# Surrogate models for uncertain dynamical systems: applications to earthquake engineering

Bruno Sudret

**Chair of Risk, Safety and Uncertainty Quantification**

**ETH Zurich**

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- Surrogate models (polynomial chaos expansions, Kriging, support vector machines)
- Bayesian model calibration and stochastic inverse problems
- Global sensitivity analysis
- Reliability-based design optimization

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# Global framework for uncertainty quantification

**Step B**
Quantification of sources of uncertainty

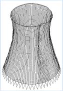
**Step A**
Model(s) of the system
Assessment criteria

**Step C**
Uncertainty propagation

Random variables



Computational model



Moments
Probability of failure
Response PDF

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B. Sudret, *Uncertainty propagation and sensitivity analysis in mechanical models – contributions to structural reliability and stochastic spectral methods (2007)*

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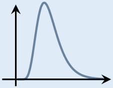

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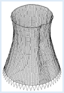

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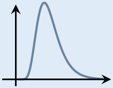

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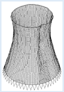

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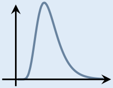

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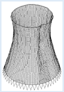

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# Surrogate models for uncertainty quantification

A surrogate model $\tilde{\mathcal{M}}$ is an approximation of the original computational model $\mathcal{M}$ with the following features:

- It is built from a limited set of runs of the original model $\mathcal{M}$ called the experimental design $\mathcal{X} = \left\{ \boldsymbol{x}^{(i)},\, i = 1, \ldots, n \right\}$
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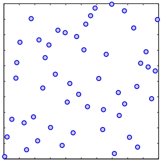
| **Name** | **Shape** | **Parameters** |
|---|---|---|
| Polynomial chaos expansions | $\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} a_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{x})$ | $\boldsymbol{a}_{\boldsymbol{\alpha}}$ |
| Low-rank tensor approximations | $\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{l=1}^{R} b_l \left( \prod_{i=1}^{M} v_l^{(i)}(x_i) \right)$ | $b_l, \; z_{k,l}^{(i)}$ |
| Kriging (a.k.a Gaussian processes) | $\tilde{\mathcal{M}}(\boldsymbol{x}) = \boldsymbol{\beta}^{\mathsf{T}} \cdot \boldsymbol{f}(\boldsymbol{x}) + Z(\boldsymbol{x}, \omega)$ | $\boldsymbol{\beta} \, , \, \sigma_Z^2 \, , \, \boldsymbol{\theta}$ |
| Support vector machines | $\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{i=1}^{n} a_i \, K(\boldsymbol{x}_i, \boldsymbol{x}) + b$ | $\boldsymbol{a} \, , \, b$ |

# Ingredients for building a surrogate model

- Select an experimental design $\mathcal{X}$ that covers at best the domain of input parameters: Latin hypercube sampling (LHS), low-discrepancy sequences



- Run the computational model $\mathcal{M}$ onto $\mathcal{X}$ exactly as in Monte Carlo simulation

- Smartly post-process the data $\{\mathcal{X}, \mathcal{M}(\mathcal{X})\}$ through a learning algorithm

| Name | Learning method |
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| Polynomial chaos expansions | sparse grid integration, least-squares, compressive sensing |
| Low-rank tensor approximations | alternate least squares |
| Kriging | maximum likelihood, Bayesian inference |
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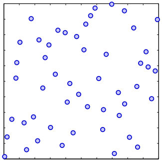

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# Advantages of surrogate models

## Usage

$$\underset{\text{hours per run}}{\mathcal{M}(\boldsymbol{x})} \quad \approx \quad \underset{\text{seconds for } 10^6 \text{ runs}}{\tilde{\mathcal{M}}(\boldsymbol{x})}$$

## Advantages

- Non-intrusive methods: based on runs of the computational model, exactly as in Monte Carlo simulation
- Construction suited to high performance computing: "embarrassingly parallel"

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## Challenges

- Need for rigorous validation
- Communication: advanced mathematical background

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Efficiency: 2-3 orders of magnitude less runs compared to Monte Carlo

# Outline

# Polynomial chaos expansions in a nutshell

Ghanem & Spanos (1991); Xiu & Karniadakis (2002); Soize & Ghanem (2004); Lemaître & Knio (2010)

- Consider the input random vector $\boldsymbol{X}$ (dim $\boldsymbol{X} = M$) with given probability density function (PDF) $f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{M} f_{X_i}(x_i)$
- Assuming that the random output $Y = \mathcal{M}(\boldsymbol{X})$ has finite variance, it can be cast as the following polynomial chaos expansion:

$$Y = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^M} y_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$$

  where :

  - $\Psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$ : basis functions
  - $y_{\boldsymbol{\alpha}}$ : coefficients to be computed (coordinates)

