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Calculus of Variations

An improved uniqueness result for the harmonic map flow in two dimensions

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Abstract Generalizing a result of Freire regarding the uniqueness of the harmonic map flow from surfaces to an arbitrary closed target manifold N, we show uniqueness of weak solutions $u \in H^1$ under the assumption that any upwards jumps of the energy function are smaller than a geometrical constant $\varepsilon^\star = \varepsilon^\star(N)$, thus establishing a conjecture of Topping, under the sole additional condition that the variation of the energy is locally finite.

1 Introduction

We consider the harmonic map flow for maps from a compact two-dimensional manifold M without boundary to an arbitrary compact manifold N. A map $u \in H^1(M \times [0, T], N)$ is said to be a weak solution of the harmonic map flow, if

$$\partial_t u - \Delta u = A(u)(\nabla u, \nabla u) \tag{1.1}$$

is solved in the weak sense, where A is the second fundamental form of N.

We define the energy of u at time t as $E(u(t)) := \frac{1}{2}\int_M |\nabla u(t)|^2\, dx$, where $u(t) = u(\cdot, t)$ is taken in the trace sense.

It was shown by Struwe [5] that for any initial condition $u_0 \in H^1(M, N)$ there exists a global weak solution $v \in H^1(M \times [0, \infty), N)$ of (1.1) which is smooth away from finitely many points (x_i, t_i) in $M \times [0, \infty)$, unique in the class of functions with $\nabla u \in L^4_{\text{loc}}(M \times [t_i, t_{i+1}))$ for every i, where $t_i \le t_{i+1}$ are the singular times, in the following called Struwe solution.

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The energy of v is monotone decreasing and for any “singular time” t_i we have $\lim_{t\nearrow t_i} E(u(t)) \geq E(u(t_i)) + K \cdot \varepsilon^\star$, where K is the number of singular points at time t_i and

$$\varepsilon^\star = \min\left\{\frac{1}{2}\int_{S^2} |\nabla u|^2\, dx,\, u : S^2 \to N \text{ is a non-constant harmonic map}\right\}.$$

It was shown by Freire [2] that any weak solution of (1.1) with non-increasing energy to an initial condition $u_0 \in H^1(M, N)$ is identical to the corresponding Struwe solution.

The goal of this paper is to show that the condition $E(t) \geq E(s)$ for all $t \leq s$ may be relaxed in the sense that we also allow sufficiently small upwards jumps in the energy without losing the uniqueness property. In fact we obtain

Theorem 1.1 *Let M be a closed Riemannian surface and let N be a compact Riemannian manifold, isometrically embedded in $\mathbb{R}^n$.*

Let $u \in H^1(M \times [0, T], N)$ be any weak solution of (1.1) to initial condition $u_0 \in H^1(M, N)$, such that the energy-function $t \mapsto E(u(t))$ fulfills

$$\overline{\lim_{s\searrow t}} E(u(s)) < E(u(t)) + \varepsilon_2 \quad \text{for every } t \in [0, T) \tag{1.2}$$

for a constant $\varepsilon_2 > 0$ depending only on the target manifold N.

Then $u = v$ on the whole domain $M \times [0, T]$, where v is the Struwe solution to initial condition u_0.

The question if non-uniqueness of weak solutions can occur at all in two dimensions was recently answered by Topping [6] and Bertsch et al. [1]. They have constructed examples of non-uniqueness for $M = D^2$ and $N = S^2$, which are based on the idea of attaching a backwards bubble at a certain point in space-time. The bubble corresponds to a harmonic map from the whole $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ to N and the energy function jumps upwards by the energy of this map. In theses examples the occurring energy-jump was exactly $\varepsilon^\star$. In [7] Topping conjectured that the condition

$$\overline{\lim_{s\searrow t}} E(u(s)) < E(u(t)) + \varepsilon^\star \tag{1.3}$$

is sufficient to prove uniqueness of the solution. In fact here we establish Topping’s conjecture under the sole additional condition that the energy function has locally finite total variation.

Theorem 1.2 *Let M and N be as in Theorem 1.1. Let $u \in H^1(M \times [0, T], N)$ be any weak solution of (1.1) to initial condition $u_0 \in H^1(M, N)$, such that the energy-function $t \mapsto E(u(t))$ has finite total variation and*

$$\overline{\lim_{s\searrow t}} E(u(s)) < E(u(t)) + \varepsilon^\star \quad \text{for every } t \in [0, T). \tag{1.4}$$

Then $u = v$ on the whole domain $M \times [0, T]$, where v is the Struwe solution to initial condition u_0.

We briefly sketch the main steps of the proof.

Note that if $u \in H^1(M \times [0, T], N)$ solves (1.1) in the weak sense, then for almost every $t \in (0, T]$ the trace $u(t)$ weakly solves

$$-\Delta u(t) = A(u(t))(\nabla u(t), \nabla u(t)) + k \text{ on } M, \tag{1.5}$$

where $k = -\partial_t u(t) \in L^2(M)$.

Applying a regularity result due to Moser [3], we prove that in the two-dimensional case the H^2-norm of a solution w of (1.5) may be estimated locally by its energy and the L^2-norm of the inhomogenity k, where k is an arbitrary function in $L^2(M)$. This estimate crucially depends on how small the concentration radius r has to be chosen, to assure that the energy in a ball of radius r is at most a given quantum ε_1^2.

