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Strings on AdS_3

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Abstract

This thesis deals with strings on three-dimensional Anti-de Sitter backgrounds and their dual conformal field theories. We analyse string theory on these backgrounds from first principles by employing worldsheet techniques. We explore the large moduli space of string backgrounds and consider in particular mixed NS-NS and R-R flux backgrounds using the hybrid formalism of Berkovits, Vafa and Witten. We solve the worldsheet description of the theory completely in the plane-wave limit. This constitutes a direct derivation of the plane-wave spectrum from the worldsheet with mixed flux. We also derive the behaviour of the string spectrum close to the pure NS-NS flux point.

We then study the tensionless limit of string theory on the backgrounds $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$. Superstring theory on these backgrounds with the smallest amount of NS-NS flux through one of the three-spheres is shown to be dual to the spacetime CFT given by the large N limit of the symmetric product orbifold $\text{Sym}^N(\mathbb{T}^4)$ and $\text{Sym}^N(\text{S}^3 \times \text{S}^1)$, respectively. To define the worldsheet theory at the tensionless point, we employ the hybrid formalism in which the $\text{AdS}_3 \times \text{S}^3(\times \text{S}^3)$ part is described by the WZW model based on $\mathfrak{psu}(1,1|2)_1$ or $\mathfrak{d}(2,1;\alpha)$ at minimal level. Unlike the case for higher background flux, it turns out that the string spectrum at minimal level does *not* exhibit the long string continuum, and perfectly matches with the large N limit of the symmetric product. We generalise the analysis away from the minimal flux case, in which case the dual CFT can be described by a symmetric product orbifold of a suitable Liouville theory. We construct a complete set of DDF operators, from which one can read off the symmetry algebra of the spacetime CFT. We also show that a similar analysis can be done for bosonic string theory on $\text{AdS}_3 \times X$.

Zusammenfassung

In dieser Arbeit studieren wir String Theorie auf drei-dimensionalen Anti-de Sitter Hintergründen und deren duale konforme Feldtheorien. Wir analysieren String Theorie auf diesen Hintergründen durch Weltflächentechniken. Der große Modulraum dieser Kompaktifizierungen wird erforscht und insbesondere gemischter NS-NS und R-R Fluss betrachtet. Diese Hintergründe können durch Berkovits, Vafas und Wittens Hybrid Formalismus studiert werden. Wir lösen diese Weltflächenbeschreibung komplett im Ebene-Wellen-Limes, was eine direkte Herleitung des String Spektrums bei gemischtem Hintergrundfluss liefert. Wir leiten auch das Verhalten des String Spektrums nahe des NS-NS Punktes her.

Wir studieren anschließend den spannungslosen Limes von String Theorie auf den Hintergründen $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ und $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$. Wir zeigen, dass Superstring Theorie auf diesen Hintergründen mit dem kleinstmöglichsten Wert von NS-NS Fluss durch eine der Drei-Sphären dual zur konformen Feldtheorie der symmetrischen Orbifaltigkeit $\mathrm{Sym}^N(\mathbb{T}^4)$ (bzw. $\mathrm{Sym}^N(\mathrm{S}^3 \times \mathrm{S}^1)$) ist. Um die Weltflächentheorie am spannungslosen Punkt zu definieren, benutzen wir den Hybrid Formalismus, in welchem der $\mathrm{AdS}_3 \times \mathrm{S}^3(\times \mathrm{S}^3)$ Teil des Hintergrundes durch ein $\mathfrak{psu}(1,1|2)_1$ (bzw. $\mathfrak{d}(2,1;\alpha)$ mit minimalem Level) Supergruppen WZW Modell beschrieben ist. Im Gegensatz zu höherem Hintergrundfluss weißt das String Spektrum bei minimalem Fluss kein Kontinuum durch lange Strings auf und stimmt direkt mit dem Spektrum der symmetrischen Orbifaltigkeit im Grenzwert von großem N überein. Wir verallgemeinern die Analyse für den Fall nicht-minimalen Flusses, in welchem die duale konforme Feldtheorie durch eine symmetrische Orbifaltigkeit einer geeigneten Liouville Theorie beschrieben werden kann. Wir konstruieren einen vollständigen Satz von DDF Operatoren, aus dem die Symmetrie Algebra der dualen konformen Feldtheorie abgelesen werden kann. Wir zeigen außerdem, dass eine ähnliche Analyse für bosonische String Theorie auf $\mathrm{AdS}_3 \times X$ durchgeführt werden kann.

Contents

Chapter 1

Introduction

1.1 String Theory

Theoretical Physics has thrived from two independent major developments taking place a century ago. *General relativity* describes well the large-scale phenomena, like the formation of galaxies, stars, black holes and even our universe as a whole. On the other hand, *quantum mechanics* is well-suited for the description of microscopic phenomena and has become indispensable for our everyday technology. Quantum mechanics describing quantum fields like the electromagnetic field is called quantum field theory and has been tested experimentally to a staggering precision [1, 2]. The standard model is the well-established quantum field theory, which describes three of the known four fundamental forces of nature: the electromagnetic force, the weak force and the strong force. The fourth force is gravity and is much weaker compared to the others.

It has long been clear that general relativity and quantum mechanics are incompatible, at least in any naive sense. Treating general relativity in the standard quantum field theory formalism is plagued by the appearance of infinities, which cannot be removed. General relativity is thus *non-renormalisable*.

There are few phenomena in nature, in which both gravitational and quantum mechanical effects play a role. One such instance are *black holes*. It was understood by Hawking that they are not completely black, but rather emit thermal radiation [3]. This effect is entirely quantum mechanical in its origin and is caused by quantum tunneling out of the black hole. Related to this is the *black hole information paradoxon* [4]. According to standard quantum mechanics lore, information can not be destroyed. And yet it seems that we lose information about objects falling into black holes irreversibly, see e.g. [5] and references therein for an overview of the possible resolutions. Another instance, where gravity and quantum mechanics clash unavoidably is the very beginning of our universe. We know quite well the chronology of our uni-

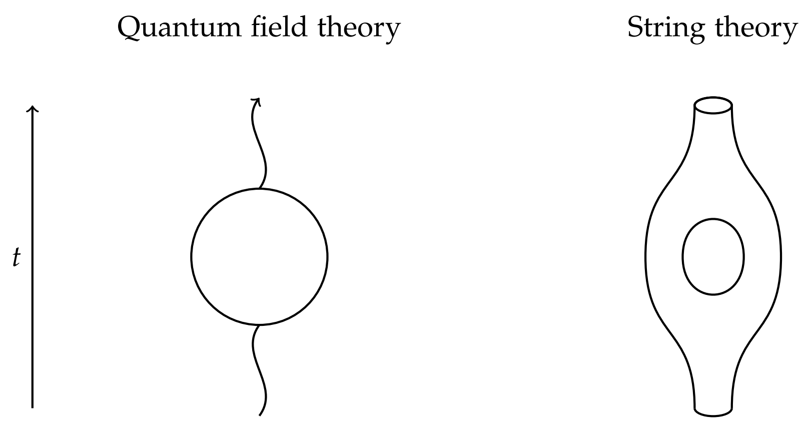

Figure 1.1: The worldline and the worldsheet in quantum field theory and string theory.

verse starting 10^{-12} seconds after the big bang till the present day, roughly 13.8 billion years later. While the unknown first 10^{-12} seconds may seem unimportant, they have an enormous influence on cosmology. The theory of *inflation* was proposed to explain various problems with standard cosmology, such as why the universe appears so homogeneous and isotropic [6]. In yet earlier periods, gravity becomes the dominant force and sets the initial conditions for the inflationary period, much in the same way as inflation sets the initial conditions for standard cosmological scenarios.

In order to remedy the situation and unite the four forces of nature, *String theory* was proposed [7–13]. Historically, it was a candidate to describe the strong force, where flux tubes at strong coupling resemble strings. It was later understood that string theory contains a graviton in its spectrum and is thus a full-fledged theory of gravity [14, 15]. In string theory, particles are replaced by strings, meaning one-dimensional extended objects. In particular, the *world line* (the trajectory of a particle through spacetime) is replaced by the *world sheet*, a two-dimensional surface traced out by the string as it moves through spacetime.

In quantum field theory, particles may be created or annihilated, which results in the worldline to branch. Similarly, in string theory, the worldsheet is allowed to have non-trivial topology, which is interpreted as particle creation or annihilation. This is sketched in Figure 1.1. Contrary to quantum field theory, the worldsheet of string theory is smooth. In quantum field theory, loops can shrink to an arbitrarily small size and this leads to the infinities mentioned before. String theory has an intrinsic length-scale (namely the size of the string) which provides a natural regularization of the integral. Thus, string theory is an inherently (UV-)finite theory and there is no need for renormalisation. The probability that the string worldsheet branches is measured by the *string coupling constant*. The picture of a string worldsheet is reliable in the regime of a small string coupling constant.

Note that in the limit where the characteristic size of the string (the *string length*) becomes small, the right picture in 1.1 degenerates again to the left one. Thus, if our accelerators are not powerful enough to resolve the string length, then we do not have a direct way of knowing whether the particles in our universe are actually strings. The expected length scale of strings can be estimated as follows. In nature, we have the fundamental constants c (the velocity of light), $\hbar$ (the *Planck constant*) and G (the *Newton constant*). These three constants control the scale where relativistic effects matter, the scale of quantum effects and the strength of gravity. The natural length scale is the *Planck length*, which is the unique combination of these constants with units of length. It takes the form

$$\ell_{\text{Planck}} = \hbar^{\frac{1}{2}} G^{\frac{1}{2}} c^{-\frac{3}{2}} \approx 1.6 \times 10^{-35}\,\text{m} \,. \tag{1.1}$$

Equivalently, we can form the unique energy combination, which is the Planck energy $E_{\text{Planck}} \approx 1.2 \times 10^{19}\,\text{GeV}$. For comparison, the LHC can probe energies of up to $1.3 \times 10^4\,\text{GeV}$, about 15 orders of magnitude below the Planck scale.

String theory contains a graviton (i.e. a massless spin-2 particle) in its spectrum [14, 15]. This particle is visible in the low-energy effective description of string theory. Thus, string theory naturally incorporates general relativity and one can recover Einstein's equation of gravity from string theory [16]. In fact, one recovers (a generalisation of) general relativity in the low-energy limit of string theory. Thus, string theory provides a natural candidate for a UV-complete unification of general relativity and quantum field theory.

A first naive version of string theory is *bosonic string theory*, which only possesses bosons in its spectrum. It turns out that this theory can be only consistently quantised in 26 spacetime dimensions. Moreover, it suffers from having a tachyon (a particle with negative mass square) in the spectrum. This renders the theory unstable. Furthermore, experience tells us that the real world contains also fermions, such as the electron. Thus, in order to construct a realistic string theory, one is led to *superstring theory* [17–24].

It was shown that there are five different consistent superstring theories. These are the open string theory of type I (with gauge group SO(32)), the type IIA and IIB string theories and the two heterotic string theories with gauge groups SO(32) and $E_8 \times E_8$ (where E_8 is the largest exceptional Lie group) [25–28]. Type I is an *open string theory*, i.e. its strings have boundaries, whereas the other theories are *closed string theories*, where the worldsheets do not have boundaries. All these theories are only consistent in 10 spacetime dimensions. Moreover, they are interconnected by a web of dualities, which is sketched in Figure 1.2 [29–37]. The string theories can become equivalent

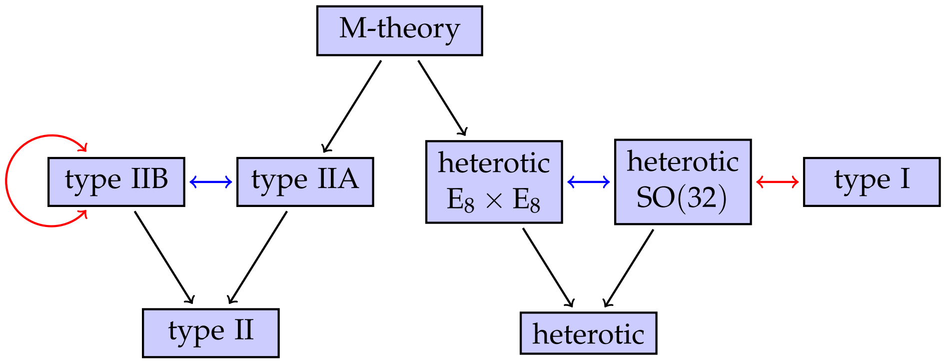

Figure 1.2: A sketch of the string dualities.

upon compactification on a circle or an interval, which is referred to as T-duality and is indicated by a blue arrow in the figure. Additionally, there is the conjectured S-duality, which is a strong/weak duality (in the string coupling constant), these are the red arrows in the figure. Finally, type IIA and the heterotic $E_8 \times E_8$ become effectively eleven-dimensional in the strong-coupling regime and appear as the compactification of *M-theory* on a circle or an interval, respectively. M-theory is a somewhat mysterious theory. The different string theories seem to be different descriptions of M-theory in different regions of its parameter space. Thus, it is thought that there is only one unique theory [35,38].

String theory can be given a good perturbative description in terms of a *worldsheet theory*. This theory is a two-dimensional theory living on the worldsheet, which describes the embedding of the string in spacetime. The worldsheet theory turns out to be a two-dimensional *conformal field theory*, which is often under good computational control.

Besides the perturbative degrees of freedom in the string coupling constant, string theory contains many more non-perturbative objects. The mass of these objects scales like an inverse power of the string coupling constant, which is very much analogous to an instanton in gauge theory. These are the so-called *p-branes* or *Dp-branes* [39].[1] Dp-branes are higher-dimensional extended objects. In much the same way as the string is a one-dimensional extended object, p-branes are p-dimensional extended objects (in space). For instance, type IIB string theory contains besides the fundamental string F1 its magnetic dual, the NS5-brane, and a D1-, a D3-, and a D5-brane.[2] Thus, while we started out with just a single string, string theory automatically yields all these additional degrees of freedom. In fact, S-duality maps strings

[1] Sometimes, the name D-brane is reserved for non-perturbative excitations in the Ramond-Ramond sector of the superstring theory, whereas p-brane can refer to a generic extended object.

[2] It also contains a $D(-1)$-instanton and D7-branes, as well as spacetime filling D9-branes.

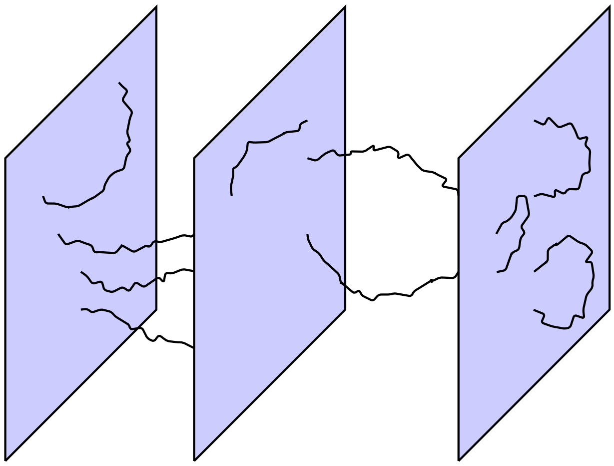

Figure 1.3: D-branes with open strings ending on them.

into branes and versa. Thus the duality chain forces the inclusion of these non-perturbative objects into the theory.

D-branes emerge naturally in string theory as the boundaries of open strings. In fact the name stems from Dirichlet, since they are associated to Dirichlet boundary conditions. As such, they feature open string excitations, for a sketch of this, see Figure 1.3. Open strings can stretch between the various D-branes. Open strings carry massless gauge fields (in the same way as closed strings have a graviton in their spectrum). Thus, the low-energy limit of the effective theory residing on a single D-brane can be described by a supersymmetric gauge theory. If multiple D-branes coincide, the strings stretching between them become massless. As a consequence, there are more fields in the low-energy limit surviving and one obtains a non-abelian $U(N)$ gauge theory, where N is the number of coincident branes [40].

1.2 The AdS/CFT correspondence

As we have mentioned above, there are a number of very powerful dualities relating the various formulations of superstring theory with each other. The duality chain of string theory is however far more impressive. It was realised at the end of the last century, that string theories can be also dual to *gauge theories*, but in one dimension lower. This duality is called the AdS/CFT correspondence.

To see this, we consider the archetypal example. Consider a stack of N D3-branes (in type IIB string theory) on top of each other. As we have discussed above, the effective theory on the D-brane can be described by a supersymmetric gauge theory, which in this case is the four-dimensional $\mathcal{N} = 4$ super

Yang-Mills (SYM) theory with gauge group $\mathrm{U}(N)$.[3] On the other hand, we can describe the low-energy limit also in terms of geometry. A large number of D3-branes backreacts on the geometry and curves it. This gives a (super)gravity solution analogous to a black hole. Since the D3-branes are extended, this is often referred to as *black branes* [42]. As such, the black brane solution has an event horizon. Taking the low-energy limit in the gauge theory picture corresponds to approaching the horizon in the geometry picture. At this point, the geometry tends to the *near-horizon geometry*, which is the space $\mathrm{AdS}_5 \times \mathrm{S}^5$. AdS_{d+1} is the $(d+1)$-dimensional *Anti-de Sitter* space, which is a space of constant negative curvature. The appearance of these spaces is typical in the near horizon limit. For instance, the near horizon geometry of the familiar Schwarzschild black hole is $\mathrm{AdS}_2 \times \mathrm{S}^2$. A sketch of this procedure can be found in Figure 1.4. Thus, one obtains the statement [41,43,44] (for a review, see also [45]):[4]

$$\text{IIB string theory on } \mathrm{AdS}_5 \times \mathrm{S}^5 \Longleftrightarrow \mathrm{SU}(N)\ \mathcal{N} = 4 \text{ SYM}\ . \tag{1.2}$$

On the left hand side, the number N controls the flux through the five-sphere and as such its size. Moreover, both sides have one further coupling constant. The string coupling constant on the left hand side is mapped to the Yang-Mills coupling constant on the right-hand side. While we have motivated the correspondence in the large N limit, it is expected to hold true irrespective of the value of N.

$\mathcal{N} = 4$ SYM theory is an example of a *conformal field theory*. Since we took the low-energy limit of the effective quantum field theory residing on the branes, it is expected in general that the theory will flow to a conformal fixed point in the IR. Thus, one obtains in general the statement that a string theory on an AdS_{d+1} background is dual to a conformal field theory on a d-dimensional space, thus giving the correspondence its name. The d-dimensional space can be interpreted as being the boundary of the Anti-de Sitter space. This is why the AdS/CFT correspondence is referred to as the *holographic principle*. The dynamics of a $(d+1)$-dimensional gravity theory can be described entirely in terms of a gauge theory on its boundary and thus gravity appears as a hologram [48,49].

In fact, the holographic principle is expected to be even more general. While we have motivated the AdS/CFT correspondence in string theory, there are by now many examples of dual AdS/CFT pairs which do not involve string theory. Moreover, there are many examples of dualities using deformed versions of AdS, which are then dual to non-conformal field theories.

[3] It turns out that the overall $\mathrm{U}(1)$ in $\mathrm{U}(N)$ actually decouples and the gauge group is $\mathrm{SU}(N)$ [41].

[4] See also [46,47] for precursors from the field theory and gravity side.

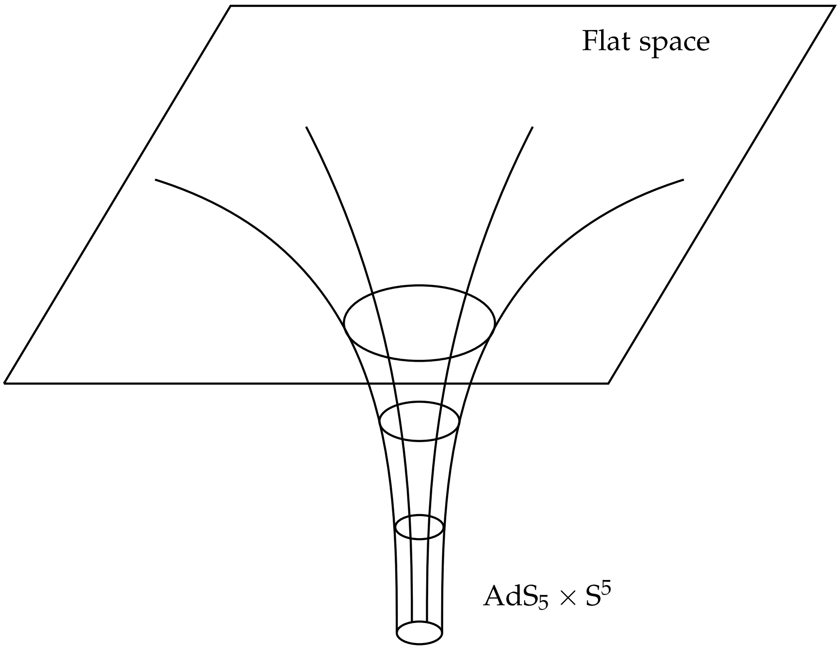

Figure 1.4: The geometry of a stack of D3-branes.

Progress in understanding the inner workings of the AdS/CFT correspondence is hampered by the insufficient understanding of string theory on AdS spaces. D-branes are sources for *Ramond-Ramond fluxes*, which are hard to include in explicit string descriptions. Happily, there is another instance of the AdS/CFT correspondence, where one can overcome this difficulty. For this, it is convenient to compactify type IIB string theory on a four-dimensional torus $\mathbb{T}^4$. Upon taking the size of the internal torus to zero, one obtains a theory, which is effectively six-dimensional. In this six-dimensional theory, one has two types of strings: the original ten-dimensional string F1 and another type of strings, which arise from wrapping NS5-branes in the original theory on the four-torus.[5] These strings form a bound state in the theory, and one can again consider the system formed of Q_1 F1-strings and Q_5 NS5-branes. In the gravitational description, this system is a *black string*, whose near horizon limit is $AdS_3 \times S^3 \times \mathbb{T}^4$. On the other hand, we can consider the effective gauge theory on the F1-NS5 brane intersection, which flows to a two-dimensional conformal field theory in the infrared. This conformal field theory has a description in terms of a deformation of the *sigma-model* on the *symmetric product orbifold* $\text{Sym}^{Q_1 Q_5}(\mathbb{T}^4)$. This consists in $Q_1 Q_5$ copies of the theory of four bosons and fermions, where we identify the copies under permutations. Thus, one obtains the statement [43], see also [50] for a

[5]There are of course more string-like objects, namely the D1-brane, wrapped D3-branes or a wrapped D5-brane. For the moment, we do not consider these.

review[6]

$$\text{IIB string theory on } \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4 \Longleftrightarrow \text{deformation of } \mathrm{Sym}^{Q_1 Q_5}(\mathbb{T}^4)\ . \tag{1.3}$$

Here, we stress that the duality is somewhat weaker than in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case. Both sides of the duality have 20 parameters (the *moduli space* of the background), which describe the shape of the torus and various fluxes in the background and the precise identification of these parameters is not clear.[7] In particular, the right hand side takes a very simple form on a codimension 4 space of this parameter space, where it becomes a symmetric orbifold. At generic points in the moduli space, the theory becomes quite intractable.

We also mention that the backgrounds we discussed so far are all supersymmetric. D-branes are BPS objects, meaning that their presence breaks half of the spacetime supersymmetry. In the $\mathrm{AdS}_5 \times \mathrm{S}^5$ example, the presence of the D3-branes in the geometry breaks half of its supersymmetry. However, the supersymmetry is again enhanced by a factor of two when taking the near-horizon limit. The original type IIB string theory in ten-dimensional flat space is $\mathcal{N} = (2,0)$ supersymmetric (thus the name II). This corresponds to 32 (real) supercharges. 32 supercharges in the dual four-dimensional CFT indeed correspond to $\mathcal{N} = 4$ superconformal symmetry. Similarly, we have introduced two types of D-branes to construct the AdS_3 background, which leaves one quarter of the supercharges. This is enhanced to twice that amount in the near-horizon limit. Thus, the background supports 16 supercharges, which corresponds to $\mathcal{N} = (4,4)$ supersymmetry in the two-dimensional dual CFT.

We have explained that the dual CFT is under very good control at the orbifold point in moduli space. Remarkably, there is also a point in moduli space, where the string theory is under good control. This happens precisely when the background is entirely supported by *Neveu-Schwarz-Neuveu-Schwarz* (NS-NS) flux. In terms of branes, this means that we do not include D1-, D3- or D5-branes in the construction, but rather only the F1-string and NS5-brane as we did above. At the pure NS-NS flux point, the worldsheet theory of the string can be described in terms of a *Wess-Zumino-Witten* model on the group manifold $\mathrm{SL}(2,\mathbb{R})$ [54–62]. This is a well-understood CFT, so that string theory on AdS_3 with pure NS-NS flux can be exactly solved in the weak string coupling limit.

Another important AdS_3 background is $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ [63–69]. This background supports also $\mathcal{N} = (4,4)$ supersymmetry. In contradistinction to $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, the relevant superconformal algebra is the so-called *large*

[6]In the literature, one often finds the same picture, but starting from a D1-D5 bound state. This is the S-dual picture of the one described.

[7]See however [51–53] for a discussion of the moduli space.

$\mathcal{N} = 4$ superconformal algebra [70]. This algebra has many peculiar features, which are not shared by the usual *small* $\mathcal{N} = 4$ superconformal algebra. The background depends on *three* quantum numbers Q_1, Q_5^+ and Q_5^-, which correspond to the number of D1-branes in the background and the amount of flux through the two three-spheres. Thus, the background can be thought of as a refined testing ground of the AdS_3/CFT_2 correspondence.

1.3 The tensionless limit

The *Planck length* sets the natural length scale of string theory. Equivalently, we can express this in terms of the *string tension* T, which measures how much energy the string stores per unit length. The relation is $T = \ell_P^{-2}$. Thus, the smaller the Planck length, the more energy is stored inside the string.

While we usually think of string theory in the low-energy limit, where the string tension is high and the string almost point-like, the true stringy nature of the theory only emerges in the high-energy regime. In this regime, the size of the string is large compared to the length-scale probed at that energy. Thus the string can be thought of as being very flabby. The excitations of the string can be conveniently visualised in the Regge-diagram, where we plot the lowest mass square of a given string excitation versus its spin. We have sketched this for bosonic string theory in Figure 1.5; the situation for the superstring is very similar. The only massless excitations are at most of spin 2, after that the mass square increases linearly with the spin. The slope of this line is given by the string tension T. Thus, when taking the string tension to zero (with respect to the relevant energy scale), also higher excitations may become massless [71].

This is a sign of a higher gauge symmetry in string theory, which is spontaneously broken at energies below the Planck scale. This symmetry is a higher spin gauge symmetry. It is the natural generalisation of gauge symmetry of massless spin-1 fields and diffeomorphism symmetry of massless spin-2 fields (gravitons). It was indeed found by Gross and Mende [72–74] that the high-energy limit of amplitudes shows signs of a very large symmetry.

In flat space, we can only explore the broken phase of string theory. The reason is that the string tension is dimensionful and can be sent to zero only in comparison with the relevant energy scale. The situation is much richer in AdS backgrounds. AdS backgrounds introduce a new dimensionful constant into the problem, namely the size of AdS ℓ_{AdS}. Thus, there is a dimensionless combination given by

$$\eta = \frac{\ell_{\text{AdS}}}{\ell_{\text{P}}} \, . \tag{1.4}$$

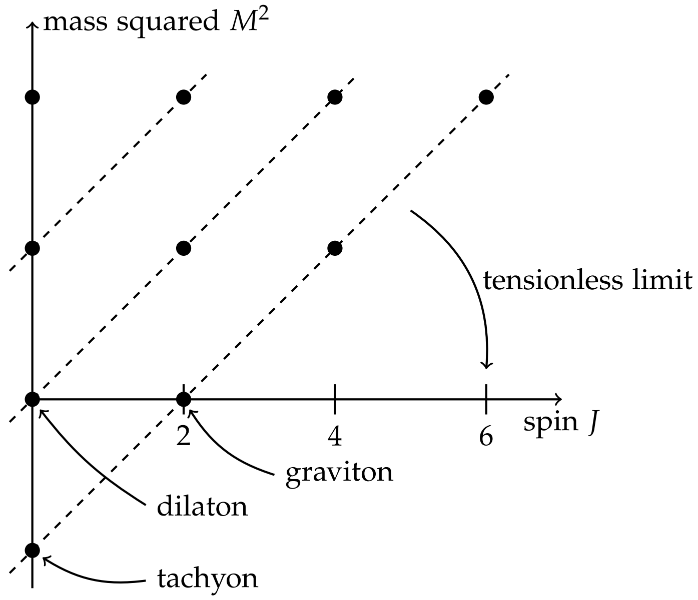

Figure 1.5: The Regge-diagram of the closed bosonic string. We plotted the mass squared of the string spectrum versus the spin. States lie on linear trajectories, the so-called Regge trajectories. We have indicated a few well-known excitations such as the graviton, which is the massless spin-2 excitation in the figure. In the tensionless limit, the slope of the trajectories becomes zero and there are many more massless excitations of higher spin.

In the example of $\mathrm{AdS}_5 \times \mathrm{S}^5$, this combination is related to the parameters of the dual CFT (the $\mathcal{N} = 4$ SYM theory) as follows:[8]

$$g_{\mathrm{s}} = g_{\mathrm{YM}}^2 \, , \qquad N g_{\mathrm{YM}}^2 = \eta^4 \, , \tag{1.5}$$

where g_{s} is the string coupling constant, g_{YM} is the gauge theory coupling constant and N is the amount of flux in the background, which is in turn identified with the rank of the gauge group. $N g_{\mathrm{YM}}^2$ is known as the 't Hooft coupling constant, since one can define a large N limit with fixed 't Hooft coupling constant. Thus, the tensionless limit $\eta \to 0$ implies a small 't Hooft coupling constant in the gauge theory. In other words, the gauge theory dual to the tensionless string simplifies significantly and is (almost) a free gauge theory.

Since the dual CFT is free, there are many more conserved quantities than in a generic interacting gauge theory. These additionally conserved quantities correspond holographically to the massless higher spin fields in the AdS bulk [75–77]. One is thus led to the statement that the high-energy limit of string theory has a dual description in terms of a free theory.

[8] Here and in the rest of the introduction, we suppress numerical factors of 2 and π, which do not influence the discussion.

While the AdS/CFT correspondence allows one to observe this higher symmetry directly from the boundary, one has not succeeded so far in computing the spectrum of string theory on $\text{AdS}_5 \times \text{S}^5$ directly in the tensionless limit. This is a somewhat unsatisfactory situation, since one does not have a direct test of these ideas. Moreover, succeeding in constructing a tensionless string theory directly would give one an example of the AdS/CFT correspondence, where many aspects of the correspondence could be tested in much detail.

As we have discussed above, AdS_3 backgrounds offer the opportunity to achieve this goal, since the corresponding string theory is under far better control, at least at the pure NS-NS flux point. In the case of $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$, the relation between the parameters is

$$k = \eta^2 \tag{1.6}$$

where k is the amount of NS-NS flux in the background. The qualitative difference of (1.5) and (1.6) comes from the different dimensionality of the AdS spaces, as well as from the fact that the $\text{AdS}_5 \times \text{S}^5$ background is supported by Ramond-Ramond flux, whereas the $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ background is supported by NS-NS flux. In particular, the flux k is quantised and has to take an integer value. This means in turn that we cannot take the tensionless limit in a continuous manner. The most tensionless point of the background appears at $k = 1$, i.e. for precisely one unit of NS-NS flux.

It will be one of the main objectives of this thesis to explore this tensionless point of string theory on AdS_3. In the dual CFT description, the embedding of higher spin theory [78–82] in string theory was understood in [83–85]. As we shall demonstrate, this point indeed realises the idea of the tensionless string and there are additional massless higher spins appearing in the theory. Moreover, the $k = 1$ point is exactly the point in the string moduli space, which is dual to the symmetric product orbifold in (1.3). In other words, we will show that the symmetric orbifold of $\mathbb{T}^4$ is dual to the pure NS-NS flux background at $k = 1$ (and not only on the same moduli space). Thus, this provides an instance of a stringy AdS/CFT duality where both sides are under complete control.

In a similar vein, we will show that

$$\text{IIB string theory on } \text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1 \text{ at } k^+ = 1 \Longleftrightarrow \text{Sym}^{Q_1}(\text{S}^3_{k^-} \times \text{S}^1)\ . \tag{1.7}$$

For this background, there are two fluxes and the tensionless limit corresponds to setting one of these fluxes to its minimal amount of one. The other flux remains arbitrary in the tensionless limit and then determines the amount of flux through the three-sphere in the dual CFT.

After successfully identifying the dual CFT for the tensionless point in moduli space, we will give also an analogue for higher values of background flux. We give very strong evidence that the correspondences generalise as follows to arbitrary values of k (resp. k^+):

$$\begin{aligned}&\text{IIB string theory on } \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4 \\ &\qquad \Longleftrightarrow \mathrm{Sym}^{Q_1}\left(\mathcal{N}=4 \text{ Liouville theory with } c = 6(k-1) \times \mathbb{T}^4\right) ,\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}&\text{IIB string theory on } \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1 \\ &\qquad \Longleftrightarrow \mathrm{Sym}^{Q_1}\left(\text{large } \mathcal{N}=4 \text{ Liouville theory with } c = \frac{6k^+k^-}{k^+ + k^-}\right) .\end{aligned} \tag{1.9}$$

(Supersymmetric) Liouville theory is a relatively well-understood two-dimensional CFT. However, the proposed dual CFT is interacting and does not possess massless higher spin fields. This mirrors the intuition we gave above for the tensionless limit of string theory.

1.4 Overview of this thesis

This work is organised as follows. We start off by introducing the necessary background material of which we will make frequent use in the following chapters. We include the conformal field theories of interest, namely Liouville theory, the $\mathrm{SL}(2,\mathbb{R})$ WZW model and the symmetric product orbifold, which arises as a dual CFT in many instances of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. We moreover explain various superconformal algebras, namely the $\mathcal{N}=1$, $\mathcal{N}=2$, small $\mathcal{N}=4$ and large $\mathcal{N}=4$ superconformal algebra. After this, we move on to explain background material on string theory on AdS_3. This section comprises the application of the $\mathrm{SL}(2,\mathbb{R})$ WZW model to string theory on AdS_3, the hybrid formalism of superstrings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}_4$ and background material on the moduli space of the compactification $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}_4$, both from a D-brane point of view and a gauge theory point of view.

The original work starts in Chapter 4. We use the hybrid formalism to explore string theory on mixed background flux, in order to explore the string theory moduli space from the string side. This involves the study of the sigma-model on the supergroup $\mathrm{PSU}(1,1|2)$. It features a non-holomorphic current algebra, which we use to constrain the spectrum as much as possible. Using this technology, we derive the plane-wave spectrum of the string in mixed flux backgrounds, which is the spectrum in the limit of large geometry and fast rotating strings. This is an important check on the hybrid formalism formulation of string theory. We are also able to use the hybrid formalism to derive the qualitative behaviour of the string spectrum as we approach the pure NS-NS point in moduli space. This limit is very interesting, since the pure NS-NS point is a singular point in moduli space. In particular, we show that the spectrum changes discontinuously, thus confirming the prediction of [53] explicitly. The results of this chapter are based on the publications [86, 87].

We then move on in Chapter 5 to the study of the tensionless string on the background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$. To this end, we make again use of the hybrid formalism, which we already studied in the previous chapter, but we will be interested in the pure NS-NS point, where the hybrid formalism is based on the WZW model on the supergroup $\mathrm{PSU}(1,1|2)$. We explore the representation theory of $\mathfrak{psu}(1,1|2)$ in detail and explain how one can define the worldsheet theory for the minimal flux $k = 1$. We show that this worldsheet theory realises the idea of a tensionless string directly. We check that its partition function is directly the partition function of the symmetric product orbifold, thereby providing very strong evidence for the duality (1.3). The results of this chapter are based on the publication [88].

We develop the duality map further in Chapter 6, where we show that the symmetry algebra we compute from the worldsheet also matches with the symmetry algebra of the symmetric product orbifold. Fixing the symmetry algebra together with the spectrum then constrains the dual CFT significantly. Hence this goes a long way towards proving the AdS/CFT duality in this instance. We will also extend the duality from the tensionless point to large NS-NS flux. In this case, we can show that the dual CFT is (1.8). We will also give a simpler bosonic version of the same duality, which involves a symmetric product of usual bosonic Liouville theory. The results of this chapter are based on the publication [89].

In Chapter 7, we then extend our findings to the more complicated background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$. To treat the tensionless limit for this background, we first develop a hybrid formalism adapted to the background. This involves a WZW model on the exceptional supergroup $\mathrm{D}(2,1;\alpha)$. By studying the representation theory of $\mathfrak{d}(2,1;\alpha)$ in detail, we are again able to formulate a consistent worldsheet theory for the background at $k^+ = 1$. We then compute the spacetime spectrum and confirm that it matches the spectrum of the symmetric product orbifold of $\mathrm{S}^3 \times \mathrm{S}^1$. As in the small $\mathcal{N} = 4$ supersymmetric case, we also determine the spacetime symmetry algebra and confirm that it agrees with the symmetry algebra of $\mathrm{S}^3 \times \mathrm{S}^1$. We finally extend the duality away from the symmetric orbifold and show (1.9). The results of this chapter are based on the publication [69].

Each of the chapters contains a short summary and conclusion. We conclude in Chapter 8 with a broader discussion of the results of this thesis.

The work is complemented by some appendices, which contain more details about various technical points. Appendix A contains the conventions used throughout the main text of the thesis. Appendix B contains details about representations used in the main text. Finally, we relegated technical details about the $\mathfrak{psu}(1,1|2)$ WZW model and the $\mathfrak{d}(2,1;\alpha)$ WZW model to Appendices C and D.

In the course of this PhD, also the publications [68, 90–93] appeared, which are not included in this thesis.

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Chapter 2

Two-dimensional CFTs

We assume that the reader is familiar with basic concepts of two-dimensional CFTs. In this chapter, we introduce some CFTs, which are central to our discussion, namely Liouville theory, the $\mathrm{SL}(2,\mathbb{R})$ WZW-model and symmetric product orbifold CFTs. Finally, we also explain various superconformal algebras relevant to our discussion.

2.1 Liouville theory

Liouville theory is one of the simplest irrational CFTs. It is characterised by the following properties [94]

(a) Liouville theory is a family of conformal field theories which exists for any central charge c.

(b) The spectrum of Liouville theory is a continuum and is given by

$$\int_{\frac{c-1}{24}}^{\infty}\!\!\!\!\!\!\!\!\!\bigoplus \mathrm{d}h\ \mathrm{Vir}_h \otimes \mathrm{Vir}_h\ . \tag{2.1}$$

By this we mean that each Virasoro representation of conformal weight $h \in [\frac{c-1}{24}, \infty)$ appears precisely once in the spectrum.[9]

(c) Correlation functions dependent analytically on c and h. For the moment, we allow branch cuts in the dependence. We will see below how to resolve them.

These properties characterise Liouville theory completely. It is convenient to parametrise c and h in terms of the following alternative quantities:

$$c = 1 + 6Q^2\ , \tag{2.2}$$

[9]This is of course a bit subtle in a theory with a continuum spectrum. More precisely, there is only one independent OPE coefficient for each conformal weight.

$$Q = b + b^{-1} , \tag{2.3}$$
$$h = \alpha(Q - \alpha) . \tag{2.4}$$

We refer to Q as *background charge*, to b as *Liouville coupling constant*, and to α as *Liouville momentum*. These quantities are not uniquely determined in terms of c and h, but the different choices do not differ physically. Note that α takes values in $\frac{1}{2}Q + i\mathbb{R}$. Correlation functions are then meromorphic functions in b and α.

The theory can be derived from a classical action, which takes the form

$$\mathcal{S}[\varphi] = \frac{1}{4\pi} \int \mathrm{d}^2 z \, \sqrt{g} \left(g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + QR\varphi + \mu \mathrm{e}^{2b\varphi} \right) . \tag{2.5}$$

Here, the fundamental field φ is the called the Liouville field. It is not holormorphic and hence does not constitute a good variable in the quantum theory. Here, R is the Ricci scalar on the worldsheet. Thus the theory couples to the curvature of the worldsheet. Since we are working on the plane and choose the flat metric, the curvature is entirely concentrated at ∞. Thus this term has the effect of placing a charge Q at ∞, hence explaining its name. Finally, there is the exponential coupling. The value of μ is inconsequential, since we can always shift φ by constant, which changes the value of μ.

It is customary to denote primary fields corresponding to α by $V_\alpha(z, \bar{z})$, where α is the Liouville momentum. In order not to overcount, the primary fields $V_\alpha(z)$ and $V_{Q-\alpha}(z)$ are identified. The corresponding constant of proportionality is called the *reflection amplide*. We are then mostly interested in computing 3-point functions

$$\begin{aligned} &\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle \\ &\qquad = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{(z_1 - z_2)^{h_1+h_2-h_3} (z_2 - z_3)^{h_2+h_3-h_1} (z_3 - z_1)^{h_1+h_3-h_1}} . \end{aligned} \tag{2.6}$$

The z dependence of this expression is entirely fixed by conformal invariance. There is a unique solution for the constants $C(\alpha_1, \alpha_2, \alpha_3)$ (which is analytic in α_i and b and satisfies the crossing equation for the 4-point function), given by the DOZZ formula [95,96]

$$\begin{aligned} C(\alpha_1, \alpha_2, \alpha_3) &= \left(\pi \nu \gamma(b^2) b^{2-2b^2} \right)^{\frac{1}{b}\left(Q - \sum_{i=1}^3 \alpha_i\right)} \\ &\times \frac{\Upsilon(0)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)} . \end{aligned} \tag{2.7}$$

Here, we have used the special function $\Upsilon(x)$, which can be defined by

$$\log \Upsilon(x) = \int_0^\infty \frac{\mathrm{d}t}{t} \left(\left(\frac{Q}{2} - x \right)^2 \mathrm{e}^{-t} - \frac{\sinh^2 \left(\frac{Q}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right) . \tag{2.8}$$

Moreover, we have used the function

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \ , \tag{2.9}$$

where $\Gamma(x)$ is the standard Gamma function. ν is an arbitrary constant, which depends on how the two-point function is normalised.

Knowing the 3-point functions solves the theory in principle completely on the plane, in the sense that all correlation functions can be computed. This follows from conformal symmetry alone. We have for instance for the four-point function

$$\begin{aligned}\langle V_{\alpha_1}(z)V_{\alpha_2}(0)V_{\alpha_3}(\infty)V_{\alpha_4}(1)\rangle \\ = \int_{\frac{Q}{2}+i\mathbb{R}_{\geq 0}} \mathrm{d}\alpha\ C(\alpha_1,\alpha_2,Q-\alpha)C(\alpha,\alpha_3,\alpha_4)|\mathcal{F}^{(s)}_\alpha(z)|^2 \ .\end{aligned} \tag{2.10}$$

We have used the OPE on the fields $V_{\alpha_1}(z)V_{\alpha_2}(0)$. This yields the intermediate primary state α and all its descendants, whose contributions is captured by the s-channel conformal block $\mathcal{F}^{(s)}_\alpha(z)$. The conformal block depends on z, c, α, as well as on α_1, α_2, α_3 and α_4. After taking the OPE of the fields $V_{\alpha_1}(z)V_{\alpha_2}(0)$, the four-point function reduces to the three-point function of the resulting field $\langle V_\alpha(0)V_{\alpha_3}(\infty)V_{\alpha_4}(1)\rangle$ (together with its descendants), which produces the factor $C(\alpha,\alpha_3,\alpha_4)$ in (2.10). Equivalently, we could have used the OPE in a different order, which would result in the equation

$$\begin{aligned}\langle V_{\alpha_1}(z)V_{\alpha_2}(0)V_{\alpha_3}(\infty)V_{\alpha_4}(1)\rangle \\ = \int_{\frac{Q}{2}+i\mathbb{R}_{\geq 0}} \mathrm{d}\alpha\ C(\alpha_1,\alpha_4,Q-\alpha)C(\alpha,\alpha_2,\alpha_3)|\mathcal{F}^{(t)}_\alpha(z)|^2 \ .\end{aligned} \tag{2.11}$$

Equality of (2.10) and (2.11) is the statement of crossing symmetry and is extremely non-trivial. In fact, it is best checked numerically.

For more details on Liouville theory, see e.g. [94, 97–99].

2.2 The $\mathrm{SL}(2,\mathbb{R})$ WZW model

Next, we introduce a model, which is of central importance for the study of string theory on AdS_3 – the WZW model based on the group $\mathrm{SL}(2,\mathbb{R})$. We take the current algebra $\mathfrak{sl}(2,\mathbb{R})_k$ to have defining commutation relations

$$[J^3_m, J^3_n] = -\frac{k}{2} m\delta_{m+n,0} \ , \tag{2.12a}$$

$$[J^3_m, J^\pm_n] = \pm J^\pm_{m+n} \ , \tag{2.12b}$$

$$[J^+_m, J^-_n] = km\delta_{m+n,0} - 2J^3_{m+n} \ . \tag{2.12c}$$

We impose moreover the hermicity conditions

$$\left(J^{+}_{m}\right)^{\dagger} = J^{-}_{-m}\,, \qquad \left(J^{3}_{m}\right)^{\dagger} = J^{3}_{-m}\,. \tag{2.13}$$

To construct a conformal field theory based on this chiral algebra, we have to study its representations. An important class of representations are those intoduced from representations of the $\mathfrak{sl}(2,\mathbb{R})$ zero-mode algebra. Thus, we start by studying $\mathfrak{sl}(2,\mathbb{R})$ representations.

2.2.1 Representations of $\mathfrak{sl}(2,\mathbb{R})$

There are two kinds of unitary representations of $\mathfrak{sl}(2,\mathbb{R})$ that will be relevant for us [60,100,101]

(a) Discrete representations. These are representations of $\mathfrak{sl}(2,\mathbb{R})$ that possess a lowest (highest) weight state. The representation is characterised by the $\mathfrak{sl}(2,\mathbb{R})$ spin j of the lowest (highest) weight state.

(b) Continuous representations. Continuous representations of $\mathfrak{sl}(2,\mathbb{R})$ do not contain a highest nor a lowest weight state. These representations are characterised by their Casimir $\mathcal{C}$, as well as the fractional part of the J^3_0-eigenvalues which we label by $\lambda \in \mathbb{R}/\mathbb{Z}$.

More specifically, the continuous representations of $\mathfrak{sl}(2,\mathbb{R})$ are defined via

$$J^{+}_{0}\,|m\rangle = |m+1\rangle\;, \tag{2.14a}$$

$$J^{3}_{0}\,|m\rangle = m\,|m\rangle\;, \tag{2.14b}$$

$$J^{-}_{0}\,|m\rangle = \big(m(m-1)+\mathcal{C}\big)\,|m-1\rangle\;. \tag{2.14c}$$

Here, $\mathcal{C}$ is the quadratic Casimir of the $\mathfrak{sl}(2,\mathbb{R})$ representation, which in these conventions takes the form

$$\mathcal{C} = -J^3_0 J^3_0 + \frac{1}{2}\left(J^+_0 J^-_0 + J^-_0 J^+_0\right)\,, \tag{2.15}$$

while m takes values in $m \in \mathbb{Z}+\lambda$. Provided that the Casimir satisfies $\mathcal{C} \geq \lambda(1-\lambda)$ where we take $\lambda \in [0,1]$, these representations are unitary, see [100,101] for useful reviews. (For the case of the discrete representations, the relevant condition is $j \geq 0$.)

The discrete representations can be found as subrepresentations of the continuous representations. Thus, we mostly discuss continuous representations. It is convenient to parametrise the Casimir of the continuous representations by j as well, i.e. to write

$$\mathcal{C} = -j(j-1) = \frac{1}{4} - \left(j-\frac{1}{2}\right)^2\,, \tag{2.16}$$

where $j \in \mathbb{R} \cup \left(\frac{1}{2} + i\mathbb{R}\right)$. In this notation, j and $1-j$ parametrise the same continuous representation. For real j, we then see by virtue of the relation (2.14c) that $J_0^- \left|j\right\rangle = 0$. Thus, the states $\left|m\right\rangle$ for which $m - j \in \mathbb{Z}_{\geq 0}$ form a subrepresentation, which is isomorphic to a lowest weight discrete representation.

We will denote the continuous representation by $\mathcal{C}^j_\lambda$ and the discrete representation by $\mathcal{D}^j_+$. There exist also highest-weight (rather than lowest-weight) discrete representations, which we will denote by $\mathcal{D}^j_-$; they are characterised by

$$J_0^+ \left|j\right\rangle = 0 \,, \qquad J_0^3 \left|j\right\rangle = -j \left|j\right\rangle \,. \tag{2.17}$$

In addition to these infinite-dimensional representations, there are also the usual finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$, although they will not be part of the worldsheet spectrum. We will simply denote the m-dimensional representation by $\mathbf{m}$. Except for the trivial representation, these are non-unitary. Later, we will need the Clebsch-Gordan coefficients of the tensor product of $\mathcal{C}^j_\lambda$ and $\mathcal{D}^j_\pm$ with $\mathbf{2}$. An explicit calculation shows that

$$\mathcal{C}^j_\lambda \otimes \mathbf{2} \cong \mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}} \oplus \mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}} \,, \qquad \mathcal{D}^j_\pm \otimes \mathbf{2} \cong \mathcal{D}^{j+\frac{1}{2}}_\pm \oplus \mathcal{D}^{j-\frac{1}{2}}_\pm \,. \tag{2.18}$$

2.2.2 The energy-momentum tensor and the Sugawara construction

After having discussed the relevant $\mathfrak{sl}(2,\mathbb{R})$ representations, we discuss how to turn the WZW model into a conformal field theory. For this, we have to construct an energy-momentum tensor out of the fundamental currents $J^\pm(z)$ and $J^3(z)$. As for any WZW model based on a simple group, this is achieved via the Sugawara construction [102]:

$$T(z) = \frac{1}{k-2}\left(-(J^3J^3) + \frac{1}{2}\left((J^+J^-) + (J^-J^+)\right)\right)(z) \,. \tag{2.19}$$

With respect to this energy-momentum tensor, the currents are primary fields of conformal weight one. The central charge of the theory is given by

$$c = \frac{3k}{k-2} \,. \tag{2.20}$$

This is formally the same central charge as for the $\mathfrak{su}(2)_{-k}$ theory. In fact, the two theories are the same, except for having different hermicity conditions on the currents.[10]

[10] In the conventions (2.12a)–(2.12c), $\mathfrak{su}(2)_{-k}$ would be given by

$$\left(J_m^+\right)^\dagger = -J^-_{-m} \,, \qquad \left(J_m^3\right)^\dagger = J^3_{-m} \,. \tag{2.21}$$

2.2.3 Spectral flow

As we have mentioned above, we can construct representations of the affine $\mathfrak{sl}(2,\mathbb{R})_k$ algebra by letting the zero-modes act in a specific representation of $\mathfrak{sl}(2,\mathbb{R})$ on a set of ground states. These ground states are annihilated by all positive modes and the negative modes act freely. These are the so-called conformal highest weight representations. For the $\mathfrak{sl}(2,\mathbb{R})_k$ WZW model, it is crucial that further representations are included. They are best described via the so-called spectral flow.

The affine algebra (2.12a)–(2.12c) possesses an outer automorphism, the so-called spectral flow automorphism σ^w. We define σ^w on the generators as follows:

$$\sigma^w(J^\pm_m) = J^\pm_{m\mp w}\ , \qquad \sigma^w(J^3_m) = J^3_m + \tfrac{kw}{2}\delta_{m,0}\ . \tag{2.22}$$

Given a representation of the affine algebra, composing it with the spectral flow automorphism will yield a new representation.

In fact, for the quantum theory to be consistent, we have to require that all spectrally flowed images of a given representation are part of the spectrum as well. First, 3-point functions can violate spectral flow conservation. Second, one needs to include spectrally flowed representations to obtain a modular invariant torus partition function.

Under spectral flow, the energy-momentum tensor transforms as follows:

$$\sigma^w(L_m) = L_m - wJ^3_m - \frac{kw^2}{4}\delta_{m,0}\ . \tag{2.23}$$

Spectrally flowed representations are no longer conformal highest weight representations (in the sense that all positive modes annihilate a given highest weight state). In particular, this will yield a spectrum which is unbounded from below. The theory is still physically sensible and stable, since the spectrum is bounded from below for a fixed $\mathfrak{sl}(2,\mathbb{R})$ charge.

2.2.4 The complete spectrum of the theory

Maldacena & Ooguri proposed a set of representations for the theory, based on semiclassical reasoning in the $k \to \infty$ limit [60]. In the $k \to \infty$ limit, the theory becomes quantum mechanics on $\mathrm{SL}(2,\mathbb{R})$ and one obtains the spectrum from decomposing $\mathrm{L}^2(\mathrm{SL}(2,\mathbb{R}))$ (with the biinvariant Haar measure) into $\mathrm{SL}(2,\mathbb{R})$ representations. To obtain the spectrum in the finite k regime, one demands that the spectrum is compatible with the spectral flow symmetry. The proposed spectrum consists of the affine representations based on

$$\text{discrete} : \mathcal{D}^j_+ \otimes \mathcal{D}^j_+\ , \qquad \tfrac{1}{2} < j < \tfrac{k-1}{2}\ , \tag{2.24}$$

$$\text{continuous}: \mathcal{C}^j_\lambda \otimes \mathcal{C}^j_\lambda\ , \qquad \lambda \in \mathbb{R}/\mathbb{Z}\ , \qquad j \in \tfrac{1}{2} + i\mathbb{R}_{\geq 0}\ , \tag{2.25}$$

together with all their spectrally flowed images. We will refer to the bound $\frac{1}{2} < j < \frac{k-1}{2}$ as Maldacena-Ooguri bound. We will see below that there is another important bound for strings on AdS_3, which is the unitarity bound arising from the requirement of physical states to have non-negative norm [55–57, 59]. It reads $0 \leq j \leq \frac{k}{2}$ and is thus slightly weaker. In the supersymmetric case, the two bounds become actually equivalent [87].

2.3 Superconformal theories

In the context of string theory, supersymmetric CFTs (SCFTs) will be of particular importance. We review the most important superconformal algebras in turn.

2.3.1 $\mathcal{N} = 1$ supersymmetry

Besides the Virasoro tensor $T(z)$, we also have one supercurrent $G(z)$. The generators satisfy the OPEs

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\ , \tag{2.26}$$

$$T(z)G(w) \sim \frac{3G(w)}{2(z-w)^2} + \frac{\partial G(w)}{z-w}\ , \tag{2.27}$$

$$G(z)G(w) \sim \frac{2c}{3(z-w)^3} + \frac{2T(w)}{z-w}\ . \tag{2.28}$$

The algebra affords only a discrete $\mathbb{Z}_2$ R-symmetry, which sends $G(z) \to -G(z)$, but no continuous R-symmetry.

$\mathcal{N} = 1$ supersymmetry is important in superstring theory, since the world-sheet CFT has to constitute an $\mathcal{N} = 1$ SCFT in order to define the physical state conditions.

Let us mention here also that the wedge algebra (consisting of all modes V_n with $-h < n < h$), which is the subalgebra formed by L_{-1}, L_0, L_1, $G_{-\frac{1}{2}}$ and $G_{\frac{1}{2}}$ is given by the Lie superalgebra $\mathfrak{osp}(1|2)$.

A simple example of a theory with $\mathcal{N} = 1$ supersymmetry is given by a free bosons and a free fermion. A somewhat more interesting example, which plays also a central role in the string theory construction later is given by an $\mathcal{N} = 1$ affine algebra $\mathfrak{g}^{(1)}_k$. This algebra consists of an affine current algebra $\mathfrak{g}_k$ generated by the currents J^a, together with $\dim(\mathfrak{g})$ free fermions ψ^a transforming in the adjoint representation of $\mathfrak{g}$. The free fermions ψ^a have defining OPEs

$$\psi^a(z)\psi^b(w) \sim \frac{k\delta^{ab}}{z-w} \ . \tag{2.29}$$

For simplicity, we chose a basis such that the metric on the Lie algebra becomes δ^{ab}. We can define $J^{(\mathrm{f})a}(z) = \frac{i}{2k} f^a{}_{bc}(\psi^b\psi^c)$, which gives another affine algebra at level $h^\vee$, under which the free fermions transform also as primary fields in the adjoint representation. Here, $h^\vee$ is the dual Coxeter number of the algebra $\mathfrak{g}$. Thus, we can define the 'decoupled' currents

$$\mathcal{J}^a(z) = J^a(z) - J^{(\mathrm{f})a}(z) \ , \tag{2.30}$$

which have regular OPEs with the free fermions and define an affine algebra at level $k - h^\vee$. Upon using the fact

$$\mathfrak{so}(N)_1 \cong N \text{ free fermions} \ , \tag{2.31}$$

we can write the isomorphism

$$\mathfrak{g}^{(1)}_k \cong \mathfrak{g}_{k-h^\vee} \oplus \mathfrak{so}(\dim(\mathfrak{g}))_1 \ . \tag{2.32}$$

This algebra has a canonical $\mathcal{N} = 1$ structure, given by

$$T(z) = \frac{1}{2k}\left((\mathcal{J}^a\mathcal{J}^a) - (\psi^a\partial\psi^a)\right) \ , \tag{2.33}$$

$$G(z) = \frac{1}{k}\left((\mathcal{J}^a\psi^a) - \frac{i}{6k}f_{abc}(\psi^a\psi^b\psi^c)\right) \ . \tag{2.34}$$

The central charge of the theory is given by

$$c = \frac{(k-h^\vee)\dim(\mathfrak{g})}{k} + \frac{1}{2}\dim(\mathfrak{g}) = \left(\frac{3}{2} - \frac{h^\vee}{k}\right)\dim(\mathfrak{g}) \ . \tag{2.35}$$

The $\mathcal{N} = 1$ algebra has an interesting representation, which extends the representation theory of the Virasoro algebra [103].

2.3.2 $\mathcal{N} = 2$ supersymmetry

Next, we discuss $\mathcal{N} = 2$ supersymmetry. Besides the Virasoro tensor, there are now two supercurrents present, which we denote by $G^+(z)$ and $G^-(z)$. There is a U(1) R-symmetry, under which the two supercurrents carry charges $+1$ and -1. The corresponding current will be denoted by $J(z)$. $T(z)$ itself satisfies the Virasoro algebra and the remaining fields are primary fields (of conformal weight $\frac{3}{2}$ in the case of the supercurrents and of conformal weight 1 in the case of the $\mathfrak{u}(1)$ current). The remaining OPEs read as follows

$$J(z)G^\pm(w) \sim \frac{\pm G^\pm(w)}{z-w} \ , \tag{2.36}$$

$$J(z)J(w) \sim \frac{c}{3(z-w)^2} \,, \tag{2.37}$$

$$G^+(z)G^-(w) \sim \frac{c}{6(z-w)^3} + \frac{J(w)}{2(z-w)^2} + \frac{2T(w) + \partial J(w)}{4(z-w)} \,, \tag{2.38}$$

while $G^+(z)G^+(w)$ and $G^-(z)G^-(w)$ are forced to be zero by $\mathrm{U}(1)$ charge conservation. By defining

$$G(z) = \sqrt{2}(\mathrm{e}^{i\varphi}G^+(z) + \mathrm{e}^{-i\varphi}G^-(w)) \tag{2.39}$$

for any phase $\mathrm{e}^{i\varphi}$, we see that the $\mathcal{N}=2$ superconformal algebra contains in particular also the $\mathcal{N}=1$ superconformal algebra.

Starting from $\mathcal{N}=2$ superconformal symmetry, there are interesting BPS representations. These are shorter representations, whose energies are protected by supersymmetry. To see this, consider a superconformal primary state (in the NS-sector), which satisfies

$$G_r^\pm \left|h,q\right\rangle = 0 \,, \qquad r > 0 \,, \tag{2.40}$$

$$L_m \left|h,q\right\rangle = J_m \left|h,q\right\rangle = 0 \,, \qquad m > 0 \,, \tag{2.41}$$

$$L_0 \left|h,q\right\rangle = h \left|h,q\right\rangle \,, \tag{2.42}$$

$$J_0 \left|h,q\right\rangle = q \left|h,q\right\rangle \,. \tag{2.43}$$

Then generically, there are two superconformal descendants at level $\frac{1}{2}$: $G^+_{-1/2}\left|h,q\right\rangle$ and $G^-_{-1/2}\left|h,q\right\rangle$. The norm of these states is

$$\|G^\pm_{-1/2}\left|h,q\right\rangle\|^2 = \left\langle h,q\right| \{G^\mp_{1/2}, G^\pm_{-1/2}\} \left|h,q\right\rangle \tag{2.44}$$

$$= \frac{1}{4}\left\langle h,q\right| 2L_0 \mp J_0 \left|h,q\right\rangle \tag{2.45}$$

$$= \frac{1}{4}(2h \mp q) \,. \tag{2.46}$$

Hence, for $h = \pm\frac{1}{2}q$, one of the descendants is absent. Moreover, unitarity imposes the BPS bound

$$h \geq \tfrac{1}{2}|q| \,. \tag{2.47}$$

A superconformal primary saturating the bound is called *chiral primary* (resp. antichiral primary). Usually, the $\mathrm{U}(1)$-charge in a given theory is quantised. Since $h = \pm\frac{1}{2}q$ for (anti)chiral primaries, it also follows that their conformal weight is quantised.[11] The set of chiral primaries provides however more structure, since it can be turned into a ring. The OPE between two chiral primaries has to be regular, since by the conservation of $\mathrm{U}(1)$ charge, any singular term would violate the BPS bound (2.47). By the same argument,

[11] For a complete analysis of null-vectors of the superconformal $\mathcal{N}=2$ algebra, see e.g. [104–109].

the constant term in the OPE is again a chiral primary and hence normal-ordering induces a ring structure on the chiral primaries. This ring structure is referred to as the *chiral ring* [110].

$\mathcal{N}=2$ superconformal symmetry plays also a central role in string theory. An $\mathcal{N}=2$ supersymmetric worldsheet theory (in the RNS-formalism) leads to a spacetime supersymmetric theory [111]. Moreover, $\mathcal{N}=2$ SCFTs are the fundamental building blocks to define topological CFTs and topological strings. This can be achieved by the so-called *topological twisting* [112–115], where the energy-momentum tensor is modified to

$$\hat{T}(z) = T(z) \pm \tfrac{1}{2}\partial J(z) \ . \tag{2.48}$$

With respect to this new energy-momentum tensor, $G^{\mp}(z)$ ($G^{\pm}(z)$) is still primary and of conformal weight 1 (2). Thus, we can view $G^{\mp}(z)$ as a BRST current and define a BRST charge as

$$Q_{\text{BRST}} = \oint \mathrm{d}z\ G^{\mp}(z) \ . \tag{2.49}$$

The BRST cohomology defines then a topological theory. The cohomology is in fact again the chiral (or antichiral) ring. In the context of string theory, this leads to topological strings [116–118].

We again mention that the wedge algebra (or global algebra) for this superconformal algebra is given by the Lie superalgebra $\mathfrak{osp}(2|2) \cong \mathfrak{sl}(2|1)$.

Interesting examples of $\mathcal{N}=2$ superconformal field theories include the minimal models, which can be represented as a coset

$$\frac{\mathfrak{su}(2)^{(1)}_{k+2}}{\mathfrak{u}(1)^{(1)}} \cong \frac{\mathfrak{su}(2)_k \oplus \mathfrak{so}(2)_1}{\mathfrak{u}(1)} \ , \tag{2.50}$$

as well as their generalisations, the Kazama-Suzuki models [119]

$$\frac{\mathfrak{su}(N+1)^{(1)}_{k+N+1}}{\mathfrak{su}(N)^{(1)}_{k+N+1} \oplus \mathfrak{u}(1)^{(1)}} \cong \frac{\mathfrak{su}(N+1)_k \oplus \mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1} \oplus \mathfrak{u}(1)} \ . \tag{2.51}$$

We used here the $\mathcal{N}=1$ superconformal affine algebras from above. There is no pure WZW-model based on a simple group, which has $\mathcal{N}=2$ supersymmetry. Another prominent example is given by sigma-models on Calabi-Yau manifolds, which is the standard route to compactify string theory to four dimensions [120]. See e.g. [121] for the vast literature on this topic. Special points in the moduli space can be represented by Gepner models, which are an orbifold of a product of minimal models [122]. This has in particular led to the development of *Mirror symmetry*, which is the generalisation of T-duality to Calabi-Yau manifolds [123, 124]. For sigma-models, the BPS spectrum (i.e. the set of chiral primaries) agrees with the Dolbeau cohomology of the target space [125]. Moreover, the chiral ring equals a deformation of the cohomology ring of the target space [110].

2.3.3 Small $\mathcal{N} = 4$ supersymmetry

Next, we discuss small $\mathcal{N} = 4$ supersymmetry, which has four supercurrents $G^{\alpha\beta}(z)$, where α, $\beta = \pm$. The R-symmetry is given by $\mathfrak{su}(2)$, whose currents we shall denote by $J^a(z)$, $a = \pm, 3$. Moreover, there is an outer automorphism algebra $\mathfrak{su}(2)$. The four supercharges transform as bispinors under the two $\mathfrak{su}(2)$'s. The OPEs read

$$J^3(z)J^\pm(w) \sim \frac{\pm J^\pm(w)}{z-w} , \tag{2.52}$$

$$J^3(z)J^3(w) \sim \frac{k}{2(z-w)^2} , \tag{2.53}$$

$$J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w} , \tag{2.54}$$

$$J^a(z)G^{\alpha\beta}(w) \sim \frac{(\sigma^a)^\alpha{}_\gamma G^{\gamma\beta}(w)}{z-w} , \tag{2.55}$$

$$\begin{aligned} G^{\alpha\beta}(z)G^{\gamma\delta}(w) \sim &\frac{2\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}k}{(z-w)^3} + \frac{2(\sigma_a)^{\alpha\gamma}\varepsilon^{\beta\delta}J^a(w)}{(z-w)^2} \\ &+ \frac{(\sigma_a)^{\alpha\gamma}\varepsilon^{\beta\delta}\partial J^a(w) + \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}T(w)}{z-w} . \end{aligned} \tag{2.56}$$

See Appendix A for the conventions of σ-matrices. The central charge of the algebra is given in terms of the level k of the $\mathfrak{su}(2)_k$ R-symmetry as

$$c = 6k . \tag{2.57}$$

Since we are interested in unitary models where $k \in \mathbb{Z}_{\geq 1}$, the central charge is quantised in $6\mathbb{Z}$. This is contrary to what happens in the $\mathcal{N} = 2$ case, where for any $c \geq 3$, $\mathcal{N} = 2$ Liouville theory provides an example of a unitary $\mathcal{N} = 2$ SCFT.

One can again define an $\mathcal{N} = 2$ subalgebra of the $\mathcal{N} = 4$ superconformal algebra.[12] Thus, $\mathcal{N} = 4$ implies $\mathcal{N} = 2$, which implies $\mathcal{N} = 1$.

One can also topologically twist the algebra, which results in topological $\mathcal{N} = 4$ CFTs. Applying this to string theory leads to $\mathcal{N} = 4$ topological strings [127].

The global subalgebra forms the Lie supergroup $\mathfrak{psu}(1,1|2)$ (where we already specified an appropriate real form). This fact plays a crucial role in the later chapters.

For small $\mathcal{N} = 4$ supersymmetry, one can again define chiral primaries, where the relevant BPS condition is now $h \geq \ell$, and ℓ is the $\mathfrak{su}(2)$ spin of the

[12] In fact, there is a $\mathrm{SU}(2)/\mathrm{U}(1)$ worth of such choices [126].

R-symmetry representation.[13] Chiral primaries form again a ring. $\mathcal{N}=4$ supersymmetry does not only protect the conformal weight of the chiral primaries, but also the ring structure (i.e. their three-point functions) [128, 129]. In the context of AdS_3/CFT_2 holography, this allows for dynamical tests of the correspondence [130, 131].

Examples of $\mathcal{N}=4$ theories are given by the sigma-models on hyperkähler manifolds, in particular on $\mathbb{T}^4$ and K3. Furthermore, $\mathcal{N}=4$ Liouville theory provides another example, which plays a central role in Chapter 6. Finally, the symmetric orbifold construction described in the next section generates new $\mathcal{N}=4$ theories. Of particular importance will be the symmetric orbifolds

$$\mathrm{Sym}^N(\mathbb{T}^4) \quad \text{and} \quad \mathrm{Sym}^N(\mathrm{K3})\ . \tag{2.58}$$

2.3.4 Large $\mathcal{N}=4$ supersymmetry

We also introduce the lesser-known (linear) large $\mathcal{N}=4$ superconformal algebra [70]. It has again four supercurrents $G^{\alpha\beta}(z)$, but the R-symmetry is extended to $\mathfrak{su}(2)_{k^+}\oplus\mathfrak{su}(2)_{k^-}\oplus\mathfrak{u}(1)$. In particular, the algebra depends on two parameters k^+ and k^-, which are the two levels of the two $\mathfrak{su}(2)$'s. Moreover, the algebra contains four free fermions $Q^{\alpha\beta}(z)$. There is also a non-linear version of the large $\mathcal{N}=4$ algebra, in which the $\mathfrak{u}(1)$ current and the four free fermions are factored out [132].

The defining OPEs of the algebra take the form

$$U(z)U(w)\sim \tfrac{k^++k^-}{2(z-w)^2}\ , \tag{2.59a}$$

$$A^{(+)a}(z)Q^{\alpha\beta}(w)\sim \frac{(\sigma^a)^{\alpha}{}_{\gamma}\,Q^{\gamma\beta}(w)}{2(z-w)}\ , \tag{2.59b}$$

$$A^{(-)a}(z),Q^{\alpha\beta}(w)\sim \frac{(\sigma^a)^{\beta}{}_{\gamma}\,Q^{\alpha\gamma}}{2(z-w)}\ , \tag{2.59c}$$

$$Q^{\alpha\beta}(z)Q^{\gamma\delta}(w)\sim \frac{(k^++k^-)\,\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}}{z-w}\ , \tag{2.59d}$$

$$A^{(\pm)3}(z),A^{(\pm)3}(w)\sim \frac{k^\pm}{2(z-w)^2}\ , \tag{2.59e}$$

$$A^{(\pm)3}(z)A^{(\pm)\pm}(w)\sim \pm\frac{A^{(\pm)\pm}(w)}{z-w}\ , \tag{2.59f}$$

$$A^{(\pm)+}(z)A^{(\pm)-}(w)\sim \frac{k^\pm}{(z-w)^2}+\frac{2A^{(\pm)3}(w)}{z-w}\ , \tag{2.59g}$$

$$U(z)G^{\alpha\beta}(w)=\frac{i\,Q^{\alpha\beta}(w)}{2(z-w)} \tag{2.59h}$$

[13] In fact, $\mathcal{N}=4$ chiral primaries are simply chiral primaries with respect to the $\mathcal{N}=2$ subalgebra.

$$A^{(+)a}(z)G^{\alpha\beta}(w) \sim -\frac{(1-\gamma)\,(\sigma^a)^\alpha{}_\gamma\, Q^{\gamma\beta}(w)}{2(z-w)^2} + \frac{(\sigma^a)^\alpha{}_\gamma\, G^{\gamma\beta}(w)}{2(z-w)} \tag{2.59i}$$

$$A^{(-)a}(z)G^{\alpha\beta}(w) \sim \frac{\gamma\,(\sigma^a)^\beta{}_\gamma\, Q^{\alpha\gamma}(w)}{2(z-w)^2} + \frac{(\sigma^a)^\beta{}_\gamma\, G^{\alpha\gamma}(w)}{2(z-w)}\,, \tag{2.59j}$$

$$Q^{\alpha\beta}(z)G^{\gamma\delta}(w) \sim \frac{(\sigma_a)^{\alpha\gamma}\varepsilon^{\beta\delta}\, A^{(+)a}(w) - \varepsilon^{\alpha\gamma}(\sigma_a)^{\beta\delta}\, A^{(-)a}(w)}{z-w} + \frac{i\,\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\, U(w)}{z-w}\,, \tag{2.59k}$$

$$\begin{aligned} G^{\alpha\beta}(z)G^{\gamma\delta}(w) \sim & -\frac{2k\,\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}}{(z-w)^3} \\ & - \frac{2\gamma\,(\sigma_a)^{\alpha\gamma}\varepsilon^{\beta\delta}\, A^{(+)a}(w) + 2(1-\gamma)\,\varepsilon^{\alpha\gamma}(\sigma_a)^{\beta\delta}A^{(-)a}(w)}{(z-w)^2} \\ & - \frac{\gamma\,(\sigma_a)^{\alpha\gamma}\varepsilon^{\beta\delta}\,\partial A^{(+)a}(w) + (1-\gamma)\,\varepsilon^{\alpha\gamma}(\sigma_a)^{\beta\delta}\partial A^{(-)a}(w)}{z-w} \\ & - \frac{\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\, T(w)}{z-w}\,. \end{aligned} \tag{2.59l}$$

In terms of the levels of the two $\mathfrak{su}(2)$ algebras, we have

$$\gamma = \frac{k^-}{k^+ + k^-}\,, \qquad c = \frac{6k^+k^-}{k^+ + k^-}\,. \tag{2.60}$$

See Appendix A for the conventions for the Pauli matrices etc.

Representations of the linear large $\mathcal{N} = 4$ algebra are described by their conformal weight of the highest weight state, as well as the R-symmetry representation. We denote such a representation for short by $[h, \ell^+, \ell^-, u]$, where the four labels are the conformal weight of the highest weight state, the two $\mathfrak{su}(2)$ spins and the $\mathfrak{u}(1)$ charge. There is a BPS bound on representations, which reads [65,67,133,134]

$$h \geq h_{\text{BPS}}(\ell^+, \ell^-, u) \equiv \frac{k^-\ell^+ + k^+\ell^- + (\ell^+ - \ell^-)^2 + u^2}{k^+ + k^-}\,. \tag{2.61}$$

The large $\mathcal{N} = 4$ superconformal algebra possesses a $\mathcal{N} = 2$ subalgebra.[14] Chiral primaries of the $\mathcal{N} = 2$ subalgebra correspond to BPS states with $\ell^+ = \ell^-$ and $u = 0$. In particular, only for these states a chiral ring structure can be defined. For this reason, we will refer to them as chiral primaries. Note that only chiral primaries are stable with respect to increasing the levels k^+ and k^-. In particular, the classical limit $k^\pm \to \infty$ of the BPS bound reads only

$$h \geq \frac{k^-\ell^+ + k^+\ell^-}{k^+ + k^-}\,. \tag{2.62}$$

[14] It also possesses a small $\mathcal{N} = 4$ subalgebra, but the embedding does not preserve the energy-momentum tensor, nor the hermitian structure of the algebra [70].

When going to the quantum theory, BPS states with $\ell^+ \neq \ell^-$ or $u \neq 0$ *have* to acquire quantum corrections in order to even satisfy the BPS bound (2.61). This is important in the context of the AdS_3/CFT_2 correspondence, where the classical limit on the bulk side corresponds to supergravity.

Examples of large $\mathcal{N} = 4$ superconformal theories are given by the supersymmetric WZW model on $S^3 \times S^1 \cong SU(2) \times U(1)$ [70]. This theory will be frequently called $\mathcal{S}_\kappa$ in Chapter 7, where $k^+ = \kappa + 1$ and $k^- = 1$. In particular, the large $\mathcal{N} = 4$ theory given by the symmetric product orbifold

$$\text{Sym}^N(S^3 \times S^1) \cong \text{Sym}^N(\mathcal{S}_\kappa) \tag{2.63}$$

will play an important role. Moreover, there is again a family of cosets supporting large $\mathcal{N} = 4$ supersymmetry. These are the Wolf-space cosets, whose $\mathfrak{su}(N)$ family takes the form

$$\frac{\mathfrak{su}(N+2)^{(1)}_{k+N+2}}{\mathfrak{su}(N)^{(1)}_{k+N+2} \oplus \mathfrak{u}(1)} \oplus \mathfrak{u}(1) \ . \tag{2.64}$$

The 't Hooft limit of this coset is dual to a large $\mathcal{N} = 4$ higher spin theory on AdS_3 [134].

2.4 The symmetric product orbifold CFT

We now explain an example of an orbifold CFT in some detail. This CFT plays a crucial role in the AdS_3/CFT_2 holography, where it serves as a dual CFT. We start with some arbitrary CFT of central charge c, which we denote for now by X. We then want to define the CFT

$$\text{Sym}^N(X) \equiv X^N/S_N \ , \tag{2.65}$$

which consists of N copies of X identified under permutations. The replacement $X \longrightarrow \text{Sym}^N(X)$ has a similar effect as going from a single particle to a multi-particle system, which is why this CFT is interesting as a potential candidate for a dual CFT to string theory.

2.4.1 Orbifold CFTs

In any orbifold CFT, gauging of the orbifold group consists of two steps. First, we impose the orbifold projection, which means that we keep only states invariant under the orbifold group. This forms a natural subsector of the theory and is referred to as the *untwisted* sector. The theory obtained in this way is however not yet complete. Since the obtained symmetry algebra is now smaller as before, there are further possible representations, the so-called twisted sector. Geometrically, it corresponds to solutions which are only periodic modulo a group element $g \in \mathrm{G}$:

$$X(z\,\mathrm{e}^{2\pi i}) = g \cdot X(z)\ , \tag{2.66}$$

where G is the orbifold group and X is a generic field of the theory. Twisted sectors are labelled by conjugacy classes of the orbifold group G. Modular invariance of the CFT dictates the inclusion of these sectors as well.

