The Nobel Prize in Physics 2016

The Nobel Prize in Physics 2016

Gian Michele Graf, Theoretische Physik, ETH Zürich

The Physics Nobel Prize 2016 was bestowed upon three theoretical physicists, David J. Thouless, F. Duncan M. Haldane, and J. Michael Kosterlitz. The citation read "for theoretical discoveries of topological phase transitions and topological phases of matter".

Several accounts of their results have been written, among them of course the document [1] released by the Royal Swedish Academy on the occasion of the Prize, but also the article [21] in the magazine of the EPS, which is delivered to all members of the SPS.

The present scope is different and perhaps more limited, in that we focus on a presentation of the results which warranted the Prize, while going at quite some length in explaining them in a way that is hopefully understandable to physicists of all kinds. On the other hand we shall dwell only selectively on the impact those works had and still have. More about that is found in the references just mentioned, as well as in the review [11] and in recent articles [14, 12, 22, 24] in the Mitteilungen/Communications of the SPS.

The XY-model and the Kosterlitz-Thouless phase transition

The achievement of Kosterlitz and Thouless applies to a number of models in statistical mechanics in dimension 2, through which it became clear that superfluidity and superconductivity are possible in that dimension, though not being associated with a spontaneously broken symmetry, as they are in higher dimensions. (Though we do not enter on the relation with experiments, we cannot refrain from mentioning the work of Martinoli [18].)

Here we will focus on just one such model, which is simpler than others. The XY-model is a classical ferromagnetic spin model. The system consists of spins, one at each point of a lattice. A single spin is planar, like a compass needle, and the interaction is invariant under a common rotation of all of them. Each spin is actually given by a unit vector $\vec{s}$ which points in any direction of the plane, or equivalently by the corresponding angle θ (see Fig. 1, left).

The Hamiltonian, i.e. the total energy of the system, is

$$H = -J \sum_{\langle x,y \rangle} \vec{s}_x \cdot \vec{s}_y = -J \sum_{\langle x,y \rangle} \cos(\theta_x - \theta_y)\,, \tag{1}$$

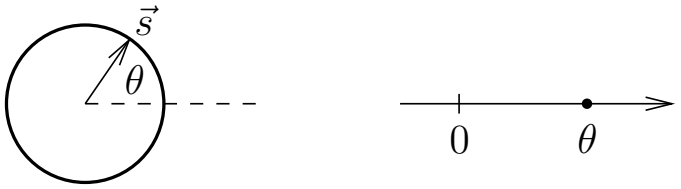

Figure 1: Left: A single spin of the XY-model. It is important that the angles θ and $\theta + 2\pi$ represent the same configuration $\vec{s}$. Right: The unwound spin becomes an unbounded real variable. Note that the real axis is simply connected, unlike the circle.

where $\vec{s}_x$ is the spin at the lattice site x and the sum ranges over pairs $\langle x, y\rangle$ of nearest neighbors. The coupling $J > 0$ favors alignment between spins. The lattice should be finite (say, of sides of length $L \gg a$ with a the lattice spacing) in order for the energy H to be so too, though it is ultimately the infinite lattice we have in mind. The model is defined for arbitrary dimension d of the lattice, which is *a priori* unrelated to the dimension 2 of the plane hosting the spin, but the result honored with the Prize is specific to $d = 2$.

Some digression is in order before that result, let alone the reason thereof, can be stated. Because of the rotational symmetry the system has many groundstates, each of them consisting of all spins pointing in one common but arbitrary direction, $\theta_x = \theta$. In any of those states, the direction of the spin at any site can be inferred from that at any other, arbitrarily distant site (long-range order), the two being in fact equal. Moreover, any groundstate breaks the symmetry since it singles out a particular direction, θ, which may serve as a label distinguishing between them (spontaneous symmetry breaking). But not so at positive temperature, no matter how small! Let us spell this out (Mermin-Wagner Theorem [20]): Even if the spins θ_x at boundary sites x were prescribed to point in one same direction θ, a spin at a site well inside the lattice would still point in all directions with almost equal probabilities.

Let us go as far as to give a heuristic rationale of this background result too, since it will prove useful later. (It will by the way also rightly suggest that in $d = 3$ spontaneous symmetry breaking *does* persist at moderate positive temperatures.) The energy of small fluctuations about one of the above groundstates is well approximated by

$$H = \frac{J}{2} \sum_{\langle x,y \rangle} (\theta_x - \theta_y)^2 \,, \tag{2}$$

as seen by expanding the cosine and then dropping an additive constant, which is independent of the field configuration and hence irrelevant. Unlike (1), the expression is no longer invariant under a 2π-shift of the angle at one site. The degrees of freedom θ_x have thus factually been replaced by those seen on the r.h.s. of Fig. 1. The model admits an electrostatic interpretation which is best seen in the continuum (or long-wavelength) approximation. There the Hamiltonian becomes

$$H = \frac{J'}{2} \int (\nabla\theta)^2 d^d x \tag{3}$$

with $J' = Ja^{2-d}$; so, if $\theta(x)$ is viewed as the electric potential, whence $-\nabla\theta$ as the electric field, then H is the electrostatic energy.

