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Random Steiner systems and bounded degree coboundary expanders of every dimension

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Abstract We introduce a new model of random d-dimensional simplicial complexes, for $d \geq 2$, whose $(d-1)$-cells have bounded degrees. We show that with high probability, complexes sampled according to this model are coboundary expanders. The construction relies on Keevash's recent result on designs (The existence of designs; arXiv:1401.3665, 2014), and the proof of the expansion uses techniques developed by Evra and Kaufman in (Bounded degree cosystolic expanders of every dimension; arXiv:1510.00839, 2015). This gives a full solution to a question raised in Dotterer and Kahle (J Topol Anal 4(4): 499–514, 2012), which was solved in the two-dimensional case by Lubotzky and Meshulam (Adv Math 272: 743–760, 2015).

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1 Introduction

The concept of expansion in graphs has proven to be extremely useful in both theoretical and practical applications. Given $\varepsilon > 0$, a finite graph $G = (V, E)$ is called an ε-expander, if for every set $S \subseteq V$ whose size is at most $|V|/2$ it holds that

$$|\{e \in E : e \cap S = 1\}| \geq \varepsilon |S|. \tag{1.1}$$

For an introduction to this vast topic, see [2,19,27] and the references therein.

A *simplicial complex* is a natural topological and combinatorial generalization of the notion of graphs. The success of expander graphs has prompted researchers to ask: what does it mean for a simplicial complex to be an expander? Several definitions have been proposed and much work has been done on elucidating the relations between these definitions as well as for presenting constructions of high dimensional expanders, cf. [3,5,6,8,9,11,15–17,20,21,24–26,30–39,41–43]. For a survey on some of these works see [28].

This paper focuses on *coboundary expansion*, a concept that came up independently in the work of Linial and Meshulam [26], where the homology groups of random complexes analogous to Erdős–Rényi graphs were studied, and in Gromov's work on topological expansion [16]. Meshulam and Wallach [33] calculated the coboundary expansion of the complete simplicial complex (see also Gromov's work [16]), and found the threshold for the random simplicial complexes defined in [26] to be coboundary expanders. Their work implies the existence of coboundary expanders whose $(d-1)$-cells have degrees logarithmic in the number of vertices. Dotterer and Kahle [5] asked whether there exist coboundary expanders whose $(d-1)$-cells have bounded degrees. Indeed, in the case of graphs, most of the work on expanders has focused on expanders with bounded degrees, which makes this a very natural question.

As a partial answer, Lubotzky and Meshulam [29] presented a model of random 2-dimensional complexes whose 1-cells have bounded degrees and are with high probability coboundary expanders, namely they are expanders with probability tending to one as the number of vertices tends to infinity. Their model made use of random *Latin squares*, which are combinatorial objects closely related to *designs*.

In this paper we present a new model of random d-dimensional simplicial complexes whose $(d-1)$-cells have bounded degrees. Our model is based on *Steiner systems*, which are specific types of *designs*.

Informally, given $k \in \mathbb{N}$, we define X to be the union of k Steiner systems, chosen randomly and independently according to a certain distribution (see Sect. 2.3 for further details).

Our main result is that for every $d \geq 2$, there exists $k_0 = k_0(d) \in \mathbb{N}$ such that for every $k \geq k_0$, the complex X is a coboundary expander with high probability.

Being coboundary expanders, the complexes are also *topological expanders*, i.e., they satisfy Gromov's topological overlapping property (see [6,16]). These are the first known coboundary expanders of dimension $d \geq 3$ (for $d = 2$ see [29]) of bounded upper-degree, i.e., complexes in which the codimension 1 cells have a uniformly bounded degree. It is still an open problem whether there exist complexes which are coboundary expanders in which all the cells are of a uniformly bounded degree. For $d = 2$, it is shown in [21], that such 2-complexes do exist if one accepts an unproven conjecture of Serre on the congruence subgroup problem. Such bounded degree topological expanders do exist, see [21] for $d = 2$ and [10] for general dimension.

2 Results

2.1 Preliminaries

Let X be a finite simplicial complex with vertex set $[n] := \{1, 2, \ldots, n\}$. This means that X is a finite collection of subsets of $[n]$, called cells, which is closed under inclusion, i.e., if $\tau \in X$ and $\sigma \subseteq \tau$, then $\sigma \in X$. The dimension of a cell σ is $|\sigma| - 1$, and X^j denotes the set of j-cells (cells of dimension j) for $j \geq -1$. The dimension of X, which we denote by d, is the maximal dimension of a cell in it. We use the abbreviation d-complex for a simplicial complex of dimension d. Given a d-complex X and $-1 \leq j \leq d$, we define the j-skeleton of X, denoted $X^{(j)}$, to be the set of cells in X of dimension at most j, that is $X^{(j)} := \bigcup_{i=-1}^{j} X^i$. All of the d-complexes considered in this paper will have a complete $(d-1)$-skeleton, by which we mean that they contain all subsets of $[n]$ whose size is at most d. For a $(j+1)$-cell $\tau = \{\tau_0, \ldots, \tau_{j+1}\}$, its boundary $\partial\tau$ is defined to be the set of j-cells $\{\tau \backslash \{\tau_i\}\}_{i=0}^{j+1}$. The degree of a j-cell σ, denoted $\deg(\sigma)$, is defined to be the number of $(j+1)$-cells τ which contain σ in their boundary.

For $j \geq -1$, let $C^j(X; \mathbb{F}_2)$ denote the space of $\mathbb{F}_2$-valued functions on X^j. The elements of C^j are also called cochains. Using the natural bijection between elements of $C^j(X; \mathbb{F}_2)$ and subsets of X^j given by $A \subseteq X^j \leftrightarrow \mathbb{1}_A \in C^j(X; \mathbb{F}_2)$, we will use a slight abuse of notation and write $A \in C^j(X; \mathbb{F}_2)$ for $A \subseteq X^j$.

The jth coboundary map $\delta_j^X : C^j(X; \mathbb{F}_2) \to C^{j+1}(X; \mathbb{F}_2)$ of the d-complex X is given by

$$\delta_j^X A = \bigl\{\tau \in X^{j+1} : |\partial\tau \cap A| \text{ is odd}\bigr\}, \qquad \text{for } A \in C^j(X; \mathbb{F}_2). \tag{2.1}$$

We will usually omit the indexes j and X from the notation when no confusion may occur. In particular, δ means δ_{d-1}^X unless otherwise stated.

For $j \geq 0$, denote by $Z^j(X; \mathbb{F}_2) = \ker(\delta_j)$ and by $B^j(X; \mathbb{F}_2) = \operatorname{im}(\delta_{j-1})$ the spaces of j-dimensional $\mathbb{F}_2$-cocycles and j-dimensional $\mathbb{F}_2$-coboundaries respectively. One can verify that (C^j, δ_j) is a cochain complex, that is $B^j \subseteq Z^j$ for every $j \geq 0$. The jth reduced $\mathbb{F}_2$-cohomology of X is $\widetilde{H}^j(X; \mathbb{F}_2) = Z^j(X; \mathbb{F}_2)/B^j(X; \mathbb{F}_2)$. For a cochain $A \in C^j(X; \mathbb{F}_2)$, let $[A]$ denote the equivalence class of A under the projection from $C^j(X; \mathbb{F}_2)$ to $C^j(X; \mathbb{F}_2)/B^j(X; \mathbb{F}_2)$.

