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ORIGINAL PAPER

Cutoff on hyperbolic surfaces

Konstantin Golubev[1] · **Amitay Kamber**[2]

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Abstract

In this paper, we study the common distance between points and the behavior of a constant length step discrete random walk on finite area hyperbolic surfaces. We show that if the second smallest eigenvalue of the Laplacian is at least $1/4$, then the distances on the surface are highly concentrated around the minimal possible value of the diameter, and that the discrete random walk exhibits cutoff. This extends the results of Lubetzky and Peres (Geom Funct Anal 26(4):1190–1216, 2016. https://doi.org/10.1007/s00039-016-0382-7) from the setting of graphs to the setting of hyperbolic surfaces. By utilizing density theorems of exceptional eigenvalues from Sarnak and Xue (Duke Math J 64(1):207–227, 1991), we are able to show that the results apply to congruence subgroups of $SL_2(\mathbb{Z})$ and other arithmetic lattices, without relying on the well-known conjecture of Selberg (Proc Symp Pure Math 8:1–15, 1965), thus relaxing the condition on the Laplace spectrum of a surface. Conceptually, we show the close relation between the cutoff phenomenon and temperedness of representations of algebraic groups over local fields, partly answering a question of Diaconis (Proc Natl Acad Sci 93(4):1659–1664, 1996), who asked under what general phenomena cutoff exists.

Keywords First keyword · Second keyword · More · Hyperbolic surfaces · Random Walks · Cutoff

1 Introduction

Let $\mathbb{H}$ be the hyperbolic plane equipped with the standard metric d and the standard measure μ. Let $\Gamma \subset PSL_2(\mathbb{R})$ be a lattice and let $X = \Gamma\backslash\mathbb{H}$ be the quotient space, which is a hyperbolic surface if Γ is torsion-free, and an orbifold in general. The measure μ descends to a finite measure on X, and let $d_X : X \times X \to \mathbb{R}_{\geq 0}$ be the induced distance on X. The injectivity radius of a point $x_0 \in X$ is $\frac{1}{2} \inf_{1\neq\gamma\in\Gamma} d(\tilde{x}_0, \gamma\tilde{x}_0)$, where $\tilde{x}_0 \in \mathbb{H}$ is a lift of x_0 to $\mathbb{H}$. Denote $R_X = \operatorname{acosh}(\mu(X)/2\pi + 1)$, this is the radius of the hyperbolic ball whose volume equals the volume $\mu(X)$ of X.

✉ Konstantin Golubev
konstantin.golubev@math.ethz.ch

Amitay Kamber
amitay.kamber@mail.huji.ac.il

1 Department of Mathemathics, ETH Zurich, Zurich, Switzerland

2 Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

Definition 1.1 We say that $X = \Gamma\backslash\mathbb{H}$ is Ramanujan[1] if the non-trivial spectrum of the Laplacian on $L^2(X)$ is bounded from below by $1/4$.

An equivalent condition, stated in terms of representation theory, is that every non-trivial subrepresentation of $G = PSL_2(\mathbb{R})$ on $L^2(\Gamma\backslash G)$, which generated by its $K = PSO_2(\mathbb{R})$-fixed vectors, is tempered (see Sect. 2 for an explanation of this notation).

We write $C = C(t)$ if C is a constant depending only on t. We write $a \ll_t b$ if there is $C = C(t)$ such that $a \le C \cdot b$ holds, and $a \asymp_t b$ if both $a \ll_t b$ and $b \ll_t a$ take place.

1.1 Common distance

Theorem 1.2 *Let $\Gamma \subset PSL_2(\mathbb{R})$ be a lattice, $X = \Gamma\backslash\mathbb{H}$, and denote $R_X = \operatorname{acosh}(\mu(X)/2\pi + 1) \ge 1$ (this is the radius of the hyperbolic ball of volume $\mu(X)$). Then for every point $x_0 \in X$ and for all $\gamma > 0$, the following inequality holds*

$$\mu\left(x \in X : d_X(x_0, x) \le R_X - \gamma \ln(R_X)\right)/\mu(X) \ll R_X^{-\gamma}.$$

If X is Ramanujan and $x_0 \in X$ has injectivity radius at least r_0, then for all $\gamma > 0$, the following inequality holds

$$\mu\left(x \in X : d_X(x_0, x) \ge R_X + \gamma \ln(R_X)\right)/\mu(X) \ll_{r_0} \left(1+\gamma^2\right) R_X^{2-\gamma}.$$

In other words, for a point x_0 on a Ramanujan surface X, the distance from it to almost every other point is within the window of size $(2+\epsilon)\ln(R_X)$ around R_X. We emphasize that the result is mainly interesting for a sequence of Ramanujan quotients with volume increasing to infinity, which is not known to exist. However, the well-known conjecture of Selberg asserts that the quotients defined by the congruence subgroups of $SL_2(\mathbb{Z})$ form a sequence of such quotients (see [31,33] and also Theorem 1.4 below). Alternatively, one may conjecture that as in case of graphs, a "random" surface is almost Ramanujan with a proper choice of the random model (see Conjecture 1.6 below).

1.2 Cutoff of random walks

In the second result, we consider the speed of convergence in the L^1-norm of two different random walks on X. The first one is the hyperbolic Brownian motion on X, which we consider as an operator $B_t : C(X) \to C(X)$ for $t \in \mathbb{R}_{\ge 0}$, where $C(X)$ is the space of continuous functions on X. The second one is the discrete time random walk with the step of a fixed length, i.e., at each step the walker rotates at a uniformly chosen angle and makes a step of some fixed length $r_1 > 0$. The corresponding operator $A_{r_1} : C(X) \to C(X)$ is the distance r_1 averaging operator. By duality, we consider both random walks as acting on measures on X.

Specifically, for a point $x_0 \in X$ we consider the continuous time random walk $B_t\delta_{x_0}$, and the discrete time random walk $A_{r_1}^k\delta_{x_0}$, both as measures on X. One can show that the measures defined by the two random walks, for $t > 0$ or $k \ge 3$, are represented by some L^1-functions, which converge in the L^1-norm to the constant function π on X normalized as $\pi(x) = \mu(X)^{-1}$ for all $x \in X$. The following theorem gives an exact estimate on the rate of convergence for points with injectivity radius bounded away from 0. For simplicity

[1] It seems that the notion of a Ramanujan surface (or more generally, a Ramanuajan manifold or a Ramanujan orbifold) does not appear in literature, but is natural given the standard notions of a Ramanujan graph ([25]) and a Ramanujan complex ([26]).

of notations we state it here for the discrete time random walk only. See Sect. 6 for a full statement of the theorem.

Theorem 1.3 *Fix $r_0 > 0$, $r_1 > 0$, $\lambda > 0$ and a point $x_0 \in X$. Assume $R_X \gg_{r_0,r_1,\lambda} 1$ and write*

$$\alpha = \frac{1}{\pi r_1} \int_0^{\pi} \ln\left(e^{r_1} \cos^2\theta + e^{-r_1} \sin^2\theta\right) d\theta \in (0, 1).$$

Then there exist constants $c = c(r_1) > 0$, and $C = C(r_0, r_1)$, such that

1. *If k satisfies $k\alpha r_1 < R_X - \lambda\sqrt{R_X}$ then $\left\| A_{r_1}^k \delta_{x_0} - \pi \right\|_1 > 2 - Ce^{-c\lambda^2}$;*
2. *If k satisfies $k\alpha r_1 > R_X + \lambda\sqrt{R_X}$, if $X = \Gamma\backslash\mathbb{H}$ is Ramanujan, and if x_0 has injectivity radius at least r_0, then $\left\| A_r^k \delta_{x_0} - \pi \right\|_1 < Ce^{-c\lambda^2}$;*

The above behavior of the random walk is a cutoff phenomenon, which is defined in general as follows (see [8]). Let $\left(P_n^k(x, y), X_n\right)$ be a sequence of Markov random walks on a probability space X_n, which converge as $k \to \infty$ to the uniform probability π_n on X_n. Let $f, g : \mathbb{N} \to \mathbb{R}$ be functions such that $f(n) \to \infty$ and $g(n) = o(f(n))$ as $n \to \infty$. We say that the sequence $\left(P_n^k(x, y), X_n\right)$ exhibits a cutoff at time $f(n)$ with window of size $g(n)$, if for every $1 > \epsilon > 0$, the time

$$k_n = \inf\left\{k \mid \sup_{x_0} \left\| P_n^k(x_0, \cdot) - \pi_n \right\|_1 < \epsilon\right\} \tag{1}$$

satisfies $k_n = f(n) + O_\epsilon(g(n))$.

Determining whether a sequence of random walks exhibit a cutoff is a fundamental problem (see [8]). Theorem 1.3 says that if a sequence of Ramanujan surfaces X_n have injectivity radius at least r_0 at every point of every surface then the discrete random walks on them exhibit a cutoff at time $R_X/\alpha r_1$ with window of size $\sqrt{R_X}/\alpha r_1$.

1.3 Arithmetic subgroups

As said in the discussion after Theorem 1.2, Selberg's conjecture implies that the quotients X of $\mathbb{H}$ by congruence subgroups of $SL_2(\mathbb{Z})$ satisfy the hypotheses of Theorems 1.2 and 1.3. Using the current knowledge, we can give a slightly weaker version of Theorem 1.2.

Theorem 1.4 *Let $\Gamma = SL_2(\mathbb{Z})$ or any cocompact arithmetic lattice in $SL_2(\mathbb{R})$ and let $X_0 = \Gamma\backslash\mathbb{H}$ be the corresponding quotient. For every $q \in \mathbb{N}$, let $\Gamma(q)$ the principal congruence subgroup of Γ, let $X_q = \Gamma(q)\backslash\mathbb{H}$ be the corresponding quotient, and let $\rho_q : X_q \to X_0$ be the cover map.*

Let $x_0^{(q)} \in X_q$ be a point such that its projection $\rho_q\left(x_0^{(q)}\right)$ to X_0 has injectivity radius at least a constant r_0. Then for every $\epsilon_0 > 0$

$$\mu\left(x \in X_q : d_{X_q}\left(x, x_0^{(q)}\right) \geq R_{X_q}(1 + \epsilon_0)\right) / \mu\left(X_q\right) \to 0$$

as $q \to \infty$.

A similar theorem was proven independently by Sarnak in his letter [29].

1.4 Methods of proof

The proofs of Theorems 1.2–1.4 exploit the following proposition:

Proposition 1.5 *The surface X is Ramanujan if and only if for every $r \geq 0$ the norm of A_r on $L_0^2(X) = \left\{ f \in L^2(X) : \int f(x)dx = 0 \right\}$ is bounded by $(r+1)e^{-r/2}$.*

A similar proposition, but of greater generality, plays a crucial role in the work of Harish-Chandra (see [12, Theorem 3]). Proposition 1.5 is the essential ingredient in the proof of Theorem 1.2.

The proof of Theorem 1.3 combines Proposition 1.5 with two other results. The first one, Lemma 5.4, says that after 3 steps the random walk measure $A_{r_1}^3 \delta_{x_0}$ (respectively, the Brownian motion measure $B_{t_0}\delta_{x_0}$ at a fixed time $t_0 > 0$) is represented an L^2-function on X, with a bounded L^2-norm, depending only on the injectivity radius r_0. The second result, Corollary 4.7, may be described as a concentration of measure theorem for the rate of escape of the random walk $A_{r_1}^k$ on $\mathbb{H}$. Informally, we may write $A_{r_1}^k \cong \int_r f_k(r) A_r dr$, where most of the measure $f_k(r)dr$ is concentrated around $k\alpha r_1$ with deviation of size $\sqrt{k}\alpha r_1$.

The proof of Theorem 1.4 depends on the following facts:

- $\Gamma(q)$ is normal in Γ;
- There exists a lower bound on the smallest eigenvalue of X_q which is independent of q;
- The number of exceptional eigenvalues of X_q can be carefully bounded depending on their distance from the non-exceptional spectrum and q (see below);

Surprisingly, the bound on the number of exceptional eigenvalues that is required for the proof is exactly the "elementary" density bound discussed in the work of Sarnak and Xue ([32]). The bound states the number of eigenvalues of X_q with corresponding matrix coefficients not in L^p for $p > 2$ is $\ll_\epsilon [\Gamma : \Gamma(q)]^{2/p+\epsilon}$ (see also [30]). Note that while in [32] cocompactness is assumed, this assumption was later removed in [15], and in any case stronger results for $SL_2(\mathbb{Z})$ were also proven earlier by different methods in [16,17]. Theorem 1.4 also holds for $SL_2(\mathbb{Z})$ for non-prime q, as a non-elementary bound on the smallest eigenvalue was proven already by Selberg in [33]. See the discussion in Sect. 8 for full details.

This work is similar in spirit to the results of [23], and shows the general connection between the common distance and cutoff phenomena in quotients of symmetric spaces (infinite regular trees in [23] and the hyperbolic plane here) and temperedness of representations (or the generalized Ramanujan conjecture).

1.5 Open questions

We expect that the results of this article can be extended to quotients of higher dimensional symmetric spaces, and also to other contexts (e.g. the action of Hecke operators on $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ and its covers). Theorems analogous to Theorem 1.2 for quotients of p-adic Lie groups, i.e., Ramanujan complexes, are proven in [18, Theorem 1.9], and [22, Theorem 1.ii]. Theorem 1.2 is also closely related to the optimal covering properties of the Golden-Gates of [28]. See also [29].

