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Convex Hulls of Random Order Types

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Abstract

We establish the following two main results on order types of points in general position in the plane (realizable simple planar order types, realizable uniform acyclic oriented matroids of rank 3):

(a) The number of extreme points in an n-point order type, chosen uniformly at random from all such order types, is on average $4 + o(1)$. For labeled order types, this number has average $4 - \frac{8}{n^2-n+2}$ and variance at most 3.

(b) The (labeled) order types read off a set of n points sampled independently from the uniform measure on a convex planar domain, smooth or polygonal, or from a Gaussian distribution are concentrated, i.e., such sampling typically encounters only a vanishingly small fraction of all order types of the given size.

Result (a) generalizes to arbitrary dimension d for labeled order types with the average number of extreme points $2d + o(1)$ and constant variance. We also discuss to what extent our methods generalize to the abstract setting of uniform acyclic oriented matroids. Moreover, our methods allow to show the following relative of the Erdős-Szekeres theorem: for any fixed k, as $n \to \infty$, a proportion $1 - O(1/n)$ of the n-point simple order types contain a triangle enclosing a convex k-chain over an edge.

For the unlabeled case in (a), we prove that for any antipodal, finite subset of the 2-dimensional sphere, the group of orientation preserving bijections is cyclic, dihedral or one of A_4, S_4 or A_5 (and each case is possible). These are the finite subgroups of $SO(3)$ and our proof follows the lines of their characterization by Felix Klein.

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1 Introduction

Two finite subsets P and Q of the plane are said to *have the same order type* if there exists a bijection $f\colon P \to Q$ that preserves orientations: for any three points p, q, r in P, r is to the left (resp. to the right) of the line (pq) oriented from p to q if and only if $f(r)$ is to the left

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(resp. to the right) of the line $(f(p)f(q))$ oriented from $f(p)$ to $f(q)$. To have the same order type is an equivalence relation, and an *order type* is an equivalence class for that relation. For each n, properties that depend solely on orientations can be established for the infinitely many n-point sets by proving it for (one representative of) each of the finitely many order types of size n. This even allows proofs by automated case analysis, see for instance [2] for an application to crossing numbers and asymptotic estimates on numbers of triangulations. This notion was studied in discrete and computational geometry as a higher-dimensional analogue of ordering on a line, but also as a geometric relative of oriented matroids, see e.g., [18, 26, 9, 13].

In this paper, we investigate the expected number of *extreme* points in a typical order type. (Since the number of extreme points is the same for all representatives of an order type, we speak of the number of extreme points of the order type; we do the same for every notion independent of the choice of representative, e.g., the size.) Here we consider only simple order types, i.e., with no three points on a line; by "typical" we mean an order type chosen equiprobably among all simple order types of a given size n. As an illustration, for $n = 4$, the only two simple order types are the convex quadrilateral and the triangle with an interior point, so the quantity we are after is $\frac{4+3}{2} = \frac{7}{2}$. For $n = 5$, it is $\frac{5+4+3}{3} = 4$, see Figure 1.

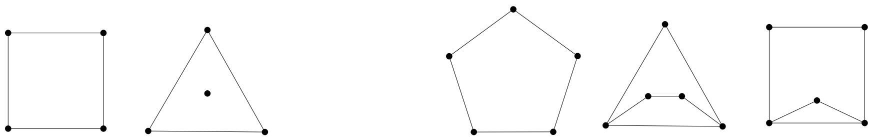

Figure 1 Left: The two simple 4-point order types. Right: The three simple 5-point order types.

1.1 Motivations

Let us say a word on our motivations.

Testing. We are interested in statistics of the uniform distribution on the space of order types. Broadly speaking, this distribution is relevant whenever one wants to *test* a property of finite point sets. Consider the two following examples:

(a) The largest point set in general position with no empty hexagon is known to have size between 29 and 1716 [32, 16], and it is tempting to try and improve the lower bound by testing order types of size 30 or so.

(b) The CGAL library [38] stresses the need, when implementing geometric algorithms, to rely solely on predicates that depend on the input of the algorithm, so as to encapsulate the numerical issues (critical for robustness [25]) into the correct evaluation of signs of polynomials. Thus, the implementation of an algorithm that depends only on orientation predicates can be assessed by running it on a realization of each possible order type.

In both cases, we want to avoid repeating the same order type, as this is redundant computation, and to be able in principle to reach every existing order type without uncontrolled bias. The uniform distribution is natural to consider for that purpose.

Random polytopes. Counting extreme points relates to the study of face vectors of random polytopes, a classical line of research in stochastic geometry initiated by Sylvester in 1865, who asked for "the probability that 4 points in the plane are in convex position". A standard model of random polytope K_n is the convex hull of n random points chosen uniformly and independently in some fixed convex body K. In this setting, the number of extreme points, i.e., the vertices of K_n, is well understood. Its average is asymptotically proportional to $n^{\frac{d-1}{d+1}} + o\left(n^{\frac{d-1}{d+1}}\right)$ if K is smooth and to $\log^{d-1} n + o\left(\log^{d-1} n\right)$ if K is a polytope [34, 35] (see [33, §2.2.2]), and up to multiplicative constant these are the two extremes [6, Theorems 1–3]. There are also estimates on the variance, concentration inequalities, central limit theorems, and large deviation inequalities. We refer the interested reader to the survey of Reitzner [33].

This model of random polytope naturally generalizes to arbitrary probability measures μ, or even to the convex hull of random non-independent point sets such as determinantal point processes. Much less is known in this direction, aside from the occasional extensively-studied model such as Gaussian polytopes (see [33, §2.3]). In a sense, what we investigate is the average number of extreme points in a random polytope for a *combinatorially defined* probability distribution on point sets.