We apply these results to solutions of (1.1). The main step is to establish, that we can bound the concentration radius from below on small time intervals, which will lead to the proof of the theorems.

To begin with, we give an alternative proof of the regularity result in [3]. Our proof is based on the paper of Rivière and Struwe [4] about the regularity of harmonic maps, which uses the fact, that the second fundamental form may be written as an antisymmetric 1-form applied to the gradient of u.

Notations

We use the short hand notation $B_r = B_r(0)$ and $B = B_1$ for balls in $\mathbb{R}^m$ and for the energy on a subset $M' \subset M$ we set

$$E(f, M') := \frac{1}{2}\int_{M'} |\nabla f|^2\, dx \text{ in particular } E_r(f; x_0) = E(f, B_r(x_0))$$

and we write $u(t) := u(\cdot, t)$.

2 Regularity of almost harmonic maps

We consider weak solutions of the equation

$$-\Delta w = A(w)(\nabla w, \nabla w) + k \tag{2.1}$$

on an open set $U \subset \mathbb{R}^m$, where A is the second fundamental form of N and k is a function in $L^s(U, \mathbb{R}^n)$. Equations of this kind were first considered by Moser [3], who used them to prove partial regularity for the harmonic map flow in small dimensions. He proved that for $k \in L^s(U, \mathbb{R}^n)$, where $s > \frac{m}{2}$ any solution $w \in H^1(U)$ to (2.1) is locally Hölder continuous, if it satisfies an appropriate Morrey estimate. His proof is based on properties of a moving tangent frame field.

Following [4], we give an alternative proof of this result by rewriting the Eq. (2.1) in the form

$$-\Delta w = \Omega \cdot \nabla w + k. \tag{2.2}$$

where Ω is the antisymmetric 1-form

$$\Omega^{ij} = (\sigma_l^i \nabla \sigma_l^j - \sigma_l^j \nabla \sigma_l^i), \quad i, j = 1, \ldots, n \text{ with } \sigma_l = \nu_l \circ w$$

for an orthonormal frame field $\nu_1, \ldots, \nu_{n-k} : N \to S^{n-1}$ of $T^{\perp}N$.

We then obtain the following generalisation of Moser's Theorem 1 in [3] analogous to Theorem 1.1 of [4].

Proposition 2.1 *For every $m \in \mathbb{N}$ there exists a number $\varepsilon_0 = \varepsilon_0(m) > 0$ such that for every ball $B_R(x_0)$, any antisymmetric 1-form $\Omega \in L^2(B_R(x_0), so(n) \otimes \Lambda^1\mathbb{R}^m)$ and for every function $k \in L^s(B_R(x_0), \mathbb{R}^n)$ with*

$$s > \frac{m}{2}$$

the following statement holds.

Every weak solution $w \in H^1(B_R(x_0), N)$ *of Eq.* (2.2) *on* $B_R(x_0)$*, satisfying the Morrey growth assumption*

$$\frac{1}{2} \sup_{x \in B_R(x_0), r>0} \left(\frac{1}{r^{m-2}} \int_{B_r(x) \cap B_R(x_0)} |\nabla w|^2 + |\Omega|^2 \, dx \right) < \varepsilon_0^2 \tag{2.3}$$

is Hölder continuous in $B_{R/2}(x_0)$*. More precisely* $w \in C^\alpha(B_{R/2}(x_0))$ *for every* $0 < \alpha < 1$ *with* $\alpha \le 2 - \frac{m}{s}$ *with estimate*

$$R^\alpha [w]_{C^\alpha(B_{R/2}(x_0))} \le C \cdot \left(\varepsilon_0 + R^{2-\frac{m}{s}} \, \|k\|_{L^s(B_R(x_0))} \right).$$

Proof Using a scaling argument, it suffices to consider the case $B = B_1(0)$.

We may assume that $s < m$ and set $\alpha = 2 - \frac{m}{s}$.

By a translation of the function w and the manifold N we can assume without loss of generality that

$$\int_B w dx = 0. \tag{2.4}$$

We choose a number $1 < p < m/(m-1)$ with $p \cdot \alpha < 1$ and a cut-off function $\varphi \in C_0^\infty(B)$ with $\varphi \equiv 1$ on $B_{3/4}$ and set $\tilde{w} = \varphi \cdot w$. We show

$$\sup_{x \in B_{1/2}} \sup_{r>0} \left(r^{-(m-p+\alpha p)} \cdot \int_{B_r(x)} |\nabla \tilde{w}|^p \, dx \right) \le C \cdot (\varepsilon_0 + \|k\|_{L^s})^p.$$

This implies the proposition by the use of Morrey's lemma.

For any given $\gamma > 0$, (2.4) and the Poincaré inequality imply this estimate for any $r > \gamma$ and $x \in B_{1/2}$.

In particular, we can assume $r \le 1/4$ and thus $w = \tilde{w}$.

The following calculations are completely analogous to these in [4], we simply have an additional term with k but no error term.