While we were unable to show it, we believe that it is possible to prove in the notations of Theorem 1.4 that (at least for $SL_2(\mathbb{Z})$) there exists a constant $C > 0$ such that

$$\mu\left(x \in X_q : d_{X_q}\left(x, x_0^{(q)}\right) \geq R_{X_q} + C \ln\left(R_{X_q}\right)\right) / \mu\left(X_q\right) \to_{q\to\infty} 0.$$

Selberg's conjecture would give $C = 2 + \epsilon$, $\epsilon > 0$ by Theorem 1.2.

The following conjectures are natural continuous analogs of well-known combinatorial results, in the spirit of this article. Assume that the lattice Γ is a free group (for example, the principal congruence subgroup $\Gamma = \Gamma(2) = \ker\left\{PSL_2(\mathbb{Z}) \overset{\text{mod}}{\to} PSL_2(\mathbb{Z}/2\mathbb{Z})\right\}$, which is freely generated by the images in $PSL_2(\mathbb{Z})$ of $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$). Then every homomorphism $\phi\colon \Gamma \to S_n$ whose image acts transitively on $\{1, \ldots, n\}$ defines an index n subgroup $\Gamma' \subset \Gamma$, by $\Gamma' = \{\gamma \in \Gamma : \phi(\gamma)(1) = 1\}$, and every index n subgroup of Γ can be defined this way. Since each homomorphism is defined using the generators, there is a finite number of such homomorphism, and it defines a probability measures on the index n subgroups of Γ, or equivalently, the n-covers of X.

Conjecture 1.6 Assume that Γ is a free group. Then:

1. For every $\epsilon > 0$, the probability that every new eigenvalue λ of an n-cover X' of X satisfies $\lambda \geq 1/4 + \epsilon$ is $1 - o(1)$ as $n \to \infty$.
   An analogous statement for graphs is called Alon's conjecture, and was proved in [9].
2. There exists a 2-cover X' of X, such that every new eigenvalue λ of X' satisfies $\lambda \geq 1/4$. In the graph setting this statement is called Bilu-Linial's conjecture, and was solved for the bipartite case in [27]. A generalization to n-covers for every $n \in \mathbb{N}$ of bipartite graphs was proved in [11].

Remark 1.7 A weaker version of (1) in a slightly different random model were proved in [1].

1.6 Outline of the article

In Sect. 2 we set notations and discuss the harmonic analysis on $\mathbb{H}$, and its relation to the operator A_r and the Laplacian. We also prove Proposition 1.5. In Sect. 3 we prove Theorem 1.2. In Sect. 4 we prove some versions of the central limit theorem for the random walks. For the discrete random walk, we reduce the problem to the standard central limit theorem. For the Brownian motion, this result is well known. In Sect. 5 we prove that after a short time the random walks turns the delta measure on a point to an L^2-function of bounded norm. In Sect. 6 we prove Theorem 1.3.

In the rest of the article, we prove a generalized version of Theorem 1.4. In Sect. 7 we generalize Proposition 1.5 which provides bounds for the spectrum to the non-Ramanujan case. We also give a weak version of Theorem 1.2, which depends on the smallest non-trivial eigenvalue of the Laplacian. In Sect. 8 we discuss covers of a fixed quotient X_0, and in particular normal covers. The condition on the spectra of normal covers is stated somewhat abstractly in Theorem 8.1. However, we then discuss density theorems and known results about them, and show that the density theorems satisfy this condition, thus proving Corollary 8.3, which implies Theorem 1.4.

The paper contains two appendices. In Appendix I, we prove that for every fixed point x_0 on a surface X there exists a distance $R_{x_0,X}$ such that the distances from x_0 to all the other points are concentrated around $R_{x_0,X}$, within a window of a constant size, where the constant depends on the smallest non-trivial eigenvalue of the Laplacian. Theorem 1.2 implies that if X is Ramanujan and x_0 has a lower bound on its injectivity radius, then $R_X \leq R_{X,x_0} \leq R_X + (2+\epsilon)\ln R_X$. The proof involves some interesting isoperimetric inequalities.

In Appendix II, we show that the Brownian random walk on the flat surfaces $\left(a\mathbb{Z}^2\right)\backslash\mathbb{R}^2$ does not exhibit a cutoff as $a \to \infty$.

2 Preliminaries

2.1 The hyperbolic plane

There are several models for the hyperbolic plane $\mathbb{H}$ of constant curvature -1, and we stick to the upper half-plane model. That is the complex half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ endowed with the metric $ds^2 = dz^2/(\mathrm{Im}(z))^2$. The distance $d(z, z')$ between $z = x+iy$, $z' = x' + iy' \in \mathbb{H}$ is

$$d\left(z, z'\right) = \operatorname{acosh}\left(1 + \frac{\left(x' - x\right)^2 + \left(y' - y\right)^2}{2yy'}\right).$$

The group $G = PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by Mobius transformations, i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d},$$

and constitutes the group of orientation preserving isometries of $\mathbb{H}$. Here and elsewhere we identify an element $g \in G = PSL_2(\mathbb{R})$ with its preimage in $G = SL_2(\mathbb{R})$. The group G acts transitively on the points of $\mathbb{H}$, with the subgroup $K = PSO_2(\mathbb{R}) \subset G$ being the stabilizer of the point i, to which we refer as the origin of $\mathbb{H}$. The subgroup K acts on $\mathbb{H}$ by rotations around i. The plane $\mathbb{H}$ can be identified with the quotient G/K, and in particular, the circle of radius r around i identifies with the double coset $K \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} K$. The Haar measure on G which is normalized so that the measure of K is equal to 1 agrees with the standard measure μ on $\mathbb{H}$.

2.2 Harmonic analysis on $\mathbb{H}$

For $f \in L^1(\mathbb{H})$, its Helgason–Fourier transform $\widehat{f}(s,k) \in C(\mathbb{C} \times K)$, is defined as

$$\widehat{f}(s,k) = \int\limits_{\mathbb{H}} f(z)\overline{(\mathrm{Im}(kz))^{\frac{1}{2}+is}}dz,$$

for $s \in \mathbb{C}$ and $k \in K = PSO_2(\mathbb{R})$ whenever the integral exists.

In the case when f is K-invariant, i.e., $f(kz) = f(z)$ for all $z \in \mathbb{H}$ and $k \in K$, its transform is independent of k and can be written with the help of the spherical functions. For every $s \in \mathbb{C}$, the corresponding spherical function is a K-invariant function on $\mathbb{H}$ defined as

$$\varphi_{\frac{1}{2}+is}(z) = \int_K \overline{(\mathrm{Im}\,(kz))^{\frac{1}{2}+is-2}}dk.$$

Since $\varphi_{\frac{1}{2}+is}$ is K-invariant, it depends solely on the hyperbolic distance from a point to the origin i, and can be written as

$$\varphi_{\frac{1}{2}+is}(z) = \varphi_{\frac{1}{2}+is}(ke^{-r}i) = P_{-\frac{1}{2}+is}(\cosh r),$$

where $k \in K$, $r \in \mathbb{R}_{\geq 0}$ is the distance from z to i, and $P_s(r)$ is the Legendre function of the first kind. We also denote $\phi(s,r) = \varphi_{\frac{1}{2}+is}(e^{-r}i)$, and note that for $s \in \mathbb{R}$ ([6, Lemma 7], or [34, Exercise 3.2.28])

$$\phi(s,r) = \frac{\sqrt{2}}{\pi} r \int_0^1 \frac{\cos(srx)}{\sqrt{\cosh r - \cosh rx}}dx.$$

The Helgason–Fourier transform of a K-invariant function f reads as

$$\widehat{f}(s) = \int_{\mathbb{H}} f(z)\varphi_{\frac{1}{2}+is}(z)dz = \int_0^\infty f(e^{-r}i)P_{-\frac{1}{2}+is}(\cosh r)\sinh r dr.$$

For two functions $f_1, f_2 \in L^1(\mathbb{H})$, their convolution is defined as

$$f_1 * f_2(z) = \int_G f_1(gi) f_2(g^{-1}z)dg.$$

We exploit of the following properties of the Helgason–Fourier transform on $\mathbb{H}$. For an extensive presentation of the theory, see [13,34].

Proposition 2.1 ([34, Theorem 3.2.3])

1. *(Plancherel Formula) The map* $f \to \widehat{f}$ *extends to an isometry of* $L^2(\mathbb{H}, d\mu)$ *with* $L^2\left(\mathbb{R}\times K, \frac{1}{4\pi} s\tanh \pi s\, ds dk\right)$*, where* K *is identified with* $\mathbb{R}/\mathbb{Z}$*.*
2. *(Convolution property) For* $f, g \in L^1(\mathbb{H})$*, where* g *is* K*-invariant,*

$$\widehat{f * g} = \widehat{f}\cdot\widehat{g},$$

where $*$ *stands for convolution, and* $\cdot$ *for pointwise multiplication.*

The Helgason–Fourier transform can be extended to compactly supported measures on $\mathbb{H}$. Namely, for such a measure ν, its transform $\widehat{\nu}(s,k) \in C(\mathbb{C}\times K)$, is defined for $s \in \mathbb{C}$ and $k \in K = PSO_2(\mathbb{R})$ as

$$\widehat{\nu}(s,k) = \int_{\mathbb{H}} \overline{(\operatorname{Im}(k(z)))^{\frac{1}{2}+is}} d\nu,$$

and, if the measure is K-invariant, its transform is independent of k, and can be written as

$$\widehat{\nu}(s) = \int_{\mathbb{H}} \varphi_{\frac{1}{2}+is}(z)d\nu.$$

We will need the following claim, which follows from Proposition 2.1.

Corollary 2.2 *Let* ν *be a compactly supported measure on* $\mathbb{H}$*, and assume that* $\widehat{\nu} \in L^2\left(\mathbb{R}\times K, \frac{1}{4\pi} s\tanh \pi s\, dt dk\right)$*. Then* ν *can be represented as an* L^2*-function on* $\mathbb{H}$*, i.e., there exists* $f_\nu \in L^2(\mathbb{H})$ *such that for every* $f \in C_c(\mathbb{H})$*,* $\nu(f) = \int f_\nu(z)f(z)dz$*.*

2.3 The averaging operator A_r

For $r > 0$, let A_r denote the operator on $C(\mathbb{H})$ that averages a function over a circle of radius r, i.e., for a function $f \in C(\mathbb{H})$ and $z = gi \in \mathbb{H}$ $(g \in G)$,

$$(A_r f)(z) = \int_K f\left(gk\begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2}\end{pmatrix} i\right)dk.$$

The operator A_r is bounded and self-adjoint with respect to the L^2-norm on $L^2(\mathbb{H})\cap C(\mathbb{H})$, so it extends to a self-adjoint operator $A_r : L^2(\mathbb{H}) \to L^2(\mathbb{H})$. By duality, we may also extend A_r to an operator on the compactly supported measures on $\mathbb{H}$. Note that the operator A_r can be written as a convolution from the right with a uniform K-invariant probability measure δ_{S_r} supported on the double coset $K\begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}K$, i.e.,

$$A_r f = f * \delta_{S_r}.$$

Also note that the Laplace-Beltrami operator $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ can be written on $C^\infty(\mathbb{H})$ as the limit

$$\Delta = 2\lim_{r\to 0}\frac{1}{r^2}(I - A_r),$$

where I stands for the identity operator. However, we are mainly concerned with the behavior of A_r when r is either fixed or tends to infinity.

The spherical functions $\varphi_{\frac{1}{2}+is}$ on $\mathbb{H}$ are eigenfunctions of Δ and of A_r for every $r > 0$, namely,

$$\Delta\varphi_{\frac{1}{2}+is} = \left(\frac{1}{4} + s^2\right)\varphi_{\frac{1}{2}+is}$$
$$A_r\varphi_{\frac{1}{2}+is} = \varphi_{\frac{1}{2}+is}(e^{-r}i)\cdot\varphi_{\frac{1}{2}+is}.$$

In particular, the following lemma follows from Proposition 2.1:

Lemma 2.3 *The L^2-spectrum of Δ on $\mathbb{H}$ is $\left[\frac{1}{4},\infty\right)$ and the L^2-spectrum of A_r on $\mathbb{H}$ is the set $\left\{\varphi_{\frac{1}{2}+is}(e^{-r}i) \mid s\in\mathbb{R}\right\} = \left\{P_{-\frac{1}{2}+is}(\cosh r) \mid s\in\mathbb{R}\right\}$.*

2.4 Spectrum on the quotients and the Ramanujan condition

Consider the actions of A_r and of Δ on a dense subspace of $L_0^2(\Gamma\backslash\mathbb{H}) = \left\{f \in L^2(\Gamma\backslash\mathbb{H}) : \int f = 0\right\}$, where $\Gamma \subseteq PSL_2(\mathbb{R})$ is a lattice. In both cases the spectrum is not necessarily discrete, but one may still associate to every point of the spectrum a spherical function $\varphi_{\frac{1}{2}+is}$. The value $\frac{1}{2}+is \in \mathbb{C}$ is called a "unitary dual parameter" and the union of all the unitary dual parameters across the spectrum is call the unitary dual of $X = \Gamma\backslash\mathbb{H}$. Namely, if $\frac{1}{2}+is\in\mathbb{C}$ appears in the unitary dual of $X = \Gamma\backslash\mathbb{H}$, then $P_{-\frac{1}{2}+is}(\cosh r)$ is in the spectrum of A_r and $\frac{1}{4}+s^2$ is an eigenvalue of the Laplacian. It is well-known that, in general, the unitary dual of $X = \Gamma\backslash\mathbb{H}$ is contained in the set $\left\{\frac{1}{2}+is \mid s\in\mathbb{R}\right\} \cup \left\{\frac{1}{2}+is \mid is \in \left(-\frac{1}{2},\frac{1}{2}\right)\right\} \cup \{0,1\}$ (see e.g., [24, Section 5.2]). The set $\left\{\frac{1}{2}+is \mid s\in\mathbb{R}\right\}$ is called the principal series, the set $\left\{\frac{1}{2}+is \mid is \in \left(-\frac{1}{2},\frac{1}{2}\right)\right\}$ is called the complementary series, and $\{0, 1\}$ is called trivial. The trivial part corresponds to the constant function on X.

