Diss. ETH No. 25544

Satake compactification of analytic Drinfeld modular varieties

A thesis submitted to attain the degree of

DOCTOR OF SCIENCES of ETH ZURICH

(Dr. sc. ETH Zurich)

presented by

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2018

Summary

The irreducible components of rigid analytic moduli spaces of Drinfeld A-modules with level structure may be described as quotients of Drinfeld's period domains by arithmetic groups. We provide comprehensive proofs of well-known fundamental facts about such quotients.

We then use this description for the construction of a compactification of any such component that is analogous to Satake's classical compactification. The compactification is a priori defined as a Grothendieck ringed space whose boundary, as a set, consists of finitely many irreducible components of moduli spaces of smaller dimensions. Its underlying topological space coincides with the one of Kapranov's compactification when A is a polynomial ring.

The compactifications of the analytic moduli spaces are constructed in a natural way as the disjoint union of the compactifications of their components. For this, we use adelic language and find a uniform description for the natural morphisms between these compactifications.

We further construct projective modular compactifications of some algebraic moduli spaces of Drinfeld modules whose boundary is stratified by moduli spaces of smaller dimension. They generalize Pink's algebraic Satake compactifications in the case of the moduli problem of Drinfeld $\mathbb{F}_q[t]$-modules with level (t) structure. Pink's compactification in the general case is the quotient by a finite group of the normalization of one of these modular compactifications.

By means of explicit morphisms, we then show that the compactifications of the analytic moduli spaces are the normalizations of the analytifications of the compactifications of the algebraic moduli spaces. In particular, the former are normal projective rigid analytic varieties.

We finally view the analytic Drinfeld modular forms as global sections of ample invertible sheaves on these projective spaces. From this, we conclude finiteness results on the algebras and vector spaces of such modular forms.

Zusammenfassung

Die irreduziblen Komponenten von rigid analytischen Modulräumen von Drinfeld A-Moduln mit Niveaustruktur lassen sich beschreiben als Quotienten von Drinfeld's oberen Halbräumen durch arithmetische Gruppen. Wir geben ausführliche Beweise bekannter fundamentaler Resultate über solche Quotienten.

Wir verwenden diese Beschreibung dann zur Konstruktion einer Kompaktifizierung einer beliebigen solchen Komponente analog zu Satake's klassischer Kompaktifizierung. A priori ist die Kompaktifizierung definiert als Grothendieck geringter Raum, dessen Rand sich mengentheoretisch aus irreduziblen Komponenten von Modulräumen niedrigerer Dimension zusammensetzt. Der zugrunde liegende topologische Raum stimmt mit jenem von Kapranov's Kompaktifizierung überein, wenn A ein Polynomring ist.

Die Kompaktifizierungen der analytischen Modulräumen konstruieren wir auf natürliche Weise als disjunkte Vereinigung der Kompaktifizierungen ihrer Komponenten. Hierfür verwenden wir adelische Sprache und finden eine einheitliche Beschreibung der natürlichen Morphismen zwischen diesen Kompaktifizierungen.

Zudem konstruieren wir projektive modulare Kompaktifizierungen gewisser algebraischer Modulräumen von Drinfeld-Moduln, deren Rand stratifiziert ist durch Modulräume niedrigerer Dimension. Diese verallgemeinern Pink's algebraische Satake Kompaktifizierung im Fall von Drinfeld $\mathbb{F}_q[t]$-moduln mit Niveau (t) Struktur. Pink's Kompaktifizierung im allgemeinen Fall ist der Quotient nach einer endlichen Gruppe der Normalisierung einer solchen modularen Kompaktifizierung.

Wir zeigen dann mittels expliziter Morphismen, dass die Kompaktifizierungen der analytischen Räume die Normalisierungen sind der Analytifizierungen der modularen Kompaktifizierung. Insbesondere sind Erstere normale projektive rigid analytische Varietäten.

Schliesslich fassen wir die analytischen Drinfeld'schen Modulformen auf als globale Schnitte ampler invertierbarer Garben auf diesen projektiven Kompaktifizierungen. Daraus folgern wir Endlichkeitsresultate über die Algebren und Vektorräume solcher Modulformen.

Acknowledgments

I wish to express my deep gratitude towards my advisor Richard Pink for his guidance of this thesis. He suggested its topic, that is, to construct compactifications à la Satake-Baily-Borel of Drinfeld modular varieties generalizing Kapranov's compactification. His expertise in these subjects, his willingness to share his knowledge as well as his support and patience have ensured the success of this project. The algebraic part of this thesis has been profoundly inspired by his articles on algebraic compactifications. His course on Drinfeld modules has been a helpful introduction to the theory. I am grateful to him for providing me with many indispensable ideas and questions, and for stimulating me to strive for natural and precise mathematical formulations.

I wish to thank Gebhard Böckle for being co-examiner and for many helpful remarks on earlier versions of the thesis. I am grateful to him for showing interest for this project from early on, for inviting me to Heidelberg and giving me the opportunity to give a talk.

I wish to thank Brian Conrad and Urs Hartl for kindly and helpfully answering my questions.

I wish to thank Dirk Basson for sharing helpful thoughts about the topic of this project with me.

I wish to thank Hadi Hedayatzadeh for helpful discussions about this project and beyond.

I wish to thank Georg Oberdieck and Alexandre Puttick for many suggestions which helped improve the introduction of the thesis.

This thesis largely builds on the classical book by Bosch, Güntzer and Remmert in which they systematically developed a language of rigid analytic geometry. It further makes essential use of some of Bartenwerfer's, Conrad's, Kiehl's, Köpf's and Lütkebohmert's contributions to this geometry. I am grateful to them for their work.

I wish to thank Özlem Imamoglu for giving a very enjoyable algebra course during my Bachelor studies. It greatly contributed to my fascination for pure abstract mathematics.

I wish to thank my high school teacher Hansueli Hösli for presenting mathematics in a new and tempting way to me and for encouraging me to pursue mathematical studies.

I wish to thank Aarti, Alexandre, Bledar, Fabrizio, Giuseppe, Georg, Hadi, Maria, Michael, Nadine, Réda, Thomi, my siblings, my parents and my grandmothers for their supports during my Phd.

Contents

1 Introduction

Drinfeld modules with level structure, introduced by Drinfeld [15] in 1974, are a function field analogue to elliptic curves with level structure. They give rise to fine algebraic moduli spaces and to coarse rigid analytic modular varieties. Any analytic irreducible component is the quotient of a Drinfeld upper half space by an arithmetic group Γ and thus carries natural spaces of weak modular forms with respect to Γ of varying integral weight.

In analogy to the weak modular forms on the complex upper half space with respect to congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, such weak modular forms have various Fourier expansions by means of which modular forms may be defined. Such expansions were studied and used to define modular forms by Gekeler [18] and Goss [22, 23] mainly in the case of Drinfeld modules of rank 2 and in general by Basson, Breuer and Pink [3, 4, 5, 6].

In order to prove finiteness results for the spaces of modular forms, it is a classical approach to construct compactifications of the modular varieties which carry ample invertible sheaves whose global sections may be identified with the spaces of modular forms of interest. In a special case, such a compactification of an analytic modular variety was constructed by Kapranov [30] up to explicitly specifying the invertible sheaves. Pink [34] provided a normal compactification of a general algebraic moduli scheme and defined algebraic modular forms as global sections of ample invertible sheaves thereof.

In this thesis, we construct compactifications of general analytic modular varieties endowed with natural ample invertible sheaves whose global sections correspond bijectively to modular forms. We further construct new projective modular compactifications of algebraic modular schemes generalizing Pink and Schieder's [36]. We show that any analytic compactification is the quotient by a finite group of the normalization of the analytification of such a modular compactification. We apply these results to deduce finiteness results for general spaces of analytic modular forms.

Drinfeld modular varieties and analytic modular forms

Consider any global function field F of characteristic $p > 0$ and any place ∞ of F. Denote by E the completion of F with respect to ∞. Let $A \subset F$ be the subring of elements that are regular outside of ∞. The basic example for A is the polynomial ring over a finite field.

Let $d \geq 1$ be any positive integer. Consider any ring R over F and denote by $\iota\colon A \to R$ the structure morphism. Denote by $R\{\tau\} \subset R[T]$, with $\tau := T^p$, the subgroup of additive polynomials and equip it with the ring structure for which multiplication is given by composition. A *Drinfeld A-module* of rank d over R is a ring homomorphism

(1) $$\varphi\colon A \to R\{\tau\}, 0 \neq a \mapsto \varphi_a = \sum_{0 \leq i \leq d \cdot \deg(a)} \varphi_{a,i}\tau^i$$

with $\varphi_{a,0} = \iota(a)$ and $\varphi_{a,d\cdot\deg(a)} \in R^\times$, where $\deg(a) := \dim_{\mathbb{F}_p}(A/(a))$. Consider any non-zero non-unital $t \in A$. Set $V := (A/(t))^d$ and $\mathring{V} := V \setminus \{0\}$. A *level* (t) *structure* for such a φ is a map $\lambda\colon V \to R$ with $\lambda(\mathring{V}) \subset R^\times$ and

$$\varphi_t(T) = t \cdot T \prod_{0 \neq v \in V} \left(1 - \frac{T}{\lambda(v)}\right)$$

for which the induced map $V \to \mathrm{Ker}(R \xrightarrow{\varphi_t} R)$ is an A-linear isomorphism.

Consider any ideal $0 \neq I \subsetneq A$. More generally, one defines (see Section 8.1) Drinfeld A-modules with level I structures over arbitrary schemes S over F. This is done in a way such that the functor which associates to such an S the set of isomorphism classes of Drinfeld A-modules of rank d over S with level I structure is represented by an irreducible smooth affine variety X_I^d of dimension $d-1$ over F (see Drinfeld's [15, Section 5]).

Consider any non-Archimedean complete algebraically closed valued field C containing E as a valued subfield. A projective A-submodule $\Lambda \subset C$ of finite rank is called an *A-lattice* if the natural homomorphism $\Lambda \otimes_A E \to C$ is injective. A *level I structure* of an A-lattice Λ of rank d is an A-linear isomorphism $(A/I)^d \to I^{-1}\Lambda/\Lambda$. Drinfeld [15, Prop. 3.1] showed that the isomorphism classes of Drinfeld A-modules over C of rank d with level I structure are in natural bijective correspondence with A-lattices in C of rank d with level I structure up to homothety.

This correspondence is a function field analogue of the correspondence between $\mathbb{Z}$-lattices in the complex numbers $\mathbb{C}$ of rank 2 with level structure and elliptic curves over $\mathbb{C}$ with level structure. However, by contrast, as C is of infinite dimension over E, there exist A-lattices in C, and hence Drinfeld A-modules over C, of arbitrary rank.

Drinfeld's *upper half space* of dimension $d-1$ is the $\mathrm{PGL}_d(E)$-invariant subset $\Omega^{d-1} \subset \mathbb{P}_C^{d-1}$ of the standard projective space over C consisting of those elements lying in no E-linear hyperplane. Consider any irreducible component Y of the rigid analytic variety $X_I^d(C)$ of C-valued points. The above correspondence yields the description of Y as the quotient

$$\Omega_\Gamma := \Gamma\backslash\Omega^{d-1}$$

of Ω^{d-1} by an arithmetic subgroup Γ isomorphic to the kernel of the natural morphism $\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(\Lambda/I\Lambda)$ for a projective A-module Λ of rank d.

In analogy to the quotients of the complex upper half plane by arithmetic groups, the quotient Ω_Γ is naturally equipped with an invertible sheaf $\mathcal{O}_\Gamma(k)$ of analytic weak modular forms with respect to Γ of weight k for any integer $k \geq 0$. Weak modular forms admit a Fourier expansion with respect to certain irreducible components of modular varieties of codimension 1, called *cusps*. Using such expansions, one defines modular forms. They form a C-subspace

$$\mathcal{M}_\Gamma(k) \subset \mathcal{O}_\Gamma(k)(\Omega_\Gamma).$$

Kapranov's compactification

The component Ω_Γ of $X_I^d(C)$ is affine and non-compact if $d \geq 2$. This is, for instance, reflected in the fact that the space of weak modular forms $\mathcal{O}_\Gamma(0)(\Omega_\Gamma)$ is of infinite dimension over C. On the other hand, among the weak modular forms, the modular forms are precisely the ones which extend to all cusps. It therefore seems natural to hope for a compactification Ω_Γ^* of Ω_Γ whose boundary consists of irreducible components of smaller dimensional modular varieties including all cusps and on which any $\mathcal{O}_\Gamma(k)$ is extended by an ample invertible sheaf $\mathcal{O}_\Gamma^*(k)$ whose global sections bijectively correspond to modular forms of weight k via the restriction map. Such compactifications then force the spaces of modular forms to be finite dimensional.

Theorem 1.1 (Kapranov [30]). *If A is a polynomial ring, then Ω_Γ admits a normal compactification Ω_Γ^* whose boundary is stratified by finitely many copies of irreducible components $\Omega_{\Gamma'}$ of $X_I^{d'}(C)$ for all $1 \leq d' < d$.*

Furthermore, Kapranov along with Goss [20] sketched how to view modular forms as global sections of invertible sheaves on Ω_Γ^*.

Kapranov first constructed Ω_Γ^* as the disjoint union of Ω_Γ and certain irreducible components $\Omega_{\Gamma'}$ of smaller rank modular varieties and endowed it with a suitable topology. He then specified a projective embedding of Ω_Γ^*

using Eisenstein series of high weight and then defined the Satake compactification of Ω_Γ as the normalization of the image of Ω_Γ^*. The construction is largely analogous to Satake's [38] construction of his compactification of the analytic moduli space of abelian varieties of rank $2g$ but with a crucial difference when $g > 1$: In that case, all boundary components in Satake's compactification have codimension > 1 so that by normality all – a priori – weak modular forms extend to global sections.

When A is the polynomial ring, Gekeler [19] has recently, and independently of this thesis, improved on Kapranov's approach by carrying out an embedding defined only by the Eisenstein series of weight 1.

Pink's compactifications

In [34], Pink introduced the notion of *generalized Drinfeld A-module* of rank $\leq d$ over any scheme S over F. It generalizes the notion of Drinfeld A-module over S in that its fibres over the points of S – which are Drinfeld A-modules of the form (1) – are allowed to have rank $\leq d$ rather than only $= d$. A generalized Drinfeld A-module over S is *weakly separating* if for any Drinfeld A-module φ over any field extension $F' \supset F$ at most finitely many fibres of the generalized Drinfeld A-module over F'-valued points of S are isomorphic to φ.

Theorem 1.2 (Pink [34])**.** *Uniquely up to unique isomorphism, there exists an integral normal projective algebraic variety $\overline{X}_I^d$ over F together with an embedding $X_I^d \to \overline{X}_I^d$ and a weakly separating generalized Drinfeld module on $\overline{X}_I^d$ extending the universal family on X_I^d.*

The notion of level structure does not directly generalize to generalized Drinfeld modules in a satisfying way so as to turn $\overline{X}_I^d$ in a fine moduli space. In the case where A is the polynomial ring $\mathbb{F}_q[t]$ over a finite field $\mathbb{F}_q$ and where $I = (t)$ and thus $V = \mathbb{F}_q^d$, Pink and Schieder [36] instead introduced and studied the notion of a *reciprocal map*. Over any ring R over $\mathbb{F}_q$, the injective reciprocal maps are precisely the ones that arise from the injective $\mathbb{F}_q$-linear morphisms $\lambda\colon V \to R$ with $\lambda(\mathring{V}) \subset R^\times$ by the rule

$$\rho_\lambda\colon \mathring{V} \to R^\times, v \mapsto \frac{1}{\lambda(v)}.$$

The maps ρ_λ thus obtained are the injective maps $\rho\colon \mathring{V} \to R^\times$ such that

- $\forall \alpha \in \mathbb{F}_q^\times, v \in \mathring{V}\colon \rho(\alpha \cdot v) = \alpha^{-1} \cdot \rho(v)$,

- $\forall v, v' \in \mathring{V}\colon [v + v' \in \mathring{V} \Rightarrow \rho(v)\cdot\rho(v') = \rho(v+v')\cdot(\rho(v)+\rho(v'))]$.

A general reciprocal map over R is then defined to be any map $\rho\colon \mathring{V} \to R$ satisfying these polynomial conditions. Globally, reciprocal maps are defined more generally to be certain maps from $\mathring{V}$ to the set of global sections $\Gamma(S,\mathcal{L})$ of invertible sheaves $\mathcal{L}$ over schemes S over $\mathbb{F}_q$.

Theorem 1.3. *([36, Theorems 1.7 and 7.10]) Consider the functor that associates with any scheme S over $\mathbb{F}_q$ the set of isomorphism classes of reciprocal maps $\mathring{V} \to \Gamma(S,\mathcal{L})$ whose induced morphism $\mathring{V} \to \mathcal{L}\otimes_{\mathcal{O}_S} k(s)$ is non-zero for every point $s \in S$. It is represented by a normal projective scheme Q_V over $\mathbb{F}_q$.*

Using the fact that a Drinfeld $\mathbb{F}_q[t]$-module over a scheme over F is uniquely determined by a level (t) structure, Pink deduced from Theorem 1.3:

Theorem 1.4. *([34, Section 7]) If $A = \mathbb{F}_q[t]$, then $\overline{X}^d_{(t)}$ equals the pullback of Q_V to $\mathrm{Spec}(F)$ and is stratified by copies of $X^{d'}_{(t)}$ for all $1 \le d' \le d$ indexed by the non-zero $\mathbb{F}_q$-subspaces of V.*

In fact, Pink proved Theorem 1.2 by reduction to the case $A = \mathbb{F}_q[t]$ and $I = (t)$ which he proved jointly with Theorem 1.4 using Theorem 1.3.

Main results

On the algebraic side, we prove versions of Theorems 1.3 and 1.4 for general A and more general level. Namely, for suitable $t \in A$ we construct a projective modular compactification $Q_{V,F}$ of $X^d_{(t)}$ which is stratified by finitely many copies of the $X^{d'}_{(t)}$ for all $1 \le d' \le d$ as follows:

Using the fact that A is finitely generated, choose t such that its divisors

$$\mathrm{Div}_A(t) := \{a \in A \mid t \in (a)\}$$

generate A. In this case, too, a Drinfeld A-module over any ring R over F with level (t) structure λ is uniquely determined by λ and hence by the reciprocal map

$$\rho\colon \mathring{V} \to R^\times, v \mapsto \frac{1}{\lambda(v)}.$$

Here as well, there is a set of necessary and sufficient polynomial conditions for an injective map $\rho\colon \mathring{V} \to R^\times$ to arise from such a λ. An *A-reciprocal map* over R is then any map $\rho\colon \mathring{V} \to R$ satisfying this set of conditions. Globally,

A-reciprocal maps will be defined (see Definition 8.14) more generally to be certain maps from $\mathring{V}$ to the set of global sections $\Gamma(S, \mathcal{L})$ of invertible sheaves $\mathcal{L}$ over schemes S over $\mathrm{Spec}(A)$.

Theorem 1.5. *(See Theorem 8.16 and Corollary 8.24) Suppose that* $\mathrm{Div}_A(t)$ *generates A. Consider the functor which assigns to a scheme S over* $\mathrm{Spec}(A)$ *the set of isomorphism classes of A-reciprocal maps* $\mathring{V} \to \Gamma(S, \mathcal{L})$ *whose induced morphism* $V \to \mathcal{L} \otimes_{\mathcal{O}_S} k(s)$ *is non-zero for every* $s \in S$.

i) This functor is represented by a projective scheme Q_V *over* $\mathrm{Spec}(A)$.

ii) The pullback $Q_{V,F}$ *of* Q_V *to* $\mathrm{Spec}(F)$ *contains* $X^d_{(t)}$ *as an open subscheme and is stratified by locally closed subschemes* Ω_W *for all free* $A/(t)$*-submodules* $0 \neq W \subset V$ *each of which is isomorphic to* $X^{d'}_{(t)}$*, where* $d' := \mathrm{rank}_{A/(t)}(W)$.

iii) If $t \in I$*, then* $\overline{X}^d_I$ *is the quotient by a finite group of the normalization of* $Q_{V,F}$.

In fact, the universal family on $Q_{V,F}$ induces a generalized Drinfeld module on $Q_{V,F}$ and the pullback of this generalized Drinfeld module to the normalization descends, if $t \in I$, to the one on $\overline{X}^d_I$ (see Corollary 8.24). Note that, as long as I is fixed, one may choose $t \in I$ such that $\mathrm{Div}_A(t)$ generates A.

After the work presented here was done, Pink [35] modified the notion of A-reciprocal maps by using defining conditions [35, Def. 2.3.1] that are homogeneous equations solely of weight 1. His conditions are stronger (see [35, Prop. 1.3.4 (b) and 2.4.4]) and more explicit than the ones here and enabled him to generalize computations from his and Schieder's article [36]. However, the reduced scheme underlying the modular compactification that he obtains coincides with the one underlying Q_V and this is the scheme that we use in the comparison with the following analytic compactification.

On the analytic side, we generalize Theorem 1.1 to arbitrary A. More precisely, we construct a Grothendieck ringed space $(\Omega^*_\Gamma, \mathcal{O}^*_\Gamma)$ whose underlying topological space coincides with Kapranov's when $A = \mathbb{F}_q[t]$. We further define a natural sheaf

$$\mathcal{R}^*_\Gamma = \sum_{k \geq 0} \mathcal{O}^*_\Gamma(k)$$

of graded $\mathcal{O}^*_\Gamma$-algebras, where $\mathcal{O}^*_\Gamma(0) = \mathcal{O}^*_\Gamma$ and where the $\mathcal{O}^*_\Gamma$-module $\mathcal{O}^*_\Gamma(k)$ of the homogeneous sections of weight k extends $\mathcal{O}_\Gamma(k)$ for any $k \geq 0$.

Theorem 1.6. *(See Corollary 10.4) The Grothendieck ringed space* $(\Omega_\Gamma^*, \mathcal{O}_\Gamma^*)$ *is a normal projective rigid analytic variety over* C.

Theorem 1.7. *(See Corollary 10.6)* $\mathcal{O}_\Gamma^*(k)$ *is ample invertible for any* $k \geq 1$.

Before discussing their proofs, we state an application.

In Sections 5.4 and 6.6, we recall the definition of Fourier expansions of weak modular forms and provide in detail everything required for it. Using these expansions as well as the normality of Ω_Γ^*, we show

Proposition 1.8. *(See Proposition 10.10) The restriction morphism*

$$\mathcal{O}_\Gamma^*(k)(\Omega_\Gamma^*) \to \mathcal{O}_\Gamma(k)(\Omega_\Gamma)$$

is injective with image $\mathcal{M}_\Gamma(k)$ *for any* $k \geq 0$.

By standard arguments, Theorems 1.6 and 1.7 via Proposition 1.8 imply

Corollary 1.9. *(See Cor. 10.7,10.11 and Prop. 10.10) The graded* C*-algebra* $\mathcal{M}_\Gamma := \sum_{k\geq 0} \mathcal{M}_\Gamma(k)$ *of modular forms with respect to* Γ *is finitely generated and* Ω_Γ^* *is the rigid analytification of* $\mathrm{Proj}(\mathcal{M}_\Gamma)$.

The interpretation of modular forms as global sections via Proposition 1.8 is useful beyond these corollaries. For instance, we introduce analogues of the classical Poincaré-Eisenstein series by defining them, without technical difficulties, directly as global sections. Their Fourier expansions, on the other hand, seem difficult to deal with. As an application, we show

Proposition 1.10. *(See Proposition 6.37) Any two points* $p, p' \in \Omega_\Gamma^*$ *admit a Poincaré-Eisenstein series* P *for which* $P(p) \neq 0 = P(p')$.

The basic examples of modular forms with respect to Γ are the Eisenstein series of weight 1 indexed by $(A/I)^d \setminus \{0\}$. In this case, too, we directly write down global sections E_α of $\mathcal{O}_\Gamma^*(1)$ for all $\alpha \in (A/I)^d \setminus \{0\}$ which, a posteriori, uniquely restrict to these series. They play a fundamental role in our proof of Theorems 1.6 and 1.7 to which we turn now.

Suppose without loss of generality that $t \in I$. By construction, then Ω_Γ^* is the quotient of $\Omega_{\Gamma'}^*$ by the finite group Γ/Γ' , where Γ' is the kernel of $\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(\Lambda/t \cdot \Lambda)$. Moreover, $\mathcal{O}_\Gamma^*(k)$ is the subsheaf of (Γ/Γ')-invariants of $\mathcal{O}_{\Gamma'}^*(k)$ for any $k \geq 0$. Thus, by standard arguments, Theorems 1.6 and 1.7 are reduced to the case where $I = (t)$. In this case, they are consequences of the following central result of the thesis which relates the analytic compactification Ω_Γ^* with the algebraic compactification Q_V.

Theorem 1.11. *(See Theorem 9.1 and Corollary 10.6) Suppose that* $\mathrm{Div}_A(t)$ *generates* A *and that* $I = (t)$*. Then the* $(E_\alpha)_{\alpha \in \mathring{V}}$ *define a morphism of Grothendieck ringed spaces* $E\colon \Omega_\Gamma^* \to Q_V(C)$ *onto an irreducible component* X *of the rigid analytic variety* $Q_V(C)$ *of* C*-valued points which is the normalization morphism of* X *in the sense of Conrad [12]. Moreover,* $\mathcal{O}_\Gamma^*(k)$ *is the pullback of the* k*-th twisting sheaf under this morphism for any* $k \geq 0$*.*

We finally outline our proof of Theorem 1.11. Consider any free $A/(t)$-submodule $0 \neq W \subset V$ and the restriction $E^{-1}(\Omega_W(C)) \to \Omega_W(C)$ of E via Theorem 1.5, ii). Via the isomorphism $\Omega_W \cong X_{(t)}^{d'}$, where $d' = \mathrm{rank}_{A/(t)}(W)$, this restriction is Drinfeld's isomorphism from the analytically defined modular variety to $\Omega_W(C)$ if $W \subsetneq V$. If $W = V$, it is the restriction of this isomorphism to the irreducible component Ω_Γ. Using these isomorphisms and elementary inequalities of Drinfeld's exponential functions, we prove the following result as a step towards Theorem 1.11:

Proposition 1.12. *(See Prop. 9.8 and Cor. 9.12) The morphism between Grothendieck topological spaces underlying* E *is an isomorphism onto an irreducible component* X *of* $Q_V(C)$*. Moreover,* $X \cap \Omega_V(C)$ *is an irreducible component of* $\Omega_V(C)$ *and*

$$X = (X \cap \Omega_V(C)) \cup (Q_V(C) \setminus \Omega_V(C)).$$

We further define a sheaf of rings $\tilde{\mathcal{O}}_X$ on X in terms of the stratification by the $\Omega_W(C)$ provided by Theorem 1.5, ii) for which the following holds:

Corollary 1.13. *(See Cor. 9.13) The isomorphism between Grothendieck topological spaces underlying* E *induces an isomorphism of Grothendieck ringed spaces*

$$(\Omega_\Gamma^*, \mathcal{O}_\Gamma^*) \xrightarrow{\sim} (X, \tilde{\mathcal{O}}_X). \tag{2}$$

The stratification of X may be described in terms of vanishing and non-vanishing loci of subsets of some finite set of global sections of the first twisting sheaf on X. More generally, with any finite set of global sections of an invertible sheaf on a rigid analytic variety Z may be associated (see Section 3) a stratification of Z by locally closed subvarieties and a natural sheaf of rings $\tilde{\mathcal{O}}_Z$ in terms of the stratification together with a morphism of Grothendieck ringed spaces $n_Z\colon (Z, \tilde{\mathcal{O}}_Z) \to (Z, \mathcal{O}_Z)$. In Section 3.3 we specify conditions under which n_Z is the normalization morphism.

We show these conditions in the case $Z = X$ using the isomorphism in (2). The hardest condition to show is that any point in Ω_Γ^* admits a fundamental set of neighborhoods whose intersections with Ω_Γ are irreducible;

this is essentially done in Section 5.3. Having proved all conditions, we then deduce that n_X is the normalization morphism. Via Corollary 1.13, this yields the first part of Theorem 1.11. The second part will then follow directly from the first and the construction of E.

Outline of the thesis

Let C be an algebraically closed complete non-Archimedean valued field. From Section 6 on, suppose that the characteristic of C is finite for otherwise the theory will be empty. For any module M over any ring R and any ring extension $R \subset R'$ let

$$M_{R'} := M \otimes_R R'$$

denote the module over R' obtained by extension of scalars.

Section 2. In this section we recall some definitions and results that are fundamental to what will follow.

In Section 2.1 we recall the notion of Grothendieck ringed space.

In Section 2.2 we define rigid analytic varieties over C as did Bosch, Güntzer and Remmert [8] and recall some fundamental results about them.

In Section 2.3 we determine necessary conditions for the quotient of a rigid analytic variety by a group to be again a rigid analytic variety.

In Section 2.4 we consider any non-Archimedean local field E and any finite dimensional vector space $\mathcal{V} \neq 0$ over E. We recall the structure of $\mathcal{G} := \mathrm{Aut}_E(\mathcal{V})$ as locally profinite group and characterize the discrete subgroups of $\mathcal{G}$ as well as of its quotient $\mathcal{PG} := \mathrm{PGL}(\mathcal{V}) = \mathcal{G}/E^\times$.

In Section 2.5 we call a subring $A \subset C$ an *admissible coefficient subring* if it is a Dedekind domain and finitely generated over a finite subfield and if its intersection with any ball in C is finite. For example, if A and C are as at the beginning of the introduction, then $A \subset C$ is an admissible coefficient subring. We recall some basic facts about A-lattices in C.

Section 3. Let S be a finite set of global sections of an invertible sheaf on a rigid analytic variety Z over C. For any $T \subset S$ denote by $\Omega(T) \subset Z$ the intersection of the non-vanishing locus of T with the vanishing locus of $S \setminus T$. These $\Omega(T)$ for all $T \subset S$ form a *stratification* of Z, i.e., a covering of Z by pairwise disjoint, locally closed subvarieties. In this section we characterize the topology of Z and, in some cases, the normalization of Z in terms of the stratification. The results of this section will be applied in the proof of Theorem 1.11 to the case where S consists of global sections of the first twisting sheaf on $Q_V(C)$.

In Section 3.1 we characterize the Grothendieck topology on Z in terms of this stratification.

In Section 3.2 we consider the special case in which a certain morphism $\mathcal{U}(T) \to \Omega(T)$ is given for any $\Omega(T) \neq \emptyset$, where $\mathcal{U}(T) \subset Z$ denotes the non-vanishing locus of T, and reformulate the result in Section 3.1 in terms of such morphisms. Suitable such morphisms are for instance given when S is a basis of the space of global sections of the first twisting sheaf on any standard projective space (see Example 3.3).

In Section 3.3 we describe, under some conditions, the normalization (in the sense of Conrad's [12]) of Z in terms of the stratification. The criterion obtained is analogous to a special case of [1, Theorem 9.2] by Baily and Borel in the complex analytic setting.

Section 4. Let E, $\mathcal{V}$, $\mathcal{G}$ and $\mathcal{PG}$ be as in Section 2.4 and suppose that E is contained in C as a valued subfield. We recall the Bruhat-Tits building for $\mathcal{PG}$ and the set of homothety classes of norms on $\mathcal{V}$ as well as the $\mathcal{PG}$-equivariant bijection between them that was considered by Drinfeld [15, Section 6]. We try to give natural and rigorous arguments for all steps involved. In the context of this thesis, this section serves as a preparation for Section 5, where we will recall Drinfeld's approach to endow the quotient by any discrete subgroup of his period domain with the structure of rigid analytic variety.

In Section 4.1 we furnish the geometric realization of an arbitrary simplicial complex with a covering whose nerve is the barycentric subdivision of the complex. This is an abstraction of parts of Drinfeld's [15, Section 6].

In Section 4.2 we define in a usual way the Bruhat-Tits building $\mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ for $\mathcal{PG}$ as the geometric realization of some simplicial complex whose vertices are the $E^{\times}$-classes of the free $\mathcal{O}_E$-submodules of $\mathcal{V}$ of maximal rank.

In Section 4.3 we consider the set $\mathrm{N}_{\mathcal{V}}$ of norms on $\mathcal{V}$ and the set $\mathrm{SN}_{\mathcal{V}}$ of seminorms on $\mathcal{V}_C$ that restrict to norms on $\mathcal{V}$ and the natural $\mathbb{R}_{>0}$- and $\mathcal{G}$-actions on these sets. We define a right inverse $i_{\mathrm{N}}\colon \mathrm{N}_{\mathcal{V}} \to \mathrm{SN}_{\mathcal{V}}$ to the natural $\mathbb{R}_{>0}$- and $\mathcal{G}$-equivariant restriction map $r_{\mathrm{N}}\colon \mathrm{SN}_{\mathcal{V}} \to \mathrm{N}_{\mathcal{V}}$.

To any $\nu \in \mathrm{SN}_{\mathcal{V}}$ uniquely corresponds the $\mathcal{O}_C$-submodule $\nu^{-1}([0,1]) \subset \mathcal{V}_C$. Denote by $\mathrm{M}_{\mathcal{V}}$ the set of $\mathcal{O}_C$-submodules of $\mathcal{V}_C$ arising in this way. In Section 4.4 we give an intrinsic definition, in terms of modules, of $\mathrm{M}_{\mathcal{V}}$, of its induced $\mathbb{R}_{>0}$- and $\mathcal{G}$-action, and of the idempotent map $r_{\mathrm{L}}\colon \mathrm{M}_{\mathcal{V}} \to \mathrm{M}_{\mathcal{V}}$ corresponding to $i_{\mathrm{N}} \circ r_{\mathrm{N}}$.

In Section 4.5 we define a set $\mathrm{T}_{\mathcal{V}}$ acted upon by $\mathcal{G}$ and freely by $\mathbb{R}_{>0}$ and define a $\mathcal{G}$-equivariant map $\mathrm{T}_{\mathcal{V}} \to \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ which induces an isomorphism

$$\mathbb{R}_{>0}\backslash\, \mathrm{T}_{\mathcal{V}} \stackrel{\sim}{\longrightarrow} \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0}). \tag{3}$$

Using the intrinsic definition of $\mathrm{M}_{\mathcal{V}}$, we define an $\mathbb{R}_{>0}$- and $\mathcal{G}$-equivariant map $r_{\mathrm{T}}\colon \mathrm{M}_{\mathcal{V}} \to \mathrm{T}_{\mathcal{V}}$ and a right-inverse $i_{\mathrm{T}}\colon \mathrm{T}_{\mathcal{V}} \to \mathrm{M}_{\mathcal{V}}$ with $i_{\mathrm{T}} \circ r_{\mathrm{T}} = r_{\mathrm{L}}$. Identifying $\mathrm{M}_{\mathcal{V}}$ and $\mathrm{SN}_{\mathcal{V}}$, by then it is thus established that

$$i_{\mathrm{T}} \circ r_{\mathrm{T}} = i_{\mathrm{N}} \circ r_{\mathrm{N}}.$$

In particular, the right-inverses i_{T} and i_{N} have the same image and thus yield an $\mathbb{R}_{>0}$- and $\mathcal{G}$-equivariant isomorphism $\mathrm{T}_{\mathcal{V}} \to \mathrm{N}_{\mathcal{V}}$. Its induced isomorphism

$$\mathbb{R}_{>0}\backslash \mathrm{T}_{\mathcal{V}} \xrightarrow{\sim} \mathbb{R}_{>0}\backslash \mathrm{N}_{\mathcal{V}}$$

is, up to (3), precisely the one considered by Drinfeld.

In Section 4.6 we recall the metric on $\mathbb{R}_{>0}\backslash \mathrm{N}_{\mathcal{V}}$ studied by Goldmann and Iwahori in [25, Section 2]. We express in terms of modules the metric induced on $\mathbb{R}_{>0}\backslash i_{\mathrm{N}}(\mathrm{N}_{\mathcal{V}})$. We compute the distance between an arbitrary element and any vertex in $\mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ with respect to the induced metric. This computation makes the covering of $\mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ provided by Section 4.1 expressible in terms of the metric.

Section 5. We give comprehensive proofs of well-known fundamental facts about the rigid analytic structure of Drinfeld's period domain associated with $\mathcal{V}$ and its quotients by discrete subgroups of $\mathcal{PG}$. For some parts, we may proceed more generally: Consider any integer $k \geq 1$. Then

$$\mathbb{P}_{\mathrm{Hom}_C(\mathcal{V}_C, C^k)} := (\mathrm{Hom}_C(\mathcal{V}_C, C^k) \setminus \{0\})/C^\times$$

is equipped with a structure of projective rigid analytic variety over C. Consider the $\mathcal{PG}$-invariant subset

$$\Omega_{\mathcal{V},k} \subset \mathbb{P}_{\mathrm{Hom}_C(\mathcal{V}_C, C^k)}$$

of those $C^\times$-classes $[l]$ of C-linear maps $l\colon \mathcal{V}_C \to C^k$ with $\mathrm{Ker}(l) \cap \mathcal{V} = 0$; if $k = 1$, this is Drinfeld's *period domain* for $\mathcal{V}$ which we denote by $\Omega_{\mathcal{V}}$. Let

$$\lambda_{\mathcal{V},k}\colon \Omega_{\mathcal{V},k} \to \mathbb{R}_{>0}\backslash \mathrm{N}_{\mathcal{V}}$$

be the $\mathcal{PG}$-equivariant map that sends any $[l] = [(l_i)_{1\leq i\leq k}]$ to the class of the norm

$$v \mapsto |l(v)| := \max\nolimits_{1\leq i\leq k} |l_i(v)|.$$

In Section 5.1 we show that $\Omega_{\mathcal{V},k} \subset \mathbb{P}_{\mathrm{Hom}_C(\mathcal{V}_C, C^k)}$ is an admissible subset and that its covering by the preimages of all closed balls under $\lambda_{\mathcal{V},k}$ is admissible and consists of quasi-compact, resp. affinoid if $k = 1$, subsets.

If $k = 1$, this is Drinfeld's [15, Proposition 6.1]. Our proof specializes to the one given by Schneider and Stuhler in [40].

In Section 5.2 we consider any discrete subgroup $\Gamma \subset \mathcal{PG}$. If $k > 1$, we assume that its action on $\Omega_{\mathcal{V},k}$ is free. The quotient $\Omega_{\Gamma,k}$ of $\Omega_{\mathcal{V},k}$ by Γ is naturally equipped with a structure of Grothendieck ringed space via the quotient map. We show that it is a normal rigid analytic variety over C. If $k = 1$, this is Drinfeld's [15, Proposition 6.2]. We generalize his proof as follows: Let $\mathcal{C}$ be the covering of $\Omega_{\mathcal{V},k}$ defined as the preimage under $\lambda_{\mathcal{V},k}$ of the $\mathcal{PG}$-invariant covering of $\mathbb{R}_{>0} \setminus \mathrm{N}_{\mathcal{V}}$ provided by Sections 4.1 and 4.2 via the the identification $\mathbb{R}_{>0} \setminus \mathrm{N}_{\mathcal{V}} = \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ discussed in Section 4. It is admissible and $\Omega_{\Gamma,k}$ is locally isomorphic to the quotient of some element of $\mathcal{C}$ by some finite subgroup of Γ. Moreover, in the case $k = 1$, the elements of $\mathcal{C}$, and hence their quotients by finite groups, are normal affinoid varieties. In the case $k > 1$, the elements of $\mathcal{C}$ are no longer affinoid but still normal quasi-compact; in order for their quotients to still be normal rigid analytic varieties, we assume Γ to act freely which, in particular, allows us to apply Conrad and Temkin's criterion [14, Theorem 5.1.1].

We will use this section in Section 7.4 in order to show that the isomorphism classes of A-lattices in C^k with level I structures are parametrized by rigid analytic varieties.

In Section 5.3 we prove, inspired by van der Put's [43], a result on the connectedness of certain subsets of $\Omega_{\mathcal{V}}$. It implies that Drinfeld's period domain itself and hence its quotient by any discrete subgroup is connected and hence, by normality, irreducible. The result furthermore implies that any point in any of the Satake compactifications in Section 6 admits a fundamental set of irreducible admissible neighborhoods.

In Section 5.4 we suppose that $\dim_E(\mathcal{V}) > 1$ and consider a natural action on $\Omega_{\mathcal{V}}$ by any discrete subgroup of any codimension 1 subspace $\mathcal{W} \subset \mathcal{V}$. We prove that a certain map, defined using exponential functions, from its quotient to $\Omega_{\mathcal{W}} \times C$ is an open embedding between rigid analytic varieties. Such a result is fundamental in order to define the Fourier expansion of weak modular forms and, in particular, to define modular forms.

Section 6. In this section we construct the compactification of any irreducible component of the analytic modular variety viewing it as a quotient by a congruence subgroup. Consider any admissible coefficient subring $A \subset C$ as in Section 2.5 and denote by $E \subset C$ the completion of its quotient field. Consider any projective A-module $\Lambda \neq 0$ of finite rank, set $\mathcal{V} := \Lambda_E$ and

$$\Omega_\Lambda := \Omega_{\mathcal{V}}.$$

Let Ω_Λ^* denote the disjoint union of the sets Ω_L for all direct summands $0 \neq L \subset \Lambda$. Consider any congruence subgroup $\Gamma \subset \mathrm{Aut}_A(\Lambda)$. Then Ω_Λ^* is naturally equipped with a natural Γ-action which is compatible with the action of Γ on the direct summands of Λ and which restricts to the action on Ω_Λ of its image in $\mathcal{PG}$ studied in Section 5.2.

In Section 6.1 we endow Ω_Λ^* with the structure of Grothendieck topological space which induces on any stratum Ω_L the rigid analytic Grothendieck topology and which contains Ω_Λ as a dense admissible subset.

In Section 6.2 we study the induced Grothendieck topology on

$$\Omega_\Gamma^* := \Gamma \backslash \Omega_\Lambda^*$$

and endow this quotient with a structure sheaf $\mathcal{O}_\Gamma^*$ and a sheaf of graded $\mathcal{O}_\Gamma^*$-algebras $\mathcal{R}_\Gamma^*$ whose homogeneous components restrict on Ω_Γ to the sheaf of weak modular forms with respect to Γ.

In Section 6.3 we define the natural morphisms between the Grothendieck graded ringed spaces as Γ varies. In fact, we define a category of such tripples (A, Λ, Γ) and understand $(A, \Lambda, \Gamma) \mapsto \Omega_\Gamma^*$ as a functor.

In Section 6.4 we define Eisenstein series and Poincaré-Eisenstein series as explicit global sections of homogeneous components of $\mathcal{R}_\Gamma^*$.

In Section 6.5 we show that any two points $p, p' \in \Omega_\Gamma^*$ admit a Poincaré-Eisenstein series P for which $P(p) \neq 0 = P(p')$.

In Section 6.6 jointly with Section 5.4, we provide a comprehensive proof that any weak modular form has a Fourier expansion at any cusp.

Section 7. In this section we define the compactifications of the analytic modular varieties using adelic language. Consider any $A \subset C$ as before and let $\hat{A}$ be its profinite completion. Let $M \neq 0$ be any free finitely generated $\hat{A}$-module. We define Ω_M, resp. Ω_M^*, to be the disjoint union of (copies of) the Grothendieck topological spaces Ω_Λ, resp. the Ω_Λ^*, for all A-submodules $\Lambda \subset M$ for which $\Lambda_{\hat{A}} \to M$ is an isomorphism.

Consider any congruence subgroup $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$. Then $\mathcal{K}$ acts in a natural way on Ω_M^* which is compatible with the $\mathcal{K}$-action on such A-submodules Λ and which restricts to an action on Ω_M. Then the quotient

$$\Omega_\mathcal{K} := \mathcal{K} \backslash \Omega_M$$

has the structure of rigid analytic variety and in fact any $X_I^d(C)$ is isomorphic to such a quotient.

In Section 7.1, we endow the quotient

$$\Omega_\mathcal{K}^* := \mathcal{K} \backslash \Omega_M^*$$

with a structure of Grothendieck graded ringed space and show that it is, as such, the disjoint union of finitely many spaces Ω_Γ^* as in Section 6.

In Sections 7.3 and 7.4 we study the case where $\mathcal{K}$ is the kernel of the natural morphism $\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_A(M/IM)$ for some ideal $0 \neq I \subset A$; in particular, we provide a proof that $\Omega_\mathcal{K}$ then parametrizes A-lattices in C of rank $\mathrm{rank}_{\hat{A}}(M)$ with a level I structure.

Section 8. In Section 8.1 we recall the notion of (generalized) Drinfeld module and Pink's compactifications of the algebraic moduli spaces.

In Section 8.2 we define A-reciprocal maps and prove Theorem 1.5.

Section 9. Here we prove Theorem 1.11 and then deduce various consequences such as Theorems 1.6 and 1.7 as well as Corollary 1.9.

2 Preliminaries and preparation

Let C be an algebraically closed complete non-Archimedean valued field.

2.1 Grothendieck ringed spaces

This thesis builds on the notion of Grothendieck ringed spaces and morphisms between them as recalled in this section. In fact, the category of rigid analytic varieties is defined (see Section 2.2) as a subcategory of the the category of Grothendieck ringed spaces. Moreover, our compactifications will a priori be defined (see Sections 6 and 7) as Grothendieck (graded) ringed spaces and will only after quite some work turn out to be rigid analytic varieties.

Definition 2.1. *i) A family $\{U_i\}_{i\in I}$ of subsets U_i of a set U is called a* covering *of U if $U = \bigcup_{i\in I} U_i$.*

ii) A covering $\{U'_j\}_{j\in J}$ of a set U is called a refinement *of a covering $\{U_i\}_{i\in I}$ of U if there exists a map $\tau\colon J \to I$ with $U'_j \subset U_{\tau(j)}$ for any $j \in J$.*

iii) The intersection *of a covering $\{U_i\}_{i\in I}$ of a set U with a subset $U' \subset U$ is the covering $\{U_i \cap U'\}_{i\in I}$ of U'.*

iv) The intersection *of a covering $\{U_i\}_{i\in I}$ of a subset $U \subset X$ with a covering $\{U'_j\}_{j\in J}$ of a subset $U' \subset X$ is the covering $\{U_i \cap U'_j\}_{i\in I, j\in J}$ of $U \cap U'$.*

v) The preimage *of a covering $\{U_i\}_{i\in I}$ of a subset $U \subset X$ under a map $f\colon Y \to X$ is the covering $\{f^{-1}(U_i)\}_{i\in I}$ of $f^{-1}(U)$.*

Definition 2.2. *A* Grothendieck topology *on a set X consists of*

- *a system $\mathcal{S}$ of subsets of X and*
- *a family $\mathcal{C} = \{\mathrm{Cov}(U)\}_{U\in S}$ of systems of coverings, where $\mathrm{Cov}(U)$ contains coverings of U by elements in S for any $U \in S$,*

subject to the following conditions:

i) $U, U' \in \mathcal{S} \Rightarrow U \cap U' \in \mathcal{S}$.

ii) $U \in \mathcal{S} \Rightarrow \{U\} \in \mathrm{Cov}(U)$.

iii) If $U \in \mathcal{S}, \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ and $\{U_{ij}\}_{j\in J_i} \in \mathrm{Cov}(U_i)$ for any $i \in I$, then $\{U_{ij}\}_{i\in I, j\in J_i} \in \mathrm{Cov}(U)$.

iv) *If $U, U' \in \mathcal{S}$ with $U' \subset U$ and if $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, then $\{U_i \cap U'\}_{i\in I} \in \mathrm{Cov}(U')$.*

v) *$\emptyset, X \in \mathcal{S}$.*

vi) *If $U \in \mathcal{S}$ and $U' \subset U$ such that there exists $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ with $U_i \cap U' \in \mathcal{S}$ for any $i \in I$, then $U' \in \mathcal{S}$.*

vii) *Consider any $U \in \mathcal{S}$ and any covering $\{U_i\}_{i\in I}$ of U with $U_i \in \mathcal{S}$ for any $i \in I$. If $\{U_i\}_{i\in I}$ has a refinement in $\mathrm{Cov}(U)$, then it is itself in $\mathrm{Cov}(U)$.*

If a Grothendieck topology $(\mathcal{S}, \mathcal{C})$ on X is understood, then the elements of $\mathcal{S}$ are called the admissible *subsets of X and the elements of any $\mathrm{Cov}(U)$ are called the* admissible *coverings of U. In this case, the topology (in the usual sense) of X whose open sets are the unions of admissible sets, is called the* canonical topology *of X.*

Definition 2.3. *A morphism of Grothendieck topological spaces is a map under which the preimage of any admissible subset and of any admissible covering is admissible.*

Definition 2.4. *Consider any Grothendieck topological space X and any ring R.*

i) *A* presheaf *of (graded) R-algebras on X is a contravariant functor from the category of all admissible subsets of X with inclusions as morphisms into the category of (graded) R-algebras.*

ii) *Given any presheaf $\mathcal{F}$ on X, we denote by*

$$\mathcal{F}(U) \to \mathcal{F}(U'),\ f \mapsto f|_{U'}$$

the morphism associated with any admissible subsets $U' \subset U \subset X$.

iii) *A presheaf $\mathcal{F}$ on X is called a* sheaf *if any admissible subset U of X and any admissible covering $\mathcal{C}$ of U satisfy:*

- *If $f, g \in \mathcal{F}(U)$ are such that $f|_{U'} = g|_{U'}$ for any $U' \in \mathcal{C}$, then $f = g$.*
- *For any family $(f_{U'})_{U'\in\mathcal{C}} \in (\mathcal{F}(U'))_{U'\in C}$ with*

$$\forall\, U', U'' \in C\colon\ f_{U'}|_{U'\cap U''} = f_{U''}|_{U'\cap U''}$$

there exists an $f \in \mathcal{F}(U)$ such that $f|_{U'} = f_{U'}$ for any $U' \in \mathcal{C}$.

iv) *The morphisms between sheaves on X are the morphisms between the underlying presheaves.*

Definition-Proposition 2.5. *[8, Proposition 9.2.2.4] Any presheaf $\mathcal{F}$ on a Grothendieck topological space admits a* sheafification, *i.e., a homomorphism $i\colon \mathcal{F} \to \mathcal{F}'$ into a sheaf $\mathcal{F}'$ such that any homomorphism $\mathcal{F} \to \mathcal{G}$ into a sheaf $\mathcal{G}$ equals $\varphi \circ i$ for a unique morphism $\varphi\colon \mathcal{F}' \to \mathcal{G}$.*

Definition 2.6. *A* Grothendieck (graded) ringed space *over a ring R is a pair $(X, \mathcal{F})$, where X is a Grothendieck topological space and $\mathcal{F}$ is a sheaf of (graded) R-algebras on X.*

Definition 2.7. *A morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ of Grothendieck (graded) ringed spaces over a ring R is a pair $(f, f^\#)$, where $f\colon X \to Y$ is a morphism of Grothendieck topological spaces and where $f^\#$ is a collection of (graded) R-algebra homomorphisms*

$$f_U^\# \colon \mathcal{G}(U) \to \mathcal{F}(f^{-1}(U))$$

compatible with restriction homomorphisms, where U ranges over all admissible subsets of Y.

2.2 Rigid analytic varieties

In this section we briefly recall the language of rigid analytic varieties over C as developed by Bosch, Güntzer and Remmert in [8] and some results about such varieties that will be used repeatedly.

Definition 2.8. *A C-*algebra norm *on a C-algebra R is a map $|\cdot|\colon R \to \mathbb{R}_{\geq 0}$ which restricts to the norm on C such that every $r, s \in R$ satisfy*

- $|r| = 0 \Leftrightarrow r = 0$,
- $|r \cdot s| \leq |r| \cdot |s|$,
- $|r - s| \leq \max\{|r|, |s|\}$.

Definition 2.9. *A C-*Banach algebra *is a C-algebra R together with a C-algebra norm whose induced topology on R is complete.*

Definition-Proposition 2.10. *[8, Proposition 5.1.1.1] For any integer $n \geq 0$ the* Tate algebra *over C in n variables is the subalgebra T_n of $C[[X_1, \ldots, X_n]]$ of elements*

$$f = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} \cdot X_1^{i_1} \cdot \cdots \cdot X_n^{i_n}$$

for which $|a_{i_1, \ldots, i_n}| \to 0$ as $i_1 + \cdots + i_n \to \infty$. The Gauss norm

$$|f| := \max_{i_1, \ldots, i_n \geq 0} |a_{i_1, \ldots, i_n}|$$

is a C-algebra norm on T_n by means of which T_n is a C-Banach algebra.

Definition 2.11. *A C-Banach algebra R is called C-*affinoid *if there exists an integer $n \geq 0$ and a continuous epimorphism $T_n \to R$.*

Definition 2.12. *i) A C-*affinoid variety *is a pair $\mathrm{Sp}(R) = (\mathrm{Max}(R), R)$, where R is any C-affinoid algebra and $\mathrm{Max}(R)$ is the* maximal spectrum *of R, i.e., the set of maximal ideals of R equipped with the Zariski topology.*

ii) A morphism *$\mathrm{Sp}(S) \to \mathrm{Sp}(R)$ of C-affinoid varieties is a pair $(\sigma, \sigma^{\#})$, where $\sigma^{\#}\colon R \to S$ is any C-algebra homomorphism and*

$$\sigma\colon \mathrm{Max}(S) \to \mathrm{Max}(R), \mathfrak{m} \mapsto (\sigma^{\#})^{-1}(\mathfrak{m})$$

is the induced continuous map.

Definition-Proposition 2.13. *[8, Proposition 7.2.2.1]*

i) A morphism $(i, i^{\#})\colon \mathrm{Sp}(R') \to \mathrm{Sp}(R)$ is called an open immersion *if for any morphism*

$$(\sigma, \sigma^{\#})\colon \mathrm{Sp}(S) \to \mathrm{Sp}(R) \text{ with } \sigma(\mathrm{Max}(S)) \subset i(\mathrm{Sp}(R'))$$

there exists a unique morphism $(\psi, \psi^{\#})\colon \mathrm{Sp}(S) \to \mathrm{Sp}(R')$ with

$$(\sigma, \sigma^{\#}) = (i, i^{\#}) \circ (\psi, \psi^{\#}).$$

In this case, i is injective.

ii) Any composition of open immersions is an open immersion.

iii) A subset $U \subset \mathrm{Max}(R)$ is called affinoid *if it is the image of i of an open immersion $(i, i^{\#})\colon \mathrm{Sp}(R') \to \mathrm{Sp}(R)$. In this case, U is (uniquely up to unique isomorphism) endowed with the structure of C-affinoid variety and we identify U with $\mathrm{Sp}(R')$.*

iv) The preimage of any affinoid subset under any morphism between C-affinoid varieties is an affinoid subset.

Definition-Proposition 2.14. *[8, Proposition 9.1.4.2] The following specifies a structure of Grothendieck topology on any C-affinoid variety $\mathrm{Sp}(R)$:*

i) A subset $X \subset \mathrm{Max}(R)$ is admissible *if it admits a covering C by affinoid subsets of $\mathrm{Max}(R)$ whose preimage under any morphism $\mathrm{Sp}(S) \to \mathrm{Sp}(R)$ has a finite refinement by affinoid subsets of $\mathrm{Max}(S)$. In particular, the union of any finitely many affinoid subsets of $\mathrm{Max}(R)$ is admissible.*

ii) A covering $\mathcal{C}$ of an admissible subset $X \subset \mathrm{Max}(R)$ by admissible subsets is admissible *if its preimage under any morphism $\mathrm{Sp}(S) \to \mathrm{Sp}(R)$ has a finite refinement by affinoid subsets of $\mathrm{Max}(S)$.*

Definition-Proposition 2.15. *[8, Proposition 9.2.3.1] Consider any C-affinoid variety $Y = \mathrm{Sp}(R)$. Then there exists a unique sheaf $\mathcal{O}_Y$ of C-algebras on Y with $\mathcal{O}_Y(\mathrm{Sp}(R')) = R'$ for any affinoid subset $\mathrm{Sp}(R') \subset Y$ and such that for any composition of open immersions*

$$\mathrm{Sp}(R'') \overset{(j,j^{\#})}{\longrightarrow} \mathrm{Sp}(R') \subset X$$

the restriction homomorphism $\mathcal{O}_Y(\mathrm{Sp}(R')) \to \mathcal{O}_Y(\mathrm{Sp}(R''))$ equals $j^{\#}$. In particular, the pair $(Y, \mathcal{O}_Y)$ is a Grothendieck ringed space over C.

Definition 2.16. *A Grothendieck ringed space $(X, \mathcal{O})$ over C is a* rigid analytic variety *over C if X admits an admissible covering $\mathcal{C}$ and any $U \in \mathcal{C}$ possesses an isomorphism $(U, \mathcal{O}|_U) \cong (Y, \mathcal{O}_Y)$ of Grothendieck ringed spaces for some C-affinoid variety Y.*

As C is algebraically closed, the elements of any affinoid C-algebra A uniquely give rise to functions $\mathrm{Sp}(A) \to C$ (see [8, Section 7.1]). The global sections of any rigid analytic variety $(X, \mathcal{O})$ over C may thus be viewed as the functions $f \colon X \to C$ whose restriction to any admissible affinoid subset $\mathrm{Sp}(A)$ are induced by elements of A.

Definition 2.17. *Any such $f \colon X \to C$ is called* regular.

Definition-Proposition 2.18. *For any affinoid varieties X, Y*

$$\mathrm{Mor}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \to \mathrm{Mor}(X, Y),\ (f, f^{\#}) \mapsto (f, f^{\#}_Y)$$

constitutes a bijection by means of which we view the category of C-affinoid varieties as a full subcategory of the category of rigid analytic varieties over C.

Proposition 2.19. *[8, Theorem 6.2.4.1] For any affinoid algebra A over C the map $A \to |C|,\ f \mapsto \sup_{x \in \mathrm{Sp}(A)} |f(x)|$ is a complete norm on A.*

Example 2.20. *[8, Example 9.3.4.1] For any $n \geq 0$ the affine space C^n has a unique structure of rigid analytic variety for which the covering by all closed balls with radius in $|C|$ is admissible affinoid, where any such ball is naturally isomorphic to $\mathrm{Sp}(T_n)$.*

Example 2.21. *[8, Example 9.3.4.3] For any $n \geq 1$ the projective space P^n_C over C has a unique structure of rigid analytic variety which is compatible with the ones of all affine subspaces.*

Proposition 2.22. *Let Z be the product of any affine with any projective rigid analytic variety. Then the intersection of finitely many affinoid subsets of Z is again affinoid.*

Proof. Both affine and projective rigid analytic varieties and hence their products are separated in the sense of [8, Definition 9.6.1.1]. The proposition then holds by [8, Proposition 9.6.1.6]. □

Definition 2.23. *A morphism of rigid analytic varieties is called a* locally closed immersion *if the underlying map is injective and the induced homomorphisms on stalks are surjective.*

Proposition 2.24. *[8, Proposition 9.5.3.5] A morphism $f\colon Y \to X$ of rigid analytic varieties is a closed immersion if and only if*

i) it is a locally closed immersion,

ii) its image is an analytic subset of X and

iii) there exists an admissible affinoid covering $(X_i)_{i\in I}$ of X and, for each $i \in I$, a finite admissible affinoid covering of $f^{-1}(X_i)$.

