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Virtual rigid motives of semi-algebraic sets

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Abstract
Let k be a field of characteristic zero containing all roots of unity and $K = k((t))$. We build a ring morphism from the Grothendieck ring of semi-algebraic sets over K to the Grothendieck ring of motives of rigid analytic varieties over K. It extends the morphism sending the class of an algebraic variety over K to its cohomological motive with compact support. We show that it fits inside a commutative diagram involving Hrushovski and Kazhdan's motivic integration and Ayoub's equivalence between motives of rigid analytic varieties over K and quasi-unipotent motives over k; we also show that it satisfies a form of duality. This allows us to answer a question by Ayoub, Ivorra and Sebag about the analytic Milnor fiber.

Keywords Motivic integration · Rigid motives · Rigid analytic geometry · Motivic Milnor fiber · Analytic Milnor fiber

Mathematics Subject Classification 14C15 · 14F42 · 03C60 · 14G22 · 32S30

1 Introduction

Let k be a field of characteristic zero containing all roots of unity and $K = k(\!(t)\!)$ the field of Laurent series. Morel and Voevodsky build in [27] the category $\mathrm{SH}(k)$ of stable $\mathbb{A}^1$-invariant motivic sheaves without transfers over k. More generally, for S a k-scheme they build the category of S-motives $\mathrm{SH}(S)$. Following an insight by Voevodsky, see Deligne's notes [13], Ayoub developed in [1] a six functors formalism for the categories $\mathrm{SH}(-)$, mimicking Grothendieck's six functors formalism for étale cohomology. See also in [11] an alternative construction by Cisinski and Déglise. For $f : X \to Y$ a morphism of schemes, in addition to the direct image $f_* : \mathrm{SH}(X) \to \mathrm{SH}(Y)$ and pull-back $f^* : \mathrm{SH}(Y) \to \mathrm{SH}(X)$, one has the extraordinary direct image $f_! : \mathrm{SH}(X) \to \mathrm{SH}(Y)$ and extraordinary pull-back $f^! : \mathrm{SH}(Y) \to \mathrm{SH}(X)$.

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This allows in particular to define for any S-scheme $f : X \to S$ an object $\mathrm{M}^\vee_{S,c}(X) = f_! f^* \mathbb{1}_k \in \mathrm{SH}(S)$, the so-called cohomological motive with compact support of X.

Denote by $\mathbf{K}(\mathrm{Var}_k)$ the Grothendieck ring of k-varieties. It is the abelian group generated by isomorphism classes of k-varieties, with the scissors relations

$$[X] = [Y] + [X \backslash Y]$$

for Y a closed subvariety of X. The cartesian product induces a ring structure on $\mathbf{K}(\mathrm{Var}_k)$.

As $\mathrm{SH}(k)$ is a triangulated category, we can consider its Grothendieck ring $\mathbf{K}(\mathrm{SH}(k))$, which is the abelian group generated by isomorphism classes of its compact (also called constructible) objects, with relations $[B] = [A]+[C]$ whenever there is a distinguished triangle

$$A \to B \to C \xrightarrow{+1} .$$

Elements of $\mathbf{K}(\mathrm{SH}(k))$ are called virtual motives and the tensor product on $\mathrm{SH}(k)$ induces a ring structure on $\mathbf{K}(\mathrm{SH}(k))$. The locality principle implies that the assignment $X \in \mathrm{Var}_k \mapsto [\mathrm{M}^\vee_{k,c}(X)] \in \mathbf{K}(\mathrm{SH}(k))$ satisfies the scissors relations, hence induces a morphism

$$\chi_k : \mathbf{K}(\mathrm{Var}_k) \to \mathbf{K}(\mathrm{SH}(k))$$

which is a ring morphism. Such a morphism was first considered by Ivorra and Sebag [22].

Ayoub builds in [3] the category $\mathrm{RigSH}(K)$ of rigid analytic motives over K, in a similar fashion of $\mathrm{SH}(K)$ but instead of K-schemes, he starts with rigid analytic K-varieties in the sense of Tate. The analytification functor from algebraic K-varieties to rigid K-varieties induces a functor

$$\mathrm{Rig}^* : \mathrm{SH}(K) \to \mathrm{RigSH}(K).$$

For any rigid K-variety X, Ayoub defines $\mathrm{M}_{\mathrm{Rig}}(X)$ and $\mathrm{M}^\vee_{\mathrm{Rig}}(X)$, respectively, the homological and cohomological rigid motives of X. However, to our knowledge there is no general notion of cohomological rigid motive with compact support.

One can also consider $\mathbf{K}(\mathrm{VF}_K)$, the Grothendieck ring of semi-algebraic sets over K. If $X = \mathrm{Spec}(A)$ is an affine variety over K, a semi-algebraic subset of X^{an} is a boolean combination of subsets of the form $\{x \in X^{\mathrm{an}} \mid \mathrm{v}(f(x)) \leq \mathrm{v}(g(x))\}$, for $f, g \in A$ (where v is the valuation on K). The ring $\mathbf{K}(\mathrm{VF}_K)$ is then the abelian group of isomorphism classes of semi-algebraic sets (for semi-algebraic bijections) with relations $[X] = [U]+[V]$ if X is the disjoint union of U and V. We could also consider $\mathbf{K}(\mathrm{VF}^{\mathrm{an}}_K)$, the Grothendieck ring of subanalytic sets over K. It is isomorphic to $\mathbf{K}(\mathrm{VF}_K)$ by a byproduct of Hrushovski and Kazhdan's theory of motivic integration [20].

In this situation it is rather natural to ask about the existence of a ring morphism

$$\chi_{\mathrm{Rig}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}(K))$$

extending the morphism $\chi_K : \mathbf{K}(\mathrm{Var}_K) \to \mathbf{K}(\mathrm{SH}(K))$.

Ayoub, Ivorra and Sebag ask in [4, Remark 8.15] about the existence of a morphism similar to χ_{Rig} and speculate that one should be able to recover from it their comparison result about the motivic Milnor fiber. We will show that it is indeed the case, see below.

If X is an algebraic K-variety smooth and connected of dimension d, then $[\mathrm{M}^{\vee}_{K,c}(X)] = [\mathrm{M}_K(X)(-d)]$, where $(-d)$ is the Tate twist (iterated d times). We would like to define for X a quasi-compact rigid K-variety smooth and connected of dimension d, $\chi_{\mathrm{Rig}}([X]) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)]$. Such classes generate $\mathbf{K}(\mathrm{VF}_K)$. If χ_{Rig} is well-defined, it will be the unique morphism satisfying such conditions. The main objective of this paper is to show the existence of such a morphism.

The strategy of proof is to use alternative descriptions of $\mathbf{K}(\mathrm{VF}_K)$ and $\mathrm{RigSH}(K)$, the former being established by Hrushovski and Kazhdan, the latter by Ayoub. Let us describe them briefly.

From a model-theoretic point of view, semi-algebraic sets over K are definable sets in the (first order) theory of algebraically closed valued fields over K. If L is a valued field, with ring of integers $\mathcal{O}_L$ of maximal ideal $\mathcal{M}_L$, we set $\mathrm{RV}(L) = L^{\times}/(1+\mathcal{M}_L)$. Observe that RV fits in the following exact sequence, where $\mathbf{k}$ is the residue field and Γ the value group:

$$1 \to \mathbf{k}^{\times} \to \mathrm{RV} \to \Gamma \to 0.$$

Working in a two sorted language, with one sort VF for the valued field and one sort RV, Hrushovski and Kazhdan establish in [20] the following isomorphism of rings:

$$\oint : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}},$$

where $\mathbf{K}(\mathrm{RV}_K[*])$ is the Grothendieck ring of definable sets of RV, the $[*]$ meaning that some grading is considered and I_{sp} is an ideal generated by a single explicit relation, see Sect. 2.1. Set $\hat{\mu} = \lim_{\leftarrow n} \mu_n$, with μ_n the group of n-th roots of unity in k and $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ the Grothendieck of varieties equipped with a good $\hat{\mu}$-action, see Definition 2.6 for the precise definition.

The ring $\mathbf{K}(\mathrm{RV}_K[*])$ can be further decomposed into a part generated by $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ and a part generated by definable subsets of the value group. The latter being polytopes, one can apply Euler characteristic with compact supports to get a ring morphism

$$\Theta \circ \mathcal{E}_c : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}).$$

Ayoub on his side defines the category of quasi-unipotent motives $\mathrm{QUSH}(k)$ as the triangulated subcategory of $\mathrm{SH}(\mathbb{G}_{m\,k})$ with infinite sums generated by homological motives (and their twists) of $\mathbb{G}_{m\,k}$-varieties of the form

$$X[T, T^{-1}, V]/(V^r - Tf) \to \operatorname{Spec}(k[T, T^{-1}]) = \mathbb{G}_{mk}$$

where X is a smooth k-variety, $r \in \mathbb{N}^*$, and $f \in \Gamma(X, \mathcal{O}_X^\times)$. Let $q : \operatorname{Spec}(K) \to \mathbb{G}_{mk}$ be the morphism defined by $T \in k[T, T^{-1}] \mapsto t \in K = k(\!(t)\!)$. Ayoub shows in [3] that the functor

$$\mathfrak{F} : \mathrm{QUSH}(k) \xrightarrow{q^*} \mathrm{SH}(K) \xrightarrow{\mathrm{Rig}^*} \mathrm{RigSH}(K)$$

is an equivalence of categories, denote by $\mathfrak{R}$ a quasi-inverse.

We will define a morphism

$$\chi_{\hat{\mu}} : \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{QUSH}(k))$$

compatible with χ_k in the sense that it commutes with the morphism $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_k)$ induced by the forgetful functor and $1^* : \mathbf{K}(\mathrm{SH}(\mathbb{G}_{mk})) \to \mathbf{K}(\mathrm{SH}(k))$, where $1 : \operatorname{Spec}(k) \to \mathbb{G}_{mk}$ is the unit section, see Sect. 3.3.

Here is our main theorem.

Theorem 1.1 *Let k be a field of characteristic zero containing all roots of unity and set $K = k(\!(t)\!)$. Then there exists a unique ring morphism*

$$\chi_{\mathrm{Rig}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}(K))$$

such that for any quasi-compact rigid K-variety X, smooth and connected of dimension d, $\chi_{\mathrm{Rig}}([X]) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)]$.

Moreover, all the squares in the following diagram commute:

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\chi_K"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\oint", "\simeq"'] \arrow[d, "\chi_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \arrow[r, "\Theta\circ\mathcal{E}_c"] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\chi_{\hat{\mu}}"] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[rr, "\simeq", "\mathfrak{R}"'] & & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)).
\end{tikzcd}$$

Observe that with this diagram in mind, defining χ_{Rig} is easy since $\mathfrak{R}$ is an isomorphism, it is the equality $\chi_{\mathrm{Rig}}(X) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)]$ that we will have to prove. We will rely for this on an explicit computation of $\oint[X]$ when a semi-stable formal $R = k[\![t]\!]$-model of X is chosen.

Two choices are made in this construction. The first is when applying the compactly supported Euler characteristic $\mathcal{E}_c$, where we also could have used the Euler characteristic $\mathcal{E}$, the second is when we apply the morphism $\chi_{\hat{\mu}}$, where we can also consider the morphism sending the class of a variety to its homological motive with compact support. Varying these choices leads to three other ring morphisms

$$\chi'_{\mathrm{Rig}}, \widetilde{\chi_{\mathrm{Rig}}}, \widetilde{\chi'_{\mathrm{Rig}}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

satisfying properties analogous to Theorem 1.1. In particular, we will show that $\widetilde{\chi'_{\mathrm{Rig}}}$ also extends the morphism χ_K.

We claim that $\chi_{\mathrm{Rig}}(X)$ is the virtual incarnation of a hypothetical cohomological rigid motive with compact support of X. Hence, we expect some duality to appear. Here is what we prove in this direction.

Theorem 1.2 *Let X be a quasi-compact smooth rigid variety, $\mathcal{X}$ an formal R-model of X, D a proper subscheme of its special fiber $\mathcal{X}_\sigma$. Consider the tube $]D[$ of D in $\mathcal{X}$, it is a (possibly non-quasi-compact) rigid subvariety of X. Then*

$$\chi_{\mathrm{Rig}}(]D[) = [\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)].$$

In particular, if X is a smooth and proper rigid variety,

$$\chi_{\mathrm{Rig}}(X) = [\mathrm{M}^\vee_{\mathrm{Rig}}(X)].$$

To prove this theorem, we will once again rely on a choice of a semi-stable formal R-model of X and compute explicitly $[\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)]$ in terms of homological motives of $]D[$ and some subsets of $]D[$. Our approach is inspired by parts of Bittner's works [6,7] where she defines duality involutions in $\mathbf{K}(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ and shows that a toric variety associated to a simplicial fan satisfies an instance of Poincaré's duality.

Theorem 1.2 allows us to answer the question asked by Ayoub et al. [4, Remark 8.15] in relation to the motivic Milnor fiber. Fix X a smooth connected k-variety and let $f : X \to \mathbb{A}^1_k$ be a non-constant morphism. Define X_σ to be the closed subvariety of X defined by the vanishing of f. Denef and Loeser define in [14–16], see also [25], the motivic nearby cycle of f as an element $\psi_f \in \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_{X_\sigma})$. If $x : \mathrm{Spec}(k) \to X_\sigma$ is a closed point of X_σ, fiber product induces a morphism $x^* : \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_{X_\sigma}) \to \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_k)$, and $\psi_{f,x} = x^*\psi_f \in \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_k)$ is the motivic Milnor fiber of f at x.

Denef and Loeser justify their definition by showing that known additive invariants associated to the classical nearby cycle functor can be recovered from ψ_f and $\psi_{f,x}$, the Euler characteristic for example.

Ivorra and Sebag study a new instance of such a principle in [22] where they show (with our notations) that $\chi_{X_\sigma}(\psi_f) = [\Psi_f \mathbb{1}] \in \mathbf{K}(\mathrm{SH}(X_\sigma))$, where Ψ_f is the motivic nearby cycle functor constructed by Ayoub [2, Chapitre 3]. Literally speaking they only prove it in $\mathbf{K}(\mathrm{DA}^{\text{ét}}(X_\sigma, \mathbb{Q}))$, but it is observed in [4, Section 8.2] that their result generalizes to $\mathbf{K}(\mathrm{SH}(X_\sigma))$.

It was first observed by Nicaise and Sebag [30] that one can relate the motivic Milnor fiber to a rigid analytic variety. Consider the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{A}^1_k)$ induced by $T \in k[T] \mapsto t \in k[[t]]$. Still denote $X \to \mathrm{Spec}(R)$ the base change of f along this morphism, and let $\mathcal{X}$ be the formal t-adic completion of X. For $x \in X_\sigma$ a closed point, let $\mathcal{F}^{\mathrm{an}}_{f,x}$ be the tube of $\{x\}$ in $\mathcal{X}$. It is called the analytic Milnor fiber.

Ayoub, Ivorra and Sebag show in [4] that

$$[1^* \circ \mathfrak{R}\mathrm{M}^\vee_{\mathrm{Rig}}(\mathcal{F}^{\mathrm{an}}_{f,x})] = \chi_k(\psi_{f,x}) \in \mathbf{K}(\mathrm{SH}(k)).$$

In our context, we have $\Theta \circ \mathcal{E}_c \circ \oint \mathcal{F}^{\mathrm{an}}_{f,x} = \psi_{f,x} \in \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$, we can see it either by a direct computation using resolution of singularities as in [28,29] or by adapting results by Hrushovski and Loeser [21]. Now Theorem 1.2 shows that $\chi_{\mathrm{Rig}}(\mathcal{F}^{\mathrm{an}}_{f,x}) = [\mathrm{M}^{\vee}_{\mathrm{Rig}}(\mathcal{F}^{\mathrm{an}}_{f,x})]$ hence by Theorem 1.1,

$$[\mathfrak{R}\mathrm{M}^{\vee}_{\mathrm{Rig}}(\mathcal{F}^{\mathrm{an}}_{f,x})] = \chi_{\hat{\mu}}(\psi_{f,x}) \in \mathbf{K}(\mathrm{QUSH}(k)).$$

We then have refined the result of Ayoub, Ivorra and Sebag to an equivariant setting.

The paper is organized as follows. See the beginning of each section for the precise content. Section 2 is devoted to what we need on Hrushovski and Kazhdan motivic integration. In Sect. 3, we settle what we will use on motives, rigid analytic geometry and rigid motives. In Sect. 4 we build the realization map χ_{Rig} and prove Theorem 1.1. The last Sect. 5 is devoted to duality and the proof of Theorem 1.2.

2 Preliminaries on motivic integration

In this section we will introduce Hrushovski and Kazhdan's theory of motivic integration in Sect. 2.1 and use it to define two maps from the Grothendieck ring of semi-algebraic sets over K to the equivariant Grothendieck ring of varieties over k in Sect. 2.2.

2.1 Recap on Hrushovski and Kazhdan's integration in valued fields

We outline here the construction of Hrushovski and Kazhdan's motivic integration [20], focusing on the universal additive invariant since this is the only part that we will use. See also the papers [35,36] by Yin who gives an account of the theory in ACVF.

We will work in the first order theory ACVF of algebraically closed valued fields of equicharacteristic zero in the two-sorted language $\mathcal{L}$. The two sorts are VF and RV. We put the ring language on VF, with symbols $(0, 1, +, -, \cdot)$, on RV we put the group language $(\cdot, ()^{-1})$, a unary predicate $\mathbf{k}^{\times}$ for a subgroup, and operations $+ : \mathbf{k}^2 \to \mathbf{k}$ where $\mathbf{k}$ is the union of $\mathbf{k}^{\times}$ and a symbol 0. We add also a unary function $\mathrm{rv} : \mathrm{VF}^{\times} = \mathrm{VF}\backslash\{0\} \to \mathrm{RV}$.

We will also consider the imaginary sort Γ defined by the exact sequence

$$1 \to \mathbf{k}^{\times} \to \mathrm{RV} \to \Gamma \to 0,$$

together with maps $\mathrm{v}_{\mathrm{rv}} : \mathrm{RV} \to \Gamma$ and $\mathrm{v} : \mathrm{VF}^{\times} \to \Gamma$. We extend v to K by setting $\mathrm{v}(0) = +\infty$.

If L is a valued field, with valuation ring $\mathcal{O}_L$ and maximal ideal $\mathcal{M}_L$, define an $\mathcal{L}$-structure by $\mathrm{VF}(L) = L$, $\mathrm{RV}(L) = L^{\times}/(1 + \mathcal{M}_L)$, $\mathbf{k}(L) = \mathcal{O}_L/\mathcal{M}_L$, $\Gamma(L) = L^{\times}/\mathcal{O}_L^{\times}$. Note that the valuation ring is definable in this language because $\mathcal{O}_L^{\times} = \mathrm{rv}^{-1}(\mathbf{k}^{\times}(L))$.