A quotient $X = \Gamma\backslash\mathbb{H}$ is called Ramanujan if its non-trivial unitary dual is contained solely in $\left\{\frac{1}{2}+is \mid s\in\mathbb{R}\right\}$. Equivalently, X is Ramanujan if and only if all the non-trivial eigenvalues of the Laplacian are greater or equal to $\frac{1}{4}$.

The Ramanujan condition can also be stated in terms of representation theory, and in particular the representation of G on $L_0^2(\Gamma\backslash G)$, defined by $(gf)(x) = f(xg)$, for $f \in L_0^2(\Gamma\backslash G)$ and $g \in G$. We avoid giving the full definitions and statements and refer the reader to [24, Chapter 5] for the connection between representation theory and the spectrum of the Laplacian, to [3] for the notion of temperedness, and also to [20, Section VII.11].

A (unitary) representation of G is (ρ, V), where V is a Hilbert space and $\rho : G \to U(V)$ is a continuous group homomorphism into the group of unitary operators of V, with the topology defined by the operator norm. Given a representation (ρ, V), we let $V^K = \{v \in V : \rho(k)(v) = v \forall k \in K\}$. If $\rho(G)V^K$ is dense in V we say that (ρ, V) is generated by its K-fixed vectors. A representation (ρ, V) that is generated by its K-fixed vectors is tempered if for every $v, v' \in V$ which are K-fixed it holds that

$$\int_G \left|\langle v, \rho(g)v' \rangle\right|^{2+\epsilon} dg < \infty.$$

See [3] for the general definition of temperedness.

It is well-known (see, e.g., [24, Chapter 5, and Appendix, Proposition 2.4]) that $X = \Gamma \backslash \mathbb{H}$ is Ramanujan if and only if the subrepresentation of $L_0^2(\Gamma \backslash G)$ which is generated by the $K = PSO_2(\mathbb{R})$-fixed vectors of $L_0^2(\Gamma \backslash G)$, is tempered.

Note that if $f \in L_0^2(X)$ then it can be lifted to a K-fixed function $\tilde{f} \in L_0^2(\Gamma \backslash G)$. Moreover, every K-fixed function in $L_0^2(\Gamma \backslash G)$ can be generated this way.

Lemma 2.4 *Let $f, f' \in L_0^2(X)$, and let $\tilde{f}, \tilde{f}' \in L_0^2(\Gamma \backslash G)$ be their lifts. Let $g \in G$ and let $k_1 \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} k_2$ be a Cartan decomposition of g. Then*

$$\left\langle \tilde{f}, g\tilde{f}' \right\rangle = \langle f, A_r f' \rangle.$$

Proof We first note that if $f'' = A_r f'$, then the lift $\tilde{f}'' \in L_0^2(\Gamma \backslash G)$ satisfies

$$\tilde{f}'' = \int_K k \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \tilde{f}' dk.$$

Since the representation is unitary and $\tilde{f}, \tilde{f}'$ are K-fixed,

$$\begin{aligned} \left\langle \tilde{f}, g\tilde{f}' \right\rangle &= \left\langle \tilde{f}, k_1 \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} k_2 \tilde{f}' \right\rangle \\ &= \left\langle k_1^{-1} \tilde{f}, \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \tilde{f}' \right\rangle = \left\langle \tilde{f}, \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \tilde{f}' \right\rangle. \end{aligned}$$

Since $\tilde{f}$ is K-fixed we have $\tilde{f} = \int_K k^{-1} \tilde{f} dk$. Therefore

$$\begin{aligned} \left\langle \tilde{f}, g\tilde{f}' \right\rangle &= \left\langle \int_K k^{-1} \tilde{f} dk, \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \tilde{f}' \right\rangle \\ &= \left\langle \tilde{f}, \int_K k \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \tilde{f}' dk \right\rangle \\ &= \left\langle \tilde{f}, \tilde{f}'' \right\rangle = \langle f, A_r f' \rangle. \end{aligned}$$

□

We therefore deduce:

Lemma 2.5 *The surface X is Ramanujan if and only if for every $f, f' \in L_0^2(X)$ and for every $\epsilon > 0$,*

$$\int_{r \geq 0} e^r \left|\langle f, A_r f' \rangle\right|^{2+\epsilon} dr < \infty.$$

Proof From the equivalent conditions of temperedness of representations with K-fixed vectors, we know that the surface X is Ramanujan if and only if for every $\tilde{f}, \tilde{f}' \in L_0^2(\Gamma \backslash G)$, it holds that

$$\int_G \left|\left\langle \tilde{f}, g\tilde{f}' \right\rangle\right|^{2+\epsilon} dg < \infty.$$

Now consider the Cartan decomposition $G = \cup_{r\geq 0} K \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} K$, which corresponds to the polar coordinates in $\mathbb{H}$. The measure on the group in this coordinates reads as $dg = \sinh r dk dr$. If $f, f' \in L_0^2(X)$ and $\tilde{f}, \tilde{f}' \in L_0^2(\Gamma \backslash G)$ are their lifts. Then

$$\begin{aligned}\int_G \left|\left\langle \tilde{f}, g\tilde{f}' \right\rangle\right|^{2+\epsilon} dg &= \int_{r\geq 0} \sinh r \left|\left\langle \tilde{f}, \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \tilde{f}' \right\rangle\right|^{2+\epsilon} dr \\ &= \int_{r\geq 0} \sinh r \left|\left\langle f, A_r f' \right\rangle\right|^{2+\epsilon} dr.\end{aligned}$$

Finally, using the fact that for r large $\sinh r \asymp e^r$, and the fact that every $\tilde{f}, \tilde{f}' \in L_0^2(\Gamma \backslash G)$ are lifts of functions $f, f' \in L_0^2(X)$, we arrive to the statement of the lemma. □

2.5 Harish-Chandra bounds

Proposition 2.6 *The spectrum of A_r on $L^2(\mathbb{H})$ is bounded in absolute value by $(r+1)e^{-r/2}$.*

Proof The L^2-spectrum is composed of eigenvalues of A_r on the principal series spherical functions, and hence is equal to the range of the function $\phi(s,r) = \frac{\sqrt{2}}{\pi} r \int_0^1 \frac{\cos(srx)}{\sqrt{\cosh r - \cosh rx}} dx$, for $s \in \mathbb{R}$. Since $\cosh r - \cosh(rx) \geq (\cosh r - 1)\left(1 - x^2\right)$, for $0 \leq x \leq 1$, (which follows from the Taylor expansion of cosh), the following inequalities hold

$$\begin{aligned}|\phi(s,r)| &= \frac{\sqrt{2}}{\pi} r \left| \int_0^1 \frac{\cos(srx)}{\sqrt{\cosh r - \cosh rx}} dx \right| \\ &\leq \frac{\sqrt{2}}{\pi} r \frac{1}{\sqrt{\cosh r - 1}} \int_0^1 \frac{1}{\sqrt{1-x^2}} dx \\ &= \frac{1}{\sqrt{2}} r (\cosh r - 1)^{-1/2} = \frac{r}{2} \left(\sinh \frac{r}{2} \right)^{-1} \leq (r+1)e^{-r/2}.\end{aligned}$$

□

Corollary 2.7 *If X is Ramanujan then the norm of A_r on $L_0^2(X)$ is bounded by $(r+1)e^{-r/2}$.*

The inverse direction can be proven in a similar way, by analyzing the complementary series. Let us present a more conceptual proof of it:

Proposition 2.8 *If for every $r \geq 0$ the norm of A_r on $L_0^2(X)$ is bounded by $(r+1)e^{-r/2}$ then X is Ramanujan.*

Proof If the bound on the norm of A_r holds then for every $f, f' \in L_0^2(X)$ and every $r > 0$ it holds that $\left|\langle f, A_r f'\rangle\right| \le (r+1)e^{-r/2}\left|\langle f, f'\rangle\right|$, so

$$\int_{r\ge 0} e^r \left|\langle f, A_r f'\rangle\right|^{2+\epsilon} dr \le \int_{r\ge 0} e^r e^{(-1-\epsilon/2)r}(r+1)^{2+\epsilon}\left|\langle f, f'\rangle\right|^{2+\epsilon} dr$$
$$= \left|\langle f, f'\rangle\right|^{2+\epsilon}\int_{r\ge 0}(r+1)^{2+\epsilon}e^{-\epsilon r} dr < \infty,$$

and the proposition follows from Lemma 2.5. □

3 Proof of theorem 1.2

Proof of Theorem 1.2. Let $r \le R_X - \gamma \ln(R_X)$. The measure of

$$Y_< = \{x \in X : d(x, x_0) < r\}$$

is at most the volume of the ball of radius r in the hyperbolic plane, i.e.,

$$\mu(Y_<) \le \mu(B_r) \ll e^r \le e^{R_X} e^{-\gamma \ln(R_X)} \ll \mu(X) R_X^{-\gamma}, \tag{2}$$

which implies the lower bound of the theorem (note that we assume that $\mu(X) \asymp e^{R_X}$ since $R_X \ge 1$).

Now let $r' = R_X + \gamma \ln(R_X) - r_0$, and $Y_> = \left\{x \in X : d_X(y, x_0) > r'\right\}$. Let b_{x_0,r_0} be the characteristic function of $B_{x_0}(r_0) \subset X$, normalized as follows

$$b_{x_0,r_0}(x) = \begin{cases} 1/\mu\left(B_{r_0}\right), & x \in B_{x_0}(r_0); \\ 0, & x \notin B_{x_0}(r_0). \end{cases}$$

It is well-defined since x_0 has injectivity radius at least r_0. Then $Y_> \subset Z$ where $Z = \left\{x \in X :\ A_{r'} b_{x_0,r_0}(x) = 0\right\}$. Denote by $\pi \in L^2(X)$ the constant function with $\pi(x) = 1/\mu(X)$ for every $x \in X$. For every point $x \in Z$, one has $\left|\left(A_{r'} b_{x_0,r_0} - \pi\right)(x)\right| = \pi(x) = \frac{1}{\mu(X)}$, so $\mu(Z)\mu^{-2}(X) \le \left\|A_{r'} b_{x_0,r_0} - \pi\right\|_2^2$, and therefore

$$\mu(Y_>) \le \mu(Z) \le \mu^2(X)\left\|A_{r'_n} b_{x_0,r_0} - \pi\right\|_2^2.$$

Since $b_{x_0,r_0} - \pi \perp \pi$ in the space $L^2(X)$, it holds that

$$\left\|b_{x_0,r_0} - \pi\right\|_2 \le \left\|b_{x_0,r_0}\right\|_2 = \mu\left(B_{r_0}\right)^{-1/2} \ll_{r_0} 1.$$

The bounds on the norm of $A_{r'}$ of Proposition 1.5 imply the following inequality

$$\left\|A_{r'} b_{x_0,r_0} - \pi\right\|_2 = \left\|A_{r'}\left(b_{x_0,r_0} - \pi\right)\right\|_2 \le \left(r'+1\right)e^{-r'/2}\left\|b_{x_0,r_0} - \pi\right\|_2$$
$$\ll_{r_0} \left(\frac{R_X + \gamma \ln(R_X) - r_0 + 1}{R_X}\right) R_X e^{-\frac{1}{2}R_X - \frac{1}{2}\gamma \ln(R_X) + \frac{1}{2}r_0}$$
$$\ll_{r_0} (1+\gamma)e^{-R_X/2}R_X^{1-\gamma/2} \ll (1+\gamma)\mu(X)^{-1/2} R_X^{1-\gamma/2}.$$

The following inequality completes the proof

$$\mu(Y_>) \le \mu^2(X)\left\|A_{r'} b_{x_0,r_0} - \pi\right\|_2^2 \ll_{r_0} \mu(X)\left(1+\gamma^2\right)R_X^{2-\gamma}.$$

□

4 Deviations of the random walk

Let $r_1 > 0$ be fixed. Consider the random walk on $\mathbb{H}$, emanating from $z_0 = i$ and having z_{k+1} equidistributed on the sphere of radius r_1 around z_k. In other words, z_k distributes according to the measure $A_{r_1}^k \delta_{z_0}$, where δ_{z_0} is the Dirac delta-measure at z_0. Write $z_k = x_k + y_k i$ for $k \in \mathbb{N} \cup \{0\}$.

