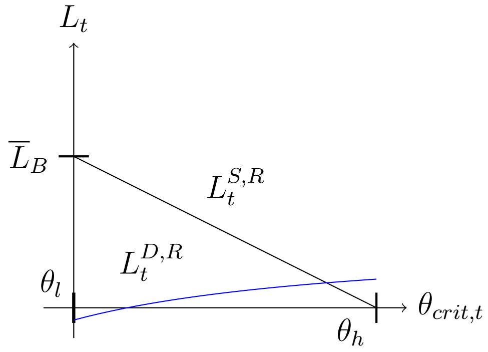

Figure 2.1: Labor market and optimism equilibrium.

until $\theta_{crit,t}$ reaches θ_h. Labor demand is negative at θ_l and increases in a concave fashion, intersecting the θ-axis only once. While the first observation results from our choice of values for θ_h and θ_l, the latter results from the strict concavity of labor demand. The intersection of the curves shows the labor market equilibrium. To obtain it analytically, we set demand equal to supply and solve for $\theta_{crit,t}$:

$$\theta_{crit,t}^2 + \theta_{crit,t}\frac{\beta(\theta_h - \theta_l)}{(1+\beta)\overline{L}_B} - \theta_h^2\left(1 - \frac{\beta}{(1+\beta)\overline{L}_B}\right) - \frac{\beta\theta_h\theta_l - 2(\theta_h - \theta_l)}{(1+\beta)\overline{L}_B} = 0. \tag{2.19}$$

This yields a second degree polynomial in $\theta_{crit,t}$. It depends on the boundaries of the belief interval θ_l and θ_h, the discount factor β, and the share of agents that can work in research $\overline{L}_B$. However it does not depend on the true productivity θ. As all parameters are constant, $\theta_{crit,t}$ is also constant over time.

The above polynomial has two real solutions at the most. We can show that if two solutions exist, one can be excluded. To see this, consider the demand function, which is strictly increasing for $\theta_{crit} \in \mathbb{R}$ but is not continuous everywhere, because the following holds:

$$\lim_{\theta_{crit} \overset{+}{\to} -\theta_h} L^{D,R}(\theta_{crit}) = -\infty, \quad \text{while} \quad \lim_{\theta_{crit} \overset{-}{\to} -\theta_h} L^{D,R}(\theta_{crit}) = +\infty,$$

i.e. the limits to $\theta_{crit} = -\theta_h$ from left and right are not identical. Given that demand is a strictly increasing function, we can conclude that the function becomes infinitely large on the left of $-\theta_h$, while it falls to $-\infty$ on the right of it and then increases with θ_{crit}. This explains how two intersections are possible. One of them has to lie to the left of $-\theta_h$ and is thus irrelevant.

With fixed $L_t^{D,R}$, w_t^P is given and the government needs to set w_t^R according to (2.14). By setting the wage in the research sector correctly, the government can implement its demand as the market equilibrium so that, as before, $L^{D,R} = L^{R,H}$, where $L^{R,H}$ is the equilibrium value in the market for research labor in the steady state. The superscript H stands for "heterogeneous beliefs".

The economy reaches the described equilibrium in the following way: First, the government hires a number of researchers and obtains the estimate $\widetilde{\theta}_t$. The expected productivity determines the government's optimal labor demand. If the optimal demand turns out to be greater, the government increases w_t^R, and hence labor supply, to lower θ_{crit}. By doing so, it hires additional, less optimistic agents and obtains, in turn, a lower average for θ. This adjustment of labor demand continues until the government hires exactly as many researchers as are justified by their aggregated belief.

In this economy, the tax rate τ_t differs from the previous one, as w_t^P and w_t^R are related by (2.14) and not simply $(1-\tau_t)w_t^P = w_t^R$. From labor supply, we know that any market equilibrium $L^{R,H}$ implies the following critical belief:

$$\theta_{crit} = \theta_h - \frac{(\theta_h - \theta_l)L^{R,H}}{\overline{L}_B}.$$

Substituting this expression into (2.14) yields

$$\theta_h - \frac{(\theta_h - \theta_l)L^{R,H}}{\overline{L}_B} = \frac{(1-\tau_t)w_t^P}{w_t^R} - (1-\theta_l),$$

or equivalently

$$w_t^R = \frac{(1-\tau_t)w_t^P}{1 + (\theta_h - \theta_l)(1 - \frac{L^{R,H}}{\overline{L}_B})}. \tag{2.20}$$

We find that, unlike before, researchers are now paid at a markdown. This follows, of course, from the fact that researchers have their belief as an additional source of utility and thus require less compensation for working in the research sector. Additionally, we can see that market mechanisms determine this markdown. A greater demand for research, expressed by bigger $L^{R,H}$, will lower the markdown, while a greater overall supply of researchers, expressed by a bigger $\overline{L}_B$, will increase it. Hence, the tax rate now reads

$$\tau_t = \frac{L^{R,H}}{1 + (\theta_h - \theta_l)\left(1 - \frac{L^{R,H}}{\overline{L}_B}\right)(1 - L^{R,H})},$$

which is obtained by substituting (2.20) into the government's budget constraint, $L^{R,H} w_t^R = \tau_t (1 - L^{R,H}) w_t^P$.

2.3.5 Social planner solution

To derive the social planner optimum for this economy, one would have to include the following term in the previous welfare function W:

$$\sum_{t=0}^{\infty} \beta_s^t \int_{\theta_{crit,t}}^{\theta_h} \log(1-\theta_l+\theta_i)di, \text{ with}$$
$$\theta_{crit,t} = \theta_h - \frac{(\theta_h-\theta_l)L_t^R}{\overline{L}_B},$$

which captures the additional utility for those working in research. Note that $\theta_l < \theta_{crit,t}$, so that the integral is finite. If, however, the social planner communicates the true parameter θ and it replaces the individual belief θ_i, then the integral is zero and the problem collapses to the one studied above.

Next, we compare the socially optimal outcome to the decentralized outcomes in economies with and without research bubbles. Table 2.1 provides the parameter values that we use.

α	β	θ	$\overline{L}_B$	θ_l	θ_h
0.3	0.85	1.5	1	1	2

Table 2.1: Parameter values.

We derive research labor for the social planner, $L^{O,R}$ and the government, $L^{R,H}$ under a research bubble, and the equilibrium without research bubbles, L^R.

L^R	$L^{R,H}$	$L^{O,R}$
0.0991	0.1767	0.75

Table 2.2: Labor input in research for the social planner and government.

Table 2.2 provides our findings, which we can summarize in one inequality: $L^R < L^{R,H} < L^{O,R}$. We find the following: First, there is a research bubble in the decentralized economy, as can be seen in the first inequality. Although the true productivity of research θ has remained the same, we find more labor dedicated to knowledge production. Second, the research bubble moves the decentralized amount of investment closer to the socially optimal one: We observe an increase of roughly 8 percentage points, when comparing the two outcomes. Third, even in the presence of a research bubble, the decentralized economy remains below the optimal outcome, as can be seen in the second inequality. We summarize our findings in Proposition 5.

Proposition 5

The steady state levels of the critical belief value θ_{crit} and of research labor $L^{R,H}$ in the decentralized economy are given by Equations (2.15), (2.18)*, and Equation* (2.19)*. Research labor $L^{O,R}$ in the social optimum is given by Equation* (2.11) *. We observe a welfare-improving research bubble.*

Several remarks are in order. The government is optimistic since $\theta_{crit,t} > \theta_l$. Thus its estimate $\frac{\theta_h+\theta_{crit,t}}{2}$ is higher than the true productivity. As described above, this over-optimism and the ensuing research bubble are generated by two mechanisms, self-selection of researchers and information aggregation by the government. By self-selection we mean that agents decide themselves which sector they want to work in. More optimistic agents are willing to work in research even if w_t^R is small. Only with greater w_t^R does labor demand increase thus also attracting less optimistic researchers . Consequently, researchers are hired, beginning at the higher end of the belief distribution. The least optimistic ones are not hired, as employing all agents in research is prohibitively costly. By information aggregation we mean that the government forms an estimate about θ based only on the beliefs of the agents hired. Hence, the estimate of the government does not yield the true productivity and it demands more research than in the previous model. This is a research bubble.

If the government asked all agents about their respective beliefs, its estimate would be exactly θ, as with $\theta_l = 1$ and $\theta_h = 2$, we have $\widetilde{\theta} = 1.5 = \theta$. Yet the government receives information from a non-representative sample of the population, as only agents with $\theta_i \geq \theta_{crit}$ work in research.

One can imagine the government as an econometrician tries to measure θ. It faces random differences in the parameter because of the random distribution of beliefs. Although the government's methods are sophisticated, it overestimates the parameter, because it does not take the self-selection bias into account.

2.4 Implementing the socially optimal solution

In this section, we explore how the socially optimal solution can be implemented by the decentralized solution in the steady state.

First we focus on whether and how the decentralized solution can implement the socially optimal solution through research bubbles. If the true productivity in the economy is θ, then Equation (2.12) shows us the necessary size of the research bubble. The equation also provides a necessary condition for the implementation of the social optimum, given by $1/\beta^2 - 1 > \theta$. It continues to hold. However, it is not a sufficient condition, as $\widetilde{\theta} = \eta\theta_h + (1-\eta)\theta_{crit}$. Hence, we have

$$\eta\theta_h + (1-\eta)\theta_{crit} = \frac{\theta}{1-\beta^2(1+\theta)},$$

which implies that

$$\hat{\theta}_{crit} = \frac{1}{1-\eta}\left[\frac{1}{1-\beta^2(1+\theta)} - \eta\theta_h\right] \text{ and thus } \theta > \frac{\theta_h(1-\beta^2)-1}{\beta^2\theta_h} \tag{2.21}$$

must hold for implementation. Expression (2.21) provides the sufficient condition for a positive value of θ_{crit}. Also, it yields the value $\hat{\theta}_{crit}$ which is the critical belief that is socially optimal. This yields the following proposition:

Proposition 6 *An optimistic view of the government* $\widetilde{\theta} = \eta\theta_h + (1-\eta)\theta_{crit}$ *can implement the social optimum if* $1/\beta^2 - 1 > \theta$ *and Expression* (2.21) *are satisfied.*

Note that the equilibrium θ_{crit} of the decentralized economy does not have to coincide with $\hat{\theta}_{crit}$, even if the inequality from Expression (2.21) is fulfilled. The mere possibility of implementation does not mean that the economy's research bubble will have precisely the optimal size. If $\theta_{crit} < \hat{\theta}_{crit}$ the economy's research bubble will be too large. In the opposite case it will be too small.

As a numerical example, consider the parameter values from Table 2.1 and the fact that the government forms an average of the researchers' beliefs, i.e. $\eta = 1/2$. In this case implementation is not possible, as $1/\beta^2 - 1 > \theta$ does not hold. However, if we consider the following set of parameter values: $\theta = 0.3$, $\theta_h = 4$, $\beta = 0.8$, and $\eta = 0.5$, we find that the necessary and sufficient conditions for implementation are met. Under these parameter values we obtain $\hat{\theta}_{crit} = 1.9524$.

However, if the government forms biased estimates of active researchers' beliefs, the socially optimal steady state can be implemented as a decentralized balanced growth path, as we show next. For this purpose we consider the steady state solution given in Equation (2.11).

Proposition 7 *If* $L^{D,R}(\eta = \frac{1}{2}) < L^{O,R} < \frac{\beta}{1+\beta}$, *i.e. if the social planner solution lies between the government's demand for research labor with* $\eta = \frac{1}{2}$ *and the government's maximal demand,* $\frac{\beta}{1+\beta}$, *there exists an* $\eta^* > \frac{1}{2}$ *and an associated research bubble such that the decentralized solution can implement the social optimum. If* $L^{O,R} < L^{D,R}(\eta = \frac{1}{2}) < \frac{\beta}{1+\beta}$, *there exists an* $\eta^* < \frac{1}{2}$ *that implements the social optimum.*

We stress that the existence of such an η^* hinges on the aforementioned condition. If $L^{O,R} > \frac{\beta}{1+\beta}$, no amount of optimism will elevate the government's demand to the socially optimal level.

But if such an η^* exists, it can be found as follows: First, the implementation of the social optimum as a market outcome requires $L^{S,R} = L^{D,R} = L^{O,R}$. Hence, we set research labor supply equal to the socially optimal level and solve for θ_{crit}:

$$\theta_{crit} = \theta_h - \frac{\theta_h - \theta_l}{\overline{L}_B} L^{O,R}. \tag{2.22}$$

Next we equate demand to $L^{O,R}$ and solve for η^*:

$$\frac{1}{1+\beta}\left(\beta - \frac{2}{(\eta\theta_h + (1-\eta)\theta_{crit})}\right) = L^{O,R}, \tag{2.23}$$

which yields

$$\eta^* = \frac{2}{(\theta_h - \theta_{crit})(\beta - L^{O,R}(1+\beta))} - \frac{\theta_{crit}}{\theta_h - \theta_{crit}}, \tag{2.24}$$

where θ_{crit} is given by (2.22). Note the factor $(\beta - L^{O,R}(1+\beta))$ in the denominator of the expression on the right hand side of (2.24). Slightly rewritten, it reads $\beta\left(1 - \frac{1+\beta}{\beta}L^{O,R}\right)$, meaning that η can only be positive if and only if $L^{O,R}$ is indeed smaller than $\frac{\beta}{1+\beta}$. Note that this is a necessary but not a sufficient condition for $\eta^* > 0$.

Let us turn again to a numerical illustration under our baseline parametrization. We already know from Table 2.2 that $L^{O,R} = 0.75 > \frac{\beta}{1+\beta} = 0.4595$, so that implementation is not possible. This is also reflected in the negative value of $\eta^* = -0.664627$. A simple way to achieve implementation is to assume a smaller value for the true productivity θ. A decrease in θ will lower the socially optimal level of research labor but will not affect the market equilibrium, as the latter depends on the distribution of beliefs and on other model parameters, but not on the actual research productivity. Setting $\theta = 0.3$ instead of 1.5 reduces the socially optimal solution to 0.35. It implies $\theta_{crit} = 1.65$ and $\eta^* = 25.044$, meaning that the market economy can achieve the social optimum given that the government is more than twenty times as optimistic as the most optimistic researcher.

Another way of achieving implementation is to increase the government's discount factor. There is no reason why the government's discount factor (call it β_g) should be equal to that of the social planner or the households. As the government discounts only one future period, β_g can even be greater than one. In qualitative terms this possibility produces the same results as changing θ.

2.5 Bursting research bubbles and prevention

2.5.1 Drawbacks

In this subsection we explore two possible reasons why the implementation of the socially optimal solution through research bubbles may fail.

Overstatement of beliefs

One potential drawback is that active researchers may all want to express θ_h and not $[\theta_{crit}, \theta_h]$, since overstating their belief might lead to higher research wages. This can be a drawback if the research bubble is very large, i.e. decentralized demand is equal to, or already larger than, the social optimum. If active researchers all report θ_h, then $\widetilde{\theta} = \theta_h$ instead of $\widetilde{\theta} = \eta\theta_h + (1-\eta)\theta_{crit}$ and labor demand is

$$L^{D,R} = \frac{1}{1+\beta}\left(\beta - \frac{1}{\theta_h}\right).$$

Graphically, the demand curve is shifted upwards, while supply remains unchanged, as can be seen in Figure 2.2. The blue curve represents the old demand for research labor while the red curve indicated the new shifting demand. Because of the upward shift, θ_{crit} is lower, and the equilibrium level of labor in research is higher.

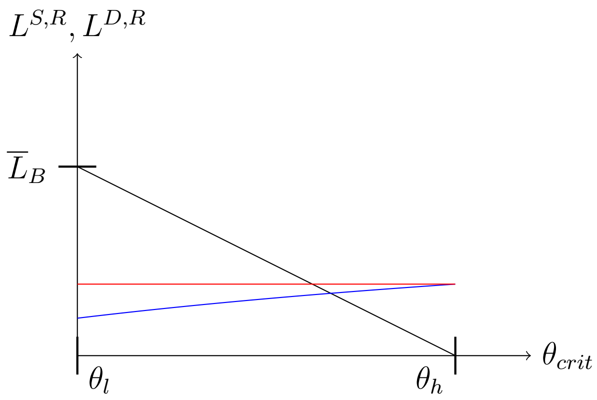

Figure 2.2: Labor market and optimism equilibrium with overstatement. For the purpose of illustration, we use the following parameter values: $\beta = 0.85$, $\theta_l = 1$, $\theta_h = 2$, and $\overline{L}_B = 0.5$.

The increase in research labor is not infinite. Even if all hired researchers over-report their belief, not all agents of the economy will be hired in the research sector. This new equilibrium is obtained by equating labor supply with the new labor demand:

$$\theta_{crit} = \theta_h - \frac{(\theta_h - \theta_l)}{\overline{L}_B(1+\beta)} \left(\beta - \frac{1}{\theta_h}\right).$$

Collapse

Governments might learn that beliefs are too optimistic. This could happen from observing past outcomes of basic research investments by previous governments or—recognizing the selection of optimistic researchers into the research activities—by discounting the assessment of researchers. While the first source of learning might be difficult, since basic research activities are very different across time,[7] the second source of learning is more plausible. Such learning might lead to a collapse of the bubble, as we demonstrate next.

Assume a situation in which the social optimum has been achieved due to over-optimism and the government's η is equal to η^*. Now suppose a government becomes less optimistic and lowers its η to $\overline{\eta} < \eta^*$. We ask whether it will be optimal for the government to lower its demand from $L^{O,R}$ to $L^{D,R}(\overline{\eta})$, which is the labor demand associated to some $\overline{\eta}$. Once its optimism decreases, the government has two options. On the one hand, it can continue to demand $L^{O,R}$, while $\overline{\eta}$ and not η will enter its utility function. We assume that the government knows θ^*_{crit}, which implements the social optimum. The government can calculate its utility from maintaining the social optimum. Another thing it can do is to reassess the productivity, which will lead to a higher θ_{crit} and a lower demand for research labor. Doing this, the government does not internalize how a change in demand

[7] Moreover, if productivity is affected by macroeconomic shocks—and many varieties of such shocks are discussed in the literature—inferring the impact of basic research on GDP may be inherently difficult or impossible.

for labor impacts θ_{crit}. It will thus believe that, if it changes its demand, θ_{crit} will remain the same. This is a utility that the government expects to obtain. Clearly, its choice will influence θ_{crit} and influence the level of utility it actually achieves.

As the real productivity of research is always θ and not the believed value $\eta\theta_h + (1-\eta)\theta_{crit}$, we distinguish two different levels of utility. On the one hand, there is the utility that the government active in period t expects to obtain, based on the anticipated productivity. On the other, there is the actually realized utility, based on the real productivity. If the government chooses to maintain the social optimum, it will expect to obtain the following utility:

$$\widetilde{u}_t^{G,O} = \log(1 - L^{O,R}) + \beta \log\left(1 + (\overline{\eta}\theta_h + (1-\overline{\eta})\theta^*_{crit})L^{O,R}\right),$$

where θ^*_{crit} is the critical value that implements the social optimum. The tilde indicates the expected value, the superscript indicates the social optimum. Actually, the government will achieve

$$u_t^G = \log(1 - L^{O,R}) + \beta \log\left(1 + \theta L^{O,R}\right).$$

If it chooses to deviate, it believes that its deviation will not influence θ_{crit}, which will remain at the level θ^*_{crit}. Hence, the government maximizes the following expression:

$$\max_{L_t^{D,R}} \widetilde{u}_t^G(L_t^{D,R}) = \log\left(1 - L_t^{D,R}\right) + \beta \log\left(1 + (\overline{\eta}\theta_h + (1-\overline{\eta})\theta^*_{crit})\, L_t^{D,R}\right),$$

which yields

$$\overline{L}_t^{D,R} = \overline{L}^{D,R} = \frac{1}{1+\beta}\left(\beta - \frac{1}{(\overline{\eta}\theta_h + (1-\overline{\eta})\theta^*_{crit})}\right).$$

Note that $\overline{L}^{D,R}$ is not a function but a specific value that the government believes to be the market equilibrium.

In the next step we examine whether the government will deviate from the social optimum. We have shown that the government's maximization problem is strictly concave over the domain of $L^{D,R}$, so that a maximizer of the objective function is unique and is also the global maximum. Therefore it remains to show that $L^{O,R} \neq \overline{L}^{D,R}$. To see that this is indeed the case, recall that

$$\begin{aligned}
&L^{O,R} = L^{D,R}(\eta^*, \theta^*_{crit}), \quad \overline{L}^{D,R} = L^{D,R}(\overline{\eta}, \theta^*_{crit}), \text{ where} \\
&L^{D,R}(\eta, \theta^*_{crit}) = \frac{1}{1+\beta}\left(\beta - \frac{1}{(\eta\theta_h + (1-\eta)\theta^*_{crit})}\right) \\
&\Leftrightarrow \frac{\partial L^{D,R}(\eta, \theta^*_{crit})}{\partial \eta} > 0,
\end{aligned}$$

so that $\overline{L}^{D,R} < L^{O,R}$, which concludes the proof. Note that we have assumed $\beta > 1/\theta$ and that $\theta_h \geq \theta$, so that $\beta > 1/\theta_h$ holds. We can be certain that the government will deviate from the social optimum if it becomes less optimistic.

The government makes its decision believing that θ_{crit} will not adjust. However, we know that θ_{crit} is likely to change. The reason is that lower optimism on the part of the government, $\overline{\eta}$, shifts the demand curve downwards and leads to a larger θ_{crit}. Therefore it is unlikely that $\overline{L}^{D,R}$ will be the new market equilibrium $L^R(\overline{\eta})$ that is associated to $\overline{\eta}$. Hence, the government will receive the following utility:

$$u^G_t(\overline{\eta}) = \log\left(1 - L^R(\overline{\eta})\right) + \beta \log\left(1 + \theta L^R(\overline{\eta})\right),$$

which we write as a function of $\overline{\eta}$ because it determines the market clearing $\theta_{crit}(\overline{\eta})$ and the labor market equilibrium $L^R(\overline{\eta})$. Thus deviation will be profitable *ex post* if $u^{G,O}_t < u^G_t(\overline{\eta})$, i.e. if the decentralized equilibrium provides higher utility than the social optimum. This is the case, as the L^R that maximizes the expression

$u_t^G = \log\left(1 - L^R\right) + \beta \log\left(1 + \theta L^R\right)$ is, of course, the labor demand from the simple model. Also, remember that u_t^G is a strictly concave function and thus is decreasing on $(L^R, 1)$, so that $L^R < L^R(\overline{\eta}) < L^{O,R}$ implies $u_t^G(L^R) > u_t^G(L^R(\overline{\eta})) > u_t^{G,O}$. Hence deviation is profitable *ex post*.

2.5.2 Institutional remedies

We have observed that governments may resort to more realistic assessments, thus lowering basic research investments below socially optimal levels. Of course, it is not the more realistic assessments of the impact of basic research that should be prevented, but the lowering of basic research investments as a consequence. There are three possible ways of preventing such attempts. First, one could give optimistic researchers a strong say in decisions about basic research investments. Of course, these views have to be balanced to prevent excessive basic research investments. Second, one could allow the government to issue public debt the amount of which is dependent on the level of research activities. This would provide generations with more incentives to undertake the socially optimal amount of research. We will explore this case in the next subsection. Third, attempts to lower research investment could also be prevented by traditional constitutional means making deviations from the social optimum difficult for a single generation. This can be achieved, by say, committing to longer term funding plans that cannot be rapidly changed by one individual generation.

2.5.3 Debt financing

We have shown that the decentralized solution implies less research than the social optimum. The reason is straightforward. The decentralized government does not internalize the marginal benefits of research for future generations. To increase the the government's labor demand, one could allow to issue debt. More precisely, our goal is study whether there is some amount of debt d_t enabling the steady state social planner solution to be implemented in the decentralized case. We assume that the government has access to financial markets and can borrow at the rate r_t without any frictions. Also, we assume that debt can be fully rolled over to the next generation, i.e. if the government in period t borrows d_t, in it can always borrow at least $r_{t+1}d_t$ in $t+1$.

We propose a debt contract made up of two parts. First, the government is allowed to borrow an amount equal to the total debt level times interest in t . As described above, this allows to roll over debt. Second, the government can borrow some amount $D_t(L_t^{D,R})$, which depends on its demand for research labor. We can show that if

$$D_t(L_t^{D,R}) = \left(\frac{1}{(1 - L_t^{D,R})^{\beta(1-\alpha)}} - 1 \right) Y_t,$$

then the social optimum can be implemented in the market economy. To prove this, we write the modified maximization problem of the government as

$$\max_{L_t^{D,R}} \log \Big(Y_t + D_t(L_t^{D,R}) + r_t d_{t-1} - r_t d_{t-1} \Big) + \beta \log \left(Y_{t+1} + r_{t+1} d_t - r_{t+1} d_t \right),$$

where $d_t = D_t(L_t^{D,R}) + d_{t-1} r_t$ is the accumulated debt stock, i.e. the sum of debt taken over from the previous generation plus the additional debt from period t.

We can simplify the problem to

$$\max_{L_t^{D,R}} \ \log\left(\frac{Y_t}{(1-L_t^{D,R})^{\beta(1-\alpha)}}\right) + \beta\log\left(Y_{t+1}\right) =$$
$$\max_{L_t^{D,R}} \ \log\left(\frac{((1-L_t^{D,R})B_t)^{1-\alpha}K_t^{\alpha}}{(1-L_t^{D,R})^{\beta(1-\alpha)}}\right) + \beta\log\left(((1-L_{t+1}^{D,R})B_t(1+\theta L_t^{D,R}))^{1-\alpha}K_{t+1}^{\alpha}\right),$$

where as before, K_t and B_t are state variables in period t. In t, the government perceives L_{t+1} and K_{t+1} as independent of its choice. Due to the logarithmic utility function, the problem is a sum in which the components depending on the aforementioned variables can be omitted. This enables us to reduce the problem to

$$\max_{L_t^{D,R}}(1-\beta)\log\left(1-L_t^{D,R}\right) + \beta\log\left(1+\theta L_t^{D,R}\right),$$

which yields

$$\frac{1-\beta}{1-L_t^{D,R}} - \frac{\beta\theta}{1+\theta L_t^{D,R}} = 0$$
$$\Rightarrow L_t^{D,R} = \beta - \frac{1-\beta}{\theta} = L^{O,R}.$$

It is thus possible to implement the socially optimal steady state as labor demand in every period by allowing the government to issue debt. In the Appendix for Chapter 2, we show that such a debt contract leads to a constant ratio of debt to output if the interest rate satisfies $r < 1 + \theta L^{O,R}$.

The socially optimal demand must be financed in order to become the new market equilibrium. For this, we assume that all debt income is used to finance wages in the research sector. At best, this would allow the government to lower income taxes τ_t to zero and pay researchers the amount $D_t(L^{O,R})$. We cannot however be certain whether $D_t(L^{O,R})$ will actually cover the government's financing need, which has increased, because the government wants to employ more researchers than before. It may even be the case that the government will have to raise the income tax rate. Therefore, when the government implements the optimal amount $L^{O,R}$ and issues debt of magnitude $D_t(L^{O,R})$, balancing the budget requires

$$L^{O,R}w_t^R = \tau_t^G(1 - L^{O,R})w_t^P + D_t(L^{O,R}),$$

with τ_t^G being the new tax rate. The right-hand side of this equation shows the two sources of government income: Taxes and debt. Substituting w_t^R from (2.20) and simplifying yields

$$w_t^P \kappa \left(\frac{L^{O,R}}{1 + (1 - L^{O,R})(\theta_h - \theta_l)\left(1 - \frac{L^{O,R}}{\overline{L}_B}\right)} - \tau_t^G \right) = D_t(L^{O,R}), \qquad \text{with}$$
$$\kappa := \left(\frac{1 + (1 - L^{O,R})(\theta_h - \theta_l)\left(1 - \frac{L^{O,R}}{\overline{L}_B}\right)}{1 + (\theta_h - \theta_l)\left(1 - \frac{L^{O,R}}{\overline{L}_B}\right)} \right).$$

Note that the term

$$\frac{L^{O,R}}{1 + (1 - L^{O,R})(\theta_h - \theta_l)\left(1 - \frac{L^{O,R}}{\overline{L}_B}\right)}$$

is exactly the tax rate that would be required to finance the optimal labor demand in the absence of debt. Thus, we call it τ^S and have

$$w_t^P \kappa(\tau^S - \tau_t^G) = D_t(L^{O,R}).$$

By plugging in the definition of $D_t(L^{O,R})$, and making use of $w_t^P = (1-\alpha)Y_t/(1-$

$L^{O,R}$), we arrive at

$$\tau^S - \tau^G = \left((1 - L^{O,R})^{-\beta(1-\alpha)} - 1\right) \frac{Y_t(1 - L^{O,R})}{(1-\alpha)\kappa Y_t},$$
$$\tau^G = \tau^S - \frac{1 - L^{O,R}}{(1-\alpha)\kappa} \left((1 - L^{O,R})^{-\beta(1-\alpha)} - 1\right),$$

and find that the government does not have to increase the tax rate to τ^S thanks to the presence of debt financing. However, it is not clear whether τ^G will be greater or smaller than the previous τ. Their relative size depends on the difference between the market and the social planner outcome and hence on parameter constellations.

2.6 Extensions

The model allows a number of extensions that shed further light on the role of research bubbles.

2.6.1 Effort in knowledge production

In this extension, the output of the research sector depends not only on the number of researchers but also on the effort they invest. The expected production function for knowledge changes to

$$\widetilde{B}_{t+1} = B_t(1 + \widetilde{\theta}_t L_t^{D,R} E_t),$$

where, as before, B_t stands for the knowledge stock in t, $\widetilde{B}_{t+1}$ for the expected knowledge stock in $t+1$, $\widetilde{\theta}_t$ for the governments' estimate of the productivity of the research sector, given by $\widetilde{\theta}_t = \eta\theta_h + (1-\eta)\theta_{crit,t}$, and $L_t^{D,R}$ for the demand for research labor. The new variable is E_t, which is the aggregate effort of researchers, i.e.

$$E_t = \int_{\theta_{crit,t}}^{\theta_h} e_{t,i} di,$$

where $e_{t,i}$ is the effort of the individual researcher i. Every θ_i corresponds to a finite value of $e_{t,i}$, making the integral finite. Researchers choose $e_{t,i}$ by maximizing utility. Unlike before, utility from research depends on the product of how efficient the agent believes his research to be, θ_i and the effort $e_{t,i}$ he invests. Effort is also associated with costs, which we capture by the cost function $C(e_{t,i}) = q\frac{e_{t,i}^2}{2}$, where $q \geq 0$ is a scaling parameter. The effort-augmented utility function for a researcher thus writes as

$$U_{t,i}^R = \log(w_t^R - s_{t,i}) + (1+\beta)\log(\theta_i e_{t,i} - q\frac{e_{t,i}^2}{2}) + \beta \log(r_{t+1} s_{t,i}).$$

Maximizing utility with respect to effort and savings yields

$$e_{t,i} = \frac{\theta_i}{q}, \quad s_{t,i}^P = \beta\frac{(1-\tau_t)w_t^P}{1+\beta} \quad \text{and} \quad s_{t,i}^R = \beta\frac{w_t^R}{1+\beta}.$$

Plugging these optimal choices into the utility of a researcher and setting it equal to the utility of a worker, we obtain the critical value for the belief θ_i:

$$\theta_{crit,t} = \sqrt{2q\frac{(1-\tau_t)w_t^P}{w_t^R}}.$$

In this extension, the government faces the following maximization problem:

$$\max_{L_t^{D,R}} \log\left(((1-L_t^{D,R})B_t)^{1-\alpha}K_t^\alpha\right) + \\ \beta\log\left(((1-L_{t+1}^{D,R})B_t(1+\widetilde{\theta}_t E_t L_t^{D,R}))^{1-\alpha}K_{t+1}^\alpha\right),$$

where it takes B_t, K_t and E_t as given and both K_{t+1} and $L_{t+1}^{D,R}$ as outside its sphere of influence. Hence, the problem can be written as

$$\max_{L_t^{D,R}} \log\left(1 - L_t^{D,R}\right) + \beta \log\left(1 + \widetilde{\theta}_t E_t L_t^{D,R}\right),$$

which gives the following demand function:

$$\begin{aligned} L_t^{D,R} &= \frac{1}{1+\beta}\left(\beta - \frac{1}{E_t \widetilde{\theta}_t}\right), \quad \text{with} \\ E_t &= \int_{\theta_{crit,t}}^{\theta_h} \frac{\theta_i}{q} di = \frac{\theta_h^2 - \theta_{crit,t}^2}{2q}. \end{aligned} \tag{2.25}$$

Plugging in the definitions of E_t and $\widetilde{\theta}_t$, we can write demand as a function of $\theta_{crit,t}$ only:

$$L_t^{D,R} = \frac{1}{1+\beta}\left(\beta - \frac{4q}{(\theta_h^2 - \theta_{crit,t}^2)(\theta_h + \theta_{crit,t})}\right).$$

It is possible to show that this demand function is strictly concave in $\theta_{crit,t}$, which yields two possible labor market equilibria, given that labor supply is the same linear function as in the previous model[8]. Moreover, we demonstrate that the market equilibrium with greater labor in research L^R is preferred by the government.

[8]Proofs for this claim, as well as for all others in this section, are available on request.

We also explore the solution of a social planner who knows the true value of θ:

$$\max_{c^1_{t,i}, c^2_{t+1,i}, e_{t,i}, L^R_t, B_{t+1}, K_{t+1}} W(c^1_{t,i}, c^2_{t+1,i}, e_{t,i}, L^R_t, B_{t+1}, K_{t+1}),$$

with

$$W = \sum_{t=0}^{\infty} \beta_s^t \left(\int_0^1 \log(c^1_{t,i}) + \beta \log(c^2_{t+1,i}) \, di + \int_0^1 \log(\theta e_{t,i} - q\frac{e^2_{t,i}}{2}) \, di \right),$$

subject to

$$((1 - L^R_t)B_t)^{1-\alpha} K^\alpha_t = \int_0^1 (c^1_{t,i} + c^2_{t,i}) \, di + K_{t+1},$$

and

$$B_{t+1} = (1 + \theta \cdot \int_0^1 e_{t,i} \, di \cdot L^R_t)B_t.$$

For a symmetric equilibrium with $e_{t,i} = E_t$ and $c^1_{t,i} = c^1_t = c^2_{t,i} = c^2_t \; \forall i$, due to $\beta_s = \beta$, we find the following optimality conditions:

$$\frac{1}{c_t^1} = \frac{\alpha\beta Y_{t+1}}{K_{t+1}c_{t+1}^2}, \tag{2.26}$$

$$\frac{Y_t}{1-L_t^R}\frac{1}{E_t} = \frac{K_{t+1}}{\alpha Y_{t+1}}\left(\frac{\theta Y_{t+1}}{(1+\theta L_t^R E_t)} + \frac{Y_{t+1}}{(1-L_{t+1}^R)E_{t+1}}\frac{1+\theta L_{t+1}^R E_{t+1}}{(1+\theta L_t^R E_t)}\right), \tag{2.27}$$

and

$$\frac{E_t}{L_t^R}\frac{\theta - qE_t}{\theta E_t - q\frac{E_t^2}{2}} = -\frac{2(1-\alpha)}{(1-\frac{K_{t+1}}{Y_t})(1-L_t^R)}, \tag{2.28}$$

which, in a steady state, yield

$$L^R = \beta - \frac{1-\beta}{\theta E}, \quad \text{and} \tag{2.29}$$

$$E^2 q\left[\frac{(1-\alpha)\beta}{1-\alpha\beta} + 1 - \beta\right] - (1-\beta)\left[1 - \frac{2(1-\alpha)}{1-\alpha\beta}\right] \tag{2.30}$$
$$- E\left[(1-\beta)\left(\theta - \frac{q}{\theta}\right) + \frac{2(1-\alpha)}{1-\alpha\beta}\left(\beta\theta + \frac{(1-\beta)q}{2\theta}\right)\right] = 0.$$

We find that the socially optimal labor input equation, L^R, is structurally equivalent to what it was before, but that θ is replaced by θE. The same holds for the decentralized labor demand. Hence, if θ and E were the same in the decentralized economy and the social planner solution, L^R would be too low in the decentralized economy. However, if aggregate effort E was greater in the market equilibrium, this could mitigate the government's myopia. To investigate this possibility, we compare the first order condition for the individual effort of some agent i in the decentralized economy to that of the social planner:

$$\frac{\theta_i - qe_{t,i}}{\theta_i e_{t,i} - q\frac{e_{t,i}^2}{2}} + 0 = 0, \tag{2.31}$$

$$\frac{\theta - qe_t}{\theta e_t - q\frac{e_t^2}{2}} + \frac{2(1-\alpha)Y_t}{e_t C_t}\frac{L_t^R}{1-L_t^R} = 0. \tag{2.32}$$

The first equation is the first-order condition in the decentralized equilibrium. We obtain two differences. First, the individual agent bases his effort on his belief θ_i

and not on θ. Second, he does not internalize the positive externality of his effort on knowledge production, which is captured by the second term in Equation (2.32). Hence it is not clear whether an individual agent supplies more or less effort than would be socially optimal. On the one hand, the individual belief might be greater than θ, implying more effort, while on the other hand, the agent might not be aware of the externality, implying less effort. It is likely, however, that the effort externality is greater than the over-optimism, especially for more conservative agents, so that aggregation of individual effort would provide an overall level of effort that is too small compared to the social optimum.

With this finding, the implementation of the social optimum in the decentralized economy cannot be achieved by introducing government debt alone, even when the government knows θ. It is also necessary to create incentives for scientists to provide the optimum amount of effort. Hence the following wage contracts must be offered to a researcher with belief θ_i:

Proposition 8 *The social optimum can be obtained in the decentralized economy by offering researcher i with belief θ_i the wage*

$$
\begin{aligned}
w_{t,i}^R &= \widetilde{w}_t{}^R \left(\frac{\hat{\theta} e_{t,i} - q \frac{e_{t,i}^2}{2}}{\theta_i e_{t,i} - q \frac{e_{t,i}^2}{2}} \right)^{\frac{1}{1+\beta}} e^{\frac{\hat{G}_t e_{t,i}}{1+\beta}}, \quad \textit{with} \\
\widetilde{w}_t^R &= w_t^P \left(\hat{\theta} e_t - q \frac{e_t^2}{2} \right)^{\frac{-1}{1+\beta}} e^{-\hat{G}_t \frac{e_t}{1+\beta}}, \\
e_t &= \frac{\theta_{crit} - \theta_l}{\theta_h - \theta_l} E^{SOC},
\end{aligned}
$$

and allowing the government to issue debt of magnitude

$$
\frac{Y_t}{(1 - L^{D,R})^{\beta(1-\alpha)}} + r_t d_{t-1},
$$

where $\widetilde{w}_t^R$ is a fixed-wage component that is equal for all researchers. E^{SOC} stands for the social planner steady state levels of aggregate effort. Furthermore, $\hat{\theta} = \frac{\theta_h - \theta_l}{\theta_h - \theta_{crit}}\theta$ and $\hat{G}_t = \frac{\theta_h - \theta_{crit}}{\theta_h - \theta_l} G_t$. θ_{crit} corresponds to the steady state level of the critical belief value that leads to the socially optimal supply of research labor. e_t is individual effort, which is the same for all researchers.