Fix a field k of characteristic zero containing all roots of unity and set $K = k(\!(t)\!)$. Viewing K as a fixed base structure, for the rest of the paper, we will only consider

$\mathcal{L}(K)$-structures, where $\mathcal{L}(K)$ is the language obtained by adjoining to $\mathcal{L}$ constants symbols for elements of K. Any valued field extending K can be interpreted as a $\mathcal{L}(K)$-structure. Denote by ACVF_K the $\mathcal{L}(K)$-theory of such algebraically closed valued fields. The theory ACVF_K admits quantifier elimination in the language $\mathcal{L}(K)$.

We will use the notation $\mathrm{VF}^\bullet$ for VF^n for some n. The $\mathcal{L}(K)$-definable subsets of $\mathrm{VF}^\bullet$ are semi-algebraic sets, that is boolean combinations of sets of the form

$$\left\{x \in \mathrm{VF}^n \mid \mathrm{v}(f(x)) \geq \mathrm{v}(g(x))\right\},$$

where f and g are polynomials with coefficients in K. Observe that constructible sets are semi-algebraic, since one can take $g = 0$ in the definition.

Denote by $\mathbf{K}(\mathrm{VF}_K)$ the free group of $\mathcal{L}(K)$-definable subsets of $\mathrm{VF}^\bullet$, with the following relations:

- $[X] = [Y]$ if there is a semi-algebraic bijection $X \to Y$
- $[X] = [U] + [V]$ if X is the disjoint union $X = U \dot{\cup} V$.

The cartesian product endows $\mathbf{K}(\mathrm{VF}_K)$ with a ring structure.

Remark 2.1 Note that this framework allows us to consider general semi-algebraic subsets of K-varieties as studied for example by Martin [26]. We say that S is a semi-algebraic subset of a k-scheme X, if S is a finite union $S = \cup S_i$ such that for every i, there is an open affine subset $U_i = \mathrm{Spec}(A_i)$ of U such that $S_i \subseteq U_i$ is defined by a boolean combination of subsets of the form $\left\{y \in U_i^{\mathrm{an}} \mid \mathrm{v}(f(y)) \leq \mathrm{v}(g(y))\right\}$, with $f, g \in A_i$. Hence, we can consider its class $[S] \in \mathbf{K}(\mathrm{VF}_K)$.

Remark 2.2 Hrushovski and Kazhdan use a slightly different definition for $\mathbf{K}(\mathrm{VF}_K)$. They define it as the group generated by isomorphism classes of definable sets $X \subseteq \mathrm{VF}^\bullet \times \mathrm{RV}^\bullet$, such that for some $n \in \mathbb{N}$, there is some definable function $f : X \to \mathrm{VF}^n$ with finite fibers, with cut-and-paste relations (the function f is not part of the data). We can show that for such an X, there is some definable $X' \subseteq \mathrm{VF}^\bullet$, with a definable bijection $X \simeq X'$, see [20, Lemma 8.1], hence the rings are isomorphic.

The category $\mathrm{RV}_K[n]$ is the category of pairs (Y, f), with $Y \subseteq \mathrm{RV}^\bullet$ definable and $f : Y \to \mathrm{RV}^n$ a definable finite-to-one function. A morphism between (Y, f) and (Y', f') is a definable function $g : Y \to Y'$. One defines $\mathrm{RES}_K[n]$ to be the full subcategory of $\mathrm{RV}[n]$ whose objects (Y, f) are such that $\mathrm{v}_{\mathrm{rv}}(Y)$ is finite. From those categories one forms the graded categories

$$\mathrm{RV}_K[*] := \coprod_{i \in \mathbb{N}} \mathrm{RV}_K[i], \mathrm{RES}_K[*] := \coprod_{i \in \mathbb{N}} \mathrm{RES}_K[i].$$

For later purpose, we will also need a category related to the value group. One defines $\Gamma[n]$ to be the category with objects subsets of Γ^n defined by piecewise linear equations and inequations with $\mathbb{Z}$-coefficients. A morphism between Y and Y' is a bijection defined piecewise by composite of $\mathbb{Q}$-translations and $\mathrm{GL}_n(\mathbb{Z})$ morphisms. From this one forms $\Gamma[*] := \coprod_{n\in\mathbb{N}} \Gamma[n]$. One defines also $\Gamma^{\mathrm{fin}}[n]$ and $\Gamma^{\mathrm{fin}}[*]$ to be the full subcategories of $\Gamma[n]$ and $\Gamma[*]$ whose objects are finite.

Each of these categories $\mathcal{C}$ has disjoint unions, induced by disjoint unions of definable sets. We can form the associated Grothendieck ring $\mathbf{K}(\mathcal{C})$. It is the abelian group generated by isomorphism classes of objects of $\mathcal{C}$, with relations induced by disjoint unions and the product induced by the cartesian product.

For a fixed definable set $X \subseteq \mathrm{RV}^m$, we can view X as an object in $\mathrm{RV}_K[n]$, for any $n \geq m$. Hence for each $n \geq m$, X induces a class denoted $[X]_n \in \mathbf{K}(\mathrm{RV}_K[*])$. If X is non-empty, we then have $[X]_n \neq [X]_{n'}$ for $n \neq n'$.

Note that he Cartesian product induces graded ring structures on $\mathbf{K}(\mathrm{RV}_K[*])$, $\mathbf{K}(\mathrm{RES}_K[*])$ and $\mathbf{K}(\Gamma[*])$. We can also forget the grading and obtain rings $\mathbf{K}(\mathrm{RV}_K)$ and $\mathbf{K}(\mathrm{RES}_K)$.

Set $(X, f) \in \mathrm{RV}_K[n]$. Define $\mathfrak{L}(X, f)$ to be the fiber product

$$\mathfrak{L}(X, f) = \left\{(x, y) \in \mathrm{VF}^n \times X \mid \mathrm{rv}(x) = f(y)\right\}.$$

As f is finite-to-one, the projection of $\mathfrak{L}(X, f)$ to VF^n is finite-to-one, hence we can view it as an object in VF_K by Remark 2.2.

If $(X, f), (X', f') \in \mathrm{RV}_K[n]$, with a definable bijection $X \simeq X'$, then there is a definable bijection $\mathfrak{L}(X, f) \simeq \mathfrak{L}(X', f')$ by [20, Proposition 6.1], hence we have a ring morphism $\mathfrak{L} : \mathbf{K}(\mathrm{RV}_K[*]) \to \mathbf{K}(\mathrm{VF}_K)$.

Set $\mathrm{RV}^{>0} = \{x \in \mathrm{RV} \mid v_{\mathrm{rv}}(x) > 0\}$. Denote by I_{sp} the ideal of $\mathbf{K}(\mathrm{RV}_K[*])$ generated by $[\mathrm{RV}^{>0}] + [1]_0 - [1]_1$. The main theorem of [20] is the following.

Theorem 2.3 *The morphism $\mathfrak{L}$ is surjective and its kernel is I_{sp}.*

Denote by $\oint$ the inverse: $\mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}}$.

Remark 2.4 We will also consider the theory $\mathrm{ACVF}_K^{\mathrm{an}}$ in the language $\mathcal{L}_{\mathrm{an}}(K)$. This language is an enrichment of $\mathcal{L}$ where we add symbols for restricted analytic functions with coefficients in K, see [12,24] for details. A maximally complete algebraically closed valued field containing K can be enriched as an $\mathcal{L}_{\mathrm{an}}(K)$-structure. Denote $\mathrm{ACVF}_K^{\mathrm{an}}$ their $\mathcal{L}_{\mathrm{an}}(K)$-theory. We shall refer to $\mathcal{L}_{\mathrm{an}}(K)$-definable subsets of $\mathrm{VF}^{\bullet}$ as subanalytic sets. We can form similarly the Grothendieck ring of sub-analytic sets $\mathbf{K}(\mathrm{VF}_K^{\mathrm{an}})$.

As $\mathrm{ACVF}_K^{\mathrm{an}}$ is an enrichment of ACVF_K, we have a canonical map $\mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{VF}_K^{\mathrm{an}})$, which is an isomorphism.

Indeed Hrushovski and Kazhdan establish the isomorphism $\oint$ for any first order theory T which is V-minimal. The theory $\mathrm{ACVF}^{\mathrm{an}}$ being an example of such a theory, we get also an isomorphism

$$\oint^{\mathrm{an}} : \mathbf{K}(\mathrm{VF}_K^{\mathrm{an}}) \to \mathbf{K}(\mathrm{RV}_K^{\mathrm{an}}[*])/\mathrm{I}_{\mathrm{sp}}.$$

Quantifier elimination shows that $\mathbf{K}(\mathrm{RV}_K^{\mathrm{an}}[*]) \simeq \mathbf{K}(\mathrm{RV}_K[*])$, hence in particular $\mathbf{K}(\mathrm{VF}_K) \simeq \mathbf{K}(\mathrm{VF}_K^{\mathrm{an}})$.

The above isomorphism allows to consider the class of any subanalytic set in $\mathbf{K}(\mathrm{VF}_K)$. If X is a quasicompact rigid analytic K-variety, it determines a subanalytic set X^{VF} and we can then consider its class in $\mathbf{K}(\mathrm{VF}_K)$. From now on, we will implicitly use this convention when referring to classes of subanalytic sets.

2.2 Landing in $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$

Our goal here is to relate the target ring of motivic integration $\mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}}$ to the Grothendieck ring of k-varieties equipped with a $\hat{\mu}$-action.

Recall from [20, Corollary 10.3] that there is an isomorphism of rings

$$\mathbf{K}(\mathrm{RES}_K[*]) \otimes_{\mathbf{K}(\Gamma^{\mathrm{fin}}[*])} \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RV}_K[*]).$$

As the theory of Γ is o-minimal, one can use o-minimal Euler characteristic to define an additive map $\mathrm{eu} : \mathbf{K}(\Gamma[n]) \to \mathbb{Z}$. Any $X \subseteq \Gamma^n$ can be finitely partitioned into pieces definably isomorphic to non-empty open cubes $\prod_{i=1,\dots,k}(\alpha_i, \beta_i)$, with $\alpha_i, \beta_i \in \Gamma \cup \{-\infty, +\infty\}$. One sets $\mathrm{eu}((\alpha, \beta)^k) = (-1)^k$ and then defines $\mathrm{eu}(X)$ by additivity. One can show that this does not depends on the chosen partition of X, see [33, Chapter 4]. One can also show that when $M \to +\infty$, $\mathrm{eu}(X \cap [-M, M]^n)$ stabilizes and one defines the bounded Euler characteristic to be

$$\mathrm{eu}_c(X) := \lim_{M \to +\infty} \mathrm{eu}(X \cap [-M, M]^n).$$

The Euler characteristics eu and eu_c do coincide on bounded sets, but not in general. For example, $eu((0, +\infty)) = -1$ but $\mathrm{eu}_c((0, +\infty)) = 0$.

For $a \in \mathbb{Q}$, set $e_a = [\mathrm{v}_{\mathrm{rv}}^{-1}(a)]_1 \in \mathbf{K}(\mathrm{RES}_K[1])$. Let !I be the ideal of $\mathbf{K}(\mathrm{RES}_K)$ spanned by all differences $e_a - e_0$ and set $!\mathbf{K}(\mathrm{RES}_K) := \mathbf{K}(\mathrm{RES}_K)/!\mathrm{I}$. Define also $\mathbb{L} = [\mathbb{A}_k^1]$.

Proposition 2.5 ([20, Theorem 10.5 (2) and (4)]) *There are ring morphisms*

$$\mathcal{E} : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \to !\mathbf{K}(\mathrm{RES}_K)[\mathbb{L}^{-1}]$$

and

$$\mathcal{E}_c : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \to !\mathbf{K}(\mathrm{RES}_K),$$

such that for $[X]_n \in \mathrm{RES}_K[n]$, $\mathcal{E}([X]_n) = [X]/\mathbb{L}^n$ *and* $\mathcal{E}_c([X]_n) = [X]$, *and for* $\Delta \in \Gamma[n]$, $\mathcal{E}(\mathrm{v}^{-1}(\Delta)) = \mathrm{eu}(\Delta)[\mathbb{G}_{m_k}^n]/\mathbb{L}^n$ *and* $\mathcal{E}_c(\mathrm{v}^{-1}(\Delta)) = \mathrm{eu}_c(\Delta)[\mathbb{G}_{m_k}^n]$.

Definition 2.6 Let μ_n be the group of n-th roots of unity in k and $\hat{\mu} = \lim_{\substack{\leftarrow \\ n}} \mu_n$. Define $\mathrm{Var}_k^{\hat{\mu}}$ to be the category of quasi-projective k-varieties equipped with a good $\hat{\mu}$-action, that is, a $\hat{\mu}$-action that factors through some μ_n-action. Since the varieties are assumed to be quasi-projective, such an action is automatically good in the usual sense, *i.e.* the orbit of every point is contained in an affine open subset stable by the action.

Let $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$ be the abelian group generated by isomorphism classes of quasi-projective k-varieties X equipped with good $\hat{\mu}$-action, with the scissors relations. Let $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ be the quotient of $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$ by additional relations $[(V,\rho)] = [(V,\rho')]$ if V is a finite dimensional k-vector space and ρ, ρ' two good linear $\hat{\mu}$-actions on V. Note that the Cartesian product induces ring structures on $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$ and $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$.

We want to define a map $\mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$. Fix a set of parameters $t_a \in K((t))^{\mathrm{alg}}$ for $a \in \mathbb{Q}$ such that $t_1 = t$ and $t_{ab} = t_b^a$ for $a \in \mathbb{N}^*$ and denote $\mathbf{t}_a := \mathrm{rv}(t_a)$. Set $V_\gamma^* = \mathrm{v}^{-1}(\gamma)$ and $V_\gamma = V_\gamma^* \cup \{0\}$. If $X \in \mathrm{RES}$, then $X \subseteq \mathrm{RV}^n$ and the image of $\mathrm{v}_{\mathrm{rv}} : X \to \Gamma^n$ is finite. Working piecewise we can suppose this image is a singleton. In this case, there are $m, k_1, \dots, k_n \in \mathbb{N}^*$ such that $X \subseteq V_{k_1/m} \times \cdots \times V_{k_n/m}$. The function $g : (x_1, \dots, x_n) \in X \mapsto (x_1/\mathbf{t}_{k_1/m}, \dots, x_n/\mathbf{t}_{k_n/m}) \in k^n$ is $K((t^{1/m}))$-definable and its image $g(X)$ inherits a μ_n-action from the one on X. Moreover $g(X)$ is a definable subset of k^n, hence constructible by quantifier elimination. So, we get a map $\Theta : \mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$, and it induces also a map $!\mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$. Hrushovski and Loeser prove the following proposition.

Proposition 2.7 ([21, Proposition 4.3.1]) *The ring morphisms*

$$\Theta : \mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat}) \text{ and } \Theta : !\mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$$

are isomorphisms.

Set $\mathbf{t} = \mathrm{rv}(t)$. If $U \subseteq \mathbb{A}_k^n$ is a smooth subvariety of $\mathbb{A}_k^n$, $f \in \Gamma(U, \mathcal{O}_U^\times)$ an invertible regular function on U and $r \in \mathbb{N} \setminus \{0\}$, set

$$Q_r^{\mathrm{RV}}(U, f) = \left\{(u, v) \in V_0^n \times V_{1/r} \mid u \in U, v^r = \mathbf{t} f(u)\right\}.$$

Corollary 2.8 *The ring* $\mathbf{K}(\mathrm{RES}_K[*])$ *is generated by classes of sets of the form* $[Q_r^{\mathrm{RV}}(U, f)]_n \in \mathbf{K}(\mathrm{RES}_K[n])$.

Corollary 2.9 *There is a unique ring morphism* $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}})$ *that satisfies the following condition. For X a k-variety, $f \in \mathcal{O}_X^\times(X)$ and $m \in \mathbb{N}^*$, it sends the class of $X[V]/(V^m - f)$ with the μ_m-action on V to the class of*

$$X[V, V^{-1}, T, T^{-1}]/(V^m - Tf) \to \mathbb{G}_{mk} = \mathrm{Spec}(k[T, T^{-1}]).$$

Proof By Corollary 2.8 and Proposition 2.7, the classes in the statement of the corollary generate $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$, hence uniqueness is clear. To show that the morphism is well defined, one proceeds by induction on the dimension as in the proof of Proposition 2.7. This leads to a well-defined map $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}})$. Indeed, given Y as above, X and m are uniquely determined. The function f is only determined up to a factor in $\mathcal{O}_X^{\times n}$, but all different choices of representatives will lead to isomorphic Z.

Finally, note that the relations added when dropping the flat are in the kernel of the above map. Indeed, once again, because a linear action of μ_n on k^r is diagonalizable, it suffices to show that the image of $[(k, \mu_n)]$, where μ_n acts on k by multiplication by n-th roots of unity, is independent of n. As 0 is a fixed point, we can restrict the action on $k^\times$. The image in $\mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}})$ is then $[\mathrm{Spec}(k[U, U^{-1}, T, T^{-1}, V]/(V^n - TU))]$. But this variety is isomorphic to $\mathrm{Spec}(k[V, V^{-1}, T, T^{-1}])$ over $\mathbb{G}_{mk}$, the isomorphism being defined by $U \mapsto V^n T^{-1}$, $V \mapsto V$. □

3 Preliminaries on motives

This section is devoted to fix notations about motives. After a brief recap on Grothendieck rings of triangulated categories in Sect. 3.1, we introduce the category of motives in Sect. 3.2. We then build a map from the equivariant Grothendieck ring of varieties to the Grothendieck ring of quasi-unipotent motives in Sect. 3.3. Finally, we introduce motives of rigid analytic varieties in Sect. 3.4.

3.1 Triangulated categories

A triangulated category, as introduced by Verdier in his thesis [34], is an additive category endowed with an autoequivalence, denoted $-[1]$ and called the suspension, and a class of distinguished triangles, of the form $A \to B \to C \overset{+1}{\to}$, satisfying some axioms.

Recall from [31, Tag 09SM] the notion of compact object. Let $\mathcal{T}_{\mathrm{cp}}$ be the full subcategory of compact objects of $\mathcal{T}$. It is a triangulated subcategory of $\mathcal{T}$.