Proposition 2.25. *(Maximum Modulus Principle) [8, Lemma 9.1.4.6] Consider any affinoid algebra A and any $f \in A$. Then there exists $c > 0$ with $|f(x)| \leq c$ for any $x \in X := \mathrm{Sp}(A)$. Moreover, if f vanishes nowhere on X, then there exists $\delta > 0$ with $|f(x)| \geq \delta$ for any $x \in X$.*

Definition 2.26. *[8, Definition 7.2.3.5] A subset X of an affinoid variety $Y = \mathrm{Sp}(R)$ over C is called* rational *if*

$$X = \{y \in Y \mid \forall 1 \leq i \leq n\colon |r_i(y)| \leq |r(y)|\}$$

for some $r, r_1, \ldots, r_n \in R$ generating the unit ideal.

Proposition 2.27. *[8, Proposition 7.2.3.4] Any rational subset of any affinoid variety is admisible affinoid.*

Theorem 2.28. *(Gerritzen and Grauert) [8, Cor. 7.3.5.3] Any affinoid subvariety of any affinoid variety X is the union of finitely many rational subdomains of X.*

Corollary 2.29. *Consider any $n \geq 1$, any closed subvariety $Z \subset C^n$ and any quasi-compact admissible subset $X \subset Z$. Then there exists an $\varepsilon > 0$ such that*

$$\forall x \in X\ \forall z \in Z\colon |x - z| \leq \varepsilon \Rightarrow z \in X.$$

Proof. As $X \subset Z$ is admissible quasi-compact, it is a finite union of affinoid subsets of Z. We thus assume without loss of generality that $X \subset Z$ itself is affinoid. Moreover, X is contained in some closed ball $B \subset C^n$ around the origin. We assume without loss of generality that the radius of B is 1. Let $Y \subset B$ be an affinoid subvariety with $Y \cap Z = X$. By Theorem 2.28, Y is a finite union of rational subdomains of B. We thus assume without loss of generality that Y is itself a rational subdomain of B, i.e., that there exist regular functions $f_0, f_1, \ldots, f_k$ on B without a common zero such that

$$Y = \{b \in B \mid \forall 1 \leq i \leq k \colon |f_i(b)| \leq |f_0(b)|\}.$$

In particular, f_0 has no zero on Y. By Proposition 2.25, there exists thus $\delta > 0$ with $|f_0(y)| > \delta$ for any $y \in Y$. Moreover, by [8, Prop. 7.2.1.1], there exists $c > 0$ with $|f_i(b) - f_i(b')| \leq c \cdot |b - b'|$ for any $b, b' \in B$ and any $0 \leq i \leq k$. Choose any $0 < \varepsilon < 1$ with $c \cdot \varepsilon < \delta$. Any $x \in X, z \in Z$ with $|x - z| \leq \varepsilon$ then satisfy that $z \in Y \cap Z = X$ since

$$\begin{aligned}|f_i(z)| &= |f_i(z) - f_i(x) + f_i(x)| \leq \max\{c \cdot |z - x|, |f_i(x)|\} \leq \max\{\delta, |f_i(x)|\} \\ &\leq |f_0(x)| = |f_0(x) - f_0(z) + f_0(z)| = |f_0(z)|\end{aligned}$$

for any $1 \leq i \leq k$, where in the last step we have used that

$$|f_0(x) - f_0(z)| \leq c \cdot |x - z| \leq c \cdot \varepsilon < \delta < |f_0(x)|.$$

□

Corollary 2.30. *Consider any rigid analytic variety Y and and any closed subvariety $Z \subset Y$. For any admissible quasi-compact $O \subset Z$ exists an admissible quasi-compact subset $U \subset Y$ with $U \cap Z = O$. Moreover, if Y is affinoid, then for any rational subdomain $O \subset Z$ exists a rational subdomain $U \subset Y$ with $U \cap Z = O$.*

Proof. By means of an admissible affinoid covering of Y, the first part is reduced to the case where Y is affinoid. In this case the first part is reduced to the second part by by Theorem 2.28. We thus assume that Y is affinoid and consider first any rational subdomain $O \subset Z$. Let $f_1, \ldots, f_n, g$ be regular functions on Z without common zeroes such that

$$O = \{z \in Z \mid \forall 1 \leq i \leq n : |f_i(z)| \leq |g(z)|\}.$$

In particular, g has no zero on O so that, by Proposition 2.25, it is bounded from below on O by some $\varepsilon \in |C^\times|$. In particular, we may assume that

$f_n = \varepsilon$. Choose then any lifts $\tilde{f}_1, \ldots, \tilde{f}_n, \tilde{g}$ to Y of $f_1, \ldots, f_n, g$ with $\tilde{f}_n = \varepsilon$. As $\varepsilon \neq 0$, these lifts have no common zero. Hence

$$U := \{y \in Y \mid \forall 1 \leq i \leq n : |\tilde{f}_i(y)| \leq |\tilde{g}(y)|\} \subset Y$$

is a rational subdomain with $U \cap Z = O$. □

The equivalence of i) and iii) in the next proposition is the Riemann extension theorem for affinoid varieties, which was proved by Bartenwerfer.

Theorem 2.31. *[2, Section 3] Consider any normal quasi-compact rigid analytic variety Y, any closed subvariety $Z \subsetneq Y$ which is everywhere of positive codimension and any regular function $s \colon Y \setminus Z \to C$. Then the following are equivalent:*

i) s extends uniquely to a regular function $Y \to C$,

ii) s extends uniquely to a morphism $Y \to C$ of Grothendieck topological spaces whose restriction to Z is regular,

iii) s is bounded.

Proof. That i) implies ii) follows immediately from the definitions and that ii) implies iii) follows from the quasi-compactness of Y and the fact that covering of C by all closed balls of integer radius around the origin is admissible. By means of an admissible affinoid covering of Y, the implication i) ⇒ iii) is reduced the case where Y is affinoid which it is the content of [2, Section 3]. □

In his proof of Proposition 2.31, Bartenwerfer proved and used the following result on Laurent series.

Proposition 2.32. *[2, Satz 12] Consider any separated quasi-compact variety O, any $\varepsilon \in |C^\times|$ and any regular function $s \colon O \times \dot{B}_\varepsilon \to C$. Then there exist unique regular functions $s_i \colon O \to C$ over all $i \in \mathbb{Z}$ such that*

$$s((o,z)) = \sum_{i \in \mathbb{Z}} s_i(o) z^i, \quad \textit{for any } (o,z) \in O \times \dot{B}_\varepsilon.$$

Moreover, the following are equivalent:

i) s extends uniquely to a regular function $O \times B_\varepsilon \to C$,

ii) s extends uniquely to a morphism $O \times B_\varepsilon \to C$ of Grothendieck topological spaces whose restriction to $O \times \{0\}$ is regular,

iii) s is bounded.

iv) $\forall i < 0 : s_i = 0$,

2.3 On some quotients of rigid analytic varieties

Consider any group Γ of C-linear automorphisms of any rigid analytic variety Y over C. Let

$$p\colon Y \to \Gamma\backslash Y$$

be the quotient morphism, where $\Gamma\backslash Y$ is endowed with the structure of Grothendieck ringed space induced by the quotient map, that is, a subset (resp. a covering of a subset) of $\Gamma\backslash Y$ is admissible precisely when its preimage is admissible and the sections on an admissible subset of $\Gamma\backslash Y$ are the Γ-invariant sections on its preimage.

Proposition 2.33. *Suppose that $Y = \mathrm{Sp}(A)$ is the affinoid variety associated with any affinoid variety A and suppose that Γ is finite. Then the subalgebra $A^\Gamma \subset A$ of Γ-invariant elements is affinoid and induces an isomorphism of affinoid varieties*

$$\Gamma\backslash \mathrm{Sp}(A) \to \mathrm{Sp}(A^\Gamma).$$

Moreover, A is a finite A^Γ-module and if A is normal, then so is A^Γ.

Proof. For the first part see [26, Theorem 1.3]. Moreover, A is a finite A^Γ-module by [8, Proposition 6.3.3.3]. Finally suppose that A is normal. Then the irreducible components of $\mathrm{Sp}(A)$ are disjoint and permuted by G. We may thus assume without loss of generality that $\mathrm{Sp}(A)$ is irreducible. Then A and hence $A^\Gamma \subset A$ is a domain. Consider any $s, t \in A^\Gamma$ with $t \neq 0$ such that s/t is integral over A^Γ and let us show that then $s/t \in A^\Gamma$. Since A is integrally closed, there exists an $a \in A$ such that $s = t \cdot a$. It remains to be shown that in fact $a \in A^\Gamma$. As s and t are Γ-invariant, a and γa coincide outside the zero locus of s for any $\gamma \in \Gamma$. If $s \neq 0$, then normality of A yields via Proposition 2.31 that $a = \gamma a$ for any $\gamma \in \Gamma$ and hence that $a \in A^\Gamma$. On the other hand, $a = 0 \in A^\Gamma$ if $s = 0$. $\square$

We will use the following generalization of Proposition 2.33.

Proposition 2.34. *Suppose that Y is separated (see [8, Definition 9.6.1.1.]). Denote by $p\colon Y \to \Gamma\backslash Y$ the quotient map. Consider any admissible affinoid covering $(Y_n)_{n\geq 1}$ of Y and finite subgroups $(\Gamma_n)_{n\geq 1}$ of Γ such that*

i) $\forall n' \geq n \geq 1\colon \Gamma_n \subset \Gamma_{n'} \wedge Y_n \subset Y_{n'}$,

ii) $\forall n \geq 1, \forall \gamma \in \Gamma_n\colon \gamma(Y_n) = Y_n$

iii) and any $n \geq 1$ admits an $n' \geq 1$ such that $\forall \gamma \in \Gamma \setminus \Gamma_{n'} : \gamma(Y_n) \cap Y_n = \emptyset$.

Then $(p(Y_n))_{n\geq 1}$ is an admissible covering of $\Gamma\backslash Y$ and any $p(Y_n)$ is admissibly covered by finitely many affinoid varieties. In particular, $\Gamma\backslash Y$ is a rigid analytic variety. Moreover, if Y is normal, then so is $\Gamma\backslash Y$.

Proof. Consider any $n \geq 1$ and choose any $n' \geq n \geq 1$ satisfying the property in iii). Let I be a set of representatives of $\Gamma/\Gamma_{n'}$. Set

$$\forall \gamma \in I\colon U_\gamma := \bigcup_{\gamma' \in \Gamma_{n'}} (\gamma\gamma')(Y_n).$$

Then the U_γ are pairwise disjoint and they cover $U := p^{-1}(p(Y_n))$. We claim that $U \subset Y$ is admissible and admissibly covered by the U_γ and, in particular, that $p(Y_n) \subset \Gamma\backslash Y$ is admissible. In order to prove the claim, it is enough, since $(Y_k)_{k\geq 1}$ is an admissible covering of Y, to check for any $k \geq 1$ that $U \cap Y_k \subset Y_k$ is admissible and admissibly covered by $(U_\gamma \cap Y_k)_{\gamma \in I}$. Consider any such k. Since Y is separated, the intersection of any finitely many affinoid subsets of Y is again affinoid [8, Proposition 9.6.1.6]. As U_γ is the union of finitely many admissible affinoid subsets, thus so is $U_\gamma \cap Y_k$ for any $\gamma \in I$. Moreover, iii) provides a $k' \geq 1$ such that $U_\gamma \cap Y_k = \emptyset$ for any $\gamma \in I \setminus \Gamma_{k'}$. Hence $U \cap Y_k$ is the union of finitely many admissible affinoid subsets and hence an admissible subset of Y_k and the covering $(U_\gamma \cap Y_k)_{\gamma \in I}$ has the finite affinoid, and thus admissible, refinement $(U_\gamma \cap Y_k)_{\gamma \in I \cap \Gamma_{k'}}$ and is thus itself admissible. This yields the claim.

As $\Gamma_{n'}$ is finite and acts on the affinoid $Y_{n'}$ by ii), Proposition 2.33 yields that $\Gamma_{n'}\backslash Y_{n'}$ is an affinoid variety and that its admissible subsets are precisely those whose preimages in $Y_{n'}$ are admissible. Let $\gamma_0 \in I$ represent the identity. By i), U_{γ_0} is the union of finitely many affinoid subsets of $Y_{n'}$, and hence quasi-compact, and $\Gamma_{n'}$-invariant. Hence its image $\Gamma_{n'}\backslash U_{\gamma_0}$ in $\Gamma_{n'}\backslash Y_{n'}$ is an admissible quasi-compact subset or, equivalently, the union of finitely many admissible affinoid subsets. As the U_γ are pairwise disjoint and form an admissible covering, the inclusion morphism $U_{\gamma_0} \to U$ induces an isomorphism $\Gamma_{n'}\backslash U_{\gamma_0} \to \pi(Y_n)$ of Grothendieck ringed spaces. Thus $p(Y_n)$ is indeed admissibly covered by finitely many affinoid varieties. Moreover, if Y is normal, then so is $Y_{n'}$ and hence $\Gamma_{n'}\backslash Y_{n'}$ by Proposition 2.33 and hence $\Gamma_{n'}\backslash U_{\gamma_0}$ and hence $p(Y_n)$.

It remains to be checked that the covering $(p(Y_n))_{n\geq 1}$ of $\Gamma\backslash Y$ is admissible. Using that $(Y_k)_{k\geq 1}$ is an admissible covering of Y, it suffices to check for any $k \geq 1$ that the covering $(p^{-1}(p(Y_n)) \cap Y_k)_{n\geq 1}$ of Y_k is admissible. But the latter covering has as admissible refinement the covering given by the single subset $p^{-1}(p(Y_k)) \cap Y_k$, i.e., by Y_k, and is thus itself admissible. □

2.4 PGL over a non-Archimedean local field

Consider any non-Archimedean local field E. Denote by $\mathcal{O}_E$ the ring of integers of E. Choose a prime element $\pi \in \mathcal{O}_E$ and set $q := |\frac{1}{\pi}|$. Consider any finite dimensional E-vector space $\mathcal{V} \neq 0$ and set $d := \dim_E(\mathcal{V})$. Set

$$\mathcal{G} := \mathrm{Aut}_E(\mathcal{V}) \text{ and } \mathcal{PG} := \mathrm{PGL}(\mathcal{V}) = \mathcal{G}/E^{\times}.$$

We shall recall the structure of $\mathcal{V}$ and of $\mathcal{G}$ as locally profinite groups and describe the discrete subgroups of $\mathcal{G}$ and of $\mathcal{PG}$.

Definition 2.35. *i) A topological group is* profinite *if it is the inverse limit of an inverse system of discrete finite groups or, equivalently, if it is Hausdorff, compact and totally disconnected.*

ii) A topological group is locally profinite *if it is Hausdorff and if the identity has a fundamental sytems of profinite open neighborhoods.*

Definition 2.36. *Let $\mathrm{S}_{\mathcal{V}}$ be the set of free $\mathcal{O}_E$-submodules $m \subset \mathcal{V}$ of maximal rank together with the action of $E^{\times}$ by* dilation*, i.e., the one induced by scalar multiplication.*

Definition-Proposition 2.37. *i) Any free $\mathcal{O}_E$-module m of finite rank together with the natural morphisms $m \to m/\pi^n m$ for all $n \geq 0$ is the inverse limit of the natural projections between these $m/\pi^n m$; as such, endow m with the structure of profinite topological group.*

ii) Let $\mathcal{V}$ be endowed with the unique structure of locally profinite topological group inducing on any free $m \in \mathrm{S}_{\mathcal{V}}$ the profinite topology and containing m as an open subgroup. Let $E^{\times}\backslash\mathcal{V}$ be endowed with the quotient topology.

Proof. As the field E is non-Archimedean local, $\mathcal{O}_E$ is the inverse limit of the finite $\mathcal{O}_E/\pi^n\mathcal{O}_E$ for all $n \geq 0$ and their projection morphisms. From this, i) directly follows. Any $m \in \mathrm{S}_{\mathcal{V}}$ induces a unique structure of topological group on $\mathcal{V}$ which contains m as an open subgroup and induces the profinite topology on it. This induced topology does not depend on the choice of any such m since, as is directly checked, for any further $m' \in \mathrm{S}_{\mathcal{V}}$ the fundamental sets of neighborhoods $(\pi^n m)_{n\geq 0}$ of 0 in m and $(\pi^n m')_{n\geq 0}$ of 0 in m' are cofinal. Moreover, the intersection for all $n \geq 0$ of the $\mathcal{O}_E\pi^n$, and hence of the $\pi^n m$, is 0. Since a topological group is Hausdorff if and only if some open subgroups have trivial intersection, this topology is thus also Hausdorff which yields ii). □

Lemma 2.38. *Consider any non-negative integer n, any free $\mathcal{O}_E$-module m of finite rank and any free $\mathcal{O}_E$-submodules $m', m'' \subset m$ each having a free direct complement in m and such that $\overline{m}' := m'/\pi^n m'$ and $\overline{m}'' := m''/\pi^n m''$ coincide as $\mathcal{O}_E/\pi^n\mathcal{O}_E$-submodules of $\overline{m} := m/\pi^n m$. For any $\mathcal{O}_E$-linear isomorphism $\tau : m' \to m''$ and any $\epsilon \in \mathrm{Aut}_{\mathcal{O}_E/\pi^n\mathcal{O}_E}(\overline{m})$ whose restriction to $\overline{m}'$ is the isomorphism $\overline{m}' \to \overline{m}''$ induced by τ there exists a $\sigma \in \mathrm{Aut}_{\mathcal{O}_E}(m)$ that induces ϵ and restricts to τ.*

Proof. Consider any such τ and ε. Using surjectivity of the natural morphism $m \to \overline{m}$ and using any $\mathcal{O}_E$-basis of m that extends an $\mathcal{O}_E$-basis of m', we may choose a $\sigma \in \mathrm{Hom}_{\mathcal{O}_E}(m, m)$ that induces ϵ and restricts to τ. Then the determinant of σ modulo π^n equals the determinant of ϵ and is thus a unit. If $n \geq 1$, then the determininant of σ is thus itself a unit since $\mathcal{O}_E$ is a discrete valuation ring so that σ is in fact an automorphism. If $n = 0$, then $\epsilon = 0$, so that σ may be *chosen* to further be an automorphism. ☐

Definition-Proposition 2.39. *Consider any free $\mathcal{O}_E$-module m of finite rank. Then $\mathrm{Aut}_{\mathcal{O}_E}(m)$ together with the natural surjective morphisms*

$$\mathrm{Aut}_{\mathcal{O}_E}(m) \to \mathrm{Aut}_{\mathcal{O}_E/\pi^n\mathcal{O}_E}(m/\pi^n m) \tag{4}$$

for all $n \geq 0$ is the inverse limit of all natural morphisms between these targets; as such, endow $\mathrm{Aut}_{\mathcal{O}_E}(m)$ with the structure of profinite topological group.

Proof. By Lemma 2.38, these morphisms are indeed surjective. The assertion is then directly checked. ☐

Definition-Proposition 2.40. *Let $\mathcal{G}$ be endowed with the unique structure of locally profinite topological group such that for any $m \in \mathrm{S}_{\mathcal{V}}$ the natural embedding $\mathrm{Aut}_{\mathcal{O}_E}(m) \to \mathcal{G}$ is a homeomorphism onto an open subgroup. Let $\mathcal{PG}$ be endowed with the quotient topology.*

Proof. Any $m \in \mathrm{S}_{\mathcal{V}}$ induces a unique structure of topological group on $\mathcal{G}$ for which the embedding $\mathrm{Aut}_{\mathcal{O}_E}(m) \to \mathcal{G}$ is a homeomorphism onto an open subgroup. It remains to be shown that the induced such topology is independent of the choice of m and that it is Hausdorff. Denote by $K(m, n)$ the kernel of the homomorphism in (4). Then $\mathcal{F}_m := (K(m, n))_{n \geq 0}$ is a fundamental system of open neighborhoods of the identity in $\mathrm{Aut}_{\mathcal{O}_E}(m)$ for any $m \in \mathrm{S}_{\mathcal{V}}$. In order to see, that the topologies on $\mathcal{G}$ induced by any $m, m' \in \mathrm{S}_{\mathcal{V}}$ coincide, it suffices to show that $\mathcal{F}_m$ and $\mathcal{F}_{m'}$, viewing any of its elements as a subset of $\mathcal{G}$, are cofinal. Consider any such m, m' and any $n \geq 0$. To find is an $n' \geq 0$ for which $K(m', n') \subset K(m, n)$. Using that any

$K(m', n') \subset \mathcal{G}$ is invariant under dilating m', assume that $m \subset m'$. Any $n' \geq 0$ for which $\pi^{n'} m' \subset \pi^n m$ is then as desired. Finally, a topological group is Hausdorff if and only if some open subgroups have trivial intersection. As

$$\bigcap_{n \geq 0} \pi^n m = 0 \text{ and hence } \bigcap_{n \geq 0} K(m, n) = 0$$

for any $m \in \mathrm{S}_{\mathcal{V}}$, thus $\mathcal{G}$ is Hausdorff. □

In the sequel, view any such $\mathrm{Aut}_{\mathcal{O}_E}(m)$ as an open subgroup of $\mathcal{G}$.

Lemma 2.41. *A subgroup of $\mathcal{G}$, resp. of $\mathcal{PG}$, is discrete if and only if its intersetion with $\mathrm{Aut}_{\mathcal{O}_E}(m)$, resp. $\mathrm{Aut}_{\mathcal{O}_E}(m) \cdot E^\times / E^\times$, is finite for any $m \in \mathrm{S}_{\mathcal{V}}$.*

Proof. This follows directly from the facts that $\mathcal{G}$ is locally profinite and thus Hausdorff and that any $\mathrm{Aut}_{\mathcal{O}_E}(m)$ is open and profinite and thus compact. □

Example 2.42. *Consider any subring $A \subset E$ and any discrete A-submodule $\Lambda \subset \mathcal{V}$ for which the natural homomorphism $\Lambda \otimes_A E \to \mathcal{V}$ is an isomorphism. Then $\mathrm{Aut}_A(\Lambda)$ embeds naturally into $\mathcal{G}$ onto a discrete subgroup whose image in $\mathcal{PG}$ is discrete.*

Proof. By virtue of the isomorphism $\Lambda \otimes_A E \to \mathcal{V}$, view $\Gamma := \mathrm{Aut}_A(\Lambda)$ as a subgroup of $\mathcal{G}$. Let $m \in \mathrm{S}_{\mathcal{V}}$ and set $G := \mathrm{Aut}_{\mathcal{O}_E}(m)$. We shall show that

$$\Gamma \cap E^\times \cdot G = \Gamma \cap G$$

and that this group is finite; in view of Lemma 2.41, this will yield the assertion. Using that $G \subset \mathcal{G}$ is invariant under dilating m and that $\Lambda \otimes_A E \to \mathcal{V}$ is an isomorphism, we assume m to be such that $m \cap \Lambda$ contains a basis of $\mathcal{V}$. The natural homomorphism

$$\Gamma \cap G \to \mathrm{Aut}_{A \cap \mathcal{O}_E}(\Lambda \cap m)$$

is then injective. As the discrete subset $\Lambda \cap m$ of the compact m is finite, the target, and hence the domain, of this homomorphism are finite. It remains to be shown that $\Gamma \cap E^\times \cdot G \subset G$. Let $g \in \Gamma \cap E^\times \cdot G$. Choose an $e \in E^\times$ for which $g(m) = e \cdot m$. Then $g^n(\Lambda \cap m) = \Lambda \cap e^n \cdot m$ for any integer n. Thus $e \in \mathcal{O}_E^\times$; indeed, otherwise the e^n could become arbitrarily small so that, Λ being discrete, $\Lambda \cap e^n \cdot m = 0$ for some n whereas $\Lambda \cap m \neq 0$. Hence $g(m) = m$ as desired. □

In Section 4.3 we will recall the usual definition of *norm* on $\mathcal{V}$.

Lemma 2.43. *Any norm on $\mathcal{V}$ induces the locally profinite topology.*

Proof. This is Corollary 4.8 below. □

2.5 On lattices over admissible coefficient subrings

Suppose that the characteristic of C is finite.

Definition 2.44. *A subset $S \subset C$ is called* strongly discrete *if its intersection with every ball of finite radius is finite.*

Definition 2.45. *We call a subring $A \subset C$ an* admissible coefficient subring *if it is strongly discrete and if it is a Dedekind domain that is finitely generated over a finite subfield of C.*

Example 2.46. *Consider any finite subfield $\mathbb{F}_q \subset C$ and any $t \in C$ with $|t| > 1$. Then $\mathbb{F}_q[t]$ is a polynomial ring over $\mathbb{F}_q$ and an admissible coefficient subring of C.*

Proof. As the norm of C is non-Archimedean, as $|x| = 1$ for any $0 \neq x \in \mathbb{F}_q$ and as $|t| > 1$, any polynomial of degree $n \geq 0$ over $\mathbb{F}_q$ evaluated at t has norm $|t|^n$ in C. This implies both that $\mathbb{F}_q[t]$ is a polynomial ring and that it is strongly discrete in C. That any polynomial ring in one variable over a field is a Dedekind domain, is a classical fact. □

Remark 2.47. *Any admissible coefficient subring $A \subset C$ is the ring of sections on $X \setminus \{x\}$ for some closed point x in some projective smooth irreducible algebraic curve X over a finite field such that the completion of the quotient field of A with respect to the valuation corresponding to x is contained in C as a valued subfield.*

Proof. (Sketch) Consider any admissible coefficient subring $A \subset C$. For instance by Harder's [27, Volume 2, Section 9.1-3], then A is the ring of sections on $X \setminus Y$ for some finite set Y of closed points in some projective smooth irreducible algebraic curve X over a finite field. Moreover, from the strong discreteness of A it follows that its unit group is finite. Then use that the ideal class group of A is finite in order to conclude that $|Y| = 1$. □

The following is an example to Example 2.42.

Example 2.48. *Consider any admissible subring $A \subset C$. Denote by E the completion of its quotient field. Let $\mathcal{V}$ be a finite dimensional vector space over E. Any projective A-submodule $\Lambda \subset \mathcal{V}$ for which the natural homomorphism $\Lambda \otimes_A E \to \mathcal{V}$ is injective is then discrete in $\mathcal{V}$.*

Proof. As A is discrete in C, it is discrete in E. Consider any such A-submodule $\Lambda \subset \mathcal{V}$ and choose any free A-submodule Λ' of Λ of maximal rank. Then Λ' is discrete in $\mathcal{V}$. Moreover, Λ/Λ' is a torsion A-module by maximality of Λ'. As Λ/Λ' further is finitely generated, it is in fact finite as A is a Dedekind ring. Thus Λ is the union of finitely many translates of the discrete Λ' and hence itself discrete. □

Lemma 2.49. *Consider any non-Archimedean local field E contained in C as a valued subfield. Let $\mathcal{V} \subset C$ be any finite dimensional E-subspace. A subset $S \subset \mathcal{V}$ then is discrete with respect to the locally profinite topology if and only if it is strongly discrete as a subset of C.*

Proof. This follows from Lemma 2.43. □

Definition 2.50. *Consider any admissible coefficient subring $A \subset C$ and let E be the completion of its quotient field. A finitely generated projective A-submodule $\Lambda \subset C$ is an A-*lattice *if the natural homomorphism $\Lambda \otimes_A E \to C$ is injective.*

Proposition 2.51. *Any A-lattice $\Lambda \subset C$ is strongly discrete.*

Proof. Combine Example 2.48 and Lemma 2.49. □

Definition 2.52. *Consider any admissible coefficient subring $A \subset C$, any projective A-module Λ of finite rank $d > 0$ and any norm $|\cdot|$ on $\Lambda \otimes_A E$ in the sense of Section 4.3, where E is the completion of the quotient field of A. For any $1 \leq i \leq d$ call*

$$\mu_i(\Lambda) := \inf\{\max\{|\lambda_1|, \ldots, |\lambda_i|\} \mid \lambda_1, \ldots, \lambda_i \in \Lambda \textit{ linearly independent }\}$$

the i-th successive minimum *of Λ. Set $\mu_{\max}(\Lambda) := \mu_d(\Lambda)$.*

Definition-Proposition 2.53. *Let $A = \mathbb{F}_q[t]$ be as in Example 2.46. Consider any A-module Λ and any norm $|\cdot|$ as in Definition 2.52. Then there exists a* minimal reduced *basis of Λ, i.e., an ordered basis $(\lambda_1, \ldots, \lambda_d)$ of Λ such that $(|\lambda_1|, \ldots, |\lambda_d|) = (\mu_1(\Lambda), \ldots, \mu_d(\Lambda))$ and such that*

$$\forall a \in A^d\colon \left| \sum_{1 \leq i \leq d} a_i \lambda_i \right| = \max_{1 \leq i \leq d} |a_i| \cdot |\lambda_i|.$$

Moreover, $|\lambda_i| = \inf_{\lambda \in \Lambda} |\lambda_i + t \cdot \lambda|$ for any λ_i in any such basis.

Proof. Up to the last assertion, this is [7, Theorem 2.2.8]. Consider then any λ_i in any minimal reduced basis $(\lambda_1, \ldots, \lambda_n)$ of Λ. Any $\lambda = \sum_{1=j}^{d} a_j \cdot \lambda_j \in \Lambda$ then satisfies as desired that

$$|\lambda_i + t\lambda| = \max_{j \neq i}\{|(1 + t \cdot a_i) \cdot \lambda_i|, |t \cdot a_j \cdot \lambda_j|\} \geq |(1 + t \cdot a_i) \cdot \lambda_i| \geq |\lambda_i|.$$

□

For any subset S of any normed vector space $\mathcal{V}$ set

$$\mathrm{d}(S) := \inf_{0\neq s\in S} |s|;$$

this measures the distance of $S \setminus \{0\}$ to the origin.

Corollary 2.54. *Consider any $A = \mathbb{F}_q[t]$, any Λ and any norm on $\Lambda \otimes_A E$ as in Def.-Prop. 2.53. Consider any direct summand $0 \neq L \subset \Lambda$. Consider the projection $\pi\colon t^{-1}\Lambda \to \overline{\Lambda} := t^{-1}\Lambda/\Lambda$ and set $\overline{L} := t^{-1}L/L \subset \overline{\Lambda}$. Then*

$$\max_{\alpha\in\overline{L}} \mathrm{d}(\pi^{-1}(\alpha)) \leq \mu_{\max}(L). \tag{5}$$

Moreover, choose a minimal reduced basis $(\lambda_1, \ldots, \lambda_n)$ of $t^{-1}\Lambda$. Let $L' \subset t^{-1}\Lambda$ be the submodule generated by the λ_i with $|\lambda_i| < \mathrm{d}(\pi^{-1}(\overline{\Lambda} \setminus \overline{L}))$. Set $\overline{L}' := \pi(L')$. If

$$\max_{\alpha\in\overline{L}} \mathrm{d}(\pi^{-1}(\{\alpha\})) < \mathrm{d}(\pi^{-1}(\overline{\Lambda} \setminus \overline{L})), \tag{6}$$

then $\overline{L}' = \overline{L}$ and $\mathrm{d}(t^{-1}\Lambda \setminus L') = \mathrm{d}(\pi^{-1}(\overline{\Lambda} \setminus \overline{L}))$.

Proof. Choose a minimal reduced basis $\lambda'_1, \ldots, \lambda'_m$ of $t^{-1}L$. Any $0 \neq \alpha \in \overline{L}$ admits an $\mathbb{F}_q$-linear combination $\mu \neq 0$ of the λ'_i in $\pi^{-1}(\alpha)$; then

$$\mathrm{d}(\pi^{-1}(\alpha)) = |\mu| \leq |\lambda'_m| = |t|^{-1}\mu_m(L) \leq \mu_m(L)$$

Moreover, $\mathrm{d}(\pi^{-1}(0)) \leq |t| \cdot |\lambda'_m| = |t| \cdot \mu_m(t^{-1}L) = \mu_m(L)$ This yields (5).

Suppose (6). Then $\overline{L}' \subset \overline{L}$. Conversely, let $\alpha \in \overline{L}$ and choose any $\lambda \in \pi^{-1}(\alpha)$ with $|\lambda| = \mathrm{d}(\pi^{-1}(\alpha))$. Write $\lambda = \sum_{i=1}^n a_i \cdot \lambda_i$ for some $a_1, \ldots, a_n \in A$. Then $\lambda \in L'$ and hence $\alpha \in \overline{L}'$: Indeed, if λ was not in L', then we could choose a $\lambda_j \notin L'$ for which $a_j \neq 0$ and get the contradiction that

$$|\lambda| = \max_{1\leq i\leq n} |a_i \cdot \lambda_i| \geq |a_j \cdot \lambda_j| \geq |\lambda_j| \geq \mathrm{d}(\pi^{-1}(\overline{\Lambda} \setminus \overline{L})) \overset{(6)}{>} \max_{\alpha\in\overline{L}} \mathrm{d}(\pi^{-1}(\alpha)) \geq |\lambda|.$$

Hence $\overline{L}' = \overline{L}$. This and the defining property of L' then imply that

$$\mathrm{d}(t^{-1}\Lambda \setminus L') = |\lambda_{m+1}| = \mathrm{d}(\pi^{-1}(\overline{\Lambda} \setminus \overline{L}')) = \mathrm{d}(\pi^{-1}(\overline{\Lambda} \setminus \overline{L})).$$

□

Definition-Proposition 2.55. *For any strongly discrete subgroup $\Lambda \subset C$ the formula*

$$e_\Lambda(T) := T \cdot \prod_{0\neq\lambda\in\Lambda} \left(1 - \frac{T}{\lambda}\right)$$

defines a regular function $e_\Lambda \colon C \to C$ that is a surjective homomorphism with kernel Λ. Moreover,

$$\forall c \in C \setminus \Lambda \colon \frac{1}{e_\Lambda(c)} = \sum_{\lambda \in \Lambda} \frac{1}{c+\lambda}.$$

Proof. This is explained for instance in [16, Chapter 2, Section 1] up to the last part. The last part follows from logarithmic differentiation using that $\frac{d}{dT} \exp_\Lambda(T) = 1$. □

Proposition 2.56. *Consider any A-lattice $\Lambda \subset C$ and any $0 \neq c, c' \in C$ such that $|c| < |\lambda|$ and $|c'| \leq |c' + \lambda|$ for every $0 \neq \lambda \in \Lambda$. Then*

$$\left|\frac{c'}{c}\right| \leq \left|\frac{e_\Lambda(c')}{e_\Lambda(c)}\right| \leq \left|\frac{c'}{c}\right|^{\left|\frac{c'}{c}\right| \cdot q \cdot \operatorname{rank}_{\mathbb{F}_q[t]}(\Lambda)}$$

for any polynomial ring $\mathbb{F}_q[t] \subset A$ over any finite field with q elements.

Proof. The assumptions yield for any $0 \neq \lambda \in \Lambda$ that

$$|c + \lambda| = |\lambda| = |(c' + \lambda) - c'| \leq \max\{|c' + \lambda|, |c'||\} \leq |c' + \lambda|$$

and that

$$\left|\frac{c' + \lambda}{c + \lambda}\right| = 1 \text{ if } |\lambda| > |c'|$$

and that

$$\left|\frac{c' + \lambda}{c + \lambda}\right| \leq \left|\frac{c'}{c + \lambda}\right| \leq \left|\frac{c'}{c}\right| \text{ if } |\lambda| \leq |c'|.$$

Hence

$$\left|\frac{c'}{c}\right| \leq \prod_{\lambda \in \Lambda} \left|\frac{c' + \lambda}{c + \lambda}\right| = \left|\frac{e_\Lambda(c')}{e_\Lambda(c)}\right| \leq \prod_{\substack{\lambda \in \Lambda \\ |\lambda| \leq |c'|}} \left|\frac{c'}{c}\right|.$$

Consider any polynomial ring $A' := \mathbb{F}_q[t] \subset A$. It remains to be shown that the number $\lambda \in \Lambda$ with $|\lambda| \leq |c'|$ is bounded by $\left|\frac{c'}{c}\right| \cdot q \cdot d$, where $d := \operatorname{rank}_{A'}(\Lambda)$. By means of Proposition 2.53, choose a reduced basis $\lambda_1, \ldots, \lambda_d$ of the A'-module Λ. Any $\lambda \in \Lambda$ then admits unique $a_1, \ldots, a_d \in A'$ with $\lambda = \sum_{i=1}^d a_i \cdot \lambda_i$ and, if $|\lambda| \leq |c'|$, then $|a_j \cdot \lambda_j| \leq \max_{1 \leq i \leq d} |a_i \cdot \lambda_i| = |\lambda| \leq |c'|$ and hence $|a_j| \leq \left|\frac{c'}{\lambda_j}\right| < \left|\frac{c'}{c}\right|$ for any $1 \leq j \leq d$. This yields the remaining bound. □

3 On stratifications of rigid analytic varieties by global sections

Throughout this section we consider any reduced rigid analytic variety Z over an algebraically closed complete non-Archimedean field C and any finite set S of global sections of an invertible sheaf on Z. With any $T \subset S$ and any $\varepsilon \in |C^\times|$ associate the reduced Zariski open, resp. admissible, resp. locally closed subvariety

$$\begin{aligned} \mathcal{U}(T) &:= \{z \in Z \mid \forall t \in T : t(z) \neq 0\} \subset Z, \\ \mathcal{U}(T,\varepsilon) &:= \{z \in \mathcal{U}(T) \mid \forall s \in S \setminus T,\ \forall t \in T : \left|\frac{s}{t}(z)\right| \leq \varepsilon\} \subset \mathcal{U}(T), \\ \Omega(T) &:= \{z \in \mathcal{U}(T) \mid \forall s \in S \setminus T,\ \forall t \in T : \frac{s}{t}(z) = 0\} \subset Z. \end{aligned}$$

This yields a stratification of Z by locally closed subvarieties

$$Z = \dot{\bigcup_{T \subset S}} \Omega(T).$$

3.1 Characterization of the Grothendieck topology

Proposition 3.1. *A subset $X \subset Z$ is admissible if and only if any $T \subset S$ with $\Omega(T) \neq \emptyset$ satisfies that*

i) the subset $X \cap \Omega(T) \subset \Omega(T)$ is admissible and that

ii) any admissible quasi-compact $U \subset \mathcal{U}(T)$ with $U \cap \Omega(T) \subset X$ admits an $\varepsilon \in |C^\times|$ with $U \cap \mathcal{U}(T,\varepsilon) \subset X$.

Moreover, a covering of an admissible $X \subset Z$ by admissible subsets is admissible if and only if its intersection with $X \cap \Omega(T)$ is admissible for any $T \subset S$.

Proposition 3.1 will essentially be a formal consequence of

Proposition 3.2 (Kisin)**.** *For any affinoid algebra A over C, any admissible $U \subset X := \mathrm{Sp}(A)$ and any $a_1, \dots, a_n \in A$ whose common zeroes lie in U exists an $\varepsilon > 0$ such that $\{x \in X | \forall 1 \leq i \leq n \colon |a_i(x)| \leq \varepsilon\} \subset U$.*

Proof. See [13, after Remark 5.2.9] for Conrad's short proof using Berkovich spaces. ☐

Proof of Proposition 3.1. Consider any subset $X \subset Z$. Suppose first that X is admissible and consider any $T \subset S$. Then i) follows from the fact that $\Omega(T)$ is a locally closed subvariety of Z. Consider further any admissible quasi-compact (a.q.c.) $U \subset \mathcal{U}(T)$ with $U \cap \Omega(T) \cap X$. Then apply Proposition 3.2 to the admissible subset $U \cap X \subset U$ and the restrictions to Y of the $\frac{s}{t}$ over all $s \in S \setminus T$ and $t \in T$ to get a desired $\varepsilon \in |C^\times|$ fulfilling $U \cap \mathcal{U}(T, \varepsilon) \subset U \cap X \subset X$.

Conversely, suppose that i) and ii) hold. For any $0 \leq i \leq |S|$ let $\mathcal{U}(i)$ be the union of the $\mathcal{U}(T)$ for all $T \subset S$ with $|S \setminus T| \geq i$ and let $\mathcal{P}(i)$ be the claim that $X \cap \mathcal{U}(i) \subset \mathcal{U}(i)$ is admissible. Then $\mathcal{P}(|S|)$ is precisely the desired statement. We shall prove $\mathcal{P}(i)$ by induction on i.

As $\mathcal{U}(0) = \Omega(S)$, Condition i) implies $\mathcal{P}(0)$. Consider then any $0 < i \leq |S|$ and suppose that $\mathcal{P}(i-1)$ holds. As the $\mathcal{U}(T)$ over all $T \subset S$ with $|S \setminus T| = i$ form a Zariski open and hence admissible covering of $\mathcal{U}(i)$, we may and do thus choose any such T, set $\mathcal{U} := \mathcal{U}(T)$ and $\Omega := \Omega(T)$ and restrict ourselves to showing that $X \cap \mathcal{U} \subset \mathcal{U}$ is admissible. In particular, we may and do assume that $X \subset \mathcal{U}$. By means of i) choose any admissible affinoid covering $\mathcal{C}$ of $X \cap \Omega$. Applying Corollary 2.30 to the closed subvariety $\Omega \subset \mathcal{U}$ choose for any $O \in \mathcal{C}$ an admissible quasi-compact $U(O) \subset \mathcal{U}$ with $U(O) \cap \Omega = O$ and, furthermore using Condition ii), with $U(O) \subset X$. Using that $X \setminus \Omega \subset \mathcal{U} \setminus \Omega$ is admissible by the induction hypothesis, we further choose any admissible covering $\mathcal{D}$ of $X \setminus \Omega$, for instance $\{X \setminus \Omega\}$. Let $\mathcal{E}$ be the covering of X consisting of all elements in $\mathcal{D}$ and the $U(O, \varepsilon(O))$ for all $O \in C$. We claim that X is an admissible subset by means of $\mathcal{E}$, i.e., that for any morphism $\varphi\colon Y \to \mathcal{U}$ with image in X, where Y is affinoid, the preimage of $\mathcal{E}$ under φ has a finite subcovering.

Consider thus any such $\varphi\colon Y \to \mathcal{U}$. Let $Y' \subset Y$ denote the common zero locus of the $f_{s/t} := \frac{s}{t} \circ \varphi\colon Y \to C$ for all $s \in S \setminus T$ and all $t \in T$. Then φ restricts to a morphism $\psi\colon Y' \to X \cap \Omega$. As Y is quasi-compact, the image of ψ is contained in the union of finitely many $O_1, \ldots, O_k \in C$. Being the union of finitely many adissible quasi-compact subsets, $U := U(O_1) \cup \cdots \cup U(O_k)$ is admissible. Its preimage $\varphi^{-1}(U) \subset Y$ is thus admissible, too, and, by construction, contains Y'. Proposition 3.2 thus provides an $\varepsilon \in |C^\times|$ with

$$Y_\varepsilon := \left\{ y \in Y \mid \forall s \in S \setminus T, \forall t \in T\colon |f_{s/t}(y)| < \varepsilon \right\} \subset \varphi^{-1}(U).$$

Since $Y \setminus Y_\varepsilon$ is quasi-compact and since $\mathcal{D}$ is admissible, $Y \setminus Y_\varepsilon$ is covered by finitely many elements of the preimage of $\mathcal{D}$ under φ. Together with the $\varphi^{-1}(U(O_i))$, such finitely many subsets yield a desired finite subcovering of the preimage of $\mathcal{E}$ under φ. This finishes the induction step and hence yields the first equivalence of the proposition.

Finally, suppose that X is admissible and consider any covering $\mathcal{F}$ of X by admissible subsets. Suppose that the intersection of $\mathcal{F}$ with $X \cap \Omega(T)$ is admissible for any $T \subset S$ and let us show that $\mathcal{F}$ is then itself admissible. By means of a similar induction argument as above, we assume that $X \subset \mathcal{U}(T)$ for some $T \subset S$ and that the intersection $\mathcal{D}$ of $\mathcal{F}$ with $X \setminus \Omega(T)$ is admissible. Let $\mathcal{D}$ be an admissible affinoid, and hence quasi-compact, refinement of the intersection of $\mathcal{F}$ with $X \cap \Omega(T)$. For any $O \in \mathcal{C}$ choose an $X_O \in \mathcal{F}$ such that $O \subset X_O$ and, similarly as before, an admissible $U(O) \subset X_O$ with $U(O) \cap \Omega(T) = O$. Let $\mathcal{E}$ be the covering of X consisting of all elements in D and of the $U(O)$ for all $O \in \mathcal{C}$. It is enough to show that $\mathcal{E}$ is admissible since, by construction, it refines $\mathcal{F}$. By a previous argument, the preimage of $\mathcal{E}$ under any morphism $\varphi\colon Y \to X$, where Y is affinoid, has a finite subcovering. Thus $\mathcal{E}$ and hence $\mathcal{F}$ are indeed admissible.

Conversely, the intersection of any admissible covering of X with any $\Omega(T)$ is admissible since $\Omega(T)$ is a locally closed subvariety of Z. □

3.2 The characterization in a special case

Assume for any $\Omega(T) \neq \emptyset$ the existence and choice of a morphism

$$\rho_T \colon \mathcal{U}(T) \to \Omega(T) \tag{7}$$

such that $\rho_T|_{\Omega(T)} = \mathrm{id}_{\Omega(T)}$ and such that

$$\mathcal{U}(O, \varepsilon) := \rho_T^{-1}(O) \cap \mathcal{U}(T, \varepsilon)$$

is quasi-compact for any quasi-compact $O \subset \Omega(T)$ and any $\varepsilon \in |C^\times|$.

Example 3.3. *Let for example S be a C-basis of the global sections of the first twisting sheaf of any standard projective space Z over C and let ρ_T be the natural projection for any $\emptyset \neq T \subset S$. Consider for any $t \in T \subset S$ the isomorphism*

$$i_t \colon \mathcal{U}(T) \to \Omega(T) \times C^{S \setminus T}, q \mapsto \left(\rho_T(q), (\frac{s}{t}(q))_{s \in S \setminus T}\right).$$

For any $\emptyset \neq T \subset S$, any $O \subset \Omega(T)$ and any $\varepsilon \in |C^\times|$ then

$$\mathcal{U}(O, \varepsilon) = \bigcap_{t \in T} i_t^{-1}(O \times B_\varepsilon).$$

In particular, such $\mathcal{U}(O, \varepsilon)$ is quasi-compact, resp. affinoid, whenever O is.

Corollary 3.4. *Consider any Z, S and morphisms ρ_T as in (7). Let $Y \subset Z$ be any closed subvariety. Then a subset $X \subset Y$ is admissible if and only if any $T \subset S$ with $\Omega(T) \cap Y \neq \emptyset$ satisfies that*

i) the subset $X \cap \Omega(T) \subset Y \cap \Omega(T)$ is admissible and that

ii) any admissible quasi-compact $O \subset \Omega(T)$ with $O \cap Y \subset X$ admits an $\varepsilon(O) \in |C^{\times}|$ with $\mathcal{U}(O, \varepsilon(O)) \cap Y \subset X$.

Moreover, a covering of an admissible $X \subset Y$ by admissible subsets is admissible if and only if its intersection with $X \cap \Omega(T)$ is admissible for any $T \subset S$.

Proof. Consider any subset $X \subset Y$ and any $T \subset S$ with $\Omega(T) \cap Y \neq \emptyset$. Set $\Omega := \Omega(T)$ and $\mathcal{U} := \mathcal{U}(T)$. In view of Proposition 3.1 applied to the restrictions of the sections in S to Y, it is enough to assume that $\Omega \cap Y \neq \emptyset$, that $X \cap \Omega \subset Y \cap \Omega$ is admissible and to show the equivalence of

(A) Condition ii) of this corollary and

(B) the condition that any admissible quasi-compact $U \subset \mathcal{U} \cap Y$ with $U \cap \Omega \subset X$ admits an $\varepsilon \in |C^{\times}|$ with $U \cap \mathcal{U}(T, \varepsilon) \subset X$.

First consider any such U assuming (A). Then $U \cap \Omega$ is an admissible quasi-compact of $Y \cap \Omega$ and hence, by Corollary 2.30, the intersection with Y of an admissible quasi-compact $O \subset \Omega$. Choose such an O. Then (1) provides an $\varepsilon' \in |C^{\times}|$ with $X' := \mathcal{U}(O, \varepsilon') \cap Y \subset X$. By construction, then $U \cap \Omega = O \cap \Omega = X' \cap \Omega$. Applying Proposition 3 to the admissible subset $X' \subset Y$, thus yields an $\varepsilon \in |C^{\times}|$ with $U \cap \mathcal{U}(T, \varepsilon) \subset X' \subset X$.

Conversely, assume (B) and consider any admissible quasi-compact $O \subset \Omega$ with $O \cap Y \subset X$. Choose any $\varepsilon' \in |C^{\times}|$. By assumption on ρ_T, the subset $\mathcal{U}(O, \varepsilon') \subset \mathcal{U}$, and hence $U := \mathcal{U}(O, \varepsilon') \cap Y \subset \mathcal{U} \cap Y$, is then admissible quasi-compact Condition (2) thus provides an $\varepsilon' \geq \varepsilon \in |C^{\times}|$ with $\mathcal{U}(O, \varepsilon) \cap Y = U \cap \mathcal{U}(T, \varepsilon) \subset X$ as desired. □

Corollary 3.5. *Consider any rigid analytic variety R and any integer $n \geq 0$. Let $Y \subset R \times C^n$ be any closed subvariety. Then a subset $X \subset Y$ is admissible if and only if*

i) the subset $X \setminus R \times \{0\} \subset Y \setminus R \times \{0\}$ is admissible,

ii) the subset $X \cap R \times \{0\} \subset Y \cap R \times \{0\}$ is admissible and

iii) for any admissible quasi-compact $O \subset R$ with $O \times \{0\} \subset X$ exists an $\varepsilon > 0$ such that $(O \times B_{\varepsilon}) \cap Y \subset X$.

Moreover, a covering of an admissible subset $X \subset Y$ by admissible subsets is admissible if and only if both its intersection with $X \setminus R \times \{0\}$ and its intersection with $X \cap R \times \{0\}$ is admissible.

Proof. Suppose that $Z = R \times C^n$ and S consists of the regular function 1 and the i-th projection $p_i \colon Z \to C$ for all $1 \leq i \leq n$. For any $T \subset S$ then $\Omega(T) \neq \emptyset \Leftrightarrow 1 \in T$ in which case we assume $\rho_T \colon \mathcal{U}(T) \to \Omega(T)$ to be the natural projection. Moreover, let $Z' := Z \setminus R \times \{0\}$, $Y' := Y \setminus R \times \{0\}$ and $S' := \{p'_1, \ldots, p'_n\}$, where any p'_i is the restriction of p_i to Z'. For any $T' \subset S'$ define $\mathcal{U}(T')$ and $\Omega(T')$ analogously with respect to Z' and S'. Then $\Omega(T') \neq \emptyset \Leftrightarrow T' \neq \emptyset$ for any $T' \subset S'$. For any such $\emptyset \neq T' \subset S'$ let $T \subset S$ be the subset with $1 \in T$ and $\forall 1 \leq i \leq n \colon p_i \in T \Leftrightarrow p'_i \in T'$; then

$$\mathcal{U}(T') = \mathcal{U}(T) \ \text{ and } \ \Omega(T') = \Omega(T)$$

and we set $\rho'_{T'} := \rho_T$. Using that $\Omega(\{1\}) = R \times \{0\}$, the corollary then follows by applying Corollary 3.4 twice; once as stated and once for Z, Y, S, and the ρ_T replaced by Z', Y', S', and the $\rho'_{T'}$. □

Corollary 3.6. *Let R be any separated rigid analytic variety. Consider any admissible subset $X \subset R \times C$ and any regular function $s \colon X \setminus R \times \{0\} \to C$. Then there exist unique regular functions $s_i \colon X \cap R \times \{0\} \to C$ such that*

$$s((o,z)) = \sum_{i \in \mathbb{Z}} s_i(o,0) z^i \ \text{ for any } (o,z) \in O \times B_\varepsilon \setminus O \times \{0\}$$

for any admissibe affinoid $O \times \{0\} \subset X \cap R \times \{0\}$ and any $\varepsilon \in |C^\times|$ with $O \times B_\varepsilon \subset X$. Moreover, the following statements are equivalent:

i) s extends to a regular function $X \to C$.

ii) s extends to a morphism $X \to C$ of Grothendieck topological spaces whose restriction to $X \cap R \times \{0\}$ is regular.

iii) Any admissible affinoid $O \times \{0\} \subset X \cap R \times \{0\}$ admits an $\varepsilon \in |C^\times|$ with $O \times B_\varepsilon \subset X$ and such that s is bounded on $O \times B_\varepsilon \setminus O \times \{0\}$.

iv) $\forall i < 0 : s_i = 0$.

Moreover, the extension in i), resp. ii), is unique if it exists.

Proof. Let $\mathcal{C}'$ denote the covering of $X \cap R \times \{0\}$ by all its admissible affinoid subsets. This is admissible since any admissible affinoid covering refines it. By means of the first part of Corollary 3.5, let $\mathcal{C}$ be the covering of X

consisting of $X \setminus R \times \{0\}$ and the $O \times B_{\varepsilon(O)}$ for all $O \times \{0\} \in \mathcal{C}'$ and any choice of $\varepsilon(O) \in |C^{\times}|$ with $O \times B_{\varepsilon(O)} \subset X$. Since the intersection of $\mathcal{C}$ with $X \setminus R \times \{0\}$ is refined by the admissible covering $\{X \setminus R \times \{0\}\}$ and since the intersection of $\mathcal{C}$ with $X \cap R \times \{0\}$ equals $\mathcal{C}'$, the second part of Corollary 3.5 yields that $\mathcal{C}$ is admissible. Proposition 2.32 yields unique regular functions $s_{i,O} \colon O \to C$, one for each $i \in \mathbb{Z}$, for any such O such that

$$s((o,z)) = \sum_{i \in \mathbb{Z}} s_{i,O}(o) z^i \text{ for any } (o,z) \in O \times B_{\varepsilon(O)} \setminus O \times \{0\}$$

By Proposition 2.22, $O \cap O'$ is affinoid for any admissible affinoid $O \times \{0\}, O' \times \{0\} \subset X \cap R \times \{0\}$ so that

$$s_{i,O}|_{O \cap O'} = s_{i, O \cap O'} = s_{i,O'}|_{O \cap O'} \text{ for any } i \in \mathbb{Z}$$

by the previous uniqueness property. As $\mathcal{C}'$ is admissible, any $i \in \mathbb{Z}$ thus admits a unique regular function $s_i \colon X \cap R \times \{0\} \to C$ with $s_i|_{O \times \{0\}}(o,0) = s_{i,O}(o)$ for any $o \in O \in \mathcal{C}'$. By the admissibility of $\mathcal{C}$ and again by Proposition 2.32, these s_i therefore satisfy the desired properties. □

Proposition 3.7. *Let Z and S and the ρ_T be as in Example 3.3. Consider the natural left-action on Z of any subgroup G of the symmetric group of S. Then for any G-invariant closed subvariety $Y \subset Z$ the quotient $G \backslash Y$ is a rigid analytic variety and it is normal if Y is.*

Proof. For any $0 \neq T \subset S$ and any $r \in |C|$ set

$$O(T,r) := \left\{ z \in \Omega(T) \mid \forall t, t' \in T : \left| \frac{t'}{t}(z) \right| \leq r \right\} \subset \Omega(T)$$

and for any further $\varepsilon \in |C^{\times}|$ set $\mathcal{U}(T,r,\varepsilon) := \mathcal{U}(O(T,r),\varepsilon) \subset \mathcal{U}(T)$. By Example 3.3 and the construction, any such $\mathcal{U}(T,r,\varepsilon)$ is a G_T-invariant admissible affinoid subvariety of Z, where G_T denotes the stabilizer of T in G. Fix any $1 > \varepsilon \in |C^{\times}|$. The construction yields for any $T' \subset S$ with $T' \not\subset T \not\subset T'$ and any $g \in G$ that

$$\mathcal{U}(T,\varepsilon) \cap \mathcal{U}(T',\varepsilon) = \emptyset \text{ and that } g(\mathcal{U}(T,r,\varepsilon)) = \mathcal{U}(g(T),r,\varepsilon).$$

Hence the G-invariant subvariety

$$G(\mathcal{U}(T,r,\varepsilon)) = \bigcup_{g \in G} \mathcal{U}(g(T),r,\varepsilon) \subset Z$$

is a disjoint union of finitely many admissible affinoids; in particular, it is itself admissible affinoid. Finally, let $\mathcal{C}$ be the covering of Z by the $G(\mathcal{U}(T, r, \varepsilon))$ for varying $\emptyset \neq T \subset S$ and $r \in |C|$. Its intersection with any $\Omega(T) \neq \emptyset$ is refined by the admissble covering of $\Omega(T)$ by the $O(T, r)$, for varying r, so that it is itself admissible. By Proposition 3.4, thus $\mathcal{C}$ is admissible. In particular, the intersection of $\mathcal{C}$ with any G-invariant closed subvariety $Y \subset Z$ is an admissible covering by G-invariant affinoids. The proposition then follows from Proposition 2.33. □

3.3 Stratification and normalization

Consider first a general reduced rigid analytic variety X. We refer to Conrad's [12] for the definition of the normalization of X and a proof that it uniquely exists. Conrad uses it to define the irreducible components of X as the images of the connected components under the normalization morphism [12, Def. 2.2.2]. The irreducible components are then the maximal irreducible Zariski closed subsets of X [12, Thm. 2.2.4.(2)]. If X is the analytification of an algebraic variety X' over C, then its normalization, resp. its irreducible components, are the analytification of the normalization of X', resp. of the irreducible components of X' [12, Thm. 2.1.3, resp. 2.3.1].

Denote then by $\mathcal{O}_Z$ the structure sheaf of Z. Consider the Grothendieck ringed space $(Z, \tilde{\mathcal{O}}_Z)$ whose underlying Grothendieck topological space coincides with the one underlying $(Z, \mathcal{O}_Z)$ and whose section on any admissible $U \subset Z$ are precisely the functions $f \colon U \to C$ that are continuous with respect to the canonical topologies, that are bounded on any admissible affinoid subset of U and that restrict to regular functions $U \cap \Omega(T) \to C$ for any $T \subset S$. Consider the morphism of Grothendieck ringed spaces

$$n_Z \colon (Z, \tilde{\mathcal{O}}_Z) \to (Z, \mathcal{O}_Z) \tag{8}$$

whose underlying topological morphism is the identity and whose homomorphism $\mathcal{O}_Z(U) \to \tilde{\mathcal{O}}_Z(U)$ for any admissible $U \subset Z$ is the natural injection by means of the Maximum Modulus Principle, i.e., Proposition 2.25.

Theorem 3.8. *Consider the morphism n_Z defined in (8). Suppose that*

i) Z is irreducible,

ii) the Zariski open subvariety $\Omega(S) \subset Z$ is normal,

iii) $Z \setminus \Omega(S)$ is of everywhere positive codimension in Z.

iv) any function $f\colon X \to C$ on any admissible $X \subset Z$ which is continuous with respect to the canonical topology and restricts to a regular function on $X \cap \Omega(S)$ restricts to a regular function on $X \cap \Omega(T)$ for any $T \subset S$ and

v) any $z \in Z$ has a fundamental basis of admissible neighborhoods U such that $U \cap \Omega(S)$ is connected and, in particular, non-empty.

Then n_Z is the normalization morphism in the sense of Conrad [12]. In particular, $(Z, \tilde{\mathcal{O}}_Z)$ is a normal rigid analytic variety.

We shall deduce Theorem 3.8 from Proposition 3.1 and the following lemma. We have learnt through Conrad's [12, End of proof of Theorem 1.1.3] about the following lemma and how to deduce it from Proposition 3.2.