Following [10,14,21], we define the weighted norm $\|\cdot\|^j$ on $C^j(X; \mathbb{F}_2)$ by

$$\|A\|^j := \sum_{\sigma \in A} w(\sigma), \qquad \text{where} \qquad w(\sigma) := \frac{\bigl|\bigl\{\tau \in X^d : \sigma \subseteq \tau\bigr\}\bigr|}{\binom{d+1}{|\sigma|}|X^d|}. \tag{2.2}$$

The norm above is not the one usually defined on $C^j(X;\mathbb{F}_2)$, that is the counting norm $A \mapsto |A|$, but it has several advantages: it is always bounded by 1, it induces a probability measure on X^j, i.e., $\sum_{\sigma\in X^j} w(\sigma) = 1$, it makes it easier to compare the norm of cochains of different dimension and it simplifies the comparison of dense complexes versus sparse complexes. We will usually abbreviate the notation by writing $\|\cdot\|$ instead of $\|\cdot\|^j$. The induced norm on the space of equivalence classes is defined by

$$\|[A]\| = \min\bigl\{\|B\| : [B] = [A]\bigr\}, \quad \text{for } A \in C^j(X;\mathbb{F}_2). \tag{2.3}$$

In particular $\|[A]\| = 0$ if and only if $A \in B^j(X;\mathbb{F}_2)$.

For a cochain $A \in C^j(X;\mathbb{F}_2)/B^j(X;\mathbb{F}_2)$ we define its coboundary expansion by

$$h_j(A) = \frac{\|\delta_j A\|}{\|[A]\|}. \tag{2.4}$$

Note that a cochain's expansion is constant on equivalence classes. The jth coboundary expansion constant of X is defined to be the minimum of the expansion among all cochains in $C^j(X;\mathbb{F}_2)/B^j(X;\mathbb{F}_2)$, i.e.,

$$h_j(X) = \min\bigl\{h_j(A) : A \in C^j(X;\mathbb{F}_2)/B^j(X;\mathbb{F}_2)\bigr\}. \tag{2.5}$$

A d-dimensional complex X is called a (j,ε)-coboundary expander if $h_i(X) \ge \varepsilon$ for all $0 \le i \le j$. Similarly, X is called a (j,k,ε)-coboundary expander if in addition $\max_{\sigma\in X^{j-1}} \deg(\sigma) \le k$.

Remark 2.1 If X is a d-complex such that $1 \le \deg(\sigma) \le k$ for all $\sigma \in X^{d-1}$, then the definition of (d,k,ε)-coboundary expansion is equivalent to

$$|\delta_{d-1}A| \ge \widetilde{\varepsilon}\cdot \min\bigl\{|B| : [B] = [A]\bigr\}, \quad \text{for } A \in C^{d-1}(X;\mathbb{F}_2) \tag{2.6}$$

for some $\varepsilon/(d+1) \le \widetilde{\varepsilon} \le k\varepsilon/(d+1)$. The inequality (2.6) is the original definition of coboundary expansion, see [26].

Given $\rho \in X$, the link of ρ in X is a simplicial complex of dimension $d - |\rho|$ on the vertex set $[n]\setminus\rho$, defined by

$$X_\rho := \bigl\{\sigma \subseteq ([n]\setminus\rho) : \rho\cup\sigma \in X\bigr\}. \tag{2.7}$$

In addition, let $\delta_\rho := \delta^{X_\rho}_{d-|\rho|-1} \colon C^{d-|\rho|-1}(X_\rho;\mathbb{F}_2) \to C^{d-|\rho|}(X_\rho;\mathbb{F}_2)$ be the top coboundary operator on X_ρ. For $-1 \le j \le d-|\rho|$, we will denote by $\|\cdot\|^j_\rho$, and abbreviate $\|\cdot\|_\rho$, the norm defined by (2.2) on the space $C^j(X_\rho;\mathbb{F}_2)$.

Remark 2.2 (Notation) Throughout this paper small Greek letters (except for σ, τ and ρ) as well as the letter c are used to denote positive constants that might depend on certain parameters. The notation $c = c(d,k)$ is used to state that c depends only on d and k. The Greek letters τ, σ and ρ are used to denote cells in a complex.

2.2 A General Strategy for Proving Coboundary Expansion

The goal of this paper is to introduce (for every fixed $d \geq 2$ and sufficiently large but fixed $k = k(d) \in \mathbb{N}$) a model of random d-complexes which are with high probability (d, k, ε)-coboundary expanders, for some positive $\varepsilon > 0$. The general philosophy of the proof follows Lubotzky and Meshulam [29], that is, we consider separately expansion for small cochains, i.e., cochains $A \in C^{d-1}(X; \mathbb{F}_2)$ such that $\|[A]\| \leq c$ for some small fixed constant $c > 0$, and the remaining cochains, which are called large cochains.

In a recent paper [10], Evra and Kaufman gave sufficient conditions for the coboundary expansion of small cochains.

Theorem 2.3 ([10, Thm. 3.2]) *Given $d \geq 2$ and $\beta > 0$, there exist constants $\overline{\gamma} = \overline{\gamma}(d, \beta) > 0$, $c_0 = c_0(d, \beta) > 0$ and $\varepsilon_0 = \varepsilon_0(d, \beta) > 0$ such that the following holds: Let Y be a d-dimensional complex*[1] *satisfying:*

(a) *For every $\emptyset \neq \rho \in Y^{(d-2)}$, the link Y_ρ is a $(d - |\rho|, \beta)$-coboundary expander.*
(b) *For any $\rho \in Y^{(d-2)}$, the link Y_ρ satisfies*

$$\|E_\rho(B, B)\|_\rho \leq 4\big(\|B\|_\rho^2 + \overline{\gamma}\|B\|_\rho\big), \quad \text{for } B \subseteq Y_\rho^0, \tag{2.8}$$

where $E_\rho(B, B) \subseteq Y_\rho^1$ is the set of edges in $Y_\rho^{(1)}$ with both vertices in B.

Then,

$$\|\delta A\| \geq \varepsilon_0 \|[A]\|, \quad \text{for } A \in C^{d-1}(Y; \mathbb{F}_2) \text{ satisfying } \|[A]\| \leq c_0. \tag{2.9}$$

Remark 2.4 Note that in [10], the conclusion of Theorem 3.2 refers to locally minimal cochains (see [21, Defn. 2.4] for the definition). However, by replacing $\|A\|$ by $\|[A]\|$, the conclusion holds for all cochains with sufficiently small norm, as minimal cochains are locally minimal, see [21] for further details.

Theorem 2.3 suggests a strategy for proving coboundary expansion of d-complexes. In order to state it some additional definitions are needed. Given a graph $G = (V, E)$, we denote by $\mathbf{P} = D^{-1}\mathbf{A}$ its normalized adjacency matrix, where D is the diagonal matrix whose entries are the degrees of the vertices and $\mathbf{A}$ is the standard adjacency matrix $\mathbf{A}_{v,w} = \mathbb{1}_{\{v,w\}\in E}$. Note that we implicitly assumed here that each vertex is contained in at least one edge. One can verify that $\mathbf{P}$ is self-adjoint with respect to the inner product $\langle f, g\rangle = \sum_{v\in V} f(v)g(v)\deg(v)$ and hence has real eigenvalues. Furthermore, the eigenvalues of $\mathbf{P}$ lie within the interval $[-1, 1]$ and 1 is always an eigenvalue with eigenfunction $v \mapsto 1$. Denoting by $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{|V|}$ the eigenvalues of $\mathbf{P}$ in decreasing order, let $\lambda(G) := \max\{|\lambda_2|, |\lambda_{|V|}|\}$ be its second largest eigenvalue in absolute value.