In $d = 3$ ($d = 2$) the potential outside of a ball (resp. disk) of charge Q and radius R is

$$\theta(x) = \frac{Q}{4\pi r}\,, \quad (d=3)\,; \qquad \theta(x) = -\frac{Q}{2\pi}\log\frac{r}{R}\,, \quad (d=2) \tag{4}$$

with $r = |x|$. (The relevance of this field configuration for the lattice model (2) is limited to $R \gtrsim a$, with the lattice spacing a serving as a lower cutoff for distances.) Let us compare this with the elementary relation $Q = CV$, where C is the capacitance of the ball, or disk, and V is the potential difference to spatial infinity, i.e. $V = Q/4\pi R$ and $V = \infty$ in the two cases at hand. We so see that C is finite in $d = 3$ (in fact, $C = 4\pi R$), but vanishes in $d = 2$, and we conclude that, in the first case, it takes a strictly positive minimal energy $E = CV^2/2$ to create a finite field $\theta(x) = V$ in a finite region of space (in fact, growing with its extension R), but also that the cost of the same fluctuation is arbitrarily small in the second case, no matter how extended it is. We will stick to the case $d = 2$.

What matters more than the energy cost is, at positive temperature $T > 0$, the cost in *free energy*

$$F = E - TS\,,$$

where $S = k \log N$ is the entropy associated to the number N of fluctuations (of energy E) affecting a given site well inside the lattice. Since $S > 0$ is growing with the extension R of the fluctuation, whereas $E = 0$, we get $F \ll -kT$ and it becomes clear that the system favors extended random fluctuations, thus obliterating at that given site any influence of the boundary value of the field. In the case of the XY-model that restores the rotational symmetry in the thermal average.

Though there is no long-range order, as just seen, there is quasi-long-range order. In the case of the unwound models (2, 3) that feature is expressed by the correlations

$$\langle \exp \mathrm{i}(\theta_x - \theta_y)\rangle = \langle \cos(\theta_x - \theta_y)\rangle \propto |x-y|^{-kT/2\pi J} \tag{5}$$

at large separations $x - y$ between points. Here $\langle \cdot \rangle$ denotes the thermal average, wherein field configurations θ are weighted by the usual Boltzmann factor $\exp(-H[\theta])$. (Equation (5) is derived quite easily, since by (2) or (3) the average results in a Gaussian integral.) The point to be noticed is that the correlations decay by a power law, and thus fairly slowly, and they do so throughout the phase, meaning for a whole range of temperatures (here $T > 0$). By contrast, most often correlations decay exponentially in absence of long-range order, whereas power-law decay is limited to critical temperatures corresponding to phase transitions.

Let us finally return to the XY-model, where the correlations are given as $\langle \vec{s}_x \cdot \vec{s}_y \rangle = \langle \cos(\theta_x - \theta_y) \rangle$. Based on the analogy with the models (2, 3), one would guess the same behavior as in Eq. (5). But not quite so! The truth, compellingly established by Berezinskii [5], Kosterlitz and Thouless [16], entails a phase transition: There is a critical temperature $T_c > 0$ such that

$$\langle \vec{s}_x \cdot \vec{s}_y \rangle \propto \begin{cases} |x-y|^{-kT'/2\pi J}, & (T < T_c), \\ \mathrm{e}^{-|x-y|/\xi}, & (T > T_c), \end{cases}$$

where $T' = T'(T)$ is a renormalized temperature with $T'/T \to 1$ as $T \to 0$, and $T'/T \to \infty$ as $T \uparrow T_c$; moreover $\xi = \xi(T/J)$ defines the correlation length with $\xi \to \infty$ as $T \downarrow T_c$, and $\xi \to 0$ as $T \to \infty$.

The result says that, in comparison with the models with unwound spin, the quasi-long-range order survives only at low enough temperatures. The reason of the discrepancy is that those models sweep it under the rug that the spin is actually a periodic variable (see Fig. 1, left).