The Appendix for Chapter 2 contains the proof that this payment scheme will implement the social optimum. The intuition behind it is the following: First, we want every researcher to base his effort decision on the true parameter and not on his beliefs. Therefore $w_{t,i}^R$ depends on θ_i and θ, i.e. on the individual belief and on the scaled true parameter.[9] Second, every researcher is supposed to internalize the effect of his effort on research productivity. Therefore $\hat{G}_t$ is incorporated in the wage scheme. Third, the government is supposed to increase its labor demand, so that the debt it can issue depends on $L_t^{D,R}$. The first of our measures simplifies the task of implementing the social optimum. The second and third fulfill the task.

2.6.2 Linear utility

Another variant of the model is one with effort and a linear utility function for the government. Accordingly, the government's problem is

$$\max_{L_t^{D,R}} ((1 - L_t^{D,R})B_t)^{1-\alpha} K_t^{\alpha} + \beta((1 - L_{t+1}^{D,R})B_t(1 + \widetilde{\theta}_t E_t L_t^{D,R}))^{1-\alpha} K_{t+1}^{\alpha}.$$

In the following, we use $\widetilde{Y}_{t+1}$ to denote the output that the government believes will be created, while Y_{t+1} is the true future output. Maximizing with respect to

[9] Why the true parameter needs to be scaled is set out in the Appendix for Chapter 2.

$L_t^{D,R}$ yields

$$\frac{Y_t}{1-L_t^{D,R}} = \beta \frac{\widetilde{\theta}_t E_t \widetilde{Y}_{t+1}}{1+\widetilde{\theta}_t E_t L_t^{D,R}}, \text{ with} \tag{2.33}$$

$$E_t = \int_{\theta_{crit,t}}^{\theta_h} \frac{\theta_i}{q} di = \frac{\theta_h^2 - \theta_{crit,t}^2}{2q} \tag{2.34}$$

and $\widetilde{\theta}_t$ given by (2.16).

Unlike before, the first order condition of the government is a dynamic equation in L_t^R and does not yield a time-constant value for research labor demand. However, a steady state with constant labor demand can be found

$$L^{D,R} = 1 - \frac{1}{\beta \widetilde{\theta} E} = 1 - \frac{4q}{\beta(\theta_h^2 - \theta_{crit}^2)(\theta_h + \theta_{crit})}.$$

In this case, it is possible that the government may demand more research labor than is socially optimal. Furthermore, the demand function is not strictly increasing in θ_{crit}, as can be seen from plugging in $\widetilde{\theta}$ and E. We can show that $L^{D,R}$ is a concave function in θ_{crit}, so that two market equilibria are possible. It can be demonstrated, though, that one of them yields more utility for the government.[10]

The dynamic demand function allows for a convergence analysis. We find that the decentralized market outcome is saddle-path stable. The implementation of the social optimum steady state as a balanced growth path for the decentralized economy is possible. However, this is only the case if the initial market allocation implies a lower level of research labor. If that is so, then a combination of public debt and wage contracts for researchers allows for implementation, as discussed

[10]By plugging in $L^{D,R}$ into the government's utility function, one can analyze the governments utility as a function of θ_{crit}. It turns out to be a concave function over the interval $[\theta_l, \theta_h]$ with a local maximum. Hence, the desirability of the equilibria can be studied by their proximity to the local maximum. This hold for parameter constellations under which the function behaves rather symmetrically around its maximum.

above.

2.7 Conclusion

We have developed a model that provides a rationale and a microfoundation for research bubbles. Such research bubbles emerge when researchers self-select into those activities they believe to be most promising, and when the assessments of these researchers are aggregated by the government. Furthermore, bubbles can implement socially desirable allocations. Thus specific forms of research bubbles are desirable from a long-term welfare perspective. Numerous extensions require further scrutiny, as they have the potential to shed further light on the emergence, social desirability, and downside of research bubbles, which may be a key factor of modern knowledge economies.

Chapter 3

Untimely destruction: pestilence, war, and accumulation in the long run

Abstract

This chapter analyses the effects of disease and war on the accumulation of human and physical capital.[1] We employ an overlapping-generations framework in which young adults, confronted with such hazards and motivated by old-age provision and altruism, make decisions about investments in schooling and reproducible capital. A poverty trap exists for a wide range of stationary war losses and premature adult mortality. If parents are altruistic and their sub-utility function for own consumption is more concave than that for the children's human capital, the only possible steady-state growth path involves full education. Otherwise, steady-state paths with incompletely educated children may exist, some of them stationary ones. We also examine, analytically and with numerical examples, a growing economy's robustness in a stochastic environment. The initial boundary conditions have a strong influence on outcomes in response to a limited sequence of destructive shocks.

[1]This chapter is joint work with Prof. Dr. Clive Bell and Prof. Dr. Hans Gersbach.

3.1 Introduction

Motivation

Dürer's woodcut, 'The Four Horsemen of the Apocalypse', is a terrifying vision of the great scourges of humanity from time immemorial. This chapter deals with three of them – pestilence, war and death, with their accompanying destruction of human and physical capital. Its particular concern is how these calamities affect the accumulation of capital, with special reference to the existence of growth paths and poverty traps. The treatment is necessarily stylized, simple and, in contrast to Dürer's masterpiece, desiccated.

In such a setting, the distinction between human and physical capital is vital. Not only are they complementary in production, but they are also, in general, subject to different, albeit not fully independent, hazard rates. The attendant risks are not, moreover, equally insurable. These considerations weigh heavily in the decision of how much to invest and in what form, with all the ensuing consequences for material prosperity over the long run.

A few selected examples of such calamities will convey some flavor of the historical dimensions of what is involved. The Black Death carried off about one-third of the entire European population between 1347 and 1352. The so-called "Spanish influenza" pandemic of 1918-1920 is estimated to have caused at least 50 million deaths globally, with exceptionally high mortality among young adults. In recent times, the AIDS pandemic, far slower in its course like the disease itself, still threatens to rival that figure, despite the improved availability of anti-retroviral therapies. Pestilence and war also ride together. Half a million died in an outbreak of smallpox in the Franco-Prussian War of 1870-71 (Morgan (2002)). For every British soldier killed in combat in the Crimean War (1854-56), another ten died of dysentery, and in the Boer War (1899-1902), the ratio was still one to five.

War losses in the 20th Century make for especially grim reading. Between 15 and 20 million people died in the First World War, the majority of them young men. Almost two million French soldiers fell, including nearly 30 per cent of the conscript classes of 1912-15. Joining this companionship of death were over 2 million Germans, including almost two of every five boys born between 1892 and 1895 (Keegan (1990a)), almost a million members of the British Empire's armed forces, and many millions more in those of Imperial Austria, Russia and Turkey. Its continuation, the Second World War, was conducted, in every respect, on a much vaster scale. Most estimates suggest that it resulted in at least 50 million deaths, directly and indirectly. Among them were 15 million or more Soviet soldiers and civilians, 6 million Poles (20 per cent of that country's pre-war population) and at least 4 million Germans (Keegan (1990b)). With these staggering human losses went the razing of German and Japanese cities and massive destruction in the western part of the Soviet Union as well as the states of Eastern Europe. The catalog of conflicts in the second half of the 20th Century is also unbearably long, with particularly appalling casualties in South-east Asia and Rwanda.

Great epidemics and wars capture the headlines and grip the imagination, but the majority of those adults who die prematurely fall victim to low-level, 'everyday' causes, especially in poor countries: notable killers are endemic communicable diseases, accidents, violence and childbirth. These are competing hazards—one dies only once—, but their combined effect is not wholly negligible even in contemporary O.E.C.D. countries. In many poorer ones, it is quite dismaying. According to the W.H.O. (2007), those who had reached the age of 20 in the O.E.C.D. group could expect to live, on average, another 60 years or so, their counterparts in China and India another 50-55 years, and those in sub-Saharan Africa but 30-40. The odds that a 20-year old in the O.E.C.D. group would not live to see his or her

40th birthday were 1 or 2 in a 100, rising to 2.5-5 in a 100 for the 50th birthday. These odds were just a little worse for young Chinese, decidedly worse for young Indians, and for young Africans much less favorable than those of Russian roulette – in some countries where the AIDS epidemic was raging, indeed, scarcely better than the toss of a fair coin.

Approach and results

The human and material losses so inflicted, whether they are caused by great epidemics and wars, or endemic communicable diseases and low-level conflicts, have long-run as well as immediate economic consequences. Taking as given agents' preferences and the technologies for producing output and human capital in the presence of these hazards, we address the following questions.

1. Under what conditions are steady-state growth paths outcomes in equilibrium?
2. Are such paths possible when parents are moved by altruism; and if so, is stronger altruism conducive to faster steady-state growth?
3. If mortality and destruction rates do not vary over time, are both secular, low-level stagnation and steady-state growth possible equilibria, thus establishing the existence of a poverty trap?
4. If mortality and destruction rates are stochastic, under what conditions would the economy fall into such a trap when it would otherwise be growing?

The overlapping generations model (OLG) offers the natural framework within which to analyze the long-run consequences of economic behavior in such environments. In the variant adopted here, there are children, young (working) adults and the old. Young adults decide how much schooling the children will receive and how much to put aside to yield a stock of physical capital in the next period. In doing so, they are bound by certain social norms, which govern the distribution of aggregate current consumption among the three generations. Untimely destruction can undo these plans, however carefully laid. The children may die prematurely at some point in young adulthood; and war can wreak havoc on the newly formed capital stock. These losses, if they occur, will reduce the resources available to satisfy claims on consumption in old age in the period that follows. Parents may also be motivated by altruism towards their children, so that their premature deaths will be felt as a distinct loss quite independently of the ensuing reduction in old-age consumption under the prevailing social norms – and arguably all the more keenly if the children have been well educated. The institutional form within which all this takes place is assumed to be a very large extended family, in which the surviving young adults raise all surviving children. Given such pooling, the law of large numbers makes the level of consumption in old age – for those who survive to enjoy it – virtually certain when mortality and war loss rates are forecast unerringly, but even then, the idiosyncratic risk of dying earlier remains. War losses are wholly uninsurable and operate much like cohort-specific mortality. When these rates are stochastic, as is wholly plausible, they constitute unavoidable systemic risks, with consequent effects on investment in both forms of capital.

Our main insights are as follows. Since balanced growth paths with endogenous physical and human capital may not exist – as Uzawa (1961) pointed out in his classic contribution – we first establish conditions for the existence of two extreme steady states, namely, permanent backwardness with no education and unbounded growth with a fully educated population, which we term 'progress'. Without altruism the well-known poverty trap always exists under standard conditions. Moreover, both backwardness and progress may both exist as equilibria for a wide range of mortality and destruction rates. Parents' altruism influences the set of balanced growth paths in two ways. First, if sufficiently strong, it can rule out backwardness. Yet with a robust numerical example, we show that even under quite strong altruism, a poverty trap can exist. Second, in the presence of altruism, progress is the only steady-state path other than backwardness if the sub-utility function for own consumption is more strongly concave than that for parents' evaluation of their children's human capital. If this latter condition is reversed, other steady-state paths with incompletely educated children may exist, some of them stationary, even if altruism is strong.

We also establish conditions for the local stability of a poverty trap. In some settings, both extreme states may be locally stable equilibria, which contrasts with results from corresponding models in which only human capital accumulation matters. We also provide conditions for balanced growth paths with intermediate levels of schooling.

Finally, we explore whether a growing economy can withstand an outbreak of war, a severe epidemic, or a combination of both, as stochastic events; for such destructive events, even if temporary, may pitch a growing economy into backwardness. We establish that these risks depress investment in both physical and human capital; and only extreme destruction of physical capital could induce an increase in schooling. We also establish thresholds for human and physical capital above which an economy can withstand a particular configuration of shocks. We show with simulations that the duration of adverse events – wars or epidemics – is often decisive in determining whether an economy can regain growth.

Relation to the literature

There is a substantial literature on the relationship between the health of populations and aggregate economic activity. Notable is the general empirical observation that good health has a positive and statistically significant effect on aggregate output (Barro and Sala-I-Martin (1995); Bloom and Canning (2000); Bloom et al. (2001)). What is especially relevant for present purposes, however, is a body of work on the macroeconomic effects of AIDS, in which there are varying points of emphasis. Corrigan et al. (2005a,b), for example, adopt a two-generation OLG framework in which the epidemic can affect schooling and the accumulation of physical capital, but expectations about future losses play no role. In two contrasting studies of South Africa, Young (2005) uses a Solovian model to estimate the epidemic's impact on living standards through its effects on schooling and fertility, with a constant savings rate; whereas Bell et al. (2006) apply a two-generation OLG model with pooling through extended families and a vital role for expectations, but no role for physical capital.

Closely related theoretical contributions include Chakraborty (2004), in whose OLG framework endogenous mortality is at centre-stage. Better health promotes growth by improving longevity, and investment in health emerges as a prerequisite for sustained growth. Individual investment in health is also the prime mechanism in Augier and Yaly (2013). Young adults, whose only income is wages, pay a fixed fraction thereof as taxes into a fund managed by the government. This fund provides all capital for the next period, with the gross returns going to the survivors. In Boucekkine and Laffargue‘s (2010) two-period framework with heterogeneous levels of human capital, a rise in mortality among adults in the first period reduces the proportion of young adults with low human capital in the second period because the mortality rate among children at the end of the first rises more sharply in poor families. The number of orphans in the first period increases, however, so that the proportion of young adults with low human capital in the second period will increase if orphans go poorly educated. Bell and Gersbach (2013) analyze growth paths and poverty traps when epidemics take the form of two-period shocks to mortality, paying particular attention to their effects on inequality in nuclear family systems, albeit without a place for physical capital.

A salient feature of these studies is the central importance, if only implicitly, of premature adult mortality. Physical capital, when it does appear, is not subject to similar hazards. Voigtländer and Voth (2013, 2009) take a Malthusian position in explaining the rise of growth in early modern Europe. Disease and war rode together, but '[war] destroyed human life quickly while not wreaking havoc on infrastructure on a scale comparable to modern wars.' (Voigtländer and Voth (2013)). In contrast, the possibility of destruction on such a scale is an essential element of the present chapter, in which there are no fixed factors like land. Furthermore, the second part of the chapter deals with the robustness of a growing economy to shocks: both destruction rates are stochastic. In this connection, exponential depreciation at a constant rate in Solovian models does not lend itself to the task of representing the shocks of war losses. To our knowledge, no other contribution addresses the possibilities of long-term growth and stagnation when both forms of premature destruction are salient features of the environment wherein agents make decisions about accumulation.

The chapter's theme is also broadly related to the existence and relevance of 'balanced growth paths'. The classic problem examined by Uzawa (1961) is whether such paths exist in neoclassical growth models with capital accumulation, population growth and labor- or capital-augmenting technological progress. Grossman et al. (2016) establish that balanced growth requires either an absence of capital-augmenting technological change or a unitary elasticity of substitution between physical and human capital, in which case the forms of factor-augmenting technical change are all equivalent. In this connection, we explore a complementary balanced growth problem: does balanced growth exist in an OLG framework with endogenous physical and human capital accumulation, with or without altruism? We establish conditions on the utility functions with respect to altruism and own consumption that allow balanced growth without imposing very strong restrictions on the production technology.

Structure

The plan of the chapter is as follows: Section 3.2 lays out the model and specifies the general problem to be solved. There follows an analysis of steady states, which necessarily involves unchanging mortality and destruction rates. Sections 3.3 and 3.4 not only establish the conditions for the existence of a stable, low-level equilibrium in which all generations go uneducated, but also that these conditions and those under which steady-state growth is also an equilibrium in the environment in question can be satisfied simultaneously, thus establishing the existence of a poverty trap. Settings in which the destruction rates are stochastic are treated in Sections 3.5 to 3.7. Section 3.8 briefly draws together the chief conclusions.

3.2 The model

There are three overlapping generations: children, who split their time between schooling and work; young adults, who work full time; and the old, who are active neither economically nor in raising children. The timing of events within each period t relates to a generation born in period $t-1$, thus becoming young adults at the start of period t. It is displayed in Figure 3.1. Those individuals who survive into full old age in the following period $t+1$ therefore live for the three periods

$t-1$ to $t+1$.

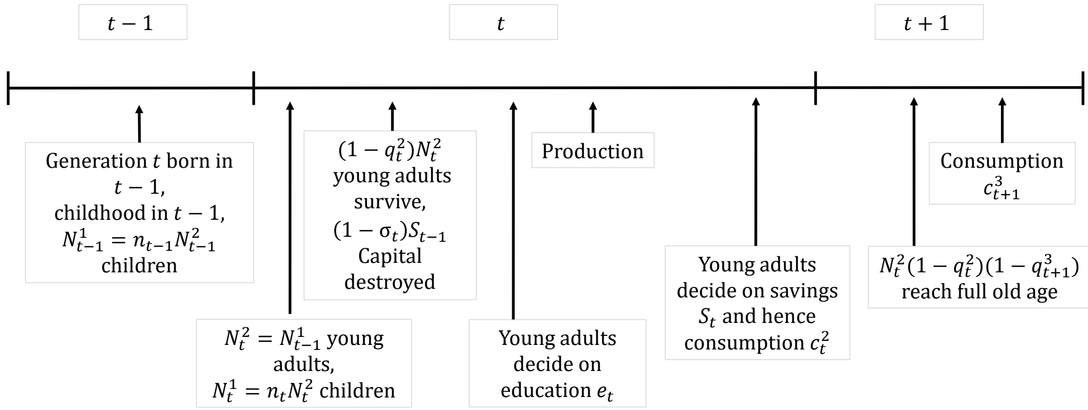

Figure 3.1: Sequence of events for the generation born in period $t-1$.

All individuals belong to numerous, identical and very large extended families. The number of young adults in each family at the beginning of period t is N_t^2. They marry and have children at once. Mortality among children only occurs in infancy, and any child who dies is replaced immediately. After such replacement fertility, each couple within the extended family has $2n_t$ children, all of whom survive into adulthood in the next period. Thus, n_t is the net reproduction rate (NNR). Death then claims some young adults and some of those who have just entered old age. The surviving young adults rear all children collectively and decide how to allocate the children's time between schooling and work, and the resulting aggregate output between consumption and savings, whereby certain social rules govern the claims of children and the old in relation to the consumption of young adults. The numbers of young adults and their offspring who reach maturity are, therefore,

$$N_t^2 = n_{t-1}N_{t-1}^2 \text{ and } N_t^1 = n_t N_t^2,$$

respectively. The numbers of young and old adults who make claims on output in period t are as follows:

$(1 - q_t^2)N_t^2$ young adults survive to raise all children, and

$(1 - q_t^3)N_t^3$ old adults survive to full old age, where $N_t^3 = (1 - q_{t-1}^2)N_{t-1}^2$

and q_t^a denotes the premature mortality rate among age group $a(= 2, 3)$. All adults who do reach full old age in period t die at the end of that period.

Two social rules govern consumption-sharing in the extended family:

(i) When each surviving young adult consumes c_t^2, each child consumes βc_t^2 ($\beta < 1$).

(ii) All surviving old adults receive the share ρ of the family's current 'full income', $\bar{Y}_t$, which is the level of output that would result if all children were to work full time.[2] Since the extended family is very large, each surviving old adult will consume

$$c_t^3 = \frac{\rho \bar{Y}_t}{(1 - q_t^3)N_t^3}. \tag{3.1}$$

Output is produced under constant returns to scale by means of labor augmented by human capital (that is, labor is measured in efficiency units) and physical capital, which is made of the same stuff as output. All individuals are endowed with one unit of time. The time the child spends in school in period t is denoted by $e_t \in [0, 1]$. Each young adult possesses λ_t efficiency units of labor, each child γ units. Each fully educated child ($e_t = 1$) requires w (< 1) young adults as teachers, so that the direct cost of providing each child with schooling in the amount e_t is $w\lambda_t e_t$, measured in units of human capital. The total endowment of the surviving

[2] The determination of β and ρ is discussed in Section 3.3.3. A variant of the rule governing old-age provision is discussed in Section 3.6.

young adults' human capital is $\Lambda_t \equiv (1-q_t^2)N_t^2\lambda_t$; $\bar{L}_t \equiv \Lambda_t+\gamma N_t^1$ is the household's endowment of labor (measured in efficiency units) at time t; and the amount of labor supplied to the production of the aggregate good is

$$L_t \equiv [(1-q_t^2 - wn_te_t)\lambda_t + n_t\gamma(1-e_t)]N_t^2.$$

The aggregate savings of the previous period, S_{t-1}, like the cohort of children entering adulthood, are also subject to losses early in the current one, and what does remain has a lifetime of one period. The capital stock available for current production is therefore $K_t = \sigma_t S_{t-1}$, where $\sigma_t \in (0,1]$ is the survival rate in period t. The current levels of aggregate output and full income are, respectively,

$$Y_t = F(L_t, \sigma_t S_{t-1}) \tag{3.2}$$

and, putting $e_t = 0$,

$$\bar{Y}_t \equiv Y_t(e_t = 0) = F(\Lambda_t + \gamma N_t^1, \sigma_t S_{t-1}),$$

where the function F is assumed to be monotonically increasing in both arguments, continuously differentiable and homogeneous of degree 1, with both inputs necessary in production.

Full income is available to finance the consumption of all three generations in keeping with the social rules, savings to provide the capital stock in the next period, and investment in the children's education.

$$P_tc_t^2 + S_t + \rho\bar{Y}_t = Y_t, \tag{3.3}$$

where $P_t \equiv [1-q_t^2+\beta n_t]N_t^2$ is effectively the price of one unit of a young adult's consumption in terms of output, the numéraire.

The formation of human capital involves the contributions of parents' human capital as well as formal education. The human capital attained by a child on reaching adulthood is assumed to be given by

$$\lambda_{t+1} = z_t h(e_t)\lambda_t + 1. \tag{3.4}$$

The multiplier $z_t(> 0)$ represents the strength with which capacity is transmitted across generations; and it may depend on the number of children each surviving young adult must raise. The function $h(\cdot)$ may be thought of as representing the educational technology, albeit with the fixed pupil-teacher ratio of $1/w$. Let $h(\cdot)$ be an increasing, differentiable function on $[0, 1]$, with $h(0) = 0$ and $\lim_{e\to 0+} h'(e) < \infty$. The property $h(0) = 0$ implies that unschooled children attain, as adults, only some basic level of human capital, which has been normalized to unity.

3.2.1 Preferences and choices

Young adults, who make all allocative decisions, have preferences over lotteries involving current consumption, consumption in old age and, if they are altruistic, the human capital attained by the children in their care. When deciding on an allocation (c_t^2, e_t, S_t), young adults must forecast mortality and destruction rates in the coming period. If these forecasts are unerring, as would be the case in a steady state, those who survive into old age will obtain c_{t+1}^3, from (3.1), which the law of large numbers renders virtually non-stochastic. The stochastic element in the lotteries in question therefore only arises from the individual risks of failing to reach old age and, where altruism towards the children is concerned, that the latter will suffer the misfortune to die prematurely in young adulthood. In this connection, let there be full altruism towards adopted children. If, in contrast, the outbreaks of war and disease in the future are viewed as stochastic events, there will be systemic risks. The analysis of such environments is deferred to Sections 3.5-3.7.

The surviving young adults' preferences are assumed to be additively separable in $(c_t^2, c_{t+1}^3, \lambda_{t+1})$ and von Neumann-Morgenstern in form:

$$V_t = u(c_t^2) + \delta(1 - q_{t+1}^3)u(c_{t+1}^3) + \frac{b(1 - q_{t+1}^2)}{(1 - q_t^2)} n_t v(\lambda_{t+1}), \tag{3.5}$$

where δ is the pure impatience factor and b is a taste parameter for altruism.

The term $\frac{1}{1-q_t^2}$ accounts for the children in the extended family whose parents have died.[3] The sub-utility functions u and v are assumed to be strictly concave, where u satisfies $\lim_{c\to 0} u'(.) = \infty$. In view of the considerations they represent, there are strong reasons to suppose that these functions are not the same.

The surviving young adults' decision problem is as follows:

$$\max_{(c_t^2, e_t, S_t)} V_t \ \text{ s.t. } (3.1) - (3.4),\ c_t^2 \geq 0, e_t \in [0,1], S_t \geq 0. \tag{3.6}$$

When solving it, they note the current state variables, $(n_t, z_t, N_t^1, N_t^2, N_t^3, q_t^2, q_t^3, \lambda_t, K_t)$, and form beliefs about all relevant future levels. Note that these decisions in period t are not influenced by their successors in subsequent periods. Let (c_t^{20}, e_t^0, S_t^0) solve (3.6).

The evolution of the economy is governed by the following difference equations:

$$\lambda_{t+1}^0 = z_t h(e_t^0)\lambda_t + 1 \text{ and } K_{t+1} = \sigma_{t+1} S_t^0.$$

[3] If only natural children count, the 'adjustment' for adopted children $1/(1 - q_t^2)$ drops out.

In what follows, the superscript '0' will be dropped if no confusion would arise.

A preliminary step is to normalize the system by the size of the cohort N_t^2, exploiting the assumption that F is homogeneous of degree one. Let $l_t \equiv L_t/N_t^2$ and $s_t \equiv S_t/N_t^2$, so that (3.1) and (3.3) can be written respectively as

$$c_{t+1}^3 = \frac{\rho n_t}{(1-q_{t+1}^3)(1-q_t^2)} \cdot F\left[(1-q_{t+1}^2)\lambda_{t+1}(e_t) + n_{t+1}\gamma, \frac{\sigma_{t+1}s_t}{n_t}\right] \tag{3.7}$$

and

$$[1-q_t^2+\beta n_t]c_t^2 + s_t + \rho F\left[(1-q_t^2)\lambda_t + n_t\gamma, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right] = F\left(l_t, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right). \tag{3.8}$$

Normalized output is

$$y_t \equiv F\left((1-q_t^2-wn_te_t)\lambda_t + n_t\gamma(1-e_t), \sigma_t s_{t-1}/n_{t-1}\right).$$

The analogous definition of normalized full income is $\bar{y}_t \equiv F\left(\bar{l}_t, \sigma_t s_{t-1}/n_{t-1}\right)$, where $\bar{l}_t \equiv \bar{L}_t/N_t^2$ denotes the normalized endowment of labor at time t. Closely associated with these normalization is the ratio $\zeta_t \equiv \lambda_t/s_{t-1}$, which arises from investment decisions in the previous period.

Together with the constraints $c_t^2 \geq 0, e_t \in [0,1]$ and $s_t \geq 0$, the budget identity (3.8) defines the set of all feasible allocations (c_t^2, e_t, s_t). Upon substitution for c_{t+1}^3 from (3.7) into (3.5), it is seen that V_t is likewise defined in the same space.

3.3 Steady states

In a steady state, the levels of inputs, output and (dated) consumption all grow at a constant rate. Thus, the parameters $\mathbf{q}_t, n_t, \sigma_t$, and z_t are constant, as is the level of the children's education e_t.[4] A special case is that wherein all per capita levels are constant, though population may be growing. In a slight abuse of terminology, this will be called a stationary state, even if the population is not constant. If all per capita levels are growing at the same, positive rate, the economy is said to be on a steady-state growth path. There are two notable steady states, which involve the extreme values of education. If a whole generation of children goes uneducated ($e_t = 0$), so that $\lambda_{t+1} = 1$, the state of backwardness is said to rule in period $t + 1$. If such a state, once reached, becomes permanent, the associated stationary equilibrium implies the existence of a poverty trap. If, at the other extreme, children born in period t enjoy a full education ($e_t = 1$), and all generations that follow them do likewise, this will be called the progressive state, or simply 'progress'. It may, under certain conditions to be explored below, be an equilibrium which exhibits steady-state growth. A fundamental question to be answered is whether permanent backwardness and a progressive growth path are both possible equilibria of an economy of the kind treated here.

So much for the extremes, but are there also steady-state equilibria in which there is some constant level of education short of a full one? If both backwardness and progress are possible equilibria, considerations of continuity suggest that there exists at least one stationary state with $e_t = e^s \in (0, 1)\ \forall t$. If, moreover, steady-state growth with $e_t^0 = 1$ is not an equilibrium path, are there such growth paths with e_t^0 constant and sufficiently close to 1?

It will be helpful to rewrite V_t as a function of the decision variables:

$$V_t = u(c_t^2) + \chi_t u\left(\frac{\rho n_t \bar{y}_{t+1}}{(1 - q_{t+1}^3)(1 - q_t^2)}\right) + \nu_t v(z_t h(e_t)\lambda_t + 1), \tag{3.9}$$

[4]Since $\lambda_{t+1} = z_t h(e_t)\lambda_t + 1$, the latter property is implied by the former.

where

$$\chi_t \equiv \delta(1 - q^3_{t+1}) \text{ and } \nu_t \equiv \frac{b(1 - q^2_{t+1})n_t}{(1 - q^2_t)}.$$

The budget constraint (3.8) can be expressed as $y_t = [1 - q^2_t + \beta n_t]c^2_t + s_t + \rho \bar{y}_t$. Hence, the associated Lagrangian is

$$\Phi_t = V_t + \mu_t[y_t - [1 - q^2_t + \beta n_t]c^2_t - s_t - \rho \bar{y}_t], \tag{3.10}$$

whose multiplier is μ_t. Note that y_t depends on the amount of child labor $1 - e_t$. The assumptions on u (the Inada condition at $c^2_t = 0$) ensure that, at the optimum, $c^2_t > 0$. By assumption, physical capital is necessary in production. Hence, if some young adults are forecast to survive into full old age ($q^3_{t+1} < 1$), so that $\chi_t > 0$, then $s^0_t > 0$.

The associated f.o.c. are set out in the Appendix for Chapter 3. Those w.r.t. c^2_t and s_t yield

$$\frac{u'(c^2_t)}{u'(c^3_{t+1})} = \frac{\sigma_{t+1}\delta\rho[(1 - q^2_t) + \beta n_t]}{(1 - q^2_t)} \cdot F_2\left[\bar{l}_{t+1}, \frac{\sigma_{t+1}s_t}{n_t}\right], \tag{3.11}$$

which holds for all $e_t \in [0, 1]$. Those with respect to c^2_t and e_t yield:

$$\begin{gathered}\delta\rho u'(c^3_{t+1}) \cdot F_1\left[\bar{l}_{t+1}, \frac{\sigma_{t+1}s_t}{n_t}\right] zh'(e_t) + bv'(\lambda_{t+1})zh'(e_t) \geq \\ \frac{(1 - q^2_t)(w\lambda_t + \gamma)u'(c^2_t)}{(1 - q^2_{t+1})(1 - q^2_t + \beta n_t)\lambda_t} F_1\left[l_t, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right],\ e \leq 1,\end{gathered} \tag{3.12}$$

where the inequality holds in the other direction for $e \geq 0$. Substituting from

(3.11) in (3.12), we obtain, for all interior solutions $e_t \in (0,1)$,

$$\frac{v'(\lambda_{t+1})}{u'(c_t^2)} = \frac{\left(w + \frac{\gamma}{\lambda_t}\right) \frac{1-q_t^2}{1-q_{t+1}^2} F_1\left[l_t, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right] - \left((1-q_t^2) z h'(e_t)/\sigma_{t+1}\right) F_1\left[\bar{l}_{t+1}, \frac{\sigma_{t+1} s_t}{n_t}\right] / F_2\left[\bar{l}_{t+1}, \frac{\sigma_{t+1} s_t}{n_t}\right]}{b(1 - q_t^2 + \beta n_t) z h'(e_t)}. \tag{3.13}$$

3.3.1 Conditions for backwardness

Given stationary demographic conditions, output per head can increase only if there is some form of technical progress. If time t does not appear as an explicit argument of F, the only possible form of technical progress in the present framework is the labor-augmenting kind, which is expressed by an increase in the average level of human capital possessed by those supplying labor to production. The first question to be answered, therefore, is whether allocations in which no generation receives any schooling can be equilibria, with the result that $\lambda_t = 1 \; \forall t$. The second, related question is whether such a state is locally stable. If it is, then backwardness – should it once occur – will persist: there will be a poverty trap. The third question, which is of central importance, is whether, in a given stationary setting, both backwardness and progress can be equilibria.

We examine young adults' choice of e_t when they expect the next generation to choose $e_{t+1} = 0$. Given this expectation, $\lambda_t = 1 \; \forall t$ will be a steady state of the economy if, and only if, each and every generation's optimal choice is $e_t = 0$. We therefore seek to establish conditions that yield a steady-state path $e_t^0 = 0 \; \forall t$. Along such a path,

$$\bar{y}_t = y_t(e_t^0 = 0) = F\left[(1 - q_t^2) + n_t \gamma, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right] \forall t,$$

since $\lambda_t = z_t h(0)\lambda_{t-1} + 1 = 1 \; \forall t$.

Dropping the index t in Equation (3.11), we have

$$\frac{\delta\rho\sigma[1-q^2+\beta n]}{(1-q^2)} F_2\left[(1-q^2)+n\gamma, \frac{\sigma s}{n}\right] u'(c^3) - u'(c^2) = 0.$$

The budget constraint (3.8) specializes to

$$[1-q^2+\beta n]c^2 + s = (1-\rho)F[(1-q^2)+\gamma n, \sigma s/n],$$

and (3.7) to

$$c^3 = \frac{n\rho}{(1-q^2)(1-q^3)} \cdot F[(1-q^2)+\gamma n, \sigma s/n].$$

Remark: $F\left[(1-q^2)+\gamma n, \sigma s/n\right]$ is the output per young adult at the *start* of each period. Each of them has n children, but only the fraction $(1-q^2)$ of these adults survive early adulthood. The deceased make no claims on full income in the following period.

Substituting for c^2 and c^3 in (3.11), we obtain an equation in s, given the constellation (n, q^2, q^3, σ) and the parameters $(\rho, \beta, \gamma, \delta)$. Denote the smallest positive value of s that satisfies this equation by $s^b = s^b(n, q^2, q^3, \sigma)$.

The final step is to examine the counterpart of (3.13) when $e_t = 0 \; \forall t$. Rearranging terms, we obtain

$$\left((\gamma+w) - \frac{1-q^2}{\sigma} \cdot \frac{zh'(0)}{F_2\left[\bar{l}, \frac{\sigma s}{n}\right]}\right) u'(c^2) F_1\left[\bar{l}, \frac{\sigma s}{n}\right] \geq \left[(1-q^2)+\beta n\right] bv'(1)zh'(0), \tag{3.14}$$

where the derivatives are evaluated at the arguments $((1-q^2)+\gamma n, \sigma s^b/n)$.

If parents are at all altruistic, the r.h.s. of (3.14) will be positive, so the said condition can hold as a strict inequality at the hypothesized $e_t^0 = 0$ only if

$$(1 - q^2)zh'(0) < \sigma(\gamma + w) \cdot F_2\big[(1 - q^2) + \gamma n, \sigma s^b/n\big]. \tag{3.15}$$

A small investment in a child's education will yield $zh'(0)$ units of human capital, over and above the basic endowment of unity, in the next period, with the fraction $1 - q^2$ of all children surviving early adulthood, and so contributing to output.

The cost of this investment involves the sum of the opportunity and direct costs of education at the margin, measured in units of human capital. When $\lambda_t = 1$, this combined direct cost is $(\gamma + w)$ for each child, which is surely less than unity. For a child is much less productive than an uneducated adult and w is the teacher-pupil ratio, with some allowance for an administrative overhead. The alternative is to invest in physical capital. The marginal product thereof, F_2, is a pure number, since capital is made of the same stuff as output. When adjusted by the survival rate σ, it measures the yield of investing a little more in physical capital, the proportional claim on future full income being ρ for both forms of investment. Hence, σF_2 is the opportunity cost of investing a little in education, only considering making provision for one's old age.

We make the following assumption, which will be relaxed in Section 3.4:

Assumption 1

$u(c_t) = \ln c_t$ *and* $v(\lambda_{t+1}) = \ln \lambda_{t+1}$.

Under Assumption 1, a sufficient condition for (3.15) to hold is derived as follows.

$$\frac{u'(c_t^2)}{u'(c_{t+1}^3)} = \frac{c_{t+1}^3}{c_t^2} = \frac{[(1 - q^2) + \beta n]n\rho}{(1 - q^2)(1 - q^3)[(1 - \rho) - s^b/F]},$$

where F is evaluated at the arguments $((1-q^2)+\gamma n, \sigma s^b/n)$. Recalling (3.11) and noting that $s^b > 0$, it is seen that (3.15) will hold if

$$\frac{n}{\delta(1-q^2)(1-q^3)(1-\rho)} > \frac{zh'(0)}{\gamma+w}. \tag{3.16}$$

The l.h.s. only depends on fertility and mortality rates, and the social norm and preference parameters ρ and δ; the r.h.s. only on those representing the costs of education and the associated marginal yield of human capital at $e_t = 0$. This separation establishes the existence of a measurable subset of all these parameters such that (3.15) will indeed hold. Since $n \geq 1$, $\delta < 1$ and both mortality rates and ρ are positive, this is not a very exacting condition, even though $\gamma + w < 1$. In particular, it creates some scope for z to exceed 1, and hence of fulfilling the growth requirement $zh(1) > 1$.

In the absence of altruism ($b = 0$), condition (3.15) is also sufficient to ensure the existence of a locally stable, steady-state equilibrium in which there is no investment in human capital, children work full time, and output per head is stationary. It does not, however, rule out $zh(1) > 1$, and hence the possible existence of a steady-state path along which output per head grows without limit. If condition (3.15) holds strongly, then by continuity, the same conclusions will also hold if the altruism motive is sufficiently weak, since the latter implies that the r.h.s of (3.14) will be small and hence that (3.14) will hold as a strict inequality. If, however, altruism is strong, such a low-level equilibrium may well not exist. We summarize our findings in Proposition 9.

Proposition 9

Under Assumption 1, conditions (3.14) and $zh(1) > 1$ are compatible, especially if altruism is not too strong and the survival rates for investments in both forms *of capital are similar. If (3.14) holds as a strict inequality, backwardness will be a locally stable state. If both conditions hold, an escape can be followed by an asymptotic approach to a steady-state growth path along which output per head increases without bound.*

3.3.2 Conditions for both a poverty trap and progress

On any steady-state growth path, λ_t and s_{t-1} will increase without bound, and when they are sufficiently large, the contribution of γ in the relevant terms can be neglected. The (asymptotic) rate of growth of λ_t and s_t at any fixed e, denoted by $g(e)$, is given by (3.4): $1 + g(e) = zh(e)$. A growth path with $e_t = e$ is feasible, therefore, only if the education technology and intergenerational transmission of human capital satisfy the condition $zh(e) > 1$. Each path is effectively defined by the value of e. The state of progress is a steady state with $e = 1 \forall t$.