- The PCE basis $\left\{\Psi_{\boldsymbol{\alpha}}(\boldsymbol{X}),\ \boldsymbol{\alpha} \in \mathbb{N}^M\right\}$ is made of multivariate orthonormal polynomials

# Computing the coefficients by least-square minimization

Isukapalli (1999); Berveiller, Sudret & Lemaire (2006)

## Principle

The exact (infinite) series expansion is considered as the sum of a truncated series and a residual:

$$Y = \mathcal{M}(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} y_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) + \varepsilon_P \equiv \mathbf{Y}^{\mathsf{T}} \boldsymbol{\Psi}(\boldsymbol{X}) + \varepsilon_P(\boldsymbol{X})$$

where : $\mathbf{Y} = \{y_{\boldsymbol{\alpha}}, \, \boldsymbol{\alpha} \in \mathcal{A}\} \equiv \{y_0, \, \dots \, , y_{P-1}\}$ ($P$ unknown coef.)

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \{\Psi_0(\boldsymbol{x}), \, \dots \, , \Psi_{P-1}(\boldsymbol{x})\}$$

## Least-square minimization

The unknown coefficients are estimated by minimizing the mean square residual error:

$$\hat{\mathbf{Y}} = \arg\min \mathbb{E}\left[\left(\mathbf{Y}^{\mathsf{T}} \boldsymbol{\Psi}(\boldsymbol{X}) - \mathcal{M}(\boldsymbol{X})\right)^2\right]$$

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# Discrete (ordinary) least-square minimization

An estimate of the mean square error (sample average) is minimized:

$$\hat{\mathbf{Y}} = \arg\min_{\mathbf{Y}\in\mathbb{R}^P} \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{Y}^{\mathsf{T}}\boldsymbol{\Psi}(\boldsymbol{x}^{(i)}) - \mathcal{M}(\boldsymbol{x}^{(i)})\right)^2$$

## Procedure

- Select a truncation scheme, *e.g.* $\mathcal{A}^{M,p} = \left\{\boldsymbol{\alpha}\in\mathbb{N}^M \;:\; |\boldsymbol{\alpha}|_1 \le p\right\}$
- Select an experimental design and evaluate the model response

$$\mathbf{M} = \left\{\mathcal{M}(\boldsymbol{x}^{(1)}), \dots, \mathcal{M}(\boldsymbol{x}^{(n)})\right\}^{\mathsf{T}}$$

- Compute the experimental matrix

$$\mathbf{A}_{ij} = \Psi_j\left(\boldsymbol{x}^{(i)}\right) \quad i=1,\dots,n\;;\; j=0,\dots,P-1$$

- Solve the resulting linear system

$$\hat{\mathbf{Y}} = (\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{M}$$

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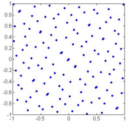


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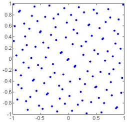

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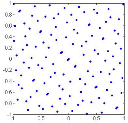

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Simple is beautiful !

# Error estimators

- In least-squares analysis, the generalization error is defined as:

$$E_{gen} = \mathbb{E}\left[\left(\mathcal{M}(\boldsymbol{X}) - \mathcal{M}^{\mathsf{PC}}(\boldsymbol{X})\right)^2\right] \qquad \mathcal{M}^{\mathsf{PC}}(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} y_{\boldsymbol{\alpha}}\, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$$

- The empirical error based on the experimental design $\mathcal{X}$ is a poor estimator in case of overfitting

$$E_{emp} = \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{M}(\boldsymbol{x}^{(i)}) - \mathcal{M}^{\mathsf{PC}}(\boldsymbol{x}^{(i)})\right)^2$$

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## Leave-one-out cross validation

- From statistical learning theory, model validation shall be carried out using independent data

$$E_{LOO} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mathcal{M}(\boldsymbol{x}^{(i)}) - \mathcal{M}^{PC}(\boldsymbol{x}^{(i)})}{1-h_i}\right)^2$$

where $h_i$ is the $i$-th diagonal term of matrix $\mathbf{A}(\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}$

# Outline

# Why brute-force PCE fails in dynamics?

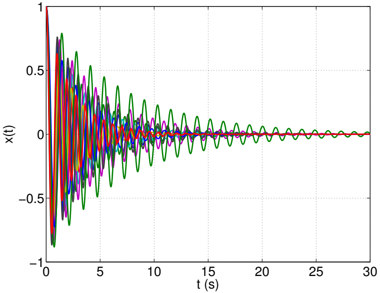
## Non-linear SDOF Duffing oscillator

$$\ddot{x}(t) + 2\,\omega\,\zeta\,\dot{x}(t) + \omega^2\,\left(x(t) + \varepsilon\,x^3(t)\right) = 0$$