To see this in more detail, we start with the original partition of the theory, evaluated on the torus. Its fields are periodic in both the space and time cycle, which we may denote by

$$Z_{1,1}(\tau,\bar{\tau}) = 1\underset{1}{\square}\ , \tag{2.67}$$

the 1 indicating that the fields do not pick up a group element when going around the respective cycle. In the untwisted sector, we want to project on orbifold invariant states, which amounts to inserting the projector

$$\frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} g \tag{2.68}$$

in the trace of the partition function. In the path integral formalism, this changes the periodicity conditions of the fields along the time-cycle of the torus. We hence have

$$Z^{\text{untwisted}}(\tau,\bar{\tau}) = \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} Z_{g,1}(\tau,\bar{\tau})\ , \qquad Z_{g,1}(\tau,\bar{\tau}) = g\underset{1}{\square}\ . \tag{2.69}$$

Under a S-modular transformation, the space and time cycle are interchanged, which shows that the twisted sectors described by $Z_{1,h}(\tau,\bar{\tau})$ are also part of the theory. To obtain the modular completion, we note that the two generators of the modular group act as follows on the cycles:

$$S:\ g\underset{h}{\square} \longrightarrow h\underset{g}{\square}\ , \tag{2.70}$$

$$T:\ g\underset{h}{\square} \longrightarrow gh\underset{h}{\square}\ . \tag{2.71}$$

Only sectors with commuting g and h are well-defined, since otherwise the boundary conditions are ambiguous (following first the time-cycle and then the space-cycle has to give the same result as first following the space-cycle and then the time-cycle). Thus, the appropriate modular completion is given by

$$Z(\tau,\bar{\tau}) = \frac{1}{|\mathrm{G}|}\sum_{gh=hg} Z_{h,g}(\tau,\bar{\tau})\ . \tag{2.72}$$

This defines the full content of the orbifold CFT. See also [135–137] for more details.

Let us now describe these steps in detail for the symmetric product orbifold. We first assume the theory to be bosonic and treat the fermionic case below.

2.4.2 The untwisted sector

Let $X_1^{(1)}(z,\bar{z}),\ldots,X_k^{(1)}(z,\bar{z})$ be some fields in the first copy X of the symmetric product orbifold. (We denote by the superscript (i) the i-th copy in the symmetric product.) We can then produce a field invariant under the symmetric group by

$$\sum_{i_1,i_2,\ldots,i_k=1}^{N}\left(X_1^{(i_1)}\cdots X_k^{(i_k)}\right)(z,\bar{z})\ . \tag{2.73}$$

We will refer to such a field as a k-sum field. The collection of these fields comprises the untwisted sector.

The number of such fields can be counted as follows. It is convenient to change to a 'grand canonical' ensemble

$$\bigoplus_{N=0}^{\infty}\mathrm{Sym}^N(X)\ , \tag{2.74}$$

where $\mathrm{Sym}^0(X)$ is by definition the trivial theory. The grand canoncial partition function is defined by

$$\mathcal{Z}^{\text{untwisted}}(\tau,\bar{\tau})=\sum_{N=0}^{\infty}p^N\,\mathrm{tr}_{\mathrm{Sym}^N(X)}^{\text{untwisted}}\left(q^{L_0}\bar{q}^{\bar{L}_0}\right)\ . \tag{2.75}$$

Here, $q=\mathrm{e}^{2\pi i\tau}$. Note that we have *not* included the usual shift $-\frac{cN}{24}$ in the definition of the partition function, as this would diverge in the limit $N\to\infty$. Denoting by

$$Z(\tau,\bar{\tau})=\mathrm{tr}_X\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right)=q^{-\frac{c}{24}}\bar{q}^{-\frac{c}{24}}\sum_{h,\bar{h}}c(h,\bar{h})q^h\bar{q}^{\bar{h}} \tag{2.76}$$

the partition function of X, we can then compute the grand canonical partition function as

$$\mathcal{Z}^{\text{untwisted}}(\tau,\bar{\tau})=\prod_{h,\bar{h}}\left(1-pq^h\bar{q}^{\bar{h}}\right)^{-c(h,\bar{h})}\ . \tag{2.77}$$

This looks indeed like a multi-particle partition function. We shall be particularly interested in 'single-particle' fields, which cannot written as a product of other fields. These correspond precisely to the single-sum fields. Thus, single-particle fields are in one-to-one correspondence to fields in X.

2.4.3 The twisted sector

As mentioned above, the twisted sectors of an orbifold are labelled by the conjugacy classes of the orbifold group. In the case of the symmetric group, the conjugacy class of a permutation is entirely determined by the cycle type of the permutation π. Consider a permutation with n_1 cycles of length 1, n_2 cycles of length 2, etc. Permutations with multiple cycles are again products of smaller cycles and will hence correspond to multi-particle states. Let us focus on a single cycle of length n (i.e. a permutation conjugate to $(1\,2\cdots n)$). As we shall see below, the states in this sector can be constructed by a fractional action of the symmetry algebra. We will discuss here the construction on the level of the partition function.

We have mentioned above that the twisted sector can be constructed by requiring modular invariance. More specifically, since it corresponds to the twisted boundary conditions (2.66), we can construct it by performing an S-modular transformation on twisted boundary conditions in time. In the operator formalism, this means that we should consider

$$\mathrm{tr}\left(g\, q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) \tag{2.78}$$

in the untwisted sector and perform an S-modular transformation. Since the group action for the symmetric group acts by a permutation, this trace localises on the fixed point of the group action. For $g=(1\,2\cdots n)$ and the symmetric group, fields of the form

$$X^{(1)}(z,\bar{z})\cdots X^{(n)}(z,\bar{z}) \tag{2.79}$$

are fixed under g. Moreover, we can add arbitrary fields for the other $N-n$ copies of X. Thus,

$$\mathrm{tr}\left((1\,2\cdots n)\, q^{L_0-\frac{cN}{24}}\bar{q}^{\bar{L}_0-\frac{cN}{24}}\right)=Z(n\tau,n\bar{\tau})Z(\tau,\bar{\tau})^{N-n}\ . \tag{2.80}$$

Upon performing an S-modular transformation and using modular invariance of X, we hence learn that the twisted sector contributes

$$Z\left(\tfrac{\tau}{n},\tfrac{\bar{\tau}}{n}\right)Z(\tau,\bar{\tau})^{N-n} \tag{2.81}$$

to the partition function. We have now constructed the twisted sector. However, it is not yet invariant under the orbifold group G. The orbifold projection can be imposed by demanding full modular invariance. Performing successive T-modular transformations inserts the group element g^m in the time-cycle of the torus. Since under a T-modular transformation $q^h\bar{q}^{\bar{h}}\longrightarrow q^h\bar{q}^{\bar{h}}e^{2\pi i(h-\bar{h})}$, this modular completion will just impose the constraint of integer spins, $h-\bar{h}\in\mathbb{Z}$.

The complete single-particle partition function of the theory then reads

$$\sum_{n=1}^{N} q^{\frac{cn}{24}}\bar{q}^{\frac{cn}{24}}Z\left(\tfrac{\tau}{n},\tfrac{\bar{\tau}}{n}\right)\Bigg|_{h-\bar{h}\in\mathbb{Z}}\ . \tag{2.82}$$

Here, we have corrected again the ground state energies (remember that we did not include $q^{-\frac{cN}{24}}\bar{q}^{-\frac{cN}{24}}$ in the partition function for the symmetric orbifold). In (2.80) we have included the ground state contribution to obtain good modular properties. So to remove it, one has to multiply (2.81) by $q^{\frac{cN}{24}}\bar{q}^{\frac{cN}{24}}$. The total ground state energy of the n twisted sector is given by

$$\Delta_n = \frac{c(n^2-1)}{24n} \,, \tag{2.83}$$

where the additional contribution comes from $Z\left(\frac{\tau}{n}, \frac{\bar{\tau}}{n}\right)$.

Finally, we can also write down the complete partition function of the theory in the grand canonical ensemble. It reads [138, 139]

$$\mathcal{Z}(\tau,\bar{\tau}) = \prod_{n=1}^{\infty} \prod_{\substack{h,\bar{h} \\ h-\bar{h}\in n\mathbb{Z}}} \left(1 - p^n\, q^{\frac{c(n^2-1)}{24n}+\frac{h}{n}}\, \bar{q}^{\frac{c(n^2-1)}{24n}+\frac{\bar{h}}{n}}\right)^{-c(h,\bar{h})} \,, \tag{2.84}$$

which is the completion of (2.77).

This construction has the following microscopic interpretation. Since the modular parameter τ is rescaled by a factor $\frac{1}{n}$ in the twisted sector, the modes of the symmetry algebra take now values in $\frac{1}{n}\mathbb{Z}$. Fields in the twisted sector still have trivial monodromies in the complex plane, since the monodromy due to the modes being fractional is compensated by cyclically permuting the fields, which generates a compensating phase.

2.4.4 The fermionic case

Now we also include fermions in the construction, which cause some additional subtleties. We work in the NS-sector of the theory. In the untwisted sector, almost nothing changes, and (2.77) still holds, provided that we include $(-1)^{\mathrm{F}}$ in the partition functions.

On the other hand, there is a complication arising in the previous computation for the twisted sector. Essentially, we now have

$$\mathrm{tr}\left((1\,2\cdots n)\, q^{L_0-\frac{cN}{24}}\bar{q}^{\bar{L}_0-\frac{cN}{24}}\right) = Z_{\mathrm{NS}}^{(-1)^{(n-1)\mathrm{F}}}(n\tau, \bar{n}\tau) Z_{\mathrm{NS}}(\tau,\bar{\tau})^{N-n} \,, \tag{2.85}$$

where we denoted by Z_{NS} the partition function of the seed theory X in the NS-sector without $(-1)^{\mathrm{F}}$ insertion and by $Z^{(-1)^{\mathrm{F}}}$ the version with $(-1)^{\mathrm{F}}$ insertion. Let us explain how this comes about. The trace again localises on fixed points of the cyclic group action, but the fixed points can have a minus sign due to the fermionic nature. When cyclically permuting n fermions, one has to shift one fermion past the other $n-1$ fermions, which results in the sign $(-1)^{n-1}$. Thus, if n is odd, the result is always $+1$, also for fermions. When n is even, we get however a minus sign for fermionic fields. Correspondingly, the twisted sector partition function can now be found by applying an S-modular transformation. For this, we remember that the NS-sector maps to itself under an S-modular transformation, whereas the NS-sector with $(-1)^{\mathrm{F}}$ insertion maps to the R-sector. Thus, the single-particle contribution to the twisted sector is

$$n \text{ odd}: \qquad q^{\frac{cn}{24}} \bar{q}^{\frac{cn}{24}} Z_{\mathrm{NS}}\left(\tfrac{\tau}{n}, \tfrac{\bar{\tau}}{n}\right) , \tag{2.86}$$

$$n \text{ even}: \qquad q^{\frac{cn}{24}} \bar{q}^{\frac{cn}{24}} Z_{\mathrm{R}}\left(\tfrac{\tau}{n}, \tfrac{\bar{\tau}}{n}\right) . \tag{2.87}$$

Finally, the orbifold projection has to be imposed again. This is now a bit more subtle, since we cannot simply demand invariance under the T-modular transformation, as also the original theory was not invariant under the T-modular transformation. Instead, we have to demand that $h - \bar{h} \in \mathbb{Z}$ for bosonic states and $h - \bar{h} \in \mathbb{Z} + \frac{1}{2}$ for fermionic states.

The ground state energies of the twisted sector are now

$$\Delta_n = \begin{cases} \frac{c(n^2-1)}{24n} , & n \text{ odd} , \\ \frac{c(n^2-1)}{24n} + \frac{\Delta_{\mathrm{R}}}{n} , & n \text{ even} . \end{cases} \tag{2.88}$$

Here, Δ_{R} is the ground state energy of the R-sector of the seed theory X. For free fermions, we have $\Delta_{\mathrm{R}} = \frac{1}{16} N_{\mathrm{f}}$, where N_{f} is the number of fermions in the original theory. We will later consider the case where X has small $\mathcal{N} = 4$ supersymmetry, where we have always $\Delta_{\mathrm{R}} = \frac{c}{24}$. This can be violated for different amounts of supersymmetry [68, 93, 140], for instance for large $\mathcal{N} = 4$ supersymmetry.

Chapter 3

String theory on AdS_3

In this chapter, we introduce string theory on AdS_3. We discuss the string sigma-model on AdS_3 and how the pure NS-NS background can be described in terms of a (supersymmetric) WZW model on $SL(2,\mathbb{R})$. We also recall some general properties about BRST quantisation of (super)string theory and the hybrid formalism.

3.1 Strings on AdS_3 and the $SL(2,\mathbb{R})$ WZW model

Recall that the general string sigma-model for NS-NS backgrounds has the following worldsheet action

$$\mathcal{S}[X] = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2 z \, \sqrt{g}\big(g^{\alpha\beta} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu + i\varepsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' \Phi(X) R\big) \ . \quad (3.1)$$

Here, $G_{\mu\nu}$ is the spacetime metric, $B_{\mu\nu}$ the background Kalb-Ramond field and Φ the dilaton field. R denotes the curvature scalar on the worldsheet Σ. We consider a background with a constant dilaton, so that the dilaton term in the action produces the factor $g_s^{-\chi(\Sigma)}$, where $g_s = \mathrm{e}^{-\Phi}$ and $\chi(\Sigma)$ is the Euler characteristic of the worldsheet in correlation functions.

In order for the Weyl anomaly to vanish, the background fields have to satisfy the (super)gravity equations of motion (to leading order in α'). Considering a background of the form $AdS_3 \times X$, this requires a constant B-field flux through AdS_3. The resulting model has the action of the $SL(2,\mathbb{R})$ WZW model, see e.g. [60].

The $SL(2,\mathbb{R})$ WZW model was already discussed in Section 2.2; here we discuss the string theoretic interpretation of the model.

3.1.1 Bosonic strings on AdS_3

Let us start with bosonic strings. We treat the case of superstrings below (which is largely identical). Recall that there are two kinds of $\mathfrak{sl}(2,\mathbb{R})$ representations appearing in the $\mathrm{SL}(2,\mathbb{R})$ model and of this, the spectral flow operation produces new representations. We start with unflowed representations.

(a) Discrete representations. The mass-shell condition $L_0 = 1$ reads for these representations

$$-\frac{j_0(j_0-1)}{k-2}+h_{\text{int}}+N=1\ , \tag{3.2}$$

where h_{int} denotes the conformal weight of the representation of the internal CFT described by X and N the excitation number inside the representation. This can be solved for j_0 and one obtains

$$j_0=\frac{1}{2}+\sqrt{\frac{9}{4}+4(k-2)(h_{\text{int}}+N)}\ . \tag{3.3}$$

Note that j_0 does not have to be the actual spin of the physical state, since it can be changed by the use of the oscillators J^+_n or J^-_n. The Maldacena-Ooguri bound (2.24) is satisfied, provided that $h_{\text{int}}+N$ does not grow too large.

These string solutions are the so-called short strings, which can be thought of as (close to) point particle string solutions. The existence of the Maldacena-Ooguri bound shows that there are only finitely many such string solutions.

(b) Continuous representations. The mass-shell condition now reads

$$\frac{1+4p^2}{4(k-2)}+h_{\text{int}}+N=1\ . \tag{3.4}$$

Here, we set $j_0=\frac{1}{2}+ip$. Clearly, this only admits a solution for

$$h_{\text{int}}+N\le 1-\frac{1}{4(k-2)}\ . \tag{3.5}$$

These solutions yield to representations in space-time, which are unbounded from below. This is the moral analogue of the flat-space tachyon in the AdS_3 background. In the supersymmetric case, there are no solutions in this sector.

Now, we include spectral flow in our discussion. Spectrally flowed representations give physical states, both from the discrete and the continuous sector. They have the interpretation of being winding strings, which asymptotically wind around AdS_3. The mass-shell condition for a continuous representation becomes

$$\frac{1+4p^2}{4(k-2)} - wm - \frac{k}{4}w^2 + h_{\text{int}} + N = 1 \ , \tag{3.6}$$

which can be solved for m as long as $w \neq 0$. Here, we have used (2.23). As we shall explain in Chapter 4, continuous representations are a special phenomenon occurring only for pure NS-NS flux. As soon as a tiny amount of R-R flux is switched on, they vanish from the spectrum.

Similarly, the mass-shell condition for a discrete representation in the w spectrally flowed sector reads

$$-\frac{j_0(j_0-1)}{k-2} - wm - \frac{k}{4}w^2 + h_{\text{int}} + N = 1 \ , \tag{3.7}$$

where j_0 satisfies the Maldacena-Ooguri bound (2.24). Moreover, m has to satisfy $m - j_0 + N \in \mathbb{Z}_{\geq 0}$, since m has to be an affine descendant of the discrete representation. In particular, this quantises its $\mathfrak{sl}(2,\mathbb{R})$ charge to be equal to j_0, modulo an integer. The quantisation condition makes it more complicated to solve the mass-shell condition in this case.

3.1.2 Superstrings on AdS_3

In this thesis, we only consider type IIB superstrings. For the supersymmetric case, the relevant model becomes the $\mathcal{N}=1$ supersymmetric WZW model on $SL(2,\mathbb{R})$, based on the algebra

$$\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \cong \mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus 3 \text{ free fermions} \ , \tag{3.8}$$

as discussed in Section 2.3.1. Thus, most formulas are identical, up to a shift $k \to k+2$. Moreover, we have to sum in the final theory over the different spin structures of the worldsheet and impose the GSO projection.

While, any internal $\mathcal{N}=1$ SCFT X with the correct central charge will give rise to a possible superstring background $AdS_3 \times X$, we would like to consider backgrounds, which are also spacetime supersymmetric. There are three maximally supersymmetric backgrounds (where the internal CFT X has discrete spectrum) given by

$$AdS_3 \times S^3 \times \mathbb{T}^4 \ , \qquad AdS_3 \times S^3 \times K3 \quad \text{and} \qquad AdS_3 \times S^3 \times S^3 \times S^1 \ . \tag{3.9}$$

The former two backgrounds support spacetime small $\mathcal{N}=(4,4)$ supersymmetry, whereas the latter background supports so-called large $\mathcal{N}=(4,4)$ supersymmetry [64–68,70]. There is also a plethora of interesting backgrounds with less supersymmetry in the literature (see [90, 91, 93, 141–146] for an incomplete list).

We will mostly be concerned with the background $AdS_3 \times S^3 \times \mathcal{M}_4$ with $\mathcal{M}_4 = \mathbb{T}^4$, K3 or $S^3 \times S^1$, which is easily tractable with worldsheet methods.

3.2 The hybrid formalism

Here, we discuss the background $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ in some more detail. We explain how to redefine the fields in order to pass to the hybrid formalism.

3.2.1 The RNS formalism

In the RNS formalism, the worldsheet theory takes the form [60,147]

$$\mathfrak{sl}(2,\mathbb{R})_k^{(1)} \oplus \mathfrak{su}(2)_k^{(1)} \oplus \left(\mathfrak{u}(1)^{(1)}\right)^4 , \tag{3.10}$$

where $\mathfrak{sl}(2,\mathbb{R})_k^{(1)}$ describes the AdS_3 factor, $\mathfrak{su}(2)_k^{(1)}$ the S^3 factor and the flat torus directions are represented by four $\mathfrak{u}(1)$ currents. The superscript (1) indicates that these are the corresponding $\mathcal{N} = 1$ WZW models. We can decouple the free fermions from the affine generators and hence get

$$\mathfrak{sl}(2,\mathbb{R})_k^{(1)} \cong \mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus 3 \text{ free fermions} , \tag{3.11}$$

$$\mathfrak{su}(2)_k^{(1)} \cong \mathfrak{su}(2)_{k-2} \oplus 3 \text{ free fermions} . \tag{3.12}$$

We shall denote the decoupled $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ currents by $\mathcal{J}^a$, the decoupled $\mathfrak{su}(2)_{k-2}$ currents by $\mathcal{K}^a$, and the free bosons of $\mathbb{T}^4$ by ∂X^α and $\partial \bar{X}^\alpha$. Here, $a = \pm, 3$ is an adjoint index of $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$, respectively, while $\alpha = \pm$. Note that we have paired the four bosons of $\mathbb{T}^4$ into two complex bosons. The index $\alpha = \pm$ will turn out to be a spinor index of the outer automorphism group of small $\mathcal{N} = 4$ supersymmetry.

Moreover, we have ten fermions on the worldsheet transforming in the adjoint representation of the bosonic groups. We denote the $\mathfrak{sl}(2,\mathbb{R})$ fermions by ψ^a, and the $\mathfrak{su}(2)$ fermions by χ^a, where again $a = \pm, 3$. Finally, we pair the four fermions of $\mathbb{T}^4$ together into two complex fermions, which we denote by λ^α and $\bar{\lambda}^\alpha$. λ^α is the $\mathcal{N} = 1$ superpartner of the bosons ∂X^α on the worldsheet, and similarly for $\bar{\lambda}^\alpha$ and $\partial \bar{X}^\alpha$. The α-index is again a spinor index of the outer automorphism $\mathfrak{su}(2)$ of the small $\mathcal{N} = 4$ spacetime supersymmetry. We will see below that the fermions λ^α and $\bar{\lambda}^\alpha$ give rise to four fermions in spacetime which transform as doublets under the $\mathcal{N} = 4$ R-symmetry. The (anti)commutation relation of the modes of these fields are summarised in Appendix A.2.

We again want to quantise this theory via BRST quantisation. For this, we need the $\mathcal{N} = 1$ superconformal structure on the worldsheet, which is explicitly given by

$$\begin{aligned} T(z) = \frac{1}{k}\Big(& -\mathcal{J}^3\mathcal{J}^3 + \tfrac{1}{2}\left(\mathcal{J}^+\mathcal{J}^- + \mathcal{J}^-\mathcal{J}^+\right) \\ & + \psi^3\partial\psi^3 - \tfrac{1}{2}\left(\psi^+\partial\psi^- + \psi^-\partial\psi^+\right)\Big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{k}\Big(\mathcal{K}^3\mathcal{K}^3+\tfrac{1}{2}(\mathcal{K}^+\mathcal{K}^-+\mathcal{K}^-\mathcal{K}^+) \\
&\qquad -\chi^3\partial\chi^3-\tfrac{1}{2}(\chi^+\partial\chi^-+\chi^-\partial\chi^+)\Big) \\
&+\varepsilon_{\alpha\beta}(\partial X^\alpha\partial\bar{X}^\beta)+\frac{1}{2}\varepsilon_{\alpha\beta}(\partial\lambda^\alpha\bar{\lambda}^\beta-\lambda^\alpha\partial\bar{\lambda}^\beta)\ , \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
G(z) = &-\frac{1}{k}\Big(-\mathcal{J}^3\psi^3+\tfrac{1}{2}(\mathcal{J}^+\psi^-+\mathcal{J}^-\psi^+)-\tfrac{1}{k}(\psi^3\psi^+\psi^-)\Big) \\
&-\frac{1}{k}\Big(\mathcal{K}^3\chi^3+\tfrac{1}{2}(\mathcal{K}^+\chi^-+\mathcal{K}^-\chi^+)+\tfrac{1}{k}(\chi^3\chi^+\chi^-)\Big) \\
&+\frac{1}{2}\varepsilon_{\alpha\beta}(\partial X^\alpha\bar{\lambda}^\beta-\partial\bar{X}^\alpha\lambda^\beta)\ , \qquad (3.14)
\end{aligned}
$$

where here, as in the following, normal-ordering is always understood. Moreover, we introduce the standard ghosts of the superstring, i.e. a bc system with $\lambda = 2$, and a $\beta\gamma$ system with $\lambda = \frac{3}{2}$, satisfying (see Appendix A.3 for our conventions)[15]

$$
b(z)c(w) \sim \frac{1}{z-w}\ , \qquad \hat{\beta}(z)\hat{\gamma}(w) \sim -\frac{1}{z-w}\ . \qquad (3.15)
$$

These fields also generate an $\mathcal{N}=1$ superconformal structure with

$$
T_{\text{gh}}(z) = -2b(\partial c)-(\partial b)c-\tfrac{3}{2}\hat{\beta}(\partial\hat{\gamma})-\tfrac{1}{2}(\partial\hat{\beta})\hat{\gamma}\ , \qquad (3.16)
$$

$$
G_{\text{gh}}(z) = (\partial\hat{\beta})c+\tfrac{3}{2}\hat{\beta}(\partial c)-\tfrac{1}{2}b\hat{\gamma}\ . \qquad (3.17)
$$

The standard BRST operator of the superstring is then given by

$$
Q = \oint \mathrm{d}z\Big(c\big(T+\tfrac{1}{2}T_{\text{gh}}\big)+\hat{\gamma}\big(G+\tfrac{1}{2}G_{\text{gh}}\big)\Big)\ . \qquad (3.18)
$$

We split Q into three pieces according to their $\hat{\beta}\hat{\gamma}$ ghost number [23]

$$
Q = Q_0+Q_1+Q_2\ , \qquad (3.19)
$$

where

$$
Q_0 = \oint \mathrm{d}z\ c\big(T+T_{\text{gh}}\big)+b(\partial c)c+\tfrac{3}{4}\partial\big(\hat{\gamma}\hat{\beta}c\big) = \oint \mathrm{d}z\ c\big(T+T_{\text{gh}}\big)+b(\partial c)c\ , \qquad (3.20)
$$

$$
Q_1 = \oint \mathrm{d}z\ \hat{\gamma}G\ , \qquad (3.21)
$$

$$
Q_2 = -\tfrac{1}{4}\oint \mathrm{d}z\ b\hat{\gamma}\hat{\gamma}\ . \qquad (3.22)
$$

Q_0 is the BRST operator of the bosonic string, for which the $\hat{\beta}\hat{\gamma}$ ghosts are treated as additional matter fields. BRST invariance under Q_2 is usually

[15] In order to distinguish the superconformal ghosts from the $\beta\gamma$ system that appears in the Wakimoto realisation of $\mathfrak{sl}(2,\mathbb{R})$, we denote them with a hat.

trivial (at least in the canonical picture), and the important piece of the BRST charge is Q_1.

In the following, we will always use the bosonised form of the superconformal ghosts, i.e. we introduce free bosons with background charge $Q_\varphi = 2$ and $Q_\chi = -1$ and OPEs

$$\varphi(z)\varphi(w) \sim -\log(z-w)\ , \qquad \chi(z)\chi(w) \sim \log(z-w)\ , \tag{3.23}$$

and write

$$\hat{\beta} = \mathrm{e}^{-\varphi}\mathrm{e}^{\chi}\partial\chi\ , \qquad \hat{\gamma} = \mathrm{e}^{-\chi}\mathrm{e}^{\varphi}\ . \tag{3.24}$$

The φ-boson then has $c = 13$, while the χ-boson yields $c = -2$, see Appendix A.3 for our conventions. (This then reproduces the central charge $c = 11$ of the $\hat{\beta}\hat{\gamma}$ system.) We also define the (ξ, η) pair via

$$\xi = \mathrm{e}^{\chi}\ , \qquad \eta = \mathrm{e}^{-\chi}\ , \tag{3.25}$$

where $h(\xi) = 0$ and $h(\eta) = 1$. Finally, the picture raising operation on vertex operators is defined by $Z = -2[Q, \xi\,\cdot\,]$.

3.2.2 The hybrid formalism

In order to rewrite these degrees of freedom in terms of the hybrid formalism, we now bosonise the ten fermions, i.e. we introduce bosons via

$$\partial H_1 = \frac{1}{k}(\psi^+\psi^-)\ , \qquad \partial H_2 = \frac{1}{k}(\chi^+\chi^-)\ , \qquad \partial H_3 = \frac{2}{k}(\psi^3\chi^3)\ , \tag{3.26a}$$

$$\partial H_4 = (\lambda^+\bar{\lambda}^-)\ , \qquad \partial H_5 = -(\lambda^-\bar{\lambda}^+)\ . \tag{3.26b}$$

These bosons satisfy the standard OPEs (with vanishing background charge $Q_H = 0$)

$$H_i(z)H_j(w) \sim \delta_{ij}\log(z-w)\ , \tag{3.27}$$

and we can express the fermions in terms of them as

$$\psi^\pm = \sqrt{k}\mathrm{e}^{\pm H_1}\ , \quad \chi^\pm = \sqrt{k}\mathrm{e}^{\pm H_2} \quad \psi^3 \mp \chi^3 = \sqrt{k}\mathrm{e}^{\pm H_3}\ , \tag{3.28a}$$

$$\lambda^+ = \mathrm{e}^{H_4}, \qquad \lambda^- = \mathrm{e}^{-H_5}\ , \qquad \bar{\lambda}^+ = \mathrm{e}^{H_5}, \qquad \bar{\lambda}^- = \mathrm{e}^{-H_4}\ . \tag{3.28b}$$

Here and in the following we will suppress cocycle factors. The final step consists of refermionising these bosons, i.e. by considering the fermionic generators that can be constructed out of these bosons as

$$p^{\alpha\beta} = \mathrm{e}^{\frac{\alpha}{2}H_1+\frac{\beta}{2}H_2+\frac{\alpha\beta}{2}H_3+\frac{1}{2}H_4+\frac{1}{2}H_5-\frac{1}{2}\varphi}\ , \tag{3.29a}$$

$$\theta^{\alpha\beta} = \mathrm{e}^{\frac{\alpha}{2}H_1+\frac{\beta}{2}H_2-\frac{\alpha\beta}{2}H_3-\frac{1}{2}H_4-\frac{1}{2}H_5+\frac{1}{2}\varphi}\ , \tag{3.29b}$$

$$\Psi^+ = \mathrm{e}^{H_4-\varphi+\chi} \,, \tag{3.29c}$$
$$\Psi^- = \mathrm{e}^{-H_5+\varphi-\chi} \,, \tag{3.29d}$$
$$\bar{\Psi}^+ = \mathrm{e}^{H_5-\varphi+\chi} \,, \tag{3.29e}$$
$$\bar{\Psi}^- = \mathrm{e}^{-H_4+\varphi-\chi} \,. \tag{3.29f}$$

These fields constitute again a collection of fermionic first-order systems

$$p^{\alpha\beta}(z)\theta^{\gamma\delta}(w) \sim \frac{\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}}{z-w} \,, \tag{3.30}$$
$$\Psi^\alpha(z)\bar{\Psi}^\beta(w) \sim \frac{\varepsilon^{\alpha\beta}}{z-w} \,, \tag{3.31}$$

where $p^{\alpha\beta}$ and Ψ^α have conformal dimension equal to one, while $\theta^{\alpha\beta}$ and $\bar{\Psi}^\alpha$ have conformal dimension equal to zero. In fact the four fields $p^{\alpha\beta}$ describe four of the eight spacetime supercharges in the canonical ghost picture, so the hybrid formalism makes half of spacetime supersymmetry manifest. We will see in the next subsection that the other four supercharges are also (almost) manifest.

The fermionic first-order system (3.30) describes six bc pairs each with $\lambda = 1$, thus giving rise altogether to $c = -12$. On the other hand, we started with ten fermions (giving $c = 5$), as well as the $\hat{\beta}\hat{\gamma}$ superconformal ghosts with $c = 11$. Thus we are missing central charge $c = 28$, which is accounted for by the boson

$$\rho = 2\varphi - H_4 - H_5 - \chi \,, \tag{3.32}$$

which has background charge $Q = 3$ in the conventions of Appendix A.3, and will serve as a bosonised ghost in the hybrid formalism. Thus we have rewritten the fermionic degrees of freedom of the NS-R formalism in terms of the fermionic first-order system (3.30) and (3.31), as well as the boson (3.32).

3.2.3 Supergroup generators

The final step consists of assembling the (unchanged) bosonic fields $\mathcal{J}^a$, $\mathcal{K}^a$ together with the fermions $p^{\alpha\beta}$ and $\theta^{\alpha\beta}$ into the current algebra for the superalgebra $\mathfrak{psu}(1,1|2)_k$; in fact, this is just the Wakimoto representation for this superalgebra. More specifically, we define

$$J^{(\mathrm{f})a} = \tfrac{1}{2}c_a(\sigma^a)_{\alpha\mu}\varepsilon_{\beta\nu}(p^{\alpha\beta}\theta^{\mu\nu}) \,, \tag{3.33a}$$
$$K^{(\mathrm{f})a} = \tfrac{1}{2}\varepsilon_{\alpha\mu}(\sigma^a)_{\beta\nu}(p^{\alpha\beta}\theta^{\mu\nu}) \,, \tag{3.33b}$$

which generate the $\mathfrak{sl}(2,\mathbb{R})_{-2} \oplus \mathfrak{su}(2)_2$ algebra. The full $\mathfrak{psu}(1,1|2)_k$ generators are then given as

$$J^a = \mathcal{J}^a + J^{(\mathrm{f})a} \,, \tag{3.34a}$$

$$K^a = \mathcal{K}^a + K^{(\mathrm{f})a} \;, \tag{3.34b}$$

$$S^{\alpha\beta+} = p^{\alpha\beta} \;, \tag{3.34c}$$

$$S^{\alpha\beta-} = k\partial\theta^{\alpha\beta} + c_a(\sigma_a)^{\alpha}{}_{\gamma}\big(\mathcal{J}^a + \tfrac{1}{2}J^{(\mathrm{f})a}\big)\theta^{\gamma\beta} - (\sigma_a)^{\beta}{}_{\gamma}\big(\mathcal{K}^a + \tfrac{1}{2}K^{(\mathrm{f})a}\big)\theta^{\alpha\gamma} \;. \tag{3.34d}$$

One checks by a direct calculation, see also [126, 148, 149], that these generators then satisfy the relations of $\mathfrak{psu}(1,1|2)_k$ that are spelled out in Appendix A.1.2. Moreover, we have used the conventions of (A.3c) for the sigma-matrices.

Thus we conclude that the worldsheet theory in the hybrid formalism is generated by

$$\mathfrak{psu}(1,1|2)_k \oplus \text{topologically twisted } \mathbb{T}^4 \oplus \text{ ghosts} \;. \tag{3.35}$$

Here the topologically twisted $\mathbb{T}^4$ is described by the bosons ∂X^α, $\partial\bar{X}^\alpha$, together with the (topologically twisted) fermions Ψ^α and $\bar{\Psi}^\alpha$, while the ghosts consist of the bosonic (b,c) ghosts together with the ρ ghost. In order for this to make sense one must also be able to rewrite the BRST operator (as well as the (ξ,η) pair) in terms of these redefined fields, and this is indeed possible, see [126] for details.

3.2.4 Mixed flux

The hybrid formalism has another advantage besides making spacetime supersymmetry manifest: it is conceptually relatively simple to add R-R flux to the background. In the next chapter, we will explore the mixed-flux background in some detail.

As will be explained in the next chapter, the vanishing of the dual Coxeter number of $\mathfrak{psu}(1,1|2)$ implies that the sigma-model on the supergroup $\mathrm{PSU}(1,1|2)$ is a CFT. In other words, the coefficient of the kinetic term (denoted by f^{-2} in the following) and the coefficient of the WZW term (denoted by $k \in \mathbb{Z}$ in the following) can be chosen independently. In string theory, k is interpreted as the number of NS5-branes creating the background geometry, and therefore also their total charge $Q_5^{\mathrm{NS}} \equiv k$. On the other hand, f^{-1} describes the radius of AdS_3 and S^3, which we denote by R_{AdS}. The relation with the D5-brane charge Q_5^{RR} can be found for large radii using supergravity, and reads [50, 126, 150, 151]:

$$\frac{1}{f^2} = \frac{R_{\mathrm{AdS}}^2}{\alpha'} = \sqrt{\big(Q_5^{\mathrm{NS}}\big)^2 + g_s^2\big(Q_5^{\mathrm{RR}}\big)^2} \,. \tag{3.36}$$

Here g_s is the ten-dimensional string coupling constant. This explains why f^{-2} is not quantised in the worldsheet description: since we are treating the string perturbatively, g_s is small and hence Q_5^{RR} has to be of order g_s^{-1}

to have a visible effect on f^{-2}. Thus, it is effectively continuous in the worldsheet theory. In a full non-perturbative description of string theory, also f^{-2} would become quantised. Note that (3.36) restrict the parameters to the range

$$-1 \leq kf^2 \leq 1\,. \tag{3.37}$$

Negative values of kf^2 correspond to anti-branes; we will not consider them in the following.

The FS1- and D1-brane charges enter as follows in the hybrid formalism. Supersymmetry imposes that the ratios Q_5/Q_1 agree for NS- and R-R-fields:

$$\frac{Q_5^{\mathrm{NS}}}{Q_1^{\mathrm{NS}}} = \frac{Q_5^{\mathrm{RR}}}{Q_1^{\mathrm{RR}}}\,. \tag{3.38}$$

Finally, Q_1^{RR} determines the volume of the compactification manifold $\mathbb{T}^4$ as

$$v = f^2 g_s Q_1^{\mathrm{RR}}\,, \tag{3.39}$$

but it does not enter directly in the $\mathrm{PSU}(1,1|2)$-sigma model.

Upon introducing R-R flux to the background, also the action of the ghosts has to be modified and ghost couplings have to be included [126]. This makes the theory quite intractable (apart for certain limits explained in the next chapter).

3.3 The sigma-model description of $\mathbf{AdS_3} \times \mathbf{S^3} \times \boldsymbol{\mathcal{M}_4}$

Here, we discuss the D-brane construction of the background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}_4$, where $\mathcal{M}_4 = \mathbb{T}^4$ or K3. This is mostly a review of the material appearing in [53,152,153].

3.3.1 The D-brane setup

We consider the D1-D5 system on compactified on $\mathcal{M}_4$. The D-branes are wrapped as follows:

	0	1	2	3	4	5	6	7	8	9
Q_5 D5-branes	$\times$					$\times$	$\times$	$\times$	$\times$	$\times$
Q_1 D1-branes	$\times$					$\times$	$\sim$	$\sim$	$\sim$	$\sim$

(3.40)

The manifold $\mathcal{M}_4$ is located in the directions 6789, $\times$ denotes directions in which the brane extends, while $\sim$ denotes directions in which the brane is smeared. We can also consider the inclusion of F1-strings and NS5-branes, and moreover D3-branes can wrap any of the $n+6$ two-cycles of $\mathcal{M}_4$, where $n = 0$ for $\mathbb{T}^4$ and $n = 16$ for K3. The charge vector parametrising different configurations of the system takes values in the even self-dual lattice $\Gamma_{5,5+n}$. The U-duality group is the orthogonal group $O(\Gamma_{5,5+n})$, under which the charge vector transforms in the fundamental representation. In the following we will assume that this charge vector is primitive, i.e. not a non-trivial multiple of another charge vector. If this is not so, the brane system can break into subsystems at no cost of energy at any point in the moduli space, which renders the dual CFT singular. Note that the U-duality group acts transitively on the set of primitive charge vectors of a fixed norm. Therefore we can always apply a U-duality transformation to bring the charge vector into the standard form $Q_1' = N = Q_1 Q_5$ and $Q_5' = 1$, with all other charges vanishing [51].

The moduli space is provided by the scalars of the compactification. Locally, they parametrise the homogeneous space

$$\frac{O(5,5+n)}{O(5)\times O(5+n)}\,, \tag{3.41}$$

on which U-duality acts and which leads to global identifications. In the near-horizon limit some of the moduli freeze out and the charge vector becomes fixed. The remaining scalars parametrise locally the moduli space

$$\frac{O(4,5+n)}{O(4)\times O(5+n)}\,, \tag{3.42}$$

and U-duality is reduced to the little group fixing the charge vector [51, 52].

Seiberg and Witten studied under what circumstances the system can break apart at no cost of energy [53, 154]. For a primitive charge vector, this happens on a codimension 4 subspace of the moduli space. On this sublocus, the instability should be reflected as a singularity in the dual CFT. In particular, the pure NS-NS flux background lies on this locus and is hence a singular region in the moduli space. In this way, for pure NS-NS flux fundamental strings can leave the system and can reach the boundary of AdS_3 at a finite cost of energy. These are the so-called long strings. These considerations predict the existence of a continuum of states above a certain threshold for pure NS-NS flux. Such states indeed exist in the worldsheet description of string theory, and are associated with continuous representations of the $\mathfrak{sl}(2,\mathbb{R})_k$-current algebra [60].

3.3.2 The gauge theory description

In this part we review the gauge theory worldvolume description of the D1-D5 system. For simplicity, we work in the case in which neither D3-branes, F1-strings nor NS5-branes are present. The worldvolume theory of the D5-branes is given by a $U(Q_5)$ gauge theory coupling to the two-dimensional defects given by the D1-branes. In the low-energy limit, the dynamics becomes essentially a two-dimensional gauge theory which lives on the intersection of the D1-D5 branes [45], and which flows to an $\mathcal{N} = (4,4)$ superconformal field theory in the IR. In fact, the IR fixed-point is described by *two* superconformal field theories – one corresponding to the Coulomb branch and one to the Higgs branch of the theory.[16] There are a number of ways of justifying this, the simplest being the comparison of central charges and R-symmetries [152]. Indeed, these two SCFTs have different sets of massless fields, and hence different central charges. Furthermore, since the scalars transform non-trivially under the various $\mathfrak{su}(2)$ R-symmetries and obtain non-trivial vacuum expectation values, the R-symmetry is generically broken down to different $\mathfrak{su}(2)$'s.

Let us have a closer look at the different central charges. On the Coulomb branch, the gauge group is generically broken to $\mathrm{U}(1)^{Q_5}$, while all other fields are massive. The central charge is then given by the Q_5 massless gauge vector multiplets, that is $c = 6Q_5$. On the other hand, on the Higgs branch only $n_\mathrm{H} - n_\mathrm{V}$ hypermultiplets remain massless, while all other fields become massive. The central charge is then $c = 6(n_\mathrm{H} - n_\mathrm{V})$, where n_H is the number of hypermultiplets and n_V the number of vector multiplets. Evaluating this number gives

$$c = \begin{cases} 6Q_1Q_5\,, & \mathcal{M}_4 = \mathbb{T}^4\,, \\ 6(Q_1Q_5 + 1)\,, & \mathcal{M}_4 = \mathrm{K3}\,. \end{cases} \tag{3.43}$$

We hence conclude that the central charges on the Higgs and Coulomb branches are generically different, and therefore the IR fixed-point is described by two decoupled SCFTs.[17]

These two branches meet classically at the small instanton singularity of the gauge theory. In the quantum theory, the Coulomb branch metric is corrected and develops a tube near the small instanton singularity [53]. Hence the Coulomb branch moves infinitely far away from the Higgs branch. For the Higgs branch the story is more subtle: since it is hyperkähler, it is not renormalised at the quantum level. Nevertheless, the description of the Higgs branch SCFT as a sigma-model on the classical Higgs branch breaks down near the singularities of the moduli space, and one has to use a different set of variables. In those variables, the small instanton singularity exhibits also a tube-like behaviour on the Higgs branch [53].[18] This implies that an instanton can travel through the tube and come out on the Coulomb

[16] There can of course be also mixed branches.

[17] The same result can be obtained semi-classically by using the Brown-Henneaux central charge [47], which yields $c = 6Q_1Q_5$. The correction in the K3 case is a supergravity one-loop effect [155].

[18] This tube can be described by a Liouville field in the gauge theory, and the energy gaps can be seen to match [53,153].

branch. This is the gauge theory description of the emission of a D1-brane, i.e. of the long strings. In this process the central charge does not change since, for example for $\mathcal{M}_4 = \mathbb{T}^4$,

$$c_{\text{tot}} = 6Q_1Q_5 = 6(Q_1 - 1)Q_5 + 6Q_5\,, \tag{3.44}$$

where we have used the central charge for the Coulomb and Higgs branch.

Let us slowly move away from the singular locus in the moduli space of the theory. From the gauge theory picture we learn that the tube disappears from the moduli space, since the sigma-model description is always a good description. Note that this happens immediately at the slightest perturbation away from the singular locus. This means that when perturbing the theory slightly, the continuum provided by the long strings should completely disappear. The situation is depicted schematically in Figure 3.1.

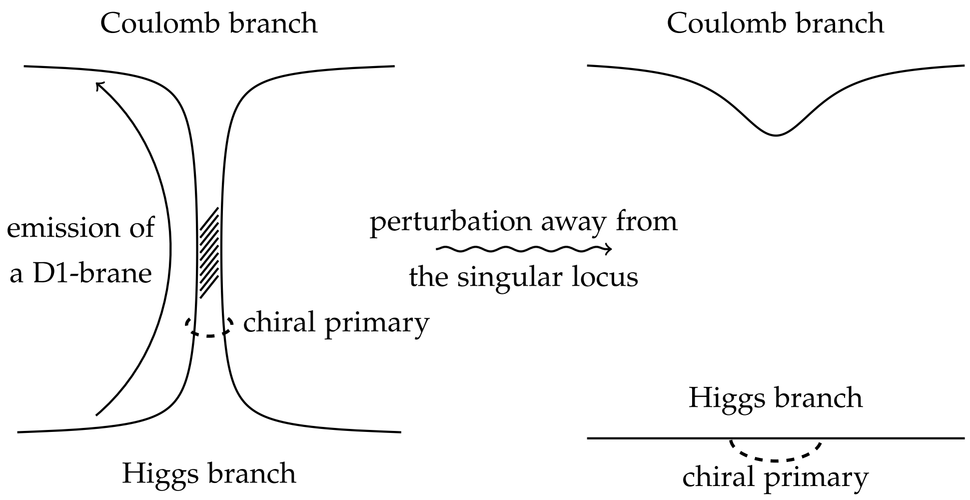

Figure 3.1: The structure of the moduli space on the singular locus and when slightly perturbed away from it. On the singular locus (left-hand picture), chiral primaries can escape to the Coulomb branch and are emitted as D1-branes from the system. The Higgs branch and the Coulomb branch are connected by an infinitely long tube, with the string coupling constant blowing up in the middle. Associated with the tube are long strings, which give rise to a continuum in the spectrum. When slightly perturbing the system away from the singular locus (right-hand picture), the moduli space approximation becomes good and the non-renormalization theorem makes the Higgs branch flat. The tube disappears and all chiral primaries are confined to the Higgs branch.

There is one related phenomenon occurring. Starting at a non-singular point in the moduli space, as we slowly approach the singular locus the small instanton singularity will form at some places on the Higgs branch. The support of the cohomology cycles associated with the instanton shrinks to zero size in this process and, as the tube forms, these cycles will move down the tube and disappear from the Higgs branch, see Figure 3.1. As cohomology classes correspond to chiral primaries in the CFT description, this means that these chiral primaries are missing on the singular locus. In this way, all cohomology classes which are obtained by multiplication in the chiral ring vanish from the spectrum. From a string theory point of view, this means that all multi-particle chiral primary states obtained from a given chiral primary are missing.[19] It is hard to say which cycles are these from a gauge theory perspective, since there is no good explicit description of the instanton moduli space on $\mathbb{T}^4$ or K3. In [53, 156] it was argued that the first missing chiral primary should have degree $(Q_5 - 1, Q_5 - 1)$, i.e. conformal weights $h = \bar{h} = \frac{1}{2}(Q_5 - 1)$. However, more chiral primaries are expected to be missing from the spectrum. We will argue in the worldsheet description that all cohomology classes of degrees

$$((w+1)Q_5 - 1, (w+1)Q_5 - 1) \tag{3.45}$$

are in fact missing, where $w \in \{0, 1, \ldots, Q_1\}$ corresponds to the spectral flow parameter on the worldsheet.[20] It would be interesting to confirm this directly from the instanton moduli space side.

These statements are true for $Q_5 \geq 2$, since only for these values of the flux the RNS-formalism exists. The $Q_5 = 1$ ($k = 1$) case is special and is discussed in Chapter 5. The $Q_5 = 1$ case does not seem to be singular, since there is no continuum in the spectrum and there are no missing chiral primaries. In the gauge theory picture, this corresponds to a $\mathrm{U}(1)$ gauge theory. The IR description of this gauge theory is drastically different and the picture 1.4 breaks down. In particular, the tube metric is absent. This will be directly visible from the analysis in Chapter 5. See also [53, 152, 157] for further discussions on this issue from the brane-picture.

[19] Furthermore, since the chiral primaries always come in Hodge diamonds of $\mathcal{M}_4$, the whole diamond will be missing.

[20] The importance of spectral flow in the worldsheet description of AdS_3 was not yet realised when [53] was published, which explains the differences between our statement and the one in [53, 156].

Chapter 4

Strings in mixed flux backgrounds

In this Chapter, we systematically analyse mixed flux backgrounds using the hybrid formalism. We analyse the supergroup sigma-model on $\mathrm{PSU}(1,1|2)$ and its algebraic structure in detail, which leads to a derivation of the plane-wave string spectrum in the background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ with mixed flux. We also derive the qualitative behaviour of the string spectrum close to the pure NS-NS point in moduli space.

4.1 Semiclassical analysis

In this section we perform a semiclassical analysis of the worldsheet sigma-model that will be our focus throughout the paper. We will do this by finding the worldsheet conformal weight of some classical solutions, and interpreting them semiclassically.

4.1.1 Classical action and conserved currents

We consider the two-parameter sigma model on a (super)group G

$$\mathcal{S}[g] = -\frac{1}{4\pi f^2} \int d^2z \, \mathrm{Tr}\left(\partial g g^{-1} \, \bar{\partial} g g^{-1}\right) + k \, \mathcal{S}_{\mathrm{WZ}}[g] \, , \tag{4.1}$$

with $g \in \mathrm{G}$, and where $\mathcal{S}_{\mathrm{WZ}}[g]$ denotes the Wess-Zumino term. The points $kf^2 = \pm 1$ on parameter space correspond to the usual WZW-model. At these points (4.1) possesses a local $\mathrm{G} \times \mathrm{G}$ symmetry, while at $k = 0$ we recover the principal chiral model [151].

Away from the WZW-point, the model still has a global $\mathrm{G} \times \mathrm{G}$ symmetry, which gives rise to two local conserved currents.[21] Let us focus on the sym-

[21] There is another deformation of the WZW-model which preserves conformal symmetry and gives rise to a current algebra, see [158]. However, it only preserves the diagonal global G symmetry.

metry by left-multiplication, with associated current $j(z,\bar{z})$.[22] This current has the following components in complex coordinates:

$$j_z = -\frac{1+kf^2}{2f^2}\partial g g^{-1} \;, \quad j_{\bar{z}} = -\frac{1-kf^2}{2f^2}\bar{\partial} g g^{-1} \;, \tag{4.2}$$

and the equations of motion of (4.1) are equivalent to the conservation law

$$\bar{\partial} j_z + \partial j_{\bar{z}} = 0 \;. \tag{4.3}$$

At the WZW-point $kf^2 = 1$, $j_{\bar{z}} \equiv 0$, and conservation implies the holomorphicity of the j_z component. However, we stress that in general j_z is neither holomorphic nor anti-holomorphic. We will henceforth write $j_z(z)$ but it is understood that no quantity is assumed to be purely holomorphic or anti-holomorphic. The associated Noether charges are given by the integral of the time component of the currents over a constant time slice

$$Q \equiv \oint_{|z|=R} \frac{\mathrm{d}z}{z} j^t(z) = \oint_{|z|=R} \mathrm{d}z \left(j_z(z) + \frac{\bar{z}}{z} j_{\bar{z}}(z) \right) \;. \tag{4.4}$$

Note that, in contrast with the usual techniques in CFT, the integration contour cannot be deformed since the currents are not holomorphic. The conservation of the current ensures the independence of the charge on the radius R, so we will fix $R \equiv 1$ from here on. The action (4.1) has conformal symmetry for any values of k and f^2 since the energy-momentum tensor is holomorphic:

$$T(z) = \frac{2f^2}{(1+kf^2)^2}\,\mathrm{Tr}\left(j_z(z) j_z(z)\right) = \frac{2f^2}{(1-kf^2)^2}\,\mathrm{Tr}\left(\bar{j}_z(z)\bar{j}_z(z)\right) \;, \tag{4.5}$$

where the second equality follows from the existing conjugacy relation between the components of the left and right currents. Throughout this paper L_n will denote the modes of the expansion of the energy-momentum tensor, as usual. If the dual Coxeter number of G vanishes (as for $\mathrm{PSU}(1,1|2)$), several non-renormalisation theorems on two- and three-point functions ensure that this symmetry is preserved at the quantum level [126,151,159,160].

4.1.2 Ground state solutions

The classical ground state solution is given by

$$g(z,\bar{z}) = \exp\left(-kf^2 \;\log(z\bar{z})\; (\nu \cdot t)\right) \;, \tag{4.6}$$

[22] The right-multiplication can be treated similarly, and in fact the components of its associated current $\bar{j}(z,\bar{z})$ are conjugate to those of $j(z,\bar{z})$. In particular, they will give rise to the same energy-momentum tensor.

where t^i is an element of the Cartan subalgebra of $\mathfrak{g}$, the Lie algebra of G, and $i = 1, \ldots, \text{rank}(\mathfrak{g})$. Furthermore exp denotes the Lie algebra exponential. We use here the inner product $\nu \cdot t \equiv \nu^i \kappa_{ij} t^j$, where κ_{ij} is an invariant form on the Lie algebra. It can easily be shown that this solution indeed satisfies the equations of motion. The conserved charges are given by $Q = \bar{Q} = k\nu^i \kappa_{ij} t^j$, so that ν^i is interpreted semiclassically as $\nu^i = \ell_0^i / k$, where $Q^i = \bar{Q}^i = \ell_0^i$ are the charges of the ground state solution. Finally, the energy-momentum tensor is

$$T(z) = \frac{f^2(\ell_0 \cdot \ell_0)}{2z^2} \,, \tag{4.7}$$

and similarly for $\bar{T}(\bar{z})$. In Section 4.5 we will find the quantum correction to this result.

4.1.3 Excited solutions

Consider now the following excited solution:

$$g(z, \bar{z}) = \exp\left(\frac{1}{\sqrt{k}} \left(\mu\, z^\alpha \bar{z}^\beta t^a - \mu^*\, z^{-\alpha} \bar{z}^{-\beta} t^{-a}\right)\right) \exp\left(-kf^2 \log(z\bar{z})(\nu \cdot t)\right) \,, \tag{4.8}$$

where t^a is a step operator or a Cartan-element of $\mathfrak{g}$, and μ is a coefficient to be fixed. Furthermore, t^{-a} denotes the step operator associated with the opposite root. This has to be included to ensure the reality of the solution. Single-valuedness requires $\beta - \alpha = n$, with n an integer. Finally, the equations of motion (4.3) are obeyed provided that[23]

$$\alpha = \frac{1}{2}\left(-n - (a \cdot \nu)\, kf^2 + \sqrt{n^2 - 2(a \cdot \nu)\, nk^2 f^4 + (a \cdot \nu)^2 k^2 f^4}\right) \,, \tag{4.9}$$

where a^i denotes α^i if a is a root, and 0 if a is a Cartan-index. Plugging (4.8) into (4.4) and using (4.2), the charges associated with these excited solutions can be explicitly computed. The expressions for these charges in terms of μ, ν, n are quite involved, so we will not reproduce them here. Instead, we will use the parametrisation

$$Q = \ell_0 \cdot t + N_n (a \cdot t) \,, \qquad \bar{Q} = \ell_0 \cdot t \,, \tag{4.10}$$

which allows us to trade ν^i for ℓ_0^i and μ for N_n. The plane-wave limit may now be obtained by considering $n, N_n \ll k$, ℓ_0^i, f^{-2}. In this limit the zero-mode of the energy-momentum tensor (4.5) becomes

$$L_0 = \frac{f^2(\ell_0 \cdot \ell_0)}{2} + \frac{N_n}{2}\left(n + (a \cdot \ell_0) f^2 + \sqrt{n^2 - 2(a \cdot \ell_0) nkf^4 + (a \cdot \ell_0)^2 f^4}\right)$$

[23] For this solution we have chosen a specific branch of the square-root, by assuming that $n \geq 2\nu$. The other branch can be obtained from the first by considering $n \leq 2\nu$.

$$+\,\mathcal{O}\big(k^{-1}\big)\ . \quad (4.11)$$

This expression is to be compared with (4.45), which will be obtained from the full quantum treatment developed in the following sections. In Section 4.5 these results will be applied to string theory, and the full BMN formula of [161] will be obtained. This preliminary result shows that worldsheet methods based on (4.1) may give us access to the plane-wave spectrum with mixed flux.

Semiclassically, the parametrisation (4.10) suggests that this state is obtained from the groundstate of charge ℓ_0 by the application of N_n generators of the left symmetry, with mode number n. This interpretation is reinforced by the observation that, to all orders in k^{-1},

$$L_0 - \bar{L}_0 = nN_n\ , \tag{4.12}$$

which is indeed quantised in the quantum theory.

Finally, we stress that this is an exact classical solution and is hence expected to yield the correct conformal weight in the classical limit. The classical limit is given by $k,\ \ell_0^i,\ f^{-2} \to \infty$, with all their ratios fixed, and for any n and N_n. In particular, this is a much more powerful limit than the plane-wave limit, and even the decompactification limit [162]. However, we are so far limited in our computations, in that we have only managed to find a single-excitation solution.[24]

4.2 Review of the current algebra

In this section we review the current algebra introduced in [163] and further analysed in [160, 164], which will be the main tool of this work. In particular, its applications to string theory via the hybrid formalism [126] will be described in Section 4.5.

4.2.1 Conformal current algebra

A non-chiral current algebra in two-dimensions compatible with conformal symmetry was first formulated in [163] in all generality. This algebra was constructed at the level of OPE's by requiring their consistency with locality, and Lorentz and parity-time reversal symmetries. The non-linear sigma models of the kind (4.1) were then seen to consistently realise the constructed general OPE structure, by computing the current-current correlators and the OPE's of the models in conformal perturbation theory. This result holds for sigma models based on Lie supergroups whose superalgebra has vanishing Killing form, such as PSU(1, 1|2) (see Appendix A.1.1

[24] In particular our solution will not be level-matched in string theory.

for a review of the relevant properties of Lie superalgebras). For those, a non-renormalisation theorem [151] allows one to do the computation to all orders in perturbation theory.

The OPE's between the components of the currents were found to be as follows:

$$
\begin{aligned}
j_z^a(z) j_z^b(w) &\sim \frac{(1+kf^2)^2 \kappa^{ab}}{4f^2(z-w)^2} + \frac{i}{4} f^{ab}{}_c \left(\frac{(3-kf^2)(1+kf^2)}{z-w}\, j_z^c(w) \right. \\
&\qquad \left. + \frac{(1-kf^2)^2(\bar{z}-\bar{w})}{(z-w)^2}\, j_{\bar{z}}^c(w) \right) , \qquad (4.13a) \\
j_{\bar{z}}^a(z) j_{\bar{z}}^b(w) &\sim \frac{(1-kf^2)^2 \kappa^{ab}}{4f^2(\bar{z}-\bar{w})^2} + \frac{i}{4} f^{ab}{}_c \left(\frac{(3+kf^2)(1-kf^2)}{\bar{z}-\bar{w}}\, j_{\bar{z}}^c(w) \right. \\
&\qquad \left. + \frac{(1-kf^2)^2(z-w)}{(\bar{z}-\bar{w})^2}\, j_z^c(w) \right) , \qquad (4.13b) \\
j_z^a(z) j_{\bar{z}}^b(w) &\sim (1-kf^2)^2 f^{ab}{}_c \left(\frac{j_z^c(w)}{(\bar{z}-\bar{w})} + \frac{j_{\bar{z}}^c(w)}{(z-w)} \right) , \qquad (4.13c)
\end{aligned}
$$

where $\sim$ denotes equality up to regular and contact terms. A regular term is by definition less divergent than a pole, in particular there are logarithmic corrections to these OPE's. Their explicit form can be found in [160]. Here κ^{ab} and $f^{ab}{}_c$ are the components of the invariant tensor and the structure constants of $\mathfrak{g}$, respectively (see Appendix A.1.1). Notice that at the WZW-point this current algebra reduces to a Kač-Moody algebra.

4.2.2 Energy-momentum tensor

The holomorphic energy-momentum tensor is as usual the regularisation of its classical counterpart:

$$
T(z) = \frac{2f^2}{(1+kf^2)^2}\, \kappa_{ab} (j_z^a j_z^b)(z) = \frac{2f^2}{(1-kf^2)^2}\, \kappa_{\bar{a}\bar{b}} (\bar{j}_z^{\bar{a}} \bar{j}_z^{\bar{b}})(z) \,, \qquad (4.14)
$$

It was shown in [160] that this energy-momentum tensor is indeed holomorphic. In fact,

$$
W^{(s)}(z) = d_{a_1 \cdots a_s} (j_z^{a_1} \cdots j_z^{a_s})(z) \qquad (4.15)
$$

is holomorphic for every Casimir $d_{a_1 \cdots a_s} t^{a_1} \cdots t^{a_s}$ of $\mathfrak{g}$. These fields generate the full chiral algebra of the CFT.[25] This chiral algebra is much too small to constrain the spectrum of the CFT and is hence not very useful for our purpose. In particular, the CFT is not rational.

[25] The algebra $\mathfrak{psu}(1,1|2)$ possesses one further Casimir of order 6 for which the result applies, so the chiral algebra of this theory is a $\mathcal{W}(2,6)$-algebra. For an explicit construction of this algebra, see [165].

Despite the fact that $j_z(z)$ and $j_{\bar{z}}(z)$ are not holomorphic nor anti holomorphic, their OPE's with $T(z)$ are those of primary fields of dimension one and zero, respectively:

$$T(z) j_z^a(w) \sim \frac{j_z^a(w)}{(z-w)^2} + \frac{\partial j_z^a(w)}{z-w} \,, \tag{4.16a}$$

$$T(z) j_{\bar{z}}^a(w) \sim \frac{\partial j_{\bar{z}}^a(w)}{z-w} \,, \tag{4.16b}$$

possibly with logarithmic corrections. We take this as an indication that it is useful to think of the currents and their OPE's as the spectrum-generating algebra, even away from the WZW-point. We will see in Section 4.4 that this is true in a BMN-like limit.

4.2.3 Conserved charges and the mode algebra

As usual in quantum field theory, the symmetry algebra must be realised on the Hilbert space of the theory through a set of conserved charges.[26] These charges were introduced in (4.4), and their bracket $[Q^a, Q^b]$ may now be computed. Here and in the following it is implicit that if both a and b are fermionic indices the bracket $[Q^a, Q^b]$ is to be understood as an anti-commutator. Moreover, for the sake of simplicity we suppress possible signs arising from the fermionic nature of the supercurrents. Nevertheless, our final results hold for bosonic as well as for fermionic currents. The computation is subtle since we cannot rely on usual CFT techniques like contour deformation. However the commutator can be written as

$$[Q^a, Q^b] = \lim_{\varepsilon \downarrow 0} \left(\oint_{|z|=R+\varepsilon} \mathrm{d}z \oint_{|z|=R} \mathrm{d}w - \oint_{|z|=R-\varepsilon} \mathrm{d}z \oint_{|z|=R} \mathrm{d}w \right) \times \left(j_z^a(z) + \frac{\bar{z}}{z} j_{\bar{z}}^a(z) \right) \left(j_z^a(w) + \frac{\bar{w}}{w} j_{\bar{z}}^a(w) \right) \,. \tag{4.17}$$

Inserting the OPE's (4.13a)–(4.13c) and performing the integrals we indeed obtain

$$[Q^a, Q^b] = i f^{ab}{}_c Q^c \,. \tag{4.18}$$

This is a very good consistency check on the construction. Similarly, one can compute the commutators of Q^a with the modes of the energy-momentum tensor L_n and $\bar{L}_n$. Holomorphicity of $T(z)$ simplifies the computation considerably, and yields the expected result

$$[L_n, Q^a] = [\bar{L}_n, Q^a] = 0 \,, \tag{4.19}$$

[26] The symmetry could also be anomalous, but we will see shortly that this is not the case.

i.e. the internal and conformal symmetries commute. In particular, this shows that the charge is indeed conserved, since it commutes with the Hamiltonian $L_0 + \bar{L}_0$.

Motivated by this construction, we now define a convenient set of operators (of which the conserved charges above form a subset) which allows us to build the spectrum of our model. In analogy with the usual chiral currents in CFT we define[27]

$$X_n^a \equiv \oint_{|z|=R} \frac{\mathrm{d}z}{R}\, z^n j_z^a(z)\,, \qquad Y_n^a \equiv \oint_{|z|=R} \frac{\mathrm{d}z}{R}\, z^{n-1}\bar{z}\, j_{\bar{z}}^a(z)\,, \tag{4.20}$$

and analogously for the right-current $\bar{j}(z)$, which give rise to operators $\bar{X}_n^a$, $\bar{Y}_n^a$. As before, the commutation relations of these quantities can be worked out. It is quite convenient to define the combinations

$$Q_n^a = X_n^a + Y_n^a\,, \qquad P_n^a = 2kf^2\left(\frac{X_n^a}{1+kf^2} - \frac{Y_n^a}{1-kf^2}\right)\,, \tag{4.21a}$$

$$\bar{Q}_n^{\bar{a}} = \bar{X}_n^{\bar{a}} + \bar{Y}_n^{\bar{a}}\,, \qquad \bar{P}_n^{\bar{a}} = -2kf^2\left(\frac{\bar{X}_n^{\bar{a}}}{1-kf^2} - \frac{\bar{Y}_n^{\bar{a}}}{1+kf^2}\right)\,, \tag{4.21b}$$

for which we find the commutation relations

$$[Q_m^a, Q_n^b] = km\kappa^{ab}\delta_{m+n,0} + if^{ab}{}_c\, Q_{m+n}^c\,, \qquad [Q_m^a, \bar{P}_n^{\bar{b}}] = kmA_{m+n}^{a\bar{b}}\,, \tag{4.22a}$$

$$[Q_m^a, P_n^b] = km\kappa^{ab}\delta_{m+n,0} + if^{ab}{}_c\, P_{m+n}^c\,, \qquad [\bar{Q}_m^{\bar{a}}, A_n^{b\bar{b}}] = if^{\bar{a}\bar{b}}{}_{\bar{c}}\, A_{m+n}^{b\bar{c}}\,, \tag{4.22b}$$

$$[\bar{Q}_m^{\bar{a}}, \bar{Q}_n^{\bar{b}}] = -km\kappa^{ab}\delta_{m+n,0} + if^{\bar{a}\bar{b}}{}_{\bar{c}}\, Q_{m+n}^{\bar{c}}\,, \qquad [Q_m^a, A_n^{b\bar{b}}] = if^{ab}{}_c\, A_{m+n}^{c\bar{b}}\,, \tag{4.22c}$$

$$[\bar{Q}_m^{\bar{a}}, \bar{P}_n^{\bar{b}}] = -km\kappa^{\bar{a}\bar{b}}\delta_{m+n,0} + if^{\bar{a}\bar{b}}{}_{\bar{c}}\, \bar{P}_{m+n}^{\bar{c}}\,, \qquad [\bar{Q}_m^{\bar{b}}, P_n^a] = -kmA_{m+n}^{a\bar{b}}\,, \tag{4.22d}$$

with all other commutators vanishing, and in particular $[P_m^a, P_n^b] = 0$. The barred and unbarred modes constitute two non-commuting non-semisimple super-Kač-Moody algebras at level k and $-k$.[28] A rescaled version of this algebra appears already in [163]. The non-commutativity of these algebras is of course related to the non-holomorphicity of the currents, and it is encoded in the bi-adjoint field

$$A^{a\bar{a}} = \mathrm{STr}\left(g^{-1}t^a g\, t^{\bar{a}}\right)\,, \tag{4.23}$$

where t^a, $t^{\bar{a}}$ are the generators of each of the two copies of $\mathfrak{g}$ in the adjoint representation. It has conformal weight zero, since the Casimir of the adjoint representation of $\mathfrak{g}$ vanishes.

[27]In contrast with the usual conventions in CFT, all operators are defined via contour z-integrals. The contour relation $z = R^2\bar{z}^{-1}$ will lead to some unusual signs in our modes. On the other hand, in line with usual QFT results, no physical implication stems from the actual value of R.

[28]The negative sign of one of the levels is immaterial: it is simply a consequence of our unusual conventions for barred modes.

Since P_n^a commutes with itself, its scaling is arbitrary.[29] Therefore the only meaningful parameter which appears and which is subject to possible unitarity restrictions is k. At the WZW-point, the Y_n^a become null fields and Q_n^a reduce to the modes of the chiral currents of the WZW model.[30] Finally, note that the conserved charges constructed above are simply the zero-modes Q_0^a.

It is important to notice that not all the modes of the mode algebra are independent, i.e. this mode algebra does not act faithfully on the Hilbert space. The relations between the different modes can be found in [160], and their precise form is mostly irrelevant for our results.

4.2.4 The Virasoro modes

Since $T(z)$ is holomorphic, the computation of the commutation relations of the Virasoro modes with the current modes can be simplified by contour-deformation techniques, and by ignoring non-singular terms in the OPE's. Alternatively, we can use (4.5) and (4.20) to first write

$$L_n = \frac{2f^2}{(1+kf^2)^2}\kappa_{ab}\left(X^a X^b\right)_n = \frac{2f^2}{(1-kf^2)^2}\kappa_{\bar{a}\bar{b}}\left(\bar{X}^{\bar{a}}\bar{X}^{\bar{b}}\right)_n . \tag{4.24}$$

and then take commutators of normal-ordered products as usual. The two methods yield the same result, namely the following commutation relations:

$$[L_m, Q_n^a] = -\frac{1+kf^2}{2} n Q_{n+m}^a - \frac{1-k^2f^4}{4kf^2} n P_{m+n}^a , \tag{4.25a}$$

$$[L_m, P_n^a] = -kf^2 n Q_{n+m}^a - \frac{1-kf^2}{2} n P_{n+m}^a - \mathrm{i} f^2 f^a{}_{bc}\left(Q^b P^c\right)_{n+m} , \tag{4.25b}$$

$$[L_m, \bar{Q}_n^{\bar{a}}] = -\frac{1-kf^2}{2} n \bar{Q}_{n+m}^{\bar{a}} + \frac{1-k^2f^4}{4kf^2} n \bar{P}_{m+n}^{\bar{a}} , \tag{4.25c}$$

$$[L_m, \bar{P}_n^{\bar{a}}] = kf^2 n \bar{Q}_{n+m}^{\bar{a}} - \frac{1+kf^2}{2} n \bar{P}_{n+m}^{\bar{a}} - \mathrm{i} f^2 f^{\bar{a}}{}_{\bar{b}\bar{c}}\left(\bar{Q}^{\bar{b}} \bar{P}^{\bar{c}}\right)_{n+m} . \tag{4.25d}$$

These results can be derived from both expressions for the Virasoro modes in (4.24). Note that the result above is independent of the normal-ordering scheme we use, since $f^a{}_{bc}\, Q_m^b P_n^c = f^a{}_{bc}\, P_m^c Q_m^b$ because of the vanishing of the dual Coxeter number. It is important to notice that, due to the appearance of normal-ordered operators in (4.25a)–(4.25d), the Virasoro tensor does not act diagonally. Therefore the spectrum-generating currents are not (combinations of) quasi-primary fields, which hinders the computation of the conformal weights of the states on the Hilbert space. In Section 4.4 a BMN-like limit which simplifies this issue will be presented.

[29] Note that $P = -k\mathrm{Tr}\left(g^{-1}\partial_\varphi g\right)$, where φ is the compact direction on the worldsheet. It is then natural that P commutes with itself, since it contains no time derivatives.

[30] Analogously, at the WZW-point $\bar{X}_n^a$ become null and $\bar{Q}_n^a$ reduce to modes of the anti-chiral currents.

4.3 Representations

After having established that an extension of the affine Lie superalgebra $\mathfrak{g}_k \oplus \mathfrak{g}_{-k}$ (see (4.22a)–(4.22d) for the complete commutation relations) naturally acts on the Hilbert space of the theory, we go on and study possible representations of this algebra.

There is immediately a severe problem arising, which hinders us from solving the complete theory. At the WZW-point, the representation theory of the algebra $\mathfrak{g}_k \oplus \mathfrak{g}_{-k}$ is very-well understood, see [149] for the case of $\mathfrak{psu}(1,1|2)$. In particular the modes Q^a_m define lowest weight representations on the Hilbert space,[31] while the modes $\bar{Q}^{\bar{a}}_m$ define highest weight representations. Since the algebra depends only on k, this should not change when going away from the WZW-point. When adding the modes P^a_m and $\bar{P}^{\bar{a}}_m$, it is natural to assume that they define the same kind of representations, since they form an affine algebra together with the modes Q^a_m and $\bar{Q}^{\bar{a}}_m$. This however implies, by virtue of the commutation relations (4.22a)–(4.22d), that the modes $A^{a\bar{a}}_m$ define neither highest nor lowest weight representations on the Hilbert space. This fact prevents us from computing conformal weights of excitations with both barred and unbarred oscillators. We will explain in the next section how to circumvent this problem in a BMN-like limit.

4.3.1 Affine primaries

Similarly to [160, 163] and analogously with the WZW-point, we define an affine primary state $|\Phi\rangle$ transforming in the representation $\mathcal{R}_0$ as follows:

$$Q^a_m |\Phi\rangle = 0 \ , \ m > 0 \ , \quad Q^a_0 |\Phi\rangle = t^a_{\mathcal{R}_0} |\Phi\rangle \ , \quad P^a_m |\Phi\rangle = 0 \ , \ m \geq 0 \tag{4.26a}$$

$$\bar{Q}^{\bar{a}}_m |\Phi\rangle = 0 \ , \ m < 0 \ , \quad \bar{Q}^{\bar{a}}_0 |\Phi\rangle = t^{\bar{a}}_{\mathcal{R}_0} |\Phi\rangle \ , \quad \bar{P}^{\bar{a}}_m, |\Phi\rangle = 0 \ , \ m \leq 0 \ , \tag{4.26b}$$

where $t^a_{\mathcal{R}_0}$ are the generators of $\mathfrak{g}$ in the representation $\mathcal{R}_0$. As we have mentioned, we cannot impose a highest or lowest weight condition on $A^{a\bar{a}}_m$. These conditions are consistent with the Jacobi identity.

One might also be worried with the fact that the anti-holomorphic Virasoro modes $\bar{L}_n$ can be expressed in terms of the unbarred oscillators, similarly to (4.24). Our definition of affine primary states implies that the anti-holormophic Virasoro modes act in the opposite way than usual. Thus, it seems as if the spectrum is unbounded from below. However, due to the various identifications among the modes, several other states are removed from the spectrum. In particular negative energy states are consistently removed from the physical spectrum.

[31] This is of course not quite true for spectrally flowed representations, but for the sake of this argument, we restrict to the unflowed sector.

We note in particular that the conformal weight of the ground state is now very easy to compute:

$$L_0 \left|\Phi\right\rangle = \frac{2f^2}{(1+kf^2)^2}(X^a X^a)_0 \left|\Phi\right\rangle = \frac{1}{2} f^2 \; t^a_{\mathcal{R}_0} \; t^a_{\mathcal{R}_0} \left|\Phi\right\rangle = \frac{1}{2} f^2 \mathcal{C}(\mathcal{R}_0) \left|\Phi\right\rangle \; . \tag{4.27}$$

Thus, the conformal weight of an affine primary is given by

$$h(\left|\Phi\right\rangle) = \tfrac{1}{2} f^2 \mathcal{C}(\mathcal{R}_0) \; , \tag{4.28}$$

where $\mathcal{C}(\mathcal{R}_0)$ denotes the quadratic Casimir of $\mathfrak{g}$ in $\mathcal{R}_0$. This matches with [151, 160], and is the quantum analogue of (4.7).

4.3.2 Spectral flow

We can also define so-called spectrally flowed representations of the mode algebra. For this, we introduce the following notation for Lie (super)algebras. Cartan-indices will be denoted by latin letters $i, j, \ldots$, while roots will be denoted by greek letters $\alpha, \beta, \ldots$. Hence Q^i_0 denote the Cartan-generators of $\mathfrak{g}$, while Q^α_0 denote the step operators. We assume for ease of presentation that $\mathfrak{g}$ is simply-laced (this is in particular true for $\mathfrak{psu}(1,1|2)$), but the same analysis goes also through in the non simply-laced case. To match the usual conventions for $\mathfrak{su}(2)$ and $\mathfrak{psu}(1,1|2)$, all roots are assumed to have length 1. The commutation relations of the Q^a_m with themselves take the following form in this basis [166]:

$$[Q^i_m, Q^j_n] = \kappa^{ij} k m \delta_{n+m,0} \; , \tag{4.29}$$

$$[Q^i_m, Q^\alpha_n] = \alpha^i Q^\alpha_{m+n} \; , \tag{4.30}$$

$$[Q^\alpha_m, Q^\beta_n] = \begin{cases} km\delta_{m+n,0} + \kappa_{ij}\alpha^i Q^j_{m+n} \; , & \alpha+\beta = 0 \; , \\ \mathcal{N}_{\alpha,\beta} Q^{\alpha+\beta}_{m+n} \; , & \alpha+\beta \text{ is a root} \; , \\ 0 \; , & \text{otherwise} \; , \end{cases} \tag{4.31}$$

where $\kappa^{ij} = \frac{1}{2}\delta^{ij}$, and we used $\kappa^{\alpha\beta} = \delta_{\alpha+\beta,0}$. Here $\mathcal{N}_{\alpha,\beta}$ are constants whose precise values do not play a rôle in the following. Similarly, all other commutation relations of (4.22a)–(4.22d) can be written in this form. The action of the spectral flow on the modes is as follows:

$$\hat{Q}^i_m = Q^i_m + \tfrac{1}{2} k w^i \delta_{m,0} \; , \qquad \hat{P}^i_m = P^i_m + \tfrac{1}{2} k w^i \delta_{m,0} - \tfrac{1}{2} k \kappa_{\bar{\imath}\bar{\jmath}} \bar{w}^{\bar{\imath}} A^{i\bar{\jmath}}_m \; , \tag{4.32a}$$

$$\hat{Q}^\alpha_m = Q^\alpha_{m+\alpha\cdot w/2} \; , \qquad \hat{P}^\alpha_m = P^\alpha_{m+\alpha\cdot w/2} - \tfrac{1}{2} k \kappa_{\bar{\imath}\bar{\jmath}} \bar{w}^{\bar{\imath}} A^{\alpha\bar{\jmath}}_{m+\alpha\cdot w/2} \; , \tag{4.32b}$$

$$\hat{\bar{Q}}^{\bar{\imath}}_m = \bar{Q}^{\bar{\imath}}_m - \tfrac{1}{2} k \bar{w}^{\bar{\imath}} \delta_{m,0} \; , \qquad \hat{\bar{P}}^{\bar{\imath}}_m = \bar{P}^{\bar{\imath}}_m - \tfrac{1}{2} k \bar{w}^{\bar{\imath}} \delta_{m,0} + \tfrac{1}{2} k \kappa_{ij} w^i A^{j\bar{\jmath}}_m \; , \tag{4.32c}$$

$$\hat{\bar{Q}}^{\bar{\alpha}}_m = \bar{Q}^{\bar{\alpha}}_{m+\bar{\alpha}\cdot\bar{w}/2} \; , \qquad \hat{\bar{P}}^{\bar{\alpha}}_m = \bar{P}^{\bar{\alpha}}_{m+\bar{\alpha}\cdot\bar{w}/2} + \tfrac{1}{2} k \kappa_{ij} w^i A^{j\bar{\alpha}}_{m+\bar{\alpha}\cdot\bar{w}/2} \; , \tag{4.32d}$$

$$\hat{A}^{a\bar{a}}_m = A^{a\bar{a}}_{m+a\cdot w/2+\bar{a}\cdot\bar{w}/2} \; , \tag{4.32e}$$

where w^i, $\bar{w}^{\bar{\imath}}$ are the spectral flow parameters. Here, $\alpha \cdot w = \kappa_{ij}\alpha^i w^j$ is the inner product on the root space. We also used the notation $a \cdot w$, which equals $\alpha \cdot w$ if a is a root index and zero if a is a Cartan-index. One can check that this indeed leaves the algebra (4.22a)–(4.22d) invariant. Note that the modes $A_m^{a\bar{a}}$ play a crucial rôle in defining this automorphism.