We use the gauge transformation introduced in Lemma 3.1 in [4] to rewrite Eq. (2.2). If $\varepsilon_0 = \varepsilon_0(m)$ is small enough there exists $P \in H^1(B, SO(m))$ and $\xi \in H^1(B, so(m) \otimes \Lambda^{m-2})$ such that

$$P^{-1} dP + P^{-1} \Omega P = \star d\xi$$

with

$$\sup_{x \in B, r>0} \left(\frac{1}{r^{m-2}} \int_{B_r(x) \cap B} |dP|^2 + |d\xi|^2 \, dx \right) < C \cdot \varepsilon_0^2.$$

Using this gauge transformation shows that Eq. (2.2) is equivalent to

$$\begin{aligned} -\operatorname{div}(P^{-1} \nabla w) &= -d(P^{-1}) \nabla w - P^{-1} \Delta w \\ &= (P^{-1} dP P^{-1} + P^{-1} \Omega P) P^{-1} \nabla w + P^{-1} k \\ &= \star d\xi \cdot P^{-1} dw + P^{-1} k. \end{aligned} \tag{2.5}$$

We fix a ball $B_R(x_0)$ with $R \leq 1/4$ and $x_0 \in B_{1/2}$ and use the Hodge decomposition

$$P^{-1}dw = df + \star dg + h,$$

where $f \in H^1_0(B_R(x_0))$ and where g is a co-closed $(m-2)$-form of class $H^1(B_R(x_0))$ whose restriction to the boundary $\partial B_R(x_0)$ vanishes, and with a harmonic 1-form $h \in L^2(B_R(x_0))$.

Using (2.5) for f and g we obtain the equations

$$\begin{aligned} -\Delta f &= -\mathrm{div}(P^{-1}\nabla w) = \star d\xi \cdot P^{-1}dw + P^{-1}k, \\ -\Delta g &= \star d(P^{-1}dw) = \star(dP^{-1} \wedge dw). \end{aligned} \tag{2.6}$$

In the following all the norms are computed with respect to $B_R(x_0)$. Let $q > m$ be the conjugate exponent of p.

Since $f = 0$ on $\partial B_R(x_0)$ we can use duality between L^p and L^q to bound

$$\|df\|_{L^p} \leq C \cdot \sup_{\{\varphi \in W^{1,q}_0(B_R(x_0)), \|\varphi\|_{W^{1,q}} \leq 1\}} \int_{B_R(x_0)} df \cdot d\varphi dx.$$

Fix any such $\varphi \in W^{1,q}_0(B_R(x_0))$ with $\|\varphi\|_{W^{1,q}} \leq 1$.

As $q > m$, we have $\varphi \in L^\infty$ with

$$\|\varphi\|_{L^\infty} \leq CR^{1-m/q}\|\varphi\|_{W^{1,q}} \leq CR^{1-m/q}.$$

Integrating by parts, we obtain

$$\begin{aligned} \int_{B_R(x_0)} df \cdot d\varphi dx &= -\int_{B_R(x_0)} \Delta f \cdot \varphi dx \\ &= \int_{B_R(x_0)} d\xi \wedge P^{-1}\varphi dw + \int_{B_R(x_0)} P^{-1}k \cdot \varphi dx = I + II. \end{aligned}$$

Exactly as in [4] we can estimate

$$|I| \leq R^{m/p-1}\varepsilon_0[w]_{\mathrm{BMO}(B_R(x_0))}.$$

Using $\frac{m}{s} = 2 - \alpha$, the second term may be estimated as

$$\begin{aligned} |II| &= \left| \int_{B_R(x_0)} P^{-1}k \cdot \varphi dx \right| \leq \|\varphi\|_{L^\infty} \cdot \|k\|_{L^1} \\ &\leq CR^{1-m/q} \cdot \|k\|_{L^s} \cdot (R^m)^{1-1/s} = C \cdot \|k\|_{L^s} \cdot R^{m/p-1+\alpha}. \end{aligned}$$

The equation for g in (2.6) is identical to the one for g in [4] and we have the same estimates, i.e.

$$\|dg\|_{L^p} \leq C \cdot R^{m/p-1} \cdot \varepsilon_0 \cdot [w]_{\mathrm{BMO}(B_R(x_0))}.$$

Using the Campanato estimates for harmonic functions to estimate h we can thus conclude that for $r < R < \frac{1}{4}$

$$\begin{aligned}
\int_{B_r(x_0)} |dw|^p\,dx &\le \int_{B_r(x_0)} |h|^p\,dx + \int_{B_r(x_0)} |df|^p + |dg|^p\,dx \\
&\le C\cdot\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |h|^p\,dx + \int_{B_R(x_0)} |df|^p + |dg|^p\,dx \\
&\le C\cdot\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |dw|^p\,dx + C\int_{B_R(x_0)} |df|^p + |dg|^p\,dx \\
&\le C\cdot\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |dw|^p\,dx + C\cdot R^{m-p+p\cdot\alpha}\cdot\|k\|_{L^s}^p \\
&\quad + C\cdot R^{m-p}\varepsilon_0^p\cdot[w]^p_{\mathrm{BMO}(B_R(x_0))}.
\end{aligned} \tag{2.7}$$