Recall that in the upper half-plane model, the points at infinity of $\mathbb{H}$ are $\mathbb{R} \cup \{\infty\}$. In the following lemma we show that the random walk $A_{r_1}^k \delta_{z_0}$ moves away from ∞ at a constant speed.

Lemma 4.1 *Let $f\colon\ [0,\pi] \to [-1,1]$ be the function defined as*

$$f(\theta) = -\frac{1}{r_1} \ln\left(e^{r_1}\cos^2\theta + e^{-r_1}\sin^2\theta\right).$$

Let m be the uniform probability measure on $[0,\pi]$ and let $\nu = f^*m$ be the induced probability measure on $[-1,1]$ (i.e. for $A \subset [-1,1]$, $\nu(A) = m\left(f^{-1}(A)\right)$). Then $\frac{1}{r_1}\ln(y_k)$ distributes according to $\nu * \nu * \ldots * \nu$ (k times). In other words, $\ln(y_k) = \ln(y_{k-1}) + r_1 Y$, where Y is a random variable, independent of y_{k-1}, that distributes according to ν.

Proof One should show that for a given point $z \in \mathbb{H}$, the logarithm of the imaginary part of the measure $A_{r_1}\delta_z$ is distributed according to $\ln\left(\mathrm{Im} z'\right) = \ln(\mathrm{Im} z) - \ln\left(e^{r_1}\cos^2\theta + e^{-r_1}\sin^2\theta\right)$, for $0 \le \theta \le \pi$ equidistributed.

In the case of $z = i$, the sphere of radius r_1 around z can be parameterized as

$$S_{r_1}(i) = \left\{ \begin{pmatrix} e^{\frac{r_1}{2}}\sin\theta & e^{-\frac{r_1}{2}}\cos\theta \\ -e^{\frac{r_1}{2}}\cos\theta & e^{-\frac{r_1}{2}}\sin\theta \end{pmatrix} i = \frac{i + \sin\theta\cos\theta\left(e^{-r_1} - e^{r_1}\right)}{e^{2r_1}\cos^2\theta + \sin^2\theta} \mid \theta \in [0,\pi) \right\}.$$

The logarithm of the imaginary part y' of $z' = A_{r_1} z$ distributes according to $\ln(y') = -\ln\left(e^{r_1}\cos^2\theta + e^{-r_1}\sin^2\theta\right)$.

To prove the claim for points other than $z = i$, notice that an isometry g of $\mathbb{H}$ maps $A_{r_1}\delta_z$ to $A_{r_1}\delta_{g\cdot z}$. Since the action of $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, $s \in \mathbb{R}$ does not change the imaginary coordinate of a point and maps a point $z \in \mathbb{H}$ to $z' = z + s$, if the claim holds for z, it also holds for $z + s$. Similarly, the action of $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, $t \in \mathbb{R}$, maps a point z to $z' = e^t z$, and, in particular, multiplies its imaginary coordinate by e^t, hence if the claim is true for z, it is true for $e^t z$ as well. Therefore it is holds for every point $z \in \mathbb{H}$. □

Corollary 4.2 *The random variables $\sqrt{k}^{-1}\left(k^{-1} r_1 \ln(y_k) + \alpha_{r_1}\right)$ converges in distribution to the normal distribution $N(0, \sigma_{r_1}^2)$, where*

$$\alpha_{r_1} = \frac{1}{\pi r_1}\int_0^\pi \ln\left(e^{r_1}\cos^2\theta + e^{-r_1}\sin^2\theta\right) d\theta,$$

$$\sigma_{r_1}^2 = \frac{1}{\pi}\int_0^\pi \left(\frac{1}{r_1}\ln\left(e^{r_1}\cos^2\theta + e^{-r_1}\sin^2\theta\right) - \alpha_{r_1}\right)^2 d\theta.$$

Also, these numbers satisfies $0 < \alpha_{r_1} < 1$ and that $\sigma_{r_1}^2 \le 4$.

Moreover, the Hoeffding inequality holds: there exist $c > 0$ such that for every $\lambda \ge 0$ and $k \ge 0$

$$Pr\left(|\ln(y_k) + \alpha_r k r_1| \ge \lambda r_1 \sqrt{k}\right) \ll e^{-c\lambda^2}.$$

Proof The statement is a direct application of the central limit theorem and Hoeffding's inequality for independent bounded random variables. The expectancy is equal to α_{r_1} and the variance is equal to $\sigma_{r_1}^2$. The fact that $0 < \alpha_{r_1} < 1$ follows from the fact that logarithm is a concave function. □

The random walk operator A_{r_1} commutes with the action by isometries on $\mathbb{H}$. The stabilizer of i acts transitively on the points at infinity of $\mathbb{H}$. Therefore, just as the random walk $A_{r_1}^k \delta_{z_0}$ moves away from ∞, it moves away from any other point at infinity.

Corollary 4.3 *Let $g \in G$ be an isometry of $\mathbb{H}$ fixing i, then Corollary 4.2 holds if we replace $y_k = Im z_k$ by $Im\,(g \cdot z_k)$, i.e., $\sqrt{k}^{-1}\left(k^{-1} r_1 \ln\left(Im\,(g \cdot z_k)\right) + \alpha_{r_1}\right)$ converges in distribution to the normal distribution $N(0, \sigma_{r_1}^2)$ with α_1 and σ_1^2 as in Corollary 4.2.*

In the following lemma, we make a particular use of the above corollary for the isometry $g : z \mapsto -1/z$.

Lemma 4.4 *There exists $c > 0$ such that $Pr\left(x_k^2 \geq \exp\left(\lambda r_1 k^{1/2}\right)\right) \ll e^{-c\lambda^2}$ for all $\lambda > 0$ and $k \geq 0$.*

Proof By Corollary 4.2 there exists $c_0 > 0$ such that

$$\Pr\left(\left|\ln(y_k) + \alpha_{r_1} k r_1\right| \geq \lambda r_1 \sqrt{k}\right) \ll e^{-c_0 \lambda^2}.$$

By Corollary 4.3 applied for $-\mathrm{Im} z_k^{-1} = \frac{y_k}{x_k^2 + y_k^2}$, there exists $c_1 > 0$ such that

$$\Pr\left(\left|\ln\left(\frac{y_k}{x_k^2 + y_k^2}\right) + r_1 \alpha_{r_1} k\right| \geq \lambda r_1 \sqrt{k}\right) \ll e^{-c_1 \lambda^2},$$

and hence

$$\Pr\left(\left|\ln\left(x_k^2 + y_k^2\right)\right| \geq 2 r_1 \lambda \sqrt{k}\right) \ll e^{-c_0 \lambda^2} + e^{-c_1 \lambda^2}.$$

Therefore there exists $c > 0$ such that

$$\Pr\left(x_k^2 \geq \exp\left(r_1 \lambda \sqrt{k}\right)\right) \leq \Pr\left(x_k^2 + y_k^2 \geq \exp\left(r_1 \lambda \sqrt{k}\right)\right) \ll e^{-c\lambda^2}.$$

□

Corollary 4.5 *Let z_k be distributed according to $A_{r_1}^k \delta_{z_0}$. Then there exists $c = c(r_1) > 0$, such that for every $k \geq 0$ and $\lambda \geq 0$*

$$Pr\left(\left|d\left(z_k, z_0\right) - \alpha_{r_1} r_1 k\right| \geq \lambda \sqrt{k}\right) \ll_{r_1} e^{-c\lambda^2}. \tag{3}$$

Proof Let us start by proving that there exists $c > 0$, such that for $k \geq 0$, $\lambda \geq 0$,

$$\Pr\left(\left|d\left(z_k, z_0\right) - \alpha_{r_1} r_1 k\right| \geq 1 + \lambda r_1 \sqrt{k}\right) \ll e^{-c\lambda^2} \tag{4}$$

For any point $z = x + iy \in \mathbb{H}$, the triangle inequality implies that

$$\begin{aligned} \left|d\left(z, i\right) - d\left(z, x + i\right)\right| &\leq d\left(x + i, i\right) = \operatorname{acosh}\left(1 + \frac{x^2}{2}\right) \\ &\leq \max\left\{1, 1 + 10 \ln\left(x^2\right)\right\}. \end{aligned}$$

Hence by Lemma 4.4 there exists $c_0 > 0$, such that

$$\Pr\left(|d(z,i) - d(z,x+i)| \le 1 + \lambda r_1\sqrt{k}\right) \ll e^{-c_0\lambda^2}. \tag{5}$$

And by Corollary 4.2 there exists $c_1 > 0$, such that

$$\Pr\left(\left|\ln y + \alpha_{r_1} k r_1\right| \ge \lambda r_1\sqrt{k}\right) \ll e^{-c_1\lambda^2}.$$

Since $\left|d(z,x+i) - \alpha_{r_1} k r_1\right| = \left||\ln y| - \alpha_{r_1} k r_1\right|$, if $\left||\ln y| - \alpha_{r_1} k r_1\right| \ge \lambda r_1\sqrt{k}$ then also $\left|\ln y + \alpha_{r_1} k r_1\right| \ge \lambda r_1\sqrt{k}$, and

$$\Pr\left(\left|d(z,x+i) - \alpha_{r_1} k r_1\right| \ge \lambda r_1\sqrt{k}\right) \ll e^{-c_1\lambda^2}, \tag{6}$$

which completes the proof.

Equation 3 follows from Eq. 4, as for $\lambda \ge r_1^{-1}$ and $k > 0$, $1 + \lambda r_1\sqrt{k} \le 2\lambda r_1\sqrt{k}$, and we can choose $c'(r_1) = 2c/r_1$ and choose the constant of $\ll_{r_1}$ in such a way that Eq. 3 holds for $\lambda \le r_1^{-1}$. □

Remark 4.6 One cannot hope to change $\left|d(z_k,z_0) - \alpha_{r_1} k r_1\right| \ge 1 + \lambda r_1\sqrt{k}$ to $\left|d(z_k,z_0) - \alpha_{r_1} k r_1\right| \ge \lambda r_1\sqrt{k}$ in Eq. 4 without assuming the dependency of $\ll$ on r_1, since for $r_1 \to 0$, $k \to \infty$ and $kr_1 \to 0$ the random walk behaves like the distance r_1 random walk in $\mathbb{R}^2$, and in particular it will not diverge at a constant speed.

Note that for $f \in L^2(\mathbb{H})$ ($f \in L^2(X)$, resp.), and for $x \in \mathbb{H}$ ($x \in X$, resp.), the following equality holds

$$A_{r_1}^k f(x) = \int_0^{kr_1} (A_r f)(x) dm_k^{r_1}(r),$$

for some probability measure $m_k^{r_1}$ supported on $[0, kr_1]$ and $k \in \mathbb{N}$.

Corollary 4.7 *There exists $c = c(r_1) > 0$ such that for every $k \ge 0$*

$$\int_{r:\left|r - kr_1\alpha_{r_1}\right| \le \lambda r_1\sqrt{k}} dm_k^{r_1}(r) \ll_{r_1} \exp\left(-c\lambda^2\right).$$

Proof Follows directly from Corollary 4.5. □

In the next section we will prove that the measure m_k for $k \ge 3$ is actually defined by an L^2-function $M(r_1, r)$, and $dm_k^{r_1}(r) = M(r_1, r)\,dr$.

4.1 The Brownian motion

The Brownian motion is the random walk on $\mathbb{H}$ defined by the operator $B_t = \exp(-\Delta t)$. The Brownian motion was studied by many authors, and can be analyzed either by the Helgason–Fourier transform, or by the "distance to infinity" approach used to study the discrete random walk. In any case, based on [2,5], we may write $B_t f(x) = \int p(t,r)(A_r f)(x) dr$, with

$$p(t,r) \asymp \frac{t^{-1} r}{\sqrt{1+r+t}} \exp\left(-\frac{(r-t)^2}{4t}\right) \ll t^{-1} r \exp\left(-\frac{(r-t)^2}{4t}\right).$$

Proposition 4.8 *There exist $c > 0$, $t_0 \geq 0$ such that for every $\lambda > 0$ and every $t > t_0$*

$$\int\limits_{r:|r-t|\geq\lambda\sqrt{t}} p(t,r)dr \ll_{t_0} e^{-c\lambda^2}.$$

Proof We have

$$\int\limits_{r:|r-t|\geq\lambda\sqrt{t}} p(t,r)dr \leq \int\limits_{-\infty}^{-\lambda} p(t,t+\lambda\sqrt{t})d\lambda' + \int\limits_{\lambda}^{\infty} p(t,t+\lambda'\sqrt{t})d\lambda'$$

For $r = t - \lambda'\sqrt{t} \leq t$, we have

$$p\,(t,r) \ll e^{-\frac{\lambda'^2}{4}},$$

so by the standard bound for $\lambda \geq 0$

$$\int\limits_{-\infty}^{-\lambda} e^{-x^2}dx = \int\limits_{-\infty}^{0} e^{-(-\lambda+x)^2}dx \leq e^{-\lambda^2}\int\limits_{-\infty}^{0} e^{-x^2} \ll e^{-\lambda^2},$$

we have

$$\int\limits_{-\infty}^{-\lambda} p(t,t+\lambda'\sqrt{t})d\lambda' \ll e^{-\frac{\lambda^2}{4}}.$$

For $r = t + \lambda'\sqrt{t} \geq t$

$$p\,(t,r) \ll \left(1+\frac{\lambda'}{\sqrt{t}}\right)e^{-\frac{\lambda'^2}{4}},$$

so for $t \geq t_0$

$$\int\limits_{\lambda}^{\infty} p(t,t+\lambda'\sqrt{t})d\lambda' \ll e^{-\frac{\lambda^2}{4}} + \frac{1}{\sqrt{t_0}}\int\limits_{\lambda}^{\infty} \lambda' e^{-\frac{\lambda'^2}{4}}d\lambda'$$

$$\ll_{t_0} e^{-\frac{\lambda^2}{4}} + \left(e^{-\frac{\lambda'^2}{4}}\right)\Bigg|_{\lambda}^{\infty} \ll e^{-\frac{\lambda^2}{4}}.$$

□

5 Short time bound on the random walks

In this section we show that after a short time both random walks on X can be described by an L^2-function of bounded norm, given the injectivity radius of x_0 is bounded away from 0.