One can in particular investigate the effect of this automorphism on the energy-momentum tensor. For this, we observe that the spectral-flow symmetry in terms of the X_m^a reads as follows:

$$\hat{L}_n = L_n + \frac{1}{2}\kappa_{ij} w^i X_n^j + \frac{1}{2}\kappa_{\bar{\imath}\bar{\jmath}}\bar{w}^{\bar{\imath}}\bar{X}_n^{\bar{\jmath}} + \kappa_{ij} w^i w^j \frac{(1+kf^2)^2}{32 f^2}\delta_{n,0} + \kappa_{\bar{\imath}\bar{\jmath}}\bar{w}^{\bar{\imath}}\bar{w}^{\bar{\jmath}}\frac{(1-kf^2)}{32f^2}\delta_{n,0} - \kappa_{ij}\kappa_{\bar{\imath}\bar{\jmath}} w^i \bar{w}^{\bar{\imath}} \frac{1-k^2f^4}{16f^2} A_n^{j\bar{\jmath}} \,. \quad (4.33)$$

One may check that this indeed still satisfies the Virasoro algebra. The appearance of X_n^j and $\bar{X}_n^{\bar{\jmath}}$ in a symmetric way is a very satisfying feature of this spectral flow symmetry. Unfortunately, also the modes $A_n^{j\bar{\jmath}}$ appear, which makes it generally hard to compute the effect of this spectral flow on states. Note also that these expressions reduce to the ones of [60, 149] at the WZW-point.

Similarly to the simplification in the representation theory, the spectral flow simplifies considerably when flowing only with the unbarred algebra, i.e. $\bar{w} = 0$. Then the field $A_n^{j\bar{\jmath}}$ disappears and the effect becomes computable. However, the physical spectrum seems to rather require $w = \bar{w}$ [149], so it is not clear whether it makes sense to look at states which are only partially spectrally flowed. This deserves a better understanding.

4.4 The large charge limit

In this section we will consider a limit where all charges are sent to infinity. Since for affine algebras the charges are at most of the same order as their level k, we also require $k \to \infty$ at the same rate. Finally, we require that kf^2 remains constant in the limit. In its applications to string theory (see Section 4.5), this will precisely correspond to the BMN-limit [161]. In this limit, the theory simplifies drastically, as we will see below.

4.4.1 The contraction of the mode algebra

Let us consider the effect of this limit on the mode algebra (4.22a)–(4.22d). The eigenvalues of the Cartan-generators Q_0^i are of order $\mathcal{O}(k)$, since we assumed that all charges are of this order. The step-operators Q_0^α are of order

$\mathcal{O}\big(k^{\frac{1}{2}}\big)$ (since their commutator gives back the Cartan-generators). From (4.26a) we know that $P^i_0\,|\Phi\rangle = 0$, and so P^i_0 is not large even though it is a Cartan-generator. The modes Q^a_m and P^a_m for $m \neq 0$ are then of order $\mathcal{O}\big(k^{\frac{1}{2}}\big)$ in this limit, since their commutator gives the Cartan-generators and central terms. Hence, the Cartan-generators Q^i_0 are of order $\mathcal{O}\big(k\big)$, while all other oscillators are of order $\mathcal{O}\big(k^{\frac{1}{2}}\big)$. Keeping only the leading terms gives the following contraction of the mode algebra (4.22a)–(4.22d)

$$[Q^a_m, Q^b_n] = \Big(mk\kappa^{ab} + i f^{ab}{}_i\, Q^i_0\Big)\, \delta_{m+n,0} \,, \tag{4.34a}$$

$$[Q^a_m, P^b_n] = mk\kappa^{ab}\delta_{m+n,0} \,, \tag{4.34b}$$

$$[P^a_m, P^b_n] = 0 \,, \tag{4.34c}$$

and similarly for the barred oscillators. Furthermore note that the Cartan zero-modes Q^i_0 become central extensions of this almost-abelian algebra. Likewise, the field $A^{a\bar{a}}$ appears solely as a central extension. Since the Q^i_0 are central, we may replace them with their eigenvalues ℓ^i in the given representation.

Let us now look into the action of the modes $A^{a\bar{a}}_m$ in this limit. In [160] it was found that these modes are not all independent. In fact, we have the following relation:

$$mA^{a\bar{a}}_m = \frac{i}{k} f^a{}_{bc}\,(P^c A^{b\bar{a}})_m = -\frac{i}{k} f^{\bar{a}}{}_{\bar{b}\bar{c}}\,(\bar{P}^{\bar{c}} A^{a\bar{b}})_m \,, \tag{4.35}$$

From this relation we conclude that any non-zero mode is of order $\mathcal{O}\big(k^{-\frac{1}{2}}\big)$, whereas $A^{a\bar{a}}_0$ is of order $\mathcal{O}(1)$, see (4.23). Therefore all non-zero modes are subleading in this limit. Evaluating (4.23) on the classical ground state (4.6) yields $A^{i\bar{\imath}} = \kappa^{i\bar{\imath}}$. It is then natural to assume

$$A^{i\bar{\imath}}_0\,|\Phi\rangle = \kappa^{i\bar{\imath}}\,|\Phi\rangle \;. \tag{4.36}$$

In particular, this is consistent with all commutation relations, as well as with all the identifications between the modes.

4.4.2 The spectrum-generating algebra

We now look at the commutation relations of L_m with Q^a_n and P^a_n, which follow from taking the appropriate limit of (4.25a)–(4.25d). Indeed, the commutator of L_m with Q^a_n does not change, while the commutator of L_m with P^a_n becomes

$$\begin{aligned}[L_m, P^a_n] &= -kf^2 n Q^a_{m+n} - \tfrac{1}{2}(1-kf^2)nP^a_{m+n} - if^2 f^a{}_{ic}\,\ell^i P^c_{m+n} \\ &= -kf^2 n Q^a_{m+n} - \tfrac{1}{2}(1-kf^2)nP^a_{m+n} + f^2\kappa_{ij}a^j\ell^i P^a_{m+n} \,.\end{aligned} \tag{4.37}$$

In the second line, a^j denotes α^j if a is a root and 0 if a is a Cartan-index. Note that the coefficients of all three terms in (4.37) are of the same order $\mathcal{O}(1)$. Thus we see that in this limit L_m only mixes Q_n^a and P_n^a, so we can simply find the eigenvectors. For $n \neq 0$, they are given by:

$$J^a_{\pm,n} \equiv Q^a_n + \frac{-f^2(a\cdot\ell + kn) \pm \sqrt{n^2 + 2(a\cdot\ell)kf^4 n + (a\cdot\ell)^2 f^4}}{2kf^2 n} P^a_n \,. \qquad (4.38)$$

where $a \cdot \ell \equiv \kappa_{ij} a^j \ell^i$. Their commutation relations with L_m are given by

$$[L_m, J^a_{\pm,n}] = \frac{1}{2}\Big((a\cdot\ell)f^2 - n \mp \sqrt{n^2 + 2(a\cdot\ell)kf^4 n + (a\cdot\ell)^2 f^4}\Big) J^a_{\pm,m+n} \,. \quad (4.39)$$

We can similarly diagonalise the barred modes and compute commutators with $\bar{L}_m$. The other relevant commutation relations are given by

$$[L_m, \bar{J}^{\bar{a}}_{\pm,n}] = \frac{1}{2}\Big((\bar{a}\cdot\bar{\ell})f^2 - n \mp \sqrt{n^2 - 2(\bar{a}\cdot\bar{\ell})kf^4 n + (\bar{a}\cdot\bar{\ell})^2 f^4}\Big) \bar{J}^{\bar{a}}_{\pm,m+n} \,, \qquad (4.40)$$

Moreover, $[L_m - \bar{L}_m, J^a_{\pm,n}] = -nJ^a_{\pm,n}$ and similarly for the barred oscillators. This was expected, since $L_m - \bar{L}_m$ measures the spin of the state which should be an integer.

We thus seem to obtain four oscillators $J^a_{\pm,n}$, $\bar{J}^a_{\pm,n}$ generating the CFT spectrum in this limit, but in fact only two are independent. Indeed, knowing the two currents j_z and $j_{\bar{z}}$, for example, is enough to completely determine (up to an isometry transformation) the classical solution $g(z,\bar{z})$ using (4.2). The quantum version of this statement is the relation [160]

$$\bar{Q}^a_m = -\kappa_{ab}(Q^a A^{b\bar{a}})_m + \kappa_{ab}(P^a A^{b\bar{a}})_m \qquad (4.41a)$$

$$\bar{P}^a_m = \kappa_{ab}(P^a A^{b\bar{a}})_m \,, \qquad (4.41b)$$

between the modes introduced in (4.21a) and (4.21b). This relation allows us to express the actions of $\bar{Q}^a_m$, $\bar{P}^a_m$ on an affine primary, for example, in terms of the actions of Q^a_m and P^a_m.[32] More concretely, the semiclassical solutions suggests that we should identify the following two states up to a phase:

$$J^a_{\varepsilon,n}|\ell^i, \bar{\ell}^i + \bar{a}^i\rangle \longleftrightarrow \bar{J}^{\bar{a}}_{\varepsilon,n}|\ell^i + a^i, \bar{\ell}^i\rangle \,, \qquad (4.42)$$

with $(a\cdot\ell) = -(\bar{a}\cdot\bar{\ell})$. Here $|\ell^i, \bar{\ell}^i + \bar{a}^i\rangle$ and $|\ell^i + a^i, \bar{\ell}^i\rangle$ are affine primary states with charges $(\ell^i,\ \bar{\ell}^i + \bar{a}^i)$ and $(\ell^i + a^i,\ \bar{\ell}^i)$, respectively. These two affine primary states can be obtained from each other by the action of the

[32] Note that the action of $A^{a\bar{a}}_m$ can be likewise, in principle, expressed in terms of the modes Q^a_m and P^a_m. For our purposes it is enough to notice that the number of oscillators is reduced from four to two.

zero-modes. It is easy to see that the charges of the states (4.42) are the same, as well as their conformal dimension.

Thus, we can now generate the spectrum by considering solely the states

$$|\Psi\rangle \equiv \prod_{n=1}^{\infty}\prod_{i_n=1}^{N_n} J^{a_{i_n}}_{-,-n} \prod_{\bar{n}=1}^{\infty}\prod_{\bar{\imath}_{\bar{n}}=1}^{\bar{N}_{\bar{n}}} \bar{J}^{\bar{a}_{\bar{\imath}_{\bar{n}}}}_{-,\bar{n}} |\Phi\rangle \ . \tag{4.43}$$

The conformal weight of these states follows from (4.39) and (4.40):

$$\begin{aligned} h(|\Psi\rangle) &= h(|\Phi\rangle) \\ &+ \frac{1}{2}\sum_{n=1}^{\infty}\sum_{i_n=1}^{N_n}\Big((a_{i_n}\cdot\ell)f^2 + n + \sqrt{n^2 - 2(a_{i_n}\cdot\ell)kf^4 n + (a_{i_n}\cdot\ell)^2 f^4}\Big) \\ &+ \frac{1}{2}\sum_{\bar{n}=1}^{\infty}\sum_{\bar{\imath}_{\bar{n}}=1}^{\bar{N}_{\bar{n}}}\Big((\bar{a}_{\bar{\imath}_{\bar{n}}}\cdot\bar{\ell})f^2 + \bar{n} + \sqrt{\bar{n}^2 - 2(\bar{a}_{\bar{\imath}_{\bar{n}}}\cdot\bar{\ell})kf^4\bar{n} + (\bar{a}_{\bar{\imath}_{\bar{n}}}\cdot\bar{\ell})^2 f^4}\Big) \ . \end{aligned} \tag{4.44}$$

In later applications, we will always take the left- and right-moving representations of $|\Phi\rangle$ to coincide, i.e. we will consider the diagonal modular invariant. Let us now restrict to this case, where $\ell = \bar{\ell}$. Then we use the notation $\bar{n} = -n$ for $n < 0$ to write the formula in a compact way as follows:

$$\begin{aligned} h(|\Psi\rangle) &= \frac{1}{2}f^2\mathcal{C}(\mathcal{R}_0) \\ &+ \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\sum_{i_n=1}^{N_n}\Big((a_{i_n}\cdot\ell)f^2 + n + \sqrt{n^2 - 2(a_{i_n}\cdot\ell)kf^4|n| + (a_{i_n}\cdot\ell)^2 f^4}\Big) \ , \end{aligned} \tag{4.45}$$

where we also inserted the conformal weight of the affine primary (4.28). This is the main result of this section.

The spin $s = h - \bar{h}$ of the state is given by

$$s(|\Psi\rangle) = h(|\Psi\rangle) - \bar{h}(|\Psi\rangle) = \sum_{n=-\infty}^{\infty} nN_n \ . \tag{4.46}$$

In particular, it is integer, which is a consistency check of our analysis. One may also check that this formula reduces to the correct conformal weight at the WZW-point $kf^2 = 1$.

4.4.3 Characters

We can work out the characters of the representation we just found. Since k is large, the Verma-module does not contain any null-vectors. We can directly read from (4.45) the character of such a representation:

$$\chi(\tau,\bar{\tau}) = |\chi_0|^2 \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\prod_{a}\Big(1 - |q|^{(a\cdot\ell)f^2 + \sqrt{n^2 - 2(a\cdot\ell)kf^4|n| + (a\cdot\ell)^2 f^4}}\, q^{\frac{n}{2}}\bar{q}^{-\frac{n}{2}}\Big)^{-|a|} \ . \tag{4.47}$$

Here a runs over the complete Lie superalgebra and $|a| = 1$ if the index is bosonic and $|a| = -1$ if it is fermionic. Also, χ_0 denotes the character of the zero-mode algebra of the representation $\mathcal{R}_0$. We have not included chemical potentials in the formula, their inclusion is straightforward.

4.5 Applications to string theory

In this section, we will apply the formalism we constructed in the previous sections to string theory on the backgrounds $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$.[33] For this, our starting point is the hybrid formalism for $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ [126], in which the sigma-model on the supergroup $\text{PSU}(1,1|2)$ features prominently.

4.5.1 The BMN limit

The plane-wave or Berenstein-Maldacena-Nastase (BMN) limit [161] is the following limiting case of the theory:

$$j,\ \ell,\ k,\ f^{-2} \to \infty\ , \tag{4.48}$$

with all their ratios remaining constant in the limit. Here, j and ℓ are the eigenvalues of the Cartan-generators of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$, respectively. Also, the BMN limit is near-BPS, meaning that $j - \ell$ is kept finite in the limit.

The complete action for the worldsheet theory reads

$$\mathcal{S} = f^{-2}(\mathcal{S}_0 + \mathcal{S}_1) + k\mathcal{S}_{\text{WZ}} + \mathcal{S}_{\text{ghost}}\ , \tag{4.49}$$

where $f^{-2}\mathcal{S}_0 + k\mathcal{S}_{\text{WZ}}$ is the action of the $\text{PSU}(1,1|2)$-sigma model as in eq. (4.1). Furthermore, $\mathcal{S}_1$ are ghost couplings [126, eq. (8.39)]. These ghost couplings are bilinear in the fermionic currents, and the ghosts which appear are at worst of order $\mathcal{O}(1)$. As discussed in Subsection 4.4.1, the fermionic currents scale in the BMN limit as $\mathcal{O}(k^{\frac{1}{2}})$. Therefore $\mathcal{S}_1$ is of order $\mathcal{O}(k)$. On the other hand, $\mathcal{S}_0$ is bilinear in the bosonic currents, which can be of order $\mathcal{O}(k)$. Hence $\mathcal{S}_0$ is of order $\mathcal{O}(k^2)$. Thus, the ghost couplings are very much suppressed in the BMN limit and can be neglected. This was to be expected, since the ghost couplings vanish in flat space and the BMN limit is an almost-flat space approximation.

Berenstein, Maldacena and Nastase derived in [161] a formula for the string spectrum in this limit:

$$\Delta - L = \sum_{n=-\infty}^{\infty} N_n \sqrt{1 \pm \frac{2nk}{L} + \frac{n^2}{L^2 f^4}} + \frac{1}{Lf^2}\left(L_0^{\mathbb{T}^4} + \bar{L}_0^{\mathbb{T}^4}\right) + \mathcal{O}(k^{-1})\ . \tag{4.50}$$

[33] A similar treatment applies to $\text{AdS}_3 \times \text{S}^3 \times \text{K3}$.

Here, $L_0^{\mathbb{T}^4}$ is the conformal weight coming from the torus excitations, $L = \ell + \bar{\ell}$ is the total $\mathfrak{su}(2)$-spin from both left- and right-movers, and $\Delta = j + \bar{\jmath}$ is the scaling dimension of the dual CFT. Since Δ and L are both large, but their difference is finite, this is a near-BPS limit. The summation goes over the different worldsheet oscillators, where $n < 0$ refers to right-movers and $n > 0$ to left-movers. Also, N_n is the occupation number of the respective mode. Level-matching translates into

$$\sum_{n=-\infty}^{\infty} n N_n = \bar{L}_0^{\mathbb{T}^4} - L_0^{\mathbb{T}^4} \ , \tag{4.51}$$

in this language. Notice that while the RHS of (4.50) contains L, we could have also written Δ since these quantities differ only by subleading terms. We also assume that only finitely many occupation numbers N_n are non-zero.

4.5.2 Reproducing the BMN formula of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ from the worldsheet

We are finally in the position to reproduce (4.50) from the worldsheet. For this, we note that the BMN limit coincides on the worldsheet precisely with the large charge limit we considered in Section 4.4. So we may start with (4.45), which we derived in the last section. Requiring the state to be level-matched, i.e. (4.51) to be satisfied, (4.45) simplifies to

$$\begin{aligned} h(|\Psi\rangle) &= \frac{1}{2} f^2 \mathcal{C}(\mathcal{R}_0) + L_0^{\mathbb{T}^4} \\ &\quad + \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{i_n=1}^{N_n} \Big((a_{i_n} \cdot \ell) f^2 + n + \sqrt{n^2 - 2(a_{i_n} \cdot \ell) k f^4 |n| + (a_{i_n} \cdot \ell)^2 f^4} \Big) \\ &= \frac{1}{2} f^2 \mathcal{C}(\mathcal{R}_0) + \frac{1}{2} \big(L_0^{\mathbb{T}^4} + \bar{L}_0^{\mathbb{T}^4} \big) \\ &\quad + \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{i_n=1}^{N_n} \Big((a_{i_n} \cdot \ell) f^2 + \sqrt{n^2 - 2(a_{i_n} \cdot \ell) k f^4 |n| + (a_{i_n} \cdot \ell)^2 f^4} \Big) \ . \end{aligned} \tag{4.52}$$

We included a possible conformal weight from the torus. For the case of $\mathfrak{psu}(1,1|2)$, the Casimir equals (A.5):

$$\mathcal{C}(\mathcal{R}_0) = -2 j_0 (j_0 - 1) + 2\ell_0 (\ell_0 + 1) \ , \tag{4.53}$$

where j_0 and ℓ_0 denote the $\mathfrak{sl}(2,\mathbb{R})$-spin and the $\mathfrak{su}(2)$-spin of the ground state $|\Phi\rangle$, respectively. Note that the generic charges ℓ^i of Section 4.4 correspond now to j_0 and ℓ_0. Furthermore, notice that solving the mass-shell condition $h(|\Psi\rangle) = 0$ will imply that $j_0 = \ell_0 + \mathcal{O}(1)$. To this order, we may therefore replace j_0 everywhere by ℓ_0, except in (4.53). Thus we have $a \cdot \ell = a \ell_0$, where a takes the following values for the generators of $\mathfrak{psu}(1,1|2)$:

$$J^3 : 0\,, \quad J^\pm : \mp 2\,, \quad K^3 : 0\,, \quad K^\pm : \pm 2\,, \quad S^{\pm\pm\alpha} : 0\,, \quad S^{\pm\mp\alpha} : \mp 2\,. \tag{4.54}$$

For the complete commutation relations of the affine algebra $\mathfrak{psu}(1,1|2)_k$ in this basis, see Appendix A.1.2. With all this in mind, we now solve (4.52) for j_0 and expand the result in orders of the characteristic scale k to obtain:

$$j_0 = \ell_0 + 1 + \frac{1}{4} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{i_n=1}^{N_n} \left(a_{i_n} + \sqrt{a_{i_n}^2 - \frac{2a_{i_n} k|n|}{\ell_0} + \frac{n^2}{\ell_0^2 f^4}} \right) + \frac{1}{4\ell_0 f^2} \big(L_0^{\mathbb{T}^4} + \bar{L}_0^{\mathbb{T}^4} \big) + \mathcal{O}(k^{-1})\,. \tag{4.55}$$

The summand 1 comes from the fact that j_0 and ℓ_0 measure the $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$-spin of the highest weight state. As one can see from the structure of a typical multiplet of $\mathfrak{psu}(1,1|2)$ (see A.6), the state with the lowest $j-\ell$ is not the highest weight state and it has precisely $j-\ell$ lowered by one.[34]

Finally, we have to take into account the contribution of the oscillators to the $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$-spins. We notice that a measures precisely the difference of $\mathfrak{sl}(2,\mathbb{R})$-spin with $\mathfrak{su}(2)$-spin of every oscillator:

$$a = 2 \times (\mathfrak{su}(2)\text{-spin}) - 2 \times (\mathfrak{sl}(2,\mathbb{R})\text{-spin})\,. \tag{4.56}$$

Combining these observations, we find

$$j - \ell = j_0 - \ell_0 - 1 - \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i_n=1}^{N_n} a_{i_n}\,, \quad \bar{\jmath} - \bar{\ell} = j_0 - \ell_0 - 1 - \frac{1}{2} \sum_{n=-\infty}^{-1} \sum_{i_n=1}^{N_n} a_{i_n}\,, \tag{4.57}$$

where j, ℓ denote the $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ spins of the state $|\Psi\rangle$, respectively. Defining $\Delta = j + \bar{\jmath}$ and $L = \ell + \bar{\ell}$, and combining all the ingredients, we finally obtain

$$\begin{aligned} \Delta - L &= 2j_0 - 2\ell_0 - 2 - \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{i_n=1}^{N_n} a_{i_n} \\ &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{i_n=1}^{N_n} \sqrt{\frac{a_{i_n}^2}{4} - \frac{a_{i_n} k|n|}{2\ell_0} + \frac{n^2}{4\ell_0^2 f^4}} + \frac{1}{2\ell_0 f^2} \big(L_0^{\mathbb{T}^4} + \bar{L}_0^{\mathbb{T}^4} \big) + \mathcal{O}(k^{-1}) \\ &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{i_n=1}^{N_n} \sqrt{\frac{a_{i_n}^2}{4} - \frac{a_{i_n} k|n|}{L} + \frac{n^2}{L^2 f^4}} + \frac{1}{L f^2} \big(L_0^{\mathbb{T}^4} + \bar{L}_0^{\mathbb{T}^4} \big) + \mathcal{O}(k^{-1})\,. \end{aligned} \tag{4.58}$$

[34] In fact there are four such states.

Finally, we impose the remaining physical state conditions which have the effect of removing all oscillators with $a_{i_n} = 0$. This can be most easily seen by comparing the pure NS-NS case with the partition function derived in [61, 167]. The cohomological argument was given in [168]. Therefore the physical oscillators are the ones with $a_{i_n} = \pm 2$, and thus the physical spectrum reads

$$\Delta - L = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} N_n \sqrt{1 \pm \frac{2kn}{L} + \frac{n^2}{L^2 f^4}} + \frac{1}{Lf^2}\left(L_0^{\mathbb{T}^4} + \bar{L}_0^{\mathbb{T}^4}\right) + \mathcal{O}\left(k^{-1}\right) , \quad (4.59)$$

which matches the BMN formula (4.50). This concludes the derivation of the BMN formula from the worldsheet.

One can furthermore confirm that the analysis holds also true in the spectrally flowed sectors. For this, we choose $w \equiv w^{\mathfrak{sl}(2,\mathbb{R})} = -\bar{w}^{\mathfrak{sl}(2,\mathbb{R})} = w^{\mathfrak{su}(2)} = -\bar{w}^{\mathfrak{su}(2)}$.[35] We spectrally flow the ground state on top of which we build the spectrum, e.g. the state in the $\mathfrak{psu}(1,1|2)$-multiplet with quantum numbers $(j_0 - 1, \ell_0)$. From (4.33), we conclude that on this state

$$\hat{L}_0 \left|\Phi\right\rangle = f^2(-\hat{j}_0(\hat{j}_0 - 1) + \hat{\ell}_0(\hat{\ell}_0 + 1)) \left|\Phi\right\rangle = \tfrac{1}{2} f^2 \mathcal{C}\left(\hat{\mathcal{R}}_0\right) \left|\Phi\right\rangle , \quad (4.60)$$

where $\hat{j}_0 = j_0 + \frac{kw}{2}$ and $\hat{\ell}_0 = \ell_0 + \frac{kw}{2}$ are the spectrally flowed spins of the ground state, see eq. (4.32a)–(4.32e). After this, we can apply hatted oscillators on this spectrally flowed ground state to generate a state in this new representation. Since the spectral flow is an automorphism of the spectrum-generating algebra, the derivation is from hereon exactly the same as before, except that everything is replaced by spectrally flowed quantities. We obtain precisely (4.58), except that all quantities are now spectrally flowed. So we conclude that (4.58) continues to hold true in the spectrally flowed sectors.

4.5.3 The case of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$

We can similarly treat the background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$, which has recently attracted considerable attention [65–68, 140, 169, 170].

Currently, there is no hybrid formalism à la Berkovits, Vafa and Witten for this background and for mixed flux.[36] Nevertheless we expect that in the BMN limit, in analogy with $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, the theory can be described by a sigma model on $\mathrm{D}(2,1;\alpha)$ together with the theory on S^1 and free ghosts. The bosonic part of the Lie supergroup $\mathrm{D}(2,1;\alpha)$ is $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3$, with the parameter α giving the ratio of the radii of the two spheres (for more details,

[35]The sign for the barred spectral flow parameters seem peculiar, but this is again related to our mode conventions for the barred modes.

[36]For the pure NS-NS flux case, we develop a hybrid formalism in Chapter 7.

see Appendix A.1.3). Representations are now labelled by the three spins $(j_0, \ell_0^+, \ell_0^-)$. The Casimir of such a representation is given by

$$\mathcal{C}(j_0, \ell_0^+, \ell_0^-) = -2j_0(j_0-1) + 2\cos^2\varphi\, \ell_0^+(\ell_0^+ + 1) + 2\sin^2\varphi\, \ell_0^-(\ell^- + 1)\ , \tag{4.61}$$

where we introduced the angle $0 \le \varphi \le \frac{\pi}{2}$ such that $\alpha = \cot^2\varphi$.[37] In the limit we are taking, we set

$$\ell_0^+ = \frac{\cos\omega}{\cos\varphi}\ell_0 \quad \ell_0^- = \frac{\sin\omega}{\sin\varphi}\ell_0\ , \tag{4.62}$$

where $0 \le \omega \le \frac{\pi}{2}$ is another angle parametrising the ratio of the spins as these are taken to infinity. Then as before we have $j_0 = \ell_0 + \mathcal{O}(1)$, and hence again $a \cdot \ell$ can be replaced with $a\ell_0$, where a takes the following values for the different elements of the superalgebra $\mathfrak{d}(2,1;\alpha)$:

$$K^{(+)\pm} :\ \pm 2\cos(\varphi)\cos(\omega)\ , \quad K^{(-)\pm} :\ \pm 2\sin(\varphi)\sin(\omega)\ , \quad J^{\pm} :\ \mp 2\ , \tag{4.63a}$$

$$S^{\pm\pm\pm} :\ \mp 2\sin^2\left(\frac{\varphi-\omega}{2}\right), \quad S^{\pm\mp\pm} :\ \mp 2\cos^2\left(\frac{\varphi+\omega}{2}\right), \tag{4.63b}$$

$$S^{\pm\pm\mp} :\ \mp 2\sin^2\left(\frac{\varphi+\omega}{2}\right), \quad S^{\pm\mp\mp} :\ \mp 2\cos^2\left(\frac{\varphi-\omega}{2}\right). \tag{4.63c}$$

All three Cartan-generators J^3 and $K^{(\pm)3}$ have $a = 0$. Solving the mass-shell condition for j_0 as in (4.55) yields

$$j_0 = \ell_0 + \cos^2\left(\frac{\varphi-\omega}{2}\right) + \frac{1}{4}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\sum_{i_n=1}^{N_n}\left(a_{i_n} + \sqrt{a_{i_n}^2 - \frac{2a_{i_n}k|n|}{\ell_0} + \frac{n^2}{\ell_0^2 f^4}}\right) + \frac{1}{4\ell_0 f^2}\left(L_0^{S^1} + \bar{L}_0^{S^1}\right) + \mathcal{O}(k^{-1})\ , \tag{4.64}$$

where we again used the mass-shell condition to simplify the result. We recognise that in this case, a measures

$$a = 2\cos(\varphi)\cos(\omega) \times (\mathfrak{su}(2)^+\text{-spin}) + 2\sin(\varphi)\sin(\omega) \times (\mathfrak{su}(2)^-\text{-spin}) - 2 \times (\mathfrak{sl}(2,\mathbb{R})\text{-spin})\ . \tag{4.65}$$

Defining $\ell = \cos(\varphi)\cos(\omega)\ell^+ + \sin(\varphi)\sin(\omega)\ell^-$ and $L = \ell + \bar{\ell}$, $\Delta = j + \bar{\jmath}$, the same steps as before yield the final result

$$\Delta - L = -\sin^2\left(\frac{\varphi-\omega}{2}\right) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\sum_{i_n=1}^{N_n}\sqrt{\frac{a_{i_n}^2}{4} - \frac{a_{i_n}k|n|}{L} + \frac{n^2}{L^2 f^4}}$$

[37] We thank Andrea Dei for bringing this parametrisation to our attention.

$$+\frac{1}{Lf^2}\big(L_0^{S^1}+\bar{L}_0^{S^1}\big)+\mathcal{O}\big(k^{-1}\big)\ , \quad (4.66)$$

which formally coincides with (4.58), except for the constant squared sine. As before, in order to choose the state in a typical $\mathfrak{d}(2,1;\alpha)$ multiplet with smallest $\Delta - L$, a constant term 1 was included in the relations between $j-\ell$ and $j_0-\ell_0$.

Notice that the BPS condition for $\mathfrak{d}(2,1;\alpha)$ takes the form (A.9)

$$\Delta_{\rm BPS}=\cos^2(\varphi)L^+ +\sin^2(\varphi)L^- =\cos(\varphi-\omega)L\ , \quad (4.67)$$

and so

$$\Delta-\Delta_{\rm BPS}=(2L-1)\sin^2\Big(\frac{\varphi-\omega}{2}\Big)+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\sum_{i_n=1}^{N_n}\sqrt{\frac{a_{i_n}^2}{4}-\frac{a_{i_n}k|n|}{L}+\frac{n^2}{L^2f^4}}$$
$$+\frac{1}{Lf^2}\big(L_0^{S^1}+\bar{L}_0^{S^1}\big)+\mathcal{O}\big(k^{-1}\big)\ . \quad (4.68)$$

In particular, since all terms on the right-hand side are positive,[38] we see that this is only a near-BPS expansion if $\varphi=\omega$, i.e. $L^+=L^-$. Hence, all BPS states on the background $AdS_3\times S^3\times S^3\times S^1$ have (in the large k limit) $\ell^+=\ell^-$. This was recently shown in [67,68,169], our calculation confirms the result again. The squared sine of (4.66) then vanishes for near-BPS states. After discussing the BMN-limit of the model, we turn towards a different regime. We

4.6 The spectrum of the sigma-model

In general we would like to determine the conformal weight of states obtained by the action of normal-ordered products on a primary state $|\Phi\rangle$, such as for example $Q^a_n|\Phi\rangle$, $(Q^aP^b)_n|\Phi\rangle$, $(Q^a\bar{Q}^{\bar{a}})_n|\Phi\rangle$, $(A^{a\bar{a}})_n|\Phi\rangle$, and others. In the following we will be able to compute the conformal weight of a state containing either solely unbarred oscillators or solely barred oscillators, and no $A^{a\bar{a}}_m$. The reason for this is that L_0 mixes only finitely many states constructed using solely unbarred oscillators, say, and in this way its eigenvalues can be computed. In this case, we will be able to make use of the definition of affine primary states (4.26a) associated with the 'chiral' lowest weight representations introduced in the previous section. When including also barred oscillators or the $A^{a\bar{a}}$-field, infinitely many states get mixed under the action of L_0, and its eigenvalues cannot be extracted with a finite amount of calculation. This is a difficulty which we have not been able to overcome.

[38] Since $L\gg 1$, $2L-1$ is also positive.

We are then interested in the conformal weights of the single-sided states of the type

$$\prod_{n=1}^{\infty}\left(\prod_{i_n=1}^{N_n} Q^{a_{i_n}}_{-n}\prod_{i_n=1}^{M_n} P^{b_{i_n}}_{-n}\right)|\Phi\rangle \quad \text{or} \quad \prod_{n=1}^{\infty}\left(\prod_{i_n=1}^{N_n} \bar{Q}^{\bar{a}_{i_n}}_{n}\prod_{i_n=1}^{M_n} \bar{P}^{\bar{b}_{i_n}}_{n}\right)|\Phi\rangle \ . \tag{4.69}$$

For simplicity, in the following we illustrate the computation of the conformal weights of such states using single-oscillator excitations, i.e. using states of the type

$$Q^a_{-n}|\Phi\rangle \ , \quad P^a_{-n}|\Phi\rangle \ . \tag{4.70}$$

The multi-oscillator states can be treated using the same methods, but we have not managed to find a closed form solution. Nevertheless, we will be able to derive strong results concerning the expected qualitative behaviour of the spectrum described in Section 3.3 using only (4.70). In particular, in Subsection 4.6.3 we will derive a unitarity bound on the values that kf^2 can take, in Subsection 4.6.4 we will argue that the continuous representations cannot be part of the CFT spectrum, and in Subsection 4.6.5 we will retrieve the chiral primaries that are missing at the pure NS-NS point.

4.6.1 The spectrum at the first level

The states in the spectrum at the first level are

$$Q^a_{-1}|\Phi\rangle \ , \quad P^a_{-1}|\Phi\rangle \ . \tag{4.71}$$

They mix under the application of L_0 as follows:

$$L_0 Q^a_{-1}|\Phi\rangle = \left(h(\Phi)+\frac{1}{2}(1+kf^2)Q^a_{-1}+\frac{1-k^2f^4}{4kf^2}P^a_{-1}\right)|\Phi\rangle \ , \tag{4.72}$$

$$\begin{aligned} L_0 P^a_{-1}|\Phi\rangle &= \left(h(\Phi)+kf^2Q^a_{-1}+\frac{1}{2}(1-kf^2)P^a_{-1}-if^2 f^a{}_{bc}\left(Q^bP^c\right)_{-1}\right)|\Phi\rangle \\ &= \left(h(\Phi)+kf^2Q^a_{-1}+\frac{1}{2}(1-kf^2)P^a_{-1}-if^2 f^a{}_{bc}P^c_{-1}t^b\right)|\Phi\rangle \ . \end{aligned} \tag{4.73}$$

We have used the definition of affine primary (4.26a) and the commutation relations (4.25a)–(4.25d). Note that the structure constants $if^{bc}{}_a=(t^b_{\mathbf{ad}})^a{}_c$ are the generators in the adjoint representation and hence

$$if^a{}_{bc}t^b=-\kappa_{bd}(t^d_{\mathbf{ad}})^a{}_ct^c \ . \tag{4.74}$$

This can expresses as a difference of Casimirs:

$$\kappa_{bd}t^b_{\mathbf{ad}}t^d=\frac{1}{2}\Big(\kappa_{bd}(t^b_{\mathbf{ad}}+t^b)(t^d_{\mathbf{ad}}+t^d)-\kappa_{bd}t^b_{\mathbf{ad}}t^d_{\mathbf{ad}}-\kappa_{bd}t^bt^d\Big)$$

$$= \frac{1}{2}\Big(\mathcal{C}\big(\mathcal{R}_0 \otimes \mathbf{ad}\big) - \mathcal{C}\big(\mathcal{R}_0\big)\Big) \,, \tag{4.75}$$

where we have used that the Casimir of the adjoint representation vanishes $\mathcal{C}\big(\mathbf{ad}\big) = 0$. Note that the states (4.71) transform in the (reducible) representation $\mathcal{R}_0 \otimes \mathbf{ad}$. Restricting to an irreducible subrepresentation $\mathcal{R}_1 \subset \mathcal{R}_0 \otimes \mathbf{ad}$ we find

$$if^2 f^a{}_{bc}\, t^b = -\frac{1}{2} f^2 \Big(\mathcal{C}(\mathcal{R}_1) - \mathcal{C}(\mathcal{R}_0)\Big)\delta^a{}_c = -\frac{1}{2} f^2 \Delta\mathcal{C}\delta^a{}_c \,, \tag{4.76}$$

where we denoted by $\Delta\mathcal{C}$ the difference of Casimirs.[39] Thus L_0 mixes only $Q^a_{-1}\,|\Phi\rangle$ and $P^a_{-1}\,|\Phi\rangle$, and in this basis L_0 takes the form

$$L_0 = h(|\Phi\rangle)\mathbb{1} + \begin{pmatrix} \frac{1}{2}(1+kf^2) & kf^2 \\ \frac{1-k^2f^4}{4kf^2} & \frac{1}{2}(1-kf^2) + \frac{1}{2}f^2\Delta\mathcal{C} \end{pmatrix} \,, \tag{4.77}$$

where we used (4.28). The associated eigenvalues are

$$\begin{aligned} h_\pm\big(Q^a_{-1}\,|\Phi\rangle\,,\, P^a_{-1}\,|\Phi\rangle\,\big) &= h\big(\,|\Phi\rangle\,\big) \\ &\quad + \frac{1}{4}\Big(f^2\Delta\mathcal{C} + 2 \pm \sqrt{4 - 4kf^4\Delta\mathcal{C} + f^4(\Delta\mathcal{C})^2}\Big) \,. \end{aligned} \tag{4.78}$$

Notice that this result is similar to the large-charge formula found above (4.45), except that $2(a \cdot \ell)$ has been replaced by $\Delta\mathcal{C}$. It is easy to confirm that in the large-charge limit $\Delta\mathcal{C}$ indeed becomes $2(a \cdot \ell)$, and the exact formula (4.78) is therefore consistent with the one found in the large-charge limit. Furthermore, it was argued above that only the solution h_+ is physical. In fact, due to the identifications between the modes of the algebra, the solution h_- can be interpreted as the application of a barred oscillator with the wrong mode number. On the other hand, only the solution h_+ reduces to the correct result $h_+ = h(|\Phi\rangle) + 1$ at the WZW-point $kf^2 = 1$. Hence we will discard the state with eigenvalue h_- from the physical spectrum. It is not clear at this point if this should be the only effect of the physical constraints on the one-sided worldsheet spectrum.

4.6.2 The spectrum at the n-th level

Now we generalise the analysis to level n excitations of the form

$$Q^a_{-n}\,|\Phi\rangle \ , \quad P^a_{-n}\,|\Phi\rangle \ . \tag{4.79}$$

As we will see, under the action of L_0 these states mix with multi-oscillator states such as $f^a{}_{bc}\, Q^b_{-n+1} P^c_{-1}\,|\Phi\rangle$. However, L_0 behaves as follows: under the

[39] The pertinence of the difference of Casimirs to the computation of conformal weights was already noticed in [160].

action of L_0 the number of oscillators either increases or stays the same, but never decreases.

To prove this assertion, we start with a state of the form

$$g^a{}_{b_1\cdots b_m} J^{b_1}_{-n_1} \cdots J^{b_m}_{-n_m} |\Phi\rangle \ . \tag{4.80}$$

Here $g^a{}_{b_1\cdots b_m}$ is an invariant tensor of $\mathfrak{g}$ of the form

$$g^a{}_{b_1\cdots b_m} = f^a{}_{b_1 a_1} f^{a_1}{}_{b_2 a_2} \cdots f^{a_{m-2}}{}_{b_{m-1} b_m} \ , \tag{4.81}$$

up to possible permutations of the free indices. In the expression (4.80), each $J^{b_i}_{-n_i}$ can stand either for $Q^{b_i}_{-n_i}$ or $P^{b_i}_{-n_i}$. Moreover, we require that the state is at level n,

$$\sum_{i=1}^{m} n_i = n \ . \tag{4.82}$$

The invariant tensor (4.81) has the property

$$g^a{}_{b_1\cdots b_m} \kappa^{b_i b_j} = 0 \ , \quad g^a{}_{b_1\cdots b_m} f^{b_i b_j}{}_c = 0 \ , \tag{4.83}$$

thanks to the vanishing of all Casimirs of the adjoint representation, see [151]. This implies that normal ordering in (4.80) is not relevant: the oscillators can freely be reordered, since the commutator produces structure constants. They vanish because of the second relation in (4.83). We compute L_0 on the state (4.80). There will be two types of terms appearing, corresponding to the two types of terms in the commutation relations (4.25a) and (4.25b). The first type of terms are linear in the modes and obviously preserve the number of modes. The second type of terms yields the following expression:

$$g^a{}_{b_1\cdots b_m} f^{b_i}{}_{cd} J^{b_1}_{-n_1} \cdots J^{b_{i-1}}_{-n_{i-1}} (Q^c P^d)_{-n_i} J^{b_{i+1}}_{-n_{i+1}} \cdots J^{b_m}_{-n_m} |\Phi\rangle \ . \tag{4.84}$$

The invariant tensor $g^a{}_{b_1\cdots b_m} f^{b_i}{}_{cd}$ still has the same property as (4.83), so we may still freely reorder the oscillators. In the normal-ordered product term $(Q^c P^d)_{-n_i}$ in (4.84), either both oscillators have negative modes or one is a zero-mode (a term with positive mode vanishes, since we can commute it through to the right, where it then annihilates $|\Phi\rangle$). In the former case, we obtain a term with $m+1$ oscillators, whereas in the latter case, the zero mode on $|\Phi\rangle$ gives a generator t^c or t^d and hence the number of oscillators remains the same. Also, we note that the action of L_0 closes on the set (4.80), we do not have to consider other invariant tensors. This proves the above assertion that the number of oscillators can never be decreased by the action of L_0.

When computing the matrix-representation of L_0 on all level n states which can be mixed by the action of L_0, we hence get the following block structure:

$$\begin{matrix} \text{1 oscillator} \\ \text{2 oscillators} \\ \text{3 oscillators} \\ \vdots \\ n-1 \text{ oscillators} \\ n \text{ oscillators} \end{matrix} \quad \begin{pmatrix} \star & 0 & 0 & \cdots & 0 & 0 & 0 \\ \star & \star & 0 & \cdots & 0 & 0 & 0 \\ 0 & \star & \star & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \star & \star & 0 \\ 0 & 0 & 0 & \cdots & 0 & \star & \star \end{pmatrix} . \tag{4.85}$$

Thus for the purpose of computing the spectrum of L_0 on single-oscillator excitations, we can simply ignore multi-oscillator excitations, since they do not contribute to the eigenvalue. They do however contribute to the precise eigenvector.

With this at hand, the computation is completely analogous to the computation in (4.6.1): L_0 acts on $Q^a_{-n} |\Phi\rangle$ and $P^a_{-n} |\Phi\rangle$ as follows:

$$L_0 = h(|\Phi\rangle)\mathbb{1} + \begin{pmatrix} \frac{1}{2}(1+kf^2)n & kf^2 n \\ \frac{1-k^2f^4}{4kf^2} n & \frac{1}{2}(1-kf^2)n + \frac{1}{2}f^2\Delta\mathcal{C} \end{pmatrix} , \tag{4.86}$$

where we ignored all multi-oscillator terms. The correction to the eigenvalues with respect to the ground state is given by

$$\delta h_\pm\big(Q^a_{-n}|\Phi\rangle\,,\; P^a_{-n}|\Phi\rangle\big) = \frac{1}{4}\Big(f^2\Delta\mathcal{C} + 2n \pm \sqrt{4n^2 - 4kf^4 n\Delta\mathcal{C} + f^4(\Delta\mathcal{C})^2}\Big) . \tag{4.87}$$

We again expect only the positive sign eigenvalue to be part of the physical spectrum. This reduces again to the BMN-like limit (4.45) for large values of the charges. Furthermore, at the pure NS-NS point $kf^2 = 1$ we retrieve the WZW result.

This result makes it seem as if the structure is always so simple. However, once one tries to compute the conformal weight of multioscillator excitations, the computations become quickly very complicated.

4.6.3 A unitarity bound

There is one very interesting consequence of (4.78). Classically, we know from (3.36) that $-1 \leq kf^2 \leq 1$, and we will see that also holds at the quantum level, assuming that $k \geq 2$.[40] According to [149, eq. (6.3)], for $k \geq 2$

[40] The $k = 1$ theory behaves quite differently. Since $\mathfrak{su}(2)_1 \subset \mathfrak{psu}(1,1|2)_1$ has no affine representation based on the adjoint representation of $\mathfrak{su}(2)$, the theory cannot have a field in the adjoint representation. In particular, the biadjoint field $A^{a\bar{a}}$ does not transform in a valid representation of $\mathfrak{psu}(1,1|2)_1 \times \mathfrak{psu}(1,1|2)_1$ at the WZW-point. Hence it is not clear whether we can deform the model away from the WZW-point. The $k = 1$ theory at the WZW-point is discussed in [140, 171] and in Chapter 5.

the spectrum of the sigma-model on $\mathfrak{psu}(1,1|2)$ should contain the representation $\mathcal{R}_0 = \big(j, \ell = \frac{k}{2} - 1\big)$, where j is the $\mathfrak{sl}(2,\mathbb{R})$-spin and $\ell = \frac{k}{2} - 1$ the $\mathfrak{su}(2)$-spin, see also Appendix A.1.2 for the conventions of $\mathfrak{psu}(1,1|2)$. In this way, we can choose

$$\mathcal{R}_1 = \big(j, \tfrac{k}{2}\big) \subset \big(j, \tfrac{k}{2} - 1\big) \otimes \mathbf{ad} \ . \tag{4.88}$$

This choice of representations yields $\Delta\mathcal{C} = 2k$, and inserting this into (4.78) we obtain the following conformal weight of the excited state:

$$h = \frac{1}{2}\Big(kf^2 + 1 + \sqrt{1 - k^2 f^4}\Big) \ . \tag{4.89}$$

An obvious requirement of any CFT is that the conformal weights are real. We see that this is only the case provided that

$$-1 \leq kf^2 \leq 1 \ . \tag{4.90}$$

4.6.4 Continuous representations

We found that the conformal weight of states constructed with a single oscillator depend on the difference of Casimirs $\Delta\mathcal{C}$ between the ground state representation and the representation of the state. Consider then a ground state representation with $\mathfrak{su}(2)$-spin ℓ and $\mathfrak{sl}(2,\mathbb{R})$ spin $j = \frac{1}{2} + ip$, i.e. the $\mathfrak{sl}(2,\mathbb{R})$ part transforms in a continuous representation. Its Casimir is then $\mathcal{C} = -2j(j-1) + 2\ell(\ell+1) = \frac{1}{2} + 2p^2 + 2\ell(\ell+1)$. At first excitation level, we have states in the representations with spin $j-1$, j and $j+1$ of $\mathfrak{sl}(2,\mathbb{R})$ appearing. The respective differences of Casimirs are

$$\Delta\mathcal{C} = 2 - 4ip \ , \quad 0 \ , \quad \text{and} \quad 2 + 4ip \ . \tag{4.91}$$

Plugging this result into the formula for the conformal weight at level one (4.78), we realise that the conformal weights for $p \neq 0$ generated by charged oscillators become generically complex.

Since the appearance of complex conformal weights implies that the energy momentum tensor is not self-adjoint in these representations, these representations are forbidden and hence cannot be part of the spectrum. The only exception to this statement is the WZW-point, where the conformal dimensions do not depend explicitly on the difference of Casimirs $\Delta\mathcal{C}$. This result should continue to hold once we consider complete representations of the mode algebra, and not just of its 'chiral' version. Since already the 'chiral' continuous representations contain complex conformal weights, the full representations must be ruled out. Hence we confirm the fact that long strings disappear from the spectrum in a mixed-flux background.

4.6.5 Missing chiral primaries

We are also in the position to retrieve the chiral primaries that are missing from the spectrum at the WZW-point. In the following we review this phenomenon in the worldsheet description. For simplicity, we focus on the background $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$. The $\mathfrak{psu}(1,1|2)_k$ WZW-model has [149] discrete representations (j, ℓ, w) with $\frac{1}{2} < j < \frac{k+1}{2}$ and $\ell \in \{0, \frac{1}{2}, \ldots, \frac{k-2}{2}\}$, where $w \in \mathbb{Z}$ is the spectral flow number. Every discrete representation of the form $(\ell+1, \ell, w)$ yields four chiral primary states [50, 53, 68, 172]. These are the four $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$ representations in (A.6) for which $j - \ell$ is minimal.[41] They have the followings $\mathfrak{su}(2)$-spins:

$$\big(\ell + kw\big) \ , \quad 2 \times \big(\ell + \tfrac{1}{2} + kw\big) \ , \quad \big(\ell + 1 + kw\big) \ . \tag{4.92}$$

Combining with the right-movers, we obtain the complete Hodge-diamond of $\mathbb{T}^4$, with the lowest state having left- and right-moving $\mathfrak{su}(2)$ spin $\ell + kw$. It has the following form:

$$\begin{array}{ccccc} & & (1,1) & & \\ & 2\times(\frac{1}{2},1) & & 2\times(1,\frac{1}{2}) & \\ (0,1) & & 4\times(\frac{1}{2},\frac{1}{2}) & & (1,0) \\ & 2\times(0,\frac{1}{2}) & & 2\times(\frac{1}{2},0) & \\ & & (0,0) & & \end{array} \tag{4.93}$$

where $(\delta, \bar{\delta})$ denotes an $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ representation with spin $(\ell + kw + \delta, \ell + kw + \bar{\delta})$. Note that because of the restriction $\ell \in \{0, \frac{1}{2}, \ldots, \frac{k-2}{2}\}$, $\ell + kw$ takes values in $\frac{1}{2}\mathbb{Z} \setminus \big(\frac{k}{2}\mathbb{Z} - \frac{1}{2}\big)$ and thus every k-th Hodge diamond is missing. This was alluded to in Section 3.3.2 and is what we mean by 'missing chiral primary'.

The absence of the chiral primaries on the worldsheet is caused by the unitarity bounds constraining the worldsheet theory. The main bounds are the restriction to $j < \frac{k+1}{2}$ and $\ell \leq \frac{k-2}{2}$, whose origin we will briefly review in the following. We will only treat the unflowed sector $w = 0$. Consider the state $J^-_{-1} S_0^{-++} |j, \ell\rangle$, whose norm at the WZW-point is (see Appendix A.1.2):

$$\langle j, \ell | S_0^{+--} J_1^+ J_{-1}^- S_0^{-++} | j, \ell \rangle = \big(-2(j - \tfrac{1}{2}) + k \big) \langle j - \tfrac{1}{2}, \ell | j - \tfrac{1}{2}, \ell \rangle \ . \tag{4.94}$$

This norm is non-negative if

$$j \leq \frac{k+1}{2} \ , \tag{4.95}$$

[41] In fact, these representations saturate the $\mathfrak{psu}(1,1|2)$ BPS bound and are therefore atypical representations. Thus the representation splits up into four atypical representations, each of which yielding one BPS state.

which is the Maldacena-Ooguri bound. For this to be a unitarity restriction, the state $J^-_{-1}S_0^{-++}|j\rangle$ has to be physical in string theory, which is in fact the case. This can be seen from the fact that there is no state at level zero with the same quantum numbers.[42] Hence all positive modes of uncharged operators have to annihilate the state and so it lies in particular in the BRST-cohomology of physical states. This is then the most stringent bound possible. In the RNS formalism, it arises from considering the no-ghost theorem in the R-sector [87].

Similarly, the unitarity constraint for $\mathfrak{su}(2)$ representations can be obtained by requiring the norm of the state $K^+_{-1}S_0^{-++}S_0^{+++}|j,\ell\rangle$ to be non-negative. This yields

$$\ell \le \frac{k-2}{2}\ , \tag{4.96}$$

which is the familiar bound from the RNS formalism. The considered state is again physical. These are the bounds we mentioned above.

Let us move away from the WZW-point and see how these bounds change. For this we first find the eigenvectors of L_0 at the first level, which are $(Q^a_{-1} + b_\pm P^a_{-1})\,|\Phi\rangle$, where

$$b_\pm = \frac{\Delta\mathcal{C}f^2 - 2kf^2 \pm \sqrt{4 - 4\Delta\mathcal{C}kf^4 + (\Delta\mathcal{C})^2 f^4}}{4kf^2}\ . \tag{4.97}$$

These have L_0 eigenvalues $h_\pm$ as in (4.78), respectively. As noted before, only the state with conformal weight h_+ is part of the physical spectrum. The analogue of the state $J^-_{-1}S_0^{-++}\,|j,\ell\rangle$ in the mixed flux case is

$$(J^{Q,-}_{-1} + b_\pm J^{P,-}_{-1})S_0^{Q,-++}\,|j,\ell\rangle\ , \tag{4.98}$$

where we use the notation J^Q for the J-currents of the Q-modes and J^P for the J-currents of the P-modes. Using the algebra (4.22a)–(4.22d) and the explicit form of $b_\pm$ with $\Delta\mathcal{C} = 4j-4$, the norm of this state can be computed. Requiring this norm to be non-negative gives the constraint

$$j \le \frac{k+1}{2} + \frac{1}{2} - \frac{\sqrt{f^4 + k^2 f^4 - 1}}{2f^2} < \frac{k+2}{2}\ , \tag{4.99}$$

which is less constraining than the usual bound (4.95), and reduces to it at $kf^2 = 1$. We see that the bound changes slightly when going away from the WZW-point, but nothing spectacular happens.

[42] This would not be true for the state $J^-_{-1}|j\rangle$, since at level zero there is a state with the same quantum numbers, namely $S_0^{-++}S_0^{--+}|j\rangle$.

The situation is entirely different when looking at the corresponding state for the $\mathfrak{su}(2)$-spin bound

$$|\Psi\rangle \equiv (K^{Q,+}_{-1} + b_\pm K^{P,+}_{-1}) S^{Q,-++}_0 S^{Q,+++}_0 |j,\ell\rangle \ . \tag{4.100}$$

Asking for $|\Psi\rangle$ to have positive norm led at the WZW-point to the constraint $\ell \leq \frac{k-2}{2}$, which in turn excluded the missing chiral primary at $\ell = \frac{k-1}{2}$ from the spectrum. Now we find that the norm of this state is in general[43]

$$\begin{aligned}\langle\Psi|\Psi\rangle &= \pm\sqrt{f^{-4} - 4(\ell+1)(k-\ell-1)} \\ &\xrightarrow{\text{WZW-point}} \pm\sqrt{\big(k-2\ell-2\big)^2} = \pm\big(k-2\ell-2\big) \ .\end{aligned} \tag{4.101}$$

As indicated, the term under the square root becomes a perfect square at the WZW-point. From this description it is not clear which sign should be chosen in the last equality, but from the WZW-description we know that we should take the positive sign. The two branches for the norm of are plotted in Figure 4.1. We see that away from the WZW-point, the two branches no longer cross. In particular, the first branch has always positive norm and there is no unitarity bound on ℓ!

In summary, away from the pure NS-NS point we found that the upper bound on j is slightly shifted upwards, but always strictly less than $\frac{k+2}{2}$. On the other hand, the bound on ℓ completely disappears. This has the following consequences for the chiral primaries. As we discussed above, chiral primaries come from representations with $\ell = j-1 \in \frac{1}{2}\mathbb{N}_0$. While there is no longer an upper bound on ℓ, there is such a bound on j, which now allows the values $\ell \in \{0, \frac{1}{2}, \ldots, \frac{k-1}{2}\}$. Thus, we see that there is one new chiral primary compared to the WZW-point, namely $\ell = \frac{k-1}{2}$. Combining this with spectral flow, it precisely fills the gaps (3.45) in the BPS spectrum. We conclude that the missing chiral primaries are indeed reinstated by any perturbation away from the WZW-point.

4.7 Summary and Conclusion

In this Chapter, we started to explore the moduli space of the compactifications $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{K3}$ from the string side. We analysed the string theory in a mixed flux background in detail, which is described by a sigma model on the supergroup $\text{PSU}(1,1|2)$. The theory is governed by a non-holomorphic current algebra, which extends the affine symmetry $\mathfrak{psu}(1,1|2)_k \times \mathfrak{psu}(1,1|2)_k$, which is present at the pure NS-NS flux point. We used this symmetry to organise the spectrum and succeeded to use it to

[43]Notice the state with conformal weight h_- has negative norm and is therefore unphysical, as argued before.

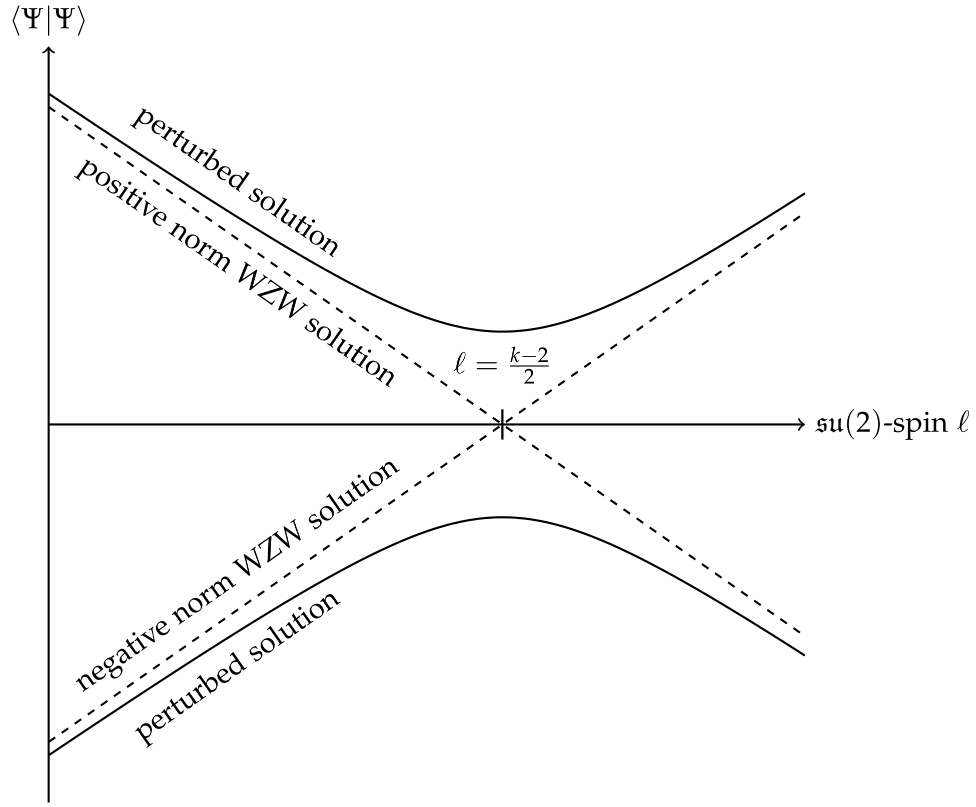

Figure 4.1: The two branches of the norm of $|\Psi\rangle$. At the WZW-point, the two branches intersect at $\ell = \frac{k-2}{2}$. For a slight perturbation away from the WZW-point, we have an 'avoided crossing' and the first branch has always positive norm.

compute the string spectrum in the plane-wave limit, thus reproducing the spectrum derived in [161]. Our results provide a direct link between Green-Schwarz-like computations and worldsheet methods to determine the spectrum. The tools developed in here seem much more powerful than necessary to derive the plane-wave spectrum. We can in principle derive the exact conformal weights of arbitrary one-sided excitations (i.e. constructed using only unbarred modes), at least level by level, as discussed in Section 4.6. With the help of this, we can confirm some well-known conjectures explicitly, such as the fact that long strings disappear from the string spectrum away from the WZW-point. We can also retrieve the missing chiral primaries in the spacetime BPS spectrum [53, 167, 173].

We have presented in Section 4.1 an exact one-excitation solution of the classical theory. One can hope to extend this result to multi-particle excitations by employing integrability methods [174–176]. In particular, the presented solution corresponds to a one-cut solution of the spectral curve. In principle, the spectral curve can be used to extend the result to multi-cut solutions. This would provide a way to compute the spectrum of string theory beyond the plane-wave limit.

We expect that the analysis can be extended to other backgrounds like $AdS_5 \times S^5$, $AdS_4 \times \mathbb{CP}^3$ and $AdS_2 \times S^2 \times \mathbb{T}^6$, where similar supergroup actions exist [177–181]. They feature the supergroups $PSU(2,2|4)$, $OSP(6|2,2)$ and $PSU(1,1|2)$, which all have vanishing dual Coxeter numbers. However, the backgrounds require us to consider cosets of these supergroups, so one should effectively consider a coset of the current algebra considered in Chapter 4. We expect that this can be worked out, but have not tried to do so.

The dispersion relation obtained using integrability methods [182], in the decompactification limit, contains a term which is linear in the mode number and a squared sine term. In [183] the comparison with the giant magnon solution suggested a transcendental analytic structure of the string spectrum. On the other hand, our conformal weights arise through diagonalisation, so the current algebra approach we have presented can only produce an algebraic structure for the spectrum. In particular, we cannot reproduce the giant magnon solution. We believe this is not a contradiction: the giant magnon solution is not physical, since it is not level-matched, and likewise we can so far only compute non level-matched conformal weights, as mentioned above. So there is no a priori reasons for the two formulas to agree. It would be very interesting to establish a connection between the two approaches.

Chapter 5

The symmetric product from the worldsheet

In this chapter, we focus on the $AdS_3 \times S^3 \times \mathbb{T}^4$, where we have much more tools to analyse the worldsheet theory at our disposal. We can solve the worldsheet theory completely. A particularly interesting point occurs when the background is supported by exactly one unit of NS-NS flux. This point in moduli space is not directly accessible in the RNS-formalism. We show that it is the point, where the string theory becomes precisely dual to the symmetric product orbifold.

5.1 Representations of $\mathfrak{psu}(1,1|2)$

For the following, it will be important to study various aspects of the representation theory of $\mathfrak{psu}(1,1|2)_k$. We will denote the generators of the bosonic affine subalgebras $\mathfrak{sl}(2,\mathbb{R})_k$ and $\mathfrak{su}(2)_k$ by J^a_m and K^a_m, respectively, while the fermionic generators are labelled by $S^{\alpha\beta\gamma}_m$. Our conventions for the (anti)commutators of $\mathfrak{psu}(1,1|2)_k$ are given in Appendix A.1.2.

5.1.1 Long representations of $\mathfrak{psu}(1,1|2)$

Next we describe the representations of $\mathfrak{psu}(1,1|2)$. We first consider the long (typical) representations, which come in the form of continuous and discrete representations for the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra. Let us concentrate on the continuous case, since the discrete representations arise as subrepresentations.

The eight supercharges of $\mathfrak{psu}(1,1|2)$, i.e. the generators $S^{\alpha\beta\gamma}_0$, generate a Clifford module. We can find a highest-weight state of the supercharges which is annihilated by half of them. Let us assume that the highest weight state transforms in the representation $(\mathcal{C}^j_\lambda, \mathbf{n})$ with respect to the bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$, where $\mathbf{n}$ here refers to the dimensionality of the $\mathfrak{su}(2)$-representation. The supercharges transform in the bispinor representation $2 \cdot (\mathbf{2},\mathbf{2})$ of the bosonic subalgebra. Thus we conclude that a typical multiplet takes the form:

$$\begin{array}{ccccc} & & (\mathcal{C}^{j}_{\lambda},\mathbf{n}) & & \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{n}+\mathbf{1}) & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{n}-\mathbf{1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{n}+\mathbf{1}) & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{n}-\mathbf{1}) \\ (\mathcal{C}^{j+1}_{\lambda},\mathbf{n}) \quad (\mathcal{C}^{j}_{\lambda},\mathbf{n}+\mathbf{2}) & & 2\cdot(\mathcal{C}^{j}_{\lambda},\mathbf{n}) & & (\mathcal{C}^{j}_{\lambda},\mathbf{n}-\mathbf{2}) \quad (\mathcal{C}^{j-1}_{\lambda},\mathbf{n}) \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{n}+\mathbf{1}) & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{n}-\mathbf{1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{n}+\mathbf{1}) & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{n}-\mathbf{1}) \\ & & (\mathcal{C}^{j}_{\lambda},\mathbf{n}) & & \end{array} \tag{5.1}$$

Here, the top state is the highest weight state of the Clifford module, and the action of the supercharges moves between the different bosonic representations.

For the important cases of $\mathbf{n} = \mathbf{1}$ and $\mathbf{n} = \mathbf{2}$ some shortenings occur. For $\mathbf{n} = \mathbf{2}$, the representation involving $\mathbf{n} - \mathbf{2}$ is absent, i.e.

$$\begin{array}{ccccc} & & (\mathcal{C}^{j}_{\lambda},\mathbf{2}) & & \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{3}) & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{3}) & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{1}) \\ (\mathcal{C}^{j+1}_{\lambda},\mathbf{2}) \quad (\mathcal{C}^{j}_{\lambda},\mathbf{4}) & & 2\cdot(\mathcal{C}^{j}_{\lambda},\mathbf{2}) & & (\mathcal{C}^{j-1}_{\lambda},\mathbf{2}) \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{3}) & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{3}) & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{1}) \\ & & (\mathcal{C}^{j}_{\lambda},\mathbf{2}) & & \end{array} \tag{5.2}$$

while for $\mathbf{n} = \mathbf{1}$ even more representations are missing,

$$\begin{array}{ccccc} & & (\mathcal{C}^{j}_{\lambda},\mathbf{1}) & & \\ & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{2}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{2}) & \\ (\mathcal{C}^{j+1}_{\lambda},\mathbf{1}) & (\mathcal{C}^{j}_{\lambda},\mathbf{3}) & & (\mathcal{C}^{j}_{\lambda},\mathbf{1}) & (\mathcal{C}^{j-1}_{\lambda},\mathbf{1}) \\ & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{2}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{2}) & \\ & & (\mathcal{C}^{j}_{\lambda},\mathbf{1}) & & \end{array} \tag{5.3}$$

All of these representations exist also in the discrete version; they can be obtained by replacing the continuous by the corresponding discrete representations,

$$\mathcal{C}^{j}_{\lambda} \longrightarrow \mathcal{D}^{j}_{\pm} \, . \tag{5.4}$$

5.1.2 Short representations of $\mathfrak{psu}(1,1|2)$

Below we will be interested in the affine algebra of $\mathfrak{psu}(1,1|2)_k$ at level $k = 1$. Then the $\mathfrak{su}(2)_k$ factor also has level $k = 1$, and as a consequence, the affine highest weight states are only allowed to transform in the $\mathbf{n} = \mathbf{1}$ and $\mathbf{n} = \mathbf{2}$ representations of $\mathfrak{su}(2)$.[44] Thus it is clear that all of the long representations we have presented above are not allowed at $k = 1$. Let us therefore look systematically for short multiplets. Specifically, we will consider shortening conditions for the multiplets (5.2) and (5.3).

Starting with (5.2), we require that the two representations with a $\mathbf{3}$ in the second line are null. This will remove also all other representations that appear further below in the multiplet. Thus, the multiplet would reduce to

$$\begin{array}{ccc} & (\mathcal{C}^{j}_{\lambda}, \mathbf{2}) & \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}) \end{array} \tag{5.5}$$

Actually, we will see below that this requires $j = \frac{1}{2}$. Similarly, for the multiplet (5.3), the only way to eliminate the representation involving the $\mathbf{3}$ is to require one of the representations in the second line to be null. This gives then the following two possibilities:

$$\begin{array}{ccc} (\mathcal{C}^{j}_{\lambda}, \mathbf{1}) & & \\ & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{2}) & \\ & & (\mathcal{C}^{j-1}_{\lambda}, \mathbf{1}) \end{array} \quad \text{or} \quad \begin{array}{ccc} & & (\mathcal{C}^{j}_{\lambda}, \mathbf{1}) \\ & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{2}) & \\ (\mathcal{C}^{j+1}_{\lambda}, \mathbf{1}) & & \end{array} \tag{5.6}$$

However, after redefining $j \to j \pm \frac{1}{2}$ (and rearranging the picture),[45] these representations become equivalent to (5.5). Thus, there is at most one such multiplet, and we shall describe it using the conventions of (5.5). The above discussion works similarly for the discrete case where we replace $\mathcal{C}^{j}_{\lambda}$ by $\mathcal{D}^{j}_{\pm}$.

Next, we want to analyse the conditions under which this shortening can happen. For the discrete case $\mathcal{D}^{j}_{+}$, these multiplets are well-known in the context of superconformal field theories with small $\mathcal{N} = 4$ superconformal symmetry, since $\mathfrak{psu}(1,1|2)$ is the global subalgebra of this superconformal algebra. In particular, this algebra has the well-known BPS bound $h = j \geq \ell$, where ℓ is the $\mathfrak{su}(2)$-spin. In this context, the $\mathfrak{sl}(2,\mathbb{R})$-spin is identified with the conformal weight. Thus, for a BPS-representation, we need $j = \ell = \frac{1}{2}$ in the discrete case. (Formally, $j = \ell = 0$ is also possible, but this just corresponds to the lower representations in (5.5) with $j = \ell = \frac{1}{2}$.)

Since there is no $\mathfrak{sl}(2,\mathbb{R})$ highest weight state for the continuous representation, the analysis for the continuous case is a bit more involved. First,

[44] In this section we are discussing the representations of the finite-dimensional Lie superalgebra $\mathfrak{psu}(1,1|2)$. The affine highest weight states of the corresponding affine algebra will therefore transform in representations of this algebra. We will also see the shortening of the $\mathfrak{psu}(1,1|2)$ representations at $k = 1$, from the affine viewpoint, in the next section.

[45] 'Rearranging' means here that we change which state we regard as the highest weight state of the Clifford module.

we note that the $\mathfrak{psu}(1,1|2)$-Casimir decomposes into its bosonic and its fermionic components as

$$\mathcal{C}^{\mathfrak{psu}(1,1|2)} = \mathcal{C}^{\mathfrak{psu}(1,1|2)}_{\text{bos}} + \mathcal{C}^{\mathfrak{psu}(1,1|2)}_{\text{ferm}} \ , \tag{5.7}$$

$$\mathcal{C}^{\mathfrak{psu}(1,1|2)}_{\text{bos}} = \mathcal{C}^{\mathfrak{sl}(2,\mathbb{R})} + \mathcal{C}^{\mathfrak{su}(2)} \ , \tag{5.8}$$

$$\mathcal{C}^{\mathfrak{psu}(1,1|2)}_{\text{ferm}} = -\frac{1}{2}\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}\varepsilon_{\gamma\rho}S_0^{\alpha\beta\gamma}S_0^{\mu\nu\rho} \ . \tag{5.9}$$

The fermionic component of the Casimir commutes by construction with the bosonic subalgebra. It is not difficult to compute its value on the different constituents of the short representation (5.5). For instance, on the representation $(\mathcal{C}^j_\lambda, \mathbf{2})$, we determine its value on the highest weight state of the $\mathfrak{su}(2)$-algebra, which we denote by $|m,\uparrow\rangle$. (Here m labels the state in the $\mathfrak{sl}(2,\mathbb{R})$ representation $\mathcal{C}^j_\lambda$, see eq. (2.14).) We have

$$\begin{aligned}
\mathcal{C}^{\mathfrak{psu}(1,1|2)}_{\text{ferm}}\,|m,\uparrow\rangle &= -\frac{1}{2}\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}\varepsilon_{\gamma\rho}S_0^{\alpha\beta\gamma}S_0^{\mu\nu\rho}\,|m,\uparrow\rangle & (5.10)\\
&= -\frac{1}{2}\varepsilon_{\alpha\mu}\varepsilon_{\gamma\rho}\{S_0^{\alpha+\gamma}, S_0^{\mu-\rho}\}\,|m,\uparrow\rangle & (5.11)\\
&= -\frac{1}{2}\varepsilon_{\alpha\mu}\varepsilon_{\gamma\rho}\big(-\varepsilon^{\gamma\rho}c_a\sigma_a{}^{\alpha\mu}\,J_0^a + \varepsilon^{\alpha\mu}\varepsilon^{\gamma\rho}\sigma_a{}^{+-}\,K_0^a\big)\,|m,\uparrow\rangle & (5.12)\\
&= -2K_0^3\,|m,\uparrow\rangle = -\,|m,\uparrow\rangle \ . & (5.13)
\end{aligned}$$

Thus,

$$\mathcal{C}^{\mathfrak{psu}(1,1|2)}_{\text{ferm}}\big(\mathcal{C}^j_\lambda, \mathbf{2}\big) = -1 \ , \qquad \mathcal{C}^{\mathfrak{psu}(1,1|2)}_{\text{ferm}}\big(\mathcal{C}^{j\pm\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}\big) = 0 \ , \tag{5.14}$$

where the second equality follows by a similar computation. Since the complete $\mathfrak{psu}(1,1|2)$-Casimir must be equal on all the representations appearing in (5.5), we conclude that the $\mathfrak{sl}(2,\mathbb{R})$-Casimir must satisfy

$$\mathcal{C}^{\mathfrak{sl}(2,\mathbb{R})}\big(\mathcal{C}^{j\pm\frac{1}{2}}_{\lambda+\frac{1}{2}}\big) = \mathcal{C}^{\mathfrak{sl}(2,\mathbb{R})}\big(\mathcal{C}^j_\lambda\big) - \frac{1}{4} \ . \tag{5.15}$$

(Here we have used that the Casimir of $\mathfrak{su}(2)$ equals $\mathcal{C}^{\mathfrak{su}(2)} = 0$ on $\mathbf{n} = \mathbf{1}$ and $\mathcal{C}^{\mathfrak{su}(2)} = \frac{3}{4}$ on $\mathbf{n} = \mathbf{2}$.) Together with (2.16), this then implies that $j = \frac{1}{2}$, see the comment after eq. (5.5) above. (Incidentally, this is also the same condition as for the discrete case.) Note that, as a consequence,

$$\mathcal{C}^{\mathfrak{psu}(1,1|2)} = 0 \tag{5.16}$$

on these representations.

This is the only condition for the shortening to occur. The details of this short representation are spelled out in Appendix B.2. As we also explain there, the case $\lambda = \frac{1}{2}$ is special since then the $\mathfrak{sl}(2,\mathbb{R})$ representation $\mathcal{C}^j_\lambda$ (with $j = \frac{1}{2}$) becomes indecomposable.

5.2 The $\mathfrak{psu}(1,1|2)_1$ WZW model

This section is devoted to a detailed study of the $\mathfrak{psu}(1,1|2)_1$ WZW model. Our main aim is to show how to define a consistent CFT for this chiral algebra. We will discuss, in particular, the fusion rules and modular invariance. Subtleties appear due to the fact that this CFT is logarithmic.

5.2.1 $\mathfrak{psu}(1,1|2)_k$ for $k \geq 2$

Let us first review the WZW model based on $\mathfrak{psu}(1,1|2)_k$ for $k \geq 2$. The super Wakimoto representation states the equivalence [126, 148, 149]

$$\begin{aligned}\mathfrak{psu}(1,1|2)_k \cong\ & \mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k-2} \\ & \oplus 8 \text{ topologically twisted fermions in the } 2 \cdot (\mathbf{2},\mathbf{2})\ . \end{aligned} \tag{5.17}$$

This looks then similar to what one would obtain from the RNS formulation upon rewriting it in GS-like language, i.e. applying the abstruse identity. Here, the 8 fermions transform in the $2 \cdot (\mathbf{2},\mathbf{2})$ with respect to the bosonic zero-mode algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \subset \mathfrak{psu}(1,1|2)$. The fermions will lead to an $2^{\frac{8}{2}} = 16$-dimensional Clifford module, and there cannot be any shortenings in the Clifford module since the fermions are free. Thus only long representations of $\mathfrak{psu}(1,1|2)$ appear in the theory. Furthermore, for the continuous representations, every $\mathfrak{sl}(2,\mathbb{R})$-Casimir $\mathcal{C} \geq \frac{1}{4}$ (corresponding to $j = \frac{1}{2} + is$ with s real) is allowed in the spectrum; the corresponding states describe the continuum of long strings in the spectrum of string theory on $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ [60].

5.2.2 Representations of $\mathfrak{psu}(1,1|2)_1$

Let us now consider the case of $k = 1$. Then the equivalence (5.17) does not hold any longer since the affine $\mathfrak{su}(2)_{k-2}$ algebra has negative level -1, leading to an additional non-unitary factor. This is also reflected by the fact that the long representations lead to $\mathbf{n} = \mathbf{3}$ representations for $\mathfrak{su}(2)$, that are not allowed for $\mathfrak{su}(2)_1 \subset \mathfrak{psu}(1,1|2)_1$. (In particular, the $\mathbf{n} = \mathbf{3}$ representation is non-unitary at $\mathfrak{su}(2)_1$.)