Defining

$$\Phi(x_0,r) = r^{p-m}\int_{B_r(x_0)} |dw|^p\,dx$$

and letting

$$\psi(R) = \sup_{x_0\in B_{1/2},\,0<r<R} \Phi(x_0,r),$$

using the Poincaré-inequality we can bound

$$\sup_{x_0\in B_{1/2}} [w]^p_{\mathrm{BMO}(B_R(x_0))} \le C\cdot\psi(R).$$

From (2.7) we then get

$$\Phi(x_0,r) \le C\cdot\left(\frac{r}{R}\right)^p \Phi(x_0,R) + C\cdot\left(\frac{r}{R}\right)^{p-m}\cdot\varepsilon_0^p\cdot\psi(R) + C\cdot\left(\frac{r}{R}\right)^{p-m} R^{p\cdot\alpha}\cdot\|k\|_{L^s}^p.$$

For a fixed ratio $0<\gamma = r/R < 1$ this gives

$$\Phi(x_0,\gamma R) \le C_1\gamma^p\Phi(x_0,R) + C\gamma^{p-m}\varepsilon_0^p\psi(R) + C\gamma^{p-m}R^{p\cdot\alpha}\,\|k\|_{L^s}^p$$

uniformly in $0 < R < \frac{1}{4}$ and $x_0 \in B_{1/2}$. Thus for any $R_0 < \frac{1}{4}$ and $0 < R < R_0$

$$\Phi(x_0,\gamma R) \le C_1\gamma^p(1+\varepsilon_0^p\gamma^{-m})\psi(R_0) + C\gamma^{p-m}\cdot R_0^{p\cdot\alpha}\,\|k\|_{L^s}^p.$$

Passing to the supremum with respect to x_0 and $R < R_0$ we find

$$\psi(\gamma R_0) \le C_1\gamma^p(1+\varepsilon_0^p\gamma^{-m})\psi(R_0) + C\gamma^{p-m}\cdot R_0^{p\cdot\alpha}\,\|k\|_{L^s}^p.$$

Thus, if we fix $\gamma > 0$ such that

$$C_1\gamma^{(p-1)/2} \le 1/2$$

and choose $\varepsilon_0 = \varepsilon_0(m)$ with $\varepsilon_0^p \le \gamma^m$, we obtain (writing again R instead of R_0)

$$\psi(\gamma R) \le \gamma^{(p+1)/2}\psi(R) + C\cdot R^{p\cdot\alpha}\,\|k\|_{L^s}^p. \tag{2.8}$$

Given $r \in (0, \gamma)$, choose $l \in \mathbb{N}$ such that $\gamma^{l+1} < r \leq \gamma^l$. Iterating (2.8), we get the following estimate

$$\begin{aligned}\psi(r) &\leq \psi(\gamma^l) \leq \gamma^{(p+1)/2}\psi(\gamma^{l-1}) + C \cdot (\gamma^l)^{p\alpha} \|k\|_{L^s}^p \\ &\leq (\gamma^{(p+1)/2})^l \cdot \psi(\gamma) + C \cdot \|k\|_{L^s}^p \sum_{j=1}^{l} (\gamma^j)^{p\alpha}(\gamma^{\frac{p+1}{2}})^{l-j} \\ &\leq C \cdot r \cdot \psi(\gamma) + C \cdot r^{p\cdot\alpha} \cdot \|k\|_{L^s}^p \leq C \cdot (\varepsilon_0^p + \|k\|_{L^s}^p) r^{p\alpha},\end{aligned}$$

as $\varphi(\gamma) < C \cdot \varepsilon_0^p$ and $p\alpha < 1$. □

Remark 2.2 A standard argument shows that for a solution $w \in H^1(U) \cap C^0(U)$ of (2.1) with $k \in L^2(U)$, $U \subset \mathbb{R}^m$ open we have $w \in H^2_{\text{loc}}(U) \cap W^{1,4}_{\text{loc}}(U)$ and thus w solves Eq. (2.1) in the strong sense.

We will show now that for $m = 2$ the H^2-norm of a solution $w \in H^1$ of (2.1) may be estimated locally by quantities depending only on the concentration radius of w and the L^2-norm of k.

For this we use the following Sobolev interpolation inequality due to Gagliardo–Nirenberg and Ladyzhenskaya.

Proposition 2.3 *For any function $g \in H^1_{\text{loc}}(\mathbf{R}^2)$, any $R > 0$ and any function $\varphi \in C_0^\infty(B_R)$ with $0 \leq \varphi \leq 1$ and $|\nabla\varphi| \leq \frac{c_1}{R}$ there holds*

$$\int_{B_R} |g|^4 \varphi^2 dx \leq c_0 \left(\int_{B_R} |g|^2 \, dx \right) \cdot \left(\int_{B_R} |\nabla g|^2 \varphi^2 dx + c_1^2 R^{-2} \int_{B_R} |g|^2 \, dx \right) \tag{2.9}$$

with a constant c_0 independent of R and g.

We use this proposition to show the following H^2-estimate.

Proposition 2.4 *Let $U \subset \mathbb{R}^2$ be open, let $k \in L^2(U, \mathbb{R}^n)$ and let $w \in H^2(U, \mathbb{R}^2)$ be a solution of the equation*

$$-\Delta w = B(w)(\nabla w, \nabla w) + k \quad \text{on } U \tag{2.10}$$

where B is a bounded bilinear form with bounded first derivatives.

There exists a constant $\varepsilon_1 = \varepsilon_1(B)$ (independent of k) with the following property.