The following classical results on expansion in graphs (see [1,2,4,19] for the case of regular graphs) will be of use to us in the proof of Theorem 2.7. For completeness we provide its proof in the Appendix.

[1] Not necessarily with a complete $(d-1)$-skeleton.

Proposition 2.5 (Expander mixing lemma for irregular graphs) *Let $G=(V,E)$ be a finite graph. Then for any two sets $B, C \subseteq V(G)$*

$$\left|\widehat{E}(B,C) - \frac{\left(\sum_{v\in B}\deg(v)\right)\left(\sum_{v\in C}\deg(v)\right)}{2|E(G)|}\right| \le \lambda(G)\sqrt{\sum_{v\in B}\deg(v)}\sqrt{\sum_{v\in C}\deg(v)}\,. \tag{2.10}$$

Here we denote by $\widehat{E}(B,C)$ the number of edges between B and C, where edges with both endpoints in $B\cap C$ are counted twice.

Proposition 2.6 (Discrete Cheeger inequality for irregular graphs) *Let $G=(V,E)$ be a finite graph. Then $h_0(G)\ge 1-\lambda_2$.*

We are now ready to state and prove the general strategy for showing coboundary expansion in d-complexes with a complete skeleton. Informally, by assuming sufficiently high spectral expansion of all links of $(d-2)$-dimensional cells (which are graphs) and using the fact that the $(d-1)$-skeleton is complete, one can guarantee that condition (b) in Theorem 2.3 holds. By assuming in addition coboundary expansion for big chains (with big chosen according to Theorem 2.3) one can prove by induction that all links (including the link of the empty cell) are coboundary expanders.

Theorem 2.7 (General strategy for proving coboundary expansion) *Fix $d\ge 2$ and a function $\varphi\colon (0,1]\to(0,1]$. There exist positive constants $c_{d-3}, c_{d-4},\dots,c_{-1}$, $\overline{\lambda}$ and ε depending only on d and φ such that the following holds. Let X be a d-complex with a complete $(d-1)$-skeleton. Assume further that*

(A) *For any $\rho\in X^{(d-3)}$ the complex X_ρ satisfies*

$$\|\delta_\rho A\|_\rho \ge \varphi(c_j)\|[A]\|_\rho,\quad \text{for } A\in C^{d-|\rho|-1}(X_\rho,\mathbb{F}_2) \text{ such that } \|[A]\|_\rho\ge c_j. \tag{2.11}$$

(B) *For every $\rho\in X^{d-2}$, $\lambda(X_\rho)\le\overline{\lambda}$.*

Then, $h_{d-1}(X)\ge\varepsilon$. In particular, if X also satisfies $\max_{\sigma\in X^{d-1}}\deg(\sigma)\le k$, then X is a (d,k,ε)-coboundary expander.

Proof The proof follows by induction on the following hypothesis:

$$\text{There exists } \varepsilon_j=\varepsilon_j(d,\varphi)>0 \text{ such that for all}\\ \rho\in X^j, \text{ the link } X_\rho \text{ satisfies } h_{d-|\rho|-1}(X_\rho)\ge\varepsilon_j,$$

by letting j run from $d-2$ to -1. Indeed, the case $j=-1$ gives the result with $\varepsilon=\varepsilon_{-1}$.

Starting with the case $j=d-2$, assume $\overline{\lambda}<1/2$, and note that for every $\rho\in X^{d-2}$, the link X_ρ is a graph and is thus equal to its 1-skeleton. Due to assumption (B), it is also a spectral expander relative to $\mathbf{P}$. Consequently, by the Cheeger inequality (see Proposition 2.6) $h_0(X_\rho)\ge 1-\overline{\lambda}>1/2$, so $h_0(X_\rho)\ge\varepsilon_{d-2}$ with $\varepsilon_{d-2}=1/2$.

Assuming the statement holds for $j+1, j+2,\dots,d-2$, we turn to prove it for j. Let $\rho\in X^j$. We will apply Theorem 2.3 to $Y=X_\rho$. Due to the induction hypothesis we know that condition (a) of Theorem 2.3 holds with $\beta_{j+1}=$

$\min\{\varepsilon_{d-2}, \dots, \varepsilon_{j+1}\}$, which only depends on d and φ. Furthermore, we claim that for every $\rho' \in X_\rho^{(\dim X_\rho - 2)} = X^{(d-|\rho|-2)}$, the 1-skeleton of the link $(X_\rho)_{\rho'} = X_{\rho\cup\rho'}$ satisfies condition (b) of Theorem 2.3. Indeed, if $\rho \cup \rho' \in X^{(d-3)}$, then due to the assumption that X has a complete $(d-1)$-skeleton it follows that $X^{(1)}_{\rho\cup\rho'}$ is the complete graph on $n - |\rho\cup\rho'|$ vertices and hence satisfies (2.8) with $\overline{\gamma} = 0$. Conversely, if $\rho\cup\rho' \in X^{d-2}$, then it follows from assumption (b) that $\lambda\big(X^{(1)}_{\rho\cup\rho'}\big) = \lambda(X_{\rho\cup\rho'}) \le \overline{\lambda}$, and therefore, due to the expander mixing lemma (see Proposition 2.5) for every $B, C \subset X^0_{\rho\cup\rho'}$

$$\begin{aligned}\left|\widehat{E}_{\rho\cup\rho'}(B,C) - \frac{\big(\sum_{v\in B}\deg_{\rho\cup\rho'}(v)\big)\big(\sum_{v\in C}\deg_{\rho\cup\rho'}(v)\big)}{2|X^1_{\rho\cup\rho'}|}\right| \\ \le \overline{\lambda}\sqrt{\sum_{v\in B}\deg_{\rho\cup\rho'}(v)}\sqrt{\sum_{v\in C}\deg_{\rho\cup\rho'}(v)},\end{aligned} \tag{2.12}$$

where for $v \in X^0_{\rho\cup\rho'}$, we denote by $\deg_{\rho\cup\rho'}(v)$ the vertex degree in the graph $X^{(1)}_{\rho\cup\rho'}$. Taking $B = C$ and dividing both sides by $2|X^1_{\rho\cup\rho'}|$, this implies

$$\|E(B,B)\|_{\rho\cup\rho'} \le \|B\|^2_{\rho\cup\rho'} + \overline{\lambda}\|B\|_{\rho\cup\rho'}, \quad \text{for } B \subseteq X^0_{\rho\cup\rho'}. \tag{2.13}$$

Thus, if $\overline{\lambda} < 4\overline{\gamma}(d-|\rho|, \beta_{j+1})$, the conditions of Theorem 2.3 hold and one can find $\varepsilon'_j > 0$ and $c_j > 0$ depending only on β_{j+1} (and thus only on d and φ) so that

$$\|\delta_\rho A\|_\rho \ge \varepsilon'_j \|[A]\|_\rho, \quad \text{for } A \in C^{d-|\rho|-1}(X_\rho; \mathbb{F}_2) \text{ satisfying } \|[A]\|_\rho \le c_j. \tag{2.14}$$

Exploiting assumption (A), it follows that

$$\|\delta_\rho A\|_\rho \ge \varphi(c_j)\|[A]\|_\rho, \quad \text{for } A \in C^{d-|\rho|-1}(X_\rho; \mathbb{F}_2) \text{ satisfying } \|[A]\|_\rho \ge c_j. \tag{2.15}$$

Combining (2.14) and (2.15) we conclude that

$$\|\delta_\rho A\|_\rho \ge \varepsilon_j \|[A]\|_\rho, \quad \text{for } A \in C^{d-|\rho|-1}(X_\rho; \mathbb{F}_2), \tag{2.16}$$

where $\varepsilon_j := \min\{\varphi(c_j), \varepsilon'_j\} > 0$. Since ε_j and c_j depend only on d and φ, and in particular are independent of $\rho \in X^j$ the result follows by setting $\overline{\lambda}$ to be the minimum between $1/2$ and

$$\min\left\{2\overline{\gamma}(d-j-1, \beta_{j+1}) : -1 \le j \le d-3\right\} > 0. \qquad \square$$

2.3 The Model and the Main Result

In this subsection we present a new model for random simplicial complexes and show that it satisfies the conditions of Theorem 2.7. Thus, we get d-complexes of arbitrary dimension $d \geq 2$, whose $(d-1)$-cells are of bounded degree, and are coboundary expanders with high probability. The construction is based on the notion of designs which we now recall.