As we have explained before, there is a natural way around this problem: at $k = 1$ we need to consider short representations of $\mathfrak{psu}(1,1|2)$ that do not involve the $\mathbf{n} = \mathbf{3}$ representation of $\mathfrak{su}(2)$. Such short representations exist, and they take the form of (5.5) with $j = \frac{1}{2}$

$$\begin{array}{ccc} & (\mathcal{C}^{\frac{1}{2}}_{\lambda},\mathbf{2}) & \\ (\mathcal{C}^{1}_{\lambda+\frac{1}{2}},\mathbf{1}) & & (\mathcal{C}^{0}_{\lambda+\frac{1}{2}},\mathbf{1}) \end{array} \tag{5.18}$$

Since the shortening condition fixes j to $j = \frac{1}{2}$, in particular also the Casimir is fixed. Thus the continuum of states (corresponding to arbitrary values of the Casimir for the continuous representations) is not allowed any longer, but only the bottom component (corresponding to $j = \frac{1}{2} + is$ with $s = 0$) survives. These $\mathfrak{psu}(1,1|2)$-representations can then be extended to consistent affine representations.

This is the main mechanism for how the problem with the RNS formalism at $k = 1$ is circumvented in the hybrid description. In the RNS formalism only long representations of $\mathfrak{psu}(1,1|2)$ appear, since the fermions are free and transform in the adjoint representation of $\mathfrak{su}(2)$. This is reflected in the equivalence (5.17). On the other hand, in the hybrid formalism it is possible to consider instead the short representations of $\mathfrak{psu}(1,1|2)$.[46] The introduction of short representations has another drastic consequence: it makes the string spectrum significantly smaller than in the generic case. In particular, the final spectrum will seem to have effectively only four bosonic and fermionic oscillators on the worldsheet, instead of the usual eight oscillators.

We should mention that the structure of the representations can also be deduced from the null-vector of $\mathfrak{psu}(1,1|2)_1$. The generating null-vector may be taken to be the vector $K^+_{-1}K^+_{-1}|0\rangle = 0$, which sits in the same $\mathfrak{psu}(1,1|2)$ multiplet as the null-vector

$$\left(L^{\mathfrak{psu}(1,1|2)}_{-2} - L^{\mathfrak{sl}(2,\mathbb{R})}_{-2} - L^{\mathfrak{su}(2)}_{-2}\right)|0\rangle = 0 \ . \tag{5.19}$$

This equation just means that we have a conformal embedding

$$\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{su}(2)_1 \subset \mathfrak{psu}(1,1|2)_1 \ , \tag{5.20}$$

see also [184] for a related discussion. Evaluated on affine highest weight states, we therefore conclude that

$$\mathcal{C}^{\mathfrak{psu}(1,1|2)} = -\mathcal{C}^{\mathfrak{sl}(2,\mathbb{R})} + \frac{1}{3}\,\mathcal{C}^{\mathfrak{su}(2)} \ , \tag{5.21}$$

which together with the condition that the only possible $\mathfrak{su}(2)$ representations are $\mathbf{n} = \mathbf{1}$ and $\mathbf{n} = \mathbf{2}$, fixes the allowed representations. On the other hand, for $k \geq 2$, the null-vector, i.e. the analogue of (5.19), appears at higher mode number (conformal dimension), and hence we do not get a constraint on the quadratic Casimir.

We shall denote the affine representations that are generated from the affine highest weights in (5.18) by $\mathcal{F}_\lambda$. These representations will be the main focus

[46] In [140], a proposal was made for how to make sense of the theory at $k = 1$ in the RNS formalism. It was proposed there that the $\mathfrak{su}(2)_{-1}$-factor can be represented by symplectic bosons, which effectively cancel half of the fermions. This prescription yields essentially the same spectrum as the introduction of the short representations in the hybrid description.

of study. For $\lambda = \frac{1}{2}$, the affine representation is not irreducible (see below), and we also need the affine representations associated to (5.18) where $\mathcal{C}^j_\lambda$ has been replaced by $\mathcal{D}^j_\pm$ with $j = \frac{1}{2}$ (see also Appendix B.2); the corresponding affine representations will be denoted by $\mathcal{G}_\pm$. Finally, we also need the affine representation based on the trivial representation (i.e. the vacuum representation), which is also consistent; it will be denoted by $\mathcal{L}$.

5.2.3 Spectral flow

$\mathfrak{psu}(1,1|2)_k$ possesses a spectral flow automorphism σ. On the bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R})_k \oplus \mathfrak{su}(2)_k$, it acts by a simultaneous spectral flow on both components. Explicitly, we have

$$\sigma^w(J^3_m) = J^3_m + \tfrac{kw}{2}\delta_{m,0}\ , \tag{5.22a}$$
$$\sigma^w(J^\pm_m) = J^\pm_{m\mp w}\ , \tag{5.22b}$$
$$\sigma^w(K^3_m) = K^3_m + \tfrac{kw}{2}\delta_{m,0}\ , \tag{5.22c}$$
$$\sigma^w(K^\pm_m) = K^\pm_{m\pm w}\ , \tag{5.22d}$$
$$\sigma^w(S^{\alpha\beta\gamma}_m) = S^{\alpha\beta\gamma}_{m+\frac{1}{2}w(\beta-\alpha)}\ . \tag{5.22e}$$

In addition, the energy-momentum tensor transforms as

$$\sigma^w(L_m) = L_m + w(K^3_m - J^3_m)\ . \tag{5.23}$$

Notice that the simultaneous spectral flow in $\mathfrak{sl}(2,\mathbb{R})_k \oplus \mathfrak{su}(2)_k$ keeps the supercharges integer moded. As we shall see, it will be necessary to include also spectrally flowed representations into the theory so that the fusion rules close, see also [60].[47] Thus, we are considering the set of representations (for $k = 1$)

$$\sigma^w(\mathcal{F}_\lambda)\ , \qquad w \in \mathbb{Z}\ . \tag{5.24}$$

As regards the discrete representations and the vacuum representation, we have in fact the identity

$$\sigma(\mathcal{L}) \cong \mathcal{G}_+\ , \qquad \sigma^{-1}(\mathcal{L}) \cong \mathcal{G}_-\ . \tag{5.25}$$

Thus, it suffices to consider the spectrally flowed versions of the vacuum,

$$\sigma^w(\mathcal{L})\ , \qquad w \in \mathbb{Z}\ . \tag{5.26}$$

There is one final complication: since the CFT is actually logarithmic,[48] additional (indecomposable) representations will appear. This can already be

[47] The necessity to add spectrally flowed representations of $\mathfrak{sl}(2,\mathbb{R})$ was first noticed, using arguments based on modular invariance, in [58].

[48] In fact, the same phenomenon also appears for $\mathfrak{psu}(1,1|2)_k$ with $k > 1$, see [185,186].

seen at the zero-mode level, see also Appendix B.2: $\mathcal{C}^j_\lambda$ is not irreducible for $\lambda = j$ (since it contains $\mathcal{D}^j_+$ as a subrepresentation), but indecomposable. Hence, we expect that $\lambda = \frac{1}{2}$ will play a special role. In fact, it turns out that $\mathcal{F}_{1/2}$ is *not* separately part of the spectrum. Instead, $\sigma(\mathcal{F}_{1/2})$, two copies of the representation $\mathcal{F}_{1/2}$, as well as $\sigma^{-1}(\mathcal{F}_{1/2})$ join up to form one indecomposable representation, which we denote by $\mathcal{T}$. Thus, the representations appearing in the spectrum are in fact

$$\sigma^w(\mathcal{F}_\lambda)\ , \quad \lambda \neq \tfrac{1}{2} \qquad \text{and} \qquad \sigma^w(\mathcal{T})\ , \qquad w \in \mathbb{Z}\ . \tag{5.27}$$

While the emergence of the indecomposable representation $\mathcal{T}$ leads to many technical complications, it will turn out that the resulting physical spectrum is largely unaffected by this subtlety, see also [185, 186]. Furthermore, for many considerations (in particular, for the analysis of the partition function) we may work with the so-called *Grothendieck ring* of modules, where modules related by short exact sequences are identified, i.e.

$$\mathcal{C} \sim \mathcal{A} \oplus \mathcal{B} \quad \Longleftrightarrow \quad 0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{C} \longrightarrow \mathcal{B} \longrightarrow 0\ . \tag{5.28}$$

This equivalence relation therefore forgets the indecomposablity of modules. On this, level $\mathcal{T}$ becomes then equivalent to $\sigma(\mathcal{F}_{1/2}) \oplus 2 \cdot \mathcal{F}_{1/2} \oplus \sigma^{-1}(\mathcal{F}_{1/2})$. (Similarly, $\mathcal{F}_{1/2}$ becomes equivalent to $\mathcal{G}_+ \oplus 2 \cdot \mathcal{L} \oplus \mathcal{G}_-$, see the end of Appendix B.2.) There is no material difference between the $\lambda \neq \frac{1}{2}$ contributions and that for $\lambda = \frac{1}{2}$ in (5.27), except potentially for a factor of 4 that will also be resolved below, see eq. (C.73). A more careful treatment of the indecomposable representations is given in Appendix C.2.

5.2.4 The fusion rules of $\mathfrak{psu}(1,1|2)_1$

Let us next discuss the fusion rules of the theory. For this, we use the well-tested conjecture that spectral flow respects fusion [187]. More precisely, for two modules $\mathcal{A}$ and $\mathcal{B}$, we have

$$\sigma^{w_1}(\mathcal{A}) \times \sigma^{w_2}(\mathcal{B}) \cong \sigma^{w_1+w_2}(\mathcal{A} \times \mathcal{B})\ . \tag{5.29}$$

In particular, since $\mathcal{L}$ is the identity of the fusion ring, this determines the fusion of $\sigma^w(\mathcal{L})$ with any representation. Furthermore, it follows that it is sufficient to compute $\mathcal{A} \times \mathcal{B}$ without worrying about spectral flow.

In order to motivate our ansatz for the fusion of $\mathcal{F}_\lambda$, we note that, on the level of the Grothendieck ring, we have

$$\mathcal{F}_{1/2} \sim \mathcal{G}_+ \oplus 2 \cdot \mathcal{L} \oplus \mathcal{G}_- \cong \sigma(\mathcal{L}) \oplus 2 \cdot \mathcal{L} \oplus \sigma^{-1}(\mathcal{L})\ , \tag{5.30}$$

and hence

$$\mathcal{F}_{1/2} \times \mathcal{F}_{1/2} \sim \Big(\sigma(\mathcal{L}) \oplus 2 \cdot \mathcal{L} \oplus \sigma^{-1}(\mathcal{L})\Big) \times \mathcal{F}_{1/2} \tag{5.31}$$

$$\cong \sigma(\mathcal{F}_{1/2}) \oplus 2 \cdot \mathcal{F}_{1/2} \oplus \sigma^{-1}(\mathcal{F}_{1/2}) \,. \tag{5.32}$$

Assuming that the general structure is similar for generic λ, this then suggests that (on the level of the Grothendieck ring, i.e. ignoring indecomposability issues)

$$\mathcal{F}_\lambda \times \mathcal{F}_\mu = \sigma(\mathcal{F}_{\lambda+\mu+\frac{1}{2}}) \oplus 2 \cdot \mathcal{F}_{\lambda+\mu+\frac{1}{2}} \oplus \sigma^{-1}(\mathcal{F}_{\lambda+\mu+\frac{1}{2}}) \,, \tag{5.33}$$

where the dependence on λ and μ follows by requiring that the J^3_0 eigenvalues add up correctly — this requires that the right-hand-side must only depend on $\lambda + \mu$ — together with the requirement that (5.33) reduces to (5.31) for $\lambda = \mu = \frac{1}{2}$. Note that the J^3_0 charges of the middle term differ by $1/2$ with respect to those on the left-hand-side; since only the fermionic generators of $\mathfrak{psu}(1,1|2)$ have half-integer charges, it follows that the middle term has opposite fermion number relative to the left-hand-side (and indeed opposite fermion number relative to the other two terms on the right-hand-side, since one unit of spectral flow shifts the J^3_0 eigenvalue by $\frac{1}{2}$, see eq. (5.22a)). In terms of fusion rules, this means that the middle term arises in the 'odd' fusion rules, while the other two terms are part of the 'even' fusion rules, see e.g. [188]. This will play an important role in Section 5.4.2.

5.2.5 A free field construction and the full fusion rules

We can in fact deduce these fusion rules (including the correct indecomposable structure, see Appendix C.2), using a free field realisation of $\mathfrak{psu}(1,1|2)_1$. To start with, we have the free field constructions

$$\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1) \cong 2 \text{ complex fermions} \,, \tag{5.34}$$

$$\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{u}(1) \cong 2 \text{ pairs of symplectic bosons} \,. \tag{5.35}$$

The first equivalence is well-known: if we denote the two complex fermions by ψ^α and $\bar{\psi}^\alpha$ with $\alpha = \pm$, then the $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)_V$ generators come from the bilinears $\psi^\alpha \bar{\psi}^\beta$. The second equivalence was first discussed in [189] and is probably less familiar. Recall that a pair of symplectic bosons consists of the two fields ξ and $\bar{\xi}$, whose modes satisfy the commutation relations

$$[\bar{\xi}_m, \xi_n] = \delta_{m+n,0} \,, \qquad [\xi_m, \xi_n] = [\bar{\xi}_m, \bar{\xi}_n] = 0 \,. \tag{5.36}$$

(Thus, the fields are bosons of spin $\frac{1}{2}$.) Considering two such pairs ξ^α and $\bar{\xi}^\alpha$ with $\alpha = \pm$, the bilinears $\xi^\alpha \bar{\xi}^\beta$ generate the Lie algebra $\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{u}(1)_U$.[49] If we consider in addition the (neutral) bilinear generators involving one fermion and one symplectic boson, i.e. the generators $\psi^\alpha \bar{\xi}^\beta$ and $\bar{\psi}^\alpha \xi^\beta$, we

[49] This is exactly the same construction as described in [140], except that it was interpreted there in terms of $\mathfrak{sl}(2,\mathbb{R})_1 \cong \mathfrak{su}(2)_{-1}$.

obtain eight supercharges. Altogether, we thus generate the superalgebra $\mathfrak{u}(1,1|2)_1$

$$\mathfrak{u}(1,1|2)_1 \cong 2 \text{ pairs of symplectic bosons and } 2 \text{ complex fermions} . \tag{5.37}$$

In order to reduce this to $\mathfrak{psu}(1,1|2)_1$, we thus only need to quotient out by the two $\mathfrak{u}(1)$ currents $\mathfrak{u}(1)_U$ and $\mathfrak{u}(1)_V$, i.e. we have[50]

$$\mathfrak{psu}(1,1|2)_1 \cong \frac{\mathfrak{u}(1,1|2)_1}{\mathfrak{u}(1)_U \oplus \mathfrak{u}(1)_V} \tag{5.38}$$

$$\cong \frac{2 \text{ pairs of symplectic bosons and } 2 \text{ complex fermions}}{\mathfrak{u}(1)_U \oplus \mathfrak{u}(1)_V} . \tag{5.39}$$

The details and our precise conventions for the free fields are summarised in Appendix C.1.2. As expected, this free field construction only has short representations of $\mathfrak{psu}(1,1|2)_1$, since it makes use of only four fermions.

While representations of complex fermions are standard, the fusion rules of the symplectic boson theory were worked out in detail in [190]. One should note that even though this is a free field construction, the fusion rules are highly non-trivial. (In particular, the symplectic boson theory is also a logarithmic CFT.) Translating the fusion rules of the free fields leads then to the fusion rules of the $\mathfrak{psu}(1,1|2)_1$-theory, see Appendix C.1

$$\mathcal{F}_\lambda \times \mathcal{F}_\mu = \begin{cases} \sigma^{-1}(\mathcal{F}_{\lambda+\mu+\frac{1}{2}}) \oplus 2 \cdot \mathcal{F}_{\lambda+\mu+\frac{1}{2}} \oplus \sigma(\mathcal{F}_{\lambda+\mu+\frac{1}{2}}) , & \lambda+\mu \neq 0 , \\ \mathcal{T} , & \lambda+\mu = 0 , \end{cases} \tag{5.40a}$$

$$\mathcal{F}_\lambda \times \mathcal{T} = \sigma^{-2}(\mathcal{F}_\lambda) \oplus 4 \cdot \sigma^{-1}(\mathcal{F}_\lambda) \oplus 6 \cdot \mathcal{F}_\lambda \oplus 4 \cdot \sigma(\mathcal{F}_\lambda) \oplus \sigma^2(\mathcal{F}_\lambda) , \tag{5.40b}$$

$$\mathcal{T} \times \mathcal{T} = \sigma^{-2}(\mathcal{T}) \oplus 4 \cdot \sigma^{-1}(\mathcal{T}) \oplus 6 \cdot \mathcal{T} \oplus 4 \cdot \sigma(\mathcal{T}) \oplus \sigma^2(\mathcal{T}) . \tag{5.40c}$$

In particular, this argument shows that the set of representations given in (5.27) closes under fusion. As we shall see below, the chiral fields of the dual CFT come from the representation $\sigma(\mathcal{T})$, whose fusion with itself indeed contains $\sigma(\mathcal{T})$ again.

One can also obtain these fusion rules from a Verlinde formula; this is explained in Appendix C.1.6.

5.2.6 The partition function and modular invariance

Next we will demonstrate that these representations give rise to a modular invariant spectrum, thus making the $\mathfrak{psu}(1,1|2)_1$ model also well-defined on

[50] More abstractly, while $\mathfrak{su}(1,1|2)$ is a subalgebra of $\mathfrak{u}(1,1|2)$, $\mathfrak{psu}(1,1|2)$ is obtained from $\mathfrak{su}(1,1|2)$ by quotienting out the ideal generated by the identity, i.e. $\mathfrak{su}(1,1|2)$ is a central extension of $\mathfrak{psu}(1,1|2)$.

the torus. The relevant modular invariant is the 'diagonal modular invariant' with spectrum

$$\mathcal{H} \cong \bigoplus_{w\in\mathbb{Z}} \oint_{[0,1)\setminus\{\frac{1}{2}\}} \mathrm{d}\lambda \; \sigma^w(\mathcal{F}_\lambda) \otimes \overline{\sigma^w(\mathcal{F}_\lambda)} \; . \tag{5.41}$$

Including the indecomposable module $\mathcal{T}$ makes the structure of the Hilbert space slightly more complicated. In particular, an ideal has to be factored out to make the action of $L_0 - \bar{L}_0$ diagonalisable and ensure locality [191]; this is again described in more detail in Appendix C.2. Once this ideal is factored out, and working on the level of the Grothendieck ring (as appropriate for the discussion of the partition function), the above factor of $16 = 4 \times 4$ is removed (see the discussion below eq. (5.28)), and the indecomposable representation just fills in the contribution for $\lambda = \frac{1}{2}$.

To show that (5.41) is indeed modular invariant, we have to determine the characters of the representations $\mathcal{F}_\lambda$. This is done in Appendix C.1 with the help of the free field realisation (5.39), and leads to (see eq. (C.38))

$$\mathrm{ch}[\sigma^w(\mathcal{F}_\lambda)](t,z;\tau) = q^{\frac{w^2}{2}} \sum_{r\in\mathbb{Z}+\lambda} x^r q^{-rw} \frac{\vartheta_2\left(\frac{t+z}{2};\tau\right)\vartheta_2\left(\frac{t-z}{2};\tau\right)}{\eta(\tau)^4} \; , \tag{5.42}$$

where our conventions for theta-functions are spelled out in Appendix A.4. Here $x = \mathrm{e}^{2\pi i t}$ is the chemical potential of $\mathfrak{sl}(2,\mathbb{R})$, while $y = e^{2\pi i z}$ is the chemical potential of $\mathfrak{su}(2)$. In particular, t will play the role of the modular parameter of the boundary torus of AdS_3. The characters are treated as formal power series and not as meromorphic functions. Indeed, the sum over r formally leads to the factor

$$\sum_{r\in\mathbb{Z}+\lambda} x^r q^{-rw} = e^{2\pi i\lambda(t-\tau w)} \sum_{m\in\mathbb{Z}} \delta(t - \tau w + m) \; , \tag{5.43}$$

and thus modular invariance is a somewhat formal property. Note that this problem is not specific to $k = 1$, but also arises for generic k in the original discussion of [60], see Appendix B.4 of that paper. Incidentally, the delta-functions that appear in (5.43) arise precisely at the points in the τ-plane where the worldsheet torus can be mapped holomorphically to the boundary torus, see eq. (74) of [61]. Unlike the situation described there (where for these values of τ there was a pole in partition function), the partition function localises in our case to these maps, thus suggesting that the $AdS_3 \times S^3$ factor has become topological.

Under modular transformations, the characters transform into one another. Since invariance of (5.41) under the T-modular transformation is clear, we focus on the S-modular transformation. As usual in string partition functions, to get a good modular behaviour, we have to include a $(-1)^{\mathrm{F}}$ into the character. With this, the S-modular transformation of the characters is described by the formal S-matrix

$$S_{(w,\lambda),(w',\lambda')} = -i\,\mathrm{sgn}(\mathrm{Re}(\tau))\,\mathrm{e}^{2\pi i(w'\lambda+w\lambda')}\,, \tag{5.44}$$

see Appendix C.1.5 for more details. The fact that the S-matrix depends on τ is typical of logarithmic conformal field theories [192, 193], and it will cancel out once left- and right-movers are correctly combined. The S-matrix is formally unitary and symmetric. This allows us to deduce that (5.41) is at least formally modular invariant.

Up to the zero modes, the character (5.42) agrees precisely with the character of four R-sector fermions and two bosons (where the fermions transform in the $(\mathbf{2},\mathbf{2})$ with respect to $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2)$). Morally, they originate from the four free bosons and fermions of the free field construction, of which two bosons have been factored out by the coset (5.39). We should note that, for generic k, we should have expected to find six bosonic oscillators corresponding to the six-dimensional bosonic subalgebra (capturing the 6-dimensional space $\mathrm{AdS}_3\times\mathrm{S}^3$), and eight fermionic oscillators, one for each supercharge. Thus the character (5.42) has four bosonic and fermionic oscillators fewer than in the generic case. (In particular, these representation have therefore many null-states!) This feature will carry through and is responsible for the fact that also in the final string theory answer, we will only have four bosonic and four fermionic oscillators.

5.3 The string theory spectrum

In the final step we now combine the $\mathfrak{psu}(1,1|2)_1$ WZW model with the other ingredients of the hybrid formalism and discuss the physical state conditions.

5.3.1 Physical state conditions

In addition to the $\mathfrak{psu}(1,1|2)_1$ WZW model, we have the sigma model corresponding to $\mathcal{M}_4$, as well as the ghosts. In this subsection we will first deal with the ghost contribution.

For any $k\geq 2$, we can obtain the physical string spectrum of the hybrid string by comparison to the RNS formalism.[51] The characters of the $\mathfrak{psu}(1,1|2)_k$ WZW model for $k\geq 2$ are known [149], and thus the spectrum before imposing the physical state conditions can be computed in the hybrid formalism. By comparison to the known physical spectrum as determined

[51]One could also attempt to calculate the physical spectrum directly in the hybrid formalism, using the cohomological description of the physical state condition, but this calculation seems to be difficult. In fact, even for generic k, this has only been done for the first few energy levels and in the unflowed sector, see [168].

in the RNS formalism, we then conclude that the ghost contribution to the partition function in the hybrid formalism cancels four fermionic oscillators transforming in the $(\mathbf{2},\mathbf{2})$ of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$, and two bosonic oscillators.[52] Thus, we have for $k \geq 2$

$$Z_{\text{ghost}}(t,z;\tau) = \left| \frac{\eta(\tau)^4}{\vartheta_2\left(\frac{z+t}{2};\tau\right)\vartheta_2\left(\frac{z-t}{2};\tau\right)} \right|^2 . \tag{5.45}$$

Since this is independent of k and the ghosts are free fields not interacting with the WZW model at the pure NS-NS point, the ghost contribution should remain the same also for $k = 1$. Comparing with (5.42), we note that at the end of the day only the zero mode contribution survives after the physical state conditions have been imposed. This will, in turn, be fixed by the mass-shell condition. We should note that this structure is strongly reminiscent of a topological theory.

5.3.2 The sigma-model on $\mathbb{T}^4$

Let us concentrate in the following on the case where $\mathcal{M}_4$ is described by the sigma-model on $\mathbb{T}^4$ with small $\mathcal{N} = (4,4)$ supersymmetry. As discussed in [126], we actually need a topologically twisted version of the sigma-model. The topological twist will effectively amount to evaluating the partition function in the R-sector, for which we then find

$$Z^{\text{R}}_{\mathbb{T}^4}(z,t;\tau) = \left| \frac{\vartheta_2\left(\frac{z+t}{2};\tau\right)\vartheta_2\left(\frac{z-t}{2};\tau\right)}{\eta(\tau)^6} \right|^2 \Theta_{\mathbb{T}^4}(\tau) . \tag{5.46}$$

Here, the two theta-functions account for the four fermions in the R-sector, which transform in the $(\mathbf{2},\mathbf{2})$ with respect to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$, and the eta-functions in the denominator describe the four free bosons. We have also included the lattice theta-function

$$\Theta_{\mathbb{T}^4}(\tau) = \sum_{(p,\bar{p})\in\Gamma_{4,4}} q^{\frac{1}{2}p^2}\bar{q}^{\frac{1}{2}\bar{p}^2} , \tag{5.47}$$

which accounts for the non-zero winding and momentum states. Here $\Gamma_{4,4}$ is the Narain lattice of the torus.

Combining the three ingredients (5.42), (5.45) and (5.46), we see that the representation $\sigma^w(\mathcal{F}_\lambda)$ contributes altogether

$$\left| \sum_{m\in\mathbb{Z}+\lambda} x^m q^{-mw+\frac{w^2}{2}} \right|^2 Z^{\text{R}}_{\mathbb{T}^4}(z,t;\tau) \tag{5.48}$$

[52] Note that, prior to imposing the physical state condition, the hybrid string has $8+4$ fermions (8 from the $\mathfrak{psu}(1,1|2)_k$ WZW model and 4 from the $\mathbb{T}^4$) but only $6+4$ bosons. Thus this is the expected number.

to the worldsheet spectrum. Next we have to impose the mass-shell and level-matching conditions, i.e. we need to demand that

$$h_{\text{osc}} - mw + \frac{w^2}{2} = 0 \quad \Rightarrow \quad m = \frac{w}{2} + \frac{h_{\text{osc}}}{w} \ , \tag{5.49}$$

where h_{osc} is the conformal dimension coming from the $\mathbb{T}^4$ sigma model, and similarly for the right-movers. Thus only one term in the sum survives, and the string partition function becomes

$$Z_{\text{string}}(t,z) = \sum_{w=1}^{\infty} x^{\frac{w}{2}} \bar{x}^{\frac{w}{2}} Z^{\text{R}'}_{\mathbb{T}^4}\big(z,t;\tfrac{t}{w}\big) \ . \tag{5.50}$$

Here, we have performed already the sum over the spectrally flowed sectors.[53] We should note that there is one additional constraint coming from the physical state conditions: since the left- and right-movers are both in $\mathcal{F}_\lambda$ (for the same λ), we have to have

$$h_{\text{osc}} - \bar{h}_{\text{osc}} \equiv 0 \bmod w \ . \tag{5.51}$$

We have indicated this constraint by a prime in (5.50). Next we can use the theta-function identity

$$\vartheta_2\big(\tfrac{z\pm t}{2};\tfrac{t}{w}\big) = y^{\mp\frac{w}{4}} x^{-\frac{w}{8}} \begin{cases} \vartheta_2\big(\tfrac{z}{2};\tfrac{t}{w}\big) \ , & w \text{ even} \ , \\ \vartheta_3\big(\tfrac{z}{2};\tfrac{t}{w}\big) \ , & w \text{ odd} \end{cases} \tag{5.52}$$

to simplify the torus partition function (5.46). Thus, we finally arrive at the complete string partition function

$$Z_{\text{string}}(t,z) = \sum_{w=1,\text{ even}}^{\infty} x^{\frac{w}{4}} \bar{x}^{\frac{w}{4}} Z^{\text{R}'}_{\mathbb{T}^4}\big(z,0;\tfrac{t}{w}\big) + \sum_{w=1,\text{ odd}}^{\infty} x^{\frac{w}{4}} \bar{x}^{\frac{w}{4}} Z^{\text{NS}'}_{\mathbb{T}^4}\big(z,0;\tfrac{t}{w}\big) \ , \tag{5.53}$$

where $Z^{\text{NS}'}_{\mathbb{T}^4}$ is the NS-sector version of (5.46), for which the ϑ_2 factors have been replaced by ϑ_3. This then reproduces precisely the single-particle partition function of the symmetric orbifold of $\mathbb{T}^4$, see [140]. The spectral flow index w is here identified with the length of the single cycle twisted sector of the orbifold CFT. We expect the analysis to work similarly for $\mathcal{M}_4 = \text{K3}$.

5.3.3 The chiral fields

Given that w corresponds to the length of the twisted sector cycle, the untwisted sector arises for $w = 1$. In particular, the chiral fields of the dual CFT therefore come from the $w = 1$ sector, as was already anticipated in [194]

[53] We have restricted the spectral flow to $w > 0$, see Section 5.4 below for an interpretation of the other states.

and [195]. We also see from (5.49), that for $w = 1$ the quantum number m must be a half-integer, i.e. that $\lambda = \frac{1}{2}$. Thus the chiral fields come in fact from $\sigma(\mathcal{T})$. While $\mathcal{T}$ is the indecomposable representation discussed in Appendix C.2 in detail, this is largely invisible on the level of the physical spectrum. Indeed, as shown in the Appendix, we have to divide the space $\bigoplus_{w\in\mathbb{Z}} \sigma^w(\mathcal{T}) \otimes \sigma^w(\mathcal{T})$ by an ideal to obtain the true atypical contribution to the Hilbert space. In the resulting quotient space, we can choose a gauge such that $\mathcal{T}$ becomes the moral analogue of $\mathcal{F}_{1/2}$, i.e.

$$(\mathcal{T})^{\text{gauge-fixed}} \sim \sigma^{-1}(\mathcal{L}) \oplus 2 \cdot \mathcal{L} \oplus \sigma(\mathcal{L}) \sim \mathcal{F}_{1/2}\ , \tag{5.54}$$

where $\mathcal{L}$ denotes the vacuum representation of $\mathfrak{psu}(1,1|2)_1$. After the gauge-fixing, $\sigma(\mathcal{T})$ consists then of the modules $\mathcal{L}$, $2 \cdot \sigma(\mathcal{L})$ and $\sigma^2(\mathcal{L})$.

The vacuum module $\mathcal{L}$ yields exactly one physical state, namely the vacuum itself. (Any excited state has positive conformal weight on the worldsheet and hence cannot satisfy the worldsheet mass-shell condition.) This state has vanishing spacetime conformal dimension and hence corresponds to the spacetime vacuum. Thus, as one might have anticipated, the spacetime vacuum comes directly from a vacuum module on the worldsheet — which sits however in a larger indecomposable module.

Next, $\sigma(\mathcal{L}) = \mathcal{G}_+$ is part of a continuous representation. In particular, its L_0 spectrum is bounded from below and hence only the ground states of the representation survive the mass-shell condition. In spacetime, $\sigma(\mathcal{L})$ hence yields exactly one $\mathfrak{psu}(1,1|2)$-representation, which corresponds to a $h = \frac{1}{2}$ BPS-representation. The only such representation in the vacuum sector of the symmetric orbifold are two of the fermions together with their superconformal descendants. Accounting for the multiplicity, we thus see that $2 \cdot \sigma(\mathcal{L})$ yields in spacetime the fundamental fields, i.e. the four fermions and the four bosons together with their derivatives.

Finally, the remaining chiral fields come from the module $\sigma^2(\mathcal{L})$. Since its L_0 eigenvalue on the worldsheet is unbounded from below, there are many excited states which satisfy the mass-shell condition. In particular, this is the sector where the higher spin square (HSS) symmetry generators of [83, 84] sit.

It is also instructive to understand where the exactly marginal operators come from. In the untwisted sector they sit in the sector

$$4 \cdot \sigma(\mathcal{L}) \otimes \sigma(\mathcal{L}) \subset \sigma(\mathcal{T}) \otimes \sigma(\mathcal{T})\ , \tag{5.55}$$

corresponding to the $4 \times 4 = 16$ moduli which deform the 4-torus $\mathbb{T}^4$. The theory has one more exactly marginal operator that comes from the 2-cycle twisted sector, and hence arises from

$$\sigma(\mathcal{L}) \otimes \sigma(\mathcal{L}) \subset \sigma^2(\mathcal{T}) \otimes \sigma^2(\mathcal{T})\ . \tag{5.56}$$

We should mention that each worldsheet representation $\sigma(\mathcal{L})$ gives only rise to one physical $\mathfrak{psu}(1,1|2)$ multiplet in the dual CFT, and hence each $\sigma(\mathcal{L}) \otimes \sigma(\mathcal{L})$ factor yields four moduli. This structure therefore reflects the SO$(4,5)$ symmetry of the moduli space of the theory [196].

Finally, it is worth mentioning that $\mathcal{T}$ plays also another special role in our construction: all spacetime (quarter)-BPS states come on the worldsheet from the spectrally flowed images of $\mathcal{T}$! This is simply a consequence of the fact that the spacetime conformal weight of the BPS states is in $\frac{1}{2}\mathbb{Z}$, and hence they have to come from $\sigma^w(\mathcal{T})$ on the worldsheet. (A more careful argument shows that they cannot come from $\sigma^w(\mathcal{F}_0)$.) In particular, the chiral ring sits entirely in $\sigma^w(\mathcal{T})$, which ties together with the fact that $\bigoplus_{w\in\mathbb{Z}} \sigma^w(\mathcal{T})$ closes under fusion on itself, see eq. (5.40c). Furthermore, each summand in (5.54) contains precisely one BPS state, and similarly for the w-flowed versions.

5.3.4 A subtlety at $w = 0$

Finally, we notice that there are in fact also 'physical' states for $w = 0$. Looking back at (5.49) we see that for $w = 0$ the ground states of $\mathfrak{psu}(1,1|2)$ satisfy the physical state condition, without any excitation along either $\mathfrak{psu}(1,1|2)$ or $\mathbb{T}^4$. (This is a direct consequence of (5.16), and therefore independent of λ.) From the perspective of the spacetime CFT, these states therefore transform as in (5.18). In particular, this representation is non-unitary since it contains the summand

$$\big(\mathcal{C}^0_{\lambda+\frac{1}{2}}, \mathbf{1}\big) \ , \tag{5.57}$$

which is a non-unitary representation of $\mathfrak{sl}(2,\mathbb{R})$ (unless $\lambda = \frac{1}{2}$). Since (5.57) is the 'zero-momentum' ground state of $\text{AdS}_3 \times \text{S}^3$, these states are the natural analogue of the state in, say, bosonic string theory given by

$$\alpha^0_{-1}|p = 0\rangle \ , \tag{5.58}$$

which is also physical despite having negative norm. (Here α^0_{-1} denotes the time-like oscillator, and $|p = 0\rangle$ is the ground state with zero momentum.) These states should therefore be discarded. We note that this is consistent since the $w = 0$ states can never be produced in OPEs of physical states with $w > 0$, which follows, as we shall see, from the fusion rules (5.66).

5.4 The fusion rules

In the previous sections we have shown that the spectrum of the hybrid string on $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ with a single unit of NS-NS flux ($k = 1$) agrees precisely with that of the symmetric orbifold of $\mathbb{T}^4$. However, on the face of it, it seems that the fusion rules of the worldsheet theory, see in particular eq. (5.29), are not compatible with those of the symmetric orbifold theory. (This issue was already alluded to in [171].) Indeed, the above dictionary implies that we should identify w with the length of the twisted cycle ℓ in the symmetric orbifold. However, as was shown in [197–199], the fusion rules of single cycle twisted sectors at leading order in $1/N$ take the form

$$[\ell_1] \times [\ell_2] = \bigoplus_{\ell=|\ell_1-\ell_2|+1}^{\ell_1+\ell_2-1} [\ell] \,, \tag{5.59}$$

where ℓ denotes the length of the cycle, and the sum on the right-hand-side runs over every other value, i.e. $\ell_1 + \ell_2 + \ell$ is odd.

5.4.1 The x-basis

In order to see how to reconcile this with the worldsheet description, we recall from Section 2.2.1 and Appendix B.1 that the representation theory of the Möbius group of the dual CFT requires us to work in the so-called x-basis of [60].[54] In that 'basis' the fusion rules were worked out at the end of Appendix D of [62], where it was argued that they only lead to the constraint (see eq. (D.8) of [62])

$$w \leq w_1 + w_2 + 1 \,, \tag{5.60}$$

but that the lower bound (that is visible in the m-basis, see for example eq. (D.7) of that paper) is not present any longer. As a consequence, the fusion rules of the symmetric orbifold are then compatible with those of the worldsheet theory.

In the following we want to explain the fusion rules in the x-basis more conceptually. To start with we recall that the spectral flow in eq. (5.22a) and (5.22b) can be understood as arising from conjugation by a loop in the J_0^3 direction [200, 201]. Under the action of the spacetime Möbius group $\mathrm{SL}(2,\mathbb{R})$, a state in a spectrally flowed representation is hence mapped to one in which the spectral flow direction has been conjugated, see also the paragraph below eq. (D.8) in [62]. Thus there is really a moduli space of (isomorphic) spectrally flowed representations that are characterised by the direction of the spectral flow. This direction is described by an element in the Lie algebra of $\mathfrak{sl}(2,\mathbb{R})$, and hence transforms in the $\mathbf{3}$ of $\mathfrak{sl}_2$.

[54] As is also explained there, this is not just a basis change of a given representation, but rather considers certain direct sums of representations. In fact, the x-basis only depends on j, i.e. the value of the Casimir $\mathcal{C} = -j(j-1)$, but does not fix λ; it therefore includes all representations corresponding to the different values of λ. We should also note that in the m-basis the fusion rules preserve the J_0^3 eigenvalue modulo integers. If the m-basis was the correct basis for the description of the dual CFT, this would imply that conformal dimensions in OPEs of the dual CFT would add modulo integers, which is not true in general.

The above analysis applies directly to the case of unit spectral flow ($w = 1$). If we combine two spectral flow automorphisms corresponding to $w = 1$, we may in principle choose them to point in different directions. The resulting spectral flow automorphism thus transforms in the tensor product

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \ . \tag{5.61}$$

Note that the special case where the two spectral flow directions point in the same direction describes the highest weight state of this tensor product, and hence lies in the $\mathbf{5}$ of this $\mathfrak{sl}_2$; this is therefore the spectral flow that should be identified with the $w = 2$ sector. Recursively proceeding in this manner we thus conclude that the w-spectrally flowed sector is characterised by transforming in the $\mathbf{2w+1}$ dimensional representation of $\mathfrak{sl}_2$.

We note in passing that this prescription naturally incorporates the constraint $w > 0$: the states with negative w lie in the same representation of the spacetime Möbius group $\mathrm{SL}(2,\mathbb{R})$ since the conformal transformation $\gamma(x) = -\frac{1}{x}$ inverts the sign of w. Indeed, γ induces the inner automorphism of $\mathfrak{sl}(2,\mathbb{R})$ corresponding to

$$J^3 \mapsto -J^3 \ , \qquad J^\pm \mapsto J^\mp \ , \tag{5.62}$$

which inverts the sign of w in (5.22a) and (5.22b). This also ties in with the fact that, as already argued in [62], $w > 0$ describes the 'in'-states at $x = 0$, while $w < 0$ corresponds to the 'out'-states that are inserted at $x = \infty = \gamma(0)$.

The fusion rules of the spectral flow (in the x-basis) are therefore constrained by the representation theory of this $\mathfrak{sl}_2$, and hence take the form

$$[w_1] \times [w_2] = \bigoplus_{w=|w_1-w_2|}^{w_1+w_2} [w] \ . \tag{5.63}$$

This selection rule replaces (5.29), which was derived under the assumption that all spectral flow automorphisms point in the same direction — this is the situation that arises for $w = w_1 + w_2$. Together with the shift by ± 1 in spectral flow that comes directly from the fusion rules, see eq. (5.40a), the upper limit of (5.63) thus reproduces (5.60).

Since the different spectral flow directions lead to isomorphic representations — they are related to one another by an inner automorphism — one may wonder why one cannot always choose them to lie in the same direction. The reason why this is not possible is that this is a singular 'gauge' choice; indeed, from the viewpoint of the dual CFT, the spectral flow direction is related to the position x in the dual CFT since the Möbius group that maps the 'in' states at $x = 0$ to some generic point also rotates the spectral flow direction. Thus requiring the spectral flow directions to align corresponds to the (singular) configuration where the points in the dual CFT coincide, see also the discussion leading to (5.66) below.

5.4.2 Symmetric orbifold fusion rules

While these fusion rules are now compatible with the symmetric orbifold answer of (5.59), they do not quite match precisely yet. In particular, the upper and lower bounds on w are $w = w_1 + w_2 + 1$ and $w = |w_1 - w_2| - 1$, respectively, while from the symmetric orbifold we would expect $w = w_1 + w_2 - 1$ and $w = |w_1 - w_2| + 1$, respectively. In addition, in the symmetric orbifold we have the parity constraint that $\ell_1 + \ell_2 + \ell \equiv w_1 + w_2 + w$ is odd, which is not visible in the above fusion rules.

As regards the second point, we note that spectral flow by one unit changes the fermion number by one, see e.g. [195]. If we use the convention that the ground state of the vacuum representation $\mathcal{L}$ is bosonic (as is natural), then the ground state of $\sigma(\mathcal{L}) = \mathcal{G}_+ \subset \mathcal{F}_{1/2}$ is fermionic, and hence the ground state of $\sigma^w(\mathcal{F}_\lambda)$ has fermion number

$$\text{fermion number}\Big[\sigma^w(\mathcal{F}_\lambda)\Big] = (-1)^{w+1} \ . \tag{5.64}$$

Here by the 'ground state' of $\sigma^w(\mathcal{F}_\lambda)$ we mean the (spectral flow) of the affine primaries transforming in the $(\mathcal{C}^{\frac{1}{2}}_\lambda, \mathbf{2})$ of $\mathcal{F}_\lambda$, see eq. (5.18). Incidentally, the fermion number may also be read off from the $\mathfrak{su}(2)$ spin: since the fermionic generators transform in the $\mathbf{2}$ with respect to $\mathfrak{su}(2)$, we note that the states in $\mathbf{1}$ are bosonic, while those in $\mathbf{2}$ are fermionic. Together with the fact that each single spectral flow exchanges the two $\mathfrak{su}(2)$ representations, this also leads to (5.64).

Thus the term $\sigma^w(\mathcal{F}_\lambda)$ has the same fermion number as $\sigma^{w_1}(\mathcal{F}_\lambda) \times \sigma^{w_2}(\mathcal{F}_\mu)$ (and hence appears in the 'even' fusion rules) provided that

$$(-1)^{w_1+1}(-1)^{w_2+1} = (-1)^{w+1} \ , \qquad \text{i.e. if } w_1 + w_2 + w = \text{odd} \ . \tag{5.65}$$

On the other hand, as we have seen in Section 5.3 above, the ghost contribution removes the entire $\mathfrak{psu}(1,1|2)_1$ descendants, including the fermionic zero modes, since only the bosonic zero modes survive, see eq. (5.48). Thus the physical states all come from 'bosonic' fields in $\mathfrak{psu}(1,1|2)_1$, and hence in correlation functions of physical states only the even fusion rules contribute. This then implies that for the OPE of physical states we need to have $w_1 + w_2 + w = \text{odd}$.

This leaves us with understanding the extremal values $w = w_1 + w_2 + 1$ and $w = |w_1 - w_2| - 1$. Looking back at (5.63), it is clear that $w = w_1 + w_2 + 1$ can only arise if the two spectral flow directions point in exactly the same direction, i.e. it comes from the $w = w_1 + w_2$ term in (5.63).[55] In terms of the dual CFT this means that two of the fields are inserted at the same

[55] The analysis for $w = |w_1 - w_2| - 1$ is similar since again the three spectral flow directions all have to align.

point, which we should exclude. Incidentally, the same issue is also visible from the viewpoint of [198, 199], where it is assumed that the branched coverings are regular in the sense that the branch points do not coincide; if we relaxed this condition, $w = w_1 + w_2$ would also appear in the analogue of (5.63). Thus we conclude that the even fusion rules (that are relevant for the physical states) are

$$\sigma^{w_1}(\mathcal{F}) \times \sigma^{w_2}(\mathcal{F}) = \bigoplus_{w=|w_1-w_2|+1}^{w_1+w_2-1} \sigma^{w}(\mathcal{F}) \ , \tag{5.66}$$

where $w_1 + w_2 + w$ is odd, and we have suppressed the dependence on λ and μ — as we have mentioned before, see footnote 54, the actual representations of the spacetime Möbius group involve all values of λ. This is then in precise agreement with the symmetric product fusion rules (5.59).

5.5 Summary and Conclusion

In this Chapter, we went on to study the tensionless limit of string theory on $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$. We have systematically analysed the hybrid formalism on the background and studied in particular the tensionless point described by one unit of NS-NS flux ($k = 1$). By studying the representation theory of the relevant affine algebra $\mathfrak{psu}(1,1|2)_1$, we have discovered that only short $\mathfrak{psu}(1,1|2)$ multiplets are allowed to survive at this minimum value of the background flux. This had several important and dramatic consequences for the string spectrum on the background. First, *no* discrete representations are allowed at this value of the flux (which would correspond to conventional short string solutions). This in particular also implies that there are no string states originating from the unflowed sector of the $\text{PSU}(1,1|2)_1$ WZW model. Second, only the 'bottom of the continuum', i.e. the continuous representations with lowest energy are allowed to survive in the string spectrum. Finally, since the relevant representations are all short, the string spectrum is much smaller than in the generic case. There are effectively only four bosonic and fermionic physical oscillators on the worldsheet instead of the usual eight. We have computed the string partition function for $k = 1$ and found that the previous observations conspire such that we obtain precisely the partition function of the symmetric orbifold $\text{Sym}^N(\mathbb{T}^4)$, thereby giving strong credence to the idea that the symmetric orbifold is *precisely* dual to the $k = 1$ point with pure NS-NS flux.

Hence, we have brought together several of the observations in [194, 195], and more specifically [140], into a coherent picture of the tensionless limit of AdS_3 superstring theory (with NS-NS flux) and its precise equivalence to the free symmetric product CFT. The evidence for this equivalence consists of matching, not only the spectra, but also the fusion rules governing vertex operators on the worldsheet with the selection rules for correlators in the orbifold CFT. It will be nice to actually compare a set of three point correlators on both sides such as for the extremal ones of [199] but especially those of non-BPS operators.

The resulting picture that emerges fits in with many of the expectations one has on the tensionless limit of AdS string theory. The match of the spectra directly implies that there are enhanced unbroken symmetries: they arise from the massless higher spin gauge fields which are dual to the additional conserved currents in the dual free CFT [74,75,83,202,203]. Indeed, as explained in [84, 85], the tensionless string theory dual to the symmetric product CFT must have an enlarged stringy symmetry — the Higher Spin Square (HSS) — which organises the entire perturbative spectrum. We identified, in Section 5.3.3, the sector of the worldsheet spectrum which corresponds to the chiral currents generating the HSS. Given the explicit description that we have now proposed for the string theory, it should be possible to investigate the properties of the HSS and its representations from the worldsheet viewpoint. We find it quite striking that the dual of the spacetime free CFT is also given in terms of free fields on the worldsheet. Note that the free field description on the worldsheet arises precisely at $k = 1$ whereas the supergroup sigma model is generically an interacting theory for $k \geq 2$. This is perhaps a reflection of the general phenomenon whereby additional symmetries in spacetime are mirrored on the worldsheet.

There have been indications from several directions that the tensionless limit in AdS is a topological string theory. For instance, this is the natural way in which one can reproduce correlators of a dual free CFT. Thus, in the proposed general scheme of [204–206] to obtain the string worldsheet theory from the free CFT, one sees signatures of localisation of correlators on the worldsheet [207–210], a property common to topological string theories. In particular, a similar feature was also noticed in [198] in their attempt to construct worldsheets dual to the symmetric orbifold CFT. It will be very interesting to connect our worldsheet description with these approaches that start from the field theory.

As remarked at several places, we see independent signatures of an underlying topological string description of our worldsheet theory. We have only short representations of the worldsheet CFT contributing and as a result there are no net string oscillator degrees of freedom in the $\text{AdS}_3 \times \text{S}^3$ directions after including the ghost contributions (see eq. (5.48) which has only the zero mode contributions in these directions). Yet another indication comes from the worldsheet partition function of the $\mathfrak{psu}(1,1|2)_1$ theory which is formally a sum of contact terms — see eq. (5.43). This kind of localisation to maps which are holomorphic from the worldsheet to the boundary spacetime torus is seen in A-model topological string theories [211]. It will be interesting to relate our worldsheet description to proposals made by Berkovits et.al. for an A-model topological string dual for the free $\mathcal{N} = 4$ Super Yang-Mills theory [212–215] (see also [216] for the AdS_4 case). These works are similar in spirit to the present case in that they start from the corresponding supercoset sigma models. We also note the topological sector of AdS_3 superstring theory studied in [217, 218] though their specific proposal, based on the RNS formalism, appears to be for $k \geq 2$.

Other directions also open up through having a worldsheet dual to the spacetime CFT. These include the possibility of much more refined tests of the AdS/CFT correspondence in this background, going beyond tree level in string coupling. One should also be able to study specific marginal perturbations away from the free theory and study their possible integrability, thus connecting with the growing literature of integrable spin chain descriptions for this system — see the recent works [219–221] and references therein. Finally, studying the effect of the specific marginal perturbation which corresponds to the blowup mode of the symmetric orbifold can shed light on the stringy higgsing of the higher spin symmetries — see [222] for a study from the orbifold CFT point of view.

Chapter 6

Spacetime DDF operators and Liouville theory

After having discussed the tensionless point $k = 1$ for the pure NS-NS background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, we analyse now also the case with higher flux. We also develop the theory further and construct the symmetry algebra of the dual CFT directly on the worldsheet. We will show that the dual CFT can in general be described by the symmetric product orbifold

$$\mathrm{Sym}^N \Big((\mathcal{N} = 4 \text{ Liouville theory with } c = 6(k-1)) \times \mathbb{T}^4 \Big) . \qquad (6.1)$$

At $k = 1$, the Liouville factor disappears and we reproduce the previous result. We also show that there is a similar statement for bosonic strings on AdS_3. More precisely, we sow that string theory on $\mathrm{AdS}_3 \times X$ are dual to the symmetric product orbifold of

$$\mathrm{Sym}^N \left(\left(\text{Liouville theory with } c = 1 + \frac{6(k-3)^2}{k-2} \right) \times X \right) . \qquad (6.2)$$

Since many of the complications of superstring theory are absent in this example, we start with the bosonic case.

6.1 Bosonic strings on AdS_3

As a warm-up to the technically more complex situation with supersymmetry, let us begin by analysing bosonic string theory on

$$\mathrm{AdS}_3 \times X . \qquad (6.3)$$

6.1.1 The $\mathfrak{sl}(2,\mathbb{R})_k$ WZW model and its free field realisation

The AdS_3 part of the background can be described by an $\mathfrak{sl}(2,\mathbb{R})_k$ WZW-model. Criticality of the background imposes then

$$\frac{3k}{k-2} + c_X = 26 \ . \tag{6.4}$$

In order to describe the vertex operators of the background, it is useful to employ the Wakimoto representation of $\mathfrak{sl}(2,\mathbb{R})_k$. Let us introduce a pair of bosonic ghosts with $\lambda = 1$ (see Appendix A.3 for our conventions)

$$\beta(z)\gamma(w) \sim -\frac{1}{z-w} \ , \tag{6.5}$$

as well as a free boson

$$\partial\Phi(z)\partial\Phi(w) \sim -\frac{1}{(z-w)^2} \ . \tag{6.6}$$

The free boson has background charge $Q = \sqrt{\frac{1}{k-2}}$ so that the total central charge equals

$$c = 2 + 1 + \frac{6}{k-2} = \frac{3k}{k-2} = c\big(\mathfrak{sl}(2,\mathbb{R})_k\big) \ . \tag{6.7}$$

We then have the Wakimoto representation

$$J^+ = \beta \ , \tag{6.8a}$$

$$J^3 = \sqrt{\tfrac{k-2}{2}}\partial\Phi + (\beta\gamma) \ , \tag{6.8b}$$

$$J^- = \sqrt{2(k-2)}(\partial\Phi\gamma) + (\beta\gamma\gamma) - k\partial\gamma \ . \tag{6.8c}$$

We should mention that treating Φ as a free field is only adequate near the boundary of AdS_3, see [147] for a discussion. More generally, it should be understood as a Liouville field. However, since we are mainly interested in constructing the boundary CFT, the above description will be sufficient for our purposes.

6.1.2 Vertex operators

The spectrum of the WZW model consists of affine highest weight representations of $\mathfrak{sl}(2,\mathbb{R})_k$ (whose ground states we label by $|j,m\rangle$), together with their spectrally flowed images. In the above free field realisation of $\mathfrak{sl}(2,\mathbb{R})_k$, the affine highest weight state $|j,m\rangle$ is described by

$$|j,m\rangle = \oint \mathrm{d}z \ \gamma^{-j-m} \mathrm{e}^{j\sqrt{\frac{2}{k-2}}\Phi}(z)\,|0\rangle \ . \tag{6.9}$$

Indeed, this state is annihilated by the positive modes of (6.8a)–(6.8c), and it transforms under the zero mode algebra $\mathfrak{sl}(2,\mathbb{R})$ in a representation with Casimir $\mathcal{C} = -j(j-1)$

$$J_0^+|j,m\rangle = (m+j)|j,m+1\rangle \ , \tag{6.10a}$$
$$J_0^3|j,m\rangle = m|j,m\rangle \ , \tag{6.10b}$$
$$J_0^-|j,m\rangle = (m-j)|j,m-1\rangle \ . \tag{6.10c}$$

Here j can either be real, in which case we are dealing with a discrete representation, and then either $m-j \in \mathbb{Z}_{\geq 0}$ or $m+j \in \mathbb{Z}_{\leq 0}$ so that the representation truncates. Alternatively, j can also take the value $j = \frac{1}{2} + ip$ where $p \in \mathbb{R}$, in which case the Casimir is still real. In this case, $m \in \mathbb{Z} + \lambda$ for an arbitrary $\lambda \in \mathbb{R}/\mathbb{Z}$ and the representation does not truncate (except for $p = \frac{1}{2}$ and $\lambda = \frac{1}{2}$). These are the continuous representations of $\mathfrak{sl}(2,\mathbb{R})$, which we shall denote by $\mathcal{C}^j_\lambda$. In the following we shall be treating both cases simultaneously.

We should mention that the continuous representations with $j = \frac{1}{2} + ip$ correspond to the usual representations of the Liouville field Φ, whereas the discrete representations with j real describe non-normalisable representations. Indeed, the conformal dimension of the highest weight state $|j,m\rangle$ is

$$h\big(|j,m\rangle\big) = -\frac{j(j-1)}{k-2} = -\alpha(\alpha - Q) \ , \qquad Q = \frac{1}{\sqrt{k-2}} \ , \tag{6.11}$$

where we have written $j = \frac{\alpha}{Q}$, so that $j = \frac{1}{2} + ip$ translates into $\alpha = \frac{Q}{2} + i\hat{p}$. In fact, this will be a common theme throughout this paper, namely that the continuous representations on the world-sheet behave much better than the discrete (non-normalisable) representations. Finally, we should mention that the representations with j and $1-j$ are in fact identified thanks to the reflection formula of Liouville theory [97].

Spectral flow

We will also need the behaviour of the Wakimoto representation under the spectral flow automorphism σ of $\mathfrak{sl}(2,\mathbb{R})_k$, which transforms the fields according to

$$\sigma^w(J^\pm)(z) = J^\pm(z) z^{\mp w} \ , \tag{6.12a}$$
$$\sigma^w(J^3)(z) = J^3(z) + \tfrac{kw}{2z} \ . \tag{6.12b}$$

This forces the fields of the Wakimoto representation to transform as

$$\sigma^w(\beta)(z) = \beta(z) z^{-w} \ , \tag{6.13a}$$
$$\sigma^w(\gamma)(z) = \gamma(z) z^{w} \ , \tag{6.13b}$$

$$\sigma^w(\partial\Phi)(z) = \partial\Phi(z) + \sqrt{\tfrac{k-2}{2}}\,\frac{w}{z} \ . \tag{6.13c}$$

Applying spectral flow to the affine highest weight state (6.9) thus leads to

$$|j,m,w\rangle = \oint \mathrm{d}z\, z^{-mw}\gamma^{-j-m}\mathrm{e}^{j\sqrt{\frac{2}{k-2}}\Phi}(z)|0\rangle \ . \tag{6.14}$$

6.1.3 The DDF operators

The next step of our analysis consists of constructing the DDF operators [223], i.e. the spectrum generating operators of the spacetime CFT from the world-sheet.

The Virasoro algebra

The most generic spacetime generators are the Virasoro generators that can be constructed following [147]. They can be formulated purely in terms of the $\mathfrak{sl}(2,\mathbb{R})_k$ worldsheet currents; this reflects that the conformal symmetry of the spacetime CFT is a direct consequence of the AdS_3 factor. To this end we make the ansatz

$$\mathcal{L}_m = \oint \mathrm{d}z \left(\alpha_3(m)\gamma^m J^3 + \alpha_+(m)\gamma^{m+1}J^+ + \alpha_-(m)\gamma^{m-1}J^-\right)(z) \ , \tag{6.15}$$

where we have introduced the curly $\mathcal{L}$ symbol in order to distinguish the spacetime Virasoro algebra from the world-sheet Virasoro generators. Here the exponents of γ are chosen such that the $\mathfrak{sl}(2,\mathbb{R})$-charges are homogeneous. In order for $\mathcal{L}_m$ to be a DDF operator, it has to satisfy the following requirements:

(i) It has to commute with the physical state conditions, i.e. the string theory BRST operator. In the context of bosonic string theory this requirement is equivalent to the condition that the integrand is a primary field of conformal dimension 1. A direct computation shows that this requires

$$m\alpha_3(m) + (m+1)\alpha_+(m) + (m-1)\alpha_-(m) = 0 \ . \tag{6.16}$$

(ii) It must not be BRST trivial, i.e. the integrand must not be a Virasoro descendant. The BRST trivial combination of the integrand is

$$\gamma^m J^3 - \tfrac{1}{2}\gamma^{m+1}J^+ - \tfrac{1}{2}\gamma^{m-1}J^- = \tfrac{k}{2}\,\gamma^{m-1}\partial\gamma \ , \tag{6.17}$$

which is a total derivative unless $m = 0$ (in which case this will only shift the zero mode $\mathcal{L}_0$ by the spacetime identity (6.21), see the discussion in Section 6.2.3). Thus we are free to redefine the DDF operator by adding a multiple of (6.17).

(iii) Finally, we may require that the three global (Möbius) generators correspond to the global $\mathfrak{sl}(2,\mathbb{R})$ charges,

$$\mathcal{L}_0 = \oint \mathrm{d}z \, J^3(z) \, , \quad \mathcal{L}_{-1} = \oint \mathrm{d}z \, J^+(z) \, , \quad \mathcal{L}_1 = \oint \mathrm{d}z \, J^-(z) \, . \tag{6.18}$$

These requirements admit the following (symmetrical) solution

$$\mathcal{L}_m = \oint \mathrm{d}z \left((1-m^2)\gamma^m J^3 + \frac{m(m-1)}{2}\gamma^{m+1}J^+ + \frac{m(m+1)}{2}\gamma^{m-1}J^- \right)(z) \, . \tag{6.19}$$

By construction, these operators then map physical states onto physical states. One can directly compute their algebra as

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\,\mathcal{L}_{m+n} + \tfrac{k}{2}\,\mathcal{I}\, m(m^2-1)\,\delta_{m+n,0} \, , \tag{6.20}$$

where

$$\mathcal{I} = \oint \mathrm{d}z \, \big(\gamma^{-1}\partial\gamma\big)(z) \, . \tag{6.21}$$

We will discuss the meaning of $\mathcal{I}$ below. For now, we remark that $\mathcal{I}$ commutes with all Virasoro generators $\mathcal{L}_m$ and is hence a central element of the algebra. One way to see this is to note that

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{I}] &= \left(\oint\limits_{|z|>|w|} \mathrm{d}z \oint \mathrm{d}w - \oint\limits_{|z|<|w|} \mathrm{d}z \oint \mathrm{d}w \right) \big(\gamma^{-1}\partial\gamma\big)(w) \\
&\qquad \Big((1-m^2)\gamma^m J^3 + \frac{m(m-1)}{2}\gamma^{m+1}J^+ \frac{m(m+1)}{2}\gamma^{m-1}J^- \Big)(z) \\
&= \oint_0 \mathrm{d}w \oint_w \mathrm{d}z \, \left(-\frac{\gamma^m(w)}{(z-w)^2} - \frac{m(\gamma^{m-1}\partial\gamma)(w)}{z-w} \right) && (6.22) \\
&= -m \oint \mathrm{d}w \big(\gamma^{m-1}\partial\gamma\big) \, , && (6.23)
\end{aligned}$$

where the OPE in (6.22) follows directly by inserting the Wakimoto representation (6.8a)–(6.8c) and using the free field OPEs (6.5) and (6.6). The integrand in the last expression is a total derivative for $m \neq 0$ and hence vanishes, whereas for $m = 0$ the prefactor vanishes. In either case we conclude $[\mathcal{L}_m, \mathcal{I}] = 0$.

Kac-Moody algebras

In the superconformal situation to be discussed below, the internal manifold X will contain an S^3 factor, which can be described by an $\mathfrak{su}(2)$ WZW model. Whenever the world-sheet theory contains a WZW factor, we have an affine Kac-Moody algebra $\mathfrak{g}_{k_\mathrm{G}}$ on the world-sheet whose generators we denote by

$$[K^a_m, K^b_n] = k_\mathrm{G}\, m\delta^{ab}\delta_{m+n,0} + i f^{ab}{}_c K^c_{m+n} \, , \tag{6.24}$$

where $f^{ab}{}_c$ are the structure constants of the Lie algebra $\mathfrak{g}$ corresponding to the group G. We may then construct DDF operators realising the corresponding symmetry in spacetime via [147]

$$\mathcal{K}^a_m = \oint \mathrm{d}z \, \big(\gamma^m K^a\big)(z) \ . \tag{6.25}$$

These operators are BRST invariant and lead to a Kac-Moody algebra in spacetime with commutation relations

$$[\mathcal{K}^a_m, \mathcal{K}^b_n] = k_{\mathrm{G}} \, \mathcal{I} \, m \delta^{ab} \delta_{m+n,0} + i f^{ab}{}_c \mathcal{K}^c_{m+n} \ . \tag{6.26}$$

Here $\mathcal{I}$ is again defined by (6.21).

6.2 Higher spin fields in spacetime

In this appendix we explain how to construct DDF operators associated to the higher spin generators on the world-sheet. Since this construction does not seem to have been discussed in the literature before, we will be fairly explicit.

6.2.1 Internal Virasoro algebra

Let us now construct the DDF operators associated to the (internal) Virasoro algebra arising from X, whose Virasoro tensor we denote by $T^{\mathrm{m}}(z)$. We define the corresponding spacetime Virasoro generators via

$$\mathcal{L}^{\mathrm{m}}_m \equiv \oint \mathrm{d}z \, \big((\partial\gamma)^{-1}\gamma^{m+1}T^{\mathrm{m}}\big)(z) \\ + \frac{c^{\mathrm{m}}}{12} \oint \mathrm{d}z \, \gamma^{m+1} \left(\frac{3}{2}(\partial^2\gamma)^2(\partial\gamma)^{-3} - \partial^3\gamma(\partial\gamma)^{-2}\right) \ . \tag{6.27}$$

This definition is a bit formal, but as we shall see it makes sense. In particular, since $\partial\gamma$ is a primary field on the worldsheet, the first term in the definition is a quasi-primary field on the worldsheet. The second term corrects for the fact that T^{m} is only quasi-primary and makes the expression primary. It is essentially the Schwarzian derivative of the transformation $z \mapsto \gamma(z)$. There are no normal-ordering ambiguities in the definition (6.27), since $\gamma(z)$ has regular OPE with itself (as well as with all its derivatives).

We can calculate the algebra of modes via

$$[\mathcal{L}^{\mathrm{m}}_m, \mathcal{L}^{\mathrm{m}}_n] = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z \, (\partial\gamma(z))^{-1}(\partial\gamma(w))^{-1}\gamma(z)^{m+1}\gamma(w)^{n+1} \\ \times \left(\frac{c^{\mathrm{m}}/2}{(z-w)^4} + \frac{2\,T^{\mathrm{m}}(w)}{(z-w)^2} + \frac{\partial T^{\mathrm{m}}(w)}{z-w}\right) \tag{6.28}$$

$$= \oint_0 \mathrm{d}w \left(\frac{c^{\mathrm{m}}}{12} \partial^3 \big(\gamma^{m+1}(\partial\gamma)^{-1}\big) \gamma^{n+1}(\partial\gamma)^{-1} \right.$$
$$\left. + 2\partial\big(\gamma^{m+1}(\partial\gamma)^{-1}\big)\gamma^{n+1}(\partial\gamma)^{-1} T^{\mathrm{m}} + \gamma^{m+n+2}(\partial\gamma)^{-2}\partial T^{\mathrm{m}} \right) \tag{6.29}$$
$$= \oint_0 \mathrm{d}w \left(\frac{c^{\mathrm{m}}}{12} \partial^3 \big(\gamma^{m+1}(\partial\gamma)^{-1}\big) \gamma^{n+1}(\partial\gamma)^{-1} \right.$$
$$\left. + (m-n)\gamma^{m+n+1}(\partial\gamma)^{-1} T^{\mathrm{m}} \right) \tag{6.30}$$
$$= (m-n)\mathcal{L}^{\mathrm{m}}_{m+n} + \frac{c^{\mathrm{m}}}{12} m(m^2-1) \oint_0 \mathrm{d}w\ \gamma^{m+n-1}\partial\gamma$$
$$+ \oint_0 \mathrm{d}w\, \partial \left(\gamma^{m+n+2} \left(\frac{3}{2}(\partial^2\gamma)^2(\partial\gamma)^{-4} - \partial^3\gamma(\partial\gamma)^{-3} \right) \right) . \tag{6.31}$$

The last term vanishes upon integration. In the second term, the same integral which defined the identity, see eq. (6.21), appears. Thus we obtain indeed a Virasoro algebra. When identifying $\mathcal{I} = w\mathbb{1}$, we see that the central charge equals $c^{\mathrm{m}}w$, but the generators are again fractionally moded as in the discussion of Section 6.2.4.

6.2.2 Higher spin fields

It should now be clear how to generalise the discussion to an arbitrary chiral field on the world-sheet. Given a primary field $W^{(s)}(z)$ of spin s on the worldsheet, we define its spacetime analogue as

$$\mathcal{W}^{(s)}_m \equiv \oint \mathrm{d}z\ \big((\partial\gamma)^{1-s}\gamma^{m+s-1}W^{(s)}\big)(z)\ . \tag{6.32}$$

Note that this expresses basically the coordinate transformation $z \mapsto \gamma(z)$ of the field $W^{(s)}(z)$, except that the coordinate transformation itself is described by a dynamical field. If the field in question is not primary, one can add extra terms as we did in the definition (6.27). The commutation relations of $\mathcal{W}^{(s)}_m$ can be computed as follows. First, we note that we can organise the OPE of $W^{(s)}(z)$ with another primary $V^{(t)}(w)$ in terms of Virasoro representations, i.e. that we can restrict ourselves to the primary fields appearing the OPE. Thus,

$$[\mathcal{W}^{(s)}_m, \mathcal{V}^{(t)}_n] = \sum_{u\geq 0} \oint \mathrm{d}z \Big(c_u(m,n)\, (\partial\gamma)^{1-u}\gamma^{m+u-1}U^{(u)}(z) \Big) + \text{descendants}\ , \tag{6.33}$$

where $c_u(m,n)$ are at this stage arbitrary coefficients, and we have summed over all primary fields of spin u appearing in the OPE on the right hand side. The fields $U^{(u)}(z)$ have to appear in this combination with γ, since this is the only combination which is primary and of conformal weight one on the worldsheet. Moreover, the exponent of γ is fixed by noting that γ carries charge -1 under the $\mathfrak{sl}(2,\mathbb{R})$ on the worldsheet and hence the number of γ's has to be conserved in the expression.

In order to determine the $c_u(m,n)$, we note that they do not depend on γ, and hence we can compute them by setting $\gamma(z) = z$. Then $\mathcal{W}^{(s)}_m$ agrees with the modes $W^{(s)}_m$ of the worldsheet field, and thus the coefficients are exactly the same as those that appear in the commutation relations of the algebra on the worldsheet. We have therefore shown that the commutation relations of the spacetime algebra are identical to the commutation relations of the modes of the respective fields on the worldsheet.

The argument we have presented and the definition (6.32) holds in particular also for the identity field, in which case it reduces to the definition of $\mathcal{I}$, see eq. (6.21). Thus, the central terms of the commutation relations are replaced by $\mathcal{I}$, which can be identified with the spectral flow parameter w, as discussed in the main text, see Section 6.2.5.

One can also check that these generators transform indeed as primary fields of spin s in spacetime,

$$[\mathcal{L}_m, \mathcal{W}^{(s)}_n] = \big(m(s-1)-n\big)\, \mathcal{W}^{(s)}_{m+n} \,, \tag{6.34}$$

where $\mathcal{L}_m$ is the Virasoro algebra (6.19).

6.2.3 The identity operator

For the following it will be important to understand the structure of the central extension $\mathcal{I}$ of eq. (6.21). As we have seen above, $\mathcal{I}$ commutes with all DDF operators and hence acts as a constant in a given representation of the spacetime algebra. Since its definition only involves γ, its value can only depend on the given $\mathfrak{sl}(2,\mathbb{R})_k$ representation. To determine the relevant constants, we recall that the highest weight states of the $\mathfrak{sl}(2,\mathbb{R})$ WZW model can be described by (6.9)

$$|j,m\rangle = \oint \mathrm{d}z\; \gamma^{-j-m} \mathrm{e}^{j\sqrt{\frac{2}{k-2}}\Phi}(z)|0\rangle \tag{6.35}$$

in the unflowed sector. Since $|j,m\rangle$ does not contain β (and γ has regular OPE with itself as well as with $\partial\Phi$), it follows directly that $\mathcal{I}$ has trivial action in the unflowed sector

$$\mathcal{I}\,|j,m\rangle = 0 \,. \tag{6.36}$$

On the other hand, upon spectral flow

$$\sigma^w(\mathcal{I}) = \oint \mathrm{d}z\; \big(\gamma^{-1} z^{-w} \partial(\gamma z^w)\big)(z) = \mathcal{I} + w\,\mathbb{1} \,. \tag{6.37}$$

Thus, we conclude that $\mathcal{I}$ acts as w times the identity in the w-th spectrally flowed sector.

6.2.4 The moding of the spacetime algebra

Next we want to analyse the structure of the algebra of DDF operators, in particular, that of the central terms. For this it will be important to analyse carefully the conditions under which the action of these DDF operators is well-defined.

Let us first consider the unflowed sector. Because of (6.36) together with the explicit formula for (6.21), it follows that $\log(\gamma)(z)$ can be consistently defined without branch cut by[56]

$$\log(\gamma)(z) \equiv \int_1^z \mathrm{d}w \, \left(\gamma^{-1}\partial\gamma\right)(w) \ . \tag{6.38}$$

Thus we conclude that

$$\gamma^m(z) = \exp(m \log(\gamma)(z)) \tag{6.39}$$

is a single-valued field for any (real) number $m \in \mathbb{R}$.

Next we consider the w-th spectrally flowed sector. Because of (6.13b), the m-th power of γ becomes

$$\left(\sigma^w(\gamma)\right)^m(z) = z^{mw}\gamma^m(z) \ . \tag{6.40}$$

This therefore defines a single-valued field provided that $m \in \frac{1}{w}\mathbb{Z}$. Since it is γ^m that appears in the definition of the various DDF operators, see eqs. (6.19), (6.25), as well as (6.32), we conclude that we may take the mode numbers of the DDF operators to be fractional, with the fractional part being determined by the spectral flow,[57]

$$w\text{-th spectrally flowed sector:} \qquad n \in \tfrac{1}{w}\mathbb{Z} \ . \tag{6.41}$$

We should stress that these fractionally moded operators map in general different $\mathfrak{sl}(2,\mathbb{R})$ representations into one another. In particular, an oscillator $\mathcal{Z}_n$ with mode number n carries charge $-n$ under J^3_0, and hence $\mathcal{Z}_n$ acts on the continuous representations as

$$\mathcal{Z}_n : \mathcal{C}^j_\lambda \to \mathcal{C}^j_{\lambda-n} \ . \tag{6.42}$$

Thus for $n \notin \mathbb{Z}$, the two representations are inequivalent, but since both are part of the world-sheet spectrum, these operators are still well-defined. Note that for the diagonal world-sheet spectrum (that is appropriate for

[56]Strictly speaking, this argument only shows that $\log(\gamma)(z)$ can be defined without branch cut in a neighborhood of 0. However, this is all what is needed in the following.

[57]In the unflowed sector we may take $n \in \mathbb{R}$. We should remind the reader that the only physical states (except for the tachyon) that arise from the unflowed sector come from discrete representations.

the description of AdS_3), the left- and right-moving values of λ agree; this turns out to incorporate the orbifold projection in the w-cycle twisted sector, see [140, 171] and Chapter 5 and the comments below in Section 6.4.4.

We should also mention that on the discrete representations we necessarily need $n \in \mathbb{Z}$ (since otherwise the image lies in a representation that is not part of the world-sheet spectrum). In the following we shall therefore consider the continuous world-sheet representations; we will comment on the role of the discrete world-sheet representations in Section 6.2.6.

Untwisting

As we have just seen, the spacetime generators are naturally fractionally moded when acting in the w-th spectrally flowed continuous representations, see eq. (6.41). This suggests that these worldsheet representations give rise to the w-cycle twisted sector of a symmetric product orbifold from the viewpoint of the spacetime CFT, see also [140, 171] and Chapter 5. In order to read off the structure of the underlying seed theory we can 'untwist' these generators. Let us explain this for the case of the (overall) Virasoro generators. We propose to define the untwisted generators $\widehat{\mathcal{L}}_m$ via

$$\mathcal{L}_{\frac{m}{w}} = \frac{1}{w}\widehat{\mathcal{L}}_m + \frac{k\,(w^2-1)}{4w}\,\delta_{m,0}\ , \tag{6.43}$$

where m now takes values in $\mathbb{Z}$. While the central term for the original $\mathcal{L}_n$ modes depends on w (because $\mathcal{I} = w\,\mathbb{1}$), the above correction term ensures that the $\widehat{\mathcal{L}}_m$ satisfy a Virasoro algebra with $c = 6k$, independently of w. Indeed, it follows from (6.20) that $[\widehat{\mathcal{L}}_m, \widehat{\mathcal{L}}_{-m}]$ equals

$$[\widehat{\mathcal{L}}_m, \widehat{\mathcal{L}}_{-m}] = 2mw\,\mathcal{L}_0 + \frac{k}{2}w^2\frac{m}{w}\Big(\frac{m^2}{w^2}-1\Big)w\,\delta_{m+n,0} \tag{6.44}$$

$$= 2m\,\widehat{\mathcal{L}}_0 + m\,\frac{k}{2}\Big[(w^2-1)+(m^2-w^2)\Big]\,\delta_{m+n,0} \tag{6.45}$$

$$= 2m\,\widehat{\mathcal{L}}_0 + m\,(m^2-1)\,\frac{k}{2}\,\delta_{m+n,0}\ . \tag{6.46}$$

Note that the relation between the two sets of modes in eq. (6.43) has exactly the same form as for the case of a symmetric orbifold, where the $\mathcal{L}_n$ modes act on the covering space, while the $\widehat{\mathcal{L}}_m$ generators are those of the seed theory, see e.g. [224–226].

The analysis for the other DDF operators is similar. For example, for the current algebra (6.26), the untwisted generators are defined via

$$\mathcal{K}^a_{\frac{m}{w}} = \widehat{\mathcal{K}}^a_m\ , \tag{6.47}$$

leading to an affine Kac-Moody algebra of level k_G for the seed theory. This formula generalises directly also to the higher spin DDF operators of Appendix 6.2: for a primary field of spin s, the untwisted generators are defined via

$$\mathcal{W}^{(s)}_{\frac{m}{w}} = w^{1-s}\,\widehat{\mathcal{W}}^{(s)}_{m} \,, \tag{6.48}$$

and these untwisted generators then have exactly the same commutation relations as the original world-sheet chiral algebra. Note that (6.47) and (6.48) are simply the transformation rules of a primary field of conformal weight s under the map $z \mapsto z^w$, which relates the modes defined on the covering space to those of the base space. The fact that the correction term in (6.43) appears is due to the fact that the stress-energy tensor is only quasiprimary, and that its transformation rule therefore also involves the Schwarzian derivative.

The seed theory

The above considerations suggest that the $w = 1$ sector corresponds to the untwisted sector of a symmetric product orbifold in spacetime. As a consequence, the DDF operators in the $w = 1$ sector give describe the chiral algebra of the seed theory of the symmetric product. The construction we have discussed so far shows that this chiral algebra contains the chiral algebra of X, together with the overall Virasoro tensor of central charge $c = 6k$, under which the primary fields of X transform also as primary fields. In order to elucidate the structure of this algebra, it is convenient to decouple the X factor by defining the (coset) Virasoro tensor

$$\mathcal{L}^{\mathrm{L}}_m = \mathcal{L}_m - \mathcal{L}^{\mathrm{m}}_m \,, \tag{6.49}$$

where we have subtracted the 'matter' Virasoro algebra associated to X, see Appendix 6.2 for its explicit construction from the worldsheet. The modes of $\mathcal{L}^{\mathrm{L}}_m$ commute by construction with all modes of X and lead to a Virosoro algebra of central charge [53]

$$c^{\mathrm{L}} = 6k - c_X = 6k - \left(26 - \frac{3k}{k-2}\right) = 1 + \frac{6(k-3)^2}{k-2} \,. \tag{6.50}$$

Here, we have used that the string background is critical, see eq. (6.4). Thus, the spacetime algebra is a Virasoro algebra of central charge c^{L} together with the (decoupled) chiral algebra of the internal CFT. We note that the Virasoro algebra can be represented by a Liouville field with

$$Q' = \frac{k-3}{\sqrt{k-2}} \,. \tag{6.51}$$

This will play an important role in the following.