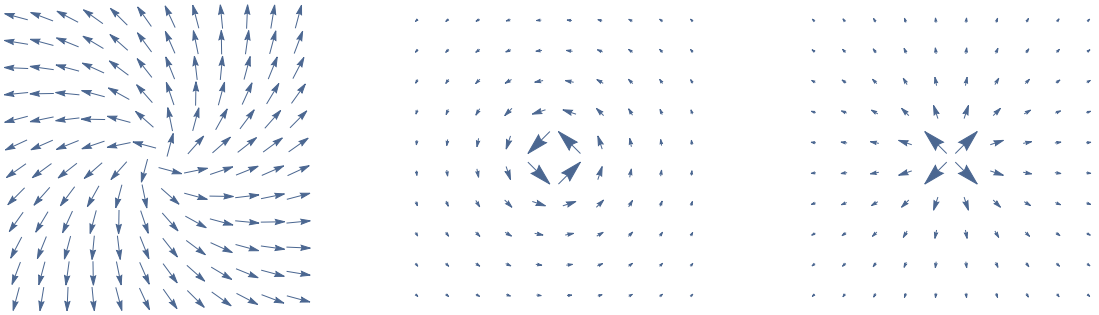

Figure 2: Left: A vortex θ_x with $\alpha = \pi/6$. Center: The field $\nabla\theta$, which has circulation 2π and is not a gradient. Right: The field $(\nabla\theta)_\perp$ with vectors rotated by $\pi/2$.

In more detail it has to do with vortex configurations (see Fig. 2, left) and given by $\theta_x = \alpha + \arg x$, which is the direction of the site x in polar coordinates, rotated by some angle α. That configuration is slowly varying with x away from the origin if viewed as a configuration of the XY-model, where θ_x is understood up to multiples of 2π. As a result, the energy densities $(\theta_x - \theta_y)^2$ seen in (2), and likewise in (3), remain accurate for large $r = |x|$, those being local expressions in x. (This is true, although the same configuration θ_x is not smooth in the sense of the unwound model (2), since a unique assignment of $\arg x$ requires introducing a discontinuity cut.)

The gradient $\nabla\theta_x$ which, as noticed, isn't one globally, equals $1/r$ in magnitude and points in azimuthal direction (see Fig. 2, center). Let us turn the vectors pointwise and clockwise by 90° and then denote them by $(\nabla\theta_x)_\perp$ (see Fig. 2, right). The move does not affect the energy density, $(\nabla\theta_x)^2 = (\nabla\theta_x)_\perp^2$, yet results in an honest gradient field, $(\nabla\theta)_\perp = -\nabla(\theta_\perp)$, (as any radial, rotationally symmetric field is) where $\theta_\perp = -\log(r/a)$ is in fact the electric potential of a quantized charge $Q = 2\pi$, cf. (4). Based on the electrostatic analogy we conclude that the energy of two vortices (of core size a) of opposite circulation is

$$E = 2\pi J \log \frac{l}{a}\,,$$

when they are a distance l apart. Incidentally, the divergence for $l \to \infty$ implies that (twice) the energy of a lone vortex is infinite.

Let us consider the low-temperature phase and ask: How large does a system (or the size L of a subsystem) have to be, so that it likely contains a vortex pair of given separation l? This will happen roughly as soon as the energy cost and the entropy gain break even, $F = 0$. We qualitatively have

$$F = E - TS \approx 2\pi J \log \frac{l}{a} - kT \log \frac{L^2 l}{a^3}$$

because the number of ways $N = \exp(S/k)$ of placing the pair results from that of picking its midpoint ($\sim (L/a)^2$) and its orientation ($\sim l/a$). The condition yields

$$L/l = (l/a)^\alpha\,, \quad (\alpha = \pi J/kT - 3/2)\,.$$

When T is small ($\alpha \gg 1$), L/l grows fast with l/a, meaning that vortex pairs are all the rarer the larger their separation is. As T grows and α decreases, the suppression of large pairs weakens. Finally, when $\alpha \to 0$ we have $L/l \approx 1$, which means that vortex pairs of all sizes are now abundant, heralding the onset of a new, high-temperature phase. In summary: For $T < T_c$ the system is populated by vortex

pairs, which grow in density and separation with T. Like dipoles in a medium they affect (or renormalize) its dielectric constant. As T approaches T_c the vortices, or the charges of the dipoles, break loose; for $T > T_c$ they screen each other (Debye screening) with some screening length ξ, decreasing in T.

It must though be said that the above is in essence a very compelling scenario, but not a proof. It is thus comforting to know that McBryan and Spencer [19] and Fröhlich and Spencer [6] turned much of the above story into theorems. It is, perhaps, a little surprising that this is not mentioned in the document [1] of the Royal Swedish Academy.