Lemma 3.9. *Consider any morphism $f\colon Y := \mathrm{Sp}(B) \to \mathrm{Sp}(A) =: X$ between affinoid varieties over C and any $x \in X$. For any admissible $V \subset Y$ containing $f^{-1}(x)$ there exists an admissible $U \subset X$ containing x with $f^{-1}(U) \subset V$. In particular, if $f^{-1}(x)$ is finite, then the natural $\mathcal{O}_{X,x}$-homomorphism*

$$(f_*\mathcal{O}_Y)_x \to \prod_{y \in f^{-1}(x)} \mathcal{O}_{Y,y}$$

is an isomorphism.

Proof. Using that A is noetherian [8, Prop. 6.1.1.3], we choose finitely many generators $a_1, \ldots, a_n \in A$ of the vanishing ideal of x in A. Then the $b_i := f^*(a_i)$ generate the vanishing ideal of $f^{-1}(x)$. Consider any admissible $V \subset Y$ containing $f^{-1}(x)$. Proposition 3.2 then provides an $\varepsilon \in |C^\times|$ such that

$$f^{-1}(X(a_1 \leq \varepsilon, \ldots, a_n \leq \varepsilon)) = Y(b_1 \leq \varepsilon, \ldots, b_n \leq \varepsilon) \subset V$$

which shows the first part. Suppose that, moreover, $f^{-1}(x)$ is finite. Since for any $y \in Y$ the admissible affinoid subsets containing y form a fundamental system of open neighborhoods of y with respect to the canonical topology on Y and since the latter is Hausdorff, we may then choose for each $y \in f^{-1}(x)$ a fundamental system $\mathcal{C}_y$ of admissible affinoid neighborhoods of y such that $O_y \cap O_{y'} = \emptyset$ for any $O_y \in \mathcal{C}_y$ and any $O_{y'} \in \mathcal{C}_{y'}$ with $y \neq y' \in f^{-1}(x)$. As the union of any finitely many admissible affinoid subsets is again admissible, the second part of the lemma then follows by applying the first part to subsets V of the form

$$\bigcup_{y \in f^{-1}(x)} O_y$$

for various $O_y \in \mathcal{C}_y$. ∎

Proof of Theorem 3.8. Consider the normalization morphism in the sense of [12]

$$(n, n^{\#})\colon (\tilde{Z}, \mathcal{O}_{\tilde{Z}}) \to (Z, \mathcal{O}_Z).$$

We shall show that $(n, n^{\#})$ induces an isomorphism

$$(n, n^{+})\colon (\tilde{Z}, \mathcal{O}_{\tilde{Z}}) \to (Z, \tilde{\mathcal{O}}_Z)$$

whose composition with n_Z is $(n, n^{\#})$; this will then yield the theorem.

For any $T \subset S$ let $\tilde{T} := n^{+}(T)$ denote the set of global sections on $\tilde{Z}$ obtained by pulling back the elements of T by (n, n^{+}). Analogously as for Z and S, this yields a stratification of $\tilde{Z}$ by reduced locally closed subvarieties $\Omega(\tilde{T}) \subset \tilde{Z}$ for various $T \subset S$. Then $\Omega(\tilde{T}) = n^{-1}(\Omega(T))$ for any $T \subset S$; let

$$(n_T, n_T^{\#})\colon \Omega(\tilde{T}) \to \Omega(T)$$

be the morphism induced by $(n, n^{\#})$. Abbreviate $\Omega := \Omega(S)$ and $\tilde{\Omega} := \Omega(\tilde{S})$.

We first show that n is bijective. As it underlies a normalization morphism, it is surjective. We then consider any $z \in Z$ and claim that $|n^{-1}(z)| = 1$. Since any normalization morphism is finite, Lemma 3.9 applies to $(n, n^{\#})$ and yields that the natural homomorphism

$$(n_*\mathcal{O}_{\tilde{Z}})_z \to \prod_{y \in n^{-1}(z)} \mathcal{O}_{\tilde{Z},y}$$

is an isomorphism. It thus suffices to show that $(n_*\mathcal{O}_{\tilde{Z}})_z$ is integral. As $\Omega(S)$ is normal by ii), its irreducible and its connected subsets coincide. Assumptions v) and ii) thus provide a fundamental system $\mathcal{F}$ of admissible open neighborhoods $U \subset Z$ of z such that $U \cap \Omega$ is irreducible or, equivalently, such that $\mathcal{O}_Z(U \cap \Omega)$ is integral. As Ω is normal by ii), $(n_S, n_S^{\#})$ is an isomorphism so that $(n_*\mathcal{O}_{\tilde{Z}})(U \cap \Omega)$ is integral, too, for any $U \in \mathcal{F}$. Assumption i) implies that $\tilde{Z}$ is irreducible. Assumption v) implies that $\Omega \neq \emptyset$ if $Z \neq \emptyset$. Thus the Zariski open subvariety $\tilde{\Omega}$ of the irreducible $\tilde{Z}$ is dense. Consequently, the restriction homomorphism

$$(n_*\mathcal{O}_{\tilde{Z}})(U) \to (n_*\mathcal{O}_{\tilde{Z}})(U \cap \Omega)$$

is injective for any $U \in \mathcal{F}$ so that, in fact, $(n_*\mathcal{O}_{\tilde{Z}})(U)$ is integral. Since $\mathcal{F}$ is a fundamental system of admissible neighborhoods of z, this implies that $(n_*\mathcal{O}_{\tilde{Z}})_z$ is indeed integral. We have thus shown that n is bijective.

Since, furthermore, $(n, n^{\#})$ is finite, n is a homeomorphism with respect to the canonical topologies by [8, Lemma 9.5.3.6].

Let us then define n^+. Consider first any admissible affinoid $U \subset Z$ and set $\tilde{U} := n^{-1}(U)$. Let $n^+(U)$ be the composition

$$\tilde{\mathcal{O}}_Z(U) \hookrightarrow \tilde{\mathcal{O}}_Z(U \cap \Omega)^b = \mathcal{O}_Z(U \cap \Omega)^b \xrightarrow{\cong} \mathcal{O}_{\tilde{Z}}(\tilde{U} \cap \tilde{\Omega})^b \xrightarrow{\cong} \mathcal{O}_{\tilde{Z}}(\tilde{U}),$$

where $(\cdot)^b$ denotes the operator that associates the subalgebra of bounded elements and where the arrows are defined as follows: The first arrow is the restriction homomorphism; it is injective since $\Omega \subset Z$ is dense. As Ω is normal, the homomorphism $n_S^{\#}(U \cap \Omega)$ is an isomorphism. The second arrow is the restriction of this isomorphism to the subalgebra of bounded elements. Finally, we claim that the restriction homomorphism

$$R := \mathcal{O}_{\tilde{Z}}(\tilde{U}) \to \mathcal{O}_{\tilde{Z}}(\tilde{U} \cap \tilde{\Omega}) =: S$$

induces an isomorphism onto S^b; the last arrow is then defined to be the induced inverse. As n is finite and U is affinoid, its preimage $\tilde{U}$ is affinoid too by [8, Proposition 9.4.4.1]. The Maximum modulus principle thus yields the boundedness of any element in R and hence of its image in S. Conversely, any element in S^b extends uniquely to an element in R: Indeed, by normality of $\tilde{U}$ and the Riemann extension theorem (see Theorem 2.31), this holds true if $\tilde{U} \setminus \tilde{\Omega}$ is of everywhere positive codimension in $\tilde{U}$. But the latter condition is guaranteed by iii) since n is finite. This shows the claim and thus finishes the definition of $n^+(U)$.

In fact, $n^+(U)$ is surjective and hence, by the above, an isomorphism. Indeed, consider any $\tilde{f} \in \mathcal{O}_{\tilde{Z}}(\tilde{U})$. As n is a homeomorphism,

$$f := \tilde{f} \circ n^{-1}|_U : U \to C$$

is continuous with respect to the canonical topologies. As $\mathcal{O}_{\tilde{Z}}(\tilde{U})$ is affinoid, the Maximum Modulus Principle (see Propostion 2.25) yields that $\tilde{f}$, and hence f, is bounded. In order to show that $f \in \tilde{\mathcal{O}}_Z(U)$, it remains to be checked that the restriction f_T of f to $U \cap \Omega(T)$ is regular for any $T \subset S$. Since f_S corresponds to the restriction of $\tilde{f}$ to $\tilde{U} \cap \tilde{\Omega}$ via the isomorphism $\mathcal{O}_Z(U \cap \Omega)^b \to \mathcal{O}_{\tilde{Z}}(\tilde{U} \cap \tilde{\Omega})^b$, it is regular. The regularity of an arbitrary f_T then follows from Assumption iv). Hence $n^+(U)$ is indeed surjective.

For an arbitrary admissible subset $X \subset Z$, the homomorphism $n^+(X)$ is then defined in the natural way by means of the admissible covering of X by all its admissible affinoid subsets using the sheaf property of $\mathcal{O}_{\tilde{Z}}$; by the affinoid case above, it is an isomorphism as well.

It remains to be shown that n is an isomorphism of Grothendieck topologial spaces. We first consider any $T \subset S$ with $\Omega(T) \neq \emptyset$ and show that

$(n_T, n_T^\#)$ is an isomorphism. As $(n, n^\#)$ is finite and a homeomorphism with respect to the canonical topologies, so is $(n_T, n_T^\#)$. In order to see that the latter is an isomorphism, it thus suffices, by Proposition 2.24, to show that $n_T^\#$ induces isomorphisms on stalks. Consider any $z \in \Omega(T)$ and set $\tilde{z} := n_T^{-1}(z)$. As n_T is surjective and $\Omega(T)$ is reduced, the homomorphism on stalks $\mathcal{O}_{\Omega(T),z} \to \mathcal{O}_{\Omega(\tilde{T}),\tilde{z}}$ is injective. In order to see that it is also surjective, consider any $\tilde{g} \in \mathcal{O}_{\Omega(\tilde{T}),\tilde{z}}$ and choose, using that n is a homeomorphism, an admissible affinoid $U \subset \mathcal{U}(T)$ containing z such that $\tilde{g}$ is defined on $\tilde{U} \cap \Omega(\tilde{T})$, where $\tilde{U} := n^{-1}(U)$. As n is finite, also $\tilde{U}$ is affinoid. Thus we may and do choose an $\tilde{f} \in \mathcal{O}_{\tilde{Z}}(\tilde{U})$ that restricts to $\tilde{g}$ on the Zariski closed affinoid subvariety $\tilde{U} \cap \Omega(\tilde{T}) \subset \tilde{U}$. Let $f \in \tilde{\mathcal{O}}_Z(U)$ correspond to $\tilde{f}$ under the isomorphism $n^+(U)$ discussed above. In particular, f restricts to a regular function g on $U \cap \Omega(T)$. By continuity of n and the construction, then $n_T^\#(g) = \tilde{g}$. This yields surjectivity of the above map on stalks. We have thus shown that $(n_T, n_T^\#)$ is an isomorphism.

That n is an isomorphism, then follows from the fact that the preimage under the finite morphism $(n, n^\#)$ of any quasi-compact is quasi-compact and from applying Proposition 3.1 once as stated and once to Z and S replaced by $\tilde{Z}$ and $\tilde{S}$ using that $\mathcal{U}(\tilde{T}, \varepsilon) = n^{-1}(\mathcal{U}(T, \varepsilon))$ for any $T \subset S$ and any $\varepsilon \in |C^\times|$. This finishes the proof. $\square$

4 On the building for PGL over a non-Archimedean local field

Consider any non-Archimedean local field E. Denote by $\mathcal{O}_E$ the ring of integers of E. Choose a prime element $\pi \in \mathcal{O}_E$ and set $q := |\frac{1}{\pi}|$. Consider any finite dimensional E-vector space $\mathcal{V} \neq 0$ and set $d := \dim_E(\mathcal{V})$. Set

$$\mathcal{G} := \mathrm{Aut}_E(\mathcal{V}) \text{ and } \mathcal{PG} := \mathrm{PGL}(\mathcal{V}) = \mathcal{G}/E^{\times}.$$

In this section we give comprehensive proofs of results concerning the Bruhat-Tits building for $\mathcal{PG}$ and related concepts.

Consider further any algebraically closed complete non-Archimedean valued field C containing E as a valued subfield.

4.1 On a geometric covering attached to any simplicial complex

Consider any set S and any *simplicial complex* I whose set of vertices is S, i.e., I is any set of non-empty finite subsets of S, called *simplices*, such that

- $\forall s \in S\colon \{s\} \in I$ and
- $\forall \emptyset \neq \Delta' \subset \Delta \subset S\colon \Delta \in I \Rightarrow \Delta' \in I$.

Denote by $\overline{I}$ the barycentric subdivision of I, i.e., the simplicial complex whose set of vertices is I and whose general simplices are the sets $\{\Delta_1, \dots, \Delta_k\}$ with $\Delta_i \in I$ such that

$$\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_k.$$

Denote by $I(\mathbb{R})$ the set underlying the geometric realisation of I; it consists of the functions $\alpha : S \to [0,1]$ for which

$$\Delta(\alpha) := \{s \in S \mid \alpha(s) \neq 0\} \in I \text{ and } \sum_{s \in S} \alpha(s) = 1. \tag{9}$$

Consider any real $0 < c < 1$. Associate with any $\Delta \in I$ the subset

$$V_\Delta := \left\{ \alpha \in I(\mathbb{R}) \;\middle|\; \sum_{s \in \Delta} \alpha(s) \geq 1 - \frac{1+c}{4^{\#\Delta}} \text{ and } \forall s \in \Delta : \alpha(s) \geq \frac{3-c}{4^{\#\Delta}} \right\} \subset I(\mathbb{R}),$$

where $\#\Delta$ denotes the cardinality of Δ. The *nerve* of $(V_\Delta)_{\Delta \in I}$ is the set of $I' \subset I$ such that

$$\bigcap_{\Delta \in I'} V_\Delta \neq \emptyset.$$

An abstract version of Drinfeld's [15, Proposition 6.2] is

Proposition 4.1. *$(V_\Delta)_{\Delta\in I}$ is a covering of $I(\mathbb{R})$ whose nerve is $\overline{I}$.*

Proof. Consider any $\alpha \in I(\mathbb{R})$. Denote by $s_1, \dots, s_d$ the elements of $\Delta(\alpha)$ such that $\alpha(s_1) \geq \alpha(s_2) \geq \cdots \geq \alpha(s_d)$. By means of (9), let $1 \leq k \leq d$ be such that

$$\alpha(s_k) \geq \frac{3-c}{4^k} \text{ and } \forall d \geq i > k\colon \alpha(s_i) < \frac{3-c}{4^i}.$$

Then $\alpha \in V_{\{s_1,\dots,s_k\}}$ as $\alpha(s_i) \geq \alpha(s_k) \geq \frac{3-c}{4^k}$ for any $1 \leq i \leq k$ and as

$$1 - \sum_{1\leq i\leq k} \alpha(s_i) = \sum_{k<i\leq d} \alpha(s_i) < 3 \cdot \sum_{k<i} \frac{1}{4^i} < \frac{1}{4^k} < \frac{1+c}{4^k}.$$

Hence $(V_\Delta)_{\Delta\in I}$ is a covering of $I(\mathbb{R})$. It remains to be shown that its nerve is $\overline{I}$. Consider first any $\Delta \in I$ and any ordering $s_1, \dots, s_d$ of the elements in Δ. Let $\alpha \in I(\mathbb{R})$ be the element that vanishes on $S \setminus \Delta$ and on Δ is defined by

$$\forall 1 < i \leq d\colon \alpha(s_i) := \frac{3-c}{4^i} \text{ and } \alpha(s_1) := 1 - \sum_{1<i\leq d} \alpha(s_i).$$

Then $\alpha \in \bigcap_{1\leq k\leq d} V_{\{s_1,\dots,s_k\}}$ as $\alpha(s_i) \geq \frac{3-c}{4^k}$ for any $1 < i \leq k \leq d$ and as

$$\forall 1 \leq k \leq d\colon \sum_{1\leq i\leq k} \alpha(s_i) > 1 - \sum_{k<i} \frac{3-c}{4^i} = 1 - \frac{(3-c)}{3}\frac{1}{4^k} \geq 1 - \frac{1+c}{4^k}$$

which further implies that $\alpha(s_1) \geq \frac{1+c}{4} \geq \frac{1+c}{4^k}$ for any $1 \leq k \leq d$. It follows that $\overline{I}$ is contained in the nerve.

Conversely, consider any $\Delta, \Delta' \in I$ such that $\Delta \not\subset \Delta' \not\subset \Delta$. It remains to be shown that $V_\Delta \cap V_{\Delta'} = \emptyset$. Assume without loss of generality that $k := \#\Delta \leq \#\Delta' =: k'$. Choose then any $s_0 \in \Delta \setminus \Delta'$. Assume, by contradiction, the existence and choice of an $\alpha \in V_\Delta \cap V_{\Delta'}$. Then

$$\sum_{s_0\neq s\in S} \alpha(s) \geq \sum_{s\in\Delta'} \alpha(s) \geq 1 - \frac{1+c}{4^{k'}}, \text{ resp. } \alpha(s_0) \geq \frac{3-c}{4^k},$$

because $\alpha \in V_{\Delta'}$, resp. $\alpha \in V_\Delta$. Hence

$$1 = \sum_{s\in S} \alpha(s) \geq 1 - \frac{1+c}{4^{k'}} + \frac{3-c}{4^k} > 1 - \frac{2}{4^{k'}} + \frac{2}{4^k}$$

which contradicts our assumption that $k' \geq k$ as desired. □

Corollary 4.2. *Consider any left-action of any group G on I and the induced left-action of G on $I(\mathbb{R})$. For any $g \in G$ and any $\Delta \in I$ then $g(V_\Delta) = V_{g(\Delta)}$ and*

$$g(\Delta) \neq \Delta \Rightarrow g(V_\Delta) \cap V_\Delta = \emptyset.$$

Proof. The first property follows from the construction and implies jointly with Proposition 4.1 the second. □

4.2 The Bruhat-Tits building for PGL

Recall from Definition 2.36 that $\mathrm{S}_\mathcal{V}$ is defined as the set of free $\mathcal{O}_E$-submodules $m \subset \mathcal{V}$ of maximal rank together with the action of $E^\times$ by dilation.

Let $\mathrm{I}_\mathcal{V}$ be the simplicial complex in the sense of Section 4.1 whose simplices are the non-empty subsets $\Delta = \{s_1, \ldots, s_k\} \subset E^\times \backslash \mathrm{S}_\mathcal{V}$ admitting representatives $m_1 \in s_1, \ldots, m_k \in s_k$ such that

$$m_1 \supsetneq m_2 \supsetneq \cdots \supsetneq m_k \supsetneq \pi m_1. \tag{10}$$

We call any sequence as in (10) a *presimplex* of $\mathrm{S}_\mathcal{V}$ for Δ.

Remark 4.3. The presimplices as in (10) for fixed m_1 correspond bijectively to strictly increasing sequences of $\mathcal{O}_E/\pi\mathcal{O}_E$-vector subspaces of $m_1/\pi m_1$. In particular, $k \leq d$ for any such presimplex.

The set of vertices of $\mathrm{I}_\mathcal{V}$ then coincides with $E^\times \backslash \mathrm{S}_\mathcal{V}$ and the natural left action of $\mathcal{PG}$ on $E^\times \backslash \mathrm{S}_\mathcal{V}$ naturally extends to one on $\mathrm{I}_\mathcal{V}$.

Lemma 4.4. *The stabilizer $\{\gamma \in \Gamma \colon \gamma(\Delta) = \Delta\}$ of any $\Delta \in \mathrm{I}_\mathcal{V}$ in any discrete subgroup $\Gamma \subset \mathcal{PG}$ is finite.*

Proof. The stabilizer $\{g \in \mathcal{PG} \colon g([m]) = [m]\}$ of any $[m] \in E^\times \backslash \mathrm{S}_\mathcal{V}$ in $\mathcal{PG}$ equals $\mathrm{Aut}_{\mathcal{O}_E}(m) \cdot E^\times/E^\times$; its intersection with Γ is thus finite by Lemma 2.41. As any $\Delta \in \mathrm{I}_\mathcal{V}$ is a finite subset of $E^\times \backslash \mathrm{S}_\mathcal{V}$, the lemma follows. □

The following definition is a special case of the definition of the building in the sense of Bruhat and Tits [11] of a general semi-simple group over a local field. However, by contrast, we only define it as a set and will not use its natural topology.

Definition 4.5. *The set $\mathrm{I}_\mathcal{V}(\mathbb{R}_{>0})$ of functions $\alpha \colon E^\times \backslash \mathrm{S}_\mathcal{V} \to [1, q]$ for which*

$$\Delta(\alpha) := \{s \in E^\times \backslash \mathrm{S}_\mathcal{V} \mid \alpha(s) \neq 1\} \in \mathrm{I}_\mathcal{V} \text{ and } \prod_{s \in E^\times \backslash \mathrm{S}_\mathcal{V}} \alpha(s) = q$$

together with its naturally induced $\mathcal{PG}$-action is called the building *for $\mathcal{PG}$.*

Remark 4.6. Up to $[0,1] \to [1,q], x \to q^x$, the building coincides with the set underlying the geometric realization $\mathrm{I}_{\mathcal{V}}(\mathbb{R})$ attached to $\mathrm{I}_{\mathcal{V}}$ by Section 4.1.

In Def.-Prop.'s 4.19 and 4.21 below, we will define a natural $\mathcal{G}$-equivariant $\mathbb{R}_{>0}$-torsor $\mathrm{T}_{\mathcal{V}} \to \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$.

4.3 A set of seminorms

For any vector space X over any $D \in \{E, C\}$ a map $\nu\colon X \to \mathbb{R}_{\geq 0}$ is called a *seminorm* on X if

- $\forall d \in D, \forall x \in X\colon \nu(d \cdot x) = |d| \cdot \nu(x)$ and
- $\forall x, x' \in X\colon \nu(x + x') \leq \max\{\nu(x), \nu(x')\}$;

it is called a *norm* if furthermore

- $\forall x \in X\colon \nu(x) = 0 \Leftrightarrow x = 0$.

Any such seminorm ν is called *cartesian* if it is induced by the norm

$$X/\nu^{-1}(0) \to \mathbb{R}_{\geq 0}, \sum_{1 \leq i \leq k} d_i x_i \mapsto \max_{1 \leq i \leq k} |d_i| \cdot r_i \tag{11}$$

for a basis $\underline{x} := (x_1, \ldots, x_k)$ of $X/\nu^{-1}(0)$ and an $\underline{r} = (r_1, \ldots, r_k) \in (\mathbb{R}_{>0})^k$.

Denote by $\mathrm{SN}_{\mathcal{V}}$ the set of seminorms on $\mathcal{V}_C$ that restrict to norms on $\mathcal{V}$. It is naturally acted by $\mathbb{R}_{>0}$ and $\mathcal{G}$. For any $1 \leq k \leq d$ let

$$(\mathrm{CSN}_{\mathcal{V},k} \subset)\, \mathrm{SN}_{\mathcal{V},k} \subset \mathrm{SN}_{\mathcal{V}}$$

denote the $\mathbb{R}_{>0}$- and $\mathcal{G}$-invariant subsets of those (cartesian) seminorms ν for which the C-subspace $\nu^{-1}(0) \subset \mathcal{V}_C$ has codimension k.

Denote by $\mathrm{N}_{\mathcal{V}}$ the set of norms on $\mathcal{V}$. It is naturally acted by $\mathbb{R}_{>0}$ and $\mathcal{G}$.

Lemma 4.7. *[25, Proposition 1.1] Any $\nu \in \mathrm{N}_{\mathcal{V}}$ is cartesian.*

Corollary 4.8. *Any $\nu \in \mathrm{N}_{\mathcal{V}}$ induces the locally profinite topology on $\mathcal{V}$.*

By Lemma 4.7, choose for any $\nu \in \mathrm{N}_{\mathcal{V}}$ an $(\underline{x}, \underline{r})$ as in (11) in order to define, viewing $\underline{x}$ as a basis of $\mathcal{V}_C$, a cartesian norm $\nu_C \in \mathrm{CSN}_{\mathcal{V},d}$ on $\mathcal{V}_C$ extending ν.

Lemma 4.9. *For any $\nu \in \mathrm{N}_{\mathcal{V}}$ the norm $\nu_C \in \mathrm{CSN}_{\mathcal{V},d}$ is independent of the choice of $(\underline{x}, \underline{r})$.*

We shall prove Lemma 4.9 after Lemma 4.17 below.

Definition-Proposition 4.10. *Restricting any $\nu \in \mathrm{SN}_{\mathcal{V}}$ to $\mathcal{V}$ induces an $\mathbb{R}_{>0}$- and $\mathcal{G}$-equivariant map $r_N \colon \mathrm{SN}_{\mathcal{V}} \to \mathrm{N}_{\mathcal{V}}$ having as a right-inverse*

$$i_N \colon \mathrm{N}_{\mathcal{V}} \to \mathrm{CSN}_{\mathcal{V},d}, \nu \mapsto \nu_C.$$

Proof. This is directly verified. □

4.4 Interpretation of seminorms through submodules

For any C-vector space X and any $\mathcal{O}_C$-submodule $M \subset X$ denote by $K(M) \subset X$ the maximal C-vector subspace contained in M and set

$$\forall r \in \mathbb{R}_{\geq 0} \colon \; rM := \bigcap_{r<|c|\in|C^\times|} c \cdot M. \tag{12}$$

As $|C^\times|$ contains $q^{\mathbb{Q}}$, it is dense in $\mathbb{R}_{>0}$. Hence $(rr')M = r(r'M)$ for any such M and any $r, r' \in \mathbb{R}_{>0}$. But, in general, $1M$ need not equal M; for instance, when $X = C$ and M is the maximal ideal of $\mathcal{O}_C$, it does not.

Definition-Proposition 4.11. *Let $\mathrm{M}_{\mathcal{V}}$ be the set of $\mathcal{O}_C$-submodules $M \subset \mathcal{V}_C$ with*

i) $K(M) \cap \mathcal{V} = 0$ *and*

ii) $1M = M$ *and* $\bigcup_{c \in C^\times} c \cdot M = \mathcal{V}_C$.

Then (12) defines an $\mathbb{R}_{>0}$-action on $\mathrm{M}_{\mathcal{V}}$ such that $|c|M = c \cdot M$ for every $c \in C$ and every $M \in \mathrm{M}_{\mathcal{V}}$. Moreover, $\mathrm{M}_{\mathcal{V}}$ is closed under finite intersections.

Proof. This is directly checked using again that $|C^\times| \subset \mathbb{R}_{>0}$ is dense. □

Example 4.12. *Let $\nu \in \mathrm{SN}_{\mathcal{V}}$. Then $\nu^{-1}([0,1)) \subset \mathcal{V}_C$ satisfies i) and it satisfies ii) if and only if $\nu^{-1}([0,1)) = \nu^{-1}([0,1])$, i.e., if and only if $|C^\times| \cap \nu(\mathcal{V}_C) = \emptyset$.*

Proof. If it satisfies ii), then indeed

$$\nu^{-1}([0,1)) = 1\nu^{-1}([0,1)) = \bigcap_{1<|c|\in|C^\times|} c \cdot \nu^{-1}([0,1)) = \nu^{-1}([0,1]).$$

The converse direction follows similarly. □

Caution 4.13. *Defining rM in (12) with $<$ replaced by $\leq$ would not yield an $\mathbb{R}_{>0}$-action on the set of* all *$\mathcal{O}_C$-submodules $M \subset \mathcal{V}_C$ neither; indeed, it would yield for any $\nu \in \mathrm{SN}_{\mathcal{V}}$ and any $r, r' \in \mathbb{R}_{>0} \setminus |C^\times|$ with $rr' \in |C^\times| \cap \nu(\mathcal{V}_C)$ that*

$$(rr')\nu^{-1}([0,1)) = \nu^{-1}([0,rr')) \neq \nu^{-1}([0,rr']) = r(\nu^{-1}([0,r'])) = r(r'\nu^{-1}([0,1)).$$

Definition-Proposition 4.14. *Associating with any $M \in \mathrm{M}_{\mathcal{V}}$ the semi-norm*

$$\nu_M \colon \mathcal{V}_C \to \mathbb{R}_{\geq 0},\ v \mapsto \inf\{r \in \mathbb{R}_{\geq 0} : v \in rM\}$$

yields an $\mathbb{R}_{>0}$- and $\mathcal{G}$-equivariant bijection $\nu_\circ \colon \mathrm{M}_{\mathcal{V}} \to \mathrm{SN}_{\mathcal{V}}$ inverse to

$$\mathrm{SN}_{\mathcal{V}} \to \mathrm{M}_{\mathcal{V}}, \nu \mapsto \nu^{-1}([0,1]).$$

Moreover, $\forall M, M' \in \mathrm{M}_{\mathcal{V}}, \forall v \in \mathcal{V}_C \colon \nu_{M \cap M'}(v) = \max\{\nu_M(v), \nu_{M'}(v)\}$.

Proof. This is directly checked. □

Any $M \subset X$ as in (12) is called a *G-lattice* of X, where $G \subset \mathbb{R}_{>0}$ is any subgroup, if

$$M = \bigoplus_{x \in \beta} g_x(\mathcal{O}_C \cdot x)$$

for some basis β of X and some $(g_x)_{x\in\beta} \in G^\beta$, where $g_x(\mathcal{O}_C \cdot x)$ is defined by means of (12). For instance, the $|C^\times|$-lattices of such an X are the free $\mathcal{O}_C$-submodules of X whose C-span is X.

Definition-Proposition 4.15. *Any $\mathcal{O}_C$-submodule $M \subset \mathcal{V}_C$ for which $M/K(M)$ is an $\mathbb{R}_{>0}$-lattice of $\mathcal{V}_C/K(M)$ satisfies Def.-Prop. 4.11, ii). Denote by*

$$\mathrm{L}_{\mathcal{V}} \subset \mathrm{M}_{\mathcal{V}}$$

the subset of those M for which $M/K(M)$ is an $\mathbb{R}_{>0}$-lattice of $\mathcal{V}_C/K(M)$. Let

$$\mathrm{M}_{\mathcal{V},k} \subset \mathrm{M}_{\mathcal{V}} \text{ and } \mathrm{L}_{\mathcal{V},k} \subset \mathrm{L}_{\mathcal{V}}$$

denote the subsets of those M with $\dim_C(\mathcal{V}_C/K(M)) = k$ for any $1 \leq k \leq d$. All these subsets are $\mathbb{R}_{>0}$- and $\mathcal{G}$-invariant and for any $1 \leq k \leq d$ holds that

$$\nu_\circ(\mathrm{M}_{\mathcal{V},k}) = \mathrm{SN}_{\mathcal{V},k} \text{ and } \nu_\circ(\mathrm{L}_{\mathcal{V},k}) = \mathrm{CSN}_{\mathcal{V},k}\,. \tag{13}$$

Proof. This is directly checked. □

Denote the free $\mathcal{O}_C$-submodule of $\mathcal{V}_C$ generated by any $m \in \mathrm{S}_\mathcal{V}$ by

$$\langle m \rangle \in \mathrm{L}_{\mathcal{V},d}\,. \tag{14}$$

For any $r \in \mathbb{R}_{>0}$ denote by $\overline{r}$ the unique representative in $[1, q)$ of the class of r in $\mathbb{R}_{>0}/q^\mathbb{Z}$ and by $\lfloor r \rfloor$ the largest element in $q^\mathbb{Z}$ being $\leq r$; then

$$r = \overline{r} \cdot \lfloor r \rfloor. \tag{15}$$

Lemma 4.16. *i) $\forall M \in \mathrm{M}_\mathcal{V} \colon M \cap \mathcal{V} \in \mathrm{S}_\mathcal{V}$.*

ii) $\forall m \in \mathrm{S}_\mathcal{V}, \forall r \in \mathbb{R}_{>0} \colon \mathcal{V} \cap r\langle m \rangle = \lfloor r \rfloor \cdot m$.

Proof. Let $M \in \mathrm{M}_\mathcal{V}$. Then $M \cap \mathcal{V}$ is the unit ball of $r_\mathrm{N}(\nu_M)$. As $r_\mathrm{N}(\nu_M)$ is strictly cartesian, $M \cap \mathcal{V}$ is contained in some free $\mathcal{O}_E$-submodule of $\mathcal{V}$ of maximal rank. As $\mathcal{O}_E$ is a principal ideal domain, thus $M \cap \mathcal{V}$ is itself a free $\mathcal{O}_E$-module of $\mathcal{V}$ and, being a unit ball of a norm, of maximal rank, i.e., $M \cap \mathcal{V} \in \mathrm{S}_\mathcal{V}$. In order to see Part ii), choose an $\mathcal{O}_E$-basis for any $m \in \mathrm{S}_\mathcal{V}$ and view it as an $\mathcal{O}_C$-basis of $\langle m \rangle$. □

Lemma 4.17. *Consider any basis $v_1 \ldots, v_d$ of $\mathcal{V}$ and any $s_1, \ldots, s_d \in (1, q]$ and set*

$$M := \bigoplus_{1 \leq j \leq d} s_j \mathcal{O}_C \cdot v_j.$$

Then $M = q \bigcap_{r \in \mathbb{R}_{>0}} \frac{1}{r} \langle \mathcal{V} \cap rM \rangle$.

Proof. Denote by $1 < t_1 < \cdots < t_k \leq q$ the ordered elements of $\{s_j \colon 1 \leq j \leq d\}$ and set $t_{k+1} := q \cdot t_1$. For any $1 \leq i \leq k$ and any $t_i < r \leq t_{i+1}$ then

$$\mathcal{V} \cap \frac{1}{r} M = \bigoplus_{j \colon s_j > t_i} \mathcal{O}_E \cdot v_j \oplus \bigoplus_{j \colon s_j \leq t_i} \frac{1}{q} \cdot \mathcal{O}_E \cdot v_j$$

using that $\mathcal{V} \cap \frac{s_j}{r}\mathcal{O}_C \cdot v_j = \lfloor \frac{s_j}{r} \rfloor \cdot \mathcal{O}_E \cdot v_j$ for any $1 \le j \le d$. Hence

$$\begin{aligned}
q \bigcap_{r \in \mathbb{R}_{>0}} \frac{1}{r}\langle \mathcal{V} \cap rM \rangle &= \bigcap_{t_1 < r \le t_{k+1}} rq\langle \mathcal{V} \cap \frac{1}{r}M \rangle \\
&= \bigcap_{1 \le i \le k} \bigcap_{t_i < r \le t_{i+1}} rq\langle \mathcal{V} \cap \frac{1}{r}M \rangle \\
&= \bigcap_{1 \le i \le k} t_i q \left(\bigoplus_{j\colon s_j > t_i} \mathcal{O}_C \cdot v_j \oplus \bigoplus_{j\colon s_j \le t_i} \frac{1}{q}\mathcal{O}_C \cdot v_j \right) \\
&= \bigcap_{1 \le i \le k} \left(\bigoplus_{j\colon s_j > t_i} t_i q \mathcal{O}_C \cdot v_j \oplus \bigoplus_{j\colon s_j \le t_i} t_i \mathcal{O}_C \cdot v_j \right) \\
&= \bigoplus_{1 \le j \le d} \min\left(\{t_i q\colon s_j > t_i\} \cup \{t_i\colon s_j \le t_i\}\right) \mathcal{O}_C \cdot v_j \\
&= \bigoplus_{1 \le j \le d} s_j \mathcal{O}_C \cdot v_j = M
\end{aligned}$$

as desired. □

Proof of Lemma 4.9. Consider any $\nu \in \mathrm{N}_{\mathcal{V}}$ and any $(\underline{x}, \underline{r})$ and $(\underline{x}', \underline{r}')$ both satisfying (11) for ν. We have to show that the cartesian norms $\nu_C, \nu'_C \in \mathrm{CSN}_{\mathcal{V},d}$ defined using $(\underline{x}, \underline{r})$, resp. $(\underline{x}', \underline{r}')$, coincide. Denote by $M, M' \in \mathrm{L}_{\mathcal{V},d}$ the lattices corresponding to ν_C, ν'_C. By Def.-Prop. 4.15, it suffices to show that $M = M'$. However, both M, M' are of the form considered in Lemma 4.17 and

$$\forall r \in \mathbb{R}_{>0}\colon \mathcal{V} \cap rM = \nu^{-1}([0, r]) = \mathcal{V} \cap rM'$$

by construction. By Lemma 4.17, thus $M = M'$ as desired. □

Definition-Proposition 4.18. *The $\mathbb{R}_{>0}$- and $\mathcal{G}$-equivariant map*

$$r_{\mathrm{L}} := \nu_\circ^{-1} \circ i_{\mathrm{N}} \circ r_{\mathrm{N}} \circ \nu_\circ\colon \mathrm{M}_{\mathcal{V}} \to \mathrm{L}_{\mathcal{V},d}$$

satisfies that $r_{\mathrm{L}} \circ r_{\mathrm{L}} = r_{\mathrm{L}}$ and for any $M \in \mathrm{M}_{\mathcal{V}}$ that

$$r_{\mathrm{L}}(M) = q \bigcap_{r \in \mathbb{R}_{>0}} \frac{1}{r}\langle \mathcal{V} \cap rM \rangle = q \bigcap_{c \in C^\times} \langle M \cap c\mathcal{V} \rangle. \tag{16}$$

where $\langle M \cap c\mathcal{V} \rangle \in \mathrm{L}_{\mathcal{V},d}$ is the $\mathcal{O}_C$-submodule generated by $M \cap c\mathcal{V} \in \mathrm{S}_{c\mathcal{V}}$.

Proof. Let us first argue for (16). Denote by $r'(M)$ the middle term of (16) for any $M \in \mathrm{M}_{\mathcal{V}}$. Consider any $M \in \mathrm{M}_{\mathcal{V}}$. Then ν_M and $\nu_{r_{\mathrm{L}}(M)}$ restrict to the same norm on $\mathcal{V}$. In particular, the unit balls $\mathcal{V} \cap M$ and $\mathcal{V} \cap r_{\mathrm{L}}(M)$ of their restrictions coincide. By $\mathbb{R}_{>0}$-equivariance of r_{L}, in fact $\mathcal{V} \cap r \cdot M = \mathcal{V} \cap r \cdot r_{\mathrm{L}}(M)$ for any $r \in \mathbb{R}_{>0}$. Hence $r'(M) = r'(r_{\mathrm{L}}(M))$. Moreover, $r'(r_{\mathrm{L}}(M)) = r_{\mathrm{L}}(M)$ by Lemma 4.17 using that, by construction, any module in the image of $\nu_\circ^{-1} \circ i_{\mathrm{N}}$ is of the form considered in Lemma 4.17. Hence $r'(M) = r_{\mathrm{L}}(M)$ as desired. In the proof of Def.-Prop. 4.22,ii) below, we will further show that $(i_{\mathrm{T}} \circ r_{\mathrm{T}})(M)$ is equal to both the middle and the right term of (16), where the definition of $i_{\mathrm{T}} \circ r_{\mathrm{T}}$ is independent of this def.-prop.. This yields (16).

That $r_{\mathrm{L}} \circ r_{\mathrm{L}} = r_{\mathrm{L}}$ follows from Def.-Prop. 4.10 but it may also be checked using (16): Indeed, for any $c \in C^\times$ holds that

$$\begin{aligned}\bigcap_{c'' \in C^\times} \langle M \cap c''\mathcal{V} \rangle \cap c\mathcal{V} &\overset{c' = \frac{c''}{c}}{=} \bigcap_{c' \in C^\times} c' \langle \frac{1}{c'} M \cap c\mathcal{V} \rangle \cap c\mathcal{V} \\ &= \bigcap_{c' \in C^\times, 1 \le |c'| < q} c' \langle \frac{1}{c'} M \cap c\mathcal{V} \rangle \cap c\mathcal{V} \\ &\overset{*}{=} \bigcap_{c' \in C^\times, 1 \le |c'| < q} \frac{1}{c'} M \cap c\mathcal{V} = \pi M \cap c\mathcal{V} = \pi(M \cap c\mathcal{V}),\end{aligned}$$

where $\overset{*}{=}$ holds by Lemma 4.16,ii) upon replacing $\mathcal{V}$ by $c\mathcal{V}$. Hence

$$\frac{1}{q^2} r_{\mathrm{L}}(r_{\mathrm{L}}(M)) = \bigcap_{c \in C^\times} \left\langle \bigcap_{c'' \in C^\times} \langle M \cap c''\mathcal{V} \rangle \cap c\mathcal{V} \right\rangle = \bigcap_{c \in C^\times} \pi \langle M \cap c\mathcal{V} \rangle = \frac{1}{q^2} r_{\mathrm{L}}(M).$$

□

4.5 The building as a quotient by a free $\mathbb{R}_{>0}$-action

Using the notation in (14) set $\mathrm{S}_{\mathcal{V}}(\mathbb{R}_{>0}) := \{ r\langle m \rangle : r \in \mathbb{R}_{>0}, m \in \mathrm{S}_{\mathcal{V}} \} \subset \mathrm{L}_{\mathcal{V},d}$.

Definition-Proposition 4.19. *Define* $\mathrm{T}_{\mathcal{V}}$ *to be the set of subsets*

$$\{ t_{i-1} \langle m_i \rangle : 1 \le i \le k \} \subset \mathrm{S}_{\mathcal{V}}(\mathbb{R}_{>0}) \tag{17}$$

for all $1 \le k \le d$*, all presimplices* $m_1 \supsetneq \cdots \supsetneq m_k \supsetneq \pi m_1$ *of* $\mathrm{S}_{\mathcal{V}}$ *and all* $1 < t_1 < \cdots < t_k \le q$ *with* $t_0 := |\pi| \cdot t_k$*. Then*

i) any such subset uniquely determines such m_i *and* t_i *and* *ii) the action of $\mathcal{G}$, resp. the free action of $\mathbb{R}_{>0}$, on the set of all subsets of $\mathrm{S}_{\mathcal{V}}(\mathbb{R}_{>0})$ induced by its natural action on $\mathrm{S}_{\mathcal{V}}(\mathbb{R}_{>0})$ restricts to $\mathrm{T}_{\mathcal{V}}$.*

Proof. Part i) is directly checked using Lemma 4.16, ii). Consider then any $T = \{t_{i-1}\langle m_i\rangle \colon 1 \le i \le k\} \in \mathrm{T}_{\mathcal{V}}$ as in (17) and any $r \in \mathbb{R}_{>0}$. Then

$$rT := \{r \cdot t_{i-1}\langle m_i\rangle \colon 1 \le i \le k\} = \left\{t'_{i-1}\langle m'_i\rangle \colon 1 \le i \le k\right\},$$

where for any $0 \le i \le k$, using the notation of (15),

$$t'_i := \left\{ \begin{array}{ll} \overline{r} \cdot t_i, & \text{if } \overline{r} \cdot t_i \le q \\ |\pi| \cdot \overline{r} \cdot t_i, & \text{if } \overline{r} \cdot t_i > q \end{array} \right\} \text{ and } m'_i := \left\{ \begin{array}{ll} \lfloor r \rfloor m_i, & \text{if } \overline{r} \cdot t_i \le q \\ \frac{1}{\pi}\lfloor r \rfloor m_i, & \text{if } \overline{r} \cdot t_i > q \end{array} \right\}.$$

There is thus a unique cyclic permutation σ of $\{1, \ldots, k\}$ with $1 < t'_{\sigma(1)} < \cdots < t'_{\sigma(k)} \le q$ and such that $m'_{\sigma(1)} \supsetneq \cdots \supsetneq m'_{\sigma(k)} \supsetneq \pi m'_{\sigma(1)}$ is a presimplex. Hence $rT \in \mathrm{T}_{\mathcal{V}}$ as desired. The statement for $\mathcal{G}$ is directly checked. □

Caution 4.20. *The $\mathbb{R}_{>0}$-action on any $T \in \mathrm{T}_{\mathcal{V}}$ respects the ordering of the elements of T only up to cyclic permutations.*

Definition-Proposition 4.21. *For any $T = \{t_{i-1}\langle m_i\rangle \colon 1 \le i \le k\} \in \mathrm{T}_{\mathcal{V}}$ let*

$$\alpha_T \colon E^\times \backslash \mathrm{S}_{\mathcal{V}} \to [1, q], [m] \mapsto \left\{ \begin{array}{ll} \frac{t_i}{t_{i-1}}, & \text{if } [m] = [m_i] \text{ for some } 1 \le i \le k, \\ 1, & \text{otherwise.} \end{array} \right\}$$

This yields an $\mathbb{R}_{>0}$-invariant and $\mathcal{G}$-equivariant map $\mathrm{T}_{\mathcal{V}} \to \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0}), T \mapsto \alpha_T$ which induces a $\mathcal{PG}$-equivariant bijection

$$\mathbb{R}_{>0} \backslash \mathrm{T}_{\mathcal{V}} \cong \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0}).$$

Proof. That the map is well-defined, invariant on $\mathbb{R}_{>0}$-classes and that the induced map $\mathbb{R}_{>0} \backslash \mathrm{T}_{\mathcal{V}} \to \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ is injective and $\mathcal{PG}$-equivariant is directly checked. Consider then any $\alpha \in \mathrm{I}_{\mathcal{V}}(\mathbb{R})$ and choose any presimplex $m_1 \supsetneq \cdots \supsetneq m_k \supsetneq \pi m_1$ for $\Delta(\alpha)$. Set

$$\forall 0 \le i \le k \colon t_i := \prod_{1 \le j \le i} \alpha([m_j]).$$

Then $T := \{t_{i-1}\langle m_i\rangle \colon 1 \le i \le k\} \in \mathrm{T}_{\mathcal{V}}$ and $\alpha_T = \alpha$. Hence the map is also surjective. □

Definition-Proposition 4.22. *i) Let $M \in \mathrm{M}_{\mathcal{V}}$ with $\mathbb{R}_{>0}$-class $[M]$. Then*

$$\Delta_{[M]} := \{[\mathcal{V} \cap rM] \colon r \in \mathbb{R}_{>0}\} = \{[\mathcal{V} \cap rM] \colon 1 \leq r < q\}$$

is a simplex admitting a unique presimplex $m_1 \supsetneq \cdots \supsetneq m_{k_M} \supsetneq \pi m_1$ with

$$\{V \cap rM \colon 1 \leq r < q\} = \{m_1, \ldots, m_{k_M}\}. \tag{18}$$

Moreover, any $\{1 \leq r < q \colon \mathcal{V} \cap rM = m_i\}$ contains its infimum $r_{M,i}$; in particular, $m_i = \mathcal{V} \cap r_{M,i}M$. Set $r_{M,0} := q \cdot r_{M,k_M}$ and

$$T_M := \left\{ \frac{q}{r_{M,i-1}} \langle \mathcal{V} \cap r_{M,i} M \rangle \colon 1 \leq i \leq k_M \right\} \in \mathrm{T}_{\mathcal{V}}\,.$$

ii) The map $r_{\mathrm{T}} \colon \mathrm{M}_{\mathcal{V}} \to \mathrm{T}_{\mathcal{V}}, M \mapsto T_M$ is $\mathbb{R}_{>0}$- and $\mathcal{G}$-equivariant and has

$$i_{\mathrm{T}} \colon \mathrm{T}_{\mathcal{V}} \to \mathrm{L}_{\mathcal{V},d},\ T \mapsto \bigcap_{M' \in T} M'$$

as right inverse. Moreover, $i_{\mathrm{T}} \circ r_{\mathrm{T}} = r_{\mathrm{L}}$.

Remark 4.23. See (22) below for a rather explicit description of the lattice structure of any $i_{\mathrm{T}}(T)$.

Proof. Part i): From Lemma 4.16, i) and Remark 4.3 follows that the left hand side of (18) is of the desired form. Moreover, as $M \in \mathrm{M}_{\mathcal{V}}$, the set $\{1 \leq r < q \colon w \in rM\}$ contains its infimum i_w for any $w \in \mathcal{W}$. For any $\mathcal{O}_E$-basis β of any m_i thus $\{1 \leq r < q \colon \mathcal{V} \cap rM = m_i\}$ has $\max_{w \in \beta} i_w$ as its infimum which it contains. The remaining assertions are directly checked.

Part ii): That r_{T} is $\mathbb{R}_{>0}$- and $\mathcal{G}$-equivariant is directly checked. Consider any

$$T = \{t_{i-1} \langle m_i \rangle \colon 1 \leq i \leq k\} \in \mathrm{T}_{\mathcal{V}}\,.$$

Then $i_{\mathrm{T}}(M) \in \mathrm{M}_{\mathcal{V},d}$ since $\mathrm{M}_{\mathcal{V},d}$ is closed under finite intersections. That in fact $i_{\mathrm{T}}(M) \in \mathrm{L}_{\mathcal{V},d}$, follows from Def.-Prop. 4.18 and the equality $i_{\mathrm{T}} \circ r_{\mathrm{T}} = r_{\mathrm{L}}$ that we prove below. That $(r_{\mathrm{T}} \circ i_{\mathrm{T}})(T) = T$ holds true since any $r \in [1, q)$ admits a unique $1 \leq j \leq k$ for which $\frac{q}{t_j} \leq r < \frac{q}{t_{j-1}}$ and hence

$$\mathcal{V} \cap r \cdot (i_{\mathrm{T}}(T)) = \bigcap_{1 \leq i \leq k} \mathcal{V} \cap r t_{i-1} \langle m_i \rangle \overset{4.16, ii)}{=} \bigcap_{\substack{1 \leq i \leq k \\ r t_{i-1} \geq q}} q m_i \cap \bigcap_{\substack{1 \leq i \leq k \\ r t_{i-1} < q}} m_i = m_j.$$

Let $M \in \mathrm{M}_{\mathcal{V}}$ and set $r_i := r_{M,i}$ for any $0 \leq i \leq k_M =: k$. We show that $(i_{\mathrm{T}} \circ r_{\mathrm{T}})(M)$ is equal to both the middle and the right term of (16) and hence to $r_{\mathrm{L}}(M)$. Using, in the first of the subsequent equations, that $|E^\times| = q^{\mathbb{Z}}$ and that any $c\mathcal{V}$ depends only on the class of c in $C^\times/E^\times$, yields that

$$\begin{aligned}\bigcap_{c\in C^\times}\langle M\cap c\mathcal{V}\rangle &= \bigcap_{\substack{c\in C^\times\\ r_k\leq|c|<r_0}}\langle M\cap \frac{1}{c}\mathcal{V}\rangle = \bigcap_{\substack{c\in C^\times\\ r_k\leq|c|<r_0}}\frac{1}{c}\langle\mathcal{V}\cap cM\rangle\\ &= \bigcap_{1\leq i\leq k}\ \bigcap_{\substack{c\in C^\times\\ r_i\leq|c|<r_{i-1}}}\frac{1}{c}\langle\mathcal{V}\cap cM\rangle = \bigcap_{1\leq i\leq k}\ \bigcap_{\substack{c\in C^\times\\ |c|<r_{i-1}}}\frac{1}{c}\langle\mathcal{V}\cap r_iM\rangle\\ &= \bigcap_{1\leq i\leq k}\frac{1}{r_{i-1}}\langle\mathcal{V}\cap r_iM\rangle = \frac{1}{q}(i_{\mathrm{T}}\circ r_{\mathrm{T}})(M).\end{aligned}$$

Hence $(i_{\mathrm{T}}\circ r_{\mathrm{T}})(M)$ equals the right term of (16). Moreover,

$$\bigcap_{r\in\mathbb{R}_{>0}}\frac{1}{r}\langle\mathcal{V}\cap rM\rangle = \bigcap_{r_k\leq r<r_0}\frac{1}{r}\langle\mathcal{V}\cap rM\rangle = \frac{1}{q}(i_{\mathrm{T}}\circ r_{\mathrm{T}})(M),$$

where the first equation is satisfied since any $\frac{1}{r}\langle\mathcal{V}\cap rM\rangle$ depends only on the class of r in $\mathbb{R}_{>0}/q^{\mathbb{Z}}$ and where the second equation follows by the same argument as in the previous sentence upon replacing $C^\times$ by $\mathbb{R}_{>0}$. Hence $(i_{\mathrm{T}}\circ r_{\mathrm{T}})(M)$ also equals the middle term of (16) as desired. □

4.6 Metrization

Throughout this section, a *metric* on a set X is any symmetric map

$$\rho\colon X\times X\to\mathbb{R}_{\geq 1}$$

such that $\rho(x,y)=1\Leftrightarrow x=y$ and $\rho(x,y)\leq\rho(x,z)\cdot\rho(z,y)$ for all $x,y,z\in X$.

Definition-Proposition 4.24. *[25, Theorem 2.3] Setting*

$$\forall[\nu],[\nu']\in\mathbb{R}_{>0}\backslash\,\mathrm{N}_{\mathcal{V}}\colon\ \rho_{\mathrm{N}}([\nu],[\nu']) := \max_{0\neq v,v'\in\mathcal{V}}\frac{\nu'(v)}{\nu(v)}\cdot\frac{\nu(v')}{\nu'(v')}$$

defines a metric ρ_{N} on $\mathbb{R}_{>0}\backslash\,\mathrm{N}_{\mathcal{V}}$ for which any closed ball is compact.

Definition-Proposition 4.25. *A metric ρ_{L} on $\mathbb{R}_{>0}\backslash\,\mathrm{L}_{\mathcal{V},d}$ is given by setting*

$$\rho_{\mathrm{L}}(S,S') := \inf\{r\in\mathbb{R}_{\geq 1}\mid \exists M\in S, M'\in S'\colon M\subset M'\subset rM\}$$

for any $S,S'\in\mathbb{R}_{>0}\backslash\,\mathrm{L}_{\mathcal{V},d}$, where the set contains its infimum.

Proof. This is directly checked using $\mathcal{O}_C$-bases of the elements in $\mathrm{L}_{\mathcal{V},d}$. $\square$

Proposition 4.26. *The injection $(\nu_\circ)^{-1} \circ i_{\mathrm{N}}$ induces an isometric embedding $E^\times \backslash \mathrm{N}_{\mathcal{V}} \to \mathbb{R}_{>0} \backslash \mathrm{L}_{\mathcal{V},d}$.*

Proof. Consider any $\nu, \nu' \in \mathrm{N}_{\mathcal{V}}$, denote by $M, M' \in \mathrm{L}_{\mathcal{V},d}$ their images under $(\nu_\circ)^{-1} \circ i_{\mathrm{N}}$ and let us show that

$$r := \rho_{\mathrm{L}}([M],[M']) = \rho_{\mathrm{N}}([\nu],[\nu']). \tag{19}$$

Using the $\mathbb{R}_{>0}$-equivariance of $(\nu_\circ)^{-1} \circ i_N$, we assume without loss of generality that M and M' realize r, i.e., that $M \subset M' \subset rM$ or, equivalently, that the global inequalities $\nu_M \leq \nu_{M'} \leq \nu_{rM}$ hold and that $\nu_M(v) = \nu_{M'}(v)$ and $\nu_{M'}(v') = \nu_{rM}(v')$ for some $v, v' \in \mathcal{V}_C$. As, by construction of $(\nu_\circ)^{-1} \circ i_N$,

$$M = \bigoplus_{1 \leq i \leq d} r_i \mathcal{O}_C v_i, \text{ resp. } M' = \bigoplus_{1 \leq i \leq d} r'_i \mathcal{O}_C v'_i,$$

for some E-basis $v_1, \ldots, v_d$, resp. $v'_1, \ldots, v'_d$, of $\mathcal{V}$ and some $r_1, \ldots, r_d \in \mathbb{R}_{>0}$, resp. $r'_1, \ldots, r'_d \in \mathbb{R}_{>0}$, such a v, resp. v', may in fact be chosen among the v_i, resp. the v'_i. Using that $\nu_{rM} = r \cdot \nu_M$ and that ν_M, resp. $\nu_{M'}$, restricts to ν, resp. ν', on $\mathcal{V}$, the equality in (19) is then directly checked. $\square$

Lemma 4.27. *i) $\forall m, m' \in \mathrm{S}_{\mathcal{V}}\colon \rho_{\mathrm{L}}([\langle m \rangle],[\langle m' \rangle]) \in q^{\mathbb{Z}_{\geq 0}}$.*

ii) Let $S \subset E^\times \backslash \mathrm{S}_{\mathcal{V}}$. Then S is a simplex if and only if $\rho_{\mathrm{L}}([\langle m \rangle],[\langle m' \rangle]) \leq q$ for any $[m],[m'] \in S$.

Proof. This is directly checked; for instance, Part ii) follows directly from Part i) which in turn follows directly from Lemma 4.16, ii). $\square$

Proposition 4.28. *Consider any $T = \{t_{i-1}\langle m_i \rangle \colon 1 \leq i \leq k\} \in \mathrm{T}_{\mathcal{V}}$ as in (17) and any $T' = \{q\langle m \rangle\} \in \mathrm{T}_{\mathcal{V}}$ for any $m \in \mathrm{S}_{\mathcal{V}}$. Then*

$$\rho_{\mathrm{L}}([i_{\mathrm{T}}(T)],[i_{\mathrm{T}}(T')]) = \left\{ \begin{array}{ll} q \cdot \frac{t_{i-1}}{t_i}, & \textit{if } [m] = [m_i] \textit{ for some } 1 \leq i \leq k, \\ \geq q, & \textit{otherwise.} \end{array} \right\}$$

Proof. Set $S := [i_{\mathrm{T}}(T)]$ and $S' := [i_{\mathrm{T}}(T')] = [\langle m \rangle]$ and $\rho := \rho_{\mathrm{L}}$. Then $\rho(S,S')$ is the minimal quotient $\frac{r}{s}$ of all $r, s \in \mathbb{R}_{>0}$ for which

$$s\langle m \rangle \subset i_{\mathrm{T}}(T) = \bigcap_{1 \leq i \leq k} t_{i-1}\langle m_i \rangle \subset r\langle m \rangle. \tag{20}$$

We first consider the case where $\{[m]\} \cup \{[m_i] \colon 1 \leq i \leq k\}$ is a simplex. In order to show the proposition in this case, we may replace m by $\pi^n \cdot m$ for any integer n and thus assume that

$$m_1 \supset \cdots \supset m_j \supset m \supsetneq m_{j+1} \supset \cdots \supset m_{k+1} := \pi m_1$$

for a unique $1 \leq j \leq k$. We further choose an $\mathcal{O}_E$-basis β of m_1 such that

$$\beta = \bigcup_{1 \leq l \leq k} \beta_l, \text{ where } \beta_l := \beta \cap (m_l \setminus m_{l+1}),$$

using that the $m_l/\pi m_1$ for all $1 \leq l \leq k$ form a flag of $\mathcal{O}_E/(\pi)$-vector subspaces of $m_1/\pi m_1$. For any $1 \leq i \leq k$ then

$$m_i = \left(\bigoplus_{1 \leq l < i} \bigoplus_{v \in \beta_l} \mathcal{O}_E \pi v \right) \oplus \left(\bigoplus_{i \leq j \leq k} \bigoplus_{v \in \beta_l} \mathcal{O}_E v \right) \tag{21}$$

so that

$$\bigcap_{1 \leq i \leq k} t_{i-1} \langle m_i \rangle = \pi \cdot \bigoplus_{1 \leq l \leq k} \bigoplus_{v \in \beta_l} t_l \cdot \mathcal{O}_C v. \tag{22}$$

If $m_j = m$, one directly deduces from (21) and (22) that (20) holds true if and only if $s \geq |\pi| \cdot t_j$ and $r \leq t_{j-1}$ so that then $\rho(S, S') = \frac{t_{j-1}}{|\pi| t_j} = q \cdot \frac{t_{j-1}}{t_j}$.