Initial conditions: $x(0) = 1, \quad \dot{x}(0) = 0$

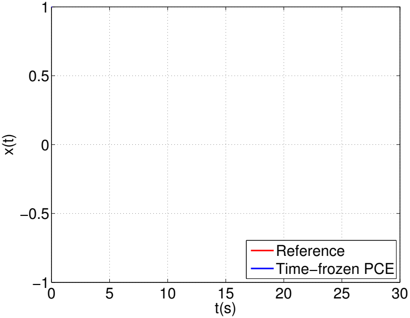
Input: 3 uniform random variables

| RV | Distribution | Values |
|---|---|---|
| $\zeta$ | Uniform | $\mathcal{U}[0.015,\, 0.045]$ |
| $\omega$ | Uniform | $\mathcal{U}[\pi,\, 3\,\pi]$ |
| $\varepsilon$ | Uniform | $\mathcal{U}[-0.25,\, -0.75]$ |

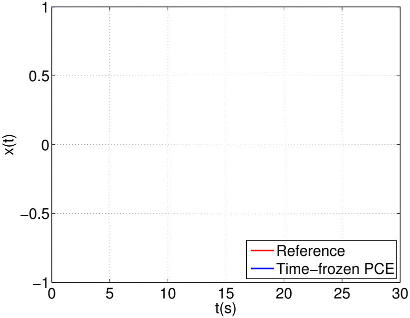



**Samples of trajectories**

# Time-frozen PCE



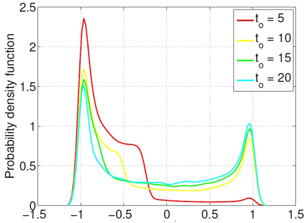

$(\zeta, \omega, \varepsilon) = (0.03, 8.92, -0.34)$

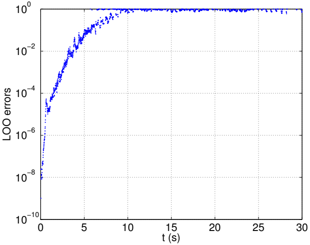



$(\zeta, \omega, \varepsilon) = (0.04, 3.18, -0.33))$

# Why time-frozen PCE does not work?

- The map $\boldsymbol{\xi} \mapsto \mathcal{M}(\boldsymbol{\xi}, t)$ becomes increasingly non linear with time
- The time-frozen distribution of the output at time $t_0$ becomes more complex (*e.g.* multimodal)
- Expansions of higher degree would be required to keep sufficient accuracy with time
- For a fixed experimental design, the LOO error blows up



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# Some literature

- Multi-elements PCEs: decomposition of the random space into non-overlapping sub-elements
  Wan & Karniadakis, 2005
- Constant phase interpolation: responses interpolated in the phase space
  Witteveen & Bijl, 2008
- Asynchronous time integration: intrusive transformed time variable introduced to reduce variability
  Le Maître et al., 2010
- Time-dependent PCEs: new random variables added on-the-fly
  Gerritsma et al., 2010
- PC flow map composition: long-term response obtained by composing intermediate PCE-based flow maps
  Luchtenburg et al., 2014
- PC-NARX: future state determined by current and past states
  Spiridonakos & Chatzi, 2015

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# Nonlinear AutoRegressive with eXogenous input model

## NARX model

Billings, 2013

Based on a time-dependent input excitation $x(t)$ and corresponding system response $y(t)$, the dynamics is captured through:

$$y(t) = \mathcal{F}\left(x(t), \ldots, x(t-n_x), y(t-1), \ldots, y(t-n_y)\right) + \varepsilon_t$$

where:

- $\boldsymbol{z}(t) = \left(x(t), \ldots, x(t-n_x), y(t-1), \ldots, y(t-n_y)\right)^{\mathsf{T}}$ is the vector of current and past values
- $n_x$ and $n_y$ denote the maximum input and output time lags
- $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2(t))$ is the residual error
- $\mathcal{F}(\cdot)$ is a functional of NARX terms, usually linear-in-parameters:

$$y(t) = \sum_{i=1}^{n_g} \vartheta_i \, g_i(\boldsymbol{z}(t)) + \varepsilon_t$$

# PC-NARX model

Spiridonakos *et al.* , 2015a,2015b

## Computational model with uncertainties

$$y(t, \boldsymbol{\xi}_x, \boldsymbol{\xi}_s) \stackrel{\text{def}}{=} \mathcal{M}(x(t, \boldsymbol{\xi}_x), \boldsymbol{\xi}_s)$$

- $\boldsymbol{\xi}_x$ : uncertainty in the input excitation
- $\boldsymbol{\xi}_s$ : uncertainty in the system