Quantum antiferromagnets and gapped spin phases

The prime example of a quantum antiferromagnet is the Heisenberg spin chain. Each spin has quantum number $s = 1/2, 1, 3/2, \ldots$, meaning that the spin vector $\vec{S}$ satisfies $\vec{S}^{\,2} = s(s+1)$. The Hamiltonian is

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} \tag{6}$$

and formally resembles (1) except for the antiferromagnetic coupling ($J = -1$). A basic question about the model is as to whether there is a strictly positive minimal energy to pay for exciting the system above its groundstate (a *gap* for short). In the classical case, which corresponds to $s = \infty$, there is no gap, as can be seen by an expansion in small fluctuations similar to (3). True, that expression was derived for a ferromagnet, but that is of no importance because the two situations are connected by the transformation $\vec{S}_i \mapsto -\vec{S}_i$ applied at every other site (staggering). Quantum-mechanically however that sign flip is not allowed, because it conflicts with the commutation relation $[S_x, S_y] = \mathrm{i}\hbar S_z$. In the quantum realm things might thus be different, but need not. Actually in the extreme quantum case, $s = 1/2$, the model is also gapless, as known from the Bethe ansatz. Common belief was that the same would hold true at all intermediate values of s.

Haldane formulated [9, 8] a conjecture, now named after him, that in light of the above is quite surprising: Chains with half-integer spin are gapless, but those with integer spin are gapped.

Evidence of the sort Haldane gave can be presented here only in very sketchy terms: It rests on a path-integral formulation, which is a representation in terms of classical paths, thus allowing for a comparison between the quantum groundstate and the classical one, which is staggered. This manifests itself in two ways. First, the

(classical) paths contribute different (quantum) phases depending on spin, somehow in the same way that turning a single spin by 2π contributes a sign only for half-integer spin. Second, staggering remains allowed, though with the effect that phase differences (not sums) between neighboring spins matter. They give rise to a so-called topological term in the action, on top of its classical expression.

Note that ultimately it is the spin chain of *integer* spin, i.e. the case where the above (quantum) sign effect is *absent*, which is apparently at variance with classical behavior. This may seem puzzling. The better viewpoint however is that for antiferromagnets the comparison should be done with a classical spin chain at *positive* temperature (which has a gap), because the quantum system has fluctuations even in its groundstate in view of the non-commuting spin components.

Evidence of a completely different kind was provided for $s = 1$ by Affleck, Kennedy, Lieb, and Tasaki [2], who considered Hamiltonians depending on a parameter,

$$H = \sum_i \big(\vec{S}_i \cdot \vec{S}_{i+1} + \alpha(\vec{S}_i \cdot \vec{S}_{i+1})^2\big)\,, \tag{7}$$

that generalize (6) but are supposed to behave the same way for moderate values of the parameter α. Remarkably, for $\alpha = 1/3$ the model is solvable, as we momentarily explain. Let us recall that two spins $s = 1$ add up to a spin $\vec{S}_1 + \vec{S}_2$ with quantum number among $S = 0, 1, 2$. Let $P^{(S)}$ denote the projection onto the corresponding subspace of their joint Hilbert space. Then

$$H = \sum_i \big(\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{2}{3}\big) = 2\sum_i P^{(2)}_{i,i+1}\,. \tag{8}$$

This simply follows from $(\vec{S}_1 + \vec{S}_2)^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$ by using $\vec{S}_i^2 = 2$ and the values $S(S+1)$ of the l.h.s. associated with the different projections $P^{(S)}$; the coefficients on the l.h.s. of (8) are chosen in such a way that only $S = 2$ survives.

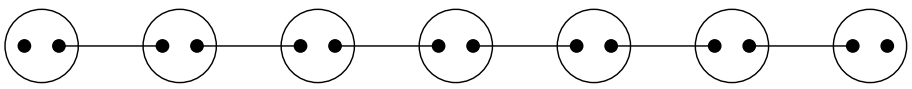

Figure 3: Each dot, line, and circle represents a spin 1/2, a singlet pair, and the projection constraining two spin 1/2's into a spin 1 (after [2]).

The model is antiferromagnetic in that it penalizes maximal spin alignment ($S = 2$). Moreover it has an explicit eigenstate of zero energy, which must be a groundstate because the Hamiltonian is a sum of (positive) projections. The construction of that

state goes as follows: Each spin $\vec{S}_i$ can be thought of as the sum of a pair of spins 1/2, subject to the constraint that they add up to $S = 1$, not $S = 0$ (see Fig. 3). We can however postpone that constraint, since the corresponding projection commutes with the Hamiltonian. Two neighboring spin 1's now involve four spin 1/2's. The middle two are put into a singlet state ($S = 0$), whence all four spins have $S = 1$ at most. In particular the state is annihilated by the projections in (8). Finally, the constraint projection is applied to the state, which remains an eigenstate with eigenvalue 0.

In a semi-infinite chain the unpaired spin 1/2 at the one remaining end of the chain provides an example of fractionalization, since all the fundamental degrees of freedom are spin 1's. It also accounts for a 2-fold degeneracy of the groundstate. In the (two-sided) infinite chain the groundstate is unique and, more importantly, gapped as proven in [2]. The main reason is that the Hamiltonian is free of frustration, meaning that the groundstate minimizes all terms on the r.h.s. of (8) one by one.