If $m_j \supsetneq m \supsetneq m_{j+1}$, assume without loss of generality that $\beta_j = \beta_j' \cup \beta_j''$, where $\beta_j' := \beta_j \cap (m_j \setminus m)$ and $\beta_j'' := \beta_j \cap (m \setminus m_{j+1})$, in order to write that

$$m = \left(\bigoplus_{1 \leq l < i} \bigoplus_{v \in \beta_l} \mathcal{O}_E \pi v \right) \oplus \bigoplus_{v \in \beta_j'} \mathcal{O}_E \pi v \oplus \bigoplus_{v \in \beta_j''} \mathcal{O}_E v \oplus \left(\bigoplus_{i < j \leq k} \bigoplus_{v \in \beta_l} \mathcal{O}_E v \right);$$

from this, jointly with (22), one directly deduces that (20) holds true if and only if $s \geq |\pi| \cdot t_j$ and $r \leq t_j$ so that then $\rho(S, S') = \frac{t_j}{|\pi| t_j} = q$.

Consider finally the case, where $\{[m]\} \cup \{[m_i] \colon 1 \leq i \leq k\}$ is not a simplex. Using Lemma 4.27,ii), choose an $1 \leq i \leq k$ for which $q^2 \leq \rho([\langle m_i \rangle], S')$. As $\rho(S, [\langle m_i \rangle]) = q \cdot \frac{t_{i-1}}{t_i} < q$ by the previous case, then

$$q^2 \leq \rho([\langle m_i \rangle], S') \leq \rho([\langle m_i \rangle], S) \cdot \rho(S, S') < q \cdot \rho(S, S')$$

so that $\rho(S, S') > q$ in this case. This finishes the proof. □

Definition-Proposition 4.29. *The injections* i_{T} *and* $\nu^{-1}\circ\circ\,\mathrm{i}_{\mathrm{N}}$*, whose images coincide, induce a* $\mathcal{G}$*- and* $\mathbb{R}_{>0}$*-equivariant bijection* $\mathrm{T}_{\mathcal{V}}\to\mathrm{N}_{\mathcal{V}}$ *and thus, by Def.-Prop. 4.21, a* $\mathcal{PG}$*-equivariant bijection* $i_{\mathcal{V}}\colon\ \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})\to\mathbb{R}_{>0}\backslash\,\mathrm{N}_{\mathcal{V}}$ *such that*

$$\forall\alpha\in\mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0}),s\in E^{\times}\backslash\,\mathrm{S}_{\mathcal{V}}\colon\ \rho_{\mathrm{N}}(i_{\mathcal{V}}(\alpha),i_{\mathcal{V}}(s))=\left\{\begin{array}{ll}\frac{q}{\alpha(s)}, & \textit{if } s\in\Delta(\alpha),\\ \geq q, & \textit{otherwise.}\end{array}\right\}$$

Proof. Def.-Prop.'s 4.18, 4.21 and 4.22,ii) yield the desired map and the metric is computed via Def.-Prop 4.21 by Proposition 4.28. □

Definition 4.30. *Denote by* $U_\nu\subset\mathcal{V}$ *the unit ball with respect to any* $\nu\in\mathrm{N}_{\mathcal{V}}$*.*

Definition-Proposition 4.31. *For any* $m\in\mathrm{S}_{\mathcal{V}}$ *denote by* $\nu_m\in\mathrm{N}_{\mathcal{V}}$ *the norm* $v\mapsto\inf\{|e|\in|E^{\times}|\colon v\in e\cdot m\}$*; then* $U_{\nu_m}=m\setminus\pi m$ *and for arbitrary* $\nu'\in\mathrm{N}_{\mathcal{V}}$*:*

$$\rho_{\mathrm{N}}([\nu'],[\nu])=\frac{\max_{u\in U_{\nu_m}}\nu'(u)}{\min_{u\in U_{\nu_m}}\nu'(u)}. \tag{23}$$

Proof. This is directly checked. □

5 Quotients of Drinfeld's period domain by discrete groups

Let E, π, q, $\mathcal{V}$, $\mathcal{G}$, $\mathcal{PG}$ and C be as in Section 4 and k a positive integer. Then

$$\mathbb{P}_{\mathrm{Hom}_C(\mathcal{V}_C,C^k)} := (\mathrm{Hom}_C(\mathcal{V}_C, C^k) \setminus \{0\})/C^\times$$

is equipped with a structure of projective rigid analytic variety over C (see Example 2.21). Consider the $\mathcal{PG}$-invariant subset

$$\Omega_{\mathcal{V},k} \subset \mathbb{P}_{\mathrm{Hom}_C(\mathcal{V}_C,C^k)}$$

of those $C^\times$-classes $[l]$ of C-linear maps $l\colon \mathcal{V}_C \to C^k$ for which $\mathrm{Ker}(l) \cap \mathcal{V} = 0$; if $k = 1$, this is Drinfeld's *period domain* for $\mathcal{V}$ which we denote by $\Omega_{\mathcal{V}}$.

5.1 Admissibility of the period domain

Continue to denote by $\mathrm{N}_{\mathcal{V}}$ the set of norms on $\mathcal{V}$, by $\mathrm{S}_{\mathcal{V}}$ the set of free $\mathcal{O}_E$-submodules of $\mathcal{V}$ of maximal rank and by $\nu_m \in \mathrm{N}_{\mathcal{V}}$ the norm associated in Def.-Prop. 4.31 with any $m \in \mathrm{S}_{\mathcal{V}}$. Denote by $B_r([\nu])$ the closed ball of radius $r \geq 1$ around any $[\nu] \in \mathbb{R}_{>0}\backslash \mathrm{N}_{\mathcal{V}}$ with respect to the metric ρ_{N} introduced in Def.-Prop. 4.24. Consider the $\mathcal{PG}$-equivariant map

$$\lambda_{\mathcal{V},k}\colon \Omega_{\mathcal{V},k} \to \mathbb{R}_{>0}\backslash \mathrm{N}_{\mathcal{V}}$$

that sends any $[l] = [(l_i)_{1\leq i\leq k}]$, where $l_i\colon \mathcal{V}_C \to C$ is the i-th coefficient function, to the class of the norm

$$v \mapsto |l(v)| := \max_{1\leq i\leq k} |l_i(v)|.$$

Proposition 5.1. *Let $m_1, \ldots, m_t \in \mathrm{S}_{\mathcal{V}}$ and for any $1 \leq c \in |C|$ set*

$$X(c) := \lambda_{\mathcal{V},k}^{-1}\left(\left\{[\nu] \in \mathbb{R}_{>0}\backslash \mathrm{N}_{\mathcal{V}} \colon \prod_{1\leq s\leq t} \rho_{\mathrm{N}}([\nu],[\nu_{m_s}]) \leq c\right\}\right) \subset \Omega_{\mathcal{V},k}.$$

Then $\Omega_{\mathcal{V},k} \subset \mathbb{P}_{\mathrm{Hom}_C(\mathcal{V}_C,C^k)}$ is admissible, any such $X(c) \subset \Omega_{\mathcal{V},k}$ is an admissible quasi-compact, resp. affinoid if $k = 1$, subset and the covering $(X(c_n))_{n\geq 1}$ of $\Omega_{\mathcal{V}}$, for any unbounded sequence $(c_n)_{n\geq 1}$ in $|C|$, is admissible.

Remark 5.2. If $k = 1$, this is Drinfeld's [15, Proposition 6.1]. However, our proof does not exactly specialize to his. In the case where $k = t = 1$, our proof specializes to the one given by Schneider and Stuhler in [40, Section 1, Proof of Proposition 4], where they denote $X(q^n)$ by $\overline{\Omega}_n$ for any positive integer n. In fact, by means of such a covering, they show that $\Omega_{\mathcal{V}}$ is a Stein space.

Proof of Proposition 5.1. Set $\mathbb{P} := \mathbb{P}_{\mathrm{Hom}_C(\mathcal{V}_C, C^k)}$. Using Definition 4.30 and Def.-Prop. 4.31, set $U_s := U_{\nu_{m_s}}$ for any $1 \leq s \leq t$. Set $m := \prod_{1\leq s\leq t} m_s$ and $U := \prod_{1\leq s\leq t} U_s$ and $J := \{1, \ldots, k\}^t$. Let $1 \leq c \in |C|$. Set

$$X(c, u, j) := \left\{ [l] \in \mathbb{P} \;\middle|\; c \cdot \prod_{1\leq s\leq t} |l_{j_s}(u_s)| \geq \prod_{1\leq s\leq t} \max_{u'_s \in U_s}(|l(u'_s)|) \right\} \tag{24}$$

for any $(u, j) \in U \times J$. By (23) and as $\mathcal{V} \setminus \{0\} = E^\times \cdot U_s$ for any $1 \leq s \leq t$,

$$X(c) = \bigcap_{u\in U} \bigcup_{j\in J} X(c, u, j); \tag{25}$$

this is in fact a finite intersection since any $X(c, u, j)$ depends only on $u \bmod \pi^n m$ for any $n \geq 1$ with $q^n > c$. In order to see that any $X(c)$ is admissible quasi-compact, resp. affinoid if $k = 1$, it thus suffices, by Proposition 2.22, to show that any such subset $X(c, u, j) \subset \mathbb{P}$ is admissible affinoid. Consider any $(u, j) \in U \times J$ and for any $1 \leq s \leq t$ the admissible subset

$$X(u_s, j_s) := \{[l] \in \mathbb{P} \colon l_{j_s}(u_s) \neq 0\} \subset \mathbb{P};$$

any basis β of $\mathcal{V}$ containing u_s yields the rigid analytic isomorphism

$$i_{u_s,j_s,\beta} \colon X(u_s, j_s) \to \mathbb{A}_C^{\beta\times\{1,\ldots,k\}\setminus\{(u_s,j_s)\}}, [l] \mapsto \left(\frac{l_n(v)}{l_{j_s}(u_s)}\right)_{(v,n)\neq(u_s,j_s)}.$$

Let $X(u, j) \subset \mathbb{P}$ be the intersection of these $X(u_s, j_s)$. Choose any $\mathcal{O}_E$-basis β_s of m_s that contains u_s for every $1 \leq s \leq t$. As any of the maxima in (24) is attained at an element of such a β_s, thus

$$X(c, u, j) = \bigcap_{v\in\prod_{1\leq s\leq t}\beta_s} \bigcap_{i\in J} \left\{ [l] \in X(u, j) \;\middle|\; c \geq \prod_{1\leq s\leq t} \left| \frac{l_{i_s}(v_s)}{l_{j_s}(u_s)} \right| \right\}. \tag{26}$$

In particular, for any $1 \leq s \leq t$ then $X(c, u, j)$ is contained in the affinod $i^{-1}_{u_s,j_s,\beta_s}(B_c)$, where B_c denotes the closed polydisc of radius c around 0. Denote by $X'(c, u, j) \subset \mathbb{P}$ the intersection of all $i^{-1}_{u_s,j_s,\beta_s}(B_c)$ for all $1 \leq$

$s \leq t$; it is again affinoid by Proposition 2.22 and satisfies that $X(c, u, j) \subset X'(c, u, j) \subset X(u, j)$. In particular, the equality in (26) remains valid if $X(u, j)$ is replaced by $X'(c, u, j)$. Thus $X(c, u, j)$ is an admissible affinoid subset of the admissible affinoid subset $X'(c, u, j) \subset \mathbb{P}$ and hence itself an admissible affinoid subset of $\mathbb{P}$. As argued before, thus $X(c) \subset \mathbb{P}$ is an admissible quasi-compact, resp. affinoid if $k = 1$, subset. Moreover, the covering $(X(q^n, u, j))_{n\geq 1}$ of $X(u, j)$ is admissible as the image of any morphism $Z \to X(u, j)$ from an affinoid Z is already contained in some $X(q^n, u, j)$ by the Maximum Modulus Principle (see Proposition 2.25) applied to the composition of φ with any of the products in (26).

For any $u \in U$ let $X(u) \subset \mathbb{P}$ be the union of the $X(u, j)$ for all $j \in J$; then $(X(u, j))_{j\in J}$ is a Zariski open, and hence admissible, covering of the Zariski open, and hence admissible, $X(u)$. Thus $(X(q^n, u, j)_{j\in J, n\geq 1}$ is an admissibly covering of any such $X(u)$ and it refines the covering $(X(q^n, u))_{n\geq 1}$ of $X(u)$, where

$$\forall c \in |C|\colon X(c, u) := \bigcup_{j\in J} X(c, u, j)$$

is admissible quasi-compact. Thus $(X(q^n, u))_{n\geq 1}$ is admissible.

Consider then any unbounded sequence $(c_n)_{n\geq 1}$ in $|C|$. Consider an arbitrary morphism $\varphi\colon Z \to \mathbb{P}$ from an affinoid variety Z whose image is contained in $\Omega_{\mathcal{V},k}$. In order to show that $\Omega_{\mathcal{V},k} \subset \mathbb{P}$ is admissible and admissibly covered by the $X(c_n)$, it suffices (see [8, Prop. 9.1.4.2]) to show that the image of φ is already contained in some $X(c_n)$. Since $\Omega_{\mathcal{V},k} \subset X(u)$ for any $u \in U$ and since $(X(q^n, u))_{n\geq 1}$ is an admissible covering of $X(u)$, the image of φ is contained in $X(q^{n_u}, u)$ for some $n_u \geq 1$. Choose such an $n_u \geq 1$ for any $u \in U$. By means of the quasi-compactness of U, choose a finite subset $U_0 \subset U$ such that U is covered by the $u + \pi^{n_u+1}m$ for all $u \in U_0$. Choose an $n \geq 1$ such that $c_n \geq q^{n_u}$ for all $u \in U_0$. Any $u' \in U$ thus admits an $u \in U_0$ for which $u' - u \in \pi^{n_u+1}m$ and hence

$$X(c_n, u') \supset X(q^{n_u}, u') = X(q^{n_u}, u) \supset \mathrm{Im}(\varphi).$$

Hence the image of φ is contained in $X(c_n)$ by (25) as desired. □

Corollary 5.3. *For any $O \subset \Omega_{\mathcal{V},k}$ the following are equivalent:*

i) O is contained in an admissible quasi-compact subset of $\Omega_{\mathcal{V},k}$.

ii) $\lambda_{\mathcal{V},k}(O)$ is bounded.

iii) $\exists \kappa > 0\colon \forall\, [l], [l'] \in O, \forall\, 0 \neq x, y \in \mathcal{V}\colon\ \frac{|l(y)|}{|l(x)|} \leq \kappa \cdot \frac{|l'(y)|}{|l'(x)|}$.

If $k = 1$, these conditions are equivalent to O being contained in an admissible affinoid subset of $\Omega_{\mathcal{V}}$.

Proof. Condition iii) is a reformulation of ii) (c.f. Def-Prop. 4.24). The Corollary then directly follows from Proposition 5.1. $\square$

Corollary 5.4. *Consider any $O \subset \Omega_{\mathcal{V},k}$ that is contained in an admissible quasi-compact subset of $\Omega_{\mathcal{V},k}$ and consider any non-empty discrete subset $\Lambda \subset \mathcal{V}$. Then there exists a finite subset of Λ in which every $[l] \in O$ attains $\inf_{\lambda\in\Lambda} |l(\lambda)|$.*

Proof. For any $[l] \in \Omega_{\mathcal{V}}$ the infimum $i(l) := \inf_{\lambda\in\Lambda} |l(\lambda)|$ is attained at some element of Λ as $l(\Lambda)$ is strongly discrete (see Example 2.48 and Lemma 2.49). Assume without loss of generality that $O \neq \emptyset$ and that $0 \notin \Lambda$. Choose any $\kappa > 0$ satisfying the property in Corollary 5.3, iii) for O. Choose any $[l_0] \in O$ and any $\lambda_0 \in \Lambda$ at which $i(l_0)$ is attained. Consider any further $[l] \in O$ and any $\lambda \in \Lambda$ at which $i(l)$ is attained. Then

$$\frac{|l_0(\lambda)|}{|l_0(\lambda_0)|} \leq \kappa \cdot \frac{|l(\lambda)|}{|l(\lambda_0)|} \leq \kappa$$

and hence $|l_0(\lambda)| \leq |l_0(\lambda_0)|$. As $l_0(\Lambda)$ is strongly discrete and as $l_0|_{\mathcal{V}}$ is injective, the last inequality requires λ to lie in a finite subset of Λ that depends only on $[l_0]$ and λ_0 and κ. This yields the corollary. $\square$

Lemma 5.5. *Any fiber of $\lambda_{\mathcal{V},k}$ is open with respect to the canonical topology.*

Proof. Set $\mathbb{P} := \mathbb{P}_{\mathrm{Hom}_C(\mathcal{V}_C, C^k)}$. Let $[l] \in \Omega_{\mathcal{V},k}$. By means of Lemma 4.7, choose a basis β of $\mathcal{V}$ such that

$$\forall (\mu_w)_{w\in\beta} \in E^\beta \colon \left| l\left(\sum_{w\in\beta} \mu_w \cdot w\right)\right| = \max_{w\in\beta} |\mu_w| \cdot |l(w)|. \tag{27}$$

Choose any $w_0 \in \beta$ and any $1 \leq i_0 \leq k$ for which $l_{i_0}(w_0) \neq 0$. We further choose an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ the closed ball

$$B_{\varepsilon,\beta}([l]) := \left\{ [l'] \in \mathbb{P} \;\middle|\; l'_{i_0}(w_0) \neq 0 \wedge \forall w \in \beta, \forall 1 \leq i \leq k \colon \left| \frac{l'_i(w)}{l'_{i_0}(w_0)} - \frac{l_i(w)}{l_{i_0}(w_0)} \right| \leq \varepsilon \right\}$$

around $[l]$ is contained in $\Omega_{\mathcal{V},k}$, where we use that such balls form a basis of neighborhoods of $[l] \in \mathbb{P}$ and that $\Omega_{\mathcal{V},k} \subset \mathbb{P}$ is admissible. Consider any $0 < \varepsilon < \varepsilon_0$ such that

$$\forall w \in \beta \colon \varepsilon < \left| \frac{l(w)}{l_{i_0}(w_0)} \right|. \tag{28}$$

We claim that $\lambda_{\mathcal{V},k}(B_{\varepsilon,\beta}([l])) = \lambda_{\mathcal{V},k}([l])$; this will then directly yield that $\lambda_{\mathcal{V},k}^{-1}(\lambda_{\mathcal{V},k}([l]))$ is indeed open. It suffices to show that

$$\forall [l'] \in B_{\varepsilon,\beta}([l]), \forall v \in \mathcal{V}\colon \left|\frac{l'(v)}{l'_{i_0}(w_0)}\right| = \left|\frac{l(v)}{l_{i_0}(w_0)}\right|; \tag{29}$$

indeed, any such $[l']$ then gives rise to the same class of norms on $\mathcal{V}$ as $[l]$. Consider any such $[l']$ and v and write $v = \sum_{w\in\beta} \mu_w \cdot w$ with $\mu_w \in E$. Then

$$\begin{aligned}
\left|\frac{l'(v)}{l'_{i_0}(w_0)} - \frac{l(v)}{l_{i_0}(w_0)}\right| &= \max_{1\le i\le k}\left|\frac{l'_i(v)}{l'_{i_0}(w_0)} - \frac{l_i(v)}{l_{i_0}(w_0)}\right| \\
&\le \max_{1\le i\le k}\max_{w\in\beta} |\mu_w| \cdot \left|\frac{l'_i(w)}{l'_{i_0}(w_0)} - \frac{l_i(w)}{l_{i_0}(w_0)}\right| \\
&\le \max_{1\le i\le k}\max_{w\in\beta} |\mu_w| \cdot \varepsilon \;=\; \max_{w\in\beta}\; |\mu_w| \cdot \varepsilon \\
&\overset{(28)}{<} \max_{w\in\beta} |\mu_w| \cdot \left|\frac{l(w)}{l_{i_0}(w_0)}\right| \overset{(27)}{=} \left|\frac{l(v)}{l_{i_0}(w_0)}\right|.
\end{aligned}$$

As the norm on C^k is non-Archimedean, this yields (29) as desired. □

5.2 Quotients by discrete subgroups

By means of Def.-Prop. 4.29, identify $\mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ with $\mathbb{R}_{>0}\backslash \mathrm{N}_{\mathcal{V}}$ and consider for any $0 < \varepsilon < 1$ its $\mathcal{PG}$-invariant covering $(V_\Delta^\varepsilon)_{\Delta\in \mathrm{I}_{\mathcal{V}}}$ from Proposition 4.1 via Remark 4.6 whose nerve is the barycentric subdivision $\bar{\mathrm{I}}_{\mathcal{V}}$ of $\mathrm{I}_{\mathcal{V}}$, where

$$V_\Delta^\varepsilon := \left\{\alpha \in \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0}) \;\middle|\; \prod_{s\in\Delta} \rho_{\mathrm{N}}(\alpha, s) \le c' \text{ and } \forall s \in \Delta : \rho_{\mathrm{N}}(\alpha,s) \le c\right\},$$

where $c' = q^{|\Delta|-1-\frac{1+\varepsilon}{4\#\Delta}}$ and $c = q^{1-\frac{3-\varepsilon}{4\#\Delta}}$. For any such ε and Δ set

$$U_{\Delta,k}^\varepsilon := \lambda_{\mathcal{V},k}^{-1}(V_\Delta^\varepsilon),$$

where $\lambda_{\mathcal{V},k}\colon \Omega_{\mathcal{V},k} \to \mathbb{R}_{>0}\backslash \mathrm{N}_{\mathcal{V}}$ is the $\mathcal{PG}$-equivariant map from Section 5.1.

Proposition 5.6. *[15, Propositions 6.1 and 6.2] Let $0 < \varepsilon < 1$ be rational. Then $(U_{\Delta,k}^\varepsilon)_{\Delta\in\mathrm{I}_{\mathcal{V}}}$ is an admissible covering of $\Omega_{\mathcal{V},k}$ by quasi-compact, resp. affinoid if $k = 1$, subsets which has nerve $\bar{\mathrm{I}}_{\mathcal{V}}$; in particular,*

$$\forall \Delta, \Delta' \in \mathrm{I}_{\mathcal{V}}\colon U_{\Delta,k}^\varepsilon \cap U_{\Delta',k}^\varepsilon \ne \emptyset \Leftrightarrow \Delta \subset \Delta' \vee \Delta \supset \Delta'.$$

Moreover, $\forall g \in \mathcal{PG}, \Delta \in \mathrm{I}_{\mathcal{V}}\colon g(U_{\Delta,k}^\varepsilon) = U_{g(\Delta),k}^\varepsilon$.

Proof. By Proposition 5.1 and since $q^{\mathbb{Q}} \subset |C|$, any $U^{\varepsilon}_{\Delta,k}$ is the intersection of finitely many admissible quasi-compact, resp. affinoid if $k = 1$, subsets of $\Omega_{\mathcal{V},k}$ and is thus, by Proposition 2.22, itself an admissible quasi-compact, resp. affinoid if $k = 1$, subset of $\Omega_{\mathcal{V},k}$.

In order to see that the covering $(U^{\varepsilon}_{\Delta,k})_{\Delta\in \mathrm{I}_{\mathcal{V}}}$ of $\Omega_{\mathcal{V},k}$ is admissible, we consider any closed ball $B \subset \mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ around any $[\nu_m]$ for any $m \in \mathrm{S}_{\mathcal{V}}$, set $X := \lambda^{-1}_{\mathcal{V},k}(B)$ and are reduced to showing, by Proposition 5.1, that the quasi-compact X is admissibly covered by the $U^{\varepsilon}_{\Delta,k} \cap X$ or, equivalently, that X is covered by finitely many of the $U^{\varepsilon}_{\Delta,k}$ or, equivalently, that B is covered by finitely many of the V^{ε}_{Δ}. The latter follows from the quasi-compactness of B (c.f. Def.-Prop. 4.24): Indeed, for any $\Delta \in \mathrm{I}_{\mathcal{V}}$ let $\mathring{V}^{\varepsilon}_{\Delta}$ be defined like V^{ε}_{Δ} upon replacing $\leq$ by $<$ everywhere. Then $V^{\varepsilon'}_{\Delta} \subset \mathring{V}^{\varepsilon}_{\Delta}$ for any $\Delta \in \mathrm{I}_{\mathcal{V}}$ and any $0 < \varepsilon' < \varepsilon$. Hence the open $\mathring{V}^{\varepsilon}_{\Delta}$ for all $\Delta \in \mathrm{I}_{\mathcal{V}}$ cover $\mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ as well. The quasi-compact B is thus covered by finitely many of the $\mathring{V}^{\varepsilon}_{\Delta}$ and hence by finitely many of the V^{ε}_{Δ} as desired. The remaining assertions of the proposition follow directly from the discussion preceding it. □

Let $\Gamma \subset \mathcal{PG}$ be any subgroup which is discrete with respect the locally profinite topology on $\mathcal{PG}$ defined in Section 2.4. Consider the quotient map

$$p_{\Gamma,k} \colon \Omega_{\mathcal{V},k} \to \Gamma\backslash\Omega_{\mathcal{V},k} =: \Omega_{\Gamma,k}$$

and endow its target with the structure of Grothendieck ringed space which it induces, that is, a subset (resp. a covering of a subset) of $\Omega_{\Gamma,k}$ is admissible precisely when its preimage is admissible and the sections on an admissible subset of $\Omega_{\Gamma,k}$ are the Γ-invariant sections on its preimage.

Lemma 5.7. *For any quasi-compact $U, U' \subset \Omega_{\mathcal{V},k}$ the set $\{\gamma \in \Gamma \colon U' \cap \gamma(U) \neq \emptyset\}$ is finite.*

Proof. Consider any rational $0 < \varepsilon < 1$. As the covering $(U^{\varepsilon}_{\Delta,k})_{\Delta\in \mathrm{I}_{\mathcal{V}}}$ of $\Omega_{\mathcal{V}}$ is admissible by Proposition 5.6, any quasi-compact subset of $\Omega_{\mathcal{V}}$ is covered by finitely many of its elements. It thus suffices to show for any $\Delta, \Delta' \in \mathrm{I}_{\mathcal{V}}$ that $\{\gamma \in \Gamma \colon U^{\varepsilon}_{\Delta',k} \cap \gamma(U^{\varepsilon}_{\Delta,k}) \neq \emptyset\}$ is finite. However, this follows from Proposition 5.6 and, using that Γ is discrete, from Lemma 4.4. □

Proposition 5.8. *If $k > 1$, suppose that the action of Γ on $\Omega_{\mathcal{V},k}$ is free. Then $\Omega_{\Gamma,k}$ is a normal rigid analytic variety over C and an admissible covering of $\Omega_{\Gamma,k}$ by quasi-compact, resp. affinoid if $k = 1$, subsets is given by $(p_{\Gamma,k}(U^{\varepsilon}_{\Delta,k})_{\Delta\in \mathrm{I}_{\mathcal{V}}}$ for any rational $0 < \varepsilon < 1$.*

Proof. Consider any such ε. Set $p_\Gamma := p_{\Gamma,k}$. For any $\Delta \in \mathrm{I}_\mathcal{V}$ set $U_\Delta := U^\varepsilon_{\Delta,k}$ and

$$\Gamma U_\Delta := \bigcup_{\gamma\in\Gamma} \gamma(U_\Delta) = p_\Gamma^{-1}(p_\Gamma(U_\Delta)).$$

From Propositions 2.22 and 5.6 and Lemma 5.7 follows that $(U_{\Delta'} \cap \Gamma U_\Delta)_{\Delta\in\mathrm{I}_\mathcal{V}}$ is a system of admissible subsets of $U_{\Delta'}$ for any $\Delta' \in \mathrm{I}_\mathcal{V}$; it is then in fact an admissible covering since it is refined by $(U_{\Delta'} \cap U_\Delta)_{\Delta\in\mathrm{I}_\mathcal{V}}$ which is an admissible covering by Proposition 5.6. As $(U_{\Delta'})_{\Delta'\in\mathrm{I}_\mathcal{V}}$ is an admissible covering of $\Omega_{\mathcal{V},k}$, thus $(\Gamma U_\Delta)_{\Delta\in\mathrm{I}_\mathcal{V}}$ is an admissible covering of $\Omega_{\mathcal{V},k}$ and, equivalently, $(p_\Gamma(U_\Delta))_{\Delta\in\mathrm{I}_\mathcal{V}}$ is an admissible covering of $\Omega_{\Gamma,k}$.

Consider any $\Delta \in \mathrm{I}_\mathcal{V}$. It remains to be shown that any $(p_\Gamma(U_\Delta))$ is a quasi-compact, resp. affinoid if $k = 1$, rigid analytic variety over C. The covering $(\gamma(U_\Delta))_{\gamma\in\Gamma}$ of ΓU_Δ is admissible since, by Propositions 2.22 and 5.6 and Lemma 5.7, its intersection with any element of the admissible covering $(U_{\Delta'})_{\Delta'\in\mathrm{I}_\mathcal{V}}$ of $\Omega_{\mathcal{V},k}$ is admissible. Denote by Γ_Δ the stabilizer of Δ in Γ; it is finite by Lemma 4.4. By Proposition 5.6, then $\gamma(U_\Delta) = U_\Delta$ for any $\gamma \in \Gamma_\Delta$ and $\gamma(U_\Delta) \cap U_\Delta = \emptyset$ for any $\gamma \in \Gamma \setminus \Gamma_\Delta$. The inclusion $U_\Delta \to \Gamma U_\Delta$ thus induces an isomorphism of Grothendieck ringed spaces $\Gamma_\Delta \backslash U_\Delta \to p_\Gamma(U_\Delta)$. It thus suffices to show that the domain of this isomorphism is a normal quasi-compact, resp. affinoid if $k = 1$, rigid analytic variety over C. If $k = 1$, this follows from Proposition 2.33. If $k > 1$, except for the normality, this follows from [14, Theorem 5.1.1 and Remark before Lemma 5.2.1] since U_Δ is separated and the finite Γ_Δ acts freely on it. If $k > 1$, normality follows from Corollary 5.10 below whose proof does not depend on it. □

Proposition 5.9. *Consider any $\omega \in \Omega_{\mathcal{V},k}$ and denote by Γ_ω its stabilizer in Γ. Then there exists a basis of admissible affinoid neighborhoods U of ω such that*

i) $\forall \gamma \in \Gamma_\omega : \ \gamma(U) = U$ *and*

ii) $\forall \gamma \in \Gamma \setminus \Gamma_\omega : \ \gamma(U) \cap U = \emptyset$.

Proof. Consider the fiber $f := \lambda^{-1}_{\mathcal{V},k}(\lambda_{\mathcal{V},k}(\omega))$. Let $S \subset \Gamma$ be the subset of elements γ for which $\gamma(f) \cap f \neq \emptyset$; it is finite by Proposition 5.1 and Lemma 5.7. Using that the canonical topology on $\Omega_{\mathcal{V},k}$ is Hausdorff and that $f \subset \Omega_{\mathcal{V},k}$ is open by Lemma 5.5, we choose for any $\gamma \in S \setminus \Gamma_\omega$ an admissible affinoid neighborhood $U_\gamma \subset f$ of ω for which $\gamma(U_\gamma) \cap U_\gamma = \emptyset$. For any neighborhood U' of ω then

$$U := \left(\bigcap_{\gamma'\in\Gamma_\omega} \bigcap_{\gamma\in S\setminus\Gamma_\omega} \gamma'(U_\gamma) \right) \cap \left(\bigcap_{\gamma\in\Gamma_\omega} \gamma(U') \right)$$

is an neighborhood of ω that is contained in U' and satisfies i) and ii). If such an U' is affinoid, then U is the intersection of finitely many affinoid subsets and hence, by Proposition 2.22, itself affinoid. As the affinoid neighborhoods of ω form a basis of neighborhoods of ω, this yields the proposition. □

Corollary 5.10. *The morphism $p_{\Gamma,k}$ is open with respect to the canonical topologies and, if Γ acts fixed-points free on $\Omega_{\mathcal{V},k}$, it induces isomorphisms on the stalks and the stalks on $\Omega_{\Gamma,k}$ are then regular.*

Example 5.11. *Consider any admissible coefficient subring $A \subset C$ such that the completion of its quotient field is E. Consider any projective A-submodule $\Lambda \subset \mathcal{V}$ for which the natural homomorphism $\Lambda \otimes_A E \to \mathcal{V}$ is an isomorphism. Let $0 \neq I \subset A$ be an ideal. Then the kernel of the natural homomorphism*

$$\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(I^{-1}\Lambda\backslash\Lambda)$$

has discrete image in $\mathcal{PG}$ and, if $I \neq A$, its action on $\Omega_{\mathcal{V},k}$ is fixed-point free.

Proof. As $\Lambda \subset \mathcal{V}$ is discrete, the image in $\mathcal{PG}$ of $\mathrm{Aut}_A(\Lambda)$ itself is discrete by Examples 2.42 and 2.48. Consider any $[l] \in \Omega_{\mathcal{V},k}$ and any γ in the kernel of $\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(I^{-1}\Lambda\backslash\Lambda)$ with $\gamma[l] = [l]$. Hence $\gamma l = c \cdot l$ for some $c \in C^\times$. Using Example 2.48 and Lemma 2.49, choose an $0 \neq \lambda \in \Lambda$ for which $|l(\lambda)|$ is minimal among $|l(\Lambda)|\backslash\{0\}$. Then $|c \cdot l(\lambda)|$ is minimal among $|c \cdot l(\Lambda)|\backslash\{0\}$. As $l(\Lambda) = c \cdot l(\Lambda)$, thus $|l(\lambda)| = |c \cdot l(\lambda)| = |(\gamma l)(\lambda)| = |l(\lambda) + l(\gamma^{-1}\lambda - \lambda)|$ and hence

$$|l(\gamma^{-1}\lambda - \lambda)| \leq |l(\lambda)|$$

Moreover, $\gamma^{-1}\lambda - \lambda \in I\Lambda$ by definition of Γ. If $I \neq A$, then the smallest non-zero element of $l(I\Lambda)$ is strictly larger than the one of $l(\Lambda)$. In this case, thus $\gamma^{-1}\lambda - \lambda = 0$ and hence $c \cdot l(\lambda) = (\gamma l)(\lambda) = l(\lambda)$ and hence $c = 1$ and hence $\gamma l = l$. As $l|_{\mathcal{V}}$ is injective, thus γ is the identity as desired. □

Remark 5.12. Let $I \subset A$ and Γ be as in Example 5.11. If $k > 1$, suppose that $I \subsetneq A$. In Proposition 7.31 and Corollary 7.32, we will use Proposition 5.8 in order to construct a normal rigid analytic variety over C that parametrizes the isomorphism classes of A-lattices in C^k with a level-I-structure.

5.3 Some connected subsets of Drinfeld's period domain

Consider any E-subspace $\mathcal{W} \subset \mathcal{V}$ and any discrete subset $\Lambda \subset \mathcal{V}$ such that $\Lambda \cap \mathcal{W}$ contains a non-zero element. For any $O \subset \Omega_{\mathcal{W}}$ and any $r \in |C|$ set

$$\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) := \left\{ [l] \in \Omega_{\mathcal{V}} \,\middle|\, [l|_{\mathcal{W}_C}] \in O \wedge \inf_{\lambda \in \Lambda\backslash\mathcal{W}} |l(\lambda)| \geq r \cdot \inf_{0 \neq \lambda \in \Lambda \cap \mathcal{W}} |l(\lambda)| \right\}.$$

Lemma 5.13. *Consider any $r \in |C|$ and any admissible $O \subset \Omega_{\mathcal{W}}$. For any admissible affinoid $U \subset \Omega_{\mathcal{V}}$ then $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) \cap U \subset U$ is admissible and, if O is quasi-compact, quasi-compact. Moerover, $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) \subset \Omega_{\mathcal{V}}$ is admissible.*

Proof. Let $U \subset \Omega_{\mathcal{V}}$ be affinoid. We first show that the intersection of

$$\mathcal{U}(r) := \left\{ [l] \in \Omega_{\mathcal{V}} \,\middle|\, \inf_{\lambda \in \Lambda \setminus \mathcal{W}} |l(\lambda)| \geq r \cdot \inf_{0 \neq \lambda \in \Lambda \cap \mathcal{W}} |l(\lambda)| \right\} \subset \Omega_{\mathcal{V}}$$

with U is a quasi-compact admissible subset of U. Suppose without loss of generality that $\Lambda \setminus \mathcal{W} \neq \emptyset$; otherwise $\mathcal{U}(r) = \Omega_{\mathcal{V}}$ and then the intersection equals the affinoid U. Corollary 5.4 provides a finite subset $S_1 \subset \Lambda \setminus \mathcal{W}$, respectively $S_2 \subset \Lambda \cap \mathcal{W} \setminus \{0\}$, in which every $[l] \in U$ attains

$$\inf_{\lambda \in \Lambda \setminus \mathcal{W}} |l(\lambda)|, \text{ respectively } \inf_{0 \neq \lambda \in \Lambda \cap \mathcal{W}} |l(\lambda)|.$$

Hence

$$\mathcal{U}(r) \cap U = \bigcup_{\mu \in S_2} \bigcap_{\lambda \in S_1} \{[l] \in U \colon |l(\lambda)| \geq r \cdot |l(\mu)|\}.$$

By [8, Prop. 7.2.3.7], thus the subset $\mathcal{U}(r) \cap U \subset U$ is the union of finitely many rational subdomains and hence quasi-compact admissible as desired.

Consider the morphism $\pi \colon \Omega_{\mathcal{V}} \to \Omega_{\mathcal{W}}, [l] \mapsto [l|_{\mathcal{W}_C}]$. Then $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) \cap U$ is the intersection of the admissible $\mathcal{U}(r) \cap U$ with the admissible $\pi^{-1}(O)$ and is thus itself admissible. Since U was arbitrary, the admissibility of $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r)$ follows by virtue of an admissible affinoid covering of $\Omega_{\mathcal{V}}$.

Suppose then that O is quasi-compact. By means of Corollary 5.3, choose an admissible affinoid $O' \subset \Omega_{\mathcal{W}}$ such that $\pi(U) \subset O'$ and $O \subset O'$. Then π restricts to a morphism $\mathcal{U}(r) \cap U \to O'$ from a quasi-compact to an affinoid variety. By [8, Proposition 7.2.2.4], the preimage $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) \cap U$ of the affinoid O under this restriction is thus quasi-compact as desired. $\square$

The following definition and proposition concerning connected varieties is due to Conrad [12, Below Theorem 2.1.3] except that we furthermore ask them to be non-empty.

Definition 5.14. *A rigid analytic variety X is* connected *if it is non-empty and if any admissible covering $\{U, U'\}$ of X satisfies that*

$$U \cap U' = \emptyset \Rightarrow U = \emptyset \vee U' = \emptyset.$$

Proposition 5.15. *A non-empty rigid analytic variety X is connected if and only if any $x, x' \in X$ admit connected admissible subvarieties $X_1, \ldots, X_n$ of X such that $x \in X_1$ and $x' \in X_n$ and $X_i \cap X_{i+1} \neq \emptyset$ for any $1 \leq i < n$; in this case, such X_i can in fact be chosen to be affinoid.*

Theorem 5.16. *Suppose that $\Lambda \subset \mathcal{V}$ is a discrete subgroup such that*

$$\Lambda = \underbrace{(\Lambda \cap \mathcal{W})}_{\neq 0} \oplus (\Lambda \cap E \cdot v_1) \oplus \cdots \oplus (\Lambda \cap E \cdot v_k)$$

for some $0 \neq v_i \in \mathcal{V}$ such that $\mathcal{V} = \mathcal{W} \oplus E \cdot v_1 \oplus \cdots \oplus E \cdot v_k$. Then $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r)$ is connected for any connected admissible $O \subset \Omega_{\mathcal{W}}$ and any $1 \leq r \in |C|$.

We shall prove Theorem 5.16 at the end of this section. First, we apply it: If $\Lambda \subset \mathcal{W}$, then $\mathcal{U}_{\mathcal{V}}(\Lambda, \Omega_{\mathcal{W}}, r) = \Omega_{\mathcal{V}}$ for any $r \in |C|$. If $\dim_E(\mathcal{W}) = 1$, then $\Omega_{\mathcal{W}}$ is a point and thus connected. Theorem 5.16 thus specializes to

Corollary 5.17. *Drinfeld's period domain $\Omega_{\mathcal{V}}$ is connected.*

Corollary 5.18. *The quotient of Drinfeld's period domain by any discrete subgroup of $\mathcal{PG}$ is irreducible.*

Proof. As such a quotient is a normal rigid analytic variety by Proposition 5.8, it is irreducible if and only if it is connected (see Conrad [12, Definition 2.2.2]). However, any quotient of a connected variety is connected. □

Remark 5.19. In Example 6.18, we will consider the case where $A \subset C$ is an admissible coefficient subring, where $\Lambda \subset \mathcal{V}$ is a projective A-submodule of maximal rank and where $\mathcal{W}$ is generated by a direct summand of Λ and show that then the image of $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r)$ in $\Gamma \backslash \Omega_{\Gamma}$ is connected admissible for any connected admissible quasi-compact $O \subset \Omega_{\mathcal{W}}$, any $1 < r \in |C|$ and any subgroup $\Gamma \subset \mathrm{Aut}_A(\Lambda)$.

The proof below of Theorem 5.16 is inspired by van der Put's [43, Example 3.5] and builds on the following results. In the case of Corollary 5.17, it in fact specializes to a variation of the proof that van der Put outlines there.

Proposition 5.20. *(Bosch, Lütkebohmert [9, Corollary 5.11]) Let $p\colon X \to Y$ be a flat morphism between quasi-compact rigid analytic varieties over C. Then the image under p of any admissible quasi-compact subset is admissible quasi-compact.*

Corollary 5.21. *Consider any flat morphism $p\colon X \to Y$ between quasi-compact rigid analytic varieties over C. Suppose that Y and every fiber of p is connected. Then X is connected.*

Proof. By assumption, every fibre of p lies in a connected component of X. Thus the images under p of the connected components of X are disjoint and, by surjectivity of p, cover Y. By Proposition 5.20, this covering of Y is admissible. The connectedness of Y then yields that X has only one connected component, i.e., that X is connected. □

Definition 5.22. *A subset S of the projective line $\mathbb{P}^1_C$ is a* closed ball *if it is the image of the closed unit ball of the affine line $\mathbb{A}^1_C$ under an element of $\mathrm{PGL}_2(C)$.*

Proposition 5.23. *A subset $S \subset \mathbb{P}^1_C$ is a closed ball if and only if it equals*

$$\{z \in \mathbb{A}^1_C \colon |z - c| \le |c'|\} \text{ or } \{z \in \mathbb{A}^1_C \colon |z - c| \ge |c'|\} \cup (\mathbb{P}^1_C \setminus \mathbb{A}^1_C) \tag{30}$$

for some $0 \ne c', c \in \mathbb{A}^1_C$.

Proof. That any subset as in (30) is a closed ball is directly checked. We consider then any $g = (a, b; c, d) \in \mathrm{GL}_2(C)$ and need to show that

$$B_g := \{z \in \mathbb{A}^1_C \colon |az + b| \le |cz + d|\}$$

is a subset of $\mathbb{A}^1_C$ of the first kind in (30) if $|a| > |c|$, resp. of the second if $|a| \le |c|$. If $a = 0$ or $c = 0$, this is directly checked. Thus assume that $a \ne 0 \ne c$. Let $z \in \mathbb{A}^1_C$ and set $z' := cz + d$ and $\mu := \frac{bc-ad}{c}$. Then $az + b = \frac{a}{c}z' + \mu$ and

$$z \in B_g \Leftrightarrow \left|\frac{a}{c}z' + \mu\right| \le |z'|.$$

If $|a| \le |c|$, thus $z \in B_g \Leftrightarrow |\mu| \le |z'|$. If $|a| > |c|$, then

$$\left|\frac{a}{c}z' + \mu\right| \le |z'| \Leftrightarrow \left|\frac{a}{c}z' + \mu\right| \le \left|\frac{c}{a}\mu\right|$$

since both sides imply that $|\mu| = |\frac{a}{c}z'|$. From this is directly checked that B_g is of the desired form in both cases. ☐

Proposition 5.24. *Any non-empty intersection of any finitely many closed balls in $\mathbb{P}^1_C$ is connected.*

Proof. Consider any such intersection I. The connectedness of $\mathbb{P}^1_C$ yields the proposition in the case where I is the intersection over the empty set. Suppose then that I is contained in a ball. Then the image of I under the transformation by a suitable element of $\mathrm{PGL}_2(C)$ is in $\mathbb{A}^1_C$. We thus assume without loss of generality that $I \subset \mathbb{A}^1_C$. By Proposition 5.23 and since any non-empty intersection of finitely many closed balls in $\mathbb{A}^1_C$ is again a closed ball, there exist a $0 \le k \le n-1$ and some $c_0, c_0' \ldots, c_k, c_k' \in \mathbb{A}^1_C$ such that

$$I = \{z \in \mathbb{A}^1_C \colon |z - c_0| \le |c_0'| \wedge \forall 1 \le j \le k \colon |z - c_j| \ge |c_j'|\}.$$

By [8, Theorem 9.7.2.2], any non-empty such set is connected. ☐

Proof of Theorem 5.16. Let $1 \leq r \in |C|$ and set $\mathcal{U}(O) := \mathcal{U}_{\mathcal{V}}(\Lambda, O, r)$ for any admissible $O \subset \Omega_{\mathcal{W}}$. We shall show that $\mathcal{U}(O)$ is connected in the case where $O \subset \Omega_{\mathcal{W}}$ is any connected admissible and affinoid subset. In particular, $\mathcal{U}(O)$ is then non-empty for any non-empty admissible $O \subset \Omega_{\mathcal{W}}$ since the latter can be covered by connected subsets. Given this affinoid case, the theorem thus directly follows: Indeed, for an arbitrary connected admissible $O \subset \Omega_{\mathcal{W}}$ use that any admissible affinoid $O', O'' \subset O$ with $O' \cap O'' \neq \emptyset$ satisfy that $\mathcal{U}(O') \cap \mathcal{U}(O'') = \mathcal{U}(O' \cap O'') \neq \emptyset$ and that, by the affinoid case, both $\mathcal{U}(O')$ and $\mathcal{U}(O'')$ are connected if O' and O'' are.

Consider thus any connected admissible affinoid $O \subset \Omega_{\mathcal{W}}$. Choose any free $\mathcal{O}_E$-submodule of $m_0 \subset \mathcal{V}$ of maximal rank such that $\Lambda \cap m_0 \neq 0$ and any v_i as in the theorem and, using that Λ is discrete, such that

$$\forall 1 \leq i \leq k\colon\ [\Lambda \cap E \cdot v_i \neq 0 \ \Rightarrow\ \Lambda \cap \mathcal{O}_E \cdot v_i \neq 0 = \Lambda \cap \mathcal{O}_E \cdot \pi \cdot v_i]. \tag{31}$$

For any $0 \leq i \leq k$ set $m_i := m_0 \oplus \mathcal{O}_E \cdot v_1 \oplus \cdots \oplus \mathcal{O}_E \cdot v_i$ and $U_i := m_i \setminus \pi m_i$ and $\mathcal{W}_i := E \cdot m_i$ and for any linear $l\colon (\mathcal{W}_i)_C \to C$ set

$$\mu_i(l) := \max\nolimits_{u \in U_i} |l(u)|. \tag{32}$$

Any $\Omega_{\mathcal{W}_i}$ is admissibly covered by the ascending affinoid subsets

$$\Omega_i^n := \{[l] \in \mathbb{P}_{(\mathcal{W}_i)_C^*} \mid \forall u \in U_i\colon |l(u)| \geq |\pi^n| \cdot \mu_i(l)\}$$

for all $n \geq 1$ by Lemma 5.1 and Def.-Prop. 4.31. Consider the morphism

$$p_i\colon \Omega_{\mathcal{W}_i} \to \Omega_{\mathcal{W}_{i-1}}, [l] \mapsto [l|_{(\mathcal{W}_{i-1})_C}]$$

for any $1 \leq i \leq k$. Choose any $j \geq 1$ for which $|\pi|^{-j} > r$. Define $\underline{\Omega}_0^n := \Omega_0^n$ for any $n \geq 1$ and iteratively

$$\forall 1 \leq i \leq k, \forall n > i \cdot j\colon\ \underline{\Omega}_i^n := p_i^{-1}(\underline{\Omega}_{i-1}^{n-j}) \cap \Omega_i^n.$$

Since, by construction, $p_i(\Omega_i^n) \subset \Omega_{i-1}^n$ for any $n \geq 1$ and since the preimage of an affinoid subset under an affinoid morphism is affinoid by [8, Proposition 7.2.2.4], any such $\underline{\Omega}_i^n \subset \Omega_i^n$ is an affinoid subset. Moreover, being cofinal with $(\Omega_i^n)_{n \geq 1}$, the system $(\underline{\Omega}_i^n)_{n > i \cdot j}$ of ascending subsets is an admissible covering of $\Omega_{\mathcal{W}_i}$ for any $0 \leq i \leq k$. Set

$$\forall 0 \leq i \leq k, \forall n > i \cdot j\colon\ Y_i^n := \underline{\Omega}_i^n \cap \underbrace{\mathcal{U}_{\mathcal{W}_i}(\Lambda \cap \mathcal{W}_i, O, r)}_{=: \mathcal{U}_i(O)}.$$

Thus $\mathcal{U}(O) = \mathcal{U}_k(O)$ is admissibly covered by the ascending subsets Y_k^n for all $n > k \cdot j$. It thus suffices to show that Y_k^n is connected for any large enough n. We choose, by means of Corollary 5.3, any $n_0 \geq 1$ such that $O \subset \Omega_0^{n_0}$. We are reduced to showing that Y_i^n is connected for any $0 \leq i \leq k$ and any $n \geq n_0 + i \cdot j$. We prove this by induction on i. If $i = 0$, it follows directly from the assumption on O using that $Y_0^n = O$ for any $n \geq n_0$. Consider then any $i > 0$ and any $n \geq n_0 + i \cdot j$ and suppose by induction hypothesis that Y_{i-1}^{n-j} is connected. By construction, p_i restricts to a morphism

$$p \colon Y_i^n \to Y_{i-1}^{n-j}.$$

As both $\underline{\Omega}_i^n$ and $\underline{\Omega}_{i-1}^{n-j}$ are affinoid, Lemma 5.13 yields that both Y_i^n and Y_{i-1}^{n-j} are quasi-compact. Being admissible subvarieties of standard projective spaces, they are further regular. Let $[l'] \in Y_{i-1}^{n-j}$. In view of Corollary 5.21, it remains to show that $p^{-1}([l'])$ is isomorphic to a connected admissible subvariety of $\mathbb{P}_C^1$. In view of Proposition 5.24, this follows from the following lemmas.

Lemma 5.25. *$p^{-1}([l']) \neq \emptyset$.*

Proof. Use that $|C|$contains $|\pi|^{\mathbb{Q}}$, that $|l'(W_{i-1})|$ is the union in $|C|$ of finitely many translates of $|\pi|^{\mathbb{Z}}$ by Lemma 4.7 and that $|\pi|^{-j} > r$ in order to choose a linear form $l \colon (\mathcal{W}_i)_C \to C$ such that

i) $l|_{(\mathcal{W}_{i-1})_C} = l'$,

ii) $|l(v_i)| \in |C| \setminus |l'(\mathcal{W}_{i-1})|$,

iii) $|\pi|^{-j} \cdot \mu_{i-1}(l') \overset{(*)}{\geq} |l(v_i)| \overset{(**)}{\geq} r \cdot \mu_{i-1}(l')$.

We show that $[l] \in p^{-1}([l'])$. By i), it suffices to show that $[l] \in Y_i^n$. As $r \geq 1$, Condition $(**)$ implies that $|l(v_i)| \geq \mu_{i-1}(l')$ and hence that

$$\mu_i(l) = \max\{|l(v_i)|, \mu_{i-1}(l')\} = |l(v_i)|, \tag{33}$$

where the first equality holds true, as $\mu_i(l)$ is attained by an element of any basis of m_i, so for instance of a basis consisting of v_i and a basis of m_{i-1}. From ii) it follows, as $|\cdot|$ is non-Archimedean, that

$$\forall e \in E, \forall w' \in \mathcal{W}_{i-1} \colon \; |l(e \cdot v_i + w')| = \max\{|e| \cdot |l(v_i)|, |l'(w')|\}. \tag{34}$$

As $[l'] \in \underline{\Omega}_{i-1}^{n-j}$, that $[l] \in \underline{\Omega}_i^n$ is equivalent to $[l] \in \Omega_i^n$, i.e., to

$$\forall u \in U_i \colon \; |l(u)| \geq |\pi|^n \cdot \mu_i(l).$$

Consider any $u \in U_i$ and write $u = e \cdot v_i + w'$ for some $e \in \mathcal{O}_E$ and some $w' \in m_{i-1}$. If $w' \in U_{i-1}$, then

$$|l(u)| \overset{(34)}{\geq} |l'(w')| \overset{[l'] \in \Omega_{i-1}^{n-j}}{\geq} |\pi|^{n-j} \cdot \mu_{i-1}(l') \overset{(*) \wedge (33)}{\geq} |\pi|^n \cdot \mu_i(l).$$

If $w' \notin U_{i-1}$, then $e \in \mathcal{O}_E^\times$ as $u \in U_i$; in this case, thus

$$|l(u)| \overset{(34)}{\geq} |e| \cdot |l(v_i)| = |l(v_i)| \overset{(33)}{=} \mu_i(l) \geq |\pi|^n \mu_i(l).$$

Hence $[l] \in \underline{\Omega}_i^n$. It remains to show that $[l] \in \mathcal{U}_i(O)$, i.e., that

$$\forall \lambda \in \Lambda \cap \mathcal{W}_i \setminus \mathcal{W} \colon |l(\lambda)| \geq r \cdot \min_{0 \neq w \in \Lambda \cap \mathcal{W}} |l'(w)|.$$

Consider any $\lambda \in \Lambda \cap \mathcal{W}_i \setminus \mathcal{W}$. Write $\lambda = e \cdot v_i + \lambda'$ for some $e \in E$ and $\lambda' \in \mathcal{W}_{i-1}$. By assumption on Λ, both $e \cdot v_i$ and λ' lie in Λ. If $\lambda' \notin \mathcal{W}$, then

$$|l(\lambda)| \overset{(34)}{\geq} |l'(\lambda')| \overset{[l'] \in \mathcal{U}_{i-1}(O)}{\geq} r \cdot \min_{0 \neq w \in \Lambda \cap \mathcal{W}} |l'(w)|.$$

If $\lambda' \in \mathcal{W}$, then $e \cdot v_i \neq 0$ as $\lambda \notin \mathcal{W}$; in this case, $|e| \geq 1$ by (31) and hence

$$|l(\lambda)| \overset{(34)}{\geq} |e| \cdot |l(v_i)| \overset{(**)}{\geq} r \cdot \mu_{i-1}(l') \overset{\Lambda \cap m_0 \neq 0}{\geq} r \cdot \min_{0 \neq w \in \Lambda \cap m_0} |l'(w)| \geq r \cdot \min_{0 \neq w \in \Lambda \cap \mathcal{W}} |l'(w)|.$$

Hence $[l] \in \mathcal{U}_i(O)$. As argued before, thus $[l] \in p^{-1}([l'])$ as desired. $\square$

Lemma 5.26. *The fibre $p^{-1}([l'])$ is isomorphic to the intersection of finitely many closed balls of $\mathbb{P}_C^1$.*

Proof. Denote by A the set of $C^\times$-classes of linear forms $l \colon (\mathcal{W}_i)_C \to C$ for which $[l|_{(\mathcal{W}_{i-1})_C}] = [l']$. Then $p^{-1}([l']) = \mathcal{U}_i(O) \cap \Omega_i^n \cap A$. Choose any $0 \neq w_0 \in \mathcal{W}$ and consider the isomorphism

$$\varphi \colon A \to \mathbb{A}_C^1,\ [l] \mapsto \frac{l(v_i)}{l(w_0)}.$$

We first show that $\varphi(\Omega_i^n \cap A)$ is a closed ball of $\mathbb{P}_C^1$ and then that so is $\varphi(\mathcal{U}_i(O) \cap \Omega_i^n \cap A)$. For any $[l] \in P$ set

$$\mu_{i-1}(l) := \mu_{i-1}(l|_{(W_{i-1})_C}) \overset{(32)}{:=} \max\nolimits_{u \in U_{i-1}} |l(u)|.$$

Choose a finite set of representatives S of U_i modulo $\pi^{n+1}m_i$, respectively S' of $(U_i \setminus U_{i-1})$ modulo $\pi^{n+1}m_i$. Using that $[l'] \in \Omega_{i-1}^{n-j} \subset \Omega_{i-1}^n$ and that $\mu_i(l) = \max\{|l(v_i)|, \mu_{i-1}(l)\}$ by the same reason as in (32), then

$$\Omega_i^n \cap A = \{[l] \in A | \, \forall u \in S\colon |l(u)| \geq |\pi^n l(v_i)| \wedge \forall u' \in S'\colon |l(u')| \geq |\pi^n| \mu_{i-1}(l)\}.$$

Write any element u of S (resp. of S') in the form $e_u \cdot v_i + w_u$ for some $w_u \in m_{i-1}$ and some (non-zero) $e_u \in \mathcal{O}_E$ such that $w_u \in U_{i-1}$ or $e_u \in \mathcal{O}_E^\times$ and set

$$c_u := \frac{l'(w_u)}{l'(w_0)} = \frac{l(w_u)}{l(w_0)} \text{ and } c := \frac{\mu_{i-1}(l')}{l'(w_0)} = \frac{\mu_{i-1}(l)}{l(w_0)}$$

for any $[l] \in A$. Then $\varphi(\Omega_i^n \cap A)$ equals

$$\{z \in \mathbb{A}_C^1 | \, \forall u \in S\colon |e_u \cdot z + c_u| \overset{(*)}{\geq} |\pi^n \cdot z| \wedge \forall u' \in S'\colon |e_{u'} \cdot z + c_{u'}| \geq |\pi^n \cdot c|\}.$$

We may and do assume that $e_u = 0$ for some $u \in S$. For such a u then $\overset{(*)}{\geq}$ defines a closed ball in $\mathbb{P}_C^1$ which is already contained in $\mathbb{A}_C^1$. By Proposition 5.23, thus $\varphi(\Omega_i^n \cap A)$ is a closed ball in $\mathbb{P}_C^1$.