The f.o.c for positive investment in education may be written as

$$\begin{aligned}\delta\rho u'(c^3_{t+1})F_1\left[(1-q^2)\zeta, \frac{\sigma}{n}\right] zh'(e) + bv'(\lambda_{t+1})zh'(e) \geq \\ \frac{wu'(c^2_t)}{1-q^2+\beta n}F_1\left[(1-q^2-wne)\zeta, \frac{\sigma}{n}\right], \ e \leq 1,\end{aligned} \tag{3.17}$$

so that (3.13) becomes

$$\frac{v'(\lambda_{t+1})}{u'(c^2_t)} \geq \frac{wF_1\left[l_t, \frac{\sigma s_{t-1}}{n}\right] - ((1-q^2)zh'(e)/\sigma)\, F_1\left[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\right] / F_2\left[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\right]}{b[(1-q^2)+\beta n]zh'(e)}, \ e \leq 1, \tag{3.18}$$

The following conditions must be satisfied if both states are to be equilibria.

(i) Condition (3.15), which must hold for backwardness ($e^0_t = 0 \ \forall t$) to be a locally stable equilibrium.

(ii) $zh(1) > 1$, so that unbounded growth results when $e_t = 1\,\forall t$.

(iii) $e_t^0 = 1$ along the steady-state path $e = 1$.

In order to ensure that V_t is concave over the feasible set, we also impose

(iv) $Z(e_t) \equiv v(zh(e_t)\lambda_t + 1)$ is concave $\forall e_t \in [0, 1]$.

This technical requirement is satisfied if $h(e)$ is concave.[5] We now examine whether these conditions can be met simultaneously.

The social norms, as represented by the values of the parameters β and ρ, play an important role. In the state of backwardness, $\lambda_t = 1$, and although a child's endowment of human capital, γ, is smaller, his or her potential contribution to output will be relatively important. If $\beta < \gamma$, the (relative) claim on the common pot is, in a sense, less than the child's potential contribution, thus favoring child labor over education. This consideration argues for keeping β fairly close to γ, n being exogenous.

The old-age generation's claim to the fraction ρ of current full income can be regarded as stemming from its investments in the previous period. Under pure individualism, with no family considerations other than pooling for insurance purposes, this claim comprises the imputed share of physical capital in current output and the return to investments in educating their children. In the state of backwardness, there are no investments in education, so that ρ would then be the said imputed share. In the state of progress, the direct cost of educating each child is $w\lambda_t$ and full income is larger than output, the actual input of human capital being $(1-q^2-wn)\lambda_t$. Thus, ρ is a weighted average of physical capital's imputed share in current output and the combined imputed share of physical capital and, neglecting the opportunity cost of the children's endowment, $wn\lambda_t$ units of human capital.

[5]This analytically convenient restriction on h is not easy to square with the fact that there is the need to lay secure foundations early on in schooling in order to enable the rapid development of wider abilities later on.

Since wn is unlikely to be much greater than 0.1 and altruism enters through v, this way of regarding the norm expressed by ρ argues for keeping its value fairly close to physical capital's imputed share of output in the state of progress. With this preliminary settled, we turn to conditions (i)-(iii).

Condition (i). A sufficient condition for (3.15) to hold is (3.16), which is independent of η, b, σ and F and only imposes a mild restriction on $h(e)$, thereby leaving considerable scope to satisfy conditions (ii) and (iii).

Condition (ii). To illustrate, let $h = e$, so that $h'(e) = h(1) = 1$ and $z > 1$ yields $g(1) > 0$.

Condition (iii). Let us define $\zeta(1)$ as the constant ratio of λ_t and s_{t-1} along the state of progress with $e = 1$. Then in this state, $F[(1 - q^2 - wn)\zeta(1), \sigma/n]$ is obtained using (6.16), which is derived in the proof of Lemma 1 below (see the Appendix for Chapter 3). When F is Cobb-Douglas, $y_t = A \cdot l_t^{1-\alpha} k_t^{\alpha}$, and $u = \ln c_t$, this condition specializes to

$$A\zeta(1)^{1-\alpha}(\sigma/n)^{\alpha}[(1 - q^2 - wn)^{1-\alpha} - \rho(1 - q^2)^{1-\alpha}] = \left(1 + \frac{1}{(1 - q^3)\alpha\delta}\right) zh(1),$$

that is,

$$F[(1 - q^2 - wn)\zeta(1), \sigma/n] = \frac{(1 - q^2 - wn)^{1-\alpha}}{[(1 - q^2 - wn)^{1-\alpha} - \rho(1 - q^2)^{1-\alpha}]} \cdot \left(1 + \frac{1}{(1 - q^3)\alpha\delta}\right) zh(1) \equiv F[\cdot^p], \tag{3.19}$$

where it should be noted that the right-hand side is independent of the TFP-parameter A.

It is proved in the Appendix for Chapter 3, that the condition for progress to be an equilibrium is

$$zh'(1) \geq \frac{w(1-\alpha)}{(1-q^2-wn)} \cdot F[.^p] \cdot \left(\frac{b}{(1-q^3)\alpha\delta} + \frac{1-\alpha}{n\alpha} \right)^{-1}. \tag{3.20}$$

In the absence of altruism ($b = 0$), (3.15) is both necessary and sufficient to ensure the existence of backwardness as a stable equilibrium. Given $u = \ln c_t$ and $h(e) = e$, conditions (i), (ii) and (iii) will be satisfied if there exists a $z > 1$ such that

$$\frac{n(\gamma + w)}{\delta(1-q^2)(1-q^3)(1-\rho)} > z \geq \frac{\alpha n w}{(1-q^2-wn)} \cdot F[(1-q^2-wn)\zeta(1), \sigma/n], \tag{3.21}$$

whereby the weak inequality is also a necessary condition. The left-hand inequality is readily satisfied by very large ranges of plausible parameter values, with $z > 1$. The same holds for that on the right. For it is seen that although $F[(1-q^2-wn)\zeta(1), \sigma/n] > zh(1) > z$, the term αnw takes values quite close to zero, plausibly in the range $[0.02, 0.07]$. We summarize our findings in proposition 10.

Proposition 10

If $u = \ln c_t$ and F is Cobb-Douglas, both backwardness and progress will be equilibria if there exists an h and a z such that conditions (3.16) and (3.20) are satisfied, whereby altruism is sufficiently weak. In the absence of altruism ($b = 0$), condition (3.21) is both necessary and sufficient when $h(e) = e$.

The robust numerical examples that follow in Section 3.3.3 confirm that a poverty trap, coupled with steady-state growth as an alternative equilibrium, will exist for a wide range of functional forms and plausible parameter values. The function $h(e)$, for example, may be sufficiently weakly convex. If it is strictly convex for all e close to zero, but weakly concave thereafter, it will restrict $h'(0)$ without necessarily making $h'(1)$ too small. Technologies close to Cobb-Douglas will also serve, as will sub-utility functions close to $u = \ln c_t$.

3.3.3 Numerical examples

Let $h(e) = d_1 \cdot e - d_2 \cdot e^{d_3}$. Table 3.1 sets out the whole constellation of parameter values: $h(e)$ is fairly weakly concave ($d_3 = 1.5$), with $h(1) = 0.8, h'(0) = 1$ and $h'(1) = 0.7$. Long-run growth at a steady rate is feasible: $zh(1) = 1 + g(1) = 1.2$. In the first variant, there is no altruism ($b = 0$). Backwardness is an equilibrium; for

$$zh'(0) = 1.5 < \frac{n(\gamma + w)}{\delta(1 - q^2)(1 - q^3)(1 - \rho)} = \frac{1.2(0.6 + 0.075)}{0.85(1 - 0.1)(1 - 0.3)(1 - 0.35)} = 2.327,$$

this inequality being itself a sufficient condition for (3.15) to hold. The progressive state will also be an equilibrium if, and only if,

$$zh'(1) \geq \frac{\alpha n w}{(1 - q^2 - wn)} \cdot F[(1 - q^2 - wn)\zeta(1), \sigma/n].$$

Using the chosen parameter values, we have $zh'(1) = 1.05$, which substantially exceeds

$$\begin{aligned} &\frac{\alpha n w}{(1 - q^2 - wn)} \cdot F[(1 - q^2 - wn)\zeta(1), \sigma/n] \\ =&\frac{(1.2 \cdot 0.075/3)}{(1 - 0.1 - 0.09)} \frac{(1 - 0.1 - 0.09)^{2/3}}{(1 - 0.1 - 0.09)^{2/3} - 0.35(1 - 0.1)^{2/3}} \\ &\cdot \left(1 + \frac{3}{(1 - 0.3)0.85}\right) 1.2 = 0.4300. \end{aligned}$$

Table 3.1: Poverty traps and progress: A constellation of parameter values.

Parameter	Value	Variable
n	1.2	net reproduction rate
q^2	0.1	mortality rate at the start of young adulthood
q^3	0.3	mortality rate at the close of young adulthood
σ	0.75	survival rate of physical capital
γ	0.6	a child's endowment of human capital
d_1	1	a parameter of $h(e)$
d_2	0.2	a parameter of $h(e)$
d_3	1.5	a parameter of $h(e)$
z	1.5	transmission factor for human capital formation
w	0.075	teacher-pupil ratio
A	1	TFP parameter
α	1/3	elasticity of output w.r.t. physical capital
δ	0.85	pure impatience factor
b	$(0, 0.1)$	taste parameter for altruism
ρ	0.35	share of current full income accruing to the old
β	0.325	share parameter for a child's consumption

The scope for substantial changes to this constellation of values, while satisfying the conditions in question, is evidently large. Of particular interest is its robustness to altruism. When $b = 0.1$, the said condition is easily satisfied: $zh'(1) = 1.05 \geq 0.3301$, whereby 0.3301 is smaller than the value 0.4300 when $b = 0$. We conclude that even quite strong altruism is compatible with a poverty trap.

3.3.4 The choice of schooling

To establish whether there are also other, 'intermediate', equilibria, we now analyze how the choice of the level of schooling depends on the stocks of human and physical capital. The optimal level, e_t^0, depends on the state of the world today, expectations about the state that will rule in the next period, and various state variables, including λ_t and ζ_t.

Let F be Cobb-Douglas. Substituting (3.7) and (3.8) into (3.11) and recalling

Assumption 1, we obtain

$$\frac{\zeta_{t+1}}{\zeta_t^{\alpha}} = \left(A \left(1 - q^2 - wne_t + \frac{n\gamma(1-e_t)}{\lambda_t} \right)^{1-\alpha} - \rho A \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha} \right)^{-1} \cdot \\ \left(zh(e_t) + \frac{1}{\lambda_t} \right) \left(1 + \frac{1}{\alpha\delta(1-q^3)} \right) \left(\frac{n}{\sigma} \right)^{\alpha} \\ \equiv \psi(e_t; \lambda_t; \cdot). \tag{3.22}$$

Likewise, we can use $u(c_t^2)$ from (3.11) and substitute it into (3.12). Using the definition of c_{t+1}^3 then yields

$$\frac{\zeta_{t+1}}{\zeta_t^{\alpha}} = \left(1 - q^2 - wne_t + \frac{n\gamma(1-e_t)}{\lambda_t} \right)^{\alpha} \left(\frac{zh'(e_t)}{\alpha A \delta n \left(w + \frac{\gamma}{\lambda_t} \right) (1-q^3)} \right) \cdot \\ \left(\frac{\delta(1-q^3)(1-q^2)}{1 - q^2 + \frac{n\gamma}{\lambda_{t+1}(e_t, \lambda_t)}} + \frac{bn}{1-\alpha} \right) \left(\frac{n}{\sigma} \right)^{\alpha} \\ \equiv \phi(e_t; \lambda_t; \cdot). \tag{3.23}$$

It is seen that this particular iso-elastic combination of preferences and technology yields an optimal choice that only depends on the state variable λ_t and the various parameters; for the optimum satisfies $\psi(e_t^0; \lambda_t; \cdot) = \psi(e_t^0; \lambda_t; \cdot)$, which is independent of the physical capital stock inherited from the decision $s_{t-1} (> 0)$. The extreme values of e_t are covered by noting that if $\psi(e_t; \cdot) = \phi(e_t; \cdot)$ is satisfied by $e_t' \leq 0$ or $e_t' \geq 1$, then $e_t^0 = 0$ or $e_t^0 = 1$, respectively. With e_t^0 thus determined, $\psi(e_t^0; \lambda_t; \cdot) = \zeta_{t+1}/\zeta_t^{\alpha}$ yields s_t^0.

To summarize these results precisely:

Proposition 11

If $u = \ln(c_t)$ and F is Cobb-Douglas, the functions ψ and ϕ yield the optimum values of e_t and ζ_{t+1} as follows:

(i) If $e_t < 0$ satisfies $\psi(e_t;\cdot) = \phi(e_t;\cdot)$, then $e_t^0 = 0$ and $\psi(0;\cdot) = \zeta_{t+1}^1/\zeta_t^\alpha$ yields the value of ζ_{t+1} $(= \lambda_{t+1}(e_t^0 = 0)/s_t^0 = 1/s_t^0)$.

(ii) If $e_t \in (0,1)$ satisfies $\psi(e_t;\cdot) = \phi(e_t;\cdot)$, then both (3.36) and (3.37) yield the value of ζ_{t+1}.

(iii) If $e_t > 1$ satisfies $\psi(e_t;\cdot) = \phi(e_t;\cdot)$, then $e_t^0 = 1$ and $\psi(1;\cdot) = \zeta_{t+1}^1/\zeta_t^\alpha$ yields the value of ζ_{t+1}.

In Figure 3.2, we illustrate the determination of e_t^0 with the parameter values given above. We plot the difference between $\psi(e_t;\cdot)$ and $\phi(e_t;\cdot)$, which we call $G(e_t, \lambda_t)$, over the values of $e_t \in [0,1]$ for the values of $\lambda_t \in \{1, 2, 3, ..., 20\}$. An increase in λ_t shifts $G(e_t, \lambda_t)$ upwards, and so increases e_t^0, the value of e_t satisfying $G = 0$. This explains how a poverty trap and a progressive state can coexist in our model. A state with a low level of human capital implies a low level of education and the opposite holds for a state with high values for λ_t. This means that once one generation is schooled sufficiently for human capital to increase, the next generation will receive even more schooling and the economy will converge to the progressive state.

3.3.5 Stationary paths with incomplete schooling

The possibility of paths with stationary levels of s_t and λ_t is of particular interest. If λ_t is stationary, $(1 - zh(e_t))\lambda_t = 1 \; \forall t$, where $zh(e_t) < 1$. We seek an $e^* \in (0,1)$ such that $e_t = e^*, \lambda^* = 1/(1 - zh(e^*))$ and $s^* = s_t^0(\lambda^*)$. Substituting for λ^* in

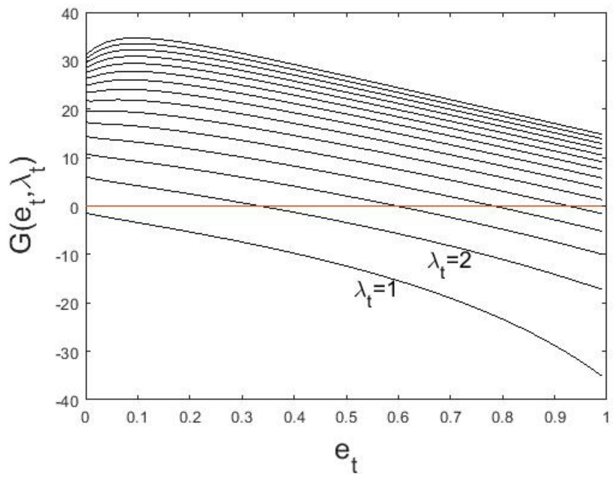

Figure 3.2: $G(e_t, \lambda_t)$ and the optimal values of $e_t(\lambda_t)$.

(3.22) and rearranging, we obtain

$$\left(\frac{1}{s^*}\right)^{1-\alpha} = \left(A\left(\frac{1-q^2-wne^*}{1-zh(e^*)} + n\gamma(1-e^*)\right)^{1-\alpha} - \rho A\left(\frac{1-q^2}{1-zh(e^*)} + n\gamma\right)^{1-\alpha}\right)^{-1}$$
$$\left(1+\frac{1}{\alpha\delta(1-q^3)}\right)\left(\frac{n}{\sigma}\right)^{\alpha}.$$

All pairs $(e^* \in (0, h^{-1}(1/z)), s^*)$ satisfying this equation are stationary configurations of the system. For any such pair to be an equilibrium, however, it must also satisfy (3.13). In the absence of altruism, the latter becomes

$$(1-q^2)zh'(e_t) = \left(w + \frac{\gamma}{\lambda_t}\right)\frac{\sigma F_2\left[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\right]}{F_1\left[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\right]} F_1\left[l_t, \sigma s_{t-1}/n\right],$$

where $l_t = [(1-q^2-wne^*)/(1-zh(e^*))] + n\gamma(1-e^*)$, $\bar{l}_t$ and $s_t = s^*$ are stationary. It is seen that if the derivative $h'(e^*)$ can be chosen independently of the level $h(e^*)$, there is considerable scope to satisfy both conditions. If, in contrast, h is linear or nearly so, then it is far from clear that there exists an equilibrium path with $e_t = e^* \in (0,1)$. It appears, therefore, that some fairly strong restrictions must be imposed on h to ensure the existence of such stationary paths, with or without altruism.

3.4 A generalization: Isoelastic functions

We now relax the assumptions on u and F regarding the existence of steady state paths with positive education. The state called progress is of central importance, so the social norm represented by ρ should not make it infeasible. A (weak) necessary condition for them to be compatible is $y_t(e=1) > \rho\bar{y}_t$, or $F\left[(1-q^2-wn)\zeta, \frac{\sigma}{n}\right] > \rho F\left[(1-q^2)\zeta, \frac{\sigma}{n}\right]$.

We now turn to (3.18). The numerator on the r.h.s. of the left weak inequality may be written as a function of e and ζ. Where the latter is also constant along any such path:

$$D(e,\zeta) \equiv wF_1\left[(1-q^2-wne)\zeta, \frac{\sigma}{n}\right] - \frac{(1-q^2)zh'(e)}{\sigma} \cdot \frac{F_1\left[(1-q^2)\zeta, \frac{\sigma}{n}\right]}{F_2\left[(1-q^2)\zeta, \frac{\sigma}{n}\right]}, \quad (3.24)$$

and hence the ratio $v'(\lambda_{t+1})/u'(c_t^2)$ must be likewise. Condition (3.11) specializes to

$$\frac{u'(c_t^2)}{u'(c_{t+1}^3)} = \frac{\sigma\delta\rho[1-q^2+\beta n]}{(1-q^2)} \cdot F_2\left[(1-q^2)\zeta, \frac{\sigma}{n}\right], \quad (3.25)$$

so that $u'(c_{t+1}^3)/u'(c_t^2)$ must also be constant, a requirement that motivates the following restriction on preferences:

Assumption 2

$u(c_t) = c_t^{1-\xi}/(1-\xi)$.

Hence, (3.25) may be written

$$F_2\left[(1-q^2)\zeta, \frac{\sigma}{n}\right] = \frac{(1-q^2)[(1+g(e))(c_t^3/c_t^2)]^\xi}{\delta\rho\sigma[(1-q^2)+\beta n)]}. \tag{3.26}$$

The following lemma enables comparisons to be made across steady-state growth paths.

Lemma 1

Let e vary parametrically to yield steady-state growth paths. Then ζ is increasing in e for all F that are:

(i) Sufficiently close to Cobb-Douglas in form, provided $\xi \leq 1$; or

(ii) Members of the CES family the absolute value of whose elasticity of substitution, $|(\epsilon-1)^{-1}|$, is at most 1, provided $\xi + \epsilon \leq 1$.

Proof. See the Appendix for Chapter 3 .

Remark: The condition $\xi \leq 1$ in part (i) can be weakened to include values exceeding, but sufficiently close to, 1. Regarding part (ii), if, for example, $\epsilon = -1$, the elasticity of substitution is -0.5, and the result holds for all $\xi \leq 2$.

Corollary 1

If $h(e)$ is concave or sufficiently weakly convex, $D(e, \zeta(e))$ is increasing in e across paths.

Proof. See the Appendix for Chapter 3.

The whole yield of human capital in the next period, $zh(e)$, as well as the marginal yield $zh'(e)$ in optimization, plays a central role. Let e^p denote the smallest value of e satisfying $zh(e) = 1$, where $e^p > 0$ in virtue of $h(0) = 0$. If $e^p \geq 1$, there will exist no steady-state growth path. Corollary 1 yields:

Corollary 2

If h is strictly concave for all $e \in [e^p, 1]$ and $D(e^p, \zeta(e^p)) > 0$, the expression $D(e, \zeta(e))/[b[(1-q^2)+\beta n]zh'(e)]$ on the r.h.s. of the left weak inequality in (3.18) will be continuous, positive and increasing in e for all $e \in (e^p, 1]$.

Under the hypothesis that growth is occurring at a steady rate, the said expression in (3.18) is a constant. If $e < 1$, the l.h.s. of the left weak inequality must be likewise. If, however, (3.18) holds as a strict inequality at $e = 1$, the behavior of $v'(\lambda_{t+1})/u'(c_t^2)$ is not so restricted. That is to say, the requirement that c_t^2 and λ_t grow at the same rate imposes certain restrictions on both v and u.

Assumption 3

The sub-utility function v is iso-elastic: $v(\lambda) = \lambda^{1-\eta}/(1-\eta)$.

3.4.1 No altruism

A special case of particular interest is the absence of altruism ($b = 0$), wherein v plays no role. Condition (3.17) then specializes to $0 \geq D(e),\ e \leq 1$. Note that this condition is completely independent of t so that once a generation has chosen the level of e that fulfills this condition all future generations will chose the same value. Hence, the choice of families will be sustained over time. In virtue of Corollary In virtue of Corollary 1, this yields:

Proposition 12

If F satisfies the conditions in Lemma 1 and $e^p < 1$, then in the absence of altruism, there are just three possibilities when e is parametric:

(i) If $D(e^p) > 0$, there exists no steady-state growth path.

(ii) If $D(e^p) \leq 0 < D(1)$, there exists a unique, steady-state growth path such

that $e \in (e^p, 1)$.

(iii) If $D(1) \leq 0$, *the only such path is the progressive state.*

(iv) The parametric growth paths defined by parts (ii) and (iii) will be sustained by families' optimal choices.

The direct costs of education, as represented by the parameter w, exert a strong influence on which of these holds. If w is sufficiently close to zero, it follows from (3.24) that $D < 0$, so that progress is the only possible outcome, a result which accords with intuition. In fact, the educational system is a fairly heavy user of its own output, so that the other outcomes are then distinctly possible.

Another way to show that families will choose to maintain the value of e everywhere along the path in question is the following: Suppose the economy is on such a path. The pairwise marginal rates of transformation among c_t^2, e_t and s_t are obtained from the budget constraint (3.8). For any value of $e_t \in [0, 1]$,

$$MRT_{ce} = -\frac{1 - q^2 + \beta n}{n(w\lambda_t + \gamma)F_1\big[l_t, \frac{\sigma s_{t-1}}{n}\big]}, \tag{3.27}$$

where $F_1\big[l_t, \frac{\sigma s_{t-1}}{n}\big]$ is constant and terms involving γ can be neglected along the hypothesized path.

Total differentiation of (3.9) yields the corresponding marginal rate of substitution:

$$\begin{aligned} MRS_{ce} &= -\frac{u'(c_t^2)}{n\big[\delta\rho F_1\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3) + bv'(\lambda_{t+1})\big]zh'(e_t)\lambda_t} \\ &\equiv -\frac{u'(c_t^2)}{Q_t zh'(e_t)\lambda_t} \equiv -R_t\,. \end{aligned} \tag{3.28}$$

We now compare the levels of MRT_{ce} and MRS_{ce} along the path in question, noting that e_t and $F_1\left[\bar{l}_t, \frac{\sigma s_{t-1}}{n}\right]$ are constant. A continuous approximation yields:

$$ds_t/s_t = dk_t/k_t = d\lambda_t/\lambda_t = dc_t^2/c_t^2 = dc_t^3/c_t^3 = zh(e_t) - 1 = g(e_t). \tag{3.29}$$

In the absence of altruism ($b = 0$), it is seen from Assumption 2 that $\lambda_t R_t$ is constant, so that R_t is falling at the rate $g(e_t)$. From (3.27), the same holds for $|MRT_{ce}|$. Hence, if the optimality condition $|MRT_{ce}| \geq |MRS_{ce}|,\ e_t \leq 1$ is once established, it will hold thereafter.

3.4.2 Altruism

Altruism introduces the additional term $bv'(\lambda_{t+1})$ into Q_t.

Differentiating (3.28) totally and recalling (3.29) and Assumption 2, we obtain

$$\frac{dR_t}{R_t} = -\left(\xi + \frac{dQ_t}{d\lambda_t}\frac{\lambda_t}{Q_t} + 1\right)\frac{d\lambda_t}{\lambda_t}. \tag{3.30}$$

It is shown in the Appendix for Chapter 3 that the elasticity of Q_t w.r.t. λ_t can be expressed in the form

$$\frac{dQ_t}{d\lambda_t} \cdot \frac{\lambda_t}{Q_t} = -\frac{\xi A + \eta b \cdot B(1+g)^{-(\eta-\xi)t}}{A + b \cdot B(1+g)^{-(\eta-\xi)t}}, \tag{3.31}$$

where $A = \delta\rho F_1\left(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\right)$ is constant along such a path and B is also a positive constant.

The final step is to establish conditions under which the choice $e_t = e$, once attained, remains optimal as λ_t grows without bound at the rate $g(e)$. As noted above from (3.27), $|MRT_{ce}|$ falls at the rate $g(e)$.

To maintain the optimality of $e_t = e$ when $e < 1$, however, the $|MRS_{ce}|\,(= R_t)$

must fall at exactly the same rate. If, in any period t, $|MRS_{ce}(e = 1)| < |MRT_{ce}(e = 1)|$, then $e_t^0 = 1$, which will remain optimal thereafter if $|MRS_{ce}(e = 1)|$ falls at least as fast as $|MRT_{ce}(e = 1)|$.

Rewriting (3.30) as

$$\frac{dR_t}{d\lambda_t}\frac{\lambda_t}{R_t} = -\xi + \frac{\xi A + \eta b \cdot B(1+g(e))^{-(\eta-\xi)t}}{A + b \cdot B(1+g(e))^{-(\eta-\xi)t}} - 1,$$

we obtain:

Lemma 2

With altruism ($b > 0$), there exists a steady-state growth path with $e_t^0 = e$ if

$$1 \geq \frac{A + (\eta/\xi) b \cdot B(1+g(e))^{-(\eta-\xi)t}}{A + b \cdot B(1+g(e))^{-(\eta-\xi)t}}, \tag{3.32}$$

which must hold as an equality if $e < 1$.

If $e < 1$, (3.32) will hold iff $\xi = \eta$, whereas if $e = 1$, then $\eta \leq \xi$. The reason is that, when $\eta \leq \xi$ and t is large, the condition approaches $1 \geq \frac{\eta}{\xi}$.

This leaves open what paths are possible if $\eta > \xi$. For if $|MRS_{ce}(e = 1)| < |MRT_{ce}(e = 1)|$ the former may still fall more slowly than the latter without necessarily violating the condition itself. Suppose, therefore, that $e_t^0 = 1$, with growth proceeding at the steady rate $g(1) > 0$. For t sufficiently large, the child-labor parameter γ can be neglected, and $|MRS_{ce}(e = 1)| < |MRT_{ce}(e = 1)|$ if, and only if,

$$\frac{1 - q^2 + \beta n}{wF_1\left[l_t, \frac{\sigma s_{t-1}}{n}\right]} > \frac{u'(c_t^2)}{\left[\delta \rho F_1\left(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\right) u'(c_{t+1}^3) + b v'(\lambda_{t+1})\right] z h'(1)}.$$

By hypothesis, c_t^2, c_t^3 and λ_t are all growing at the same steady rate, and the derivatives $F_1(\cdot)$ are constant. Thus the l.h.s. is constant. By Assumptions 2 and

3, the r.h.s. may be written $A_1/[B_1+b\cdot B_2(1+g(1))^{(\xi-\eta)t}]zh'(1)$, where A_1, B_1 and B_2 are positive constants. If $\eta > \xi$, then in the limit as $t \to \infty$, the r.h.s. goes to $A_1/B_1 zh'(1)$: the altruism term $b \cdot B(1+g(1))^{(\xi-\eta)t}$ goes to zero. It then follows from Propositions 12 that the postulated path of progress can hold if, and only if, $D(1) \leq 0$. The same argument holds if the postulated steady-state growth path is such that $e_t < 1$, whereby $D(e^p) \leq 0 < D(1)$ must now hold.

The only remaining possible steady states are stationary ones, wherein $e_t^0 = e_{t-1}^0, \lambda_t = \lambda_{t-1}$ and $s_t^0 = s_{t-1}^0$ for all t. Denoting stationary values by a *, we have $\lambda^* = 1/[1 - zh(e^*)]$. In virtue of the assumption $zh(1) > 1$, there exists an $e^* \in (0,1)$ satisfying the latter condition. If such an equilibrium exists, then $MRS_{ce}(e^*) = MRT_{ce}(e^*)$ and, from (3.13), $D(e^*) > 0$. The altruism term is now operative for all t, and $Q_t = n\big[\delta\rho F_1\big(\bar{l}^*, \frac{\sigma s^*}{n}\big)u'(c^{3*}) + bv'(\lambda^*)\big]$ is constant. There may exist more than one such $e^* \in (0,1)$.

These results are summarized as:

Proposition 13

With iso-elastic preferences, the possible steady-state paths with positive education are as follows.

(i) If u and v differ, with $\eta < \xi$, progress ($e = 1$) is the sole steady-state path that can be supported by families' optimizing decisions.

(ii) If $\eta = \xi$, there may exist steady-state growth paths with a less than fully educated population ($e < 1$); but the state of progress is also possible as a limiting case.

(iii) If $\eta > \xi$ and $D(1) \leq 0$, then progress is the sole steady-state path that can be supported by families' optimizing decisions. If $D(1) > 0$, progress is ruled out. If $D(e^p) \leq 0 < D(1)$, then by part (ii) of Proposition 12, steady-state

growth is possible with $e_t < 1$. If $D(e^p) > 0$, there is no steady-state growth path, but there may exist a stationary state wherein $e > 0$.

What is the intuition for these findings? If R_t is falling at the rate $g(e)$ and $e < 1$, both terms in Q_t must fall at the same rate to preserve $MRS_{ce} = MRT_{ce}$, which imposes $\xi = \eta$ – a very strong restriction on preferences. In relaxing it, consider the path $e = 1$, along which R_t may fall at a rate faster than $g(1)$ without violating the conditions for optimality. Investing in education provides both for old age and the children's well-being in adulthood, as expressed by the two terms comprising Q_t. Since λ_{t+1} and c^3_{t+1} are growing at the rate $g(1)$, it suffices that $v'(\lambda_{t+1})$ fall no faster than $u'(c^3_{t+1})$. That is to say, if v is less strongly concave than u, then steady-state growth will be maintained. If parents are perfectly selfish, this consideration does not arise.

If, however, u is less strongly concave than v ($\eta > \xi$), potential difficulties arise; for along any postulated growth path, $v'(\lambda_{t+1})$ is falling faster than $u'(c^2_t)$ and $u'(c^3_{t+1})$. With iso-elastic preferences, there is a common growth term in the numerator and denominator of MRS_{ce}, but with different exponents. If $\eta > \xi$, then upon division, the altruism term, which only appears in the denominator, goes asymptotically to zero as t becomes arbitrarily large, and the results of Section 3.4.1 apply.

We have already established that altruism can rule out backwardness, which is not surprising. It now emerges that it may also fail to yield steady-state growth when u and v are iso-elastic and v is more concave than u.

For diminishing marginal returns then set in faster where the evaluation of the children's human capital is concerned than that of own consumption, and growth renders altruism effectively inoperative over the long run.

With steady growth ruled out when the direct costs of education are so high that

$D(e^p) > 0$, the only remaining possible steady states are stationary ones.

3.5 War and pestilence as stochastic events

In reality, mortality and destruction rates are, in some degree, stochastic; for the outbreak of war or a severe epidemic are events that cannot be forecast with certainty. This fact rules out steady-state growth, and if there is a poverty trap, such shocks may pitch a growing economy into backwardness. In order to analyze this possibility, a preliminary step is needed, namely, to establish how such events influence consumption and investment. We formulate the shock as the actual outbreak of war, coupled with the (prior) probability of its occurrence. This prior is assumed to be sharp.[6]

Let $I_t \in \{0, 1\}$ denote the states of peace and war, respectively, in period t, and let $\pi_{t+1} = Pr(I_{t+1} = 0)$ denote the probability of peace in period $t+1$. The survival rate of physical capital is $\sigma_t(I_t)$, where $\sigma_t(1) < \sigma_t(0) \leq 1$. Mortality rates $\mathbf{q}_t$ are likewise dependent on I_t. It is almost surely the case that $q_t^a(1) > q_t^a(0)$ $(a = 2, 3)$, and this much will be assumed. By assumption, I_t is known when decisions are taken in period t. Consumption in old age, denoted by $c_{t+1}^3(I_{t+1})$ for those who survive to enjoy it, now depends on the state ruling in period $t+1$. The large extended family cannot provide insurance against this particular risk, which does not exist in a steady state.

The young adults' preferences now not only involve the compound lottery arising

[6] For a vigorous argument that rational actors must have sharp priors, see Elga (2010).

from the future state I_{t+1}, but also the current realization of I_t if this affects q_t^2.

$$V_t(I_t) = u(c_t^2) + \delta\left[\pi_{t+1}(1-q_{t+1}^3(0))u(c_{t+1}^3(0)) + (1-\pi_{t+1})(1-q_{t+1}^3(1))u(c_{t+1}^3(1))\right] + \frac{b\left[\pi_{t+1}(1-q_{t+1}^2(0)) + (1-\pi_{t+1})(1-q_{t+1}^2(1))\right]}{(1-q_t^2(I_t))} \cdot nv(\lambda_{t+1})\,,\ I_t = 0,1. \tag{3.33}$$

Exploiting as before the assumption that F is homogeneous of degree one, we have

$$c_{t+1}^3(I_{t+1}; I_t) = \frac{\rho n \cdot F[(1-q_{t+1}^2(I_{t+1}))\lambda_{t+1}(e_t) + n\gamma, \sigma_{t+1}(I_{t+1})s_t/n]}{(1-q_t^2(I_t))(1-q_{t+1}^3(I_{t+1}))}. \tag{3.34}$$

The budget constraint becomes

$$[(1-q_t^2(I_t)) + \beta n]c_t^2 + s_t + \rho F\left[(1-q_t^2(I_t))\lambda_t + n\gamma, \frac{\sigma_t(I_t)s_{t-1}}{n}\right] \le F\left[(1-q_t^2(I_t) - wne_t)\lambda_t + n\gamma(1-e_t), \frac{\sigma_t(I_t)s_{t-1}}{n}\right],\ I_t = 0,1, \tag{3.35}$$

where the dependence of current decision variables on the current realized state can be (notationally) suppressed without ambiguity.

To analyze the economy's behavior in the face of systemic shocks, we proceed essentially as before, noting that the choices of s_t and e_t determine the productive endowments in the next period and hence ζ_{t+1}. Let F be Cobb-Douglas. We provide the generalized versions of (3.22) and (3.23) in a stochastic setting:

$$\frac{\zeta_{t+1}}{\zeta_t^\alpha} = \left(A\left(1-q_t^2(I_t) - wne_t + \frac{n\gamma(1-e_t)}{\lambda_t}\right)^{1-\alpha} - \rho A\left(1-q_t^2(I_t) + \frac{n\gamma}{\lambda_t}\right)^{1-\alpha}\right)^{-1} \cdot \left(zh(e_t) + \frac{1}{\lambda_t}\right)\left(\frac{1}{\alpha\delta E_t[1-q_{t+1}^3(I_{t+1})]} + 1\right)\left(\frac{\sigma_t(I_t)}{n}\right)^{-\alpha} \equiv \psi(e_t, I_t; \cdot) \tag{3.36}$$

and, if $e_t \in (0,1)$,

$$\frac{\zeta_{t+1}}{\zeta_t^{\alpha}} = \left(1 - q_t^2(I_t) - wne_t + \frac{n\gamma(1-e_t)}{\lambda_t}\right)^{\alpha} \left(\frac{zh'(e_t)}{\alpha A\delta n\left(w + \frac{\gamma}{\lambda_t}\right) E_t[1 - q_{t+1}^3(I_{t+1})]}\right)$$
$$\left(E_t\left[\frac{(1-q_{t+1}^3(I_{t+1}))(1-q_{t+1}^2(I_{t+1}))}{1-q_{t+1}^2(I_{t+1}) + n\gamma/\lambda_{t+1}}\right]\delta + \frac{\widetilde{b}n}{1-\alpha}\right)\left(\frac{n}{\sigma_t(I_t)}\right)^{\alpha} = \phi(e_t, I_t; \cdot), \tag{3.37}$$

where $\widetilde{b} = b \cdot [\pi_{t+1}(1-q_{t+1}^2(0)) + (1-\pi_{t+1})(1-q_{t+1}^2(1))]/(1-q_t^2(I_t)) = b \cdot E_t[1 - q_{t+1}^2]/1 - q_t^2$.

The forms of ψ and ϕ are highly complicated, even under the assumption that F is Cobb-Douglas, so it would be as well to untangle their elements, relying rather on intuition. We therefore discuss how the various factors in play influence the final outcome, but without the said restriction on F.

3.5.1 The occurrence of war

The first step is to examine how I_t and q_t^2 affect the set of current feasible choices, which is independent of π_{t+1}. This set is denoted by

$$S(I_t) = \{c_t^2, e_t, s_t : (3.35), c_t^2 \geq 0, e_t \in [0,1], s_t \geq 0\}.$$

It is seen that if the ratio of survival rates, $\sigma_t(I_t)/(1 - q_t^2(I_t))$, is independent of the current state, then the outer frontier of the feasible set is only affected by the mortality rate $q_t^2(I_t)$, though the latter may certainly depend on the current state. If the said ratio is indeed independent of I_t, an increase in q_t^2, whether associated with war or not, also makes c_t^2 cheaper relative to s_t.

The extreme allocations of $S(I_t)$'s outer frontier, which are depicted as the points

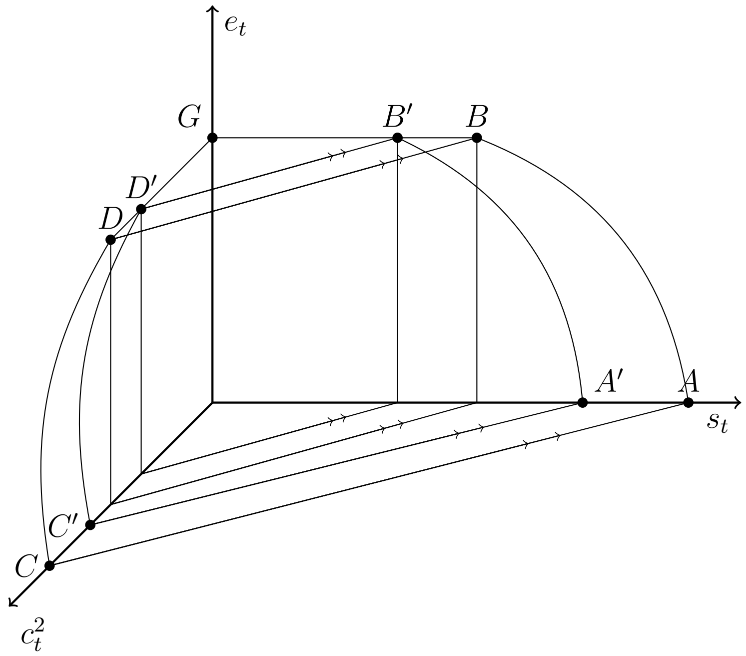

Figure 3.3: Feasible sets of consumption and investment.