Exploration of order types. The space of order types is generally not well understood. Already, its size is not known precisely, not even asymptotically. The most precise bounds are given for *labeled* order types, which declare two point *sequences* $P = (p_1, p_2, \ldots)$ and $Q = (q_1, q_2, \ldots)$ equivalent if the monotone map $p_i \mapsto q_i$ preserves orientations: there are $n^{4n}\phi(n)$ labeled order types, where $2^{-cn} \leq \phi(n) \leq 2^{c'n}$ for some positive constants c, c' [19, 3]; factoring out the labelling is not immediate as the number of labeled order types corresponding to a given unlabeled one depends on the symmetries of the latter. We show that in the plane, every unlabeled order type corresponds to at least $(n-1)!$ (and clearly at most $n!$) different labeled ones. Order types have been tabulated up to size 11 [1, 2], for which they are already counted in billions.

Random sampling of order types is also quite unsatisfactory. First, the standard methods in discrete random generation such as Boltzmann samplers are unlikely to work here, as they require structural results (such as recursive decompositions) that usually make counting a routine task. It is of course easy to produce a random order type by merely reading off the order type of n random points; standard models include points chosen independently from the uniform distribution in a square or a disk, from a Gaussian distribution, as well as points obtained as a random 2-dimensional projection of a n-dimensional simplex[1]. There are no results, however, on how well or badly distributed the order types of such random point sets are. More generally, no random generation method is known to be both efficient (say, taking polynomial time per sample) and with controlled bias. This sad state of affairs can perhaps be explained by two fundamental issues: when working with order types symbolically (say as orientation maps to $\{-1, 0, 1\}$, see Section 1.4 below), one has to work around the NP-hardness (actually, $\exists\mathbb{R}$-completeness) of membership testing [37, 29, 36]. When working with explicit point sets, one has to account for the exponential growth of the worst-case number of coordinate bits required to realize an order type of size n [20]. It turns out that our bounds on the expected number of extreme points in an order type imply that several standard models of random point sets typically explore only a vanishingly small fraction of the space of order types (Theorem 3).

Order types with forbidden patterns. Given two order types ω and τ, we say that ω *contains* τ if any point set that realizes ω contains a subset that realizes τ. (Of course this needs only be checked for a single realization of ω.) By the Erdös-Szekeres theorem [14],

[1] This is called the Goodman-Pollack model and is statistically equivalent to points chosen independently from a Gaussian distribution [7, Theorem 1].

almost all order types contain the order type of k points in convex position. Similarly, Carathéodory's theorem implies that almost all order types contain the order type of a triangle with one interior point. Could it be that for any *fixed* order type τ, the number of order types of size n that *do not* contain τ is vanishingly small as $n \to \infty$? This question may seem quite bold given the limited number of observations, but it is also motivated by an analogous phenomenon for permutations: the Marcus-Tardos theorem [27] asserts that for every fixed permutation π, the number of size-n permutations that *do not* contain π is at most exponential in n (see [27] for the definition of containment). We are not aware of any result on such a Marcus-Tardos phenomenon for order types besides the two simple cases mentioned above. It turns out that along the way, we prove some new results in this direction as well (Theorem 4).

1.2 Results

Our first result is on labeled order types. Two affine point sequences $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n)$ are defined to be of the same *labeled order type* if the map $p_i \mapsto q_i$ preserves orientations: for any indices $1 \leq i, j, k \leq n$, p_k is to the left (resp. to the right) of the line $(p_i p_j)$ oriented from p_i to p_j if and only if q_k is to the left (resp. to the right) of the line $(q_i q_j)$ oriented from q_i to q_j. The labeled order type of a point sequence is *simple* if no three points of that sequence are aligned.

▶ **Theorem 1.** *For $n \geq 3$, the number of extreme points in a random simple labeled order type chosen uniformly among the simple, labeled order types of size n in the plane has average $4 - \frac{8}{n^2-n+2}$ and variance at most 3.*

A set of n points gives rise to $n!$ point sequences, but different sequences may have the same labeled order type. The exact number of labeled order types corresponding to a given order type actually depends on the number of order-preserving bijections, that is *symmetries*, of that order type. We show that the symmetries of a simple affine order type form a (possibly trivial) cyclic group (Theorem 6) and we bound from above the number of simple affine order types with many, but not too many, symmetries. We then prove a non-labeled analogue of Theorem 1:

▶ **Theorem 2.** *For $n \geq 3$, the number of extreme points in a random simple order type chosen uniformly among the simple order types of size n in the plane has average $4 + O\left(n^{-3/4+\varepsilon}\right)$ for any $\varepsilon > 0$.*

Our proof of Theorem 1 extends to arbitrary dimension, but not our proof of Theorem 2. A large part of our methods and results extend to *abstract order types*, that is uniform oriented matroids, where lines are replaced by pseudo-line arrangements. In particular, Theorem 1 holds in the abstract setting with the same bound, also in arbitrary dimension. The proof of Theorem 2 does not completely carry over to the abstract setting, but our methods yield an analogue statement with a bound of $10 + o(1)$.

Theorems 1 and 2 are in sharp contrast with the $\Omega(\log n)$, and possibly polynomially many, extreme points in a uniform random sample of a convex planar domain. Theorems 1 and 2 can actually be used to turn concentration bounds on the number of extreme points in a random point set into concentration results on the distribution of order types produced by these random point sets.