Choose then a $\lambda_0 \in \Lambda \cap \mathcal{W}$ for which $|l'(\lambda_0)| = \min_{0 \neq \lambda \in \Lambda \cap \mathcal{W}} |l'(\lambda)|$. Then

$$\forall [l] \in A\colon [l] \in \mathcal{U}_i(O) \Leftrightarrow \forall \lambda \in \Lambda \cap \mathcal{W}_i \setminus \mathcal{W}\colon \left|\frac{l(\lambda)}{l(w_0)}\right| \geq r \cdot \left|\frac{l'(\lambda_0)}{l'(w_0)}\right|. \tag{35}$$

By Corollar 5.4, the affinoid Ω_i^n admits a finite subset $T \subset \Lambda \cap \mathcal{W}_i \setminus \mathcal{W}$ in which the infimum of the $|l(\lambda)|$ for all $\lambda \in \Lambda \cap \mathcal{W}_i \setminus \mathcal{W}$ is attained for any $[l] \in \Omega_i^n$. Choose such a T. Write any $\lambda \in T$ in the form $e_\lambda \cdot v_i + w_\lambda$ for some $e_\lambda \in E$ and some $w_\lambda \in \mathcal{W}_{i-1}$ and set

$$c_\lambda := \frac{l'(w_\lambda)}{l'(w_0)} = \frac{l(w_\lambda)}{l(w_0)} \text{ and } c_0 := \frac{l'(\lambda_0)}{l'(w_0)}$$

for any $[l] \in A$. Then

$$\varphi(p^{-1}([l'])) = \varphi(\mathcal{U}_i(O) \cap \Omega_i^n \cap A) = \{z \in \varphi(\Omega_i^n \cap A) | \, \forall \lambda \in T\colon |e_\lambda \cdot v_i + c_\lambda| \geq r \cdot |c_0|\}.$$

As $\varphi(\Omega_i^n \cap A)$ is a closed ball of $\mathbb{P}_C^1$ which is already contained in $\mathbb{A}_C^1$, thus so is $\varphi(p^{-1}([l']))$ by Proposition 5.23. This yields the lemma. □

As argued before Lemmas 5.25 and 5.26, they finish the proof. □

5.4 Quotient by a discrete subgroup of a codimension 1 vector subspace

Suppose that $d := \dim_E(\mathcal{V}) \geq 2$. Consider any E-subspace $\mathcal{W} \subset \mathcal{V}$ of codimension 1, any $0 \neq w \in \mathcal{W}$, any $v \in \mathcal{V} \setminus \mathcal{W}$ and any discrete subgroup $\Gamma \subset \mathrm{Aut}_E(\mathcal{V})$ such that any $\gamma \in \Gamma$ restricts to the identity on $\mathcal{W}$ and satisfies that $\gamma(v) - v \in \mathcal{W}$.

If $\mathrm{Aut}_E(\mathcal{V})$ is identified with $\mathrm{GL}_d(E)$ via the choice of an ordered basis of $\mathcal{V}$ whose first $d-1$ vectors are an ordered basis of $\mathcal{W}$, then any $\gamma \in \Gamma$ is of the form

$$\begin{pmatrix} \mathrm{id} & * \\ 0 & 1 \end{pmatrix}.$$

Consider the admissible subvariety $\mathcal{E} \subset \mathbb{P}_{\mathcal{V}_C^*}$ of those $[l]$ for which $[l|_{\mathcal{W}_C}] \in \Omega_{\mathcal{W}}$; it is isomorphic to $\Omega_{\mathcal{W}} \times C$ via

$$i \colon \mathcal{E} \to \Omega_{\mathcal{W}} \times C,\ [l] \mapsto \left([l|_{\mathcal{W}_C}], \frac{l(v)}{l(w)} \right). \tag{36}$$

For any $O \subset \Omega_{\mathcal{W}}$ and any integer $n \geq 1$ set

$$\mathcal{E}(O) := i^{-1}(O \times C) \text{ and } \mathcal{E}(O,n) := i^{-1}(O \times B_n),$$

where $B_n \subset C$ denotes the closed ball of radius n around the origin. Thus $(\mathcal{E}(O,n))_{n\geq 1}$ is an admissible affinoid covering of $\mathcal{E}(O)$ for any admissible affinoid $O \subset \Omega_{\mathcal{W}}$. By construction, Γ acts on $\mathcal{E}$. Consider the quotient map

$$p_\Gamma \colon \mathcal{E} \to \Gamma\backslash\mathcal{E}$$

and endow its target with the structure of Grothendieck ringed space induced by p_Γ, that is, a subset (resp. a covering of a subset) of $\Gamma\backslash\mathcal{E}$ is admissible precisely when its preimage is admissible and the sections on an admissible subset of $\Gamma\backslash\mathcal{E}$ are the Γ-invariant sections on its preimage. Thus p_Γ restricts to the quotient map $\Omega_{\mathcal{V}} \to \Omega_\Gamma$ considered in Section 5.9 and $\Gamma\backslash\mathcal{E}$ contains the rigid analytic variety Ω_Γ as a Grothendieck ringed subspace. By Lemma 5.30 below, $\Gamma\backslash\mathcal{E}$ is in fact itself a rigid analytic variety.

Denote by $v_\Gamma \subset \mathcal{W}$ the image of the injective continuous homomorphism

$$\Gamma \to \mathcal{W},\ \gamma \mapsto v_\gamma := \gamma(v) - v;$$

it is a discrete subgroup of $\mathcal{W}$ as Γ is discrete in $\mathrm{Aut}_E(\mathcal{V})$. By Example 2.48 and Lemma 2.49, thus $l(v_\Gamma) \subset C$ is strongly discrete for any $[l] \in \Omega_{\mathcal{W}}$, i.e., its intersection with every ball of finite radius is finite. Set

$$\forall [l] \in \mathcal{E} \colon e([l]) := \frac{\mathrm{e}_{l(v_\Gamma)}(l(v))}{l(w)} = \mathrm{e}_{\frac{l(v_\Gamma)}{l(w)}}\left(\frac{l(v)}{l(w)}\right),$$

where $e_{l(v_\Gamma)} \colon C \to C$ is the analytic surjective group homomorphism with kernel $l(v_\Gamma)$ defined in Def.-Prop. 2.55. We are thus given a bijective map

$$\mathrm{e}_\Gamma \colon \Gamma \backslash \mathcal{E} \to \Omega_{\mathcal{W}} \times C, p_\Gamma([l]) \mapsto ([l|_{\mathcal{W}_C}], e([l])).$$

Lemma 5.27. *The map $e \colon \mathcal{E} \to C, [l] \mapsto e([l])$ is regular.*

Proof. Any admissible affinoid covering $\mathcal{C}$ of $\Omega_{\mathcal{W}}$ yields via (36) the admissible affinoid covering $(\mathcal{E}(O,n))_{O \in \mathcal{C}, n \geq 1}$ of $\mathcal{E}$. Consider any admissible affinoid non-empty $O \subset \Omega_{\mathcal{W}}$ and any integer $n \geq 1$. It thus suffices to show that the restriction of e to $Y := \mathcal{E}(O,n)$ is regular. Choose any $[l_0] \in O$. As $L := v_\Gamma \subset \mathcal{W}$ is discrete, the subset $\frac{1}{l_0(w)} \cdot l_0(L) \subset C$ is strongly discrete by Lemma 2.49. For any integer $k \geq 1$ thus

$$L_k := \left\{ \lambda \in L \colon \left| \frac{l_0(\lambda)}{l_0(w)} \right| \leq k \right\} \subset L$$

is finite and hence the function

$$e_k \colon Y \to C, [l] \mapsto \frac{l(v)}{l(w)} \cdot \prod_{0 \neq \lambda \in L_k} \left(1 - \frac{l(v)}{l(\lambda)}\right)$$

is a finite product of regular functions and thus regular. As the sup-norm on the ring of regular functions on Y is complete by [8, Theorem 6.2.4.1], it thus suffices to show that the e_k for all $k \geq 1$ converge uniformly to e. By means of Corollary 5.3, choose a $\kappa > 0$ such that

$$\forall [l], [l'] \in O, \forall 0 \neq x, y \in \mathcal{W} \colon \left| \frac{l(y)}{l(x)} \right| \leq \kappa \cdot \left| \frac{l'(y)}{l'(x)} \right|.$$

For any $k \geq 1$, any $\lambda \in L \setminus L_k$ and any $[l] \in Y$ then

$$\left| \frac{l(v)}{l(\lambda)} \right| \leq \left| \frac{l(v)}{l(w)} \right| \cdot \left| \frac{l(w)}{l(\lambda)} \right| \leq n \cdot \kappa \cdot \left| \frac{l_0(w)}{l_0(\lambda)} \right| < \frac{n \cdot \kappa}{k}$$

and, if $\frac{n \cdot \kappa}{k} < 1$, hence $\left|1 - \frac{l(v)}{l(\lambda)}\right| = 1$ and

$$\left| 1 - \prod_{\lambda \in L \setminus L_k} \left(1 - \frac{l(v)}{l(\lambda)}\right) \right| \leq \frac{n \cdot \kappa}{k}. \tag{37}$$

Choose a $k_0 > n \cdot \kappa$. Using Proposition 2.25 and that Y is affinoid, choose a $c_0 > 0$ which bounds e_{k_0}. For any $k \geq k_0$ and any $[l] \in Y$ thus $|e_k([l])| = |e_{k_0}([l])| \leq c_0$ and, further using (37), hence

$$|e_k([l]) - e([l])| = |e_k([l])| \cdot \left| 1 - \prod_{\lambda \in L \setminus L_k} \left(1 - \frac{l(v)}{l(\lambda)} \right) \right| \leq c_0 \cdot \frac{n \cdot \kappa}{k}.$$

This shows as desired that the e_k converge uniformly to e. □

Proposition 5.28. *The map* e_Γ *is an isomorphism of rigid analytic varieties. In particular, it restricts to an open immersion on* Ω_Γ.

Since any $[l] \in \Omega_\mathcal{V}$ satisfies that $l(v) \notin l(v_\Gamma)$ and hence that $e([l]) \neq 0$, Proposition 5.28 will thus directly yield

Definition-Proposition 5.29. *The map*

$$q_\Gamma \colon \Omega_\Gamma \to \Omega_\mathcal{W} \times C^\times, p_\Gamma([l]) \mapsto \left([l|_{\mathcal{W}_C}], \frac{1}{e([l])} \right)$$

is an open immersion.

In order to prove Proposition 5.28 we need the following lemmas.

Lemma 5.30. *Consider any admissible affinoid covering* $\mathcal{C}$ *of* $\Omega_\mathcal{W}$. *Then*

$$(p_\Gamma(\mathcal{E}(O, n)))_{O \in \mathcal{C}, n \geq 1}$$

is an admissible covering of $\Gamma \backslash \mathcal{E}$ *and any* $p_\Gamma(\mathcal{E}(O, n))$ *is admissibly covered by finitely many affinoid varieties. In particular,* $\Gamma \backslash \mathcal{E}$ *is a rigid analytic variety.*

Proof. The covering $(\pi_\Gamma(\mathcal{E}(O)))_{O \in \mathcal{C}}$ of $\Gamma \backslash \mathcal{E}$ is the preimage of $\mathcal{C}$ under the natural morphism $\Gamma \backslash \mathcal{E} \to \Omega_\mathcal{W}$ and hence admissible. We consider any admissible affinoid $O \subset \Omega_\mathcal{W}$, set $Y := \mathcal{E}(O)$ and $Y_n := \mathcal{E}(O, n)$ for any $n \geq 1$ and are thus reduced to showing the claim that $\Gamma \backslash Y$ is admissibly covered by the $p_\Gamma(Y_n)$ and that each of them is admissibly covered by finitely many affinoid varieties. In order to prove the claim, we shall apply Proposition 2.34 to the following setting: For any $n \geq 1$ denote by $\Gamma_n \subset \Gamma$ the subgroup of those elements γ such that

$$\forall\, [l] \in O : \left| \frac{l(v_\gamma)}{l(w)} \right| \leq n;$$

it is finite as $\frac{l(v_\Gamma)}{l(w)} \subset C$ is strongly discrete for any $[l] \in O$. Moreover, any Y_n is Γ_n-invariant. Furthermore, as $\Omega_{\mathcal{W}}$ and C are both separated, so is their product and hence $\mathcal{E}$ and hence the admissible subvariety $Y \subset \mathcal{E}$. It remains to verify the remaining Condition iii) of Proposition 2.34; i.e., that any $n \geq 1$ admits an $n' \geq n$ such that

$$\forall \gamma \in \Gamma \setminus \Gamma_{n'} : \gamma(Y_n) \cap Y_n = \emptyset. \tag{38}$$

In order to do so, choose, by means of Corollary 5.3, a $\kappa > 0$ such that

$$\forall\, [l], [l'] \in O, \forall\, 0 \neq x, y \in \mathcal{W} : \left| \frac{l(y)}{l(x)} \right| \geq \kappa \cdot \left| \frac{l'(y)}{l'(x)} \right|.$$

Consider any $n \geq 1$, choose any $n' \geq \frac{n}{\kappa}$ and consider any $\gamma \in \Gamma \setminus \Gamma_{n'}$. Thus

$$\exists\, [l'] \in O : \left| \frac{l'(v_\gamma)}{l'(w)} \right| > n'$$

which implies that

$$\forall\, [l] \in O : \left| \frac{l(v_\gamma)}{l(w)} \right| > \kappa \cdot n' \geq n$$

and hence that

$$\forall\, [l] \in Y_n : \left| \frac{l(\gamma v)}{l(w)} \right| = \left| \frac{l(v_\gamma)}{l(w)} + \frac{l(v)}{l(w)} \right| > n$$

or, equivalently, that $\gamma(Y_n) \cap Y_n = \emptyset$ as desired. □

Lemma 5.31. *Any $[l] \in \mathcal{E}$ admits a basis of admissible neighborhoods such that $\gamma(U) \cap U = \emptyset$ for any U in this basis and any $\mathrm{id} \neq \gamma \in \Gamma$ and such that $(\gamma(U))_{\gamma\in\Gamma}$ is an admissible covering of an admissible subset of $\mathcal{E}$.*

Proof. Let $[l] \in \mathcal{E}$ and set $l_0 := l|_{\mathcal{W}_C}$. Associate with any admissible neighborhood O of $[l_0]$ in $\Omega_{\mathcal{W}}$ and any $\varepsilon \in |C^\times|$ the admissible neighborhood

$$X(O, \varepsilon) := \left\{ [l'] \in \mathcal{E} \colon [l'|_{\mathcal{W}_C}] \in O \wedge \left| \frac{l'(v)}{l'(w)} - \frac{l(v)}{l(w)} \right| \leq \varepsilon \right\}$$

of $[l]$ in $\mathcal{E}$. Using that $l_0(v_\Gamma) \subset C$ is strongly discrete, we choose an $\varepsilon_0 > 0$ such that 0 is the only element in $l_0(v_\Gamma)$ whose norm is $\leq \varepsilon_0 \cdot |l_0(w)|$. Moreover, by Lemma 5.5, $[l_0]$ admits an admissible affinoid neighborhood O such that all elements in O induce the same class of norms on $\mathcal{W}$. Then the $X(O,\varepsilon)$ for all such O and all $\varepsilon_0 \geq \varepsilon \in |C^\times|$ form a desired basis of admissible neighborhoods of $[l]$. Indeed, consider any such O and ε and any $[l'] \in X(O,\varepsilon)$ and $\gamma \in \Gamma$ such that $\gamma[l'] \in X(O,\varepsilon)$. Then

$$\left|\frac{l_0(\gamma v - v)}{l_0(w)}\right| = \left|\frac{l'(\gamma v - v)}{l'(w)}\right| = \left|\frac{(\gamma^{-1}l')(v)}{(\gamma^{-1}l')(w)} - \frac{l(v)}{l(w)} + \frac{l(v)}{l(w)} - \frac{l'(v)}{l'(w)}\right| \leq \varepsilon$$

so that $\gamma v - v = 0$. As γ further restricts to the identity on $\mathcal{W}$, it is the identity as desired. In order to see that $(\gamma(X(O,\varepsilon))_{\gamma\in\Gamma}$ is an admissible covering of an admissible subset of $\mathcal{E}(O)$, it suffices, as $(\mathcal{E}(O,n))_{n\geq 1}$ is an admissible covering of $\mathcal{E}(O)$, to show for any $n \geq 1$ that $(\gamma(X(O,\varepsilon)) \cap \mathcal{E}(O,n))_{\gamma\in\Gamma}$ is an admissible covering of an admissible subset of $\mathcal{E}(O,n)$. However, this holds true for any $n \geq 1$ since, by Proposition 2.22, the intersection of the affinoid $\gamma(X(O,\varepsilon))$ with the affinoid $\mathcal{E}(O,n)$ is again affinoid for any $\gamma \in \Gamma$ and, by (38), empty for all but finitely many $\gamma \in \Gamma$. □

Lemma 5.32. *Consider any admissible $O \subset \Omega_{\mathcal{W}}$ and any integer $n \geq 1$. Then*

$$\mathrm{e}_\Gamma(p_\Gamma(\mathcal{E}(O,n))) \supset O \times B_n.$$

Moreover, if O is affinoid, then there exists an $m \geq 1$ with

$$\mathrm{e}_\Gamma(p_\Gamma(\mathcal{E}(O,n))) \subset O \times B_m.$$

Proof. Set $\mathcal{L} := v_\Gamma$. As $|\cdot|$ is non-Archimedean, any $[l] \in \mathcal{E}$ satisfies that

$$(39) \quad |e([l])| = \left|\frac{l(v)}{l(w)} \cdot \prod_{0\neq\lambda\in\mathcal{L}} \frac{l(v)+l(\lambda)}{l(\lambda)}\right| = \left|\frac{l(v)}{l(w)}\right| \cdot \prod_{\substack{0\neq\lambda\in\mathcal{L} \\ |l(\lambda)|\leq|l(v)|}} \left|\frac{l(v)+l(\lambda)}{l(\lambda)}\right|.$$

As $l(\mathcal{L})$ is strongly discrete for any $l \in \tilde{\Omega}_L$, any $x \in \Gamma\backslash\mathcal{E}$ is represented by some $[l] \in \mathcal{E}$ such that $|l(v)| \leq |l(v) + l(\lambda)|$ for any $\lambda \in \mathcal{L}$ and hence, by (39), such that $|\mathrm{e}_\Gamma(x)| = |e([l])| \geq \left|\frac{l(v)}{l(w)}\right|$. As e_Γ is surjective, this shows the first part. Suppose then that O is affinoid. By (39), any $[l] \in \mathcal{E}$ satifies that

$$(40) \quad |\mathrm{e}([l])| \leq \left|\frac{l(v)}{l(w)}\right| \cdot \prod_{\substack{0\neq\lambda\in\mathcal{L} \\ |l(\lambda)|\leq|l(v)|}} \left|\frac{l(v)}{l(\lambda)}\right| \leq \left|\frac{l(v)}{l(w)}\right| \cdot \prod_{\substack{0\neq\lambda\in\mathcal{L} \\ |l(\lambda)|\leq|l(v)|}} \left|\frac{l(v)}{l(w)}\right| \cdot \left|\frac{l(w)}{l(\lambda)}\right|.$$

Moreover, any $[l] \in \mathcal{E}(O,n)$ and any $\lambda \in \mathcal{L}$ with $|l(\lambda)| \leq |l(v)|$ satisfy that $\left|\frac{l(\lambda)}{l(w)}\right| \leq \left|\frac{l(v)}{l(w)}\right| \leq n$. By (40), it thus suffices to show that for any $[l] \in \mathcal{E}(O,n)$

both the number of $\lambda \in \mathcal{L}$ with $\left|\frac{l(\lambda)}{l(w)}\right| \leq n$ as well as the norm $\left|\frac{l(w)}{l(\lambda)}\right|$ for any such λ is bounded from above by a constant depending only on O and n. Since O is affinoid, Corollary 5.3 provides a $\kappa > 0$ such that

$$\forall [l'], [l] \in O, \forall \lambda \in \mathcal{L}\colon \left|\frac{l'(\lambda)}{l'(w)}\right| \leq \kappa \cdot \left|\frac{l(\lambda)}{l(w)}\right|.$$

From this thus follows the second part as any $[l] \in O$ admits only finitely many $\lambda \in \mathcal{L}$ with $\left|\frac{l(\lambda)}{l(w)}\right| \leq \kappa \cdot n$ as $\frac{l(\mathcal{L})}{l(w)} \subset C$ is strongly discrete. □

Proof of Proposition 5.28. By Lemma 5.30, e_Γ is a morphism between reduced rigid analytic varieties. As argued above, it is bijective. By Proposition 2.24, it thus remains to be shown that e_Γ induces isomorphism on the stalks and that there exists an admissible affinoid covering of $\Omega_\mathcal{W} \times C$ such that the preimage under e_Γ of any of its elements is a finite union of affinoids. As

$$\frac{d}{dT}\left(e_{\frac{l(v_\Gamma)}{l(w)}}(T)\right) = 1,$$

the tangent map of $\mathrm{e}_\Gamma \circ p_\Gamma$ at any point is a triangular matrix with only ones on the diagonal with respect to a suitable basis and thus an isomorphism; thus it induces isomorphisms on the stalks (see [39, Part 2, Chapter 3.9, Theorem 2]). By Lemma 5.31, the quotient morphism p_Γ induces isomorphism on the stalks, too. Hence so does e_Γ. Moreover, the $O_n := O \times B_n$ for all admissible affinoid subsets $O \subset \Omega_\mathcal{W}$ and all $n \geq 1$ form an admissible affinoid covering of $\Omega_\mathcal{W} \times C$. Consider any such O_n and set $X := p_\Gamma(\mathcal{E}(O,n))$. By means of Lemma 5.32 choose an $m \geq n$ with $O_n \subset \mathrm{e}_\Gamma(X) \subset O_m$. For any affinoid $X' \subset X$ then $\mathrm{e}_\Gamma^{-1}(O_n) \cap X'$ is the preimage of the affinoid subset O_n under the morphism $X' \to O \times B_m$ between affinoid varieties induced by e_Γ and is thus [8, Proposition 7.2.2.4] itself affinoid. As, by Lemma 5.30, X is admissibly covered by finitely many such affinoid subvarieties X', the preimage $\mathrm{e}_\Gamma^{-1}(O_n)$ is thus a finite union of affinoid subsets. As argued before, e_Γ is thus an isomorphism. Finally, by Lemma 5.1, the Γ-invariant $\Omega_\mathcal{V} \subset \mathcal{E}$ is an admissible subvariety. Hence so is $\Omega_\Gamma \subset \Gamma\backslash\mathcal{E}$. Thus the restriction of e_Γ to Ω_Γ is an open immersion. □

Proposition 5.33. *Let A and $\Lambda \subset \mathcal{V}$ be as in Example 5.11 and suppose that $b \cdot L \subset v_\Gamma$ for some $0 \neq b \in A$. Let $O \subset \Omega_\mathcal{W}$ be admissible affinoid. Then*

i) any $\varepsilon > 0$ admis an $r > 0$ such that $q_\Gamma^{-1}(O \times B_\varepsilon) \supset p_\Gamma(\mathcal{U}_\mathcal{V}(\Lambda, O, r))$,

ii) any $r > 0$ admits an $\varepsilon > 0$ such that $q_\Gamma^{-1}(O \times B_\varepsilon) \subset p_\Gamma(\mathcal{U}_\mathcal{V}(\Lambda, O, r))$, *where $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) \subset \Omega_{\mathcal{V}}$ is the subset defined before Lemma 5.13.*

Proof. Set $L := \Lambda \cap \mathcal{W}$ and $\mathcal{L} := v_\Gamma \subset L$ and $\mathcal{U}(O, r) := \mathcal{U}_{\mathcal{V}}(\Lambda, O, r)$ for any $r \in |C|$. As O is affinoid, Corollary 5.3 provides a $\kappa > 0$ such that

$$\forall [l], [l'] \in O, \forall w'', w' \in \mathcal{W} \setminus \{0\}\colon \left|\frac{l(w'')}{l(w')}\right| \leq \kappa \cdot \left|\frac{l'(w'')}{l'(w')}\right|. \tag{41}$$

Choose for any $[l] \in O$ an $0 \neq w_l \in L$ such that

$$|l(w_l)| = \inf_{0 \neq \lambda \in L} |l(\lambda)|.$$

using that $l(L)$ is strongly discrete by Proposition 2.51. Then

$$\forall [l], [l'] \in O\colon \left|\frac{l(w)}{l(w_l)}\right| \leq \kappa \cdot \left|\frac{l'(w)}{l'(w_l)}\right| \leq \kappa \cdot \left|\frac{l'(w)}{l'(w_{l'})}\right|.$$

In particular, there exists an $s > 0$ such that

$$\forall [l] \in O\colon \left|\frac{l(w)}{l(w_l)}\right| \leq s.$$

For any $[l] \in \mathcal{E}(O)$ set $w_l := w_{l|_{\mathcal{W}_C}}$ and choose a $v_l \in \Lambda \setminus L$ such that

$$|l(v_l)| = \inf_{\lambda \in \Lambda \setminus L} |l(\lambda)|.$$

Part i) holds true as for any for any $r \in |C^\times|$ and any $[l] \in \mathcal{U}(O, r)$ holds that

$$\left|\frac{l(w)}{l(v+\lambda)}\right| \leq \left|\frac{l(w)}{l(v_l)}\right| = \left|\frac{l(w_l)}{l(v_l)} \cdot \frac{l(w)}{l(w_l)}\right| \leq \frac{s}{r}$$

and hence that

$$\left|\frac{1}{e([l])}\right| = \left|\frac{l(w)}{e_{l(\mathcal{L})}(l(v))}\right| = \left|\sum_{\lambda \in \mathcal{L}} \frac{l(w)}{l(v+\lambda)}\right| \leq \frac{s}{r}.$$

Let us then show Part ii). As $\frac{1}{l(w_l)} \cdot l(L)$ is co-compact in $l(\mathcal{W})$ for any $[l] \in O$, there exists an $r([l]) > 0$ such that

$$\forall x \in \mathcal{W}\, \exists \lambda \in L\colon \left|\frac{l(x-\lambda)}{l(w_l)}\right| \leq r([l]). \tag{42}$$

Using (41) and that $|l'(w_l)| \geq |l'(w_{l'})|$ for any $[l], [l'] \in O$, we may and do choose the $r([l])$ to be uniformly bounded. As $\mathcal{U}(O, r) \subset \mathcal{U}(O, r')$ for any

$r \geq r' > 0$, it thus suffices to show Part ii) only for any $r \in |C|$ such that $r > r([l])$ for any $[l] \in O$. Consider any such r. By surjectivity of e_Γ, it suffices to find an $\varepsilon > 0$ such that

$$p_\Gamma(\mathcal{E}(O) \setminus \mathcal{U}(O, r)) \subset \mathrm{e}_\Gamma^{-1}(O \times B_{\frac{1}{\varepsilon}}).$$

By the second part of Lemma 5.32, it thus suffices to find an $n \geq 1$ with

$$\mathcal{E}(O) \setminus \mathcal{U}(O, r) \subset \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{E}(O, n)). \tag{43}$$

Using the assumption, choose $0 \neq b \in A$ such that $b \cdot L \subset \mathcal{L}$. Using that $\Lambda/(A \cdot v + L)$ is a torsion A-module, we further choose a $c \in A$ such that $c \cdot \Lambda \subset A \cdot v + L$. We claim that any $n \geq \max\{|c|, |b|\} \cdot r$ satisfies (43). Consider any $[l] \in \mathcal{E}(O) \setminus \mathcal{U}(O, r)$. Then

$$\left| \frac{l(v_l)}{l(w_l)} \right| < r;$$

indeed, if $[l] \in \Omega_\Lambda$, this follows from the definition of $\mathcal{U}(O, r)$ and if $[l] \notin \Omega_\Lambda$, then $l(\mathcal{V}) = l(\mathcal{W})$ so that (42) provides for any $x \in \mathcal{W}$ with $l(v_l) = l(x)$ an $\lambda \in L$ with $\left| \frac{l(x-\lambda)}{l(w_l)} \right| < r$ so that

$$|l(v_l)| \leq |l(v_l - \lambda)| = |l(x - \lambda)| < r \cdot |l(w_l)|.$$

Let $a \in A$ and $\lambda' \in L$ be such that $c \cdot v_l = a \cdot v + \lambda'$. Using (42), choose a $\lambda \in \mathcal{L}$ with $\left| \frac{l(\frac{\lambda'}{a} - \lambda)}{l(w_l)} \right| \leq |b| \cdot r$. Write $\lambda = \gamma(v) - v$ for a unique $\gamma \in \Gamma$. Then

$$\begin{aligned} \left| \frac{(\gamma^{-1}l)(v)}{(\gamma^{-1}l)(w)} \right| &= \left| \frac{l(v+\lambda)}{l(w)} \right| \leq \left| \frac{l(v+\lambda)}{l(w_l)} \right| = \left| \frac{l(v + \frac{\lambda'}{a} + \lambda - \frac{\lambda'}{a})}{l(w_l)} \right| \\ &\leq \max\left\{ \frac{|c|}{|a|} \cdot \left| \frac{l(v_l)}{l(w_l)} \right|, \left| \frac{l(\frac{\lambda'}{a} - \lambda)}{l(w_l)} \right| \right\} \leq \max\{|c|, |b|\} \cdot r \end{aligned}$$

which yields the claim and hence Part ii). □

6 Compactification of analytic irreducible components

Consider any algebraically closed complete non-Archimedean valued field C of finite characteristic.

Definition 6.1. *Let $\mathcal{A}$ be the category of triples (A, Λ, Γ), where*

i) A is an admissible coefficient subring of C (see Definition 2.45),

ii) Λ is a non-zero finitely generated projective A-module,

iii) Γ is a subgroup of $\mathrm{Aut}_A(\Lambda)$,

whose morphisms from any (A', Λ', Γ') to any (A, Λ, Γ) are the injective A-linear maps $\varphi : \Lambda' \to \Lambda$, where $A' \supset A$, such that $\varphi(\Lambda') \subset \Lambda$ is a direct summand and

$$\varphi^*(\Gamma') := \big\{g \in \mathrm{Aut}_A(\varphi(\Lambda'))\big|\, \exists \gamma' \in \Gamma' : g \circ \varphi = \varphi \circ \gamma'\big\} \subset \overline{\Gamma}_{\varphi(\Lambda')}, \tag{44}$$

where $\overline{\Gamma}_L := \{g \in \mathrm{Aut}_A(L) |\, \exists \gamma \in \Gamma \colon \gamma|_L = g\}$ for any A-submodule $L \subset \Lambda$.

Ultimately, we are interested only congruence subgroups $\Gamma \subset \mathrm{Aut}_A(\Lambda)$. However, in Section 6.6 (see Def.-Prop. 6.42) we will apply the formalism of the next two sections also to a non-congruence subgroup of $\mathrm{Aut}_A(\Lambda)$.

By means of Example 2.42 and Proposition 5.8, with any $(A, \Lambda, \Gamma) \in \mathcal{A}$ is associated the rigid analytic variety $\Omega_\Gamma := \Gamma \backslash \Omega_\Lambda$ over C, where $\Omega_\Lambda := \Omega_{\Lambda_E}$, where $\Lambda_E := \Lambda \otimes_A E$, for the smallest local field $E \subset C$ containing A.

In this section we construct a functor $(A, \Lambda, \Gamma) \mapsto \Omega^*_\Gamma$ from $\mathcal{A}$ to the category of Grothendieck graded ringed spaces such that, as a set,

$$\Omega^*_\Gamma = \coprod_{L \in S} \Omega_{\overline{\Gamma}_L},$$

where S is a set of representatives for the natural Γ-action on the set of non-zero direct summands of Λ.

Notation 6.2. *For any $(A, \Lambda, \Gamma) \in \mathcal{A}$ and any direct summand $0 \neq L \subset \Lambda$ let*

- $\Gamma_L := \{\gamma \in \Gamma |\, \gamma(L) = L\} \subset \Gamma$, *resp.*
- $\mathring{\Gamma}_L := \{\gamma \in \Gamma_L |\, \gamma|_L = \mathrm{id}_L\} \subset \Gamma_L$,

denote the normalizer, *resp.* centralizer, *of L in Γ; thus $\mathring{\Gamma}_L$ is the kernel of the restriction homomorphism $\Gamma_L \to \mathrm{Aut}_A(L)$ whose image is $\overline{\Gamma}_L$ (see Definition 6.1).*

6.1 Grothendieck topology on the pre-quotient

Consider any $(A, \Lambda, \Gamma) \in \mathcal{A}$. In this section we construct a Grothendieck topological space Ω_Λ^* which is acted by Γ and whose quotient by Γ will be the Grothendieck topological space underlying Ω_Γ^*.

Definition 6.3. *For any subset $S \subset C$ let $\mathrm{d}(S) := \inf_{0 \neq s \in S} |s|$.*

Definition 6.4. *Denote by $\mu_{\max}(\mathcal{L})$ the largest successive minimum of any A-lattice $\mathcal{L} \subset C$ (see Definition 2.52).*

Definition 6.5. *Let Ω_Λ^* be the set-theoretic disjoint union of all the Ω_L for all direct summands $0 \neq L \subset \Lambda$.*

Definition-Proposition 6.6. *For any direct summand $0 \neq L \subset \Lambda$ and any admissible $O \subset \Omega_L$ and any $r \in |C|$ denote by*

$$\mathcal{U}(\Lambda, O, r), \ \text{ resp. } \ \mathcal{U}'(\Lambda, O, r), \tag{45}$$

the subset of Ω_Λ^* of all elements $[l]$ with $[l] \in \Omega_{L'}$ for some $L \subset L' \subset \Lambda$ for which

i) $[l|_{L_C}] \in O$ and

ii) $\frac{\mathrm{d}(l(L' \setminus L))}{\mathrm{d}(l(L))} \geq r$, resp. $\frac{\mathrm{d}(l(L' \setminus L))}{\mu_{\max}(l(L))} \geq r$.

Any such L and any admissible quasi-compact $O \subset \Omega_L$ admit a $c > 0$ such that

$$\forall\, c^2 \leq r \in |C| : \ \mathcal{U}'(\Lambda, O, r) \subset \mathcal{U}(\Lambda, O, r) \subset \mathcal{U}'(\Lambda, O, \frac{r}{c}). \tag{46}$$

Proof. Such constants c are provided by Corollary 5.3. □

Definition-Proposition 6.7. *Endow Ω_Λ^* with the following Grothendieck topology: A subset $Y \subset \Omega_\Lambda^*$ is* admissible *if for every direct summand $0 \neq L \subset \Lambda$*

i) the subset $Y \cap \Omega_L \subset \Omega_L$ is admissible and

ii) if every affinoid $O \subset Y \cap \Omega_L$ and one, and hence every, $U \in \{\mathcal{U}, \mathcal{U}'\}$ admit an $r \in |C|$ with $U(\Lambda, O, r) \subset Y$.

Moreover, a covering of an admissible subset of Ω_Λ^* by admissible subsets is admissible *if its intersection with every Ω_L is admissible.*

Proof. All properties required by Definition 2.2 follow directly from the corresponding ones of the Grothendieck toplogical spaces Ω_L and the fact that any admissible covering of an affinoid subset has a finite subcovering. □

Example 6.8. *Any subset as in (45) is admissible.*

Proof. Consider any admissible $O \in \Omega_L$ and any $r \in |C|$. By Lemma 5.13, the intersection of $U := \mathcal{U}(\Lambda, O, r)$ with any $\Omega_{L'}$ is an admissible subset of $\Omega_{L'}$ for any further direct summand $L \subset L' \subset \Lambda$. By a similar proof as in Lemma 5.13, also the intersection of $U' := \mathcal{U}'(\Lambda, O, r)$ with any such $\Omega_{L'}$ is admissible. Consider then any affinoid $O' \subset U \cap \Omega_{L'}$ for any such L'. We shall show that $\mathcal{U}(\Lambda, O', r') \subset U$ for some $r' \in |C|$. Using Corollaries 5.3 and 5.4 and that O' is affinoid, choose an $\varepsilon > 0$ such that

$$\forall [l'] \in O' \colon\ \mathrm{d}(l'(L')) \geq \varepsilon \cdot \mathrm{d}(l'(L)).$$

Let $\frac{r}{\varepsilon} \leq r' \in |C|$ and $[l] \in \mathcal{U}(\Lambda, O', r') \cap \Omega_{L''}$ for any $L' \subset L'' \subset \Lambda$. Then

$$\begin{aligned} \mathrm{d}(l(L'' \setminus L)) &\geq \min\{\mathrm{d}(l(L'' \setminus L')), \mathrm{d}(l(L' \setminus L))\} \\ &\geq \min\{r' \cdot \mathrm{d}(l(L')), r \cdot \mathrm{d}(l(L))\} \geq r \cdot \mathrm{d}(l(L)). \end{aligned}$$

As $[l|_{L'}] \in O' \subset U$, also $[l|_L] \in O$. Thus $[l] \in U$. Hence $\mathcal{U}(\Lambda', O, r') \subset U$. Using (46), this yields the corresponding property for U', too. Hence U and U' are indeed admissible. □

Corollary 6.9. *Consider any admissible $Y \subset \Omega^*_\Lambda$ and for any direct summand $0 \neq L \subset \Lambda$ an admissible covering $\mathcal{C}_L$ of $Y \cap \Omega_L$ and an $r_O \in |C|$ for any $O \in \mathcal{C}_L$ such that $U_O := \mathcal{U}(\Lambda, O, r_O) \subset Y$, resp. $U'_O := \mathcal{U}'(\Lambda, O, r_O) \subset Y$. Then the covering $\mathcal{C}$ of Y by all these U_O, resp. U'_O, is admissible.*

Proof. By Example 6.8, any U_O, resp. U'_O, is an admissible subset of Y. Moreover, the intersection of $\mathcal{C}$ with any boundary component Ω_L is refined by the admissible $\mathcal{C}_L$ and is thus, by Property vii) of Definition 2.2, itself admissible. Hence $\mathcal{C}$ is indeed admissible. □

Corollary 6.10. *For any direct summand $0 \neq L \subset \Lambda$, any $[l] \in \Omega_L$ and any countable neighborhood basis $(O_n)_{n\geq 1}$ of $[l]$ in Ω_L the system $(\mathcal{U}(\Lambda, O_n, r_n))_{n\geq 1}$ is a countable neighborhood basis of $[l]$ in Ω^*_Λ for any unbounded sequence $\{r_n\}_{n\geq 1} \subset |C|$.*

Proof. This follows directly from Example 6.8 and Definition 6.7, i). □

Corollary 6.11. *The canonical topology on Ω^*_Λ is first countable.*

Corollary 6.12. *Let $Y \subset \Omega^*_\Lambda$ be admissible. With respect to the canonical topologies, then a function $f \colon Y \to C$ is continuous if and only if it is sequentially continuous.*

Proposition 6.13. *[30, Proposition 1.4] Consider any $1 < r \in |C|$ and any direct summands $L_1, L_2 \subset \Lambda$ for which $L_1 \not\subset L_2 \not\subset L_1$. Then*

$$\mathcal{U}'(\Lambda, \Omega_{L_1}, r) \cap \mathcal{U}'(\Lambda, \Omega_{L_2}, r) = \emptyset.$$

Proof. Suppose, by contradiction, the existence and choice of an $[l] \in \Omega_L$ in the intersection in (6.13). By means of Def.-Prop. 2.52, choose for any $1 \leq i \leq 2$ a subset $\beta_i \subset L_i$ whose image under l is a realization of the set of successive minima of $l(L_i)$. If

$$\mu := \mu_{\max}(l(L_1)) \leq \mu_{\max}(l(L_2)) =: \nu,$$

then choose, by means of the assumption, a $\lambda \in \beta_1 \setminus L_2$ to get the contradiction

$$|l(\lambda)| \leq \mu \leq \nu < r \cdot \nu \leq \mathrm{d}(l(L \setminus L_2)) \leq |l(\lambda)|.$$

By symmetry, we also get a contradiction if $\mu \geq \nu$. □

6.2 Structure of Grothendieck graded ringed space

Let $(A, \Lambda, \Gamma) \in \mathcal{A}$. Let F be the quotient field of A and $E \subset C$ its completion.

For any direct summand $0 \neq L \subset \Lambda$ denote by $\tilde{\Omega}_L$ the preimage of Ω_L under the quotient-by-$C^\times$ morphism $\mathbb{A}_{L_C^*} \setminus \{0\} \to \mathbb{P}_{L_C^*}$; it consists precisely of the C-linear maps $l\colon L_C \to C$ for which $\mathrm{Ker}(l) \cap L_E = 0$.

Definition 6.14. *Let $\tilde{\Omega}_\Lambda^*$ be the set-theoretic disjoint union of all such $\tilde{\Omega}_L$ equipped with the induced $C^\times$-action.*

Definition-Proposition 6.15. *Set $G := \mathrm{Aut}_F(\Lambda_F)$. Equip $\tilde{\Omega}_\Lambda^*$ with a $C^\times$-equivariant G-action by the rule*

$$g(l) := l \circ (g^{-1}|_{(gL)_C}) \in \tilde{\Omega}_{gL}, \;\; \textit{where } gL := g(L_F) \cap \Lambda \subset \Lambda,$$

for any $g \in G$ and any l in any $\tilde{\Omega}_L$; its induced G-action on Ω_Λ^* is continuous and

$$\forall g \in \mathrm{Aut}_A(\Lambda)\colon\; g(\mathcal{U}(\Lambda, O, r)) = \mathcal{U}(\Lambda, g(O), r). \tag{47}$$

Moreover, for any direct summand $0 \neq L \subset \Lambda$ holds that

$$\forall g \in \mathrm{Aut}_A(\Lambda)\colon\; gL = g(L). \tag{48}$$

Proof. It is directly checked that the rule defines a $C^\times$-equivariant G-action on Ω^*_Λ and that then (47) and (48) hold. Let $g \in G$. Then the bijection $\Omega^*_\Lambda \to \Omega^*_\Lambda$ given by the induced action of g restricts stratawise to isomorphisms of rigid analytic varieties. Consider then any admissible affinoid $O \subset \Omega_L$ and any $r \in |C|$. It remains to find an $r' \in |C|$ for which

$$\mathcal{U}(\Lambda, g(O), r') \subset g(\mathcal{U}(\Lambda, O, r)). \tag{49}$$

Using that both Λ and $g(\Lambda)$ are projective A-submodules of maximal rank of Λ_F, choose an $f \in F$ for which $g(\Lambda) \subset f \cdot \Lambda$. Then $g(L) \subset f \cdot gL$ and

$$g(\Lambda) \setminus g(L) = g(\Lambda) \setminus g(L)_F \subset f \cdot \Lambda \setminus g(L)_F = f \cdot (\Lambda \setminus g(L)_F) = f \cdot (\Lambda \setminus gL)$$

from which (49) is directly deduced for any $\frac{r}{|f|^2} \le r' \in |C|$. $\square$

Definition 6.16. *By means of Def.-Prop- 6.15, consider the quotient map*

$$p_\Gamma \colon \Omega^*_\Lambda \to \Gamma \backslash \Omega^*_\Lambda =: \Omega^*_\Gamma$$

and endow its target with the structure of Grothendieck topological space which it induces, that is, a subset (resp. a covering of a subset) of Ω^*_Γ is admissible precisely when its preimage is admissible.

Remark 6.17. The induced canonical topology on Ω^*_Γ was introduced by Kapranov [30] in the case, where A is a polynomial ring.

Example 6.18. *Consider any direct summand $0 \neq L \subset \Lambda$, any admissible quasi-compact $O \subset \Omega_L$ and any $1 < r \in |C|$. Then the $\gamma(\mathcal{U}(\Lambda, O, r))$ for all $\gamma \in \Gamma$ form an admissible covering of an admissible subset of Ω_Λ. In particular, $p_\Gamma(\mathcal{U}(\Lambda, O, r)) \subset \Omega^*_\Gamma$ is admissible. Moreover, if O is connected, then so is $p_\Gamma(\mathcal{U}(\Lambda, O, r)) \cap \Omega_\Gamma \subset \Omega_\Gamma$.*

Proof. Set $U := \mathcal{U}(\Lambda, O, r)$. Let us show the first assertion. By construction, any $\gamma(U)$ depends only on γ modulo $\mathring{\Gamma}_L$. By means of Example 6.8, Lemma 5.13 and any admissible affinoid covering of any stratum $\Omega_{L'}$, it suffices to show for any admissible affinoid subset O' of any $\Omega_{L'}$ that $\gamma(U) \cap O' = \emptyset$ for all but finitely many $\gamma \in \Gamma$ modulo $\mathring{\Gamma}_L$. Thus consider any such admissible affinoid $O' \subset \Omega_{L'}$. We assume that $L' = \Lambda$ and that $U \cap O' \neq \emptyset$; the general case is directly reduced to this case. Using that O, resp. O', is quasi-compact, choose a $\kappa > 0$, resp. $\kappa' > 0$, which satisfies the property in Corollary 5.3, iii) with respect to O and L_E, resp. O' and Λ_E. Choose any basis β of L_F that is contained in L and choose any $[l] \in U \cap O'$. Choose a $\lambda \in L$ such that $|l(\lambda)| = \mathrm{d}(l(L))$. Consider any $\gamma \in \Gamma$ for which

$\gamma(U) \cap O' \neq \emptyset$ and choose any $[l'] \in \gamma(U) \cap O'$. Choose a $\lambda' \in \gamma L$ with $|l'(\lambda')| = \mathrm{d}(l'(\gamma L)) \overset{r \geq 1}{=} \mathrm{d}(l'(\Lambda))$. For any $v \in \beta$ then

$$\begin{aligned}\left|\frac{l(\gamma v)}{l(\lambda)}\right| &\leq \kappa' \cdot \left|\frac{l'(\gamma v)}{l'(\lambda)}\right| \leq \kappa' \cdot \left|\frac{l'(\gamma v)}{l'(\lambda')}\right| = \kappa' \cdot \left|\frac{l'(\gamma v)}{l'(\gamma\lambda)}\right| \cdot \left|\frac{l'(\gamma\lambda)}{l'(\lambda')}\right| \\ &\overset{*}{\leq} \kappa' \cdot \kappa^2 \cdot \left|\frac{(\gamma l)(\gamma v)}{(\gamma l)(\gamma\lambda)}\right| \cdot \left|\frac{(\gamma l)(\gamma\lambda)}{(\gamma l)(\lambda')}\right| = \kappa' \cdot \kappa^2 \cdot \left|\frac{l(v)}{l(\lambda)}\right| \cdot \left|\frac{(l(\lambda)}{l(\gamma^{-1}\lambda')}\right| \\ &\leq \kappa' \cdot \kappa^2 \cdot \left|\frac{l(v)}{l(\lambda)}\right|,\end{aligned}$$

where at $\overset{*}{\leq}$ we have used that $[(\gamma l)|_{(\gamma L)_C}] \in \gamma(O)$ as $[l|_{L_C}] \in O$ and that, as is directly checked, the constant κ also satisfies the property in Corollary 5.3, iii) for $\gamma(O)$ and $(\gamma L)_E$. As $l(\Lambda)$ is strongly discrete, thus $l(\gamma(\beta))$ lies in a finite subset of $l(\Lambda)$ that depends not on γ. As such a γ modulo $\mathring{\Gamma}_L$ is uniquely determined by its action on β, there exists thus indeed only finitely many γ modulo $\mathring{\Gamma}_L$ satisfying the above inequality and hence that $\gamma(U) \cap O' \neq \emptyset$. Moreover, if O is connected, then so is $U \cap \Omega_\Lambda$ by Theorem 5.16 and hence so is its admissible image $p_\Gamma(U) \cap \Omega_\Gamma = p_\Gamma(U \cap \Omega_\Lambda) \subset \Omega_\Gamma$. □

Corollary 6.19. *The map p_Γ is open with respect to the canonical topologies.*

Proof. This follows from Def.-Prop 6.7 and Examples 6.8 and 6.18. □

Corollary 6.20. *Any point in Ω_Γ^* has a fundamental basis of admissible neighborhoods whose intersection with Ω_Γ is connected and irreducible.*

Proof. As Ω_Γ is normal by Proposition 5.8, its irreducible and connected subsets coincide by [12, Definition 2.2.2]. The corollary thus follows from Example 6.18 and Corollary 6.10 using that any point in any stratum Ω_L has a basis of connected admissible affinoid neighborhoods in Ω_L. □

Example 6.21. *Consider any admissible subset $X \subset \Omega_\Gamma^*$. For any direct summand $0 \neq L \subset \Lambda$ choose an admissible affinoid covering $\mathcal{C}_L$ of $p_\Gamma^{-1}(X) \cap \Omega_L$ and for any $O \in \mathcal{C}_L$ an $r_O \in |C|$ for which $\mathcal{U}(\Lambda, O, r_O) \subset p_\Gamma^{-1}(X)$. Then the covering $\mathcal{C}$ of X by the $p_\Gamma(\mathcal{U}(\Lambda, O, r_O))$ for all O in all $\mathcal{C}_L$ is admissible.*

Proof. By Example 6.18, any element of $\mathcal{C}$ is admissible. Let $\mathcal{D}$ be the covering of $p_\Gamma^{-1}(X)$ defined as the preimage of C under p_Γ. It remains to check that the intersection of $\mathcal{D}$ with any Ω_L is an admissible covering of $p_\Gamma^{-1}(X) \cap \Omega_L$. However, by construction, such an intersection is refined by $\mathcal{C}_L$ and is thus admissible as desired. □

Definition-Proposition 6.22. *For any orbit $\mathfrak{O}$ of the natural Γ-action on the set of non-zero direct summands let*

$$\Omega_{\mathfrak{O}} := p_\Gamma\left(\bigcup_{L\in\mathfrak{O}} \Omega_L\right)$$

be equipped with the structure of Grothendieck ringed space turning the natural map $\Omega_{\overline{\Gamma}_L} \to \Omega_{\mathfrak{O}}$ into an isomorphism for every $L \in \mathfrak{O}$. Then a subset $X \subset \Omega_\Gamma^*$ is admissible if and only if for every such orbit $\mathfrak{O}$:

i) $X \cap \Omega_{\mathfrak{O}} \subset \Omega_{\mathfrak{O}}$ is admissible and

ii) every admissible quasi-compact $Y \subset \Omega_{\mathfrak{O}}$ with $Y \subset X$ admits an $r \geq 0$ with

$$\mathcal{U}(\Lambda, Y, r) := p_\Gamma\left(\bigcup_{L\in\mathfrak{O}} \mathcal{U}(L, p_\Gamma^{-1}(Y) \cap \Omega_L, r)\right) \subset X.$$

Moreover, a covering of an admissible subset $X \subset \Omega_\Gamma^*$ by admissible subsets is admissible precisely if its intersection with $X \cap \Omega_{\mathfrak{O}}$ is admissible for every orbit $\mathfrak{O}$.

Proof. Consider any $X \subset \Omega_\Gamma^*$. For any orbit $\mathfrak{O}$ the subset $X \cap \Omega_{\mathfrak{O}} \subset \Omega_{\mathfrak{O}}$ is admissible if and only if for every $L \in \mathfrak{O}$ the subset $p_\Gamma^{-1}(X) \cap \Omega_L$ is admissible. Thus i) holds true for every orbit $\mathfrak{O}$ if and only if $p_\Gamma^{-1}(X) \cap \Omega_L \subset \Omega_L$ is admissible for every direct summand $0 \neq L \subset \Lambda$. We assume that these equivalent statements hold true. Similarly, it directly follows that a covering of X by admissible subsets is admissible if and only if its intersection with $X \cap \Omega_{\mathfrak{O}}$ is admissible for every such $\mathfrak{O}$. We are thus reduced to showing that ii) holds true for every such $\mathfrak{O}$ if and only if every such L and every affinoid $O \subset p_\Gamma^{-1}(X) \cap \Omega_L$ admit an $r \in |C|$ for which $\mathcal{U}(\Lambda, O, r) \subset p_\Gamma^{-1}(X)$. First assume the first property and consider any such affinoid $O \subset \Omega_L$ and denote by $\mathfrak{O}$ the Γ-orbit of L. For any $r \in |C|$ then

$$p_\Gamma(\mathcal{U}(\Lambda, O, r)) = \mathcal{U}(\Lambda, p_\Gamma(O), r)$$

is admissible by Example 6.18; in particular, $p_\Gamma(O) \subset \Omega_{\mathfrak{O}}$ is admissible and, being the image of O, quasi-compact. Thus ii) provides a desired $r \in |C|$. Conversely assume the second property and consider any admissible quasi-compact $Y \subset \Omega_{\mathfrak{O}}$. Choose any $L \in \mathfrak{O}$ and an admissible affinoid covering $\mathcal{C}_L$ of $p_\Gamma^{-1}(Y) \cap \Omega_L$ and for any $O \in \mathcal{C}_L$ an $r_O \in |C|$ such that $\mathcal{U}(\Lambda, O, r_O) \subset p_\Gamma^{-1}(X)$. Then $(p_\Gamma(O))_{O\in\mathcal{C}_L}$ is a covering of Y; it is in fact admissible since any of its elements is admissible by Example 6.18 since the preimage in Ω_L of the covering has the admissible refinement $\mathcal{C}_L$ and is thus itself admissible. By quasi-compactness of Y, we may thus choose finitely many $p_\Gamma(O_1), \ldots, p_\Gamma(O_n)$ which cover Y. For any $r \in |C|$ greater than the r_{O_i} for all $1 \leq i \leq n$ then

$$\mathcal{U}(\Lambda, Y, r) \subset \bigcup_{1 \leq i \leq n} \mathcal{U}(\Lambda, p_\Gamma(O_i), r_{O_i}) = p_\Gamma\left(\bigcup_{1 \leq i \leq n} \mathcal{U}(\Lambda, O_i, r_{O_i})\right) \subset X.$$

This yields the converse direction and finishes the proof. □

Definition-Proposition 6.23. *Consider any integer k and any admissible $Y \subset \Omega^*_\Lambda$ with preimage $\tilde{Y} \subset \tilde{\Omega}^*_\Lambda$. A function $f\colon \tilde{Y} \to C$ is called* weight k regular *if*

i) $\forall c \in C^\times, l \in \tilde{Y}\colon f(c \cdot l) = c^{-k} \cdot f(l)$

ii) and every direct summand $0 \neq L \subset \Lambda$, every admissible affinoid $O \subset Y \cap \Omega_L$ and one, and hence every, $0 \neq \lambda \in L$, admit an $r \in |C|$ such that $\mathcal{U} := \mathcal{U}(\Lambda, O, r) \subset Y$ and

(50) $$\mathcal{U} \to C, [l] \mapsto f(l) \cdot l(\lambda)^k$$

is bounded, continuous with respect to the canonical topologies and restricts to a regular (see Definition 2.17) function $\mathcal{U} \cap \Omega_{L'} \to C$ for every direct summand $0 \neq L' \subset \Lambda$.

Proof. If for some $0 \neq \lambda \in L$ the function in (50) is bounded and continuous, then the same holds true for any such λ since for any $0 \neq \lambda', \lambda \in L$ the regular function

$$\mathcal{U} \to C, [l] \mapsto \frac{l(\lambda)}{l(\lambda')}$$

is continuous and bounded; indeed the function is continuous as it factors through the continuous restriction morphism $\mathcal{U} \to O$ and it is bounded by Proposition 2.25 applied to the affinoid O. □

Definition-Proposition 6.24. *For any admissible $X \subset \Omega^*_\Gamma$ and any integer k let $\mathcal{O}^*_\Gamma(k)(X)$ be the set of Γ-invariant weight k regular functions $\pi_\Gamma^{-1}(X) \to C$, where*

$$\pi_\Gamma\colon \tilde{\Omega}^*_\Lambda \to \Omega^*_\Gamma = \Gamma\backslash\tilde{\Omega}^*_\Lambda/C^\times$$

is the double quotient map. By means of the ring structure on C, then *i) the $\mathcal{O}^*_\Gamma(0)(X)$ for all admissible subsets $X \subset \Omega^*_\Gamma$ together with the natural restriction homomorphisms form a sheaf $\mathcal{O}^*_\Gamma$ of rings on Ω^*_Γ, called* structure sheaf *on Ω^*_Γ, and*

ii) for any integer k the $\mathcal{O}^*_\Gamma(k)(X)$ for all admissible $X \subset \Omega^*_\Gamma$ together with the natural restriction homomorphisms form a sheaf $\mathcal{O}^*_\Gamma(k)$ of $\mathcal{O}^*_\Gamma$-modules on Ω^*_Γ, called k-th twisting $\mathcal{O}^*_\Gamma$-module *and*

iii) a sheaf $\mathcal{R}^*_\Gamma$ of graded $\mathcal{O}^*_\Gamma$-algebras on Ω^*_Γ is formed by the

$$\mathcal{R}^*_\Gamma(X) := \sum_{k\in\mathbb{Z}} \mathcal{O}^*_\Gamma(k)(X)$$

for all admissible $X \subset \Omega^*_\Gamma$ together with the natural restriction homomorphisms.

In particular, $(\Omega^*_\Gamma, \mathcal{O}^*_\Gamma)$ (resp. $(\Omega^*_\Gamma, \mathcal{R}^*_\Gamma)$) is a Grothendieck (graded) ringed space containing the rigid analytic variety Ω_Γ as an admissible Grothendieck ringed subspace.

Proof. This is directly checked. □

Remark 6.25. In fact, in Corollaries 10.4 and 10.6 below we will show the following: If Γ is a congruence subgroup of $\mathrm{Aut}_A(\Lambda)$, then $(\Omega^*_\Gamma, \mathcal{O}^*_\Gamma)$ is a normal integral projective rigid analytic variety over C. If, furthermore, Γ is fine, then $\mathcal{O}^*_\Gamma(k)$ is an ample invertible sheaf for any $k \geq 1$.

Proposition 6.26. *Consider any admissible $X \subset \Omega^*_\Gamma$. Then precomposition with the restriction $\pi_\Gamma^{-1}(X) \to X$ of π_Γ induces a bijection to $\mathcal{O}^*_\Gamma(X)$ from the set $\mathcal{O}'_\Gamma(X)$ of functions $s\colon X \to C$ that are continuous with respect to the canonical topologies, that restrict to a regular function on $X \cap \Omega_{\mathfrak{O}}$ for every Γ-orbit $\mathfrak{O}$ and that are bounded on $\mathcal{U}(\Lambda, Y, r)$ for every admissible quasi-compact $Y \subset \Omega_{\mathfrak{O}}$ and every $r \in |C|$ for which $\mathcal{U}(\Lambda, Y, r) \subset X$.*

Proof. Consider first any direct summand $0 \neq L \subset \Lambda$ in any orbit $\mathfrak{O}$. As already argued in the proof of Def.-Prop. 6.22, for any admissible quasi-compact $O \subset L$ then $p_\Gamma(O) \subset \Omega_{\mathfrak{O}}$ is admissible quasi-compact and

$$p_\Gamma(\mathcal{U}(\Lambda, O, r)) = \mathcal{U}(\Lambda, p_\Gamma(O), r)$$

for any $r \in |C|$. Using this, it directly follows that precomposition with π_Γ defines an injective map $\mathcal{O}'_\Gamma(X) \to \mathcal{O}^*_\Gamma(X)$. On the other hand, consider any $f \in \mathcal{O}^*_\Gamma(X)$. Being $C^\times$- and Γ-invariant, it induces a function $s\colon X \to C$

which, by Corollary 6.19, is continuous with respect to the canonical topologie since f is. By construction of the $\Omega_{\mathfrak{D}}$, moreover, s restricts stratawise to a regular function since f does. The boundedness property for s follows from the one of f via the argument at the end of the proof of Def.-Prop. 6.22. $\square$

6.3 Morphisms

Consider any $(A', \Lambda', \Gamma') \xrightarrow{\varphi} (A, \Lambda, \Gamma) \in \mathcal{A}$ and the map

$$\tilde{\Omega}^*_\varphi \colon \tilde{\Omega}^*_{\Lambda'} \to \tilde{\Omega}^*_\Lambda, [L'_C \xrightarrow{l'} C] \mapsto [\varphi(L')_C \xrightarrow{\varphi_C^{-1}} L'_C \xrightarrow{l'} C].$$

Proposition 6.27. *By $\tilde{\Omega}^*_\varphi$ is induced a morphism of Grothendieck graded ringed spaces*

$$(\Omega^*_\varphi, \mathcal{R}^*_\varphi) : (\Omega^*_{\Gamma'}, \mathcal{R}^*_{\Gamma'}) \to (\Omega^*_\Gamma, \mathcal{R}^*_\Gamma)$$

which on sections of $\mathcal{R}^*_\Gamma$ is defined by precomposition with $\tilde{\Omega}^*_\varphi$.