6.2.5 Identifying Liouville theory on the world-sheet

The previous discussion now suggests that the seed theory of the spacetime symmetric orbifold is given by

$$\left[\text{Liouville with } c^{\mathrm{L}} = 1 + \frac{6(k-3)^2}{k-2}\right] \times X \, . \tag{6.52}$$

Since we have already shown that the spacetime theory has the corresponding symmetry generators, it remains to match the spectrum of the two theories. Recall that bosonic Liouville theory is believed to be uniquely characterised by having Virasoro symmetry, together with the full spectrum of primary fields [94, 227].

To match the spectrum, we look at the string theory mass shell condition, which reads in the bosonic case

$$\frac{-j(j-1)}{k-2} - wh + \frac{k}{4}w^2 + h_{\text{int}} + N = 1 \, , \tag{6.53}$$

where h is the conformal weight of the state in the dual CFT, N the excitation number on the worldsheet and h_{int} the conformal weight of the state in the internal CFT. For $w > 0$, we can immediately solve for h,[58] which gives

$$h = -\frac{\alpha(\alpha - Q)}{w} + \frac{kw^2 - 4}{4w} + \frac{h_{\text{int}} + N}{w} \, , \tag{6.54}$$

where we have rewritten the conformal dimension of the $\mathfrak{sl}(2,\mathbb{R})_k$ part using eq. (6.11). Next we observe that

$$-\frac{\alpha(\alpha - Q)}{w} + \frac{kw^2 - 4}{4w} = -\frac{1}{w}\left(\alpha - \frac{Q}{2}\right)^2 + \frac{Q^2}{4w} + \frac{kw^2 - 4}{4w} \tag{6.55}$$

$$= -\frac{1}{w}\left(\alpha - \frac{Q}{2}\right)^2 + \frac{Q'^2}{4w} + \frac{k}{4w}(w^2 - 1) \tag{6.56}$$

$$\equiv -\frac{\alpha'(\alpha' - Q')}{w} + \frac{c}{24w}(w^2 - 1) \, , \tag{6.57}$$

where we have used that (6.11) and (6.51) imply that

$$Q^2 = Q'^2 - (k-4) \, , \tag{6.58}$$

and identified

$$\alpha' - \frac{Q'}{2} = \pm\left(\alpha - \frac{Q}{2}\right) \, . \tag{6.59}$$

[58] This step only works in this manner for the continuous world-sheet representations since the J_0^3 eigenvalues (modulo one) are not already determined by j, but are parametrised by the independent parameter λ.

Moreover, we have used the fact that the seed theory has central charge $c = 6k$. Thus the spacetime seed theory has exactly the spectrum of (6.52), where the last term $\frac{c}{24w}(w^2 - 1)$ is exactly the ground state conformal weight in the w-cycle twisted sector. Furthermore, the continuous representations on the world-sheet (that correspond precisely to the usual representation of the Liouville field Φ), see the discussion below eq. (6.11), map one-to-one to the usual Liouville representations of the spacetime Q' theory.

We have therefore shown that the continuous representations on the world-sheet lead precisely to the symmetric orbifold

$$\mathrm{Sym}^N \left(\left[\text{Liouville with } c^{\mathrm{L}} = 1 + \frac{6(k-3)^2}{k-2} \right] \times X \right) \tag{6.60}$$

in spacetime.

Let us mention that for $k = 3$, the Liouville part has $c^{\mathrm{L}} = 1$,[59] and in particular there is no gap in the spectrum. This ties in with the observation made in [194], that there are massless higher spin fields appearing for this special amount of flux. Contrary to what happens in the supersymmetric setting Chapter 5, this does, however, not mean that the long string continuum disappears.

6.2.6 Discrete representations

We end this bosonic analysis with a brief discussion of the role of the discrete world-sheet representations. As we have seen above, the continuous world-sheet representations lead precisely to the spacetime spectrum of the symmetric orbifold (6.60), which defines a well-defined spacetime CFT by itself. One may therefore wonder what role the states from the world-sheet discrete representations should play?

It follows from the analysis in Section 6.2.5 that the discrete representations can give rise to physical states that lie below the Liouville gap, see in particular eq. (6.59). However, the analysis of the discrete representations is somewhat complicated since the J^3_0 eigenvalue, modulo integers, equals the spin j, and hence one cannot just solve the mass-shell condition as in eq. (6.54). As a consequence, the existence of a physical state also depends on the precise value of h_{int} (and N). Furthermore, the Maldacena-Ooguri bound [60] constrains j to lie in the interval $(\frac{1}{2}, \frac{k-1}{2})$.

[59] There are actually two theories with this spectrum for $c^{\mathrm{L}} = 1$, one being Liouville theory and the other being the Runkel-Watts theory [228, 229], see [230–232]. From what we have shown, it is not entirely clear what the correct theory should be. However, Liouville theory exists for any real k, whereas non-analytic Liouville theory (of which the Runkel-Watts theory is a particular case) only exists for $b^2 \in \mathbb{Q}$, where $c = 1 + 6(b + b^{-1})^2$. Since we expect a continuous behaviour in k, it is natural that the correct theory should always be Liouville theory, also at $c^{\mathrm{L}} = 1$.

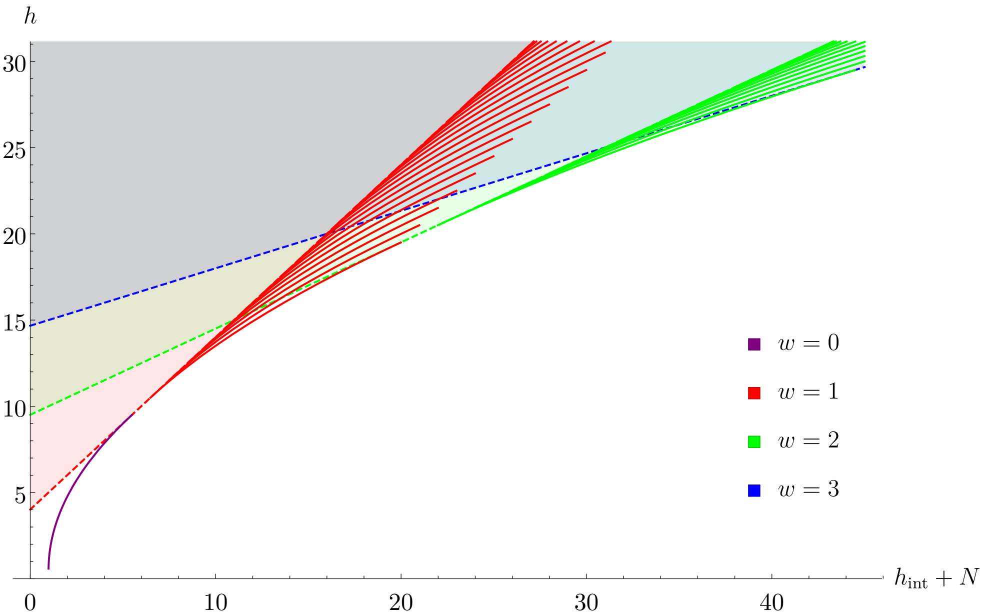

Figure 6.1: The spacetime conformal weight in dependence of the internal conformal weight $h_{\text{int}} + N$. For definiteness, we have chosen $k = 20$. Dashed lines indicate the bottom of the continuum in the respective sector. We have plotted the discrete representation as solid lines. There are several lines which correspond to the choice of state in a particular $\mathfrak{sl}(2,\mathbb{R})$ representation.

We have analysed systematically which spacetime states arise from the discrete representations (as a function of $h_{\text{int}} + N$), and the picture that emerges is the following,[60] see Fig. 6.1: the discrete representations with spectral flow w lead to spacetime states that either lie inside the continuum — this is the case for the vast majority of these states — or are a number of isolated states just below the Liouville gap. (For large k, the distance to the Liouville gap scales as k.) These isolated states in turn lie above a line that interpolates between the gaps coming from the w'th and $w+1$'st twisted sector of the symmetric orbifold. The fact that these states lie below the symmetrically orbifolded Liouville gap will be important below in the susy setting, since the discrete representations account for the BPS states of the spactime theory (which are not part of the symmetric orbifold of $\mathcal{N} = 4$ Liouville theory). However, apart from these special cases, the resulting spacetime states do not seem to correspond to special Liouville representations — in fact, given that they depend sensitively on the value of $h_{\text{int}} + N$, this cannot be otherwise.

As explained in [62], the discrete representations on the world-sheet lead to non-normalisable operators in the spacetime CFT that are not really part

[60] This also ties in with the findings of [195].

of the spacetime CFT. In particular, not all of their correlation functions are well-defined; for example, in the unflowed sector the condition for the n-point correlator to be well-defined (and make physical sense) is that

$$\sum_i j_i < k + n - 3 \,, \tag{6.61}$$

see footnote 19 of [62]. As we have seen above, see eq. (6.59), the dictionary to the spacetime CFT implies that the discrete representation with spin $j = \frac{\alpha}{Q}$, corresponds to a field in the spacetime Liouville theory with

$$\alpha' - \tfrac{Q'}{2} = \big(j - \tfrac{1}{2}\big) Q \,. \tag{6.62}$$

Thus the above bound (6.61) becomes

$$\sum_i (\alpha_i' - \tfrac{Q'}{2}) = \sum_i P_i < \big(k + \tfrac{1}{2}(n-6)\big) Q \cong Q' \,, \tag{6.63}$$

where P_i are the so-called Liouville momenta, and we have used that $(k - 3)Q = Q'$. In the semiclassical limit, k and hence Q' are large, and hence

$$Q' = b' + (b')^{-1} \cong b' \,, \tag{6.64}$$

the parameter that appears in the exponential of the Liouville potential. Thus the condition on the Liouville momenta implies that the Liouville potential dominates over the excitations of the fields in the correlator. This therefore realises the idea of the quantum mechanical toy model in Section 3.2 of [62], and thus nicely ties in with their findings.

6.3 The $\mathfrak{psu}(1,1|2)_k$ WZW model

In the following sections we shall essentially repeat the above analysis for the supersymmetric setting. We shall concentrate on the case of superstring theory on $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$. There are essentially two formalisms for describing the background, the RNS formalism [60] and the hybrid formalism [126], see Chapter 3 for a review. The RNS formalism is technically simpler, since it involves only bosonic WZW models and free fermions on the worldsheet. On the other hand, the hybrid formalism makes spacetime supersymmetry manifest. Moreover as demonstrated in Chapter 5, it is well-defined for all values of NS-NS background flux k, in particular for $k = 1$. In the following, we shall actually use a mixture of both formalisms, treating the fields of $\mathbb{T}^4$ in the RNS-formalism, and the fields of $\text{AdS}_3 \times \text{S}^3$ in the hybrid formalism.

In the next step we discuss the $\mathfrak{psu}(1,1|2)_k$ WZW-model and its vertex operators. Details about the OPEs are collected in Appendix A.2.

6.3.1 Wakimoto representation of $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ and vertex operators

We recall from Section 6.1.2 that

$$|j,m\rangle = \oint \mathrm{d}z\ \gamma^{-j-m} e^{j\sqrt{\frac{2}{k}}\Phi}(z)|0\rangle \qquad (6.65)$$

defines an affine highest weight state of a $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ representation.[61] These states can also be viewed as part of a $\mathfrak{psu}(1,1|2)$ multiplet. Since in the Wakimoto representation $S^{\alpha\beta+} = p^{\alpha\beta}$, these states are annihilated by all supercharge zero modes of that form. In order to match the conventions of Chapter 5, we now shift j by one half, and define

$$|j,m,0,\uparrow\rangle = \oint \mathrm{d}z\ \gamma^{-j-m+\frac{1}{2}} e^{(j-\frac{1}{2})\sqrt{\frac{2}{k}}\Phi}(z)|0\rangle\ , \qquad m \in \mathbb{Z}+\lambda+\tfrac{1}{2}\ . \qquad (6.66)$$

Here the penultimate entry in the ket indicates the transformation properties under the R-symmetry $\mathfrak{su}(2)_{\mathrm{R}}$, while the last entry describes the representation with respect to the outer automorphism. Thus these states transform in the representation $(\mathcal{C}^{j+\frac{1}{2}}_{\lambda}, \mathbf{1})$ with respect to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)_{\mathrm{R}}$, but are the highest weight states of a non-trivial representation of the outer automorphism.

We can now find the other vertex operators of the $\mathfrak{psu}(1,1|2)$ multiplet by applying the supercharges $S^{\alpha\beta-}_0$. First we consider the four states $S^{\alpha\beta-}_0|j,m,0,\uparrow\rangle$. Diagonalising the $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)_{\mathrm{R}}$ action leads to the linear combinations

$$|j,m,\uparrow,0\rangle = S^{++-}_0|j,m-\tfrac{1}{2},0,\uparrow\rangle + S^{-+-}_0|j,m+\tfrac{1}{2},0,\uparrow\rangle\ , \qquad (6.67a)$$

$$|j,m,\uparrow,0\rangle' = (-j+m+1)S^{++-}_0|j,m-\tfrac{1}{2},0,\uparrow\rangle + (j+m-1)S^{-+-}_0|j,m+\tfrac{1}{2},0,\uparrow\rangle\ . \qquad (6.67b)$$

The states in the first line are the spin up component (with respect to $\mathfrak{su}(2)_{\mathrm{R}}$) of the representation $(\mathcal{C}^{j}_{\lambda}, \mathbf{2})$, whereas the states in the second line are the spin up component of $(\mathcal{C}^{j-1}_{\lambda}, \mathbf{2})$. In the explicit Wakimoto representation, we thus have

$$|j,m,\uparrow,0\rangle = \left(j-\tfrac{3}{2}\right) \oint \mathrm{d}z\ \left(\theta^{++}\gamma^{-j-m+1} + \theta^{-+}\gamma^{-j-m}\right) e^{(j-\frac{1}{2})\sqrt{\frac{2}{k}}\Phi}(z)|0\rangle\ , \qquad (6.68a)$$

$$|j,m,\uparrow,0\rangle' = \left(j-\tfrac{1}{2}\right) \oint \mathrm{d}z\ \Big((j-m-1)\theta^{++}\gamma^{-j-m+1} - (j+m-1)\theta^{-+}\gamma^{-j-m}\Big) e^{(j-\frac{1}{2})\sqrt{\frac{2}{k}}\Phi}(z)|0\rangle\ . \qquad (6.68b)$$

We can similarly construct the vertex operators for the other descendants.

[61] Note that the decoupled $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ algebra is now at level $k+2$, and hence k is shifted by $+2$ relative to the formulae of Section 6.1.

6.3.2 The short representation

As an aside we should mention that the above multiplet shortens for $j = \frac{1}{2}$, which was one of the key insights in Chapter 5. Indeed, (6.68b) becomes null for $j = \frac{1}{2}$. In this special case, the $\mathfrak{psu}(1,1|2)$ representation is particularly simple and the complete list of vertex operators reads (we suppress the label $j = \frac{1}{2}$ in the following):

$$|m,0,\uparrow\rangle = \oint \mathrm{d}z\ \gamma^{-m}|0\rangle\ , \tag{6.69a}$$

$$|m,\uparrow,0\rangle = -\oint \mathrm{d}z\ \left(\theta^{++}\gamma^{-m+\frac{1}{2}} + \theta^{-+}\gamma^{-m-\frac{1}{2}}\right)|0\rangle\ , \tag{6.69b}$$

$$|m,\downarrow,0\rangle = -\oint \mathrm{d}z\ \left(\theta^{+-}\gamma^{-m+\frac{1}{2}} + \theta^{--}\gamma^{-m-\frac{1}{2}}\right)|0\rangle\ , \tag{6.69c}$$

$$\begin{aligned}|m,0,\downarrow\rangle = \oint \mathrm{d}z\ \Big(&-\theta^{--}\theta^{-+}\gamma^{-m-1} + \theta^{-+}\theta^{+-}\gamma^{-m} \\ &-\theta^{--}\theta^{++}\gamma^{-m} - \theta^{+-}\theta^{++}\gamma^{-m+1}\Big)|0\rangle\ .\end{aligned} \tag{6.69d}$$

We should stress that this shortening happens regardless of the value of k. While this multiplet is usually not part of the string theory spectrum as conjectured in [60], at $k = 1$ it is the only consistent multiplet Chapter 5. This follows from the fact that any long multiplet contains a spin ≥ 1 representation of R-symmetry $\mathfrak{su}(2)_k$, which is not allowed at level $k = 1$.

6.3.3 Spectral flow

We will also need the behaviour of the Wakimoto representation under the spectral flow σ of $\mathfrak{psu}(1,1|2)_k$ (which acts both on $\mathfrak{sl}(2,\mathbb{R})_k$ and $\mathfrak{su}(2)_k$). The generating fields transform under spectral flow as

$$\sigma^w(p^{\alpha\beta})(z) = z^{\frac{1}{2}w(\beta-\alpha)}p^{\alpha\beta}(z)\ , \tag{6.70a}$$

$$\sigma^w(\theta^{\alpha\beta})(z) = z^{-\frac{1}{2}w(\beta-\alpha)}\theta^{\alpha\beta}(z)\ , \tag{6.70b}$$

$$\sigma^w(\beta)(z) = \beta(z)z^{-w}\ , \tag{6.70c}$$

$$\sigma^w(\gamma)(z) = \gamma(z)z^{w}\ , \tag{6.70d}$$

$$\sigma^w(\partial\Phi)(z) = \partial\Phi(z) + \sqrt{\tfrac{k}{2}}\tfrac{w}{z}\ , \tag{6.70e}$$

$$\sigma^w(\mathcal{K}^\pm)(z) = \mathcal{K}^\pm(z)z^{\pm w}\ , \tag{6.70f}$$

$$\sigma^w(\mathcal{K}^3)(z) = \mathcal{K}^3(z) + \tfrac{kw}{2z}\ . \tag{6.70g}$$

6.4 The spacetime symmetry algebra

In this Section we define the boundary symmetry generators for the background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ following [147, 164, 233, 234]. These boundary symmetry generators are DDF operators [223] that commute with the BRST operator and map physical states to physical states. We will first show how to construct the operators that realise an extended spacetime $\mathcal{N} = 4$ algebra.

In the following it will be convenient to treat the torus excitations (that are independent of k) in the RNS formalism, while the $\text{AdS}_3 \times \text{S}^3$ part will be analysed in the hybrid formalism. This then also remains well-defined at $k = 1$ (where only the $\text{AdS}_3 \times \text{S}^3$ part becomes ill-defined in the RNS formalism).

6.4.1 Spacetime operators in the RNS-formalism

We begin by constructing the spacetime operators for the torus directions in the RNS-formalism. Since these are independent of k, there is no need to invoke the hybrid formalism for them. In any case, we can always rewrite them in terms of the hybrid variables if we want to.

Free bosons

The spacetime operators for the free bosons of the torus are given in the canonical (-1)-picture as

$$\partial \mathcal{X}_m^{(-1)\alpha} = \oint \mathrm{d}z\, \lambda^\alpha \mathrm{e}^{-\varphi} \gamma^m \,, \qquad \partial \bar{\mathcal{X}}_m^{(-1)\alpha} = \oint \mathrm{d}z\, \bar{\lambda}^\alpha \mathrm{e}^{-\varphi} \gamma^m \,. \tag{6.71}$$

One readily shows that these operators are BRST invariant under the BRST charge (3.19). Indeed, since the integrand has conformal weight $h = 1$, it is invariant under the bosonic piece Q_0 (3.20). On the other hand, both Q_1 and Q_2 have regular OPEs with the integrand.

For the following it will also be convenient to evaluate these operators in the (0)-picture, where they become

$$\begin{aligned} \partial \mathcal{X}_m^{(0)\alpha} &= -2 \oint \mathrm{d}z\, G_{-1/2}(\lambda^\alpha \gamma^m)(z) &(6.72)\\ &= \oint \mathrm{d}z\, \Big(\partial X^\alpha \gamma^m - \frac{m}{k} \lambda^\alpha \big(2\psi^3 \gamma^m - \psi^+ \gamma^{m+1} - \psi^- \gamma^{m-1} \big) \Big)(z) \,, &(6.73) \end{aligned}$$

and similarly for the barred bosons. These generators satisfy then the commutation relations[62]

$$[\partial \mathcal{X}_m^{(0)\alpha}, \partial \bar{\mathcal{X}}_n^{(0)\beta}] = -n \varepsilon^{\alpha\beta} \mathcal{I}_{m+n}^{(0)} \,, \tag{6.74}$$

where we have defined

$$\mathcal{I}_m^{(0)} \equiv \oint \mathrm{d}z\, (\gamma^{m-1} \partial \gamma)(z) \,. \tag{6.75}$$

[62]This is most easily evaluated by taking one of the generators in the (-1) picture and the other in the (0) picture, and then changing the picture, but one can also work this out directly with both of them in the (0) picture.

For $m \neq 0$, the integrand is a total derivative and vanishes, and thus $\mathcal{I}^{(0)}_m = \mathcal{I}\,\delta_{m,0}$, where $\mathcal{I}$ is the 'identity' operator that was already introduced in the bosonic analysis, see eq. (6.21). Thus we arrive at

$$[\partial\mathcal{X}^{(0)\alpha}_m, \partial\bar{\mathcal{X}}^{(0)\beta}_m] = m\delta_{m+n}\varepsilon^{\alpha\beta}\mathcal{I} \ . \tag{6.76}$$

Of course, this relation now holds in any picture. Similarly, one checks that

$$[\partial\mathcal{X}^{(0)\alpha}_m, \partial\mathcal{X}^{(0)\beta}_m] = 0 \ , \qquad [\partial\bar{\mathcal{X}}^{(0)\alpha}_m, \partial\bar{\mathcal{X}}^{(0)\beta}_m] = 0 \ . \tag{6.77}$$

Free fermions

Next we come to the fermionic operators. The free fermions are best constructed in the canonical $(-\frac{1}{2})$-picture, where the relevant vertex operators are given as[63]

$$\begin{aligned}\Lambda^{(-\frac{1}{2})\alpha}_r = k^{-\frac{1}{4}} \oint \mathrm{d}z \, \Big(&\mathrm{e}^{\frac{1}{2}H_1+\frac{\alpha}{2}H_2-\frac{\alpha}{2}H_3+\frac{1}{2}H_4-\frac{1}{2}H_5-\frac{\varphi}{2}}\gamma^{r+\frac{1}{2}} \\ &+ \mathrm{e}^{-\frac{1}{2}H_1+\frac{\alpha}{2}H_2+\frac{\alpha}{2}H_3+\frac{1}{2}H_4-\frac{1}{2}H_5-\frac{\varphi}{2}}\gamma^{r-\frac{1}{2}}\Big) \ ,\end{aligned} \tag{6.78}$$

and similarly for the complex conjugates

$$\begin{aligned}\bar{\Lambda}^{(-\frac{1}{2})\alpha}_r = k^{-\frac{1}{4}} \oint \mathrm{d}z \, \Big(&\mathrm{e}^{\frac{1}{2}H_1+\frac{\alpha}{2}H_2-\frac{\alpha}{2}H_3-\frac{1}{2}H_4+\frac{1}{2}H_5-\frac{\varphi}{2}}\gamma^{r+\frac{1}{2}} \\ &+ \mathrm{e}^{-\frac{1}{2}H_1+\frac{\alpha}{2}H_2+\frac{\alpha}{2}H_3-\frac{1}{2}H_4+\frac{1}{2}H_5-\frac{\varphi}{2}}\gamma^{r-\frac{1}{2}}\Big) \ .\end{aligned} \tag{6.79}$$

Here α will be an R-symmetry index, and we have suppressed the cocycle factors that are necessary for locality. These vertex operators (anti)commute again trivially with Q_0 and Q_2. To demonstrate invariance with respect to Q_1, one has to show that $G(z)$ has no $(z-w)^{-3/2}$ singularity with the integrand. There are two potential contributions to this singularity, one arising from the cubic fermion terms in (3.14), and one from the contraction of the $\mathfrak{sl}(2,\mathbb{R})$ part in the first line of (3.14) with the γ's in the integrand of (6.78) or (6.79). As it turns out the two contributions cancel precisely, thus proving that these operators are indeed BRST invariant. Furthermore, since the exponents of the H_i involve an even number of $(-)$-signs, the operators also respect the GSO projection.

The anticommutators of these fermionic operators can be computed directly, and one finds

$$\{\Lambda^{(-\frac{1}{2})\alpha}_r, \Lambda^{(-\frac{1}{2})\beta}_s\} = 0 \ , \tag{6.80}$$

[63]Since they describe spacetime fermions they come from the R-sector on the world-sheet; their structure can then be read off from (3.26a) and (3.26b).

$$\{\bar{\Lambda}_r^{(-\frac{1}{2})\alpha}, \bar{\Lambda}_s^{(-\frac{1}{2})\beta}\} = 0\,, \tag{6.81}$$

$$\{\Lambda_r^{(-\frac{1}{2})\alpha}, \bar{\Lambda}_s^{(-\frac{1}{2})\beta}\} = \frac{\varepsilon^{\alpha\beta}}{k}\oint \mathrm{d}z\, \mathrm{e}^{-\varphi}\Big(\psi^+\gamma^{r+s+1} - 2\psi^3\gamma^{r+s} + \psi^-\gamma^{r+s-1}\Big)(z) \tag{6.82}$$

$$= \varepsilon^{\alpha\beta}\mathcal{I}^{(-1)}_{r+s}\,, \tag{6.83}$$

where we have used picture changing in the last step.[64] Thus, the fermions behave as free fields in spacetime.

6.4.2 The spacetime operators in the hybrid formalism

Next we want to construct the spacetime $\mathcal{N}=4$ algebra whose supercharges will transform these boson and fermion fields into one another. Since the superconformal symmetry arises from the $\mathrm{AdS}_3\times\mathrm{S}^3$ part of the background, we should now switch to the hybrid formalism.

The Virasoro algebra

Let us begin with the spacetime Virasoro algebra that was already (in the zero picture) given in [147]

$$\mathcal{L}_n^{(0)} = \oint \mathrm{d}z\,\Big((1-n^2)J^3\gamma^n + \tfrac{n(n-1)}{2}J^+\gamma^{n+1} + \tfrac{n(n+1)}{2}J^-\gamma^{n-1}\Big)(z)\,. \tag{6.84}$$

Incidentally, this formula is the same in the NSR and the hybrid formalism — the calculations of [147] were done in the RNS formalism — since the $\mathfrak{sl}(2,\mathbb{R})_k$ currents J^a are the same in both descriptions.[65] A direct computation similar to the bosonic case shows that

$$[\mathcal{L}_m^{(0)}, \mathcal{L}_n^{(0)}] = (m-n)\mathcal{L}^{(0)}_{m+n} + \frac{k}{2}m(m^2-1)\,\mathcal{I}^{(0)}_{m+n}\,, \tag{6.85}$$

where $\mathcal{I}^{(0)}_m = \mathcal{I}\,\delta_{m,0}$ is again given by (6.75). One also checks that the spacetime free fields from the torus transform as primary fields of conformal weight 1 and $\frac{1}{2}$ (for the bosons and fermions, respectively) with respect to this Virasoro algebra, which is the analogue of (6.34).

The supercharges

Next we want to find the DDF operators for the supercharges. In the $(-1/2)$ picture they are given as

[64] Strictly speaking, since we have not kept track of the cocycle factors, our calculation only shows that the last anti-commutator is given by the right-hand-side up to a sign (which could in principle also depend on $\alpha=-\beta$). However, given that α and β are spinor indices with respect to the outer $\mathfrak{su}(2)$ symmetry, the dependence must be proportional to $\varepsilon^{\alpha\beta}$.

[65] These generators describe spacetime symmetries, and hence should agree. One can also check this explicitly by inserting (3.29a) and (3.29b) into (3.34a).

$$\mathcal{G}_r^{(-\frac{1}{2})\alpha\beta} = k^{\frac{1}{4}} \oint \mathrm{d}z \, \Big(\big(r - \tfrac{1}{2}\big) \mathrm{e}^{\frac{1}{2}H_1 + \frac{\alpha}{2}H_2 + \frac{\alpha}{2}H_3 + \frac{\beta}{2}H_4 + \frac{\beta}{2}H_5 - \frac{\varphi}{2}} \gamma^{r+\frac{1}{2}} + \big(r + \tfrac{1}{2}\big) \mathrm{e}^{-\frac{1}{2}H_1 + \frac{\alpha}{2}H_2 - \frac{\alpha}{2}H_3 + \frac{\beta}{2}H_4 + \frac{\beta}{2}H_5 - \frac{\varphi}{2}} \gamma^{r-\frac{1}{2}} \Big)(z) \, . \quad (6.86)$$

We have written this formula in terms of the RNS fields since then the expressions are simpler (and more symmetrical). Note that for the $\mathcal{G}^{(-\frac{1}{2})\alpha+}_{\mp\beta/2}$ generators, the integrand is equal to $p^{\beta\alpha}$, see (3.29a), and hence these generators also have a simple description in the hybrid formalism; the $\mathcal{G}^{\alpha-}_{\mp\beta/2}$ generators are more easily described in terms of the hybrid fields in the $(+\frac{1}{2})$ picture, see below.

One can again check that these operators commute with the BRST charge and preserve the GSO-projection. Furthermore, they indeed transform the bosonic and fermionic spacetime operators into one another, i.e.

$$\{\mathcal{G}_r^{(-\frac{1}{2})\alpha\beta}, \Lambda_s^{(-\frac{1}{2})\gamma}\} = \varepsilon^{\alpha\gamma} \partial \mathcal{X}_{r+s}^{(-1)\beta} \, , \quad (6.87)$$

$$[\mathcal{G}_r^{(-\frac{1}{2})\alpha\beta}, \partial \mathcal{X}_m^{(0)\gamma}] = m \varepsilon^{\beta\gamma} \Lambda_{m+r}^{(-\frac{1}{2})\alpha} \, , \quad (6.88)$$

and similarly for the barred free fields.

In order to confirm that the supercharges generate the $\mathcal{N} = 4$ superconformal algebra, it is convenient to take the supercharges $\mathcal{G}_r^{\alpha+}$ in the $(-\frac{1}{2})$ picture, and the supercharges $\mathcal{G}_r^{\alpha-}$ in the $(+\frac{1}{2})$ picture. Applying picture changing on $\mathcal{G}^{(-\frac{1}{2})\alpha-}_{\pm\frac{1}{2}}$ gives rise to (3.34d) with two extra terms,[66]

$$\mathcal{G}^{(+\frac{1}{2})\alpha-}_{\pm\frac{1}{2}} = \pm \tilde{S}_0^{\mp\alpha-} \quad (6.89)$$

$$\equiv \pm S_0^{\mp\alpha-} \pm \oint \mathrm{d}z \, \Big(p^{\mp\alpha} \mathrm{e}^{\rho} (\partial \bar{X}^- \Psi^+ - \partial X^- \bar{\Psi}^+) + \frac{1}{2} p^{\mp\alpha} b \mathrm{e}^{\rho} \Big) \, , \quad (6.90)$$

where we have now written the generators in terms of the hybrid fields. We should mention that the two extra terms will not modify the $\mathfrak{psu}(1,1|2)_k$ algebra.

The expression for general mode number r can be obtained by taking the commutator with the Virasoro generators, and one finds

$$\mathcal{G}_r^{(+\frac{1}{2})\alpha-} = \oint \mathrm{d}z \, \Big(\big(r + \tfrac{1}{2}\big) \tilde{S}^{-\alpha-} \gamma^{r-\frac{1}{2}} + \big(r - \tfrac{1}{2}\big) \tilde{S}^{+\alpha-} \gamma^{r+\frac{1}{2}} + \big(r^2 - \tfrac{1}{4}\big) \big(2J^3 \theta^{-\alpha} \gamma^{r-\frac{1}{2}} + 2J^3 \theta^{+\alpha} \gamma^{r+\frac{1}{2}} - J^+ \theta^{-\alpha} \gamma^{r+\frac{1}{2}} - J^- \theta^{-\alpha} \gamma^{r-\frac{3}{2}} - J^+ \theta^{+\alpha} \gamma^{r+\frac{3}{2}} - J^- \theta^{+\alpha} \gamma^{r-\frac{1}{2}} \big) \Big) \, , \quad (6.91)$$

[66] Thus the Wakimoto representation (3.34d) essentially implements picture changing.

i.e. there are further correction terms that are spelled out in the second and third line. Incidentally, this expression can also be obtained directly by applying picture changing to the expressions for the supercharges in the $(-\frac{1}{2})$ picture.

The spacetime $\mathfrak{su}(2)$-currents

With these expressions at hand we can now calculate the anti-commutators of the supercharges, and thereby read off the form of the spacetime affine $\mathfrak{su}(2)$ algebra generators,

$$\mathcal{K}_m^{(0)a} = \oint \mathrm{d}z \, \Big(K^a \gamma^m - \tfrac{m}{2} (\sigma^a)_{\alpha\beta} \big(S^{+\alpha+} \theta^{+\beta} \gamma^{m+1} + S^{-\alpha+} \theta^{-\beta} \gamma^{m-1} + S^{+\alpha+} \theta^{-\beta} \gamma^m + S^{-\alpha+} \theta^{+\beta} \gamma^m \big) \Big) \,. \quad (6.92)$$

This agrees with what one obtains from [147], upon rewriting the RNS fields in terms of the hybrid fields.

6.4.3 The complete spacetime algebra

It remains to check that the generators (6.84), (6.86), (6.90) and (6.92) satisfy the (anti-)commutation relations of the small $\mathcal{N} = 4$ algebra,

$$[\mathcal{L}_m, \mathcal{L}_n] = \tfrac{k}{2} \mathcal{I}\, m(m^2-1) \delta_{m+n,0} + (m-n) \mathcal{L}_{m+n} \,, \quad (6.93a)$$

$$[\mathcal{L}_m, \mathcal{G}_r^{\alpha\beta}] = \big(\tfrac{1}{2} m - r\big) \mathcal{G}_{m+r}^{\alpha\beta} \,, \quad (6.93b)$$

$$[\mathcal{L}_m, \mathcal{K}_n^a] = -n \mathcal{K}_{m+n}^a \,, \quad (6.93c)$$

$$[\mathcal{K}_m^3, \mathcal{K}_n^3] = \tfrac{k}{2} \mathcal{I}\, m \delta_{m+n,0} \,, \quad (6.93d)$$

$$[\mathcal{K}_m^3, \mathcal{K}_n^\pm] = \pm \mathcal{K}_{m+n}^\pm \,, \quad (6.93e)$$

$$[\mathcal{K}_m^+, \mathcal{K}_n^-] = k \mathcal{I}\, m \delta_{m+n,0} + 2 \mathcal{K}_{m+n}^3 \,, \quad (6.93f)$$

$$[\mathcal{K}_m^a, \mathcal{G}_r^{\alpha\beta}] = \tfrac{1}{2} (\sigma^a)^\alpha{}_\gamma \mathcal{G}_{r+m}^{\gamma\beta} \,, \quad (6.93g)$$

$$\{\mathcal{G}_r^{\alpha\beta}, \mathcal{G}_s^{\gamma\delta}\} = k\big(r^2 - \tfrac{1}{4}\big) \varepsilon^{\alpha\gamma} \varepsilon^{\beta\delta} \mathcal{I}\, \delta_{r+s,0} + \varepsilon^{\alpha\gamma} \varepsilon^{\beta\delta} \mathcal{L}_{r+s} + (r-s) \varepsilon^{\beta\delta} (\sigma_a)^{\alpha\gamma} \mathcal{K}_{r+s}^a \,, \quad (6.93h)$$

and this turns out to be the case. Furthermore, the free boson and fermion fields extend this algebra to the so-called extended small $\mathcal{N} = 4$ algebra,

$$\{\mathcal{G}_r^{\alpha\beta}, \Lambda_s^\gamma\} = \varepsilon^{\alpha\gamma} (\partial \mathcal{X})_{r+s}^\beta \,, \quad (6.94a)$$

$$[\mathcal{G}_r^{\alpha\beta}, \partial \mathcal{X}_m^\gamma] = m \varepsilon^{\beta\gamma} \Lambda_{r+m}^\alpha \,, \quad (6.94b)$$

$$\{\mathcal{G}_r^{\alpha\beta}, \bar{\Lambda}_s^\gamma\} = \varepsilon^{\alpha\gamma} (\partial \bar{\mathcal{X}})_{r+s}^\beta \,, \quad (6.94c)$$

$$[\mathcal{G}_r^{\alpha\beta}, \partial \bar{\mathcal{X}}_m^\gamma] = m \varepsilon^{\beta\gamma} \bar{\Lambda}_{r+m}^\alpha \,, \quad (6.94d)$$

$$[\partial \mathcal{X}_m^\alpha, \partial \bar{\mathcal{X}}_n^\beta] = m \varepsilon^{\alpha\beta} \mathcal{I}\, \delta_{m+n,0} \,, \quad (6.94e)$$

$$\{\Lambda^\alpha_r, \bar{\Lambda}^\beta_s\} = \varepsilon^{\alpha\beta}\, \mathcal{I}\, \delta_{r+s,0} \,. \tag{6.94f}$$

We have checked these relations in specific pictures, but they remain then also true in general (and we have therefore not written the picture numbers explicitly). We should also mention that the identity operator $\mathcal{I}$ commutes with the $\mathcal{N}=4$ generators and the free fields, and that the free fields transform as Virasoro primaries.

6.4.4 The action of the spacetime algebra on physical states

As in the bosonic case discussed in Section 6.2.4 it remains to understand the values the various mode numbers can take. The argument that was given there continues to hold essentially unmodified — the γ field has the property that γ^m is single-valued for any $m \in \mathbb{R}$, see eq. (6.39). Given the form of the spectral flow on γ, see eq. (6.70d), as well as the form of the various DDF operators, it follows that the mode numbers of bosonic DDF operators may be taken to lie in[67]

$$w\text{-th spectrally flowed sector:} \qquad n \in \tfrac{1}{w}\mathbb{Z} \,, \tag{6.95}$$

while the condition for the fermionic generators is instead

$$w\text{-th spectrally flowed sector:} \qquad r \in \tfrac{1}{w}\mathbb{Z} + \tfrac{1}{2} \,. \tag{6.96}$$

This is again reminiscent of the fractionally moded algebra in the symmetric orbifold [235], and indeed the untwisting that was done in the bosonic case, see Section 6.2.4, can be similarly performed. As in the bosonic case, the DDF operators map then different continuous representations $\mathcal{C}^j_\lambda$ into one another, see eq. (6.42). While both $\mathcal{C}^j_\lambda$ and $\mathcal{C}^j_{\lambda-n}$ are part of the world-sheet spectrum, there is actually a non-trivial constraint in that in the 'diagonal modular invariant' we are considering, only the combinations $\mathcal{C}^j_\lambda \otimes \mathcal{C}^j_\lambda$ appear in the Hilbert space, i.e. the left- and right-moving λ always agree. This means that, in order to map physical states to physical states, we need to combine left- and right-moving DDF operators such that the total left- and right-moving mode numbers differ by an integer (for bosons). This condition reflects precisely the orbifold invariance condition of the spacetime CFT [140, 171] and Chapter 5.

6.5 The symmetric product orbifold

As we have seen above, the spectrally flowed continuous world-sheet representations give rise to the different single-cycle twisted sectors of a symmetric orbifold. In the previous section we have identified some of the generators of the corresponding seed theory; in particular, we have shown that

[67] For $w = 0$ the modes can a priori take any real value.

the seed theory contains the extended small $\mathcal{N} = 4$ superconformal algebra with $c = 6k$, see eq. (6.93a). In this section we show that the full spectrum generating algebra of the spacetime CFT is

$$\Big[\text{small } \mathcal{N} = 4 \text{ Liouville with } c = 6(k-1)\Big] \oplus \mathbb{T}^4 \ . \tag{6.97}$$

In particular, for $k = 1$, the Liouville part vanishes and we recover the symmetric orbifold of $\mathbb{T}^4$ Chapter 5, see Section 6.5.5 below.

6.5.1 The $\mathbb{T}^4$ algebra

The first step of our argument consists of separating out the torus degrees of freedom from the rest (which is the $\mathcal{N} = 4$ analogue of (6.49)). In order to do so, we note that we can construct small $\mathcal{N} = 4$ generators out of the free fields as

$$(\mathcal{K}^a_{\mathbb{T}^4})_m = \tfrac{1}{2}(\sigma^a)_{\alpha\beta}(\Lambda^\alpha \bar{\Lambda}^\beta)_m \ , \tag{6.98a}$$
$$(\mathcal{G}^{\alpha\beta}_{\mathbb{T}^4})_r = (\Lambda^\alpha \partial \bar{\mathcal{X}}^\beta)_r - (\bar{\Lambda}^\alpha \partial \mathcal{X}^\beta)_r \ , \tag{6.98b}$$
$$(\mathcal{L}_{\mathbb{T}^4})_m = \varepsilon_{\alpha\beta}(\partial \mathcal{X}^\alpha \partial \bar{\mathcal{X}}^\beta)_m + \tfrac{1}{2}\varepsilon_{\alpha\beta}\big((\partial \Lambda^\alpha \bar{\Lambda}^\beta)_m - (\Lambda^\alpha \partial \bar{\Lambda}^\beta)_m\big) \ . \tag{6.98c}$$

These generators satisfy the small $\mathcal{N} = 4$ algebra with $c = 6$. We can then decouple the $\mathbb{T}^4$ part from the small $\mathcal{N} = 4$ algebra by considering the differences $\mathcal{L}_m - (\mathcal{L}_{\mathbb{T}^4})_m$ and similarly for $\mathcal{G}^{\alpha\beta}_r$ and $\mathcal{K}^a_m$. These difference satisfy again the small $\mathcal{N} = 4$ algebra, but commute with the torus modes. This shows that the chiral algebra of the seed theory of the symmetric product orbifold is

$$\Big[\text{small } \mathcal{N} = 4 \text{ with } c = 6(k-1)\Big] \oplus 4 \text{ free bosons and } 4 \text{ free bosons} \ . \tag{6.99}$$

6.5.2 $\mathcal{N} = 4$ Liouville theory

Next we want to show that the first term should be thought of as $\mathcal{N} = 4$ Liouville theory. Since $\mathcal{N} = 4$ Liouville theory is not very well known, we briefly review its main features below.

We start by recalling that bosonic Liouville theory is a marginal deformation of a single free boson (with background charge). Similarly, $\mathcal{N} = 4$ Liouville theory can be constructed starting from an $\mathcal{N} = 1$ supersymmetric WZW-model based on $\text{SU}(2) \times \text{U}(1)$ [53, 70, 236, 237], together with some background charge for the U(1) factor so that the total central charge is $c = 6\kappa$. (For the application we have in mind, we will later identify $\kappa = k - 1$.) We denote the generating fields by[68]

$$\psi^{\alpha\beta}\ , \quad \partial\varphi \quad \text{and} \quad J^a\ . \tag{6.100}$$

Here, $\partial\varphi$ is the bosonic generator of the U(1) factor. The indices α and β are spinor indices while a is an adjoint index of $\mathfrak{su}(2)$, and the currents J^a generate the affine $\mathfrak{su}(2)$ algebra at level $\kappa - 1$. Our conventions for their OPEs can be found in Appendix A.2.3. The chiral algebra has actually large $\mathcal{N} = 4$ superconformal symmetry, but it also contains a small $\mathcal{N} = 4$ algebra [70, 237], whose generators take the form

$$\begin{aligned} T &= \frac{1}{\kappa+1}\Big(J^3J^3 + \tfrac{1}{2}\big(J^+J^- + J^-J^+\big)\Big) + \frac{1}{2}\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\partial\psi^{\alpha\beta}\psi^{\gamma\delta} \\ &\qquad + \frac{1}{2}\partial\varphi\partial\varphi + \frac{i\kappa}{\sqrt{2(\kappa+1)}}\partial^2\varphi\ , \qquad (6.101\text{a}) \\ G^{\alpha\beta} &= \frac{1}{\sqrt{2}}(\partial\varphi\psi^{\alpha\beta}) + \frac{i}{\sqrt{\kappa+1}}\Big(-(\sigma_a)^{\alpha}{}_{\gamma}\big(J^a + \tfrac{1}{3}J^{(\mathrm{f},+)a}\big)\psi^{\gamma\beta} \\ &\qquad + \tfrac{1}{3}(\sigma_a)^{\beta}{}_{\gamma}J^{(\mathrm{f},-)a}\psi^{\alpha\gamma} + \kappa\partial\psi^{\alpha\beta}\Big)\ , \\ &\qquad (6.101\text{b}) \\ K^a &= J^a + J^{(\mathrm{f},+)a}\ , \qquad (6.101\text{c}) \end{aligned}$$

where the fermionic currents are

$$J^{(\mathrm{f},+)a} = \frac{1}{4}(\sigma_a)_{\alpha\gamma}\varepsilon_{\beta\delta}(\psi^{\alpha\beta}\psi^{\gamma\delta}) \quad \text{and} \quad J^{(\mathrm{f},-)a} = \frac{1}{4}\varepsilon_{\alpha\gamma}(\sigma_a)_{\beta\delta}(\psi^{\alpha\beta}\psi^{\gamma\delta})\ . \tag{6.102}$$

The vertex operators of this theory can be described by

$$\mathrm{e}^{\frac{1}{\sqrt{2(\kappa+1)}}(2p-i\kappa)\varphi}\mathcal{V}_\ell\ , \tag{6.103}$$

where $p \in \mathbb{R}$, and $\mathcal{V}_\ell$ denotes the $\mathfrak{su}(2)_{\kappa-1}$ primary of spin ℓ. It has conformal weight

$$h = \frac{\big(\ell+\frac{1}{2}\big)^2 + p^2}{\kappa+1} + \frac{\kappa-1}{4}\ . \tag{6.104}$$

In particular, the lowest conformal dimension of spin ℓ is therefore

$$\Delta_\ell^{\mathrm{NS}} = \frac{\big(\ell+\frac{1}{2}\big)^2}{\kappa+1} + \frac{\kappa-1}{4}\ , \tag{6.105}$$

above which there is a continuum of conformal weights. Here the $\mathfrak{su}(2)$ spin ℓ takes values in $\{0, \frac{1}{2}, \ldots, \frac{\kappa-1}{2}\}$, in agreement with the unitarity bound for the small $\mathcal{N} = 4$ superconformal algebra [238].

[68]These fields should not be confused with the world-sheet fields we were describing earlier: from now on we shall only talk about the fields of the (seed theory) of the spacetime CFT. We use the same symbols as before since this is the usual convention for $\mathcal{N} = 4$ Liouville theory.

The above discussion applies to the NS-sector. In the R-sector there is an additional contribution from the ground state energy of the four fermions, whose zero modes generate the $\mathfrak{su}(2)$ representation $\mathbf{2} \oplus 2 \cdot \mathbf{1}$. As a consequence, the value of the $\mathfrak{su}(2)$ spin ℓ is shifted by one unit, and the gap in the R-sector is

$$\Delta_\ell^{\mathrm{R}} = \frac{\ell^2 + p^2}{\kappa + 1} + \frac{\kappa - 1}{4} + \frac{1}{4} \, . \tag{6.106}$$

In the R-sector, the $\mathfrak{su}(2)$ spin ℓ takes values in $\{\frac{1}{2}, 1, \ldots, \frac{\kappa}{2}\}$.

6.5.3 Identifying Liouville theory on the world-sheet

In the bosonic case, Liouville theory is believed to be uniquely characterised by having Virasoro symmetry, together with the full spectrum of Liouville fields [94,227]. It is tempting to speculate that a similar statement should be true for $\mathcal{N} = 4$ Liouville theory. Since we have already shown that the first factor has small $\mathcal{N} = 4$ superconformal symmetry, it only remains to show that the world-sheet theory gives rise to the full spectrum of Liouville theory. Since the single-particle perturbative part of the spacetime theory only has a NS-sector, we need to show that the spacetime spectrum exhibits the gaps (6.105), together with a continuum above. Similarly, we show that the same is true for the twisted sectors of the symmetric orbifold (where for even twist w we also need the R-sector ground state energy (6.106), see [140].) This mirrors then precisely the analysis of Section 6.2.5 for the bosonic case.

In order to establish this, we first note that the restriction of the $\mathfrak{su}(2)$ spins to $\ell \in \{0, \frac{1}{2}, \ldots, \frac{k-2}{2}\}$ is correctly implemented in the world-sheet theory, since the bosonic $\mathfrak{su}(2)$ algebra (which we denoted by $\mathcal{K}^a$ above) is at level $k-2$. To determine the gap predicted by the world-sheet theory, we simply have to solve the mass-shell condition on the worldsheet for a continuous representation. In the w spectrally flowed NS-sector, it takes the form [140]

$$\frac{\frac{1}{4} + p^2}{k} - hw + \frac{kw^2}{4} + \frac{\ell(\ell+1)}{k} = \frac{1}{2} \, , \tag{6.107}$$

where the first term comes from the Casimir of the $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ representation, the next two terms arise from the spectral flow and the last term is the $\mathfrak{su}(2)_{k-2}$ ground state energy. Finally, the right-hand side is the appropriate normal ordering constant in the NS-sector. Here, h denotes the spectrally flowed J_0^3 eigenvalue, which corresponds to the conformal weight in the dual CFT. From this, we solve

$$h = \frac{\left(\ell + \frac{1}{2}\right)^2 + p^2}{wk} + \frac{kw^2 - 2}{4w} = \frac{6k}{24w}(w^2 - 1) + \frac{\Delta_\ell^{\mathrm{NS}}}{w} + \frac{p^2}{wk} \, , \tag{6.108}$$

which matches precisely with (6.105), provided that $\kappa = k - 1$. This state is allowed to exist only for an odd unit of spectral flow, because of the GSO projection.[69]

For even spectral flow, we have to apply another fermion, which we take to be $\chi^+_{-1/2}$, i.e. the positively charged $\mathfrak{su}(2)$ fermion. The mass-shell condition then reads

$$\frac{\frac{1}{4}+p^2}{k} - hw + \frac{kw^2}{4} + \frac{\ell(\ell-1)}{k} + \frac{1}{2} = \frac{1}{2}\,, \tag{6.109}$$

where ℓ is the actual $\mathfrak{su}(2)$ spin of the state. (Since we have applied $\chi^+_{-1/2}$, it differs by one unit from the spin of the ground state, which is therefore $\ell_0 = \ell - 1$.) Solving the mass-shell condition yields now

$$h = \frac{\ell(\ell-1)+\frac{1}{4}+p^2}{kw} + \frac{kw}{4} = \frac{6k}{24w}(w^2-1) + \frac{\Delta^{\mathrm{R}}_{\ell-\frac{1}{2}}}{w} + \frac{p^2}{wk} + \frac{1}{4w}\,. \tag{6.110}$$

This matches with the R-sector ground state energy of Liouville theory, using that the symmetric orbifold in even twist sectors behaves effectively as in the R-sector, see [50, 235]. Note that the additional contribution $+\frac{1}{4w}$ in (6.110) comes from the fact that also the additional $\mathbb{T}^4$ in (6.97) is now in the R-sector for which the ground state energy is $\frac{1}{4}$. Furthermore, the $\mathfrak{su}(2)$ spin ℓ is shifted by $\frac{1}{2}$ with respect to pure Liouville theory, because of the additional zero modes of the torus theory $\mathbb{T}^4$. Thus, the representation content matches exactly.

6.5.4 Spectrum generating algebra

So far we have only shown that the w-spectrally flowed continuous representations give rise to a spacetime spectrum on which the w-cycle twisted sector operators of the generators in (6.97) act. Now we want to show that these twisted sector operators generate in fact the entire spectrum.

We shall first consider the case $k \geq 2$; the case $k = 1$ will be discussed separately in the following section. For $k \geq 2$, the argument works essentially as in flat space. For $k \geq 2$, we have 8 bosonic and fermionic DDF operators as follows. For the bosonic operators, 4 of them come from $\mathcal{N} = 4$ Liouville theory (namely from the J^a and $\partial\varphi$ in (6.100)), while the other 4 are the 4 torus modes. For the fermions, we have 4 fermionic generators from $\mathcal{N} = 4$ Liouville theory (namely the $\psi^{\alpha\beta}$ in (6.100)), while the other 4 generators are the 4 torus fermions.

These DDF operators can now be compared to the world-sheet description. The matching of the fermions is straightforward since they define free fields

[69] Here we have only spectrally flow in the $\mathfrak{sl}(2,\mathbb{R})$ sector; then the GSO projection depends on the cardinality of the sepctral flow, see e.g. [195].

(and hence do not contain any null-vectors). As regards the bosons, we have before imposing the physical state conditions, 10 bosonic generators on the world-sheet: 3 from $\mathfrak{sl}(2)_{k+2}$, 3 from $\mathfrak{su}(2)_{k-2}$, and 4 from the torus. $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ does not have any null-vectors (for $k \geq 2$), and hence the physical state condition removes two of the bosonic generators, leaving essentially one boson behind (that we may identify with $\partial\varphi$ in (6.100)). The $\mathfrak{su}(2)_{k-2}$ generators of the world-sheet can be directly identified with the $\mathfrak{su}(2)_{\kappa-1}$ generators of (6.100) — in particular, their characters agree precisely, including null-vectors — while the remaining 4 bosons are torus bosons in both descriptions. The fact that our DDF operators generate the entire spectrum then follows by the usual character argument. This is to say, we can easily calculate the character of the physical spectrum from the world-sheet, and it manifestly agrees with the corresponding character of the DDF operators. This works separately for each w, and for each ground-state representation. Thus the DDF operators we have constructed generate the full spacetime spectrum.

6.5.5 The case of $k = 1$

The case $k = 1$ is very special. In particular, the Liouville part of the seed theory (6.99) now has $c = 0$, and we would expect that it vanishes entirely from the spectrum. In fact, this is precisely in agreement with what was shown in Chapter 5, where we determined the world-sheet characters at $k = 1$, and demonstrated that they are generated by 4 free bosons and fermions. Thus we should only expect to have $4 + 4$ DDF operators, and these are precisely the ones associated to $\mathbb{T}^4$ (that will always exist). Furthermore, at $k = 1$ the continuous representations account for the complete worldsheet theory, since there are no discrete representations on the worldsheet in this case Chapter 5. In view of the discussion in Section 6.2.6, this also has a natural interpretation from the dual CFT perspective: since the Liouville part is absent, there are no new fields propagating in intermediate channels and hence no additional representations should appear in the symmetric orbifold.

We should emphasise that relative to Chapter 5, where 'only' the spectrum was matched, we have now established that the algebraic structure of the spacetime theory is indeed that of the symmetric orbifold of $\mathbb{T}^4$: we have shown that the spacetime CFT contains the spectrum generating operators of the symmetric orbifold with the correct commutation relations. This essentially amounts to proving that the spacetime theory is indeed the symmetric orbifold of $\mathbb{T}^4$.

6.6 Summary and Conclusions

In this Chapter, we have continued our study of the background $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ and in particular its tensionless limit described by $k = 1$. By considering a complete set of DDF operators, we have shown that the spacetime theory is given by a symmetric orbifold of Liouville theory together with the internal CFT. We have established this for bosonic string theory on $\text{AdS}_3 \times X$, as well as for superstrings on $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$; in the latter case, the dual CFT is the symmetric orbifold of the product of $\mathcal{N} = 4$ Liouville theory with the $\mathbb{T}^4$ theory. We have moreover seen that the $k = 1$ limit comes about naturally, since in this case the $\mathcal{N} = 4$ Liouville part (together with its long string continuum) disappears.

This gives a fairly complete picture of holography on AdS_3 with pure NS-NS flux. The background is indeed 'singular', but this does not hinder the existence of a well-defined dual CFT. In the general case, the proposed dual CFTs contain also a continuum of states, and in particular the vacuum is non-normalisable. As we have seen, the entire spacetime spectrum is accounted for by the continuous representations on the world-sheet. We have argued in Section 6.2.6 that the discrete representations on the world-sheet give rise to non-normalisable operators in the dual CFT that are not directly part of the CFT spectrum, see also [62].

While our discussion in the bosonic case was general, we focused on the specific example of $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ in the supersymmetric case. This is because the fermions couple the AdS_3 factor to the rest of the background, and one cannot easily treat the general case uniformly. For instance, in the case of K3, the spacetime theory has also small $\mathcal{N} = 4$ supersymmetry and at least at the orbifold point of K3, one easily sees that the general answer for the dual CFT will be

$$\text{Sym}^N \Big(\Big[\mathcal{N} = 4 \text{ Liouville with } c = 6(k-1) \Big] \oplus \text{ K3} \Big) \ . \tag{6.111}$$

In particular, for $k = 1$, one simply recovers the symmetric orbifold of K3 [239].

It is interesting to note that the seed theories of the dual CFTs we have given are essentially the Drinfel'd Sokolov (quantum Hamiltonian) reductions of the respective worldsheet theories, in close analogy to the higher spin setting [240]. Indeed, it is well-known that the quantum Hamiltonian reduction of $\mathfrak{sl}(2,\mathbb{R})_k$ yields Liouville theory with central charge [241]

$$c = 13 - 6(-k+2) - \frac{6}{-k+2} = 1 + \frac{6(k-1)^2}{k-2} \ , \tag{6.112}$$

which differs by 24 from (6.50). This is related to the fact that in bosonic string theory, the ghosts contribute central charge $c = -26$, whereas in the quantum Hamiltonian reduction, they only contribute $c = -2$. Thus, while our construction is certainly related to quantum Hamiltonian reduction, it is not exactly clear what the precise relation should be.

Chapter 7

$\mathbf{AdS_3 \times S^3 \times S^3 \times S^1}$

In this chapter, we explore the background $AdS_3 \times S^3 \times S^3 \times S^1$, which supports large $\mathcal{N} = (4,4)$ spacetime supersymmetry. We will develop a hybrid formalism for the background. We will see that a similar phenomenon occurs as in the background $AdS_3 \times S^3 \times \mathbb{T}^4$: When one of the fluxes through the three-spheres attains its minimal value, the long string continuum disappears. We will show that the background with one unit of NS-NS flux is precisely dual to the symmetric orbifold of the WZW model on the space $S^3 \times S^1$. In the higher flux case, we demonstrate in the same spirit as in the previous chapter that the theory admits a dual description in terms of the symmetric product orbifold of a large $\mathcal{N} = 4$ Liouville theory.

7.1 The worldsheet theory

In the RNS-formalism, the worldsheet theory is described by the WZW-model

$$\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \oplus \mathfrak{su}(2)^{(1)}_{k^+} \oplus \mathfrak{su}(2)^{(1)}_{k^-} \oplus \mathfrak{u}(1)^{(1)} , \tag{7.1}$$

together with the usual superconformal ghost system. Here, $\mathfrak{g}^{(1)}_k$ denotes the $\mathcal{N} = 1$ superconformal affine algebra of $\mathfrak{g}$ at level k. Criticality of the background requires the three levels to be related according to

$$\frac{1}{k} = \frac{1}{k^+} + \frac{1}{k^-} . \tag{7.2}$$

As is well known, the fermions of this algebra can be decoupled, leading to

$$\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \cong \mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus 3 \text{ free fermions} , \tag{7.3}$$

$$\mathfrak{su}(2)^{(1)}_{k^\pm} \cong \mathfrak{su}(2)_{k^\pm-2} \oplus 3 \text{ free fermions} . \tag{7.4}$$

We shall denote the decoupled currents by $\mathcal{J}^a$ and $\mathcal{K}^{(\pm)a}$ with levels $k+2$ and $k^\pm - 2$, respectively. Here, $a \in \{+,-,3\}$ is an adjoint index of $\mathfrak{sl}(2,\mathbb{R})$

or $\mathfrak{su}(2)$. Similarly, we denote the corresponding fermionic partners as ψ^a and $\chi^{(\pm)a}$. Finally, the free boson of the S^1 will be denoted by $\partial\Phi$ and the corresponding fermion by λ. The relevant commutation relations are spelled out in Appendix A.2. The $\mathcal{N}=1$ superconformal currents on the worldsheet are defined by

$$\begin{aligned}
T(z) = \frac{1}{k}\Big(& -\mathcal{J}^3\mathcal{J}^3 + \tfrac{1}{2}(\mathcal{J}^+\mathcal{J}^- + \mathcal{J}^-\mathcal{J}^+) + \psi^3\partial\psi^3 - \tfrac{1}{2}(\psi^+\partial\psi^- + \psi^-\partial\psi^+)\Big) \\
&+ \frac{1}{k^+}\Big(\mathcal{K}^{(+)3}\mathcal{K}^{(+)3} + \tfrac{1}{2}(\mathcal{K}^{(+)+}\mathcal{K}^{(+)-} + \mathcal{K}^{(+)-}\mathcal{K}^{(+)+}) \\
&\qquad - \chi^{(+)3}\partial\chi^{(+)3} - \tfrac{1}{2}(\chi^{(+)+}\partial\chi^{(+)-} + \chi^{(+)-}\partial\chi^{(+)+})\Big) \\
&+ \frac{1}{k^-}\Big(\mathcal{K}^{(-)3}\mathcal{K}^{(-)3} + \tfrac{1}{2}(\mathcal{K}^{(-)+}\mathcal{K}^{(-)-} + \mathcal{K}^{(-)-}\mathcal{K}^{(-)+})\Big) \\
&\qquad - \chi^{(-)3}\partial\chi^{(-)3} - \tfrac{1}{2}(\chi^{(-)+}\partial\chi^{(-)-} + \chi^{(-)-}\partial\chi^{(-)+})\Big) \\
&+ \frac{1}{2}(\partial\Phi\partial\Phi) - \frac{1}{2}(\lambda\partial\lambda)\ , \qquad (7.5)
\end{aligned}$$

$$\begin{aligned}
G(z) = -\frac{1}{k}\Big(& -\mathcal{J}^3\psi^3 + \tfrac{1}{2}(\mathcal{J}^+\psi^- + \mathcal{J}^-\psi^+) - \tfrac{1}{k}(\psi^3\psi^+\psi^-)\Big) \\
&- \frac{1}{k^+}\Big(\mathcal{K}^{(+)3}\chi^{(+)3} + \tfrac{1}{2}(\mathcal{K}^{(+)+}\chi^{(+)-} + \mathcal{K}^{(+)-}\chi^{(+)+})\Big) \\
&- \frac{1}{k^-}\Big(\mathcal{K}^{(-)3}\chi^{(-)3} + \tfrac{1}{2}(\mathcal{K}^{(-)+}\chi^{(-)-} + \mathcal{K}^{(-)-}\chi^{(-)+})\Big) \\
&- \frac{1}{(k^+)^2}(\chi^{(+)3}\chi^{(+)+}\chi^{(+)-}) - \frac{1}{(k^-)^2}\frac{1}{k^-}(\chi^{(-)3}\chi^{(-)+}\chi^{(-)-}) \\
&+ \frac{1}{2}(\partial\Phi\lambda)\ . \qquad (7.6)
\end{aligned}$$

The $\mathcal{N}=1$ superconformal structure on the worldsheet allows us to define the BRST charge as

$$Q_{\mathrm{BRST}} = \oint \mathrm{d}z\, \Big(c\big(T + \tfrac{1}{2}T_{\mathrm{gh}}\big) + \gamma\big(G + \tfrac{1}{2}G_{\mathrm{gh}}\big)\Big)\ . \qquad (7.7)$$

Here, T^{gh} and G^{gh} are the $\mathcal{N}=1$ generators of the superconformal ghost system; this consists of a bc system with $\lambda=2$ and a $\beta\gamma$ system with $\lambda=\frac{3}{2}$, whose OPE's we take to be (see also Appendix A.3)

$$b(z)c(w) \sim \frac{1}{z-w}\ , \qquad \beta(z)\gamma(w) \sim -\frac{1}{z-w}\ . \qquad (7.8)$$

In these conventions, the $\mathcal{N}=1$ superconformal algebra of the ghost system is then

$$T_{\mathrm{gh}}(z) = -2b(\partial c) - (\partial b)c - \tfrac{3}{2}\hat{\beta}(\partial\hat{\gamma}) - \tfrac{1}{2}(\partial\hat{\beta})\hat{\gamma}\ , \qquad (7.9)$$

$$G_{\mathrm{gh}}(z) = (\partial\hat{\beta})c + \tfrac{3}{2}\hat{\beta}(\partial c) - \tfrac{1}{2}b\hat{\gamma}\ , \qquad (7.10)$$

which realises the $\mathcal{N}=1$ superconformal algebra with central charge $c=-15$.

7.1.1 Bosonisation

In order to relate this description to the hybrid formalism it is convenient to bosonise the fermions as

$$\partial H_1(z) = \frac{1}{k}(\psi^+\psi^-)(z)\ , \qquad \partial H_2(z) = \frac{1}{k^+}(\chi^{(+)+}\chi^{(+)-})(z)\ , \tag{7.11a}$$

$$\partial H_3(z) = \frac{2}{\sqrt{kk^+}}(\psi^3\chi^{(+)3})(z)\ , \quad \partial H_4(z) = \frac{1}{k^-}(\chi^{(-)+}\chi^{(-)-})(z)\ , \tag{7.11b}$$

$$\partial H_5(z) = i\sqrt{\frac{2}{k^-}}(\lambda\chi^{(-)3})(z)\ . \tag{7.11c}$$

This bosonisation scheme reduces to that of the previous chapter in the limit $\gamma \to 1$, in which the geometry degenerates to $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$. The bosons are normalised as

$$\partial H_i(z)\partial H_j(w) \sim \frac{\delta_{ij}}{(z-w)^2}\ . \tag{7.12}$$

We also choose the same bosonisation of the superconformal ghost system (the $\beta\gamma$ system) as there, i.e. we write

$$\beta(z) = \mathrm{e}^{-\varphi(z)+\chi(z)}\partial\chi(z)\ , \qquad \gamma = \mathrm{e}^{\varphi(z)-\chi(z)}\ , \tag{7.13}$$

where the two bosons $\varphi(z)$ and $\chi(z)$ have background charge $Q_\varphi = 2$ and $Q_\chi = -1$, respectively, and OPEs

$$\varphi(z)\varphi(w) \sim -\log(z-w)\ , \qquad \chi(z)\chi(w) \sim \log(z-w)\ . \tag{7.14}$$

The energy-momentum tensor for the free-field representation then takes the form

$$T = T^\varphi + T^\chi\ , \tag{7.15}$$

$$T^\varphi = -\tfrac{1}{2}(\partial\varphi)^2 + \partial^2\varphi\ , \tag{7.16}$$

$$T^\chi = \tfrac{1}{2}(\partial\chi)^2 + \tfrac{1}{2}\partial^2\chi\ . \tag{7.17}$$

Finally, the picture charge is defined as

$$Q_{\mathrm{pic}} = \oint \mathrm{d}z\ \big(\partial\chi - \partial\varphi\big)\ . \tag{7.18}$$

7.2 The hybrid formalism

Next we want to rewrite these degrees of freedom in a way that makes spacetime supersymmetry manifest. This can be done by passing to a $\mathfrak{d}(2,1;\alpha)_k$ WZW-model, thereby leading to the natural analogue of the 'hybrid formalism' for this background; as far as we are aware, the hybrid formalism for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ has not been developed before.

7.2.1 Defining free field variables

We start by defining the vertex operators, cf. (3.29a) and (3.29b) in the case of $AdS_3 \times S^3 \times \mathbb{T}^4$

$$p^{\alpha\beta} = e^{\frac{1}{2}(\alpha H_1 + \beta H_2 + \alpha\beta H_3 + H_4 + H_5 - \varphi)} , \tag{7.19a}$$

$$\theta^{\alpha\beta} = e^{\frac{1}{2}(\alpha H_1 + \beta H_2 - \alpha\beta H_3 - H_4 - H_5 + \varphi)} , \tag{7.19b}$$

which obey the free field OPEs

$$p^{\alpha\beta}(z)\theta^{\gamma\delta}(w) \sim \frac{\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}}{z-w} . \tag{7.20}$$

We have suppressed the cocycle factors in the expressions. These fields have conformal weight 1 and 0, and picture numbers $(-\frac{1}{2})$ and $(\frac{1}{2})$, respectively. The indices $\alpha, \beta \in \{+,-\}$ are spinor indices of $\mathfrak{sl}(2,\mathbb{R})_k \oplus \mathfrak{su}(2)_{k^+}$. Note that we have explicitly broken the second $\mathfrak{su}(2)$ symmetry: $p^{\alpha\beta}$ carries charge $+\frac{1}{2}$ under $\mathfrak{su}(2)_{k^-}$, while that of $\theta^{\alpha\beta}$ is $-\frac{1}{2}$.

In the case of $AdS_3 \times S^3 \times \mathbb{T}^4$, one can construct out of these fields the affine algebra $\mathfrak{psu}(1,1|2)_k$. Analogously, as we shall now explain, we can define a $\mathfrak{d}(2,1;\alpha)_k$ affine algebra in our case (and it will be part of the hybrid formulation). To start with, we define

$$S^{\alpha\beta+} = p^{\alpha\beta} - \frac{k^+}{2(k^+ + k^-)}(J^{(-)+}\theta^{\alpha\beta}) , \tag{7.21}$$

which define half of the supercurrents in $\mathfrak{d}(2,1;\alpha)_k$. They are also part of the Wakimoto construction of $\mathfrak{d}(2,1;\alpha)_k$ that is described in detail in Appendix D.1, see eq. (D.7).

7.2.2 Remaining fields

In order to construct the remaining fields of the hybrid formalism (and complete the construction of $\mathfrak{d}(2,1;\alpha)_k$) we now recall that the bosonic generators of $\mathfrak{d}(2,1;\alpha)_k$ form the Lie algebra $\mathfrak{sl}(2,\mathbb{R})_k \oplus \mathfrak{su}(2)_{k^+} \oplus \mathfrak{su}(2)_{k^-}$. The original boson $\partial\Phi$ corresponding to S^1 commutes with $\mathfrak{d}(2,1;\alpha)_k$, and can be directly added to the theory. (It is naturally defined in the (0)-picture.) This accounts for all bosonic degrees of freedom. Furthermore, we have not used the *bc* ghosts in our reformulation and they simply continue to be also part of the hybrid description.

As regards the fermions, we can define four more fermions which commute with the $p^{\alpha\beta}$'s as well as with the $\theta^{\alpha\beta}$'s. As in the standard hybrid formalism (see Section 3.2.2), they are given by

$$e^{H_4 - \varphi + \chi} , \qquad e^{H_5 - \varphi + \chi} , \qquad e^{-H_4 + \varphi - \chi} , \qquad e^{-H_5 + \varphi - \chi} , \tag{7.22}$$

where we have also made use of the boson χ that was introduced in the bosonisation of the superconformal ghosts, see eq. (7.13). The conformal weights of the first two fermions is one, while that of the last two fermions is zero; thus they define 2 pairs of topologically twisted fermions (i.e. two *bc* systems of conformal weight 1 and 0). Finally, we replace the other boson φ from the bosonisation of the superconformal ghosts by the combination

$$\rho = 2\varphi - H_4 - H_5 - \chi \,, \tag{7.23}$$

that commutes with all the above fields, and defines the new ghost field of the hybrid formalism. As in Section 3.2.2 one then checks that the central charge of all of these fields is equal to zero, i.e. that we have accounted for all degrees of freedom. Thus, we have reassembled the RNS degrees of freedom as

$$\begin{aligned}\mathfrak{d}(2,1;\alpha)_k \oplus \mathfrak{u}(1) \oplus 2 \text{ pairs of topologically twisted fermions from eq. (7.22)}\\ \oplus\, bc \text{ and } \rho \text{ ghosts}\,.\end{aligned} \tag{7.24}$$

While this construction is fairly parallel to the case of $\mathbb{T}^4$, there is one important difference: the $\mathfrak{su}(2)_{k^-}$ currents that appear in $\mathfrak{d}(2,1;\alpha)_k$, see eqs. (D.8), (D.9) and (D.10a), do *not* correspond to the correct spacetime supersymmetry currents. One can repair this by redefining the generators of $\mathfrak{d}(2,1;\alpha)_k$ as

$$\tilde{K}^{(-)3} := K^{(-)3} + \partial\varphi - \partial H_5 \,, \tag{7.25}$$

$$\tilde{K}^{(-)-} := K^{(-)-} - 2(\partial\varphi - \partial H_5)\hat{\gamma} \,, \tag{7.26}$$

$$\tilde{S}^{\alpha\beta-} := S^{\alpha\beta-} + \frac{k^+}{k^- + k^-}(\partial\varphi - \partial H_5)\theta^{\alpha\beta} \,, \tag{7.27}$$

without changing the commutation relations of $\mathfrak{d}(2,1;\alpha)_k$. Here, $\hat{\gamma}$ is the free field appearing in the Wakimoto representation of $\mathfrak{sl}(2,\mathbb{R})_{k^--2}$, see Appendix D.1 for details. However, this redefined $\mathfrak{d}(2,1;\alpha)_k$ algebra does not commute any longer with the remaining free fermions (7.22).

We should mention that we can also express the physical state conditions in terms of these new variables, which entails rewriting the BRST operator (7.7). The explicit expressions are quite complicated (as already in the $\mathbb{T}^4$ case [126, 186]) but since we will not need them for our purposes, we have not written them out explicitly.

7.3 Representations of $\mathfrak{d}(2,1;\alpha)$

For the following, it is important to understand representations of $\mathfrak{d}(2,1;\alpha)$ in detail. The bosonic subalgebra of $\mathfrak{d}(2,1;\alpha)$ is $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

While the representations of $\mathfrak{su}(2)$ that appear are the familiar finite-dimensional spin ℓ representations, the representations of $\mathfrak{sl}(2,\mathbb{R})$ that are relevant are either discrete lowest (or highest) weight representations that we denote by $\mathcal{D}^j_+$ (or $\mathcal{D}^j_-$ in the case of lowest weight); the other class of $\mathfrak{sl}(2,\mathbb{R})$ representations that appear are the continuous representations that are neither highest nor lowest weight and that will be denoted by $\mathcal{C}^j_\lambda$. In either case, j determines the value of the quadratic Casimir of $\mathfrak{sl}(2,\mathbb{R})$

$$\mathcal{C} = -J^3_0 J^3_0 + \frac{1}{2}\left(J^+_0 J^-_0 + J^-_0 J^+_0\right) , \tag{7.28}$$

as $\mathcal{C} = -j(j-1)$, and in the case of the continuous representations $\lambda \in \mathbb{R}/\mathbb{Z}$ denotes the fractional part of the J^3_0-eigenvalues. Since the Casimir $\mathcal{C}$ is invariant under $j \to 1-j$, we may assume without loss of generality that $\mathrm{Re}(j) \geq 1/2$. More details about our conventions can be found in Appendix B.

7.3.1 Long representations

Next we want to understand the structure of the representations of $\mathfrak{d}(2,1;\alpha)$. The fermionic generators of $\mathfrak{d}(2,1;\alpha)$ form a Clifford algebra, and the representations of $\mathfrak{d}(2,1;\alpha)$ are thus generated from an irreducible representation of the bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ by the action of these fermionic modes. We shall mainly focus on the case where the representation with respect to $\mathfrak{sl}(2,\mathbb{R})$ is a continuous representation,[70] and we shall label the representations of $\mathfrak{su}(2)$ by their dimension $\mathbf{m}^\pm$. A generic (long) multiplet decomposes then with respect to the bosonic subalgebra as

$$\begin{array}{ccc} & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{m}^+, \mathbf{m}^-) & \\ & (\mathcal{C}^{j}_{\lambda}, \mathbf{m}^+ \pm \mathbf{1}, \mathbf{m}^- \pm \mathbf{1}) & \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{m}^+ \pm \mathbf{2}, \mathbf{m}^-) & 2\cdot(\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{m}^+, \mathbf{m}^-) & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{m}^+, \mathbf{m}^- \pm \mathbf{2}) \\ & (\mathcal{C}^{j+1}_{\lambda}, \mathbf{m}^+ \pm \mathbf{1}, \mathbf{m}^- \pm \mathbf{1}) & \\ & (\mathcal{C}^{j+\frac{3}{2}}_{\lambda+\frac{1}{2}}, \mathbf{m}^+, \mathbf{m}^-) & \end{array} . \tag{7.29}$$

[70]The situation for the discrete representations is essentially identical.

For small $\mathbf{m}^{\pm}$, additional shortenings occur; for example if $\mathbf{m}^{+} = 1$ — this case will be important below — the representation shortens to

$$\begin{array}{ccc} & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{m}^-) & \\ & (\mathcal{C}^{j}_{\lambda}, \mathbf{2}, \mathbf{m}^- \pm \mathbf{1}) & \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{3}, \mathbf{m}^-) & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{m}^-) & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{m}^- \pm \mathbf{2}) \\ & (\mathcal{C}^{j+1}_{\lambda}, \mathbf{2}, \mathbf{m}^- \pm \mathbf{1}) & \\ & (\mathcal{C}^{j+\frac{3}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{m}^-) & \end{array} \ . \tag{7.30}$$

However, even in this case, the multiplet still contains a representation with $\mathbf{m}^{+} \geq \mathbf{3}$.

In the following we shall mainly be interested in the $\mathfrak{d}(2,1;\alpha)$ representations that can appear as (Virasoro) highest weights of an affine $\mathfrak{d}(2,1;\alpha)$ representation at $k^{+} = 1$. Then, because of the usual representation theory of $\mathfrak{su}(2)_1$, see also the analogous discussion in Chapter 5, only $\mathfrak{d}(2,1;\alpha)$ representations with $\mathbf{m}^{+} \leq \mathbf{2}$ are allowed. The above argument therefore shows that only 'short' representations of $\mathfrak{d}(2,1;\alpha)$ are then possible.