## PC-NARX expansion

$$y(t, \boldsymbol{\xi}) = \sum_{i=1}^{n_g} \vartheta_i(\boldsymbol{\xi})\, g_i(\boldsymbol{z}(t)) + \varepsilon_g(t, \boldsymbol{\xi}) \qquad \boldsymbol{\xi} = (\boldsymbol{\xi}_x, \boldsymbol{\xi}_s)$$

The NARX stochastic coefficients $\vartheta_i(\boldsymbol{\xi})$ are represented by PCEs:

$$\vartheta_i(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}_i} \vartheta_{i,\boldsymbol{\alpha}}\, \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$$

# PC-NARX model

Spiridonakos *et al.* , 2015a,2015b

## Computational model with uncertainties

$$y(t, \boldsymbol{\xi}_x, \boldsymbol{\xi}_s) \stackrel{\text{def}}{=} \mathcal{M}(x(t, \boldsymbol{\xi}_x), \boldsymbol{\xi}_s)$$

- $\boldsymbol{\xi}_x$ : uncertainty in the input excitation
- $\boldsymbol{\xi}_s$ : uncertainty in the system

## PC-NARX expansion

$$y(t, \boldsymbol{\xi}) = \sum_{i=1}^{n_g} \vartheta_i(\boldsymbol{\xi})\, g_i(\boldsymbol{z}(t)) + \varepsilon_g(t, \boldsymbol{\xi}) \qquad \boldsymbol{\xi} = (\boldsymbol{\xi}_x, \boldsymbol{\xi}_s)$$

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# PC-NARX model

$$y(t,\boldsymbol{\xi}) = \sum_{i=1}^{n_g} \sum_{\boldsymbol{\alpha}\in\mathcal{A}_i} \vartheta_{i,\boldsymbol{\alpha}}\, \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\, g_i(\boldsymbol{z}(t)) + \varepsilon(t,\boldsymbol{\xi})$$

## Interpretation

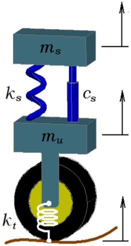
- PC-NARX is a NARX model in which each (random) coefficient is expanded as a PCE
- Compared to time-frozen PCE, a specific dynamics of the random coefficients is imposed
- Similar to flow map composition since the response at current instant is used to predict the response at future instants

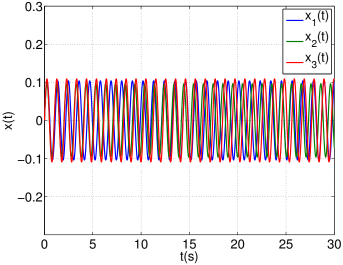
# Experimental design

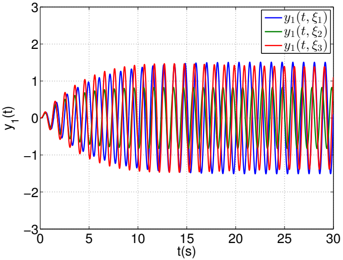
## Data

- $N$ realizations of the input excitation, cast as $\left(x_k[1], \ldots, x_k[T]\right)^{\mathsf{T}}, \ k = 1, \ldots, N$ ($T$ time instants)
- The corresponding system response computed by a simulator, cast as $\left(y_k[1], \ldots, y_k[T]\right)^{\mathsf{T}}$

## Example: quarter car model



# Deterministic NARX calibration

For a particular realization $\boldsymbol{\xi}_k$

- Select NARX model (candidate terms):

$$\boldsymbol{z}(t) = (x(t), \ldots, x(t-n_x), y(t-1), \ldots, y(t-n_y))^{\mathsf{T}}$$
$$\boldsymbol{\phi}(t) = \{g_i(\boldsymbol{z}(t)),\ i = 1, \ldots, n_g\}^{\mathsf{T}}$$

- Use least angle regression (LARS) to select the best explanatory subset of terms Efron *et al.* , 2004

- Compute the coefficients $\boldsymbol{\vartheta}_k$ by ordinary least-squares

Prediction error (of model $\#k$ on trajectory $l$)

$$\varepsilon_l^{\#k} = \frac{\sum_{t=1}^{T} (y(t,\boldsymbol{\xi}_l) - \hat{y}^{\#k}(t,\boldsymbol{\xi}_l))^2}{\sum_{t=1}^{T} (y(t,\boldsymbol{\xi}_l) - \bar{y}(t,\boldsymbol{\xi}_l))^2}$$

# Common NARX basis

## Premise

To expand the NARX coefficients onto a PC basis, it is necessary to have a common NARX model for all trajectories