Proof. We first show that the map $\rho \colon \Omega^*_{\Lambda'} \to \Omega^*_\Lambda$ induced by $\tilde{\Omega}^*_\varphi$ is a morphism of Grothendieck topological spaces. We consider any admissible $Y \subset \Omega^*_\Lambda$ and claim that

$$Y' := \rho^{-1}(Y) \subset \Omega^*_{\Lambda'}$$

is admissible, i.e., that for any discrete summand $0 \neq L' \subset \Lambda'$, the subset $Y' \cap \Omega_{L'} \subset \Omega_{L'}$ is admissible and admits an admissible covering $\mathcal{C}'$ and for any $O' \in \mathcal{C}'$ an $r \in |C|$ with $\mathcal{U}(\Lambda', O', r) \subset Y'$. Consider any such L', and set $L := \varphi(L')$. By assumption, $Y \cap \Omega_L \subset \Omega_L$ is admissible and admits an admissible covering $\mathcal{C}$ and for any $O \in \mathcal{C}$ an $r \in |C|$ with $\mathcal{U}(\Lambda, O, r) \subset Y$. Since the restriction $\rho_{L'} \colon \Omega_{L'} \to \Omega_L$ of ρ is a morphism,

$$Y' \cap \Omega_{L'} = \rho_{L'}^{-1}(Y \cap \Omega_L) \subset \Omega_{L'}$$

is admissible and admissibly covered by the preimage $\mathcal{C}'$ of $\mathcal{C}$ under $\rho_{L'}$. The claim then follows from the straightforwardly checked fact that

$$\mathcal{U}(\Lambda', \rho_{L'}^{-1}(O), r) = \rho^{-1}(\mathcal{U}(\Lambda, O, r)) \subset \rho^{-1}(Y) = Y'$$

for any $O \in \mathcal{C}$ and any $r \in |C|$. The regularity of any such restriction $\rho_{L'}$ further shows that the preimage of any admissible covering of Y is an admissible covering of Y'. Hence ρ is indeed a morphism of Grothendieck topological spaces.

Since $\varphi^*(\Gamma') \subset \overline{\Gamma}_{\varphi(L')}$ by assumption, ρ induces a map $\Omega^*_\varphi : \Omega^*_{\Gamma'} \to \Omega^*_\Gamma$ which is straightforwardly checked to be a morphism of Grothendieck topological spaces using that ρ is one.

Finally, using the above properties of ρ and that $\tilde{\Omega}^*_\varphi$ restricts stratawise to a morphism, it is directly checked that for any admissible $X \subset \Omega^*_\Gamma$ pre-composition with $\tilde{\Omega}^*_\varphi|_{\tilde{X}'}$ yields a a graded ring homomorphism

$$\mathcal{R}^*_\varphi(X)\colon \mathcal{R}^*_\Gamma(X) \to \mathcal{R}^*_{\Gamma'}(X'),$$

where $X' := (\Omega^*_\varphi)^{-1}(X)$ and $\tilde{X}' := \pi^{-1}_{\Gamma'}(X')$, and that these $\mathcal{R}^*_\varphi(X)$ are compatible with the restriction homomorphisms of $\mathcal{R}^*_\varphi$ and $\mathcal{R}^*_{\Gamma'}$ as X varies. □

Proposition 6.28. *Consider any further* $(A'', \Lambda'', \Gamma'') \xrightarrow{\psi} (A', \Lambda', \Gamma') \in \mathcal{A}$. *Then*

$$(\Omega^*_\varphi, \mathcal{R}^*_\varphi) \circ (\Omega^*_\psi, \mathcal{R}^*_\psi) = (\Omega^*_{\varphi\circ\psi}, \mathcal{R}^*_{\varphi\circ\psi}).$$

Proof. This is straightforwardly checked using that $\tilde{\Omega}^*_\varphi \circ \tilde{\Omega}^*_\psi = \tilde{\Omega}^*_{\varphi\circ\psi}$. □

Via restriction of the maps of $\mathcal{R}^*_\varphi$ to weight zero, $(\Omega^*_\varphi, \mathcal{R}^*_\varphi)$ induces a morphism of Grothendieck ringed spaces

$$(\Omega^*_\varphi, \mathcal{O}^*_\varphi)\colon (\Omega^*_{\Gamma'}, \mathcal{O}^*_{\Gamma'}) \to (\Omega^*_\Gamma, \mathcal{O}^*_\Gamma).$$

Remark 6.29. In Corollary 10.5 below we will show the following: If $\Gamma \subset \mathrm{Aut}_A(\Lambda)$ and $\Gamma' \subset \mathrm{Aut}_A(\Lambda')$ are congruence subgroups, then $(\Omega^*_\varphi, \mathcal{O}^*_\varphi)$ is a proper morphism of rigid analytic varieties. If, furthermore, the index of $\varphi^*(\Gamma') \subset \overline{\Gamma}_{\varphi(\Lambda')}$ is finite, then the morphism is even finite.

Consider any integer k. Let $(\mathcal{O}^*_\varphi)^{-1}\mathcal{O}^*_\Gamma(k)$ be the sheaf on Ω^*_Γ associated with the presheaf

$$X' \mapsto \varinjlim_{X \supset \Omega^*_\varphi(X')} \mathcal{O}^*_\Gamma(k)(X),$$

where X' is any admissible subset of $\Omega^*_{\Gamma'}$ and the limit is taken over all admissible subsets $X \subset \Omega^*_\Gamma$ containing $\Omega^*_\varphi(X')$. Moreover, $\mathcal{R}^*_\varphi$ induces a morphism of sheaves

$$(\mathcal{O}^*_\varphi)^{-1}\mathcal{O}^*_\Gamma(k) \to \mathcal{O}^*_{\Gamma'}(k) \tag{51}$$

by means of which we define the *preimage* of $\mathcal{O}^*_\Gamma(k)$ under $\mathcal{O}^*_\varphi$ to be the sheaf $(\mathcal{O}^*_\varphi)^*\mathcal{O}^*_\Gamma(k)$ of $\mathcal{O}^*_{\Gamma'}$-modules associated with the presheaf

$$X' \mapsto (\mathcal{O}^*_\varphi)^{-1}\mathcal{O}^*_\Gamma(k)(X') \otimes_{(\mathcal{O}^*_\varphi)^{-1}\mathcal{O}^*_\Gamma(X')} \mathcal{O}^*_{\Gamma'}(X').$$

The morphism in (51) further yields a natural morphism of $\mathcal{O}^*_{\Gamma'}$-modules

$$(\mathcal{O}^*_\varphi)^*\mathcal{O}^*_\Gamma(k) \to \mathcal{O}^*_{\Gamma'}(k). \tag{52}$$

6.4 Examples of global sections of twisting modules

Consider any $(A, \Lambda, \Gamma) \in \mathcal{A}$. Denote by F the quotient field of A.

6.4.1 Eisenstein series

Denote by $\pi\colon \Lambda_F \to \Lambda_F/\Lambda$ the quotient homomorphism. Consider any $\alpha \in \Lambda_F/\Lambda$ and set $L(\alpha) := \pi^{-1}(\alpha) \cap L_F$ for any direct summand $0 \neq L \subset \Lambda$. Let k be any positive integer. Consider the sum

$$E_{\Lambda,\alpha,k}\colon \tilde{\Omega}_\Lambda^* \to C,\ l \mapsto \sum_{0\neq\lambda\in L(\alpha)} \frac{1}{l(\lambda)^k},\ \text{ if } l \in \tilde{\Omega}_L.$$

Proposition 6.30. *$E_{\Lambda,\alpha,k}$ converges everywhere and, if Γ fixes α, is in $\mathcal{O}_\Gamma^*(k)(\Omega_\Gamma^*)$.*

Proof. Consider any direct summand $0 \neq L_0 \subset \Lambda$, any affinoid $O \subset \Omega_{L_0}$, any $0 \neq \lambda_0 \in L_0$ and any $r \in |C|$. Set $U := \mathcal{U}(\Lambda, O, r)$ and consider the sum

$$E\colon U \to C, [l] \mapsto E_{\Lambda,\alpha,k}(l) \cdot l(\lambda_0)^k.$$

We first show via the following lemmas that for every further direct summand $L_0 \subset L \subset \Lambda$ the sum E converges to a regular function on $U_L := U \cap \Omega_L$ and that E is continuous with respect to the canonical topologies and bounded.

Lemma 6.31. *On every U_L the sum E converges to a regular function.*

Proof. By means of an admissible affinoid covering of any such U_L, it suffices to show that the restriction $E_{O'}$ of E to every admissible affinoid $O' \subset U_L$ converges to a regular function. Consider any such $O' \subset U_L$ and choose any $[l] \in O'$. As $L(\alpha) \subset L_E$ is discrete, where E is the completion of F, the subset $l(L(\alpha)) \subset C$ is strongly discrete by Example 2.48 and Lemma 2.49. In particular, for any integer $m \geq 1$ the subset $L(\alpha)_m \subset L(\alpha)$ of those λ for which $\left|\frac{l(\lambda)}{l(\lambda_0)}\right| \leq m$ is finite; thus

$$E_{O',m}\colon O' \to C, [l'] \mapsto \sum_{0\neq\lambda\in L(\alpha)_m} \frac{l'(\lambda_0)^k}{l'(\lambda)^k}$$

is a regular function. As the ring of regular functions on O' is complete with respect to the sup-norm by Proposition 2.19, it suffices to show that the $E_{O',m}$ for all $m \geq 1$ form a Cauchy-sequence; indeed, as $L(\alpha)$ is covered by the $L(\alpha)_m$ for all $m \geq 1$, their limit must then be $E_{O'}$. Applying Corollary 5.3 to the affinoid O', we choose a $\kappa' > 0$ such that

$$\forall m \geq 1, \forall [l'] \in O', \forall \lambda \in L(\alpha) \setminus L(\alpha)_m \colon \left| \frac{l'(\lambda_0)}{l'(\lambda)} \right| \leq \kappa' \cdot \left| \frac{l(\lambda_0)}{l(\lambda)} \right| \leq \frac{\kappa'}{m}.$$

This directly yields that the $E_{O',m}$ indeed form a Cauchy-sequence. □

Lemma 6.32. *E is continuous with respect to the canonical topologies and bounded.*

Proof. As for continuity, it suffices, by Corollary 6.12, to show that E is sequentially continuous. Consider thus any $[l] \in U$ and any sequence $\{[l_n]\}_{n\geq 1} \subset U$ converging to $[l]$ and let us show that

$$\lim_{n\to\infty} E([l_n]) = E([l]). \tag{53}$$

Let $L \supset L_0$, resp. $L_n \supset L_0$, be such that $[l] \in \Omega_L$, resp. $[l_n] \in \Omega_{L_n}$ for any $n \geq 1$. Choose a fundamental basis of admissible affinoid neighborhoods $(O_n)_{n\geq 1}$ of $[l]$ in Ω_L such that $O_n \supset O_{n+1}$ for all $n \geq 1$. Using Corollary 6.10, we choose a sequence $\{r_n\}_{n\geq 1} \subset |C|$ converging to infinity and an $n_0 \geq 1$ such that $[l_n] \in U_n := \mathcal{U}(\Lambda, O_n, r_n)$ for every $n \geq n_0$. The choice of the O_n and the regularity of the restriction E_{O_1} of E to O_1 by Lemma 6.31 imply that

$$\lim_{n\to\infty} E([l_n|_{L_C}]) = \lim_{n\to\infty} E_{O_1}([l_n|_{L_C}]) = E_{O_1}([l]) = E([l]).$$

It thus remains to show that $E([l_n]) - E([l_n|_{L_C}])$ converges to 0 for $n \to \infty$. Applying Corollary 5.4 and Proposition 2.25 to the affinoid O_1, we choose an $s > 0$ such that $|l(\lambda_0)| \leq s \cdot \mathrm{d}(l(L))$ for every $[l] \in O_1$. Choose any $f \in F$ for which $\pi^{-1}(\alpha) \subset f \cdot \Lambda$. For any $n \geq n_0$ and any $\lambda \in L_n(\alpha) \setminus L_0$ then

$$\begin{aligned} |l_n(\lambda)| &\geq \mathrm{d}(l_n(L_n(\alpha) \setminus L_0)) \geq |f| \cdot \mathrm{d}(l_n(L_n \setminus L_0)) \overset{[l_n]\in U_n}{\geq} |f| \cdot r_n \cdot \mathrm{d}(l_n(L)) \\ &\geq \frac{|f| \cdot r_n}{s} \cdot |l_n(\lambda_0)|. \end{aligned}$$

For any $n \geq n_0$, as $O_1 \supset O_n$, thus

$$|E([l_n]) - E([l_n|_{L_C}])| = \left| \sum_{\lambda \in L_n(\alpha)\setminus L} \frac{l_n(\lambda_0)^k}{l_n(\lambda)^k} \right| \leq \frac{s^k}{(|f| \cdot r_n)^k} \xrightarrow[n\to\infty]{} 0.$$

Thus E is indeed sequentially continuous. In order to see that E is bounded, we use the preceding calculations in the case, where $O_1 = O$ and $n_0 = 1$

and $r_1 = r$, in order to see that

$$\forall [l] \in U = U_1\colon |E([l]) - E_O([l|_{L_C}])| = |E([l]) - E([l|_{L_C}])| \leq \frac{s^k}{(|f| \cdot r)^k}.$$

Moreover, as E_O is regular by Lemma 6.31, it is bounded by Proposition 2.25. Thus E is indeed bounded. □

By Lemma 6.31, the sum E converges everywhere on O. Hence $E_{\Lambda,\alpha,k}$ converges everywhere on O and thus, as O was arbitrary, on $\tilde{\Omega}^*_\Lambda$. By construction, it further holds that $E_{\Lambda,\alpha,k}(c \cdot l) = c^{-k} \cdot E_{\Lambda,\alpha,k}(l)$ for any $c \in C^\times$ and any $l \in \tilde{\Omega}^*_\Lambda$ and, if Γ fixes α, that $E_{\Lambda,\alpha,k}$ is Γ-invariant. Jointly with Lemmas 6.31 and 6.32, this yields the proposition. □

6.4.2 Poincaré-Eisenstein series

Consider any direct summand $0 \neq L \subset \Lambda$, any basis β of L_F and for every $w \in \beta$ an integer $k_w > 0$. Associate with any further direct summand $0 \neq L' \subset \Lambda$ the set

$$\Gamma_{L,L'} := \big\{\gamma \in \Gamma \mid L \subset \gamma(L')\big\}$$

with its natural left action of Γ_L and with any $\gamma \in \Gamma_{L,L'}$ the function

$$P^\gamma_{L'} : \tilde{\Omega}_{L'} \to C,\ l \mapsto \prod_{w \in \beta} \frac{1}{l(\gamma^{-1}w)^{k_w}}. \tag{54}$$

Using that $P^\gamma_{L'} = P^\sigma_{L'}$ for any $\gamma, \sigma \in \Gamma_{L,L'}$ whose classes $\overline{\gamma}, \overline{\sigma}$ in

$$\overline{\Gamma}_{L,L'} := \mathring{\Gamma}_L \backslash \Gamma_{L,L'}$$

coincide, we set $P^{\overline{\gamma}}_{L'} := P^\gamma_{L'}$. Associate with any such L' the sum

$$P_{L'} := \sum_{\overline{\gamma} \in \overline{\Gamma}_{L,L'}} P^{\overline{\gamma}}_{L'}.$$

Proposition 6.33. *Any $P_{L'}$ converges everywhere. The function $P_{\Lambda^*} : \tilde{\Omega}^*_\Lambda \to C$ whose restriction to any $\tilde{\Omega}_{L'}$ equals $P_{L'}$ is in $\mathcal{O}^*_\Gamma(k)(\Omega^*_\Gamma)$, where $k := \sum_{w \in \beta} k_w$.*

Definition 6.34. *Any C-linear combination of global sections of the above form P_{Λ^*} of the same weight is called a* Poincaré-Eisenstein series *with respect to Γ.*

Proof of Proposition 6.33. For a suitable $f \in F^\times$ the basis $f \cdot \beta$ of L_F is contained in L. Using that $P^\gamma_{L'}(f \cdot l) = f^{-k} \cdot P^\gamma_{L'}(l)$ for any $f \in F^\times$, any $\gamma \in \Gamma_{L,L'}$ and any $l \in \tilde{\Omega}_{L'}$ and using the linearity of any such l, we assume without loss of generality that β is contained in L. Consider then any direct summand $0 \neq L_0 \subset \Lambda$, any affinoid $O \subset \Omega_{L_0}$, any $0 \neq \lambda_0 \in L_0$ and any $1 \leq r \in |C|$. Set $U := \mathcal{U}(\Lambda, O, r)$ and consider the sum

$$P\colon U \to C, [l] \mapsto P_{L'}(l) \cdot l(\lambda_0)^k, \text{ if } [l] \in \Omega_{L'}.$$

We first show via the following lemmas that for every further direct summand $L_0 \subset L' \subset \Lambda$ the sum P converges to a regular function on $U_{L'} := U \cap \Omega_{L'}$ and that P is continuous with respect to the canonical topologies and bounded.

Lemma 6.35. *On every $U_{L'}$ the sum P converges to a regular function.*

Proof. By means of an admissible affinoid covering of any such $U_{L'}$, it suffices to show that the restriction $P_{O'}$ of P to every admissible affinoid $O' \subset U_{L'}$ converges to a regular function. Consider any such $O' \subset U_{L'}$. Set

$$\overline{\Gamma} := \overline{\Gamma}_{L,L'}.$$

Choose any $[l] \in O'$. As $l(L') \subset C$ is strongly discrete and as any $\overline{\gamma} \in \overline{\Gamma}$ is uniquely determined by the action of γ^{-1} on β, for any integer $m \geq 1$ the subset

$$\overline{\Gamma}_m := \left\{ \overline{\gamma} \in \overline{\Gamma} \,\middle|\, \forall w \in \beta\colon \left| \frac{l(\gamma^{-1} w)}{l(\lambda_0)} \right| \leq m \right\} \subset \overline{\Gamma}$$

is finite. With any $m \geq 1$ may thus be associated the regular function

$$P_{O',m}\colon O' \to C, [l'] \mapsto \sum_{\overline{\gamma} \in \overline{\Gamma}_m} \frac{l'(\lambda_0)^k}{\prod_{w \in \beta} l'(\gamma^{-1} w)^{k_w}}.$$

As the ring of regular functions on O' is complete with respect to the supnorm by Proposition 2.19, it suffices to show that the $P_{O',m}$ for all $m \geq 1$ form a Cauchy-sequence; indeed, as $\overline{\Gamma}$ is the union of the Γ_m for all $m \geq 1$, their limit must then be $P_{O'}$. Applying Corollary 5.3 to the affinoid O', we choose a $\kappa' > 0$ such that

$$\forall [l'] \in O', \forall 0 \neq \lambda \in L'\colon \left| \frac{l'(\lambda_0)}{l'(\lambda)} \right| \leq \kappa' \cdot \left| \frac{l(\lambda_0)}{l(\lambda)} \right|.$$

For any $m \geq 1$, any $\overline{\gamma} \in \overline{\Gamma} \setminus \overline{\Gamma}_m$ and any $[l'] \in O'$ then holds that

$$\left| \frac{l'(\lambda_0)^k}{\prod_{w\in\beta} l'(\gamma^{-1}w)^{k_w}} \right| \leq \kappa^k \cdot \left| \frac{l(\lambda_0)^k}{\prod_{w\in\beta} l(\gamma^{-1}w)^{k_w}} \right| \leq \kappa^k \cdot \left(\frac{|l(\lambda_0)|}{\mathrm{d}(l(L'))} \right)^{k-1} \cdot \frac{1}{m}.$$

This directly yields that the $P_{O',m}$ indeed form a Cauchy-sequence. □

Lemma 6.36. *P is continuous with respect to the canonical topologies and bounded.*

Proof. As for continuity, it suffices, by Corollary 6.12, to show that P is sequentially continuous. Consider thus any $[l] \in U$ and any sequence $\{[l_n]\}_{n\geq 1} \subset U$ converging to $[l]$ and let us show that

$$\lim_{n\to\infty} P([l_n]) = P([l]). \tag{55}$$

Let $L' \supset L_0$, resp. $L_n \supset L_0$, be such that $[l] \in \Omega_{L'}$, resp. $[l_n] \in \Omega_{L_n}$ for any $n \geq 1$. Choose a fundamental basis of admissible affinoid neighborhoods $(O_n)_{n\geq 1}$ of $[l]$ in $\Omega_{L'}$ such that $O_n \supset O_{n+1}$ for all $n \geq 1$. Using Corollary 6.10, we choose a sequence $\{r_n\}_{n\geq 1} \subset |C|$ converging to infinity with $r_n \geq 1$ for any $n \geq 1$ and an $n_0 \geq 1$ such that $[l_n] \in U_n := \mathcal{U}(\Lambda, O_n, r_n)$ for every $n \geq n_0$. The choice of the O_n and the regularity of the restriction P_{O_1} of P to O_1 by Lemma 6.35 imply that

$$\lim_{n\to\infty} P([l_n|_{L'_C}]) = \lim_{n\to\infty} P_{O_1}([l_n|_{L'_C}]) = P_{O_1}([l]) = P([l]).$$

It thus remains to show that $P([l_n]) - P([l_n|_{L'_C}])$ converges to 0 for $n \to \infty$. Applying Corollary 5.4 and Proposition 2.25 to the affinoid O_1, we choose an $s > 0$ such that $|l(\lambda_0)| \leq s \cdot \mathrm{d}(l(L))$ for every $[l] \in O_1$. Consider any $n \geq n_0$ and any $\overline{\gamma} \in \overline{\Gamma}_{L,L_n} \setminus \overline{\Gamma}_{L,L'}$ and set $\beta_\gamma := \{w \in \beta |\, \gamma^{-1}(w) \in L'\} \subset \beta$. Then $\beta_\gamma \subsetneq \beta$ and hence $k > k_\gamma := \sum_{w\in\beta_\gamma} k_w$. Moreover

$$\forall w \in \beta_\gamma \colon |l_n(\gamma^{-1}(w))| \geq \mathrm{d}(l_n(l(L'))) \geq \frac{1}{s} \cdot |l_n(\lambda_0)|$$

and

$$\forall w \in \beta \setminus \beta_\gamma \colon |l_n(\gamma^{-1}(w))| \geq \mathrm{d}(l_n(L_n \setminus L')) \overset{[l_n]\in U_n}{\geq} r_n \cdot \mathrm{d}(l_n(L')) \geq \frac{r_n}{s} \cdot |l_n(\lambda_0)|.$$

Thus

$$\left| \frac{l_n(\lambda_0)^k}{\prod_{w\in\beta} l(\gamma^{-1}w)^{k_w}} \right| \leq s^{k_\gamma} \cdot \left(\frac{s}{r_n} \right)^{k-k_\gamma} \leq \frac{s^k}{r_n},$$

where the last inequality holds true as $k > k_\gamma$ and $r_n \geq 1$. Thus

$$|P([l_n]) - P([l_n|_{(L')_C}])| = \left| \sum_{\overline{\gamma} \in \overline{\Gamma}_{L,L_n} \setminus \overline{\Gamma}_{L,L'}} \frac{l(\lambda_0)^k}{\prod_{w \in \beta} l(\gamma^{-1} w)^{k_w}} \right| \leq \frac{s^k}{r_n} \underset{n \to \infty}{\longrightarrow} 0.$$

Thus P is indeed sequentially continuous. In order to see that P is bounded, we use the preceding calculations in the case, where $O_1 = O$ and $n_0 = 1$ and $r_1 = r$, in order to see that

$$\forall [l] \in U = U_1 \colon |P([l]) - P_O([l|_{L'_C}])| = |P([l]) - P([l|_{L'_C}])| \leq \frac{s^k}{r}.$$

Moreover, as P_O is regular by Lemma 6.35, it is bounded by Proposition 2.25. Thus P is indeed bounded. □

By Lemma 6.35, the sum P, and hence P_{L_0}, converges everywhere on O. As L_0 and O are arbitrary, thus $P_{L'}$ converges everywhere for every direct summand $0 \neq L' \subset \Lambda$. The function $P_{\Lambda^*} \colon \tilde{\Omega}^*_\Lambda \to C$ whose restriction to any $\tilde{\Omega}_{L'}$ equals $P_{L'}$ is, by construction, Γ-invariant and satisfies that $P_{\Lambda^*}(c \cdot l) = c^{-k} \cdot P_{\Lambda^*}(l)$ for any $c \in C^\times$ and any $l \in \tilde{\Omega}^*_\Lambda$. Jointly with Lemmas 6.35 and 6.36, this yields the proposition. □

6.5 Separation of points

Consider any $(A, \Lambda, \Gamma) \in \mathcal{A}$. Inspired by Godement's [24] via its application in Baily's and Borel's [1], we show

Proposition 6.37. *Consider any $l, l' \in \tilde{\Omega}^*_\Lambda$ with $\pi_\Gamma(l) \neq \pi_\Gamma(l')$. Then there exists an integer $k > 0$ and for any integer $n > 0$ a Poincaré-Eisenstein series $P \in \mathcal{O}^*_\Gamma(n \cdot k)$ such that $P(l) \neq 0$ and $P(l') = 0$.*

Lemma 6.38. *Consider any direct summands $0 \neq L_1, L_2 \subset \Lambda$ such that*

$$r_1 := \mathrm{rank}_A(L_1) \geq \mathrm{rank}_A(L_2) =: r_2$$

and any $l_1 \in \tilde{\Omega}_{L_1}$ and $l_2 \in \tilde{\Omega}_{L_2}$ with $\pi_\Gamma(l_1) \neq \pi_\Gamma(l_2)$. Then there exists an integer $k > 0$ and for any integer $n > 0$ a Poincaré-Eisenstein series $P \in \mathcal{O}^*_\Gamma(n \cdot k)$ such that $|P(l_1)| > |P(l_2)|$ and, if $r_1 > r_2$, such that $P(l_2) = 0$.

Proof of Lemma 6.38. Set $L := L_1$. Choose a basis β of L_F such that $|l_1(w)| \leq 1$ for every $w \in \beta$. Using the notation of Section 6.4.2, associate with any direct summand $0 \neq L' \subset \Lambda$, any $l \in \tilde{\Omega}_{L'}$ and any $\overline{\gamma} \in \overline{\Gamma}_{L,L'}$ the product

$$J_{\overline{\gamma}}(l) := \prod_{w \in \beta} l(\gamma^{-1}(w)).$$

Choose any $0 < \varepsilon < 1$ and consider for any $i = 1, 2$ the set

$$\overline{\Gamma}_i^{\varepsilon} := \left\{ \overline{\gamma} \in \overline{\Gamma}_{L,L_i} \left| \frac{1}{|J_{\overline{\gamma}}(l_i)|} \geq \varepsilon \vee \exists w \in \beta \colon \left| \frac{l_i(\gamma^{-1}(w))}{J_{\overline{\gamma}}(l_i)^2} \right| \geq \varepsilon^2 \right. \right\};$$

it is finite as $l_i(L_i) \subset C$ is strongly discrete by Example 2.48 and Lemma 2.49 and as any $\overline{\gamma} \in \overline{\Gamma}_{L,L_i}$ is determined by the action of γ^{-1} on β. Set

$$\Gamma_{[l_1]} := \{\gamma \in \Gamma |\, \gamma([l_1]) = [l_1]\} \text{ and } \overline{\Gamma}_{[l_1]} := \mathring{\Gamma}_L \backslash \Gamma_{[l_1]}.$$

As $\overline{\Gamma}_{[l_1]} \subset \mathrm{Aut}_A(L)$, there is a natural embedding

$$\overline{\Gamma}_{[l_1]} \hookrightarrow \{c \in C^{\times} |\, c \cdot l_1(L) = l_1(L)\}$$

whose target is the multiplicative group of a finite subfield of C. Thus

$$k_1 := |\overline{\Gamma}_{[l_1]}|$$

is finite and is not a multiple of the characteristic of C and

$$\forall \gamma \in \overline{\Gamma}_{[l_1]}, \forall \lambda \in L_C \colon ((\gamma l_1)(\lambda))^{k_1} = (l_1(\lambda))^{k_1} \tag{56}$$

Using that for any $\overline{\sigma} \in \overline{\Gamma}_1^{\varepsilon} \setminus \overline{\Gamma}_{[l_1]}$ the kernels of l_1 and σl_1 are different C-subspaces of L_C of codimension 1, we choose for any such $\overline{\sigma}$ a $\lambda_1^{\overline{\sigma}} \in L_C$ such that $l_1(\lambda_1^{\overline{\sigma}}) \neq 0$ and $(\sigma l_1)(\lambda_1^{\overline{\sigma}}) = 0$. Further consider any $\overline{\sigma} \in \overline{\Gamma}_2^{\varepsilon}$. As $r_1 \geq r_2$, then $L = \sigma(L_2)$ and hence $\sigma(l_2) \in \tilde{\Omega}_L$. As $\pi_\Gamma(l_1) \neq \pi_\Gamma(l_2) = \pi_\Gamma(\sigma(l_2))$, the kernels of l_1 and σl_2 are different C-subspaces of L_C of codimension 1. Using this, we choose a $\lambda_2^{\overline{\sigma}} \in L_C$ such that $l_1(\lambda_2^{\overline{\sigma}}) \neq 0$ and $(\sigma l_2)(\lambda_2^{\overline{\sigma}}) = 0$. Consider the homogeneous function

$$Q \colon \tilde{\Omega}_L \to C,\ l \mapsto \left(\prod_{\overline{\sigma} \in \overline{\Gamma}_1^{\varepsilon} \setminus \overline{\Gamma}_{[l_1]}} l(\lambda_1^{\overline{\sigma}}) \cdot \prod_{\overline{\sigma} \in \overline{\Gamma}_2^{\varepsilon}} l(\lambda_2^{\overline{\sigma}}) \right)^{k_1}.$$

Any $\lambda_i^{\overline{\sigma}}$ is uniquely a C-linear combination in β. Let $\mu > 0$ be greater than the norms of all coefficients of all $\lambda_i^{\overline{\sigma}}$. Choose an integer k_0 such that

i) k_0 is a multiple of k_1,

ii) $k_0 > \max_{w \in \beta} \deg_w(Q)$ and

iii) $|Q(l_1)| > \mu^{\deg(Q)} \cdot \varepsilon^{k_0}$ using that, by construction, $Q(l_1) \neq 0$.

Set $k := |\beta| \cdot k_0 - \deg(Q) \overset{ii)}{>} 0$. Consider any integer $n > 0$. Associate with any direct summand $0 \neq L' \subset \Lambda$ the function

$$P^{\overline{\gamma}}_{L'} \colon \tilde{\Omega}_{L'} \to C, l \mapsto \left(\frac{Q((\gamma l)|_{L_C})}{(J_{\overline{\gamma}}(l))^{k_0}} \right)^n .$$

Writing any $\lambda_i^{\overline{\sigma}}$ as a C-linear combination of β and using ii), it is directly checked that any such $P^{\overline{\gamma}}_{L'}$ is a C-linear combination of homogeneous functions, all being of the same weight, of the form (54) and that the coefficients of this combination are independent of L' and $\overline{\gamma}$. Proposition 6.33 and Definition 6.34 thus provide a Poincaré-Eisenstein series $P \in \mathcal{O}^*_\Gamma(n \cdot k)$ whose restriction to any stratum $\tilde{\Omega}_{L'}$ is

$$P_{L'} := \sum_{\overline{\gamma} \in \overline{\Gamma}_{L,L'}} P^{\overline{\gamma}}_{L'}.$$

If $r_1 > r_2$, then $\overline{\Gamma}_{L_1,L_2} = \emptyset$ and hence $P_{L_2} = 0$ and hence $P(l_2) = 0$. We are thus reduced to showing that $|P(l_1)| > |P(l_2)|$. By construction, for any $i = 1, 2$, any $\overline{\gamma} \in \overline{\Gamma}_{L,L_i} \setminus \overline{\Gamma}^\varepsilon_i$ and any $\overline{\sigma} \in \overline{\Gamma}^\varepsilon_2 \cup (\overline{\Gamma}^\varepsilon_1 \setminus \overline{\Gamma}_{[l_1]})$ holds that

$$\frac{1}{|J_{\overline{\gamma}}(l_i)|} < \varepsilon \;\wedge\; \left| \frac{(\gamma l_i)(\lambda_i^{\overline{\sigma}})}{J_{\overline{\gamma}}(l_i)^2} \right| \leq \mu \cdot \max_{w \in \beta} \left\{ \left| \frac{(\gamma l_i)(w|}{J_{\overline{\gamma}}(l_i)^2} \right| \right\} < \mu \cdot \varepsilon^2$$

and hence that

$$|P^{\overline{\gamma}}_{L_i}(l_i)| < (\mu^{\deg(Q)} \cdot \varepsilon^{k_0})^n =: s.$$

By construction of Q, moreover $P^{\overline{\gamma}}_{L_2}(l_2) = 0$ for any $\overline{\gamma} \in \overline{\Gamma}^\varepsilon_2$. Hence

$$|P(l_2)| < s. \tag{57}$$

By construction of Q, also $P^{\overline{\gamma}}_{L_1}(l_1) = 0$ for any $\overline{\gamma} \in \overline{\Gamma}^\varepsilon_1 \setminus \overline{\Gamma}_{[l_1]}$. Hence

$$\left| \sum_{\overline{\gamma} \in \overline{\Gamma}_{L,L_1} \setminus \overline{\Gamma}_{[l_1]}} P^{\overline{\gamma}}_{L_1}(l_1) \right| < s. \tag{58}$$

By (56) and i) and the construction of Q, for any $\overline{\gamma} \in \overline{\Gamma}_{[l_1]}$ holds that

$$\frac{Q(\gamma(l_1))}{(J_{\overline{\gamma}}(l_1))^{k_0}} = \frac{Q(l_1)}{(J_{\overline{\mathrm{id}}}(l_1))^{k_0}}$$

and hence that $P^{\overline{\gamma}}_{L_1}(l_1) = P^{\overline{\mathrm{id}}}_{L_1}(l_1)$. As $k_1 = |\overline{\Gamma}_{[l_1]}|$ is not a multiple of the characteristic of C and as $|J_{\mathrm{id}}(l_1)|^n \leq 1$ by the choice of β, thus

$$(59) \qquad \left| \sum_{\overline{\gamma}\in\overline{\Gamma}_{[l_1]}} P^{\overline{\gamma}}_{L_1}(l_1) \right| = |P^{\overline{\mathrm{id}}}_{L_1}(l_1)| \geq |(Q(l_1))^n| \overset{iii)}{>} s.$$

Thus $|P(l_1)| \overset{(58),(59)}{>} s \overset{(57)}{>} |P(l_2)|$ as desired. $\square$

Proof of Proposition 6.37. Let $L, L' \subset \Lambda$ be such that $l \in \tilde{\Omega}_L$ and $l' \in \tilde{\Omega}_{L'}$.

Consider first the case, where $r := \mathrm{rank}_A(L) \geq \mathrm{rank}_A(L') =: r'$. If $r > r'$, then the proposition follows directly from Lemma 6.38. Thus suppose further that $r = r'$, Then Lemma 6.38 provides an integer $i > 0$, resp. $i' > 0$, and for any integer $n > 0$ a Poincaré-Eisenstein series $P_{n\cdot i}$ of weight $n \cdot i$, resp. $P_{n\cdot i'}$ of weight $n \cdot i'$, satisfying $|P_{n\cdot i}(l)| > |P_{n\cdot i}(l')|$, resp. $|P'_{n\cdot i'}(l)| < |P'_{n\cdot i'}(l')|$. Set $k := i \cdot i'$. Thus for any integer $n > 0$ hols that

$$P := P_{n\cdot k} - \frac{P_{n\cdot k}(l')}{P'_{n\cdot k}(l')} \cdot P'_{n\cdot k}$$

is a Poincaré-Eisenstein series of weight $n \cdot k$ for which $P(l) \neq 0 = P(l')$. This yields the proposition in the case, where $r \geq r'$.

Consider then the case, where $r < r'$. The previous case provides an integer $j > 0$ and for any integer $n > 0$ a Poincaré-Eisenstein series $P_{n\cdot j}$ of weight $n \cdot j$ with $P_{n\cdot j}(l') \neq 0 = P_{n\cdot j}(l)$. Lemma 6.38 further provides an integer $j' > 0$ and for any integer $n > 0$ a Poincaré-Eisenstein series $P'_{n\cdot j'}$ of weight $n \cdot j'$ with $P'_{n\cdot j'}(l) \neq 0$. Set $k := j \cdot j'$. Thus for any integer $n > 0$ holds that

$$P := P'_{n\cdot k} - \frac{P'_{n\cdot k}(l')}{P_{n\cdot k}(l')} \cdot P_{n\cdot k}$$

is a Poincaré-Eisenstein series of weight $n \cdot k$ for which $P(l) \neq 0 = P(l')$. This yields the proposition in the case, where $r < r'$. $\square$

6.6 Fourier expansion of weak modular forms

Let $(A, \Lambda, \Gamma) \in \mathcal{A}$ such that $d := \mathrm{rank}_A(\Lambda) \geq 2$ and such that Γ is a congruence subgroup of $\mathrm{Aut}_A(\Lambda)$. Denote by $E \subset C$ the completion of the quotient field of A.

Definition 6.39. *For any integer k the sections in $\mathcal{O}^*_\Gamma(k)(\Omega_\Gamma)$ are called* weak modular forms *of weight k with respect to Γ.*

The following remark discusses a bijective correspondence between weak modular forms as above and weak modular forms as defined classically in terms of coordinates by Goss [22], [23] and by Gekeler [18] in the case $d = 2$ and in general by Basson, Breuer and Pink (see e.g. [3, Definitions 3.1.7 and 3.3.1]). We have learned about the correspondence through Goss' [22, Corollary 1.40 and Proposition 1.43].

Remark 6.40. Let $w_1, \ldots, w_d$ be an ordered basis of Λ_E. By means of this, identify $\mathcal{G} := \mathrm{Aut}_E(\Lambda_E)$ with $\mathrm{GL}_d(E)$. Let Ω^{d-1} denote the image of the injection

$$i \colon \Omega_\Lambda \to \mathbb{A}^d_C, [l] \mapsto \frac{1}{l(w_d)}(l(w_1), \ldots, l(w_d)).$$

The unique $\mathcal{G}$-action on Ω^{d-1} with respect to which i is $\mathcal{G}$-equivariant is given by

$$\forall g \in \mathcal{G}, \forall z = (z_1, \ldots, z_{d-1}, 1) \in \Omega^{d-1} \colon gz := \frac{zg^{-1}}{(zg^{-1})_d},$$

where zg^{-1} is the matrix product of z with g^{-1} and $(zg^{-1})_d$ its d-th coordinate. Consider any integer k. For any $f \in \mathcal{O}^*_\Gamma(k)(\Omega_\Gamma)$ then

$$h \colon \Omega^{d-1} \to C, z \mapsto f(l), \text{ where } z = i([l]) \text{ and } l(w_d) = 1,$$

is a *weak modular form* on Ω^{d-1} of weight k and type 0 for Γ as in [3, Definition 3.1.7], i.e., it is regular and satisfies that

$$\forall \gamma \in \Gamma, \forall z \in \Omega^{r-1} \colon h(\gamma z) = h(z) \cdot ((z\gamma^{-1})_d)^k.$$

Conversely, any such weak modular form h induces the weak modular form $f \in \mathcal{O}^*_\Gamma(k)(\Omega_\Gamma)$, where $f(l) := h(i([l])) \cdot l(w_d)^{-k}$ for any $l \in \tilde{\Omega}_\Lambda$. This yields mutually inverse isomorphisms of C-vector spaces between $\mathcal{O}^*_\Gamma(k)(\Omega_\Gamma)$ and the space of such weak modular forms on Ω^{d-1}.

Proof. All of this is directly checked. □

Consider any direct summand $L \subset \Lambda$ of corank 1. Choose any $v \in \Lambda \setminus L$. Denote by

$$\mathring{\Gamma} \subset \mathring{\Gamma}_L$$

the subgroup of elements γ such that $\gamma(v) - v \in L$. Thus $\mathring{\Gamma}$ is a discrete subgroup of $\mathrm{Aut}_E(\Lambda_E)$ of the form considered in Section 5.4 and

$$\mathring{\Gamma} \to L, \; \gamma \mapsto v_\gamma := \gamma(v) - v$$

is a continuous injective group homomorphism. Denote by $v_{\mathring{\Gamma}} \subset L$ its image and set

$$\forall l \in \tilde{\Omega}_\Lambda \colon u(l) := \frac{1}{\mathrm{e}_{l(v_{\mathring{\Gamma}})}(l(v))}.$$

Denote by $\Gamma_{L,v} \subset \Gamma_L$ the subgroup of elements γ such that $\gamma(v_{\mathring{\Gamma}}) = v_{\mathring{\Gamma}}$ and $\gamma(v) - v \in v_{\mathring{\Gamma}}$ and by $\overline{\Gamma}_{L,v} \subset \overline{\Gamma}_L$ its image under the restriction homomorphism $\Gamma_L \to \overline{\Gamma}_L$. Thus $\mathring{\Gamma}$ is the kernel of the homomorphism $\Gamma_{L,v} \to \overline{\Gamma}_{L,v}$. In fact, the index of $\overline{\Gamma}_{L,v} \subset \overline{\Gamma}_L$ is finite as is directly checked.

In this section we show (see Corollaries 6.43, 6.44 and 6.45 below applied to $X = \mathcal{U}_{\mathring{\Gamma}}$) that any $f \in \mathcal{O}^*_\Gamma(k)(\Omega_\Gamma)$, where k is any integer, admits unique $f_i \in \mathcal{O}_{\overline{\Gamma}_{L,v}}(k-i)(\Omega_{\overline{\Gamma}_{L,v}})$ for all $i \in \mathbb{Z}$ with the following property: Any affinoid $O \subset \Omega_L$ is contained in an admissible $Y \subset \Omega^*_{\mathring{\Gamma}}$ such that

$$\forall l \in \tilde{\Omega}_\Lambda \cap \pi_{\mathring{\Gamma}}^{-1}(Y) \colon \; f(l) = \sum_{i \in \mathbb{Z}} f_i(l|_{L_C}) u(l)^i; \tag{60}$$

in fact, any such O admits such a Y which satisfies (60) for all such f.

Proofs of versions of this fact were already given by many people. Up to the correspondence in Remark 6.40 and up to the homogeneity of the f_i, a proof of it was already given by Goss [22, Theorem 1.76] in the case where $d = 2$ and later by Kapranov [30, Proof of Proposition 1.19] by an extension of Goss's arguments in the case where d is arbitrary and A is a polynomial ring. In the general case, a generalized proof was recently given by Basson, Breuer and Pink (see [3, Proposition 3.2.5]). Basson [3, Proposition 3.2.7] was the first to note that any f_i is homogeneous of weight $k - i$.

Essential in each of their work is some version of the following Definition-Proposition 6.42. We believe to furthermore have taken rigorous care of the rigid analytic aspects involved in showing it.

Definition 6.41. $\mathcal{U}_{\mathring{\Gamma}} := \Omega_L \cup \Omega_{\mathring{\Gamma}} = p_{\mathring{\Gamma}}(\mathcal{U}(\Lambda, \Omega_L, 0))$.

Definition-Proposition 6.42. *Choose any* $0 \neq w \in L$. *Then*

$$q \colon \mathcal{U}_{\mathring{\Gamma}} \to \Omega_L \times C, p_{\mathring{\Gamma}}([l]) \mapsto \begin{cases} ([l|_{L_C}], l(w) \cdot u(l)) & \textit{if } [l] \in \Omega_\Lambda \\ ([l], 0) & \textit{if } l \in \Omega_L. \end{cases}$$

is an open immersion of regular rigid analytic varieties.

Proof. Set $\mathcal{V} := \Lambda_E$ and $\mathcal{W} := L_E$. Then q restricts to the open immersion $q_{\mathring{\Gamma}} \colon \Omega_{\mathring{\Gamma}} \to \Omega_{\mathcal{W}} \times C^{\times}$ provided by Def.-Prop. 5.29. Moreover, by assumption, Γ contains a principal congruence subgroup of some level $0 \neq (b) \subsetneq A$. But $b \cdot L \subset v_{\mathring{\Gamma}}$ for any such b. Hence Proposition 5.33 applies and yields, jointly with Corollary 3.5 and the fact that the restriction $q_{\mathring{\Gamma}}$ of q is an open immersion, that q is an isomorphism of Grothendieck topological spaces onto an admissible subset of $\Omega_L \times C$. Jointly with Corollary 3.6, this implies that q is an isomorphism of Grothendieck ringed spaces onto an admissible subvariety of $\Omega_L \times C$. □

For any $\varepsilon \in |C^{\times}|$ denote by $B_\varepsilon \subset C$ the closed disc around 0 of radius ε.

Corollary 6.43. *Any admissible $X \subset \mathcal{U}_{\mathring{\Gamma}}$ and any admissible affinoid $O \subset X \cap \Omega_L$ admit an $\epsilon \in |C^{\times}|$ such that $O \times B_\varepsilon \subset q(X)$.*

Proof. This follows directly from Proposition 6.42 and Corollary 3.5. □

For any $O \subset \Omega_L$ and any $\varepsilon \in |C^{\times}|$ set $\tilde{\mathcal{U}}(O,\varepsilon) := \tilde{\Omega}_\Lambda \cap (q \circ \pi_{\mathring{\Gamma}})^{-1}(O \times B_\varepsilon)$.

Corollary 6.44. *Any admissible $X \subset \mathcal{U}_{\mathring{\Gamma}}$, any $k \in \mathbb{Z}$ and any $f \in \mathcal{O}_{\mathring{\Gamma}}(k)(X \cap \Omega_{\mathring{\Gamma}})$ admit unique $f_i \in \mathcal{O}_{\{\mathrm{id}\}}(k-i)(X \cap \Omega_L)$ for all $i \in \mathbb{Z}$ such that*

$$\forall l \in \tilde{\mathcal{U}}(O,\varepsilon)\colon\ f(l) = \sum_{i \in \mathbb{Z}} f_i(l|_{L_C})u(l)^i \tag{61}$$

for every admissible affinoid $O \times B_\varepsilon \subset q(X)$. Moreover, the following are equivalent:

i) $\forall i < 0 : f_i = 0$.

ii) *f extends to an element in $\mathcal{O}_{\mathring{\Gamma}}(k)(X)$ which restricts to f_0 on $\pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_L)$.*

iii) *The section $\left[g\colon \pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_{\mathring{\Gamma}}) \to C \ \ l \mapsto f(l) \cdot l(w)^k\right] \in \mathcal{O}_{\mathring{\Gamma}}(0)(X \cap \Omega_{\mathring{\Gamma}})$ extends to a morphism of Grothendieck topological spaces $\pi_{\mathring{\Gamma}}^{-1}(X) \to C$ whose restriction to $\pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_L)$ is in $\mathcal{O}_{\{\mathrm{id}\}}(0)(X \cap \Omega_L)$.*

iv) *g is bounded on $\tilde{\mathcal{U}}(O,\varepsilon)$ for any admissible affinoid $O \times B_\varepsilon \subset q(X)$.*

Proof. Proposition 6.42 and Corollaries 3.5 and 3.6 yield unique

$$g_i \in \mathcal{O}_{\{\mathrm{id}\}}(-i)(X \cap \Omega_L)$$

for all $i \in \mathbb{Z}$ that satisfy the desired properties when f is replaced by the section g defined in iii) and k by 0. It is then directly checked that the

$$\Big[f_i \colon \pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_L) \to C \ \ l \mapsto g_i([l|_{L_C}]) \cdot l(w)^{-k}\Big] \in \mathcal{O}_{\{\mathrm{id}\}}(k-i)(X \cap \Omega_L)$$

satisfy the desired properties. □

Corollary 6.45. *(Basson [3, Proposition 3.2.7]) Let $f \in \mathcal{O}_{\Gamma_{L,v}}(k)(\Omega_{\Gamma_{L,v}})$. Then*

$$\forall i \in \mathbb{Z}\colon \ f_i \in \mathcal{O}_{\overline{\Gamma}_{L,v}}(k-i)(\Omega_{\overline{\Gamma}_{L,v}}).$$

Proof. Consider any $\tau \in \overline{\Gamma}_{L,v}$, any lift $\gamma \in \Gamma_{L,v}$ of τ and any affinoid subset $O \subset \Omega_L$. By means of Proposition 6.42, choose an $\varepsilon \in |C^\times|$ such that

$$O \times B_\varepsilon \subset q(\mathcal{U}_{\mathring{\Gamma}}) \ \text{ and } \ \gamma(O) \times B_\varepsilon \subset q(\mathcal{U}_{\mathring{\Gamma}}).$$

Invariance of f and u under $\Gamma_{L,v}$ and (61) yield that

$$\forall l \in \tilde{\mathcal{U}}(O,\varepsilon)\colon f(l) = f(\gamma l) = \sum_{i\in\mathbb{Z}} f_i((\gamma l)|_{L_C}) u(\gamma l)^i = \sum_{i\in\mathbb{Z}} f_i(\tau(l|_{L_C}))u(l)^i.$$

The $f_i \circ \tau$ are thus further coefficients satisfying (61) and, by uniqueness, thus coincide with the f_i. Hence the f_i are indeed $\overline{\Gamma}_{L,v}$-invariant. □

That the following Definition 6.46, ii) is independent of the choice of $v \in \Lambda \setminus L$ follows from the equivalence of iv) and i) in Proposition 6.44.

Definition 6.46. *Consider any X and f as in Proposition 6.44.*

i) For any $i \in \mathbb{Z}$ the element f_i in Proposition 6.44 is called the i-th (Fourier) u-coefficient *of f and*

$$\sum_{i\in\mathbb{Z}} f_i \cdot u^i$$

is called the (Fourier) u-expansion *of f.*

ii) The order *of f is defined to be* $\inf\{i \in \mathbb{Z} \mid f_i \neq 0\}$. *Moreover, f is called* meromorphic, *resp.* holomorphic, *resp.* cuspidal, *resp.* double-cuspidal, *if its order is $> -\infty$, resp. ≥ 0, resp. ≥ 1, resp. ≥ 2.*

Remark 6.47. In [3, Sections 3.4 and 3.5], Basson has computed the Fourier expansion and the order of for instance the Eisenstein series from Section 6.4.1.

A further consequence of Proposition 6.42 is

Corollary 6.48. *i)* $\mathcal{U}_\Gamma := p_\Gamma(\mathcal{U}(\Lambda, \Omega_L, 0))$ *is a normal rigid analytic variety.*

ii) If $\mathring{\Gamma} = \mathring{\Gamma}_L$ *and the action of* Γ *on* Ω_Λ *and the action of* $\overline{\Gamma}_L$ *on* Ω_L *are both fixed-point free, then the composition*

$$q(\mathcal{U}_{\mathring{\Gamma}}) \xrightarrow{q^{-1}} \mathcal{U}_{\mathring{\Gamma}} \xrightarrow{\pi} \mathcal{U}_\Gamma,$$

where π *denotes the natural quotient morphism, induces isomorphisms on stalks; in particular,* $\mathcal{U}_\Gamma$ *is then regular.*

Example 6.49. *Set* $\Lambda' := L \oplus A \cdot v$*. As* $\mathrm{Aut}_A(\Lambda) \cap \mathrm{Aut}_A(\Lambda')$ *is a congruence subgroup of both* $\mathrm{Aut}_A(\Lambda)$ *and* $\mathrm{Aut}_A(\Lambda')$*, the kernel* $\Gamma(I)$ *of* $\mathrm{Aut}_A(\Lambda') \to \mathrm{Aut}_A(\Lambda'/I\Lambda')$ *is contained in* $\mathrm{Aut}_A(\Lambda)$ *for some ideal* $0 \neq I \subsetneq A$*. If* $\Gamma = \Gamma(I)$ *for such* I*, then* $\mathring{\Gamma} = \mathring{\Gamma}_L$ *and, by Example 5.11,* Γ *acts fixed point free on* $\Omega_{\Lambda'} = \Omega_\Lambda$ *and* $\overline{\Gamma}_L$*, being the kernel of* $\mathrm{Aut}_A(L) \to \mathrm{Aut}_A(L/IL)$*, acts fixed point free on* Ω_L*.*

Proof of Corollary 6.48. We first need some preparation. Consider any admissible affinoid $O \subset \Omega_L$ such that

(62) $$\forall \gamma \in \Gamma \setminus \Gamma_O \colon \gamma(O) \cap O = \emptyset, \text{ where } \Gamma_O := \{\gamma \in \Gamma |\, \gamma(O) = O\}.$$

Using (62) and Proposition 6.13 and Def.-Prop. 6.6, choose an $r_1 \in |C|$ with

(63) $$\forall \gamma \in \Gamma \setminus \Gamma_O \colon \gamma(\mathcal{U}(\Lambda, O, r_1)) \cap \mathcal{U}(\Lambda, O, r_1) = \emptyset.$$

By (63) and Example 6.18, the inclusion

$$\mathcal{U}(\Lambda, O, r_1) \to \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{U}(\Lambda, O, r_1))$$

induces an isomorphism of Grothendieck ringed spaces

(64) $$i_O \colon p_{\Gamma_O}(\mathcal{U}(\Lambda, O, r_1)) \to p_\Gamma(\mathcal{U}(\Lambda, O, r_1)).$$

Using Corollary 6.43, choose an $\varepsilon \in |C^\times|$ such that

$$q(p_{\mathring{\Gamma}}(\mathcal{U}(\Lambda, O, r_1))) \subset O \times B_\varepsilon.$$

The quotient $\Gamma_O/\mathring{\Gamma}$ is finite as $\Gamma_O/\mathring{\Gamma}_L$ is finite by Lemma 5.7 and as $\mathring{\Gamma}_L/\mathring{\Gamma}$ may naturally be embedded in the finite $\mathrm{Aut}_A(\Lambda \cap E \cdot v)$. By Proposition 6.42, thus

$$\mathcal{U}_{\mathring{\Gamma}}(O, \varepsilon) := \bigcap_{\overline{\gamma} \in \Gamma_O/\mathring{\Gamma}} \gamma(q^{-1}(O \times B_\varepsilon)) \subset \mathcal{U}_{\mathring{\Gamma}}$$

is the intersection of finitely many normal affinoid subvarieties and hence, by Proposition 2.22, itself affinoid. Moreover, it is invariant under the finite group $\Gamma_O/\mathring{\Gamma}$. Thus

$$\mathcal{U}_{\Gamma_O}(O,\varepsilon) := (\Gamma_O/\mathring{\Gamma})\backslash\mathcal{U}_{\mathring{\Gamma}}(O,\varepsilon)$$

is a normal rigid analytic variety by Proposition 2.33. Hence so is its image

$$\mathcal{U}_{\Gamma}(O,\varepsilon) := \pi(\mathcal{U}_{\mathring{\Gamma}}(O,\varepsilon)) = i_O(\mathcal{U}_{\Gamma_O}(O,\varepsilon))$$

under the isomorphism (64). As $O \subset \mathcal{U}_{\mathring{\Gamma}}(O,\varepsilon)$ and as $\Gamma_O/\mathring{\Gamma}$ is finite, Proposition 5.33 provides an $r_2 \in |C|$ with $p_{\mathring{\Gamma}}(\mathcal{U}(\Lambda,O,r_2)) \subset \mathcal{U}_{\mathring{\Gamma}}(O,\varepsilon)$. Thus

$$p_\Gamma(\mathcal{U}(\Lambda,O,r_2)) \subset \mathcal{U}_\Gamma(O,\varepsilon)$$

is a normal rigid analytic subvariety.

Part i): By means of Proposition 5.6, choose an admissible affinoid covering $\mathcal{C}$ of Ω_L such that any $O \in \mathcal{C}$ satisfies (62). By the above, choose for every $O \in \mathcal{C}$ some $r_2(O) \in |C|$ such that $p_\Gamma(\mathcal{U}(\Lambda,O,r_2(O)))$ is a normal rigid analytic variety. By Proposition 5.8, also Ω_Γ is a normal rigid analytic variety. As, by Example 6.21, $\mathcal{U}_\Gamma$ is admissibly covered by Ω_Γ and the $p_\Gamma(\mathcal{U}(\Lambda,O,r_2))$ for all $O \in \mathcal{C}$, thus $\mathcal{U}_\Gamma$ is itself a normal rigid analytic variety.

Part ii): assume that $\mathring{\Gamma} = \mathring{\Gamma}_L$ and that the action of Γ on Ω_Λ and the action of $\overline{\Gamma}_L$ on Ω_L are both fixed-point free. By Proposition 6.42, it suffices to show that π induces isomorphism on stalks. As $\mathring{\Gamma}$ is a subgroup of Γ, it also acts fixed-point free on Ω_Λ. By Corollary 5.10, thus the restriction $\Omega_{\mathring{\Gamma}} \to \Omega_\Gamma$ of π induces isomorphism on stalks. Consider then any $[l] \in \Omega_L$. Using Proposition 5.9, choose a fundamental basis $(O_n)_{n\geq 1}$ of admissible affinoid neighborhoods of $[l]$ in Ω_L such that O_n satisfies (62) and such that $\Gamma_{O_n} = \Gamma_{[l]}$ for every $n \geq 1$. For every $n \geq 1$ choose an $r_1(O_n) \in |C|$ that satisfies (63) for O_n and such that $\{r_1(O_n)\}_{n\geq 1}$ is unbounded. By Example 6.18 and Corollary 6.10, then

$$(p_{\mathring{\Gamma}}(\mathcal{U}(\Lambda,O_n,r_1(O_n))))_{n\geq 1} \text{ resp. } (p_\Gamma(\mathcal{U}(\Lambda,O_n,r_1(O_n))))_{n\geq 1},$$

is a fundamental basis of admissible neighborhoods of $p_{\mathring{\Gamma}}([l])$ in $\mathcal{U}_{\mathring{\Gamma}}$, resp. of $p_\Gamma([l])$ in $\mathcal{U}_\Gamma$. Now use the isomorphisms in (64) for all $n \geq 1$ and that

$$\forall n \geq 1\colon \Gamma_{O_n} = \Gamma_{[l]} \overset{*}{=} \mathring{\Gamma}_L = \mathring{\Gamma},$$

where $\overset{*}{=}$ holds true as $\overline{\Gamma}_L$ acts fixed-point free, in order to directly deduce that π induces an isomorphism at the stalks of $p_{\mathring{\Gamma}}([l])$ and $p_\Gamma([l])$. □

7 Compactification of analytic moduli spaces

Let C be the algebraically closed complete non-Archimedean valued field of finite characteristic from Section 6.

Notation 7.1. *Consider any admissible coefficient subring $A \subset C$ (see Definition 2.45) and any module M over its profinite completion $\hat{A}$. For any $B \in \{A, \hat{A}\}$, any B-submodule $P \subset M$ and any subgroup $G \subset \mathrm{Aut}_{\hat{A}}(M)$ set*

i) $G_P := \{g \in G |\, g(P) = P\}$,

ii) $\overline{G}_P := \{\gamma \in \mathrm{Aut}_B(P) |\, \exists g \in G_P \colon g|_P = \gamma\}$.