It was shown in Sect. 2 that the L^2-spectrum of the operator A_r constitutes of the values of $\varphi_{\frac{1}{2}+is}(e^r i)$ for $s \in \mathbb{R}$. Recall that $\phi(s,r) = \varphi_{\frac{1}{2}+is}(e^{-r} i)$.

Lemma 5.1 *For any r, the following inequality holds*

$$|\phi(s,r)| \ll_r |s|^{-1/2}.$$

Proof Up to a constant, the function $\phi(s,r)$ is equal to $\int_0^1 \frac{\cos(sr(1-x))}{\sqrt{\cosh r - \cosh r(1-x)}}dx$. This function is continuous in s, hence we may assume that $|s|$ is large enough. Write

$$\int_0^1 \frac{\cos(sr(1-x))}{\sqrt{\cosh r - \cosh r(1-x)}}dx = \int_0^{|s|^{-1}} \frac{\cos(sr(1-x))}{\sqrt{\cosh r - \cosh r(1-x)}}dx + \int_{|s|^{-1}}^1 \frac{\cos(sr(1-x))}{\sqrt{\cosh r - \cosh r(1-x)}}dx.$$

Then since $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\cosh r - \cosh r(1-x)}} = c_r > 0$, for $|s|$ large enough we have

$$\left| \int_0^{|s|^{-1}} \frac{\cos(sr(1-x))}{\sqrt{\cosh r - \cosh r(1-x)}}dx \right| \leq \int_0^{|s|^{-1}} \frac{1}{\sqrt{\cosh r - \cosh r(1-x)}}dx \ll_r \int_0^{|s|^{-1}} \frac{1}{\sqrt{x}}dx \ll \frac{1}{\sqrt{|s|}}.$$

Analogously,

$$\int_{|s|^{-1}}^1 \frac{1}{(\cosh r - \cosh r(1-x))^{3/2}}dx \ll_r \sqrt{|s|}.$$

Write $G(x) = -\frac{1}{sr}\sin(sr(1-x))$ and $F(x) = 1/\sqrt{\cosh r - \cosh r(1-x)}$, then by integration by parts,

$$\int_{|s|^{-1}}^1 G'(x)F(x)dx = G(1)F(1) - G\left(|s|^{-1}\right)F(|s|^{-1}) - \int_{|s|^{-1}}^1 G(x)F'(x)dx,$$
$$= -G\left(|s|^{-1}\right)F(|s|^{-1}) - \int_{|s|^{-1}}^1 G(x)F'(x)dx$$

and hence,

$$\left| \int_{|s|^{-1}}^1 \frac{\cos(sr(1-x))}{\sqrt{\cosh r - \cosh r(1-x)}}dx \right| \ll \left|\frac{1}{sr}\right| \left| \frac{1}{\sqrt{\cosh r - \cosh r\left(1-|s|^{-1}\right)}} \right| + \int_{|s|^{-1}}^1 \frac{1}{|s|\,r\,(\cosh r - \cosh r(1-x))^{3/2}}dx$$
$$\ll_r |s|^{-1}|s|^{1/2} + \frac{1}{|s|}\cdot\sqrt{|s|} \ll \frac{1}{\sqrt{|s|}},$$

which completes the proof. □

Lemma 5.2 *For any $x_0 \in \mathbb{H}$, we have $A_{r_1}^3 \delta_{x_0} \in L^2(\mathbb{H})$.*

Proof By Theorem 2.1, the Helgason–Fourier transform of $A_{r_1}^3$ satisfies $\widehat{A_{r_1}^3(s)} = \left(\widehat{A_{r_1}}(s)\right)^3 = \phi^3(s, r_1)$. Applying Lemma 5.1, and using the fact that $\tanh(\pi s) \le 1$ implies that

$$\int_{-\infty}^{\infty} \left|\widehat{A_{r_1}^3(s)}\right|^2 \frac{s}{4\pi} \tanh(\pi s)\, ds = \int_{-\infty}^{\infty} |\phi(s, r_1)|^6 \frac{s}{4\pi} \tanh(\pi s)\, ds$$

$$\ll_r 1 + \int_{|s|>1} |s|^{-3} |s|\, ds < \infty.$$

By the inverse Fourier transform we conclude by Corollary 2.2 that $A_{r_1}^3 = \int_r f(r) A_r dr$, with $f(r)$ an L^2-function on $\mathbb{R}_{\ge 0}$. In particular, $\left\|A_{r_1}^3 \delta_{x_0}\right\|_2 = \int_r |f(r)|^2 dr < \infty$, as needed. □

Remark 5.3 For $k = 0, 1, 2$ the analogous statement is not true. For $k = 0, 1$, $A_{r_1}^k \delta_{x_0}$ cannot be considered as a function. For $k = 2$, $A_{r_1}^2 = \int_0^{r_1} g(r) A_r dr$, where $g(r)$ is a function on $\mathbb{R}_{\ge 0}$, but not an L^2-function.

Lemma 5.4 *For $k_0 \ge 3$ (respectively for $t_0 > 0$) there exists a constant $C = C(r_0, r_1, k_0)$ (resp. $C = C(r_0, t_0)$) such that if $x_0 \in X$ has an injectivity radius at least r_0 then $A_{r_1}^{k_0} \delta_{x_0} \in L^2(X)$ and $\left\|A_{r_1}^{k_0} \delta_{x_0}\right\|_2 \le C$ (resp. $B_{t_0} \delta_{x_0} \in L^2(X)$ and $\left\|B_{t_0} \delta_{x_0}\right\|_2 \le C$).*

Proof We start with the discrete random walk $A_{r_1}^{k_0} \delta_{y_0}$. Since $\left\|A_{r_1}\right\|_2 \le 1$ it is enough to assume that $k_0 = 3$.

Let $y_0 \in \mathbb{H}$ be a fixed point covering $x_0 \in X$. Let $x_1 \in X$ be a point different from x_0. Denote by $B(y_0, r)$ the ball of radius r around y, and by $B(r)$ some ball of radius r. We claim that it has a bounded number $D \ll_{r_0, k_0, k_1} 1$ of points $z_1, \dots, z_D \in B(y_0, k_0 r_0)$ covering x_1. Since $A_{r_1}^{k_0} \delta_{y_0} \in L^2(\mathbb{H})$, it is supported on $B(y_0, k_0 r_0)$ and $A_{r_1}^{k_0} \delta_{x_0}$ is the push-forward of $A_{r_1}^{k_0} \delta_{y_0}$ to X, this claim will give the lemma for the discrete random walk. We may assume that $d_0 = d(x_0, x_1) < k_0 r_1$. Let therefore $z_1, z_2, \dots, \in B(y_0, k_0 r_0)$ be a sequence of different points covering x_1. Then each such point $z_i \in \mathbb{H}$ can be associated with another point $y_i \in \mathbb{H}$, covering x_0, with $d(y_i, z_i) = d_0$. Moreover, we may choose y_i such that $y_i \ne y_j$ for $z_i \ne z_j$. By the injectivity radius assumption, $d\left(y_i, y_j\right) \ge 2r_0$ for $i \ne j$. All the y_i's are contained in the ball $B(y_0, 2k_0 r_0)$, and their number is therefore bounded by $\frac{\mu(B(2k_0 r_0))}{\mu(B(r_0))} \ll_{r_0, k_0, k_1} 1$.

Now we turn to the Brownian motion. Since $\|B_t\|_2 \le 1$ and $B_{t+t'} = B_t B_{t'}$ we may assume that t_0 is small enough so that $p_2(r, t_0)$ is decreasing for $r > r_0$ and $B_{t_0} \delta_{y_0}(z) \le e^{-cd(y_0, z)^2}$ for some $c = c(r_0, t_0) > 0$ and $d(y_0, z) > r_0$.

Let $y_0 \in \mathbb{H}$ be again a fixed point covering $x_0 \in X$. Let $x_1 \in X$ be another point and let $d_1 = d(x_0, x_1)$. Each point z_i covering x_1 satisfies $d(y_0, z_i) \ge d_1$ and the number of points z_i covering x_1 of distance $d(y_0, z_i) \le r$ is at most $D_r \le \frac{\mu\left(B_{r+d_1}\right)}{\mu\left(B_{r_0}\right)} \ll_{r_0} e^{d_1 + r}$. Therefore we get the bound:

$$B_{t_0} \delta_{x_0}(x) = \sum_{z_i} B_{t_0} \delta_{y_0}(z_i) = \sum_{k=0}^{\infty} \sum_{z_i : d_1 + r_0 k \le d(y_0, z_i) \le d_1 + r_0(k+1)} B_{t_0} \delta_{y_0}(z_i)$$

$$\leq \sum_{k=0}^{\infty} D_{d_1+r_0(k+1)} \cdot e^{-c(d_1+r_0k)^2} \ll_{r_0} \sum_{k=0}^{\infty} e^{d_1+d_1+r_0(k+1)-c(d_1+r_0k)^2}$$
$$\ll_{r_0,t_0} e^{2d_1-c'd_1^2}.$$

For some constant $c' > 0$ depending on r_0, t_0. Finally, using the fact that the volume of $x \in X$ with $d(x_0, x) \leq d_1$ is $\ll e^{d_1}$,

$$\| B_{t_0}\delta_{x_0}\|_2^2 = \int_X \left|B_{t_0}\delta_{x_0}(x)\right|^2 dx \ll_{t_0,r_0} \int_{d_1\geq 0} e^{2\left(2d_1-c'd_1^2\right)} \cdot e^{d_1} dt \ll_{t_0,r_0} 1,$$

and the lemma is proved. □

6 Proof of theorem 1.3

The following theorem is the full statement of Theorem 1.3.

Theorem 6.1 *Fix $r_0 > 0$, $r_1 > 0$, $\lambda > 0$ and a point $x_0 \in X$. Assume $R_X \gg_{r_0,r_1,\lambda} 1$ and write*

$$\alpha = \frac{1}{\pi r_1} \int_0^{\pi} \ln\left(e^{r_1} \cos^2\theta + e^{-r_1} \sin^2\theta\right) d\theta \in (0, 1).$$

1. *There exist constants $c = c(r_1) > 0$, and $C = C(r_0, r_1)$, such that*
   (a) *If k satisfies $k\alpha r_1 < R_X - \lambda\sqrt{R_X}$ then $\left\|A_{r_1}^k\delta_{x_0} - \pi\right\|_1 > 2 - Ce^{-c\lambda^2}$;*
   (b) *If k satisfies $k\alpha r_1 > R_X + \lambda\sqrt{R_X}$, if $X = \Gamma\backslash\mathbb{H}$ is Ramanujan, and if x_0 has injectivity radius at least r_0, then $\left\|A_r^k\delta_{x_0} - \pi\right\|_1 < Ce^{-c\lambda^2}$;*
2. *There exist constants $c > 0$, $C = C(r_0)$ such that*
   (a) *If t satisfies $t < R_X - \lambda\sqrt{R_X}$ then $\left\|B_t\delta_{x_0} - \pi\right\|_1 > 2 - Ce^{-c\lambda^2}$;*
   (b) *If t satisfies $t > R_X + \lambda\sqrt{R_X}$, if $X = \Gamma\backslash\mathbb{H}$ is Ramanujan, and if x_0 has injectivity radius at least r_0, then $\left\|B_t\delta_{x_0} - \pi\right\|_1 < Ce^{-c\lambda^2}$;*

   for every $\lambda > 0$, assuming $R_X \gg_{r_0,\lambda} 1$.

We prove the theorem for the discrete random walk only. The proof for the Brownian motion is analogous, and exploits Corollary 4.8 instead of Corollary 4.7 and the Brownian motion part of Lemma 5.4 instead of its discrete part.

Proof Suppose that $rk\alpha < R_X - \lambda\sqrt{R_X}$. Let $Y = \left\{y \in X : d(x_0, y) > R_X - \frac{\lambda}{2}\sqrt{R_X}\right\}$. As $R_X \to \infty$,

$$1 \geq \mu(Y)/\mu(X) \geq \left(\mu(X) - \mu\left(B\left(R_X - \frac{\lambda}{2}\sqrt{R_X}\right)\right)\right)/\mu(X) \to 1,$$

so $\mu(Y)/\mu(X) \to 1$ (Recall that $B(r)$ stands for the hyperbolic ball of radius r).