7.3.2 Short representations

We have analysed systematically the (short) representations with $\mathbf{m}^{+} \leq \mathbf{2}$. The analysis is fairly parallel to the case discussed in detail in Chapter 5, and up to relabelling, the only representations with this property have the form

$$\begin{array}{ccc} & (\mathcal{C}^{j}_{\lambda}, \mathbf{2}, \mathbf{m}) & \\ (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{m}+\mathbf{1}) & & (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{m}-\mathbf{1}) \end{array} \ , \tag{7.31}$$

where j (with $\mathfrak{Re}(j) \geq 1/2$) will be determined momentarily. Note that if the multiplet was a discrete multiplet (i.e. if the continuous representations $\mathcal{C}^{j}_{\lambda}$ were replaced by the discrete representations $\mathcal{D}^{j}_{+}$), we could easily determine the relevant shortening condition: it requires that the lowest weight state, i.e. the state in $(\mathcal{D}^{j-1/2}_{+}, \mathbf{1}, \mathbf{m}+\mathbf{1})$ is BPS, and hence saturates the familiar BPS bound [65, 133], which in the above parametrisation (see also [67, 134]) takes the form

$$j = (1-\gamma)\big(\ell + \tfrac{1}{2}\big) + \tfrac{1}{2} \ . \tag{7.32}$$

Here we have defined $\gamma = \frac{\alpha}{1+\alpha}$, and expressed the $\mathfrak{su}(2)$-representation via its spin, $\mathbf{m} = 2\ell + 1$.

The same result is also true in the continuous case, as we shall now explain. One way of seeing this is to decompose the $\mathfrak{d}(2,1;\alpha)$-Casimir into its bosonic and its fermionic pieces

$$\mathcal{C}^{\mathfrak{d}(2,1;\alpha)} = \mathcal{C}^{\mathfrak{d}(2,1;\alpha)}_{\text{bos}} + \mathcal{C}^{\mathfrak{d}(2,1;\alpha)}_{\text{ferm}} , \tag{7.33}$$

$$\mathcal{C}^{\mathfrak{d}(2,1;\alpha)}_{\text{bos}} = \mathcal{C}^{\mathfrak{sl}(2,\mathbb{R})} + \gamma \mathcal{C}^{\mathfrak{su}(2)_+} + (1-\gamma)\mathcal{C}^{\mathfrak{su}(2)_-} , \tag{7.34}$$

$$\mathcal{C}^{\mathfrak{d}(2,1;\alpha)}_{\text{ferm}} = -\frac{1}{2}\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}\varepsilon_{\gamma\rho}S_0^{\alpha\beta\gamma}S_0^{\mu\nu\rho} . \tag{7.35}$$

The fermionic Casimir can be computed explicitly on the representations of the bosonic subalgebra with the result[71]

$$\mathcal{C}^{\mathfrak{d}(2,1;\alpha)}_{\text{ferm}}\Big|_{(\mathcal{C}^j_\lambda,\mathbf{2},\mathbf{m})} = -\gamma , \qquad \mathcal{C}^{\mathfrak{d}(2,1;\alpha)}_{\text{ferm}}\Big|_{(\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}},\mathbf{1},\mathbf{m+1})} = -(1-\gamma)(2\ell+1) . \tag{7.36}$$

On these two representations of the bosonic subalgebra, the full $\mathfrak{d}(2,1;\alpha)$ Casimir therefore takes the values

$$\mathcal{C}^{\mathfrak{d}(2,1;\alpha)}\Big|_{(\mathcal{C}^j_\lambda,\mathbf{2},\mathbf{m})} = -j(j-1) + \frac{3\gamma}{4} + (1-\gamma)\ell(\ell+1) - \gamma , \tag{7.37}$$

$$\begin{aligned}\mathcal{C}^{\mathfrak{d}(2,1;\alpha)}\Big|_{(\mathcal{C}^{j-\frac{1}{2}}_\lambda,\mathbf{1},\mathbf{m+1})} &= -\left(j-\tfrac{1}{2}\right)\left(j-\tfrac{3}{2}\right) + (1-\gamma)\left(\ell+\tfrac{1}{2}\right)\left(\ell+\tfrac{3}{2}\right) \\ &\quad - (1-\gamma)(2\ell+1) .\end{aligned} \tag{7.38}$$

Demanding the two expressions to be equal reproduces then (7.32), in which case the Casimir simplifies to

$$\mathcal{C}^{\mathfrak{d}(2,1;\alpha)} = \gamma(1-\gamma)\left(\ell+\tfrac{1}{2}\right)^2 . \tag{7.39}$$

We should mention that for the minimal value of $\ell = 0$, the third term in (7.31) is absent, i.e. the representation is ultrashort, and takes the form

$$(\mathcal{C}^j_\lambda, \mathbf{2}, \mathbf{1}) \oplus (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{2}) \tag{7.40}$$

with $j = 1 - \frac{\gamma}{2}$.

In the limit $\gamma \to 1$, $\mathfrak{d}(2,1;\alpha)$ degenerates to $\mathfrak{psu}(1,1|2)$ and the second $\mathfrak{su}(2)$ becomes an outer automorphism. Then the short representation reduces as

$$\begin{array}{ccc} & (\mathcal{C}^j_\lambda, \mathbf{2}, \mathbf{m}) & \\ (\mathcal{C}^{j+\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{m+1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{m-1}) \end{array} \xrightarrow{\gamma\to 1} m \times \left((\mathcal{C}^{\frac{1}{2}}_\lambda, \mathbf{2}) \oplus 2\cdot(\mathcal{C}^0_{\lambda+\frac{1}{2}}, \mathbf{1}) \right) . \tag{7.41}$$

The expression in the bracket on the right hand side is the short representation of $\mathfrak{psu}(1,1|2)$ that was discussed in Chapter 5. Similarly, the shortening

[71] One can also work this out on the third representation $(\mathcal{C}^{j+\frac{1}{2}}_\lambda, \mathbf{1}, \mathbf{m-1})$, but the analysis is more complicated in that case.

condition (7.32) just becomes $j = \frac{1}{2}$ in this limit, again in agreement with the analysis of Chapter 5.

The continuous representations $\mathcal{C}^j_\lambda$ of $\mathfrak{sl}(2,\mathbb{R})$ are indecomposable for $j = \lambda$ and $1 - j = \lambda$, i.e. for $\lambda = \pm j \,(\mathrm{mod}\, \mathbb{Z})$. The same property carries, of course, also through to the $\mathfrak{d}(2,1;\alpha)$ representations. For the above short representations this becomes

$$\lambda = \pm j = \pm\lambda_\ell \,, \qquad \lambda_\ell = (1-\gamma)(\ell + \tfrac{1}{2}) + \tfrac{1}{2} \,. \tag{7.42}$$

In each case, there is a discrete subrepresentation, and we can define the continuous representation such that the discrete subrepresentation is either highest of lowest weight. Altogether there are therefore four cases.

In addition there is another degeneration that occurs if the Casimir of a $\mathfrak{sl}(2,\mathbb{R})$ representation vanishes, since it contains then the trivial representation as a subrepresentation. There are two ways in which this may occur in (7.31). First, we can formally set $\mathbf{m} = \mathbf{0}$, in which case we just keep the left-hand-factor

$$(\mathcal{C}^0_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{1}) \,, \tag{7.43}$$

where we have used that $\mathbf{m} = \mathbf{0}$ leads to $\ell = -\frac{1}{2}$ and hence to $j = \frac{1}{2}$ in (7.32). This then contains the trivial representation for $\lambda = \frac{1}{2}$. The other case arises for

$$\mathbf{m} = \frac{1}{1-\gamma} \in \mathbb{Z}_{\geq 0} \,, \tag{7.44}$$

since then (7.32) leads to $j = 1$. (This is obviously only possible provided that $(1-\gamma)^{-1}$ is an integer.) In this case the middle representation in (7.31) can contain the trivial $\mathfrak{sl}(2,\mathbb{R})$ representation, and then the two other terms will be absent. Thus, we conclude that also

$$\left(\mathcal{C}^1_\lambda, \mathbf{2}, \mathbf{m} = \frac{1}{1-\gamma}\right) \tag{7.45}$$

is a consistent multiplet. Note that there is no analogue of this in the limit $\gamma \to 1$.

7.4 The $\mathfrak{d}(2,1;\alpha)$ WZW-model at $k^+ = 1$

In the following we shall concentrate on the WZW-model based on $\mathfrak{d}(2,1;\alpha)$ with $k^+ = 1$. We shall set $k^- = \kappa + 1$ with $\kappa \in \mathbb{Z}_{\geq 0}$, as this will be convenient below. With this choice of parameters, we then have

$$k = \gamma = \frac{\kappa+1}{\kappa+2} \qquad \text{so that} \qquad (1-\gamma) = \frac{1}{\kappa+2} \,, \tag{7.46}$$

see (7.2).

7.4.1 The affine short representations

As we have explained in the previous section, the only allowed ground state representations are the short continuous representations of $\mathfrak{d}(2,1;\alpha)$ of eq. (7.31). We will denote the resulting affine representations by $\mathcal{F}^{\ell}_{\lambda}$, where $\ell = 2\mathbf{m}+1$. Since the $\mathfrak{su}(2)_{k^-}$ ground state representations must have spin less or equal to $\frac{1}{2}k^-$, ℓ is allowed to take only the values $\ell \in \{0, \frac{1}{2}, \ldots, \frac{\kappa}{2}\}$ — note that the multiplet (7.31) also contains a representation with $\mathbf{m}+\mathbf{1}$. This bound was noted in the discrete case already in [133].

As we have explained above, the ground state representations of $\mathcal{F}^{\ell}_{\lambda}$ become indecomposable for $\lambda = \pm\lambda_\ell$ (7.42) and the same is, of course, also true for the affine representations. We shall denote the corresponding discrete subrepresentations by

$$\mathcal{G}^{\ell}_{>,\pm} \subset \mathcal{F}^{\ell}_{\lambda_\ell} \,, \qquad \mathcal{G}^{\ell}_{<,\pm} \subset \mathcal{F}^{\ell}_{-\lambda_\ell} \,, \tag{7.47}$$

where $\pm$ refers to whether the representation is lowest weight $(+)$, i.e. runs to the right, or whether it is highest weight $(-)$, i.e. runs to the left. Note that for $\gamma \neq 0,1$, the parameter $\lambda_\ell \notin \frac{1}{2}\mathbb{Z}$, and hence λ_ℓ and $-\lambda_\ell$ never differ by an integer (and hence never define the same representation).

The other representations that will be relevant for us is the vacuum representation $\mathcal{L}$ of $\mathfrak{d}(2,1;\alpha)_\kappa$ — this is the affine representation based on the trivial representation of $\mathfrak{d}(2,1;\alpha)$ — as well as the representation $\mathcal{L}'$, whose ground state representation is (7.45). Note that, because of (7.46), $(1-\gamma)^{-1} = \kappa+2$ is an integer, and hence this representation exists for all κ. As we shall see below, see eq. (7.50c), $\mathcal{L}'$ arises naturally by applying the joint spectral flow in the two affine $\mathfrak{su}(2)$'s to the vacuum representation.

Thus, up to now, we have the following irreducible modules of $\mathfrak{d}(2,1;\alpha)_k$ for $k^+ = 1$

$$\mathcal{F}^{\ell}_{\lambda} \,, \quad \mathcal{G}^{\ell}_{<,\pm} \,, \quad \mathcal{G}^{\ell}_{>,\pm} \,, \quad \mathcal{L} \,, \quad \mathcal{L}' \,, \tag{7.48}$$

where ℓ runs over $\ell \in \{0, \frac{1}{2}, \ldots, \frac{\kappa}{2}\}$ and $\lambda \in \mathbb{R}/\mathbb{Z}$ with $\lambda \neq \pm\lambda_\ell$.

7.4.2 Spectral flow

For the following it will be important that $\mathfrak{d}(2,1;\alpha)_k$ possesses a spectral flow automorphism σ. On the bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R})_k \oplus \mathfrak{su}(2)_{k^+} \oplus \mathfrak{su}(2)_{k^-}$, we define it to act by a simultaneous spectral flow on $\mathfrak{sl}(2,\mathbb{R})_k \oplus \mathfrak{su}(2)_{k^+}$,

$$\sigma^w(J^3_m) = J^3_m + \tfrac{kw}{2}\delta_{m,0} \,, \tag{7.49a}$$

$$\sigma^w(J^\pm_m) = J^\pm_{m\mp w} \,, \tag{7.49b}$$

$$\sigma^w(K^{(+)3}_m) = K^{(+)3}_m + \tfrac{k^+ w}{2}\delta_{m,0} \,, \tag{7.49c}$$

$$\sigma^w(K_m^{(+)\pm}) = K_{m\pm w}^{(+)\pm} , \tag{7.49d}$$

$$\sigma^w(K_m^{(-)a}) = K_m^{(-)a} , \tag{7.49e}$$

$$\sigma^w(S_m^{\alpha\beta\gamma}) = S_{m+\frac{1}{2}w(\beta-\alpha)}^{\alpha\beta\gamma} . \tag{7.49f}$$

In particular, this spectral flow keeps the supercharges integer moded. As we will see below, see also [60], these spectrally flowed representations will have to be included in order to get a well-defined worldsheet theory; we therefore need to extend (7.48) by their spectrally flowed images.

We should mention that we have made an artificial choice in flowing in $\mathfrak{su}(2)_{k^+}$, and not in $\mathfrak{su}(2)_{k^-}$. This is reflected by the existence of another spectral flow ρ, which flows simultaneously in the two $\mathfrak{su}(2)$'s. This spectral flow does not generate any new representations, and it satisfies $\rho^2 = 1$.[72]

Since spectral flow maps representations to representations, there are in fact a number of identifications. In particular, we have

$$\rho(\mathcal{F}_\lambda^\ell) = \mathcal{F}_{\lambda+\frac{1}{2}}^{\frac{\kappa}{2}-\ell} , \qquad \rho(\mathcal{G}_{>,\pm}^\ell) = \mathcal{G}_{<,\pm}^{\frac{\kappa}{2}-\ell} , \qquad \rho(\mathcal{L}) = \mathcal{L}' , \tag{7.50a}$$

$$\sigma(\mathcal{L}) \cong \mathcal{G}_{<,+}^0 , \qquad \sigma^{-1}(\mathcal{L}) \cong \mathcal{G}_{>,-}^0 , \tag{7.50b}$$

$$\sigma(\mathcal{L}') \cong \mathcal{G}_{>,+}^{\frac{\kappa}{2}} , \qquad \sigma^{-1}(\mathcal{L}') \cong \mathcal{G}_{<,-}^{\frac{\kappa}{2}} , \tag{7.50c}$$

$$\sigma(\mathcal{G}_{>,-}^{\ell+\frac{1}{2}}) \cong \mathcal{G}_{>,+}^\ell , \qquad \sigma^{-1}(\mathcal{G}_{<,+}^{\ell+\frac{1}{2}}) \cong \mathcal{G}_{<,-}^\ell . \tag{7.50d}$$

Finally, as in the case studied in Chapter 5, the CFT is actually logarithmic, and one also needs to consider indecomposable representations. We have already seen that for $\lambda = \pm\lambda_\ell$ the module $\mathcal{F}_\lambda^\ell$ becomes indecomposable and contains a discrete subrepresentation. As it turns out — this is typical for logarithmic CFTs — $\mathcal{F}_\lambda^\ell$ itself does not appear in the spectrum of the theory, but it is instead part of an even larger indecomposable module. While these indecomposable modules lead to many technical complications, most of our results are largely unaffected by this subtlety, see also [185, 186]. We have therefore relegated the analysis of these indecomposable representations to Appendix D.4.

7.4.3 The fusion rules

Next we want to describe the fusion rules of the model. For the case of $\mathfrak{psu}(1,1|2)_1$ that was discussed in Chapter 5, there exists a free field realisation from which the fusion rules can be deduced. We are not aware of such a free-field representation in the present case, except for $\kappa = 0$; this free-field realisation for $\kappa = 0$ is discussed in Appendix D.3.

[72]This is to say, ρ^2 is an inner automorphism that maps each representation to itself. However, ρ^2 does not act trivially on the individual states.

We therefore have to resort to other methods. In particular, we can use a continuum version of the Verlinde formula to determine the typical fusion rules, i.e. those that do not involve indecomposable representations.[73] The calculation is somewhat lengthy, see Appendix D.2, but it leads to the simple result

$$\mathcal{F}^{\ell_1}_{\lambda_1} \times \mathcal{F}^{\ell_2}_{\lambda_2} \cong \bigoplus_{\ell_3=0}^{\frac{\kappa}{2}} N^{\ell_3}_{\ell_1\ell_2} \Big(\sigma\big(\mathcal{F}^{\ell_3}_{\lambda_1+\lambda_2-\frac{\gamma}{2}}\big) \oplus \mathcal{F}^{\ell_3+\frac{1}{2}}_{\lambda_1+\lambda_2+\frac{1}{2}} \oplus \mathcal{F}^{\ell_3-\frac{1}{2}}_{\lambda_1+\lambda_2+\frac{1}{2}} \oplus \sigma^{-1}\big(\mathcal{F}^{\ell_3}_{\lambda_1+\lambda_2+\frac{\gamma}{2}}\big) \Big) \ . \tag{7.51}$$

Here, $N^{\ell_3}_{\ell_1\ell_2}$ are the $\mathfrak{su}(2)_\kappa$ fusion rules, and, by definition, $\mathcal{F}^{-\frac{1}{2}}_{\lambda}$ and $\mathcal{F}^{\frac{\kappa+1}{2}}_{\lambda}$ are considered to be zero. Since the Verlinde formula is blind to indecomposability issues, it is conceivable that some modules on the right hand side are actually part of a bigger indecomposable module. In fact, if $\lambda_1 + \lambda_2 + \frac{1}{2} = \pm\lambda_{\ell_3 \pm \frac{1}{2}}$ for some ℓ_3, then we expect indecomposable modules to appear. While it is difficult to derive this for general κ, we can use our knowledge from the free-field realisation at $\kappa = 0$, see Appendix D.3, and from the $\mathfrak{psu}(1,1|2)_1$ analysis (which arises for $\kappa \to \infty$) to make a reasonable guess for the indecomposable structure in general. This is also described in Appendix D.4.

As in Chapter 5, the fusion rules are compatible with spectral flow,

$$\sigma^{w_1}\big(\mathcal{F}^{\ell_1}_{\lambda_1}\big) \times \sigma^{w_2}\big(\mathcal{F}^{\ell_2}_{\lambda_2}\big) \cong \sigma^{w_1+w_2}\big(\mathcal{F}^{\ell_1}_{\lambda_1} \times \mathcal{F}^{\ell_2}_{\lambda_2}\big) \ , \tag{7.52}$$

and they reduce to the ones for $\mathfrak{psu}(1,1|2)_1$ in the limit $\kappa \to \infty$. In that limit, $\mathfrak{su}(2)_{\kappa+1}$ becomes an outer automorphism, and we therefore get from (7.51)

$$\mathcal{F}^{0}_{\lambda_1} \times \mathcal{F}^{0}_{\lambda_2} \cong \sigma\big(\mathcal{F}^{0}_{\lambda_1+\lambda_2+\frac{1}{2}}\big) \oplus \mathcal{F}^{\frac{1}{2}}_{\lambda_1+\lambda_2+\frac{1}{2}} \oplus \sigma^{-1}\big(\mathcal{F}^{0}_{\lambda_1+\lambda_2+\frac{1}{2}}\big) \tag{7.53}$$

$$\cong \sigma\big(\mathcal{F}^{0}_{\lambda_1+\lambda_2+\frac{1}{2}}\big) \oplus 2\cdot\mathcal{F}^{0}_{\lambda_1+\lambda_2+\frac{1}{2}} \oplus \sigma^{-1}\big(\mathcal{F}^{0}_{\lambda_1+\lambda_2+\frac{1}{2}}\big) \ , \tag{7.54}$$

where the isomorphism breaks the outer automorphism $\mathfrak{su}(2)$; this then reproduces (5.33). As a second cross-check, we notice that they reduce, for $\kappa = 0$, to

$$\mathcal{F}^{0}_{\lambda_1} \times \mathcal{F}^{0}_{\lambda_2} \cong \sigma\big(\mathcal{F}^{0}_{\lambda_1+\lambda_2-\frac{1}{4}}\big) \oplus \sigma^{-1}\big(\mathcal{F}^{0}_{\lambda_1+\lambda_2+\frac{1}{4}}\big) \ , \tag{7.55}$$

thereby reproducing the special case derived in Appendix D.3 from the free field realisation.

[73]For the case of $\mathfrak{psu}(1,1|2)_1$, this is also done in Appendix C.1.6.

7.4.4 The Hilbert space and modular invariance

With these preparations at hand, we can now write down the complete worldsheet spectrum. It takes the form

$$\mathcal{H} = \bigoplus_{w\in\mathbb{Z}} \bigoplus_{\ell=0,\bar{\ell}=0}^{\frac{\kappa}{2}} M_{\ell\bar{\ell}} \oint_{[0,1)} \mathrm{d}\lambda \, \sigma^w\big(\mathcal{F}^\ell_\lambda\big) \otimes \overline{\sigma^w\big(\mathcal{F}^\ell_\lambda\big)} \, , \tag{7.56}$$

where $M_{\ell\bar{\ell}}$ is any $\mathfrak{su}(2)_\kappa$ modular invariant. In Appendix D.2, we determine the S-matrix for the modular transformations of the characters, see eq. (D.52)

$$S_{(w,\lambda,\ell),(w',\lambda',\ell')} = -i\,\mathrm{sgn}(\mathrm{Re}(\tau))\, \mathrm{e}^{2\pi i\left(w'\lambda + w\lambda' - \frac{ww'}{2(\kappa+2)}\right)} S^{\mathfrak{su}(2)}_{\ell\ell'} \, , \tag{7.57}$$

where $S^{\mathfrak{su}(2)}_{\ell\ell'}$ is the standard modular S-matrix of $\mathfrak{su}(2)_\kappa$. The S-matrix in (7.57) is formally unitary, and hence the spectrum (7.56) is (formally) modular invariant. This is true for any modular invariant of $\mathfrak{su}(2)_\kappa$, since the S-matrix is of tensor product form, i.e. the spectral flow part and the $\mathfrak{su}(2)_\kappa$ part are independent.

In writing down (7.56) we have ignored the subtlety that the fusion rules require us to consider also some indecomposable modules. There is a general recipe for how to deal with this issue that was already explained in some detail in Appendix C.2; we have sketched some aspects of this in Appendix D.4.

In Appendix D.2, we have also derived the characters of the spectrally flowed representation $\sigma^w\big(\mathcal{F}^\ell_\lambda\big)$, which take the form

$$\begin{aligned} \mathrm{ch}\big[\sigma^w\big(\mathcal{F}^\ell_\lambda\big)\big](t,u,v;\tau) &= q^{\frac{w^2}{4(\kappa+2)}} x^{\frac{\kappa+1}{2(\kappa+2)}w} y^{\frac{w}{2}} \\ \times \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi i(\lambda+\frac{1}{2})n}\, \delta(t-w\tau-n)\, & \frac{\vartheta_2\big(\frac{t+u+v}{2};\tau\big)\vartheta_2\big(\frac{t+u-v}{2};\tau\big)}{\eta(\tau)^4}\, \chi^{(\ell)}_\kappa(v;\tau) \, . \end{aligned} \tag{7.58}$$

Here, u, v and t are the chemical potentials of $\mathfrak{su}(2)_1$, $\mathfrak{su}(2)_{\kappa+1}$, and $\mathfrak{sl}(2,\mathbb{R})_k$, respectively, which we write as

$$q = \mathrm{e}^{2\pi i\tau} \, , \qquad x = \mathrm{e}^{2\pi i t} \, , \qquad y = \mathrm{e}^{2\pi i u} \, , \qquad z = \mathrm{e}^{2\pi i v} \, . \tag{7.59}$$

We have also included a $(-1)^{\mathrm{F}}$ factor in the character, and $\chi^{(\ell)}_\kappa(v;\tau)$ is the $\mathfrak{su}(2)_\kappa$ affine character, for more details see Appendix D.2.3. At this point, the appearance of $\mathfrak{su}(2)_\kappa$ is somewhat mysterious, since we started out with $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_{\kappa+1} \subset \mathfrak{d}(2,1;\alpha)_k$. However, its appearance is very natural from a spacetime perspective since the dual theory is expected [68] to be the symmetric orbifold of $\mathcal{S}_\kappa$, which also contains a $\mathfrak{su}(2)_\kappa$ algebra; this will be explained in more detail in Section 7.5.

We should also draw attention to the delta function which appears in the character. As in the case discussed in Chapter 5, it means that the character localises on solutions which map the worldsheet torus (with modular parameter τ) *holomorphically* to the boundary torus (with modular parameter t). This is the hallmark of a topological string theory and hence suggests that also $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3$ becomes essentially topological at $k^+ = 1$.

7.5 Physical states in string theory

Now we are ready to compute the full string theory spectrum of our theory. As we shall see, it will turn out to equal the partition function of the symmetric orbifold of $\mathcal{S}_\kappa$, nicely confirming the prediction of [68], see also [242]. Here $\mathcal{S}_\kappa$ is the $\mathcal{N} = 1$ supersymmetric WZW-model on $\mathrm{S}^3 \times \mathrm{S}^1$ (with κ units of flux through the S^3), which exhibits in fact large $\mathcal{N} = (4,4)$ supersymmetry.

7.5.1 The theory $\mathcal{S}_\kappa$ and its symmetric orbifold

Let us begin by reviewing briefly the $\mathcal{S}_\kappa$ theory [65, 68, 70]. The $\mathcal{S}_\kappa$ theory is defined by

$$\mathfrak{su}(2)^{(1)}_{\kappa+2} \oplus \mathfrak{u}(1)^{(1)} \cong \mathfrak{su}(2)_\kappa \oplus \mathfrak{u}(1) \oplus 4 \text{ free fermions} \ , \tag{7.60}$$

and possesses large $\mathcal{N} = (4,4)$ superconformal symmetry whose R-symmetry group is $\mathfrak{su}(2)_{\kappa+1} \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$. Some background material about the large $\mathcal{N} = 4$ superconformal algebra can be found in [65, 67, 134].

For the comparison with the worldsheet answer, we will need the partition function of the $\mathcal{S}_\kappa$ theory, which is explicitly given (in the NS sector) as

$$Z^{\mathrm{NS}}_{\mathcal{S}_\kappa}(u,v;t) = \left| \frac{\vartheta_3\big(\frac{u+v}{2};t\big)\,\vartheta_3\big(\frac{u-v}{2};t\big)}{\eta(t)^3} \right|^2 Z_{\mathfrak{su}(2)_\kappa}(v;t)\,\Theta(\tau) \ . \tag{7.61}$$

Here, $\Theta(\tau)$ is the momentum-winding sum of the free boson, u and v are the chemical potentials for $\mathfrak{su}(2)_{\kappa+1}$ and $\mathfrak{su}(2)_1$, respectively,[74] while t is the modular parameter, and

$$Z_{\mathfrak{su}(2)_\kappa}(v;t) = \sum_{\ell=0}^{\kappa} M_{\ell\bar{\ell}}\, \chi^{(\ell)}_\kappa(v;t)\, \overline{\chi^{(\ell)}_\kappa(v;t)} \tag{7.62}$$

is the partition function of $\mathfrak{su}(2)_\kappa$. The central charge of this theory equals

$$c = \frac{6(\kappa+1)}{\kappa+2} \ , \tag{7.63}$$

[74] To keep the notation simple, we have not introduced a chemical potential for the $\mathfrak{u}(1)$ factor. It is straightforward to include it and in fact the analysis of this paper carries through directly.

and the formula in the R-sector is obtained upon replacing ϑ_3 by ϑ_2.

Given the partition function of the seed theory, it is straightforward to work out the partition function of the N-fold symmetric product [68,138,139], and the partition function of the single particle states equals

$$Z_{\mathrm{Sym}^N(\mathcal{S}_\kappa)}(u,v;t) = x^{-\frac{Nc}{24}}\bar{x}^{-\frac{Nc}{24}}\left(\sum_{w=1 \text{ odd}}^{N} x^{\frac{cw}{24}}\bar{x}^{\frac{cw}{24}} Z^{\mathrm{NS}'}_{\mathcal{S}_\kappa}\big(u,v;\tfrac{t}{w}\big) + \sum_{w=1 \text{ even}}^{N} x^{\frac{cw}{24}}\bar{x}^{\frac{cw}{24}} Z^{\mathrm{R}'}_{\mathcal{S}_\kappa}\big(u,v;\tfrac{t}{w}\big)\right) . \quad (7.64)$$

Here $'$ denotes the orbifold projection, which ensures that only states with $h-\bar{h} \in \mathbb{Z}$ are kept (resp. $h-\bar{h} \in \mathbb{Z}+\frac{1}{2}$ for fermions in the NS-sector). Since we are interested in the large N limit, we will strip off the prefactor $x^{-\frac{Nc}{24}}\bar{x}^{-\frac{Nc}{24}}$; in the holographic setting, it corresponds to the divergent vacuum contribution.

7.5.2 Adding the remaining matter and ghost fields

Now we want to reproduce this answer from our worldsheet description. Recall that the complete worldsheet theory has in addition to $\mathfrak{d}(2,1;\alpha)_k$ an additional $\mathfrak{u}(1)$ current, four topologically twisted fermions, as well as the bc and ρ ghost system, see eq. (7.24). The additional fields are all free, so it is a trivial matter to compute their partition functions. For the free bosons describing S^1, we have

$$Z_{\mathrm{S}^1}(\tau) = \frac{\Theta(\tau)}{|\eta(\tau)|^2} , \quad (7.65)$$

where $\Theta(\tau)$ is the momentum-winding sum. We have already accounted for eight fermions on the worldsheet (since we constructed $\mathfrak{d}(2,1;\alpha)$ out of 8 fermions). So there should not be any additional fermionic contributions to the partition function, and indeed the ρ ghost cancels the four topologically twisted fermions, as was discussed in Chapter 5. Finally, the bosonic ghosts remove two neutral oscillators. Thus the full partition function of the worldsheet theory is simply obtained by multiplying the partition function of $\mathfrak{d}(2,1;\alpha)_k$ with $\Theta(\tau)\cdot|\eta(\tau)|^2$.

7.5.3 The mass shell condition

Finally, we need to impose the mass shell condition on the worldsheet, i.e. we need to demand that $L_0 = 0$. For this it is convenient to rewrite the delta function in (7.58) as an infinite sum — this is in fact how the delta function was obtained in the first place — so that the character reads

$$\begin{aligned}\mathrm{ch}\big[\sigma^w(\mathcal{F}^\ell_\lambda)\big](t,u,v;\tau) &= q^{\frac{w^2}{4(\kappa+2)}} x^{\frac{\kappa+1}{2(\kappa+2)}w} y^{\frac{w}{2}} \\ &\times \sum_{m\in\mathbb{Z}+\lambda+\frac{1}{2}} x^m q^{-mw} \frac{\vartheta_2\big(\frac{t+u+v}{2};\tau\big)\vartheta_2\big(\frac{t+u-v}{2};\tau\big)}{\eta(\tau)^4} \chi^{(\ell)}_\kappa(v;\tau)\ . \end{aligned} \tag{7.66}$$

Imposing the mass shell condition now amounts to solving

$$\frac{w^2}{4(\kappa+2)} - mw + h_{\text{osc}} = 0 \qquad \Rightarrow \qquad m = \frac{w}{4(\kappa+2)} + \frac{h_{\text{osc}}}{w}\ , \tag{7.67}$$

where h_{osc} is the conformal weight coming from the oscillator part (i.e. the theta-functions, the eta-functions and the affine $\mathfrak{su}(2)_\kappa$ character). Thus one term in the infinite sum of (7.66) is picked out, for a specific choice of λ (which is thereby also fixed). We correspondingly solve the mass shell condition for the right-movers. Since λ is the same for both left- and right-movers, this imposes the additional condition

$$h_{\text{osc}} - \bar{h}_{\text{osc}} \equiv 0 \text{ mod } w\ . \tag{7.68}$$

In terms of the character, imposing the two mass shall conditions can thus be implemented by removing the infinite sum, replacing $\tau \to \frac{t}{w}$, including the appropriate prefactor (coming from the first term in (7.67)), and imposing the constraint (7.68). Using the theta-function identities

$$\vartheta_2\big(\tfrac{t+u\pm v}{2};\tfrac{t}{w}\big) = x^{-\frac{w}{8}} y^{-\frac{w}{4}} z^{\mp\frac{w}{4}} \begin{cases} \vartheta_2\big(\frac{u\pm v}{2};\frac{t}{w}\big)\ , & w \text{ even}\ , \\ \vartheta_3\big(\frac{u\pm v}{2};\frac{t}{w}\big)\ , & w \text{ odd}\ . \end{cases} \tag{7.69}$$

the partition function of the physical spectrum can thus be written as

$$Z_{\text{string}}(u,v;t) = \sum_{w=1 \text{ odd}}^{\infty} x^{\frac{cw}{24}} \bar{x}^{\frac{cw}{24}} Z^{\text{NS}'}_{\mathcal{S}_\kappa}\big(u,v;\tfrac{t}{w}\big) + \sum_{w=1 \text{ even}}^{\infty} x^{\frac{cw}{24}} \bar{x}^{\frac{cw}{24}} Z^{\text{R}'}_{\mathcal{S}_\kappa}\big(u,v;\tfrac{t}{w}\big)\ , \tag{7.70}$$

where c is given by (7.63). This then agrees precisely with the large N limit of (7.64). We note in passing that this works for any modular invariant of $\mathfrak{su}(2)_\kappa$.

We should mention that we have restricted the calculation here to the $w \geq 1$ sector. It is easy to see that there are no physical states in the $w = 0$ sector, while the states from the $w \leq -1$ sector have the interpretation of out-states in the dual CFT [62], and hence should not be included in the partition function.

7.5.4 The BPS spectrum

It is instructive to understand how the BPS spectrum arises from the world-sheet. Recall that the single-particle BPS spectrum of the symmetric orbifold of $\mathcal{S}_\kappa$ is [65,68][75]

$$\bigoplus_{\ell=0}^{\frac{cN}{12}}[h=\ell,\ell^+=\ell,\ell^-=\ell,u=0]\otimes\overline{[h=\ell,\ell^+=\ell,\ell^-=\ell,u=0]}\ . \qquad (7.71)$$

Here, $[h=h_{\text{BPS}}(\ell^+,\ell^-,u),\ell^+,\ell^-,u]$ denotes a large $\mathcal{N}=4$ BPS multiplet in the representation (ℓ^+,ℓ^-,u) of the R-symmetry algebra $\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)$. This BPS spectrum also agrees with the supergravity BPS spectrum for $\text{AdS}_3\times\text{S}^3\times\text{S}^3\times\text{S}^1$ [67,169].

The different states in eq. (7.71) arise as follows. There is a BPS representation in every w-twisted sector, provided that $w\notin(\kappa+2)\mathbb{Z}$. In order to describe it, we write

$$w=m(\kappa+2)+2\ell+1 \qquad (7.72)$$

for some $m\in\mathbb{Z}$ and $\ell\in\{0,\frac{1}{2},\dots,\frac{\kappa}{2}\}$; this is possible since w is not divisible by $(\kappa+2)$. Then we consider the $(2\ell+m(\kappa+1),m)$-fold spectral flow of the ground state representation of spin $(0,\ell)$ of $\mathfrak{su}(2)_1\oplus\mathfrak{su}(2)_{\kappa+1}$. This gives a state in the w twisted sector which is indeed BPS. It was furthermore shown in [68] that all BPS states arise in this manner.

This structure can be directly translated to the worldsheet: BPS states come from the representations

$$\sigma^{m(\kappa+2)+2\ell+1}\big(\mathcal{F}^{\ell}_{\lambda_\ell}\big)\quad(m\text{ even})\ ,\qquad\text{and}\qquad\sigma^{m(\kappa+2)+2\ell+1}\big(\mathcal{F}^{\frac{\kappa}{2}-\ell}_{-\lambda_{\frac{\kappa}{2}-\ell}}\big)\quad(m\text{ odd})\ . \qquad (7.73)$$

To see this, we first recall that the m-fold spectral flow on $\mathfrak{su}(2)_\kappa$ maps the spin-ℓ representation back to itself if m is even, and to $\frac{\kappa}{2}-\ell$ if m is odd; the resulting state therefore sits in the correct representation of $\mathfrak{su}(2)_{\kappa+1}$. This leaves us with determining the λ-parameters, which can be computed by requiring that the $\mathfrak{sl}(2,\mathbb{R})$ weights agree with the BPS bound up to an integer. We see that we obtain precisely the values at which the modules become indecomposable. (Strictly speaking, we should therefore replace $\mathcal{F}^{\ell}_{\lambda_\ell}$ by its indecomposable analogue $\mathcal{T}^{\ell+\frac{1}{2}}_{>}$ and $\mathcal{F}^{\frac{\kappa}{2}-\ell}_{-\lambda_{\frac{\kappa}{2}-\ell}}$ by $\mathcal{T}^{\frac{\kappa+1}{2}-\ell}_{<}$, see Appendix D.4 for more details). The fact that BPS states live in indecomposable representations is typical for supergroup theories [185,186].

Finally, we discuss the moduli of the theory. Moduli of large $\mathcal{N}=4$ theories are superconformal descendants of $(\ell^+,\ell^-,u)=(\frac{1}{2},\frac{1}{2},0)$ BPS states [65]. These can come from the vacuum representation or the large $\mathcal{N}=4$ BPS representation labelled by $[h=\frac{1}{2},\ell^+=\frac{1}{2},\ell^-=\frac{1}{2},u=0]$. These states in turn come from the worldsheet representations

$$\sigma(\mathcal{F}^0_{\lambda_0})\sim\mathcal{L}\oplus\sigma(\mathcal{G}^0_{>,+})\ , \qquad (7.74)$$

[75] There are some additional BPS states in the N-twisted sector, which disappear in the large N limit [68]. We therefore do not consider them here.

$$\sigma^2(\mathcal{F}^{1/2}_{\lambda_{1/2}}) \sim \sigma(\mathcal{G}^0_{>,+}) \oplus \sigma^2(\mathcal{G}^{1/2}_{>,+}) \ . \tag{7.75}$$

The module $\mathcal{L}$ contains a single physical state, namely the vacuum itself, which corresponds to the spacetime vacuum. The actual moduli therefore come from the representation $\sigma(\mathcal{G}^0_{>,+})$ which indeed appears twice. This reflects the situation in the dual CFT, where one of the moduli comes from the untwisted sector and changes the radius of S^1, whereas the other modulus carries one away from the symmetric orbifold point. The two moduli in the theory are exactly on the same footing, in agreement with the fact that the geometry of the two-dimensional moduli space is the upper half plane [65].

7.6 The spacetime DDF operators

In the previous sections we have shown that the spacetime partition function of string theory on $AdS_3 \times S^3 \times S^3 \times S^1$ coincides with the partition function of the symmetric orbifold of $\mathcal{S}_\kappa$ if $k^+ = 1$. In this section we want to establish that also the algebraic structure of the two sides agree, thus extending the analysis of Chapter 6 to the present setting. Moreover, we show that the correspondence can be extended to the case of $k^+ > 1$, in which case the dual CFT becomes the symmetric orbifold of large $\mathcal{N} = 4$ Liouville theory. Most of the arguments are very similar to what was done in Chapter 6, and we shall therefore be rather brief.

7.6.1 Spacetime operators

In [242] the DDF operators generating the large $\mathcal{N} = 4$ superconformal algebra were constructed for the background $AdS_3 \times S^3 \times S^3 \times S^1$. The analysis was performed in the RNS formalism assuming that $k^\pm \geq 2$, and it is a priori not clear whether the construction continues to make sense also for $k^+ = 1$. Using similar argument as in Chapter 6, we have checked that the DDF operators of [242] are also well-defined for $k^+ = 1$.

Let us denote the large $\mathcal{N} = 4$ spacetime algebra generators (our conventions follow [68]) by

$$\mathcal{L}_m \ , \qquad \mathcal{G}^{\alpha\beta}_r \ , \qquad \mathcal{K}^{(\pm)a}_m \ , \qquad \mathcal{U}_m \ , \qquad \mathcal{Q}^{\alpha\beta}_r \ , \tag{7.76}$$

where $\mathcal{L}_m$ are the modes of the spacetime energy momentum tensor, $\mathcal{G}^{\alpha\beta}_r$ those of the spacetime supercharges, while $\mathcal{K}^{(\pm)a}_m$ and $\mathcal{U}_m$ define the R-symmetry generators. In addition, there are four free fermions that are denoted by $\mathcal{Q}^{\alpha\beta}_r$.

As was explained in Chapter 6 — the argument is essentially the same here — the modes of this algebra can take values in $\frac{1}{w}\mathbb{Z}$ (or $\frac{1}{w}\big(\mathbb{Z} + \frac{1}{2}\big)$ in the case of fermions). By the same reasoning as in Chapter 6 this then suggests that the spacetime states that arise from the continuous representations on the worldsheet are in general (i.e. for arbitrary k^+ and k^-) described by the symmetric product orbifold of

$$\text{large } \mathcal{N} = 4 \text{ Liouville theory with } (k^+, k^-) \,. \tag{7.77}$$

We shall review the construction of large $\mathcal{N} = 4$ Liouville theory in the following section, and explain the crucial steps in this derivation in Section 7.6.3. In Section 7.6.4 we will then demonstrate that large $\mathcal{N} = 4$ Liouville theory reduces, for $k^+ = 1$, to $\mathcal{S}_\kappa$. Furthermore, since for $k^+ = 1$ the entire worldsheet spectrum comes from the continuous representations, this is in fact a complete description of the theory.

7.6.2 Large $\mathcal{N} = 4$ Liouville theory

Let us first discuss large $\mathcal{N} = 4$ Liouville theory, which does not seem to be well-known. We shall first assume $k^\pm \geq 2$, and study the case of $k^+ = 1$ in Section 7.6.4. To motivate the construction of this theory, we consider a free boson coupled to the curvature of the worldsheet (i.e. with background charge), together with the $\mathfrak{su}(2)_{k^+-2} \oplus \mathfrak{su}(2)_{k^--2} \oplus \mathfrak{u}(1)$ R-symmetry and 8 free fermions. (This is basically the same field content as for the worldsheet theory in the RNS formalism, except that the $\mathfrak{sl}(2,\mathbb{R})$ factor has been replaced by a boson with screening charge.) It was noticed in [243] that this theory supports large $\mathcal{N} = 4$ supersymmetry with levels k^+ and k^- for the two $\mathfrak{su}(2)$ currents. The free boson with screening charge $Q = \frac{(k-1)}{\sqrt{k}}$ leads to a continuous spectrum, whose gap above the vacuum equals

$$\Delta_\varphi = \frac{c^\varphi - 1}{24} = \frac{(k-1)^2}{4k} \,. \tag{7.78}$$

We can combine this with arbitrary $\mathfrak{su}(2)_{k^\pm-2}$ and $\mathfrak{u}(1)$ representations, thus leading to the general formula for the gap

$$\Delta_{\ell^+,\ell^-,u} = \frac{(k-1)^2}{4k} + \frac{\ell^+(\ell^++1)}{k^+} + \frac{\ell^-(\ell^-+1)}{k^-} + \frac{u^2}{k^++k^-} \tag{7.79}$$

$$= \frac{(\ell^+ + \frac{1}{2})^2}{k^+} + \frac{(\ell^- + \frac{1}{2})^2}{k^-} + \frac{k-2}{4} + \frac{u^2}{k^++k^-} \,, \tag{7.80}$$

where we have used (7.2). We should note that, generically, all the BPS representations lie below this gap since

$$\Delta_{\ell^+,\ell^-,u} - h_{\text{BPS}}(\ell^+,\ell^-,u) = \frac{k}{4}\left(1 - \frac{2\ell^-+1}{k^-} - \frac{2\ell^++1}{k^+}\right)^2 \geq 0 \,, \tag{7.81}$$

where we have used the expression for the BPS bound, see e.g. eq. (A.13) of [68]. The only BPS states that appear in large $\mathcal{N} = 4$ Liouville theory therefore arise if

$$\frac{2\ell^- + 1}{k^-} + \frac{2\ell^+ + 1}{k^+} = 1 \ . \tag{7.82}$$

The fact that such solutions exist is related to the fact that also the continuous sector of the worldsheet theory of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ contributes to the BPS spectrum [68].

The full spectrum of large $\mathcal{N} = 4$ Liouville theory is obtained by taking the diagonal modular invariant of all of the representations that lie above the gap (and have allowed $\mathfrak{su}(2)_{k^\pm-2}$ and $\mathfrak{u}(1)$ representations). We should note that in large $\mathcal{N} = 4$ Liouville, each representation appears precisely once, whereas in the free bosons realisation from above, each representation appears twice since opposite values of the momentum lead to the same Virasoro representation.

7.6.3 The Liouville spectrum from the worldsheet

Next, we want to reproduce this Liouville spectrum directly from the worldsheet. Solving the mass-shell condition in the spectrally flowed sector leads to

$$\frac{\frac{1}{4} + p^2}{k} - wh + \frac{k}{4}w^2 + \frac{\ell^+(\ell^+ + 1)}{k^+} + \frac{\ell^-(\ell^- + 1)}{k^-} + \frac{u^2}{k^+ + k^-} + N = \frac{1}{2} \ . \tag{7.83}$$

Here, N is the conformal weight which is contributed by the oscillator part. Solving this equation for the conformal weight h of the dual CFT yields

$$h = \frac{k}{4w}(w^2 - 1) + \frac{\Delta_{\ell^+,\ell^-,u}}{w} + \frac{N}{w} + \frac{p^2}{kw} \ . \tag{7.84}$$

This matches exactly the form expected from the symmetric orbifold of large $\mathcal{N} = 4$ Liouville: the first term is the universal ground state energy of the twisted sector, which equals $\frac{c}{24w}(w^2 - 1)$, where $c = 6k$ is the central charge of the 'seed theory', while the second term describes the gap in the w-cycle twisted sector. Since the modes are $\frac{1}{w}$-fractionally moded in the w-cycle twisted sector, the contribution of N has to be divided by w. Finally, the term $\frac{p^2}{kw}$ leads to a continuum in the spectrum (since p is any real number corresponding to the momentum of the long string). Furthermore, the representations belonging to p and $-p$ are identified on the worldsheet — they describe the same $\mathfrak{sl}(2,\mathbb{R})$ representation — and appear only once in the spectrum, as appropriate for $\mathcal{N} = 4$ Liouville, see the comment at the end of the previous section.

In order to conclude from this that the complete spectrum matches we use again a character argument. To compute the relevant characters, we again make use of the free-field construction of [243]. Both the worldsheet theory as well as Liouville theory has 8 free fermions (after imposing the physical state conditions on the worldsheet). In addition also the bosonic degrees of freedom match: the $\mathfrak{su}(2)_{k^+-2} \oplus \mathfrak{su}(2)_{k^--2} \oplus \mathfrak{u}(1)$ algebra is the same on both sides and the $\mathfrak{sl}(2,\mathbb{R})_k$ factor has the character of a free boson after imposing the physical state conditions. This reproduces the contribution of the Liouville boson.

Thus, we have matched the spectrum as well as the chiral algebras on both sides of the duality. Since Liouville theory is believed to be uniquely characterised by this data (and the same should be true for large $\mathcal{N} = 4$ Liouville), this goes a long way towards proving the duality in this case.

We should stress that the 'symmetric orbifold of Liouville theory' contains single-particle states for which only one copy is in the ground state of Liouville theory, while the other copies are in the 'vacuum' — this is part of the spectrum as determined from the dual worldsheet analysis. (This is different from the naive definition of the symmetric orbifold where the 'vacuum' would not be allowed for any copy.) As a consequence, the effective central charge scales as $6N\frac{k^+k^-}{k^++k^-}$, and the spectrum has the correct density at large conformal dimension.

7.6.4 The case of $k^+ = 1$

Upon setting $k^+ = 1$, the construction of $\mathcal{N} = 4$ Liouville theory breaks down since the level of the corresponding bosonic algebra is -1, which makes the theory non-unitary. Instead, the superconformal algebra collapses to $\mathcal{S}_\kappa$, i.e. as chiral algebras we have the equivalence [132]

$$A_\gamma(k^+ = 1, k^- = \kappa + 1) = \mathfrak{su}(2)_\kappa \oplus \mathfrak{u}(1) \oplus 4 \text{ free fermions}\ , \tag{7.85}$$

mirroring exactly what happens on the worldsheet.

Contrary to the $k^\pm \geq 2$ case, the $\mathcal{S}_\kappa$ theory (and hence also A_γ at $k^+ = 1$) contains *only* BPS representations. This just follows from the fact that the conformal weight of a representation with $\mathfrak{su}(2)$ spin ℓ^- and $\mathfrak{u}(1)$-charge u is

$$\Delta_{\ell^-,u} = \frac{\ell^-(\ell^-+1)}{\kappa+2} + \frac{u^2}{\kappa+2} = h_{\text{BPS}}(0,\ell^-,u)\ . \tag{7.86}$$

As a consequence, any large $\mathcal{N} = (4,4)$ theory at $k^+ = 1$ *cannot* have a continuum (such as the one that appears in Liouville theory). Furthermore, the above DDF analysis predicts that the CFT dual of string theory on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ must be a symmetric orbifold, whose seed theory has large $\mathcal{N} = 4$ superconformal symmetry with levels $k^\pm = k^\pm_{\text{worldsheet}}$. For $k^+ = 1$, the seed theory must therefore be $\mathcal{S}_\kappa$, thus inevitably leading to the proposal of [68] (for $k^+ = 1$).

7.7 Summary and Conclusions

In this Chapter, we have considered the background $AdS_3 \times S^3 \times S^3 \times S^1$ with pure NS-NS flux and minimal flux through one of the two S^3's, while the dual CFTs are symmetric orbifolds of the so-called $\mathcal{S}_\kappa$ theory, the simplest conformal field theory with large $\mathcal{N} = 4$ superconformal symmetry [65,70]. We have shown that the spacetime spectrum of the worldsheet theory agrees precisely with the dual symmetric orbifold CFT in the large N limit. We have furthermore shown that the spectrum generating fields on the worldsheet (the DDF operators) obey the same algebra as those of the symmetric orbifold. This gives strong support to the identification of the dual CFT that was proposed in [68], see also [242]. Our results are a natural generalisation of the results obtained in the previous two Chapters.

We have also analysed the situation where the NS-NS flux through both spheres is bigger than its minimal value ($k^{\pm} > 1$), and in this case, our analysis suggests that the dual CFT is the symmetric orbifold of large $\mathcal{N} = 4$ Liouville theory. In this case the spectrum of the symmetric orbifold is entirely accounted for in terms of the continuous representations on the worldsheet, while the role of the spacetimes states that originate from discrete representations on the worldsheet is less clear.[76] Again, this mirrors precisely what was found for the $\mathbb{T}^4$ case in Chapter 6.

It is suggestive that, apart from some small technical differences, the analysis (as well as the resulting picture) that we find here is quite similar to that obtained in the $\mathbb{T}^4$ case. This suggests that similar results may also hold for other backgrounds (say with less supersymmetry), and it would be interesting to explore this. It would also be interesting to probe these dual pairs in more detail, say, by comparing their 3-point functions, or by computing $1/N$ corrections (which should correspond to higher genus corrections from the worldsheet viewpoint). In any case, we feel that these three dimensional examples will provide a useful testing ground for various aspects of the AdS/CFT correspondence.

[76] Note that if one of the levels takes the minimal value, say $k^+ = 1$, then the worldsheet spectrum does not contain any discrete representations.

Chapter 8

Conclusion and Outlook

Over the last twenty years, string theory on Anti-de Sitter spaces has played a major role in theoretical high-energy physics. Via the AdS/CFT corespondence, the theory can be holographically described by a conformal field theory on the boundary. A particularly tractable instance of the AdS/CFT correspondence is the case, where the boundary CFT is two-dimensional. Then there is an efficient description of the string theory, as well as of the dual CFT. In this thesis, we made significant progress in establishing the dictionary for $\text{AdS}_3/\text{CFT}_2$ holography.

8.1 Summary

We started in Chapters 2 and 3 by reviewing background material on conformal field theory and string theory, putting particular emphasis on topics needed for the applications in the later chapters.

We then initiated a systematic study of the string spectrum in the background $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4$ (and its cousins $\text{AdS}_3 \times \text{S}^3 \times \text{K3}$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$). We considered in particular mixed NS-NS and R-R background flux. Only for $d = 2$ can AdS_{d+1} be supported by a mixture of the two types of fluxes and thus AdS_3 provides a unique opportunity to learn something about the technically hard R-R backgrounds by starting from the tractable NS-NS background flux. On the worldsheet, we can turn on R-R flux by considering the *hybrid formalism* [126], where the background is described by a supergroup sigma-model on $\text{PSU}(1,1|2)$ or $\text{D}(2,1;\alpha)$, respectively. We have seen that there is a natural algebra acting on the Hilbert space of these models given by an extension of two affine algebras. This algebra constrains the string spectrum significantly and a complete understanding of the representation theory of the algebra would in principle lead to the exact string spectrum in the background. Unfortunately, the algebra is quite unwieldy and we have managed only to extract certain parts of the string spectrum from it. In particular, we have seen how to derive the string spectrum in the plane-wave limit from first principles, thereby reproducing the celebrated result by Berenstein, Maldacena and Nastase [161]. We have however also seen that our formalism is capable to yield far stronger results. In particular, the conformal weight of certain excited states in the model can be computed exactly. We have used this to confirm some old conjectures by Seiberg and Witten [53] directly. We have seen that the string spectrum behaves discontinuously as we approach the pure NS-NS locus in moduli space. This is signalled by two phenomena: The long string continuum appears in the string spectrum, whose existence is forbidden away from the pure NS-NS point. Second the BPS spectrum (i.e. the spectrum of supersymmetric protected states) jumps; we called this phenomena the missing chiral primaries at the pure NS-NS point.

In Chapter 5, we considered the pure NS-NS point in more detail. At this background, we have in principle full control over the worldsheet theory. One conventionally describes string theory at pure NS-NS flux in the RNS formalism, where supersymmetry on the worldsheet is manifest, but supersymmetry in spacetime is not. This formalism is only adequate to describe string theory with at least two flux quanta. As we have motivated in the Introduction, we would like to study the tensionless limit of string theory on this background, since we expect it to be dual to the symmetric orbifold $\text{Sym}^N(\mathbb{T}^4)$. The most tensionless point in moduli space is the point with one unit of NS-NS flux. We have found that while the RNS-formalism does not appropriately describe the background, the hybrid formalism based on the sigma-model on $\text{PSU}(1,1|2)$ does. We have systematically studied the relevant $\text{PSU}(1,1|2)$ representations and found that the $k=1$ theory can be consistently defined. We confirmed our expectations that the $k=1$ theory is exactly dual to the symmetric product $\text{Sym}^N(\mathbb{T}^4)$ in the large N limit by matching the partition functions of both theories. By showing also that the selection rules for the three-point functions for both theories agree, we have given strong evidence for the duality

$$\text{IIB string theory on } \text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4 \text{ at } k=1 \quad = \quad \text{Sym}^N(\mathbb{T}^4)\ . \tag{8.1}$$

Hence, this provides us with an example of a stringy AdS/CFT duality, where both sides are under complete computational control.

We continued the study of this duality in Chapter 6 by analysing the symmetry algebra for both theories. Following the work of [141,147] and [244], we constructed a set of so-called DDF operators (after Del Giudice, Di Vecchia and Fubini [223]). These are operators which act on the physical Hilbert space of string theory (i.e. which commute with the BRST operator defining physical states in string theory) and generate the complete physical Hilbert space. In particular, one can directly define the Virasoro algebra of the dual CFT in string theory. The complete set of DDF operators on the background $AdS_3 \times S^3 \times \mathbb{T}^4$ defines an $\mathcal{N} = 4$ superconformal algebra with central charge $c = 6(k-1)$, together with four free bosons and fermions. We have checked that this statement continues to hold true for $k = 1$ by switching to the hybrid formalism. In particular, for $k = 1$, the $\mathcal{N} = 4$ superconformal algebra has vanishing central charge and decouples from the spectrum. Thus, the symmetry algebra consists of four free bosons and fermions, which represent the four-torus $\mathbb{T}^4$ of the symmetric orbifold. We have also seen that the $k > 1$ case has a natural interpretation in terms of a symmetric product of a Liouville theory, as stated in the Introduction, see eq. (1.8).

Finally, we applied this logic to a more complicated background, namely $AdS_3 \times S^3 \times S^3 \times S^1$. This background is characterised by two fluxes k^+ and k^-, one for each three-sphere. Hence, it provides a more refined testing ground for the ideas explored in the previous chapters. We have seen that the background becomes tensionless when one of the two fluxes attains its minimal value, say $k^+ = 1$. The matching of the spectrum and the DDF construction go through as in the previous chapters, but are somewhat more intricate.

8.2 Outlook

We give now an overview about future directions and applications of our work.

In Chapter 4, we have discussed the existence of a powerful algebraic structure constraining the string spectrum on $AdS_3 \times S^3 \times \mathbb{T}^4$ with mixed NS-NS and R-R background flux. Although it does not directly yield the full string spectrum on the background, it strongly constrains its structure. We expect that a similar algebra exists in more realistic settings of holography, for example on the background $AdS_5 \times S^5$, which is dual to $\mathcal{N} = 4$ Super Yang-Mills theory. Thus, AdS_3 backgrounds offer the opportunity to gain insights into string theory on R-R backgrounds, which is indispensable when studying the AdS/CFT correspondence in higher dimensions.

We have found strong evidence for the fact that the symmetric product orbifold CFT $\mathrm{Sym}^N(\mathbb{T}^4)$ is the dual description of the pure NS-NS flux $k = 1$ background on $AdS_3 \times S^3 \times \mathbb{T}^4$. A CFT is completely determined by its spectrum together with all its 3-point functions. Thus, to prove the AdS/CFT duality in this instance, we would need to compute all the three-point functions on both sides of the duality. In the symmetric product orbifold, these are well-known, for relevant work see [198, 199, 224–226, 235, 245]. On the other hand, few results have been obtained for the string theory answer (see however [62, 97, 246–248] for relevant work in this direction). We are in the process of completing the proof of the correspondence by matching the three-point functions on both sides of the duality [249].

The symmetric orbifold $\mathrm{Sym}^N(\mathbb{T}^4)$ remains tractable in the finite N regime, which corresponds to a finite string coupling constant on the string side. Thus, it seems that we can gain unique insights into string theory, since this duality allows us to consider finite coupling constants. Even in flat space string theory, higher string loop computations are extremely hard to perform. As in higher dimensions, the correlators of the symmetric product orbifold $\mathrm{Sym}^N(\mathbb{T}^4)$ admit a genus expansion (a large N expansion), which can be identified with the genus expansion of string theory [198]. Contrary to the usual genus expansion of gauge theories [46], this expansion is finite and truncates after a finite number of terms. In particular, the spectrum (2-point functions) are exact at leading order and do not receive any N^{-1} corrections. These observations make it possible to extend the proposal to the finite coupling regime. This direction would be complementary to what was achieved in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case for the spectrum using integrability [250–253] and whose extension to three-point functions is currently an active topic of research [254]. In this case, the spectrum is known exactly in the 't Hooft coupling constant, but at large N.

In [255], the first successful matching of black hole entropy [4] in the context of the AdS/CFT duality was achieved. A five-dimensional black hole obtained by compactifying type IIB string theory on $\mathrm{S}^1 \times \mathrm{K3}$ was considered. The near-horizon geometry of the black hole is given by $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^1 \times \mathrm{K3}$, which for large radii of S^1 can be well-approximated by $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{K3}$ and therefore the microstates of the black hole are described by the dual CFT $\mathrm{Sym}^N(\mathrm{K3})$. Given the more detailed understanding of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence obtained in this thesis, there is hope to understand the black hole microstates directly in string theory. Similarly, a proposal was made in [256, 257] on how to understand entanglement entropy geometrically in the context of the AdS/CFT duality, which should be revisited in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$ holography.

One can also study non-perturbative states in AdS_3 stringy theory. The symmetric product orbifold has a Ramond-sector, which describes black hole excitations in the bulk. Given that there is a complete understanding of how to map perturbative excitations in the duality, one is tempted to believe that a similar matching can be done for the remaining non-perturbative states. In any case, we feel that a continued study of this particular correspondence will prove to be a very useful guiding example in our quest for a full understanding of the holographic principle.

Appendix A

Conventions

Here, we fix the conventions used in the main text.

A.1 Affine Lie (super)algebras

Affine Lie (super)algebras play a central role in our constructions and we fix here the conventions for the algebras, which appeared.

A.1.1 Root system and classification

In this paper, basic classical Lie superalgebras with vanishing dual Coxeter number play an important rôle. These are completely classified by Kač [258] and are [259–261]:

$$\mathfrak{psl}(n|n)\,, \qquad \mathfrak{osp}(2n+2|2n) \qquad \text{and} \qquad \mathfrak{d}(2,1;\alpha)\,, \tag{A.1}$$

where $n \geq 1$ is any integer and $\alpha > 1$.[77] These algebras are called basic because they possess an invariant bilinear form, which we need to construct the action, and hence also the Virasoro tensor. For simple superalgebras a Cartan subalgebra $\mathfrak{h}$ can be chosen. For basic Lie superalgebras $\mathfrak{h}$ agrees with the Cartan subalgebra of the bosonic subalgebra, and is thus unique up to conjugation. Hence a root system can be defined.

It is well-known (and explained in the Section 4.2) that the principal model with WZW-term on a supergroup is a CFT if the respective dual Coxeter number vanishes. The dual Coxeter number is half of the Casimir of the adjoint representation, so in particular the quadratic Casimir of the adjoint representation vanishes for the above Lie superalgebras.

[77] We have the isomorphism $\mathfrak{osp}(4|2) \cong \mathfrak{d}(2,1;\alpha=1)$ and $\mathfrak{d}(2,1;\alpha) \cong \mathfrak{d}(2,1;\alpha^{-1})$ for $\alpha \in \mathbb{R}$. Since we want to choose a real form, we also restrict to $\alpha \in \mathbb{R}$. We also have an isomorphism $\mathfrak{d}(2,1;\alpha \to \infty) \cong \mathfrak{psu}(1,1|2) \rtimes \mathfrak{su}(2)$, where $\mathfrak{su}(2)$ acts as an outer automorphism on $\mathfrak{psu}(1,1|2)$. This will discussed further in Appendix A.1.3.

A.1.2 The affine $\mathfrak{psu}(1,1|2)_k$ algebra

The $\mathfrak{psu}(1,1|2)_k$ current algebra takes the following form in our conventions:

$$
\begin{aligned}
[J^3_m, J^3_n] &= -\tfrac{1}{2} k m \delta_{m+n,0} \,, && \text{(A.2a)}\\
[J^3_m, J^\pm_n] &= \pm J^\pm_{m+n} \,, && \text{(A.2b)}\\
[J^+_m, J^-_n] &= k m \delta_{m+n,0} - 2 J^3_{m+n} \,, && \text{(A.2c)}\\
[K^3_m, K^3_n] &= \tfrac{1}{2} k m \delta_{m+n,0} \,, && \text{(A.2d)}\\
[K^3_m, K^\pm_n] &= \pm K^\pm_{m+n} \,, && \text{(A.2e)}\\
[K^+_m, K^-_n] &= k m \delta_{m+n,0} + 2 K^3_{m+n} \,, && \text{(A.2f)}\\
[J^a_m, S^{\alpha\beta\gamma}_n] &= \tfrac{1}{2} c_a (\sigma^a)^\alpha{}_\mu S^{\mu\beta\gamma}_{m+n} \,, && \text{(A.2g)}\\
[K^a_m, S^{\alpha\beta\gamma}_n] &= \tfrac{1}{2} (\sigma^a)^\beta{}_\nu S^{\alpha\nu\gamma}_{m+n} \,, && \text{(A.2h)}\\
\{S^{\alpha\beta\gamma}_m, S^{\mu\nu\rho}_n\} &= k m \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\varepsilon^{\gamma\rho}\delta_{m+n,0} - \varepsilon^{\beta\nu}\varepsilon^{\gamma\rho} c_a (\sigma_a)^{\alpha\mu} J^a_{m+n} \\
&\qquad + \varepsilon^{\alpha\mu}\varepsilon^{\gamma\rho}(\sigma_a)^{\beta\nu} K^a_{m+n} \,. && \text{(A.2i)}
\end{aligned}
$$

Here, $\alpha, \beta, \ldots$ are spinor indices and take values in $\{+,-\}$. The third spinor index of the supercharges encodes the transformation properties under the outer automorphism $\mathfrak{su}(2)$ of $\mathfrak{psu}(1,1|2)$. Furthermore, a is an $\mathfrak{su}(2)$ adjoint index and takes values in $\{+,-,3\}$. The constant c_a equals -1 for $a=-$, and $+1$ otherwise. Finally, the σ-matrices are explicitly given by

$$
\begin{aligned}
&(\sigma^-)^+{}_- = 2\,, && (\sigma^3)^-{}_- = -1\,, && (\sigma^3)^+{}_+ = 1\,, && (\sigma^+)^-{}_+ = 2\,, && \text{(A.3a)}\\
&(\sigma_-)^{--} = 1\,, && (\sigma_3)^{-+} = 1\,, && (\sigma_3)^{+-} = 1\,, && (\sigma_+)^{++} = -1\,, && \text{(A.3b)}\\
&(\sigma^-)_{--} = 2\,, && (\sigma^3)_{+-} = 1\,, && (\sigma^3)_{-+} = 1\,, && (\sigma^+)_{++} = -2\,, && \text{(A.3c)}
\end{aligned}
$$

while all the other components vanish.

The representations we considered for string theory applications in Chapter 4 are lowest weight for the $\mathfrak{sl}(2,\mathbb{R})$-oscillators, and half-infinite.[78] For $\mathfrak{su}(2)$, they are finite dimensional. Hence they are characterised by

$$
\begin{aligned}
&J^3_0|j,\ell\rangle = j|j,\ell\rangle\,, && K^3_0|j,\ell\rangle = \ell|j,\ell\rangle\,,\\
&J^-_0|j,\ell\rangle = 0\,, && K^+_0|j,\ell\rangle = 0\,, \qquad\qquad \text{(A.4)}\\
&J^a_m|j,\ell\rangle = 0\,, \quad m>0\,, && K^a_m|j,\ell\rangle = 0\,, \quad m>0\,.
\end{aligned}
$$

Requiring that the zero-mode representation has no negative-norm states imposes $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, j is continuous. The Casimir of such a representation reads

$$\mathcal{C}(j,\ell) = -j(j-1) + \ell(\ell+1)\,. \qquad \text{(A.5)}$$

[78] Also so-called spectrally flowed representations occur. For non-vanishing R-R-flux, they cannot be described on the level of the algebra $\mathfrak{psu}(1,1|2)_k$ alone.

A representation $|j,\ell\rangle$ is atypical if the BPS bound $j \geq \ell+1$ is saturated, and it is otherwise typical. A typical representation $|j,\ell\rangle$ consists of the following 16 $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2)$-multiplets:

$$4(j,\ell)\,,\ (j\pm 1,\ell)\,,\ (j,\ell\pm 1)\,,\ 2(j\pm\tfrac{1}{2},\ell\pm\tfrac{1}{2})\,. \tag{A.6}$$

A.1.3 The (affine) Lie superalgebra $\mathfrak{d}(2,1;\alpha)_k$

The (affine) Lie superalgebra $\mathfrak{d}(2,1;\alpha)$ is used to describe string theory on the background $\text{AdS}_3\times\text{S}^3\times\text{S}^3\times\text{S}^1$, so we review it here and fix our conventions. The non-vanishing commutation relations for the affine algebra take the form

$$\begin{aligned}
[J^3_m,J^3_n] &= -\tfrac{1}{2}km\delta_{m+n,0}\,, && \text{(A.7a)}\\
[J^3_m,J^\pm_n] &= \pm J^\pm_{m+n}\,, && \text{(A.7b)}\\
[J^+_m,J^-_n] &= km\delta_{m+n,0}-2J^3_{m+n}\,, && \text{(A.7c)}\\
[K^{(\pm)3}_m,K^{(\pm)3}_n] &= \tfrac{1}{2}k^\pm m\delta_{m+n,0}\,, && \text{(A.7d)}\\
[K^{(\pm)3}_m,K^{(\pm)\pm}_n] &= \pm K^{(\pm)\pm}_{m+n}\,, && \text{(A.7e)}\\
[K^{(\pm)+}_m,K^{(\pm)-}_n] &= k^\pm m\delta_{m+n,0}+2K^{(\pm)3}_{m+n}\,, && \text{(A.7f)}\\
[J^a_m,S^{\alpha\beta\gamma}_n] &= \tfrac{1}{2}(\sigma^a)^\alpha{}_\mu S^{\mu\beta\gamma}_{m+n}\,, && \text{(A.7g)}\\
[K^{(+)a}_m,S^{\alpha\beta\gamma}_n] &= \tfrac{1}{2}(\sigma^a)^\beta{}_\nu S^{\alpha\nu\gamma}_{m+n}\,, && \text{(A.7h)}\\
[K^{(-)a}_m,S^{\alpha\beta\gamma}_n] &= \tfrac{1}{2}(\sigma^a)^\gamma{}_\rho S^{\alpha\beta\rho}_{m+n}\,, && \text{(A.7i)}\\
\{S^{\alpha\beta\gamma}_m,S^{\mu\nu\rho}_n\} &= km\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\varepsilon^{\gamma\rho}\delta_{m+n,0}-\varepsilon^{\beta\nu}\varepsilon^{\gamma\rho}(\sigma_a)^{\alpha\mu}J^a_{m+n}+\gamma\varepsilon^{\alpha\mu}\varepsilon^{\gamma\rho}(\sigma_a)^{\beta\nu}K^{(+)a}_{m+n}\\
&\quad+(1-\gamma)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}(\sigma_a)^{\gamma\rho}K^{(-)a}_{m+n}\,. && \text{(A.7j)}
\end{aligned}$$

Again, $\alpha,\beta,\ldots$ are spinor indices and take values in $\{\pm\}$. As before, a is an adjoint index and takes values in $\{\pm,3\}$. It is raised an lowered by the standard $\mathfrak{su}(2)$-invariant form. Finally, γ, k^+ and k^- are related to α and k by

$$\gamma=\frac{\alpha}{1+\alpha}\,,\qquad k^+=\frac{(\alpha+1)k}{\alpha}\,,\qquad k^-=(\alpha+1)k\,. \tag{A.8}$$

We note that $k^+,k^-\in\mathbb{Z}_{\geq 0}$, so the affine algebra imposes furthermore $\alpha\in\mathbb{Q}_{\geq 0}\cup\{\infty\}$.

In the limit $\alpha\to\infty$, $\gamma=1$ and the modes $K^{(-)a}_m$ decouple from the rest of the algebra. After decoupling, the algebra becomes again $\mathfrak{psu}(1,1|2)_k$.[79] There is a unique (up to rescaling) invariant form, which can be read off from the central terms in the commutation relations.

[79]Physically, this corresponds to the fact that when decompactifying one of the three-spheres, the geometry of the background becomes $\text{AdS}_3\times\text{S}^3\times\mathbb{R}^3\times\text{S}^1$, which is locally isometric to $\text{AdS}_3\times\text{S}^3\times\mathbb{T}^4$ and hence has $\mathfrak{psu}(1,1|2)$ as a symmetry algebra.

In chapter 4, we considered representations which are half-infinite in the $\mathfrak{sl}(2,\mathbb{R})$, and finite-dimensional in the two $\mathfrak{su}(2)$'s. Hence they are again characterised by (A.4), where the conditions on the K-modes apply to both $K_m^{(+)a}$ and $K_m^{(-)a}$. A representation is consequently parametrised by the three spins $|j,\ell^+,\ell^-\rangle$. A representation $|j,\ell^+,\ell^-\rangle$ is atypical if the BPS bound

$$j \geq \gamma\ell^+ + (1-\gamma)\ell^- \,, \tag{A.9}$$

is saturated, and it is otherwise typical. A typical multiplet consists of the following 16 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$-representations:

$$\begin{aligned} 2(j,\ell^+,\ell^-)\,,\ (j\pm 1,\ell^+,\ell^-)\,,\ (j,\ell^+\pm 1,\ell^-)\,, \\ (j,\ell^+,\ell^-\pm 1)\,,\ (j\pm\tfrac{1}{2},\ell^+\pm\tfrac{1}{2},\ell^-\pm\tfrac{1}{2})\,. \end{aligned} \tag{A.10}$$

Its quadratic Casimir reads

$$\mathcal{C}(j,\ell^+,\ell^-) = -j(j-1) + \gamma\ell^+(\ell^+ + 1) + (1-\gamma)\ell^-(\ell^- + 1)\,. \tag{A.11}$$

In the main text, we shall find it useful to parametrise γ by the angle φ through $\gamma = \cos^2\varphi$.

A.2 Various commutation relations

A.2.1 The RNS formalism of strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$

The bosonic part of the worldsheet algebra is given by the WZW-model $\mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k-2}$, together with four bosons. They have commutation relations

$$[\mathcal{J}^3_m, \mathcal{J}^3_n] = -\tfrac{k+2}{2} m\delta_{m+n,0}\,, \tag{A.12a}$$
$$[\mathcal{J}^3_m, \mathcal{J}^\pm_n] = \pm\mathcal{J}^\pm_{m+n}\,, \tag{A.12b}$$
$$[\mathcal{J}^+_m, \mathcal{J}^-_n] = (k+2)m\delta_{m+n,0} - 2\mathcal{J}^3_{m+n,0}\,, \tag{A.12c}$$
$$[\mathcal{K}^3_m, \mathcal{K}^3_n] = \tfrac{k-2}{2} m\delta_{m+n,0}\,, \tag{A.12d}$$
$$[\mathcal{K}^3_m, \mathcal{K}^\pm_n] = \pm\mathcal{K}^\pm_{m+n}\,, \tag{A.12e}$$
$$[\mathcal{K}^+_m, \mathcal{K}^-_n] = (k-2)m\delta_{m+n,0} + 2\mathcal{K}^3_{m+n,0}\,, \tag{A.12f}$$
$$[\partial X^\alpha_m, \partial\bar{X}^\beta_n] = m\varepsilon^{\alpha\beta}\delta_{m+n,0}\,. \tag{A.12g}$$

There are moreover ten bosons on the worldsheet, which we denote by ψ^a, χ^a, λ^α and $\bar{\lambda}^\alpha$. We take them to have anticommutation relations

$$\{\psi^3_r, \psi^3_s\} = -\tfrac{k}{2}\delta_{r+s,0}\,, \tag{A.13a}$$
$$\{\psi^+_r, \psi^-_s\} = k\delta_{r+s,0}\,, \tag{A.13b}$$
$$\{\chi^3_r, \chi^3_s\} = \tfrac{k}{2}\delta_{r+s,0}\,, \tag{A.13c}$$
$$\{\chi^+_r, \chi^-_s\} = k\delta_{r+s,0}\,, \tag{A.13d}$$
$$\{\lambda^\alpha_r, \bar{\lambda}^\beta_s\} = \varepsilon^{\alpha\beta}\delta_{r+s,0}\,. \tag{A.13e}$$

A.2.2 The RNS formalism of strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$

The bosonic generators on the worldsheet give rise to the affine algebra $\mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k^+-2} \oplus \mathfrak{su}(2)_{k^--2}$, together with one free boson. Their modes satisfy the commutation relations

$$[\mathcal{J}^3_m, \mathcal{J}^3_n] = -\tfrac{k+2}{2}\, m\delta_{m+n,0} \,, \tag{A.14a}$$
$$[\mathcal{J}^3_m, \mathcal{J}^\pm_n] = \pm\mathcal{J}^\pm_{m+n} \,, \tag{A.14b}$$
$$[\mathcal{J}^+_m, \mathcal{J}^-_n] = (k+2)m\delta_{m+n,0} - 2\,\mathcal{J}^3_{m+n,0} \,, \tag{A.14c}$$
$$[\mathcal{K}^{(\pm)3}_m, \mathcal{K}^{(\pm)3}_n] = \tfrac{k^\pm-2}{2}\, m\delta_{m+n,0} \,, \tag{A.14d}$$
$$[\mathcal{K}^{(\pm)3}_m, \mathcal{K}^{(\pm)\pm}_n] = \pm\mathcal{K}^{(\pm)\pm}_{m+n} \,, \tag{A.14e}$$
$$[\mathcal{K}^{(\pm)+}_m, \mathcal{K}^{(\pm)-}_n] = (k^\pm-2)m\delta_{m+n,0} + 2\,\mathcal{K}^{(\pm)3}_{m+n,0} \,, \tag{A.14f}$$
$$[\partial\Phi_m, \partial\Phi_n] = m\delta_{m+n,0} \,. \tag{A.14g}$$

There are moreover ten fermions on the worldsheet, which we denote by ψ^a, $\chi^{(\pm)a}$ and λ. We take them to have anticommutation relations

$$\{\psi^3_r, \psi^3_s\} = -\tfrac{k}{2}\,\delta_{r+s,0} \,, \tag{A.15a}$$
$$\{\psi^+_r, \psi^-_s\} = k\,\delta_{r+s,0} \,, \tag{A.15b}$$
$$\{\chi^{(\pm)3}_r, \chi^{(\pm)3}_s\} = \tfrac{k^\pm}{2}\,\delta_{r+s,0} \,, \tag{A.15c}$$
$$\{\chi^{(\pm)+}_r, \chi^{(\pm)-}_s\} = k^\pm\,\delta_{r+s,0} \,, \tag{A.15d}$$
$$\{\lambda_r, \lambda_s\} = \delta_{r+s,0} \,. \tag{A.15e}$$

Out of the bosonic currents at level $k+2$, k^+-2 and k^--2 and the free fermions, one can define 'supersymmetric' currents at level k, k^+ and k^-, respectively

$$J^\pm = \mathcal{J}^\pm \mp \frac{2}{k}(\psi^3\psi^\pm) \,, \tag{A.16a}$$
$$J^3 = \mathcal{J}^3 + \frac{1}{k}(\psi^+\psi^-) \,, \tag{A.16b}$$
$$K^{(\pm)\pm} = \mathcal{K}^{(\pm)\pm} \pm \frac{2}{k^\pm}(\chi^{(\pm)3}\chi^{(\pm)\pm}) \,, \tag{A.16c}$$
$$K^{(\pm)3} = \mathcal{K}^{(\pm)3} + \frac{1}{k^\pm}(\chi^{(\pm)+}\chi^{(\pm)-}) \,, \tag{A.16d}$$

whose zero modes correspond to the global (bosonic) generators of the spacetime supersymmetry algebra. Via picture changing, we can also write them in the canonical (-1) picture, where they simply read

$$J^\pm = \psi^\pm \mathrm{e}^{-\varphi} \,, \quad J^3 = \psi^3\mathrm{e}^{-\varphi} \,, \quad K^{(\pm)\pm} = \chi^{(\pm)\pm}\mathrm{e}^{-\varphi} \,, \quad K^{(\pm)3} = \chi^{(\pm)3}\mathrm{e}^{-\varphi} \,. \tag{A.17}$$

A.2.3 The $\mathcal{N}=4$ Liouville fields

To construct $\mathcal{N}=4$ Liouville theory, we start with a free field construction of the $\mathcal{N}=4$ algebra in terms of the free fields

$$\partial\varphi\,, \qquad \psi^{\alpha\beta}\,, \qquad J^a\,, \tag{A.18}$$

where $\partial\varphi$ is a free boson, $\psi^{\alpha\beta}$ are four free fermions and J^a is a $\mathfrak{su}(2)_{\kappa-1}$ current. They have defining OPEs

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{1}{(z-w)^2}\,, \tag{A.19a}$$
$$\psi^{\alpha\beta}(z)\psi^{\gamma\delta}(w) \sim \frac{\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}}{z-w}\,, \tag{A.19b}$$
$$J^3(z)J^3(w) \sim \frac{\kappa-1}{2(z-w)^2}\,, \tag{A.19c}$$
$$J^3(z)J^\pm(w) \sim \frac{J^\pm(w)}{z-w}\,, \tag{A.19d}$$
$$J^+(z)J^-(w) \sim \frac{\kappa-1}{(z-w)^2}+\frac{2J^3(w)}{z-w}\,. \tag{A.19e}$$

These fields can be cast into an $\mathcal{N}=4$ superfield.