The integer quantum Hall effect and Chern numbers

The fundamental discovery of the integer quantum Hall effect by von Klitzing has been recounted many times since 1980. We shall thus be brief (for more see e.g. [3]). In a slab, subjected to a (strong) out-of-plane magnetic field and traversed by a (weak) in-plane current, a voltage drop in the direction transverse to both is observed (Hall effect, 1879).

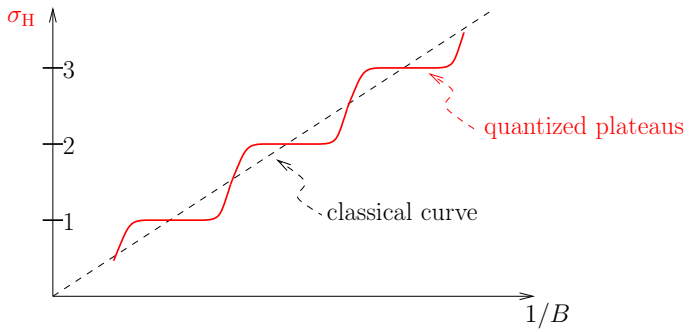

Figure 4: The Hall conductance in natural units $e^2/h = 1$ as a function of the (inverse) magnetic field (qualitative behavior).

The remarkable fact seen in two-dimensional electron gases at temperatures below 2K is that the value of the Hall conductance deviates from the classical behavior:

It is *quantized*, meaning that

$$\sigma_{\mathrm{H}} = n\frac{e^2}{2\pi\hbar}\,, \tag{9}$$

where n is an *integer*, and moreover constant within a part in 10^9 throughout some sizable range of values of the magnetic field (plateau), as shown in Fig. 4; at the same time the longitudinal voltage drop vanishes. The effect is seen in very clean, yet not perfect, samples, in which case the width of the plateaus would actually vanish.

Quite a few arguments have been put forward in order to explain this phenomenon, as well as the even more challenging fractional quantum Hall effect, where n is replaced by a rational number. (We won't say anything about the latter, except mentioning in passing the work of Fröhlich, see e.g. [7]). Common to early discussions of the integer case is the single-particle picture (but not much more than that), whence the many-body groundstate is obtained by filling states with electrons up to the Fermi level. Some of those arguments may at first sight even look unrelated to one another, whereas others recognizably lie on a logical path, along which the understanding is freed step by step from details that are peculiar to specific models and at the same time tied to more fundamental and general mathematical concepts. This has certainly played a role in the later formulation of the Haldane model [10], because it was by then clear that the breaking of time-reversal invariance was essential, more than a positive magnetic field.

But let's proceed by order: Laughlin's argument [17] is based on a (general) gauge argument and on Landau levels, which constitute the peculiar energy spectrum of an electron that is free except for being subjected to a magnetic field. In one of the works [25] for which Thouless was awarded the Prize, he and his coauthors computed σ_{H} by applying linear response theory, aka the Kubo formula (another general principle), to electrons exposed to a periodic potential (on top of the magnetic field) describing the crystalline solid. This is far from just being a feature included for "added realism"; it rather led them to place the quantization of σ_{H} in the general frame of Bloch band theory. The formula they derived for the integer is

$$n = \sum_m \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{T}} \big(\langle\partial_{k_1}\psi_{mk}|\partial_{k_2}\psi_{mk}\rangle - \langle\partial_{k_2}\psi_{mk}|\partial_{k_1}\psi_{mk}\rangle\big) d^2k\,, \tag{10}$$

where $k = (k_1, k_2)$ is the quasi-momentum ranging over the Brillouin zone $\mathbb{T}$ (a torus), Bloch bands are labeled by m, and Bloch states denoted by $\psi_{mk}(x)$ and normalized as functions of x in the unit cell; moreover ∂_{k_i} is short for $\partial/\partial k_i$ and the sum ranges over filled bands only. In particular, the Fermi level is supposed to lie in a band gap (band insulator).

It is worth at this point to warn from a pitfall: The integrand is the curl of the vector field $A(k) = \langle \psi_{mk} | \nabla_k \psi_{mk} \rangle$ (or, more polishedly, the curvature of the Berry connection), whence upon applying Stokes' theorem one is tempted to conclude that n vanishes, because the torus has no boundary at all. What saves the day is this: The state $|\psi_{mk}\rangle$ is unique only up to a phase, as it is commonplace in quantum mechanics. Changing that phase, even depending on k, does not change the integrand, nor hence n. It remains however to be seen whether there is a smooth choice $|\psi_{mk}\rangle$ for k ranging over *all* of the torus, as opposed to just patches sufficing the purpose of Eq. (10). If not, $A(k)$ isn't globally defined in k and the above argument is luckily flawed. As a matter of fact, in absence of magnetic field, such an overall smooth choice is possible (and somehow constructed in any introductory textbook on solid state physics) and in line with $n = 0$ in (9). In its presence however, n measures the obstruction to such a choice.