We define the Grothendieck group $\mathbf{K}(\mathcal{T})$ of a triangulated category $\mathcal{T}$ admitting infinite sums as the free abelian group generated by isomorphism classes of objects of $\mathcal{T}_{\mathrm{cp}}$ with relations $[B] = [A] + [C]$ for every distinguished triangle

$$A \to B \to C \overset{+1}{\to} .$$

As for every compact object A, the triangle $A \to 0 \to A[1] \overset{+1}{\to}$ is distinguished, $[A[1]] = -[A] \in \mathbf{K}(\mathcal{T})$, hence the suspension is idempotent in $\mathbf{K}(\mathcal{T})$. Moreover, since we have, for every $A, B \in \mathcal{T}_{\mathrm{cp}}$, a distinguished triangle $A \to A \oplus B \to B \overset{+1}{\to}$, we have $[A \oplus B] = [A] + [B]$. If $\mathcal{T}$ is moreover a monoidal triangulated category, then $\mathbf{K}(\mathcal{T})$ inherits a ring structure induced by tensor product.

3.2 Stable category of motives

All schemes are separated and of finite type. Fix a scheme S. Denote by $\mathrm{SH}_{\mathfrak{M}}(S)$ the stable category of motivic sheaves over S for the Nisnevich topology and coefficients $\mathfrak{M}$, as studied by Ayoub [2, Définition 4.5.21]. The two main examples are if $\mathfrak{M}$ is the category of simplicial spectra, in which case $\mathrm{SH}_{\mathfrak{M}}(S)$ is the stable homotopy category (without transfers) of Morel-Voevodsky introduced in [27]. The other one is if $\mathfrak{M}$ is the category of complexes of Λ-modules, for some ring Λ. In this case we set $\mathrm{SH}_{\mathfrak{M}}(S) = \mathrm{DA}(S, \Lambda)$.

The category $\mathrm{SH}_{\mathfrak{M}}(S)$ is triangulated, denote by $-[1]$ its suspension functor. It is also equipped with a Tate twist $-(-1)$ with is an autoequivalence. The categories $\mathrm{SH}_{\mathfrak{M}}(-)$ possess various functorialities. If $f : X \to Y$ is a morphism of schemes, then the pull-back f^* and the push-forward f_* defined at the level of sheaves induce functors $f^* : \mathrm{SH}_{\mathfrak{M}}(Y) \to \mathrm{SH}_{\mathfrak{M}}(X)$ and $f_* : \mathrm{SH}_{\mathfrak{M}}(X) \to \mathrm{SH}_{\mathfrak{M}}(Y)$, f_* is a right adjoint to f^*. Assuming we work over a base scheme of characteristic zero, Ayoub [1] has constructed a six functors formalism for $\mathrm{SH}_{\mathfrak{M}}(-)$. In particular, he defines extraordinary push-forward $f_!$ and pull-back $f^!$ that satisfy various compatibilities. See also [11] for the definition of the shriek functors in the non-projective case.

The homological (resp. cohomological, homological with compact support, cohomological with compact support) motive of X is defined as $\mathrm{M}_S(X) := f_! f^!(\mathbb{1}_S)$ (resp. $\mathrm{M}_S^\vee(X) := f_* f^*(\mathbb{1}_S)$, $\mathrm{M}_{S,c}^\vee(X) := f_! f^*(\mathbb{1}_S)$, $\mathrm{M}_{S,c}(X) := f_* f^!(\mathbb{1}_S)$). For X smooth over S, $\mathrm{M}_S(X)$ and $\mathrm{M}_S^\vee(X)$ can in fact be defined using only the suspension functor Sus_T^0 and the internal Hom $\underline{\mathrm{Hom}}$.

The following motivic realization has already been considered by Ivorra and Sebag.

Proposition 3.1 ([22, Lemma 2.1]) *Let S be a k-scheme. There is a unique ring morphism*

$$\chi_S : \mathbf{K}(\mathrm{Var}_S) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S))$$

such that $\chi_S([X]) = \mathrm{M}_{S,c}^\vee(X)$ for any S-scheme $f : X \to S$.

Proposition 3.2 *Let $f : X \to S$ be a smooth morphism of pure relative dimension d. Then*

$$[\mathrm{M}_{S,c}^\vee(X)] = [\mathrm{M}_S(X)(-d)] \in \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S)).$$

Proof By definition, $\mathrm{M}_{S,c}^\vee(X) = f_! f^*(\mathbb{1}_S)$ and $f_! = f_* \mathrm{Th}^{-1}(\Omega_f)$, where Ω_f is the bundle of relative differentials of f and $\mathrm{Th}(\Omega_f)$ its associated Thom equivalence. As $\mathrm{M}_{S,c}(-)$ is additive and Ω_f is locally free, we can assume Ω_f is free (of rank d). In that case, $\mathrm{Th}^{-1}(\Omega_f) = (-d)[-2d]$. The result now follows because the suspension function is idempotent in the Grothendieck ring. □

3.3 From $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ to $\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$

Let $X = \mathrm{Spec}(A)$ be a k-scheme of finite type, $r \in \mathbb{N}^*$ and $f \in A^\times$. We denote $Q_r^{gm}(X, f)$ the $\mathbb{G}_{mk}$-scheme

$$\mathrm{Spec}(A[T, T^{-1}, V]/(V^r - fT)) \to \mathbb{G}_{mk} = \mathrm{Spec}(k[T, T^{-1}]).$$

More generally, we define by gluing for $X = \mathrm{Spec}(A)$ a k-scheme of finite type, $r \in \mathbb{N} \backslash \{0\}$ and $f \in \Gamma(X, \mathcal{O}_X^\times)$ the $\mathbb{G}_{mk}$-scheme $Q_r^{gm}(X, f)$.

Let $\mathrm{QUSH}_{\mathfrak{M}}(k)$ be the triangulated subcategory of $\mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{mk})$ with infinite sums spanned by objects $\mathrm{Sus}_T^p(Q_r^{gm}(X, f) \otimes A_{\mathrm{cst}})$ for X smooth k-scheme and $A \in \mathcal{E}$.

Here, $\mathcal{E}$ is a set of homotopically compacts objects of $\mathfrak{M}$ generating the homotopy category of $\mathfrak{M}$, see [3, Définition 1.2.31] for details. Let $q : \mathbb{G}_{m\,k} \to \mathrm{Spec}(k)$ be the structural projection and $1 : \mathrm{Spec}(k) \to \mathbb{G}_{m\,k}$ its unit section.

Proposition 3.3 *There is a unique ring morphism*

$$\chi_{\hat{\mu}} : \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$$

such that for X a smooth k-scheme, $f \in \Gamma(X, \mathcal{O}_X^{\times})$ and $r \in \mathbb{N}\backslash\{0\}$, the class of $X[V]/(V^r - f)$ (with the μ_r-action on V) is send to $[\mathrm{M}^{\vee}_{\mathbb{G}_{m\,k},c}(Q_r^{gm}(X, f))]$.

Proof The ring morphism

$$\chi_{\hat{\mu}} : \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{m\,k}}) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{m\,k})).$$

is defined by composition of maps from Corollary 2.9 and Proposition 3.1.

It suffices to show that the image of this morphism lies in $\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$. From the proof of Proposition 2.7, $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ is generated by classes of $X[V]/(V^r = f)$ as in the statement of the proposition. Hence it suffices to show that

$$[\mathrm{M}_{\mathbb{G}_{m\,k},c}(Q_r^{gm}(X, f))] \in \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).$$

But $\mathrm{QUSH}_{\mathfrak{M}}(k)$ is the triangulated subcategory with infinite sums generated by the set of objects $\mathrm{Sus}_T^p(Q_r^{gm}(X, f) \otimes A_{\mathrm{cst}})$, which is stable by Tate twist, hence by Proposition 3.2, $[\mathrm{M}_{\mathbb{G}_{m\,k},c}(Q_r^{gm}(X, f))] \in \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$. □

Lemma 3.4 *We have a commutative diagram:*

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\chi_{\hat{\mu}}"'] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k)),
\end{tikzcd}$$

where $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_k)$ is induced by the forgetful functor and

$$1^* : \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k))$$

is the composite

$$\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \longrightarrow \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{m\,k})) \xrightarrow{1^*} \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k)).$$

Proof Recall that the composition $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{m\,k}}) \xrightarrow{1^*} \mathbf{K}(\mathrm{Var}_k)$ is the forgetful map. Hence it suffices to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_{\mathbb{G}_{m\,k}}) \arrow[rr] \arrow[d, "\chi_{\mathbb{G}_{m\,k}}"'] & & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{m\,k})) \arrow[rr, "1^*"'] & & \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k)),
\end{tikzcd}$$

where the upper map is induced by picking the fiber above 1 of a $\mathbb{G}_{m\,k}$-variety. For $X \in \mathrm{Var}_{\mathbb{G}_{m\,k}}$, we consider the following cartesian square:

$$\begin{tikzcd}
X' \arrow[r, "1'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
k \arrow[r, "1"'] & \mathbb{G}_{m\,k}.
\end{tikzcd}$$

One needs to show that $1^*\mathrm{M}_{\mathbb{G}_{m\,k},c}(X) \simeq \mathrm{M}_{k,c}(X')$. By [1, Scholie 1.4.3], there is a 2-isomorphism $f'_!1'^* \cong 1^* f_!$. Hence

$$1^*\mathrm{M}_{\mathbb{G}_{m\,k},c}(X) = 1^* f_! f^* \mathbb{1}_{\mathbb{G}_{m\,k}} \simeq f'_! 1'^* f^* \mathbb{1}_{\mathbb{G}_{m\,k}} \simeq f'_! f'^* 1^* \mathbb{1}_{\mathbb{G}_{m\,k}} = \mathrm{M}_{k,c}(X').$$

□

3.4 Rigid motives

We use the formalism of Tate's rigid analytic geometry [32]. For details and proofs, see also [9,18].

Ayoub builds in [3] a category $\mathrm{RigSH}_{\mathfrak{M}}(K)$ of rigid motives over K, in an analogous manner of $\mathrm{SH}_{\mathfrak{M}}(K)$, but with starting point rigid analytic K-varieties instead of K-schemes, and replacing $\mathbb{A}^1$-invariance by $\mathbb{B}^1$-invariance, where $\mathbb{B}^1$ represent the closed unit ball.

As in the algebraic case, one needs to choose a category of coefficients $\mathfrak{M}$, the main examples being $\mathrm{RigSH}(K)$ and $\mathrm{RigDA}(K, \Lambda)$. We also have the suspension functor $\mathrm{Sus}^r_{T^{\mathrm{an}}}(-)$, the tensor $-\otimes A$ has a right adjoint $\underline{\mathrm{Hom}}(A, -)$ and the Tate twist $-(-1)$ is defined.

We define for X a smooth rigid K-variety its homological motive by $\mathrm{M}_{\mathrm{Rig}}(X) = \mathrm{Sus}^0_{T^{\mathrm{an}}}(X \otimes \mathbb{1}_K)$ and its cohomological motive by $\mathrm{M}^\vee_{\mathrm{Rig}}(X) = \underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{Rig}}(X), \mathbb{1}_K)$.

To our knowledge a full six functor formalism is not available in this context, the missing ingredients being $f_!$ and $f^!$. Hence there is no already defined notion of compactly supported rigid motive.

The analytification functor induces a (monoidal triangulated) functor

$$\mathrm{Rig}^* : \mathrm{SH}_{\mathfrak{M}}(K) \to \mathrm{RigSH}_{\mathfrak{M}}(K).$$

Such a functor is compatible in a strong sense with the six operations defined on $\mathrm{SH}_{\mathfrak{M}}(-)$, see [3, Théorème 1.4.40].

Let X be a smooth k-scheme, $f \in \Gamma(X, \mathcal{O}_X^\times)$ and $p \in \mathbb{N}^*$. Then define $Q_p^{\mathrm{for}}(X, f)$ as the t-adic completion of the R-scheme $X \times_k R[V]/(V^p - tf)$, and $Q_p^{\mathrm{Rig}}(X, f)$ the generic fiber of $Q_p^{\mathrm{for}}(X, f)$. Define also $Q_p^{\mathrm{an}}(X, f)$ as the analytification of $X \times_k K[V]/(V^p - tf)$. There is an open immersion of rigid K-varieties $Q_p^{\mathrm{Rig}}(X, f) \to Q_p^{\mathrm{an}}(X, f)$.

Theorem 3.5 ([3, Théorème 1.3.11]) *Let X be a smooth k-scheme, $f \in \Gamma(X, \mathcal{O}_X^\times)$, and p a positive integer. Then the inclusion $Q_p^{\mathrm{Rig}}(X, f) \to Q_p^{\mathrm{an}}(X, f)$ induces an isomorphism*

$$\mathrm{M}_{\mathrm{Rig}}(Q_p^{\mathrm{Rig}}(X, f)) \simeq \mathrm{M}_{\mathrm{Rig}}(Q_p^{\mathrm{an}}(X, f)).$$

Define a functor $\mathfrak{F} : \mathrm{QUSH}_{\mathfrak{M}}(k) \to \mathrm{RigSH}_{\mathfrak{M}}(K)$ as the composite

$$\mathfrak{F} : \mathrm{QUSH}_{\mathfrak{M}}(k) \to \mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{mk}) \underset{\pi^*}{\to} \mathrm{SH}_{\mathfrak{M}}(K) \underset{\mathrm{Rig}^*}{\to} \mathrm{RigSH}_{\mathfrak{M}}(K),$$

where $\pi : \mathrm{Spec}(K) \to \mathbb{G}_{mk}$ corresponds to the ring morphism $k[T, T^{-1}] \to K = k(\!(t)\!)$ sending T to t. Observe that $\mathfrak{F}$ sends the generators $\mathrm{M}_{\mathbb{G}_{mk}}(Q_p^{\mathrm{gm}}(X, f)) \in \mathrm{QUSH}_{\mathfrak{M}}(k)$ to $\mathrm{M}_{\mathrm{Rig}}(Q_p^{\mathrm{an}}(X, f))$. One of the main results of Ayoub [3] is the following theorem.

Theorem 3.6 ([3, Scholie 1.3.26]) *The functor $\mathfrak{F} : \mathrm{QUSH}_{\mathfrak{M}}(k) \to \mathrm{RigSH}_{\mathfrak{M}}(K)$ is an equivalence of categories.*

Denote by $\mathfrak{R}$ a quasi-inverse of $\mathfrak{F}$.

4 Realization map for definable sets

This aim of this section is to define a morphism $\chi_{\mathrm{Rig}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$. We will first define it on $\mathbf{K}(\Gamma[*])$ and $\mathbf{K}(\mathrm{RES}_K[*])$ in Sects. 4.1 and 4.2. Using Hrushovski and Kazhdan's isomorphism, this will allow us to define it on $\mathbf{K}(\mathrm{VF}_K)$ in Sect. 4.3. Section 4.4 is devoted to the proof of Theorem 1.1 via the study of motives of tubes in a semi-stable situation, the main results are grouped in Sect. 4.5. The last Sect. 4.6 is devoted to the definition of two other realization maps $\mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$ and the statement of an analog of Theorem 1.1 for them.

4.1 The Γ part

Recall the o-minimal Euler characteristics eu and eu_c defined above Proposition 2.5. We use the notations from [3]. For a rigid K-variety X and $f, g \in \mathcal{O}(X)$, we denote by $\mathbb{B}_X(o, |f|)$ (resp. $\mathbb{C}r_X(o, |f|, |f|)$, $\partial\mathbb{B}_X(o, |f|)$) the family parametrized by X of closed balls centered at the origin and radius $|f|$ (resp. annuli centered at the origin and radius $|f|$ and $|g|$, thin annuli centered at the origin and radius $|f|$).

Definition 4.1 If $[X] \in \mathbf{K}(\Gamma[*])$, with $X \in \Gamma[d]$, define

$$\chi^{\Gamma}_{\mathrm{Rig}}(X) = \mathrm{eu}_c(X)[\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)^d)(-d)] \in \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

and

$$\chi'^{\Gamma}_{\mathrm{Rig}}([X]) = \mathrm{eu}(X)[\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)^d)] \in \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

Hence we get two ring morphisms

$$\chi^{\Gamma}_{\mathrm{Rig}}, \chi'^{\Gamma}_{\mathrm{Rig}} : \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

It is well defined because $\mathbf{K}(\Gamma[*])$ is naturally graded and by additivity of Euler characteristic.

Proposition 4.2 *Let $X \subseteq \Gamma^n$ be a convex bounded polytope. If X is closed, then*

$$\chi^{\Gamma}_{\mathrm{Rig}}([X]) = [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})(-n)] \text{ and } \chi'^{\Gamma}_{\mathrm{Rig}}([X]) = [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})].$$

If X is open, then

$$\chi^{\Gamma}_{\mathrm{Rig}}([X]) = (-1)^n[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})(-n)]$$

and

$$\chi'^{\Gamma}_{\mathrm{Rig}}([X]) = (-1)^n[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})].$$

Proof If X is empty, $\mathrm{eu}(X) = \mathrm{eu}_c(X) = 0$ hence the proposition is verified. Hence we can suppose X in non-empty. We have $\mathrm{eu}(X) = \mathrm{eu}_c(X) = 1$ if X is closed, and $\mathrm{eu}(X) = \mathrm{eu}_c(X) = (-1)^n$ if X is open. Hence the result follows from the following Lemma 4.3. □

Lemma 4.3 *Let $X \subseteq \Gamma^n$ be a non-empty convex polytope, either closed or open. Then*

$$\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}}) \simeq \mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)^n).$$

Proof We first assume that X is closed. We work by induction on n. If $n = 1$, then

$$X = \{x \mid \alpha \leq px \leq \beta\}$$

for $\alpha, \beta \in \mathbb{Z}$, $p \in \mathbb{N}$. Hence

$$\mathrm{v}^{-1}(X)^{\mathrm{Rig}} = \mathbb{C}r(o, \left|\pi^{\beta}\right|^{1/p}, \left|\pi^{\alpha}\right|^{1/p}).$$

By [3, Proposition 1.3.4], $\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}}) = \mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1))$.

Suppose now that the result is known for $n-1$. There are finitely many affine functions $(h_i)_{i\in I_0}$ with $\mathbb{Z}$-coefficients such that

$$X = \left\{x \in \Gamma^n \mid h_i(x) \geq 0, i \in I_0\right\}.$$

We can rewrite X as

$$X = \left\{(x, y) \in \Gamma \times \Gamma^{n-1} \mid p_i x \leq f_i(y), q_j x \geq g_j(y), i \in I, j \in J\right\}$$

for some (possibly empty) finite sets I, J, integers $p_i, q_j \in \mathbb{N}$ and affine functions $f_i, g_j : \Gamma^{n-1} \to \Gamma$ with $\mathbb{Z}$ coefficients. Now observe that the projection of X on the last $n-1$-th coordinates is

$$Y = \left\{y \in \Gamma^{n-1} \mid \forall (i, j) \in I \times J, p_i g_j(y) \leq q_j f_i(y)\right\}.$$

It satisfies the hypotheses of the proposition hence we get that $[\mathrm{M_{Rig}}(\mathrm{v}^{-1}(Y)^{\mathrm{Rig}})] = [\mathrm{M_{Rig}}(\partial\mathbb{B}(o, 1)^{n-1})]$ by induction.