## Procedure

- Select $K \leq N$ trajectories ("NARX learning set"), *e.g.* with the strongest non linear behaviour (peak displacement, velocities, etc.)
- Determine the sparse deterministic NARX models for realizations $k = 1, \ldots, K$, which leads to $P \leq K$ different possible models called $\#1, \ldots, \#P$
- Compute the NARX coefficients of the $N$ trajectories, for each model $\#p$, and evaluate an average error:

$$\varepsilon_p = \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k^{\#p}$$

- Select the final best NARX model that minimizes $\varepsilon_p$

# PCE of the NARX coefficients

## PCE calibration

- Once a common NARX basis has been found, $N$ realizations of the NARX coefficients are available:

$$\mathcal{ED} = \{\vartheta_{i,k}, i = 1, \ldots, n_g;\ k = 1, \ldots, N\}$$

- $n_g$ different sparse PC expansions are built from this experimental design, using least-angle regression (LAR) Blatman & Sudret, 2011

$$\vartheta_i(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}_i} \vartheta_{i,\boldsymbol{\alpha}}\, \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$$

## PC-NARX prediction

- For a new realization of the input parameters $\boldsymbol{\xi}_0$, the NARX coefficients are first evaluated from PCEs
- Then they are plugged into the NARX model

# Bouc-Wen model

## Governing equations

Kafali & Grigoriu (2007), Spiridonakos & Chatzi (2015)

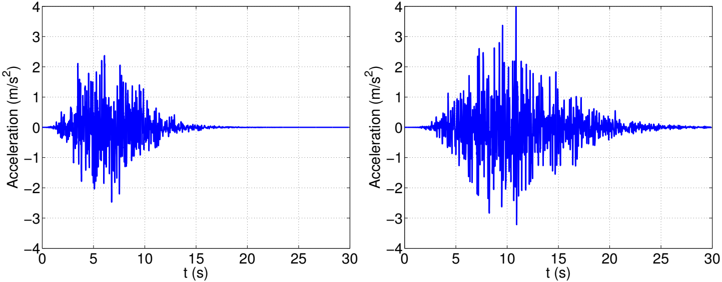
$$\ddot{y}(t) + 2\,\zeta\,\omega\,\dot{y}(t) + \omega^2(\rho\,y(t) + (1-\rho)\,z(t)) = -x(t),$$
$$\dot{z}(t) = \gamma\dot{y}(t) - \alpha\,|\dot{y}(t)|\,|z(t)|^{n-1}\,z(t) - \beta\,\dot{y}(t)\,|z(t)|^n\,,$$

## Excitation

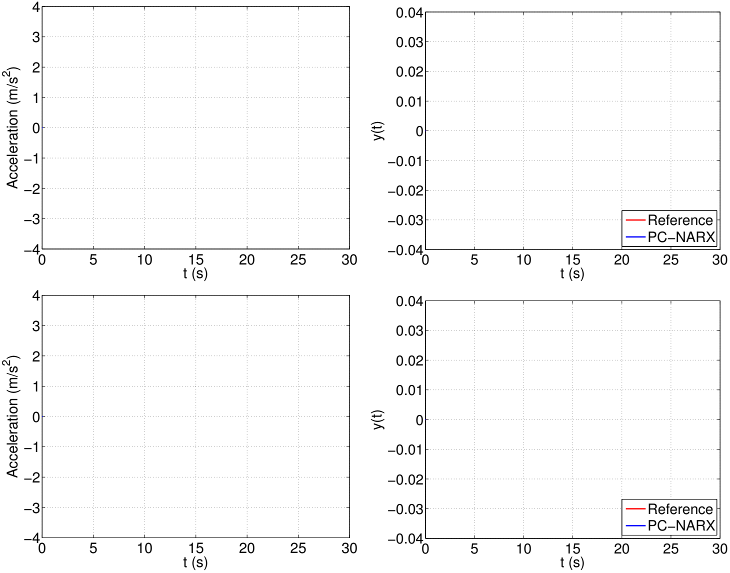
$x(t)$ is generated by a probabilistic ground motion model

Rezaeian & Der Kiureghian (2010)

$$x(t) = q(t,\boldsymbol{\alpha}) \sum_{i=1}^{n} s_i\left(t, \boldsymbol{\lambda}(t_i)\right)\, U_i$$



# Bouc-Wen model: prediction



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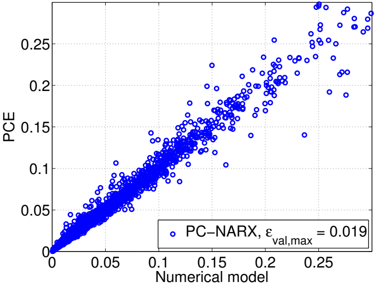
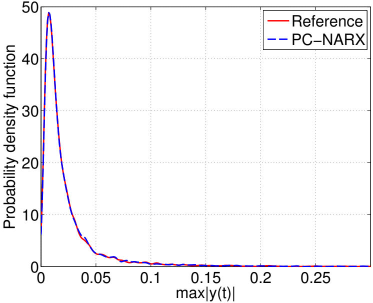