We need some definitions. Let $\mathsf{(L)OT}_n^{\text{aff}}$ denote the set of simple (labeled) affine order types. For $n \geq 3$, let μ_n be a probability measure on $\mathsf{(L)OT}_n^{\text{aff}}$. We say that the family $\{\mu_n\}_{n \geq 3}$ *exhibits concentration* if for every $n \geq 3$ there exists $A_n \subseteq \mathsf{(L)OT}_n^{\text{aff}}$ such that

$\mu_n(A_n) \to 1$ and $|A_n|/|(\mathsf{L})\mathsf{OT}_n^{\text{aff}}| \to 0$. In plain English, families of measures that exhibit concentration typically explore a vanishingly small fraction of the space of simple (labeled) order types. Devillers et al. [12] conjectured that the order types of points sampled uniformly and independently from a unit square exhibit concentration. We prove this conjecture and more:

▶ **Theorem 3.** *Let μ be a probability measure on $\mathbb{R}^2$ given by one of the following: (a) the uniform distribution on a smooth compact convex set, (b) the uniform distribution on a convex compact polygon, (c) a Gaussian distribution. The family of probabilities on $(\mathsf{L})\mathsf{OT}_n^{\text{aff}}$ defined by the (labeled) order type of n random points chosen independently from μ exhibits concentration.*

Since the random projection of the vertices of a regular n-dimensional simplex, the Goodman-Pollack model, is distributed like a set of points sampled independently from a Gaussian distribution [7] (see also [33, §2.3.1]), the distribution on random order types it produces in the plane also exhibits concentration.

As we explain in the next paragraphs, we prove Theorems 1 and 2 by recasting affine order types in a projective setting, where we study so-called projective order types. This relation between affine and projective order types reveals more examples of order types difficult to avoid.

▶ **Theorem 4.** *For any integer $k \geq 2$, the proportion of order types of size n that contain a triangle and k points forming a convex chain over one edge is $1 - O(1/n)$.*

Our final result is a classification of the symmetry groups of simple projective order types: we prove that they are exactly the finite subgroups of $SO(3)$, the group of rotations (Theorem 7).

1.3 Approach

Our proof of Theorems 1 and 2 divides up the simple planar order types into their orbits under the action of projective transforms, and averages the number of extreme points inside each orbit. Let us illustrate this "action" we consider with the two order types of Figure 2. Starting with the left hand-side convex pentagon, any projective transform $\mathbb{R}^2 \to \mathbb{R}^2$ that maps the dashed line to the line at infinity yields the triangle with two interior points on the right. Following up with any other projective transform that sends the dotted line back to infinity will turn the triangle with two interior points back into a convex pentagon. We invite the reader to check that all three simple order types of size 5 (Figure 1) form a single orbit under projective transforms.

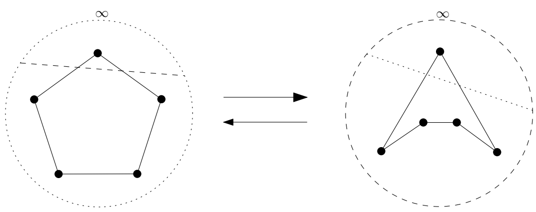

Figure 2 Two projectively equivalent planar order types.

Here is a simple example of how such projective transforms may help:

▶ **Lemma 5.** *Let A be a finite planar point set in general position and $t : \mathbb{R}^2 \to \mathbb{R}^2$ a projective transform with the line sent to infinity disjoint from A, and splitting A. Then there are at most 4 extreme vertices of A whose images are also extreme in $t(A)$.*

Proof. Let ℓ be the line sent by t to infinity. The extreme points of $t(A)$ are exactly the images of the points of A that ℓ can touch by moving continuously without crossing over a point of A. It is the union of two convex chains, on either side of ℓ, and each chain contains at most 2 points extreme in A. ◀

This essentially allows to match order types of size n so that in every pair, the size of the convex hulls add up to at most $n + 4$. Assuming one dealt with issues such as symmetries, this could provide an upper bound of $n/2 + 2$ on the average number of extreme points in a typical order type. We do not formalize this matching idea further, but recast it into a projective that makes it easier to analyze the action of projective transforms on order types.

1.4 Setting and terminology

We take all our points on the origin-centered unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, except for occasional mentions of the origin $\mathbf{0}$. Two points p and q on the sphere are called *antipodal*, if $q = -p$. A *great circle* is the intersection of the sphere with a plane containing $\mathbf{0}$, an *open hemisphere* is a connected component of the sphere in the complement of a great circle, and a *closed hemisphere* is the closure of an open one. A finite subset P of the sphere is a *projective set* if $p \in P \Leftrightarrow -p \in P$. We call a finite set of points on the sphere an *affine set* if it is contained in an open hemisphere. An affine set is in *general position* if no three points are coplanar with $\mathbf{0}$; a projective set P is in *general position* if whenever three points in P are coplanar with $\mathbf{0}$, two of them are antipodal.

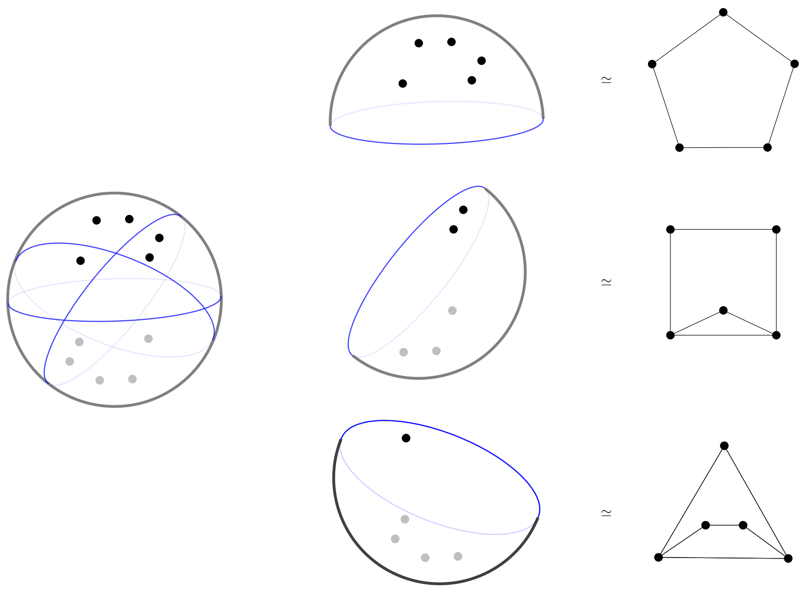

Figure 3 A projective set of size 10 (left) containing the three simple affine order types of size 5.