We set $I' = \{i \in I \mid p_i > 0\}$ and $J' = \left\{j \in J \mid q_j > 0\right\}$. Observe that

$$X = \left\{(x, y) \in \Gamma \times Y \mid p_i x \leq f_i(y), q_j x \geq g_j(y), i \in I', j \in J'\right\}.$$

Set $X_{i,j} = \mathbb{C}r_{Y^{\mathrm{Rig}}}(o, \left|\tilde{f_i}\right|^{1/p_i}, \left|\tilde{g_j}\right|^{1/q_j})$, where we used the notation that if

$$f(y_1, \ldots, y_{n-1}) = b + a_1 y_1 + \cdots + a_{n-1} y_{n-1},$$

then

$$\tilde{f}(x_1, \ldots, x_{n-1}) = t^b \cdot x_1^{a_1} \cdot \cdots \cdot x_{n-1}^{a_{n-1}}.$$

We have now

$$X^{\mathrm{Rig}} = \bigcap_{(i,j)\in I'\times J'} X_{i,j}.$$

Set

$$Y_{i,j} = \left\{y \in Y^{\mathrm{Rig}} \mid \forall (i', j') \in I \times J, \left|\tilde{f_i}\right|^{1/p_i} \geq \left|\tilde{f_{i'}}\right|^{1/p_{i'}}, \left|\tilde{g_j}\right|^{1/q_j} \leq \left|\tilde{g_{j'}}\right|^{1/q_{j'}}\right\}.$$

The $(Y_{i,j})_{(i,j)\in I\times J}$ form an admissible cover of Y^{Rig}, indeed, $Y_{i,j}$ is defined in Y^{Rig} by some non-strict valuative inequalities, if D is a rational domain, the standard cover of D induced by functions used to define D and the functions used to define the $Y_{i,j}$ gives the required refinement of $(D \cap X_{i,j})_{(i,j)\in I\times J}$.

Then it suffices to show the result for $X^{\mathrm{Rig}} \cap (Y_{i,j} \times_K \mathbb{A}_K^{1,\mathrm{an}})$. But we have then

$$X^{\mathrm{Rig}} \cap (Y_{i,j} \times_K \mathbb{A}_K^{1,\mathrm{an}}) = \mathbb{C}r_{Y_{i,j}}\left(o, \left|\tilde{f_i}\right|^{1/p_i}, \left|\tilde{g_j}\right|^{1/q_j}\right)$$

hence the result follows from [3, Proposition 1.3.4], which gives

$$\mathrm{M}_{\mathrm{Rig}}\left(\mathbb{C}r_{Y_{i,j}}\left(o,\left|\tilde{f_i}\right|^{1/p_i},\left|\tilde{g_j}\right|^{1/q_j}\right)\right)\simeq \mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}_{Y_{i,j}}(o,1)).$$

Suppose now that X is an open polyhedron. We work similarly by induction on n. If $n=1$,

$$X=\{x \mid \alpha < px < \beta\}$$

for $\alpha,\beta\in\mathbb{Z}$, $p\in\mathbb{N}$. Hence

$$\mathrm{v}^{-1}(X)^{\mathrm{Rig}}=\bigcup_{r_0\leq r<1}\mathbb{C}r(o,r^{-1}\left|\pi^{\beta}\right|^{1/p}r,r\left|\pi^{\alpha}\right|^{1/p}),$$

for some $r_0<1$ close enough to 1. It is thus enough to show that the inclusion

$$\partial\mathbb{B}(o,r_1)\hookrightarrow\mathbb{C}r\left(o,r^{-1}\left|\pi^{\beta}\right|^{1/p},r\left|\pi^{\alpha}\right|^{1/p}\right)$$

induces an isomorphism in $\mathrm{RigSH}_{\mathfrak{M}}(K)$, for $r_0\leq r<1$ and $r_1\in\mathbb{R}_+^*$ such that $r_0^{-1}\left|\pi^{\beta}\right|^{1/p}\leq r_1\leq r_0|\pi^{\alpha}|^{1/p}$. This is [3, Proposition 1.3.4].

Suppose now that the result is known for $n-1$. There are finitely many affine functions h_i, $i\in I_0$ with $\mathbb{Z}$-coefficients such that

$$X=\left\{x\in\Gamma^n \mid h_i(x)>0\right\}.$$

Proceed now as in the closed case, denote by Y the projection of X on the last $n-1$-th coordinates. By induction, it suffices to show that $\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})\simeq\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}_Y(o,1))$. Define I, J, $Y_{i,j}$, $X_{i,j}$ as above, replacing large inequalities by strict ones where needed. We will show that $\mathrm{M}_{\mathrm{Rig}}(X_{i,j})\simeq\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}_{Y_{i,j}}(o,1))$, with these isomorphisms compatible on $X_{i,j}\cap X_{i',j'}$. We can find $r_0\in\mathbb{Q}$ with $0<r_0<1$ such that for each $(i,j)\in I\times J$, and $y\in Y_{i,j}$, $r_0{}^{-1}\left|\tilde{f_i}(y)\right|^{1/p_i}\leq r_0\left|\tilde{g_j}(y)\right|^{1/q_j}$. We can moreover choose for each $(i,j)\in I\times J$ a monomial function $\tilde{h_{i,j}}$ on $Y_{i,j}$ such that for all $y\in Y_{i,j}$,

$$r_0{}^{-1}\left|\tilde{f_i}(y)\right|^{1/p_i}\leq\left|\tilde{h_{i,j}}(y)\right|\leq r_0\left|\tilde{g_j}(y)\right|^{1/q_j}$$

and assume that these functions coincide on $Y_{i,j}\cap Y_{i',j'}$.

We now have

$$X_{i,j}\cap(Y_{i,j}\times_K\mathbb{A}_K^{1,\mathrm{an}}=\cup_{r_0<r<1}\mathbb{C}r_{Y_{i,j}}\left(o,r^{-1}\left|\tilde{f_i}(y)\right|^{1/p_i},r\left|\tilde{g_j}(y)\right|^{1/q_j}\right),$$

hence it suffices to show that the immersion

$$\partial\mathbb{B}_{Y_{i,j}}(o, \tilde{h_{i,j}}) \hookrightarrow \mathbb{C}r_{Y_{i,j}}\left(o, r^{-1}\left|\tilde{f}_i(y)\right|^{1/p_i}, r\left|\tilde{g_j}(y)\right|^{1/q_j}\right)$$

induces an isomorphism in $\mathrm{RigSH}_{\mathfrak{M}}(K)$, which follows once again from [3, Proposition 1.3.4]. □

We will not use it, but Proposition 4.2 can be extended to all closed polyhedral complexes.

Proposition 4.4 *Let $X \subseteq \Gamma^n$ be (the realization of) a bounded closed polyhedral complex. Then*

$$\chi_{\mathrm{Rig}}^{\Gamma}([X]) = [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})(-d)].$$

Proof In view of the definition of $\chi_{\mathrm{Rig}}^{\Gamma}$, we need to show that

$$[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})] = \mathrm{eu}(X)[\partial\mathbb{B}(o, 1)^n].$$

We work by double induction on the maximal dimension of simplexes in X and the number of simplexes of maximal dimension. Let $\Delta \subset X$ be a simplex of maximal dimension. Set $Y = X\backslash\Delta^\circ$, with Δ° the interior of Δ, $\partial\Delta = \Delta\backslash\Delta^\circ$.

Then $(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}}, \mathrm{v}^{-1}(Y)^{\mathrm{Rig}})$ is an admissible cover of $\mathrm{v}^{-1}(X)^{\mathrm{Rig}}$, with intersection $\mathrm{v}^{-1}(\partial\Delta)^{\mathrm{Rig}}$ hence

$$[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})] = [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})] + [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(Y)^{\mathrm{Rig}})] - [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\partial\Delta)^{\mathrm{Rig}})].$$

By Lemma 4.3, $[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})] = [\partial\mathbb{B}(o, 1)^n]$. Apply the induction hypothesis to get $[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\partial\Delta)^{\mathrm{Rig}})] = (1 - (-1)^d)([\partial\mathbb{B}(o, 1)^n]$ and $[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(Y)^{\mathrm{Rig}})] = \mathrm{eu}(Y)[\partial\mathbb{B}(o, 1)^n]$. We have the result, since $\mathrm{eu}(X) = \mathrm{eu}(Y) + (-1)^d$. □

4.2 The RES part

Definition 4.5 Define ring morphisms

$$\chi_{\mathrm{Rig}}^{\mathrm{RES}} : \mathbf{K}(\mathrm{RES}_K)[*] \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

and

$$\chi_{\mathrm{Rig}}^{\prime\mathrm{RES}} : \mathbf{K}(\mathrm{RES}_K)[*] \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

by the formulas, for X a smooth k-variety of pure dimension r, $f \in \Gamma(X, \mathcal{O}_X^\times)$, $m \in \mathbb{N}^*$,

$$\chi_{\mathrm{Rig}}^{\mathrm{RES}}([\mathcal{Q}_m^{\mathrm{RV}}(X, f)]_n) = [\mathcal{Q}_m^{\mathrm{Rig}}(X, f)(-r)]$$

and

$$\chi'^{\mathrm{RES}}_{\mathrm{Rig}}([Q^{\mathrm{RV}}_m(X, f)]_n) = [Q^{\mathrm{Rig}}_m(X, f)(n-r)].$$

As the $[Q^{\mathrm{RV}}_m(X, f)]_n$ generate $\mathbf{K}(\mathrm{RES}_K[*])$ by Corollary 2.8, one only needs to show that the maps are well defined. But we can check that they coincide with the composite

$$\mathbf{K}(\mathrm{RES}_K[*]) \to !\mathbf{K}(\mathrm{RES}_K)[\mathbb{L}^{-1}] \xrightarrow{\Theta} \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})[\mathbb{L}^{-1}]$$
$$\xrightarrow{\chi_{\hat{\mu}}} \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \xrightarrow{\mathfrak{F}} \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

where the map $\mathbf{K}(\mathrm{RES}_K[*]) \to !\mathbf{K}(\mathrm{RES}_K)[\mathbb{A}^1_k]^{-1}$ is $[X]_n \mapsto [X]$ for $\chi^{\mathrm{RES}}_{\mathrm{Rig}}$ and $[X]_n \mapsto [X]\mathbb{L}^{-n}$ for $\chi'^{\mathrm{RES}}_{\mathrm{Rig}}$. The maps Θ, $\chi_{\hat{\mu}}$, $\mathfrak{F}$ are respectively defined in Propositions 2.7, 3.3 and Theorem 3.6. Note that this also implies that it is a morphism of rings.

4.3 Definition of $\chi^{\mathrm{RV}}_{\mathrm{Rig}}$

Recall the isomorphism $\mathbf{K}(\mathrm{RES}_K[*]) \otimes_{\mathbf{K}(\Gamma^{\mathrm{fin}}[*])} \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RV}_K[*])$. To define a ring morphism $\mathbf{K}(\mathrm{RV}_K[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$, it suffices to specify rings morphisms

$$\mathbf{K}(\mathrm{RES}_K[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \text{ and } \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

that coincide on $\mathbf{K}(\Gamma^{\mathrm{fin}}[*])$.

Definition 4.6 Define

$$\chi^{\mathrm{RV}}_{\mathrm{Rig}} : \mathbf{K}(\mathrm{RV}_K[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

using the morphisms $\chi^{\mathrm{RES}}_{\mathrm{Rig}}$ and $\chi^{\Gamma}_{\mathrm{Rig}}$ and

$$\chi'^{\mathrm{RV}}_{\mathrm{Rig}} : \mathbf{K}(\mathrm{RV}_K[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

using the morphisms $\chi'^{\mathrm{RES}}_{\mathrm{Rig}}$ and $\chi'^{\Gamma}_{\mathrm{Rig}}$.

To show that it is well defined, one needs to check that if $A \subseteq \Gamma^n$ is definable and finite, then $\chi^{\Gamma}_{\mathrm{Rig}}([A]) = \chi^{\mathrm{RES}}_{\mathrm{Rig}}([\mathrm{v_{rv}}^{-1}(A)]_n)$ and $\chi'^{\Gamma}_{\mathrm{Rig}}([A]) = \chi'^{\mathrm{RES}}_{\mathrm{Rig}}([\mathrm{v_{rv}}^{-1}(A)]_n)$. By additivity, one can assume $A = \{\alpha\}$. Hence it follows from the following lemma.

Lemma 4.7 *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma^n$ be definable. Then*

$$\chi^{\mathrm{RES}}_{\mathrm{Rig}}([\mathrm{v_{rv}}^{-1}(\{\alpha\})] = [\partial\mathbb{B}(o, 1)^n(-n)]$$

and

$$\chi'^{\mathrm{RES}}_{\mathrm{Rig}}([\mathrm{v_{rv}}^{-1}(\{\alpha\})] = [\partial\mathbb{B}(o,1)^n]$$

Proof We have $\mathrm{v_{rv}}^{-1}(\{\alpha\}) = \mathrm{v_{rv}}^{-1}(\alpha_1) \times \cdots \times \mathrm{v_{rv}}^{-1}(\alpha_n)$. Because $\chi^{\mathrm{RES}}_{\mathrm{Rig}}$ and $\chi'^{\mathrm{RES}}_{\mathrm{Rig}}$ are ring morphisms and $[\partial\mathbb{B}(o,1)^n(-n)] = [\partial\mathbb{B}(o,1)(-1)]^n$, we can assume $n = 1$. Suppose $\alpha = k/m$, with $k \in \mathbb{Z}$ and $m \in \mathbb{N}^*$ relatively prime. Let $a, b \in \mathbb{Z}$ be such that $am + bk = 1$. In this case, we have

$$\mathrm{v_{rv}}^{-1}(k/m) \simeq \left\{(z,u) \in \mathrm{v_{rv}}^{-1}(1/m) \times \mathrm{v_{rv}}^{-1}(0) \mid z^m = \mathbf{t}u^b\right\} = Q^{\mathrm{RV}}_m(\mathbb{G}_{mk}, u^b)$$

via the isomorphism $w \in \mathrm{v_{rv}}^{-1}(k/m) \mapsto (\mathbf{t}^a w^b, \mathbf{t}^{-k} w^m)$. But now, $Q^{\mathrm{Rig}}_m(\mathbb{G}_{mk}, u^b) \simeq \partial\mathbb{B}(o, k/m)$ via the isomorphism $(z,u) \mapsto z^k u^a$ and

$$\mathrm{M_{Rig}}(\partial\mathbb{B}(o,k/m)) \simeq \mathrm{M_{Rig}}(\partial\mathbb{B}(o,1))$$

by [3, Proposition 1.3.4]. □

Remark 4.8 If $X \subseteq \mathrm{RV}^n$ is definable, then $\chi_{\mathrm{Rig}}([X]_m) = \chi_{\mathrm{Rig}}([X]_n)$ for any $m \geq n$, hence χ_{Rig} does not depend on the grading in $\mathbf{K}(\mathrm{RV}[*])$, it is in fact defined on $\mathbf{K}(\mathrm{RV})$.

Proposition 4.9 *The ring morphisms* $\chi^{\mathrm{RV}}_{\mathrm{Rig}}$ *and* $\chi'^{\mathrm{RV}}_{\mathrm{Rig}}$ *of Definition 4.6 induce ring morphisms*

$$\chi^{\mathrm{RV}}_{\mathrm{Rig}}, \chi'^{\mathrm{RV}}_{\mathrm{Rig}} : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I_{sp}} \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

Proof We need to check that the generator of $\mathrm{I_{sp}}$ vanishes under $\chi^{\mathrm{RV}}_{\mathrm{Rig}}$ and $\chi'^{\mathrm{RV}}_{\mathrm{Rig}}$.

We have $\mathrm{M_{Rig}}(\{1\}) = \mathrm{M_{Rig}}(\mathrm{Spm}(K)) = \mathbb{1}_K$, hence

$$\chi^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1) = \chi^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0) = \chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0) = [\mathbb{1}_K]$$

and $\chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1) = [\mathbb{1}_K(1)]$.

Moreover, $\mathrm{RV}^{>0} = \mathrm{v}^{-1}((0,+\infty))$ and $\mathrm{eu}_c((0,+\infty)) = 0$, $\mathrm{eu}((0,+\infty)) = -1$. Hence $\chi^{\mathrm{RV}}_{\mathrm{Rig}}([\mathrm{RV}^{>0}]_1) = 0$ and $\chi^{\mathrm{RV}}_{\mathrm{Rig}}([\mathrm{RV}^{>0}]_1) = -[\mathrm{M_{Rig}}(\partial\mathbb{B}(o,1))]$, which implies that

$$\chi^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1) = \chi^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0 + [\mathrm{RV}^{>0}]_1).$$

For $\chi'^{\mathrm{RV}}_{\mathrm{Rig}}$, we have by construction

$$\mathrm{M_{Rig}}(\partial\mathbb{B}(o,1)) = \mathrm{M_{Rig}}(\mathrm{Spm}(K)) \oplus \mathrm{M_{Rig}}(\mathrm{Spm}(()K))(1)[1].$$

Hence we get that in $\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$, the equality

$$[\mathrm{M_{Rig}}(\mathrm{Spm}(K))] = [\mathrm{M_{Rig}}(\mathrm{Spm}(K))(1)] + [\mathrm{M_{Rig}}(\partial\mathbb{B}(o,1))]$$

holds, which implies that

$$\chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0) = \chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1] - \chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\mathrm{RV}^{>0}]_1),$$

hence

$$\chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1) = \chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0 + [\mathrm{RV}^{>0}]_1).$$

$\square$

Recall the isomorphism $\oint : K(\mathrm{VF}_K) \to K(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}}$.

Definition 4.10 Define χ_{Rig} and $\chi'_{\mathrm{Rig}}\mathbf{K}(\mathrm{VF}) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$ by $\chi_{\mathrm{Rig}} = \chi^{\mathrm{RV}}_{\mathrm{Rig}} \circ \oint$ and $\chi'_{\mathrm{Rig}} = \chi'^{\mathrm{RV}}_{\mathrm{Rig}} \circ \oint$.