A, B, C and D, respectively, in Figure 3.3, are examined in the Appendix for Chapter 3.

To summarize: A sufficient condition for $(dc_t^2/dq_t^2(I_t))_{e_t=s_t=0} < 0\ \forall\lambda_t$ is $\beta > \gamma$, which is not a very strong requirement. If the ratio of survival rates, $\sigma_t(I_t)/(1 - q_t^2(I_t))$, is fixed for each current state I_t and $\beta > \gamma$, the outer frontier of the feasible set $S(I_t)$ will contract inwards everywhere as the mortality rate $q_t^2(I_t)$ rises. If the said ratio is the same for both states, the contraction from $S(0)$ to $S(1)$ represents the effects of an outbreak of war.

To complete the argument, consider the case where $e_t = 1$ is infeasible for sufficiently large values of $q_t^2(I_t)$. Suppose that when $q_t^2(I_t) = 0$, the maximal values of c_t^2 and s_t, respectively, are both positive, as depicted in the figure when ABCD corresponds to a zero level of such premature mortality. As $q_t^2(I_t)$ progressively increases, BD will shift towards G until the allocations B and D coincide at G ($e_t = 1, c_t^2 = s_t = 0$). Further increases in $q_t^2(I_t)$ will reduce the maximal feasible level of e_t below one, with the associated allocation moving downwards along the e_t-axis towards the origin O. Since AC also shifts progressively inwards towards O, the outer frontier of $S(I_t)$ contracts everywhere as $q_t^2(I_t)$ increases.

The contraction of the feasible set established above points to unambiguous income effects, c_t^2, e_t and s_t being all normal goods; but changes in survival rates also imply changes in marginal rates of transformation, which are now examined. As noted above, given the current state I_t, an increase in $q_t^2(I_t)$ makes c_t^2 cheaper relative to s_t, as does an outbreak of war if this event leaves the ratio of survival rates unchanged. Turning to the marginal rate of transformation between s_t and e_t, we have

$$MRT_{se}(I_t) = -\left([n(w\lambda_t+\gamma)]F_1\left[(1-q_t^2(I_t)-wne_t)\lambda_t+n\gamma(1-e_t),\, \sigma_t(I_t)\cdot\frac{s_{t-1}}{n}\right]\right)^{-1}.$$

For any given I_t and e_t, an increase in $q_t^2(I_t)$ will increase F_1 and so reduce $|MRT_{se}(I_t)|$: s_t will be come cheaper relative to e_t, as intuition would suggest.

An outbreak of war, however, has ambiguous effects on $MRT_{se}(I_t)$. Since F_1 is homogeneous of degree zero, $MRT_{se}(I_t)$ can be expressed in the form

$$MRT_{se}(I_t) = -\left([n(w\lambda_t+\gamma)]F_1\left[\lambda_t+\frac{n[\gamma-(w\lambda_t+\gamma)e_t]}{1-q_t^2(I_t)},\, \frac{\sigma_t(I_t)}{1-q_t^2(I_t)}\cdot\frac{s_{t-1}}{n}\right]\right)^{-1}.$$

Suppose, as before, that the ratio of survival rates is independent of I_t, but with $q_t^2(1) > q_t^2(0)$. It is seen that the associated increase in mortality reduces, or increases, the (normalized) input of human capital according as $e_t \gtrless \gamma/(\gamma+w\lambda_t)$. For sufficiently large values of λ_t, the latter ratio will be very small, so that war in the current period will make s_t cheaper relative to e_t for all values of e_t except those very close to zero. The converse holds when λ_t is close to one; for $\gamma/(\gamma+w\lambda_t)$ is then close to one, and the $n/(1-q_t^2(I_t))$ children cared for by each surviving young adult constitute a potentially large pool of labor, relatively speaking. The outbreak of war reduces the opportunity cost of their labor and so makes investment in their education more attractive relative to investment in physical capital. We summarize our findings in Proposition 14:

Proposition 14

The contraction of the feasible set caused by war in the current period reduces both current consumption and investment in both forms of capital. Consumption also becomes cheaper relative to investment in physical capital. Investment in education is likely to suffer especially when λ_t is large, but not when λ_t is small.

It seems rather unlikely that the associated changes in the marginal rates of transformation will offset the reduction in investment in general arising from the adverse income effect.

3.5.2 The probability of war

Intuition suggests that an increase in the prior probability of war in the future will depress investment in the present. It will now be demonstrated that this is indeed so in our framework provided an additional – and plausible – condition holds.

The feasible set in period t, as defined by (3.35), is independent of π_{t+1}, so that changes in the latter will affect decisions only through $V(I_t)$. Inspection of (3.33) reveals that the weight on the altruism term v is increasing in π_{t+1}, since $q^2_{t+1}(1) > q^2_{t+1}(0)$. Where the terms involving old age are concerned, the probability of surviving into full old age is increasing in the probability of peace, π_{t+1}.

This does not, however, settle the matter; for the pay-off received by survivors depends on the number of claimants as well as the size of the common pot. From

(3.34), it is seen that for any given (e_t, s_t, I_t), $c^3_{t+1}(I_t; 0) \gtrless c^3_{t+1}(I_t; 1)$ according as

$$\frac{F\left[(1-q^2_{t+1}(0))\lambda_{t+1}(e_t)+n\gamma, \frac{\sigma_{t+1}(0)s_t}{n}\right]}{F\left[(1-q^2_{t+1}(1))\lambda_{t+1}(e_t)+n\gamma, \frac{\sigma_{t+1}(1)s_t}{n}\right]} \gtrless \frac{1-q^3_{t+1}(0)}{1-q^3_{t+1}(1)}.$$

The numerator on the l.h.s. is the level of full income in period $t+1$ when peace prevails, the denominator is the corresponding level when war does so. The r.h.s. is the corresponding ratio of survival rates into old age. Both ratios exceed 1, but it is very likely that the former ratio is the larger; for war is likely to take a proportionally heavier toll on young adults, and it will surely destroy some of the capital stock. It is highly plausible, therefore, that the condition

$$\frac{F\left[(1-q^2_{t+1}(0))\lambda_{t+1}(e_t)+n\gamma, \frac{\sigma_{t+1}(0)s_t}{n}\right]}{F\left[(1-q^2_{t+1}(1))\lambda_{t+1}(e_t)+n\gamma, \frac{\sigma_{t+1}(1)s_t}{n}\right]} \geq \frac{1-q^3_{t+1}(0)}{1-q^3_{t+1}(1)} \tag{3.38}$$

holds. This suffices to ensure that the second term on the r.h.s. of (3.33), which may be expressed as $E_{I_{t+1}}u[c^3_{t+1}(I_t; I_{t+1})]$, is increasing in π_{t+1}, $I_t = 0, 1$. Hence, we have:

Proposition 15

If (3.38) holds, an increase in the (prior) probability that war will prevail in the next period will depress investment in favor of consumption in the current one.

Consumption in old age depends on, *inter alia*, the savings made when young that survive destruction at the start of old age. War and peace, respectively, affect the second arguments of F in the denominator and numerator on the l.h.s of condition (3.38). The prospect of heavy destruction of physical capital makes saving less attractive when war in the next period has rather small effects on the survival rate of those then entering old age, and hence on the number of such claimants. The argument in Section 3.5.1 indicates that how the balance between e_t and s_t will be affected thereby depends in a complicated way on the differences in survival rates between the two states. The weight on the altruism term $v(\lambda_{t+1}(e_t))$ in (3.33) is decreasing in the future mortality rate among young adults, and the stronger is altruism, as represented by b, the larger will be the absolute size of the reduction in the said weight. War would have to be extremely destructive of physical relative to human capital such that substitution between the two forms of investment could induce a net increase in e_t.

3.6 Shocks and stability

The system's stability in the face of shocks will now be examined in some detail, drawing upon the above findings. The argument proceeds in a series of taxonomic steps. It rests on the claim that that the two extremes, i.e., backwardness with $e_t = 0$ and the progressive state with $e_t = 1$ are both locally stable steady states. Sufficient conditions for this claim to hold are given in the following proposition:

Proposition 16

If $u = \ln c_t$ and F is Cobb-Douglas, both backwardness and the progressive state are locally stable steady states if:

(i) $1 - q^2 > -\alpha n\gamma$;

(ii) the production function for human capital $h(e_t)$ is concave or sufficiently weakly convex;

(iii) $1 - q^2 > \alpha wn$.

Remark. In the light of Section 3.4, the assumptions on u and F can be weakened.

Condition (ii) has been discussed in Section 3.3.2. Condition (iii) is also easily fulfilled, as the term αwn is close to zero for plausible values of w, whereas q^2 is not close to 1. It is clear that condition (i) is always fulfilled.

Suppose backwardness is a locally stable, long-run equilibrium, even when peace always reigns ($e_t^0 = 0, I_t = 0\,\forall t$). Then once in backwardness, the economy is perpetually trapped in that state, be there war or peace thereafter.

Suppose there also exists, when peace always reigns, a set of stationary states with $e_t^0 \in (0,1)$. Let $e^*(0)$ denote the smallest such value of e_t^0, so that λ_t is stationary, at $\lambda^*(0)$, where $\lambda^*(0) = zh(e^*(0))\lambda^*(0) + 1$. Associated with $e^*(0)$ there is a stationary level of k_t, denoted by $k^*(0)$. Since the state of backwardness is locally stable, the equilibrium $(\lambda^*(0), k^*(0))$ is unstable. If, at time t, the state variables are such that $(\lambda_t, k_t) << (\lambda^*(0), k^*(0))$,[7] a descent into permanent backwardness will certainly occur. This conclusion holds *a fortiori* if there is some chance of war. For given any $\pi < 1$, it is established in Section 3.5.1 that an outbreak of war in the current period will almost surely reduce current investment relative to the state of peace, and in Section 3.5.2 that an increase in the hazard rate $1 - \pi$ will do likewise, *cet. par.*

Taking a longer view where the economy's capacity to withstand shocks is concerned, a *robust* economy can be defined as one in which growth can occur even in a state of perpetual war: $e_t^0 = e^{**}(1), I_t = 1\,\forall t$, where $\lambda_{t+1} = zh(e^{**}(1))\lambda_t + 1 > \lambda_t$. This requires, *inter alia*, that (3.32) hold at $\mathbf{q}_t = \mathbf{q}(1)$, $\sigma_t = \sigma(1)$. If steady growth is possible in a state of perpetual war, growth will also be possible when peace sometimes rules, but it will not be steady.

For any growth path to be attained, the starting values of the state variables must

[7] The inequality $\mathbf{x} << \mathbf{y}$ indicates that each component of the vector $\mathbf{y}$ exceeds its counterpart in $\mathbf{x}$.

be sufficiently favorable. The said values depend on the economy's particular history of war and peace. If, at time t', the state variables (λ_t, k_t) are such that, should war become permanent, $e_t^0 \geq e^{**}(1)\, \forall t \geq t'$, then a sustained growth path will be attained for all π.

Analogously to $(\lambda^*(0), k^*(0))$, suppose there is also a pair $(\lambda^*(1), k^*(1))$ in the state of perpetual war. Since survival rates are higher in peace, $(\lambda^*(0), k^*(0)) = (\lambda^*(1), k^*(1))$ cannot hold, and it is natural to conjecture that $(\lambda^*(0), k^*(0)) << (\lambda^*(1), k^*(1))$. If war and peace are both possible, i.e. $\pi \in (0, 1)$, this conjecture introduces chance into the final outcome in the long run if the initial conditions satisfy

$$(\lambda^*(0), k^*(0)) << (\lambda_0, k_0) << (\lambda^*(1), k^*(1)).$$

For suppose (λ_0, k_0) exceeds, but lies close to, $(\lambda^*(0), k^*(0))$. With some positive probability, the economy will enjoy an uninterrupted run of peace; and if long enough, this run could yield state variables exceeding $(\lambda^*(1), k^*(1))$, and hence ultimately, if the next stationary value of e_t^0 is such that $zh(e_t^0) > 1$, sustained growth. Then again there is the grim possibility that (λ_0, k_0) falls short of, but lies close to, $(\lambda^*(1), k^*(1))$, and that this initially tantalizing prospect recedes ever farther away as the economy endures an unbroken run of wars, an event whose probability of occurrence is also strictly positive. If long enough, such a run could well yield state variables short of $(\lambda^*(0), k^*(0))$ and hence, ultimately, backwardness with certainty.

One consequence of the sharing rule for old-age provision, as specified by (3.7), in a stochastic environment is that if war is more destructive of property than life, then the aggregate payment to the old will fall disproportionately, though war will also exact an additional toll on the numbers of those making a claim on it. Consider, therefore, the variation in which all those who survive into old age are allocated, not a fixed share of total full income at that time, but each one of them a fixed proportion ρ' of the full income of each surviving young adult: $c_t^3 = \rho'\bar{Y}_t/(1-q_t^2)N_t^2$. It is seen that if mortality rates are varying over time and sharply forecast, the difference in the respective denominators can yield different incentives to invest. Under the alternative rule, we have, instead of (3.7),

$$c_{t+1}^3 = \frac{\rho'}{(1-q_{t+1}^2)} \cdot F\left[(1-q_{t+1}^2)\lambda_{t+1}(e_t) + n\gamma, \sigma_{t+1}s_t/n\right].$$

The arguments of F are formally identical, embodying the decision (e_t, s_t) in the previous period. If the environment is stationary and non-stochastic, so that steady-state equilibria may exist, choose ρ' such that $\rho' = \rho n/(1-q^3)$. The two rules then yield identical allocations.

An unexpected outbreak of war in period t after an uninterrupted state of peace will leave each of the surviving young adults less well equipped with physical capital than their elders had intended, thus producing an adverse income effect on both forms of investment in period t. The alternative sharing rule will relieve the current loss by reducing the payment to each of the old-age survivors. If, however, war takes a heavier toll on young adults than their elders, that rule will not necessarily make the economy more robust to such asymmetric shocks to life and property.

An unexpected outbreak of pestilence, such as the Black Death, is an asymmetric shock of another kind, carrying off much of the population, but leaving the capital stock untouched. This will be a windfall for the survivors, but it will avail them little if physical and human capital are poor substitutes in production – indeed, not at all if they are strict complements. If, in contrast, they are perfect substitutes, then the windfall will yield a correspondingly large income effect, which may be sufficiently strong to propel an economy out of backwardness onto a growth path, even with perpetual, but not unduly destructive warfare.[8]

3.7 Simulations

Establishing more precisely whether an economy will withstand a particular shock in the presence of a poverty trap involves some resort to simulations, whereby the initial conditions need to be specified in a tractable form. Suppose, therefore, that the economy has been proceeding along some steady-state growth path in the state of peace, when it is suddenly hit by war out of the blue. Agents then form some sharp prior $1-\pi$ that war will also occur in the next period, and make their investment decisions accordingly.

If much time has elapsed along that path, both state variables will be very large indeed, so that extremely heavy loss rates will have to occur in order to reduce the normalized endowments to levels where even $e_t = 1$ will not be optimal, let alone a certain collapse into backwardness. Suppose, therefore, that both state variables are still relatively small.

Given the resulting normalized endowments and the (sharp) prior π_{t+1}, the households will choose (e_t^0, s_t^0). If the worst occurs again in period $t+1$, the resulting normalized endowments will be, suppressing the time subscripts for n_t, σ_t and q_t^2,

$$\bar{l}_{t+1} = (1-q^2(1))\lambda_{t+1}(e_t^0) + n\gamma \text{ and } k_{t+1} = \sigma(1)s_t^0/[(1-q^2(1))n].$$

If peace is confidently expected in period $t+2$, (e_{t+1}^0, s_{t+1}^0) are chosen accordingly. Since peace also actually rules, the resulting normalized endowments in period

[8] For an analysis of this potentially liberating stroke, see Bell and Gersbach (2013), in which there is only human capital. The assumption that both inputs are necessary in production leaves the matter open.

$t+2$ will be

$$\bar{l}_{t+2} = (1 - q^2(0))\lambda_{t+2}(e^0_{t+1}) + n\gamma \text{ and } k_{t+2} = \sigma(0)s^0_{t+1}/[(1 - q^2(0))n]. \quad (3.39)$$

Given the values of the state variables in period t, $(\lambda_t(e^0_{t-1}), s^0_{t-1})$, where these conform to the ζ associated with the path under consideration, it can be checked numerically whether the economy will recover from what are, in effect, the new starting endowments under a regime of perpetual peace given by (3.39) or fail to do so.

There remains the alternative possibility that peace, not war, rules in period $t+1$. In that event, the normalized endowments will be

$$\bar{l}_{t+1} = (1 - q^2(0))\lambda_{t+2}(e^0_t) + n\gamma \text{ and } k_{t+1} = \sigma(0)s^0_t/[(1 - q^2(0))n];$$

and the calculations for period $t+2$ then proceed as before.

A particular limitation of the two-period phase t and $t+1$ during which war can ever occur is that its influence on decisions *ex ante* is confined to period t. The only possible sequences are war-war and war-peace, each followed by permanent peace. Two consecutive adverse shocks are possible, but the certainty of peace from $t+2$ onwards makes ultimate recovery more likely. It is desirable, therefore, to extend the said phase to three periods, thus yielding an *ex ante* influence in both periods t and $t+1$. The possible sequences are

$$\{1,1,1\}, \{1,1,0\}, \{1,0,1\}, \{1,0,0\}, \{0,1,1\}, \{0,1,0\}, \{0,0,1\}, \{0,0,0\},$$

with all outcomes preceded by development in the environment $(\mathbf{q}(0), \sigma(0))$ up to $t = 0$. We concentrate on the grimmest outcome: three consecutive periods of war.

The constellation of parameter values in Table 3.1 must be extended to cover the states of war and peace. Let those values hold in the state of peace, so there is a poverty trap, even with unbroken peace, {0,0,0}, as the actual outcome. The associated long-run value of ζ_t when 'progress' rules, $\zeta_t(0; e = 1)$, is 62. Let the prior probability of war in periods 1 and 2, $1 - \pi_{t+1}(t = 0, 1)$, be 0.5, and the mortality rates in that state be $q_t^2(1) = 0.25, q_t^3(1) = 0.35$, with $\sigma_t(1) = 0.4$. As noted above, the initial values of human and physical capital, λ_0 and k_0, which are inherited from period $t = -1$, must be sufficiently small for a sequence of shocks even as heavy as {1,1,1} to rule out any path to progress. Recalling Section 3.6, the stationary (critical) values of λ^* are now in play. Under perpetual peace, expected as well as realized, $\lambda^*(0) = 2.3738$. With $\pi_{t+1} = 0.5$, the critical value of λ_0 when the realized sequence is indeed $\{0, 0, 0\}$ is 2.5414, but 2.8939 when the outcome is three periods of war, $\{1, 1, 1\}$. To complete the initial conditions, let $\zeta_0 = 26$; for physical capital plausibly forms the greater part of the whole endowment in the state of backwardness than that of progress.

The trajectories of λ_t and ζ_t for each of the values $\lambda_0 = 1, 2, 2.7, 3, 5$ are depicted in Figure 3.4. As intuition suggests, three periods of warfare generate an immediate and sharp upward spike in ζ_t, even when backwardness is the ultimate outcome. The two trajectories that closely bracket the critical value of $\lambda_0 = 2.8939$, namely, those for $\lambda_0 = 2.7$ and 3, have more than one local extremum. The former follows the spike by first undershooting, and then converging from below to the value under backwardness; the latter goes on to attain a second local maximum, at $t = 12$, before converging from above to the value under progress. These oscillations indicate complex and long drawn-out transitional dynamics near critical values of the boundary conditions, though these dynamics are less apparent in the trajectories

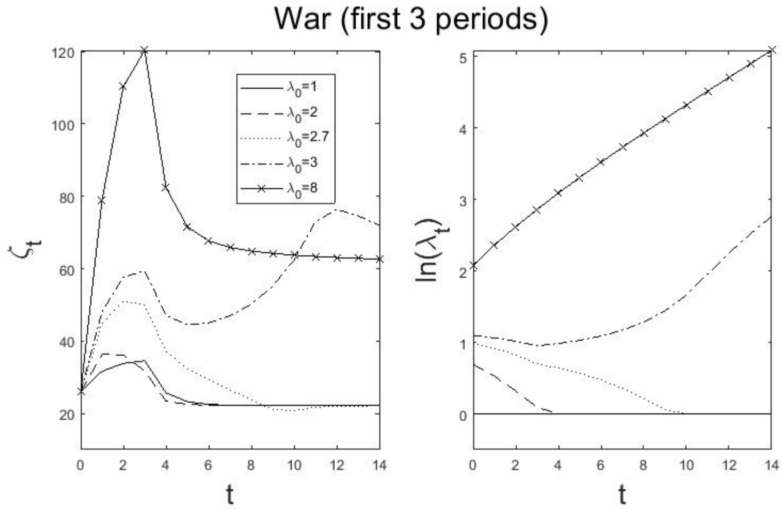

Figure 3.4: Three consecutive periods of war followed by peace.

of λ_t. The latter for $\lambda_0 = 2.7$ recovers only slowly from the three consecutive episodes of war, whereas that for $\lambda_0 = 5$ is little affected.

3.8 Conclusions

It is not difficult to think of conditions that will keep a society in a state of backwardness. Unremitting warfare and endemic communicable diseases untrammeled by public health measures, together with the privation that accompanies warfare and disease, will almost surely suffice to bring about a Hobbesian existence, even when productive technologies are available. What we have established, however, is that there are constellations of unchanging war losses and premature adult mortality such that backwardness, the state in which there is no investment in human capital through schooling, and steady growth with a fully educated population are both possible equilibria. The associated poverty trap is also precisely characterized

Parents' altruism can exert a decisive influence on the outcome. If sufficiently strong, it can rule out backwardness in environments in which the hazards of destruction are such as to keep a selfish population in that condition for good. That is no great surprise. Where attaining – and maintaining – steady-state growth is concerned, however, altruism also comes into play in a different way. If parents' preferences are such that the sub-utility functions for their own consumption and their children's well-being in adulthood differ – which is highly likely – and the former is more concave than the latter, then the only steady-state path other than backwardness is progress: There is growth with a fully educated population. If, however, the sub-utility function for own consumption is less concave than that for the children's human capital, then steady-state growth paths with an incompletely educated population may exist, as may stationary paths. The same holds if parents are perfectly selfish, so that provision for old age is the sole motive for investment. Thus, not only does altruism tend to promote growth as an outcome, as expected, but it may also lead to permanently faster growth.

The fact that outbreaks of war and pestilence are stochastic events introduces a central role for expectations. It also raises the question of whether a growing economy is sufficiently robust to withstand a series of adverse shocks. Mature economies that have experienced growth for long periods will have large per capita stocks of human and physical capital. They will be correspondingly robust, unless nuclear war destroys the environment itself. Economies at an earlier stage of development are more vulnerable. Numerical simulations in which the realized outcome is three consecutive periods of war followed by a confidently expected era of perpetual peace reveal how the boundary conditions at the start have a decisive influence on whether this series of shocks will pitch the economy into permanent backwardness.

Chapter 4

Capital flows and endogenous growth

Abstract

So-called "uphill capital flows", i.e. flows of physical capital from relatively poor to rich countries, are a new phenomenon with yet unclear impact.[1] We develop a unified framework incorporating economic institutions, human capital and physical capital to study the interaction of international capital flows and growth. Analytically, we study conditions under which a positive change of a country's economic institutions can attract inflows of physical capital from abroad, leading to long-term growth via the accumulation of human capital. Our mechanism shows how a small initial difference in the level of institutions can lead to substantial divergence in income over time. We derive conditions under which a country receives inflows of capital over time and increases its investment in human capital. Finally, we provide simulations to illustrate our results.

4.1 Introduction

Motivation

Are international capital flows a cause for growth inequality between countries?

[1]This chapter is single-authored.

In light of persistent income differences between industrialized and developing countries, and the fact that capital seems to flow from capital-poor to capital-rich countries, this questions becomes relevant. Whether developing countries, experiencing capital inflows, can benefit from capital inflows and whether they should open up to financial markets are further policy issues.

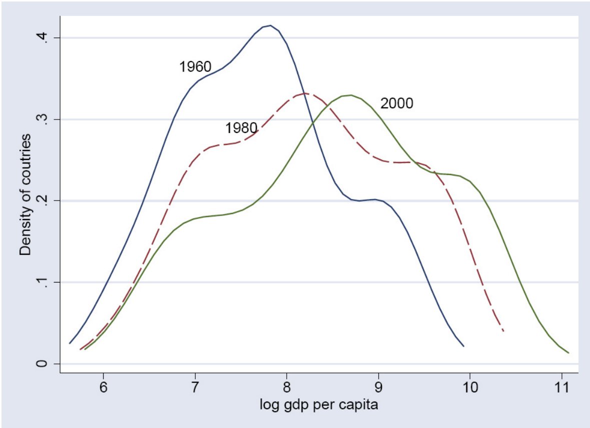

Figure 4.1: Log GDP per capita, author's own calculations.

Figure 4.1 shows the growing inequality between countries, using log GDP per capita for three points in time, 1960, 1980 and 2000. The mean of the distribution is moving to the left over time, indicating a general increase in prosperity worldwide. However, the variance is also increasing, as the decreasing peak from 1960 and the widening of the distribution show. This observation is in contrast to predictions of the neoclassical model, as income differences between countries seem to increase rather than to decrease.

Another phenomenon is the occurrence of "uphill capital flows", i.e. flows of capital from poor countries to rich ones. Such flows can be observed in Figure 4.2, which is taken from Prasad et al. (2007). The authors state "*[n]ot only is capital not flowing from rich to poor countries in the quantities the neoclassical model would predict—the famous paradox pointed out by Robert Lucas—but in the last few years it has been flowing from poor to rich countries*". They refer to the trend beginning

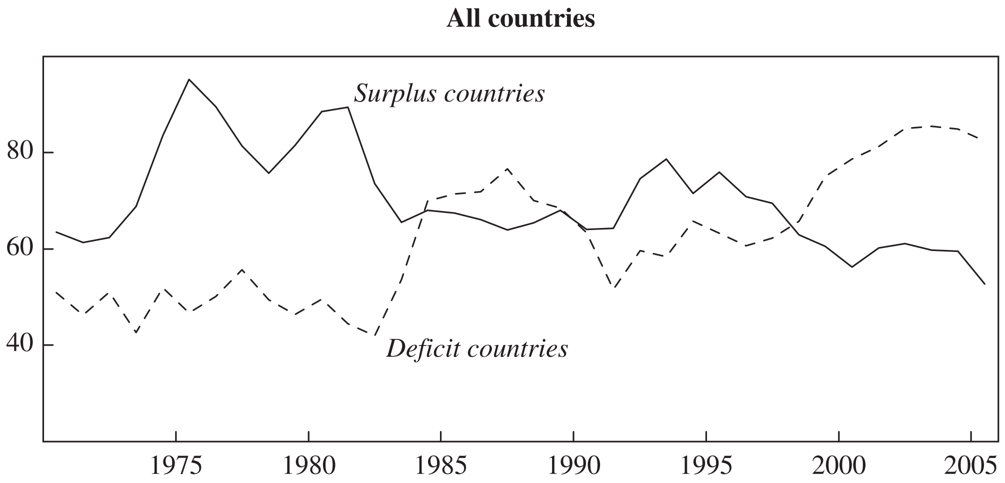

Figure 4.2: Relative GDP per capita of capital exporters and capital importers. Source: Prasad et al. (2007).

around the year 2000, after which capital-exporting countries (deficit countries) comprise more and more countries with a relatively large GDP.

While uphill capital flows are a rather new observation in comparison to the increasing inequality from Figure 4.1, the question arises whether such flows reinforce the divergence of countries. A closely related question, namely the impact of capital flows of growth, is studied by Kose et al. (2009). Their study contains a review of the recent empirical literature, finding that the literature "[...] *provides little robust evidence of a causal relationship between financial integration and growth.*" However, the authors do not claim that international capital flows have no effect on growth. The flows rather have indirect effects which might play out over a long time horizon and, among others, can take the form of increased competition and the development of a stronger financial market.[2] Hence, Kose et al. (2009) conclude "[...] *that it is not just capital inflows themselves, but what comes along with the capital inflows, that drives the benefits of financial globalization for developing*

[2]The findings of Kose et al. (2009) are corroborated in the survey by Edison et al. (2002). However, there are studies that find a clear positive impact of increased financial integration, such as Henry (2003).

countries."

The possible interaction described by Kose et al. (2009) has not been modeled up to now. Such a model is our goal in this chapter. We provide a framework where international capital flows are related to and interact with well-known causes of growth. This way, we describe some ideas concerning structures of international capital flows and endogenous growth. We aim at providing a framework for future empirical research. Following North and Thomas (1973), we assume that causes of growth can be either proximate, such as accumulation of factors and technological change, or fundamental, such as political and economic institutions. Hereby "fundamental" means that without well designed institutions, no growth-driving accumulation of factors is possible in an economy.[3] We align those two causes of growth with international capital flows in a unified framework. Our main idea is that better economic institutions, which will be the fundamental cause for growth, attract international capital flows. These, in turn, set the proximate causes in motion. The proximate source of endogenous growth will be the accumulation of human capital.

Our choice to combine economic institutions and human capital is based on the idea shown in Figure 4.3. Since there is little direct impact of capital flows on growth, we argue that capital can only have an impact via variables that interact with growth themselves. Hence, our approach requires variables correlating with international capital flows *and* growth.

There is a consensus that economic institutions and the accumulation of human capital are paramount for economic growth[4]. Their correlation with international

[3]For a thorough discussion about the reasons and ways, in which institutions are a fundamental cause for growth, see Acemoglu et al. (2004).

[4]There is an extensive literature, respectively, for the case of human capital and for the case of institutions. For the former, some examples are work like Lucas (1989), Romer (1989), Barro (1991) and Mankiw et al. (1992). Pelinsecu (2014) provides a survey, including more recent work. For the latter, some examples are work like Acemoglu et al. (2001), Acemoglu et al. (2002), and

capital flows has been demonstrated in several studies, for economic institutions in work like Olsen et al. (2000), Alfaro et al. (2008) and Kose et al. (2009), and for human capital in Lucas (1990) and Caselli and Feyrer (2007), for instance.[5]

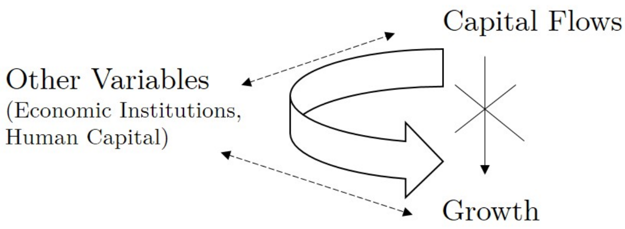

Figure 4.3: Interaction of international capital with growth. Source: Author's design.

After having chosen the variables that are relevant for our approach, the question remains how these variables interact with each other. We build on the idea that good economic institutions enable the accumulation of factors, such as physical and human capital, or the raise of the technology level. In turn, the accumulation of factors then leads to growth. We take this narrative and re-position it in an international setting to formulate the following mechanism:

1. Better economic institutions allow agents to reap the benefits of their investment in physical capital. This interpretation is in line with Acemoglu et al. (2004) that "*[e]conomic institutions matter for economic growth because they shape the incentives of key economic actors in society, in particular, they influence investments in physical and human capital* [...]".
2. The country with a larger return to capital will attract capital flows.
3. Capital inflows increase wages and hence agents have enough income to reduce labor supply and invest some time in the formation of human capital.

Hall and Jones (1999). Glaeser et al. (2004) provides a critical overview.

[5]For an overview of other determinants of international capital flows, see Taylor and Sarno (1997).

4. Human capital is not depreciated but accumulated, allowing for long-term growth.

Approach and results

We use an OLG model with three generations, consisting of children, adults and seniors. Children can either work or form human capital through schooling. Adults work, save and decide whether to school their children. Seniors only consume. We have two countries and a single good, produced with physical and human capital, embodied in labor. Capital can move freely between the countries while labor cannot. Both countries have economic institutions, determining the net return on physical capital. Initially, arbitrage ensures that returns are equal across countries. Then, a country experiences an improvement in its institutions and a subsequent increase in its return to capital. Capital flows in and wages increase. We examine whether the wage increase is sufficient for adults to send their children to school. Schooling leads to the formation of human capital that can be used without costs in the next period.

We demonstrate how a change in institutions, coupled with international capital, can generate differences in economic growth. While the country with better economic institutions moves to a balanced growth path with increasing levels of human capital, the other remains in a zero growth steady state. Furthermore, we study the existence of different steady states and convergence to them.

Throughout our work, we interpret capital flows as private flows of foreign direct investment (FDI). By doing this, our model can replicate the empirical findings of Gourinchas and Jeanne (2013) and Alfaro et al. (2014), where the authors find that public capital flows behave differently than private ones. Only private capital flows to countries with higher growth rates. In a simulation, the country with increasing investment in education experiences increasing productivity growth, attracting more capital. Hence, we establish a positive correlation between productivity growth and inflows of capital.

Relation to the literature

Our work is closely related to studies that use the neoclassical growth model in an international setting, such as Barro et al. (1992), Gourinchas and Jeanne (2006), and Aguiar and Amador (2011). The authors study the impact of capital flows on the convergence of an economy to a steady state. While Aguiar and Amador (2011) focuse on explaining the behavior of governments, Barro et al. (1992) and Gourinchas and Jeanne (2006) consider an economy where the steady state is a balanced growth path with a constant growth rate. In both studies, the production function accounts for human capital formation and growth is exogenous, driven by technological progress. These studies show that inflows of capital have a rather small effect on the convergence path. Our findings are consistent with this result. However, our study differs in third crucial ways: First, we do not assume a constant world rate of return. Instead, the flows of physical capital and the accumulation of physical and human capital in a period will determine, whether a country can also attract capital inflows in the next period. Second, we do not study the speed of convergence, but we are interested in the question, whether a country can escape a poverty trap and converge to a steady state with a constant rate. We provide analytical conditions for such an escape. Third, we consider an exogenous growth model in which growth stems from the accumulation of human capital.

Another study that analyzes a neoclassical growth model is Davenport (2018). However, Davenport (2018) studies the importance of expectation formation for international capital flows and growth. The authors are able to replicate substantial current account imbalances. However, she assumes an exogenous growth process, where countries catch up to a technological frontier. We endogenize the growth process and relate it to the structure of capital flows, making them interdependent.

Howitt (2000) provides an alternative to the neoclassical model approach. He constructs a multi-country model with elements of the Solow-Swan-model and the Schmupeterian growth model to explain cross-country differences in growth. Countries that invest in R&D will grow at a positive rate, while those who are not able to do so, are stuck with the same output. Howitt (2000) abstracts, however, from international capital flows, which are now incorporated in our work. Furthermore, we highlight the importance of economic institutions and include the accumulation of human capital instead of technological progress as the main driving force of long-term growth.

Our study is also related to Bell et al. (2019), where the authors construct an OLG model with endogenous growth. As in this work, growth is driven by the accumulation of human capital through schooling and the effects of exogenous shocks are studied in a closed economy. We take some of the central assumptions and reassess them in an international setting with capital flows. Also, we include a central role for institutions in our setting.

Finally, this chapter is related to other works about international capital flows that, however, either focus on the international asset structure, such as Caballero et al. (2008) and Cedric and Van Wincoop (2010), or on the international trade structure, such as Jin (2012). While our model incorporates international assets, it does it in a simple way, allowing us to focus on the interaction of capital flows and growth and to show the causes for international inequality.

Structure

The remainder of the chapter is structured as follows. Section 4.2 presents a simple endogenous growth model with human capital accumulation. In Section 4.3 institutions are introduced and Section 4.4 introduces intarnational capital flows. Section 4.4 presents the full model, including international capital flows. Section 4.6 contains a simulation and Section 4.7 concludes.

4.2 Simple model

The Productive Sector

We consider an economy with only a single good that is used either for consumption or investment. It is produced in two different sectors. The first is the capital-intensive sector that employs physical capital and human capital of adults.[6] The second is the child-labor sector that employs human capital of children. Their respective output is given by

$$Y_t^2 = AK_t^\alpha (H_t^2)^{1-\alpha} \quad \text{and} \tag{4.1}$$

$$Y_t^1 = H_t^1, \tag{4.2}$$

where A is some constant total factor productivity parameter, K_t physical capital, and H_t^2 the stock of human capital in the capital-intensive sector and H_t^1 the one in the child-labor sector.[7] We assume a representative firm for the capital-intensive sector that borrows both types of capital from households at the capital rental rate R_t and at the wage rate w_t^2. Under perfect competition, profits and the demand

[6] We assume that capital is complementary to the skills of adults, while it is not to the skills of children. This assumption appears plausible, as children lack the physical strength to operate machinery and the intellectual maturity to work with other types of equipment.

[7] We follow the approach by Docquier et al. (2007).

functions of the firm are

$$\Pi_t^2 = Y_t^2 - R_t K_t - w_t^2 H_t^2, \quad R_t = \alpha \frac{Y_t}{K_t}, \quad \text{and} \quad w_t^2 = (1-\alpha)\frac{Y_t}{H_t^2}. \tag{4.3}$$

We also assume perfect competition in the child-labor sector and have

$$\Pi_t^1 = Y_t^1 - w_t^1 H_t^1 \quad \text{with} \quad w_t^1 = 1. \tag{4.4}$$

Since, $w_t^1 = 1$ we will write w_t^2 simply as w_t.

Households

A household consists of three generations that are alive at the same time: children, adults, and seniors. Children and seniors only consume and do not take any economic decision. Additionally, children can use their time either for work or for education. The former earns a wage income for the family, while the latter increases the children's human capital in the next period. Consequently, the education decision involves a trade-off between wage income today and more income tomorrow. Whether and how much schooling takes place is decided by adults. They maximize their own utility, supply their own human capital to firms, decide whether and how much to school children, and how much should be saved.

Children have the level of human capital ηL^1, where L^1 is the amount of labor and η is the fixed level of human-capital per child. The adults' stock of human capital is given by $\phi_t L^2$, with L^2 being the fixed amount of labor supplied by adults and ϕ_t the amount of human capital per worker. This amount increases with schooling. If children only supply the share $(1-e_t)$ to the labor market and spend the share e_t in school, the human capital stock per worker growths, i.e. $\phi_{t+1} > \phi_t$. Hereby, we only impose positive returns to schooling: $\partial\phi_{t+1}/\partial e_t > 0$. Also, we assume that the current stock of human capital can be inherited without schooling, so that ϕ_{t+1} is always at least as large as its predecessor. For instance, if children receive education of size e_t^* then

$$\phi_{t+1}^* = \int_0^{e_t^*} \frac{\partial \phi_{t+1}}{\partial e_t} de_t + \phi_t.$$

The stock of human capital, supplied inelastically by adults to the capital-intensive sector is $\phi_t L^2$. We therefore have $H_t^2 = \phi_t L^2$ and the stock of human capital that is supplied by children to the child-labor sector is $(1 - e_t)\eta L^1$. For simplicity, we will write H_t^2 as H_t from now on.