Not surprisingly, that integer was abstractly known before to mathematicians as a homotopy invariant (Chern number) of *vector bundles*, a fact that was quickly noticed [4, 15]. It pays to use that concept further, since it is visually appealing.

Intuitively a vector bundle looks like a comb with teeth densely arranged along the shaft. Slightly more mathematically, the shaft is replaced by a manifold of arbitrary dimension d (think of a curve or a surface), called the base space, and the teeth are real or complex vector spaces E_k of common dimension r, called fibers, which depend continuously on the point k of the base space. The collection of all fibers makes the vector bundle E. For instance the (complex) vector bundle underlying (10) has the torus $\mathbb{T}$ as base space ($d = 2$) and the linear span of the Bloch states $|\psi_{mk}\rangle$, ($m = 1, \dots r$) as fiber over k, where r is the number of filled bands.

Let us investigate vector bundles beginning with a simple example: The Möbius strip (see Fig. 5, left), viewed as consisting of (real) lines ($r = 1$) arranged along the circle ($d = 1$). It is intuitively non-trivial because of the twist. We however need a precise definition, which does not rely on the strip being embedded in the ambient space and which can be generalized later. It goes as follows: Start from a point on the circle, pick a vector in its fiber, and extend that choice continuously all the way around the circle (see Fig. 5, right), the only condition being that the vectors shall not vanish anywhere, not even at the endpoints of the interval, which are in a sense awaiting to be glued back to a circle. Since the vectors there, say v^- and v^+, belong to the same 1-dimensional fiber, we have $v^+ = tv^-$ with some factor $t \neq 0$. Had we done the exercise with a trivial (untwisted) strip, we would now have $t > 0$ and we could modify the vectors along the interval so as to end up with $t = 1$ and therefore with a *globally* continuous choice on the circle. But not so for the Möbius

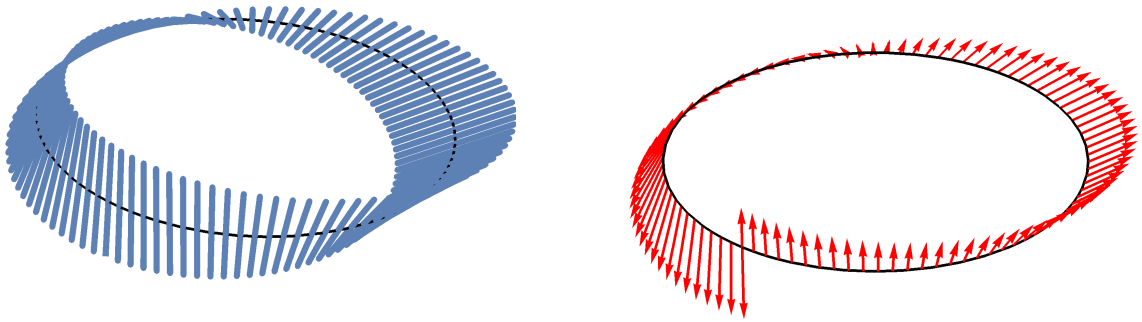

Figure 5: Left: The Möbius strip as a vector bundle with the circle as base space. Right: A nowhere vanishing vector interpolating between v^- and v^+ (see text).

strip, where we visibly get $t < 0$ and the condition $t \neq 0$ prevents any deformation from reaching $t = 1$.

A closely related example is obtained by replacing the lines with *complex* vector spaces, but still of dimension $r = 1$. In this case t is a complex number and the stated condition no longer prevents the deformation to $t = 1$, exhibiting the bundle as trivial. The point is that the removal of the origin from the complex plane does not disconnect it, unlike the real line.

A further modification is by increasing the dimension r of the complex vector space. The appropriate investigative tool is no longer a nowhere vanishing vector, but a frame $(v_1, \ldots, v_r)$ of vectors, i.e. a basis of the fiber, which continuously depends on the base point (for $r = 1$ this is the same thing as before). When joining endpoints we get $v_i^+ = \sum_{j=1}^r t_{ji} v_j^-$ with a complex $r \times r$-matrix $T = (t_{ij})$ relating the two bases of the same fiber (transition matrix). As such, $\det T \neq 0$. Any such matrix can be deformed to the unit matrix, $T = \mathbb{1}$, whence the bundle remains trivial. In the next move, let us change the dimension of the base space by fattening the circle to a cylinder (see Fig. 6, left and center). Any vector bundle above it remains trivial, since we only made continuous changes.