By Corollary 4.7, there exists $c_1(r_1)$, $C_1(r_1)$ such that for R_X large enough,

$$\int_Y \left| A_r^k \delta_{x_0}(x) \right| d\mu \le C_1 e^{-c_1\lambda^2}$$

$$\int_{X-Y} \left| A_r^k \delta_{x_0}(x) \right| d\mu \ge 1 - C_1 e^{-c_1\lambda^2}.$$

Therefore,

$$\begin{aligned}
\left\| A_r^k b_{x_0,r_0} - \pi \right\|_1 &= \int_Y \left| A_r^k \delta_{x_0}(x) - \pi(x) \right| d\mu + \int_{X-Y} \left| A_r^k \delta_{x_0}(x) - \pi(x) \right| d\mu \\
&\ge \int_Y |\pi(x)|\, d\mu - \int_Y \left| A_r^{k_n} \delta_{x_0}(x) \right| d\mu + \int_{X-Y} \left| A_r^k \delta_{x_0}(x) \right| - \int_{X-Y} |\pi(x)|\, d\mu \\
&\ge \mu(X)^{-1} \mu(Y) - C_1 e^{-c_1\lambda^2} + 1 - C_1 e^{-c_1\lambda^2} - \mu(X)^{-1} \mu(X-Y) \\
&= 2\mu(X)^{-1} \mu(Y) - 2C_1 e^{-c_1\lambda^2},
\end{aligned}$$

and the first bound follows by letting $R_X \to \infty$. Notice that it does not require the Ramanujan assumption.

For the second bound, recall that we may write

$$A_{r_1}^k b_{x_0,r_0}(x) = \int_r \left(A_r b_{x_0,r_0} \right)(x) dm_k(r).$$

Assume that $kr_1\alpha > R_X + \lambda\sqrt{R_X}$. By Corollary 4.7, for some $c_2(r_1) > 0$, for R_X large enough (depending on r_1, λ),

$$\int_{r < R_X + \frac{\lambda}{2}\sqrt{R_X}} dm_k r \ll_{r_1} e^{-c_2\lambda^2}.$$

As $x_0 \in X$ has an injectivity radius at least r_0, by Lemma 5.4, there exists a constant $C_3 = C(r_0, r_1)$ such that $\left\| A_{r_1}^3 \delta_{x_0} \right\|_2 \le C_3$.

For every $f \in L_2(X)$, Cauchy– Schwartz inequality implies that $\|f\|_1 \le \sqrt{\mu(X)}\, \|f\|_2$. Writing $R_0 = R_X + \frac{\lambda}{2}\sqrt{R_X}$, we therefore have,

$$\begin{aligned}
\left\| A_{r_1}^k f - \pi \right\|_1 &\le \int_r \left\| A_r \delta_{x_0} - \pi \right\|_1 dm_k r \\
&= \int_{r<R_0} \left\| A_r b_{x_0,r_0} - \pi \right\|_1 dm_k r + \int_{r \ge R_0} \left\| A_r b_{x_0,r_0} - \pi \right\|_1 dm_k r \\
&\le \int_{r<R_0} 2 dm_k r + \int_{r \ge R_0} \mu(X)^{1/2} \left\| A_r \left(b_{x_0,r_0} - \pi \right) \right\|_2 dm_k r \\
&\ll_{r_1} e^{-c_2\lambda^2} + \int_{r \ge R_0} \mu(X)^{1/2} (r+1) e^{-r/2} dm_k r \\
&\ll e^{-c_2\lambda^2} + \mu(X)^{1/2} (R_0 + 1) e^{-\frac{1}{2}(R_0)}
\end{aligned}$$

$$\ll e^{-c_2\lambda^2} + \mu(X)^{1/2} e^{-\frac{1}{2}R_X} (R_0+1) e^{-\frac{1}{4}\sqrt{R_X}}$$
$$\to_{R_X\to\infty} e^{-c_2\lambda^2},$$

and the second bound follows. □

Remark 6.2 The theorem holds for $\lambda > 0$ such that $R_X \gg_{r_0,r_1,\lambda} 1$. In other words, R_X has to be larger than some constant $R(r_0, r_1, \lambda)$ that depends on r_0, r_1 and λ. By fixing r_0, r_1, one can find the relation between this constant and λ, namely, it should hold that $\lambda = o\left(\sqrt{R(r_0, r_1, \lambda)}\right)$ and $\ln(R(r_0, r_1, \lambda)) = o\left(\lambda\sqrt{R(r_0, r_1, \lambda)}\right)$.

7 L^p-bounds

The above results assume X to be Ramanujan. However, similar results can be proved in a more general setting. In this section and the next one, we discuss Theorem 1.2 only, but similarly one can elaborate on Theorem 1.3 as well.

The following lemma is well-known (see [21]):

Lemma 7.1 *The spectrum of the Δ on $L_0^2(X)$ below $1/4$ is discrete, and corresponds to a finite number of eigenvalues with multiplicities.*

Eigenvalues of Δ strictly below $1/4$ are called exceptional. A nice way to measure how far is a representation V of G from being tempered is to ask what is the minimal $p \geq 2$ such that the K-finite matrix coefficients of G on V lie in $L^p(G)$. The following proposition relates this property to the spectra of the Laplacian and of the operators A_r. See [19] for the corresponding result on graphs.

Proposition 7.2 *The following are equivalent for $p \geq 2$:*

1. *For every $r \geq 0$, the norm of A_r on $L_0^2(X)$ is bounded by $(r+1)e^{-r/p}$.*
2. *Every matrix coefficient of a subrepresentation of G on $L_0^2(\Gamma\backslash G)$ with K-fixed vectors is in $L^{p+\epsilon}(G)$ for every $\epsilon > 0$.*
3. *The spectrum of Δ on $L_0^2(X)$ is bounded from below by $\frac{1}{4} - \left(\frac{1}{2} - p^{-1}\right)^2$.*

The equivalence is also true for every Δ-invariant closed subspace $V \subset L_0^2(X)$, where in (2) *we look at the G-subrepresentation generated by V.*

Proof Let V be a Δ-invariant closed subspace of $L_0^2(X)$.

The complementary series is determined by a real parameter t, $0 \leq t \leq 1$, with corresponding spherical function φ_t. Write $t = \frac{1}{2} + s_t'$ and $p_t = \left(\frac{1}{2} - |s_t'|\right)^{-1}$ (with the convention that $0^{-1} = \infty$). The corresponding eigenvalue of the Δ on φ_t is

$$\lambda_t = t(1-t) = \frac{1}{4} - s_t'^2 = \frac{1}{4} - \left(\frac{1}{2} - p_t^{-1}\right)^2.$$

The eigenvalue of A_r on φ_t is

$$\varphi_t(r) = \frac{1}{\sqrt{2}\pi} r \int_{-1}^{1} \frac{\exp(s_t' r x)}{\sqrt{\cosh r - \cosh rx}} dx.$$

Recall that for $r > 0$, $\cosh r - \cosh(rx) \geq (\cosh r - 1)\left(1 - x^2\right)$ holds. Hence

$$|\varphi_t(r)| = \frac{1}{\pi\sqrt{2}} r \left| \int_{-1}^{1} \frac{\exp\left(s_t' r x\right)}{\sqrt{\cosh r - \cosh rx}} dx \right| \leq \frac{1}{\pi\sqrt{2}} r \frac{1}{\sqrt{\cosh r - 1}} \int_{-1}^{1} \frac{\exp\left(s_t' r x\right)}{\sqrt{1 - x^2}} dx$$

$$\leq \frac{1}{\sqrt{2}\pi} r \frac{\exp\left(\left|s_t'\right| r\right)}{\sqrt{\cosh r - 1}} \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} dx = \frac{1}{\sqrt{2}} r\left(\cosh r - 1\right)^{-1/2} \exp\left(\left|s_t'\right| r\right)$$

$$\leq (r+1) e^{-r\left(\frac{1}{2} - \left|s_t'\right|\right)} = (r+1) e^{-r/p_t}.$$

This proves an implication from (3) to (1).

We also want to give a lower bound on $|\varphi_t(r)|$. Let $f(x) = \cosh r - \cosh(r(1-x))$. For $x \geq 0$, by the Taylor series

$$f(x) = xr \sinh r - \frac{1}{2} r^2 x^2 \cosh\left(r\left(1 - x'\right)\right), \text{ for some } 0 \leq x' \leq x,$$

so $f(x) \leq xr \sinh r$. Hence for a fixed $\epsilon > 0$, and $r \geq 1$,

$$\varphi_t(r) = \frac{1}{\sqrt{2}\pi} r \int_{-1}^{1} \frac{\exp\left(s_t' r x\right)}{\sqrt{\cosh r - \cosh rx}} dx \gg r \int_{0}^{2} \frac{\exp\left(s_t' r(1-x)\right)}{\sqrt{\cosh r - \cosh r(1-x)}} dx$$

$$\geq r \int_{0}^{\epsilon} \frac{\exp\left(s_t' r(1-x)\right)}{\sqrt{\cosh r - \cosh r(1-x)}} dx$$

$$\geq \frac{\sqrt{r} e^{s_t' r}}{\sqrt{\sinh r}} \int_{0}^{\epsilon} \frac{\exp\left(-s_t' r x\right)}{\sqrt{x}} dx \gg \frac{e^{s_t' r(1-\epsilon)}}{\sqrt{\sinh r}} \sqrt{\epsilon} \gg \sqrt{\epsilon} e^{-r\left(\frac{1}{2} - \left|s_t'\right|(1-\epsilon)\right)}. \tag{7}$$

This implies that if for every $r \geq 0$, $\varphi_t(r) \leq (r+1) e^{-r\left(\frac{1}{2} - S\right)}$ then $\left|s_t'\right| \leq S$. This proves the implication from (1) to (3).

Arguing as in Lemma 2.5, we see that (2) is equivalent to:

- For every $f, f' \in V$ and for every $\epsilon > 0$,

$$\int_{r \geq 0} e^r \left|\left\langle f, A_r f'\right\rangle\right|^{p+\epsilon} dr < \infty \tag{8}$$

We can immediately see that as in Proposition 2.8, this proves the implication from (1) to (2).

Assume now that (1) and (3) do not hold for $p = p_0 \geq 2$. By Lemma 7.1, there is an eigenvector $f \in V$ which satisfies $\langle f, A_r f\rangle = \varphi_t(r)$, for some φ_t, with $p_t > p_0$. By Eq. 7, for some $\delta > 0$ and for r large enough $|\langle f, A_r f\rangle| \gg_\delta e^{-r(1/p_0 + \delta)}$. Then Eq. 8 does not hold, and (2) does not hold. □

By Lemma 7.1, for each X there is a minimal p_0 satisfying the equivalent conditions of Proposition 7.2. Denote it by $p_0(X)$. For example, Selberg's lower bound $3/16$ implies that for each X corresponding to a congruence subgroup of $SL_2(\mathbb{Z})$, $p_0(X) \leq 4$. Further progress towards Selberg's conjectrure (see, for example, [31]) improves this bound as well. Without any additional assumption on X, we can say the following:

Theorem 7.3 *Let $r_0 > 0$ be fixed. Let $p = p_0(X)$ and assume $R_X \geq 1$. Let $x_0 \in X$ be a point with injectivity radius at least r_0. Then for every $\gamma > 0$*

$$\mu_X\left(x \in X : d_X(x, x_0) \geq \frac{p}{2}(R_X + \gamma \ln(R_X))\right)/\mu(X) \ll_{p,r_0} \left(1+\gamma^2\right) R_X^{2-\gamma}.$$

Proof The proof is essentially the same as the proof of Theorem 1.2, and we only write the differences. Instead of choosing $r' = R_X + \gamma \ln(R_X) - r_0$, choose $r' = \frac{p}{2}(R_X + \gamma \ln(R_X)) - r_0$. Then:

$$\begin{aligned}\left\| A_{r'} b_{x_0,r_0} - \pi \right\|_2 = \left\| A_{r'}\left(b_{x_0,r_0} - \pi\right)\right\|_2 &\leq \left(r'+1\right) e^{-r'/p} \left\| b_{x_0,r_0} - \pi \right\|_2 \\ &\ll_{r_0} \left(r'+1\right) e^{-\frac{1}{2}R_X - \frac{1}{2}\gamma \ln(R_X) + r_0/p} \\ &\ll_{r_0,p} \frac{\frac{p}{2}(R_X + \gamma \ln(R_X)) - r_0}{R_X} \mu(X)^{-1/2} e^{-\frac{1}{2}(\gamma-2)\ln(R_X)} \\ &\ll_{r_0,p} (1+\gamma)\mu(X)^{-1/2} e^{-\frac{1}{2}(\gamma-2)\ln(R_X)}.\end{aligned}$$

The rest of the proof is the same as in Theorem 1.2. □

8 Covers

Let $X_0 = \Gamma_0 \backslash \mathbb{H}$. Then a finite index subgroup $\Gamma_X < \Gamma_0$ defines a cover $X = \Gamma_X \backslash \mathbb{H}$ of X_0, with cover map $\rho\colon X \to X_0$. The pull-back $\rho^*\colon L^2(X_0) \to L^2(X)$ defines a closed subspace $\rho^* L^2(X_0) \subset L^2(X)$. Denote the orthogonal complement of $\rho^* L^2(X_0)$ in $L^2(X)$ by $L^2(X/X_0)$.