The *sign*, $\chi(p, q, r)$, of a triple (p, q, r) of points on the sphere is the sign, -1, 0, or 1, of the determinant of the matrix $(p, q, r) \in \mathbb{R}^{3\times 3}$. A bijection $f : S \to S'$ between finite subsets of the sphere is *orientation preserving* if $\chi(f(p), f(q), f(r)) = \chi(p, q, r)$ for every triple of points in S. Two affine (resp. projective) sets have *the same affine* (resp. *projective*) *order type* if there exists an orientation preserving bijection between them. An *affine* (resp. *projective*) *order type* is the equivalence class of all affine (resp. projective) sets that have the same affine (resp. projective) order type. The definitions of *labeled* affine and projective are similar: the labeling determines the bijection that is required to preserve orientations. It will sometimes be convenient to write a point sequence as $A_{[\lambda]}$, where A is the point set and $\lambda : A \to [n]$, $n = |A|$, the bijection specifying the ordering.

The plane $\mathbb{R}^2$ together with its orientation function can be mapped to any open hemisphere of $\mathbb{S}^2$ together with χ. For example, for the open hemisphere $\mathbb{S}^2 \cap \{z > 0\}$ this can be done by the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{x^2 + y^2 + 1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Hence, the planar order types discussed so far coincide with the affine order types and we, in fact, prove Theorems 1 and 2 for (labeled) affine order types. We study affine order types as subsets of projective point sets as shown in Figure 3; this inclusion requires some care and is formalized in Section 3.

Let S be a finite subset of the sphere. A *permutation* of S is a bijection $S \to S$ and a *symmetry* of S is an orientation preserving permutation of S. The symmetries of S form a group, which we call the *symmetry group* of S. This group determines the relations between labeled and non-labeled affine order types: two orderings $A_{[\lambda]}$ and $A_{[\mu]}$ of a point set A determine the same labeled order type if and only if $\mu^{-1} \circ \lambda$ is a symmetry of A. A crucial ingredient in our proof of Theorem 2 is a classification of the symmetry groups of the affine and projective sets. Here it is for affine sets. (The definitions of convex layers and lonely point are given in Section 2.3.)

▶ **Theorem 6.** *The symmetry group of any affine set A in general position is isomorphic to the cyclic group $\mathbb{Z}_k$ for some $k \in \mathbb{N}$ that divides the size of every layer of A other than its lonely point (if A has one). In particular, k divides $|A|$ (if A has no lonely point) or $|A| - 1$ (if A has a lonely point, which can happen for k odd only).*

For all values of k and n satisfying the conditions of Theorem 6, with the exception of $(k, n) = (2, 4)$, there exists an affine order type of size n with $\mathbb{Z}_k$ as symmetry group (see Figure 4).

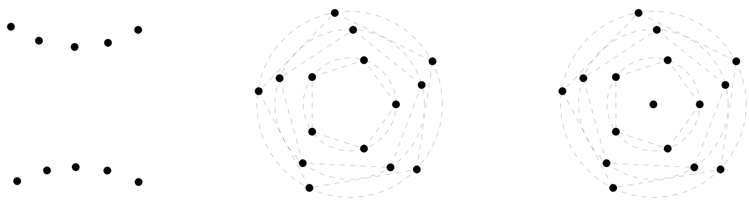

Figure 4 Left: For any even $n \geq 6$, there exists an affine set of n points with symmetry group $\mathbb{Z}_2$: take two sufficiently flat convex chains of $n/2$ points each, facing each other (so-called double chain, [31]). Center and Right: For any $3 \leq k \leq n$ where k divides n or for any odd k where k divides $n - 1$, there exists an affine set of n points with symmetry group $\mathbb{Z}_k$: just pile up regular polygons inscribed in concentric circles.

We also prove that the symmetry groups of projective sets are finite subgroups of $SO(3)$.

▶ **Theorem 7.** *The symmetry group of any projective set of $2n$ points in general position is a finite subgroup of $SO(3)$. In particular, it is one of the following groups: $\mathbb{Z}_1$ (trivial group), $\mathbb{Z}_m$ (cyclic group), D_m (dihedral, with $m \mid n$ or $m \mid n-1$), S_4 (octahedral = cubical), A_4 (tetrahedral), and A_5 (icosahedral).*

We give examples of projective point sets with symmetry groups of each of the types identified in Theorem 7.

Notation. Let us introduce or recall some notation. For $n \geq 3$ we write $\mathsf{LOT}_n^{\text{aff}}$ for the set of simple labeled affine order types of size n, $\mathsf{OT}_n^{\text{aff}}$ for the set of simple affine order types of size n, and $\mathsf{OT}_n^{\text{proj}}$ for the set of simple projective order types of size $2n$. For an affine point set A with affine order type ω, we write $\mathsf{LOT}_A^{\text{aff}} = \mathsf{LOT}_\omega^{\text{aff}}$ for the set of the labeled affine order types of the orderings of A.