At last we come back to the torus, which is obtained from the cylinder by gluing the two circles at its ends (see Fig. 6, right). Along them, one of the coordinates $k = (k_1, k_2)$ is fixed, say k_2, while k_1 is running once around the loop. We so get a transition matrix $T(k_1)$ depending on k_1. The question is no longer whether each of them can individually be deformed into the unit matrix, which it can, but whether the loop $k_1 \mapsto T(k_1)$ can be deformed into the trivial one, $k_1 \mapsto \mathbb{1}$. It not always can! Indeed, the map $k_1 \mapsto \det T(k_1)$ represents a loop in the complex plane that

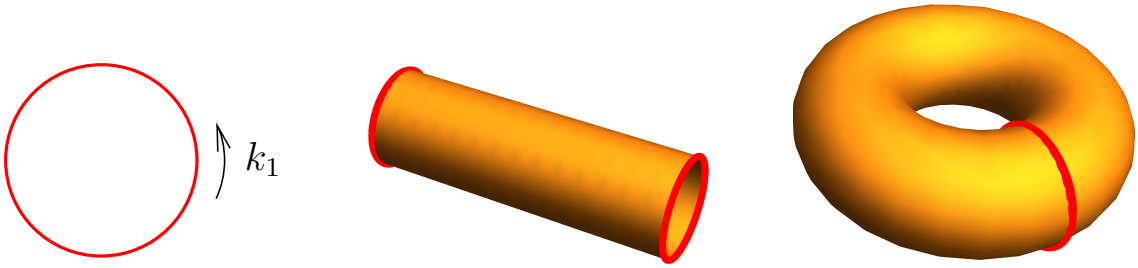

Figure 6: Left: The circle with coordinate k_1. Center: Cylinder and circles at its ends. Right: Torus obtained by gluing ends. (The base spaces are displayed, but not the fibers.)

avoids the origin. As such it may or may not wind around it. Its winding number, n, is the *Chern number* of the vector bundle, this being the same number that Eq. (10) computes.

To see this it is enough to consider a single band, thus dropping the index m. We are entitled, as we just saw, to use Stokes' theorem on the cylinder obtained by inserting a cut into the torus (see Fig. 6, right). The boundary then consists of two oppositely oriented copies of the circle $\mathbb{S}$. We obtain

$$n = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{S}} A_1\big|_-^+ dk_1 \,,$$

where A is as before and $f|_-^+$ denotes the difference of f at matching points on the two circles. There we have $|\psi_k\rangle^+ = t(k_1)|\psi_k\rangle^-$ with some complex number $t(k_1)$ of unit modulus (phase). Then $A_1|_-^+ = \langle\psi_k|\partial_{k_1}\psi_k\rangle|_-^+ = \bar{t}(dt/dk_1)$ and n is indeed the winding number of the phase.

Further developments, outlook, and conclusions

Among the three topics discussed in this article, it is the last one which has seen the strongest development in recent years, one in fact which may warrant further Nobel Prizes. Kane and Mele [13] literally brought a new twist to the story, by showing that even time reversal-invariant systems could harbor topological features, at least in the case of a time-reversal map Θ with $\Theta^2 = -1$, as appropriate to electrons and more generally to fermions. The Chern number (10) vanishes for those systems, i.e. that they are trivial in the sense discussed above. They may however not be so within their own class, meaning that their vector bundles may

not be deformable into one another if time-reversal invariance is enforced along the way, too. There is also an index which tells inequivalent bundles apart. Unlike the Chern number however, it just takes two values, ± 1 (or 0 and 1, depending on conventions). The original definition thereof was given in terms of a Pfaffian, but a pictorial account can be given as well.

To do so, let us first return briefly to the Chern number. Consider, as done before, the transition matrix $T(k_1)$ as k_1 runs along the seam joining the ends of cylinder. Without loss of generality, that matrix may be assumed to be unitary, so that its eigenvalues are points on the unit circle of the complex plane. Like the matrix $T(k_1)$ itself, the eigenvalues change with k_1 but return to their original values as k_1 runs from 0 to 2π, completing the loop (see Fig. 7, left). It takes a moment's thought to see that the Chern number, i.e. the winding number of $\det T(k_1)$, can be read off as the number of eigenvalues that cross any fiducial line; more precisely, each crossing contributes ± 1 depending on its direction.