For any irreducible smooth k-variety X of dimension d, $f \in \mathcal{O}^{\times}_X(X)$ and $r \in \mathbb{N}^*$, set

$$Q^{\mathrm{VF}}_r(X, f) = \left\{(x, y) \in X(\mathrm{VF}) \times \mathrm{VF} \mid y^r = tf(x)\right\}.$$

Proposition 4.11 *For any irreducible smooth k-variety X of dimension d, $f \in \mathcal{O}^{\times}_X(X)$ and $r \in \mathbb{N}^*$,*

$$\chi_{\mathrm{Rig}}(Q^{\mathrm{VF}}_r(X, f)) = [\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_r(X, f))(-d)].$$

Proof Since $\oint[Q^{\mathrm{VF}}_r(X, f)] = [Q^{\mathrm{RV}}_r(X, f)]_r$, it simply follows from the definition of $\chi^{\mathrm{RES}}_{\mathrm{Rig}}$. $\square$

Proposition 4.12 *Let $\Delta \subset \Gamma^n$ be defined by linearly independent affine equations $l_i > 0$ for $i = 0, \ldots, n$. Then*

$$\chi_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)) = (-1)^n[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})(-n)].$$

Proof It follows from Proposition 4.2. $\square$

Theorem 4.13 *There are commutative squares*

$$\begin{tikzcd}
\mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}_c\circ\oint"] \arrow[d, "\chi_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_k) \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}\circ\oint"] \arrow[d, "\chi'_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})[\mathbb{A}_k^1]^{-1} \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

Proof We will only show the commutativity of the first diagram, the second being similar.

We need to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{K}(\mathrm{RV}[*])/\mathrm{I_{sp}} \arrow[r, "\Theta\circ\mathcal{E}_c"] \arrow[d, "\chi^{\mathrm{RV}}_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

It suffices to show that the following diagrams are commutative:

$$\begin{tikzcd}
\mathbf{K}(\mathrm{RES}_K[*]) \arrow[r, "\Theta\circ\mathcal{E}_c"] \arrow[d, "\chi^{\mathrm{RES}}_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{K}(\Gamma) \arrow[r, "\Theta\circ\mathcal{E}_c"] \arrow[d, "\chi^{\Gamma}_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

For the first one, we already observed that $\chi^{\mathrm{RES}}_{\mathrm{Rig}}$ does not depend on the grading, hence we need to show that $\mathfrak{R}\circ\chi^{\mathrm{RES}}_{\mathrm{Rig}} = \chi_{\hat{\mu}}\circ\Theta$, as morphisms from $!\mathbf{K}(\mathrm{RES}_K)$ to $\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$. By Corollary 2.8, $!\mathbf{K}(\mathrm{RES}_K)$ is generated by classes of $Q_r^{\mathrm{RV}}(X, f)$, for X a k-variety smooth of pure dimension d, $r \in \mathbb{N}\backslash\{0\}$ and $f \in \Gamma(X, \mathcal{O}_X^{\times})$. The definition of $\chi_{\hat{\mu}}\circ\Theta$ shows that $\chi_{\hat{\mu}}\circ\Theta(Q_r^{\mathrm{RV}}(X, f)) = [\mathrm{M}_{\mathbb{G}_{mk}}(Q_r^{\mathrm{gm}}(X, f))(-\dim(X))]$. From the definition of $\chi^{\mathrm{RES}}_{\mathrm{Rig}}$, $\chi^{\mathrm{RES}}_{\mathrm{Rig}}(Q_r^{\mathrm{RV}}(X, f)) = [\mathrm{M}_{\mathrm{Rig}}(Q_r^{\mathrm{Rig}}(X, f)(-d))]$; from Theorem 3.5, $\mathrm{M}_{\mathrm{Rig}}(Q_r^{\mathrm{Rig}}(X, f) \simeq \mathrm{M}_{\mathrm{Rig}}(Q_r^{\mathrm{an}}(X, f)$, and from the definition of $\mathfrak{R}$, $\mathfrak{R}(\mathrm{M}_{\mathrm{Rig}}(Q_r^{\mathrm{an}}(X, f)) = \mathrm{M}_{\mathbb{G}_{mk}}(Q_r^{\mathrm{gm}}(X, f))$. For the second square, for any $X \subset \Gamma^n$,

$\chi^{\Gamma}_{\mathrm{Rig}}(X) = \mathrm{eu}_c(X)[\partial\mathbb{B}(o,1)^n(-n)]$ and $\mathcal{E}_c(X) = \mathrm{eu}_c(X)[{\mathbb{G}_m}^n_k]$, so it follows from the fact that $\mathfrak{R}[\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)] = [\mathrm{M}_{\mathbb{G}_{mk}}(\mathbb{G}_{mk} \times_k \mathbb{G}_{mk})]$. □

4.4 Motives of tubes

The aim of this section is to compute χ_{Rig} for a quasi-compact smooth rigid K-variety. We will use semi-stable formal models, and in particular tubes of their branches, see [5] or [23] for details on tubes. Denote by $R = k[\![t]\!]$ the valuation ring of K. All the formal R-schemes we consider are assumed to be topologically of finite type.

Let $\mathcal{X}$ be a formal R-scheme. Denote by $\mathcal{X}_\sigma \in \mathrm{Var}_k$ its special fiber and $\mathcal{X}_\eta$ its generic fiber, which is a rigid K-variety. Given an admissible formal R-scheme $\mathcal{X}$, there is a canonical map, called the specialization map (or the reduction map), defined at the level of topological spaces $\mathrm{sp} : \mathcal{X}_\eta \to \mathcal{X}_\sigma$.

Recall from [10, Theorem 4.1] that any separated quasi-compact rigid K-variety admits an admissible formal R-model.

Definition 4.14 Let $\mathcal{X}$ be a formal R-scheme. If D is a locally closed subset of the special fiber, the tube of D in $\mathcal{X}$ is the inverse image $]D[_{\mathcal{X}} := \mathrm{sp}^{-1}(D)$, with its reduced rigid variety structure. It is an open rigid analytic subvariety of $\mathcal{X}_\eta$. When there is no possible confusion, we will denote $]D[_{\mathcal{X}}$ by $]D[$.

If $\mathcal{U}$ is an open formal subscheme of $\mathcal{X}$ such that $D \subset \mathcal{U}_\sigma$, then $]D[_{\mathcal{X}} =]D[_{\mathcal{U}}$. In particular, $]\mathcal{U}_\sigma[_{\mathcal{X}} = \mathcal{U}_\eta$.

Definition 4.15 Let $\mathcal{X}$ be a formal R-scheme of finite type. Say that $\mathcal{X}$ is semi-stable if for every $x \in \mathcal{X}_\sigma$, there is a regular open formal subscheme $\mathcal{U} \subset \mathcal{X}$ containing x and elements $u, t_1, \ldots, t_r \in \mathcal{O}(\mathcal{U})$ such that the following properties hold:

1. u is invertible and there are positive integers $N_1, \ldots, N_r$ such that the following equality holds: $t = ut_1^{N_1} \cdots t_r^{N_r}$,
2. for every non empty $I \subset \{1, \ldots, r\}$, the subscheme $D_I \subseteq \mathcal{U}_\sigma$ defined by equations $t_i = 0$ for $i \in I$ is smooth over k, has codimension $|I| - 1$ in $\mathcal{U}_\sigma$ and contains x.

The irreducible components of $\mathcal{X}_\sigma$ are called its branches.

If $\mathcal{X}$ is a formal R-scheme, $f \in \Gamma(\mathcal{X}, \mathcal{O}_X)$, $\underline{N} \in \mathbb{N}^k$, we define

$$\mathrm{St}^f_{\mathcal{X},\underline{N}} = \mathcal{X}\{T_1, \ldots, T_k\}/(T_1^{N_1} \ldots T_k^{N_k} - f).$$

Let $\mathcal{X}$ be a semi-stable formal R-scheme and $(D_i)_{i\in J}$ be the branches of its special fiber $\mathcal{X}_\sigma$. For any non-empty $I \subset J$, set $D_I = \cap_{i\in I} D_i$ and $D(I) = \cup_{i\in I} D_i$. Set also for $I' \subset J\backslash I$, $D_I^{\circ I'} = D_I \backslash D(I')$ and if $I' = J\backslash I$, simply $D_I^\circ = D_I\backslash D(J\backslash I)$.

Ayoub, Ivorra and Sebag prove the following proposition.

Proposition 4.16 ([4, Theorem 5.1]) *For any non-empty $I \subset J$ and $I' \subset I'' \subset J\backslash I$, the inclusion $]D_I^{\circ I''}[\hookrightarrow]D_I^{\circ I'}[$ induces an isomorphism*

$$\mathrm{M}_{\mathrm{Rig}}(]D_I^{\circ I''}[) \simeq \mathrm{M}_{\mathrm{Rig}}(]D_I^{\circ I'}[).$$

We will mostly use this proposition in the following particular case.

Corollary 4.17 *For any non-empty $I \subseteq J$, there is an isomorphism*

$$\mathrm{M}_{\mathrm{Rig}}(]D_I^\circ[) \simeq \mathrm{M}_{\mathrm{Rig}}(]D_I[).$$

Proposition 4.18 *Let $\mathcal{X}$ be a semi-stable formal R-scheme and $D = \cup_{i\in J'} D_i$ a union of branches. Then the following equalities hold in $\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$*

$$[\mathrm{M}_{\mathrm{Rig}}(]D[)] = \sum_{I\subset J'} (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D_I[)]$$

and

$$[\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)] = \sum_{I\subset J'} (-1)^{|I|-1}[\mathrm{M}^\vee_{\mathrm{Rig}}(]D_I[)].$$

Proof The collection $(D_i)_{i\in J'}$ is a closed cover of D, hence by [5, Proposition 1.1.14], $(]D_i[)_{i\in J'}$ is an admissible cover of $]D[$. Hence by Mayer–Vietoris distinguished triangle and induction on the cardinal of I, we have the result. ◻

Using Corollary 4.17, we deduce the following formula.

Corollary 4.19 *Under the hypotheses of Proposition 4.18, we have*

$$[\mathrm{M}_{\mathrm{Rig}}(]D[)] = \sum_{I\subseteq J'} (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D_I^\circ[)]$$

and

$$[\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)] = \sum_{I\subseteq J'} (-1)^{|I|-1}[\mathrm{M}^\vee_{\mathrm{Rig}}(]D_I^\circ[)].$$

Theorem 4.20 *Let $\mathcal{X}$ be a semi-stable formal R-scheme of dimension d. Then*

$$\chi_{\mathrm{Rig}}(\mathcal{X}_\eta^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(\mathcal{X}_\eta(-d))].$$

Still denoting $\mathcal{X}_\sigma = \cup_{i\in J} D_i$ the irreducible components of $\mathcal{X}_\sigma$, the special fiber of X, we can write $\mathcal{X}_\eta^{\mathrm{VF}}$ as a disjoint union $\mathcal{X}_\eta^{\mathrm{VF}} = \dot{\bigcup}_{I\subset J}]D_I^\circ[$, hence

$$\chi_{\mathrm{Rig}}(]D[_{\mathcal{X}}^{\mathrm{VF}}) = \sum_{I\subset J} \chi_{\mathrm{Rig}}(]D_I^\circ[).$$

In view of the formula of Corollary 4.19, to prove Theorem 4.20, it suffices to prove the following proposition.

Proposition 4.21 *Let $\mathcal{X}$ be a semi-stable R-scheme. Then*

$$\chi_{\mathrm{Rig}}(]D_I^\circ[^{\mathrm{VF}}) = (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D_I^\circ[)(-d)],$$

where $d = \dim(\mathcal{X}_\sigma)$.

Before proving the proposition, we need a reduction.

Lemma 4.22 *To prove Proposition 4.21, we can assume that* $\mathcal{X} = \mathrm{St}^{u^{-1}t}_{D_I^\circ \times_k R, \underline{N}}$, *where* $\underline{N} = (N_1, \dots, N_r) \in (\mathbb{N}^\times)^r$ *(where* $r = |I|$*),* $u_I \in \mathcal{O}^\times(D_I^\circ \times_k R)$.

Proof Using Mayer–Vietoris distinguished triangles, we can also work Zariski locally, hence suppose by [3, Proposition 1.1.62] that there is an étale R-morphism

$$e : \mathcal{X}\{V, V^{-1}\} \to \mathcal{S} = \mathrm{St}^{Ut}_{\mathrm{Spec}(R[U,U^{-1}]),\underline{N}}[S_1, \dots, S_r],$$

where $\underline{N}$ is the type of $\mathcal{X}$ at $x \in]D_I^\circ[$. The irreducible components of $\mathcal{S}_\sigma$ are defined by equations $T_i = 0$, denote by C their intersection. We have $C = \mathrm{Spec}(k[U, U^{-1}, S_1, \dots, S_r])$. Up to permuting the D_i, we can assume that D_i is defined in $\mathcal{X}_\sigma$ by $T_i \circ e = 0$, inducing an étale morphism $e_\sigma : D_I[V, V^{-1}] \to C$ and a Cartesian square of R-schemes

$$\begin{tikzcd} D_I[V, V^{-1}] \arrow[r] \arrow[d, "e_\sigma"'] & X \arrow[d, "e"] \\ C \arrow[r] & S. \end{tikzcd}$$

The morphism e_σ induces an étale morphism of R-schemes

$$D_I[V, V^{-1}] \times_k R \to C \times_k R,$$

which itself induces an étale R-morphism

$$e' : \mathcal{X}' = \mathrm{St}^{u_I^{-1}t}_{D_I[V,V^{-1}]\times_k R,\underline{N}} \to \mathcal{S},$$

together with a Cartesian square

$$\begin{tikzcd} D_I[V, V^{-1}] \arrow[r] \arrow[d, "e_\sigma"'] & X' \arrow[d, "e'"] \\ C \arrow[r] & S. \end{tikzcd}$$

The fiber product $\mathcal{X}\{V, V^{-1}\} \times_{\mathcal{S}} \mathcal{X}'$ hence satisfies $\mathcal{X}\{V, V^{-1}\} \times_{\mathcal{S}} \mathcal{X}' \times_{\mathcal{S}} C \simeq D_I[V, V^{-1}] \times_C D_I[V, V^{-1}]$. Because $e_\sigma : D_I[V, V^{-1}] \to C$ is étale, the diagonal embedding $D_I[V, V^{-1}] \to D_I[V, V^{-1}] \times_C D_I[V, V^{-1}]$ is an open and closed immersion, hence induces a decomposition $D_I[V, V^{-1}] \times_C D_I[V, V^{-1}] \simeq D_I[V, V^{-1}] \cup F$. Set $\mathcal{X}'' = \mathcal{X}\{V, V^{-1}\} \times_{\mathcal{S}} \mathcal{X}' \backslash F$. We have two étale morphisms $f : \mathcal{X}'' \to \mathcal{X}\{V, V^{-1}\}$ and $f' : \mathcal{X}'' \to \mathcal{X}'$ such that $f^{-1}(D_I[V, V^{-1}]) \simeq D_I[V, V^{-1}]$ and $f'^{-1}(D_I[V, V^{-1}]) \simeq D_I[V, V^{-1}]$. We can apply twice [5, Proposition 1.3.1] to get that

$$]D_I^\circ[V, V^{-1}][_{\mathcal{X}\{V,V^{-1}\}} \simeq]D_I^\circ[V, V^{-1}][_{\mathcal{X}''}$$

and

$$]D_I^\circ[V, V^{-1}][_{\mathcal{X}'} \simeq]D_I^\circ[V, V^{-1}][_{\mathcal{X}''}.$$

By the choice of F, at the ring level both f and f' send V to the same element. Hence the isomorphism $]D_I^\circ[V, V^{-1}][_{\mathcal{X}\{V,V^{-1}\}} \simeq]D_I^\circ[V, V^{-1}][_{\mathcal{X}'}$ induces an isomorphism

$$]D_I^\circ[_{\mathcal{X}} \simeq]D_I^\circ[_{\mathrm{St}^{u_I^{-1}t}_{D_I \times_k R, \underline{N}}},$$

where the above map is the composition

$$]D_I^\circ[_{\mathcal{X}} \hookrightarrow]D_I^\circ[V, V^{-1}][_{\mathcal{X}\{V,V^{-1}\}} \simeq]D_I^\circ[V, V^{-1}][_{\mathcal{X}'} \twoheadrightarrow]D_I^\circ[_{\mathrm{St}^{u_I^{-1}t}_{D_I \times_k R, \underline{N}}},$$

with the first map the inclusion of the unit section and the last one the projection forgetting the V variable. □

Remark 4.23 The proof of Lemma 4.22 also gives a definable bijection $]D_I^\circ[^{\mathrm{VF}}_{\mathcal{X}} \simeq]D_I^\circ[^{\mathrm{VF}}_{\mathcal{X}'}$, see also [28, Theorem 2.6.1] for an alternative approach.

Proof of Proposition 4.21 We can suppose that we are in the situation of Lemma 4.22, with $\mathcal{X} = \mathrm{St}^{u^{-1}t}_{D_I^\circ \times_k R, \underline{N}}$. Let N_I be the greatest common divisor of the N_i for $i \in I$. Set $N_i' = N_i/N_I$. As the N_i' are coprime, we can form an $r \times r$ matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ which first row is constituted by the N_i'. The matrices A and A^{-1} define automorphisms of ${\mathbb{G}_m}_K^{r,\mathrm{an}}$, hence of $G = D_I^\circ(R) \times {\mathbb{G}_m}_K^{r,\mathrm{an}}$. As $]D_I^\circ[_{\mathcal{X}}$ is a rigid subvariety of G, we can consider W, its image by A. Then W is the locally closed semi-algebraic subset of G defined by

$$\left\{(x, w) \in D_I(R) \times (K^\times)^r \mid w_1^{N_I} u_I(x) = t, l_1(\mathrm{v}(w)) > 0, \ldots, l_{r-1}(\mathrm{v}(w)) > 0\right\},$$

where the $l_i : \Gamma^r \to \Gamma$ are linearly independent affine functions with integer coefficients. Hence $W = Q^{\mathrm{VF}}_{N_I}(D_I^\circ, u_I) \times \mathrm{v}^{-1}(\Delta)$, where $\Delta \subset \Gamma^{r-1}$ is defined by equations $l_i > 0$ for $i = 1, \ldots, r$.

By Propositions 4.11 and 4.12, we know that

$$\chi_{\mathrm{Rig}}(Q^{\mathrm{VF}}_{N_I}(D_I^\circ, u_I)) = [\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D_I^\circ, u_I))(-d + r - 1)]$$

and

$$\chi_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)) = (-1)^{r-1}[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})(-r+1)].$$

Hence as χ_{Rig} is multiplicative,

$$\chi_{\mathrm{Rig}}(W) = (-1)^{r-1}[\mathrm{M}_{\mathrm{Rig}}(W^{\mathrm{Rig}})(-d)].$$

After applying the isomorphism A^{-1}, we get as required

$$\chi_{\mathrm{Rig}}(]D_I^\circ[_X^{\mathrm{VF}}) = (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D_I^\circ[)(-d)].$$

□

The proof of Theorem 4.20 is now complete. For later use, note that the proofs of Proposition 4.21 and Lemma 4.22 gives the following equality.