1.5 Related work

Studying planar order types through their projective analogues is not a new idea, and appears for instance in the tabulation of planar order types of size 11 [2]. We are not aware, however, of an earlier analysis of how this relation is affected by symmetries.

Perhaps our most direct predecessor is the work of Miyata [28] on the classification of symmetry groups of oriented matroids. These structures coincide with abstract order types, and the affine order types we consider are special ("realizable") cases. Miyata classifies the symmetries of abstract order types in dimension 1 and 2. Our proof of Theorem 6 extends to the abstract setting and offers a more direct alternative to Miyata's proof [28, §6]. Also related is the $O(n^d)$ time algorithm of Aloupis et al. [4, Theorem 4.1] for computing the automorphisms of an order type (what we will call the symmetry group of orientation preserving permutations) for a set of n points in $\mathbb{R}^d$.

Several recent works have studied order types of random point sets [10, 12, 15, 21, 39], but they do not address the *equiprobable* distribution on n-point order types. The recent work of Chiu et al. [11] comes closer, as they have looked at the average size of the jth level in a random planar arrangement of n lines, chosen by fixing a projective line arrangement of size n and equiprobably choosing a random cell to contain the south-pole. This is similar to what we do, but let us stress that they do not take symmetries into account, so the actual distribution on planar arrangements they consider is not equiprobable (not even among those contained in the projective arrangement).

Order types with forbidden patterns were previously investigated in two directions. On the one hand, the Erdős-Szekeres theorem was strengthened for order types with certain forbidden patterns [30, 23, 24]. On the other hand, Han et al. [21] studied the patterns contained in random samples. We are not aware of previous results on the number of order types with a forbidden pattern.

Finally, let us point out that the study of random polytopes raises other questions close to classical questions in discrete and computational geometry. The analysis through floating bodies [6] of f-vectors of random polytopes obtained from convex bodies is close to the ϵ-net theory for halfspaces (see also [22] and [5, §3.2]). In another direction, Blaschke proved that the probability that 4 points chosen uniformly in a convex domain are in convex position is minimized when the domain is a triangle; for arbitrary planar probability measures, this merely asks for the limit as $n \to \infty$ of the rectilinear crossing number of the complete graph K_n.

1.6 Paper organization

Due to space limitation, we had to make some choices as to what to keep here. We decided to present a self-contained proof of Theorem 1 as it already gives a taste of our methods. This is essentially a prefix of the full version [17].

From here, Section 2 recalls some background material. Then, Section 3 clarifies the relation between affine and projective order types, between their symmetry groups, and between the affine subsets of a projective set and the cells of its dual arrangement. Section 4 then proves Theorem 1 by relating the number of extreme points in a random affine order type to the number of edges in a random cell of an arrangement of great circles, and by analyzing such arrangements via double counting and the zone theorem.

2 Background

We recall here some notions in finite group theory and in discrete geometry on $\mathbb{S}^2$ (duality, arrangements, convexity).

2.1 Groups

The elements of group theory we use deal with a subgroup G of the group of permutations of a finite set X. The identity map, the neutral element in G, is denoted by id or id_X. We will study such a group G through its action on X or some set of subsets of X. The *orbit* $G(x)$ of $x \in X$ is the image of x under G, *ie.* $G(x) \stackrel{\text{def}}{=} \{g(x) \mid g \in G\}$. Any two elements have disjoint or equal orbits, so the orbits partition X. The *stabilizer* of an element $x \in X$ is the set of permutations in G having x as a fixed point, *ie.* $G_x \stackrel{\text{def}}{=} \{g \in G \mid g(x) = x\}$. By the *orbit-stabilizer theorem*, $|G| = |G(x)| \cdot |G_x|$ for any $x \in X$. We write $\simeq$ for group isomorphism.

2.2 Duality and arrangements on $\mathbb{S}^2$

On the sphere, the dual of a point p is the great circle p^* contained in the plane through $\mathbf{0}$ and orthogonal to the line $\mathbf{0}p$. For any finite subset S of the sphere, we write S^* for the arrangement of the family of great circles $\{p^* \mid p \in S\}$.

Let P be a projective set of $2n$ points. Since antipodal points have the same dual great circle, P^* is an arrangement of n great circles. Observe that P is in general position if and only if no three great circles in P^* have a point in common. Any two great circles intersect in two points, so P^* has $2\binom{n}{2}$ vertices. Every vertex is incident to four edges; the total number of edges is therefore $4\binom{n}{2}$. By Euler's formula, P^* has $2\binom{n}{2} + 2$ faces of dimension 2, which we call *cells*.

Let us recall that many combinatorial quantities on arrangements of great circles on $\mathbb{S}^2$ are essentially twice their analogues for arrangements of lines in $\mathbb{R}^2$. Indeed, starting with an arrangement P^* of n great circles in general position, we can add another great circle C_∞, chosen so that $P^* \cup \{C_\infty\}$ is also in general position, and consider the two open hemispheres bounded by C_∞. Each open hemisphere can be mapped to $\mathbb{R}^2$ so that the half-circles of P^* are turned into lines, and the two line arrangements are combinatorially equivalent by antipodality. In this way, we can for instance obtain the following version of the zone theorem from the bound given in [8] for the zone of a line in an arrangement of lines[2]:

▶ **Theorem 8** (Zone Theorem). *Let P^* be an arrangement of n great circles on $\mathbb{S}^2$ and let $p^* \in P^*$. Let $Z(p^*)$ denote the* zone *of p^*, i.e., the set of cells of the arrangement incident to p^*. For a cell c, let $|c|$ denote the number of edges incident to c. Then $\sum_{c \in Z(p^*)} |c| \leq 19(n-1) - 10$.*

[2] [8] shows that the cells in the zone of a line h_0 in an arrangement of $n+1$ lines in the plane has edge-complexity at most $\lfloor 19n/2 \rfloor - 1$. For translating this bound to the zone of a great circle in an arrangement of n great circles on $\mathbb{S}^2$, (i) we replace n by $n-1$, (ii) we double for the two sides of C_∞, and (iii) we subtract 8 for the edges that get merged along C_∞ (note that the infinite edges on h_0 get merged and contribute 1 on each of their sides).