Maximal displacements




PDF of maximal displacements

# Outline

# Introduction to fragility curves

- Earthquake engineering aims at assessing the performance of structures and infrastructures w.r.t recorded or potential quakes
- Due to uncertainties in the localization, magnitude, structural behaviour and resistance, etc. probabilistic approaches are commonly used

## Fragility curves

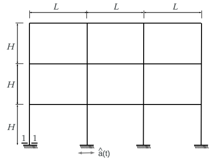
For a given performance criterion $g \leq g_{adm}$, the fragility curve represents the conditional probability of failure given an intensity measure $IM$:

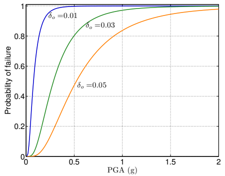
$$\text{Frag}(IM;\, g_{adm}) = \mathbb{P}\left(g \geq g_{adm} \,|\, IM\right)$$

**Example**

- $g = \max_k \max_{t_i \in [0,T]} |\delta^k_{t_i}|$ ($k$-th interstorey drift)
- $IM$: peak ground acceleration (PGA), pseudo-spectral acceleration (PSa), cumulative absolute velocity (CAV), etc.

# Fragility curves




## Classical approach

- Select a set of ground motions (recorded / synthetic)
- Compute the transient structural response (finite element analysis)
- Assume a parametric shape for the fragility curve, *e.g.* a lognormal shape:

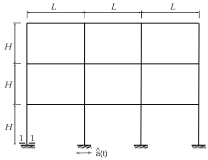
$$\text{Frag}(IM;\, \delta_o) = \mathbb{P}\left(\Delta \geq \delta_o \,|\, IM\right) = \Phi\left(\frac{\log IM - \alpha}{\beta}\right)$$

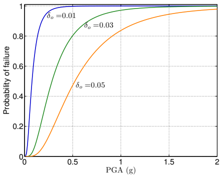
- Fit the parameters $(\alpha, \beta)$ form data

## Limitations

- Predefined shape of the curve
- Subject to epistemic uncertainties when the number of ground motions is small

# Fragility curves

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## New proposal

- Use non parametric statistics for the fragility curves
- Use surrogate models of the transient analysis based on polynomial chaos expansions

# Parametric methods

## Linear regression (LR)

Ellingwood (2001)

- Probabilistic demand model:

$$\log\Delta = A\,\log IM + B + \zeta\,Z \qquad Z\sim\mathcal{N}(0,1)$$

- $A$ and $B$ determined by ordinary least squares estimation in a log-log scale
- Results in a lognormal-like fragility curve:

$$\widehat{\text{Frag}}(IM;\,\delta_o) = \mathbb{P}\left[\log\Delta \geq \log\delta_o\right] = 1 - \mathbb{P}\left[\log\Delta \leq \log\delta_o\right]$$
$$= \Phi\left(\frac{\log IM - \left(\log\delta_o - B\right)/A}{\zeta/A}\right).$$

## Maximum likelihood estimation (ML)

Shinozuka *et al.* (2000)

- Lognormal shape:

$$\widehat{\text{Frag}}(IM;\,\delta_o) = \Phi\left(\frac{\log IM - \log\alpha}{\beta}\right)$$

- Estimation of $\alpha$ and $\beta$ by maximum likelihood for each $\delta_o$:

$$\mathcal{L}\left(\alpha,\,\beta,\,\{IM_i\}_{i=1}^{N}\right) = \prod_{IM_i:\,\Delta_i\geq\delta_o}\left[\widehat{\text{Frag}}(IM_i;\,\delta_o)\right]\prod_{IM_i:\,\Delta_i<\delta_o}\left[1-\widehat{\text{Frag}}(IM_i;\,\delta_o)\right]$$

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# Non parametric methods

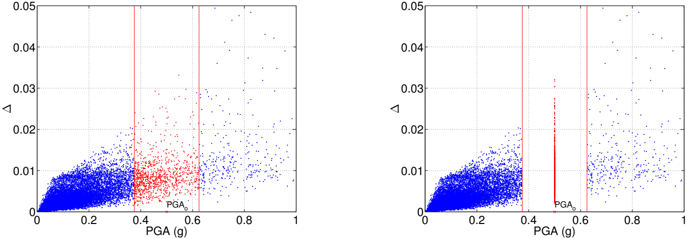
## Binned Monte Carlo estimate

Mai, Konakli & Sudret, Frontiers Struct. Civ. Eng., (2017)

- Suppose $N_s$ analyses are available for $IM = IM_o$, with $N_f$ such that $\Delta \geq \delta_o$. The fragility curve in this point could be estimated by Monte Carlo simulation:

$$\widehat{\text{Frag}}(IM_o) = \frac{N_f\,(IM_o)}{N_s\,(IM_o)}$$

- From the data cloud, a bin centered on $IM_o$ is considered, and points within the beam are "projected" onto the vertical line $IM = IM_o$ by linearization $\widetilde{\Delta_j}(IM_o) = \Delta_j \dfrac{IM_o}{IM_j}$.