Let $r \leq q \leq n$ be natural numbers and $\lambda \in \mathbb{N}$. An (n, q, r, λ)-*design* is a collection S of q-element subsets of $[n]$ such that each r-element subset of $[n]$ is contained in exactly λ elements of S. For example, an $(n, 2, 1, 6)$-design is a 6-regular graph on n vertices. Given $n, d \in \mathbb{N}$, an (n, d)-Steiner system is an $(n, d+1, d, 1)$-design, namely, a collection of subsets S of size $d+1$ of $[n]$, such that each set of size d is contained in exactly one element of S. Using the terminology from the previous section, an (n, d)-Steiner system is a collection of d-cells such that $\deg(\sigma) = 1$ for every $(d-1)$-cell. For every q, r and λ there are several necessary "divisibility conditions" which must be satisfied by n in order for designs to exist, namely that $\binom{q-i}{r-i}$ divides $\lambda\binom{n-i}{r-i}$ for every $0 \leq i \leq r-1$.

Until recently, the most important question regarding Steiner systems and more generally designs was the existence problem. Namely, for which values of d and n do (n, d)-Steiner systems exist? In a recent groundbreaking paper [22], Keevash solved this problem and gave a randomized construction of designs and in particular of Steiner systems for any fixed d and large enough n satisfying the divisability conditions (which hold for infinitely many $n \in \mathbb{N}$). He was also able to use this construction in a subsequent paper [23] in order to give an asymptotic estimate for the number of such systems. From now on, we will assume that given a fixed $d \in \mathbb{N}$, the value of n satisfies the divisibility condition from Keevash's theorem.

Keevash's construction of Steiner systems is based on a randomized algorithm which has two stages. We will explicitly describe the first stage and use the second stage as a black box.

Given a set of d-cells $A \subseteq \binom{[n]}{d+1}$, we call a d-cell τ *legal with respect to* A if no $(d-1)$-cell in its boundary belongs to the boundary of one of the d-cells in A, namely

$$\partial\tau \cap \partial\tau' = \emptyset, \quad \text{for } \tau' \in A. \tag{2.17}$$

Non-legal cells are also called forbidden cells.

In the first stage of Keevash's construction, also known as the greedy stage, one selects a sequence of d-cells according to the following procedure. In the first step, a d-cell is chosen uniformly at random from $\binom{[n]}{d+1}$. Next, at each step a legal d-cell (with respect to the set of d-cells chosen so far) is chosen uniformly at random and is added to the collection of previously chosen d-cells. If no such d-cell exists the algorithm aborts. The procedure stops when the number of $(d-1)$-cells which do not belong to the boundary of the chosen d-cells is at most $n^{d-\delta_0}$ for some fixed $\delta_0 > 0$ which only depends on d. In particular, if the algorithm does not abort the number of steps is at least $\left(\binom{n}{d} - n^{d-\delta_0}\right)/(d+1) \geq n^d/(2(d+1)!)$.

In the second stage, Keevash gives a randomized algorithm that adds additional d-cells in order to cover the remaining $(d-1)$-cells that do not belong to the boundary of any of the d-cells chosen in the greedy stage. We do not go into the details of this algorithm. The only two facts about the full algorithm we will use are that with high probability it produces an (n, d)-Steiner system and in particular does not abort, and that the distribution of the resulting Steiner system is invariant under permutations on the vertex set[2].

Fix $k \in \mathbb{N}$ and let $S_1, \ldots, S_k$ be k independent copies of (n, d)-Steiner systems chosen according to the above construction. We define

$$X_{n,k}^{(d)} = K_n^{d-1} \cup \bigcup_{i=1}^{k} S_i, \tag{2.18}$$

where K_n^{d-1} is the complete $(d-1)$-complex on the vertex set $[n]$.

We denote the probability measure describing the distribution of $X_{n,k}^{(d)}$ by $P_{n,k}^{(d)} \equiv P_{n,k}$. Note that $K_n^{d-1} \cup S_i$ for every $1 \le i \le k$ is distributed according to $P_{n,1}^{(d)}$.

The following convention is used throughout the rest of the paper. An event $\mathfrak{L}$ is said to happen with high probability if $\lim_{n\to\infty} P_{n,k}(\mathfrak{L}) = 1$ when we restrict the integers n to those satisfying the divisibility conditions.

We start with an estimation on the coboundary expansion of big chains in $X_{n,k}^{(d)}$.

Proposition 2.8 *Fix $d \ge 2$ and $c > 0$ and let $X = X_{n,k}^{(d)}$. There exist $\eta = \eta(d) > 0$ and $k_0 = k_0(c, d)$ such that for every $k \ge k_0$ the following holds with high probability: For every $-1 \le j \le d-3$ and every $\rho \in X^j$*

$$\|\delta_\rho A\|_\rho \ge \eta c \|[A]\|_\rho, \quad \textit{for } A \in C^{d-|\rho|-1}(X_\rho; \mathbb{F}_2) \textit{ satisfying } \|[A]\|_\rho > c. \tag{2.19}$$

We postpone the proof of Proposition 2.8 to the next section and turn to our main theorem.

Theorem 2.9 (The main theorem) *Let $d \ge 2$. There exist $k_0 = k_0(d) \in \mathbb{N}$ and $\eta = \eta(d) > 0$ such that the following holds. Restricting to $n \in \mathbb{N}$ satisfying the divisibility conditions, for every $k \ge k_0$ with high probability $X_{n,k}^{(d)}$ satisfies the conditions of Theorem 2.7 with respect to the function $\varphi(c) = \eta c$. In particular, for every $k \ge k_0$ there exists $\varepsilon_0 = \varepsilon_0(d) > 0$ such that with high probability, $X_{n,k}^{(d)}$ is a (d, k, ε_0)-coboundary expander.*

This is the first construction of coboundary expanders whose $(d-1)$-cells have bounded degrees in dimension $d \ge 3$.

Remark 2.10 It follows from the proof of Theorem 2.9 (see also Remark 2.1) that for every $k \ge k_0$

$$|\delta A| \ge \varepsilon k \cdot \min\big\{|B| : [B] = [A]\big\}, \tag{2.20}$$

with high probability and $\varepsilon_0' = \varepsilon_0'(d) > 0$ as in Theorem 2.9. That is, in the counting norm, the expansion grows linearly with k.

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[2] Although this property is not explicitly stated in [22,23] one can note that the algorithm is invariant under permutations. Indeed in the greedy stage the name of the vertices is not used while in the second stage the names for the vertices are chosen uniformly at random (see Template 2.3).

Proof of Theorem 2.9 Let η be as in Proposition 2.8 and denote by $\varphi_0\colon (0,1]\to(0,1]$ the function $\varphi_0(c)=\eta c$. Also, let $c_{-1},c_0,\ldots,c_{d-3}$ be the constants associated with the function φ_0 as given by Theorem 2.7. By Proposition 2.8 applied for each of the constants $c_{-1},\ldots,c_{d-3}$, it follows by a union bound argument that condition (A) holds for every $k\ge\max\{k_0(c_j,d):-1\le j\le d-3\}$ with high probability.