For $p > 2$, denote by $m(X, p)$ the dimension of the space spanned by eigenvectors of $L^2(X/X_0)$ whose matrix coefficients are not in $L^{p'}$ for every $p' \leq p$ but are in $L^{p'}$ for $p' > p$. Denote also $M(X, p) = \sum_{p' \geq p} m\left(X, p'\right)$.

A cover $\rho\colon X \to X_0$ is called normal if $\Gamma_X \lhd \Gamma_{X_0}$ is a normal subgroup. Equivalently, a cover $\rho\colon X \to X_0$ is normal if there exists a group H acting on X such that $\rho(x) = \rho(y)$ if and only if x and y are on the same H-orbit. We call H the cover group.

Our main result about covers is as the following theorem. Note that if X is an N-cover of X_0 (that is, $[\Gamma_0 : \Gamma_X] = N$) then $\mu(X) = N \cdot \mu(X_0)$. Therefore, $\mu(X) \asymp_{X_0} N$ and $R_X = \ln(N) + O_{X_0}(1)$.

Theorem 8.1 *Let $r_0 > 0$ be fixed, and let X_0 be a fixed quotient. Let $\rho_q : X_q \to X_0$ be family of normal N_q-covers, with $N_q \to \infty$ as $q \to \infty$.*

Assume that $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing function satisfying:

1. *For some fixed $\delta > 2$ and for R large enough, $g(R) \geq R + \delta \ln R$;*
2. *and either*

$$g^3\left(\ln\left(N_q\right)\right) \sum_{p: m\left(X_q, p\right) \neq 0} e^{-2g\left(\ln\left(N_q\right)\right)/p_i} m\left(X_q, p_i\right) = o(1), \tag{9}$$

or

$$g^3\left(\ln\left(N_q\right)\right) \int_2^\infty M\left(X_q, p\right) e^{-2g\left(\ln\left(N_q\right)\right)/p} p^{-2} dp = o(1), \text{ and} \tag{10}$$

$$g^2\left(\ln\left(N_q\right)\right) \lim_{p \to 2, p > 2} M\left(X_q, p\right) e^{-g\left(\ln\left(N_q\right)\right)} = o(1); \tag{11}$$

For every q, let $x_0^{(q)} \in X_q$ be a point such that its projection $\rho_q(x_0^{(q)})$ to X_0 has injectivity radius at least r_0. Then

$$\mu\left(x \in X_q : d_{X_q}\left(x, x_0^{(q)}\right) \geq g\left(\ln\left(N_q\right)\right)\right)/\mu\left(X_q\right) = o\left(1\right),$$

where the implied constant depends on X_0, $\left\{X_q\right\}$, r_0 and g.

Before proving the theorem, let us study its corollaries.

Definition 8.2 We say that a family of covers $\left\{X_q\right\}$ of X_0 satisfies a *density condition with parameter* $A \in \mathbb{R}$ if for every $\epsilon > 0$, for each $p > 2$,

$$M\left(X, p\right) \ll_{\epsilon, \{X_q\}, X_0} CN^{1-A(p-2)/p+\epsilon},$$

and furthermore

- The number of exceptional eigenvalues $\lim_{p\to 2, p>2} M\left(X, p\right) = \sum_{p>2} m\left(X, p\right)$ of X_q is $\ll_{\{X_q\}, X_0} N$.
- There exists $p_{\max}$ such that $M\left(X, p_{\max}\right) = 0$.

The assumption that the number of exceptional eigenvalues is $O(N)$ is well-known to hold in the arithmetic case (see [30]). There are two main instances of such density results:

1. The case $A = 1$: in this case we may simply write $M\left(X, p\right) \ll_{\epsilon, X_0} N^{2/p+\epsilon}$. This is known to hold for a wide range of cases, including the principal congruence subgroups of $SL_2\left(\mathbb{Z}\right)$ and all cocompact arithmetic lattices in $SL_2\left(\mathbb{R}\right)$ (See [30,32] for the uniform case and [15] for $SL_2\left(\mathbb{Z}\right)$). In this case, for prime congruence, one may find $p_{\max}$ by using lower bounds on the dimensions of representations of $SL_2\left(F_q\right)$ (see [32]). The corresponding result for LPS graphs are implicitly contained in [4, Section 4.4]
2. The case $A > 1$: this case requires deeper results in analytic number theory and the results mainly apply to congruence subgroups of $\Gamma = SL_2\left(\mathbb{Z}\right)$. Iwaniec (see [17]) proved density with $A = 2$, but only for, [14] and the references therein for recent results.

Corollary 8.3 *Let $\rho\colon X_q \to X_0$ be family of normal N_q-covers, with $N_q \to \infty$. Assume the family satisfies a density condition with parameter $A \geq 1$.*

Let $x_0^{(q)} \in X_q$ be a point such that its projection $\rho_q(x_0^{(q)})$ to X_0 has injectivity radius at least r_0. Then for every $\epsilon_0 > 0$

$$\mu\left(x \in X_q : d_{X_q}\left(x, x_0^{(q)}\right) \geq R_{X_q}\left(1+\epsilon_0\right)\right)/\mu\left(X_q\right) = o\left(1\right).$$

Proof One should verify Inequalities 10 and 11 for $g(R) = (1+\varepsilon_0)R$. We may assume $A = 1$.

For $\epsilon > 0$ small enough with respect to ϵ_0 it holds that

$$g^3\left(\ln\left(N_q\right)\right)\int_2^\infty M\left(X, p\right)e^{-2g\left(\ln\left(N_q\right)\right)/p}p^{-2}dp$$

$$\ll_{\epsilon, \{X_q\}, X_0} \left(1+e_0\right)^3\ln^3\left(N_q\right)\int_2^{p_{\max}} N_q^{2/p+\epsilon}e^{-2(1+\epsilon_0)\ln\left(N_q\right)/p}p^{-2}dp$$

$$\ll_{\epsilon_0} \ln^3\left(N_q\right)\int_2^{p_{\max}} N_q^{2/p+\epsilon}N_q^{-2(1+\epsilon_0)/p}p^{-2}dp$$

$$= \ln^3\left(N_q\right)\int_2^{p_{\max}} N_q^{\epsilon-2\epsilon_0/p_{\max}} p^{-2}dp$$
$$\ll \ln^3\left(N_q\right) N_q^{\epsilon-2/p_{\max}\epsilon_0} \to_{N_q\to\infty} 0.$$

In addition,

$$g^2\left(\ln\left(N_q\right)\right)\lim_{p\to 2, p>2} M\left(X_q, p\right)e^{-g\left(\ln\left(N_q\right)\right)}$$
$$\ll_{\{X_q\},X_0} (1+\epsilon_0)^2\ln^2\left(N_q\right)N_q^{1-(1+\epsilon_0)}$$
$$\ll_{\epsilon_0} \ln^2\left(N_q\right)N_q^{\epsilon_0} \to_{N_q\to\infty} 0.$$

□

Let us turn to the proof of Theorem 8.1. It will depend on the following two lemmas.

Lemma 8.4 *Let $\rho\colon X \to X_0$ be an N-cover, $U = \rho^* L_0^2\left(X'\right) \subset L_0^2(X)$ be the space of functions pulled back from X_0 to X and let P_U be the orthogonal projection onto U. Let $x_0 \in X$ be a point such that its projection to X_0 has injectivity radius at least r_0. Then*

$$\left\| P_U\left(b_{x_0,r_0}\right)\right\|_2 = N^{-1/2}\left\|b_{x_0,r_0}\right\|_2.$$

Proof We have

$$\left\| P_U\left(b_{x_0,r_0}\right)\right\|_2 = \max_{u\in U,\|u\|_2=1}\left\langle u, b_{x_0,r_0}\right\rangle = \max_{u'\in L^2(x'),\|\rho^*u'\|_2=1}\left\langle \rho^* u', b_{x_0,r_0}\right\rangle.$$

But $\left\|\rho^*u'\right\|_2^2 = N\left\|u'\right\|_2^2$ and $\left\langle\rho^*u', b_{x_0,r_0}\right\rangle = \left\langle u', b_{\rho(x_0),r_0}\right\rangle$. So

$$\left\| P_U\left(b_{x_0,r_0}\right)\right\|_2 = \max_{u'\in L^2(x'),\|u'\|_2=N^{-1/2}}\left\langle u', b_{\rho(x_0),r_0}\right\rangle$$
$$= N^{-1/2}\left\|b_{\rho(x_0),r_0}\right\|_2 = N^{-1/2}\left\|b_{x_0,r_0}\right\|_2.$$

□

Lemma 8.5 *Let $\rho\colon X \to X_0$ be a normal N-cover, with cover group H. Let $W \subset L^2(X)$ be a finite dimensional H-invariant subspace and P_W the orthogonal projection onto this subspace. Let $x_0 \in X$ be a point such that its projection to X_0 has injectivity radius at least r_0. Then*

$$\left\| P_W\left(b_{x_0,r_0}\right)\right\|_2 \le \sqrt{\frac{\dim W}{N}}\left\|b_{x_0,r_0}\right\|_2.$$

Proof Let $u_1, \ldots, u_{\dim W}$ be an orthonormal basis of W. Then

$$\left\| P_W\left(b_{x_0,r_0}\right)\right\|_2^2 = \sum_{i=1}^{\dim W}\left|\left\langle u_i, b_{x_0,r_0}\right\rangle\right|^2.$$

On the other hand, the points hx_0, where $h \in H$, are all distinct, the balls $B_r(hx_0)$ of radius r_0 around them are disjoint, and since W is H-invariant for each $h \in H$

$$\left\| P_W\left(b_{hx_0,r_0}\right)\right\|_2^2 = \left\| P_W\left(b_{x_0,r_0}\right)\right\|_2^2.$$

so

$$
\begin{aligned}
N \left\| P_W \left(b_{x_0,r_0} \right) \right\|_2^2 &= \sum_{h\in H} \sum_{i=1}^{\dim W} \left| \left\langle u_i , b_{hx_0,r_0} \right\rangle \right|^2 \\
&\le \sum_{i=1}^{\dim W} \sum_{h\in H} \left\| u_i |_{B_r(hx_0)} \right\|_2^2 \left\| b_{x_0,r_0} \right\|_2^2 \\
&= \left\| b_{x_0,r_0} \right\|_2^2 \sum_{i=1}^{\dim W} \sum_{h\in H} \left\| u_i |_{B_r(hx_0)} \right\|_2^2 \\
&\le \left\| b_{x_0,r_0} \right\|_2^2 \sum_{i=1}^{\dim W} \| u_i \|_2^2 = \dim W \left\| b_{x_0,r_0} \right\|_2^2 .
\end{aligned}
$$

□

Proof of Theorem 8.1. To avoid cumbersome notations we do not use the index q in the proof below.

By the proof of Theorem 1.2 one should prove the following inequality for $r = g(R_X)$,

$$\left\| A_r \left(b_{x_0,r_0} - \pi \right) \right\|_2^2 = o\left(N^{-1} \right).$$

Let $\{p_i\}_{i=1}^T$ be the set of p-values (without multiplicities) of exceptional eigenvalues of $L^2(X/X_0)$, i.e., the p such that the corresponding matrix coefficient is not in $L^{p'}$ for every $p' \le p$ but are in $L^{p'}$ for every $p' > p$. Let V_i be the vector space of eigenvectors with p-value p_i. Let $p_0 = 2$ and V_0 the orthogonal complement of the V_i in $L^2(X/X_0)$. Then by Proposition 7.2, for $i = 0, \ldots, T$, the norm of A_r on V_i is bounded by $(r+1) e^{-r/p_i}$.

We have the decomposition

$$L^2(X) = \operatorname{span}\{\pi\} \oplus \rho^* L_0^2(X_0) \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_T .$$

Decompose $b_{x_0,r_0} = \pi + u + v_0 + \cdots + v_T$. For $i = 1, \ldots, T$, denote $m(X, p_i) = \dim V_i$. We have

$$
\begin{aligned}
\|u\|_2^2 &= N^{-1} \left\| b_{x_0,r_0} \right\|_2 \ll_{r_0} N^{-1} \\
\|v_0\|_2^2 &\le \left\| b_{x_0,r_0} \right\|_2^2 \ll_{r_0} 1 \\
\|v_i\|_2^2 &\le N^{-1} m(X, p_i) \left\| b_{x_0,r_0} \right\|_2 \ll_{r_0} N^{-1} m(X.p_i) .
\end{aligned}
$$

The first equality follows from Lemma 8.4, the second inequality is straightforward, and the third inequality follows from Lemma 8.5.

Then for $r = g\left(R_{X_q} \right)$,

$$\left\| A_r \left(b_{x_0,r_0} - \pi \right) \right\|_2^2 = \|A_r u\|_2^2 + \|A_r v_0\|_2^2 + \sum_{i=1}^T \|A_r v_i\|_2^2 . \tag{12}$$

Therefore one should prove that the RHS of Eq. 12 is $O\left(N^{-1} \right)$.

Since $\|u\|_2^2 \ll_{r_0} N^{-1}$ and X_0 has some $p_0(X_0)$ and the first summand of Eq. 12 is $o\left(N^{-1} \right)$.