Definition-Proposition 7.2. *Let $\hat{\mathcal{A}}$ be the category of triples $(A, M, \mathcal{K})$ with*

i) an admissible coefficient subring $A \subset C$ (see Definition 2.45),

ii) a finitely generated free $\hat{A}$-module $M \neq 0$,

iii) a subgroup of $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$,

and whose morphisms from any $(A', M', \mathcal{K}')$ to any $(A, M, \mathcal{K})$ are the tuples (Φ, L) consisting of an injective $\hat{A}$-linear map $\Phi : M' \to M$, where $A' \supset A$, and an A-submodule $L \subset M$ such that

(a) the natural map $L \otimes_A \hat{A} \to \hat{A}L$ is an isomorphism,

(b) $\Phi(M') \oplus \hat{A}L = M$,

(c) $\Phi^*(\mathcal{K}') := \left\{k \in \mathrm{Aut}_{\hat{A}}(\Phi(M')) \middle|\, \exists k' \in \mathcal{K}' : k \circ \Phi = \Phi \circ k'\right\} \subset \overline{(\mathcal{K}_L)}_{\Phi(M')}$,

and where the composition of any such (Φ, M) with any further morphism (Ψ, L') from any $(A'', M'', \mathcal{K}'')$ to $(A', M', \mathcal{K}')$ is

$$(\Phi, L) \circ (\Psi, L') := (\Phi \circ \Psi, \Phi(L') \oplus L).$$

Proof. That this well-defines the composition of morphisms is the content of Proposition 7.15 below and is in fact directly checked. □

Ultimately, we are only interested in those triples for which $\mathcal{K}$ is a congruence subgroup of $\mathrm{Aut}_{\hat{A}}(M)$.

7.1 Structure of Grothendieck graded ringed space

Consider any object $(A, M, \mathcal{K}) \in \hat{\mathcal{A}}$. Denote by F the quotient field of A.

Definition 7.3. *An A-submodule $\Lambda \subset M$ is called an A-*structure *of M if the inclusion induces an $\hat{A}$-linear isomorphism $\Lambda_{\hat{A}} \to M$.*

Proposition 7.4. *Any A-structure of M is finitely generated projective.*

Proof. As $\hat{A}$ is a faithfully flat A-algebra, an A-module Λ is torsion-free, resp. finitely generated, if and only if $\Lambda_{\hat{A}}$ is a torsion-free, resp. finitely generated, $\hat{A}$-module (see for instance [42, 03C4]). Now use that A is a Dedekind domain. □

Definition 7.5. *Consider any A-structure Λ of M. Define the natural bijections*

i) $\Omega^*_{\{\Lambda\}} := \Omega^*_\Lambda \times \{\Lambda\} \to \Omega^*_\Lambda, ([l], \Lambda) \mapsto [l],$

ii) $\tilde{\Omega}^*_{\{\Lambda\}} := \tilde{\Omega}^*_\Lambda \times \{\Lambda\} \to \tilde{\Omega}^*_\Lambda, (l, \Lambda) \mapsto l.$

Endow $\Omega^*_{\{\Lambda\}}$ with the Grothendieck topology for which the first bijection is an isomorphism with respect to the topology of Ω^*_Λ defined in Def.-Prop. 6.7. Endow $\tilde{\Omega}^*_{\{\Lambda\}}$ with the $C^\times$-action for which the second bijection is $C^\times$-equivariant.

Definition 7.6. *Let Ω^*_M, resp. $\Omega_M \subset \Omega^*_M$, be the disjoint union of the Grothendieck topological spaces $\Omega^*_{\{\Lambda\}}$, resp. $\Omega_{\{\Lambda\}} := \Omega_\Lambda \times \{\Lambda\}$, for all A-structures Λ of M.*

Definition 7.7. *Let $\tilde{\Omega}^*_M$, resp. $\tilde{\Omega}_M \subset \tilde{\Omega}^*_M$, be the disjoint union of the $\tilde{\Omega}^*_{\{\Lambda\}}$, resp. $\tilde{\Omega}_{\{\Lambda\}} := \tilde{\Omega}_\Lambda \times \{\Lambda\}$, for all A-structures Λ of M.*

Definition-Proposition 7.8. *Set $G := \mathrm{Aut}_{\hat{A}_F}(M_F)$. For any A-structure Λ of M, any direct summand $0 \neq L \subset \Lambda$, and $l \in \tilde{\Omega}_L$ and any $g \in G$ set $g\Lambda := g(\Lambda_F) \cap M$ and $gL := g(L_F) \cap M$ and let gl be the map*

$$(gL)_C \to C, \lambda \mapsto l(g^{-1}\lambda);$$

then $g\Lambda$ is an A-structure, $gL \subset g\Lambda$ a direct summand and $gl \in \tilde{\Omega}_{gL}$. Setting

$$\forall g \in G, \forall (l, \Lambda) \in \tilde{\Omega}^*_M\colon\ g(l, \Lambda) := (gl, g\Lambda)$$

defines a $C^\times$-equivariant action of G on $\tilde{\Omega}^*_M$ whose induced action of $\mathrm{Aut}_{A_F}(\Lambda_F) \subset G$ on $\tilde{\Omega}^*_\Lambda$ coincides with the one in Def.-Prop. 6.15 for any A-structure Λ and whose induced action on Ω^*_M is continuous. Moreover, for any direct summand L of any A-structure holds that

$$\forall g \in \mathrm{Aut}_{\hat{A}}(M)\colon gL = g(L).$$

Proof. All of this is directly checked using Def.-Prop. 6.15. □

Definition 7.9. *By means of Def.-Prop- 7.8, consider the quotient map*

$$p_{\mathcal{K}}\colon \Omega_M^* \to \mathcal{K}\backslash\Omega_M^* =: \Omega_{\mathcal{K}}^*$$

and endow its target with the structure of Grothendieck topological space which it induces, that is, a subset (resp. a covering of a subset) of $\Omega_{\mathcal{K}}^*$ *is admissible precisely when its preimage is admissible.*

Denote by $\pi_{\mathcal{K}}\colon \tilde{\Omega}_M^* \to \mathcal{K}\backslash\tilde{\Omega}_M^*/C^\times = \Omega_{\mathcal{K}}^*$ the double quotient map.

Definition-Proposition 7.10. *For any admissible* $X \subset \Omega_{\mathcal{K}}^*$ *and any integer* k *let* $\mathcal{O}_{\mathcal{K}}^*(k)(X)$ *be the set of* $\mathcal{K}$*-invariant functions* $\pi_{\mathcal{K}}^{-1}(X) \to C$ *whose restriction to* $\pi_{\mathcal{K}}^{-1}(X) \cap \tilde{\Omega}_{\{\Lambda\}}^*$ *is weight* k *regular (in the sense of Def.-Prop. 6.23 via Definition 7.5, ii)) for every* A*-structure* Λ *of* M*. Then, by means of the ring structure on* C*,*

i) the $\mathcal{O}_{\mathcal{K}}^*(X) := \mathcal{O}_{\mathcal{K}}^*(0)(X)$ *for all admissible subsets* $X \subset \Omega_{\mathcal{K}}^*$ *together with the natural restriction homomorphisms form a sheaf* $\mathcal{O}_{\mathcal{K}}^*$ *of rings on* $\Omega_{\mathcal{K}}^*$*, called* structure sheaf *on* $\Omega_{\mathcal{K}}^*$*, and*

ii) for any integer k *the* $\mathcal{O}_{\mathcal{K}}^*(k)(X)$ *over all admissible* $X \subset \Omega_{\mathcal{K}}^*$ *together with the natural restriction homomorphisms form a sheaf* $\mathcal{O}_{\mathcal{K}}^*(k)$ *of* $\mathcal{O}_{\mathcal{K}}^*$*-modules on* $\Omega_{\mathcal{K}}^*$*, called* k-th twisting $\mathcal{O}_{\mathcal{K}}^*$-module *and*

iii) a sheaf $\mathcal{R}_{\mathcal{K}}^*$ *of graded* $\mathcal{O}_{\mathcal{K}}^*$*-algebras on* $\Omega_{\mathcal{K}}^*$ *is formed by the*

$$\mathcal{R}_{\mathcal{K}}^*(X) := \sum_{k\in\mathbb{Z}} \mathcal{O}_{\mathcal{K}}^*(k)(X)$$

for all admissible $X \subset \Omega_{\mathcal{K}}^*$ *and the natural restriction homomorphisms.*

In particular, $(\Omega_{\mathcal{K}}^*, \mathcal{O}_{\mathcal{K}}^*)$ *(resp.* $(\Omega_{\mathcal{K}}^*, \mathcal{R}_{\mathcal{K}}^*)$*) is a Grothendieck (graded) ringed space.*

Proof. This is directly checked. □

Remark 7.11. In fact, in Corollaries 10.2 and 10.6 below we will show the following: If $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$ is a congruence subgroup, then $(\Omega_{\mathcal{K}}^*, \mathcal{O}_{\mathcal{K}}^*)$ is a normal projective rigid analytic variety over C. If, furthermore, $\mathcal{K}$ is fine (see Definition 8.9), then $\mathcal{O}_{\mathcal{K}}^*(k)$ is an ample invertible sheaf for any $k \geq 1$.

Example 7.12. *Consider any* $\alpha \in M_F/M$ *and any integer* $k \geq 1$ *and associate with them*

$$E_{M,\alpha,k}\colon \tilde{\Omega}_M^* \to C,\ (l,\Lambda) \mapsto E_{\Lambda,\alpha,k}(l),$$

where $E_{\Lambda,\alpha,k}$ is the Eisenstein series defined in Section 6.4.1 and where α is viewed in Λ_F/Λ via the natural isomorphism $\Lambda_F/\Lambda \cong M_F/M$. If $\mathcal{K}$ fixes α, then

$$E_{M,\alpha,k} \in \mathcal{O}^*_{\mathcal{K}}(k)(\Omega^*_{\mathcal{K}}).$$

Proof. This follows directly from the construction and Proposition 6.30. □

Proposition 7.13. *Consider any complete set of representatives S of the natural $\mathcal{K}$-action on the set of A-structures of M. Then the natural maps $\Omega^*_{\Lambda} \to \Omega^*_{\{\Lambda\}} \to \Omega^*_{\{M\}}$ for all $\Lambda \in S$ induce an isomorphism of Grothendieck graded ringed spaces*

$$\coprod_{\Lambda \in S} (\Omega^*_{\overline{\mathcal{K}}_\Lambda}, \mathcal{R}^*_{\overline{\mathcal{K}}_\Lambda}) \longrightarrow (\Omega^*_{\mathcal{K}}, \mathcal{R}^*_{\mathcal{K}}) \tag{65}$$

which restricts to an isomorphism

$$\coprod_{\Lambda \in S} \Omega_{\overline{\mathcal{K}}_\Lambda} \longrightarrow \Omega_{\mathcal{K}}. \tag{66}$$

Moreover, if $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$ is a congruence subgroup, then S is finite and (66) is an isomorphism between normal rigid analytic varieties over C.

Proof. Everything except for the last part follows directly from the construction. By Proposition 5.8, $\Omega_{\overline{\mathcal{K}}_\Lambda}$ is a normal rigid analytic over C for every $\Lambda \in S$ and hence the spaces in (66) are rigid analyitic varieties over C if and only if S is finite. If $\mathcal{K}$ is a congruence subgroup of $\mathrm{Aut}_{\hat{A}}(M)$, then it contains a principal congruence subgroup of $\mathrm{Aut}_{\hat{A}}(M)$ so that S is finite by Corollary 7.28 below whose proof does not depend on this result. □

Remark 7.14. Our definition of $\Omega^*_{\mathcal{K}}$ is a coordinate-free version of the following double quotient: Let F be the quotient field of A and $G := \mathrm{Aut}_{\hat{A}_F}(M_F)$. Choose any A-structure Λ of M. Then

$$\tilde{\Omega}^*_{\Lambda} \times G \longrightarrow \tilde{\Omega}^*_{M},\ (l, g) \mapsto (gl, g\Lambda)$$

induces a natural bijection

$$\mathrm{Aut}_F(\Lambda_F) \backslash (\Omega^*_{\Lambda} \times (G/\mathcal{K})) \longrightarrow \Omega^*_{K}.$$

Moreover, any choice of F-basis of Λ_F yields an $\hat{A}_F$-basis of M_F and hence identifies $\mathrm{Aut}_F(\Lambda_F)$ with $\mathrm{GL}_d(F)$ and G with $\mathrm{GL}_d(\mathbb{A}^f_F)$, where $d := \mathrm{rank}_{\hat{A}}(M)$.

7.2 Morphisms

We first show that the composition of morphisms is a morphism. Let

$$(A'', M'', \mathcal{K}'') \xrightarrow{(\Psi, L')} (A', M', \mathcal{K}') \in \hat{\mathcal{A}} \text{ and } (A', M', \mathcal{K}') \xrightarrow{(\Phi, L)} (A, M, \mathcal{K}) \in \hat{\mathcal{A}}.$$

Proposition 7.15. *The tuple* $(\Phi \circ \Psi, \Phi(L') \oplus L)$ *satisfies Conditions (a)-(c) of Def.-Prop. 7.2, i.e., constitutes a morphism* $(A'', M'', \mathcal{K}'') \to (A, M, \mathcal{K})$.

Proof. Set $(\Theta, X) := (\Phi \circ \Psi, \Phi(L') \oplus L)$. Let us first show that (Θ, X) satisfies Conditions (a) and (b), i.e., that the natural $\hat{A}$-linear map $X_{\hat{A}} \to M$ is an isomorphism onto a direct complement of $\Theta(M'')$ in M. Conditions (a) and (b) for Φ and Ψ yield the direct sum of free $\hat{A}$-modules

$$M = \Theta(M'') \oplus \Phi(\widehat{A'} \cdot L') \oplus \hat{A} \cdot L$$

and imply that the natural map $L_{\hat{A}} \to \hat{A} \cdot L$ is an isomorphism. Via the A-linear isomorphism

$$\widehat{A'} \cong \hat{A} \otimes_A A',$$

moreover, the $\hat{A}$-linear map $\Phi(L')_{\hat{A}} \to \Phi(\widehat{A'} \cdot L')$ is the compositon

$$\hat{A} \otimes_A \Phi(L') \xrightarrow{1 \otimes \Phi^{-1}} \hat{A} \otimes_A L' \cong \widehat{A'} \otimes_{A'} L' \to \widehat{A'} L' \to \Phi(\widehat{A'} \cdot L')$$

and hence an isomorphism since, by assumption, $\widehat{A'} \otimes_{A'} L' \to \widehat{A'} \cdot L'$ is one. Conditions (a) and (b) then follow from the distributivity of tensor products over direct sums. It remains to show Condition (c), i.e., that

$$\Theta^*(\mathcal{K}'') \subset \overline{(K_X)}_{\Theta(M'')}.$$

Denote by Ψ^{-1} the inverse of the map $M'' \to \Psi(M'')$ induced by the injection Ψ and define Φ^{-1} and Θ^{-1} analogously. Let $\kappa'' \in \mathcal{K}''$. We may choose a

$$\kappa' \in (K'_{L'})_{\Psi(M'')} \text{ with } \Psi \circ \kappa'' \circ \Psi^{-1} = \kappa'|_{\Psi(M'')}$$

as $\Psi^*(\mathcal{K}'') \subset \overline{(K'_{L'})}_{\Psi(M'')}$. As $\Phi^*(\mathcal{K}') \subset \overline{(K_L)}_{\Phi(M')}$, we may further choose a

$$\kappa \in (K_L)_{\Phi(M')} \text{ with } \Phi \circ \kappa' \circ \Phi^{-1} = \kappa|_{\Phi(M')}.$$

It is directly checked that κ lies in $(K_X)_{\Theta(M'')}$ and restricts to $\Theta \circ \kappa'' \circ \Theta^{-1}$ on $\Theta(M'')$, i.e., that as desired

$$\Theta \circ \kappa'' \circ \Theta^{-1} = \kappa|_{\Theta(M'')} \in \overline{(K_X)}_{\Theta(M'')}.$$

□

Lemma 7.16. *$\Phi(\Lambda') \oplus L$ is an A-structure of M for any A'-structure L' of M'.*

Proof. Proceed as in the first part of the proof of Proposition 7.15 upon replacing X by $\Phi(\Lambda') \oplus L$ and $\widehat{A'} \cdot L'$ by M'. □

By means of Lemma 7.16, associate with (Φ, L) the map

$$\tilde{\Omega}^*_{(\Phi,L)} \colon \tilde{\Omega}^*_{M'} \to \tilde{\Omega}^*_M, (l', \Lambda') \mapsto (l' \circ \Phi_{L'}^{-1}, \Phi(\Lambda') \oplus L),$$

where $l' \in \tilde{\Omega}_{L'}$ and $\Phi_{L'} \colon L'_C \to \Phi(L')_C$ is the bijection induced by Φ.

Proposition 7.17. *The map $\tilde{\Omega}^*_{(\Phi,L)}$ induces a morphism of Grothendieck graded ringed spaces*

$$(\Omega^*_{(\Phi,L)}, \mathcal{R}^*_{(\Phi,L)}) : (\Omega^*_{\mathcal{K}'}, \mathcal{R}^*_{\mathcal{K}'}) \longrightarrow (\Omega^*_{\mathcal{K}}, \mathcal{R}^*_{\mathcal{K}})$$

which on sections of $\mathcal{R}^*_{\mathcal{K}}$ is defined by precomposition with $\tilde{\Omega}^*_{(\Phi,L)}$.

Proof. This is directly deduced from Proposition 6.27. □

Proposition 7.18.

$$(\Omega^*_{(\Phi,L)}, \mathcal{R}^*_{(\Phi,L)}) \circ (\Omega^*_{(\Psi,L')}, \mathcal{R}^*_{(\Psi,L')}) = (\Omega^*_{(\Phi,L)\circ(\Psi,L')}, \mathcal{R}^*_{(\Phi,L)\circ(\Psi,L')}).$$

Proof. It is directly checked that $\tilde{\Omega}^*_{(\Phi,L)} \circ \tilde{\Omega}^*_{(\Psi,L')} = \tilde{\Omega}^*_{(\Phi,L)\circ(\Psi,L')}$. From this, in turn, the proposition directly follows. □

Definition 7.19. *Denote by $(\Omega^*_{(\Phi,L)}, \mathcal{O}^*_{(\Phi,L)}) \colon (\Omega^*_{\mathcal{K}'}, \mathcal{O}^*_{\mathcal{K}'}) \to (\Omega^*_{\mathcal{K}}, \mathcal{O}^*_{\mathcal{K}})$ the morphism of Grothendieck ringed spaces induced by $(\Omega^*_{(\Phi,L)}, \mathcal{R}^*_{(\Phi,L)})$ via restriction of the maps of $\mathcal{R}^*_{(\Phi,L)}$ to weight zero.*

Remark 7.20. *In fact, in Corollary 10.3 below we will show the following: If $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$ and $\mathcal{K}' \subset \mathrm{Aut}_{\hat{A}'}(M')$ have finite index, then $(\Omega^*_{(\Phi,L)}, \mathcal{O}^*_{(\Phi,L)})$ is a proper morphism between rigid analytic varieties. If, furthermore, the index of $\Phi^*(\mathcal{K}') \subset \overline{\mathcal{K}}_{\Phi(M')}$ is finite, then the morphism is even finite.*

7.3 Case of principal congruence subgroups

Consider any $(A, M, \mathcal{K}) \in \hat{\mathcal{A}}$ and any ideal ideal $0 \neq I \subset A$ such that

$$\mathcal{K} = \mathrm{Ker}(\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_{\overline{A}}(\overline{M}))$$

where $\overline{A} := A/I$ and $\overline{M} := I^{-1}M/M$. More generally, associate with any A- or $\hat{A}$-module Q the $\overline{A}$-module

$$\overline{Q} := I^{-1}Q/Q.$$

Proposition 7.21. *Consider any free direct summands $N, N' \subset M$ with $\overline{N} = \overline{N'}$, any $\hat{A}$-linear isomorphism $\tau : N \to N'$ and any $\epsilon \in \mathrm{Aut}_{\overline{A}}(\overline{M})$ whose restriction to $\overline{N}$ is the isomorphism $\overline{N} \to \overline{N'}$ induced by τ. Then there exists a $\sigma \in \mathrm{Aut}_{\hat{A}}(M)$ that induces ϵ and restricts to τ.*

Proof. By means of the unique prime factorization of the non-zero ideals in the Dedekind domain A and by the Chinese remainder theorem, it is enough to show the statement of the proposition for $\hat{A}$ replaced by the $\mathfrak{p}$-adic completion $A_\mathfrak{p}$ of A at any prime ideal $\mathfrak{p} \subset A$ and for I replaced by any power $(\mathfrak{p}A_\mathfrak{p})^n$; this in turn is the statement of Lemma 2.38 in the case where $\mathcal{O}_E = A_\mathfrak{p}$. □

Corollary 7.22. *The natural morphism $\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_{\overline{A}}(\overline{M})$ is surjective.*

We further need the following consequences of Prasad's theorem [37, Theorem A] on strong approximation for semi-simple groups over function fields.

Proposition 7.23. *Let F be the quotient field of A. Consider any finitely generated projective A-module Λ. Surjective is then the natural homomorphism*

$$\mathrm{SL}_A(\Lambda) := \mathrm{Aut}_A(\Lambda) \cap \mathrm{SL}_F(\Lambda_F) \to \mathrm{SL}_{\overline{A}}(\overline{\Lambda}).$$

Proof. By Prasad's theorem [37, Theorem A], the subgroup

$$\mathrm{SL}_F(\Lambda_F) \subset \mathrm{SL}_{\hat{A}_F}(\Lambda_{\hat{A}_F})$$

is dense. Since $\mathrm{Aut}_{\hat{A}}(\Lambda_{\hat{A}})$ is open in $\mathrm{Aut}_{\hat{A}_F}(\Lambda_{\hat{A}_F})$, then also the subgroup

$$\mathrm{SL}_A(\Lambda) = \mathrm{Aut}_{\hat{A}}(\Lambda_{\hat{A}}) \cap \mathrm{SL}_F(\Lambda_F) \subset \mathrm{Aut}_{\hat{A}}(\Lambda_{\hat{A}}) \cap \mathrm{SL}_{\hat{A}_F}(\Lambda_{\hat{A}_F}) = \mathrm{SL}_{\hat{A}}(\Lambda_{\hat{A}})$$

is dense. As, by Proposition 7.21, the natural continuous group homomorphism

$$\mathrm{SL}_{\hat{A}}(\Lambda_{\hat{A}}) \to \mathrm{SL}_{\overline{A}}(\overline{\Lambda})$$

with discrete target is surjective, so is its restriction to $\mathrm{SL}_A(\Lambda)$. □

Corollary 7.24. *The determinant induces an isomorphism*

$$\mathrm{Aut}_{\overline{A}}(\overline{\Lambda}) / \mathrm{Aut}_A(\Lambda) \xrightarrow{\cong} \overline{A}^\times / A^\times. \tag{67}$$

Proposition 7.25. *Any projective module Λ of finite rank $d \geq 1$ over any Dedekind ring A admits a unique class $[J] \in \mathrm{Pic}(A)$ such that $\Lambda \cong A^{d-1} \oplus J$.*

Proof. See for instance [33, Theorems 1.32 and 1.39]. □

Corollary 7.26. *Consider any finitely generated projective A-modules L and Λ and any injective non-surjetive $\overline{A}$-linear map $\tau : \overline{L} \to \overline{\Lambda}$. Then there exists an injective A-linear map $L \to \Lambda$ onto a direct summand of Λ which induces τ.*

Proof. By the properties of τ, the rank of Λ is greater than the rank of L. By means of Proposition 7.25, we thus assume that L is a proper direct summand of Λ. Using that $\overline{L} \subsetneq \overline{\Lambda}$, we choose an extension $\rho \in \mathrm{SL}_{\overline{A}}(\overline{\Lambda})$ of τ. Proposition 7.23 then provides a desired $\sigma \in \mathrm{SL}_A(\Lambda)$ inducing ρ. □

Corollary 7.27. *Suppose that $I \subsetneq A$. Then $N \mapsto \overline{N}$ induces a bijection from the set of $\mathcal{K}$-orbits of free direct summands of M to the set of free $\overline{A}$-submodules of $\overline{M}$.*

Proof. Let $N, N' \subset M$ be free direct summands with $\overline{N} = \overline{N'}$. As $I \subsetneq A$, then
$$\mathrm{rank}_{\hat{A}}(N) = \mathrm{rank}_A(\overline{N}) = \mathrm{rank}_{\overline{A}}(\overline{N'}) = \mathrm{rank}_{\hat{A}}(N)$$
so that N and N' are $\hat{A}$-linearly isomorphic. By Corollary 7.22, the natural homomorphism $\mathrm{Aut}_{\hat{A}}(N) \to \mathrm{Aut}_{\overline{A}}(\overline{N})$ is surjective. Hence there exists an isomorphism $\tau \colon N \to N'$ inducing the identity on $\overline{N} = \overline{N'}$. By Proposition 7.21, such a τ lifts to an element in $\mathcal{K}$. This shows injectivity. Consider then any $\overline{A}$-submodule $X \subset \overline{M}$. By means of a basis of M choose a free direct summand $M' \subset M$ with
$$\mathrm{rank}_{\overline{A}}(\overline{M'}) = \mathrm{rank}_{\hat{A}}(M') = \mathrm{rank}_{\overline{A}}(X).$$
Choose an $\epsilon \in \mathrm{Aut}_A(\overline{M})$ with $\epsilon(\overline{M'}) = X$. Proposition 7.21 then provides a lift $\sigma \in \mathrm{Aut}_{\hat{A}}(M)$ of ϵ. Then $\overline{\sigma(M')} = X$ which shows surjectivity. □

Corollary 7.28. *Denote by $h(A)$ the class number of A. For any complete set S of representatives for the $\mathcal{K}$-action on the set of A-structures as in Proposition 7.13 then holds that*
$$|S| = h(A) \cdot \left| \overline{A}^{\times} / A^{\times} \right| .$$

Proof. Set $\mathcal{K}' := \mathrm{Aut}_{\hat{A}}(M)$. If any A-structures Λ, Λ' lie in the same $\mathcal{K}'$-orbit, i.e., if $\Lambda = \kappa'(\Lambda')$ for some $\kappa' \in \mathcal{K}'$, then such a κ' restricts to an A-linear isomorphism $\Lambda' \to \Lambda$. Conversely, any A-linear isomorphism $\varphi \colon \Lambda' \to \Lambda$ between any A-structures Λ, Λ' induces an automorphism
$$M \cong \Lambda'_{\hat{A}} \xrightarrow{\varphi_{\hat{A}}} \Lambda_{\hat{A}} \cong M$$

in $\mathcal{K}'$. In the case $I = A$, the corollary thus follows from Proposition 7.25. Consider any A-structure Λ. In the general case, it thus suffices to show that the number $n(\Lambda)$ of $\mathcal{K}$-orbits in the $\mathcal{K}'$-orbit of Λ equals $|\overline{A}^\times/A^\times|$. Set $\Gamma' := \mathrm{Aut}_A(\Lambda)$ and let $\Gamma \subset \Gamma'$ be its principal congruence subgroup of level I. The orbit $\mathcal{K}' \cdot \Lambda$, resp. $\mathcal{K} \cdot \Lambda$, is then in a natural bijection with $\mathcal{K}'/\Gamma'$, resp. $\mathcal{K}/\Gamma$. Via the isomorphism $\overline{\Lambda} \cong \overline{M}$ induced by $\Lambda \subset M$, then as desired

$$\begin{aligned} n(\Lambda) &= |(\mathcal{K}'/\Gamma')/(\mathcal{K}/\Gamma)| = |(\mathcal{K}'/\mathcal{K})/(\Gamma'/\Gamma)| \overset{(7.21)}{=} |\operatorname{Aut}_{\overline{A}}(\overline{M})/(\Gamma'/\Gamma)| \\ &= |\operatorname{Aut}_{\overline{A}}(\overline{\Lambda})/(\Gamma'/\Gamma)| = |\operatorname{Aut}_{\overline{A}}(\overline{\Lambda})/\Gamma'| \overset{(67)}{=} \left|(\overline{A})^\times/A^\times\right|. \end{aligned}$$

□

Suppose finally that $I \subsetneq A$ and consider any free $\overline{A}$-submodule $0 \neq W \subset \overline{M}$. For any direct summand $0 \neq N \subset M$ view Ω_N as a disjoint union of the Grothendieck ringed spaces $\Omega_{\{L\}}$ for all A-structures L of N. Consider the disjoint union of Grothendieck ringed spaces

$$\Omega_{M,W} := \coprod_{\substack{N \subset M \\ \overline{N} = W}} \Omega_N$$

being naturally acted by $\mathcal{K}$; let $\Omega_{\mathcal{K},W}$ be its quotient by $\mathcal{K}$.

Proposition 7.29. *For any free direct summand $0 \neq N \subset M$ with $\overline{N} = W$ the inclusion $\Omega_N \subset \Omega_{M,W}$ induces an isomorphism of Grothendieck ringed spaces*

$$\Omega_{\overline{\mathcal{K}}_N} \cong \Omega_{\mathcal{K},W}.$$

Proof. That it induces an injective morphism $\Omega_{\overline{\mathcal{K}}_N} \to \Omega_{\mathcal{K},W}$ follows directly from the construction. Surjectivity follows from Corollary 7.27. □

Consider further any A-structure Λ of M and identify $\overline{\Lambda}$ with $\overline{M}$ as above. Set $\Gamma := \overline{\mathcal{K}}_\Lambda$; it is the kernel of the natural homomorphism $\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_{\overline{A}}(\overline{\Lambda})$. Consider the disjoint union of Grothendieck ringed spaces

$$\Omega_{\Lambda,W} := \coprod_{\substack{L \subset \Lambda \\ \overline{L} = W}} \Omega_L$$

being naturally acted by Γ; let $\Omega_{\Gamma,W}$ be its quotient by Γ.

Proposition 7.30. *Suppose that $W \subsetneq V$. Then the injections $\Omega_L \to \Omega_{L_{\hat{A}}}, [l] \mapsto ([l], L)$ for all $L \subset \Lambda$ with $\overline{L} = W$ induce an isomorphism of Grothendieck ringed spaces $\Omega_{\Gamma,W} \cong \Omega_{\mathcal{K},W}$.*

Proof. That it induces an injective morphism $\Omega_{\Gamma,W} \to \Omega_{\mathcal{K},W}$ follows directly from the construction. Let us show that it is also surjective. Consider any direct summand $N \subset M$ with $\overline{N} = W$ and any A-structure L' of N. As $W \subsetneq V$, Corollary 7.26 provides a direct summand $0 \neq L \subsetneq \Lambda$ such that $\overline{L} = \overline{L'}$ via the canonical inclusions or identifications $\overline{L} \subset \overline{\Lambda} = \overline{M} \supset \overline{N} = \overline{L'}$ and an A-linear isomorphism $\rho\colon L \to L'$ that induces the identity map $\overline{L} \to \overline{L'}$. Proposition 7.21 then provides a $\kappa \in \mathcal{K}$ that restricts to the tensor product of ρ by $\hat{A}$ and hence restricts to ρ. Then $\kappa(\Omega_L) = \Omega_{\kappa(L)} = \Omega_{L'}$ from which surjectivity follows. $\square$

7.4 Isomorphism classes of lattices with level structures

Let $(A, M, \mathcal{K}) \in \hat{\mathcal{A}}$ and $0 \neq I \subset A$ be as in the previous Section 7.3.

Let k be any positive integer. For any A-structure Λ of M set $\Omega_{\Lambda,k} := \Omega_{\Lambda_E,k} \times \{\Lambda\}$, where E is the completion of the quotient field of A and $\Omega_{\Lambda_E,k}$ is as defined in Section 5.1; view it as a rigid analytic variety by means of Proposition 5.1. Let $\Omega_{M,k}$ be the disjoint union of the Grothendieck ringed spaces $\Omega_{\Lambda_E,k}$ for all A-structures Λ of M. Let $\Omega_{\mathcal{K},k}$ be the quotient of $\Omega_{M,k}$ by the natural action of $\mathcal{K}$. If $k = 1$, then $\Omega_{M,k} = \Omega_M$ and $\Omega_{\mathcal{K},k} = \Omega_{\mathcal{K}}$.

Proposition 7.31. *If $k > 1$, suppose that $I \subsetneq A$. Then $\Omega_{\mathcal{K},k}$ is a normal rigid analytic variety over C.*

Proof. Choose any set S of representatives of the orbits of the natural $\mathcal{K}$-action on the set of A-structures of M. The inclusions $\Omega_{\Lambda,k} \to \Omega_{\mathcal{K},k}$ for all $\Lambda \in S$ then induce an isomorphism of Grothendieck ringed spaces

$$\coprod_{\Lambda \in S} \Omega_{\overline{\mathcal{K}}_\Lambda,k} \to \Omega_{\mathcal{K},k}.$$

As any $\Omega_{\overline{\mathcal{K}}_\Lambda,k}$ is a normal rigid analytic variety by Proposition 5.8 and Example 5.11 and as S is finite by Corollary 7.28, the proposition follows. $\square$

A finitely generated projective A-submodule $Y \subset C^k$ is called an *A-lattice* if the natural homomorphism $Y_E \to E \cdot Y$ is injective. Let $d := \operatorname{rank}_{\hat{A}}(M)$. By a *level-$I$-structure* of an A-lattice $Y \subset C^k$ of rank d we mean an $\overline{A}$-linear isomorphism $i\colon \overline{M} \to \overline{Y}$. An isomorphism from any such (Y, i) to any further such tuple (Y', i') is an element $c \in C^\times$ such that multiplication by c maps Y onto Y' and such that the induced isomorphism $Y \to \overline{Y'}$ is compatible with the level structures.

For any A-structure Λ of M identify $\overline{\Lambda}$ with $\overline{M}$ via the isomorphism induced by the inclusion $\Lambda \subset M$. Corollary 7.22 essentially implies

Corollary 7.32. *A bijection between $\Omega_{\mathcal{K},k}$ and the set of isomorphism classes of A-lattices in C^k of rank d with level I-structure is induced by associating with any $([l],\Lambda) \in \Omega_{M,k}$ the class of $l(\Lambda) \subset C^k$ with level-I-structure $\bar{l}\colon \overline{M} \to \overline{l(\Lambda)}$ induced by l.*

Proof. It is directly checked for any $([l],\Lambda) \in \Omega_{M,k}$ that $(l(\Lambda),\bar{l})$ is an A-lattice with level-I-structure and that its isomorphism class depends only on its class in $\Omega_{\mathcal{K},k}$. Consider any $([l],\Lambda),([l'],\Lambda') \in \Omega_{M,k}$ whose associated isomorphism classes coincide and let us show the claim that their images in $\Omega_{\mathcal{K},k}$ coincide. Without loss of generality, we assume the representatives l,l' to be such that $l(\Lambda) = l'(\Lambda')$. Thus also the free $\hat{A}$-modules $l(\Lambda)_{\hat{A}}$ and $l'(\Lambda')_{\hat{A}}$ of rank d are equal. There exists thus a unique $\kappa \in \mathrm{Aut}_{\hat{A}}(M)$ whose composition with the tensor product $(l|_\Lambda)_{\hat{A}}$ equals $(l'|_{\Lambda'})_{\hat{A}}$ via the isomorphisms $\Lambda_{\hat{A}} \cong M \cong \Lambda'_{\hat{A}}$. As $\bar{l} = \bar{l'}$, in fact $\kappa \in \mathcal{K}$. This directly yields the claim. Consider then any A-lattice $Y \subset C^k$ of rank d with any level I-structure $i\colon \overline{M} \to \overline{Y}$. Then $Y_{\hat{A}}$ is a free $\hat{A}$-lattice of rank d and hence isomorphic to M. Further using Corollary 7.22, we choose an $\hat{A}$-linear isomorphism $\eta\colon M \to Y_{\hat{A}}$ inducing i. Then $\Lambda := \eta^{-1}(Y)$ is an A-structure of M. As Y is an A-lattice, $\eta|_\Lambda$ induces an isomorphism $\Lambda_E \to E \cdot Y$ and thus extends uniquely to a C-linear map $l\colon \Lambda_C \to C^k$ for which $\mathrm{Ker}(l) \cap \Lambda_E = 0$. Thus $[l] \in \Omega_{\Lambda,k}$ and, by construction, $(l(\Lambda),\bar{l}) = (Y,i)$ as desired. □

8 Compactifications of algebraic moduli spaces

Consider the field C from Sections 6 and 7 and any $(A, M, \mathcal{K}) \in \hat{\mathcal{A}}$ as in Definition-Proposition 7.2 such that $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$ is a congruence subgroup. Set $d := \mathrm{rank}_{\hat{A}}(M)$. Denote by F the quotient field of A. Denote by p the characteristic of F. For any $0 \neq a \in A$ set $\deg(a) := \dim_{\mathbb{F}_p}(A/(a))$. For any line bundle E over any scheme S over $\mathbb{F}_p$ denote by $\tau\colon E \to E^p, x \mapsto x^p$ the Frobenius homomorphism.

8.1 Pink's compactification

In this section we briefly recall Pink's normal algebraic Satake compactification of Drinfeld modular varieties which he introduced in [34].

Proposition 8.1. *(Drinfeld [15, Proposition 2.1]) Consider any line bundle E over any field K of characteristic p and any homomorphism*

$$\varphi\colon A \to \mathrm{End}(E), a \mapsto \varphi_a := \sum_{i \geq 0} \varphi_{a,i}\tau^i,$$

where any $\varphi_{a,i}$ is in the one-dimensional K-vector space $\Gamma(\mathrm{Spec}(K), E^{1-p^i})$ and any $\varphi_{a,0}$ is the image of a under the structure homomorphism $A \to K$. Then there exists a unique integer $r \geq 0$ such that $\varphi_{a,i} = 0$ for any $i > r \cdot \deg(a)$ and such that $\varphi_{a,r\cdot\deg(a)} \neq 0$ for any $0 \neq a \in A$ with $r \cdot \deg(a) > 0$.

Definition 8.2. *Any φ as in Proposition 8.1 with $r > 0$ is called a* Drinfeld A-module *over K of* rank r.

Let S be a scheme over F.

Definition 8.3. *(Pink [34, Definition 3.1]) A* generalized Drinfeld A-module *over S is a pair (E, φ) consisting of a line bundle E over S and a ring homomorphism*

$$\varphi\colon A \to \mathrm{End}(E), a \mapsto \varphi_a = \sum_i \varphi_{a,i}\tau^i$$

with $\varphi_{a,i} \in \Gamma(S, E^{1-p^i})$ satisfying the following conditions:

- *The derivative $d\varphi\colon A \to \varphi_{a,0}$ is the structure homomorphism $A \to \Gamma(S, \mathcal{O}_S)$.*
- *Over any point $s \in S$ the map φ defines a Drinfeld A-module of some rank $r_s \geq 1$ in the sense of Definition 8.2.*

A generalized Drinfeld A-module is of rank $\leq r$ if

$$\forall a \in A, \forall i > r \cdot \deg(a) \colon \varphi_{a,i} = 0.$$

An isomorphism of generalized Drinfeld A-modules is an isomorphism of line bundles that is equivariant with respect to the action of A on both sides.

Definition 8.4. *(Pink [34, Definition 3.2]) A generalized Drinfeld A-module over S of rank $\leq r$ with $r_s = r$ everywhere is a* Drinfeld A-module *of rank r over S.*

Lemma 8.5. *If $S = \mathrm{Spec}(R)$ is affine, then giving a Drinfeld A-module of rank r as in Definition 8.4 is equivalent to giving, as in the introduction, a ring homomorphism*

$$\varphi \colon A \to R\{\tau\}, a \mapsto \varphi_a = \sum_{0 \leq i \leq d \cdot \deg(a)} \varphi_{a,i} \tau^i$$

for which $\varphi_{a,0} = \iota(a)$, where $\iota \colon A \to R$ is the structure morphism, and for which $\varphi_{a,d\cdot\deg(a)} \in R^\times$ for any $0 \neq a \in A$.

Proof. See Pink's [34, Proposition 3.4 and its proof]. □

Notation 8.6. *For any ideal $0 \neq I \subsetneq A$ denote by $\underline{I^{-1}M/M}$ the constant group scheme over S with fibers $I^{-1}M/M$.*

Definition 8.7. *For any ideal $0 \neq I \subset A$ a* level I structure *on a Drinfeld A-module $\varphi \colon A \to \mathrm{End}(E)$ of rank d is an isomorphism of group schemes*

$$\underline{I^{-1}M/M} \longrightarrow \bigcap_{a \in I} \mathrm{Ker}(\varphi_a).$$

Lemma 8.8. *Suppose that $S = \mathrm{Spec}(R)$ is affine with structure morphism $\iota \colon A \to R$. Consider any non-zero $t \in A$ and set $V := t^{-1}M/M$. Then giving a level (t) structure on a Drinfeld A-module φ over R is is equivalent to giving, as in the introduction, a map $\lambda \colon V \to R$ for which $\lambda(V \setminus \{0\}) \subset R^\times$ and*

$$\varphi_t(T) = \iota(t) \cdot T \prod_{0 \neq v \in V} \left(1 - \frac{T}{\lambda(v)}\right) \tag{68}$$

for which the induced map $\lambda \colon V \to \mathrm{Ker}(R \xrightarrow{\varphi_t} R)$ is an A-linear isomorphism.

Proof. Consider any level (t) structure $\underline{\lambda} \colon \underline{V} \to \mathrm{Ker}(\varphi_t)$ in the sense of Definition 8.7. Then $\underline{\lambda}$ is induced by the A-linear isomorphism

$$\lambda \colon V = \underline{V}(R) \xrightarrow{\underline{\lambda}(R)} \mathrm{Ker}(R \xrightarrow{\varphi_t} R).$$

Moreover, for any maximal ideal $m \subset R$ the map $\underline{\lambda}(R_m/mR_m)$ is an isomorphism and induced by λ; thus the composition of λ with $R \to R_m/mR_m$ is injective. Thus $\lambda(V \setminus \{0\}) \subset R^\times$ and the left hand side of (68) must be a multiple in $R[T]$ of the right hand side. Moreover, both sides of (68) have the same degree in $R[T]$ and have leading coefficient in $R^\times$. Thus both sides coincide up to an element in $R^\times$. This element must be 1 since the coefficient of T of both sides equal $\iota(t)$ which is non-zero since R is over F. Thus the equality in (68) holds. Hence any level (t) structure $\underline{\lambda}$ induces a λ of the desired form. Conversely, it is directly checked that any such λ induces a unique morphism of group schemes $\underline{\lambda}\colon \underline{V} \to \mathrm{Ker}(\varphi_t)$ for which $\underline{\lambda}(R) = \lambda$. □

Denote by $X^d_{A,I}$ Drinfeld's [15, Section 5] fine moduli space over $\mathrm{Spec}(F)$ of Drinfeld A-modules of rank d with level I structure; it is an irreducible smooth affine algebraic variety of dimension $d-1$ of finite type over $\mathrm{Spec}(F)$.

Definition 8.9. *The subgroup $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$ is called fine if for some maximal ideal $\mathfrak{p} \subset A$ the image of $\mathcal{K}$ in $\mathrm{Aut}_A(p^{-1}M/M)$ is unipotent.*

Definition-Proposition 8.10. *Choose any ideal $0 \neq I \subsetneq A$ such that $\mathcal{K}$ contains the kernel $\mathcal{K}(I)$ of*

$$\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_A(I^{-1}M/M).$$

Then the natural action of $\mathcal{K}$ on level I structures induces an action on $X^d_{A,I}$ that factors through the finite group $\mathcal{K}/\mathcal{K}(I)$. Denote its quotient by

$$X^d_{A,\mathcal{K}} := (\mathcal{K}/\mathcal{K}(I))\backslash X^d_{A,I}.$$

If $\mathcal{K}$ is fine, then the universal family on $X^d_{A,I}$ descends to a Drinfeld A-module on $X^d_{A,\mathcal{K}}$ which is called the universal family *on $X^d_{A,\mathcal{K}}$. Moreover, $X^d_{A,\mathcal{K}}$ and, if $\mathcal{K}$ is fine, its universal family are, up to a natural isomorphism, independent of the choice of such I.*

Proof. See Pink's [34, (1.1)-(1.3) and Proposition 1.5]. □

Definition 8.11. *(Pink [34, Def. 3.9]) A generalized Drinfeld A-module (E,φ) over S is called* weakly separating *if for any Drinfeld A-module (E',φ') over any field L containing F, at most finitely many fibers of (E,φ) over L-valued points of S are isomorphic to (E',φ').*

Theorem 8.12. *(Pink [34, Theorem 4.2]) If $\mathcal{K}$ is fine, then there exists a normal projective algebraic variety $\overline{X}^d_{A,\mathcal{K}}$ over F together with an open embedding*

$$X^d_{A,\mathcal{K}} \to \overline{X}^d_{A,\mathcal{K}}$$

and a weakly separating generalized Drinfeld A-module $(\overline{E}, \overline{\varphi})$ *on* $\overline{X}^d_{A,\mathcal{K}}$ *extending the universal family on* $X^d_{A,\mathcal{K}}$*; moreover, such* $\overline{X}^d_{A,\mathcal{K}}$ *and* $(\overline{E}, \overline{\varphi})$ *are unique up to unique isomorphism.*

8.2 Moduli space of A-reciprocal maps

Using that A is finitely generated and that $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$ is a congruence subgroup, choose any $0 \neq t \in A$ such that A is generated by its divisors

$$\mathrm{Div}_A(t) := \{a \in A | t \in (a)\} \tag{69}$$

and $\mathcal{K} \supset \mathcal{K}(t) := \mathrm{Ker}(\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_{A/(t)}(V))$, where $V := t^{-1}M/M$.

In this section we generalize work of Pink and Schieder [36] and Pink [34, Section 7] that they did in the case where A is the polynomial ring $\mathbb{F}_q[t]$ over a finite field $\mathbb{F}_q$.

For any ideal $\mathfrak{a} \subset A$ consider the $\mathfrak{a}$-torsion submodule

$$T_{\mathfrak{a}}(V) := \{v \in V | \, \forall a \in \mathfrak{a} \colon a \cdot v = 0\} \subset V.$$

Set $T_a(V) := T_{(a)}(V)$ for any $a \in A$. For any $W \subset V$ set

$$\mathring{W} := W \setminus \{0\}.$$

With any invertible sheaf $\mathcal{L}$ on any scheme S associate the graded ring of global sections

$$R(S, \mathcal{L}) := \bigoplus_{n \geq 0} \Gamma(S, \mathcal{L}^n),$$

where any $\Gamma(S, \mathcal{L}^n)$ denotes the space of global sections of $\mathcal{L}^n$.

Definition 8.13. *A map* $\mathring{V} \to \Gamma(S, \mathcal{L})$ *is called* fibrewise non-zero, *resp.* fibrewise injective, *if for any point* $s \in S$ *the composite* $\mathring{V} \to \Gamma(S, \mathcal{L}) \to \mathcal{L} \otimes_{\mathcal{O}_S} k(s)$ *is non-zero, resp. injective.*

Definition 8.14. *Consider any invertible sheaf* $\mathcal{L}$ *on any scheme* S *over* $\mathrm{Spec}(A)$. *A map* $\rho \colon \mathring{V} \to \Gamma(S, \mathcal{L})$ *is called* A-reciprocal *if all* $a \in \mathrm{Div}_A(t)$ *and all* $v, v' \in \mathring{V}$ *satisfy that*

i) $a \cdot v \in \mathring{V} \Rightarrow \rho(v)^{|T_a(V)|} = a \cdot \rho(a \cdot v) \cdot \prod_{0 \neq w \in T_a(V)} (\rho(v) - \rho(w)),$

ii) $v + v' \in \mathring{V} \Rightarrow \rho(v) \cdot \rho(v') = \rho(v + v') \cdot (\rho(v) + \rho(v')),$

iii) there exists a ring homomorphism $\varphi^\rho\colon A \to R(S,\mathcal{L})\{\tau\} = \mathrm{End}(\mathcal{L}^{-1})$ *restricting to*

$$\mathrm{Div}_A(t) \to R(S,\mathcal{L})[T],\ a \mapsto \varphi_a(T) := a \cdot T \cdot \prod_{0 \neq l \in \rho(T_a(V))} (1 - l \cdot T).$$

Consider the polynomial ring $A_{\mathring{V}} := A[(Y_v)_{v\in\mathring{V}}]$. Let $I_{\mathring{V}} \subset A_{\mathring{V}}$ be the smallest homogeneous ideal for which

$$\sigma_V\colon \mathring{V} \to A_{\mathring{V}}, v \mapsto Y_v$$

induces an A-reciprocal map

$$\rho_V\colon \mathring{V} \to \Gamma(Q_V, \mathcal{O}_{Q_V}(1)) \subset A_{\mathring{V}}/I_{\mathring{V}},$$

where $\mathcal{O}_{Q_V}(1)$ denotes the first twisting of $Q_V := \mathrm{Proj}(A_{\mathring{V}}/I_{\mathring{V}})$. Denote by

$$\Omega_V \subset Q_V$$

the open subscheme defined as the non-vanishing locus of $\{\rho_V(v) | v \in \mathring{V}\}$.

Proposition 8.15. *The scheme Q_V, resp. Ω_V, with the universal family $(\mathcal{O}_{Q_V}(1), \rho_V)$, resp. $(\mathcal{O}_{Q_V}(1)|_{\Omega_V}, \rho_V|_{\Omega_V})$, represents the functor which associates with any scheme S over* $\mathrm{Spec}(A)$ *the set of isomorphism classes of pairs $(\mathcal{L}, \rho)$ consisting of an invertible sheaf $\mathcal{L}$ on S and a fibrewise non-zero, resp. fibrewise injective, A-reciprocal map $\rho\colon \mathring{V} \to \Gamma(S,\mathcal{L})$.*

Proof. Denote by $\mathcal{O}_{\mathring{V}}(1)$ the first twisting sheaf of $P_{\mathring{V}} := \mathrm{Proj}(A_{\mathring{V}})$. By [26, Chapter 2, Theorem 7.1], the scheme $P_{\mathring{V}}$ with the universal family $(\mathcal{O}_{\mathring{V}}(1), \sigma_V)$ represents the functor which associates with any scheme over $\mathrm{Spec}(A)$ the set of isomorphism classes of pairs $(\mathcal{L}, \rho)$ consisting of an invertible sheaf $\mathcal{L}$ on S and a fibrewise non-zero map $\rho\colon \mathring{V} \to \Gamma(S,\mathcal{L})$. The relations defining $I_{\mathring{V}}$ are precisely those that require such a ρ to be A-reciprocal. The proposition then follows by construction of Q_V and $\Omega_{\mathcal{V}}$. □

Consider any free $A/(t)$-submodule $0 \neq W \subset V$. Extending any fibrewise non-zero A-reciprocal map $\rho\colon \mathring{W} \to \Gamma(S,\mathcal{L})$ to $\mathring{V}$ by setting $\rho(v) := 0$ for any $v \in V \setminus W$ yields a fibrewise non-zero A-reciprocal map. This defines a closed embedding $\mathrm{Q}_W \to Q_V$ between the moduli schemes by means of which we identify Q_W with a closed subscheme of Q_V.

Theorem 8.16. *i) Q_V is the disjoint union of the locally closed subschemes Ω_W for all free $A/(t)$-submodules $0 \neq W \subset V$.*

ii) Consider the functor which associates with any scheme S over F the set of isomorphism classes of triples (E, φ, λ), where E is a line bundle on S and $\varphi\colon A \to \mathrm{End}(E)$ is a Drinfeld A-module of rank d over S and $\lambda\colon \underline{V} \to \mathrm{Ker}(\varphi(t))$ is a level (t)-structure. Mapping any such (E, φ, λ) to $(\mathcal{L}, \rho)$, where $\mathcal{L}$ is the inverse of the invertible sheaf on S dual to E and where $\rho\colon \mathring{V} \to \Gamma(S, \mathcal{L}), v \mapsto \frac{1}{\lambda(v)}$, induces an isomorphism of functors whose image is the functor in Proposition 8.15 represented by the pullback $\Omega_{V,F}$ of Ω_V to F.

Proof. The assertion in ii) is local in S. Consider any ring homomorphism $\iota\colon A \to R$. Via Lemma 8.5, giving a Drinfeld A-module φ of rank d over R is equivalent to giving for any $a \in \mathrm{Div}_A(t)$ a polynomial

$$\varphi_a = \sum_{0 \le i \le d\cdot\deg(a)} \varphi_{a,i}\tau^i \in R\{\tau\}$$

with $\varphi_{a,0} = \iota(a)$ and $\varphi_{a,d\cdot\deg(a)} \in R^\times$ such that

$$\mathrm{Div}_A(t) \to R\{\tau\}, a \mapsto \varphi_a \tag{70}$$

extends to a ring homomorphism $A \to R\{\tau\}$. Via Lemma 8.8, giving a level (t)-structure for such φ is equivalent to giving an injection $\lambda\colon V \to R$ for which $\lambda(\mathring{V}) \subset R^\times$ and for which (68) holds and such that

(71) $\forall a \in \mathrm{Div}_A(t)\colon \varphi_a \circ \lambda = \lambda \circ a$ and $\forall v, v' \in V\colon \lambda(v+v') = \lambda(v) + \lambda(v')$.

Let $a \in \mathrm{Div}_A(t)$. We claim that any such level (t) structure satisfies that

$$\varphi_a(T) = \iota(a) \cdot T \prod_{0 \ne v \in T_a(V)} \left(1 - \frac{T}{\lambda(v)}\right). \tag{72}$$

Indeed, (71) implies that $\lambda(T_a(V))$ is contained in the set of zeroes of φ_a. As $T_a(V)$ is a free A/a-module of rank d, moreover

$$|\lambda(T_a(V))| = |T_a(V)| = q^{d\cdot\deg(a)} = q^{\deg_\tau(\varphi_a)}.$$

Hence the left and right hand side of (72) coincide up to an element in $R^\times$. This element must be 1 since the constant coefficient of each side is $\iota(a)$ which is non-zero as R is over F. This yields the claim. From this characterization of Drinfeld A-modules over R with level (t) structure, Part ii) is directly deduced.

Consider then any $s \in Q_V$ with A-reciprocal map

$$\rho^s\colon \mathring{V} \to \mathcal{O}_{Q_V}(1) \otimes \mathcal{O}_{Q_V} k(s) =: K$$

induced by ρ. As ρ^s is non-zero by assumption, the ring homomorphism $\varphi\colon A \to K\{\tau\}$ induced by ρ^s does not coincide with the structure homomorphism $A \to K$; as K is a field, it is thus, by Proposition 8.1 a Drinfeld A-module of some rank $1 \leq d' \leq d$. Then $\mathrm{Ker}(\varphi_t)$ is a free $A/(t)$-module scheme of rank d' (see e.g. [32, Proposition 4.1]). Let

$$W := \{0\} \cup \{v \in \mathring{V} | \rho^s(v) \neq 0\}. \tag{73}$$

Properties i) and ii) in Definition 8.14 of ρ^s imply that

$$\mathring{W} \to \mathrm{Ker}(\varphi_t), w \mapsto \frac{1}{\rho^s(w)}$$

extends to an A-linear isomorphism $W \to \mathrm{Ker}(\varphi_t)$. Hence $W \subset V$ is a free $A/(t)$-submodule of rank d' and $s \in \Omega_W$. By (73), moreover, $s \notin \Omega_{W'}$ for any other free non-zero $A/(t)$-submodule $W' \subset V$. This yields Part i). $\square$

Remark 8.17. Lemma 8.8 and Theorem 8.16,ii) work more generally for schemes S over $\mathrm{Spec}(A[\frac{1}{t}])$. Thus already over $\mathrm{Spec}(A[\frac{1}{t}])$, the scheme Q_V is a compactification of Drinfeld's moduli scheme of Drinfeld A-modules of rank r with level (t) structure.

Proposition 8.18. *The pullback of Q_V to F is irreducible.*

Proof. Property v) in the proof of Theorem 9.14, which does not depend on this proposition, implies that $\Omega_V(C)$ is dense in $Q_V(C)$. Hence the pullback of Ω_V to F is dense in the pullback of Q_V to F. That the latter is irreducible, thus follows via Theorem 8.16,ii) from the irreducibility of Drinfeld's moduli scheme over F. $\square$

Definition 8.19. *A subgroup $\Delta \subset \mathrm{Aut}_A(V)$ is called* fine *if it has unipotent image in $\mathrm{Aut}_A(T_{\mathfrak{p}}(V))$ for some maximal ideal $\mathfrak{p} \subset A$ containing t.*

Proposition 8.20. *Consider any fine subgroup $\Delta \subset \mathrm{Aut}_A(V)$ by means of some maximal ideal $\mathfrak{p} \subset A$ and consider any free $A/(t)$-submodule $0 \neq W \subset V$. Then the stabilizer $\Delta_W := \{\delta \in \Delta|\ \delta(W) = W\}$ of W in Δ is a fine subgroup of $\mathrm{Aut}_A(W)$ by means of $\mathfrak{p}$ and it has a non-zero fixed point in $T_{\mathfrak{p}}(W)$, and hence in W, under the natural action.*

Proof. The first assertion is directly checked. Let us show the second assertion. The assumption that $t \in \mathfrak{p}$ implies that $T_{\mathfrak{p}}(W) \neq 0$. It then suffices to show that the image G of Δ_W in $\mathrm{Aut}_A(T_{\mathfrak{p}}(W))$ is a p-group, where p is the characteristic of A. Suppose, by contradiction, the existence of a non-trivial $g \in G$ of order k not divisible by p. Let χ, resp. m, be the characteristic, resp. minimal, polynomial of g over $A/\mathfrak{p}$ and set $r(X) := X^k - 1$. Since g is unipotent, χ is a power of $(X-1)$. Moreover, r is separable since p does not divide k. As m divides both χ and r, it thus equals $X - 1$. This implies that g is trivial and thus yields a contradiction as desired. □

Lemma 8.21. *$\mathcal{K}$ is fine if and only if its image in $\mathrm{Aut}_A(V)$ is fine.*

Proof. Denote by Δ the image of $\mathcal{K}$ in $\mathrm{Aut}_A(V)$. Suppose first that $\mathcal{K}$ is fine. Choose any maximal ideal $\mathfrak{p} \subset A$ such that the image in $\mathrm{Aut}_A(\mathfrak{p}^{-1}M/M)$ of $\mathcal{K}$ is unipotent. As $\mathcal{K} \supset \mathcal{K}(t)$, then $t \in \mathfrak{p}$. The natural morphism $T_{\mathfrak{p}}(V) \to \mathfrak{p}^{-1}M/M$ is thus an isomorphism which maps the image of Δ in $\mathrm{Aut}_A(T_{\mathfrak{p}}(V))$ onto the image of $\mathcal{K}$ in $\mathrm{Aut}_A(\mathfrak{p}^{-1}M/M)$. In particular, Δ is fine. The converse direction follows similarly from a suitable isomorphism as before. □

Proposition 8.22. *Denote by Δ the image of $\mathcal{K}$. Then the correspondence in Theorem 8.16,ii) induces an isomorphism between normal quasi-projective varieties*

$$X^d_{A,\mathcal{K}} \to \Delta\backslash\Omega_{V,F}. \tag{74}$$

Proof. Theorem 8.16,ii) provides an isomorphism $X^d_{A,\mathcal{K}(t)} \to \Omega_{V,F}$ between smooth quasi-projective varieties which is equivariant with respect to $\Delta \cong \mathcal{K}/\mathcal{K}(t)$. Its induced morphism on quotients is thus as desired. □

Denote by E_V the line bundle on Q_V dual to the minus first twisting $\mathcal{O}_{Q_V}$-module and view φ^{ρ_V} as ring homomorphism $A \to \mathrm{End}(E_V)$. Denote by $Q_{V,F}$, resp. $E_{V,F}$, resp. $\varphi^{\rho_V}_F$, the pullback of Q_V, resp. E_V, resp. φ^{ρ_V}, to F. Denote by $n\colon Q^n_{V,F} \to Q_{V,F}$ the normalization morphism. The action of Δ on Q_V induces an action on the projective variety $Q^n_{V,F}$; denote by Q^n_Δ its quotient viewed as projective algebraic variety.