Due to the definition of the model, for every $\rho\in X^{d-2}$, the one-dimensional link X_ρ is a random graph on $n-|\rho|$ vertices which is the union of k independent perfect matchings chosen uniformly at random. Indeed, since Keevash's algorithm is invariant under permutations and a random permutation of the vertices of a perfect matching yields the uniform distribution on the set of perfect matchings, the one-dimensional links of $K_n^{d-1}\cup S_i$ are uniformly random perfect matchings. It follows from Friedman's result [12,13], see also [40], that with high probability $\max_{\rho\in X^{d-2}}\lambda(X_\rho)=O_d(k^{-1/2})$. Indeed, Friedman's result combined with Markov's inequality states that the probability for the union of k independent uniformly random perfect matchings to have its second largest eigenvalue in absolute value bigger than $Ck^{-1/2}$ decays superpolynomially, provided C is large enough. Using a union bound on all the cells in X^{d-2}, the bound $\max_{\rho\in X^{d-2}}\lambda(X_\rho)=O_d(k^{-1/2})$ follows. Consequently, for large values of k such that $\max_{\rho\in X^{d-2}}\lambda(X_\rho)\le\overline{\lambda}$, with $\overline{\lambda}$ as in Theorem 2.7 condition (B) of Theorem 2.7 holds and the result follows. □

3 Proof of Proposition 2.8

Fix $d\ge 2$, $-1\le j\le d-3$ and $c>0$. Since the norm $\|\cdot\|$ is bounded by 1 the case $c\ge 1$ holds trivially, so assume $0<c<1$. Choose $\rho\in X^j$ and let $A\in C^{d-|\rho|-1}(X_\rho;\mathbb{F}_2)$ be a cochain such that $\|[A]\|_\rho\ge c$.

Denote the complete $(d-|\rho|)$-complex on the vertex set $[n]\setminus\rho$ by K_ρ. In [16,33] the coboundary expansion of the complete complex was calculated. One can verify that their result, when expressed in our norm, yields

$$\left|\delta_\rho^{K_\rho}A\right|\ge\|[A]\|_\rho\binom{n-|\rho|}{d-|\rho|+1}\ge c\binom{n-|\rho|}{d-|\rho|+1}. \tag{3.1}$$

The main idea of the proof is to show that with sufficiently high probability X_ρ has a large intersection with $\delta_\rho^{K_\rho}A$, i.e., $\left|\delta_\rho^{K_\rho}A\cap X_\rho\right|\ge\eta' ck\|[A]\|_\rho n^{d-|\rho|}$ for all sets A satisfying (3.1) and some positive constant $\eta'=\eta'(d)>0$. Noting that the number of $(d-|\rho|)$-cells in X_ρ is at most $\frac{k}{d-|\rho|+1}\binom{n-|\rho|}{d-|\rho|}$, this implies that $\|\delta_\rho A\|_\rho\ge\eta c\|[A]\|_\rho$ for some positive constant $\eta=\eta(d)$ and therefore yields the required coboundary expansion for the chain A in the link X_ρ. Applying a union bound argument on all cells ρ then completes the proof.

To this end, observe that if X_1 and X_2 are two $(d-|\rho|)$-complexes on the vertex set $[n]-\rho$ with a complete $(d-|\rho|-1)$-skeleton and $X_1\subseteq X_2$, then $|\delta^{X_1}A|\le|\delta^{X_2}A|$. Therefore, it is sufficient to prove the result when observing only those d-cells of $X_{n,k}^{(d)}$ that are obtained in the greedy phase of Keevash's construction. In fact, we only use the d-cells which are obtained in the construction of the different Steiner systems in the first

$$T := \left\lfloor \frac{cn^d}{2^{d+6}(d+1)^{2d+4}} \right\rfloor \tag{3.2}$$

steps of the greedy algorithm, because it turns out that a worst case analysis on these d-cells is sufficient for our purposes.

For $1 \le i \le k$ and $1 \le t \le T$, let $Y_i(t) \subseteq S_i$ be the set of d-cells obtained in the first t steps of the greedy algorithm constructing the ith Steiner system S_i, and set $Y_i(0) = \emptyset$. Furthermore, denote

$$Y_i^{\rho}(t) = \left\{\tau \in K_{\rho}^{d-|\rho|} : \tau \cup \rho \in Y_i(t)\right\}, \tag{3.3}$$

that is, the link at ρ induced by $Y_i(t)$, abbreviate

$$F_{\rho,A} := \delta_{\rho}^{K_{\rho}} A\,, \tag{3.4}$$

and for $1 \le i \le k$ define

$$H_i := \left(F_{\rho,A} \setminus \bigcup_{j=1}^{i-1} Y_j^{\rho}(T)\right) \cap Y_i^{\rho}(T) \tag{3.5}$$

to be the set of cells in $F_{\rho,A}$ which appear in $Y_i^{\rho}(T)$ but do not belong to the previously chosen $Y_j(T)$ for $1 \le j \le i-1$.

It follows from their definition that $\bigcup_{i=1}^k H_i = F_{\rho,A} \cap \bigcup_{i=1}^k Y_i^{\rho}(T)$, the sets H_i are disjoint and $\bigcup_{i=1}^k H_i \subseteq \delta_{\rho} A$. Consequently, for every $\widetilde{\eta} > 0$

$$\begin{aligned} P_{n,k}\Big(|\delta_{\rho}A| \le \widetilde{\eta} ck \|[A]\|_{\rho} n^{d-|\rho|}\Big) &\le P_{n,k}\bigg(\sum_{i=1}^k |H_i| \le \widetilde{\eta} ck \|[A]\|_{\rho} n^{d-|\rho|}\bigg) \\ &\le P_{n,k}\bigg(\Big|\Big\{1 \le i \le k : |H_i| \le 2\widetilde{\eta} c \|[A]\|_{\rho} n^{d-|\rho|}\Big\}\Big| \ge \frac{k}{2}\bigg). \end{aligned} \tag{3.6}$$

Denoting $Z_i^{\widetilde{\eta}} = \mathbb{1}_{|H_i| \le 2\widetilde{\eta} c \|[A]\|_{\rho} n^{d-|\rho|}}$ (the indicator function of the event $|H_i| \le 2\widetilde{\eta} c \|[A]\|_{\rho} n^{d-|\rho|}$), the right hand side of (3.6) can be rewritten as

$$\begin{aligned} P_{n,k}\Big(\sum_{i=1}^k Z_i^{\widetilde{\eta}} \ge \frac{k}{2}\Big) &= \sum_{\substack{\Gamma \in \{0,1\}^k \\ |\{i : \Gamma_i = 1\}| \ge k/2}} P_{n,k}\big(Z_i^{\widetilde{\eta}} = \Gamma_i \,:\, \forall 1 \le i \le k\big) \\ &= \sum_{\substack{\Gamma \in \{0,1\}^k \\ |\{i : \Gamma_i = 1\}| \ge k/2}} \prod_{i=1}^k P_{n,k}\Big(Z_i^{\widetilde{\eta}} = \Gamma_i : Z_j^{\widetilde{\eta}} = \Gamma_j,\ \forall 1 \le j \le i-1\Big), \end{aligned} \tag{3.7}$$

where for the second equality we used the formula for conditional probability.

The rest of the proof is based on the following large estimation for the probability of small intersection between $Y_i(T)$ and $F_{\rho,A}$.