Since $\|v_0\|_2^2 \ll_{r_0} 1$ and for some $\delta > 2$, and R large enough $g(R) \ge R + \delta \ln R$, the second summand of Eq. 12 is $o\left(N^{-1} \right)$.

For the third summand, we have

$$\sum_{i=1}^{T} \|A_r v_i\|_2^2 \leq N^{-1}(r+1)^2 \sum_{i=1}^{T} e^{-2r/p_i} m(X, p_i).$$

This proves that if Inequality 9 holds then the third summand of Eq. 12 is $o(N^{-1})$.

Notice that for $1 \leq i \leq T$, $m(X, p_i) = M(X, p_i) - M(X, p_{i+1})$, with $M(X, p_{T+1}) = 0$. Then

$$\begin{aligned}
\sum_{i=1}^{T} \|A_r v_0\|_2^2 &\leq N^{-1}(r+1)^2 \sum_{i=1}^{T} e^{-2r/p_i} m(X, p_i) \\
&= N^{-1}(r+1)^2 \sum_{i=1}^{T} e^{-2r/p_i} \left(M(X, p_i) - M(X, p_{i+1})\right) \\
&= N^{-1}(r+1)^2 \left(M(X, p_1) e^{-2r/p_1} + \sum_{i=2}^{T} M(X, p_i)\left(e^{-2r/p_i} - e^{-2r/p_{i-1}}\right) \right) \\
&\leq N^{-1}(r+1)^2 \left(M(X, p_1) e^{-2r/p_1} + \sum_{i=1}^{T} M(X, p_i)\, 2r(p_i - p_{i-1}) e^{-2r/p_i} p_{i-1}^{-2} \right),
\end{aligned}$$

Where we used $\left(e^{-2r/p_i} - e^{-2r/p_{i-1}}\right) = 2r(p_i - p_{i-1}) e^{-2r/p'} p'^{-2}$, for some $p_{i-1} \leq p' \leq p_i$.

By adding arbitrary p_i-s with $m(X, p_i) = 0$ we may conclude

$$\sum_{i=1}^{T} \|A_r v_0\|_2^2 \leq N^{-1}(r+1)^2 \left(\lim_{p_i \to 2, p_i > 2} M(X, p_i) e^{-r} + 2r \int_2^{\infty} M(X, p) e^{-2r/p} p^{-2} dp \right).$$

This proves that if Inequalities 10 and 11 hold then the third summand in Eq. 12 is $o\left(N^{-1}\right)$. □

9 Appendix I: Isoperimetric inequalities and concentration of distance from a fixed vertex

The bounds we have allows us to prove the following isoperimetric inequality. Similar bounds are well-known (see [10, Theorem 4.1]).

Lemma 9.1 *Let $X = \Gamma \backslash \mathbb{H}$ be a quotient, and $p = p_0(X)$ as defined in Proposition 7.2. For $r \geq 0$, denote $\kappa_{r,p} = (r+1)^2 e^{-2r/p}$. For a closed set $Y \subset X$, let*

$$Y_r = \{x \in X \mid d(x, Y) \leq r\},$$

and denote $c = \mu(Y)/\mu(X)$ and $c' = \mu(Y_r)/\mu(X)$. Then

$$c' \geq \frac{c}{\left(\kappa_{r,p}(1-c) + c\right)}, \text{ and hence also } c \leq \frac{\kappa_{r,p} c'}{1 - c' + \kappa_{r,p} c'}.$$

Remark 9.2 For $ck_{r,p}^{-1}$ small $c' \gg \frac{e^{2r/p}}{(r+1)^2}c$. So for $p=2$, up to an $(r+1)^{-2}$ factor, the growth of small sets is the best possible, i.e. the size of the radius r-ball.

Remark 9.3 The result of [10, Theorem 4.1], which is more general and works for all surfaces, not necessarily hyperbolic, essentially replaces the exponent $2/p = 1-\sqrt{1-4\lambda}$ by $\sqrt{\lambda}$, so the results above are asymptotically better for the relevant domain $0 \le \lambda \le 1/4$.

Proof We may assume $\mu(Y) > 0$. Let $b_Y \in L^1(Y)$ be defined by

$$b_Y = \begin{cases} \mu^{-1}(x) & x \in Y \\ 0 & x \notin Y \end{cases}.$$

Then $\|b_Y\|_1 = 1$, $\|b_Y\|_2^2 = \mu^{-1}(Y)$, $\left\|A_{r_0} b_Y\right\|_1 = 1$ and $\operatorname{supp}(A_r b_Y) \subset Y_r$, so $\|A_r b_Y\|_2^{-2} \ge \frac{1}{\mu(Y_r)}$, i.e.

$$\mu(Y_r) \ge \|A_r b_Y\|_2^2.$$

Decompose $b_Y = \pi + b$, with

$$\|b\|_2^2 = \|b_Y\|_2^2 - \|\pi\|_2^2 = \frac{1}{\mu(Y)} - \frac{1}{\mu(X)} = \frac{1-c}{\mu(Y)}.$$

We have

$$\begin{aligned} \left\|A_{r_0} b_Y\right\|_2^2 &= \|A_r b\|_2^2 + \|A_r \pi\|_2^2 \\ &\le (r+1)^2 e^{-2r/p} \|b\|_2^2 + \|\pi\|_2^2 \\ &\le (r+1)^2 e^{-2r/p}(1-c)\mu^{-1}(Y) + \mu^{-1}(X) \\ &= \left(\kappa_{r,p}(1-c)+c\right)\mu^{-1}(Y). \end{aligned}$$

Combining the two inequalities we get

$$\mu(Y_r) \ge \left\|A_{r_0} b_Y\right\|_2^{-2} \ge \frac{c}{\left(\kappa_{r,p}(1-c)+c\right)}\mu(X).$$

The other inequality in the theorem follows from the first one. □

We may now state the following concentration of distance theorem:

Theorem 9.4 *There exists $a = a(p_0(X)) > 0$ such that for each $x_0 \in X$ there exists R_{X,x_0} such that for every $\gamma > 0$:*

$$\mu\left(x \in X \mid \left|d_X(x, x_0) - R_{X,x_0}\right| \ge \gamma\right)/\mu(X) \ll_{p_0(X)} a^{-\gamma}.$$

Remark 9.5 By Theorem 1.2 if X is Ramanujan and x_0 has injectivity radius r_0 then R_{X,x_0} satisfies $R_X \le R_{X,x_0} \le R_X + (2+\epsilon)\ln R_X$.

Proof For $r \ge 0$ denote

$$Y(r) = \{x \in X \mid d(x, x_0) \le r\}.$$

Choose R_{X,x_0} to be such that

$$\mu\left(Y\left(R_{X,x_0}\right)\right) = \frac{1}{2}\mu(X).$$

Let $Y = Y(R_{X,x_0} - \gamma)$. Then $Y_\gamma = Y\left(R_{X,x_0}\right)$ and

$$\mu(Y)\,\mu(X)^{-1} \le \frac{k_{\gamma,p}\frac{1}{2}}{1-\frac{1}{2}+k_{\gamma,p}\frac{1}{2}} = \frac{k_{\gamma,p}}{1+k_{\gamma,p}} \le k_{\gamma,p}.$$

Let $Z = Y(R_{X,x_0} + \gamma)$. Then $Y\left(R_{X,x_0}\right)_\gamma = Z$ and

$$\mu(Z)\,\mu(X)^{-1} \ge \frac{\frac{1}{2}}{\left(\kappa_{r,p}(1-\frac{1}{2})+\frac{1}{2}\right)} = \frac{1}{1+k_{\gamma,p}}.$$

Hence

$$1-\mu(Z)\,\mu(X)^{-1} \le \frac{k_{\gamma,p}}{1+k_{\gamma,p}} \le k_{\gamma,p}$$

And finally,

$$\begin{aligned}&\mu\left(x\in X : \left|d_X(x,x_0)-R_{X,x_0}\right| \le \gamma\right)/\mu(X)\\ &\quad = 1-\mu(Z)\,\mu(X)^{-1}-\mu(Y)\,\mu(X)^{-1} \le 2k_{\gamma,p}.\end{aligned}$$

We finish by noting that there exists $a = a(p)$ such that

$$k_{\gamma,p} \ll_p a^{-\gamma}.$$

□

10 Appendix II: Comparison with the flat case

In [7, Section 3C], Diaconis analyzes the random walk on the Cayley graph of $\mathbb{Z}/N\mathbb{Z}$ with respect to the generators ± 1, and shows that it does not have a cutoff. Namely, he shows that the time t_0^T until the random walk satisfies $\left\|p_N^T-\pi\right\|_1 \le e^{-T}$ is $\Theta\left(N^2T\right)$.

We will similarly analyze the Brownian random walk on the torus $a\mathbb{Z}\backslash\mathbb{R}$ where $a>0$, and show it does not have a cutoff as $a\to\infty$. Namely, we will show that time until the time t_0^T until the random walk satisfies $\left\|p_a^T-\pi\right\|_1 \le e^{-T}$ is $\Theta\left(a^2T\right)$. Similar analysis shows that the Brownian random walk on quotients of $\mathbb{R}^n$ by $a\mathbb{Z}^n$ does not express a cutoff as $a\to\infty$.

It is also worth mentioning that the "distance r_1" discrete random walk on $a\mathbb{Z}\backslash\mathbb{R}$ does not even converge in L^1 to the uniform probability, since it remains discrete. For higher dimensions the "distance r_1" random walk does converge to the uniform probability (for similar reasons as in Sect. 5), but does not express a cutoff by the central limit theorem and comparison with the Brownian motion.

Let $X_a = a\mathbb{Z}\backslash\mathbb{R}$ and let $x_0\in X$. The distribution of the Brownian random walk starting at x_0 at time t for $x\in X$ is $p_t(x,x_0) = \left(\delta_{x_0} * f_t\right)(x) = \sum_{n\in\mathbb{Z}} f_t(x-x_0)$, with $f_t(x) = \frac{1}{\sqrt{2\pi t}}\exp\left(-x^2/2t\right)$.

By normalizing and choosing $\lambda = a^2$, we may consider a fixed space $X = \mathbb{Z}\backslash\mathbb{R}$, a fixed point $x_0 = \mathbb{Z}0$ and let $f_t^\lambda(x) = \frac{1}{\sqrt{2\pi\lambda^{-1}t}}\exp\left(-x^2\lambda/2t\right)$. Then $p_t^\lambda(x) = \sum_{n\in\mathbb{Z}} f_t^\epsilon(x+n)$.

Proposition 10.1 *We have for every* $\lambda>0$, $t\ge 0$.

$$\exp\left(-\lambda^{-1}t\right) \le \left\|p_t^\lambda-\pi\right\|_1 \le \sqrt{\frac{2}{1-\exp\left(-2\lambda^{-1}t\right)}}\cdot\exp\left(-\lambda^{-1}t\right).$$

The proposition says that the time until $\left\| p_t^\lambda - \pi \right\|_1 \leq e^{-T}$ takes place is $\Theta\left(T \cdot \lambda^{-1}\right)$. Therefore this random walk does not exhibit a cutoff.

Proof Let us calculate the Fourier series of p_t^λ:

$$\begin{aligned}\hat{p}_t^\lambda(m) &= \int_0^1 p_t^\lambda(x) \exp(2\pi i m x) dx \\ &= \int_0^1 \sum_{n\in\mathbb{Z}} f_t^\lambda(x+n) \exp(2\pi i m x) dx \\ &= \int_{-\infty}^{\infty} f_t^\lambda \exp\left(2\pi i m x\right) dx \\ &= \hat{f}_t^\lambda(m),\end{aligned}$$

where $\hat{f}_t^\lambda$ is the Fourier transform of f_t^λ. By a standard computation $\hat{f}_t^\lambda(\omega) = \exp\left(-\lambda^{-1} t \omega^2\right)$, so $\hat{p}_t^\lambda(m) = \exp\left(-\lambda^{-1} t m^2\right)$.

On the one hand,

$$\begin{aligned}\left\| p_t^\lambda - \pi \right\|_1 &\geq \int_0^1 \left(p_t^\lambda(x) - 1\right) exp\left(2\pi x\right) dx = \int_0^1 p_t^\lambda(x) exp\left(2\pi x\right) dx \\ &= \hat{p}_t^\lambda(1) = \exp\left(-\lambda^{-1} t\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\| p_t^\lambda - \pi \right\|_2^2 &= \sum_{m\in\mathbb{Z}} \left(\hat{p}_t^\lambda(m) - \hat{\pi}(m)\right)^2 = \sum_{m\in\mathbb{N}\setminus\{0\}} \left(\hat{p}_t^\epsilon\right)^2(m) \\ &= \sum_{m\in\mathbb{Z}\setminus\{0\}} \exp\left(-2\lambda^{-1} t m^2\right) \\ &= 2\sum_{m=1}^{\infty} \exp\left(-2\lambda^{-1} t m\right) \\ &\leq \frac{2}{1 - \exp\left(-2\lambda^{-1} t\right)} \exp\left(-2\lambda^{-1} t\right).\end{aligned}$$

Cauchy-Schwartz inequality completes the proof by

$$\left\| p_t^\epsilon - \pi \right\|_1 \leq \left\| p_t^\epsilon - \pi \right\|_2 \leq \sqrt{\frac{2}{1 - \exp\left(-2\lambda^{-1} t\right)}} \exp\left(-\lambda^{-1} t\right).$$

□

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