We model the utility of an agent in period t as in Bell et al. (2019) so that the adults receive utility from consumption in t, as well as consumption in $t + 1$. We assume a linear benevolence term that enters the utility function and depends on the level of education of children in $t + 1$.[8] Hence, the life-time utility for an adult in t reads

$$U_t^2 = \log[c_t^2] + \beta \log[c_t^3] + \beta \phi_{t+1}.$$

Adults receive their own wage income and that of children to whom, in turn, they give the share γ_1 of the total wage income. Seniors give a share of their capital return γ_3 to adults, as an additional form of income. The motivation for this behavior can be either altruism or the fact that seniors are not alive for the entire time period and bequest some of their income to adults. Hence, the income of adults is a combination of the remainder of wage income and the received capital income. Their consumption, then, is what is left of this income after savings,

[8] We choose this structural form as it allows for an analytical approach.

leading to the following utility function:

$$\begin{aligned} U_t^2 =& \log\left[(1-\gamma_1)(w_t H_t + (1-e_t)\eta L^1) + \gamma_3 K_t R_t - s_t\right] \\ &+ \beta \log\left[(1-\gamma_3) R_{t+1} s_t\right] + \beta\phi_{t+1}. \end{aligned} \tag{4.5}$$

We make an additional assumption, and say that the income that is paid to children is only used for consumption and does not enter the savings decision. The rational is that this assumption simplifies the analysis strongly. However, it can be argued that parents only then send their children to work when it is necessary, i.e. when the family goes hungry otherwise. They do not ask their children to work in order to save and accumulate capital. In the following, we will refer to the income of adults that accrues from capital and adult human capital as "income", while the sum of capital income, wage paid to adults, and wage paid to children will be called "total income". Hence, maximizing with respect to savings yields

$$\frac{\partial U_t^2}{\partial s_t} = \frac{-1}{(1-\gamma_1) w_t H_t + \gamma_3 R_t K_t - s_t} + \frac{\beta}{s_t} = 0, \tag{4.6}$$

while maximizing with respect to education yields

$$\frac{\partial U_t^2}{\partial e_t} = \frac{-(1-\gamma_1)\eta L^1}{(1-\gamma_1)(w_t H_t + (1-e_t)\eta L^1) + \gamma_3 R_t K_t - s_t} + \beta\phi'_{t+1} \leq 0 \quad \text{for} \quad e_t \geq 0, \tag{4.7}$$

$$\frac{\partial U_t^2}{\partial e_t} = \frac{-(1-\gamma_1)\eta L^1}{(1-\gamma_1)(w_t H_t + (1-e_t)\eta L^1) + \gamma_3 R_t K_t - s_t} + \beta\phi'_{t+1} \geq 0 \quad \text{for} \quad e_t \leq 1, \tag{4.8}$$

Solving (4.6) for s_t, we obtain

$$s_t = \frac{\beta}{1+\beta}\left((1-\gamma_1) w_t H_t + \gamma_3 R_t K_t\right).$$

Plugging this expression into (4.7) yields

$$\beta\phi'_{t+1} = \frac{(1+\beta)\eta L^1}{w_t\phi_t L^2 + (1+\beta)\eta(1-e_t)L^1 + \gamma R_t K_t}, \quad \text{with} \quad \gamma = \frac{\gamma_3}{1-\gamma_1} \tag{4.9}$$

if the expression holds with an equality sign. From Equation (4.9) we observe that investment in education depends on the marginal effect of education, given by ϕ'_{t+1}. Also, wealthier households, i.e households with a larger stock of human capital, $\phi_t L^2$ and ηL^1, and more capital income $R_t K_t$ are more likely to invest in human capital, as they suffer relatively less from the income loss associated with schooling. This loss is given by the numerator ηL^1.

Now we use demand for human and physical capital to substitute w_t and R_t,

$$\beta\phi'_{t+1} = \frac{(1+\beta)\eta L^1}{Y_t^2((1-\alpha)+\alpha\gamma) + (1+\beta)Y_t^1},$$

so that wealthier countries, with larger Y_t^2, are more likely to invest in human capital and are more likely to grow. We can interpret this as “history matters” or as the potential occurrence of a poverty trap in which countries with low endowment of both types of capital will not invest in education. Also, if child labor is a relevant source of income and the ratio of children’s productivity to output is large, investment in education is unlikely. Next we discuss the role of institutions in this economy.

4.3 Institutions

When modeling institutions, we follow the idea of Acemoglu et al. (2004) that institutions shape the incentives to invest. One way for institutions of doing this is by protecting agents from expropriation.

We model the risk of expropriation by assuming a government in the economy, which expropriates a constant share of capital returns in each period, so that households only receive $\psi R_t K_t$. To distinguish $1-\psi$ from a simple tax on capital income, we additionally assume that the expropriated income does not create any welfare. The government does not provide any public goods or corrects any market failures with it. Instead, one could imagine that $\psi R_t K_t$ increases the consumption of some government agents that form a vanishingly small share of the population that is negligible for total welfare. Put simply, the amount of goods $(1-\psi)R_t K_t$ is lost.[9]

With these assumptions, it is straightforward to study how institutions shape the outcome of the economy. Assume two different types of institutions, denoted by $\underline{\psi}$ and $\overline{\psi}$, with $\underline{\psi} < \overline{\psi}$. The set of institutions with higher quality has a larger value of ψ. Now, consider Equation (4.9) and multiply the term $R_t K_t$ with $\underline{\psi}$, so that the right hand side becomes

$$\frac{(1+\beta)\eta L^1}{w_t \phi_t L^2 + (1+\beta)\eta(1-e_t)L^1 + \underline{\psi}\gamma R_t K_t}.$$

With a low level of institutions, such as $\underline{\psi}$, the denominator on the right hand side of the equation is relatively large, leading to no education. Under higher quality institutions, i.e. when $\underline{\psi}$ is replaced by $\overline{\psi}$, the opposite might be the case. The increase in institutions also affects the incentive to invest in physical capital, as savings are given by

$$s_t = \frac{\beta}{1+\beta}\left[(1-\gamma_1)w_t H_t + \gamma_3 \overline{\psi} R_t K_t\right].$$

[9]Our approach is similar to Klein (2005).

4.4 International capital flows

As we pointed out above, better economic institutions can lead to investment in education and subsequently to growth. Yet, even with a high level of ψ, the marginal loss of schooling, which is given on the right hand side of (4.9), might be larger than the marginal benefit. One way to reverse that relationship is to decrease the marginal cost by providing the economy with a larger stock of physical capital K_t. Such an increase in K_t might stem from inflows of capital from another country. To model this possibility, we assume two countries indexed by $I \in \{A, B\}$, with their respective level of institutions ψ_I. Capital is internationally mobile, while labor is not. We allow for international investment and flow of goods. Also, we consider a fixed exchange regime, with an exchange rate of 1. By doing so, we abstract from nominal issues and focus on real variables.

At first, both countries are identical and have the same endowment of both types of capital and institutions. They find themselves in the zero growth steady state. Also, there are no capital flows, as the arbitrage condition

$$\psi_A R_t^A = \psi_B R_t^B \tag{4.10}$$

holds. Neither of the two countries has invested in education, so that in the two countries the following holds:

$$\beta \phi'_{t+1} \leq \frac{(1+\beta)\eta L^1}{w_t \phi_t L^2 + (1+\beta)\eta L^1 + \psi \gamma R_t K_t}. \tag{4.11}$$

The right hand side of the inequality is positive, so that $e_t = 0$ holds as an equilibrium and the total stock of human capital in a country remains at its initial level. We omitted the country-specific superscript I, as both countries are identical. The stock of physical capital is fixed and is given by

$$K = \left(\frac{\beta A}{1+\beta} \left[(1-\gamma_1)(1-\alpha) + \alpha\psi\gamma_3 \right] \right)^{\frac{1}{1-\alpha}} H, \tag{4.12}$$

where H is the initial stock of human capital of adults in the economy. Hence, we define the zero growth steady state in the following way:

Definition 3 *In a zero growth steady state, international capital flows are absent and countries do not invest in education, i.e.* (4.10) *and* (4.11) *hold. Furthermore, the stock of physical capital is constant and given by* (4.12)*. Wages, interest rates, profits and output are constant and given by* (4.1) – (4.4)*.*

Next, we allow for heterogeneous institutions across countries.

Change in institutions

In period t, before agents have decided about education, Institutions in country A improve from ψ^A to $\widetilde{\psi}^A$, so that we have $\widetilde{\psi}_A R_t^A > \psi_B R_t^B$. Hence, capital flows from country B to A until returns are equalized again, affecting the education decision.

Let us turn to the sequence of events , which can be seen in Figure 4.4. Initially, we

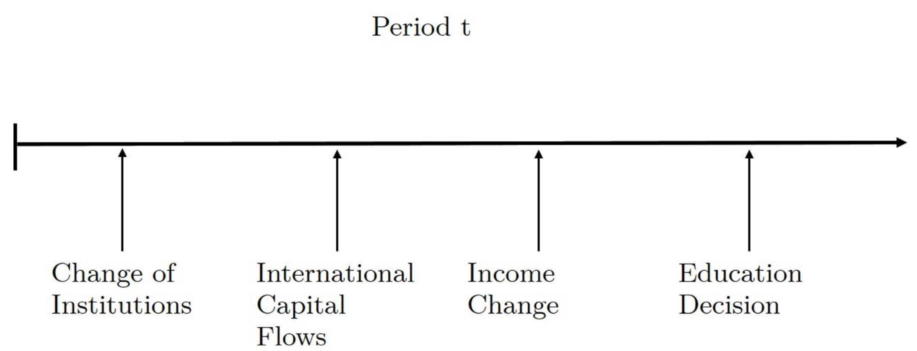

Figure 4.4: Sequence of events.

have a positive change in the quality of institutions in country A. The improvement leads to an inflow of capital and a change in income. Agents in both countries now face their respective new income. As this is the income before adults decide whether children should receive a positive amount of schooling, we call this income their pre-education income. If agents decide to school children, the total supply of human capital, given by $H_t + (1 - e_t)\eta L^1$, decreases in the respective country.

4.5 The full model

Given that capital markets pay the marginal return of capital we have

$$\widetilde{\psi}^A(\widetilde{K}_t^A)^{\alpha-1} = \psi^B(\widetilde{K}_t^B)^{\alpha-1} \Rightarrow \widetilde{K}_t^A = \left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}} \widetilde{K}_t^B, \tag{4.13}$$

where $\widetilde{K}_t^A$ and $\widetilde{K}_t^B$ are the capital stocks after capital movements from B to A. We have $\widetilde{K}_t^B = K_t^B - (\widetilde{K}_t^A - K_t^A)$, where we assume that initially, $K_t^B = K_t^A$, and where we use the result for $\widetilde{K}_t^A$ to obtain

$$\widetilde{K}_t^B = 2K_t^B - \left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}} \widetilde{K}_t^B \Rightarrow \widetilde{K}_t^B = 2\left(1 + \left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}}\right)^{-1} K_t^B.$$

Intuitively, the new capital stock $\widetilde{K}_t^B$ depends on the ratio of institutions. The number 2 on the right hand side stems from the assumption that initial capital stocks are equal and the factor with exponent -1 shows how capital is allocated between countries. Using (4.13), we obtain

$$\widetilde{K}_t^A = 2\left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}} \left(1 + \left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}}\right)^{-1} K_t^A$$

for the new capital stock in A. One of the factors on the right hand side is increasing in $\widetilde{\psi}_A$, while the other is decreasing. The increasing part stems from the arbitrage condition which yields that better economic institutions in A must increase the capital stock in A compared to the one in B. The decreasing part stems from the fact that better economic institutions in A mean that the final capital stock in B is already smaller than it initially was, leaving less capital to flow to A. These two forces offset each other in the limit, as

$$\lim_{\widetilde{\psi}_A \to \infty} \widetilde{K}_t^A = 2K_t^A,$$

which is necessary, as the total initial endowment of capital is $2K_t^A$. Of course, $\widetilde{K}_t^A$ is strictly increasing in the relative level of institutions, as one can see by slightly rewriting the expression above,

$$\widetilde{K}_t^A = 2\left(1 + \left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{-1}{1-\alpha}}\right)^{-1} K_t^A.$$

Income change

So far, a change in institutions triggers international capital flows. These flows impact the income of households in both countries. For country A, we observe the following: Before capital flows, its capital income was given by $\psi_A R_t^A K_t^A$, after capital flows, it is given by $\widetilde{\psi}_A \widetilde{R}_t^A K_t^A$. We form the ratio of both

$$\frac{\widetilde{\psi}_A \widetilde{R}_t^A K_t^A}{\psi_A R_t^A K_t^A} = \frac{\psi_B}{\psi_A} 2^{\alpha-1} \left(1 + \left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha}.$$

This term can be either larger or smaller than one. Its size depends on three things: First, on the improvement in institutions vis-à-vis the status quo ψ_A, second, on the improvement relative to the institutions of country ψ_B and third, on the initial allocation of capital, which is now represented by the number 2. These three issues decide whether the improvement of institutions is so strong that the net return rises although the marginal return to capital decreases in light of capital inflows. Under our initial assumption that $\psi_B = \psi_A$, the income ratio is 1 if no change occurs, i.e. $\widetilde{\psi}_A = \psi_B$. If $\widetilde{\psi}_A > \psi_B$ then the income from capital increases above 1, as the factor on the right hand side is a strictly increasing function in the ratio of institutions $\frac{\widetilde{\psi}_A}{\psi_B}$.

Now let us study the relative change in labor income of adults, given by

$$\frac{\widetilde{w}_t^A H_t^A}{w_t^A H_t^A} = \left(\frac{\widetilde{K}_t^A}{K_t^A}\right)^{\alpha} = 2^{\alpha}\left(1 + \left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{-1}{1-\alpha}}\right)^{-\alpha}.$$

We use $\widetilde{w}_t^A$ to denote the wage after capital flows and recall that there is no positive level of education yet. We see that the ratio is 1 for $\widetilde{\psi}_A = \psi_B$. For $\widetilde{\psi}_A > \psi_B$, it is larger than one because the expression on the right hand side is increasing in $\widetilde{\psi}_A/\psi_B$, as we have shown above.

Now let us turn to the income of agents in the capital-exporting country B. The capital income of agents in B is the same as in A, and thus increases. The wage income, however, decreases as the total local capital stock used in production in B becomes smaller. This is reflected in the following ratio:

$$\frac{\widetilde{w}_t^B H_t^B}{w_t^B H_t^B} = \left(\frac{\widetilde{K}_t^B}{K_t^B}\right)^{\alpha} = 2^{\alpha}\left(1 + \left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}}\right)^{-\alpha},$$

which is smaller than 1. With capital and wage income moving in opposite directions, we study the net effect on income in B. The disposable income of adults without earnings of their children, $I_t^B = (1-\gamma_1)\widetilde{w}_t^B H_t^B + \psi_B \gamma_3 \widetilde{R}_t^B K_t^B$, can be written as

$$\begin{aligned}I_t^B =& A(H_t^B)^{1-\alpha}(\widetilde{K}_t^B)^{\alpha}\left((1-\alpha)(1-\gamma_1)+\alpha\psi_B\gamma_3(\widetilde{K}_t^B)^{-1}K_t^B\right)\\ =& A(H_t^B)^{1-\alpha}(K_t^B)^{\alpha}2^{\alpha}\left(1+\left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}}\right)^{-\alpha}\cdot\\ &\left((1-\alpha)(1-\gamma_1)+\frac{\alpha}{2}\psi_B\gamma_3\left(1+\left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}}\right)\right).\end{aligned}$$

Taking the derivative of I_t^B with respect to $\widetilde{\psi}_A$ provides the first proposition.

Proposition 17 *Given an initially symmetric allocation of capital between country A and B, the income in B decreases after capital flows if and only if*

$$\frac{\psi_B\gamma}{2} < \left(1+\left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}}\right)^{-1}. \tag{4.14}$$

The pre-education income in country A increases unambiguously.

We provide a short proof in the Appendix for Chapter 4. Proposition 17 implies that the three factors, we encountered before, determine the change in income. First, the relative increase in institutions is crucial, as can be seen from the right hand side. Second, the initial level of institutions, represented by ψ_B on the right hand side, is also important. Third, the number 2, on the left hand side of the inequality, shows that the initial allocation of capital determines the outcome as well. If we did not assume that the initial stocks of capital are equal, the number 2 would be replaced by the term $K_t^B/(K_t^B+K_t^A)$.

To clarify, let us consider two examples. In the first, country B is relatively rich in capital and has good institutions, i.e. ψ_B is relatively large. If country A strongly improves its institutions with $\widetilde{\psi}_A$ significantly larger than ψ_B, then the income in country B actually increases. Country B loses some wage income, though, because it is rich in capital and can reap the benefits of investment. It is more than compensated by the larger return on capital and larger capital income. In the second example, Country B has a low share in world capital and comparatively weak institutions. If, in this case, country A improves its institutions slightly, country B will actually incur a decrease in income.

We also provide a more general approach to study the income change in B. For this, we write the income in B after capital flows but before any education decision, on the left hand side of the following inequality:

$$(1-\gamma_1)\widetilde{w}_t^B H_t^B + \gamma_3\psi_B \widetilde{R}_t^B K_t^B \geq (1-\gamma_1) w_t^B H_t^B + \gamma_3\psi_B R_t^B K_t^B.$$

The right hand side is the income before capital flows occur. Substituting wages and interest rates before and after capital flows yields

$$G\left(\frac{\widetilde{K}_t^B}{K_t^B}\right) \geq 1 \qquad \text{with}$$

$$G\left(\frac{\widetilde{K}_t^B}{K_t^B}\right) = \left(\frac{\widetilde{K}_t^B}{K_t^B}\right)^{\alpha} \frac{(1-\alpha)(1-\gamma_1) + \alpha\psi_B\gamma_3 \frac{K_t^B}{\widetilde{K}_t^B}}{(1-\alpha)(1-\gamma_1) + \alpha\psi_B\gamma_3}.$$

We define $\widetilde{K}_t^B / K_t^B$ as x, so that the above expression is 1 for $x = 1$, i.e. all local capital is utilized in production. The derivative of $G(x)$ is

$$\alpha(1-\alpha)x^{\alpha-1}\left(1-\gamma_1 - \frac{\psi_B\gamma_3}{x}\right).$$

It has a global minimum at $x^{min} = \psi_B \frac{\gamma_3}{1-\gamma_1}$ and is strictly increasing to the left and right of it, making $G(x)$ strictly convex. If γ_3 is similar to $1-\gamma_1$ the minimum lies to the left of $x = 1$, with $G(x^{min}) < 1$. Due to the strict convexity and the fact that $\lim_{x\to 0} G(x) \to \infty$, there must be an $\tilde{x}$ between $[0, \psi_B\gamma]$, such that $G(\tilde{x})$ is equal to 1. Hence, income in country B increases or decreases, depending on the size of capital flows and the subsequent ratio of capital used in local production to the total amount of initial capital. If capital outflows are very large, reducing $\widetilde{K}_t^B$ strongly relative to K_t^B, then country B will actually experience an increase in its income. One determining factor is certainly the change in institutions in A. The larger the change, the larger capital outflows from B will be, and the larger capital outflows from B, the better the chances for B to benefit in net from it. Let us assume for the following that income in B decreases.

Finally, we provide the income net of the earnings of children in A,

$$\begin{aligned} I_t^A =& A(H_t^A)^{1-\alpha}(K_t^A)^{\alpha}2^{\alpha}\left(1+\left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{-1}{1-\alpha}}\right)^{-\alpha} \cdot \\ & \left((1-\gamma_1)(1-\alpha)+\frac{\alpha\widetilde{\psi}_A\gamma_3}{2}\left(1+\left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{-1}{1-\alpha}}\right)\right). \end{aligned} \tag{4.15}$$

We have established that wage and capital income in country A increase and consequently lead to an increase of income. With larger income, savings in A also increase.

Education decision

A change in wealth might alter the decision about schooling. To see this, consider Equation (4.11), where we assumed that before capital flows, neither country invests in education. The marginal benefits on the left hand side of the inequality are equal to or smaller than the marginal income loss at zero education. The marginal loss will be even larger if the country's income has decreased. This is what has happened in B. Thus, we are certain that country B does not invest into education after the change in institutions in A.

We have seen that income increases in A. It enters the denominator of (4.11) allowing the right hand side to be actually lower than $\phi'_{t+1}(e_t = 0)$. If we additionally assume that the marginal benefits of education are constant, there exists a value for $e_t^* \in (0, 1]$ such that

$$\beta\phi'_{t+1}(e_t^*) = \frac{(1+\beta)(1-\gamma_1)\eta L^1}{A(\phi_t L^2)^{1-\alpha} K_t^\alpha \Gamma_t^A + (1+\beta)(1-\gamma_1)(1-e_t^*)\eta L^1}$$

is fulfilled. We let Γ_t^A capture all remaining terms from Equation (4.15). Writing the ratio $\frac{\widetilde{\psi}_A}{\psi_B}$ as χ, we note that Γ_t^A increases in χ. To see this, recall the expression for I_t^A from above. It is written as the product of income before capital flows, $A(H_t^A)^{1-\alpha}(K_t^A)^\alpha$, times the factor Γ_t^A. This factor must be larger than one, as we have shown that income in A does increase due to capital inflows. Also, we have shown that both components of income increase in χ, so that Γ_t^A must do this, too. Alternatively, we can rewrite Γ_t^A as

$$2^\alpha(1-\alpha)(1-\gamma_1)(1+\chi^{\frac{-1}{1-\alpha}})^{-\alpha} + \alpha 2^{\alpha-1}\widetilde{\psi}_A\gamma_3(1+\chi^{\frac{-1}{1-\alpha}})^{1-\alpha},$$

noting that $\widetilde{\psi}_A = \chi\psi_B$. We see that both summands increase in χ. For further analysis, it is useful to study whether households might choose a level of e_t that lowers the family income below its initial level, i.e. the level that prevailed before the inflow of capital. This can occur, as investment in education reduces the supply of human capital and also the earnings of children. We write the equation from above as $\phi'_{t+1}(e_t^*) = D(e_t^*, \Gamma^A)$, where $D(.)$ is a function of the optimal value of education and of the factor that scales output. Before capital flows, it held that $\Gamma^{A,0} = (1-\alpha)(1-\gamma_1) + \alpha\gamma_3\widetilde{\psi_A}$. In the initial steady state, we had $\phi'_{t+1}(0) \leq D(0, \Gamma^{A,0})$. After the inflow of capital, we assume that $\phi'_{t+1}(0) > D(0, \Gamma_t^A)$ holds.

This allows for some e_t^* such that $\phi'_{t+1}(e_t^*) = D(e_t^*, \Gamma_t^A)$. Now assume that the education function has a constant marginal product, i.e $\phi'_{t+1}(0) = \phi'_{t+1}(e_t^*)$. Then, $\phi'_{t+1}(e_t^*) = D(e_t^*, \Gamma_t^A) > D(0, \Gamma_t^A)$ and $D(e_t^*, \Gamma_t^A) \leq D(0, \Gamma^{A,0})$, so that the family income does not decrease below the initial level. This holds even more strongly, when ϕ_{t+1} has decreasing returns. To see this, note that $\phi'_{t+1}(0) > \phi'_{t+1}(e_t^*) = D(e_t^*, \Gamma_t^A)$. It follows that $D(e_t^*, \Gamma_t^A) < D(0, \Gamma^{A,0})$. Thus, the family income will be strictly larger for every optimal e_t^*. The opposite holds for increasing returns on education. In this case, $\phi'_{t+1}(0) < \phi'_{t+1}(e_t^*) = D(e_t^*, \Gamma_t^A) > D(0, \Gamma^A)$ and potentially $D(e_t^*, \Gamma_t^A) < D(0, \Gamma^{A,0})$. It is possible that adults invest so much in education that they have less income than before. However, in the case of increasing returns to education, $\phi'_{t+1}(e_t)$ must grow more slowly than the right hand side in e_t. Otherwise, there cannot be a steady state in which a country does not invest in education. Otherwise, the only optimal choice becomes $e_t = 1$.

Next we study the set of optimal e_t^*.We find a sufficient condition under which, given the capital inflows, no level of education reduces the income of adults below the level of the beginning of the period. This condition is

$$\frac{2^{\alpha}\left(1+\chi^{\frac{-1}{1-\alpha}}\right)^{-\alpha}\left[(1-\alpha)(1-\gamma_1)+\alpha\widetilde{\psi}_A\gamma_3(1+\chi^{\frac{-1}{1-\alpha}})\right]}{(1-\alpha)(1-\gamma_1)+\alpha\widetilde{\psi}_A\gamma_3} \geq \frac{(1-\gamma_1)\eta L^1}{(H_t^A)^{1-\alpha}(K_t^A)^{\alpha}}. \tag{4.16}$$

If it is fulfilled, then every level of $e_t \in (0,1]$ might be optimal, depending on the formation of human capital, given by $\phi'_{t+1}(e_t)$. If it is not fulfilled, then there exists a $\overline{e}_t$, such that no value for $e_t \in [\overline{e}_t, 1]$ is optimal, as otherwise, income is reduced below its initial level. This holds independently of $\phi_{t+1}(e_t)$. Whether this condition is fulfilled depends on the relative importance of the adult labor force and the size of capital inflows, and thus on the improvement of institutions. A country that improves its institutions dramatically will attract a sizable amount of capital and thus be able to choose from a variety of education levels. For a country that improves its institutions marginally, we observe relatively low levels of education being implemented, if any. We summarize our findings in the following proposition:

Proposition 18 *Under the types of education function that allow for a zero growth steady state, i.e. those with constant, decreasing or sufficiently slowly increasing returns to schooling, the country that experiences capital inflows will have a larger income than before. This even holds when agents in that country school their children.*
If Expression (4.16) *is fulfilled the upper bound for the optimal level of* e_t *is 1.*

Next period
Now, we will study the impact of institutional change, capital flows and schooling on the two countries in the next period. Above, we established an equilibrium in which capital flows from B to A. In this equilibrium, B does not invest in education while country A does, allocating the share e_t of its children's time to the formation of human capital. Hence, in $t+1$, country A is endowed with a larger stock of total human capital, given by $\phi_{t+1}L^{2,A} + \eta L^{1,A}$. Also, as shown above, income in A is larger than before. This implies

$$s_t^A > s_{t-1}^A \Rightarrow K_{t+1}^A > K_t^A \quad \text{and} \quad s_t^B < s_{t-1}^B \Rightarrow K_{t+1}^B < K_t^B,$$

as savings are a constant share of income, due to logarithmic utility. Given smaller savings and physical capital, but a constant amount of human capital, the rental rate of capital increases in B. In A, we observe an increase in both types of capital, having opposite effects on the interest rate. Hence, it is not clear whether R_{t+1}^A is smaller, equal or larger than R^B_{t+1}.

Now the following holds:

$$\begin{aligned}
R^A_{t+1} =& \alpha A(\phi_{t+1}L^2)^{1-\alpha}(K^A_{t+1})^{\alpha-1} \quad \text{and} \\
R^B_{t+1} =& \alpha A(\phi_t L^2)^{1-\alpha}(K^B_{t+1})^{\alpha-1}, \quad \text{with} \\
K^A_{t+1} =& \frac{\beta A}{1+\beta}\left(H^A_t\right)^{1-\alpha}\left(K^A_t\right)^{\alpha} 2^{\alpha}(1+\chi^{\frac{-1}{1-\alpha}})^{-\alpha} \quad \cdot \\
& \left((1-\alpha)(1-\gamma_1)+\frac{\alpha}{2}\widetilde{\psi}_A\gamma_3(1+\chi^{\frac{-1}{1-\alpha}})\right), \quad \text{and} \\
K^B_{t+1} =& \frac{\beta A}{1+\beta}(H^B_t)^{1-\alpha}(K^B_t)^{\alpha}2^{\alpha}(1+\chi^{\frac{1}{1-\alpha}})^{-\alpha} \quad \cdot \\
& \left((1-\alpha)(1-\gamma_1)+\frac{\alpha}{2}\psi_B\gamma_3\left(1+\chi^{\frac{1}{1-\alpha}}\right)\right).
\end{aligned}$$

Hence, $\widetilde{\psi}_A R^A_{t+1}$ is larger than $\psi_B R^B_{t+1}$ if

$$\chi\left(\frac{H^A_{t+1}}{K^A_{t+1}}\right)^{1-\alpha}\left(\frac{K^B_{t+1}}{H^B_{t+1}}\right)^{1-\alpha} > 1.$$

We substitute K^A_{t+1} and K^B_{t+1} to obtain:

$$\begin{aligned}
&\left(\chi\frac{\gamma_B}{\gamma_A}\right)^{1-\alpha}\left(\frac{H^A_{t+1}}{(H^A_t)^{1-\alpha}(K^A_t)^{\alpha}}\right)^{1-\alpha}\left(\frac{H^B_{t+1}}{(H^B_t)^{1-\alpha}(K^B_t)^{\alpha}}\right)^{\alpha-1} > 1, \qquad (4.17) \\
&\gamma_A = (1-\alpha)(1-\gamma_1)+\frac{\alpha}{2}\widetilde{\psi}_A\gamma_3\left(1+\chi^{\frac{-1}{1-\alpha}}\right), \quad \text{and} \\
&\gamma_B = (1-\alpha)(1-\gamma_1)+\frac{\alpha}{2}\psi_B\gamma_3\left(1+\chi^{\frac{1}{1-\alpha}}\right),
\end{aligned}$$

where γ_B and γ_A can be interpreted as the sum of income shares. Labor income of adults has the same share in both countries, $1-\alpha$, while capital has different shares in A and B. So whether the net interest rate in country A is also larger in period $t+1$ depends on the ratio of institutions, as is shown by χ and by the ratio of γ_B and γ_A. Human capital in $t+1$ in country A affects the interest rate positively, while the opposite holds for human capital in t, as it leads to more income in t. The same argument holds for the stock of physical capital in that period.

We can rewrite the above expression, using that $K_t^A = K_t^B$ and $H_{t+1}^B = H_t^B$, where we assumed the former and have shown that there is no investment in education in B due to decreasing income. This yields

$$\left(\chi\frac{\gamma_B}{\gamma_A}\right)^{1-\alpha}\left(\frac{H_{t+1}^A}{H_t^A}\right)^{1-\alpha}\left(\frac{H_t^B}{H_t^A}\right)^{\alpha(\alpha-1)} > 1. \tag{4.18}$$

First, we study the term $\chi\gamma_B/\gamma_A$. We know that $\chi > 1$ and $\gamma_B > \gamma_A$, so that the product is larger than one. To see this, note that the following holds:

$$\frac{\widetilde{\psi}_A}{\psi_B} < \chi^{\frac{1}{1-\alpha}} = \left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}} \quad \text{for all} \quad \widetilde{\psi}_A > \psi_B.$$

Next, we turn to the factor in the middle, H_{t+1}^A/H_t^A. This ratio of human capital of adults in $t+1$ to the amount of human capital of adults in t is larger than one, as the human capital stock of tomorrow will be larger than the one of today. The exact size of this term depends on the productivity of schooling. Hence, the ratio reflects the way in which education in country A in period t can ensure capital inflows in period $t+1$: Education increases H_{t+1}^A, allowing for a larger R_{t+1}^A. The last factor depends negatively on H_t^B so that a larger stock of human capital in B reduces the chances of outflow of capital in $t+1$.

In $t+1$, the capital market clears if the following condition is met:

$$\widetilde{\psi}_A\left(\frac{H_{t+1}^A}{\widetilde{K}_{t+1}^A}\right)^{1-\alpha} = \psi_B\left(\frac{H_{t+1}^B}{\widetilde{K}_{t+1}^B}\right)^{1-\alpha} \quad \Rightarrow \quad \widetilde{K}_{t+1}^A = \chi^{\frac{1}{1-\alpha}}\left(\frac{H_{t+1}^A}{H_t^B}\right)\widetilde{K}_{t+1}^B.$$

The market clearing condition has changed, now including the ratio of two different levels of human capital. We denote the ratio H_t^A/H_t^B as H_t from now on. As country A has invested in education, we observe $H^A_{t+1} > H^B_t$. Country B has not invested, remaining at the level of human capital H^B_t. While the difference in institutions is still relevant, as shown by χ, the new ratio of human capital $H_{t+1} = \frac{H^A_{t+1}}{H^B_t}$ increases the difference between capital stocks in the two countries.

To solve for the post-flow capital stocks $\widetilde{K}^A_{t+1}$ and $\widetilde{K}^B_{t+1}$, we note two things. First, from the savings decision above, we know that

$$K^A_{t+1} = K^B_{t+1} \frac{\gamma_A}{\gamma_B} \chi^{\frac{\alpha}{1-\alpha}} H_t^{1-\alpha} \quad \text{wih} \quad H_t = \frac{H^A_t}{H^B_t}. \tag{4.19}$$

Second, the international sum of physical capital after flows must equal the sum of physical capital after flows, implying

$$\widetilde{K}^B_{t+1} = K^A_{t+1} + K^B_{t+1} - \widetilde{K}^A_{t+1}.$$

We substitute the market clearing condition and (4.19) to obtain

$$\widetilde{K}^B_{t+1} = \left(1 + \frac{\gamma_A}{\gamma_B} \chi^{\frac{\alpha}{1-\alpha}}\right) \left(1 + \chi^{\frac{1}{1-\alpha}} H_{t+1}\right)^{-1} K^B_{t+1} \quad \text{and}$$
$$\widetilde{K}^A_{t+1} = \left(1 + \frac{\gamma_B}{\gamma_A} \chi^{\frac{-\alpha}{1-\alpha}}\right) \left(1 + \chi^{\frac{-1}{1-\alpha}} H^{-1}_{t+1}\right)^{-1} K^A_{t+1},$$

where we use that $H_t = 1$. In contrast to the equation that related $\widetilde{K}^B$ and K^B_t, the number 2 is replaced by the left factor. This factor is smaller than 2 and thus shows that, in period $t+1$, country B has a smaller share of the global capital stock than in t. This is due to $\gamma_A/\gamma_B < 1$ and the fact that $H_{t+1} > 1$. While this implies that $\widetilde{K}^B_{t+1} < K^B_{t+1}$, it is not clear whether $\widetilde{K}^B_{t+1}/K^B_{t+1}$ is larger or smaller than $\widetilde{K}^B_t/K^B_t$. Let us also consider the education decision for period $t+1$. In equilibrium, the optimal amount of schooling e^*_{t+1} must fulfill,

$$\beta\phi'_{t+2}(e^*_{t+1}) = \frac{(1+\beta)\eta L^1}{A(\phi_{t+1}L^2)^{1-\alpha}(K^A_{t+1})^\alpha\Gamma^A_{t+1} + (1+\beta)(1-\gamma_1)(1-e^*_{t+1})\eta L^1}, \tag{4.20}$$

with

$$\Gamma^A_{t+1} = (1-\gamma_1)\left(1+\frac{\gamma_B}{\gamma_A}\chi^{\frac{-\alpha}{1-\alpha}}\right)^\alpha \left(1+\chi^{\frac{1}{1-\alpha}}H_{t+1}\right)^{-\alpha}\chi^{\frac{\alpha}{1-\alpha}}(H^A_{t+1})^\alpha \cdot \left[1-\alpha+\alpha\widetilde{\psi}^A_t\gamma\left(1+\frac{\gamma_B}{\gamma_A}\chi^{\frac{-\alpha}{1-\alpha}}\right)^{-1}\left(1+\chi^{\frac{1}{1-\alpha}}H^A_{t+1}\right)\chi^{\frac{-1}{1-\alpha}}(H_{t+1})^{-1}\right]. \tag{4.21}$$

We know that the total endowment with both types of capital in A has increased, i.e. $\phi_t < \phi_{t+1}$ and $K^A_t < K^A_{t+1}$.

Next, we turn to Γ^A_{t+1} which is structurally similar to Γ^A_t. However, the factor 2 is replaced by $1+\frac{\gamma_B}{\gamma_A}\chi^{\frac{-\alpha}{1-\alpha}}$, which is smaller than 2, and the term $\chi^{\frac{1}{1-\alpha}}$ is replaced by $\chi^{\frac{1}{1-\alpha}}H_{t+1}$, with $H_{t+1} > 1$. This makes the derivative $\partial\Gamma^A_{t+1}/\partial\chi$ not-trivial, as the terms γ_B, γ_A, and H_{t+1} depend on χ. We cannot say whether income in A, in period $t+1$ before children are educated, is actually larger than the one in period t. However, a condition, allowing for such a case, can be found in the following way: First, we write income without the earnings of children in its general form,

$$I^A_{t+1} = (1-\alpha)(1-\gamma_1)A(H^A_{t+1})^{1-\alpha}(\widetilde{K}^A_{t+1})^\alpha + \alpha\widetilde{\psi}_A\gamma_3 A(H^A_{t+1})^{1-\alpha}(\widetilde{K}^A_{t+1})^{\alpha-1}K^A_{t+1}.$$

This expression holds for $t+1$ and t. It is clear that I_{t+1} increases in H^A_{t+1} and K^A_t. Forming the derivative with respect to $\widetilde{K}^A_{t+1}$, we find

$$\frac{\partial I^A_{t+1}}{\partial\widetilde{K}^A_{t+1}} = \alpha A\left(\widetilde{K}^A_{t+1}\right)^{\alpha-1}\left(H^A_{t+1}\right)^{1-\alpha}\left[(1-\alpha)(1-\gamma_1)+(\alpha-1)\widetilde{\psi}_A\gamma_3\frac{K^A_{t+1}}{\widetilde{K}^A_{t+1}}\right] > 0,$$

as $\widetilde{\psi}_A < 1$, $K_t^A/\widetilde{K}_t^A < 1$ due to capital inflows and that γ_3 is unlikely too different from $1-\gamma_1$. We know that $H_t^A < H_{t+1}^A$ and that $K_t^A < K_{t+1}^A$. However, it is not clear whether $\widetilde{K}_t^A < \widetilde{K}_{t+1}^A$. $\widetilde{K}_{t+1}^A$ can be written as a share of the total supply of physical capital K_{t+1}

$$\widetilde{K}_{t+1}^A = \left(1 + \frac{1}{\chi^{\frac{1}{1-\alpha}} H_{t+1}}\right)^{-1} K_{t+1}. \tag{4.22}$$

Country A obtains a larger share of total world capital. Yet, we have not established whether the global total stock of physical capital has increased, i.e. whether $K_t < K_{t+1}$. The opposite is also possible since in t, income increases in A, but decreases in B. We examine whether A's income increase compensates B's income decline. For this, we turn to Γ_t^B, given by

$$\Gamma_t^B = 2^\alpha (1 + \chi^{\frac{1}{1-\alpha}})^{-\alpha} \left((1-\alpha)(1-\gamma_1) + \frac{\alpha}{2}\psi_B\gamma_3(1+\chi^{\frac{1}{1-\alpha}})\right),$$

which, as we know, decreases in χ. The marginal change is

$$2^\alpha \alpha (1 + \chi^{\frac{-1}{1-\alpha}})^{-\alpha} \left[-\frac{1-\gamma_1}{1+\chi^{\frac{1}{1-\alpha}}} + \frac{\psi_B\gamma_3}{2} \right].$$

Next, we study the derivative of Γ_t^A. It reads

$$2^\alpha \alpha \left(1 + \chi^{\frac{-1}{1-\alpha}}\right)^{-\alpha} \chi^{-1} \left(\frac{1-\gamma_1}{1+\chi^{\frac{1}{1-\alpha}}} + \frac{\widetilde{\psi}_A\gamma_3}{2} \right).$$

It is larger than $\partial\Gamma_t^B/\partial\chi$ in absolute terms if the following expression holds[10]:

$$\frac{\gamma_3}{1-\gamma_1}\widetilde{\psi}_A > \frac{\chi-1}{1+\chi^{\frac{1}{1-\alpha}}}. \tag{4.23}$$

This inequality provides a lower bar for the value of $\widetilde{\psi}_A$. However, it is easily fulfilled if we consider small changes in institutions, so that χ is relatively close to 1.