If on $B_{2r}(x_0) \subset U$ we have $E_{2r}(w; x_0) \leq \varepsilon_1^2$, then there holds the estimate

$$\int_{B_r(x_0)} \left|\nabla^2 w\right|^2 dx + \int_{B_r(x_0)} |\nabla w|^4 \, dx \leq C \cdot \left[\frac{E_{2r}(w; x_0)}{r^2} + \|k\|^2_{L^2(B_r(x_0))} \right].$$

Proof Let $\varphi \in C_0^\infty(B_{2r}(x_0))$ be a cut-off function with $\varphi \equiv 1$ on $B_r(x_0)$ and $|\nabla\varphi|^2 + \left|\nabla^2\varphi\right| \leq \frac{C}{r^2}$.

By assumption, Eq. (2.10) is fulfilled in the strong sense and thus holds almost everywhere. We multiply (2.10) with φ and take the square to obtain

$$\begin{aligned}\int \varphi^2 |\Delta w|^2 \, dx &\leq C \cdot \int \varphi^2 |B(w)(\nabla w, \nabla w)|^2 \, dx + C \cdot \int \varphi^2 |k|^2 \, dx \\ &\leq C \cdot \int \varphi^2 |\nabla w|^4 \, dx + C \cdot \|k\|^2_{L^2(B_{2r}(x_0))},\end{aligned} \tag{2.11}$$

where $C = C(B)$ depends on the bilinear form B. Applying Proposition 2.3 to the first term on the right hand side, we find

$$\int \varphi^2 |\nabla w|^4 \, dx \le C \int_{B_{2r}(x_0)} |\nabla w|^2 \, dx \left(\int \varphi^2 \left|\nabla^2 w\right|^2 dx + \frac{1}{r^2} \int_{B_{2r}(x_0)} |\nabla w|^2 \, dx \right)$$
$$\le C \cdot E_{2r}(w; x_0) \left(\int \varphi^2 \left|\nabla^2 w\right|^2 dx + \frac{E_{2r}(w; x_0)}{r^2} \right). \tag{2.12}$$

Integrating by parts twice, one can see that

$$\int \left|\nabla^2 w\right|^2 \varphi^2 dx \le 2 \int |\Delta w|^2 \varphi^2 dx + \frac{C}{r^2} E_{2r}(w; x_0). \tag{2.13}$$

Inserting (2.12) and (2.13) into (2.11) and using $E_{2r}(w; x_0) < \varepsilon_1^2$ we get

$$\int \varphi^2 |\Delta w|^2 \, dx \le C \cdot \int \varphi^2 |\nabla w|^4 \, dx + C \cdot \|k\|^2_{L^2(B_{2r}(x_0))}$$
$$\le C\varepsilon_1^2 \left(\int \varphi^2 |\Delta w|^2 \, dx + \frac{c}{r^2} E_{2r}(w; x_0) \right) + C \cdot \|k\|^2_{L^2(B_{2r}(x_0))}.$$

Choosing $\varepsilon_1 = \varepsilon_1(B)$ small enough such that $C \cdot \varepsilon_1^2 \le \frac{1}{2}$, we can absorb the first term on the right hand side into the left hand side. This shows the proposition. □

Combining this with Proposition 2.1 we can prove a global version of the above result.

Proposition 2.5 *Let M be a two-dimensional compact Riemannian manifold without boundary, let $w \in H^1(M, N)$ be a solution of (2.1) for a function $k \in L^2(M)$. Let $r > 0$ be such that*

$$\sup_{x \in M} E_r(w; x) \le \varepsilon_1^2,$$

where ε_1 is the constant of Proposition 2.4. Then $w \in H^2(M)$ and

$$\int_M \left|\nabla^2 w\right|^2 dx + \int_M |\nabla w|^4 \, dx \le C \cdot \left(\frac{E(w)}{r^2} + \|k\|^2_{L^2(M)} \right).$$

Proof Covering M with possibly smaller balls we see from Proposition 2.1 that w is continuous and thus, by Remark 2.2, $w \in H^2(M)$. For the estimate one uses Proposition 2.4 and the fact that for every manifold M as above, there exists a $K \in \mathbb{N}$ with the property that we can cover M by balls $B_r(x_i)$ with given radius r such that at every point in M at most K of the balls overlap (see for example [5, Lemma 3.3]). □

3 Proof of the main results

As shown in [2] or [5], solutions $u \in H^1(M \times [0, T], N)$ of (1.1) with $\nabla u \in L^4(M \times [0, T])$ or $\nabla^2 u \in L^2(M \times [0, T])$ are unique:

Proposition 3.1 *Let $u, v \in L^2([0, T], H^2(M, N)) \cap H^1(M \times [0, T], N)$ be solutions of (1.1) to the same initial condition $u_0 \in H^1(M, N)$. Then u and v are identical.*

If we can show that under the assumptions of Theorem 1.1 or 1.2 we can cover M with balls $B_r(x_i)$ with $E_{2r}(u(t), x_i) \leq \varepsilon_1^2$ for all times in a small time interval I, then $\nabla^2 u \in L^2(M \times I)$ by Proposition 2.5 and uniqueness follows from Proposition 3.1.

We use the following lemma about the behavior of the energy on subsets of M.

Lemma 3.2 *Let $M' \subset M$ be open and let $u \in H^1([0, T] \times M) \cap L^\infty([0, T], H^1(M'))$ be a solution of* (1.1).