2.3 Convexity on the sphere

A point $p \in A$ is *extreme* in an affine set A if there exists a great circle C that strictly separates p from $A \setminus \{p\}$; that is, p and $A \setminus \{p\}$ lie on two different connected components of $\mathbb{S}^2 \setminus C$. An ordered pair $(p, q) \in A^2$ is a *positive extreme edge* of A if for any $r \in A \setminus \{p, q\}$ we have $\chi(p, q, r) = +1$. Assuming general position, a point $p \in A$ is extreme in A if and only if there exists $q \in A$ such that (p, q) is a positive extreme edge; in that case, the point q is unique.

A *CCW order* of the extreme points of A is an order $(p_0, p_1, \dots, p_{h-1})$ of its extreme points such that for all $i = 0, 1, \dots, h-1$, (p_i, p_{i+1}) is a positive extreme edge (indices mod h). The *convex hull* of A is

$$\mathsf{conv}(A) \stackrel{\text{def}}{=} \{r \in \mathbb{S}^2 \mid \forall \text{ positive extreme edges } (p, q), \chi(p, q, r) \geq 0\}.$$

An affine set A is *in convex position* if every point is extreme in A. The (onion) *layer sequence* of A is a sequence $(A_0, A_1, \dots, A_\ell)$ of subsets of A, partitioning A, where A_0 is the set of extreme points in A, and $(A_1, A_2, \dots, A_\ell)$ is the layer sequence of $A \setminus A_0$. The A_i's are called the *layers* of A. If the innermost layer A_ℓ consists of a sole point, then that point is called *lonely* (there is one or no lonely point).

3 Hemisets: relating affine and projective order types

Any affine set A naturally defines a projective set $A \cup -A$, which we call its *projective completion*. Going in the other direction, consider a projective set P. Any affine set whose projective completion is P must be the intersection of P with some open hemisphere. Remark, however, that the converse is not always true: the set $P = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$, the vertices of the cross polytope, intersects some open hemispheres in a single point. This reveals that for an open hemisphere to cut out an affine set that completes to P, it must be bounded by a great circle that avoids P. We therefore define a *hemiset* of P as the intersection of P with a *closed* hemisphere, and call a hemiset of P an *affine hemiset* if it is contained in an open hemisphere. With these definitions, we have:

▷ Claim 9. A projective set P is the completion of an affine set A if and only if A is an affine hemiset of P.

Notation. For a projective point set P with projective order type π, we write $\mathsf{OT}_P^{\text{proj}} = \mathsf{OT}_\pi^{\text{proj}}$ for the set of affine order types of the affine hemisets of P.

To understand how affine order types relate to projective order types, an important idea is that the symmetries of a projective point set P act on the (affine) hemisets of P:

▶ **Proposition 10.** *Let $g : P \to P'$ be an orientation preserving bijection between two projective sets in general position. If $|P| = |P'| \geq 6$, then g maps hemisets of P to hemisets of P' and affine hemisets of P to affine hemisets of P'.*

The proof of Proposition 10 starts by a simple observation of independent interest.

▶ **Lemma 11.** *Let $g : S \to S'$ be an orientation preserving bijection between two subsets of the sphere. If S contains two antipodal points $\{p, -p\}$ such that $g(-p) \neq -g(p)$, then S is contained in a great circle.*

Proof. If $g(-p)$ and $g(p)$ are not antipodal, then they are on a unique great circle, which must contain S, as for every $r \in S$ we have $0 = \chi(p, -p, g^{-1}(r)) = \chi(g(p), g(-p), r)$. ◀

Proof of Proposition 10. Let B be a hemiset of P and $E = B \cap -B$. By general position, $|B|$ is 4, 2 or it is 0 (in which case B is an affine hemiset). Since $|P| \geq 6$ and P is in general position, P is not contained in a great circle and g therefore preserves antipodality by Lemma 11. In particular, if g preserves hemisets, it also preserves affine hemisets.

If $|E| = 4$, then there are two points $p, q \in E$ such that $B = \{r \in P \mid \chi(p, q, r) \geq 0\}$. Since g preserves orientations and is bijective, it comes that

$$g(B) = \{r \in P \mid \chi(g(p), g(q), r) \geq 0\} = P \cap \{s \in \mathbb{S}^2 \mid \chi(g(p), g(q), s) \geq 0\},$$

and $g(B)$ is also a hemiset.

So assume that $|E| \leq 2$ and fix some closed hemisphere Σ such that $B = \Sigma \cap P$ and B intersects the boundary of Σ into E. We extend B into a set B'' with $|B'' \cap -B''| = 4$ as follows:

- If $|E| = 2$, then we set $B' \stackrel{\text{def}}{=} B$, $\Sigma' \stackrel{\text{def}}{=} \Sigma$ and $\{q, -q\} \stackrel{\text{def}}{=} E$. Otherwise, we fix a point p on the boundary of Σ, rotate Σ about $\mathbf{0}p$ until we first touch a point $q \in P \setminus B$ (at the same moment, $-q \in B$ moves from the interior to the boundary of the rotating hemisphere); we let Σ' denote the resulting hemisphere and put $B' \stackrel{\text{def}}{=} B \cup \{q\}$ (note $B' \cap -B' = \{q, -q\}$).
- We now rotate Σ' about $\mathbf{0}q$ until we first touch a point $r \in P \setminus B'$; we put $B'' \stackrel{\text{def}}{=} B' \cup \{r\}$.