Corollary 4.24 *With the notation of Proposition 4.21, we have*

$$\mathrm{M}_{\mathrm{Rig}}(]D_I^\circ[) \simeq \mathrm{M}_{\mathrm{Rig}}(Q_{N_I}^{\mathrm{Rig}}(D_I^\circ, u_I) \times \partial\mathbb{B}(o,1)^{|I|-1}).$$

4.5 Compatibilities of χ_{Rig}

We will now derive consequences of Theorem 4.20.

Theorem 4.25 *The morphism χ_{Rig} is the unique ring morphism*

$$\mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

such that for any quasi-compact smooth rigid K-variety X of pure dimension d,

$$\chi_{\mathrm{Rig}}(X^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)].$$

Proof By quantifier elimination in the theory ACVF_K, $\mathbf{K}(\mathrm{VF}_K)$ is generated by classes of smooth affinoid rigid K-varieties, which shows uniqueness. For the existence, fix X a quasi-compact smooth rigid K-variety of pure dimension d. We can find $\mathcal{X}$, a formal R-model of X and by Hironaka's resolution of singularities, we can assume $\mathcal{X}$ is semi-stable. We can now apply Theorem 4.20. □

Theorem 4.26 *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\chi_K"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[d, "\chi_{\mathrm{Rig}}"] \\
\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K)) \arrow[r, "\mathrm{Rig}"'] & \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).
\end{tikzcd}$$

Proof By Hironaka's resolution of singularities, the ring $\mathbf{K}(\mathrm{Var}_K)$ is generated by classes of smooth projective varieties, hence it suffices to check the compatibility for such a K-variety X. Denoting by $f : X \to K$ the structural morphism, one has by definition

$$\chi_K([X]) = [f_! f^* \mathbb{1}_K] = [f_\sharp f^* \mathbb{1}_K(-d)] = [\mathrm{M}_K(X)(-d)],$$

where $d = \dim(X)$ and we used $f_! = f_\sharp \circ \mathrm{Th}^{-1}(\Omega_f)$ for f smooth. Applying the functor Rig, one needs to show that $\chi_{\mathrm{Rig}}([X]^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(X^{\mathrm{an}})(-d)]$. As X is smooth and projective, X^{an} is a quasi-compact rigid smooth K-variety hence one can find a semi-stable formal model of X^{an} over R, denote it $\widetilde{X}$. Hence $X \simeq \widetilde{X}_K \simeq]\widetilde{X}_\sigma[_{\widetilde{X}}$, so by Theorem 4.20, $\chi_{\mathrm{Rig}}([X]^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(X^{\mathrm{Rig}})(-d)]$. □

Note that combining Theorems 4.25 and 4.26 gives Theorem 1.1.

4.6 A few more realization maps

In this section we construct in addition to χ_{Rig} and χ'_{Rig} two more realization maps $\widetilde{\chi_{\mathrm{Rig}}}$ and $\widetilde{\chi'_{\mathrm{Rig}}}$ obtained by considering homological motives with compact support instead of cohomological motives with compact support.

Recall the ring morphism of Proposition 3.1 $\chi_S : \mathbf{K}(\mathrm{Var}_S) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S))$ sending $[f : X \to S]$ to $[\mathrm{M}^\vee_{S,c}(X)] = [f_! f^* \mathbb{1}_S]$.

Working dually, we can define also a morphism $\widetilde{\chi_S} : \mathbf{K}(\mathrm{Var}_S) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S))$ sending $[f : X \to S]$ to $[\mathrm{M}_{S,c}(X)] = [f_* f^! \mathbb{1}_S]$. The proof that it respects the scissors relations is similar, using the exact triangle

$$i_* i^! A \to A \to j_* j^! A \xrightarrow{+1}$$

instead of

$$j_\sharp j^* A \to A \to i_* i^* A \xrightarrow{+1},$$

where i and j are closed and open complementary immersions. Another approach would be to use the duality involution of the following Sect. 5.

Composing with the morphism $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}})$, we get a morphism $\widetilde{\chi_{\hat{\mu}}} : \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(\mathbb{G}_{mk}))$, fitting in the following commutative square:

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\widetilde{\chi_{\hat{\mu}}}"'] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\widetilde{\chi_k}"] \\
\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

Recall that

$$\chi_{\mathrm{Rig}} = \mathfrak{F} \circ \chi_{\hat{\mu}} \circ \Theta \circ \mathcal{E}_c \circ \oint \text{ and } \chi'_{\mathrm{Rig}} = \mathfrak{F} \circ \chi_{\hat{\mu}} \circ \Theta \circ \mathcal{E} \circ \oint .$$

We can now define

$$\widetilde{\chi_{\mathrm{Rig}}} = \mathfrak{F} \circ \widetilde{\chi_{\hat{\mu}}} \circ \Theta \circ \mathcal{E}_c \circ \oint \text{ and } \widetilde{\chi'_{\mathrm{Rig}}} = \mathfrak{F} \circ \widetilde{\chi_{\hat{\mu}}} \circ \Theta \circ \mathcal{E} \circ \oint .$$

Unraveling the definitions, we see that if X is a smooth connected k-variety of dimension d, $f \in \Gamma(X, \mathcal{O}_X^\times)$, $r \in \mathbb{N}^*$ and $\Delta \subset \Gamma^n$ an open simplex of dimension n,

$$\begin{aligned}
\widetilde{\chi_{\mathrm{Rig}}}(Q_r^{\mathrm{VF}}(X, f)) &= [\mathrm{M}^\vee_{\mathrm{Rig}}(Q_r^{\mathrm{Rig}}(X, f))(d)],\\
\widetilde{\chi'_{\mathrm{Rig}}}(Q_r^{\mathrm{VF}}(X, f)) &= [\mathrm{M}^\vee_{\mathrm{Rig}}(Q_r^{\mathrm{Rig}}(X, f))],\\
\widetilde{\chi_{\mathrm{Rig}}}(\mathrm{v}^{-1}(\Delta)) &= (-1)^d[\mathrm{M}^\vee_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})(d)],
\end{aligned}$$

and

$$\widetilde{\chi'_{\mathrm{Rig}}}(\mathrm{v}^{-1}(\Delta)) = (-1)^d[\mathrm{M}^\vee_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})].$$

See the proofs of Propositions 4.11 and 4.12 for details.

Hence the proof of Theorem 4.20 can be adapted to χ'_{Rig}, $\widetilde{\chi_{\mathrm{Rig}}}$ and $\widetilde{\chi'_{\mathrm{Rig}}}$, showing in particular that if X is a quasi-compact smooth connected rigid K-variety of dimension d,

$$\begin{aligned}
\chi_{\mathrm{Rig}}(X^{\mathrm{VF}}) &= [\mathrm{M}_{\mathrm{Rig}}(X)(-d)], \quad \chi'_{\mathrm{Rig}}(X^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(X)],\\
\widetilde{\chi_{\mathrm{Rig}}}(X^{\mathrm{VF}}) &= [\mathrm{M}^\vee_{\mathrm{Rig}}(X)(d)], \quad \widetilde{\chi'_{\mathrm{Rig}}}(X^{\mathrm{VF}}) = [\mathrm{M}^\vee_{\mathrm{Rig}}(X)].
\end{aligned}$$

If X is a proper algebraic K-variety of structural morphism f, since $f_* = f_!$, we have $\mathrm{M}^\vee_{K,c}(X) = \mathrm{M}^\vee_K(X)$ and $\mathrm{M}^\vee_{K,c}(X) = \mathrm{M}_K(X)$, hence we can adapt the proof of Theorem 4.26 to get commutative diagrams similar of Theorem 1.1, the first one being the statement of Theorem 1.1.

Proposition 4.27 *The squares in the following diagrams commutes:*

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\chi_K"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}_c\circ\oint"] \arrow[d, "\chi_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\chi_{\hat{\mu}}"] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[r, "\simeq", "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\widetilde{\chi_K}"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}\circ\oint"] \arrow[d, "\chi'_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})[\mathbb{L}^{-1}] \arrow[r] \arrow[d, "\chi_{\hat{\mu}}"] & \mathbf{K}(\mathrm{Var}_k)[\mathbb{L}^{-1}] \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[r, "\simeq", "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\widetilde{\chi_K}"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}_c\circ\oint"] \arrow[d, "\widetilde{\chi_{\mathrm{Rig}}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\widetilde{\chi_{\hat{\mu}}}"] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\widetilde{\chi_k}"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[r, "\simeq", "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\chi_K"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}\circ\oint"] \arrow[d, "\widetilde{\chi'_{\mathrm{Rig}}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})[\mathbb{L}^{-1}] \arrow[r] \arrow[d, "\widetilde{\chi_{\hat{\mu}}}"] & \mathbf{K}(\mathrm{Var}_k)[\mathbb{L}^{-1}] \arrow[d, "\widetilde{\chi_k}"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[r, "\simeq", "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)).
\end{tikzcd}$$

In particular, we see that χ_{Rig} and $\widetilde{\chi'_{\mathrm{Rig}}}$ agree on the image of $\mathbf{K}(\mathrm{Var}_K)$ and similarly χ'_{Rig} and $\widetilde{\chi_{\mathrm{Rig}}}$ agree on the image of $\mathbf{K}(\mathrm{Var}_K)$.

Remark 4.28 (Volume forms) In addition of the additive morphism $\oint$, Hrushovski and Kazhdan also study the Grothendieck ring of definable sets with volume forms $\mathbf{K}(\mu_\Gamma \mathrm{VF}_K)$. Objects in $\mu_\Gamma\mathrm{VF}_K$ are pairs (X, ω) with $X \subseteq \mathrm{VF}^\bullet$ a definable set and $\omega : X \to \Gamma$ a definable function. Morphisms are measure preserving definable bijections (up to a set of lower dimension). In this context, they build an isomorphism

$$\oint^{\mu} : \mathbf{K}(\mu_\Gamma \mathrm{VF}_K) \to \mathbf{K}(\mu_\Gamma \mathrm{RV}_K)/\mu \mathrm{I}_{\mathrm{sp}},$$

see [20, Theorem 8.26].

One can further decompose $\mathbf{K}(\mu_\Gamma\mathrm{RV}_K)$ similarly to $\mathbf{K}(\mathrm{RV}_K[*])$. Using this Hrushovski and Loeser define in [21] for $m \in \mathbb{N}$ morphisms

$$h_m : \mathbf{K}(\mu_\Gamma \mathrm{RV}_K^{\mathrm{bdd}})/\mu\mathrm{I}_{\mathrm{sp}} \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})_{\mathrm{loc}}$$

with the m related to considering rational points in $k((t^{1/m}))$. Here, bdd means we consider only bounded sets. Note that there is an inaccuracy in the definition of h_m in

[21] since they use [20, Proposition 10.10 (2)] which happens to be incorrect. Using the category of bounded sets with volume forms deals with the issue.

We can further compose with the morphism $\mathfrak{F} \circ \chi_{\hat{\mu}}$ in order to get for each $m \in \mathbb{N}^*$ a morphism

$$\mathbf{K}(\mu_\Gamma \mathrm{VF}_K^{\mathrm{bdd}}) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

Such morphisms do not seem to satisfy properties similar to those of χ_{Rig}.

5 Duality

The goal of this section is to prove Theorem 1.2. We will adapt Bittner's results on duality in the Grothendieck ring of varieties in Sect. 5.1 in order to be able to compute in Sect. 5.2 explicitly the cohomological motive of some tubes in terms of homological motives. The last Sect. 5.3 is devoted to an application to the motivic Milnor fiber and analytic Milnor fiber.

5.1 Duality involutions

Bittner developed in [6] an abstract theory of duality in the Grothendieck ring of varieties. We recall here some of her results and show that they imply similar results for $\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K))$.

Using the weak factorization theorem, Bittner prove the following alternative description of $\mathbf{K}(\mathrm{Var}_X)$. We state if for a variety S above K, but it holds for varieties above any field of characteristic zero.

Proposition 5.1 ([6, Theorem 5.1]) *Fix a K-variety S. The ring $\mathbf{K}(\mathrm{Var}_S)$ is isomorphic to the abelian group generated by classes of S-varieties which are smooth over K, proper over S, subject to the relations $[\emptyset]_S = 0$ and $[\mathrm{Bl}_Y(X)]_S - [E]_S = [X]_S - [Y]_S$, where X is smooth over K, proper over S, $Y \subset X$ a closed smooth subvariety, $\mathrm{Bl}_Y(X)$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

For $f : X \to Y$ a morphism of S-varieties, composition with f induce a (group) morphism $f_! : \mathbf{K}(\mathrm{Var}_X) \to \mathbf{K}(\mathrm{Var}_Y)$ and pull-back along f induces a (group) morphism $f^* : \mathbf{K}(\mathrm{Var}_Y) \to \mathbf{K}(\mathrm{Var}_X)$. Both induces $\mathcal{M}_S$-linear morphisms $f_! : \mathcal{M}_X \to \mathcal{M}_Y$ and $f^* : \mathcal{M}_Y \to \mathcal{M}_X$, where $\mathcal{M}_X = \mathbf{K}(\mathrm{Var}_X)[\mathbb{L}^{-1}]$.

Definition 5.2 We define now a duality operator $\mathcal{D}_X : \mathbf{K}(\mathrm{Var}_X) \to \mathcal{M}_X$ for any K-variety X. Set $\mathcal{D}_X([Y]) = [Y]\mathbb{L}^{-\dim(Y)}$ if Y is an X-variety proper over X, connected and smooth over K. In view of Proposition 5.1, to show that it induces a unique (group) morphism $\mathbf{K}(\mathrm{Var}_X) \to \mathcal{M}_X$, it suffices to show that if $Y \subset Z$ is a closed immersion of X-varieties, proper over X, smooth and connected over K,

$$[\mathrm{Bl}_Y(Z)]\mathbb{L}^{-\dim(Z)} - [E]\mathbb{L}^{-\dim(Z)+1} = [Z]\mathbb{L}^{-\dim(Z)} - [Y]\mathbb{L}^{-\dim(Y)},$$

it holds since $(\mathbb{L}-1)[E] = (\mathbb{L}^{\dim(Z)-\dim(Y)} - 1)[Y]$. See [6, Definition 6.3] for details.

Observe that $\mathcal{D}_X(\mathbb{L}) = \mathbb{L}^{-1}$ and that $\mathcal{D}_X$ is $\mathcal{D}_S$-linear, hence $\mathcal{D}_X$ can be extended as a $\mathcal{D}_K$-linear morphism $\mathcal{D}_X : \mathcal{M}_X \to \mathcal{M}_X$, which is an involution.

Although $\mathcal{D}_X$ is not in general a ring morphism, $\mathcal{D}_K$ is a ring morphism.

For $f : X \to Y$, set $f^! = \mathcal{D}_X f^* \mathcal{D}_Y$ and $f_* = \mathcal{D}_Y f_! \mathcal{D}_X$. Observe that if f is proper, $f_! = f_*$ and if f is smooth of relative dimension d over S, $f^! = \mathbb{L}^{-d} f^*$.

Such a duality operator can also be defined in the Grothendieck ring of varieties equipped with a good action of some finite group G, see [6, Sections 7,8] for details.

In $\mathrm{SH}_{\mathfrak{M}}(K)$, the internal hom gives also notion of duality. Define the duality functor as $\mathbb{D}_K(A) = \underline{\mathrm{Hom}}_K(A, \mathbb{1}_K)$. Since $\mathbb{D}_K$ is triangulated, it induces a morphism on $\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S))$, still denoted $\mathbb{D}_K$. By [1, Théorème 2.3.75], $\mathbb{D}_K$ is an autoequivalence on constructible objects and its own quasi-inverse. In particular, (compactly supported) (co)homological motives of S-varieties are constructible, hence $\mathbb{D}_K$ is an involution. One can also define more generally for $a : X \to \mathrm{Spec}(K)$, $\mathbb{D}_X(A) = \underline{\mathrm{Hom}}_X(A, a^! \mathbb{1}_K)$, but we will not use those. By [1, Théorème 2.3.75], $\mathbb{D}_K(\mathrm{M}_K(X)) = \mathrm{M}_K^\vee(X)$ for any K-variety X.

The following proposition shows the compatibility between those two duality operators.

Proposition 5.3 *There is a commutative diagram*

$$\begin{tikzcd} \mathcal{M}_K \arrow[rr, "\mathcal{D}_K"] \arrow[d, "\chi_K"'] & & \mathcal{M}_K \arrow[d, "\chi_K"] \\ \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K)) \arrow[rr, "\mathbb{D}_K"'] & & \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K)). \end{tikzcd}$$

Proof It suffices to show that $\chi_K \mathcal{D}_K([X]) = \mathbb{D}_K \chi_K([X])$ for X a connected, smooth and proper K-variety of dimension d. Set $f : X \to \mathrm{Spec}(K)$. As f is smooth and proper, $[\mathrm{M}_{K,c}^\vee(X)] = [\mathrm{M}_K^\vee(X)] = [\mathrm{M}_K(X)(-d)]$. We then have

$$\chi_K(\mathcal{D}_K([X])) = \chi_K([X]\mathbb{L}^{-d}) = [\mathrm{M}_{K,c}^\vee(X)(d)] = [\mathrm{M}_K(X)]$$

and

$$\mathbb{D}_K \chi_K([X]) = \mathbb{D}_K([\mathrm{M}_{K,c}^\vee(X)]) = \mathbb{D}_K([\mathrm{M}_K^\vee(X)]) = [\mathrm{M}_K(X)].$$

□

All our duality results will ultimately boil down to the following lemma, which states that normal toric varieties satisfy Poincare duality. It is due to Bittner, see [7, Lemma 4.1]. The proof relies on toric resolution of singularities and the Dehn–Sommerville equations, see for example [19].

Lemma 5.4 *Let X be an affine toric K-variety associated to a simplicial cone, $X \to Y$ be a proper morphism, G a finite group acting on X via the torus with trivial action on Y. Then*

$$\mathcal{D}_Y([X]) = [X]\mathbb{L}^{-\dim(X)} \in \mathcal{M}_Y^G.$$

For the rest of the section, we fix a semi-stable formal R-scheme $\mathcal{X}$ and let $(D_i)_{i\in J}$ be the branches of its special fiber $\mathcal{X}_\sigma$. Fix $I \subset J$, up to reordering the coordinates, suppose $I = \{1, \ldots, k\}$. Recall that around every closed point $x \in D_I^\circ$, there is a Zariski open neighborhood $\mathcal{U}$ and regular functions $u_I, x_1, \ldots, x_k$ such that $u_I \in \mathcal{O}^\times(\mathcal{U})$ and $t = u_I x_1^{N_1} \ldots x_k^{N_k}$, with the branch D_i defined by $x_i = 0$. Still denote $u_I, x_1, \ldots, x_k$ their reductions to $U = \mathcal{U}_\sigma$. The various u_I glue to define a section $u_I \in \Gamma(D_I^\circ, \mathcal{O}^\times_{D_I^\circ}/(\mathcal{O}^\times_{D_I^\circ})^{N_I})$. Recall that N_I is the greatest common divisor of the N_i, $i \in I$.