A.3 Free field systems

A.3.1 bc system

A bc system consists of anti-commuting fields $b(z)$ and $c(z)$ with conformal dimensions

$$h(b)=\lambda\,, \qquad h(c)=1-\lambda\,, \tag{A.20}$$

and defining OPE

$$b(z)c(w) \sim \frac{1}{z-w}\,. \tag{A.21}$$

The corresponding energy momentum tensor is

$$T=(\partial b)c-\lambda\,\partial(bc)\,, \tag{A.22}$$

and it has central charge $c=1-3(2\lambda-1)^2$. In particular, the bc system with $\lambda=2$ has $c=-26$.

A.3.2 $\beta\gamma$ system

A $\beta\gamma$ system consists of commuting fields $\beta(z)$ and $\gamma(z)$ with conformal dimensions

$$h(\beta)=\lambda\,, \qquad h(\gamma)=1-\lambda\,, \tag{A.23}$$

and defining OPE

$$\beta(z)\gamma(w) \sim -\frac{1}{z-w} . \tag{A.24}$$

The corresponding energy momentum tensor is

$$T = (\partial\beta)\gamma - \lambda\,\partial(\beta\gamma) , \tag{A.25}$$

and it has central charge $c = 3(2\lambda - 1)^2 - 1$. In particular, the $\beta\gamma$ system with $\lambda = \frac{3}{2}$ has $c = 11$.

A.3.3 Free bosons

For a free boson with OPE

$$\varphi(z)\varphi(w) \sim \varepsilon \log(z-w) , \tag{A.26}$$

and background charge Q, the stress-energy tensor is

$$T = \varepsilon\Big[\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}Q\partial^2\varphi\Big] , \tag{A.27}$$

with central charge

$$c = 1 - 3\varepsilon Q^2 . \tag{A.28}$$

The conformal dimension of the field $e^{q\varphi}$ is then

$$h\big(e^{q\varphi}\big) = \frac{1}{2}\varepsilon q(q+Q) . \tag{A.29}$$

A.4 Theta functions

We follow the notation of [262] and define the theta functions as

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z;\tau) \equiv \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi i(n+\alpha)^2\tau + 2\pi i(n+\alpha)(z+\beta)} \tag{A.30}$$

$$= \mathrm{e}^{2\pi i\alpha(z+\beta)} q^{\frac{\alpha^2}{2}} \prod_{n=1}^{\infty}\big(1-q^n\big) \times \big(1+q^{n+\alpha-\frac{1}{2}}\mathrm{e}^{2\pi i(z+\beta)}\big)\big(1+q^{n-\alpha-\frac{1}{2}}\mathrm{e}^{-2\pi i(z+\beta)}\big) . \tag{A.31}$$

The four Jacobi theta functions are the special cases

$$\vartheta_1 \equiv \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} , \qquad \vartheta_2 \equiv \vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix} , \qquad \vartheta_3 \equiv \vartheta\begin{bmatrix}0\\ 0\end{bmatrix} , \qquad \vartheta_4 \equiv \vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix} . \tag{A.32}$$

In particular, we have the identity

$$\vartheta_2\big(\tfrac{u+v}{2};\tau\big)\vartheta_2\big(\tfrac{u-v}{2};\tau\big) = \vartheta_2(u;2\tau)\vartheta_3(v;2\tau) + \vartheta_3(u;2\tau)\vartheta_2(v;2\tau) , \tag{A.33}$$

$$\vartheta_1\left(\tfrac{u+v}{2};\tau\right)\vartheta_1\left(\tfrac{u-v}{2};\tau\right) = -\vartheta_2(u;2\tau)\vartheta_3(v;2\tau) + \vartheta_3(u;2\tau)\vartheta_2(v;2\tau)\ , \quad \text{(A.34)}$$

which expresses the equivalence of four free fermions to $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_1$.

We also make use of the behaviour under modular transformations, in particular of $\vartheta_1(z,\tau)$ and $\eta(\tau)$,

$$\vartheta_1\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\,\mathrm{e}^{\frac{\pi i z^2}{\tau}}\,\vartheta_1(z;\tau)\ , \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)\ . \quad \text{(A.35)}$$

Appendix B

Representations

In this Appendix, we will give some details about representations of relevant groups and algebras which appeared in the main text.

B.1 Representations of $\mathrm{SL}(2,\mathbb{R})$

Here, we describe the representations of (the universal cover of) $\mathrm{SL}(2,\mathbb{R})$ that are relevant for the description of the Maldacena & Ooguri theory [60]. We begin by reviewing the 'usual' construction of unitary representations of $\mathrm{SL}(2,\mathbb{R})$, see e.g. [100].

B.1.1 Representations of $\mathrm{SU}(1,1)$

The unitary representations of $\mathrm{SL}(2,\mathbb{R})$ are most easily constructed in terms of the group $\mathrm{SU}(1,1)$, which is isomorphic to $\mathrm{SL}(2,\mathbb{R})$. The group $\mathrm{SU}(1,1)$ consists of the complex 2×2 matrices of the form

$$D=\begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix} \tag{B.1}$$

with $|a|^2-|b|^2=1$. These matrices have a natural action on the unit disc $|z|\leq 1$ via

$$z\mapsto \gamma_D(z)=\frac{az+b}{\bar{b}z+\bar{a}}\ , \tag{B.2}$$

which in particular maps the unit circle, $|z|=1$, to itself. The irreducible unitary (continuous) representations of $\mathrm{SU}(1,1)$ can be constructed on the Hilbert space

$$\mathrm{L}^2_{j,\lambda}(\mathrm{S}^1)=\left\{f:\mathrm{S}^1\to\mathbb{C}\,|\,f(\varphi+2\pi)=\mathrm{e}^{-2\pi i(j+\lambda)}f(\varphi)\right\}\ , \tag{B.3}$$

where the action of a group element $D\in\mathrm{SU}(1,1)$ is defined via

$$(D\cdot f)(z)=(\gamma_D^{-1})'(z)^j\, f(\gamma_D^{-1}(z))\ , \tag{B.4}$$

and $\gamma_D(z)$ is given in (B.2). (Here we identify S^1 with the set $|z| = 1$, and write $z = e^{i\varphi}$.) We have denoted the spin by j (or equivalently the quadratic Casimir by $\mathcal{C} = -j(j-1)$).

It is not difficult to show that the whole group SU(1,1) acts on $L^2_{j,\lambda}(S^1)$. There is a natural set of basis functions $f_m(\varphi)$, $m \in \mathbb{Z} + \lambda$, of $L^2_{j,\lambda}(S^1)$, given by

$$f_m(\varphi) = e^{-i(j+m)\varphi} \,, \tag{B.5}$$

and on them the Lie algebra generator

$$\Lambda_0 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{su}(1,1) \tag{B.6}$$

acts diagonally with eigenvalue

$$(\Lambda_0 \cdot f_m)(\varphi) = im\, f_m(\varphi) \,. \tag{B.7}$$

Note that Λ_0 generates the compact Cartan torus $U(1) \subset SU(1,1)$. Depending on the value of j and λ, these representations therefore define the usual continuous and discrete representations of SU(1,1).

B.1.2 Representations of $SL(2,\mathbb{R})$

On the other hand, the Lie group $SL(2,\mathbb{R})$ consists of the real 2×2 matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,, \qquad ad - bc = 1 \,. \tag{B.8}$$

This group acts naturally on the upper half-plane via

$$\tau \mapsto \gamma_M(\tau) = \frac{a\tau + b}{c\tau + d} \,, \tag{B.9}$$

and it fixes the real line $\tau \in \mathbb{R}$. Its Lie algebra is generated by the elements L_0, $L_{\pm 1}$ with

$$L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,, \qquad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \,, \qquad L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \,, \tag{B.10}$$

that satisfy the Lie algebra

$$[L_m, L_n] = (m-n)L_{m+n} \,, \qquad m,n = -1,0,1 \,. \tag{B.11}$$

Note that the Cartan torus corresponding to L_0 is, in this case, the noncompact subgroup of diagonal matrices of $SL(2,\mathbb{R})$.

The Lie group $\mathrm{SL}(2,\mathbb{R})$ is isomorphic to $\mathrm{SU}(1,1)$: the isomorphism is achieved by the Cayley transform that maps the upper half-plane to the unit disc,

$$C = \frac{\mathrm{e}^{-\frac{i\pi}{4}}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} , \qquad c(\tau) = z = \frac{\tau - i}{\tau + i} . \tag{B.12}$$

Indeed, for any $M \in \mathrm{SL}(2,\mathbb{R})$,

$$D = C\, M\, C^{-1} \in \mathrm{SU}(1,1) . \tag{B.13}$$

As a consequence, any irreducible unitary representation of $\mathrm{SU}(1,1)$ gives rise to such a representation of $\mathrm{SL}(2,\mathbb{R})$, and vice versa. Under this isomorphism, the generator Λ_0 that acts diagonally on the unitary representations of $\mathrm{SU}(1,1)$ becomes

$$L_0' = C^{-1}\, \Lambda_0\, C = \frac{1}{2}\left(L_{-1} + L_1\right) . \tag{B.14}$$

In the analysis of [60] J_0^3 is taken to be the (compact) L_0' generator of the WZW algebra $\mathfrak{sl}(2,\mathbb{R})$ (which acts diagonally on the standard representations of $\mathfrak{sl}(2,\mathbb{R})$). However, from the viewpoint of the spacetime theory, J_0^3 is identified with the (non-compact) Möbius generator L_0 (rather than L_0') of the dual CFT. This has important consequences since the corresponding representations are not isomorphic as representations of $\mathrm{SL}(2,\mathbb{R})$. This is a consequence of the fact that L_0 and L_0' are *not* conjugate to one another in $\mathrm{SL}(2,\mathbb{R})$, but only in $\mathrm{SL}(2,\mathbb{C})$.

In order to understand the structure of the representations for which L_0 acts diagonally, we consider the vector space of functions

$$\mathcal{H} = \left\{ f : \mathbb{R} \to \mathbb{C} \right\} , \tag{B.15}$$

on which $M \in \mathrm{SL}(2,\mathbb{R})$ acts via

$$(M \cdot f)(x) = (\gamma_M^{-1})'(x)^j\, f(\gamma_M^{-1}(x)) , \tag{B.16}$$

where

$$\gamma_M(x) = \frac{ax+b}{cx+d} , \qquad \text{for} \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) . \tag{B.17}$$

In this representation the Lie algebra generators L_m of $\mathfrak{sl}(2,\mathbb{R})$ are the differential operators

$$L_1 = -\frac{\partial}{\partial x} , \qquad L_0 = -x\frac{\partial}{\partial x} - j , \qquad L_{-1} = -x^2\frac{\partial}{\partial x} - 2jx , \tag{B.18}$$

and the Casimir equals $\mathcal{C} = -j(j-1)$. The Lie algebra generators act on the subspace of functions that is generated by

$$g_m(x) = x^{-j-m} , \qquad m \in \mathbb{Z} + \lambda , \tag{B.19}$$

where L_0 acts diagonally with eigenvalue

$$(L_0 \cdot g_m)(x) = m\, g_m(x) \ . \tag{B.20}$$

For generic λ, the Lie algebra generators map the different $g_m(x)$ into one another, and hence realise a representation of the Lie algebra that is isomorphic to the usual continuous representation $\mathcal{C}^j_\lambda$. On the other hand, for $\lambda = -j$, we can restrict to the subspace of functions with $m + j \in \mathbb{Z}_{\leq 0}$ which leads to $\mathcal{D}^-_j$, while for $\lambda = j$ we get $\mathcal{D}^+_j$ from the functions with $m - j \in \mathbb{Z}_{\geq 0}$.

However, unlike the situation described above in the context of $\mathrm{SU}(1,1)$, this realisation does *not* actually lead to the corresponding representation of the Lie group $\mathrm{SL}(2,\mathbb{R})$. Indeed, for the inversion element of $\mathrm{SL}(2,\mathbb{R})$,

$$M_{\text{inv}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \qquad \gamma_{M_{\text{inv}}}(x) = -\frac{1}{x} \ , \tag{B.21}$$

we find

$$(M_{\text{inv}} \cdot g_m)(x) = \mathrm{e}^{-\pi i (j+m)} x^{-j+m} \ , \tag{B.22}$$

which cannot (in general, i.e. if $\lambda \notin \frac{1}{2}\mathbb{Z}$) be written in terms of the $g_m(x)$ functions. (Similarly, the discrete representations $\mathcal{D}^\pm_j$ do not define a representation of the Lie group by themselves, but the inversion element of $\mathrm{SL}(2,\mathbb{R})$ maps $\mathcal{D}^+_j$ to $\mathcal{D}^-_j$, and vice versa.) Other group elements in $\mathrm{SL}(2,\mathbb{R})$ (in particular, translations) map $g_m(x)$ also to functions that have branch-cuts originating from other points $x \neq 0$ on the real line. While one can formally write these functions in terms of Laurent polynomials around $x = 0$, the more natural way to describe the space on which (the universal covering group of) $\mathrm{SL}(2,\mathbb{R})$ acts, is as the full space (B.15), with the action being given by (B.16). This is then the so-called x-basis of [60].[80] In particular, the representation of this $\mathrm{SL}(2,\mathbb{R})$ incorporates all representations of $\mathfrak{sl}(2,\mathbb{R})$ with a given value of j (and hence of the Casimir $\mathcal{C} = -j(j-1)$), but with all values of λ, including the two discrete representations $\mathcal{D}^+_j$ and $\mathcal{D}^-_j$, see also [263] for a related observation. This observation plays an important role in Section 5.4.

B.2 The short representation of $\mathfrak{psu}(1,1|2)$

In this appendix we describe the short representation described by (5.5) explicitly. We label the states as

$$|m,\uparrow,0\rangle \ , \ |m,\downarrow,0\rangle \in (\mathcal{C}^{\frac{1}{2}}_\lambda, \mathbf{2}) \tag{B.23}$$

[80] Incidentally, the fact that both $\mathcal{D}^+_j$ and $\mathcal{D}^-_j$ appear in this basis was also already noticed there.

$$|m,0,\uparrow\rangle \in (\mathcal{C}^0_{\lambda+\frac{1}{2}},\mathbf{1}) \ , \quad |m,0,\downarrow\rangle \in (\mathcal{C}^1_{\lambda+\frac{1}{2}},\mathbf{1}) \ . \tag{B.24}$$

Thus $m \in \mathbb{Z}+\lambda$ in the first line, while $m \in \mathbb{Z}+\lambda+\frac{1}{2}$ for the states in the second line. In each case the third entry keeps track of the quantum numbers with respect to the outer automorphism $\mathfrak{su}(2)$. In particular, the two states in the second line transform actually in a doublet under the outer automophism.[81] The bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2)$ acts on these states according to the representation theory we described in Subsection 2.2.1 and the usual representation theory of $\mathfrak{su}(2)$. In our conventions, this takes the form

$$J^3_0\,|m,\updownarrow,0\rangle = m\,|m,\updownarrow,0\rangle \ , \qquad J^3_0\,|m,0,\updownarrow\rangle = m\,|m,0,\updownarrow\rangle \ , \tag{B.25a}$$
$$J^+_0\,|m,\updownarrow,0\rangle = |m+1,\updownarrow,0\rangle \ , \qquad J^+_0\,|m,0,\updownarrow\rangle = |m+1,0,\updownarrow\rangle \ , \tag{B.25b}$$
$$J^-_0\,|m,\updownarrow,0\rangle = \left(m-\tfrac{1}{2}\right)^2|m-1,\updownarrow,0\rangle \ , \ J^-_0\,|m,0,\updownarrow\rangle = m(m-1)\,|m-1,0,\updownarrow\rangle \ , \tag{B.25c}$$
$$K^3_0\,|m,\uparrow,0\rangle = \tfrac{1}{2}\,|m,\uparrow,0\rangle \ , \qquad K^3_0\,|m,\downarrow,0\rangle = -\tfrac{1}{2}\,|m,\downarrow,0\rangle \ , \tag{B.25d}$$
$$K^+_0\,|m,\downarrow,0\rangle = |m,\uparrow,0\rangle \ , \qquad K^-_0\,|m,\uparrow,0\rangle = |m,\downarrow,0\rangle \ . \tag{B.25e}$$

Here, by $\updownarrow$ we mean either of the two states corresponding to $\uparrow$ or $\downarrow$. On the other hand, the 8 supercharges act as

$$S^{---}_0\,|m,\uparrow,0\rangle = -\left(m-\tfrac{1}{2}\right)\left|m-\tfrac{1}{2},0,\downarrow\right\rangle \ , \ S^{+--}_0\,|m,\uparrow,0\rangle = \left|m+\tfrac{1}{2},0,\downarrow\right\rangle \ , \tag{B.25f}$$
$$S^{---}_0\,|m,0,\uparrow\rangle = m\left|m-\tfrac{1}{2},\downarrow,0\right\rangle \ , \qquad S^{+--}_0\,|m,0,\uparrow\rangle = -\left|m+\tfrac{1}{2},\downarrow,0\right\rangle \ , \tag{B.25g}$$
$$S^{--+}_0\,|m,\uparrow,0\rangle = -\left(m-\tfrac{1}{2}\right)\left|m-\tfrac{1}{2},0,\uparrow\right\rangle \ , \ S^{+-+}_0\,|m,\uparrow,0\rangle = \left|m+\tfrac{1}{2},0,\uparrow\right\rangle \ , \tag{B.25h}$$
$$S^{--+}_0\,|m,0,\downarrow\rangle = -m\left|m-\tfrac{1}{2},\downarrow,0\right\rangle \ , \qquad S^{+-+}_0\,|m,0,\downarrow\rangle = \left|m+\tfrac{1}{2},\downarrow,0\right\rangle \ , \tag{B.25i}$$
$$S^{-+-}_0\,|m,\downarrow,0\rangle = \left(m-\tfrac{1}{2}\right)\left|m-\tfrac{1}{2},0,\downarrow\right\rangle \ , \qquad S^{++-}_0\,|m,\downarrow,0\rangle = -\left|m+\tfrac{1}{2},0,\downarrow\right\rangle \ , \tag{B.25j}$$
$$S^{-+-}_0\,|m,0,\uparrow\rangle = m\left|m-\tfrac{1}{2},\uparrow,0\right\rangle \ , \qquad S^{++-}_0\,|m,0,\uparrow\rangle = -\left|m+\tfrac{1}{2},\uparrow,0\right\rangle \ , \tag{B.25k}$$
$$S^{-++}_0\,|m,\downarrow,0\rangle = \left(m-\tfrac{1}{2}\right)\left|m-\tfrac{1}{2},0,\uparrow\right\rangle \ , \qquad S^{+++}_0\,|m,\downarrow,0\rangle = -\left|m+\tfrac{1}{2},0,\uparrow\right\rangle \ , \tag{B.25l}$$
$$S^{-++}_0\,|m,0,\downarrow\rangle = -m\left|m-\tfrac{1}{2},\uparrow,0\right\rangle \ , \qquad S^{+++}_0\,|m,0,\downarrow\rangle = \left|m+\tfrac{1}{2},\uparrow,0\right\rangle \ , \tag{B.25m}$$

[81]Note that $\mathcal{C}^0_{\lambda+1/2} \cong \mathcal{C}^1_{\lambda+1/2}$ since both have the same Casimir $\mathcal{C}=0$. On the other hand, the Casimir of the representation $\mathcal{C}^{1/2}_{\lambda}$ equals $\mathcal{C}=\frac{1}{4}$.

while all other actions are zero. This characterises the representation completely. As we have indicated in the main text, for $\lambda = \frac{1}{2}$, the states

$$|m, \updownarrow, 0\rangle\ , \quad m \geq \tfrac{1}{2} \qquad \text{and} \qquad |m, 0, \updownarrow\rangle\ , \quad m \geq 1 \tag{B.26}$$

form an irreducible subrepresentation which is obtained from (5.5) upon replacing $\mathcal{C}^{1/2}_{1/2}$ by its subrepresentation $\mathcal{D}^{1/2}_{+}$, and $\mathcal{C}^{1}_{1} \cong \mathcal{C}^{0}_{1}$ by its subrepresentation $\mathcal{D}^{1}_{+}$. Similarly, the states

$$|m, \updownarrow, 0\rangle\ , \quad m \geq \tfrac{1}{2} \qquad \text{and} \qquad |m, 0, \updownarrow\rangle\ , \quad m \geq 0 \tag{B.27}$$

form a slightly bigger indecomposable subrepresentation. The quotient of (B.27) by (B.26) consists of the two states $|0, 0, \updownarrow\rangle$, which is isomorphic to twice the trivial representation.

B.3 The short representation of $\mathfrak{d}(2, 1; \alpha)$

In this Appendix, we will display the short representation (7.31) explicitly. We will denote the states that appear by

$$|m, m_+, m_-\rangle\ , \qquad m \in \mathbb{Z} + \lambda\ , \qquad m_+ \in \{-\tfrac{1}{2}, \tfrac{1}{2}\}\ , \qquad m_- \in \{-\tfrac{\ell}{2}, \ldots, \tfrac{\ell}{2}\}\ , \tag{B.28}$$

$$|m, 0, m_-, \pm\rangle\ , \qquad m \in \mathbb{Z} + \tfrac{1}{2} + \lambda\ , \qquad m_- \in \{-\tfrac{\ell \pm 1}{2}, \ldots, \tfrac{\ell \pm 1}{2}\}\ . \tag{B.29}$$

For the action of the bosonic subalgebra we choose the conventions that for $\mathfrak{sl}(2, \mathbb{R})$ we have

$$J^3_0\, |j, m\rangle = m\ |j, m\rangle \qquad J^{\pm}_0\, |j, m\rangle = \big(m \pm j\big)\, |j, m \pm 1\rangle\ , \tag{B.30}$$

while for the spin ℓ representation of $\mathfrak{su}(2)$ we set

$$K^3_0\, |\ell, m\rangle = m\, |\ell, m\rangle\ , \qquad K^{\pm}_0\, |\ell, m\rangle = (\ell \mp m)\ |\ell, m \pm 1\rangle\ . \tag{B.31}$$

The states are then not unit normalised, but this convention is nevertheless convenient. The action of the supercharges is

$$\begin{aligned} S^{\alpha\beta\gamma}_0\, |m, m_+, m_-\rangle = \frac{\alpha \varepsilon^{\beta, 2m_+}}{\sqrt{2\ell + 1}} \Big(&- \big|m + \tfrac{\alpha}{2}, 0, m_- + \tfrac{\gamma}{2}, +\big\rangle \\ &+ \big(m + \alpha j\big)\big(\gamma \ell - m_-\big)\, \big|m + \tfrac{\alpha}{2}, 0, m_- + \tfrac{\gamma}{2}, -\big\rangle \Big)\ , \end{aligned} \tag{B.32a}$$

$$S^{\alpha\beta\gamma}_0\, |m, 0, m_-, +\rangle = \frac{\big(\alpha m + (j - \frac{1}{2})\big)\ \big(\gamma\, (\ell + \frac{1}{2}) - m_-\big)}{\sqrt{2\ell + 1}} \Big|m + \tfrac{\alpha}{2}, \tfrac{\beta}{2}, m_- + \tfrac{\gamma}{2}\Big\rangle\ , \tag{B.32b}$$

$$S^{\alpha\beta\gamma}_0\, |m, 0, m_-, -\rangle = \frac{\alpha}{\sqrt{2\ell + 1}} \Big|m + \tfrac{\alpha}{2}, \tfrac{\beta}{2}, m_- + \tfrac{\gamma}{2}\Big\rangle\ . \tag{B.32c}$$

One can check directly that this defines a representation of $\mathfrak{d}(2,1;\alpha)$ (in the conventions of eqs. (A.7a)–(A.7j)), provided that

$$j = (1-\gamma)\left(\ell + \tfrac{1}{2}\right) + \tfrac{1}{2}\ , \tag{B.33}$$

see eq. (7.32) in the main text.

Note that for $\ell = 0$, the states $|m, 0, m_-, -\rangle$ are never produced by the action of the generators, and hence can be decoupled from the multiplet, see eq. (7.40). Similarly, the states

$$|m, m_+, m_-\rangle\ , \qquad m \in j + \mathbb{Z}_{\ge 0}\ , \tag{B.34}$$

$$|m, 0, m_-, \pm\rangle\ , \qquad m \in j \mp \tfrac{1}{2} + \mathbb{Z}_{\ge 0} \tag{B.35}$$

form a subrepresentation, which is the discrete representation on which $\mathcal{G}_{>,+}$ is based, while the states

$$|m, m_+, m_-\rangle\ , \qquad m \in -j + \mathbb{Z}_{\le 0}\ , \tag{B.36}$$

$$|m, 0, m_-, \pm\rangle\ , \qquad m \in -j \pm \tfrac{1}{2} + \mathbb{Z}_{\le 0} \tag{B.37}$$

form the discrete subrepresentation which gives rise to $\mathcal{G}_{<,-}$ in the affine algebra. In order to obtain the other two discrete representations (which give rise to $\mathcal{G}_{>,-}$ and $\mathcal{G}_{<,+}$ in the affine algebra), one has to replace j by $1-j$ in the continuous representations of $\mathfrak{sl}(2,\mathbb{R})$.

Appendix C

Details about the $\mathfrak{psu}(1,1|2)_1$ WZW model

In this appendix, we will give some details about the $\mathfrak{psu}(1,1|2)_1$ WZW model, which played an important role in the main text.

C.1 The free field representation of $\mathfrak{psu}(1,1|2)_1$

In this appendix we provide details about the free field realisation of $\mathfrak{psu}(1,1|2)_1$. We begin by reviewing the free field realisation of $\mathfrak{sl}(2,\mathbb{R})_{1/2}$ in terms of a pair of symplectic bosons [190,264].

C.1.1 The symplectic boson theory

We begin with a single pair of symplectic bosons ξ_m and $\bar{\xi}_m$, satisfying the commutation relations

$$[\xi_m, \bar{\xi}_n] = \delta_{m,-n} \ . \tag{C.1}$$

It gives rise to an $\mathfrak{sl}(2,\mathbb{R})_{1/2}$ affine algebra since we have

$$J^3_m = -\tfrac{1}{2}(\xi\bar{\xi})_m \ , \qquad J^+_m = \tfrac{1}{2}(\xi\xi)_m \ , \qquad J^-_m = \tfrac{1}{2}(\bar{\xi}\bar{\xi})_m \ . \tag{C.2}$$

Both ξ_m and $\bar{\xi}_m$ are spin-$\frac{1}{2}$ fields and possess therefore NS- and R-sector representations. The NS-sector highest weight representation is described by

$$\xi_r \,|0\rangle = 0 \ , \qquad \bar{\xi}_r \,|0\rangle = 0 \ , \qquad r \geq \tfrac{1}{2} \ , \tag{C.3}$$

and gives the vacuum representation of the theory, which we denote by $\mathcal{K}$. On the other hand, the R-sector representations of the symplectic boson pair have a zero-mode representation on the states $|m\rangle$ with action

$$\xi_0 \,|m\rangle = \sqrt{2}\,|m+\tfrac{1}{2}\rangle \ , \qquad \bar{\xi}_0 \,|m\rangle = -\sqrt{2}(m-\tfrac{1}{4})\,|m-\tfrac{1}{2}\rangle \ , \tag{C.4}$$

so that, in terms of the $\mathfrak{sl}(2,\mathbb{R})$ generators we have

$$J_0^3 \left|m\right\rangle = m \left|m\right\rangle , \qquad \mathcal{C}^{\mathfrak{sl}(2,\mathbb{R})} \left|m\right\rangle = \frac{3}{16} \left|m\right\rangle . \tag{C.5}$$

Thus the R-sector representations of the symplectic boson are labelled by $\lambda \in \mathbb{R}/\frac{1}{2}\mathbb{Z}$, describing the eigenvalues of J_0^3 mod $\frac{1}{2}\mathbb{Z}$; these representations are denoted by $\mathcal{E}_\lambda$. At $\lambda = \frac{1}{4}$, $\mathcal{E}_{1/4}$ becomes reducible, but indecomposable (as follows from the second term in (C.4)). It is not separately part of the Hilbert space, but rather combines together with other representations into an indecomposable representation $\mathcal{S}$ [190]. Its structure is best described in terms of the so-called composition series, which takes for $\mathcal{S}$ the form

$$\mathcal{S}: \quad \begin{array}{ccc} & \mathcal{K} & \\ \swarrow & & \searrow \\ \sigma^{-2}(\mathcal{K}) & & \sigma^{2}(\mathcal{K}) \\ \searrow & & \swarrow \\ & \mathcal{K} & \end{array} \tag{C.6}$$

Here the bottom line is the irreducible vacuum representation $\mathcal{K}$, and it forms a proper subrepresentation of $\mathcal{S}$. Since $\mathcal{S}$ is indecomposable, the complement of $\mathcal{K}$ does *not* form another subrepresentation of $\mathcal{S}$. However, the quotient space $\mathcal{S}/\mathcal{K}$ contains subrepresentations, namely the two irreducible representations described by the middle line of $\mathcal{S}$. Again, their complement is not another subrepresentation, so one needs to quotient again by the representations in the middle line. The resulting space is then the irreducible vacuum representation $\mathcal{K}$ appearing at the top of the diagram. In this language, the direction of the arrows indicates the symplectic boson action: symplectic bosons can map from top to bottom, but not back. The top element of the composition series is called the "head", whereas the bottom element is called the "socle".

The representation $\mathcal{S}$ is closely related to $\mathcal{E}_{1/4}$ since, on the level of the Grothendieck ring, we have

$$\mathcal{E}_{1/4} \sim \sigma(\mathcal{K}) \oplus \sigma^{-1}(\mathcal{K}) \quad \Longrightarrow \quad \mathcal{S} \sim \sigma(\mathcal{E}_{1/4}) \oplus \sigma^{-1}(\mathcal{E}_{1/4}) . \tag{C.7}$$

Here σ denotes the spectral flow of the symplectic boson theory which acts via

$$\sigma(\xi_r) = \xi_{r-\frac{1}{2}} , \qquad \sigma(\bar{\xi}_r) = \bar{\xi}_{r+\frac{1}{2}} . \tag{C.8}$$

The fusion rules of this theory were worked out in [190, eq. (5.8)],

$$\mathcal{E}_\lambda \times \mathcal{E}_\mu \cong \begin{cases} \sigma(\mathcal{E}_{\lambda+\mu+\frac{1}{4}}) \oplus \sigma^{-1}(\mathcal{E}_{\lambda+\mu+\frac{1}{4}}) , & \lambda+\mu \neq 0 , \\ \mathcal{S} , & \lambda+\mu = 0 , \end{cases} \tag{C.9a}$$

$$\mathcal{E}_\lambda \times \mathcal{S} \cong \sigma^2(\mathcal{E}_\lambda) \oplus 2 \cdot \mathcal{E}_\lambda \oplus \sigma^{-2}(\mathcal{E}_\lambda) \ , \tag{C.9b}$$
$$\mathcal{S} \times \mathcal{S} \cong \sigma^2(\mathcal{S}) \oplus 2 \cdot \mathcal{S} \oplus \sigma^{-2}(\mathcal{S}) \ . \tag{C.9c}$$

C.1.2 The explicit form of the free field representation

In order to describe the free field realisation of $\mathfrak{psu}(1,1|2)_1$ we combine together two such pairs of symplectic bosons, together with 2 complex fermions. More explicitly, let us denote the fermions by ψ^α, $\bar{\psi}^\alpha$, and the symplectic bosons by ξ^α and $\bar{\xi}^\alpha$ with $\alpha = \pm$ and (anti)-commutation relations

$$\{\bar{\psi}^\alpha_r, \psi^\beta_s\} = \epsilon^{\alpha\beta}\delta_{r,-s} \ , \qquad [\bar{\xi}^\alpha_r, \xi^\beta_s] = \epsilon^{\alpha\beta}\delta_{r,-s} \ . \tag{C.10}$$

(Anti)-commutators of barred with barred oscillators vanish, and similarly for the unbarred combinations. As is clear from the previous section, we can construct two $\mathfrak{sl}(2,\mathbb{R})_{1/2}$ algebras out of the symplectic bosons, which are explicitly given as

$$J^{(+)3}_m = -\tfrac{1}{2}(\xi^+\bar{\xi}^-)_m \ , \qquad J^{(-)3}_m = -\tfrac{1}{2}(\xi^-\bar{\xi}^+)_m \ , \tag{C.11a}$$
$$J^{(+)+}_m = \tfrac{1}{2}(\xi^+\xi^+)_m \ , \qquad J^{(-)+}_m = \tfrac{1}{2}(\bar{\xi}^+\bar{\xi}^+)_m \ , \tag{C.11b}$$
$$J^{(+)-}_m = \tfrac{1}{2}(\bar{\xi}^-\bar{\xi}^-)_m \ , \qquad J^{(-)+}_m = \tfrac{1}{2}(\xi^-\xi^-)_m \ . \tag{C.11c}$$

We define the two spectral flow symmetries via

$$\sigma^{(+)}(\xi^+_r) = \xi^+_{r-\frac{1}{2}} \ , \qquad \sigma^{(-)}(\bar{\xi}^+_r) = \bar{\xi}^+_{r-\frac{1}{2}} \ , \tag{C.12a}$$
$$\sigma^{(+)}(\bar{\xi}^-_r) = \bar{\xi}^-_{r+\frac{1}{2}} \ , \qquad \sigma^{(-)}(\xi^-_r) = \xi^-_{r+\frac{1}{2}} \ , \tag{C.12b}$$

so that $\sigma^{(+)}$ only acts on ξ^+ and $\bar{\xi}^-$ (that appear in the $J^{(+)a}$ generators), while $\sigma^{(-)}$ only acts on ξ^- and $\bar{\xi}^+$. We also define their action on the fermions via

$$\sigma^{(+)}(\psi^+_r) = \psi^+_{r+\frac{1}{2}} \ , \qquad \sigma^{(-)}(\bar{\psi}^+_r) = \bar{\psi}^+_{r+\frac{1}{2}} \ , \tag{C.13a}$$
$$\sigma^{(+)}(\bar{\psi}^-_r) = \bar{\psi}^-_{r-\frac{1}{2}} \ , \qquad \sigma^{(-)}(\psi^-_r) = \psi^-_{r-\frac{1}{2}} \ . \tag{C.13b}$$

We now realise the $\mathfrak{u}(1,1|2)_1$ algebra via the combinations

$$J^3_m = -\tfrac{1}{2}(\xi^+\bar{\xi}^-)_m - \tfrac{1}{2}(\xi^-\bar{\xi}^+)_m \ , \qquad K^3_m = -\tfrac{1}{2}(\psi^+\bar{\psi}^-)_m - \tfrac{1}{2}(\psi^-\bar{\psi}^+)_m \ , \tag{C.14a}$$
$$J^\pm_m = (\xi^\pm\bar{\xi}^\pm)_m \ , \qquad K^\pm_m = \pm(\psi^\pm\bar{\psi}^\pm)_m \ , \tag{C.14b}$$
$$S^{\alpha\beta+}_m = (\psi^\beta\bar{\xi}^\alpha)_m \ , \qquad S^{\alpha\beta-}_m = -(\xi^\alpha\bar{\psi}^\beta)_m \ , \tag{C.14c}$$
$$U_m = -\tfrac{1}{2}(\xi^+\bar{\xi}^-)_m + \tfrac{1}{2}(\xi^-\bar{\xi}^+)_m \ , \qquad V_m = -\tfrac{1}{2}(\psi^+\bar{\psi}^-)_m + \tfrac{1}{2}(\psi^-\bar{\psi}^+)_m \ . \tag{C.14d}$$

Here the generators J^a_m define $\mathfrak{sl}(2,\mathbb{R})_1$, but they are *not* just the direct sums of the $J^{(\pm)a}_m$ generators from (C.11a)–(C.11c). It is convenient to define the two linear combinations

$$Z_m = U_m + V_m \qquad \text{and} \qquad Y_m = U_m - V_m \ . \tag{C.15}$$

Then the modes J^a_m, K^a_m and $S^{\alpha\beta\gamma}_m$ satisfy (A.2a) – (A.2i) with $k=1$, except that Z_m appears as a central extension in (A.2i),

$$\{S^{\alpha\beta\gamma}_m, S^{\mu\nu\rho}_n\} = km\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\varepsilon^{\gamma\rho}\delta_{m+n,0} - \varepsilon^{\beta\nu}\varepsilon^{\gamma\rho}c_a{\sigma_a}^{\alpha\mu}J^a_{m+n} + \varepsilon^{\alpha\mu}\varepsilon^{\gamma\rho}{\sigma_a}^{\beta\nu}K^a_{m+n} + \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\delta^{\gamma,-\rho}Z_{m+n} \ . \tag{C.16}$$

Here $\delta^{\gamma,-\rho}$ denotes as usual the Kronecker delta. J^a_m, K^a_m, the supercharges $S^{\alpha\beta\gamma}_m$ and the central extension Z_m generate then the superalgebra $\mathfrak{su}(1,1|2)_1$. In particular, Z_m commutes with the modes of J^a_m, K^a_m, $S^{\alpha\beta\gamma}_m$ and its own modes. The remaining commutators involving Y_m are given by

$$[Y_m, J^a_n] = 0 \ , \qquad [Y_m, K^a_n] = 0 \ , \qquad [Y_m, S^{\alpha\beta\gamma}_n] = -\gamma S^{\alpha\beta\gamma} \ , \tag{C.17}$$

$$[Y_m, Y_n] = 0 \ , \qquad [Y_m, Z_n] = -m\delta_{m+n,0} \ . \tag{C.18}$$

From these commutation relations, we see that only the zero-charge sector of the central extension Z lifts to representations of $\mathfrak{psu}(1,1|2)_1$. Furthermore, in order to obtain complete $\mathfrak{psu}(1,1|2)_1$-representations, we have to sum over all charges of $\mathfrak{u}(1)_Y$, because the supercharges carry charge with respect to $\mathfrak{u}(1)_Y$.

We should mention that the combination of spectral flow $\sigma^{(+)} \circ (\sigma^{(-)})^{-1}$ keeps the bosonic generators J^a_m and K^a_m invariant. On the other hand, the spectral flow σ of $\mathfrak{psu}(1,1|2)_1$ can be identified with

$$\sigma \equiv \sigma^{(+)} \circ \sigma^{(-)} \ . \tag{C.19}$$

Its action on the generators coincides with (5.22a) – (5.22e).

C.1.3 The fusion rules

The coset representations of (5.39) are labelled by

$$(\sigma^{(+)m}(\mathcal{E}_\lambda), \sigma^{(-)n}(\mathcal{E}_\mu); Y, Z) \ , \qquad m+n \in 2\mathbb{Z} \ , \tag{C.20}$$

where the first entry $\sigma^{(+)m}(\mathcal{E}_\lambda)$ denotes the representation of the pair of symplectic bosons ξ^+ and $\bar{\xi}^-$, while the second entry $\sigma^{(-)n}(\mathcal{E}_\mu)$ is a representation of the pair of symplectic bosons ξ^- and $\bar{\xi}^+$. The condition $m+n \in 2\mathbb{Z}$ imposes the constraint that all four symplectic bosons are equally moded, see (C.12a) and (C.12b). Since the supercharges are bilinear expressions of one symplectic boson and one fermion, and since we require them to be integer-moded, the moding of the symplectic bosons also fixes that of the fermions. Thus the fermions will be in the R sector if m is even, and in the NS sector if m is odd. In particular, this thereby fixes the representations of the $\mathfrak{su}(2)_1$ algebra. Finally, Y and Z denote the eigenvalues of Y_0 and Z_0.

We recall that Z_0 is the central extension of $\mathfrak{psu}(1,1|2)$, while Y_0 is the other $\mathfrak{u}(1)$-charge extending $\mathfrak{su}(1,1|2)$ to $\mathfrak{u}(1,1|2)$. With these conventions, the symplectic bosons and free fermions transform with respect to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_Y \oplus \mathfrak{u}(1)_Z$ as

$$\text{symplectic bosons}: \qquad (\mathbf{2},\mathbf{1})_{1,1} \oplus (\mathbf{2},\mathbf{1})_{-1,-1} \,, \tag{C.21}$$

$$\text{fermions}: \qquad (\mathbf{1},\mathbf{2})_{1,-1} \oplus (\mathbf{1},\mathbf{2})_{-1,1} \,. \tag{C.22}$$

We have explained above that for $\mathfrak{psu}(1,1|2)$-representations we have to require Z to vanish, but should sum over all possible charges of $\mathfrak{u}(1)_Y$. Furthermore, we have the selection rules that both Z and Y must satisfy

$$Z,\, Y \in \tfrac{1}{2}\mathbb{Z} + \lambda - \mu \,. \tag{C.23}$$

Since $\lambda, \mu \in \mathbb{R}/\frac{1}{2}\mathbb{Z}$, the condition $Z = 0$ allows us to choose, without loss of generality, $\lambda = \mu$. We have the identifications

$$\mathcal{F}_\lambda \cong \bigoplus_{Y\in\mathbb{Z}} (\sigma^m(\mathcal{E}_{\frac{\lambda}{2}}),\, \sigma^{-m}(\mathcal{E}_{\frac{\lambda}{2}});\, Y,\, 0) \,, \tag{C.24a}$$

$$\mathcal{L} \cong \bigoplus_{Y\in\mathbb{Z}} (\sigma^m(\mathcal{K}),\, \sigma^{-m}(\mathcal{K});\, Y,\, 0) \,, \tag{C.24b}$$

where the spectral flow parameter m is arbitrary (and we have suppressed the indices $(\pm)$ on the spectral flows). Since the spectral flow (σ^w, σ^{-w}) leaves the bosonic subalgebra of $\mathfrak{psu}(1,1|2)_1$-algebra invariant, (C.24a) and (C.24b) define indeed highest weight representations. We can then readily compute the fusion rules: for $\lambda + \mu \neq 0$ we find from (C.9a)

$$\begin{aligned}(\mathcal{E}_{\frac{\lambda}{2}},\, \mathcal{E}_{\frac{\lambda}{2}};\, Y_1,\, 0) \otimes (\mathcal{E}_{\frac{\mu}{2}}, \mathcal{E}_{\frac{\mu}{2}};\, Y_2,\, 0) \cong\ & (\sigma(\mathcal{E}_{\frac{\lambda+\mu}{2}+\frac{1}{4}}),\, \sigma(\mathcal{E}_{\frac{\lambda+\mu}{2}+\frac{1}{4}});\, Y_1+Y_2,\, 0) \\ & \oplus (\sigma^{-1}(\mathcal{E}_{\frac{\lambda+\mu}{2}+\frac{1}{4}}),\, \sigma^{-1}(\mathcal{E}_{\frac{\lambda+\mu}{2}+\frac{1}{4}});\, Y_1+Y_2,\, 0) \\ & \oplus (\sigma(\mathcal{E}_{\frac{\lambda+\mu}{2}+\frac{1}{4}}),\, \sigma^{-1}(\mathcal{E}_{\frac{\lambda+\mu}{2}+\frac{1}{4}});\, Y_1+Y_2,\, 0) \\ & \oplus (\sigma^{-1}(\mathcal{E}_{\frac{\lambda+\mu}{2}+\frac{1}{4}}),\, \sigma(\mathcal{E}_{\frac{\lambda+\mu}{2}+\frac{1}{4}});\, Y_1+Y_2,\, 0) \,.\end{aligned} \tag{C.25}$$

Summing over the Y-charge yields then the fusion rules for the $\mathfrak{psu}(1,1|2)_1$ representations

$$\mathcal{F}_\lambda \times \mathcal{F}_\mu \cong \sigma(\mathcal{F}_{\lambda+\mu+\frac{1}{2}}) \oplus 2 \cdot \mathcal{F}_{\lambda+\mu+\frac{1}{2}} \oplus \sigma^{-1}(\mathcal{F}_{\lambda+\mu+\frac{1}{2}}) \,, \tag{C.26}$$

where the middle term arises from the last two lines in (C.25).

Let us also consider the exceptional case, where $\lambda + \mu = 0$. Then the symplectic boson language predicts the appearance of the module $(\mathcal{S}, \mathcal{S}; Y_1 + Y_2, 0)$. This module has a composition series of 16 terms. When summing over the Y-charge, we get by definition the module $\mathcal{T}$. Its composition series is (we have not specified how the arrows act on the different summands that have non-trivial multiplicities)

$$
\mathcal{T}: \quad \begin{array}{ccccccccc}
 & & & & \mathcal{L} & & & & \\
 & & & \swarrow & & \searrow & & & \\
 & & 2\sigma^{-1}(\mathcal{L}) & & & & 2\sigma(\mathcal{L}) & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \\
\sigma^{-2}(\mathcal{L}) & & & & 4\mathcal{L} & & & & \sigma^{2}(\mathcal{L}) \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & \\
 & & 2\sigma^{-1}(\mathcal{L}) & & & & 2\sigma(\mathcal{L}) & & \\
 & & & \searrow & & \swarrow & & & \\
 & & & & \mathcal{L} & & & &
\end{array} \tag{C.27}
$$

for the complete description, see Appendix C.2. In particular, because of (5.30), we have in the Grothendieck ring

$$
\mathcal{T} \sim \sigma(\mathcal{F}_{\frac{1}{2}}) \oplus 2 \cdot \mathcal{F}_{\frac{1}{2}} \oplus \sigma^{-1}(\mathcal{F}_{\frac{1}{2}}) \ . \tag{C.28}
$$

Next, we compute the fusion rules of $\mathcal{F}_\lambda$ with $\mathcal{T}$. Because of (C.28), we expect that only $\mathcal{F}_\lambda$ and spectrally flowed images can appear on the right hand side, and this indeed follows from the symplectic boson fusion rules (C.9b), leading to

$$
\mathcal{F}_\lambda \times \mathcal{T} \cong \sigma^2(\mathcal{F}_\lambda) \oplus 4 \cdot \sigma(\mathcal{F}_\lambda) \oplus 6 \cdot \mathcal{F}_\lambda \oplus 4 \cdot \sigma^{-1}(\mathcal{F}_\lambda) \oplus \sigma^{-2}(\mathcal{F}_\lambda) \ . \tag{C.29}
$$

Finally, the fusion of $\mathcal{T}$ with itself follows from (C.9c), and we find

$$
\mathcal{T} \times \mathcal{T} \cong \sigma^2(\mathcal{T}) \oplus 4 \cdot \sigma(\mathcal{T}) \oplus 6 \cdot \mathcal{T} \oplus 4 \cdot \sigma^{-1}(\mathcal{T}) \oplus \sigma^{-2}(\mathcal{T}) \ . \tag{C.30}
$$

This reproduces (5.40a) – (5.40c).

C.1.4 The characters

We can also use the free field realisation to compute the character of $\mathfrak{psu}(1,1|2)_1$-representations. For this, we recall that four free fermions in the R-sector have the character

$$
\frac{\vartheta_2\big(\frac{z+\mu-\nu}{2};\tau\big)\vartheta_2\big(\frac{z-\mu+\nu}{2};\tau\big)}{\eta(\tau)^2} = \frac{\vartheta_2(z;2\tau)\vartheta_3(\mu-\nu;2\tau)+\vartheta_3(z;2\tau)\vartheta_2(\mu-\nu;2\tau)}{\eta(\tau)^2} \ . \tag{C.31}
$$

Our conventions regarding modular functions are collected in Appendix A.4. Here, we have introduced the chemical potentials μ and ν, which are associated to Y and Z, respectively, while z is the chemical potential with respect to the $\mathfrak{su}(2)$ algebra, see eq. (C.22). The right hand side of the equation expresses the character in terms of $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_1$-characters, which separates the chemical potentials, see eq. (A.33). Similarly, the character of the continuous representation of two pairs of symplectic bosons $(\mathcal{E}_{\lambda/2}, \mathcal{E}_{\lambda/2})$ is given by

$$\sum_{m,n\in\frac{1}{2}\mathbb{Z}+\frac{\lambda}{2}} x^{m+n} \mathrm{e}^{2\pi i(\mu+\nu)(m-n)} \frac{1}{\eta(\tau)^4}$$
$$= \left(\sum_{r\in\mathbb{Z}+\lambda,\, s\in\mathbb{Z}} + \sum_{r\in\mathbb{Z}+\lambda+\frac{1}{2},\, s\in\mathbb{Z}+\frac{1}{2}} \right) x^r \mathrm{e}^{2\pi i(\mu+\nu)s} \frac{1}{\eta(\tau)^4}\ , \tag{C.32}$$

where t is the chemical potential of $\mathfrak{sl}(2,\mathbb{R})_1$ and we have set $x = \mathrm{e}^{2\pi i t}$, see eq. (C.21). The oscillator part of the character is uncharged, since any charge can be absorbed into the zero-modes of the representation. (C.31) and (C.32) can be multiplied to obtain the character of the numerator algebra in (5.39). It is then straightforward to obtain the coset character of $(\mathcal{E}_{\lambda/2}, \mathcal{E}_{\lambda/2}, Y, Z)$ by restricting to the respective exponents of $\mathrm{e}^{2\pi i\mu}$ and $\mathrm{e}^{2\pi i\nu}$. In particular, we find for $Z = 0$

$$\mathrm{ch}[(\mathcal{E}_{\lambda/2}, \mathcal{E}_{\lambda/2}, Y, 0)](t,z;\tau) = \begin{cases} q^{Y^2/4} \sum_{r\in\mathbb{Z}+\lambda} x^r \frac{\vartheta_2(z,2\tau)}{\eta(\tau)^4} & Y \text{ even}\ , \\ q^{Y^2/4} \sum_{r\in\mathbb{Z}+\lambda+\frac{1}{2}} x^r \frac{\vartheta_3(z,2\tau)}{\eta(\tau)^4} & Y \text{ odd}\ . \end{cases} \tag{C.33}$$

With the identification (C.24a), we finally deduce the character of the continuous $\mathfrak{psu}(1,1|2)_1$-representations

$$\mathrm{ch}[\mathcal{F}_\lambda](t,z;\tau) = \sum_{r\in\mathbb{Z}+\lambda} x^r \frac{\vartheta_3(2\tau)\vartheta_2(z;2\tau)}{\eta(\tau)^4} + \sum_{r\in\mathbb{Z}+\lambda+\frac{1}{2}} x^r \frac{\vartheta_2(2\tau)\vartheta_3(z;2\tau)}{\eta(\tau)^4} \tag{C.34}$$

$$= \sum_{r\in\mathbb{Z}+\lambda} x^r \frac{\vartheta_3(t;2\tau)\vartheta_2(z;2\tau)}{\eta(\tau)^4} + \sum_{r\in\mathbb{Z}+\lambda} x^r \frac{\vartheta_2(t;2\tau)\vartheta_3(z;2\tau)}{\eta(\tau)^4} \tag{C.35}$$

$$= \sum_{r\in\mathbb{Z}+\lambda} x^r \frac{\vartheta_2\left(\frac{t+z}{2};\tau\right)\vartheta_2\left(\frac{t-z}{2};\tau\right)}{\eta(\tau)^4}\ , \tag{C.36}$$

where $x = \mathrm{e}^{2\pi i t}$, and we have used that

$$\sum_{n\in\mathbb{Z}} q^{n^2} = \vartheta_3(2\tau)\ , \qquad \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{n^2} = \vartheta_2(2\tau)\ . \tag{C.37}$$

We have also used the identity (A.33). Thus, the character formally looks like 4 fermions in the $(\mathbf{2},\mathbf{2})$ of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2)$ together with two free bosons and the zero-modes from the continuous representation.

The spectrally flowed character can be obtained from this by following the rules (5.22a) – (5.22e) and (5.23). This gives

$$\mathrm{ch}[\sigma^w(\mathcal{F}_\lambda)](t,z;\tau)=q^{\frac{w^2}{2}}\sum_{r\in\mathbb{Z}+\lambda}x^r q^{-rw}\frac{\vartheta_2\big(\frac{t+z}{2};\tau\big)\vartheta_2\big(\frac{t-z}{2};\tau\big)}{\eta(\tau)^4}\ . \tag{C.38}$$

C.1.5 Modular properties

We now calculate the modular behaviour of the characters (C.38). We stress that we are treating the character as a formal object and not as a meromorphic function. Hence the following manipulations will be somewhat formal. Our calculations are inspired by [60, 265]. To obtain good modular properties, we include a $(-1)^{\mathrm{F}}$ into the character — the new characters will be denoted by $\tilde{\mathrm{ch}}$ — which corresponds to the replacement $\vartheta_2\longrightarrow\vartheta_1$ in (C.38). We first note that we can write (recall that $x=\mathrm{e}^{2\pi i t}$ and $q=\mathrm{e}^{2\pi i\tau}$)

$$\tilde{\mathrm{ch}}[\sigma^w(\mathcal{F}_\lambda)](t,z;\tau)=q^{\frac{w^2}{2}}\mathrm{e}^{2\pi i(t-w\tau)\lambda}\sum_{r\in\mathbb{Z}}\mathrm{e}^{2\pi i(t-w\tau)r}\,\frac{\vartheta_1\big(\frac{t+z}{2};\tau\big)\vartheta_1\big(\frac{t-z}{2};\tau\big)}{\eta(\tau)^4} \tag{C.39}$$

$$=q^{\frac{w^2}{2}}\mathrm{e}^{2\pi i(t-w\tau)\lambda}\sum_{m\in\mathbb{Z}}\delta(t-w\tau-m)\,\frac{\vartheta_1\big(\frac{t+z}{2};\tau\big)\vartheta_1\big(\frac{t-z}{2};\tau\big)}{\eta(\tau)^4} \tag{C.40}$$

$$=q^{\frac{w^2}{2}}\sum_{m\in\mathbb{Z}}\mathrm{e}^{2\pi i m\lambda}\delta(t-w\tau-m)\,\frac{\vartheta_1\big(\frac{t+z}{2};\tau\big)\vartheta_1\big(\frac{t-z}{2};\tau\big)}{\eta(\tau)^4}\ . \tag{C.41}$$

With this at hand, it is straightforward to compute the S-modular transformation of the characters from

$$\tilde{\mathrm{ch}}[\sigma^w(\mathcal{F}_\lambda)](t,z;\tau)\to\mathrm{e}^{\frac{\pi i}{2\tau}(t^2-z^2)}\tilde{\mathrm{ch}}[\sigma^w(\mathcal{F}_\lambda)]\big(\tfrac{t}{\tau},\tfrac{z}{\tau};-\tfrac{1}{\tau}\big) \tag{C.42}$$

$$=\frac{\mathrm{e}^{\frac{\pi i}{\tau}(t^2-w^2)}}{i\tau}\sum_{m\in\mathbb{Z}}\mathrm{e}^{2\pi i m\lambda}\delta\big(\tfrac{t+w-m\tau}{\tau}\big)\frac{\vartheta_1\big(\frac{t+z}{2};\tau\big)\vartheta_1\big(\frac{t-z}{2};\tau\big)}{\eta(\tau)^4} \tag{C.43}$$

$$=-\,i\,\mathrm{sgn}(\mathrm{Re}(\tau))\sum_{m\in\mathbb{Z}}q^{\frac{m^2}{2}}\mathrm{e}^{2\pi i m\lambda}\delta(t+w-m\tau)\times\frac{\vartheta_1\big(\frac{t+z}{2};\tau\big)\vartheta_1\big(\frac{t-z}{2};\tau\big)}{\eta(\tau)^4}\ . \tag{C.44}$$

Here the prefactor $\mathrm{e}^{\frac{\pi i}{2\tau}(t^2-z^2)}$ comes from the general transformation property of weak Jacobi forms of index 1 and -1, respectively, see e.g. [266], and we have used (A.35). In the final step we have also set $t = m\tau - w$ (because of the δ function), and used that both m and w are integers. Finally, we have inserted the formal identity

$$\delta\left(\tfrac{x}{\tau}\right) = \tau \,\mathrm{sgn}(\mathrm{Re}(\tau))\, \delta(x) \;, \tag{C.45}$$

which follows by writing

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{2\pi\varepsilon}} \mathrm{e}^{-\frac{x^2}{2\varepsilon}} \;. \tag{C.46}$$

By definition, we put the branch cut of the square root on the imaginary axis, which is the reason for the jump in (C.45) at this point. Of course, other choices for the branch cut give in the end the same physical result.[82] In particular, the sign cancels out once we combine the left-movers with the right-movers.

The expression (C.44) can now be written as

$$\sum_{w' \in \mathbb{Z}} \int_0^1 \mathrm{d}\lambda' \, S_{(w,\lambda),(w',\lambda')} \; \tilde{\mathrm{ch}}[\sigma^{w'}(\mathcal{F}_{\lambda'})](t,z;\tau) \;, \tag{C.47}$$

with

$$S_{(w,\lambda),(w',\lambda')} = -i \,\mathrm{sgn}(\mathrm{Re}(\tau))\, \mathrm{e}^{2\pi i (w'\lambda + w\lambda')} \;. \tag{C.48}$$

Thus, we have derived the formal S-matrix of the S-modular transformation, see eq. (5.44). As in [265], it is not entirely independent of τ. This dependence cancels out in every physical calculation. The S-matrix is (formally) unitary since

$$\sum_{w'' \in \mathbb{Z}} \int_0^1 \mathrm{d}\lambda'' \, S^\dagger_{(w,\lambda),(w'',\lambda'')} S_{(w'',\lambda''),(w',\lambda')} = \sum_{w'' \in \mathbb{Z}} \int_0^1 \mathrm{d}\lambda'' \, \mathrm{e}^{2\pi i (w''\lambda' + w'\lambda'' - w\lambda'' - w''\lambda)} \tag{C.49}$$

$$= \delta_{w,w'} \sum_{w'' \in \mathbb{Z}} \mathrm{e}^{2\pi i (w''\lambda' - w''\lambda)} \tag{C.50}$$

$$= \delta_{w,w'} \, \delta(\lambda - \lambda' \bmod 1) \;. \tag{C.51}$$

Moreover, it is clearly symmetric. These properties suffice to deduce that the combination (5.41) is indeed modular invariant.

[82] In particular, our formula differs from [265]. We feel that the holomorphic prescription we are using is more adequate, since in particular no $\bar{\tau}$ should appear in the S-modular transformation of (formally) holomorphic characters.

C.1.6 The Verlinde formula

We now use the formal S-matrix to derive the typical fusion rules a third time by using a continuum version of the Verlinde formula. For this, we also need the S-matrix element of the vacuum with a continuous representation. To this end, we notice that on the level of the Grothendieck ring

$$\sigma(\mathcal{F}_{1/2}) \sim \mathcal{L} \oplus 2 \cdot \sigma(\mathcal{L}) \oplus \sigma^2(\mathcal{L}) \,. \tag{C.52}$$

Using this identification repeatedly, one proves by induction

$$\begin{aligned} \mathrm{ch}[\mathcal{L}] = \sum_{m=1}^{n} (-1)^{m+1} m \, \mathrm{ch}[\sigma^m(\mathcal{F}_{1/2})] \\ + (-1)^n (n+1) \, \mathrm{ch}[\sigma^n(\mathcal{L})] + (-1)^n n \, \mathrm{ch}[\sigma^{n+1}(\mathcal{L})] \end{aligned} \tag{C.53}$$

for any n. Thus by taking $n \to \infty$ we conclude

$$\mathrm{ch}[\mathcal{L}] = \sum_{m=0}^{\infty} (-1)^{m+1} m \, \mathrm{ch}[\sigma^m(\mathcal{F}_{1/2})] \,. \tag{C.54}$$

We can then calculate the S-matrix element

$$S_{\text{vac},(w,\lambda)} = \sum_{m=0}^{\infty} (-1)^{m+1} m \, S_{(m,\frac{1}{2}),(w,\lambda)} = (-1)^w \frac{-i \, \mathrm{sgn}(\mathrm{Re}(\tau))}{\left(\mathrm{e}^{i\pi\lambda} + \mathrm{e}^{-i\pi\lambda}\right)^2} \,. \tag{C.55}$$

Finally, we use the Verlinde formula to calculate the fusion rules:

$$\mathcal{N}_{(w_1,\lambda_1)(w_2,\lambda_2)}^{(w_3,\lambda_3)} = \sum_{w \in \mathbb{Z}} \int_0^1 \mathrm{d}\lambda \, \frac{S_{(w_1,\lambda_1)(w,\lambda)} S_{(w_2,\lambda_2)(w,\lambda)} S^*_{(w_3,\lambda_3)(w,\lambda)}}{S_{\text{vac},(w,\lambda)}} \tag{C.56}$$

$$\begin{aligned} &= \sum_{w \in \mathbb{Z}} \int_0^1 \mathrm{d}\lambda \, \mathrm{e}^{2\pi i((w_1+w_2-w_3)\lambda + w(\lambda_1+\lambda_2-\lambda_3))} \\ &\qquad \times (-1)^w \left(\mathrm{e}^{\pi i \lambda} + \mathrm{e}^{-\pi i \lambda}\right)^2 \end{aligned} \tag{C.57}$$

$$\begin{aligned} &= \Big(\delta_{w_3,w_1+w_2-1} + 2\delta_{w_3,w_1+w_2} + \delta_{w_3,w_1+w_2+1}\Big) \\ &\qquad \times \delta\Big(\lambda_3 = \lambda_1 + \lambda_2 + \tfrac{1}{2}\Big) \,. \end{aligned} \tag{C.58}$$

This reproduces (5.33).

C.2 The indecomposable module $\mathcal{T}$

In this appendix, we discuss the indecomposable module $\mathcal{T}$ that appears in the fusion rules (5.40a) – (5.40c) in some more detail. In particular, we discuss how it modifies the structure of the Hilbert space. The composition diagram of $\mathcal{T}$ was given in (C.27).

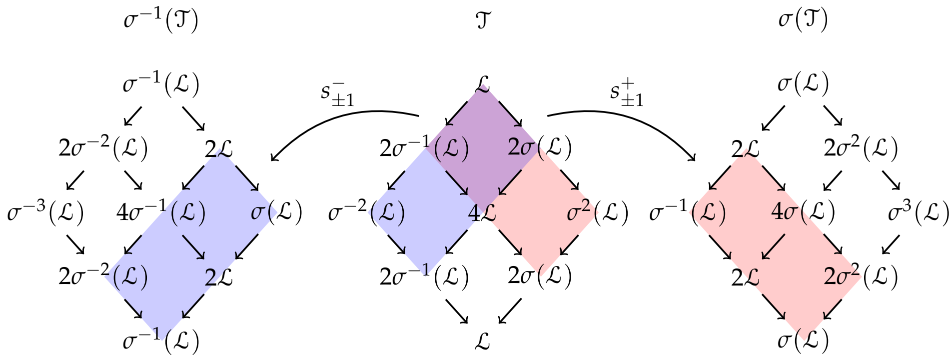

Figure C.1: The definition of the maps $s^{\pm}_{(i)}$.

In a refined version of (5.41), we should not include $\mathcal{F}_{1/2}$ in the Hilbert space, but rather $\mathcal{T}$. Thus, the naive ansatz for the Hilbert space would be

$$\mathcal{H}^{\text{naive}} \cong \bigoplus_{w\in\mathbb{Z}} \left(\sigma^w(\mathcal{T}) \otimes \overline{\sigma^w(\mathcal{T})} \oplus \oint_{[0,1)\setminus\{\frac{1}{2}\}} \mathrm{d}\lambda\ \sigma^w(\mathcal{F}_\lambda) \otimes \overline{\sigma^w(\mathcal{F}_\lambda)} \right) . \qquad \text{(C.59)}$$

We refer to the first summand as the atypical Hilbert space $\mathcal{H}^{\text{naive}}_{\text{atyp}}$, and the second term as the typical Hilbert space $\mathcal{H}_{\text{typ}}$. While this contains now only modules which close under fusion, there are two problems with this proposal. First, locality requires that $L_0 - \bar{L}_0$ acts diagonalisably, since otherwise the complete correlation functions would be multi-valued. In addition, (C.59) does not agree with (5.41) on the level of the Grothendieck ring, and hence would not be modular invariant. As explained in [267, 268], the true Hilbert space is obtained by quotienting out an ideal $\mathcal{I} \subset \mathcal{H}^{\text{naive}}_{\text{atyp}}$ from $\mathcal{H}^{\text{naive}}_{\text{atyp}}$.

There is a very natural way of defining this ideal [186]. For this, let us describe the module $\mathcal{T}$ a bit more conceptually. $\mathcal{T}$ has 16 terms in its composition series, which we shall denote by $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ for $\varepsilon_i \in \{0, 1\}$. The first line of the composition series (C.27) corresponds to $(0, 0, 0, 0)$, the second line to $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ with $\sum_i \varepsilon_i = 1$, the third line to $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ with $\sum_i \varepsilon_i = 2$, etc. $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ will correspond to a term $\sigma^{-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4}(\mathcal{L})$ in the composition series (C.27). It is useful to introduce a $\mathbb{Z}$-grading of the terms in the composition series to lift the degeneracy of the various modules, see [186]; it takes the form

$$\text{grad}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = -\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 . \qquad \text{(C.60)}$$

Furthermore, we note that there is an arrow in the composition series (C.27) from $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ to $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon'_4)$ if $\varepsilon'_i = \varepsilon_i + 1$ for exactly one i and $\varepsilon'_j = \varepsilon_j$

for all remaining $j \neq i$.[83] We can define the intertwiner maps

$$s_\rho^\pm : \sigma^w(\mathcal{T}) \to \sigma^{w\pm 1}(\mathcal{T}) \ , \tag{C.61}$$

where $\rho \in \{-1, 1\}$ denotes the grade of the map, by

$$s_{-1}^+ \sigma^w(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \sigma^{w+1}(\varepsilon_1 + 1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \ , \tag{C.62a}$$

$$s_{+1}^+ \sigma^w(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \sigma^{w+1}(\varepsilon_1, \varepsilon_2 + 1, \varepsilon_3, \varepsilon_4) \ , \tag{C.62b}$$

$$s_{-1}^- \sigma^w(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \sigma^{w-1}(\varepsilon_1, \varepsilon_2, \varepsilon_3 + 1, \varepsilon_4) \ , \tag{C.62c}$$

$$s_{+1}^- \sigma^w(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \sigma^{w-1}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 + 1) \ . \tag{C.62d}$$

By definition, the right hand side is zero if one of its argument is bigger than one. These maps preserve the grading, and we have given a graphical representation in Figure C.1. In fact, they generate the long exact sequences

$$\cdots \quad \sigma^{w-1}(\mathcal{T}) \underset{s_\rho^-}{\overset{s_\rho^+}{\rightleftarrows}} \sigma^{w}(\mathcal{T}) \underset{s_\rho^-}{\overset{s_\rho^+}{\rightleftarrows}} \sigma^{w+1}(\mathcal{T}) \underset{s_\rho^-}{\overset{s_\rho^+}{\rightleftarrows}} \sigma^{w+2}(\mathcal{T}) \quad \cdots \tag{C.63}$$

We stress that the maps are intertwiners, i.e. morphisms of $\mathfrak{psu}(1,1|2)_1$ modules. With this at hand, we define the ideal $\mathcal{I}$ as the ideal generated by $\mathcal{I}_\rho^\pm$ with

$$\mathcal{I}_\rho^\pm \equiv \bigoplus_{w \in \mathbb{Z}} \left(s_\rho^\pm \otimes \overline{\mathbb{1}} - \mathbb{1} \otimes \overline{s_{-\rho}^\mp} \right) \left(\sigma^w(\mathcal{T}) \otimes \sigma^{w\pm 1}(\mathcal{T}) \right) \ . \tag{C.64}$$

Then the Hilbert spaces becomes

$$\mathcal{H} \equiv \mathcal{H}_{\text{atyp}} \oplus \mathcal{H}_{\text{typ}} \ , \qquad \text{with} \qquad \mathcal{H}_{\text{atyp}} \equiv \mathcal{H}_{\text{atyp}}^{\text{naive}} / \mathcal{I} \ . \tag{C.65}$$

Hence we have the equivalence relations

$$\sigma^w(\varepsilon_1 + 1, \varepsilon_2, \varepsilon_3, \varepsilon_4; \bar\varepsilon_1, \bar\varepsilon_2, \bar\varepsilon_3, \bar\varepsilon_4) \sim \sigma^{w-1}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4; \bar\varepsilon_1, \bar\varepsilon_2, \bar\varepsilon_3, \bar\varepsilon_4 + 1) \ , \tag{C.66a}$$

$$\sigma^w(\varepsilon_1, \varepsilon_2 + 1, \varepsilon_3, \varepsilon_4; \bar\varepsilon_1, \bar\varepsilon_2, \bar\varepsilon_3, \bar\varepsilon_4) \sim \sigma^{w-1}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4; \bar\varepsilon_1, \bar\varepsilon_2, \bar\varepsilon_3 + 1, \bar\varepsilon_4) \ , \tag{C.66b}$$

$$\sigma^w(\varepsilon_1, \varepsilon_2, \varepsilon_3 + 1, \varepsilon_4; \bar\varepsilon_1, \bar\varepsilon_2, \bar\varepsilon_3, \bar\varepsilon_4) \sim \sigma^{w+1}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4; \bar\varepsilon_1, \bar\varepsilon_2 + 1, \bar\varepsilon_3, \bar\varepsilon_4) \ , \tag{C.66c}$$

$$\sigma^w(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 + 1; \bar\varepsilon_1, \bar\varepsilon_2, \bar\varepsilon_3, \bar\varepsilon_4) \sim \sigma^{w+1}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4; \bar\varepsilon_1 + 1, \bar\varepsilon_2, \bar\varepsilon_3, \bar\varepsilon_4) \ , \tag{C.66d}$$

[83] Geometrically, this corresponds to drawing a 4-dimensional hypercube, where the modules sit on the vertices and the arrows represent the edges. They are given by all possible shortest paths from one vertex to the opposite vertex.

where we have included the right-moving modules in a hopefully obvious way. In particular, these relations imply that the following modules are null:

$$\sigma^w(1,\varepsilon_2,\varepsilon_3,\varepsilon_4;\bar\varepsilon_1,\bar\varepsilon_2,\bar\varepsilon_3,1) \sim \sigma^w(\varepsilon_1,1,\varepsilon_3,\varepsilon_4;\bar\varepsilon_1,\bar\varepsilon_2,1,\bar\varepsilon_4) \tag{C.67}$$

$$\sim\ \sigma^w(\varepsilon_1,\varepsilon_2,1,\varepsilon_4;\bar\varepsilon_1,1,\bar\varepsilon_3,\varepsilon_4) \sim \sigma^w(\varepsilon_1,\varepsilon_2,\varepsilon_3,1;1,\bar\varepsilon_2,\bar\varepsilon_3,\bar\varepsilon_4) \sim 0\ . \tag{C.68}$$

Thus, out of the 256 terms in the composition series of $\mathcal{T} \times \overline{\mathcal{T}}$, 175 are set to zero. If $\varepsilon_1 = 0$ and $\bar\varepsilon_4 = 1$ or vice versa, (C.66a) tells us that the values of ε_1 and $\bar\varepsilon_4$ can be interchanged, if we compensate in the spectral flow,

$$\sigma^w(1,\varepsilon_2,\varepsilon_3,\varepsilon_4;\bar\varepsilon_1,\bar\varepsilon_2,\bar\varepsilon_3,0) \sim \sigma^{w-1}(0,\varepsilon_2,\varepsilon_3,\varepsilon_4;\bar\varepsilon_1,\bar\varepsilon_2,\bar\varepsilon_3,1)\ . \tag{C.69}$$

Hence, we can choose a representative by fixing either $\varepsilon_1 = 0$ or $\bar\varepsilon_4 = 0$. An analogous statement holds for the other pairs of indices from (C.66b)–(C.66d). In total, the 'gauge freedom' is fixed by setting either $\varepsilon_i = 0$ or $\bar\varepsilon_{5-i} = 0$ for $i = 1,\ldots,4$.

To see the structure of the resulting representation of the affine algebra $\mathfrak{psu}(1,1|2)_1 \times \mathfrak{psu}(1,1|2)_1$ most clearly, we first fix the representatives such that $\bar\varepsilon_i = 0$ for $i = 1,\ldots,4$. Thus, equivalence classes will be labelled by $[\sigma^w(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)]$. The left-moving action acts on these equivalence classes in the obvious way. There is an arrow for the right-moving action

$$[\sigma^w(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)] \longrightarrow [\sigma^{w'}(\varepsilon'_1,\varepsilon'_2,\varepsilon'_3,\varepsilon'_4)] \tag{C.70}$$

if $\varepsilon'_i = \varepsilon_i + 1$ for some i and $\varepsilon'_j = \varepsilon_j$ for the other $j \neq i$. Moreover, $w' = w+1$ for $i \in \{1,2\}$ and $w' = w-1$ for $i \in \{3,4\}$. Obviously, we can also choose $\varepsilon_i = 0$ for $i = 1,\ldots,4$, and obtain the same structure with the roles of left- and right-movers interchanged. Thus, the structure of the atypical Hilbert space is

$$\mathcal{H}_{\text{atyp}} \cong \bigoplus_{w\in\mathbb{Z}} \sigma^w(\mathcal{T}) \otimes \sigma^w(\mathcal{L}) \qquad \text{and} \qquad \mathcal{H}_{\text{atyp}} \cong \bigoplus_{w\in\mathbb{Z}} \sigma^w(\mathcal{L}) \otimes \sigma^w(\mathcal{T}) \tag{C.71}$$

with respect to the left (right) action of $\mathfrak{psu}(1,1|2)_1$. The module $\sigma^w(\mathcal{L})$ appearing here is always the head of the module $\sigma^w(\mathcal{T})$.

To investigate the structure of physical states in string theory, it is more convenient to choose the gauge as $\varepsilon_2 = \varepsilon_4 = \bar\varepsilon_2 = \bar\varepsilon_4 = 0$. Then equivalence classes of terms are labelled by $[\sigma^w(\varepsilon_1,\varepsilon_3;\bar\varepsilon_1,\bar\varepsilon_3)]$. Then the structure is of the form

$$\begin{array}{ccc} & \sigma^w(\mathcal{L}) & \\ \swarrow & & \searrow \\ \sigma^{w-1}(\mathcal{L}) & & \sigma^{w+1}(\mathcal{L}) \\ \searrow & & \swarrow \\ & \sigma^w(\mathcal{L}) & \end{array} \ \otimes\ \begin{array}{ccc} & \sigma^w(\mathcal{L}) & \\ \swarrow & & \searrow \\ \sigma^{w-1}(\mathcal{L}) & & \sigma^{w+1}(\mathcal{L}) \\ \searrow & & \swarrow \\ & \sigma^w(\mathcal{L}) & \end{array}\ , \tag{C.72}$$

where we drew only the 'obvious' arrows of the $\mathfrak{psu}(1,1|2)_1 \times \mathfrak{psu}(1,1|2)_1$ action. Since on the level of the Grothendieck ring, $\mathcal{F}_{1/2} \sim \sigma^{-1}(\mathcal{L}) \oplus 2\mathcal{L} \oplus \sigma(\mathcal{L})$, this equals $\sigma^w(\mathcal{F}_{1/2}) \otimes \sigma^w(\mathcal{F}_{1/2})$ in the Grothendieck ring. Thus, $\mathcal{T}$ becomes the moral analogue of $\mathcal{F}_{1/2}$ and in particular the character analysis does not differ in the atypical case from the typical case. Summarizing,

$$\mathcal{H}_{\text{atyp}} \sim \bigoplus_{w \in \mathbb{Z}} \sigma^w(\mathcal{F}_{1/2}) \otimes \overline{\sigma^w(\mathcal{F}_{1/2})} \,, \tag{C.73}$$

i.e. the quotient has removed the factor of $16 = 4 \times 4$ that was mentioned below eq. (5.28). Thus the atypical contribution precisely fills the gaps of the typical contribution in (C.59), so that in total we retrieve (5.41), which is modular invariant. Finally, one may check that $L_0 - \bar{L}_0$ now acts indeed diagonalisably, and thus correlation functions are single-valued. The resulting Hilbert space therefore defines a local consistent CFT.

Appendix D

Details about the $\mathfrak{d}(2,1;\alpha)$ WZW model at $k^+ = 1$

In this appendix, we will give some details about the $\mathfrak{d}(2,1;\alpha)$ WZW model at $k^+ = 1$, which played an important role in the main text.

D.1 The Wakimoto representation of $\mathfrak{d}(2,1;\alpha)_k$

Here, we explain the Wakimoto representation that is used in the derivation of the hybrid formalism for $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ in the main body of the paper.