Now, $E'' \stackrel{\text{def}}{=} B'' \cap -B'' = \{q, -q, r, -r\}$ and there exists a closed hemisphere Σ^* such that $g(B'') = P \cap \Sigma^*$ (by our previous analysis above for case $|E| = 4$). The boundary of Σ^* intersects P in precisely $g(E'') = \{g(-q), g(q), g(-r), g(r)\}$, and two adequate rotations kick only $g(r)$, then $g(q)$ out, witnessing that $g(B)$ is also a hemiset. ◀

Given a projective set P with symmetry group G and a hemiset B of P, we write G_B for the stabilizer of B in the action of G on hemisets of P. We also write $\mathsf{G}(B)$ for the orbit of B in that action.

▶ **Lemma 12.** *Let P be a projective set of $2n$ points, $n \geq 3$, in general position and A an affine hemiset of P.*

(a) *The symmetry group of A, as an affine set, is isomorphic to G_A.*

(b) *An affine hemiset of P has the same affine order type as A if and only if it is in $\mathsf{G}(A)$.*

Proof. Let F denote the symmetry group of A as an affine set. Since $P = A \cup -A$, we can extend any $f \in \mathsf{F}$ into a permutation $\hat{f}$ of P by setting $\hat{f}(p) \stackrel{\text{def}}{=} f(p)$ for $p \in A$ and $\hat{f}(p) \stackrel{\text{def}}{=} -f(-p)$ for $p \notin A$. Let $\hat{\mathsf{F}} \stackrel{\text{def}}{=} \{\hat{f} \colon f \in \mathsf{F}\}$. Remark that $\hat{\mathsf{F}}$ is isomorphic to F since for any two symmetries f_1, f_2 of A, we have $\widehat{f_1 \circ f_2} = \widehat{f_1} \circ \widehat{f_2}$. Moreover, any element $g \in \hat{\mathsf{F}}$ fixes A and, conversely, any symmetry $g : P \to P$ that fixes A writes $g = \widehat{g_{|A}}$. Then, $\hat{\mathsf{F}} = \mathsf{G}_A$ and statement (a) follows.

For statement (b), consider an affine hemiset A' of P with the same affine order type as A. There exists an orientation preserving bijection $f : A \to A'$. The extension $\hat{f}$ of f to P also preserves orientations, and is therefore in G. It follows that $A' \in g(A)$. The reverse inclusion follows from the fact that every symmetry of G preserves orientations. ◀

With Lemma 12, the orbit-stabilizer theorem readily implies:

▶ **Corollary 13.** *Let P be a projective set of $2n$ points, $n \geq 3$, in general position and A an affine hemiset of P. Let F and G denote the symmetry groups of A and P, respectively. There are $|\mathsf{G}|/|\mathsf{F}|$ affine hemisets of P with same affine order type as A.*

4 Analysis of labeled affine order types

Perhaps surprisingly, Corollary 13 is all we need to prove Theorem 1.

4.1 The two roles of affine symmetries

The number of symmetries of an affine order type determines both its number of labelings, and how often it occurs among the hemisets of a projective completion of one of its realizations. These two roles happen to balance each other out nicely:

▶ **Proposition 14.** *Let P be a projective set of $2n$ points, $n \geq 3$, in general position. Let R be a random affine hemiset chosen uniformly among all affine hemisets of P. Let λ be a random permutation $R \to [n]$ chosen uniformly among all such permutations. The labeled affine order type of $R_{[\lambda]}$ is uniformly distributed in $\bigcup_{\omega \in \mathsf{OT}_P^{\mathrm{aff}}} \mathsf{LOT}_\omega^{\mathrm{aff}}$.*

Proof. Let N denote the number of affine hemisets of P. Let $\omega_1, \omega_2, \ldots, \omega_k$ denote the order types of the affine hemisets of P, without repetition (that is, the ω_i are pairwise distinct). Let G denote the symmetry group of P and let F_i, $1 \leq i \leq k$, denote the symmetry group of ω_i. Let ρ denote the affine order type of R. By Corollary 13, we have

$$\mathbb{P}\left[\rho = \omega_i\right] = \frac{|\mathsf{G}|/|\mathsf{F}_i|}{N}.$$

Next, the number of distinct labelings of the order type of an affine set A is $n!/|\mathsf{F}|$, since two labelings $A_{[\lambda]}$ and $A_{[\mu]}$ of A have the same labeled order type if and only if $\mu^{-1} \circ \lambda$ is a symmetry of A. Let $\overline{\rho}$ denote the labeled affine order type of $R_{[\lambda]}$. For any $\overline{\sigma} \in \mathsf{LOT}_{\omega_i}^{\mathrm{aff}}$, we have

$$\mathbb{P}\left[\overline{\rho} = \overline{\sigma} \mid \rho = \omega_i\right] = \frac{|\mathsf{F}_i|}{n!}.$$

Altogether, for any $\overline{\sigma} \in \bigcup_{i=1}^{k} \mathsf{LOT}_{\omega_i}^{\mathrm{aff}}$, we have $\mathbb{P}\left[\overline{\rho} = \overline{\sigma}\right] = \frac{|G|}{Nn!}$ and the distribution is uniform as we claimed. ◀

4.2 Hemisets and duality

The following dualization will make counting easy.