With this, country A is richer after capital flows in $t+1$ than after flows in t, leading to a higher optimal level of education e_{t+1}^*. Hence, the stock of human capital will increase more strongly from $t+1$ to $t+2$ than it did from t to $t+1$. We summarize in the following proposition:

Proposition 19 *If Inequality* (4.23) *holds, then world income increases in* t. *This is a sufficient condition for country A to increase its level of education in* $t+1$.

More general discussion

So far, we have seen that in period t and $t+1$, two relevant things occur. First, capital flows from country B to A and second, A invests in education, and does it in $t+1$ even more than in t. We study now, whether this can also happen in the following periods, i.e. whether there is a path where two conditions are met in each period. First, country A experiences capital inflows and second, it increases its level of education e_t, so that it converges to $e_T^* = 1$, in some period T. The first condition can only be met if

$$\chi^{\frac{1}{1-\alpha}}\left(\frac{H_{t+2}^A}{H_{t+2}^B}\right)\frac{\Gamma_{t+1}^B\left(H_t^B\right)^{1-\alpha}\left(K_{t+1}^B\right)^{\alpha}}{\Gamma_{t+1}^A\left(H_{t+1}^A\right)^{1-\alpha}\left(K_{t+1}^A\right)^{\alpha}} > 1 \tag{4.24}$$

[10]To be precise, this is the relevant condition if $\frac{1-\gamma_1}{1+\chi^{\frac{1}{1-\alpha}}} > \frac{\psi_B\gamma_3}{2}$. In the opposite case, $\left|\frac{\partial\Gamma_t^A}{\partial\chi}\right| > \left|\frac{\partial\Gamma_t^B}{\partial\chi}\right|$ is always fulfilled, regardless of parameter values.

is fulfilled in $t+2$ and in every following period. For this condition to hold, it is important for country A to accumulate human capital sufficiently quickly, as K_{t+1}^A is increasing due to country A's larger wealth and K_{t+1}^B is decreasing. Also the factors Γ_{t+1}^A and Γ_{t+1}^B diverge, equalizing interest rates across the two countries.

A sufficient condition for country A to increase its education level is $1 < \Gamma_t^A < \Gamma_{t+1}^A$. It can be generalized for the following periods. If $\Gamma_{t+1}^A < \Gamma_{t+2}^A$ and so on, then $e_{t+1}^* < e_{t+2}^*$ will hold, as can be seen from (4.20). The factor Γ_{t+1}^A increases over time if $\widetilde{K}_{t+1}^A$ increases, which in turn, becomes larger over time if income rises more quickly in A than income decreases in B, in a given period. This is expressed by the following condition:

$$\begin{aligned}&(H_{t+1}^A)^{1-\alpha}(K_{t+1}^A)^\alpha\Gamma_{t+1}^A - (H_{t+1}^A)^{1-\alpha}(K_{t+1}^A)^\alpha\Gamma_{t+1}^{A,0} \geq \\ &\left|(H_{t+1}^B)^{1-\alpha}(K_{t+1}^B)^\alpha\Gamma_{t+1}^{B,0} - (H_{t+1}^B)^{1-\alpha}(K_{t+1}^B)^\alpha\Gamma_{t+1}^B\right|,\end{aligned} \tag{4.25}$$

with $\Gamma_{t+1}^{I,0} = (1-\alpha)(1-\gamma_1) + \alpha\gamma_3\psi_I$. Slightly rewritten, we have

$$\frac{Y_{t+1}^{2,A}}{Y_{t+1}^{2,B}}\left(\Gamma_{t+1}^A - \Gamma_{t+1}^{A,0}\right) > \left|\Gamma_{t+1}^{B,0} - \Gamma_{t+1}^B\right|, \tag{4.26}$$

which is a generalization of (4.23). $Y_{t+1}^{2,A}$ is country A's potential output before capital flows and education. We see that it is beneficial for A if country B is relatively poor, i.e. $Y_{t+1}^{2,B}$ is rather small.

We summarize our findings in the following proposition:

Proposition 20 *If returns to education are non-increasing or increase sufficiently slowly and in every period and* (4.24) *and* (4.26) *are fulfilled, there exists a path for country A along which it experiences inflows of capital in every period and increases its level of education.*

What happens, however, if (4.24) does not hold? To answer this question, let us consider the following. From period $t = 0$ until some period $\widetilde{T}$, capital has been flowing from B to A, and A has been investing in education, without reaching the state of full education, so that $e_{\widetilde{T}-1} < 1$. Let us first assume, that in $\widetilde{T}$, the return on capital is equal in both countries, so that there are no capital flows in this period. If capital inflows matter for country A, it might be the case that $\widetilde{K}^A_{\widetilde{T}-1} > \widetilde{K}^A_{\widetilde{T}}$, i.e. country A had more capital available for production in the previous period than in the current one, due to previous capital inflows. This might reduce the incentive to invest in education in period $\widetilde{T}$ in comparison to $\widetilde{T}-1$. However, we can exclude the possibility that $e_{\widetilde{T}}$ drops back to zero, as country A has been accumulating human capital over the previous periods, which encourages investment in education. Also, human capital is larger in the current period than in $\widetilde{T}-1$, hindering any conclusion about the relative size of $e_{\widetilde{T}}$ compared to $e_{\widetilde{T}-1}$. If capital inflows are negligible in comparison to country A's capital stock, such considerations do not matter. As before, country A will follow its path, increasing its stock of human capital. The opposite will hold in B. The country does not invest in education, as it is poorer in $\widetilde{T}$ than in the initial period t, where it already did not invest in education. Taken together, these effects enable (4.24) to hold as an inequality.

Now assume, that the return in B is actually larger than in A and that capital flows from A to B. It is more likely now that A invests less in education than in the previous period with $e_{\widetilde{T}} < e_{\widetilde{T}-1}$, but it is quite unlikely that A will completely stop investing, as the accumulation of human capital and physical capital has been increasing wealth. If inflows in B are large, providing B with more capital than it initially had, so that $K^B_t < \widetilde{K}^B_{\widetilde{T}}$, B might begin to invest in education. Even if it does, it will most likely invest less than country A, so that interest rates will equalize in the following period. However, by investing in education, B might also converge to the full education steady state.

Steady state

Let us turn to the existence of a balanced growth path. While the economy can exhibit different steady states, we concentrate on the steady state in which country A grows at the largest possible positive rate while country B does not. We define such a state as follows:

Definition 4 *Along a balanced growth path, human capital and physical capital grow at the same rate. In country, A this rate is positive and as large as possible. In B, it is zero. In the long run, the stocks of human capital and physical capital become so large that agents in A neglect the effect of capital inflows from B and the stock of children's human capital in their decisions.*

With this definition, we obtain for country A

$$\frac{K_{t+1}^A}{H_{t+1}^A} = k_{t+1}^A = \frac{\beta A((1-\alpha)(1-\gamma_1) + \alpha\gamma_3\widetilde{\psi}_A)}{(1+\beta)\Phi}(k_t^A)^\alpha \quad \text{with} \quad \Phi = \frac{\phi_{t+1}(1)}{\phi_t(1)} \quad \forall t.$$

The constant ratio of the two forms of capital is $k^A = [\beta A((1-\alpha)(1-\gamma_1)+\alpha\gamma_3)/(1+\beta)\Phi]^{1/(1-\alpha)}$. For the derivation of this equation, see the Appendix for Chapter 4. Additionally, we assume that the steady state defined above was reached along a transition path, where country B never invested in education. This might be due to country B's rental rate always being smaller than A's, i.e. Inequality (4.24) always held along the convergence, or country B experienced capital inflows which did not stimulate income sufficiently to ensure investment in education. Regardless of the transition path, we show that there exists a steady state where country A invests in education and country B does not.

Such a steady state cannot be one without capital flows, as country B has been exporting capital and has seen a continuous reduction in income along the assumed path. Hence, in some arbitrary period T, where A is already in its steady state, B faces a capital stock that is smaller than its initial endowment. With K_T^B being smaller than K_t^B and with a policy function for K_t^B that is concave, as can be seen from the savings decision, country B will accumulate capital in the absence of international capital flows. This is not consistent with a steady state, and it is not clear how this accumulation would affect the overall dynamics either.

Instead, we demonstrate the existences of a steady state with the following features. At the beginning of every period, the market clearing condition is not fulfilled, so that a time-independent amount of capital flows from B to A, clearing the arbitrage condition. In every period, country B possesses the same amount of capital K^B, of which a constant fraction flows out. Furthermore, country B's stock of capital used in production is also constant, i.e. $\widetilde{K}^B$ is fixed. Denoting the steady state ratio of capital in A by k^A, the rental rate in A is given by $A(k^A)^{\alpha-1}$, which we write as R^A. Hence, K^B and $\widetilde{K}^B$ must fulfill

$$\widetilde{\psi}_A R^A > \psi_B \left(\frac{H^B}{K^B}\right)^{1-\alpha}, \quad \widetilde{\psi}_A R^A = \psi_B \left(\frac{H^B}{\widetilde{K}^B}\right)^{1-\alpha}, \quad \text{and}$$
$$K^B = \frac{\beta A}{1+\beta}\left[(1-\alpha)(1-\gamma_1)\left(H^B\right)^{1-\alpha}\left(\widetilde{K}^B\right)^{\alpha} + \alpha\gamma_3\widetilde{\psi}_A R^A K^B\right].$$

We solve for

$$\widetilde{K}^B = H^B\left(\chi R^A\right)^{\frac{-1}{1-\alpha}}$$

and substitute this expression to obtain

$$K^B = \frac{\beta A}{1+\beta}\left[(1-\alpha)(1-\gamma_1)H^B\left(\chi R^A\right)^{\frac{-\alpha}{1-\alpha}}\right]\left(1 - \frac{\alpha\beta\gamma_3\widetilde{\psi}_A A}{1+\beta}R^A\right)^{-1}. \quad (4.27)$$

The term in the second factor on the right hand side is very likely to be positive, unless R^A is very large, for which we do not see any reason. Next, we verify whether K^B is such that the capital market clearing condition is violated at the beginning of each period, leading to

$$\frac{1}{\alpha\gamma_3\widetilde{\psi}_A} > R^A > \frac{1}{\left[(1-\alpha)(1-\gamma_1)\frac{\widetilde{\psi}_A}{\psi_B} + \alpha\gamma_3\widetilde{\psi}_A\right]}. \tag{4.28}$$

We find a lower and an upper bound on the long-term interest rate R^A, where the upper bound comes from the condition $K^B > 0$ and the lower bound can be found by plugging (4.27) into $\widetilde{\psi}_A R^A > (H^B)^{1-\alpha}(K^B)^{\alpha-1}$. The lower bound only differs with respect to the term $(1-\alpha)(1-\gamma_1)\frac{\widetilde{\psi}_A}{\psi_B}$ from the upper bound, so that the range of admissible values is not huge. We summarize in the following proposition:

Proposition 21 *A steady state as described in Definition* 4 *exists if the long-term interest rate in country A, R^A, lies between the boundaries given by* (4.28)*.*

With two countries, it is not clear whether the two arrive at their respective steady state simultaneously or whether, for instance, A reaches k^A before country B reaches K^B. One can imagine that at some T, A is in its respective steady state while B is not, i.e. $K_T^B > K^B$. Then, we know from above that capital outflows will reduce overall income in B, leading to smaller savings, so that eventually $K_t^B = K^B$. The opposite, namely that B reaches its steady state before A, is not possible, however. The reason is that the capital stock in B depends on the return in A, R^A. If R^A, and hence the effective capital stock in A, k_t^A, changes, this will have an impact on the capital stock in B. Let $K_T^B = K^B$ for some T, but $k_T^A \neq k^A$ and $R_T^A \neq R^A$. The return in A will change from period T to $T+1$, thus moving K_T^B away from K^B.

4.6 Simulation

To study whether we can actually obtain two diverging countries, we provide a simulation exercise. Our choice of parameters can be found in Table 4.1. We set α and β to the standard values of 1/3 and 0.85. The institutional parameters ψ_B and $\widetilde{\psi}_A$ take values of 0.85 and 0.95, to study a change in institutions that is consequential. ηL^1 is set smaller than $\phi_0 L^2$, reflecting that adults can supply more human capital to markets. Similar to Bell et al. (2019), we set $\gamma_1 = 0.4$ and $\gamma_3 = 0.25$. Finally, the initial level of human capital per adult ϕ_0 is set to 8. The reason is that this value allows for a an initial zero-growth steady state, where both countries find it optimal not to invest in education at all. Also, with this level of human capital, a relatively small change in the level of institution allows for divergence of the two countries.

Parameter	Value	Variable
$1-\alpha$	2/3	Factor share of human capital of adults
A	0.76	TFP-parameter
β	0.85	Discount factor
γ_1	0.4	Share of wage income going to children
γ_3	0.25	Share of capital income going to adults
δ	0.05	Productivity of human capital formation
ϕ_0	8	Initial stock of human capital per adult
ηL^1	0.56	Human capital per child
ψ_B	0.85	Initial level of institutions
$\widetilde{\psi}_A$	0.95	New level of institutions
L^2	1	Labor supplied by adults

Table 4.1: Parameter values.

We use the following production function for the formation of human capital:

$$\phi_{t+1} = (1 + \delta \log(1 + e_t))\phi_t. \tag{4.29}$$

Hence, we have decreasing returns to e_t, but linearity in ϕ_t, allowing for a balanced growth path with $e = 1$. We set δ to 0.05, which leads to a growth rate of 3.47% along the path.

The left upper panel of Figure 4.5 contains the main output of the simulation for the model. We compare the evolution of physical capital for the two countries. The full line shows the evolution for country A and the dotted line for B. We see that country A and country B diverge indeed. Both begin in the zero growth steady state, with K_0 given by (4.12). Then we observe for country A a concave convergence over the first few periods, before its stock of physical capital begins to grow exponentially. For country B, we find that the capital stock slightly declines at first and remains almost constant afterwards.

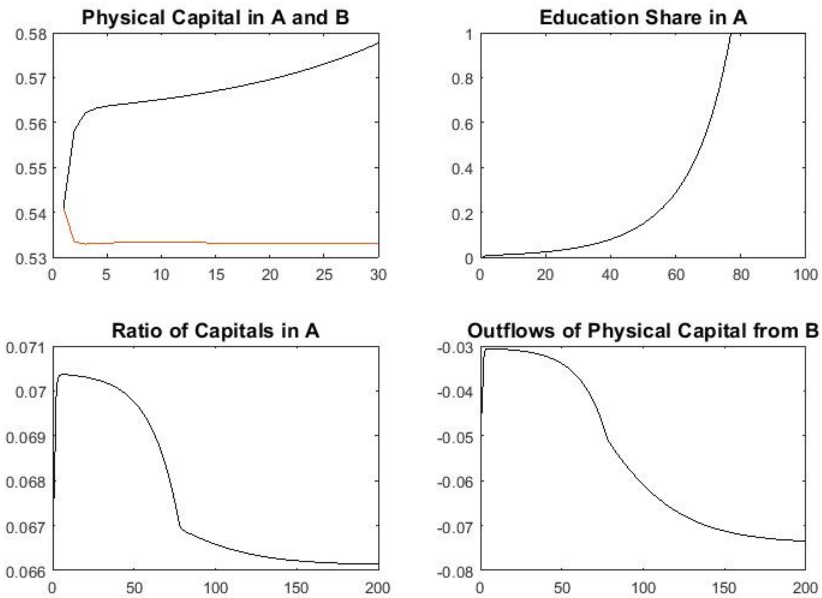

Figure 4.5: Simulation for the case that country A improves its economic institutions.

Country A's increase of physical capital is accompanied by a rising education share, as can be seen in the right upper panel of Figure 4.5. The share of time that children spend in school, e_t^A, is exponentially increasing until it reaches 1, shortly after the 50th period. Thereafter, it remains constant at that value. Investment in education in country B remains at $e^B = 0$, and is not shown.

The dynamics of the ratio of capitals in A, K_t^A/ϕ_t^A can be seen in the left lower panel of Figure 4.5. It reflects our previous findings. At first, the ratio of capital increases, due to the strong accumulation of physical capital, which we observed in the left upper panel. Then, country A accumulates human capital faster than physical capital, decreasing the ratio. Finally, after country A has reached the point in time, where $e_t^A = 1$, we see a convergence to the new steady state.

A theoretical prediction that we made was that country B converges to a steady state where it exports the same amount of physical capital in every period. In the right lower panel of Figure 4.5 we see that this indeed is the case. While flows of physical capital to country B mirror the dynamics of K_t^A/ϕ_t^A, they are always negative, indicating that B experiences outflows. The outflows decrease at first, due to A's fast accumulation of physical capital, which reduces A's return. However, outflows increase as A invests more in education. Furthermore, when country A reaches the state of full education with $e_t^A = 1$, outflows continue to increase but at a significantly lower rate, and start to converge. The convergence is quite slow, going on for over 200 periods.

4.7 Conclusion

Institutions, human capital and flows of physical capital all seem to have an impact on long-term growth. In this study, we provide a unified framework, where these forces interact, and that might explain increasing inequality between countries. We propose the idea that better economic institutions attract international capital, more physical capital increases households' income and thus allows for the formation of human capital. Human capital, then, drives growth. We analytically provide conditions under which a country can undergo these steps towards long-term growth and show that in our model endogenous separation between countries, due to an initial difference in institutions, can occur. The country that was able to improve its institutions benefits from inflows of capital and converges to a path with a positive growth rate, while the other country remains at a zero growth steady state.

There are different avenues that further research can take. First, we assume simple sharing rules for family income and do not incorporate a pension system. Implementing the latter might yield interesting insights. Second, one can extend the model to incorporate more countries and a multitude of institutional parameters to see which institutional changes are especially vital in enabling growth.

Chapter 5

Artificial intelligence and growth

Abstract

In the previous chapters, we analyzed several phenomena related to economic growth.[1] In this chapter, we turn to a trend that emerged very recently, but might have a drastic impact on growth and other macroeconomic variables—artificial intelligence (AI). By surveying a selected number of studies, we demonstrate the authors' mixed opinions about AI, and how AI can be modeled in the context of growth models. To do this, authors rely on one of the following polar approaches: On the hand, AI is considered as supporting human decision-making, thus increasing the productivity of human labor. On the other hand, AI replaces humans in the production process, since it is cheaper than conventional labor. After demonstrating these ideas, we provide a small outlook for possible future models of AI.[2]

5.1 A rather optimistic perspective

The study by Agrawal et al. (2018) discusses the unique features of modern artificial intelligence, and examines how AI may help overcome the obstacles to growth

[1]This chapter is single-authored.

[2]There is an increasing literature on the subject of AI in Economics. For a comprehensive, non-theoretical survey see Abrardi et al. (2019). In contrast, Goldfarb et al. (2019) contains a series of theoretical articles about the perspective of different economic fields on AI.

at the technological frontier. The authors begin by describing the following problems of growth: First, empirical observations show that, at the technological frontier, growth is indeed slowing down. This holds despite exponentially-increasing numbers of researchers.[3] Second, across different fields, researchers themselves experience a decline in their total factor productivity. These two observations reflect well-known phenomena in the field of knowledge creation: the "fishing-out"- and the "burden-of-knowledge"-problem. The former states that good ideas are becoming more and more difficult to find. The latter reveals that in order to create new knowledge, researchers must already have an increasing stock of knowledge.

In the face of these problems, the authors take a somewhat optimistic stance: "[O]ne reason to be hopeful about the future is the recent explosion in data availability under the rubric of 'big data' and computer-based advances in capabilities to discover and process those data." In particular, they argue that AI is becoming increasingly helpful in "needle-in-a-haystack"-problems, i.e. situations where knowledge is created by combining existing knowledge parts in a suitable way. Their idea is that AI can impact growth by simplifying the research process, and increasing the productivity of knowledge creation.

They propose two ways in which AI can support researchers. First, AI can help in the process of *search*. It can find the relevant knowledge for a specific project, easing the burden of knowledge. Examples are *Meta*$^{\alpha}$ and *BenchSci*. Second, AI can help in the process of *prediction*, by finding combinations of knowledge that will yield useful new concepts. This may be a crucial feature of AI, as with more existing knowledge, there is an combinatorial explosion of the ways that knowledge can be put together. Examples for existing AI are *Atomwise* and *DeepGenomics*.

To model these ideas, the authors construct a growth model with a combinatorial

[3]Gordon (2012) is a seminal contribution that illustrates faltering growth.

based knowledge production function. While their model is similar to the model of Jones (1995) and Romer (1990), the production of knowledge differs crucially. In Jones (1995) and Romer (1990), the research process consists of a *single step*, in which scientist use existing knowledge to create new knowledge. In Agrawal et al. (2018), however, research consists of *two steps*. First, scientists create all possible combinations of existing knowledge. Second, all possible combinations are filtered for those that are actually new and useful.

The details of Agrawal et al. (2018)'s model are as follows: Every research possesses the amount of knowledge A^{ϕ}, where A is the global stock of knowledge, and ϕ lies between 0 and 1. This parameter reflects the burden of knowledge, as $A^{\phi-1}$ is falling in A, and thus, the individual's share of global knowledge is decreasing over time. However, AI can increase ϕ, by assisting scientists in the search process. Furthermore, the knowledge of an individual researcher can be combined. This yields the following number of combinations:

$$Z_i = \sum_{a=0}^{A^{\phi}} \binom{A^{\phi}}{a} = 2^{A^{\phi}},$$

where Z_i grows exponentially in A^{ϕ}. As mentioned above, the combinations of knowledge are then transformed into new knowledge, according to the following function:

$$\dot{A}_i = \beta \left(\frac{Z_i^{\theta} - 1}{\theta} \right) = \beta \left(\frac{\left(2^{A^{\phi}}\right)^{\theta} - 1}{\theta} \right) \quad \text{for} \quad 0 < \theta \leq 1.$$

The parameter β is a positive knowledge discovery parameter and not of significant importance for the model—in contrast to θ. It stands for the elasticity of discovery and thus, reflects the fishing-out problem. θ can be increased by AI, similar to ϕ.

Unlike before, the relevant form of AI is the one helping with discovery and not with search. Hence, the creation of knowledge is aggregated in the following way:

$$\dot{A} = \beta L_A^{\lambda} \left(\frac{\left(2^{A^{\phi}}\right)^{\theta} - 1}{\theta} \right),$$

where L_A is the total amount of researchers in the economy and $0 \leq \lambda \leq 1$. The parameter λ implies that some researchers produce the same knowledge. This is called the "standing-on-toes-effect".

Above, we established that AI can increase θ. However, the authors argue that such an increase can only be temporary, and that θ must depend negatively on A. This feature reflects that even improved AI is overwhelmed by the number of possible combinations, which arise from an increasing knowledge stock. Thus, the authors assume the following:

$$\lim_{A \to \infty} \theta(A) = 0,$$

so that in the limit, the knowledge production function becomes

$$\dot{A} = \beta \ln(2) L_A^{\lambda} A^{\phi},$$

which is a Romer/Jones-type function.

To study the steady state, Agrawal et al. (2018) use that the fact that in the limit, their model is the same as in Jones (1995), and obtain the following growth rates for knowledge, per capita output, per capita consumption, and capital per capita:

$$g_A = g_y = g_c = g_K = \frac{\lambda n}{1 - \phi},$$

where n is the growth rate of the population. The long-term growth rate g_A not only depends on n, but also on the burden of knowledge parameter ϕ, and the returns to scales for researchers λ. AI affects the steady state by increasing ϕ, and thus, only through lessening the burden of knowledge. However, it also has an impact on the trajectory to the steady state via the parameter θ.

5.2 A rather pessimistic perspective

The study discussed above explicitly models the opportunities and risks provided by AI. However, there is a strand of literature that identifies AI as merely a new form of automation.[4] These studies are Hémous and Olsen (2014), Aghion et al. (2017), Acemoglu and Restrepo (2018a), and Acemoglu and Restrepo (2018b). They are motivated by two empirical observations. First, machines are replacing humans in an increasing numbers of tasks. Second, the labor share has been declining since the beginning of the century. The claim of these papers— or at least the implicit assumption— is that artificial intelligence will continue the process of automation of tasks, or that, even worse, AI might *replace* humans in tasks in which where they were considered to be irreplaceable.

Acemoglu and Restrepo (2018a)

Acemoglu and Restrepo (2018a) explain why it is necessary to model automation differently from a factor-augmenting technology. Using a production function with constant returns to scale under competitive markets, they show that capital-augmenting technology increases the labor share and does not decrease the demand

[4]In this context, we should mention the pioneering work by Zeira (1998). It laid the foundation for the studies presented in this section. However, we focus on more recent studies, as they have a richer macroeconomic setting, and seem to be related to AI more closely.

for labor, or its wage. Also, they demonstrate that labor-augmenting technology decreases the labor share for realistic parameter values, but increases the demand for labor and its wage. The production function they use is

$$Y = F(A_K K, A_L L),$$

where Y is aggregate output and A_K, K, A_L and L are the level of capital-augmenting technology, the capital stock, the level of labor-augmenting technology and labor, respectively. Hence, Acemoglu and Restrepo (2018a) state that the findings above are not consistent with empirical observations, and propose a task-based framework, which can be used instead. Here, the central assumption is that output stems from combining different tasks. The production function takes the form

$$Y = \left(\int_{N-1}^{N} y(i)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}, \tag{5.1}$$

with σ being the elasticity of substitution between tasks. $y(i)$ is a distinct task, and the measure of tasks is 1, as is reflected by the integration from $N-1$ to N. An individual task can be either non-automated or automated. In the first case, it is produced using labor only, $y(i) = \gamma(i)l(i)$. Here, $\gamma(i)$ is the productivity of labor $l(i)$ in that task. In the second case, labor and capital can be used as perfect substitutes in the production of the task, so that $y(i) = \eta(i)k(i) + \gamma(i)l(i)$, with $\eta(i)$ reflecting the productivity of capital $k(i)$ in that task. Furthermore, there exists a task $I \in [N-1, N]$, such that every task with $i > I$ is non-automated and can only be produced with labor.

Before demonstrating the results in this framework, the authors make two assumptions. First, tasks increase in complexity from $N-1$ to N, and labor has a comparative advantage for more complex tasks. Hence, $\gamma(i)/\eta(i)$ increases with i, which is an assumption that is strongly supported by empirical findings. The second assumption is that, at task I, the following holds:

$$\frac{\gamma(I)}{\eta(I)} < \frac{W}{R},$$

where W and R are the exogenously given wage and return to capital. Thus, all tasks that can be produced with capital are being produced with capital only. With these assumptions, the authors study automation, which is reflected by an increase in I. They find the following: First, automation makes production less labor-intensive and reduces the labor share. Second, automation increases productivity and output, since capital is cheaper than labor in the neighborhood of I. Hence, using capital frees up resources and increases output. Third, automation has an ambiguous net effect on the wage. On the one hand, automation increases demand for labor, through increasing productivity (productivity effect). On the other hand, capital replaces labor and reduces the wage (displacement effect).

Acemoglu and Restrepo (2018b)

The observations just mentioned are made in a simplified static model. To study the relation between automation and growth, Acemoglu and Restrepo (2018b) build an endogenous growth-model, using the task-framework as a starting point. Hereby, the most important changes are the following: The share of automated tasks is a dynamic variable. It depends not only the amount of automated tasks, but also on new and more complex tasks, which are initially performed by labor. Furthermore, savings are endogenous—as is the innovation process that drives growth. We discuss these changes in turn.

Acemoglu and Restrepo (2018b) provide empirical evidence on the evolution of tasks. They show that while old tasks are automated, new tasks are created. The way in which they incorporate this observations into their model is shown in Figure 5.1. An economy begins with the interval of tasks from $N-1$ to N, among which

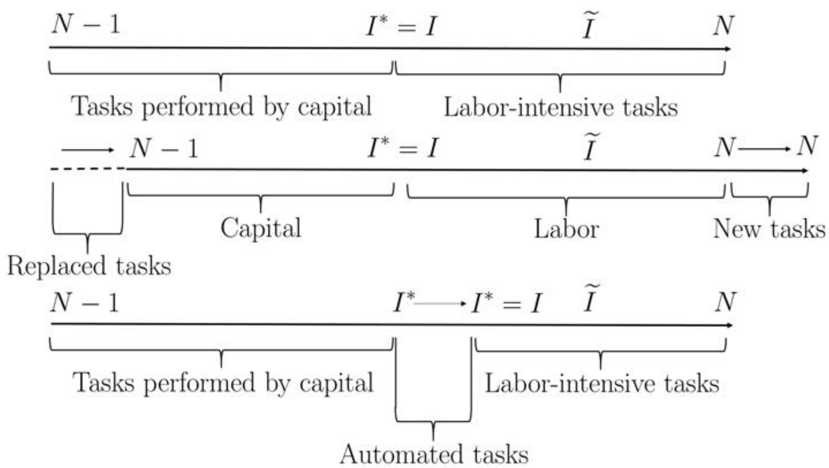

Figure 5.1: The task space and a representation of the effect of introducing new tasks (middle panel) and automating existing tasks (bottom panel) (Source Acemoglu and Restrepo (2018b)).

tasks with $i > I$ are performed by labor and the rest by capital. Research can create new tasks and replace old tasks, at the same time. As a result, the task interval is shifted to the right, so that the economy performs more complex tasks than before. To be precise, the authors assume that the productivity of labor is given by

$$\gamma(i) = e^{Ai},$$

so that when N increases, the comparative advantage of labor in performing new tasks also increases. Since some tasks were replaced, capital now performs a smaller share of tasks. However, through further automation, this share can increase again.

We omit the discussion of model features that are quite standard, such as household behavior, and turn to the innovation process in the economy instead. The aggregate production function is again a CES-function of tasks, as seen in Equation (5.1). However, Acemoglu and Restrepo (2018b) change the production function for tasks slightly. They introduce task-specific intermediates $q(i)$, so that an automated task can be produces in the following way:

$$y(i) = \overline{B}(\zeta)\left[\eta^{\frac{1}{\zeta}} q(i)^{\frac{\zeta-1}{\zeta}} + (1-\eta)^{\frac{1}{\zeta}}(k(i)+\gamma(i)l(i))^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}},$$

where $\overline{B}(\zeta)$ is just some scaling parameter. Whether the good is automated or not is determined by the intermediate $q(i)$, which is produced from final output. Hence, if a (competitive) firm wants to produce a new task that was invented recently, it must purchase the respective intermediate $q(i)$ first. The same holds for a firm producing a task that has been automated. In order to change its production function and employ capital, it must purchase the relevant intermediate. These intermediates can be purchased from so-called "technology monopolists", who sell at a mark-up over marginal costs. These monopolists can be replaced. New firms, driven by the prospect of a positive profit, employ scientists that create new tasks or automate existing tasks.[5] They do this according to the following functions:

$$\dot{I}(t) = \kappa_I S_I(t) \quad \text{and} \quad \dot{N} = \kappa_N S_N(t),$$

where κ_I and κ_N are the productivities of scientists in automation and the creation of new tasks, respectively. $S_I(t)$ is the measure of scientists in automation, while $S_N(t)$ is the measure of creation of new tasks. The sum of the two adds up to some exogenous stock of scientists S.

[5]The actual model is slightly more complicated. The authors assume that the use of a new intermediate is a infringement on the patent of an existing intermediate. For instance, consider the situation when a new monopolists emerges. It possesses the technology to produce a new tasks, and can sell it to a firm. However, the new task replaces some old task, so that the new monopolist must compensate the old monopolist that held the right to the replaced task. Thus, the new monopolist makes a take-it-or-leave-it offer that depends on its income.

Studying balanced growth paths (BGPs) with constant growth rate for the economy, the authors find that four different types of BGPs can emerge. They are shown in Figure 5.2.

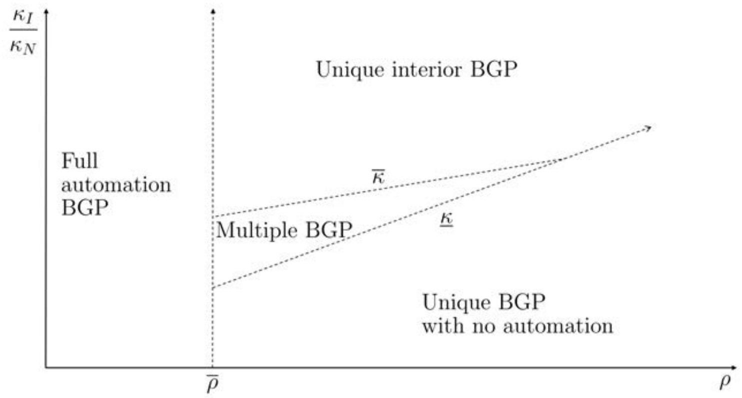

Figure 5.2: Varieties of BGPs (Source Acemoglu and Restrepo (2018b)).

The types of BGPs depend on the constellation of the household's discount rate, ρ, and the research productivities κ_I and κ_N. If the discount rate of the household is small enough, the economy finds itself in a full automation BGP, where all tasks are performed by capital. This is the first type of BGP. The other three types depend on the ratios of productivities of research in the field of new tasks and automation. Among these types, the second one includes a unique interior BGP, where the creation of new tasks and automation of tasks move at the same rate. In this type of BGP, scientists continuously create new tasks, while existing tasks are automated at the same pace. The third type is one with multiple BGPs. Here, the number of tasks performed by labor is also positive. The second and third type result from self-correcting mechanisms of the model. The authors show that the incentive to automate tasks decreases with rising automation. Thus, more automation today leads to less automation tomorrow. Also, the incentive to invent new tasks decreases with more automation. Hence, both incentives increase in the number of tasks performed by labor. Since they do this at different rates, they intersect eventually. In the third type, they intersect several types. The fourth type is, of course, the opposite of the first case, and yields a BGP without automation. It is noteworthy that in the first type, the production function becomes linear in capital, and thus, growth only stems from the accumulation of capital. In the fourth type, the economy collapses to an Aghion and Howitt (1992)-type endogenous growth model with quality improvements.

Hémous and Olsen (2014)

A closely related study is Hémous and Olsen (2014). The authors shift the focus from studying the existence of a BGP to analyzing the relationship between automation and inequality. Besides the declining labor share mentioned above, their study is motivated by the increasing skill premium that is observed in the U.S. for college graduates. The authors use a slightly different aggregate production function

$$Y_t = \left(\int_0^{N_t} y_t(i)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}},$$

where $\sigma > 1$ is the elasticity of substitution and $y_t(i)$ is not seen as a task, but simply as an intermediate input. The increase in the number of such inputs, shown by N_t, is the driving force of growth. A noteworthy difference is the production function, as it does not integrate over a fixed measure of tasks, but over an increasing number of inputs. Also, new intermediates do not come with a higher productivity for labor.

In order to study inequality, Hémous and Olsen (2014) formulate the following production function for an intermediate $y_t(i)$:

$$y(i) = \left[l(i)^{\frac{\epsilon-1}{\epsilon}} + \alpha(i)(\tilde{\phi}x(i))^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon\beta}{\epsilon-1}} h(i)^{1-\beta}.$$

The four variables are $l(i), \alpha(i), x(i)$ and $h(i)$. They respectively stand for the amount of low-skill labor, an indicator function, showing whether the firm is automated ($\alpha = 1$) or not ($\alpha = 0$), the amount of capital used in production, and the amount of high-skill labor. Hence, Hémous and Olsen (2014) distinguish between two types of labor, which are differently affected by automation. Low-skill labor can be substituted by capital, whereas high-skill labor cannot. Instead, automation can even increase the marginal productivity of high-skill workers, and thus, their wage. The parameters $\tilde{\phi}$, ϵ and β are the productivity of capital, the measure of substitutability between low-skill labor and capital, and the low-skill factor share.

Hémous and Olsen (2014) provide an endogenous growth model with endogenous capital accumulation and innovation. Hereby, innovation is twofold. First, research can create new intermediates and thus, increase N_t. This process requires high-skill workers, H_t^D, and has the Romer-type production function

$$\dot{N}_t = \gamma N_t H_t^D,$$

where γ is the productivity of research. The firm producing the intermediate is a monopolist and initially, a non-automated firm. However, the monopolist can decide to hire high-skill labor to conduct automation research. This is the second type of innovation. Automation arrives at a Poisson process with the following rate: $\eta G_t^{\tilde{\kappa}}(N_t h_t^A(i))^{\kappa}$, where G_t is the share of automated intermediate firms, and η and κ describe the productivity of this type of research. Finally, $\widetilde{\kappa}$ stands for

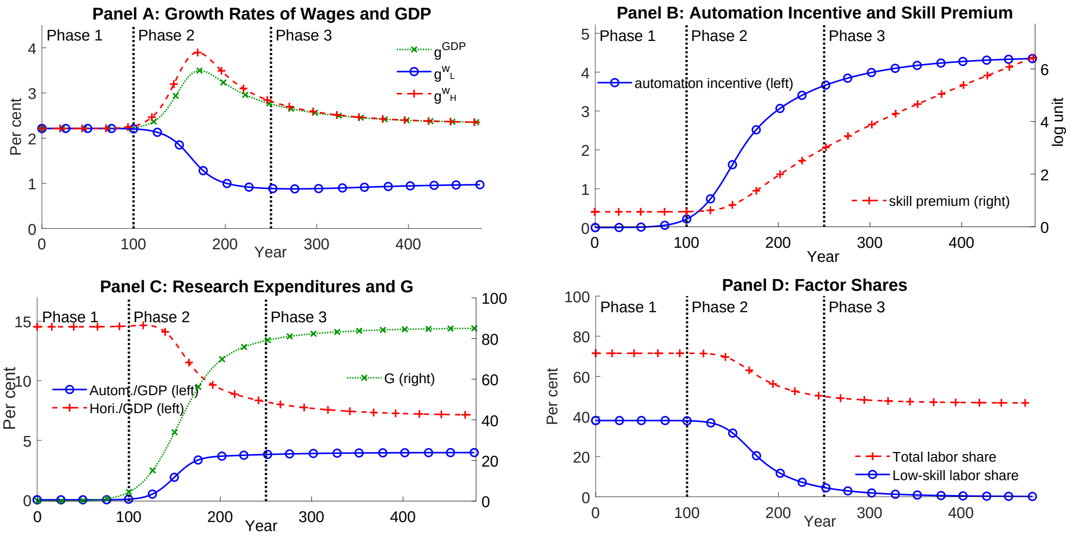

Figure 5.3: Transitional dynamics for baseline parameters (Source: Hémous and Olsen (2014)).

the importance of spillovers from automation. Hence, in this model, research is not conducted by a fixed amount of scientists, but high-skill workers allocate themselves between production, automation research, and the creation of new intermediates.