The following corollary follows from Theorem 8.16,ii) in the case where $\Delta = 0$ and then by Pink's [34, Lemma 4.4 and its proof] in the general case.

Corollary 8.23. *Suppose that $\mathcal{K}$ and its image Δ are fine. Then the pullback of $(E_{V,F}, \varphi^{\rho_V}_F)$ under n descends to a weakly separating Drinfeld A-module $(E^n_\Delta, \varphi^\Delta)$ over Q^n_Δ which extends, via (74), the universal family of the open subscheme $\Delta\backslash\Omega_V \subset Q^n_\Delta$.*

Proof. Theorem 8.16 implies that $(E_{V,F}, \varphi_F^{\rho_V})$ is a weakly separating Drinfeld A-module over $Q_{V,F}$ that extends the universal family over $\Omega_{V,F}$. By construction, it is Δ-invariant. Moreover, by means of the normality of $\Omega_{V,F}$, we identify $\Omega_{V,F}$ with its preimage under n. Hence also the pullback of $(E_{V,F}, \varphi_F^{\rho_V})$ under the finite morphism n is a Δ-invariant weakly separating Drinfeld A-module over the projective scheme $Q_{V,F}^n$ that extends the universal family over $\Omega_{V,F}$. By Pink's [34, Lemma 4.4 and its proof], as Δ is fine, then the quotient E_Δ^n of the pullback of $E_{V,F}^n$ under n by Δ is a line bundle over Q_Δ^n and the pullback of $(E_{V,F}, \varphi_F^{\rho_V})$ descends to a weakly separating Drinfeld A-module $(E_\Delta^n, \varphi^\Delta)$ over the projective scheme Q_Δ^n that extends the universal family over $\Delta\backslash\Omega_{V,F}$. □

Corollary 8.24. *Suppose that $\mathcal{K}$ and its image Δ are fine. Then Q_Δ^n and $(E_\Delta^n, \varphi^\Delta)$ coincide up to unique isomorphism with $\overline{X}_{A,\mathcal{K}}^d$ and $(\overline{E}, \overline{\varphi})$ from Theorem 8.12.*

Proof. This follows from Corollary 8.23 and the uniqueness property in Theorem 8.12. □

For any subgroup $\Delta \subset \mathrm{Aut}_A(V)$ view $\Delta\backslash Q_V(C)$ with its structure of projective rigid analytic variety. For any integer k denote by $\mathcal{O}(k)$ the analytifcation of the pullback of the k-th twisting $\mathcal{O}_{Q_V}$-module to $Q_V(C)$ under $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$. Pink's [34, Lemma 4.4 and its proof] inspired

Proposition 8.25. *Consider any fine subgroup $\Delta \subset \mathrm{Aut}_A(V)$ and any integer k. Then the subsheaf of Δ-invariants of $\mathcal{O}(k)$ is an ample invertible sheaf on the projective rigid analytic variety $\Delta\backslash Q_V(C)$ and its pullback to $Q_V(C)$ is $\mathcal{O}(k)$.*

Proof. Let $\mathcal{O}_\Delta(k)$ denote the subsheaf of $\mathcal{O}(k)$ of Δ-invariants. For any free $A/(t)$-submodule $0 \neq W \subset V$ consider the Zariski open subset $U_W \subset Q_V$ defined as the union of the $\Omega_{W'}$ for all free $A/(t)$-submodules $W \subset W' \subset V$. Choose then a $1 > \varepsilon \in |C^\times|$. For any such W consider the admissible subset

$$U(W,\varepsilon) := \left\{ [(y_\alpha)_{\alpha \in \mathring{V}}] \in U_W(C) \mid \forall \alpha \in \mathring{W}, \forall \beta \in V \setminus W \colon \left|\frac{y_\beta}{y_\alpha}\right| \leq \varepsilon \right\} \subset Q.$$

By Theorem 8.16,i) and since $\Omega_W(C) \subset U(W,\varepsilon)$ for any such W, the rigid analytic variety Q is covered by the $U(W,C)$ for all such W. As this covering is finite, it is admissible. As $\varepsilon < 1$, it holds that $U(W,\varepsilon) \cap U(W',\varepsilon) = \emptyset$ for any free submodules $W, W' \subset V$ with $W \not\subset W' \not\subset W$. Moreover, $g(U(W,\varepsilon)) = U(g(W),\varepsilon)$ for any such W and any $g \in \mathrm{Aut}_A(W)$. Consequently, any such $U(W,\varepsilon)$ is invariant under $\Delta_W := \{\delta \in \Delta | \delta(W) = W\}$

and satisfies that $\delta(U(W,\varepsilon)) \cap U(W,\varepsilon) = \emptyset$ for any $\delta \in \Delta \setminus \Delta_W$. In order to see that $\mathcal{O}_\Delta(k)$ is an invertible sheaf, it thus suffices to to show that for any such W the subsheaf $\mathcal{O}_W(k)$ of Δ_W-invariants of the restriction of $\mathcal{O}(k)$ to $U(W,\varepsilon)$ is an invertible sheaf on $\Delta_W \backslash U(W,\varepsilon)$. Consider such a W. Then Proposition 8.20 provides an $0 \neq \alpha \in W$ that is fixed by Δ_W. For such an α the restriction of the global section $(Y_\alpha)^k$ to $\mathcal{O}_W(k)$ thus induces a nowhere vanishing global section in the quotient $\Delta_W \backslash \mathcal{O}_W(k)$ and hence yields a trivialization of it as desired. Let $\mathcal{F}$ denote the pullback of $\mathcal{O}_\Delta(k)$ under the quotient morphism. Using the above trivialization, it is directly checked that the natural morphism $\mathcal{F} \to \mathcal{O}(k)$ of $\mathcal{O}(0)$-modules is an isomorphism. Having the ample pullback $\mathcal{O}(k)$ under the finite quotient map, the invertible sheaf $\mathcal{O}_\Delta(k)$ is itself ample by [28, Chapter 1, Proposition 4.4.] via Köpf's GAGA-Theorem [31, Satz 5.1]. $\square$

9 Comparison of algebraic and analytic compactifications

Let $A \subset C$ be as in Sections 6, 7, 8. Let $0 \neq t \in A$ be such that $\mathrm{Div}_A(t)$ generates A as in Section 8.2. Consider any free $\hat{A}$-module $M \neq 0$ of finite rank, set $V := t^{-1}M/M$ and $\mathring{V} := V \setminus \{0\}$ and let $\mathcal{K}$ be the kernel of the natural homomorphism $\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_A(V)$. Consider the closed subvariety

$$Q := Q_V(C) \subset P := \mathrm{Proj}(C[(Y_v)_{v\in\mathring{V}}])$$

provided by Section 8.2 with its structure of reduced rigid analytic variety over C. By Theorem 8.16,i), Q is stratified by the locally closed subvarieties $\Omega_W(C)$ for all free $A/(t)$-submodules $0 \neq W \subset V$; , any $\Omega_W(C)$ is the intersection of the non-vanishing locus in Q of $(Y_\alpha)_{0\neq\alpha\in W}$ with the vanishing locus in Q of $(Y_\alpha)_{\alpha\in V\setminus W}$. Set $\Omega := \Omega_V(C)$. Recall the Eisenstein series $E_\alpha := E_{M,\alpha,1}$ for all $\alpha \in \mathring{V}$ from Example 7.12.

Theorem 9.1. *The $(E_\alpha)_{\alpha\in\mathring{V}}$ define a morphism of Grothendieck ringed spaces $E_{\mathcal{K}}\colon \Omega^*_{\mathcal{K}} \to Q$ which is the normalization morphism (in the sense of Conrad's [12]) of Q and restricts to Drinfeld's isomorphism $\Omega_{\mathcal{K}} \to \Omega$ between normal rigid analytic varieties. Moreover, the morphism of Grothendieck topological spaces underlying $E_{\mathcal{K}}$ restricts to isomorphisms between irreducible components.*

We prove Theorem 9.1 at the end of this section. We first recall Drinfeld's correspondence between level structures of A-lattices and level structures of Drinfeld A-modules as well as the induced isomorphism between moduli spaces.

Theorem 9.2. *(Drinfeld [15, Proposition 3.1]) Consider any integer $d \geq 1$ and any ideal $0 \neq I \subset A$. For any A-lattice $Y \subset C$ of rank d with level I-structure $i\colon (I^{-1}/A)^d \to I^{-1}Y/Y$ (see Corollary 7.32) the map*

$$\varphi\colon A \to C\{\tau\}, a \mapsto \varphi_a := a \cdot T \cdot \prod_{0\neq[x]\in I^{-1}Y/Y} (1 - \frac{T}{e_Y(x)})$$

is a Drinfeld module of rank r with level I-structure

$$(I^{-1}A/A)^d \to I^{-1}Y/Y, v \mapsto e_Y(i(v))$$

satisfying $e_Y \circ a = \varphi_a \circ e_Y$ for any $a \in A$. This induces a bijection from the set of isomorphism classes of A-lattices in C of rank d with I-level structure to the set of rank d Drinfeld A-modules over C with level I-structure.

Proposition 9.3. *(Drinfeld [15, Prop. 6.6]) The rule*

$$\Omega_{\mathcal{K}} \to \Omega, \pi_{\mathcal{K}}(l, \Lambda) \mapsto \left[(E_\alpha(l, \Lambda))_{\alpha \in \mathring{V}}\right] \tag{75}$$

defines an isomorphism of rigid analytic varieties over C.

Proof. We provide some details of Drinfeld's proof. By Def.-Prop. 2.55, for any $\alpha \in \mathring{V}$ and any lift $\tilde{\alpha} \in t^{-1}\Lambda$ of α holds that

$$\forall (l, \Lambda) \in \tilde{\Omega}_M \colon E_\alpha(l, \Lambda) = \frac{1}{e_{l(\Lambda)}(l(\tilde{\alpha}))}.$$

From Proposition 7.32 and Theorem 9.2 applied to the case $I = (t)$ and from Theorem 8.16,ii) thus follows that the rule in (9) defines a bijective morphism

$$E \colon \Omega_{\mathcal{K}} \to \Omega_V(C) =: \Omega.$$

Drinfeld uses the criterion given by Proposition 2.24 in order to show that E is an isomorphism. By the criterion it remains to be shown that E induces isomorphisms at the stalks and that there exists an admissible affinoid covering $(X_j)_{j \in J}$ of Ω such that $E^{-1}(X_j)$ is an admissible quasi-compact subset for any $j \in J$. That E induces isomorphisms at the stalks follows from Part b) of Drinfeld's proof to which we refer. Consider any A-structure Λ of M and set $\Gamma := \overline{\mathcal{K}}_\Lambda$. By Proposition 7.13, it suffices to show the second property for E replaced by its restriction E' to Ω_Γ. Identify $\overline{\Lambda} := t^{-1}\Lambda/\Lambda$ with V via the isomorphism induced by the inclusion $t^{-1}\Lambda \subset t^{-1}M$. By construction,

$$\Gamma = \mathrm{Ker}(\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(\overline{\Lambda}))$$

and E' is the morphism defined by the Eisenstein series $E_\alpha := E_{\Lambda,\alpha,1}$ for all $0 \neq \alpha \in \overline{\Lambda}$. Consider the admissible affinoid covering of Ω by the

$$X_n := \left\{ [(y_\alpha)_{\alpha \in \mathring{V}}] \in \Omega \;\middle|\; \forall \alpha, \beta \in \mathring{V} \colon \left|\frac{y_\alpha}{y_\beta}\right| \leq n \right\} \subset \Omega$$

for all positive integers n. Consider any $n \geq 1$. Proposition 9.4 below implies that $X := E'^{-1}(X_n)$ is contained in some admissible quasi-compact subset $X' \subset \Omega_\Gamma$. As E_β/E_α restricts to a regular function on Ω_Γ for any $0 \neq \alpha, \beta \in \overline{\Lambda}$ by Proposition 6.30, thus X is an admissible quasi-compact subset of X' and hence of Ω_Γ. □

Proposition 9.4. *Let Γ be the image in $\mathcal{PG}$ of the kernel in Example 5.11 supposing that $I \subsetneq A$. Consider any subset $X \subset \Omega_\Gamma$. Then the following are equivalent:*

i) X is contained in a quasi-compact admissible subset.

ii) For any $0 \neq \alpha, \beta \in I^{-1}\Lambda \backslash \Lambda$ the function

$$\Omega_\Gamma \to C, \pi_\Gamma(l) \mapsto \frac{E_{\Lambda,\beta,1}(l)}{E_{\Lambda,\alpha,1}(l)}$$

is bounded on X.

iii) For any $0 \neq \alpha, \beta \in I^{-1}\Lambda \backslash \Lambda$ the function

$$\Omega_\Gamma \to |C|, \;\; \pi_\Gamma(l) \mapsto \frac{\min\{|l(v)| \mid v \in \beta\}}{\min\{|l(w)| \mid w \in \alpha\}}$$

is bounded on X.

Proof. That i) implies ii) follows directly from the Maximmum Modulus Principle (see Proposition 2.25). That ii) implies iii) follows from Proposition 2.56 and since, by Def.-Prop. 2.55, for any $0 \neq \alpha \in I^{-1}\Lambda/\Lambda$ and any lift $\tilde{\alpha} \in I^{-1}\Lambda$ of α holds that

$$\forall l \in \tilde{\Omega}_\Lambda \colon E_{\Lambda,\alpha,1}(l) = \frac{1}{e_{l(\Lambda)}(l(\tilde{\alpha}))}.$$

That iii) implies i) follows from Drinfeld's [15, Prop. 6.5] and Cor. 5.3. □

Corollary 9.5. *Consider any A-structure Λ of M, set $\Gamma := \overline{\mathcal{K}}_\Lambda$ and view Ω_Γ^* as a subspace of $\Omega_{\mathcal{K}}^*$ via Prop. 7.13. Consider any free $A/(t)$-submodule $0 \neq W \subsetneq V$ and recall the rigid analyitic variety $\Omega_{\Gamma,W}$ defined before Proposition 7.30. Then*

$$\Omega_{\Gamma,W} \to \Omega_W(C), \pi_\Gamma(l) \mapsto \left[(E_\alpha(l,\Lambda))_{\alpha \in \mathring{W}} \right]$$

defines an isomorphism of rigid analytic varieties.

Proof. By means of Corollary 7.27, choose a free direct summand $N \subset M$ such that $t^{-1}N/N = W$. By Proposition 9.3,

$$\Omega_{\overline{\mathcal{K}}_N} \to \Omega_W(C), \pi_{\overline{\mathcal{K}}_N}(l,L) \mapsto \left[(E_{\alpha,N,1}(l,L))_{\alpha \in \mathring{W}} \right]$$

defines an isomorphism of rigid analytic varieties. It is directly checked that the precomposition of this isomorphism with the isomorphisms $\Omega_{\Gamma,W} \to \Omega_{\mathcal{K},W}$ and $\Omega_{\mathcal{K},W} \to \Omega_{\overline{\mathcal{K}}_N}$ provided by Propositions 7.29 and 7.30 defines the desired isomorphism. □

Choose a finite set of representatives $\{\Lambda_i\}_{i\in I}$ of the orbits of the $\mathcal{K}$-action on the set of A-structures of M and recall from Proposition 7.13 the isomorphism

$$\coprod_{i\in I}\Omega^*_{\Gamma_i} \to \Omega^*_{\mathcal{K}},$$

where $\Gamma_i := \overline{\mathcal{K}}_{\Lambda_i}$ for every $i \in I$. For any $i \in I$ denote by Ω_i the image of Ω_{Γ_i} under the isomorphim $\Omega_{\mathcal{K}} \to \Omega$ between normal rigid analytic varieties in Proposition 9.3. By Corollary 5.18, the Ω_{Γ_i} are the irreducible components of $\Omega_{\mathcal{K}}$ and hence the Ω_i are the irreducible components of Ω.

Definition 9.6. *Set $Q_i := \Omega_i \cup (Q \setminus \Omega) \subset Q$ for any $i \in I$.*

Lemma 9.7. *Any $Q_i \subset Q$ is Zariski-closed.*

Proof. By Proposition 7.13, any Ω_{Γ_i} is Zariski closed and open in $\Omega_{\mathcal{K}}$. Hence any Ω_i is Zariski closed and open in Ω. As, furthermore, Ω is Zariski open in Q, thus any Q_i is Zariski closed in Q. □

Proposition 9.8. *For any $i \in I$ the rule*

$$E_i\colon \Omega^*_{\Gamma_i} \to Q_i, \pi_{\Gamma_i}(l) \mapsto (E_\alpha(l, \Lambda_i))_{\alpha\in\mathring{V}}$$

defines an isomorphism of Grothendieck topological spaces which restricts to a map $\Omega_{\Gamma_i,W} \to \Omega_W(C)$ underlying an isomorphism of rigid analytic varieties for any free $A/(t)$-submodule $0 \neq W \subsetneq V$.

We will prove Prop. 9.8 before Cor. 9.12 after more preparation. However, we may already see that the rule $\pi_{\Gamma_i}(l) \mapsto (E_\alpha(l, \Lambda_i))_{\alpha\in\mathring{V}}$ defines a bijective map $E_i\colon \Omega^*_{\Gamma_i} \to Q_i$ which restricts to isomorphisms $\Omega_{\Gamma_i,W} \to \Omega_W(C)$ of rigid analytic varieties: Indeed, this follows from Corollary 9.5 as well as the facts that $Q \setminus \Omega$, resp. $\Omega^*_{\Gamma_i} \setminus \Omega_{\Gamma_i}$, is the disjoint union the $\Omega_W(C)$ for all such $0 \neq W \subsetneq V$ by Theorem 8.16, resp. of the $\Omega_{\Gamma_i,W}$ by construction.

Let us recall the content of Proposition 3.4 in the present setup. Consider any subset $T \subset \mathring{V}$ and any $\varepsilon \in |C^\times|$ and associate with it the Zariski open, resp. admissible, resp. Zariski closed subvariety

$$\begin{aligned}
\mathcal{U}(T) &:= \{[(y_\alpha)_{\alpha\in\mathring{V}}] \in P \mid \forall\alpha \in T : y_\alpha \neq 0\} \subset P,\\
\mathcal{U}(T,\varepsilon) &:= \{[(y_\alpha)_{\alpha\in\mathring{V}}] \in \mathcal{U}(T) \mid \forall\alpha' \in \mathring{V} \setminus T,\ \forall\alpha \in T : \left|\frac{y_{\alpha'}}{y_\alpha}\right| \leq \varepsilon\} \subset \mathcal{U}(T),\\
\Omega(T) &:= \{[(y_\alpha)_{\alpha\in\mathring{V}}] \in \mathcal{U}(T) \mid \forall\alpha' \in \mathring{V} \setminus T,\ \forall\alpha \in T : \frac{y_{\alpha'}}{y_\alpha} = 0\} \subset \mathcal{U}(T).
\end{aligned}$$

Then $\Omega(T) \neq \emptyset \Leftrightarrow T \neq \emptyset$; in this case, denote by $\rho_T\colon \mathcal{U}(T) \to \Omega(T)$ the natural projection morphism and for any $O \subset \Omega(T)$ set

$$\mathcal{U}(O, \varepsilon) := \rho_T^{-1}(O) \cap \mathcal{U}(T, \varepsilon).$$

Proposition 9.9. *Consider any closed subvariety $P' \subset P$. Then a subset $X \subset P'$ is admissible if and only if for any $T \subset \mathring{V}$ with $\Omega(T) \cap P' \neq \emptyset$:*

i) the subset $X \cap \Omega(T) \subset P' \cap \Omega(T)$ is admissible and

ii) any admissible quasi-compact $O \subset \Omega(T)$ with $O \cap P' \subset X$ admits an $\varepsilon > 0$ such that $\mathcal{U}(O, \varepsilon) \cap P' \subset X$.

A covering of an admissible subset $X \subset P'$ by admissible subsets is admissible if and only if its intersection with $X \cap \Omega(T)$ is admissible for any $T \subset \mathring{V}$.

Proof. The present setup is a special case of Example 3.3. Hence the proposition is an instance of Proposition 3.4. □

Proposition 9.10. *Let $T \subset \mathring{V}$. If $T = \mathring{W}$ for some free $A/(t)$-submodule $0 \neq W \subset V$, then $\Omega(T) \cap Q = \Omega_W(C)$. Otherwise, $\Omega(T) \cap Q = \emptyset$. Moreover, $\Omega(\mathring{V}) \cap Q_i = \Omega_i$ for any $i \in I$.*

Proof. This follows directly from Theorem 8.16,i). The last assertion follows directly from the definition of the Q_i. □

For any $i \in I$ and any free $A/(t)$-submodule $0 \neq W \subset V$ denote by $\mathrm{Orb}(i, W)$ the finite set of orbits $\mathfrak{O}$ of the Γ_i-action on the set of direct summand $L \subset \Lambda_i$ for which $t^{-1}L/L = W$. In the notation of Def.-Prop. 6.22 and Proposition 7.30, for any $i \in I$ we have a disjoint union

$$\Omega_{\Gamma_i, W} = \coprod_{\mathfrak{O} \in \mathrm{Orb}(i,W)} \Omega_{\mathfrak{O}} \tag{76}$$

of rigid analytic varieties and for any $Y \subset \Omega_{\Gamma_i, W}$ and any $r \in |C|$ we set

$$\mathcal{U}(\Lambda_i, Y, r) := \bigcup_{\mathfrak{O} \in \mathrm{Orb}(i,W)} \mathcal{U}(\Lambda_i, Y \cap \Omega_{\mathfrak{O}}, r) \subset \Omega^*_{\Gamma_i}. \tag{77}$$

Lemma 9.11. *Consider any $i \in I$, any free $A/(t)$-submodule $0 \neq W \subsetneq V$, any admissible quasi-compact $O \subset \Omega(\mathring{W})$ and any finite field $\mathbb{F}_q \subset A$ with q elements. Then there exist $c, r_O > 0$ such that for any $r_O < r \in |C|$:*

$$E_i^{-1}(\mathcal{U}(O, r^{-r \cdot q \cdot \mathrm{rank}_{\mathbb{F}_q[t]}(\Lambda)})) \subset \mathcal{U}(\Lambda_i, E_i^{-1}(O), r) \subset E_i^{-1}(\mathcal{U}(O, \frac{c}{r}))$$

Proof. Using the quasi-compactness of O, choose a $c > 1$ such that

$$\forall [y] = [(y_\beta)_{\beta \in \mathring{V}}] \in O, \forall \alpha, \alpha' \in \mathring{W}\colon \ \left| \frac{y_{\alpha'}}{y_\alpha} \right| \leq c.$$

Choose a basis $a_1, \ldots, a_k$ of the $\mathbb{F}_q[t]$-module A and set

$$c' := c \cdot \max_{1 \leq i \leq k} |a_i|.$$

Using Cor. 2.29, choose a $\delta > 0$ such that for any $[y], [z] \in \Omega(\mathring{W})$:

$$\left[\forall \alpha, \alpha' \in \mathring{W}\colon \ \left| \frac{y_{\alpha'}}{y_\alpha} - \frac{z_{\alpha'}}{z_\alpha} \right| < \delta \right] \Rightarrow [[y] \in O \Leftrightarrow [z] \in O]\,. \tag{78}$$

Set $r_O := \max\{c', \frac{c^2}{\delta}\}$. Consider any $r_O < r \in |C|$ and set $\varepsilon := r^{-r \cdot \operatorname{rank}_{\mathbb{F}_q[t]}(\Lambda)}$. Set $\Lambda := \Lambda_i$ and $\Gamma := \Gamma_i$ and $E := E_i$. Denote by $\pi\colon t^{-1}\Lambda \to V$ the quotient morphism. Moreover, for any subset $S \subset C$ set as before

$$\mathrm{d}(S) := \inf_{0 \neq s \in S} |s|.$$

Consider any $l \in \tilde{\Omega}^*_\Lambda$, say $l \in \tilde{\Omega}_L$. Set $\mathcal{L} := l(L)$ and $n := \operatorname{rank}_{\mathbb{F}_q[t]}(\mathcal{L})$. Choose an $x_l \in t^{-1}\mathcal{L} \setminus \{0\}$ of minimal norm and let $\alpha_l := \pi(l^{-1}(x_l))$. Proposition 2.56 then yields for any further $x' \in t^{-1}\mathcal{L}$ which is non-zero modulo $\mathcal{L}$ and of minimal norm in $x' + \mathcal{L}$, with $\alpha' := \pi(l^{-1}(x'))$, that

$$\left| \frac{x'}{x_l} \right| \leq \left| \frac{e_{\mathcal{L}}(x')}{e_{\mathcal{L}}(x_l)} \right| = \left| \frac{E_{\alpha_l}(l)}{E_{\alpha'}(l)} \right| \leq \left| \frac{x'}{x_l} \right|^{\left|\frac{x'}{x_l}\right| \cdot q \cdot n}. \tag{79}$$

Suppose first that $\pi_\Gamma(l) \in E^{-1}(\mathcal{U}(O, \varepsilon))$. Then $\alpha_l \in W$; indeed, if α_l was not in W, then we could choose an x' and α' as in (79) with $\alpha' \in \mathring{W}$ and apply the assumption that then $\left| \frac{E_{\alpha_l}(l)}{E_{\alpha'}(l)} \right| \leq \varepsilon < 1$ contradicting the fact that $|x_l| \leq |x'|$ via the first inequality of (79). Consider then any x'_1 and α'_1, resp. x'_2 and α'_2, as in (79) such that $\alpha'_1 \in \mathring{W}$, resp. $\alpha'_2 \notin \mathring{W}$. The first, resp. second, inequality of (79) then yields that

$$\left| \frac{x'_1}{x_l} \right| \leq \left| \frac{E_{\alpha_l}(l)}{E_{\alpha'_1}(l)} \right| \leq c, \ \text{ resp. } \ r^{r \cdot q \cdot n} \leq \frac{1}{\varepsilon} \leq \left| \frac{E_{\alpha_l}(l)}{E_{\alpha'_2}(l)} \right| \leq \left| \frac{x'_2}{x_l} \right|^{\left|\frac{x'_2}{x_l}\right| \cdot q \cdot n}, \tag{80}$$

and, in particular, that $|x'_1| < |x'_2|$ since $r > c'$. We have thus verified condition (6) of Corollary 2.54 in the following case: Let $x_1, \ldots, x_n \in t^{-1}\mathcal{L}$ be a minimal reduced $\mathbb{F}_q[t]$-basis of $t^{-1}\mathcal{L}$ and let $L' \subset \Lambda$ be the $\mathbb{F}_q[t]$-submodule generated by the $t \cdot l^{-1}(x_i)$ for all x_i with $|x_i| < d := \mathrm{d}(l(\pi^{-1}(V \setminus W)) \cap L)$. Then $t^{-1}L'/L' = W$ and $\mathrm{d}(l(t^{-1}L \setminus t^{-1}L']) = d$ by Corollary 2.54. Hence

$$\frac{\mathrm{d}(l(L \setminus L'))}{\mathrm{d}(l(L'))} = \frac{\mathrm{d}(t^{-1}l(L \setminus L'))}{\mathrm{d}(t^{-1}l(L'))} = \frac{d}{|x_l|} \overset{(80)}{\geq} r. \tag{81}$$

In fact, $L' \subset \Lambda$ is an A-submodule and, as such, a direct summand: Indeed, the first inequality of (80) and the definition of c' and r imply that

$$\forall 1 \leq j \leq k, 1 \leq i \leq n\colon \ \ |a_j \cdot x_i| \leq c' \cdot |x_l| < r \cdot |x_l| \leq d = \mathrm{d}(l(t^{-1}L \setminus t^{-1}L'))$$

and thus, as $t^{-1}\mathcal{L}$ is an A-module, that $a_j \cdot x_i \in t^{-1}l(L')$ for any such i, j. The basis property of bot the a_j and the x_i then yields that $t^{-1}l(L') \subset t^{-1}\mathcal{L}$ and hence $L' \subset L \subset \Lambda$ are A-submodules. Moreover, as $L' \subset L$ and $L \subset \Lambda$ are direct summands as $\mathbb{F}_q[t]$-submodules, the quotient Λ/L' is torsion-free as $\mathbb{F}_q[t]$-module and hence also as A-module. In particular, Λ/L' is a projective A-module. The short exact sequence $0 \to L' \to \Lambda \to \Lambda/L' \to 0$ thus splits; equivalently, the A-submodule $L' \subset \Lambda$ is a direct summand.

Set $l' := l|_{L'_C}$. Let $\mathfrak{O}$ be the Γ-orbit of L'. As argued above, $t^{-1}L'/L' = W$. Hence $\mathfrak{O} \in \mathrm{Orb}(i, W)$. We claim that $E(\pi_\Gamma(l')) \in O$ and hence, in view of (81), that $[l] \in \mathcal{U}(\Lambda, p_\Gamma^{-1}(E^{-1}(O)) \cap \Omega_{L'}, r)$ so that as desired

$$\pi_\Gamma(l) \in \mathcal{U}(\Lambda, E^{-1}(O), r).$$

For the claim, it suffices, by (78) and since $\rho_{\mathring{W}}(E(\pi_\Gamma(l))) \in O$, to show that

$$\forall \alpha, \alpha' \in \mathring{W}\colon \ \ \left| \frac{E_{\alpha'}(l)}{E_\alpha(l)} - \frac{E_{\alpha'}(l')}{E_\alpha(l')} \right| < \delta.$$

For any $\beta \in \mathring{W}$ set

$$E_\beta = E_\beta(l) \text{ and } E'_\beta := E_\beta(l') \text{ and } \epsilon_\beta := E_\beta - E'_\beta = \sum_{\lambda \in l(\pi^{-1}(\beta) \cap L \setminus t^{-1}L')} \frac{1}{\lambda}. \tag{82}$$

We then have for any $\alpha, \alpha' \in \mathring{W}$ that

$$\left| \frac{\epsilon_{\alpha'}}{E_\alpha} \right| \overset{(81)}{\leq} \frac{1}{r \cdot |x_l| \cdot |E_\alpha|} = \frac{1}{r} \cdot \left| \frac{e_{\mathcal{L}}(x_l)}{x_l} \right| \cdot \left| \frac{E_{\alpha_l}}{E_\alpha} \right| \leq \frac{1}{r} \cdot \prod_{0 \neq \lambda \in \mathcal{L}} \left| \frac{x_l + \lambda}{\lambda} \right| \cdot c \overset{|x_l| \leq |\lambda|}{=} \frac{c}{r}$$

and hence as desired that

$$\left| \frac{E_{\alpha'}}{E_\alpha} - \frac{E'_{\alpha'}}{E'_\alpha} \right| = \left| \frac{E_{\alpha'}}{E_\alpha} - \frac{E_{\alpha'} - \epsilon_{\alpha'}}{E_\alpha - \epsilon_\alpha} \right| = \left| \frac{\frac{\epsilon_{\alpha'}}{E_\alpha} - \frac{E_{\alpha'}}{E_\alpha} \cdot \frac{\epsilon_\alpha}{E_\alpha}}{1 - \frac{\epsilon_\alpha}{E_\alpha}} \right|$$

$$\overset{1 > \frac{c}{r}}{=} \left| \frac{\epsilon_{\alpha'}}{E_\alpha} - \frac{E_{\alpha'}}{E_\alpha} \cdot \frac{\epsilon_\alpha}{E_\alpha} \right| \leq \frac{c^2}{r} < \delta.$$

This shows the claim and hence the first inclusion stated in the lemma.

Conversely, assume that $\pi_\Gamma(l) \in \mathcal{U}(\Lambda, E^{-1}(O), r)$. Thus

$$[l] \in \mathcal{U}(\Lambda, p^{-1}(E^{-1}(O)) \cap \Omega_{L'}, r)$$

for some $L' \in \mathfrak{O} \in \mathrm{Orb}(i, W)$. Choose such an L' and set $l' := l|_{L'_C} \in \tilde{\Omega}_{L'}$ and define E_β, E'_β and ϵ_β for any $\beta \in \mathring{W}$ as in (82). Using (78) similarly as before, we shall first show that $\rho_{\mathring{W}}(E(\pi_\Gamma(l))) \in O$. The assumption implies that $E(\pi_\Gamma(l')) \in O$ and, as $r > 1$, that $x_l \in l(t^{-1}L')$. For any $\alpha, \alpha' \in \mathring{W}$ thus follows that

$$\begin{aligned} \left|\frac{\epsilon_{\alpha'}}{E'_\alpha}\right| &\leq \frac{1}{r \cdot |x_l| \cdot |E'_\alpha|} = \frac{1}{r} \cdot \left|\frac{e_{l'(L')}(x_l)}{x_l}\right| \cdot \left|\frac{E'_{\alpha_l}}{E'_\alpha}\right| \\ &\leq \frac{1}{r} \cdot \prod_{0 \neq \lambda \in l'(L')} \left|\frac{x_l + \lambda}{\lambda}\right| \cdot c \overset{|x_l| < |\lambda|}{=} \frac{c}{r} \end{aligned}$$

and hence that

$$\begin{aligned} \left|\frac{E'_{\alpha'}}{E'_\alpha} - \frac{E_{\alpha'}}{E_\alpha}\right| &= \left|\frac{E'_{\alpha'}}{E'_\alpha} - \frac{E'_{\alpha'} + \epsilon_{\alpha'}}{E'_\alpha + \epsilon_\alpha}\right| = \left|\frac{\frac{E'_{\alpha'}}{E'_\alpha} \cdot \frac{\epsilon_\alpha}{E'_\alpha} - \frac{\epsilon_{\alpha'}}{E'_\alpha}}{1 + \frac{\epsilon_\alpha}{E'_\alpha}}\right| \\ &\overset{1 > \frac{c}{r}}{=} \left|\frac{E'_{\alpha'}}{E'_\alpha} \cdot \frac{\epsilon_\alpha}{E'_\alpha} - \frac{\epsilon_{\alpha'}}{E'_\alpha}\right| \leq \frac{c^2}{r} < \delta. \end{aligned}$$

Hence $\rho_{\mathring{W}}(E(\pi_\Gamma(l))) \in O$ by (78) since $E(\pi_\Gamma(l')) \in O$. We finally show that

$$\forall \alpha' \in \mathring{W}, \forall \alpha \in V \setminus W\colon \quad \left|\frac{E_\alpha(l)}{E_{\alpha'}(l)}\right| \leq \frac{c}{r}. \tag{83}$$

Consider any α, α' as in (83). Suppose without loss of generality that $E_\alpha(l) \neq 0$ so that $\pi^{-1}(\alpha) \cap L \neq \emptyset$. Choose an $x \in l(\pi^{-1}(\alpha) \cap L)$ of minimal norm. Then

$$\left|\frac{E_\alpha(l)}{E_{\alpha'}(l)}\right| = \left|\frac{E_{\alpha_l}(l)}{E_{\alpha'}(l)} \cdot \frac{E_\alpha(l)}{E_{\alpha_l}(l)}\right| \leq c \cdot \left|\frac{e_{\mathcal{L}}(x_l)}{e_{\mathcal{L}}(x)}\right| \overset{(79)}{\leq} c \cdot \left|\frac{x_l}{x}\right| \leq \frac{c}{r}.$$

Hence $E(\pi_\Gamma(l)) \in \mathcal{U}(O, \frac{c}{r})$. This establishes the second inclusion. □

Proof of Proposition 9.8. As argued after Proposition 9.8, it remains to be shown the claim that E_i induces an isomorphism of Grothendieck topologies. From Def.-Prop. 6.22 follows via (76), that a subset $X \subset \Omega^*_{\Gamma_i}$ is admissible if and only if for any free $A/(t)$-submodule $0 \neq W \subset V$ the subset $X \cap \Omega_{\Gamma_i,W} \subset \Omega_{\Gamma,i}$ is admissible and any admissible quasi-compact $Y \subset X \cap \Omega_{\Gamma_i,W}$ admits an $r \in |C|$ with $\mathcal{U}(\Lambda_i, Y, r) \subset X$. Moreover, for any such W the admissible quasi-compact subsets of $\Omega_W(C)$ are precisely the intersections with $\Omega_W(C)$ of the admissible quasi-compact subsets of $\Omega(\mathring{W})$ (see Cor. 2.29). As E_i restricts to an isomorphism $\Omega_{\Gamma_i} \to \Omega_i$ and to an isomorphism $\Omega_{\Gamma_i,W} \to \Omega_W(C)$ for any such $W \subsetneq V$ by Corollary 9.5, the claim directly follows from Proposition 9.9 applied to the case $P' = Q_i$ jointly with Proposition 9.10 and Lemma 9.11. □

Corollary 9.12. *The Q_i for all $i \in I$ are the irreducible components of Q.*

Proof. By Lemma 9.7, any $Q_i \subset Q$ is Zariski-closed. By Corollary 6.20, any Ω_{Γ_i} is dense in $\Omega^*_{\Gamma_i}$. By Proposition 9.8, thus any Ω_i is dense in Q_i. Consequently, any Q_i contains the dense irreducible subset Ω_i and is thus itself irreducible. Moreover, for any irreducible Zariski closed subset $Y \subset Q$ the intersection $Y \cap \Omega$ with the Zariski open Ω is irreducible and thus contained in some Ω_i by maximality of the irreducible components Ω_i. Hence the Q_i are maximal among the irreducible Zariski closed subsets of Q and are thus the irreducible components. □

Let $i \in I$. Consider the Grothendieck ringed space $(Q_i, \tilde{\mathcal{O}}_{Q_i})$ whose underlying Grothendieck topological space coincides with the one underlying $(Q_i, \mathcal{O}_{Q_i})$ and whose sections on any admissible $U \subset Q_i$ are the functions $f\colon U \to C$ that are continuous with respect to the canonical topologies, that are bounded on any admissible affinoid subset of U and that restrict to regular functions $U \cap \Omega_W(C) \to C$ for any free (A/t)-submodule $0 \neq W \subset V$. Denote by

$$n_{Q_i}\colon (Q_i, \tilde{\mathcal{O}}_{Q_i}) \to (Q_i, \mathcal{O}_{Q_i})$$

the morphism of Grothendieck ringed spaces whose underlying morphism of Grothendieck topological spaces is the identity and whose homomorphism $\mathcal{O}_{Q_i}(U) \to \tilde{\mathcal{O}}_{Q_i}(U)$ for any admissible $U \subset Q_i$ is the natural injection by means of the Maximum Modulus Principle, i.e., Proposition 2.25.

Corollary 9.13. *The isomorphism E_i of Grothendieck topological spaces yields an isomorphism*

$$(\Omega^*_{\Gamma_i}, \mathcal{O}^*_{\Gamma_i}) \to (Q_i, \tilde{\mathcal{O}}_{Q_i})$$

of Grothendieck ringed spaces, where the homomorphisms on sections are given by precomposition with E_i.

Proof. This directly follows from Proposition 6.26 and the construction of $\tilde{\mathcal{O}}_{Q_i}$ via Proposition 9.8, Corollary 9.5, Lemma 9.11 and (76). □

Theorem 9.14. *The morphism n_{Q_i} is the normalization of Q_i.*

Proof of Theorems 9.1 and 9.14. By means of the isomorphism in Corollary 9.13, we identify $(\Omega^*_{\Gamma_i}, \mathcal{O}^*_{\Gamma_i})$ with $(Q_i, \tilde{\mathcal{O}}_{Q_i})$ and are reduced to showing that n_{Q_i} is the normalization morphism for Q_i. Set $Z := Q_i$. We want to apply Theorem 3.8 to the present case, i.e., where the global sections S on Z are the restrictions to Z of the Y_α for all $\alpha \in \mathring{V}$. Let us verify its conditions:

i) Z is irreducible,

ii) the Zariski open subvariety $\Omega(S) \subset Z$ is normal,

iii) $Z \setminus \Omega(S)$ is of everywhere positive codimension in Z.

iv) any function $f\colon X \to C$ on any admissible $X \subset Z$ which is continuous with respect to the canonical topology and restricts to a regular function on $X \cap \Omega(S)$ restricts to a regular function on $X \cap \Omega(T)$ for any $T \subset S$ and

v) any $z \in Z$ has a fundamental basis of admissible neighborhoods U such that $U \cap \Omega(S)$ is connected and, in particular, non-empty.

Condition i) follows from Corollary 9.12. Normality of $\Omega(S) = \Omega(\mathring{V}) \cap Z = \Omega_i$ follows from Theorem 5.8 via Theorem 9.3; this yields ii). By Proposition 9.3, Example 5.11 and Corollary 5.10, moreover, $\Omega_W(C)$ is everywhere of dimension $\mathrm{rank}_{A/(t)}(W) - 1$ for any free $A/(t)$-submodule $0 \neq W \subset V$. Via Propostion 9.10, thus follows that $Z \setminus \Omega(S)$ is of everywhere positive codimension which yields iii). As for Assumption iv), consider any admissible $X \subset Z$ and any function $f\colon X \to C$ which is continuous with respect to the canonical topologies and which restricts to a regular function on $X \cap \Omega(\mathring{V})$. The regularity of the restriction of f to $X \cap \Omega(\mathring{W})$ for an arbitrary free $A/(t)$-submodule $0 \neq W \subset V$ then follows from Proposition 6.44 by descending induction on the rank of W. Taking Proposition 9.10 again into account, this yields Condition iv). Finally, Corollary 6.20 provides Condition v). We may thus apply Theorem 3.8 which concludes the proof. □

10 Consequences of the comparison

Let $A \subset C$ be as in Sections 6, 7, 8, 9. Consider any congruence subgroup $\mathcal{K} \subset \operatorname{Aut}_{\hat{A}}(M)$ as in Section 7. Choose $0 \neq t \in A$ such that $\mathcal{K}$ contains the kernel $\mathcal{K}(t)$ of the natural homomorphism $\operatorname{Aut}_{\hat{A}}(M) \to \operatorname{Aut}_A(V)$, where $V := t^{-1}M/M$, and, using that A is finitely generated, such that $\operatorname{Div}_A(t)$ generates A as in Section 8.2. Identify $\Delta := \mathcal{K}/\mathcal{K}(t)$ with the image of $\mathcal{K}$ in $\operatorname{Aut}_A(V)$. Let $Q := Q_V(C)$ and $\Omega := \Omega_V(C)$ be as in Section 9.

Theorem 10.1. *The normalization morphism $E_{\mathcal{K}(t)}$ in Theorem 9.1 is Δ-equivariant and the induced morphism $E_{\mathcal{K}} \colon \Omega^*_{\mathcal{K}} \to \Delta\backslash Q$ is the normalization morphism of $\Delta\backslash Q$ and restricts to Drinfeld's isomorphism $\Omega_{\mathcal{K}} \to \Delta\backslash\Omega$ between normal rigid analytic varieties. Moreover, the morphism of Grothendieck topological spaces underlying $E_{\mathcal{K}}$ restricts to isomorphisms between irreducible components.*

Proof. By construction, $E_{\mathcal{K}(t)}$ is Δ-equivariant and thus induces a morphism $E_{\mathcal{K}} \colon \Omega^*_{\mathcal{K}} \to \Delta\backslash Q$ between their quotients. From Theorem 9.1 follows via Köpf's GAGA-theorem [31, Satz 5.1] that the quotient $\Omega^*_{\mathcal{K}}$ of $\Omega^*_{\mathcal{K}(t)}$ by the finite group Δ is a normal projective rigid analytic variety since $\Omega^*_{\mathcal{K}(t)}$ is. Moreover, $E_{\mathcal{K}}$ is finite since $E_{\mathcal{K}(t)}$ is. Moreover, as $E_{\mathcal{K}(t)}$ restricts to an isomorphism $\Omega_{\mathcal{K}(t)} \to \Omega$, also $E_{\mathcal{K}}$ restricts to an isomorphism $\Omega_{\mathcal{K}} \to \Delta\backslash\Omega$ between their quotients. Furthermore, Corollary 6.20 yields via Proposition 7.13 that both the Zariski closed complement of $\Delta\backslash\Omega$ in $\Delta\backslash Q$ and its preimage in $\Omega^*_{\mathcal{K}}$ are nowhere dense. By [12, Theorem 2.1.2], thus $E_{\mathcal{K}}$ is indeed the normalization morphism. Moreover, as the Grothendieck topological space on each side of $E_{\mathcal{K}}$ is the quotient by Δ of the respective side of $E_{\mathcal{K}(t)}$, the last assertion, too, follows from Theorem 9.1. □

Choose any complete set S of representatives of the natural $\mathcal{K}$-action on the set of A-structures of M and recall the isomorphism

(84)
$$\coprod_{\Lambda \in S} (\Omega^*_{\mathcal{K}_\Lambda}, \mathcal{R}^*_{\mathcal{K}_\Lambda}) \longrightarrow (\Omega^*_{\mathcal{K}}, \mathcal{R}^*_{\mathcal{K}})$$

of Grothendieck graded ringed spaces provided by Proposition 7.13.

Corollary 10.2. *$\Omega^*_{\mathcal{K}}$ is a normal projective rigid analytic variety over C whose irreducible components are, via (84), the $\Omega^*_{\mathcal{K}_\Lambda}$ for all $\Lambda \in S$.*

Proof. By [12, Theorem 2.1.3], the analytification functor commutes with the normalization functor. From Theorem 10.1 thus follows that $\Omega^*_{\mathcal{K}}$ is a normal projective rigid analytic variety. Moreover, via (84), the $\Omega^*_{\overline{\mathcal{K}}_\Lambda}$ are admissible subsets of $\Omega^*_{\mathcal{K}}$ and pairwise disjoint. It thus suffices to show that each of them is irreducible. Consider any $\Lambda \in S$. Then the admissible subvariety $\Omega_{\overline{\mathcal{K}}_\Lambda} \subset \Omega^*_{\overline{\mathcal{K}}_\Lambda}$ is irreducible by Proposition 5.8 and dense by Corollary 6.20. Thus $\Omega^*_{\overline{\mathcal{K}}_\Lambda}$ is itself irreducible as desired. □

Corollary 10.3. *Let* $(A', M', \mathcal{K}') \stackrel{(\Phi,L)}{\longrightarrow} (A, M, \mathcal{K}) \in \hat{\mathcal{A}}$*, where* $\mathcal{K}' \subset \mathrm{Aut}_{\hat{A}}(M')$ *is a congruence subgroup. Then* $\Omega^*_{(\Phi,L)} \colon \Omega^*_{\mathcal{K}'} \to \Omega^*_{\mathcal{K}}$ *is a proper morphism of rigid analytic varieties; it is even finite if the index of* $\Phi^*(\mathcal{K}') \subset \overline{\mathcal{K}}_{\Phi(M')}$ *is finite.*

Proof. If the index of $\Phi^*(\mathcal{K}') \subset \overline{\mathcal{K}}_{\Phi(M')}$ is finite, then any fibre of $\Omega^*_{(\Phi,L)}$ is finite. The corollary thus follows from Corollary 10.2 and [8, Prop. 9.6.2.4 and Cor. 9.6.3.6]. □

Consider any finitely generated projective A-module $\Lambda \neq 0$ and any congruence subgroup $\Gamma \subset \mathrm{Aut}_A(\Lambda)$. For the remainder we consider the following special case of $M, t, \mathcal{K}, S$ so that we may interpret Λ as an element of S and Γ to be $\overline{\mathcal{K}}_\Lambda$: Assume that $M = \Lambda \otimes_A \hat{A}$; then Λ is an A-structure of M. Using that $\Gamma \subset \mathrm{Aut}_A(\Lambda)$ is a congruence subgroup, assume that $0 \neq t \in A$ is such that $\mathrm{Div}_A(t)$ generates A and that furthermore Γ contains the kernel of $\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(t^{-1}\Lambda/\Lambda)$. Identify $t^{-1}\Lambda/\Lambda$ with $V := t^{-1}M/M$ via the isomorphism induced by the inclucion $\Lambda \subset M$. Assume that $\mathcal{K}$ is the preimage in $\mathrm{Aut}_{\hat{A}}(M)$ of the image of Γ in $\mathrm{Aut}_A(V)$. Assume finally that S contains Λ. In this case indeed $\Gamma = \overline{\mathcal{K}}_\Lambda$.

Corollary 10.4. Ω^*_Γ *is a normal integral projective rigid analytic variety over* C *whose admissible subvariety* Ω_Γ *is dense.*

Proof. Cor. 10.2 yields the first part and Cor. 6.20 the second. □

Corollary 10.5. *Let* $(A', L', \Gamma') \stackrel{\varphi}{\to} (A, \Lambda, \Gamma) \in \mathcal{A}$*, where* $\Gamma' \subset \mathrm{Aut}_A(\Lambda')$ *is a congruence subgroup. Then* $\Omega^*_\varphi \colon \Omega^*_{\Gamma'} \to \Omega^*_\Gamma$ *is a proper morphism of rigid analytic varieties; it is even finite if the index of* $\varphi^*(\Gamma') \subset \overline{\Gamma}_{\varphi(\Lambda')}$ *is finite.*

Proof. If $\varphi^*(\Gamma') \subset \overline{\Gamma}_{\varphi(\Lambda')}$ has finite index, then Ω^*_φ is quasi-finite. The corollary thus follows from Cor. 10.7 and [8, Prop. 9.6.2.4 and Cor. 9.6.3.6]. □

Recall that Γ is called *fine* if its image in $\mathrm{Aut}_A(\Lambda/\mathfrak{p}\Lambda)$ is unipotent for some maximal ideal $\mathfrak{p} \subset A$.

Corollary 10.6. *Suppose that* Γ *is fine and let* $k \geq 0$ *be any integer. Denote by* E *the restriction of* $E_{\mathcal{K}}$ *to* Ω_Γ^* *and by* Y *its image. Then the morphism*

$$E^{-1}(\mathcal{O}_Y(k)) \otimes_{E^{-1}(\mathcal{O}_Y)} \mathcal{O}_\Gamma^* \to \mathcal{O}_\Gamma^*(k) \tag{85}$$

induced by E *from the inverse image under* E *of the* k*-th twisting module* $\mathcal{O}_Y(k)$ *(provided by Prop. 8.25) of* Y *to* $\mathcal{O}_\Gamma^*(k)$ *is an isomorphism and the natural morphism* $(\mathcal{O}_\Gamma^*(k))^{k'} \to \mathcal{O}_\Gamma^*(k \cdot k')$ *is an isomorphism for any* $k' \geq 0$*. Consequently, if* $k \geq 1$*, then* $\mathcal{O}_\Gamma^*(k)$ *is an ample invertible* $\mathcal{O}_\Gamma^*$*-module.*

Proof. The morphism of Grothendieck topological spaces underlying E is an isomorphism by Theorem 10.1. Thus

$$E^{-1}(\mathcal{O}_Y(k))(X') = \mathcal{O}_Y(k)(E(X')) \tag{86}$$

for any admissible $X' \subset \Omega_\Gamma^*$. Moreover, by construction of $\mathcal{O}_\Gamma^*$, any nowhere vanishing section in $\mathcal{O}_\Gamma(k)(X')$ is a basis for $\mathcal{O}_\Gamma(k)|_{X'}$ over $\mathcal{O}_\Gamma|_{X'}$ for any admissible $X' \subset \Omega_\Gamma^*$. Using that $\mathcal{O}_Y(k)$ is invertible, choose any admissible covering $\mathcal{C}$ of Y such that any $Y' \in \mathcal{C}$ admits a nowhere vanishing section in $\mathcal{O}_Y(k)(Y')$ which is a basis of $\mathcal{O}_Y(k)|_{Y'}$ over $\mathcal{O}_Y|_{Y'}$. Let $Y' \in \mathcal{C}$ and set $X' := E^{-1}(Y')$. Using (86) and that E sends any nowhere vanishing section in $\mathcal{O}_Y(k)(Y')$ to a nowhere vanishing section in $\mathcal{O}_\Gamma(k)(X')$, it is directly checked that (85) restricts to an isomorphism on X'. As the preimage of $\mathcal{C}$ under E is an admissible covering, this yields the first part. The second part holds true since moreover, by [29, Chapter 2, Prop. 5.12], the natural morphism $\mathcal{O}_Y(k)^{k'} \to \mathcal{O}_Y(k \cdot k')$ is an isomorphism for any $k' \geq 0$ and since the formation of tensor products and inverse images are compatible.

Suppose that $k \geq 1$. As $\mathcal{O}_Y(k)$ is ample invertible by Prop. 8.25, so is its inverse image under the finite morphism E by [28, Ch. 1, Prop. 4.4] using that E is the analytification of the algebraic normalization of Y by [12, Thm. 2.1.3]. Hence $\mathcal{O}_\Gamma(k)$ is ample invertible by the isomorphism (85). □

Corollary 10.7. *The* C*-algebra* $\mathcal{R}_\Gamma^*(\Omega_\Gamma^*)$ *is finitely generated with* $\mathcal{O}_\Gamma^*(\Omega_\Gamma^*) = C$ *and* Ω_Γ^* *is the analytification of* $\mathrm{Proj}(\mathcal{R}_\Gamma^*(\Omega_\Gamma^*))$.

Proof. By Köpf's GAGA-theorems [31, Sätze 4.7 und 5.1] and Corollaries 10.4 and 10.6, the variety Ω_Γ^* is the analytification of some normal integral projective algebraic variety X and, if Γ is fine, the ample invertible sheaf $\mathcal{O}_\Gamma^*(k)$ is the analytification of an ample invertible sheaf $\mathcal{L}_k$ on X for any $k \geq 0$, and the global sections on $\mathcal{O}_\Gamma^*(k)$ are naturally isomorphic to the ones of $\mathcal{L}_k$. If Γ is fine, thus $\mathcal{O}_\Gamma^*(\Omega_\Gamma^*) = C$ and the corollary follows using the isomorphisms $(\mathcal{O}_\Gamma^*(k))^{k'} \to \mathcal{O}_\Gamma^*(k \cdot k')$ for all $k, k' \geq 0$ as well as the fact (see [34, Theorem 5.7]) that the ring of sections in all powers of $\mathcal{L}_1$ is a finitely generated normal integral domain and that its Proj is X.

Via the choice of a fine normal subgroup $\Gamma' \subset \Gamma$, the general case is reduced to the previous case using that by Noether's theorem (see [41, Theorem 2.3.1]) the subring of invariants $\mathcal{R}^*_\Gamma(\Omega^*_\Gamma) \subset \mathcal{R}^*_{\Gamma'}(\Omega^*_{\Gamma'})$ with respect to the C-linear action by the finite group Γ/Γ' is again finitely generated. □

Definition 10.8. *For any integer $k \geq 0$ a weak modular form $f \in \mathcal{O}^*_\Gamma(k)(\Omega_\Gamma)$ (see Definition 6.39) is called a* modular form *if the negatively indexed coefficients of the Fourier expansions at all direct summands $0 \neq L \subset \Lambda$ of co-rank 1 all vanish; denote by $\mathcal{M}_\Gamma(k) \subset \mathcal{O}^*_\Gamma(k)(\Omega_\Gamma)$ the C-subspace of modular forms of weight k. Set*

$$\mathcal{M}_\Gamma := \sum_{k\geq 0} \mathcal{M}_\Gamma(k).$$

Remark 10.9. Consider any integer k. The isomorphism in Remark 6.40 from $\mathcal{O}^*_\Gamma(k)(\Omega_\Gamma)$ to the space of weak modular forms in coordinates restricts to an isomorphism from $\mathcal{M}_\Gamma(k)$ to the subspace of modular forms in coordinates in the sense of Basson's [3, Definition 3.3.1].

Proposition 10.10. *The restriction homomorphism $\mathcal{R}^*_\Gamma(\Omega^*_\Gamma) \to \mathcal{R}^*_\Gamma(\Omega_\Gamma)$ is injective with image $\mathcal{M}_\Gamma$.*

Proof. Let $\Omega^{<2}_\Lambda$ be the union of the Ω_L for all direct summands $0 \neq L \subset \Lambda$ with $\mathrm{rank}_A(L) \geq \mathrm{rank}_A(\Lambda) - 1$ and consider the admissible subset

$$\Omega^{<2}_\Gamma := p_\Gamma(\Omega^{<2}_\Lambda) \subset \Omega^*_\Gamma.$$

Corollary 6.44 applied to the various such L yields that the restriction homomorphism $\mathcal{R}^*_\Gamma(\Omega^{<2}_\Gamma) \to \mathcal{R}^*_\Gamma(\Omega_\Gamma)$ is injective with image $\mathcal{M}_\Gamma$. We claim that, moreover, the restriction morphism

$$\mathcal{R}^*_\Gamma(\Omega^*_\Gamma) \to \mathcal{R}^*_\Gamma(\Omega^{<2}_\Gamma)$$

is bijective. Consider $\Gamma' := \overline{\mathcal{K}(t)}_\Lambda \subset \Gamma$. By construction of $\Omega^{<2}_{\Gamma'}$ and Ω^*_Γ as well as of $\mathcal{O}^*_{\Gamma'}$ and $\mathcal{O}^*_\Gamma$, the claim is directly reduced to showing the claim in the case $\Gamma = \Gamma'$. Thus assume that $\Gamma = \Gamma'$. By the Riemann extension theorem [2, Satz 10], the restriction morphism is bijective if Ω^*_Γ is normal and if $\Omega^*_\Gamma \setminus \Omega^{<2}_\Gamma \subset \Omega^*_\Gamma$ is Zariski-closed of codimension ≤ 2. From Corollary 10.4 follows the normality of Ω^*_Γ. The image of Ω^*_Γ under $E_\mathcal{K}$ is then an irreducible component Q_i of Q. We are thus reduced to showing that the image U of $\Omega^*_\Gamma \setminus \Omega^{<2}_\Gamma \subset \Omega^*_\Gamma$ under the isomorphism $E_i\colon (\Omega^*_\Gamma, \mathcal{O}^*_\Gamma) \to (Q_i, \tilde{\mathcal{O}}_{Q_i})$ provided by Corollary 9.13 is Zariski-closed in Q_i and of codimension ≤ 2.

By Corollary 9.5, the image U is the union of the $\Omega_W(C)$ for all free direct summands $0 \neq W \subset V$ with $\mathrm{rank}_{A/(t)}(W) \leq \mathrm{rank}_{A/(t)}(V) - 2$. By Theorem 8.16,i), equivalently, U is the union of the $Q_W(C) \subset Q_i$ for all such W. For any such W, moreover, $Q_W(C)$ is Zariski-closed in $Q_V(C)$ and hence Zariski-closed in Q_i with respect to $\mathcal{O}_{Q_i}$ and thus also with respect to $\tilde{\mathcal{O}}_{Q_i}$. Being a finite union of Zariski-closed subsets, hence U itself is Zariski-closed. Moreover, by Theorem 8.16, ii), for any direct summand $0 \neq W \subset V$ the dimension of any irreducible component of $\Omega_W(C)$ equals $\mathrm{rank}_{A/(t)}(W) - 1$. Hence $U \subset Q_i$ is Zariski-closed of codimension ≤ 2. □

Corollary 10.11. *$\mathcal{M}_\Gamma$ is a finitely generated C-algebra with $\mathcal{M}_\Gamma(0) = C$.*

Proof. This follows from Proposition 10.10 and Corollary 10.7. □

Corollary 10.12. *The C-vector space $\mathcal{M}_\Gamma(k)$ is finite dimensional for any $k \geq 0$.*

Proof. This follows from Corollary 10.11 via induction on k. □

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