We already considered (the analytification of) the K-variety

$$\mathcal{Q}^{\mathrm{geo}}_{N_I}(D_I^\circ, u_I) = D_I^\circ \times_k K[V]/(V^{N_I} - tu_I).$$

In this section, we will denote $\widetilde{D_I^\circ} = \mathcal{Q}^{\mathrm{geo}}_{N_I}(D_I^\circ, u_I)$ to simplify the notations. We will also abuse the notations and still denote D_I°, D_I, U the base change to K of those varieties.

Let $\widetilde{D_I}$ be the normalization of D_I in $\widetilde{D_I^\circ}$. We also set for $K \subset I$, $\widetilde{D_I}_{|D_K} = \widetilde{D_I} \times_{D_I} D_K$

Proposition 5.5 *For every $I \subset K \subset J$, we have $\widetilde{D_I}_{|D_K} \simeq \widetilde{D_K}$ and $\mathcal{D}_{D_I}[\widetilde{D_I}] = \mathbb{L}^{|I|-d+1}[\widetilde{D_I}]$.*

Observe that Bittner's Lemma 5.2 in [7] is analogous, but holds in $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$. Since it is not a priori clear that dualities on $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ and $\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$ are compatible, we cannot apply directly her Lemma. We will nevertheless follow closely her proof.

Combination of Propositions 5.5 and 5.3 yields the following corollary.

Corollary 5.6 *For any $I \subset J$ such that D_I is proper, we have the equality in $\mathbf{K}(\mathrm{Var}_K)$*

$$[\widetilde{D_I}] = \sum_{I \subset K \subset J} [\widetilde{D_K^\circ}],$$

and $[\mathbb{D}_K(\mathrm{M}^\vee_{K,c}(\widetilde{D_I}))] = [\mathrm{M}^\vee_{K,c}(\widetilde{D_I})(d - |I| + 1)] \in \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K))$.

Proof The first equality is the first point of Proposition 5.5. For the second equality, since D_I is proper, setting $f : D_I \times_k \mathbb{G}_{mk} \to \mathbb{G}_m$, we have $f_!\mathcal{D}_{D_I} = \mathcal{D}_K f_!$ hence by the second point of Proposition 5.5,

$$\mathcal{D}_K([\widetilde{D_I}]) = [\widetilde{D_I}]\mathbb{L}^{-d+|I|-1}.$$

Since $\widetilde{D_I}$ is proper over K, we have $\mathrm{M}^\vee_K(\widetilde{D_I}) = \mathrm{M}^\vee_{K,c}(\widetilde{D_I})$, hence the result follows from Proposition 5.3 after applying a Tate twist. □

Proof of Proposition 5.5 Working inductively on the codimension of D_K in D_I and up to reordering the branches, we can suppose $I = \{1, \ldots, k-1\} \subset K = \{1, \ldots, k\}$.

As the statement is local we can work on an open neighborhood U of some closed point of D_K and choose a system of local coordinates $x_1, \dots, x_d$ on U such that $u_I x_1^{N_1} \dots x_{k-1}^{N_{k-1}} = u_K x_1^{N_1} \dots x_k^{N_k} = u x_1^{N_1} \dots x_d^{N_d}$, where $u \in \mathcal{O}^\times(U)$ and for $i = 1, \dots, k$, D_i is defined by equation $x_i = 0$. Define a μ_{N_1}-étale cover of U by

$$V = U[Z]/(Z^{N_1} - tu).$$

We consider V as a variety with a μ_{N_1}-action induced by multiplication of z by $\zeta \in \mu_{N_1}$.

Then $y_1 = s x_1, y_2 = x_2, \dots, x_d = y_{d+1}$ is a system of local coordinates on V. Shrinking U, we can assume the morphism $V \to \mathbb{A}_k^{d+1}$ induced by $y_1, \dots, y_{d+1}$ is étale. Denote by F_I, F_I°, F_K, F_K° the pull-backs of D_I, D_I°, D_K, D_K°.

Denote by $\widetilde{F_I^\circ}$ the following étale cover of F_I°:

$$\widetilde{F_I^\circ} = F_I^\circ[W]/(W^{N_I} - y_{k+1}^{N_{k+1}} \dots y_d^{N_d}).$$

Observe that $\widetilde{F_I^\circ}$ is isomorphic to the fiber product $(F_I^\circ) \times_{D_I^\circ} \widetilde{D_I^\circ}$. The variety $\widetilde{F_I^\circ}$ is equipped with a μ_{N_1}-action, with $\zeta \in \mu_{N_1}$ acting on w by multiplication by ζ^{N_1/N_I}.

Denote by $\widetilde{F_I}$ the normalization of F_I in $\widetilde{F_I^\circ}$ and consider the following diagram:

$$\begin{tikzcd} p^* \widetilde{D_I} \arrow[r] \arrow[d] & \widetilde{D_I} \arrow[d] \\ F_I \arrow[r, "p"'] & D_I. \end{tikzcd}$$

As $p : F_I \to D_I$ is smooth and $\widetilde{D_I}$ is normal, $p^* \widetilde{D_I}$ is normal. As $p^* \widetilde{D_I} \to F_I$ is finite and surjective, $p^* \widetilde{D_I}$ is isomorphic to $\widetilde{F_I}$. Denoting by $\widetilde{F_I}/\mu_{N_1}$ the quotient of $\widetilde{F_I}$ by the μ_{N_1}-action, we then have $\widetilde{D_I} \simeq \widetilde{F_I}/\mu_{N_1}$ and similarly $\widetilde{D_K} \simeq \widetilde{F_K}/\mu_{N_1}$. Hence it suffices to show that $\widetilde{F_I}_{|F_K} \simeq \widetilde{F_K}$ and $\mathcal{D}_{F_I}[\widetilde{F_I}] = [\widetilde{F_I}]\mathbb{L}^{-n+k-1}$, both equalities being compatible with the μ_{N_1}-actions.

Now consider the étale morphism $\pi : V \to \mathbb{A}_K^{d+1}$. Denoting by $z_1, \dots, z_{d+1}$ the coordinates of $\mathbb{A}_K^{d+1}$, define $C_I \subseteq \mathbb{A}_{\mathbb{G}_{mk}}^{d+1}$ by the equations $z_1 = \dots = z_{k-1} = 0$ and $C_I^\circ = C_I \setminus \cup_{j=k,\dots,d+1} \{z_j = 0\}$. Define similarly C_K and C_K°.

Define an étale cover of C_I° by

$$\widetilde{C_I^\circ} = C_I^\circ[S]/(S^{N_I} - z_k^{N_k} \dots z_{d+1}^{N_{d+1}}),$$

define similarly $\widetilde{C_K^\circ}$ and let $\widetilde{C_I}$ and $\widetilde{C_K}$ be the normalizations of C_I and C_K in $\widetilde{C_I^\circ}$ and $\widetilde{C_K^\circ}$.

We then have a Cartesian diagram

$$\begin{tikzcd} \widetilde{F_I} \arrow[r] \arrow[d] & \widetilde{C_I} \arrow[d] \\ F_I \arrow[r, "\pi"'] & C_I \end{tikzcd}$$

with π étale, hence it suffices to show that $\widetilde{C_I}_{|C_K} \simeq \widetilde{C_K}$ and $\mathcal{D}_{C_I}[\widetilde{C_I}] = \widetilde{C_I}\mathbb{L}^{-d+k}$, both equalities being compatible with the μ_{N_1}-actions ($\zeta \in \mu_{N_1}$ acts on $\widetilde{C_I^\circ}$ by multiplication of s by ζ^{N_1/N_I}). Indeed, since π is étale, we have $\pi^*\mathcal{D}_{C_I} = \mathcal{D}_{F_I \times_k \mathbb{G}_{mk}}\pi^*$.

Since the projection $\mathbb{A}^{d+1} \to \mathbb{A}^{d-k}$ is smooth, the result now follows from the following Lemmas 5.7 and 5.8 which correspond to Lemmas 5.3 and 5.4 in [7]. □

Lemma 5.7 *The restriction of the normalization $\widetilde{S}$ of $S = \left\{s^N = x_1^{a_1} \dots x_d^{a_d}\right\} \subset \mathbb{A}_k^1 \times_k \mathbb{A}^d_{\mathbb{G}_{mk}}$ to $\{x_1 = 0\} \subset \mathbb{A}^d_{\mathbb{G}_{mk}}$ is isomorphic to the normalization $\widetilde{S'}$ of $S' = \left\{s^{N'} = x_2^{a_2} \dots x_d^{N_d}\right\}$, where $N' = \gcd(N, a_1)$. If N divides some $q \in \mathbb{N}^*$, then the isomorphism is compatible with the μ_q-actions on $\widetilde{S}$ and $\widetilde{S'}$ where $\zeta \in \mu_q$ acts on S by multiplication of s by $\zeta^{q/N}$ and on S' by multiplication of s by $\zeta^{q/N'}$.*

Proof Assume first that $N, a_1, \dots, a_d$ are coprime. Then S is irreducible. Let M be the lattice of $\mathbb{R}^d$ spanned by $\mathbb{Z}^d$ and $v = (a_1/N, \dots, a_d/N)$. Set $M^+ = M \cap \mathbb{R}^d_+$. Then $\widetilde{S} \simeq \mathrm{Spec}(k[M^+])$. If $M_1 := \{u \in M \mid u_1 = 0\}$ and $M_1^+ = M_1 \cap \mathbb{R}^d_+$, then $\mathrm{Spec}(k[M_1^+])$ is the restriction of $\widetilde{S}$ to $\{x_1 = 0\}$.

Now consider the lattice M' generated by $v' = (0, a_2/N', \dots, a_d/N')$ and $\{0\} \times \mathbb{Z}^{d-1}$, and set $M'^+ = M' \cap \mathbb{R}^d_+$. We have $\widetilde{S'} \simeq \mathrm{Spec}(k[M'^+])$, hence it suffices to show that $M' \simeq M_1$.

Denote $e_1, \dots, e_d$ the canonical basis of $\mathbb{Z}^d$. Set $k = a_1/N'$ and $l = N/N'$. Observe that $v' = N/N'v - a_1/N'e_1 = lv - ke_1$, hence $M' \subseteq M_1$. Reciprocally, if $u = \sum_{i=1}^d \lambda_i e_i + \mu v \in M_1$, then $\lambda_1 + \mu k/l \in \mathbb{Z}$, hence $\mu' = \mu/l \in \mathbb{Z}$ (since k and ℓ are coprime), hence $u = \sum_{i=2}^d \lambda_i e_i + \mu' v' \in M'$. The μ_q-actions are compatible, since $s'^{N'} = s^N x_1^{-a_1}$.

Back to the general case, let c be the greatest common divisor of $N, a_1, \dots, a_d$. Set $e = N/c, a_i' = a_i/c, e' = N'/c$. Let $\widetilde{T}$ be the normalization of $T = \left\{s^e = x_1^{a_1'} \dots x_d^{a_d'}\right\}$ and $\widetilde{T}'$ be the normalization of $T' = \left\{s^{e'} = x_2^{a_2'} \dots x_d^{a_d'}\right\}$. Both $\widetilde{T}$ and $\widetilde{T}'$ are equipped with a μ_e-action as in the statement of the lemma.

The mapping

$$(\zeta, s, x) \in \mu_N \times T \to (\zeta s, x) \in S$$

induce an isomorphism

$$(\mu_N \times \widetilde{T})/\mu_e \simeq \widetilde{S},$$

where the μ_N-action on $\widetilde{S}$ correspond to the action on $(\mu_N \times \widetilde{T})/\mu_e$ given by multiplication on μ_N. Similarly, the mapping

$$(\zeta, s, x) \in \mu_N \times T' \to (\zeta^{N/N'} s, x) \in S'$$

induce an isomorphism

$$(\mu_N \times \widetilde{T}')/\mu_e \simeq \widetilde{S}'.$$

Hence we can apply the first case to $\widetilde{T}$ and $\widetilde{T}'$ to get $\widetilde{S}_{|x_1=0} \simeq \widetilde{S}'$ and check that the actions correspond. □

Lemma 5.8 *Denote again by $\widetilde{S}$ the normalization of $S = \{s^N = x_1^{a_1} \dots x_d^{a_d}\}$. Then $\mathcal{D}_{\mathbb{A}^d_K}[\widetilde{S}] = [\widetilde{S}]\mathbb{L}^{-d} \in \mathcal{M}^{\mu_q}_{\mathbb{A}^d_K}$, $\zeta \in \mu_q$ acting again by multiplication of s by $\zeta^{q/N}$.*

Proof It is a particular case of Lemma 5.4 when $N, a_1, \dots, a_d$ are coprime, and the general case follows as in the proof of Lemma 5.7. □

5.2 Computation of cohomological motives

Using Corollary 5.6, we can now compute the cohomological motive of $\widetilde{D_I^\circ}$ in terms of homological motives.

Proposition 5.9 *For any $I \subset J$ such that D_i is proper for $i \in I$, we have*

$$[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] = \sum_{I \subset L \subset J} (-1)^{|L|-|I|}[\mathrm{M}_K\left(\widetilde{D_L^\circ} \times_K {\mathbb{G}_m}_K^{|L|-|I|}\right)(-d+|I|-1)] \in \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K)).$$

We first prove an auxiliary formula.

Lemma 5.10 *For any $I \subset J$ such that D_i is proper for $i \in I$, we have* $[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] =$

$$\left[\mathrm{M}_K(\widetilde{D_I^\circ})(-d+|I|-1)\right] + \sum_{I \subsetneq L \subset J}\left(\left[\mathrm{M}_K(\widetilde{D_L^\circ})(-d+|L|-1)\right] - [\mathrm{M}_K^\vee(\widetilde{D_L^\circ})(|I|-|L|)]\right).$$

Proof By the first point of Corollary 5.6 and additivity of $\mathrm{M}_{K,c}^\vee(-)$, we have

$$[\mathrm{M}_{K,c}^\vee(\widetilde{D_I^\circ})] = [\mathrm{M}_{K,c}^\vee(\widetilde{D_I})] - \sum_{I \subsetneq L \subset J} [\mathrm{M}_{K,c}^\vee(\widetilde{D_L^\circ})]. \tag{5.1}$$

As each of the $\widetilde{D_L^\circ}$ is smooth of pure dimension $d - |L| + 1$, by Proposition 3.2, $[\mathrm{M}_{K,c}^\vee(\widetilde{D_L^\circ})] = [\mathrm{M}_K(\widetilde{D_L^\circ})(-d+|L|-1)]$. We apply $\mathbb{D}_K$ to Eq. 5.1. By linearity of $\mathbb{D}_K$, the fact that $\mathbb{D}_K\mathrm{M}_K(\widetilde{D_L^\circ})(-d+|L|-1) = \mathrm{M}_K^\vee(\widetilde{D_L^\circ})(+d-|L|+1)$ and second point of Corollary 5.6, we get

$$[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})(d-|I|+1)] = [\mathrm{M}_{K,c}^\vee(\widetilde{D_I})(d-|I|+1)] - \sum_{I \subsetneq L \subset J} [\mathrm{M}_K^\vee(\widetilde{D_L^\circ})(d-|L|+1)]. \tag{5.2}$$

Twisting this equation $d - |I| + 1$ times and applying again Corollary 5.6 gives the desired result. □

Proof of Proposition 5.9 We work by induction on $d - |I| + 1$. If $d - |I| + 1 < 0$, then $\widetilde{D_I^\circ}$ is empty and there is nothing to show. If $d - |I| + 1 = 0$, the formula boils down to

$$[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] = [\mathrm{M}_K(\widetilde{D_I^\circ})],$$

which holds since $\widetilde{D_I^\circ}$ is of dimension 0.

Suppose now the proposition holds for any L with $|L| > |I|$. Let $r = d - |I| + 1$. By Lemma 5.10,

$$[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] = [\mathrm{M}_K(\widetilde{D_I^\circ})(-r)] + \sum_{I\subsetneq L\subset J}([\mathrm{M}_K(\widetilde{D_L^\circ})(-d+|L|-1)] \\ -[\mathrm{M}_K^\vee(\widetilde{D_L^\circ})(|I|-|L|)]).$$

Applying the induction hypothesis to $[\mathrm{M}_K^\vee(\widetilde{D_L^\circ})]$, we get

$$[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] = [\mathrm{M}_K(\widetilde{D_I^\circ})(-r)] + \sum_{I\subsetneq L\subset J}\Bigg([\mathrm{M}_K(\widetilde{D_L^\circ})(-d+|L|-1)] \\ - \sum_{L\subset L'\subset J}(-1)^{|L'|-|L|}[\mathrm{M}_K(\widetilde{D_{L'}^\circ \times_K \mathbb{G}_{m}{}_K^{|L'|-|L|}})(-d+|I|-1)]\Bigg).$$

Interverting the sums, we get

$$[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] = [\mathrm{M}_K(\widetilde{D_I^\circ})(-r)] + \sum_{I\subsetneq L\subset J}[\mathrm{M}_K(\widetilde{D_L^\circ})(-d+|L|-1)] \\ - \sum_{I\subsetneq L'\subset J}\sum_{i=0}^{|L'|-|I|-1}\binom{|L'|-|I|}{|L'|-|I|-i}(-1)^i\left[\mathrm{M}_K(\widetilde{D_{L'}^\circ \times_K \mathbb{G}_{m}{}_K^{i}})(-r)\right].$$

Regrouping the terms, we get

$$[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] = [\mathrm{M}_K(\widetilde{D_I^\circ})(-r)] + \sum_{I\subsetneq L'\subset J}[\mathrm{M}_K(\widetilde{D_{L'}^\circ})(-r)] \\ \cdot\left([\mathrm{M}_K(\mathbb{1})(|L'|-|I|)] - \sum_{i=0}^{|L'|-|I|-1}\binom{|L'|-|I|}{i}(-1)^i[\mathrm{M}_K(\mathbb{G}_{m}{}_K^{i})]\right). \tag{5.3}$$

We need to compute the expression inside the big brackets. We have

$$[\mathrm{M}_K(\mathbb{1})(|L'|-|I|)] - \sum_{i=0}^{|L'|-|I|-1}\binom{|L'|-|I|}{i}(-1)^i[\mathrm{M}_K(\mathbb{G}_{m}{}_K^{i})] \\ = [\mathrm{M}_K(\mathbb{1})(|L'|-|I|)] + (-1)^{|L'|-|I|}[\mathrm{M}_K(\mathbb{G}_{m}{}_K^{|L'|-|I|})] \\ -([\mathrm{M}_K(\mathbb{1})]-[\mathrm{M}_K(\mathbb{G}_{m\,K})])^{|L'|-|I|} \\ = (-1)^{|L'|-|I|}[\mathrm{M}_K(\mathbb{G}_{m}{}_K^{|L'|-|I|})]$$

because $[\mathrm{M}_K(\mathbb{G}_{m\,K})] = [\mathrm{M}_K(\mathbb{1})] - [\mathrm{M}_K(\mathbb{1})(1)]$. Injecting in Eq. 5.3 gives the required expression for $[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})]$. $\square$