Lemma 3.1 (Large deviation estimate for $|H_i|$) *Fix $d \geq 2$ and $0 < c < 1$. There exist $\eta' = \eta'(d) > 0$ and $\widehat{\kappa} = \widehat{\kappa}(d,c) > 0$ such that for every $1 \leq i \leq k$ and every choice $\Gamma \in \{0,1\}^{i-1}$*

$$P_{n,k}\big(Z_i^{\eta'} = 1 : Z_j^{\eta'} = \Gamma_j,\ \forall 1 \leq j \leq i-1\big) \\ = P_{n,k}\big(|H_i| \leq 2\eta' c \|[A]\|_\rho n^{d-|\rho|} : Z_j^{\eta'} = \Gamma_j,\ \forall 1 \leq j \leq i-1\big) \leq 3e^{-\widehat{\kappa} n^{d-|\rho|}}. \tag{3.8}$$

We postpone the proof of the lemma and turn to complete the proof of Proposition 2.8. Taking $\widetilde{\eta} = \eta'$ and noting that in (3.7) the product is on at least $k/2$ terms with $Z_j^{\widetilde{\varepsilon}} = 1$, it follows from Lemma 3.1 that the right hand side of (3.7) is bounded from above by

$$\sum_{\substack{\Gamma \in \{0,1\}^k \\ |\{i : \Gamma_i = 1\}| \geq k/2}} \big(3e^{-\widehat{\kappa} n^{d-|\rho|}}\big)^{k/2} \leq (12)^{k/2} e^{-\widehat{\kappa} k n^{d-|\rho|}/2}. \tag{3.9}$$

Combining (3.6)–(3.9), we obtain for ε' as in Lemma 3.1

$$P_{n,k}\big(|\delta_\rho A| \leq \eta' c k \|[A]\|_\rho n^{d-|\rho|}\big) \leq C e^{-\widehat{\kappa} k n^{d-|\rho|}/2} \tag{3.10}$$

with $C = (12)^{k/2}$.

Applying a union bound argument over all possible $(d-|\rho|-1)$-cochains $A \in C^{d-|\rho|-1}(X_\rho; \mathbb{F}_2)$ in the link X_ρ, we get that

$$P_{n,k}\big(\exists\, A \in C^{d-|\rho|-1}(X_\rho;\mathbb{F}_2) \text{ such that } \|[A]\|_\rho \geq c \text{ and } |\delta_\rho A| \leq \eta' c k \|[A]\|_\rho n^{d-|\rho|}\big) \\ < 2^{\binom{n}{d-|\rho|}} C e^{-\widehat{\kappa} k n^{d-|\rho|}/2} < C e^{(\log(2) - \widehat{\kappa} k/2) n^{d-|\rho|}}. \tag{3.11}$$

Using an additional union bound over all $\rho \in X^j$ we obtain that

$$P_{n,k}\big(\exists\, \rho \in X^j,\ \exists\, A \in C^{d-|\rho|-1}(X_\rho;\mathbb{F}_2) \text{ such that } \|[A]\|_\rho \geq c \text{ and } \\ \|\delta_\rho A| \leq \eta' c k \|[A]\|_\rho n^{d-|\rho|}\big) \\ < \binom{n}{j+1} C e^{(\log(2) - \widehat{\kappa} k/2) n^{d-j-1}} \\ < C \exp\big((\log(2) - \widehat{\kappa} k/2) n^{d-j-1} + (j+1)\log(n)\big). \tag{3.12}$$

Recalling that $j \leq d-3$, by defining $k_0 := \lceil 2\log(2)/\widehat{\eta} \rceil + 1$ the result follows.

Proof of Lemma 3.1 Fix $1 \leq i \leq k$ and $\Gamma \in \{0,1\}^{i-1}$. Since $|F_{\rho,A}| \geq c\binom{n-|\rho|}{d-|\rho|+1}$ by (3.1) and since each $Y_j^\rho(T)$ is part of an $(n-|\rho|, d-|\rho|)$-Steiner system, which implies $|Y_j^\rho(T)| \le \binom{n-|\rho|}{d-|\rho|}$, it follows that $\left|F_{\rho,A} \setminus \bigcup_{j=1}^{i-1} Y_j^\rho(T)\right| \ge |F_{\rho,A}|/2$, provided n is sufficiently large (depending only on d and c). Therefore, it is enough to show that for an appropriate choice of $\eta' = \eta'(d) > 0$, conditioned on the event $F_{\rho,A} \setminus \bigcup_{j=1}^{i-1} Y_j^\rho(T) = B$ for some $B \subseteq F_{\rho,A}$ such that $|B| \ge |F_{\rho,A}|/2$, it holds that

$$P_{n,k}\big(|Y_i^\rho(T) \cap B| \le 2\eta' c \|[A]\|_\rho n^{d-|\rho|}\big) \le 3e^{-\widehat{\kappa} n^{d-|\rho|}}, \tag{3.13}$$

where $\widehat{\kappa} = \widehat{\kappa}(d,c) > 0$. Since Y_i^ρ are i.i.d. it follows that the probability of the last event is the same for every $1 \le i \le k$ and thus we can, without loss of generality, assume that $i = 1$. Abbreviate $Y_i^\rho(t) = Y^\rho(t)$ and $Y_i(t) = Y(t)$. For $1 \le t \le T-1$, define the forbidden set of $(d-|\rho|)$-cells for X_ρ at time t by

$$\mathrm{Forbidden}(t) = \big\{\tau \in K_\rho^{d-|\rho|} : \exists\, \tau' \in Y(t-1) \text{ such that } \partial(\tau \cup \rho) \cap \partial\tau' \ne \emptyset\big\}.$$

Note that the Forbidden cells at time t are exactly those cells in $K_\rho^{d-|\rho|}$ whose union with ρ is not legal to choose from in the greedy algorithm at time t. Also, for $0 \le j \le |\rho|$ and $t \ge 0$, let $N_j(t)$ be the number of d-cells in $Y(t)$ whose intersection with ρ is of size j. □

The proof of Lemma 3.1 is based on the following two claims:

Claim 3.2 *For every $t \ge 1$, we have*

$$|\mathrm{Forbidden}(t)| \le (d+1)nN_{|\rho|}(t-1) + (d+1)N_{|\rho|-1}(t-1). \tag{3.14}$$

Note that $N_{|\rho|}(t)$ are the number of d-cells containing ρ at time t and $N_{|\rho|-1}(t)$ are the d-cells that contain all but one vertex of ρ at time t.