# Kernel density estimation

## Fragility curves as a conditional CCDF

Mai *et al.* , Frontiers Struct. Civ. Eng., (2017)

$$\text{Frag}(a;\,\delta_o) = \mathbb{P}\left(\Delta \geq \delta_o | IM = a\right) = \int_{\delta_o}^{+\infty} f_\Delta(\delta | IM = a)\,\mathrm{d}\delta$$

where:

$$f_\Delta(\delta | IM = a) = \frac{f_{\Delta, IM}(\delta, a)}{f_{IM}(a)}$$

## Kernel density estimation

- The joint- and the marginal PDFs are estimated by:

$$\hat{f}_X\left(x\right) = \frac{1}{Nh}\sum_{i=1}^{N} K\left(\frac{x - x_i}{h}\right) \qquad \hat{f}_{\boldsymbol{X}}\left(\boldsymbol{x}\right) = \frac{1}{N\left|\boldsymbol{H}\right|^{1/2}}\sum_{i=1}^{N} K\left(\boldsymbol{H}^{-1/2}(\boldsymbol{x} - \boldsymbol{x}_i)\right)$$

NB: Use of a constant bandwidth in the logarithmic scale

Mai, C., Polynomial chaos expansions for uncertain dynamical systems – Applications in earthquake engineering, PhD Thesis, ETH Zurich, 2016

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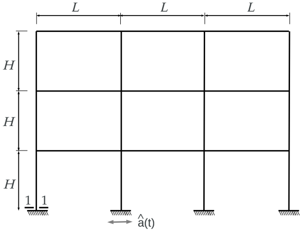
# Outline

# Application : steel frame




- 2D steel frame submitted to synthetic ground motions
- Synthetic earthquakes generated in time domain

| Parameter | Distribution | Mean | Standard deviation | C.o.V |
|---|---|---|---|---|
| $f_y$ (MPa) | Lognormal | $264.2878$ | 18.5 | $0.07$ |
| $E_0$ (MPa) | Lognormal | $210000$ | 630 | $0.03$ |

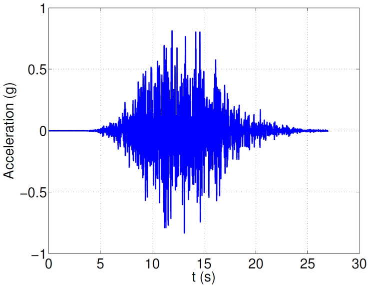
# Stochastic ground motion

## Stochastic excitation

- Obtained by a modulated filtered white noise process — Rezaeian & Der Kiureghian (2010)

$$x(t) = q(t, \boldsymbol{\alpha}) \sum_{i=1}^{n} s_i\left(t, \boldsymbol{\lambda}(t_i)\right) \cdot \xi_i \qquad \xi_i \sim \mathcal{N}(0,1)$$

- Parameters of the filter $\boldsymbol{\lambda} = \left(\omega_{mid}, \omega', \zeta_f\right)^{\mathsf{T}}$ are calibrated on recorded signals
- Global parameters (Arias intensity $I_a$, duration $D_{5-95}$, strong phase peak $t_{mid}$) are transformed into the parameters $\boldsymbol{\alpha}$ of the modulation function $q(t, \boldsymbol{\alpha})$ (e.g. gamma distribution)