Proposition 5.11 *Let $\mathcal{X}$ be a semi-stable formal R-scheme and $(D_i)_{i\in I}$ the reduced irreducible components of its special fiber. Set $J' \subset J$ such that for every $i \in J'$, D_i is proper. Then setting $D = \bigcup_{i\in J'} D_i$, we have*

$$\chi_{\mathrm{Rig}}(]D[^{\mathrm{VF}}_{\mathcal{X}}) = [\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D[_{\mathcal{X}})].$$

Proof By additivity of χ_{Rig}, Proposition 4.21 and Corollary 4.24, we have

$$\begin{aligned}\chi_{\mathrm{Rig}}(]D[^{\mathrm{VF}}_{\mathcal{X}}) &= \sum_{\substack{I\subset J\\ I\cap J'\neq\emptyset}} \chi_{\mathrm{Rig}}(]D^{\circ}_I[^{\mathrm{VF}}) \\ &= \sum_{\substack{I\subset J\\ I\cap J'\neq\emptyset}} (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D^{\circ}_I[)(-d)] \\ &= \sum_{\substack{I\subset J\\ I\cap J'\neq\emptyset}} (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D^{\circ}_I, u_I) \times \partial\mathbb{B}(o,1)^{|I|-1})(-d)]. \end{aligned} \tag{5.4}$$

We will relate the cohomological motive of the tube to this formula using the duality relations proven above. By Corollary 4.19, we have

$$\left[\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D[) = \sum_{I\subseteq J'} (-1)^{|I|-1}[\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D^{\circ}_I[)\right]. \tag{5.5}$$

By Corollary 4.24,

$$\begin{aligned}[\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D^{\circ}_I[)] &= \left[\mathrm{M}^{\vee}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D^{\circ}_I, u_I) \times \partial\mathbb{B}(o,1)^{|I|-1})\right] \\ &= \left[\mathrm{M}^{\vee}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D^{\circ}_I, u_I))\right] \cdot [\mathrm{M}^{\vee}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)^{|I|-1})\Big]. \end{aligned} \tag{5.6}$$

Combining Eqs. 5.5 and 5.6, we get

$$[\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D[)] = \sum_{I\subseteq J'} (-1)^{|I|-1}\left[\mathrm{M}^{\vee}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D^{\circ}_I, u_I))\right]\left[\mathrm{M}^{\vee}_{\mathrm{Rig}}(\partial\mathbb{B}(o, 1^{|I|-1})\right]. \tag{5.7}$$

The analytification of $\widetilde{D^{\circ}_I}$ is $Q^{\mathrm{an}}_{N_I}(D^{\circ}_I, u_I)$, hence by Theorem 3.5,

$$\mathrm{Rig}^*\mathrm{M}_K(\widetilde{D^{\circ}_I}) = \mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D^{\circ}_I, u_I))$$

and similarly for cohomological motives.

For each $I \subseteq J'$, D_I satisfies the hypothesis of Proposition 5.9, hence after applying Rig^*, we get

$$\begin{aligned}&[\mathrm{M}^{\vee}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D^{\circ}_I, u_I))] \\ &\quad= \sum_{I\subset L\subset J} (-1)^{|L|-|I|}[\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_L}(D^{\circ}_L, u_L) \times_K \partial\mathbb{B}(o,1)^{|L|-|I|})(-d+|I|-1)]. \end{aligned} \tag{5.8}$$

We also know that $[\mathrm{M}^\vee_{\mathrm{Rig}}(\partial\mathbb{B}(o,1))] = -[\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1))(-1)]$. With these two remarks, Eq. 5.7 yields

$$\begin{aligned}
[\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)] &= \sum_{\emptyset\neq I\subset J'}\sum_{I\subset L\subset J}(-1)^{|L|-|I|}\left[\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_L}(D^\circ_L,u_L)\times_K\partial\mathbb{B}(o,1)^{|L|-1})(-d)\right]\\
&= \sum_{\substack{L\subset J\\ L\cap J'\neq\emptyset}}(-1)^{|L|-1}\left[\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_L}(D^\circ_L,u_L)\times_K\partial\mathbb{B}(o,1)^{|L|-1})(-d)\right]\\
&\quad\cdot\sum_{i=1}^{|L\cap J'|}\binom{|L\cap J'|}{i}(-1)^{i-1}\\
&= \sum_{\substack{L\subset J\\ L\cap J'\neq\emptyset}}(-1)^{|L|-1}\left[\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_L}(D^\circ_L,u_L)\times_K\partial\mathbb{B}(o,1)^{|L|-1})(-d)\right].
\end{aligned}\tag{5.9}$$

Comparing to Eq. 5.4 gives the desired

$$\chi_{\mathrm{Rig}}(]D[^{\mathrm{VF}}) = [\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)].$$

□

Proposition 5.11 imply the following theorem, which is Theorem 1.2 of the introduction. All we need to do is choosing a semi-stable formal R-scheme $\mathcal{Y}$ over $\mathcal{X}$ such that $\mathcal{Y}\to\mathcal{X}$ is a composition of admissible blow-ups. Hence the induced morphism at the level of special fibers is proper and we can apply Proposition 5.11.

Theorem 5.12 *Let X be a quasi-compact smooth rigid K-variety, $\mathcal{X}$ an formal R-model of X, D a proper subscheme of its special fiber $\mathcal{X}_\sigma$. Then*

$$\chi_{\mathrm{Rig}}(]D[^{\mathrm{VF}}) = [\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)].$$

In particular, if X is a smooth and proper rigid variety,

$$\chi_{\mathrm{Rig}}([X^{\mathrm{VF}}]) = [\mathrm{M}^\vee_{\mathrm{Rig}}(X)].$$

5.3 Analytic Milnor fiber

It is suggested by Ayoub et al. [4, Remark 8.15] that one should be able to recover their comparison result between the motivic Milnor fiber and the cohomological motive of the analytic Milnor fiber using a morphism similar to χ_{Rig}. We show below that it is indeed the case and moreover generalize their comparison result to an equivariant setting.

Let X be a smooth k-variety and $f : X \to \mathbb{A}^1_k$ a non-constant regular function. Base change to R makes of X an R-scheme. Denote $\mathcal{X}_f$ the formal completion of X with respect to (t). Its special fiber $\mathcal{X}_{f,\sigma}$ is the zero locus of f in X. For any closed point $x\in\mathcal{X}_{f,\sigma}$, denote by $\mathcal{F}^{\mathrm{an}}_{f,x}$ the tube of $\{x\}$ in $\mathcal{X}_f$. It is the analytic Milnor fiber. It is a rigid subvariety of $\mathcal{X}_{f,\eta}$, the analytic nearby cycles.

Consider an embedded resolution of singularities of $\mathcal{X}_{f,\sigma}$ in X. It is a proper birational morphism $h : Y \to X$ such that $h^{-1}(\mathcal{X}_{f,\sigma})$ is a smooth strict normal crossing divisor. Denote by $(E_i)_{i\in J}$ the reductions of its (smooth) irreducible components and $N_i \in \mathbb{N}^*$ the multiplicity of E_i in $h^{-1}(\mathcal{X}_{f,\sigma})$.

For any non-empty $I \subset J$, denote by $E_I = \cap_{i\in I} E_i$ and $E_I^\circ = E_I \backslash \cup_{j\in J\backslash I} E_j$. Define as follows the étale cover $\widetilde{E_I^\circ}$ of E_I°. Let N_I be the greatest common divisor of the N_i, for $i \in I$. Working locally on some open neighborhood U of E_I° in Y, we can assume that E_i is defined by equation $t_i = 0$, for some $t_i \in \mathcal{O}(U)$ and that on U, $f = u_I t_{i_1}^{N_{i_1}} \dots t_{i_r}^{N_{i_r}}$, with $u_I \in \mathcal{O}(U)^\times$. Then set $\widetilde{E_I^\circ \cap U} = \left\{(v, x) \in \mathbb{G}_{mk} \times U \mid v^{N_I} = u_I\right\}$.

Recall the motivic Milnor fiber, defined by Denef and Loeser, see for example [17]. In an equivariant setting, for any closed point $x \in \mathcal{X}_{f,\sigma}$, it satisfies the formula

$$\psi_{f,x} = \sum_{\emptyset\neq I\subseteq J} (-1)^{|I|-1}[\mathbb{G}_{m_k}^{|I|-1}][\widetilde{E_I^\circ} \cap h^{-1}(x)] \in \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}).$$

In particular, they show that this formula is independent of the chosen resolution h.

Remark 5.13 In the literature, the motivic Milnor is defined in the localization of $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ by $\mathbb{L} = [\mathbb{A}_k^1]$. Such a localization in non-injective if $k = \mathbb{C}$, see Borisov [8]. However, Proposition 5.14 shows that it is well defined in $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$. The same fact is proven in [28, Corollary 2.6.2] using a computation similar to ours.

Proposition 5.14 *For any closed point $x \in \mathcal{X}_{f,\sigma}$,*

$$\Theta \circ \mathcal{E}_c \circ \oint (\mathcal{F}_{f,x}^{\mathrm{an,VF}}) = \psi_{f,x} \in \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}).$$

Proof The embedded resolution of $\mathcal{X}_{f,\sigma}$ induces an admissible morphism $h : \mathcal{Y} \to \mathcal{X}_f$, hence $]h^{-1}(x)[_{\mathcal{Y}} \simeq\,]\{x\}[_{\mathcal{X}_f}$. Up to changing h, we can suppose $h^{-1}(x)$ is a divisor $E = \cup_{i\in J'} E_i$ in $\mathcal{Y}_\sigma = \cup_{i\in J} D_i$, with $I' \subset I$. Then we have

$$\mathcal{F}_{f,x}^{\mathrm{an,VF}} = \dot{\bigcup_{\substack{I\subset J\\ I\cap J'\neq\emptyset}}}]E_I^\circ[^{\mathrm{VF}}.$$

We want to show that

$$\Theta \circ \mathcal{E}_c \circ \oint (]E_I^\circ[) = (-1)^{|I|-1}[\mathbb{G}_{m_k}^{|I|-1} \times_k \widetilde{E_I^\circ}].$$

By Remark 4.23 following Lemma 4.22, we can suppose $\mathcal{Y} = \mathrm{St}_{E_I^\circ\times_k R,\underline{N}}^{u_I^{-1}t}$, where $\underline{N} = (N_1, \dots, N_r) \in (\mathbb{N}^\times)^r$ (where $r = |I|$), $u_I \in \mathcal{O}(E_I^\circ \times_k R)^\times$. But now, as in the proof of Proposition 4.21, $]E_I^\circ[^{\mathrm{VF}}$ is definably isomorphic to $Q_{N_I}^{\mathrm{VF}}(E_I^\circ, u) \times \mathrm{v}^{-1}(\Delta)$. Hence $\oint]E_I^\circ[= [Q_{N_I}^{\mathrm{RV}}(E_I^\circ, u_I) \times \mathrm{v}_{\mathrm{rv}}^{-1}(\Delta)]_d$. Now $\Theta Q_{N_I}^{\mathrm{RV}}(E_I^\circ, u_I) = [\widetilde{E_I^\circ}]$ and

$$\Theta \circ \mathcal{E}_c \mathrm{v}_{\mathrm{rv}}^{-1}(\Delta) = \mathrm{eu}_c(\Delta)[\mathbb{G}_{m\,k}]^{|I|-1} = (-1)^{|I|-1}[\mathbb{G}_{m\,k}]^{|I|-1},$$

So, putting pieces together by linearity of $\Theta \circ \mathcal{E}_c \circ \oint$,

$$\Theta \circ \mathcal{E}_c \circ \oint \mathcal{F}_{f,x}^{\mathrm{an,VF}} = \sum_{\substack{I \subset J \\ I \cap J' \neq \emptyset}} (-1)^{|I|-1}[\mathbb{G}_{m\,k}^{\;|I|-1} \times_k \widetilde{E}_I^{\circ}] = \psi_{f,x} \in \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}).$$

□

From Theorem 5.12, we deduce the following corollary.

Corollary 5.15 *For any closed point $x \in \mathcal{X}_{f,\sigma}$,*

$$\chi_{\mathrm{Rig}}(\mathcal{F}_{f,x}^{\mathrm{an,VF}}) = [\mathrm{M}_{\mathrm{Rig}}^{\vee}(\mathcal{F}_{f,x}^{\mathrm{an}})].$$

Combining Corollary 5.15 and Proposition 5.14 with Theorem 4.13 gives the following result.

Corollary 5.16 *For any closed point $x \in \mathcal{X}_{f,\sigma}$, we have*

$$[\mathfrak{R}\mathrm{M}_{\mathrm{Rig}}^{\vee}(\mathcal{F}_{f,x}^{\mathrm{an}})] = \chi_{\hat{\mu}}(\psi_{f,x}) \in \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).$$

It is a generalization of Corollary 8.9 of Ayoub et al. [4] at an equivariant level. They show the same equality, but in $\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k))$, hence one can deduce their result from Corollary 5.16 using Lemma 3.4.

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References

1. Ayoub, J.: Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. Société Mathématique de France, Paris (2007)
2. Ayoub, J.: Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II. Société Mathématique de France, Paris (2007)
3. Ayoub, A.: Motifs des variétés analytiques rigides. Mém. Soc. Math. Fr., Nouv. Sér. **140–141**, 1–386 (2015)
4. Ayoub, J., Ivorra, F., Sebag, J.: Motives of rigid analytic tubes and nearby motivic sheaves. Ann. Sci. Éc. Norm. Supér. (4) **50**(6), 1335–1382 (2017)
5. Berthelot, P.: Cohomologie Rigide et Cohomologie Rigide à Supports Propres, Première Partie. Prépublication IRMAR 96-03, Université de Rennes, (1996)
6. Bittner, F.: The universal Euler characteristic for varieties of characteristic zero. Compos. Math. **140**(4), 1011–1032 (2004)
7. Bittner, F.: On motivic zeta functions and the motivic nearby fiber. Math. Z. **249**(1), 63–83 (2005)
8. Borisov, L.: The class of the affine line is a zero divisor in the Grothendieck ring. J. Algebr. Geom. **27**(2), 203–209 (2018)

9. Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean Analysis. A Systematic Approach to Rigid Analytic Geometry. Springer, New York (1984)
10. Bosch, S., Lütkebohmert, W.: Formal and rigid geometry. I: rigid spaces. Math. Ann. **295**(2), 291–317 (1993)
11. Cisinski, D.-C., Déglise, F.: Triangulated categories of mixed motives. arXiv:0912.2110 [math] (2009)
12. Cluckers, R., Lipshitz, L.: Fields with analytic structure. J. Eur. Math. Soc. (JEMS) **13**(4), 1147–1223 (2011)
13. Deligne, P.: Voevodsky's Lectures on Cross Functors (2001)
14. Denef, J., Loeser, F.: Motivic Igusa zeta functions. J. Algebraic Geom. **7**(3), 505–537 (1998)
15. Denef, J., Loeser, F.: Germs of arcs on singular algebraic varieties and motivic integration. Invent. Math. **135**(1), 201–232 (1999)
16. Denef, J., Loeser, F.: Geometry on arc spaces of algebraic varieties. In: European Congress of Mathematics, Vol. I (Barcelona, 2000), volume 201 of Progr. Math., pp. 327–348. Birkhäuser, Basel (2001)
17. Denef, J., Loeser, F.: Lefschetz numbers of iterates of the monodromy and truncated arcs. Topology **41**(5), 1031–1040 (2002)
18. Fresnel, J., van der Put, M.: Rigid Analytic Geometry and Its Applications. Birkhäuser, Boston (2004)
19. Fulton, W.: Introduction to Toric Varieties. The 1989 William H. Roever Lectures in Geometry. Princeton University Press, Princeton (1993)
20. Hrushovski, E., Kazhdan, D.: Integration in valued fields. In: Algebraic Geometry and Number Theory. In Honor of Vladimir Drinfeld's 50th Birthday, pp. 261–405. Birkhäuser, Basel (2006)
21. Hrushovski, E., Loeser, F.: Monodromie et formule des points fixes de Lefschetz. Ann. Sci. Éc. Norm. Supér. (4) **48**(2), 313–349 (2015)
22. Ivorra, F., Sebag, J.: Nearby motives and motivic nearby cycles. Sel. Math. New Ser. **19**(4), 879–902 (2013)
23. Le Stum, B.: Rigid Cohomology. Cambridge University Press, Cambridge (2007)
24. Lipshitz, L., Robinson, Z.: Uniform properties of rigid subanalytic sets. Trans. Am. Math. Soc. **357**(11), 4349–4377 (2005)
25. Loeser, F.: Seattle lectures on motivic integration. In: Algebraic Geometry, Seattle 2005. Proceedings of the 2005 Summer Research Institute, Seattle, WA, USA, July 25–August 12, 2005, pp. 745–784. American Mathematical Society (AMS), Providence, RI (2009)
26. Martin, F.: Cohomology of locally closed semi-algebraic subsets. Manuscr. Math. **144**(3–4), 373–400 (2014)
27. Morel, F., Voevodsky, V.: $\mathbb{A}^1$-homotopy theory of schemes. Publ. Math. Inst. Hautes Étud. Sci. **90**, 45–143 (1999)
28. Nicaise, J., Payne, S.: A tropical motivic Fubini theorem with applications to Donaldson–Thomas theory. Duke. Math. J. arXiv:1703.10228 (2017) (to appear)
29. Nicaise, J., Payne, S., Schroeter, F.: Tropical refined curve counting via motivic integration. Geom. Topol. **22**(6), 3175–3234 (2018)
30. Nicaise, J., Sebag, J.: Motivic Serre invariants, ramification, and the analytic Milnor fiber. Invent. math. **168**(1), 133–173 (2007)
31. Stacks project authors. The Stacks project (2018)
32. Tate, J.: Rigid analytic spaces. Invent. Math. **12**, 257–289 (1971)
33. van den Dries, L.: Tame Topology and O-Minimal Structures. Cambridge University Press, Cambridge (1998)
34. Verdier, J.-L.: Des catégories dérivées des catégories abéliennes. Société Mathématique de France, Paris (1996)
35. Yin, Y.: Special transformations in algebraically closed valued fields. Ann. Pure Appl. Logic **161**(12), 1541–1564 (2010)
36. Yin, Y.: Integration in algebraically closed valued fields. Ann. Pure Appl. Logic **162**(5), 384–408 (2011)