For this economy, Hémous and Olsen (2014) find a unique saddle-stable steady state. The steady state is asymptotic, though, and the authors thoroughly discuss the transition to this state, which is shown in Figure 5.3.

The authors argue that the transition path can be divided into three distinct phases. These phases are generated by the interplay of the incentives to innovate and automate, which, in contrast to before, do not affect each other simultaneously. It is rather the case that they are of differing importance at difference times. In Phase 1, the incentive to automate is relatively low, due to wages of low-skill workers being rather low. Hence, the incentive to innovate dominates, and the economy behaves similarly to the Romer (1990) model. In Phase 2, after low-skill wages have increased, the incentive to automate is large enough and one observes a rapid increase in the share of automated firms. Here, the growth rates of high- and low-skill wages diverge, producing a positive skill premium which begins to increase linearly. Finally, in Phase 3, the benefit from automation growths at the same rate as its cost, leading to a constant share of automated firms G_t. The economy begins to converge to the asymptotic steady state, where the economy grows at a constant rate, but wages of high-skill workers grow faster than those of low-skill ones.

5.3 An intermediate perspective

Finally, we turn to a study that not only considers the impact of AI on research, as in Agrawal et al. (2018), but also takes into account the substitution of labor by AI, similar to the studies discussed in the previous section. This study is Aghion et al. (2017). As before, the authors rely on a CES-production function

$$Y_t = A_t \left(\int_0^1 X_{it}^{\rho} di \right)^{\frac{1}{\rho}},$$

where A_t is the level of technology and X_{it} is some intermediate good, required for the production in the final good sector. An intermediate can be produced with a single unit of capital if it is automated, and with a single unit of labor otherwise. The crucial difference to previous approaches is the parameter ρ, which is assumed to be negative. This assumption reflects Baumol's "cost-disease" (see Baumol (1967)). It suggests that output is not bound by the *most* productive factors, but by the *least* productive ones. As a consequence, industries that increase productivity might experience a decline in their share of output. An example of this development is the agriculture sector.

The authors assume a symmetric allocation of capital and labor across goods, so that the production function can be written as

$$Y_t = A_t \left(\beta_t \left(\frac{K_t}{\beta_t} \right)^{\rho} + (1-\beta_t) \left(\frac{L_t}{1-\beta_t} \right)^{\rho} \right)^{\frac{1}{\rho}},$$

with β_t being the share of automated goods. The ratio of the capital share and the labor share becomes

$$\frac{\alpha_{K_t}}{\alpha_{L_t}} = \left(\frac{\beta_t}{1-\beta_t} \right)^{1-\rho} \left(\frac{K_t}{L_t} \right)^{\rho}.$$

The authors find that an increase in automation, reflected by a larger β_t, increases the share of capital relatively. However, an increase in capital itself, which goes hand in hand with automation, will decrease the price of capital, and thus, its share. Aghion et al. (2017) argue that this opens up the possibility of constant factor shares, even under continuous automation.

Furthermore, they interpret their production function as a special case of a neoclassical-production function of the following form:

$$Y_t = A_t F(B_t K_t, C_t L_t) \quad \text{where} \quad B_t \equiv \beta_t^{\frac{1-\rho}{\rho}} \quad \text{and} \quad C_t \equiv (1-\beta_t)^{\frac{1-\rho}{\rho}}.$$

As they assume that $\rho < 0$, they obtain the following: $\uparrow \beta_t$ implies $\downarrow B_t$ and $\uparrow C_t$. This means that automation is not capital-augmenting but capital-depleting, as it spreads a fixed amount of capital over a larger number of goods. The opposite holds for labor, as automation reduces the range of goods for labor, concentrating it over fewer tasks. Hence, the authors conclude that automation is labor-augmenting, and find that it is possible to obtain asymptotically constant growth and constant factor shares, even under complete automation. Figure 5.4 shows a model where in each period, a constant share of goods is automated, implying constant growth of C_t. The growth rate of the economy converges to 2% and the capital share converges to 1/3, although the economy approaches full automation.

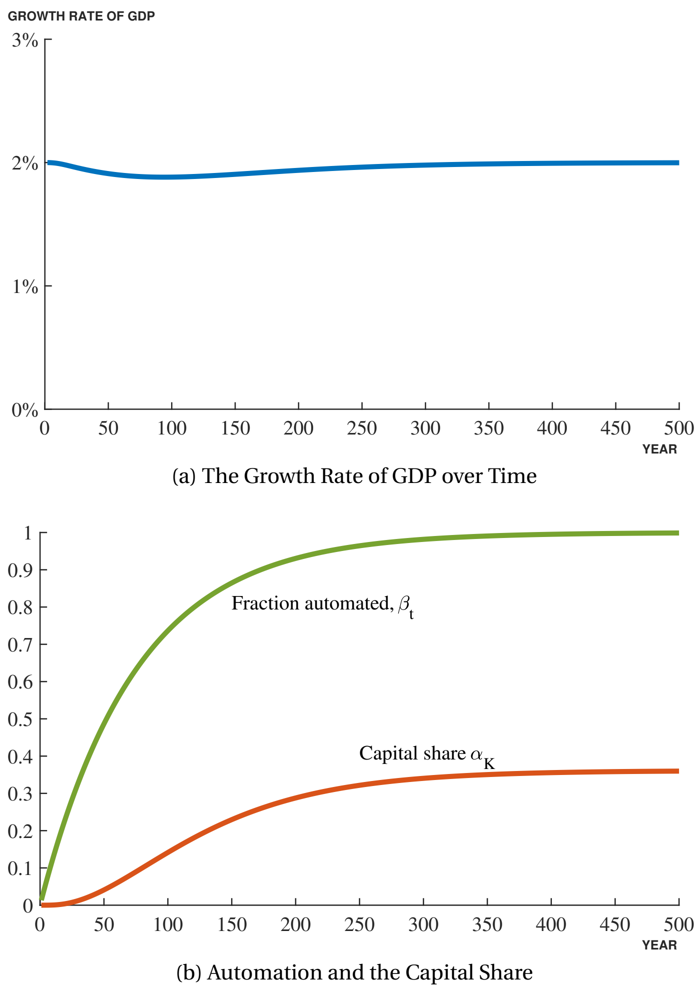

Figure 5.4: Automation and asymptotic balanced growth (Source: Aghion et al. (2017)).

After studying automation in the context of production of goods, Aghion et al. (2017) turn to automation in the realm of production of ideas, i.e. research. Hereby, they use their aggregate production function from Aghion et al. (2017). However, it now describes the growth of ideas

$$\dot{A}_t = A_t^{\phi} \left(\int_0^1 X_{ti}^{\rho} di \right)^{\frac{1}{\rho}},$$

where, as before, $\rho < 1$, X_{ti} stands for a specific task in research, and additionally $\phi < 1$. Under the same assumptions as in Aghion et al. (2017), this function becomes

$$\dot{A}_t = A_t^{\phi}((B_t K_t)^{\rho} + (C_t S_t)^{\rho})^{\frac{1}{\rho}},$$

with C_t and B_t defined above, and S_t being the measure of scientist. The authors show that if automation occurs at a constant rate, the economy will grow asymptotically at the following rate:

$$g_A = \frac{g_C + g_S}{1 - \phi}.$$

Hence, in the absence of automation, g_C would be zero and growth of ideas would be driven by the increase in the number of scientists. Thus, automation has a positive impact on growth, increasing the economy's growth rate.

Finally, the authors provide some empirical observation about the relationship between automation and factor shares. Figure 5.5 shows the change in robots against the change in capital share for different industries

The authors' main observation is that there is little correlation between the two, and they conclude that automation cannot be the single driving force behind the recent decline of the labor share.

5.4 A short outlook

We have seen that in the context of economic growth, the authors model AI as either *helping* humans in processing information and prediction, or *replacing* them. We consider this as two polar cases, that do *not* exclude each other. Furthermore,

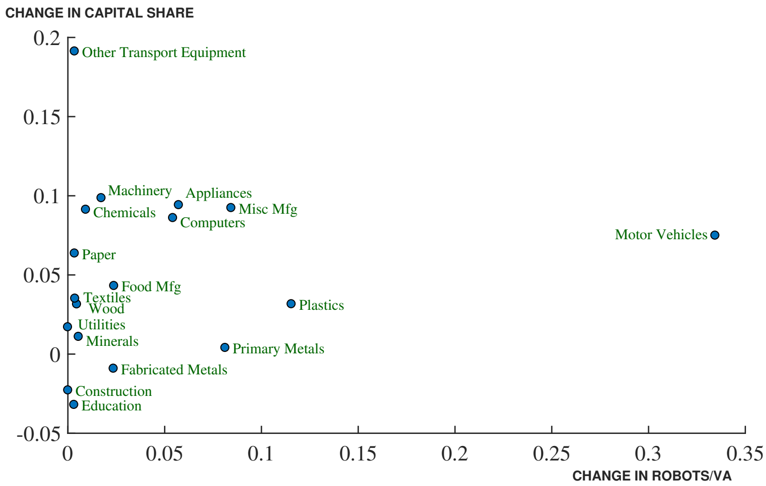

Figure 5.5: Capital share and robots 2000-2014 (Source: Aghion et al. (2017)).

while AI might further automate tasks and or jobs, it is important to distinguish it from previous automation. AI is not driven by the accumulation of physical capital. Instead, the so-called "AI-revolution", that is observed recently, is fueled by two other trends. On the one hand, there is an exponential rise in computational power, and subsequent fall in the price paid for it. On the other hand, companies have access to the huge amounts of data. Additionally, this data is generated by households at zero costs for AI-companies and so-called "Tech Giants", such as Google and Facebook. Hence, we consider the following features to be relevant for modeling AI in an endogenous growth model:

- AI interacts differently with certain labor types: While AI *replaces* some low-skill labor type, it can *increase the efficiency* of some other type of labor—in research and or production. Also, the AI sector offers employment opportunities for a particularly high-skill type of labor. Potentially, this type is even more productive in the AI-sector than in a more conventional sector.
- Physical capital and AI should be modeled as distinct variables. It is worth including three different types of capital: physical capital, robots, and AI.

Hereby, AI should not stem from the accumulation of physical capital but from the skill and effort of human labor.

- It is important not only to study AI as an intermediate in production but also as a final good, sold to the consumer.
- AI relies on data generated by other sectors and households. This dependency should be modeled, as the progress of AI might be halted by a mere lack of usable data.
- The rise of Tech Giants certainly reshapes the market-place. Hence, a rise of AI might change mark-ups and market-structure, potentially leading to inefficiencies and distortions.

This list is, of course, not exhaustive in terms of features of AI that might be modeled. In my view, these are conclusions we can infer from the literature presented.

Chapter 6

Appendix

6.1 Appendix for Chapter 2

6.1.1 Proof of Proposition 1

Assume there is a balanced growth path along which $L_t^R = \hat{L}^R \; \forall t$, where $\hat{L}^R$ is constant. The saving decision implies $s_t = \frac{\beta w_t^P (1-\tau_t)}{1+\beta} = K_{t+1}$. Furthermore, we have $w_t^P = \frac{(1-\alpha)Y_t}{1-L_t^R}$. Substituting w_t^P into the savings decision yields

$$K_{t+1} = \frac{\beta(1-\tau_t)}{(1+\beta)} \frac{(1-\alpha)Y_t}{1-L_t^R}.$$

Recall that $L_t^R = \tau_t$ has to hold, so we have

$$\frac{K_{t+1}}{Y_t} = \frac{\beta(1-\alpha)}{1+\beta}. \tag{6.1}$$

As $\frac{K_{t+1}}{Y_t}$ is constant, K_t and Y_t grow at the same rate. We show that this rate is constant and equal to $\theta \hat{L}^R$. First, we take the logarithm of the production function

$$\log(Y_t) = (1-\alpha)\log(L_t^P) + (1-\alpha)\log(B_t) + \alpha \log(K_t).$$

Then we write the analogous equation for the next period Y_{t+1} and subtract both to obtain

$$g_{Y_t} = (1-\alpha)g_{L_t^P} + (1-\alpha)g_{B_t} + \alpha g_{K_t},$$

where $g_{(.)}$ stands for the growth rate of the variable in the index. As in the steady state, the labor share is constant in both sectors and it holds that $g_{L_t^P} = 0$. We have just found that $g_{K_t} = g_{Y_t}$, even outside the steady state, so we obtain

$$g_{Y_t} = g_{B_t} = \frac{B_{t+1}}{B_t} - 1 = \theta \hat{L}^R.$$

Hence, the interest rate $\hat{r}$ is given by

$$\hat{r} = \frac{\alpha Y_{t+1}}{K_{t+1}} = \frac{\alpha Y_{t+1} Y_t}{Y_t K_{t+1}} = \frac{\alpha(1+\theta\hat{L}^R)(1+\beta)}{\beta(1-\alpha)}, \tag{6.2}$$

because $Y_{t+1}/Y_t = 1 + \theta\hat{L}^R$.

6.1.2 Steady state of the social planner solution

For clarity we write down the Lagrange function of the social planner:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta_s^t \big[\log(c_t^1) + \beta \log(c_{t+1}^2) - \lambda_t \left(((1-L_t^R)B_t)^{1-\alpha} K_t^\alpha - c_t^1 - c_t^2 - K_{t+1} \right) - \mu_t \left(B_{t+1} - (1+\theta L_t^R) B_t \right) \big],$$

where λ_t and μ_t are the Lagrange multiplier described in the main text. We derive the steady state from Equations (2.9) and (2.10) in the following way: First, we note that maximizing the Lagrangian with respect to c_t^1 and c_{t+1}^2 yields $c_{t+1}^1 \frac{\beta}{\beta_s} = c_{t+1}^2$. We assume $\beta = \beta_s$, so that $c_t^1 = c_t^2$ and consequently, $c_t^1 = C_t/2$, where C_t

stands for aggregate consumption in period t. We substitute this expression into (2.9) and obtain

$$\frac{C_{t+1}}{C_t} = \frac{\alpha\beta Y_{t+1}}{K_{t+1}}, \quad \Leftrightarrow \quad \frac{C_{t+1}Y_t}{C_t Y_{t+1}} = \frac{\alpha\beta Y_t}{K_{t+1}}.$$

In the steady state, aggregate consumption and output grow at the same rate, so that the left-hand side is 1. Hence we find

$$\frac{\overline{K}_{t+1}}{\alpha\overline{Y}_t} = \beta,$$

where $\overline{K}_{t+1}$ and $\overline{Y}_t$ stand for the constantly growing values of capital and output. Then we simplify (2.10) to

$$\frac{1}{1-L^R} = \frac{\overline{K}_{t+1}}{\alpha\overline{Y}_t}\left(\frac{\theta}{1+\theta L^R} + \frac{1}{1-L^R}\frac{1+\theta L^R}{1+\theta L^R}\right),$$

where we can substitute $\frac{\overline{K}_{t+1}}{\alpha\overline{Y}_t}$ by β,

$$\begin{aligned} 1+\theta L^R &= \beta(\theta - \theta L^R + 1 + \theta L^R), \\ \theta L^R &= \beta\theta - (1-\beta). \end{aligned}$$

6.1.3 Critical belief

Using our results from (2.13), we can write the utility of agent i if he is a researcher as

$$U_{t,i}^R = \log\left(\frac{w_t^R}{1+\beta}\right) + (1+\beta)\log\left(1-\theta_l+\theta_i\right) + \beta\log\left(\frac{\beta r_{t+1} w_t^R}{1+\beta}\right).$$

His utility as a worker is

$$U_{t,i}^P = \log\left(\frac{(1-\tau_t)w_t^P}{1+\beta}\right) + \beta \log\left(\frac{\beta r_{t+1}(1-\tau_t)w_t^P}{1+\beta}\right).$$

The individual decision of agent i has no impact on the interest rate r_{t+1} so that the respective interest rates are taken as equal in both cases. Hence we set $U_{t,i}^R = U_{t,i}^P$ for the agent with $\theta_i = \theta_{crit,t}$

$$(1+\beta)\log(w_t^R) + (1+\beta)\log(1-\theta_l+\theta_i) + \beta\log(\beta r_{t+1}) - (1+\beta)\log(1+\beta)$$

$$=$$

$$(1+\beta)\log\left((1-\tau_t)w_t^P\right) + \beta\log(\beta r_{t+1}) - (1+\beta)\log(1+\beta),$$

$$\Leftrightarrow \log(1-\theta_l+\theta_i) = \log\left(\frac{(1-\tau_t)w_t^P}{w_t^R}\right),$$

$$\Leftrightarrow \theta_{crit,t} = \frac{(1-\tau_t)w_t^P}{w_t^R} - (1-\theta_l).$$

Proof of Proposition 8

After substituting the optimal savings decision, the utility function of a researcher reads

$$U_{t,i}^R = \beta\log\left(\frac{\beta}{1+\beta}\right) - \log(1+\beta) + \log(w_t^R)(1+\beta) + \log\left(\theta_i e_{t,i} - q\frac{e_{t,i}^2}{2}\right) +$$
$$\beta\log(r_{t+1}).$$

We want to alter the utility function of the researcher so that the derivative with respect to effort will be identical to the first order condition of the social planner. One way to do this is by setting w_t^R. Substituting w_t^R from Proposition 8 changes the utility function to

$$U_{t,i}^{R} = \beta \log(\frac{\beta}{1+\beta}) + (1+\beta)\log(\widetilde{w_t}^{R}) + \log(\hat{\theta} e_{t,i} - q\frac{e_{t,i}^2}{2}) + \hat{G}_t e_{t,i} + \beta \log(r_{t+1}), \tag{6.3}$$

with the derivative

$$\frac{\partial U_{t,i}^{R}}{\partial e_{t,i}} = \frac{\hat{\theta} - q e_{t,i}}{\hat{\theta} e_{t,i} - q\frac{e_{t,i}^2}{2}} + \hat{G}_t = 0. \tag{6.4}$$

Solving for $e_{t,i}$, which is the same for all i, yields

$$e_t^2 - \frac{2}{q}\left(\hat{\theta} - \frac{q}{\hat{G}_t}\right) e_t - \frac{2\hat{\theta}}{q\hat{G}_t} = 0, \tag{6.5}$$

while the social planner FOC implies

$$e_t^{SOC\,2} - \frac{2}{q}\left(\theta - \frac{q}{G_t}\right) e_t^{SOC} - \frac{2\theta}{qG_t} = 0. \tag{6.6}$$

Due to the definition of $\hat{\theta}$ and $\hat{G}_t$ it holds that $e_t = \frac{\theta_h - \theta_l}{\theta_h - \theta_{crit}} e_t^{SOC}$. To see this, substitute e_t in Equation (6.5). This will yield Equation (6.6). In the decentralized case we purposefully demand more effort from every researcher.

If e_t^{SOC} is the socially optimal individual and aggregate effort level, recall that the mass of all agents is 1. Hence, in the steady state of the decentralized economy we have for effort

$$E^D = \int_{\frac{\theta_{crit}-\theta_l}{\theta_h-\theta_l}}^{1} e\, di = e\frac{\theta_h - \theta_{crit}}{\theta_h - \theta_l} = e^{SOC}\frac{\theta_h - \theta_l}{\theta_{crit} - \theta_l}\frac{\theta_{crit} - \theta_l}{\theta_h - \theta_l} = E^{SOC}.$$

As mentioned, substituting $e = \frac{\theta_h - \theta_l}{\theta_h - \theta_{crit}} E^D$ in the steady state version of (6.5) will yield the steady state version of Equation (6.6). Hence we have successfully altered the researchers' individual decisions to replicate one of the social planner's two optimality conditions.

Next we turn to the FOC of the decentralized government when it can issue debt,

$$\frac{1-\beta}{1-L^{D,R}} = \frac{\beta\theta E}{1+\theta E L^{D,R}}, \tag{6.7}$$

which together with Equation (6.4) replicate the two equations of the social planner problem and yield L^{SOC} and E^{SOC} as solutions.

Finally, we prove the definition of $\widetilde{w}_t{}^R$. Under Proposition 8, we demonstrated how optimal effort and labor demand can be established. Now we turn to labor supply. As before, some agent i must be indifferent between working in research and the productive sector. We show that, if $w^R_{t,i}$ is defined as above, all agents are indifferent. To see this, recall Equation (6.3). By construction, if the optimal $e_{t,i}$ is the same for all agents, $U^R_{t,i}$ is also the same. The following equality therefore holds for all i

$$(1+\beta)\log(\widetilde{w}_t{}^R) + \log\left(\hat{\theta}e_t - q\frac{e_t^2}{2}\right) + \hat{G}_t e_t = (1+\beta)\log(w_t^P),$$

which yields

$$\widetilde{w}_t{}^R = w_t^P\left(\hat{\theta}e_t - q\frac{e_t^2}{2}\right)^{\frac{-1}{1+\beta}} e^{-\frac{\hat{G}_t e_t}{1+\beta}},$$

as all researchers invest the same amount of effort. This concludes the proof.

Furthermore, note that in equilibrium, the following holds

$$w^R_{t,i} = \widetilde{w}_t{}^R\left(\frac{\hat{\theta}e_{t,i} - q\frac{e^2_{t,i}}{2}}{\theta_i e_{t,i} - q\frac{e^2_{t,i}}{2}}\right)^{\frac{1}{1+\beta}} e^{\frac{\hat{G}_t e_{t,i}}{1+\beta}}, \quad \text{and}$$

$$w^R_{t,i} = \frac{w_t^P}{e_t(\theta_i - q\frac{e_t}{2})^{\frac{1}{1+\beta}}},$$

which means that researchers are paid the wage of the productive sector with a mark-down depending on their preferences and optimal effort. This leaves agents indifferent as to sector choice and exertion of socially optimal effort.

6.1.4 Stability analysis

In this section, we analyze the stability of the steady states of the decentralized economy and the social planner solution in both models.

Decentralized solution without research bubbles

We have demonstrated that research labor is equal to the government's demand, given by

$$L_t^R = L^{D,R} = \frac{1}{1+\beta}\left(\beta - \frac{1}{\theta}\right).$$

It is therefore always constant and implies the following constant gross growth rate of the knowledge stock: $B_{t+1}/B_t = 1 + \theta L^R$. Note that savings are given by

$$s_t = K_{t+1} = \frac{\beta(1-\tau_t)w_t^P}{1+\beta},$$

so that substituting $w_t^P = (1-\alpha)Y_t/(1-L_t^R)$ yields

$$K_{t+1} = \frac{(1-\alpha)\beta(1-\tau_t)}{1+\beta}(1-L_t^R)^{-\alpha}B_t^{1-\alpha}K_t^{\alpha},$$

where $\tau_t = L_t^R = L^R$, as shown previously. We divide both sides by B_{t+1},

$$\frac{K_{t+1}}{B_{t+1}} = \frac{(1-\alpha)\beta}{(1+\beta)}(1-L^R)^{1-\alpha}\left(\frac{K_t}{B_t}\right)^{\alpha}\frac{B_t}{B_{t+1}}.$$

We write capital in terms of the knowledge stock and define $k_t = K_t/B_t$. Hence, given the constant growth rate of B_{t+1}/B_t, we have

$$k_{t+1} = \frac{(1-\alpha)\beta}{(1+\beta)(1-L^R)^{\alpha-1}(1+\theta L^R)} k_t^{\alpha},$$

which is a simple convex policy-function in capital as the factor in front of k_t^{α} is a mere constant.

Social planner solution

To study whether the economy converges to the socially optimal steady state, we rewrite (2.10):

$$\begin{aligned}
\frac{1+\theta L_t^R}{1-L_t^R} &= \frac{K_{t+1}}{\alpha Y_{t+1}} \frac{Y_{t+1}}{Y_t} \left(\theta + \frac{1+\theta L_{t+1}^R}{1-L_{t+1}^R}\right), \quad \text{which implies} \\
\frac{1+\theta L_t^R}{1-L_t^R} &= \frac{K_{t+1}}{\alpha Y_{t+1}} \frac{Y_{t+1}}{Y_t} \frac{1+\theta}{1-L_{t+1}^R} \qquad \text{or simply} \\
L_{t+1}^P = 1 - L_{t+1}^R &= \frac{K_{t+1}}{\alpha Y_{t+1}} \frac{Y_{t+1}}{Y_t} \frac{1-L_t^R}{1+\theta L_t^R}(1+\theta),
\end{aligned} \tag{6.8}$$

so that the model dynamics are governed by (6.8), the knowledge production function, and the following three equations:

$$\begin{aligned}
Y_t &= K_t^{\alpha}(B_t L_t^P)^{1-\alpha}, \\
K_{t+1} &= Y_t - c_t^1 - c_t^2, \\
c_{t+1}^2 &= \beta\alpha \frac{Y_{t+1}}{K_{t+1}} c_t^1.
\end{aligned}$$

We can rewrite aggregate consumption $C_t = c_t^1 + c_t^2$, using the fact that the maximization of the Lagrangian with respect to c_t^1 and c_{t+1}^2 yields $c_{t+1}^1 \frac{\beta}{\beta_s} = c_{t+1}^2$. Assuming $\beta = \beta_s$, we arrive at $c_t^1 = C_t/2$. Next we express the four equations in terms of effective labor $B_t L_t^P$, defining g_{B_t} and g_{L_t} as the growth rate of knowledge and of labor input in the productive sector in period t,

$$
\begin{aligned}
\frac{Y_t}{B_t L_t^P} &= y_t = k_t^{\alpha}, \\
k_{t+1}(1+g_{B_t})(1+g_{L_t}) &= y_t - c_t, \\
c_{t+1}(1+g_{B_t})(1+g_{L_t}) &= \beta \alpha k_{t+1}^{\alpha-1} c_t,
\end{aligned}
$$

where k_t and c_t without superscript indicate capital and aggregate consumption per effective labor. Equation (2.10) becomes

$$
\frac{L_{t+1}^P}{L_t^P} = 1 + g_{L_t} = \frac{K_{t+1}}{\alpha Y_{t+1}} \frac{Y_{t+1}}{Y_t} \frac{1}{1+\theta L_t^R}(1+\theta).
$$

By using $1 + g_{B_t} = 1 + \theta L_t^R$ from the knowledge production function we obtain

$$
\begin{aligned}
1 + g_{L_t} &= \frac{1}{\alpha} \frac{k_{t+1}}{y_{t+1}} \frac{y_{t+1}}{y_t}(1+g_{B_t})(1+g_{L_t}) \frac{(1+\theta)}{1+g_{B_t}}, \\
1 &= \frac{1+\theta}{\alpha} \frac{k_{t+1}}{k_t^{\alpha}}, \\
k_{t+1} &= \frac{\alpha}{1+\theta} k_t^{\alpha}. \qquad (6.9)
\end{aligned}
$$

We arrive at a concave policy-function for capital per effective labor k_t. Next we turn to the Euler equation for consumption. By using the following relationships:

$$
\begin{aligned}
1 + g_{B_t} &= 1 + \theta L_t^R, \\
1 + g_{L_t} &= \frac{K_{t+1}}{\alpha Y_t} \frac{1}{1+\theta L_t^R}(1+\theta) = \frac{Y_t - C_t}{\alpha Y_t} \frac{1}{1+\theta L_t^R}(1+\theta),
\end{aligned}
$$

we obtain

$$\frac{c_{t+1}}{c_t} = \frac{1}{(1+\theta)}\beta\alpha k_{t+1}^{\alpha-1}\frac{\alpha Y_t}{Y_t - C_t},$$
$$\frac{c_{t+1}}{c_t} = \frac{\alpha^2\beta}{1+\theta}\frac{k_{t+1}^{\alpha-1}k_t^{\alpha}}{k_t^{\alpha} - c_t}. \tag{6.10}$$

Hence it is possible to express the system in the two Equations (6.9) and (6.10) in two variables, consumption and capital per effective labor. The steady state along which g_{B_t} and all variables in terms of effective labor, and equivalently the amount of labor supplied to the productive sector are constant is given by

$$k = \left(\frac{\alpha}{1+\theta}\right)^{\frac{1}{1-\alpha}},$$
$$c = k^{\alpha}(1 - \frac{\alpha^2\beta}{1+\theta}k^{\alpha-1}) = \left(\frac{\alpha}{1+\theta}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha\beta).$$

As the policy-function for capital is concave, we know that capital will converge to the steady state. The convergence of consumption is not clear. We know that for consumption to grow at a positive rate, it must hold that

$$\frac{c_{t+1}}{c_t} > 1 \Rightarrow c_t > k_t^{\alpha} - \alpha^2\beta\left(\frac{\alpha}{1+\theta}\right)^{\alpha-1}k_t^{\alpha^2}. \tag{6.11}$$

We use the phase diagram in Figure 6.1 to discuss the convergence of the model. The black vertical line is the steady state condition for capital. The curved black line shows the respective condition for consumption. The red line divides the space into two sectors: The one above the red line indicates where consumption is growing, the one below the red line shows where consumption is decreasing. As we know that capital converges to the steady state from both sides, we can draw the appropriate arrows that indicate the movement of the variables.

The diagram shows that the model is saddle-path-stable, as convergence to the

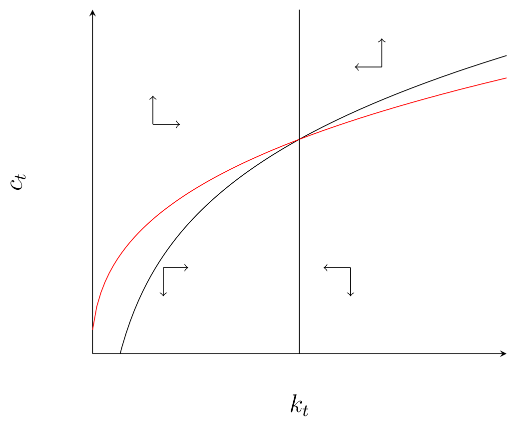

Figure 6.1: Phase diagram for consumption and capital per effective labor.

steady state, which is the intersection of all three lines, does not occur from any arbitrary initial allocation of c_t and k_t. For instance, convergence from an allocation in the right upper corner, where consumption lies above the red line and capital to the right of the vertical black line, is not possible. The economy would move to $k_t = 0$. In Figure 6.2, the space between the green lines shows all possible initial allocations of capital and consumption from which convergence to the unique steady state occurs.

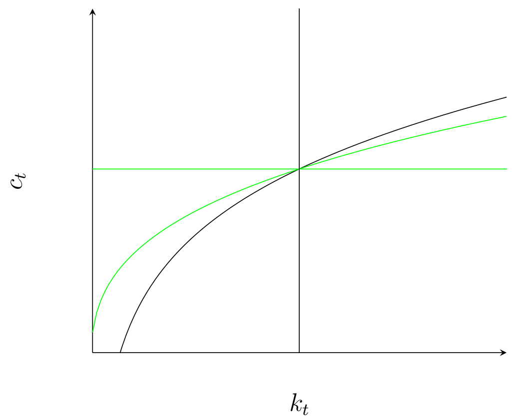

Figure 6.2: Set of initial allocations from which the economy converges to the steady state.

We find further support for the saddle-path-stability of the system by linearizing it around the steady state. The linearized versions of (6.9) and (6.10) are

$$\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \left(\frac{\alpha^2 c}{k} - \frac{\alpha c k^{\alpha-1}}{k^\alpha - c}\right) & \left(1 + \frac{c}{k^\alpha - c}\right) \end{pmatrix} \begin{pmatrix} \widehat{k}_t \\ \widehat{c}_t \end{pmatrix}.$$

As the second entry of the first row is 0, the Eigenvalues of the matrix are given by $\alpha = 0.3 < 1$ and $\left(1 + \frac{1}{k^\alpha - c}\right) > 1$. The first Eigenvalue is real and smaller in absolute value than 1, while the second is real and larger than 1. To see this, note that k^α is aggregate production per effective labor, while c is the aggregate consumption per effective labor. This difference is positive, so the second Eigenvalue is at least one.

Government solution with research bubbles

As we have demonstrated, the research bubbles lead to a greater input of labor in the research sector. Yet this value is constant over time, so that the dynamics of the model do not change in comparison to the model without bubbles. Thus our analysis from the section above carries over, the only difference being that L^R is greater.

6.1.5 Convergence under debt financing

We established above that it is theoretically possible to implement the socially optimal steady state amount of research in the decentralized economy. In this section, we investigate whether the model with heterogeneous beliefs converges to its social optimum. If the optimal amount of research is implemented, i.e. if

$L_t^R = L^{O,R} \,\forall\, t$, we have

$$K_{t+1} = \frac{\beta}{1+\beta}(w_t^R L^{O,R} + (1-\tau^G)w_t^P(1-L^{O,R})).$$

We substitute w_t^R to obtain

$$\frac{K_{t+1}}{B_{t+1}} = k_{t+1} = \frac{\beta(1-\alpha)(1-\tau^G)}{(1+\beta)(1-L^{O,R})^{\alpha-1}}\left(\frac{1+(1-L^{O,R})(\theta_h-\theta_l)\left(1-\frac{L^{O,R}}{\overline{L}_B}\right)}{1+(\theta_h-\theta_l)\left(1-\frac{L^{O,R}}{\overline{L}_B}\right)}\right)k_t^{\alpha}.$$

Note that this expression is a concave policy-function for k_t, as all terms in front of k_t^{α} are constant. Next, under conditions on the parameter values, we show that aggregate debt d_t grows at the rate of output. We have defined $D_t(L^{O,R})$ as the additional debt issued in period t. Therefore we can write aggregate debt d_t as

$$d_t = \frac{1-\gamma}{\gamma}Y_t + r_t d_{t-1},$$

where we write $\gamma = (1-L^{O,R})^{\beta(1-\alpha)}$ for convenience. Hence,

$$d_t = \frac{1-\gamma}{\gamma}\left[Y_t + r_tY_{t-1} + r_tr_{t-1}Y_{t-2} + ... + \prod_{s=1}^{t} r_sY_0\right] \quad \text{or}$$
$$d_t = \frac{1-\gamma}{\gamma}\left[Y_t + \alpha\frac{Y_t}{K_t}Y_{t-1} + \alpha^2\frac{Y_t}{K_t}\frac{Y_{t-1}}{K_{t-1}}Y_{t-2} + ... + \alpha^t\prod_{s=1}^{t}\frac{Y_s}{K_s}Y_0\right].$$

From Equation (6.1) we know that K_{t+1}/Y_t is constant for all t. This enables us to write

$$d_t = \frac{1-\gamma}{\gamma}Y_t\left[1+\frac{\alpha}{\delta}+\frac{\alpha^2}{\delta^2}+...+\frac{\alpha^t}{\delta^t}\right], \quad \text{with}$$
$$\delta = \frac{\beta(1-\alpha)}{1+\beta}.$$

For t large and if $\alpha < \delta$, i.e. $\alpha < \beta$, we can write the geometric sum as $\frac{1}{1-\frac{\alpha}{\delta}}$, so we obtain

$$\frac{d_t - d_{t-1}}{d_{t-1}} = \frac{Y_t - Y_{t-1}}{Y_{t-1}}.$$

Thus, in the long run, total debt grows at the same rate as output and the ratio of public debt to GDP becomes constant.

6.2 Appendix for Chapter 3

The f.o.c. are, noting that $0 \leq e_t \leq 1$,

$$\frac{\partial \Phi_t}{\partial c_t^2} = u'(c_t^2) - \mu_t[1 - q_t^2 + \beta n_t] = 0, \tag{6.12}$$

$$\frac{\partial \Phi_t}{\partial s_t} = \frac{\delta u'(c_{t+1}^3) \cdot n_t \rho}{(1 - q_t^2)} \cdot \frac{\partial \bar{y}_{t+1}}{\partial s_t} - \mu_t = 0, \tag{6.13}$$

$$\frac{\partial \Phi_t}{\partial e_t} = \frac{\delta u'(c_{t+1}^3) \cdot n_t \rho}{(1 - q_t^2)} \cdot \frac{\partial \bar{y}_{t+1}}{\partial \lambda_{t+1}} \frac{\partial \lambda_{t+1}}{\partial e_t} + \nu_t v'(\lambda_{t+1}) \frac{\partial \lambda_{t+1}}{\partial e_t} + \mu_t \frac{\partial y_t}{\partial e_t} \leq 0, \; e_t \geq 0, \tag{6.14}$$

$$\frac{\partial \Phi_t}{\partial e_t} = \frac{\delta u'(c_{t+1}^3) \cdot n_t \rho}{(1 - q_t^2)} \cdot \frac{\partial \bar{y}_{t+1}}{\partial \lambda_{t+1}} \frac{\partial \lambda_{t+1}}{\partial e_t} + \nu_t v'(\lambda_{t+1}) \frac{\partial \lambda_{t+1}}{\partial e_t} + \mu_t \frac{\partial y_t}{\partial e_t} \geq 0, \; e_t \leq 1, \tag{6.15}$$

where, recalling that F is homogeneous of degree 1 and $\zeta_t = \lambda_t / s_{t-1}$,

$$\frac{\partial \lambda_{t+1}}{\partial e_t} = z_t h'(e_t) \lambda_t \,,$$

$$\frac{\partial \bar{y}_{t+1}}{\partial s_t} = \frac{\sigma_{t+1}}{n_t} \cdot F_2 \left[(1 - q_{t+1}^2) \zeta_{t+1} + \frac{n_{t+1} \gamma}{s_t}, \frac{\sigma_{t+1}}{n_t} \right],$$

$$\frac{\partial y_t}{\partial e_t} = -(\gamma + w\lambda_t)n_t \cdot F_1\left[(1 - q_t^2 - wn_te_t)\zeta_t + \frac{n_t\gamma(1 - e_t)}{s_{t-1}}, \frac{\sigma_t}{n_{t-1}}\right] \text{ and}$$
$$\frac{\partial \bar{y}_{t+1}}{\partial \lambda_{t+1}} = (1 - q_{t+1}^2)F_1\left[(1 - q_{t+1}^2)\zeta_{t+1} + \frac{n_{t+1}\gamma}{s_t}, \frac{\sigma_{t+1}}{n_t}\right].$$

6.2.1 Proof of Lemma 1

Since s_t and λ_t are growing at the steady rate $g(e) = zh(e) - 1 > 0$, we have

$$\frac{c_t^2}{c_{t+1}^3} = \frac{(1 - q^2)(1 - q^3)}{\rho n(1 - q^2 + \beta n)} \cdot \left(\frac{y_t - s_t}{\bar{y}_{t+1}} - \frac{\rho}{zh(e)}\right),$$

where

$$\frac{y_t - s_t}{\bar{y}_{t+1}} - \frac{\rho}{zh(e)} = \frac{1}{zh(e)}\left(\frac{F[(1 - q^2 - wne)\zeta, \sigma/n] - zh(e)}{F[(1 - q^2)\zeta, \sigma/n]} - \rho\right),$$

which is a constant for any given e. Substituting for c_t^2/c_{t+1}^3 from (3.26), we obtain

$$\frac{\rho\delta\sigma(1 - q^2 + \beta n)F_2[(1 - q^2)\zeta, \sigma/n]}{1 - q^2}$$
$$= \left(\frac{\rho n(1 - q^2 + \beta n)zh(e)F[(1 - q^2 - wne)\zeta, \sigma/n]}{(1 - q^2)(1 - q^3)[F[(1 - q^2 - wne)\zeta, \sigma/n] - \rho F[(1 - q^2)\zeta, \sigma/n] - zh(e)]}\right)^{\xi},$$

which may be rearranged as

$$F[(1 - q^2 - wne)\zeta, \sigma/n] - \rho F[(1 - q^2)\zeta, \sigma/n]$$
$$= \left(1 + B'\frac{F[(1 - q^2)\zeta, \sigma/n]}{(F_2[(1 - q^2)\zeta, \sigma/n])^{1/\xi}}\right) zh(e), \tag{6.16}$$

where

$$B' \equiv \frac{n}{(1 - q^3)(\delta\sigma)^{1/\xi}}\left(\frac{\rho(1 - q^2 + \beta n)}{1 - q^2}\right)^{1 - 1/\xi}$$

is a positive constant.