We start with four pairs of topologically twisted fermions (or bc systems), satisfying

$$p^{\alpha\beta}(z)\,\theta^{\gamma\delta}(w) \sim \frac{\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}}{z-w}\,. \tag{D.1}$$

Furthermore, we also have the bosonic $\mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k^+-2} \oplus \mathfrak{su}(2)_{k^--2}$ currents $\mathcal{J}^a$ and $\mathcal{K}^{(\pm)a}$.

D.1.1 The root system of $\mathfrak{d}(2,1;\alpha)$

To continue systematically, let us recall the basic idea of the Wakimoto representation. Starting from the root system of a Lie (super)algebra, one first constructs a realisation of the (nilpotent) positive roots in terms of $\beta\gamma$ systems. Then one extends this construction to the positive Borel subalgebra by introducing as many free bosons as the rank of the Lie algebra. Finally, the generators for the negative roots are then uniquely fixed by requiring them to satisfy all the OPEs. This procedure requires the breaking of some symmetries. For $\mathfrak{d}(2,1;\alpha)_k$, a minimal choice is to break the $\mathfrak{su}(2)_{k^-}$ symmetry and keep the rest of the bosonic subalgebra manifest.

In the context of $\mathfrak{d}(2,1;\alpha)_k$, we pick as Cartan subalgebra J_0^3, $K_0^{(+)3}$ and $K_0^{(-)3}$, and take the simply roots to be

$$\alpha_1 = (1,0,0) \ , \qquad \alpha_2 = (0,1,0) \ , \qquad \alpha_3 = \big(-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\big) \ . \tag{D.2}$$

The first two roots are bosonic, while α_3 is fermionic, so this corresponds to the distinguished choice of simple roots.[84] The step operators corresponding to the positive roots are then

$$J^+ \ , \qquad K^{(+)+} \ , \qquad K^{(-)+} \ , \qquad S^{\alpha\beta+} \ , \tag{D.3}$$

for $\alpha, \beta \in \{+,-\}$.

D.1.2 Constructing the Borel subalgebra

We first explain how to construct J^a and $K^{(+)a}$. The topologically twisted fermions lead to the generators of $\mathfrak{sl}(2,\mathbb{R})_{-2} \oplus \mathfrak{su}(2)_2$

$$J^{(\mathrm{f})a} = \frac{1}{2} c_a (\sigma^a)_{\alpha\gamma} \varepsilon_{\beta\delta} (p^{\alpha\beta}\theta^{\gamma\delta}) \ , \tag{D.4}$$

$$K^{(\mathrm{f})a} = \frac{1}{2} \varepsilon_{\alpha\gamma} (\sigma^a)_{\beta\delta} (p^{\alpha\beta}\theta^{\gamma\delta}) \ , \tag{D.5}$$

where the different constants were explained in Appendix A.2. We then define

$$J^a = \mathcal{J}^a + J^{(\mathrm{f})a} \ , \qquad K^{(+)a} = \mathcal{K}^{(+)a} + K^{(\mathrm{f})(+)a} \ , \tag{D.6}$$

which can be checked to agree with (A.16a)–(A.16d). Next we introduce a Wakimoto representation for $\mathfrak{su}(2)_{k^- - 2}$ in terms of a $\beta\gamma$ system[85] together with a free boson $\partial\hat{\chi}$, see e.g. [166]. Then the remaining elements of the Borel subalgebra are

$$S^{\alpha\beta+} = p^{\alpha\beta} - \frac{k^+}{2(k^+ + k^-)} (\theta^{\alpha\beta}\hat{\beta}) \ , \tag{D.7}$$

$$K^{(-)+} = \hat{\beta} \ , \tag{D.8}$$

$$K^{(-)3} = \sqrt{\frac{k^-}{2}} \partial\hat{\chi} + (\hat{\beta}\hat{\gamma}) + \frac{1}{2} \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} (p^{\alpha\beta}\theta^{\gamma\delta}) \ , \tag{D.9}$$

where the explicit form of $K^{(-)3}$ is obtained by demanding the OPEs of $\mathfrak{d}(2,1;\alpha)_k$.

[84] Recall that in Lie superalgebras, there is no unique choice of simple roots, see e.g. [261].

[85] In order to distinguish this from the $\beta\gamma$ system of the superconformal ghost, see eq. (7.13), we denote the relevant fields here with a hat.

D.1.3 The complete algebra

The remaining fields are much more complicated, but they can be found by a direct computation and are uniquely determined. Explicitly they are given as

$$\begin{aligned}
K^{(-)-} &= -(\hat{\beta}\hat{\gamma}\hat{\gamma}) - k^-\partial\hat{\gamma} - \sqrt{2k^-}(\partial\chi\hat{\gamma}) - \frac{k^+(k^-+1)}{2(k^++k^-)}\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\left(\theta^{\alpha\beta}\partial\theta^{\gamma\delta}\right) \\
&\quad - \frac{k^+(k^++2k^-)}{2(k^++k^-)^2}\left(\theta^{++}\theta^{+-}\theta^{-+}\theta^{--}\hat{\beta}\right) - \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}(p^{\alpha\beta}\theta^{\gamma\delta}\hat{\gamma}) \\
&\quad + \frac{1}{2}c_a(\sigma_a)_{\alpha\gamma}\varepsilon_{\beta\delta}\left(\theta^{\alpha\beta}\theta^{\gamma\delta}\left(\mathcal{J}^{(+)a} + \tfrac{1}{3}J^{(\mathrm{f})(+)a}\right)\right) \\
&\quad - \frac{k^-}{2(k^++k^-)}\varepsilon_{\alpha\gamma}(\sigma_a)_{\beta\delta}\left(\theta^{\alpha\beta}\theta^{\gamma\delta}\left(\mathcal{K}^{(+)a} + \tfrac{1}{3}K^{(\mathrm{f})(+)a}\right)\right)\ , \qquad \text{(D.10a)} \\
S^{\alpha\beta-} &= \frac{k^+\sqrt{k^-}}{\sqrt{2}(k^++k^-)}(\theta^{\alpha\beta}\partial\hat{\chi}) + \frac{k^+}{2(k^++k^-)}(\theta^{\alpha\beta}\hat{\beta}\hat{\gamma}) - (p^{\alpha\beta}\hat{\gamma}) \\
&\quad + c_a(\sigma_a)^{\alpha}{}_{\gamma}\left(\theta^{\gamma\beta}\left(\mathcal{J}^a + \frac{3k^++2k^-}{4(k^++k^-)}J^{(\mathrm{f})a}\right)\right) \\
&\quad + (\sigma_a)^{\beta}{}_{\gamma}\left(\theta^{\alpha\gamma}\left(-\frac{k^-}{k^++k^-}\mathcal{K}^{(+)a} + \frac{k^+-2k^-}{4(k^++k^-)}K^{(\mathrm{f})(+)a}\right)\right) \\
&\quad + \frac{k^+(2k^-+1)}{2(k^++k^-)}\partial\theta^{\alpha\beta} - \frac{k^+(k^++2k^-)}{12(k^++k^-)^2}\varepsilon_{\gamma\mu}\varepsilon_{\delta\nu}\theta^{\alpha\gamma}\theta^{\delta\mu}\theta^{\nu\beta}\ . \qquad \text{(D.10b)}
\end{aligned}$$

The energy-momentum tensor becomes in terms of the defining fields

$$\begin{aligned}
T &= \frac{1}{k}\left(-\mathcal{J}^3\mathcal{J}^3 + \tfrac{1}{2}\left(\mathcal{J}^+\mathcal{J}^- + \mathcal{J}^-\mathcal{J}^+\right)\right) + \frac{1}{2}(\partial\hat{\chi}\partial\hat{\chi}) + \frac{\partial^2\hat{\chi}}{\sqrt{2k^-}} - (\hat{\beta}\partial\hat{\gamma}) \\
&+ \frac{1}{k^+}\left(\mathcal{K}^{(+)3}\mathcal{K}^{(+)3} + \tfrac{1}{2}\left(\mathcal{K}^{(+)+}\mathcal{K}^{(+)-} + \mathcal{K}^{(+)-}\mathcal{K}^{(+)+}\right)\right) - \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}(p^{\alpha\beta}\theta^{\gamma\delta})\ , \qquad \text{(D.11)}
\end{aligned}$$

which is the standard energy-momentum tensor of $\mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k^+-2} \oplus \mathfrak{su}(2)_{k^--2}$, together with the four pairs of topologically twisted fermions.

We should note that, in the limit $k^- \to \infty$, the above formulae lead to the construction for $\mathfrak{psu}(1,1|2)_k$ [126,148,149], see also Chapter 6.

D.2 Characters and modular properties at $k^+ = 1$

In this Appendix, we determine the characters of $\mathfrak{d}(2,1;\alpha)$ for $k^+ = 1$. To do so, we exploit the conformal embedding (which only exists for $k^+ = 1$) [184,269]

$$\mathfrak{sl}(2,\mathbb{R})_k \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_{k^-} \subset \mathfrak{d}(2,1;\alpha)_k\ . \qquad \text{(D.12)}$$

The modular properties of the characters will allow us to prove modular invariance of the full spectrum, see eq. (7.56). It will also allow us to compute the fusion rules via the Verlinde formula. We will use the conventions of the main text, so in particular

$$k^+ = 1 \ , \quad k^- = \alpha = \kappa + 1 \ , \quad k = \gamma = \frac{\kappa+1}{\kappa+2} \ . \tag{D.13}$$

D.2.1 Admissible $\mathfrak{su}(2)$ WZW-models

We begin by discussing $\mathfrak{sl}(2,\mathbb{R})_k$ at level $k = \frac{\kappa+1}{\kappa+2}$. On the level of the algebra (i.e. disregarding the hermitian structure, which does not matter for the calculation of the characters), this algebra is isomorphic to

$$\mathfrak{su}(2)_{-\frac{\kappa+1}{\kappa+2}} \ . \tag{D.14}$$

While the level of the $\mathfrak{su}(2)$ algebra is negative (and hence the model is non-unitary), the level is what is called admissible, see e.g. [166, 270]. (In the following we will mostly follow the notation of [271].) To explain what this means we write

$$-\frac{\kappa+1}{\kappa+2} + 2 = \frac{\kappa+3}{\kappa+2} = \frac{p}{q} \ , \tag{D.15}$$

where $p = \kappa + 3$ and $q = \kappa + 2$. Admissibility amounts then to the condition that $\gcd(p,q) = 1$ and $p \in \mathbb{Z}_{\geq 2}$, $q \in \mathbb{Z}_{\geq 1}$, all of which are obviously satisfied. The fact that the algebra is admissible means that the vacuum representation has a null-vector at level $(p-1)q = (\kappa+2)^2$. This singular vector restricts the representation theory of the admissible $\mathfrak{su}(2)$ WZW-model significantly. The admissible irreducible representation of $\mathfrak{su}(2)$ at this level are [272], see also [273]

$$\mathcal{L}_{r,0} \ , \qquad r \in \{1,\dots,\kappa+2\} \ , \tag{D.16}$$

$$\mathcal{D}^{\pm}_{r,s} \ , \qquad r \in \{1,\dots,\kappa+2\} \ , \qquad s \in \{1,\dots,\kappa+1\} \ , \tag{D.17}$$

$$\mathcal{E}_{r,s,\lambda} \ , \qquad r \in \{1,\dots,\kappa+2\} \ , \qquad s \in \{1,\dots,\kappa+1\} \ , \tag{D.18}$$

where $\lambda \in [0,1)$ encodes the quantisation of the J^3_0-eigenvalue mod $\mathbb{Z}$. We denote the conformal dimension of the ground states by $\Delta_{r,s}$, where

$$\Delta_{r,s} = \frac{((\kappa+2)r - (\kappa+3)s)^2 - (\kappa+3)^2}{4(\kappa+2)(\kappa+3)} \ , \tag{D.19}$$

and we have the field identification $\Delta_{r,s} = \Delta_{\kappa+3-r,\kappa+2-s}$. As a consequence $\mathcal{E}_{r,s,\lambda}$ and $\mathcal{E}_{\kappa+3-r,\kappa+2-s,\lambda}$ describe the same representation.

The characters of the representation $\mathcal{E}_{r,s,\lambda}$ were determined in [271]

$$\text{ch}[\mathcal{E}_{r,s,\lambda}](t;\tau) = \frac{\chi^{\text{Vir}}_{r,s}(\tau)}{\eta(\tau)^2} \sum_{m \in \mathbb{Z}+\lambda} x^m \ , \tag{D.20}$$

where $x = \mathrm{e}^{2\pi i t}$ is the chemical potential of $\mathfrak{sl}(2,\mathbb{R})$. Here, $\chi^{\text{Vir}}_{r,s}(\tau)$ is the character of the representation (r,s) of the corresponding Virasoro minimal model of central charge

$$c^{\text{Vir}} = 1 - \frac{6}{(\kappa+2)(\kappa+3)}\ , \tag{D.21}$$

which are explicitly [166][86]

$$\chi^{\text{Vir}}_{r,s}(\tau) = \frac{q^{\Delta^{\text{Vir}}_{r,s} - \frac{1}{24}(c^{\text{Vir}}-1)}}{\eta(\tau)} \sum_{\ell \in \mathbb{Z}} \left(q^{\ell(pq\ell+qr-ps)} - q^{(p\ell-r)(q\ell-s)} \right) . \tag{D.22}$$

The expression in (D.20) is a bit formal because of the infinite sum over m, which converges nowhere (and will lead to a sum over delta functions as in Chapter 5).

The theory has again a spectral flow symmetry, which we shall denote by σ. It acts on the representations as [271]

$$\sigma(\mathcal{L}_{r,0}) = \mathcal{D}^+_{\kappa+3-r,\kappa+1}\ , \tag{D.23a}$$

$$\sigma^{-1}(\mathcal{L}_{r,0}) = \mathcal{D}^-_{\kappa+3-r,\kappa+1}\ , \tag{D.23b}$$

$$\sigma(\mathcal{D}^-_{r,s}) = \mathcal{D}^+_{\kappa+3-r,\kappa+1-s}\ . \tag{D.23c}$$

Finally, there are short exact sequences analogous to (D.85a)–(D.85d), which read

$$0 \longrightarrow \mathcal{D}^+_{r,s} \longrightarrow \mathcal{E}_{r,s,\lambda_{r,s}} \longrightarrow \mathcal{D}^-_{\kappa+3-r,\kappa+2-s} \longrightarrow 0\ , \tag{D.24a}$$

$$0 \longrightarrow \mathcal{D}^-_{r,s} \longrightarrow \mathcal{E}_{r,s,-\lambda_{r,s}} \longrightarrow \mathcal{D}^+_{\kappa+3-r,\kappa+2-s} \longrightarrow 0\ , \tag{D.24b}$$

where

$$\lambda_{r,s} = \frac{r-1}{2} - \frac{\kappa+3}{2(\kappa+2)} s\ . \tag{D.25}$$

Hence, for $\lambda = \pm\lambda_{r,s}$, $\mathcal{E}_{r,s,\lambda}$ becomes indecomposable.

D.2.2 The branching rules of $\mathfrak{d}(2,1;\alpha = \kappa+1)_k$ into its bosonic subalgebra

After this interlude we now return to the case of $\mathfrak{d}(2,1;\alpha = \kappa+1)_k$ with $k^+ = 1$. We want to understand the branching rules of the representations of $\mathfrak{d}(2,1;\alpha)_k$ under the conformal embedding (D.12). For the case of the vacuum representation of $\mathfrak{d}(2,1;\alpha)_k$ (and generic $\kappa \neq \mathbb{Q}$), this was worked out in [269]. This result can be generalised to $\kappa \in \mathbb{Z}_{\geq 0}$, and we find

$$\mathcal{L} \cong \bigoplus_{r=0}^{\kappa+2} \mathcal{L}_{r,0} \otimes \mathcal{M}^{(1)}_{r \bmod 2} \otimes \mathcal{M}^{(\kappa+1)}_{r}\ , \tag{D.26}$$

[86] The modular parameter $q = \mathrm{e}^{2\pi i \tau}$ should not be confused with the parameter q of the Virasoro minimal model.

where $\mathcal{M}^{(\kappa+1)}_{2\ell+1}$ denotes the spin ℓ representation of $\mathfrak{su}(2)_{\kappa+1}$ and similarly for the level 1 factor.

By exploiting the spectral flow rules (7.50a)–(7.50d) and (D.23a)–(D.23c) together with the short exact sequences (D.85a)–(D.85d) and (D.24a)– (D.24b), we can read off from this the branching rules of all modules,

$$\mathcal{G}^{\ell}_{<,+} \cong \bigoplus_{r=1}^{\kappa+2} \mathcal{D}^{+}_{\kappa+3-r,\kappa+1-2\ell} \otimes \mathcal{M}^{(1)}_{r+2\ell+1 \bmod 2} \otimes \mathcal{M}^{(\kappa+1)}_{r} \ , \tag{D.27a}$$

$$\mathcal{G}^{\ell}_{<,-} \cong \bigoplus_{r=1}^{\kappa+2} \mathcal{D}^{-}_{r,2\ell+1} \otimes \mathcal{M}^{(1)}_{r+2\ell+1 \bmod 2} \otimes \mathcal{M}^{(\kappa+1)}_{r} \ , \tag{D.27b}$$

$$\mathcal{G}^{\ell}_{>,+} \cong \bigoplus_{r=1}^{\kappa+2} \mathcal{D}^{+}_{r,2\ell+1} \otimes \mathcal{M}^{(1)}_{r+2\ell+1 \bmod 2} \otimes \mathcal{M}^{(\kappa+1)}_{r} \ , \tag{D.27c}$$

$$\mathcal{G}^{\ell}_{>,-} \cong \bigoplus_{r=1}^{\kappa+2} \mathcal{D}^{-}_{\kappa+3-r,\kappa+1-2\ell} \otimes \mathcal{M}^{(1)}_{r+2\ell+1 \bmod 2} \otimes \mathcal{M}^{(\kappa+1)}_{r} \ , \tag{D.27d}$$

$$\mathcal{F}^{\ell}_{\lambda} \cong \bigoplus_{r=1}^{\kappa+2} \mathcal{E}_{r,2\ell+1,\lambda+\ell+\frac{r+1}{2}} \otimes \mathcal{M}^{(1)}_{r+2\ell+1 \bmod 2} \otimes \mathcal{M}^{(\kappa+1)}_{r} \ . \tag{D.27e}$$

D.2.3 The characters

Given that we know the characters of the individual factors of the above branchings, it is now straightforward to compute the complete characters of $\mathfrak{d}(2,1;\alpha)_k$ for $k^+ = 1$. In particular, the character of the spin-ℓ representation of $\mathfrak{su}(2)_\kappa$ is explicitly given as

$$\text{ch}[\mathcal{M}^{(\kappa)}_{2\ell+1}](v;\tau) = \frac{\Theta^{(\kappa+2)}_{2\ell+1}(v;\tau) - \Theta^{(\kappa+2)}_{-2\ell-1}(v;\tau)}{\Theta^{(2)}_{1}(v;\tau) - \Theta^{(2)}_{-1}(v;\tau)} \ , \tag{D.28}$$

where v is the chemical potential of $\mathfrak{su}(2)_{k^-}$ with $z = \text{e}^{2\pi i v}$, and the theta functions are explicitly

$$\Theta^{(k)}_{m}(v;\tau) = \sum_{n \in \mathbb{Z} + \frac{m}{2k}} q^{kn^2} z^{kn} \ . \tag{D.29}$$

We introduce similarly chemical potentials t and u for the subalgebras $\mathfrak{sl}(2,\mathbb{R})$, and $\mathfrak{su}(2)_{k^+}$, respectively,[87] and define

$$x = \text{e}^{2\pi i t} \ , \quad y = \text{e}^{2\pi i u} \ , \quad z = \text{e}^{2\pi i v} \ , \quad q = \text{e}^{2\pi i \tau} \ , \tag{D.30}$$

where τ is the modular parameter of the worldsheet. The characters we are interested in are

$$\text{ch}\big[\sigma^w(\mathcal{F}^{\ell}_{\lambda})\big](t,u,v;\tau) \equiv \text{tr}_{\sigma^w(\mathcal{F}^{\ell}_{\lambda})}\Big(x^{J^3_0} y^{K^{(+)3}_0} z^{K^{(-)3}_0} q^{L_0 - \frac{c}{24}}\Big) \ . \tag{D.31}$$

[87] In particular, t will play the role of the modular parameter in the dual CFT.

Because of the zero-modes, they contain in particular a sum of the form

$$\sum_{m\in\mathbb{Z}+\lambda} \mathrm{e}^{2\pi i(t-w\tau)m} = \mathrm{e}^{2\pi i(t-w\tau)\lambda} \sum_{m\in\mathbb{Z}} \mathrm{e}^{2\pi i(t-w\tau)m} \tag{D.32}$$

$$= \mathrm{e}^{2\pi i(t-w\tau)\lambda} \sum_{n\in\mathbb{Z}} \delta(t-n-w\tau) \tag{D.33}$$

$$= \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi i n\lambda} \delta(t-n-w\tau) \ . \tag{D.34}$$

We want to show that the full character can be written as

$$\mathrm{ch}[\sigma^w(\mathcal{F}^\ell_\lambda)] = q^{\frac{w^2}{4(\kappa+2)}} x^{\frac{\kappa+1}{2(\kappa+2)}w} y^{\frac{w}{2}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi i(\lambda+\frac{1}{2})n} \delta(t-n-w\tau) \times \frac{\vartheta_1\big(\frac{t+u+v}{2};\tau\big)\vartheta_1\big(\frac{t+u-v}{2};\tau\big)}{\eta(\tau)^4} \mathrm{ch}[\mathcal{M}^{(\kappa)}_{2\ell+1}](v;\tau) \ . \tag{D.35}$$

We should stress that it is, from this perspective, somewhat surprising that on the right-hand-side an $\mathfrak{su}(2)_\kappa$ character (rather than an $\mathfrak{su}(2)_{\kappa+1}$ character) appears. We should also mention that, with the exception of the additional $\mathfrak{su}(2)_\kappa$ character, this formula is almost identical to the $\mathfrak{psu}(1,1|2)_1$ characters (5.42).

In order to prove (D.35) it is sufficient to consider the unflowed sector, since the spectral flow (5.22a)–(5.23) gives immediately the generalisation to any w. To start with we rewrite the characters of $\mathfrak{su}(2)_1$ as

$$\mathrm{ch}[\mathcal{M}^{(1)}_m](u;\tau) = \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}+\frac{m+1}{2}} q^{n^2} y^n = \frac{1}{\eta(\tau)} \vartheta\begin{bmatrix} \frac{m+1}{2} \\ 0 \end{bmatrix}(u;2\tau) \ , \tag{D.36}$$

where we have used that $\mathfrak{su}(2)_1$ is equivalent to a free boson at the self-dual radius. We also rewrite the Virasoro minimal model characters (D.22) as

$$\chi^{\mathrm{Vir}}_{r,s}(\tau) = \frac{1}{\eta(\tau)} \sum_{\ell\in\mathbb{Z}} \Big(q^{\frac{(2\ell(\kappa+2)(\kappa+3)-s(\kappa+3)+r(\kappa+2))^2}{4(\kappa+2)(\kappa+3)}} - q^{\frac{(2\ell(\kappa+2)(\kappa+3)-s(\kappa+3)-r(\kappa+2))^2}{4(\kappa+2)(\kappa+3)}} \Big) \tag{D.37}$$

$$= \frac{1}{\eta(\tau)} \sum_{\varepsilon=\pm} \varepsilon \sum_{m\in\mathbb{Z}+\frac{s}{2(\kappa+2)}-\frac{\varepsilon r}{2(\kappa+3)}} q^{(\kappa+2)(\kappa+3)m^2} \ . \tag{D.38}$$

It then follows from a direct computation that

$$\mathrm{ch}[\mathcal{F}^\ell_\lambda](t,u,v;\tau) = \sum_{r=1}^{\kappa+2} \sum_{m\in\mathbb{Z}+\lambda+\ell+\frac{r+1}{2}} \frac{x^m \chi^{\mathrm{Vir}}_{r,2\ell+1}(\tau)}{\eta(\tau)^2} \times \mathrm{ch}[\mathcal{M}^{(1)}_{r+2\ell+1 \bmod 2}](u;\tau)\mathrm{ch}[\mathcal{M}^{(\kappa+1)}_r](v;\tau) \tag{D.39}$$

$$= \sum_{r=1}^{\kappa+2} \frac{\mathrm{ch}[\mathcal{M}^{(1)}_{r+2\ell+1 \bmod 2}](u;\tau)}{\eta(\tau)^3\big(\Theta^{(2)}_1(v;\tau)-\Theta^{(2)}_{-1}(v;\tau)\big)} \sum_{m\in\mathbb{Z}+\lambda+\ell+\frac{r+1}{2}} x^m$$

$$\times \sum_{\varepsilon,\eta=\pm} \varepsilon\eta \sum_{\substack{n\in\mathbb{Z}+\frac{\eta r}{2(\kappa+3)} \\ p\in\mathbb{Z}+\frac{2\ell+1}{2(\kappa+2)}-\frac{\varepsilon r}{2(\kappa+3)}}} q^{(\kappa+3)n^2+(\kappa+2)(\kappa+3)p^2} z^{(\kappa+3)n} \,. \tag{D.40}$$

The expression remains unchanged when extending the sum over r from 1 to $2\kappa+6$ and dividing the result by a factor of 2. Next we want to rewrite the sums over n and p by introducing

$$a = \varepsilon\eta p + n \in \mathbb{Z} + \frac{\varepsilon\eta(2\ell+1)}{2(\kappa+2)}\,, \qquad b = n - \varepsilon\eta(\kappa+2)p \in \mathbb{Z} + \ell + \frac{r+1}{2}\,. \tag{D.41}$$

The determinant of the matrix describing this change of variables is $\kappa+3$, which can be absorbed by restricting the summation over the $2(\kappa+3)$ values of r to just $r=1,2$. Thus (D.40) becomes

$$\begin{aligned}
\mathrm{ch}[\mathcal{F}^\ell_\lambda](t,u,v;\tau) &= \sum_{r=1,2} \frac{\mathrm{ch}[\mathcal{M}^{(1)}_{r+2\ell+1 \bmod 2}](u;\tau)}{2\eta(\tau)^3\big(\Theta^{(2)}_1(v;\tau)-\Theta^{(2)}_{-1}(v;\tau)\big)} \sum_{m\in\mathbb{Z}+\lambda+\ell+\frac{r+1}{2}} x^m \\
&\quad \times \sum_{\varepsilon,\eta=\pm} \varepsilon\eta \sum_{\substack{a\in\mathbb{Z}+\frac{\varepsilon\eta(2\ell+1)}{2(\kappa+2)} \\ b\in\mathbb{Z}+\ell+\frac{r+1}{2}}} q^{(\kappa+2)a^2+b^2} z^{(\kappa+2)a+b} && \text{(D.42)} \\
&= \sum_{r=1,2} \sum_{m\in\mathbb{Z}+\lambda+\ell+\frac{r+1}{2}} x^m \, \frac{\Theta^{(\kappa+2)}_{2\ell+1}(v;\tau)-\Theta^{(\kappa+2)}_{-2\ell-1}(v;\tau)}{\eta(\tau)^2\big(\Theta^{(2)}_1(v;\tau)-\Theta^{(2)}_{-1}(v;\tau)\big)} \\
&\quad \times \mathrm{ch}[\mathcal{M}^{(1)}_{r+2\ell+1 \bmod 2}](u;\tau)\mathrm{ch}[\mathcal{M}^{(1)}_{r+2\ell \bmod 2}](v;\tau) && \text{(D.43)} \\
&= \frac{\mathrm{ch}[\mathcal{M}^{(\kappa)}_{2\ell+1}](v;\tau)}{\eta(\tau)^2}\Bigg(\sum_{m\in\mathbb{Z}+\lambda} x^m \mathrm{ch}[\mathcal{M}^{(1)}_2](u;\tau)\mathrm{ch}[\mathcal{M}^{(1)}_1](v;\tau) \\
&\quad + \sum_{m\in\mathbb{Z}+\lambda+\frac{1}{2}} x^m \mathrm{ch}[\mathcal{M}^{(1)}_1](u;\tau)\mathrm{ch}[\mathcal{M}^{(1)}_2](v;\tau)\Bigg)\,. && \text{(D.44)}
\end{aligned}$$

Here, we have first used the fact that the expression only depends on the product $\varepsilon\eta$ and hence we can trivially perform one of the two sums. In the final expression we have rewritten the result in terms of affine $\mathfrak{su}(2)_\kappa$ characters.

Next we rewrite the two $\mathfrak{su}(2)_1$ characters in terms of free fermion characters, i.e. we use (A.33) and (A.34),

$$\mathrm{ch}[\mathcal{M}^{(1)}_2](u;\tau)\mathrm{ch}[\mathcal{M}^{(1)}_1](v;\tau) = \frac{\vartheta_2(\frac{u+v}{2};\tau)\vartheta_2(\frac{u-v}{2};\tau) - \vartheta_1(\frac{u+v}{2};\tau)\vartheta_1(\frac{u-v}{2};\tau)}{2\,\eta(\tau)^2}\,, \tag{D.45}$$

$$\mathrm{ch}[\mathcal{M}_1^{(1)}](u;\tau)\mathrm{ch}[\mathcal{M}_2^{(1)}](v;\tau) = \frac{\vartheta_2(\frac{u+v}{2};\tau)\vartheta_2(\frac{u-v}{2};\tau) + \vartheta_1(\frac{u+v}{2};\tau)\vartheta_1(\frac{u-v}{2};\tau)}{2\,\eta(\tau)^2}\,. \tag{D.46}$$

Thus we finally arrive the result

$$\begin{aligned}\sum_{m\in\mathbb{Z}+\lambda} & x^m \mathrm{ch}[\mathcal{M}_2^{(1)}](u;\tau)\mathrm{ch}[\mathcal{M}_1^{(1)}](v;\tau) \\ &+ \sum_{m\in\mathbb{Z}+\lambda+\frac{1}{2}} x^m \mathrm{ch}[\mathcal{M}_1^{(1)}](u;\tau)\mathrm{ch}[\mathcal{M}_2^{(1)}](v;\tau) \\ &= \sum_{m\in\mathbb{Z}+\lambda+\frac{1}{2}} x^m\, \frac{\vartheta_2\big(\frac{t+u+v}{2};\tau\big)\vartheta_2\big(\frac{t+u-v}{2};\tau\big)}{\eta(\tau)^2}\,,\end{aligned} \tag{D.47}$$

which reproduces (D.35) upon turning the infinite sum over m into a delta function as in (D.34).

D.2.4 Modular properties

Next we want to study the modular behaviour of the characters (D.35). To obtain good modular properties, we insert a $(-1)^{\mathrm{F}}$ in the character, which amounts to the replacement $\vartheta_2 \longrightarrow \vartheta_1$. Moreover, to match the conventions of Chapter 5, we include a $(-1)^w$ in the character. This merely defines what state in the representation is counted as being fermionic and which as bosonic. We have indicated these changes by a tilde in the character. Our calculations follow those in chapter 5 and are inspired by [60, 265]. For the S-modular transformation we find

$$\tilde{\mathrm{ch}}[\sigma^w(\mathcal{F}_\lambda^\ell)](t,u,v;\tau) \to \ \mathrm{e}^{\frac{\pi i}{2\tau}(\frac{\kappa+1}{\kappa+2}t^2-u^2-(\kappa+1)v^2)}\tilde{\mathrm{ch}}[\sigma^w(\mathcal{F}_\lambda^\ell)]\big(\tfrac{t}{\tau},\tfrac{u}{\tau},\tfrac{v}{\tau};-\tfrac{1}{\tau}\big) \tag{D.48}$$

$$\begin{aligned} &= \frac{\mathrm{sgn}(\mathrm{Re}(\tau))}{i\tau}\mathrm{e}^{\frac{i\pi(t+w)}{2(\kappa+2)\tau}(-w+2u(\kappa+2)+t(2\kappa+3))}(-1)^w \\ &\quad\times \sum_{m\in\mathbb{Z}} \mathrm{e}^{2\pi i m(\lambda+\frac{1}{2})}\delta\Big(\frac{t+w-m\tau}{\tau}\Big)\frac{\vartheta_1\big(\frac{t+u+v}{2};\tau\big)\vartheta_1\big(\frac{t+u-v}{2};\tau\big)}{\eta(\tau)^4} \\ &\quad\times \sum_{\ell'=0}^{\frac{\kappa}{2}} S^{\mathfrak{su}(2)}_{\ell\ell'}\chi_\kappa^{(\ell')}(v;\tau) \end{aligned} \tag{D.49}$$

$$\begin{aligned} &= -i\,\mathrm{sgn}(\mathrm{Re}(\tau))\sum_{m\in\mathbb{Z}} q^{\frac{m^2}{4(\kappa+2)}}x^{\frac{\kappa+1}{2(\kappa+2)}m}y^{\frac{m}{2}}\mathrm{e}^{-\frac{\pi i m w}{\kappa+2}}\mathrm{e}^{2\pi i m(\lambda+\frac{1}{2})}\delta(t+w-m\tau) \\ &\quad\times(-1)^w\frac{\vartheta_1\big(\frac{t+u+v}{2};\tau\big)\vartheta_1\big(\frac{t+u-v}{2};\tau\big)}{\eta(\tau)^4}\sum_{\ell'=0}^{\frac{\kappa}{2}} S^{\mathfrak{su}(2)}_{\ell\ell'}\chi_\kappa^{(\ell')}(v;\tau)\,. \end{aligned} \tag{D.50}$$

Here the prefactor $\mathrm{e}^{\frac{\pi i}{2\tau}(\frac{\kappa+1}{\kappa+2}t^2-y^2-(\kappa+1)z^2)}$ comes from the general transformation properties of weak Jacobi forms of index $-k = -\frac{\kappa+1}{\kappa+2}$, $k^+ = 1$ and

$k^- = \kappa + 1$, respectively, see e.g. [266], and we have used the modular transformations of the theta-functions. We have also used the modular properties of the $\mathfrak{su}(2)_\kappa$ characters. In the final step we have set $t = m\tau - w$ (because of the δ function), and used that both m and w are integers. Finally, as in Appendix C.1.5, we used the properties of the formal delta-function.

The expression (D.50) can now be written as

$$\sum_{w' \in \mathbb{Z}} \sum_{\ell'=0}^{\frac{\kappa}{2}} \int_0^1 \mathrm{d}\lambda' \, S_{(w,\lambda,\ell),(w',\lambda',\ell')} \; \tilde{\mathrm{ch}}[\sigma^{w'}(\mathcal{F}_{\lambda'})](t,u,v;\tau) \,, \tag{D.51}$$

with

$$S_{(w,\lambda,\ell),(w',\lambda',\ell')} = -i \, \mathrm{sgn}(\mathrm{Re}(\tau)) \, \mathrm{e}^{2\pi i \left(w'\lambda + w\lambda' - \frac{\pi i w w'}{2(\kappa+2)} \right)} S^{\mathfrak{su}(2)}_{\ell\ell'} \,, \tag{D.52}$$

thus obtaining the S-matrix (7.57). As in [265], it is not independent of τ, but this dependence cancels out in physical calculations. The S-matrix is (formally) unitary, meaning

$$\begin{aligned}\sum_{w'' \in \mathbb{Z}} \sum_{\ell''=0}^{\frac{\kappa}{2}} \int_0^1 \mathrm{d}\lambda'' \, S^\dagger_{(w,\lambda,\ell),(w'',\lambda'',\ell'')} S_{(w'',\lambda'',\ell''),(w',\lambda',\ell')} \\ = \delta_{w,w'} \, \delta(\lambda - \lambda' \bmod 1) \delta_{\ell,\ell'} \,. \end{aligned} \tag{D.53}$$

Moreover, it is clearly symmetric. These properties suffice to deduce that the diagonal modular invariant is indeed modular invariant.

D.2.5 The Verlinde formula

We now use the formal S-matrix to derive the typical fusion rules using a continuum version of the Verlinde formula; the following derivation is parallel to Appendix C.1.6. For this, we also need the S-matrix element of the vacuum with a continuous representation. It follows from the exact sequences (D.85a)–(D.85d), together with the identifications under spectral flow (7.50a) – (7.50d), that we have a resolution of the vacuum module as

$$\begin{aligned}
&\cdots \longrightarrow \sigma^{4\kappa+7-2\ell}\big(\mathcal{F}^{\ell}_{-\lambda_\ell}\big) \longrightarrow \cdots \longrightarrow \sigma^{3\kappa+7}\big(\mathcal{F}^{\frac{\kappa}{2}}_{-\lambda_{\frac{\kappa}{2}}}\big) \\
&\longrightarrow \sigma^{3\kappa+5}\big(\mathcal{F}^{\frac{\kappa}{2}}_{\lambda_{\frac{\kappa}{2}}}\big) \longrightarrow \cdots \longrightarrow \sigma^{2\kappa+2\ell+5}\big(\mathcal{F}^{\ell}_{\lambda_\ell}\big) \longrightarrow \cdots \longrightarrow \sigma^{2\kappa+5}\big(\mathcal{F}^{0}_{\lambda_0}\big) \\
&\longrightarrow \sigma^{2\kappa+3}\big(\mathcal{F}^{0}_{-\lambda_0}\big) \longrightarrow \cdots \longrightarrow \sigma^{2\kappa+3-2\ell}\big(\mathcal{F}^{\ell}_{-\lambda_\ell}\big) \longrightarrow \cdots \longrightarrow \sigma^{\kappa+3}\big(\mathcal{F}^{\frac{\kappa}{2}}_{-\lambda_{\frac{\kappa}{2}}}\big) \\
&\longrightarrow \sigma^{\kappa+1}\big(\mathcal{F}^{\frac{\kappa}{2}}_{\lambda_{\frac{\kappa}{2}}}\big) \longrightarrow \cdots \longrightarrow \sigma^{2\ell+1}\big(\mathcal{F}^{\ell}_{\lambda_\ell}\big) \longrightarrow \cdots \longrightarrow \sigma\big(\mathcal{F}^{0}_{\lambda_0}\big) \\
&\longrightarrow \mathcal{L} \longrightarrow 0 \,.
\end{aligned} \tag{D.54}$$

Thus we can write the vacuum character as

$$\mathrm{ch}[\mathcal{L}](t,u,v;\tau) = \sum_{m=0}^{\infty}\sum_{\ell=0}^{\frac{\kappa}{2}}(-1)^{2\ell}\Big(\mathrm{ch}\big[\sigma^{2\ell+2m(\kappa+2)+1}\big(\mathcal{F}^{\ell}_{\lambda_\ell}\big)\big](t,u,v;\tau) \\ - \mathrm{ch}\big[\sigma^{2(m+1)(\kappa+2)-2\ell-1}\big(\mathcal{F}^{\ell}_{-\lambda_\ell}\big)\big](t,u,v;\tau)\Big)\ , \quad \text{(D.55)}$$

and hence find for the S-matrix element of the vacuum and a continuous representation

$$S_{\mathrm{vac},(w,\lambda,\ell)} = \sum_{m=0}^{\infty}\sum_{j=0}^{\frac{\kappa}{2}}(-1)^{2j}\Big(S_{(2j+2m(\kappa+2)+1,\lambda_j,j),(w,\lambda,\ell)} \\ - S_{(2(m+1)(\kappa+2)-2j-1,-\lambda_j,j),(w,\lambda,\ell)}\Big)\ . \quad \text{(D.56)}$$

By using the explicit form of the S-matrix (D.52) together with the $\mathfrak{su}(2)_\kappa$ S-matrix

$$S^{\mathfrak{su}(2)}_{\ell\ell'} = \sqrt{\frac{2}{\kappa+2}}\sin\Big(\frac{\pi(2\ell+1)(2\ell'+1)}{\kappa+2}\Big)\ , \quad \text{(D.57)}$$

one finds after some algebra

$$S_{\mathrm{vac},(w,\lambda,\ell)} = -\frac{i(-1)^w\mathrm{sgn}(\mathrm{Re}(\tau))S^{\mathfrak{su}(2)}_{0\ell}}{2\cos\big(\frac{\pi(2\ell+1)}{\kappa+2}\big)+2\cos(2\pi\lambda)}\ . \quad \text{(D.58)}$$

Thus the Verlinde formula becomes

$$N^{(w_3,\lambda_3,\ell_3)}_{(w_1,\lambda_1,\ell_1)(w_2,\lambda_2,\ell_2)} = \sum_{w\in\mathbb{Z}}\sum_{\ell=0}^{\frac{\kappa}{2}}\int_0^1 \mathrm{d}\lambda\,\frac{S_{(w_1,\lambda_1,\ell_1)(w,\lambda,\ell)}S_{(w_2,\lambda_2,\ell_2)(w,\lambda,\ell)}S^*_{(w_3,\lambda_3,\ell_3)(w,\lambda,\ell)}}{S_{\mathrm{vac},(w,\lambda,\ell)}} \quad \text{(D.59)}$$

$$= \delta_{w_3,w_1+w_2}\delta\big(\lambda_3=\lambda_1+\lambda_2+\tfrac{1}{2}\big)\big(N^{\ell_3+\frac{1}{2}}_{\ell_1\ell_2}+N^{\ell_3-\frac{1}{2}}_{\ell_1\ell_2}\big) \\ + \Big(\delta_{w_3,w_1+w_2+1}\delta\big(\lambda_3=\lambda_1+\lambda_2-\tfrac{\gamma}{2}\big) \\ + \delta_{w_3,w_1+w_2-1}\delta\big(\lambda_3=\lambda_1+\lambda_2+\tfrac{\gamma}{2}\big)\Big)N^{\ell_3}_{\ell_1\ell_2}\ , \quad \text{(D.60)}$$

where, $N^{\ell_3}_{\ell_1\ell_2}$ are the $\mathfrak{su}(2)_\kappa$ rules,

$$N^{\ell_3}_{\ell_1\ell_2} = \delta_{\mathbb{Z}}(\ell_1+\ell_2+\ell_3)\begin{cases}1 & |\ell_1-\ell_2|\le\ell_3\le\min(\ell_1+\ell_2,\kappa-\ell_1-\ell_2)\\ 0 & \text{otherwise}\end{cases} \quad \text{(D.61)}$$

We take them by definition to be zero if one of the indices does not take values in $\{0,\frac{1}{2},\ldots,\frac{\kappa}{2}\}$.

D.3 The free field realisation of $\mathfrak{d}(2,1;\alpha)_k$ at $k^+ = k^- = 1$

D.3.1 The symplectic boson theory

Let us begin by explaining the free field realisation of $\mathfrak{sl}(2,\mathbb{R})_{1/2}$ in terms of a single pair of symplectic bosons. This theory and its fusion rules were analysed in detail in [190,274,275], see also Appendix C.1.1 for some background explanations.

The (pair of) symplectic bosons ξ^α_m with $\alpha = \pm$, satisfy the commutation relations

$$[\xi^\alpha_m, \xi^\beta_n] = \varepsilon^{\alpha\beta}\delta_{m,-n} \ . \tag{D.62}$$

They give rise to an $\mathfrak{sl}(2,\mathbb{R})_{1/2}$ affine algebra by setting

$$J^a_m = -\tfrac{1}{4}c_a(\sigma^a)_{\alpha\beta}(\xi^\alpha\xi^\beta)_m \ . \tag{D.63}$$

Both ξ^+_r and ξ^-_r are spin-$\frac{1}{2}$ fields and possess therefore NS- and R-sector representations. The NS-sector highest weight representation is described by

$$\xi^\alpha_r\,|0\rangle = 0\ , \qquad r \geq \tfrac{1}{2}\ , \qquad \alpha \in \{+,-\}\ , \tag{D.64}$$

and gives the vacuum representation of the theory. On the other hand, the R-sector representations of the symplectic boson pair have a zero-mode representation on the states $|m\rangle$ with action

$$\xi^+_0\,|m\rangle = \sqrt{2}\,|m+\tfrac{1}{2}\rangle\ , \qquad \xi^-_0\,|m\rangle = \sqrt{2}(m-\tfrac{1}{4})\,|m-\tfrac{1}{2}\rangle\ , \tag{D.65}$$

so that, in terms of the $\mathfrak{sl}(2,\mathbb{R})$ generators we have,

$$J^3_0\,|m\rangle = m\,|m\rangle\ , \qquad \mathcal{C}^{\mathfrak{sl}(2,\mathbb{R})}\,|m\rangle = \frac{3}{16}\,|m\rangle\ . \tag{D.66}$$

Thus the R-sector representations of the symplectic boson are labelled by $\lambda \in \mathbb{R}/\frac{1}{2}\mathbb{Z}$, describing the eigenvalues of J^3_0 mod $\frac{1}{2}\mathbb{Z}$, see also Appendix C.1.1.

Each symplectic boson representation decomposes into two $\mathfrak{sl}(2,\mathbb{R})_{1/2}$ representations, since the $\mathfrak{sl}(2,\mathbb{R})_{1/2}$ currents are bilinear in the symplectic bosons. The NS-sector representation decomposes into the two modules $\mathcal{K}_0$ and $\mathcal{K}_1$, which can be thought of as the vacuum and vector representation, respectively. Similarly, the R-sector representations decompose into the representations $\mathcal{E}_\lambda$ and $\mathcal{E}_{\lambda+1/2}$, where now $\lambda \in \mathbb{R}/\mathbb{Z}$ describes the eigenvalues of J^3_0 mod $\mathbb{Z}$. At $\lambda = \frac{1}{4}$, $\frac{3}{4}$, the modules become indecomposable, as can be seen from (D.65). The relevant modules that are required for the description of the full theory are in fact even bigger, and involve the indecomposable representations $\mathcal{S}$ and $\mathcal{S}'$ [190], whose composition series takes the form[88]

$$
\mathcal{S}: \quad \begin{array}{ccc} & \mathcal{K}_0 & \\ \swarrow & & \searrow \\ \sigma^{-2}(\mathcal{K}_1) & & \sigma^{2}(\mathcal{K}_1) \\ \searrow & & \swarrow \\ & \mathcal{K}_0 & \end{array} \quad , \qquad \mathcal{S}': \quad \begin{array}{ccc} & \mathcal{K}_1 & \\ \swarrow & & \searrow \\ \sigma^{-2}(\mathcal{K}_0) & & \sigma^{2}(\mathcal{K}_0) \\ \searrow & & \swarrow \\ & \mathcal{K}_1 & \end{array} \tag{D.67}
$$

The representation $\mathcal{S}$ is closely related to $\mathcal{E}_{1/4}$ and $\mathcal{E}_{3/4}$ since, on the level of the Grothendieck ring, we have

$$\mathcal{E}_{1/4} \sim \sigma(\mathcal{K}_0) \oplus \sigma^{-1}(\mathcal{K}_1) \quad \Longrightarrow \quad \mathcal{S} \sim \sigma(\mathcal{E}_{3/4}) \oplus \sigma^{-1}(\mathcal{E}_{1/4}) \,, \tag{D.68}$$

while the analogous statement for $\mathcal{S}'$ is

$$\mathcal{S}' \sim \sigma(\mathcal{E}_{1/4}) \oplus \sigma^{-1}(\mathcal{E}_{3/4}) \,. \tag{D.69}$$

Here σ denotes the spectral flow of the symplectic boson theory which acts via

$$\sigma(\xi^\alpha_r) = \xi^\alpha_{r-\frac{\alpha}{2}} \,. \tag{D.70}$$

The fusion rules of this theory were worked out in [190], and are explicitly

$$\mathcal{E}_\lambda \times \mathcal{E}_\mu \cong \begin{cases} \sigma(\mathcal{E}_{\lambda+\mu-\frac{1}{4}}) \oplus \sigma^{-1}(\mathcal{E}_{\lambda+\mu+\frac{1}{4}}) \,, & \lambda+\mu \neq 0 \,, \\ \mathcal{S} \,, & \lambda+\mu = 0 \,, \\ \mathcal{S}' \,, & \lambda+\mu = \frac{1}{2} \,, \end{cases} \tag{D.71a}$$

$$\mathcal{E}_\lambda \times \mathcal{S} \cong \sigma^2(\mathcal{E}_{\lambda+\frac{1}{2}}) \oplus 2\cdot\mathcal{E}_\lambda \oplus \sigma^{-2}(\mathcal{E}_{\lambda+\frac{1}{2}}) \,, \tag{D.71b}$$

$$\mathcal{E}_\lambda \times \mathcal{S}' \cong \sigma^2(\mathcal{E}_{\lambda}) \oplus 2\cdot\mathcal{E}_{\lambda+\frac{1}{2}} \oplus \sigma^{-2}(\mathcal{E}_{\lambda}) \,, \tag{D.71c}$$

$$\mathcal{S} \times \mathcal{S} \cong \mathcal{S}' \times \mathcal{S}' \cong \sigma^2(\mathcal{S}') \oplus 2\cdot\mathcal{S} \oplus \sigma^{-2}(\mathcal{S}') \,, \tag{D.71d}$$

$$\mathcal{S} \times \mathcal{S}' \cong \sigma^2(\mathcal{S}') \oplus 2\cdot\mathcal{S} \oplus \sigma^{-2}(\mathcal{S}') \,. \tag{D.71e}$$

D.3.2 The explicit form of the free field representation

In order to describe the free field realisation of $\mathfrak{d}(2,1;\alpha)_k$ at $k^+ = k^- = 1$, we now combine a symplectic boson pair with four free fermions, which we take to satisfy the anticommutation relations

$$\{\psi^{\alpha\beta}_r, \psi^{\gamma\delta}_s\} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\delta_{r+s,0} \,. \tag{D.72}$$

[88]Note that, unlike the situation discussed in Appendix C.1.1, we are considering here these modules as representations of $\mathfrak{sl}(2,\mathbb{R})_{1/2}$, not as representations of the symplectic boson theory.

The generators of $\mathfrak{d}(2,1;\alpha)_k$ are then given by

$$J^a_m = -\frac{1}{4} c_a (\sigma^a)_{\alpha\beta} (\xi^\alpha \xi^\beta)_m \ , \tag{D.73a}$$

$$K^{(+)a}_m = \frac{1}{4} (\sigma^a)_{\alpha\gamma} \varepsilon_{\beta\delta} (\psi^{\alpha\beta}\psi^{\beta\delta})_m \ , \tag{D.73b}$$

$$K^{(-)a}_m = \frac{1}{4} \varepsilon_{\alpha\gamma} (\sigma^a)_{\beta\delta} (\psi^{\alpha\beta}\psi^{\beta\delta})_m \ , \tag{D.73c}$$

$$S^{\alpha\beta\gamma}_m = \frac{1}{\sqrt{2}} (\xi^\alpha \psi^{\beta\gamma})_m \ . \tag{D.73d}$$

The spectral flow automorphism of $\mathfrak{d}(2,1;\alpha)_k$ acts on ξ^α_r as in (D.70), while on the fermions we have

$$\sigma(\psi^{\alpha\beta}_r) = \psi^{\alpha\beta}_{r+\frac{\alpha}{2}} \ . \tag{D.74}$$

D.3.3 The fusion rules

With this free field realisation at hand, we can evaluate the fusion rules directly in this case. Using the conformal embedding (D.12), the $\mathfrak{d}(2,1;\alpha)_k$ representations decompose as

$$\mathcal{L} = (\mathcal{K}_0, \mathbf{1}, \mathbf{1}) \oplus (\mathcal{K}_1, \mathbf{2}, \mathbf{2}) \ , \tag{D.75a}$$

$$\mathcal{L}' = (\mathcal{K}_0, \mathbf{2}, \mathbf{2}) \oplus (\mathcal{K}_1, \mathbf{1}, \mathbf{1}) \ , \tag{D.75b}$$

$$\mathcal{F}^0_\lambda = (\mathcal{E}_\lambda, \mathbf{2}, \mathbf{1}) \oplus (\mathcal{E}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{2}) \ , \tag{D.75c}$$

$$\mathcal{T}^{\frac{1}{2}}_{>} = \sigma^{-1}(\mathcal{S}, \mathbf{1}, \mathbf{1}) \oplus \sigma^{-1}(\mathcal{S}', \mathbf{2}, \mathbf{2}) \ , \tag{D.75d}$$

$$\mathcal{T}^{\frac{1}{2}}_{<} = \sigma^{-1}(\mathcal{S}', \mathbf{1}, \mathbf{1}) \oplus \sigma^{-1}(\mathcal{S}, \mathbf{2}, \mathbf{2}) \ , \tag{D.75e}$$

where we have denoted the $\mathfrak{su}(2)_1$ representations by the dimension of their ground state representations. Furthermore, $\mathcal{T}^{\frac{1}{2}}_{>}$ and $\mathcal{T}^{\frac{1}{2}}_{<}$ are indecomposable representations that will be introduced in Appendix D.4. If $\lambda + \mu \neq 0, \frac{1}{2}$, the fusion rules are then

$$\begin{aligned} \mathcal{F}^0_\lambda \times \mathcal{F}^0_\mu &= \big((\mathcal{E}_\lambda, \mathbf{2}, \mathbf{1}) \oplus (\mathcal{E}_{\lambda+\frac{1}{2}}, \mathbf{1}, \mathbf{2})\big) \times (\mathcal{E}_\mu, \mathbf{2}, \mathbf{1}) && \text{(D.76)} \\ &= \sigma(\mathcal{E}_{\lambda+\mu-\frac{1}{2}}, \mathbf{2}, \mathbf{1}) \oplus \sigma^{-1}(\mathcal{E}_{\lambda+\mu+\frac{1}{4}}, \mathbf{2}, \mathbf{1}) \\ &\qquad \oplus \sigma(\mathcal{E}_{\lambda+\mu+\frac{1}{4}}, \mathbf{1}, \mathbf{2}) \oplus \sigma^{-1}(\mathcal{E}_{\lambda+\mu-\frac{1}{4}}, \mathbf{1}, \mathbf{2}) \\ &= \sigma(\mathcal{F}_{\lambda+\mu-\frac{1}{4}}) \oplus \sigma^{-1}(\mathcal{F}_{\lambda+\mu+\frac{1}{4}}) \ . && \text{(D.77)} \end{aligned}$$

The other cases work similarly, and the complete fusion rules are therefore

$$\mathcal{F}^0_\lambda \times \mathcal{F}^0_\mu = \begin{cases} \sigma(\mathcal{F}_{\lambda+\mu-\frac{1}{4}}) \oplus \sigma^{-1}(\mathcal{F}_{\lambda+\mu+\frac{1}{4}}) \ , & \lambda+\mu \neq 0 \ , \\ \sigma(\mathcal{T}^{\frac{1}{2}}_{>}) \ , & \lambda+\mu = 0 \ , \\ \sigma(\mathcal{T}^{\frac{1}{2}}_{<}) \ , & \lambda+\mu = \frac{1}{2} \ , \end{cases} \tag{D.78a}$$

$$\mathcal{F}^0_\lambda \times \mathcal{T}^{\frac{1}{2}}_{>} = \sigma^{-1}(\mathcal{F}^0_{\lambda+\frac{1}{2}}) \oplus 2\sigma^{-1}(\mathcal{F}^0_\lambda) \oplus \sigma^{-3}(\mathcal{F}^0_{\lambda+\frac{1}{2}}) \,, \tag{D.78b}$$

$$\mathcal{F}^0_\lambda \times \mathcal{T}^{\frac{1}{2}}_{<} = \sigma^{-1}(\mathcal{F}^0_\lambda) \oplus 2\sigma^{-1}(\mathcal{F}^0_{\lambda+\frac{1}{2}}) \oplus \sigma^{-3}(\mathcal{F}^0_\lambda) \,, \tag{D.78c}$$

$$\mathcal{T}^{\frac{1}{2}}_{>} \times \mathcal{T}^{\frac{1}{2}}_{>} \cong \mathcal{T}^{\frac{1}{2}}_{<} \times \mathcal{T}^{\frac{1}{2}}_{<} \cong \sigma(\mathcal{T}^{\frac{1}{2}}_{<}) \oplus 2\,\sigma^{-1}(\mathcal{T}^{\frac{1}{2}}_{>}) \oplus \sigma^{-3}(\mathcal{T}^{\frac{1}{2}}_{<}) \,, \tag{D.78d}$$

$$\mathcal{T}^{\frac{1}{2}}_{>} \times \mathcal{T}^{\frac{1}{2}}_{<} \cong \sigma(\mathcal{T}^{\frac{1}{2}}_{>}) \oplus 2\,\sigma^{-1}(\mathcal{T}^{\frac{1}{2}}_{<}) \oplus \sigma^{-3}(\mathcal{T}^{\frac{1}{2}}_{>}) \,. \tag{D.78e}$$

D.3.4 The characters

Finally, it is straightforward to compute the characters using this free field realisation. Let us demonstrate how to do this for $\text{ch}[\mathcal{F}^0_\lambda](t,u,v;\tau)$. The symplectic boson R-sector representation has the character

$$\sum_{m\in\frac{1}{2}\mathbb{Z}+\lambda} \frac{x^m}{q^{\frac{1}{12}}\prod_{n=1}^\infty(1-x^{\frac{1}{2}}q^n)(1-x^{-\frac{1}{2}}q^n)} = \sum_{m\in\frac{1}{2}\mathbb{Z}+\lambda} \frac{x^m}{\eta(\tau)^2} \,, \tag{D.79}$$

where we have used that the chemical potentials of the oscillators can be absorbed into the zero modes, which allows us to rewrite the denominator in terms of the eta function. For the character of $\mathfrak{sl}(2,\mathbb{R})_{1/2}$, we have to keep every second state, and thus obtain

$$\text{ch}[\mathcal{E}_\lambda](t;\tau) = \sum_{m\in\mathbb{Z}+\lambda} \frac{x^m}{\eta(\tau)^2} \,. \tag{D.80}$$

On the other hand, the character of the four free fermions equals

$$\text{ch}[(\mathbf{2},\mathbf{1})](u,v;\tau) = \frac{\vartheta_2\big(\tfrac{u+v}{2};\tau\big)\vartheta_2\big(\tfrac{u-v}{2};\tau\big) - \vartheta_1\big(\tfrac{u+v}{2};\tau\big)\vartheta_1\big(\tfrac{u-v}{2};\tau\big)}{2\eta(\tau)^2} \,, \tag{D.81a}$$

$$\text{ch}[(\mathbf{1},\mathbf{2})](u,v;\tau) = \frac{\vartheta_2\big(\tfrac{u+v}{2};\tau\big)\vartheta_2\big(\tfrac{u-v}{2};\tau\big) + \vartheta_1\big(\tfrac{u+v}{2};\tau\big)\vartheta_1\big(\tfrac{u-v}{2};\tau\big)}{2\eta(\tau)^2} \,. \tag{D.81b}$$

Combining these ingredients according to (D.75c), we finally obtain

$$\text{ch}[\mathcal{F}^0_\lambda](t,u,v;\tau) = \sum_{m\in\mathbb{Z}+\lambda+\frac{1}{2}} \frac{x^m}{\eta(\tau)^4}\vartheta_2\big(\tfrac{t+u+v}{2};\tau\big)\vartheta_2\big(\tfrac{t+u-v}{2};\tau\big) \,, \tag{D.82}$$

which matches with the general formula (D.35).

D.4 The indecomposable modules

In this Appendix, we discuss the atypical modules appearing in the $\mathfrak{d}(2,1;\alpha)_k$ WZW-model at $k^+ = 1$. We make an educated guess for their structure, which passes many non-trivial tests.

D.4.1 The indecomposable modules

One strategy to determine the possible indecomposable modules is to study the representations that appear in fusion products. In the typical case we have, see (7.51)

$$\mathcal{F}^0_{\lambda_1} \times \mathcal{F}^\ell_{\lambda_2} \cong \sigma\big(\mathcal{F}^\ell_{\lambda_1+\lambda_2-\frac{\gamma}{2}}\big) \oplus \mathcal{F}^{\ell+\frac{1}{2}}_{\lambda_1+\lambda_2+\frac{1}{2}} \oplus \mathcal{F}^{\ell-\frac{1}{2}}_{\lambda_1+\lambda_2+\frac{1}{2}} \oplus \sigma^{-1}\big(\mathcal{F}^\ell_{\lambda_1+\lambda_2+\frac{\gamma}{2}}\big)\ . \tag{D.83}$$

If several modules on the right hand side of the fusion rules become indecomposable, we expect them to join to form one big indecomposable module. This happens when

$$\lambda_1+\lambda_2+\tfrac{1}{2} \in \big\{ \pm\lambda_{\ell+\frac{1}{2}}, \pm\lambda_{\ell-\frac{1}{2}} \big\}\ , \tag{D.84}$$

since we have the exact short sequences of modules[89]

$$0 \to \mathcal{G}^\ell_{>,+} \longrightarrow \mathcal{F}^\ell_{\lambda_\ell} \longrightarrow \mathcal{G}^\ell_{>,-} \to 0\ , \tag{D.85a}$$
$$0 \to \mathcal{G}^\ell_{>,-} \longrightarrow \mathcal{F}^\ell_{\lambda_\ell} \longrightarrow \mathcal{G}^\ell_{>,+} \to 0\ , \tag{D.85b}$$
$$0 \to \mathcal{G}^\ell_{<,+} \longrightarrow \mathcal{F}^\ell_{-\lambda_\ell} \longrightarrow \mathcal{G}^\ell_{<,-} \to 0\ , \tag{D.85c}$$
$$0 \to \mathcal{G}^\ell_{<,-} \longrightarrow \mathcal{F}^\ell_{-\lambda_\ell} \longrightarrow \mathcal{G}^\ell_{<,+} \to 0\ . \tag{D.85d}$$

The cases $\ell = 0$ and $\ell = \frac{\kappa}{2}$ are special, since then two of these values are simultaneously attained, but some modules are not present. One finds that the following modules can join up to form bigger indecomposable modules:

$$\mathcal{T}^{\frac{1}{2}}_{>} \sim \mathcal{F}^0_{\lambda_0} \oplus \sigma^{-2}\big(\mathcal{F}^0_{-\lambda_0}\big)\ , \tag{D.86a}$$
$$\mathcal{T}^{\ell}_{>} \sim \mathcal{F}^{\ell-\frac{1}{2}}_{\lambda_{\ell-\frac{1}{2}}} \oplus \sigma^{-1}\big(\mathcal{F}^{\ell-1}_{\lambda_{\ell-1}}\big)\ , \quad \ell \in \big\{1, \tfrac{3}{2}, \dots, \tfrac{\kappa+1}{2}\big\}\ , \tag{D.86b}$$
$$\mathcal{T}^{\frac{\kappa+1}{2}}_{<} \sim \mathcal{F}^{\frac{\kappa}{2}}_{-\lambda_{\frac{\kappa}{2}}} \oplus \sigma^{-2}\big(\mathcal{F}^{\frac{\kappa}{2}}_{\lambda_{\frac{\kappa}{2}}}\big)\ , \tag{D.86c}$$
$$\mathcal{T}^{\ell}_{<} \sim \mathcal{F}^{\ell-\frac{1}{2}}_{-\lambda_{\ell-\frac{1}{2}}} \oplus \sigma^{-1}\big(\mathcal{F}^{\ell}_{-\lambda_\ell}\big)\ , \quad \ell \in \big\{\tfrac{1}{2}, 1, \dots, \tfrac{\kappa}{2}\big\}\ . \tag{D.86d}$$

In order to describe the precise structures of these indecomposables, we first use the short exact sequences (D.85a) – (D.85d) to decompose the modules in the Grothendieck ring as

$$\mathcal{T}^{\frac{1}{2}}_{>} \sim \mathcal{G}^0_{>,+} \oplus \mathcal{G}^0_{>,-} \oplus \sigma^{-2}\big(\mathcal{G}^0_{<,+}\big) \oplus \sigma^{-2}\big(\mathcal{G}^0_{<,-}\big)\ , \tag{D.87}$$
$$\mathcal{T}^{\ell}_{>} \sim \mathcal{G}^{\ell-\frac{1}{2}}_{>,+} \oplus \mathcal{G}^{\ell-\frac{1}{2}}_{>,-} \oplus \sigma^{-1}\big(\mathcal{G}^{\ell-1}_{>,+}\big) \oplus \sigma^{-1}\big(\mathcal{G}^{\ell-1}_{>,-}\big)\ , \tag{D.88}$$

[89]The notation here is a bit cavalier since the modules $\mathcal{F}^\ell_{\lambda_\ell}$ that appear as the middle term in the first two lines have different indecomposable structures (since one contains a discrete highest weight and the other a discrete lowest weight representation).

$$\mathcal{T}_<^{\frac{\kappa+1}{2}} \sim \mathcal{G}^{\frac{\kappa}{2}}_{<,+} \oplus \mathcal{G}^{\frac{\kappa}{2}}_{<,-} \oplus \sigma^{-2}\big(\mathcal{G}^{\frac{\kappa}{2}}_{>,+}\big) \oplus \sigma^{-2}\big(\mathcal{G}^{\frac{\kappa}{2}}_{>,-}\big) \,, \tag{D.89}$$

$$\mathcal{T}_<^{\ell} \sim \mathcal{G}^{\ell-\frac{1}{2}}_{<,+} \oplus \mathcal{G}^{\ell-\frac{1}{2}}_{<,-} \oplus \sigma^{-1}\big(\mathcal{G}^{\ell}_{<,+}\big) \oplus \sigma^{-1}\big(\mathcal{G}^{\ell}_{<,-}\big) \,. \tag{D.90}$$

Because of the identifications under spectral flow, see eqs. (7.50b) – (7.50d), the right hand side always contains two isomorphic modules, and hence the indecomposable structure is

$$\mathcal{T}_>^{\frac{1}{2}} : \quad \begin{array}{ccccc} & & \mathcal{G}^{0}_{>,-} & & \\ & \swarrow & & \searrow & \\ \sigma^{-2}\big(\mathcal{G}^{0}_{<,-}\big) & & & & \mathcal{G}^{0}_{>,+} \\ & \searrow & & \swarrow & \\ & & \sigma^{-2}\big(\mathcal{G}^{0}_{<,+}\big) & & \end{array} \,, \qquad \mathcal{T}_>^{\ell} : \quad \begin{array}{ccccc} & & \mathcal{G}^{\ell-\frac{1}{2}}_{>,-} & & \\ & \swarrow & & \searrow & \\ \sigma^{-1}\big(\mathcal{G}^{\ell-1}_{>,-}\big) & & & & \mathcal{G}^{\ell-\frac{1}{2}}_{>,+} \\ & \searrow & & \swarrow & \\ & & \sigma^{-1}\big(\mathcal{G}^{\ell-1}_{>,+}\big) & & \end{array} \,, \tag{D.91a}$$

$$\mathcal{T}_<^{\frac{\kappa+1}{2}} : \quad \begin{array}{ccccc} & & \mathcal{G}^{\frac{\kappa}{2}}_{<,-} & & \\ & \swarrow & & \searrow & \\ \sigma^{-2}\big(\mathcal{G}^{\frac{\kappa}{2}}_{>,-}\big) & & & & \mathcal{G}^{\frac{\kappa}{2}}_{<,+} \\ & \searrow & & \swarrow & \\ & & \sigma^{-2}\big(\mathcal{G}^{\frac{\kappa}{2}}_{>,+}\big) & & \end{array} \,, \qquad \mathcal{T}_<^{\ell} : \quad \begin{array}{ccccc} & & \mathcal{G}^{\ell-\frac{1}{2}}_{<,-} & & \\ & \swarrow & & \searrow & \\ \sigma^{-1}\big(\mathcal{G}^{\ell}_{<,-}\big) & & & & \mathcal{G}^{\ell-\frac{1}{2}}_{<,+} \\ & \searrow & & \swarrow & \\ & & \sigma^{-1}\big(\mathcal{G}^{\ell}_{<,+}\big) & & \end{array} \,. \tag{D.91b}$$

Here, we have used again composition diagrams to display the indecomposable structure, see also Appendix D.3. Note that the bottom and top modules are always identical via the spectral flow identifications.

D.4.2 The atypical fusion rules

A heuristic way to determine the fusion rules of these indecomposable representations consists of writing them in terms of their summands as in (D.86a) – (D.86d). We then apply the naive generalisations of the typical fusion rules (7.51), and finally reassemble the result, so that the only representations that appear are the typical representations we considered before, together with those given in (D.86a) – (D.86d).

Unfortunately, the general formula is rather clumsy, so let us just work out one example to illustrate the idea. We consider

$$\begin{aligned} \mathcal{T}_>^{\frac{1}{2}} \times \mathcal{T}_>^{\frac{1}{2}} &\sim \big(\mathcal{F}^{0}_{\lambda_0} \oplus \sigma^{-2}\big(\mathcal{F}^{0}_{-\lambda_0}\big)\big) \times \big(\mathcal{F}^{0}_{\lambda_0} \oplus \sigma^{-2}\big(\mathcal{F}^{0}_{-\lambda_0}\big)\big) \\ &\cong \sigma\big(\mathcal{F}^{0}_{\lambda_1}\big) \oplus \mathcal{F}^{\frac{1}{2}}_{\lambda_{\frac{1}{2}}} \oplus \sigma^{-1}\big(\mathcal{F}^{0}_{\lambda_0}\big) \oplus 2\,\sigma^{-1}\big(\mathcal{F}^{0}_{\lambda_0}\big) \oplus 2\,\sigma^{-2}\big(\mathcal{F}^{\frac{1}{2}}_{\frac{1}{2}}\big) \end{aligned} \tag{D.92}$$

$$\oplus\, 2\,\sigma^{-3}\big(\mathcal{F}^0_{-\lambda_0}\big) \oplus \sigma^{-3}\big(\mathcal{F}^0_{-\lambda_0}\big) \oplus \sigma^{-4}\big(\mathcal{F}^{\frac{1}{2}}_{-\lambda_{\frac{1}{2}}}\big) \oplus \sigma^{-5}\big(\mathcal{F}^0_{-\lambda_1}\big) \tag{D.93}$$

$$\sim \mathcal{T}^1_> \oplus 2\,\sigma^{-1}\big(\mathcal{T}^{\frac{1}{2}}_>\big) \oplus \sigma^{-3}\big(\mathcal{T}^{\frac{1}{2}}_<\big) \oplus \sigma\big(\mathcal{F}^0_{\lambda_1}\big) \oplus 2\,\sigma^{-2}\big(\mathcal{F}^{\frac{1}{2}}_{\frac{1}{2}}\big) \oplus \sigma^{-5}\big(\mathcal{F}^0_{-\lambda_1}\big)\ . \tag{D.94}$$

The other cases work similarly, and the resulting products define an associative ring, which should therefore agree with the fusion ring.

D.4.3 The atypical Hilbert space

Finally, we discuss the structure of the atypical Hilbert space. Naively, we would construct an atypical Hilbert space as

$$\mathcal{H}^{\text{naive}}_{\text{atyp}} = \bigoplus_{w\in\mathbb{Z}} \bigoplus_{\ell=\frac{1}{2}}^{\frac{\kappa+1}{2}} \Big[\sigma^w\big(\mathcal{T}^\ell_>\big) \otimes \overline{\sigma^w\big(\mathcal{T}^\ell_>\big)} \ \oplus\ \sigma^w\big(\mathcal{T}^\ell_<\big) \otimes \overline{\sigma^w\big(\mathcal{T}^\ell_<\big)}\Big]\ . \tag{D.95}$$

While this contains now only modules which close under fusion, there are two problems with this proposal. First, locality requires that $L_0 - \bar{L}_0$ acts diagonalisably, since otherwise the complete correlation functions would be multi-valued. In addition, (D.95) does not agree with (7.56) on the level of the Grothendieck ring, and hence would not be modular invariant. As explained in [267,268], the true Hilbert space is obtained by quotienting out an ideal $\mathcal{I} \subset \mathcal{H}^{\text{naive}}_{\text{atyp}}$ from $\mathcal{H}^{\text{naive}}_{\text{atyp}}$.

To construct this ideal, we note that there are natural long exact sequences

$$\begin{aligned}
&\overset{s_+}{\underset{s_-}{\leftrightarrow}} \sigma^{w+1}\big(\mathcal{T}^{\frac{1}{2}}_>\big) \overset{s_+}{\underset{s_-}{\leftrightarrow}} \cdots \overset{s_+}{\underset{s_-}{\leftrightarrow}} \sigma^{w+2\ell}\big(\mathcal{T}^{\ell}_>\big) \overset{s_+}{\underset{s_-}{\leftrightarrow}} \cdots \overset{s_+}{\underset{s_-}{\leftrightarrow}} \sigma^{w+\kappa+1}\big(\mathcal{T}^{\frac{\kappa+1}{2}}_>\big) \\
&\overset{s_+}{\underset{s_-}{\leftrightarrow}} \sigma^{w+\kappa+3}\big(\mathcal{T}^{\frac{\kappa+1}{2}}_<\big) \overset{s_+}{\underset{s_-}{\leftrightarrow}} \cdots \overset{s_+}{\underset{s_-}{\leftrightarrow}} \sigma^{w+2(\kappa+2)+2\ell}\big(\mathcal{T}^{\ell}_<\big) \overset{s_+}{\underset{s_-}{\leftrightarrow}} \cdots \overset{s_+}{\underset{s_-}{\leftrightarrow}} \sigma^{w+2\kappa+3}\big(\mathcal{T}^{\frac{1}{2}}_<\big) \\
&\overset{s_+}{\underset{s_-}{\leftrightarrow}} \sigma^{w+2\kappa+5}\big(\mathcal{T}^{\frac{1}{2}}_>\big) \overset{s_+}{\underset{s_-}{\leftrightarrow}} \cdots \overset{s_+}{\underset{s_-}{\leftrightarrow}} \sigma^{w+2(\kappa+2)+2\ell}\big(\mathcal{T}^{\ell}_>\big) \overset{s_+}{\underset{s_-}{\leftrightarrow}} \cdots \overset{s_+}{\underset{s_-}{\leftrightarrow}} \sigma^{w+3\kappa+5}\big(\mathcal{T}^{\frac{\kappa+1}{2}}_>\big)\ ,
\end{aligned} \tag{D.96}$$

where s_+ maps to the right and s_- to the left. The map s_+ maps the two upper right elements of the composition diagram of (D.91a) or (D.91b) to the two lower left elements of the next term in the sequence. Similarly s_- maps the two upper left elements of the composition diagram to the two lower right elements of the previous term in the sequence. There are in fact $2(\kappa+2)$ such sequences, that are characterised by $w \in \{0,1,\ldots,2\kappa+3\}$. For each such sequence, let us denote the n-th term in the sequence by $\sigma^w\big(\mathcal{X}^n\big)$, so that

$$s_\pm : \sigma^w\big(\mathcal{X}^n\big) \longrightarrow \sigma^w\big(\mathcal{X}^{n\pm1}\big) \tag{D.97}$$

for $n \in \mathbb{Z}$. Furthermore, we denote the elements of the indecomposable modules by $\mathcal{X}^n[\varepsilon_1, \varepsilon_2]$, where $\varepsilon_i \in \{0,1\}$. $[\varepsilon_1, \varepsilon_2] = [0,0]$ denotes the top element, $[\varepsilon_1, \varepsilon_2] = [1,0]$ the left element, $[\varepsilon_1, \varepsilon_2] = [0,1]$ the right element and $[\varepsilon_1, \varepsilon_2] = [1,1]$ the bottom element. We thus have, cf. Appendix C.2

$$s_+ \sigma^w\big(\mathcal{X}^n\big)[\varepsilon_1, \varepsilon_2] = \sigma^w\big(\mathcal{X}^{n+1}\big)[\varepsilon_1 + 1, \varepsilon_2] \, , \tag{D.98}$$

$$s_- \sigma^w\big(\mathcal{X}^n\big)[\varepsilon_1, \varepsilon_2] = \sigma^w\big(\mathcal{X}^{n-1}\big)[\varepsilon_1, \varepsilon_2 + 1] \, . \tag{D.99}$$

The ideal $\mathcal{I}$ by which we have to quotient out is then generated by

$$\mathcal{I}_\pm \equiv \bigoplus_{w=0}^{2\kappa+3} \bigoplus_{n\in\mathbb{Z}} \big(s_\pm \otimes \overline{\mathbb{1}} - \mathbb{1} \otimes \overline{s_\mp}\big)\big(\sigma^w\big(\mathcal{X}^n\big) \otimes \sigma^w\big(\mathcal{X}^{n\pm1}\big)\big) \, . \tag{D.100}$$

This leads to the following identifications in the naive atypical Hilbert space,

$$\sigma^w\big(\mathcal{X}^n\big)[\varepsilon_1 + 1, \varepsilon_2] \otimes \overline{\sigma^w\big(\mathcal{X}^n\big)[\bar\varepsilon_1, \bar\varepsilon_2]} \sim \sigma^w\big(\mathcal{X}^{n-1}\big)[\varepsilon_1, \varepsilon_2] \otimes \overline{\sigma^w\big(\mathcal{X}^{n-1}\big)[\bar\varepsilon_1, \bar\varepsilon_2 + 1]} \, , \tag{D.101}$$

$$\sigma^w\big(\mathcal{X}^n\big)[\varepsilon_1, \varepsilon_2 + 1] \otimes \overline{\sigma^w\big(\mathcal{X}^n\big)[\bar\varepsilon_1, \bar\varepsilon_2]} \sim \sigma^w\big(\mathcal{X}^{n+1}\big)[\varepsilon_1, \varepsilon_2] \otimes \overline{\sigma^w\big(\mathcal{X}^{n+1}\big)[\bar\varepsilon_1 + 1, \bar\varepsilon_2]} \, . \tag{D.102}$$

This 'gauge freedom' allows us, for example, to set $\varepsilon_1 = \bar{\varepsilon_1} = 0$, so that

$$\big(\mathcal{T}^\ell_>\big)^{\text{gauge fixed}} \sim \mathcal{F}^{\ell-\frac{1}{2}}_{\lambda_{\ell-\frac{1}{2}}} \, , \quad \text{and} \quad \big(\mathcal{T}^\ell_<\big)^{\text{gauge fixed}} \sim \mathcal{F}^{\ell-\frac{1}{2}}_{-\lambda_{\ell-\frac{1}{2}}} \tag{D.103}$$

for $\ell \in \{\frac{1}{2}, 1, \ldots, \frac{\kappa+1}{2}\}$. Hence, the indecomposable modules become after gauge-fixing the moral analogue of the continuous representations for $\lambda = \pm\lambda_\ell$. The resulting space of states has then essentially the same form as (7.56), and hence is in particular modular invariant.

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