Claim 3.3 *For every $0 < \alpha < 1/(2(d+1)^{d+2})$, there exists $\kappa = \kappa(d,\alpha) > 0$ such that for sufficiently large n*

$$\begin{aligned} P_{n,1}\Bigg(N_{|\rho|}(t) \le \frac{4(d+1)^{d+1}}{n^{|\rho|}}\, t \text{ and } N_{|\rho|-1}(t) \le \frac{4(d+1)^{d+2}}{n^{|\rho|-1}}\, t \text{ for all } \frac{\alpha}{2} n^d \le t \le \alpha n^d\Bigg) \\ > 1 - 2e^{-\kappa n^{d-|\rho|}}. \end{aligned} \tag{3.15}$$

We postpone the proof of both claims and turn to complete the proof of Lemma 3.1. For every $1 \le t \le T$, the probability to choose a d-cell that belongs to the set B in the tth step is at least

$$\begin{aligned} &\frac{|B| - |\mathrm{Forbidden}(t)|}{\binom{n}{d+1}} \\ &\ge \frac{\frac{1}{2}\|[A]\|_\rho \binom{n-|\rho|}{d-|\rho|+1} - (d+1)nN_{|\rho|}(t-1) - (d+1)N_{|\rho|-1}(t-1)}{\binom{n}{d+1}}, \end{aligned} \tag{3.16}$$

where for the inequality we used the lower bound $|B| \geq |F_{\rho,A}|/2 \geq \|[A]\|_\rho \binom{n-|\rho|}{d-|\rho|+1}/2$ (see (3.1) and Claim 3.2). Consequently, by Claim 3.3, for every $\alpha < 1/(2(d+1)^{d+2})$ one can find $\kappa = \kappa(d, \alpha) > 0$ such that with probability at least $1 - 2e^{-\kappa n^{d-|\rho|}}$ for every $\alpha n^d/2 \leq t \leq \alpha n^d$, it holds that

$$(3.16) \geq \binom{n}{d+1}^{-1} n^{d+1-|\rho|}\left(\frac{\|[A]\|_\rho}{2(2(d+1))^{d+1}} - 8(d+1)^{d+3}\alpha\right). \tag{3.17}$$

Taking $\alpha = T/(2n^d)$ we can bound the last term from below by

$$\mathfrak{p} := \frac{\|[A]\|_\rho}{4(2(d+1))^{d+1}}\, n^{-|\rho|}. \tag{3.18}$$

Consequently, under the event in (3.15), the probability to choose an element from B in each of the steps between time $T/4$ and $T/2$ is at least $\mathfrak{p}$.

In particular, with $\{\chi_t\}_{1\leq t\leq T}$ denoting independent random variables distributed under $P_{n,1}$ as Bernoulli($\mathfrak{p}$) and $\mathfrak{B}$ denoting the event in (3.15), it follows from Chernoff's bound that for some $\kappa' = \kappa'(d, c) > 0$

$$\begin{aligned} P_{n,1}\big(|Y^\rho(T) \cap B| < \mathfrak{p}T/8\big) &\leq P_{n,1}\big(|Y^\rho(T) \cap B| < \mathfrak{p}T/8, \mathfrak{B}\big) + P_{n,1}(\mathfrak{B}^c) \\ &\leq P_{n,1}\Bigg(\sum_{t=\lceil T/4\rceil}^{\lfloor T/2\rfloor} \chi_t < \frac{\mathfrak{p}T}{8}\Bigg) + P_{n,1}(\mathfrak{B}^c) \leq e^{-\kappa' n^{d-|\rho|}} + P_{n,1}(\mathfrak{B}^c) \\ &\leq e^{-\kappa' n^{d-|\rho|}} + 2e^{-\kappa n^{d-|\rho|}} \leq 3e^{-\widehat{\kappa} n^{d-|\rho|}}, \end{aligned} \tag{3.19}$$

where $\widehat{\kappa} := \min\{\kappa, \kappa'\}$ and for the one before last inequality we used Claim 3.2.

Noting that $\mathfrak{p}T/4 \geq 2\eta' c\|[A]\|_\rho n^{d-|\rho|}$ for some $\eta' = \eta'(d) > 0$, the result follows. □

Proof of Claim 3.2 Let $\tau' \in Y(t-1)$. If $|\tau' \cap \rho| < |\rho| - 1$, then for every $\sigma \in \partial\tau'$ we have $|\sigma \cap \rho| < |\rho| - 1$. However, for every $\tau \in K_\rho^{d-|\rho|}$ and every $\sigma \in \partial(\tau \cup \rho)$ it holds that $|\sigma \cap \rho| \geq |\rho| - 1$. Thus $\partial\tau' \cap \partial(\tau \cup \rho) = \emptyset$. That is, the only d-cells in $Y(t-1)$ that may add cells to Forbidden(t) are $\tau' \in Y(t-1)$ such that $|\tau' \cap \rho| \in \{|\rho| - 1, |\rho|\}$. Assuming that $\tau' \in Y(t-1)$ satisfies $|\tau' \cap \rho| = |\rho|$, since each of the $(d+1)$ boundary elements in $\partial\tau'$ belongs to no more than n different d-cells, it follows that any such d-cell τ' can add to Forbidden(t) at most $(d+1)n$ elements. Similarly, if $\tau' \in Y(t-1)$ satisfies $|\tau' \cap \rho| = |\rho| - 1$, then each cell $\sigma \in \partial\tau'$ such that $|\sigma \cap \rho| = |\rho| - 1$ can contribute at most one cell to Forbidden(t), that is, the one obtained by adding to σ the missing vertex from ρ. Furthermore each cell $\sigma \in \partial\tau'$ such that $|\sigma \cap \rho| < |\rho| - 1$ does not contribute to Forbidden(t) at all. Because there are no more than $d + 1$ elements in $\partial\tau'$ the result follows. □

Proof of Claim 3.3 Observe that in each step of the process, the choice of a d-cell can make at most $(d+1) \cdot (n-d-1) + 1 \leq (d+1)n$ additional d-cells not legal for the following steps. Consequently, the number of non-legal d-cells at time t is at most

$n(d+1)t$. Thus, the probability to choose a d-cell in the tth step that contains ρ is at most

$$\frac{\binom{n-|\rho|}{d+1-|\rho|}}{\binom{n}{d+1}-n(d+1)t}, \tag{3.20}$$

which for $t \le \alpha n^d < n^d/(2(d+1)^{d+2})$ is at most $2(d+1)^{d+1}n^{-|\rho|}$. Therefore, by a Chernoff bound argument together with a union bound

$$\begin{aligned} &P_{n,1}\left(\exists\, t : \text{such that } \frac{\alpha}{2}n^d \le t \le \alpha n^d \text{ and } N_{|\rho|}(t) > \frac{4(d+1)^{d+1}}{n^{|\rho|}}t\right) \\ &\quad \le \sum_{t=\lfloor \frac{\alpha}{2}n^d \rfloor}^{\lceil \alpha n^d \rceil} P_{n,1}\left(N_{|\rho|}(t) > \frac{4(d+1)^{d+1}}{n^{|\rho|}}t\right) \le \sum_{t=\lfloor \frac{\alpha}{2}n^d \rfloor}^{\lceil \alpha n^d \rceil} e^{-\xi' t/n^{|\rho|}} \le e^{-\xi' n^{d-|\rho|}}, \end{aligned} \tag{3.21}$$

for some ξ' that only depends on α and d, and sufficiently large n.

Similarly, the probability to choose a d-cell in the tth step that contains exactly $|\rho|-1$ of the vertices of ρ is at most

$$\frac{|\rho|\binom{n-|\rho|}{d+2-|\rho|}}{\binom{n}{d+1}-n(d+1)t}, \tag{3.22}$$

which for $t \le \alpha n^d < n^d/(2(d+1)^{d+2})$ is at most $2|\rho|(d+1)^{d+1}n^{1-|\rho|} \le 2(d+1)^{d+2}n^{1-|\rho|}$. Therefore by the Chernoff bound

$$P_{n,1}\left(\exists\, t : \text{such that } \frac{\alpha}{2}n^d \le t \le \alpha n^d \text{ and } N_{|\rho|-1}(t) > \frac{4(d+1)^{d+2}}{n^{|\rho|-1}}t\right) \le e^{-\xi'' n^{d-|\rho|+1}}, \tag{3.23}$$

for some constant ξ'' that depends only on α and d and sufficiently large n.