Then for every $t_0 \in [0, T)$ and every sequence $t_m \searrow t_0$ we have

$$\varliminf_{m \to \infty} E(u(t_m), M') \geq E(u(t_0), M'). \tag{3.1}$$

Proof Choosing a subsequence we may assume that $E(u(t_m))$ converges. As the sequence $(u(t_m))_m$ is bounded in $H^1(M')$, there exists a subsequence (denoted again by t_m) and a function $u_\infty \in H^1(M')$ such that

$$\begin{aligned} u(t_m) &\to u_\infty \text{ strongly in } L^2(M') \text{ for } m \to \infty \\ \nabla u(t_m) &\rightharpoonup \nabla u_\infty \text{ weakly in } L^2(M') \text{ for } m \to \infty. \end{aligned}$$

The weak convergence implies

$$\|\nabla u_\infty\|^2_{L^2(M')} \leq \varliminf_{m \to \infty} \|\nabla u(t_m)\|^2_{L^2(M')}.$$

However, by the trace theorem we know that $u(t_m) \to u(t_0)$ in $L^2(M)$ for every sequence $t_m \to t_0$, which implies $u_\infty = u(t_0)|_{M'}$ and thus the lemma. □

Proof of Theorem 1.1 Let ε_1 be the constant of Proposition 2.4. We set $\varepsilon_2 := \min(\varepsilon_1^2, \varepsilon^\star)$ and show

Lemma 3.3 *Let $u \in H^1(M \times [0, T])$ be as in Theorem 1.1 for ε_2 as above.*

Then for every time $t_0 \in [0, T)$ there exist finitely many balls $B_{r_i}(x_i)$, $i = 1, \dots, n$ covering M and a number $\delta_1(t_0) > 0$ such that

$$\frac{1}{2} \int\limits_{B_{2r_i}(x_i)} |\nabla u(t)|^2 \, dx \leq \varepsilon_2 \quad \text{for all } t \in [t_0, t_0 + \delta_1(t_0)]$$

and thus $\nabla u \in L^4(M \times [t_0, t_0 + \delta_1(t_0)])$.

Proof Let $t_0 \in [0, T)$ and let $\rho > 0$ be such that

$$\varlimsup_{t \searrow t_0} E(u(t)) \leq E(u(t_0)) + \varepsilon_2 - \rho. \tag{3.2}$$

We choose points x_i and radii r_i, $i = 1, \dots, n$ such that the balls $B_{r_i}(x_i)$ cover M and

$$\frac{1}{2} \int\limits_{B_{2r_i}(x_i)} |\nabla u(t_0)|^2 \, dx \leq \rho/2.$$

Assume there exists a sequence $t_m \searrow t_0$ such that for every m there exists a number i with

$$\frac{1}{2} \int\limits_{B_{2r_i}(x_i)} |\nabla u(t_m)|^2 \, dx > \varepsilon_2.$$

As we have only finitely many balls we may assume (after choosing a subsequence) that the index is always the same, say $i = 1$.

We set $M' = M \setminus \overline{B_{2r_1}(x_1)}$ and apply Lemma 3.2 and (3.2) to get

$$\begin{aligned}\varepsilon_2 &\leq \overline{\lim_{m\to\infty}} \frac{1}{2} \int_{B_{2r_i}(x_1)} |\nabla u(t_m)|^2 \, dx = \overline{\lim_{m\to\infty}} E(u(t_m), M\setminus M') \\ &\leq \overline{\lim_{m\to\infty}} E(u(t_m), M) - \underline{\lim_{m\to\infty}} E(u(t_m), M') \\ &\leq E(u(t_0)) + \varepsilon_2 - \rho - E(u(t_0), M') \leq \varepsilon_2 - \rho/2.\end{aligned}$$

which is a contradiction.

As remarked in the introduction the function $u(t)$ solves an almost harmonic map equation for almost every $t_1 \in [t_0, t_0 + \delta_1(t_0)]$ and we may estimate with $r_0 = \min(r_i)$

$$\int_M |\nabla u(t_1)|^4 \, dx + \int_M \left|\nabla^2 u(t_1)\right|^2 dx \leq C \cdot n \cdot \left(\frac{E(u(t_1))}{r_0^2} + \int_M |\partial_t u(t_1)|^2 \, dx \right).$$

This implies that indeed $\nabla u \in L^4(M \times [t_0, t_0 + \delta_1(t_0)])$, as the energy $E(u(t_1))$ is bounded uniformly by $E(u(t_0)) + 2\varepsilon_2$ for δ_1 small enough. □

By Proposition 3.1 we see that $u = v_{t_0}$ on $[t_0, t_0+\delta_1(t_0)]$, where v_{t_0} is the Struwe solution to initial condition $v(\cdot, t_0) = u(\cdot, t_0)$.

This shows that the set

$$K := \{t_1 \in [0, T) : \quad u = v \quad \text{on } [0, t_1] \times M\}$$

is open in $[0, T)$.

On the other hand if $v = u$ on $[0, t_1)$, we have by the trace theorem

$$u(t_1) = \lim_{t \nearrow t_1} u(t) = \lim_{t \nearrow t_1} v(t) = v(t_1)$$

where the limits are taken in $L^2(M)$.