The assumption that F is homogeneous of degree 1, with both inputs are necessary in production, implies that ζ is differentiable in e when e is varied parametrically. For continuous changes in e produce continuous changes in the feasible set and the preference functional V_t; and the isoquant map is smooth everywhere and strictly convex to the origin, and no isoquant intersects either axis.

Part (i)

By assumption, F is Cobb-Douglas: $y_t = Al_t^{1-\alpha}k_t^{\alpha}$. Substituting into (6.16) and collecting terms, we have

$$\begin{aligned} &A\left(\frac{\sigma}{n}\right)^{\alpha}[(1-q^2-wne)^{1-\alpha}-\rho(1-q^2)^{1-\alpha}]\zeta^{1-\alpha} \\ &=\left(1+\left(\frac{n}{\alpha\sigma}\right)^{1/\xi}B'\cdot(F[(1-q^2)\zeta,\sigma/n])^{1-1/\xi}\right)zh(e). \end{aligned} \tag{6.17}$$

Differentiating (6.17) totally, noting that $\partial F/\partial\zeta = (1-\alpha)F/\zeta$, and collecting terms, we obtain

$$\begin{aligned} &\Bigg[A\left(\frac{\sigma}{n}\right)^{\alpha}[(1-q^2-wne)^{1-\alpha}-\rho(1-q^2)^{1-\alpha}](1-\alpha)\zeta^{-\alpha} \\ &-\left(1-\frac{1}{\xi}\right)\left(\frac{n}{\alpha\sigma}\right)^{1/\xi}\frac{(1-\alpha)B'}{\zeta}\cdot(F[(1-q^2)\zeta,\sigma/n])^{1-1/\xi}\Bigg]\cdot d\zeta \\ &=\Bigg[\left(1+\left(\frac{n}{\alpha\sigma}\right)^{1/\xi}B'\cdot(F[(1-q^2)\zeta,\sigma/n])^{1-1/\xi}\right)zh'(e) \\ &+A\left(\frac{\sigma}{n}\right)^{\alpha}[(1-q^2-wne)^{-\alpha}]wn(1-\alpha)\zeta^{1-\alpha}\Bigg]\cdot de. \end{aligned}$$

Now, the condition $F[(1-q^2-wn)\zeta,\sigma/n] > \rho F[(1-q^2)\zeta,\sigma/n]$ implies that $(1-q^2-wn)^{1-\alpha} > \rho(1-q^2)^{1-\alpha}$, so that ζ is increasing in e if $\xi \leq 1$. By continuity, this result also holds for all F sufficiently close to Cobb-Douglas in form and for all ξ exceeding, but sufficiently close to, 1.

Part (ii)

By assumption, $y_t = A[b_1 l_t^{\epsilon} + b_2 k^{\epsilon}]^{1/\epsilon}$, $\epsilon \leq 1$: the elasticity of substitution is $(\epsilon - 1)^{-1}$, where $\epsilon = 0$ is Cobb-Douglas. Proceeding as before,

$$\frac{F[(1-q^2)\zeta, \sigma/n]}{(F_2[(1-q^2)\zeta, \sigma/n])^{1/\xi}} = \frac{A[b_1((1-q^2)\zeta)^{\epsilon} + b_2(\sigma/n)^{\epsilon}]^{1/\epsilon}}{[b_2(\sigma/n)^{\epsilon-1} A[b_1((1-q^2)\zeta)^{\epsilon} + b_2(\sigma/n)^{\epsilon}]^{1/\epsilon - 1}]^{1/\xi}}$$
$$= B_1[b_1((1-q^2)\zeta)^{\epsilon} + b_2(\sigma/n)^{\epsilon}]^{\psi},$$

where $\psi = (1/\epsilon) + (1 - 1/\epsilon)/\xi$ and B_1 is a positive constant. Substituting into (6.16), noting the derivative of $[b_1((1-q^2)\zeta)^{\epsilon} + b_2(\sigma/n)^{\epsilon}]^{1/\psi}$ w.r.t. ζ and rearranging as before in part (i), there are two terms on the l.h.s. The first is the partial derivative of $\{F[(1 - q^2 - wne)\zeta, \sigma/n] - \rho F[(1-q^2)\zeta, \sigma/n]\}$ w.r.t. ζ, which is positive if $\epsilon \leq 0$ and $F[(1-q^2-wn)\zeta, \sigma/n] > \rho F[(1-q^2)\zeta, \sigma/n]$. The second term has the sign of $\psi \cdot \epsilon$. Now, $\psi \cdot \epsilon \leq 0$ iff $\epsilon + \xi \leq 1$. Since both inputs are assumed to be necessary in production, $\epsilon \leq 0$, which yields the required result. Q.E.D.

6.2.2 Proof of Corollary 1

Recalling that both inputs are necessary in production, it follows that for all such F, $F_1\left[(1-q^2)\zeta, \frac{\sigma}{n}\right]$ and $F_2\left[(1-q^2)\zeta, \frac{\sigma}{n}\right]$ are, respectively, decreasing and increasing in e. Now, $F_1\left[(1-q^2-wne)\zeta, \frac{\sigma}{n}\right]$ decreases more slowly than $F_1\left[(1-q^2)\zeta, \frac{\sigma}{n}\right]$ as e increases. Hence, if $h(e)$ is concave or sufficiently weakly convex, it is seen that D is increasing in e across paths. Q.E.D.

6.2.3 Derivation of Condition (3.20)

We simplify (3.18) using

$$F_1\left[l_t, \frac{\sigma s_{t-1}}{n}\right] = \frac{1-\alpha}{(1-q^2-wn)\zeta} F[(1-q^2-wn)\zeta, \sigma/n], \quad \text{and}$$
$$\frac{F_1[\bar{l}_{t+1}, \frac{\sigma s_t}{n}]}{F_2[\bar{l}_{t+1}, \frac{\sigma s_t}{n}]} = \frac{1-\alpha}{\alpha} \cdot \frac{\sigma}{n} \cdot \frac{1}{(1-q^2)\zeta}.$$

Noting that (6.16) specializes to

$$F\left[(1-q^2-wn)\zeta(1), \frac{\sigma}{n}\right] - \rho F\left[(1-q^2)\zeta(1), \frac{\sigma}{n}\right] = \left(1 + \frac{1}{(1-q^3)\alpha\delta}\right) zh(1),$$

we obtain

$$c_t^2 = s_{t-1}\left(\left(1 + \frac{1}{(1-q^3)\alpha\delta}\right) zh(1) - \frac{s_t}{s_{t-1}}\right) \frac{1}{1-q^2+\beta n} \quad \text{and}$$
$$c_t^2 = s_{t-1}\frac{1}{(1-q^3)\alpha\delta} \cdot \frac{zh(1)}{1-q^2+\beta n}.$$

Substituting into (3.18) and rearranging terms, we have

$$\lambda_{t+1}^{-1} s_{t-1} \frac{zh(1)}{(1-q^3)\alpha\delta} \geq$$
$$\left(\frac{w(1-\alpha)}{(1-q^2-wn)\zeta(1)} F[\cdot^p] - \frac{zh'(1)(1-\alpha)}{\zeta(1)n\alpha}\right) \cdot \frac{1}{bzh'(1)},$$

Since $\zeta_t = \lambda_t / s_{t-1}$ and $\lambda_{t+1}/\lambda_t \to zh(1)$ in the state of progress, a further rearrangement yields

$$zh'(1) \geq \left(\frac{w(1-\alpha)}{(1-q^2-wn)} F[\cdot^p] - \frac{zh'(1)(1-\alpha)}{n\alpha}\right) \cdot \frac{(1-q^3)\alpha\delta}{b},$$

so that

$$zh'(1) \geq \frac{w(1-\alpha)}{(1-q^2-wn)} F[\cdot^p] \cdot \left(\frac{b}{(1-q^3)\alpha\delta} + \frac{1-\alpha}{n\alpha} \right)^{-1}.$$

6.2.4 Derivation of Equation (3.31)

Total differentiation of

$$Q_t = n\big(\delta\rho F_1\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big] u'(c_{t+1}^3) + bv'(\lambda_{t+1})\big)$$

yields, noting (3.29) once more,

$$\begin{aligned} dQ_t &= \delta\rho\sigma F_{12}\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big] u'(c_{t+1}^3) ds_t + zh(e)\, d\lambda_t \cdot \\ & n\Bigg[\delta\rho\Bigg((1-q^2)F_{11}\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big] u'(c_{t+1}^3) + \frac{u''(c_{t+1}^3)c_{t+1}^3}{zh(e)\lambda_t} F_1\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big]\Bigg) + bv''(\lambda_{t+1})\Bigg] \\ &\equiv A' \cdot zh(e) \cdot d\lambda_t + \delta\rho\sigma F_{12}\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big] u'(c_{t+1}^3) ds_t \,. \end{aligned} \tag{6.18}$$

Next, we examine the expression $\frac{dQ_t}{d\lambda_t} \cdot \frac{\lambda_t}{Q_t}$ on the r.h.s. of (3.30). From (6.18), we have

$$\frac{dQ_t}{d\lambda_t} \cdot \frac{\lambda_t}{Q_t} = \frac{A' zh(e) \cdot \lambda_t + \delta\rho\sigma F_{12}\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big] u'(c_{t+1}^3) s_t}{n\big(\delta\rho F_1\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big] u'(c_{t+1}^3) + bv'(\lambda_{t+1})\big)}.$$

Collecting terms in the numerator involving $u'(c_{t+1}^3)$, the multiplicand is

$$J \equiv n\delta\rho \left[(1-q^2)\lambda_{t+1} F_{11}\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big] - \xi F_1\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big] + \frac{\sigma s_t}{n} F_{12}(\bar{l}_{t+1}, \frac{\sigma s_t}{n})\right].$$

Since $F_1\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big]$ and $F_2\big[\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big]$ are homogeneous of degree zero, it follows from

Euler's Theorem that

$$[(1-q^2)\lambda_{t+1}+n\gamma]F_{11}\big[\bar{l}_{t+1},\frac{\sigma s_t}{n}\big]+\frac{\sigma s_t}{n}F_{12}\big[\bar{l}_{t+1},\frac{\sigma s_t}{n}\big]=0,$$

so that for sufficiently large λ_t, J reduces to $-n\delta\rho\xi F_1\big[\bar{l}_{t+1},\frac{\sigma s_t}{n}\big]$. Hence, recalling that v is iso-elastic, we obtain the elasticity of Q_t w.r.t. λ_t:

$$\frac{dQ_t}{d\lambda_t}\frac{\lambda_t}{Q_t}=\frac{-\xi\delta\rho F_1\big[\bar{l}_{t+1},\frac{\sigma s_t}{n}\big]u'(c^3_{t+1})-\eta bv'(\lambda_{t+1})}{\delta\rho F_1\big[\bar{l}_{t+1},\frac{\sigma s_t}{n}\big]u'(c^3_{t+1})+bv'(\lambda_{t+1})}.$$

By hypothesis, (λ_t, s_t) are growing at the rate $g=zh(e)-1$. Hence, this elasticity can be expressed in the form

$$\frac{dQ_t}{d\lambda_t}\cdot\frac{\lambda_t}{Q_t}=-\frac{\xi A+\eta b\cdot B(1+g)^{-(\eta-\xi)}}{A+b\cdot B(1+g)^{-(\eta-\xi)}},$$

where $A=\delta\rho F_1\big[\bar{l}_{t+1},\frac{\sigma s_t}{n}\big]$ and $v'(\lambda_{t+1})/u'(c^3_{t+1})$ are positive constants along the path in question.

6.2.5 The extreme allocations of $S_t(I_t)$

In what follows, it will be useful to rewrite (3.35) in the form

$$\begin{aligned}&\left[1+\frac{\beta n}{1-q_t^2(I_t)}\right]c_t^2+\frac{s_t}{1-q_t^2(I_t)}+\rho F\left[\lambda_t+\frac{\gamma n}{1-q_t^2(I_t)},\frac{\sigma_t(I_t)}{1-q_t^2(I_t)}\cdot\frac{s_{t-1}}{n}\right]\\&\quad\leq F\left[\left(1-\frac{wne_t}{1-q_t^2(I_t)}\right)\lambda_t+\frac{\gamma n(1-e_t)}{1-q_t^2(I_t)},\frac{\sigma_t(I_t)}{1-q_t^2(I_t)}\cdot\frac{s_{t-1}}{n}\right],\ I_t=0,1.\end{aligned}\tag{6.19}$$

Allocation A: $c_t^2=e_t=0$. Given I_t, s_t is maximal. From (3.35), we have $s_t=(1-\rho)F\left[(1-q_t^2(I_t))\lambda_t+n\gamma,\frac{\sigma_t(I_t)s_{t-1}}{n}\right]$, $I_t=0,1$. Given I_t, an increase in $q_t^2(I_t)$ will induce A to shift towards the origin O, as depicted by the point A′. Given that $q_t^2(1)>q_t^2(0)$, the allocations A and A′ also represent those ruling under peace and war, respectively, in period t.

Allocation B: $c_t^2 = 0, e_t = 1$. Given I_t and maximum (full-time) investment in education, s_t is maximal. We have, for $I_t = 0, 1$,

$$s_t = F\left[(1 - q_t^2(I_t) - wn)\lambda_t, \frac{\sigma_t(I_t)s_{t-1}}{n}\right] - \rho F\left[(1 - q_t^2(I_t))\lambda_t + n\gamma, \frac{\sigma_t(I_t)s_{t-1}}{n}\right].$$

We begin by noting that the outer boundary of $S(I_t)$ in the plane defined by $c_t^2 = 0$, AB, is strictly concave in virtue of the strict concavity of F in each argument.

We next establish conditions under which the said value of s_t is positive, i.e., B lies to the right of G on the plane defined by $e_t = 1$. From (6.19), we have, for $I_t = 0, 1$,

$$\begin{aligned}(1 - q_t^2(I_t) - wn)^{-1}s_t =& F\left[\lambda_t, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t) - wn} \cdot \frac{s_{t-1}}{n}\right] \\ & - \rho F\left[\lambda_t + \frac{n(w\lambda_t + \gamma)}{1 - q_t^2(I_t) - wn}, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t) - wn} \cdot \frac{s_{t-1}}{n}\right].\end{aligned}$$

The input of human capital in the second term on the r.h.s. is larger in the proportion $n(w + \gamma/\lambda_t)/(1 - q_t^2(I_t) - wn)$. This proportion is maximal when $\lambda_t = 1$. Since F is homogeneous of degree one, $\lambda_t = 1$ will therefore yield the best chance that $s_t < 0$, as intuition would suggest. Now, w is fairly small, say about 1/20, and γ would be about 0.6. In such a state of economic backwardness, $n = 3/2$ and $q_t^2 = 0.2$ are broadly plausible, so that the said proportion of inputs of human capital would be about 4/3. Hence,

$$\lambda_t + \frac{n(w\lambda_t + \gamma)}{1 - q_t^2(I_t) - wn} \leq 7\lambda_t/3, \ \forall \lambda_t.$$

Observe, however, that F is strictly concave in each argument alone and ρ is unlikely to exceed 1/3. Comparing the two terms on the r.h.s., inputs of human capital in the second are slightly more than double those in the first, but the share in the resulting output is at most one-third. It follows that, for plausible values of parameters and demographic variables, $s_t(c_t^2 = 0, e_t = 1) > 0$ for all values of λ_t, and points B and B′ are correspondingly depicted in the diagram.

An increase in $q_t^2(I_t)$ induces a larger movement in B than in A. For the said difference in s_t is

$$F\left[(1 - q_t^2(I_t))\lambda_t + n\gamma, \frac{\sigma_t(I_t)s_{t-1}}{n}\right] - F\left[(1 - q_t^2(I_t) - wn)\lambda_t, \frac{\sigma_t(I_t)s_{t-1}}{n}\right],$$
$$I_t = 0, 1,$$

which is increasing in $q_t^2(I_t)$ in virtue of the strict concavity of F in each argument. If the cross-derivative F_{12} is sufficiently small, it is seen that the same claim will hold concerning a comparison of peace and war, respectively.

Allocation C: $e_t = s_t = 0$. Given I_t, c_t^2 is maximal. From (3.35), we have

$$c_t^2 = \frac{1 - q_t^2(I_t)}{1 - q_t^2(I_t) + \beta n} \cdot (1 - \rho)F\left[\lambda_t + \frac{n\gamma}{1 - q_t^2(I_t)}, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t)} \cdot \frac{s_{t-1}}{n}\right], \ I_t = 0, 1.$$

Suppose the ratio of survival rates is fixed for each I_t. Then

$$\frac{dc_t^2}{dq_t^2(I_t)} = \frac{(1 - \rho)n}{(1 - q_t^2(I_t) + \beta n)^2} \cdot \left[-\beta F[\cdot] + \frac{\gamma(1 - q_t^2(I_t) + \beta n)F_1[\cdot]}{1 - q_t^2(I_t)}\right], \ I_t = 0, 1,$$
$$\text{with} \quad F[\cdot] = F\left[\lambda_t + \frac{n\gamma}{1 - q_t^2(I_t)}, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t)} \cdot \frac{s_{t-1}}{n}\right]$$
$$\text{and} \quad F_1[\cdot] = F_1\left[\lambda_t + \frac{n\gamma}{1 - q_t^2(I_t)}, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t)} \cdot \frac{s_{t-1}}{n}\right],$$

so that $(dc_t^2/dq_t^2(I_t))_{e_t = s_t = 0} < 0$ iff

$$\beta > \frac{\gamma(1 - q_t^2(I_t) + \beta n)F_1[\cdot]}{(1 - q_t^2(I_t))F[\cdot]}.$$

The input of human capital is $\lambda_t + n\gamma/(1 - q_t^2(I_t))$. Its imputed share in output is $1-\alpha \equiv (\lambda_t + n\gamma(1-q_t^2(I_t))^{-1})F_1[\cdot]/F[\cdot]$, so the foregoing inequality can be written

$$\beta > \frac{(1-\alpha)\gamma(1 - q_t^2(I_t) + \beta n)}{(1 - q_t^2(I_t))\lambda_t + \gamma n},$$

which certainly holds for all sufficiently large λ_t. The denominator takes a minimum under backwardness ($\lambda_t = 1$), when the inequality becomes

$$(\beta - (1-\alpha)\gamma)(1 - q_t^2(I_t)) + \alpha\beta\gamma n > 0.$$

Since both inputs are necessary in production, F is strictly concave in both arguments and $\alpha \in (0, 1)$. It is plausible that $\alpha < 0.5$, but $n \geq 1$, so that the inequality may hold even if $\beta < \gamma$, as in Table 3.1, for which constellation the inequality holds.

Under the assumption that the ratio of survival rates is fixed for each I_t, we have established that the points C and C$'$ relate to each other as depicted in the figure, which reveals that there is damage even under a mild mortality shock, given I_t. If the ratio of survival rates is the same in both states, the points C and C$'$ also represent the respective allocations in peace and war.

Allocation D: $e_t = 1, s_t = 0$. Given I_t and maximum investment in (full-time) education, c_t^2 is maximal. Analogously to AB, the outer boundary of S in the plane defined by $s_t^2 = 0$, CD, is strictly concave in virtue of the strict concavity of F in each argument.

Given e_t and I_t, all pairs (c_t^2, s_t) on the outer frontier of S are linearly related and independent of $e_t : ds_t = -(1 - q_t^2(I_t) + \beta n)\, dc_t^2$. Hence, AC is parallel to BD, and A$'$C$'$ to B$'$D$'$. An increase in $q_t^2(I_t)$ makes c_t^2 cheaper relative to s_t; but since C$'$ lies closer to O than does C, it follows that D$'$ lies closer to G than does D. The same holds when the ratio of survival rates is the same in both states.

6.2.6 Convergence analysis

Equations (3.22) and (3.23) provide our starting point. We take their total derivatives, given that the right-hand sides depend on λ_t and e_t. We, then, equate the left-hand sides and find the sign of $de_t/d\lambda_t$, that is, of the derivative $\partial e_t^0/\partial\lambda_t$: it turns out to be positive. Hence, it follows from (3.4) that a higher value of λ_t also increases λ_{t+1} through its indirect effect on the choice e_t^0. Two further statements can be made. First, if an economy has reached the progressive state and is then hit by a sufficiently small shock, e_t^0 will fall, if at all, not much below 1, and the economy will remain in, or return to, that state. Second, if λ_t is close to some stationary level associated with $e_t^0 < 1$ and the economy is hit by a sufficiently adverse shock, there will be a descent into backwardness.

In the following, we take the derivatives of the aforementioned equations with respect to e_t and λ_t. We then analyze their signs when $(e_t, \lambda_t) = (0, 1)$ and $e_t = 1$ with λ_t large.

The derivative of the right hand side of (3.22) with respect to e_t is

$$\left[\left((1-q^2-wne_t)+\frac{n\gamma(1-e_t)}{\lambda_t}\right)^{1-\alpha}-\rho\left(1-q^2+\frac{n\gamma}{\lambda_t}\right)^{1-\alpha}\right]^{-1}\cdot$$

$$\left\{zh'(e_t)+\left(zh(e_t)+\frac{1}{\lambda_t}\right)\frac{\left[\left(1-q^2-wne_t+\frac{n\gamma(1-e_t)}{\lambda_t}\right)^{1-\alpha}-\rho(1-q^2+\frac{n\gamma}{\lambda_t})^{1-\alpha}\right]^{-1}}{\left(1-q^2-wne_t+\frac{n\gamma(1-e_t)}{\lambda_t}\right)^{\alpha}}\cdot\right.$$

$$\left.(1-\alpha)\left(w+\frac{\gamma}{\lambda_t}\right)n\right\}\left(\frac{n}{\sigma}\right)^{\alpha}A^{-1}\left(1+\frac{1}{\alpha\delta(1-q^3)}\right).$$

When $e_t = 0$ and $\lambda_t = 1$, this yields

$$\left[A(1-\rho)(1-q^2+n\gamma)^{1-\alpha}\right]^{-1}\left\{zh'(0)+\frac{(1-\alpha)(w+\gamma)n}{(1-\rho)(1-q^2+n\gamma)}\right\}\cdot$$
$$\left(\frac{n}{\sigma}\right)^{\alpha}\left(1+\frac{1}{\alpha\delta(1-q^3)}\right),$$

which is positive.

When $e_t = 1$ and λ_t is large, we have

$$\left(\frac{n}{\sigma}\right)^{\alpha}\left(1+\frac{1}{\alpha\delta(1-q^3)}\right)A^{-1}\left[(1-q^2-wn)^{1-\alpha}-\rho(1-q^2+\frac{n\gamma}{\lambda_t})^{1-\alpha}\right]^{-1}\cdot$$
$$\left\{\left(zh(1)+\frac{1}{\lambda_t}\right)\frac{\left[(1-q^2-wn)^{1-\alpha}-\rho(1-q^2+\frac{n\gamma}{\lambda_t})^{1-\alpha}\right]^{-1}(1-\alpha)\left(w+\frac{\gamma}{\lambda_t}\right)n}{(1-q^2-wn)^{\alpha}}+\right.$$
$$\left.zh'(1)\right\}$$

which is also positive.

The derivative of (3.22) with respect to λ_t is

$$\left[\left((1-q^2-wne_t)+\frac{n\gamma(1-e_t)}{\lambda_t}\right)^{1-\alpha}-\rho\left(1-q^2+\frac{n\gamma}{\lambda_t}\right)^{1-\alpha}\right]^{-1}\cdot$$
$$\left\{-1+\frac{\left(1-q^2-wne_t+\frac{n\gamma(1-e_t)}{\lambda_t}\right)^{-\alpha}(1-e_t)-\rho\left(1-q^2+\frac{n\gamma}{\lambda_t}\right)^{-\alpha}}{\left(1-q^2-wne_t+\frac{n\gamma(1-e_t)}{\lambda_t}\right)^{1-\alpha}-\rho\left(1-q^2+\frac{n\gamma}{\lambda_t}\right)^{1-\alpha}}\cdot\right.$$
$$\left.(1-\alpha)n\gamma\left(zh(e_t)+\frac{1}{\lambda_t}\right)\right\}\left(\frac{n}{\sigma}\right)^{\alpha}\left(1+\frac{1}{\alpha\delta(1-q^3)}\right)\frac{1}{A\lambda_t^2}.$$

Evaluating this expression at $e_t = 0$ and $\lambda_t = 1$ gives

$$\left(\frac{n}{\sigma}\right)^{\alpha}\left(1+\frac{1}{1+\alpha\delta(1-q^3)}\right)\left(A\left[(1-q^2+n\gamma)^{1-\alpha}(1-\rho)\right]\right)^{-1}\left\{-1+\frac{(1-\alpha)n\gamma}{1-q^2+n\gamma}\right\},$$

which can be negative if $1-q^2 > -\alpha n\gamma$.

When $e_t = 1$ and λ_t is large, we have

$$-\left(\frac{n}{\sigma}\right)^{\alpha}\left(1+\frac{1}{\alpha\delta(1-q^3)}\right)\frac{1}{A\lambda_t^2}\left[\left((1-q^2-wn)+\frac{n\gamma}{\lambda_t}\right)^{1-\alpha}-\rho\left(1-q^2+\frac{n\gamma}{\lambda_t}\right)^{1-\alpha}\right]^{-1}\cdot$$
$$\left\{1+\left(zh(1)+\frac{1}{\lambda_t}\right)\frac{(1-\alpha)\rho n\gamma\left(1-q^2+\frac{n\gamma}{\lambda_t}\right)^{-\alpha}}{\left(1-q^2-wn+\frac{n\gamma}{\lambda_t}\right)^{1-\alpha}-\rho\left(1-q^2+\frac{n\gamma}{\lambda_t}\right)^{1-\alpha}}\right\},$$

which is clearly negative.

We repeat these steps in Equation (3.23). The derivative of the right hand side with respect to e_t is

$$\frac{\left(\frac{n}{\sigma}\right)^{\alpha}zh'(e_t)\left(1-q^2-wne_t+\frac{n\gamma(1-e_t)}{\lambda_t}\right)^{\alpha}}{\alpha A\delta n(1-q^3)\left(w+\frac{\gamma}{\lambda_t}\right)}\left\{\frac{\delta(1-q^3)(1-q^2)}{\left(1-q^2+\frac{n\gamma}{\lambda_{t+1}}\right)^2}\frac{n\gamma\lambda_t}{\lambda_{t+1}^2}zh'(e_t)\right.$$
$$\left.+\left(\frac{\delta(1-q^3)(1-q^2)}{1-q^2+\frac{n\gamma}{\lambda_{t+1}}}+\frac{bn}{1-\alpha}\right)\left[\frac{h''(e_t)}{h'(e_t)}-\frac{\alpha n\left(w+\frac{\gamma}{\lambda_t}\right)}{1-q^2-wne_t+\frac{n\gamma(1-e_t)}{\lambda_t}}\right]\right\}.$$

When $e_t = 0$ and $\lambda_t = 1$, we have

$$\frac{\left(\frac{n}{\sigma}\right)^{\alpha}zh'(0)\left(1-q^2+n\gamma\right)^{\alpha}}{\alpha A\delta n(1-q^3)\left(w+\gamma\right)}\left\{\frac{\delta(1-q^3)(1-q^2)n\gamma zh'(0)}{(1-q^2+n\gamma)^2}+\left[\frac{h''(0)}{h'(0)}-\frac{\alpha n(w+\gamma)}{1-q^2+n\gamma}\right]\cdot\right.$$
$$\left.\left(\frac{\delta(1-q^3)(1-q^2)}{1-q^2+n\gamma}+\frac{bn}{1-\alpha}\right)\right\}.$$

When $e_t = 1$ and λ_t is large, we have

$$\frac{\left(\frac{n}{\sigma}\right)^{\alpha}zh'(1)(1-q^2-wn)^{\alpha}}{\alpha A\delta n(1-q^3)\left(w+\frac{\gamma}{\lambda_t}\right)}\left\{\frac{\delta(1-q^3)(1-q^2)}{\left(1-q^2+\frac{n\gamma}{\lambda_{t+1}}\right)^2}\frac{n\gamma\lambda_t zh'(1)}{\lambda_{t+1}^2}\right.$$
$$\left.+\left[\frac{h''(1)}{h'(1)}-\frac{\alpha n\left(w+\frac{\gamma}{\lambda_t}\right)}{1-q^2-wn}\right]\left(\frac{\delta(1-q^3)(1-q^2)}{1-q^2+\frac{n\gamma}{\lambda_{t+1}}}+\frac{bn}{1-\alpha}\right)\right\}.$$

Both can be negative if h is concave or sufficiently weakly convex.

Finally, we examine the derivative of the right hand side of (3.23) with respect to λ_t:

$$\frac{\left(\frac{n}{\sigma}\right)^{\alpha}\left(1-q^2-wne_t+\frac{n\gamma(1-e_t)}{\lambda_t}\right)^{\alpha} zh'(e_t)}{(\alpha A\delta n(1-q^3))\left(w+\frac{\gamma}{\lambda_t}\right)\lambda_t^2}\left\{\frac{\delta(1-q^3)(1-q^2)}{\left(1-q^2+\frac{n\gamma}{\lambda_{t+1}}\right)^2}\frac{n\gamma\lambda_t^2 zh'(e_t)}{\lambda_{t+1}^2}\right.$$
$$\left.+\left(\frac{\delta(1-q^3)(1-q^2)}{1-q^2+\frac{n\gamma}{\lambda_{t+1}}}+\frac{bn}{1-\alpha}\right)\left[\frac{\gamma}{w+\frac{\gamma}{\lambda_t}}-\frac{\alpha n\gamma(1-e_t)}{1-q^2-wne_t+\frac{n\gamma(1-e_t)}{\lambda_t}}\right]\right\}.$$

Plugging in $e_t = 0$ and $\lambda_t = 1$ yields

$$\frac{\left(\frac{n}{\sigma}\right)^{\alpha}(1-q^2+n\gamma)^{\alpha} zh'(0)}{(\alpha A\delta n(1-q^3))(w+\gamma)}\left\{\left(\frac{\delta(1-q^3)(1-q^2)}{1-q^2+n\gamma}+\frac{bn}{1-\alpha}\right)\right.$$
$$\left.\left[\frac{\gamma}{w+\gamma}-\frac{\alpha n\gamma}{1-q^2+n\gamma}\right]\right\},$$

where $1-q^2 > \alpha wn$ is a necessary condition for the derivative to be positive.

When $e_t = 1$ and λ_t is large, the derivative is

$$\left\{\frac{\delta(1-q^3)(1-q^2)}{\left(1-q^2+\frac{n\gamma}{\lambda_{t+1}}\right)^2}\frac{n\gamma\lambda_t^2 zh'(1)}{\lambda_{t+1}^2}+\left(\frac{\delta(1-q^3)(1-q^2)}{1-q^2+\frac{n\gamma}{\lambda_{t+1}}}+\frac{bn}{1-\alpha}\right)\frac{\gamma}{w+\frac{\gamma}{\lambda_t}}\right\}\cdot$$
$$\frac{\left(\frac{n}{\sigma}\right)^{\alpha}(1-q^2-wn)^{\alpha} zh(1)}{(\alpha A\delta n(1-q^3))\left(w+\frac{\gamma}{\lambda_t}\right)\lambda_t^2},$$

which is positive.

In summary, we find that the derivative of (3.22) w.r.t. e_t is positive and that w.r.t. λ_t is negative. The opposite holds for (3.23), given the previously stated conditions. Equating the total derivatives yields

$$[+]\cdot de_t+[-]\cdot d\lambda_t=[-]\cdot de_t+[+]\cdot d\lambda_t,$$

where [+] and [−] stand for the respective derivatives and their signs. It follows that $de_t/d\lambda_t > 0$ for all $e_t \in (0,1)$.

In the cases where (3.23) holds as a strict inequality, we can make the following statements. If the right hand-side of (3.23) is larger than the left-hand side, then $de_t/d\lambda_t$ is larger than some positive value. In the opposite case ($e_t = 0$), $de_t/d\lambda_t$ is smaller than this positive value, thus introducing the possibility of a negative relationship. Yet, if (3.23) holds almost with equality, we can conclude from continuity considerations that the derivative will be positive.

6.2.7 Analysis for the simulations

The optimization problem under uncertainty is specified by (3.33)-(3.35). The term $u(c_t^2)$ in the objective function is unchanged, but its derivatives with respect to s_t and e_t require close attention. We have

$$\begin{aligned}\frac{\partial u(c_{t+1}^3(I_{t+1}))}{\partial s_t} =& u'(c_{t+1}^3(I_{t+1})) \cdot \frac{\rho n}{(1-q_t^2(I_t))(1-q_{t+1}^3(I_{t+1}))} \frac{\sigma_{t+1}(I_{t+1})}{n} \cdot \\ & F_2[\bar{l}_{t+1}, \sigma_{t+1}(I_{t+1})s_t/n] \\ \frac{\partial u(c_{t+1}^3(I_{t+1}))}{\partial e_t} =& u'(c_{t+1}^3(I_{t+1})) \cdot \frac{\rho n}{(1-q_t^2(I_t))(1-q_{t+1}^3(I_{t+1}))}(1-q_{t+1}^2(I_{t+1})) \cdot \\ & zh'(e_t)\lambda_t F_1[\bar{l}_{t+1}, \sigma_{t+1}(I_{t+1})s_t/n].\end{aligned}$$

After defining $E_t[x_{t+1}] = \pi_{t+1}x_{t+1}(0) + (1-\pi_{t+1})x_{t+1}(1)$ for some variable x and substituting for the above derivatives, we obtain the following two equations:

$$\frac{\delta\rho}{1-q_t^2(I_t)}E_t\left[u'(c_{t+1}^3)\sigma_{t+1}(I_{t+1})F_2[\bar{l}_{t+1},\sigma_{t+1}(I_{t+1})s_t/n]\right]=\frac{u'(c_t^2)}{1-q_t^2(I_t)+\beta n},$$

$$\delta E_t\left[\frac{u'(c_{t+1}^3)\rho n(1-q_{t+1}^2(I_{t+1}))}{1-q_t^2(I_t)}F_1[\bar{l}_{t+1},\sigma_{t+1}(I_{t+1})s_t/n]\right]zh'(e_t)\lambda_t$$
$$+\widetilde{b}nv'(\lambda_{t+1})zh'(e_t)\lambda_t=\frac{u'(c_t^2)(wn\lambda_t+n\gamma)}{(1-q_t^2(I_t)+\beta n)}F_1[l_t,\sigma_t(I_t)s_{t-1}/n],$$

where $\widetilde{b}=b\frac{\pi_{t+1}(1-q_{t+1}^2(0))+(1-\pi_{t+1})(1-q_{t+1}^2(1))}{(1-q_t^2(I_t))}=b\cdot E_t[1-q_{t+1}^2]/(1-q_t^2)$.

In the next step, we substitute the following expressions into the two equations:

$$u'(c_t^2(I_t))=\left(\frac{A(l_t^{1-\alpha}-\rho\bar{l}_t^{1-\alpha})\left(\frac{\sigma_t(I_t)s_{t-1}}{n}\right)-s_t}{1-q_t^2(I_t)+\beta n}\right)^{-1},$$

$$u'(c_{t+1}^3(I_{t+1}))=\left(\frac{\rho n}{(1-q_t^2(I_t))(1-q_{t+1}^3(I_{t+1}))}F[\bar{l}_{t+1},\sigma_{t+1}(I_{t+1})s_t/n]\right)^{-1}.$$

This yields Equations (3.36) and (3.37).

6.3 Appendix for Chapter 4

6.3.1 Proof of Proposition 17

To see that the condition provided above is indeed necessary, consider the term $2\left(1+\left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}}\right)^{-1}$. This term is smaller than one, as $\widetilde{\psi}_A>\psi_B$ by assumption. This implies that the term $\left((1-\gamma_1)(1-\alpha)+\frac{\alpha}{2}\psi_B\gamma_3\left(1+\left(\frac{\widetilde{\psi}_A}{\psi_B}\right)^{\frac{1}{1-\alpha}}\right)\right)$ must be larger than one, which is only possible if (4.14) holds.

6.3.2 Policy function for capital

The policy function for capital can be derived in two ways. We assume that only A grows, so that $K_t^A \gg K_t^B$ and $\phi_t L^2 \gg \eta L^1$ for $t \to \infty$. The inflow of capital from B has virtually no effect on the overall income in A and the cost of sending children to school becomes negligible. Hence we have $\widetilde{K}_t^A = K_t^A$.

The first way to derive the equation is by using this assumption and consider country A as a closed economy, so that

$$K_{t+1}^A = \frac{\beta A}{1+\beta}(H_t^A)^{1-\alpha}(K_t^A)^\alpha \left((1-\alpha)(1-\gamma_1) + \alpha\gamma_3\widetilde{\psi}_A\right),$$

where $H_t^A = \phi_t L^2$. Dividing both sides by H_{t+1}^A delivers the results.

The second way is to begin with the general savings equation, in some period, where $K_t^A \neq K_t^B$, i.e. the initial allocation of physical capital is not the same, but it holds that $K_t^A + K_t^B = K_t = \widetilde{K}_t^A + \widetilde{K}_t^B$. With the stock of human capital also differing across countries, we have

$$\begin{aligned} K_{t+1}^A =& \frac{\beta A}{1+\beta}(\phi_t L^2)^{1-\alpha}(K_t^A)^\alpha \left(1 + \chi^{\frac{-1}{1-\alpha}} H_t^{-1}\right)^{-\alpha} \cdot \\ & \left[(1-\alpha)(1-\gamma_1) + \alpha\widetilde{\psi}_A\gamma_3 \left(1 + \chi^{\frac{-1}{1-\alpha}} H_t^{-1}\right) \frac{K_t^A}{K_t}\right]. \end{aligned}$$

As country A grows, it holds almost the entire share of international capital, so that $K_t^A \approx K_t$. Also, H_t becomes very large and the term $H_t(1 + \chi^{\frac{1}{1-\alpha}} H_t)^{-1}$ converges to 1, yielding the policy function.

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