Combining (3.21) and (3.23) we get the result with $\kappa = \min\{\xi', \xi''\}$. □

4 Concluding Remarks and Open Questions

4.1 Coboundary Expanders without Keevash's Construction

As one can see from the proof of Proposition 2.8, Keevash's algorithm is not really necessary and it is sufficient to consider the d-cells from the greedy stage. We choose to use Steiner systems (and thus Keevash's algorithm) since they induce the union of k independent, uniformly chosen perfect matching on the links of $(d-2)$-cells, and these are good spectral expanders by a well known result. It should be possible to show that with high probability the resulting 1-skeletons obtained by the greedy algorithm (which yields almost perfect matchings) are good spectral expanders as well. If this is indeed the case, then one can apply Theorem 2.7 to show that the union of k

independent families of d-cells obtained by the greedy algorithm are good coboundary expanders as well, without relying at all on Keevash's work.

4.2 Alternative Definitions of High-Dimensional Expansion

As mentioned in the introduction there are several competing definitions for high-dimensional expansion. Without going into details, our model yields expanders with respect to topological expansion (see [6,16]), spectral expansion (cf. [7,14,18,24]) as well as the Cheeger type expansion defined in [37,39].

4.3 Coboundary Expanders Whose Vertices Have Bounded Degree

It is a natural question whether one can construct d-complexes all of whose cells have bounded degrees and which are coboundary expanders. Such complexes would of course not have complete $(d-1)$-skeletons. An interesting open question is to have a random model of d-complexes all of whose cells are of bounded degree which are coboundary expanders, or at least topological expanders. The random model described in [11] gives random d-complexes all of whose cells are of bounded degree which are geometric expanders, but are not topological expanders.

4.4 Minimal Degree for Coboundary Expansion

It would be interesting to obtain estimates on the value of $k_0 = k_0(d)$ for which the theorem holds.

A Expander Mixing Lemma for Irregular Graphs

Proof of Proposition 2.5 Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathbb{R}^V$ given by $\langle f, g\rangle = \sum_{v\in V} f(v)g(v) \cdot \deg(v)$. Recall that with respect to this inner product, the operator $D^{-1}A$ is self-adjoint, and therefore it has an orthonormal basis of eigenvectors $u_1, \dots, u_n$, where $n = |V|$. The proof is obtained by evaluating the expression $\langle D^{-1}A \cdot \mathbb{1}_B, \mathbb{1}_C\rangle$ in two different ways.

On the one hand,

$$\begin{aligned}\langle D^{-1}A \cdot \mathbb{1}_B, \mathbb{1}_C\rangle &= \sum_{v\in V} (D^{-1}A \cdot \mathbb{1}_B)(v) \cdot (\mathbb{1}_C)(v) \cdot \deg(v)\\ &= \sum_{v\in C} \sum_{v'\in B} \mathbb{1}_{v\sim v'} = \widehat{E}(B, C)\,.\end{aligned}$$

On the other hand, using the orthonormal basis of eigenvectors $\mathbb{1}_B = \sum_{i=1}^n \alpha_i u_i$ for scalars $\alpha_i = \langle \mathbb{1}_B, u_i\rangle$, and similarly $\mathbb{1}_C = \sum_{i=1}^n \beta_i u_i$. Therefore,

$$\begin{aligned}\langle D^{-1}A\cdot \mathbb{1}_B, \mathbb{1}_C\rangle &= \sum_{i,j=1}^{n} \alpha_i\beta_j\langle D^{-1}Au_i, u_j\rangle\\ &= \sum_{i=1}^{n}\alpha_i\beta_i\lambda_i = \alpha_1\beta_1 + \sum_{i=2}^{n}\alpha_i\beta_i\lambda_i\,.\end{aligned}$$

Recall that the eigenvector of the eigenvalue 1 is given by $u_1(v) = 1/\sqrt{2|E|}$ for all $v \in V$, and note that $\alpha_1 = \langle \mathbb{1}_B, u_1\rangle = \frac{1}{\sqrt{2|E|}}\cdot\sum_{v\in B}\deg(v)$, and similarly $\beta_1 = \frac{1}{\sqrt{2|E|}}\cdot\sum_{v\in C}\deg(v)$. Combining all of the above we obtain

$$\left|\widehat{E}(B,C) - \frac{\big(\sum_{v\in B}\deg(v)\big)\big(\sum_{v\in C}\deg(v)\big)}{2|E|}\right| \le \left|\sum_{i=2}^{n}\alpha_i\beta_i\lambda_i\right| \le \lambda(G)\sum_{i=2}^{n}|\alpha_i||\beta_i|.$$

By the Cauchy–Schwarz inequality, this is bounded from above by

$$\begin{aligned}\lambda(G)\sqrt{\sum_{i=2}^{n}|\alpha_i|^2}\sqrt{\sum_{i=2}^{n}|\beta_i|^2} &\le \lambda(G)\sqrt{\langle \mathbb{1}_B, \mathbb{1}_B\rangle}\sqrt{\langle \mathbb{1}_C, \mathbb{1}_C\rangle}\\ &= \lambda(G)\sqrt{\sum_{v\in B}\deg(v)}\sqrt{\sum_{v\in C}\deg(v)}\,.\end{aligned}$$

B Cheeger's Inequality for Irregular Graphs

Proof of Proposition 2.6 Recall that the coboundary expansion of a nonempty set $B \subsetneq V(G)$ is

$$h_0(B) = \frac{\|E(B,B^c)\|}{\|[B]\|} = \frac{2|E(B,B^c)|}{\min\big\{\sum_{v\in B}\deg(v), \sum_{v\in B^c}\deg(v)\big\}}\,.$$

As in the proof of the expander mixing lemma for irregular graphs, we will use the fact that the operator $D^{-1}A$ is self-adjoint with respect to the inner product $\langle f, g\rangle = \sum_{v\in V} f(v)g(v)\deg(v)$, and therefore it has a basis of orthonormal eigenvectors. Recall that the eigenvector corresponding to the maximal eigenvalue 1 is the constant vector.

We will bound the coboundary expansion of an arbitrary set $B \subset V(G)$ in terms of λ_2, the second largest eigenvalue of $D^{-1}A$. Let $f = \alpha\mathbb{1}_B + \beta\mathbb{1}_{B^c}$, where $\alpha = -\sum_{v\in B^c}\deg(v)$ and $\beta = \sum_{v\in B}\deg(v)$. Note that $\langle f, \mathbb{1}\rangle = 0$, and therefore the Rayleigh quotient of it satisfies

$$\frac{\langle I - D^{-1}Af, f\rangle}{\langle f, f\rangle} \ge 1 - \lambda_2.$$

On the other hand, from the definition of f

$$\langle f, f\rangle = \alpha^2 \sum_{v\in B} \deg(v) + \beta^2 \sum_{v\in B^c} \deg(v)$$

and

$$\langle D^{-1}Af, f\rangle = \alpha^2 \widehat{E}(B, B) + 2\alpha\beta \widehat{E}(B, B^c) + \beta^2 \widehat{E}(B^c, B^c)\,,$$

and therefore

$$\frac{\langle I - D^{-1}Af, f\rangle}{\langle f, f\rangle} = \frac{(\alpha-\beta)^2 E(B, B^c)}{\alpha^2 \sum_{v\in B} \deg(v) + \beta^2 \sum_{v\in B^c} \deg(v)}\,.$$

Plugging in the values for α and β gives

$$\frac{4|E|^2 \cdot E(B, B^c)}{2|E|\big(\sum_{v\in B} \deg(v)\big)\cdot\big(\sum_{v\in B^c} \deg(v)\big)} \le \frac{2\cdot E(B, B^c)}{\min\big\{\sum_{v\in B} \deg(v), \sum_{v\in B^c} \deg(v)\big\}} = h_0(B).$$

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