This shows that K is closed in $[0, T)$ implying $K = [0, T)$ and thus

$$u = v \quad \text{on } M \times [0, T]$$

which proves Theorem 1.1.

Proof of Theorem 1.2 For ε_2 as above we set

$$S := \{t \in [0, T) : \overline{\lim}_{s \searrow t} E(u(s)) - E(u(t)) \geq \varepsilon_2\}. \tag{3.3}$$

Then S is finite, $|S| \leq \frac{TV(E(u))}{\varepsilon_2}$, where $\mathrm{TV}(E(u))$ is the total variation of the energy function over the interval $[0, T]$.

On every closed interval $[t_0, t_1] \subset [0, T)\setminus S$ assumption (2.5) is satisfied and thus $u = v_{t_0}$ on $[t_0, t_1]$, where v_{t_0} is the Struwe solution to initial condition $u(\cdot, t_0)$.

As the energy is bounded, say by E_0, the function v_{t_0} has a bounded number of singular times for any $t_0 \in [0, T) \setminus S$. By uniqueness of the Struwe solution we have certainly that $v_{t_0}|_{[\tilde{t}_0, t_1]} = v_{\tilde{t}_0}$ for any $t_0 < \tilde{t}_0 < t_1$ and $[t_0, t_1] \subset [0, T]\setminus S$. Thus for any $s \in S$, we can find a number $\delta > 0$ such that no singularities occur in $(s, s + \delta)$ and thus $\nabla u \in L^4(M \times [t_0, t_1])$ for any $[t_0, t_1] \subset (s, s + \delta)$.

We use this to discuss the behavior of u for times in S.

Let $s_1 = \min S$. If $s_1 > 0$ we have $u = v$ on $[0, s_1]$. We may thus assume $s_1 = 0$ and we finish the proof of Theorem 1.2 by showing

Lemma 3.4 *Let $u \in H^1(M \times [0,T])$ be a solution of (1.1) to initial condition $u_0 \in H^1(M,N)$ with*

$$\overline{\lim_{t \searrow 0}} E(u(t)) < E(u_0) + \varepsilon^\star, \tag{3.4}$$

such that the assumption of Theorem 1.1 is satisfied on any closed interval $I \subset (0, T_1]$ for a number $0 < T_1 < T$.

Then there exists a number $\delta_0 > 0$ and a radius $r > 0$ such that

$$E_r(u(t), x_0) \leq \varepsilon_2 \quad \text{for all } x_0 \in M, \quad t \in [0, \delta_0]. \tag{3.5}$$

This implies that $\nabla u \in L^4(M \times [0, \delta_0])$.

Proof We argue by contradiction and show that if the claim were false we would have a backwards bubble with energy at least $\varepsilon^\star$ which will lead to a contradiction to (3.4).

Assume there exist sequences $t_m \searrow 0$, $\tilde{r}_m \to 0$ such that

$$\sup_{x \in M} E_{\tilde{r}_m}(u(t_m), x) > \varepsilon_2 \text{ for every } m.$$

As M is compact and $\nabla u(t_m) \in L^2(M)$, we may choose slightly smaller radii r_m and a sequence x_m such that

$$E_{r_m}(u(t_m), x_m) = \sup_{x \in M} E_{r_m}(u(t_m), x) = \frac{\varepsilon_2}{2}.$$

Restricting ourselves to a subsequence we may assume that $x_m \to x_0$ for a point $x_0 \in M$.

For $M^T = M \times [0,T]$, we define the set $V(M^T, N) := \{u \in H^1(M^T, N)$ with $\nabla u \in L^\infty([0,T], L^2(M,N)) \cap L^4(M^T, N)\}$.

We use the following lemma of [5].

Lemma 3.5 *There exists a constant $c_1 = c_1(N)$ such that for every solution $u \in V(M^T, N)$ of (1.1) and every $R > 0$, $(x,t) \in M^T$ we have*

$$E_R(u(t); x) \leq E_{2R}(u(0); x) + c_1 \cdot \frac{t}{R^2} E(u(0)).$$

Applying this lemma to balls of radius $\frac{r_m}{2}$ and times $t \in [t_m, t_m + c_2 r_m^2]$, we get

$$E_{r_m/2}(u(t); x) \leq E_{r_m}(u(t_m); x) + c_1 c_2 \cdot E(u(t_m)) \leq \varepsilon_2 \tag{3.6}$$

for any $x \in M$, if we choose $c_2 \leq \frac{\varepsilon_2}{2E_0 \cdot c_1}$.

This allows us to apply Proposition 2.5, to estimate

$$\int_M \int_{t_m}^{t_m + c_2 r_m^2} \left|\nabla^2 u\right|^2 dt dx \leq C \cdot \left(\frac{E_0}{r_m^2} \cdot c_2 r_m^2 + \int_M \int_{t_m}^{t_m + c_2 r_m^2} |\partial_t u|^2 \, dt dx \right) \leq c_3$$

uniformly in m.