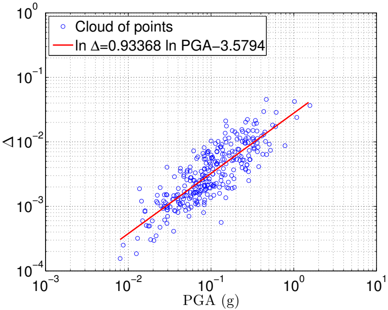
# Stochastic ground motion

## Parameters of the excitation

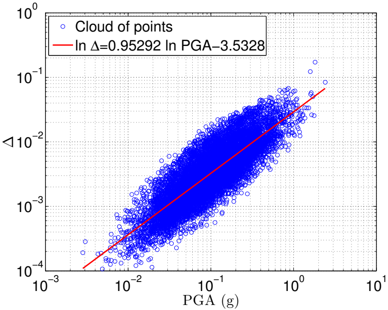
| Parameter | Distribution | Support | Mean | Standard deviation |
|---|---|---|---|---|
| $I_a$ (s.g) | Lognormal | $(0, +\infty)$ | 0.0468 | 0.164 |
| $D_{5-95}$ (s) | Beta | [5, 45] | 17.3 | 9.31 |
| $t_{mid}$ (s) | Beta | [0.5, 40] | 12.4 | 7.44 |
| $\omega_{mid}/2\pi$ (Hz) | Gamma | $(0, +\infty)$ | 5.87 | 3.11 |
| $\omega'/2\pi$ (Hz) | Two-sided exponential | [-2, 0.5] | -0.089 | 0.185 |
| $\zeta_f$ (.) | Beta | [0.02, 1] | 0.213 | 0.143 |

# Training and validation data

- Reference solution: Monte Carlo sampling of 10,000 non linear transient analyses
- PC-NARX: 300 samples



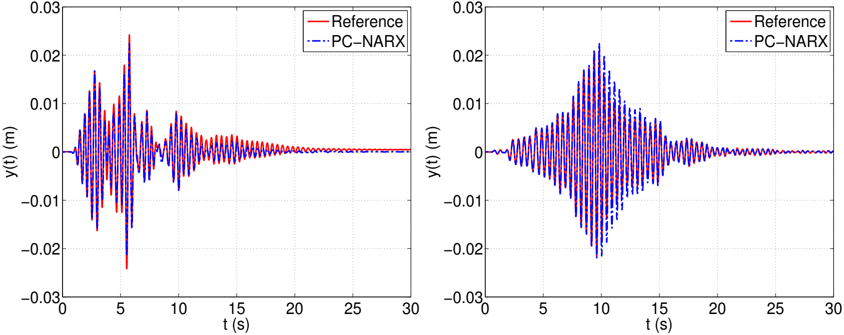

**300 simulations (training PC-NARX)**



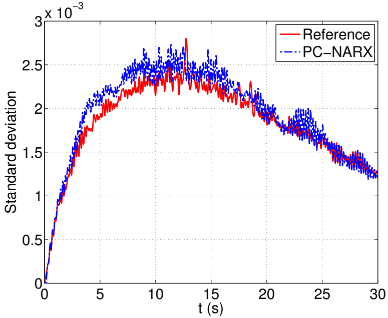

**10,000 simulations**

# Two trajectories (first floor displacement)

- Reference solution: Monte Carlo sampling of 10,000 non linear transient analyses
- PC-NARX: 300 samples



# Statistics of the first floor drift

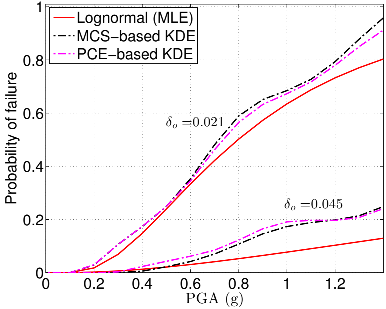



**Standard deviation** **PDF of max. drift**

# Fragility curve – maximal drift

- Reference solution: Monte Carlo sampling of 10,000 non linear transient analyses
- PC-NARX: 300 samples

## Fragility curves




**Kernel density estimation**

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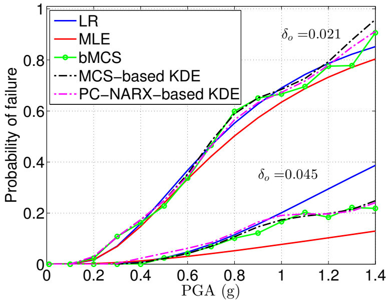
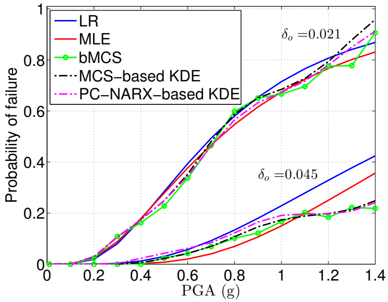



**Using 300 points**




**Using 10,000 points**

# Conclusions

- Polynomial chaos expansions are facing challenging issues when modelling time-dependent systems such as arising in structural dynamics
- A non-intrusive approach based on NARX models (from structural identification) and sparse PCE is proposed
- The accuracy is remarkable on the statistical moments (mean/std. deviation), PDF of the maximum output, but also on particular trajectories
- The method was successfully used for computing fragility curves in earthquake engineering applications

# Questions ?

**Chair of Risk, Safety & Uncertainty Quantification**

www.rsuq.ethz.ch

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www.uqlab.com

Thank you very much for your attention !

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