▶ **Lemma 15.** *There is a bijection ϕ between the affine hemisets of a projective point set P and the cells of the dual arrangement P^*, such that a point p is extreme in an affine hemiset A if and only if the great circle p^* supports an edge of $\phi(A)$.*

Proof. For any point p we write p^+ for the hemisphere centered in p, that is the closed hemisphere containing p and bounded by p^*. For any closed hemisphere H we write H^+ for its center, that is the point q with $H = q^+$. Now, a point p is in a closed hemisphere H if and only if the scalar product $\langle p, H^+ \rangle$ is nonnegative. Thus, p lies in H if and only if H^+ lies in p^+. It follows that two hemispheres H_0 and H_1 intersect P in the same hemiset if and only if H_0^+ and H_1^+ lie in the same cell of P^*. Moreover, as H^+ moves in the cell the hemisphere H also moves while enclosing the same set of points; the boundary of H touches a point p if and only if H^+ touches p^*. ◀

For example, we now see that a projective set of $2n$ points, $n \geq 3$, in general position has $2\binom{n}{2}+2$ distinct affine hemisets. Also, it should be clear from the final computations of the proof of Proposition 14 that if that projective point set has symmetry group G, then it supports $\left(2\binom{n}{2}+2\right)\frac{n!}{|G|}$ distinct labeled affine order types.

4.3 Counting extreme points: expectation and variance

We can now prove Theorem 1 on the expectation and variance of the number of extreme points in a random labeled affine order type.

▶ **Lemma 16.** *Let P be a projective set of $2n$ points, $n \geq 3$, in general position. If X_P denotes the number of extreme points in a labeled affine order type chosen uniformly among those supported by P, then*

$$\mathbb{E}\left[X_P\right] = \frac{4n(n-1)}{n(n-1)+2} = 4 - \tfrac{8}{n^2-n+2} \qquad \textit{and} \qquad \mathbb{E}\left[{X_P}^2\right] \leq \frac{19n(n-1)-10n}{n(n-1)+2} \leq 19.$$

Proof. By Lemma 15, X_P has the same distribution as the number of edges in a cell chosen uniformly at random in P^*. The arrangement P^* has $2\binom{n}{2}+2$ cells and $4\binom{n}{2}$ edges. Since every edge bounds exactly two cells, it comes that

$$\mathbb{E}\left[X_P\right] = \frac{8\binom{n}{2}}{2\binom{n}{2}+2} = \frac{4n(n-1)}{n(n-1)+2} = 4 - \tfrac{8}{n^2-n+2}.$$

Moreover, the random variable ${X_P}^2$ has the same distribution as the square of the number of edges in a random cell chosen uniformly in P^*. Let $F_2(P^*)$ denote the set of cells of P^* and for $c \in F_2(P^*)$ let $|c|$ denote its number of edges. We thus have

$$\left(2\binom{n}{2}+2\right)\mathbb{E}\left[{X_P}^2\right] = \sum_{c\in F_2(P^*)} |c|^2.$$

In the right-hand term, every edge e of P^* is counted $|c_1|+|c_2|$ times, where c_1 and c_2 are its two adjacent cells. For any point $p \in P$, the contribution of the edges supported by p^* to that sum equals $\sum_{c\in Z(p^*)}|c| \leq 19(n-1)-10$ (following notation and bound in Theorem 8). Altogether,

$$\left(2\binom{n}{2}+2\right)\mathbb{E}\left[{X_P}^2\right] \leq n(19(n-1)-10)$$

and $\mathbb{E}\left[{X_P}^2\right] \leq \dfrac{19n(n-1)-10n}{n(n-1)+2} \leq 19.$ ◀

Here comes the announced proof.

Proof of Theorem 1. Let $\overline{\rho}$ be a simple labeled order type chosen uniformly at random in $\mathsf{LOT}_n^{\mathrm{aff}}$. Let X_n denote the number of extreme points in ρ, where ρ denotes the unlabeling of $\overline{\rho}$ and let π be the projective completion of ρ. By Lemma 16, we have for any $\pi' \in \mathsf{OT}_n^{\mathrm{proj}}$

$$\mathbb{E}\left[X_n \mid \pi = \pi'\right] = \frac{4n(n-1)}{n(n-1)+2} \quad \text{and} \quad \mathbb{E}\left[{X_n}^2 \mid \pi = \pi'\right] \leq \frac{19n(n-1)-10n}{n(n-1)+2}.$$

The formula of total probability therefore yields

$$\mathbb{E}\left[X_n\right] = \frac{4n(n-1)}{n(n-1)+2} \quad \text{and} \quad \mathbb{E}\left[{X_n}^2\right] \leq \frac{19n(n-1)-10n}{n(n-1)+2}.$$

From there, $\mathrm{Var}\left[X_n\right] = \mathbb{E}\left[{X_n}^2\right] - \mathbb{E}\left[X_n\right]^2 \leq 3$. (A bound of $3+o(1)$ is readily seen from $\mathbb{E}\left[X_n\right] = 4+o(1)$ and $\mathbb{E}\left[{X_n}^2\right] = 19+o(1)$; the bound of 3 holds exploiting $n \geq 3$.) ◀

As a consequence, we obtain for instance the following estimates.

▶ **Corollary 17.** *The proportion of simple labeled affine n-point order types with* $h \geq 6$ *vertices on the convex hull is at most* $3/(h-4)^2$.

Proof. By the Bienaymé-Chebyshev inequality, for any real $t > 0$ and any random variable X with finite expected value and non-zero variance, we have

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq t\sqrt{\mathrm{Var}[X]}\right] \leq \frac{1}{t^2}.$$

Together with Theorem 1, this implies the statement. ◀

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