Finally we wish to estimate $\int_{B_{2r_m}(x_m)} |\nabla u(t)|^2 \, dx$ from below. We use

Lemma 3.6 *There exists a constant $c_4 > 0$ such that for every solution $u \in V(M^T, N)$ of (1.1), for every $x \in M$ and every $R > 0$ the following estimate holds*

$$E_{2R}(u(T); x) \geq E_R(u(0); x) - 2\,\|\partial_t u\|^2_{L^2(B_{2R}(x)\times[0,T])} - \frac{c_4 T}{R^2} \sup_{t\in[0,T]} E_{2R}(u(t); x).$$

Proof We multiply Eq. (1.1) with $\partial_t u \cdot \varphi^2$, where $\varphi \in C_0^\infty(B_{2R}(x))$ is a cut-off function with $\varphi = 1$ on $B_R(x)$ and $|\nabla\varphi| \leq \frac{C}{R}$ and integrate by parts

$$\begin{aligned} 0 &= \int_0^T \int_M \varphi^2 |\partial_t u|^2\, dxdt - \int_0^T \int_M \varphi^2 \Delta u \cdot \partial_t u dxdt \\ &= \int_0^T \int_M \varphi^2 |\partial_t u|^2\, dxdt + \int_0^T \int_M \frac{1}{2}\frac{d}{dt}\left(\varphi^2 |\nabla u|^2\right) + \partial_t u \cdot \nabla u \cdot \varphi\nabla\varphi dxdt \\ &= \int_0^T \int_M \varphi^2 |\partial_t u|^2\, dxdt + \frac{1}{2}\int_M \varphi^2 \cdot |\nabla u(T)|^2\, dxdt - \frac{1}{2}\int_M \varphi^2 \cdot |\nabla u(0)|^2\, dxdt \\ &\quad + \int_0^T \int_M \partial_t u \cdot \nabla u \cdot \varphi\nabla\varphi dxdt. \end{aligned}$$

Young's inequality implies the claim. □

Now, choosing $T_1 > 0$ such that

$$\int_0^{T_1} \int_M |\partial_t u|^2\, dxdt \leq \frac{\varepsilon_2}{16}$$

and applying Lemma 3.6 we have for m large enough

$$E_{2r_m}(u(t); x_m) \geq E_{r_m}(u(t_m); x_m) - 2\cdot\frac{\varepsilon_2}{16} - \frac{c_4}{r_m^2}\cdot c_5 r_m^2 E_0 \geq \frac{\varepsilon_2}{4} \tag{3.7}$$

for every $t \in [t_m, t_m + c_5 r_m^2]$, provided $c_5 \leq \frac{\varepsilon_2}{8c_4 E_0}$.

Setting $c_6 = \min(c_2, c_5)$, we have thus uniform estimates in m for the energy on $B_{2r_m}(x_m)$ from below as well as for the L^2-norm of the second derivative of u on M from above on the intervals $[t_m, t_m + c_6 r_m^2]$.

We can proceed by a standard bubble argument.

We choose $\rho > 0$ such that $\overline{\lim}_{t\searrow 0} E(u(t)) \leq E(u(0)) + \varepsilon^\star - \rho$ and fix a radius $R_0 > 0$ with $E_{R_0}(u_0; x_0) \leq \rho/2$. Then we may construct a sequence $\tau_m \in [0, c_6]$, such that the functions $v_m(x) := u_m(x, \tau_m)$, where $u_m(x,t) = u(x_m + r_m x, t_m + r_m^2 t)$ defined on $D_m = \{x : x_m + r_m x \in B_{R_0}(x_0)\}$ satisfy

- v_m converges to a function v_∞ weakly in $H^2_{\text{loc}}(\mathbb{R}^2)$ and strongly in $H^1_{\text{loc}}(\mathbb{R}^2)$.
- The time derivatives $\partial_t u_m(x, \tau_m)$ converge to zero in $L^2_{\text{loc}}(\mathbb{R}^2)$.
- We have $E_2(v_m) \geq \frac{\varepsilon_2}{4}$ and $E(v_m, D_m) \leq E_0$.

As the functions u_m solve Eq. (1.1) and as the times derivatives converge to zero, v_∞ is a non-constant harmonic map with finite energy and can thus be extended to S^1. By definition of $\varepsilon^\star$ we have

$$\lim_{m\to\infty} E_{R_0}(u(t_m + r_m^2\tau_m), x_0) = \frac{1}{2}\lim_{m\to\infty}\int_{D_m} |\nabla v_m|^2\,dx = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_\infty|^2\,dx \ge \varepsilon^\star.$$

Using Lemma 3.2 this leads to a contradiction, as

$$\begin{aligned} E(u_0) &\le E(u_0, M\setminus B_{R_0}(x_0)) + \rho/2 \\ &\le \varliminf_{m\to\infty} E(u(t_m + r_m^2\tau_m), M\setminus B_{R_0}(x_0)) + \rho/2 \\ &\le \varliminf_{m\to\infty} \left[E(u(t_m + r_m^2\tau_m)) - E_{R_0}(u(t_m + r_m^2\tau_m), x_0)\right] + \rho/2 \\ &< E(u_0) + \varepsilon^\star - \varepsilon^\star - \rho/2. \end{aligned} \tag{3.8}$$

This proves Lemma 3.4, as $\nabla u \in L^4(M \times [0, \delta_0])$ for $\delta_0 > 0$ small enough follows again by the use of the H^2-estimate for almost harmonic maps. This concludes the proof of Theorem 1.2, by the same kind of argument about the set where $u = v$ as in the proof of Theorem 1.1. □

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