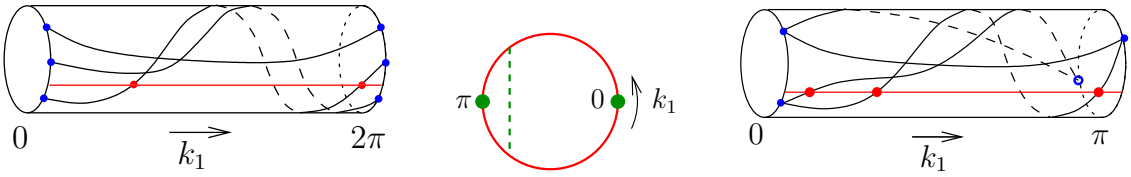

Figure 7: Left: (Chern number) The eigenvalues as functions of $k_1 \in [0, 2\pi]$ (loop) plotted as points on the unit circle. Note that they are the same (in blue) at both endpoints. As a result, the *number of signed crossings* is independent of the height of the fiducial line (in red). Center: The circle with coordinate and the link between k_1 and $-k_1$. Right: (Kane-Mele index) The eigenvalues as functions of $k_1 \in [0, \pi]$ (half-loop). Note that eigenvalues pair up at endpoints. As a result, the *parity of the number of crossings* is independent of the height of the fiducial line.

The index devised by Kane and Mele is similarly described. Time-reversal sends the point (k_1, k_2) of the torus to $(-k_1, -k_2)$, while the symmetry requires that frames at the two points are related to one another (with details omitted). The original question is sharpened by asking whether there is a global choice of frames enjoying the symmetry. The investigation proceeds as before, but with a difference: On the seam $k_2 = \pm\pi$ the transition matrix $T(k_1)$ still links the frames at $(k_1, -\pi)$ and at (k_1, π), but now they are in turn related by symmetry to those at $(-k_1, \pi)$ and at $(-k_1, -\pi)$, respectively. Therefore, the matrices $T(k_1)$ and $T(-k_1)$ must determine each other, as indicated by a dashed line in Fig. 7, center. In particular the upper half of the loop teaches us all there is to learn from the full one. (As an example,

the Chern number indeed vanishes, because during the lower half of the loop the eigenvalues just backtrack the motion they had during the upper half.) At the point $k_1 = 0$ of the loop (and likewise at $k_1 = \pi$) the dashed line represents a constraint on the matrix $T(0)$ itself; it states, as it turns out, that its eigenvalues are even degenerate (see Fig. 7, right), which is a manifestation of Kramers degeneracy. The Kane-Mele index can then be read off from the figure as the parity of the number of crossings. As a playful remark, the figures in Fig. 7 (right and left) may be interpreted as choreographies of round dances, with k_1 in the role of time and the curves in that of worldlines of dancers. The one on the right e.g. corresponds to a dance known as "rueda de casino", where dancers are supposed to pair up at the ends, but are often free in between. The rueda is thus endowed with a Kane-Mele index!

Schnyder et al. [23] pointed out that time-reversal symmetry is not the only one allowing for a finer classification; in fact particle-hole symmetry, as well as the product of both, do so too. Moreover the classification depends on the dimensionality of the material and is eventually summarized in the so-called periodic table of topological insulators and superconductors (for a review, see [11]). To conclude it may suffice to say that many more developments, both theoretical and experimental, have occurred in recent years, such as Majorana boundary states just to name another one, and more will surely follow.

References

[1] The Royal Swedish Academy. The Nobel prize in physics 2016 - Advanced information. Nobelprize.org. Nobel Media AB 2014. Web. 28 Jan 2017.

[2] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki. Rigorous results on valence-bond ground states in antiferromagnets. *Phys. Rev. Lett.*, 59:799, 1987.

[3] J. E. Avron, D. Osadchy, and R. Seiler. A topological look at the quantum Hall effect. *Physics Today*, 56:38, 2003.

[4] J. E. Avron, R. Seiler, and B. Simon. Homotopy and quantization in condensed matter physics. *Phys. Rev. Lett.*, 51:51, 1983.

[5] V. L. Berezinskii. Destruction of long-range order in one-dimensional and 2-dimensional systems possessing a continuous symmetry group, i. Classical systems. *Sov. Phys. JETP*, 32:493, 1972.

[6] J. Fröhlich and T. Spencer. The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas. *Comm. Math. Phys.*, 81:527, 1981.

[7] J. Fröhlich and E. Thiran. Integral quadratic forms, Kac-Moody algebras, and fractional quantum Hall effect. An *ADE-$\mathcal{O}$* classification. *J. Statist. Phys.*, 76:209, 1994.

[8] F. D. M. Haldane. Continuum dynamics of the 1-d Heisenberg anti-ferromagnet - identification with the $o(3)$ non-linear sigma-model. *Phys. Lett. A*, 93:464, 1983.

[9] F. D. M. Haldane. Nonlinear field theory of large-spin Heisenberg antiferromagnets: Semiclassically quantized solitons of the one-dimensional easy-axis Néel state. *Phys. Rev. Lett.*, 50:1153, 1983.

[10] F. D. M. Haldane. Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the "parity anomaly". *Phys. Rev. Lett.*, 61:2015, 1988.

[11] M. Z. Hasan and C. L. Kane. Colloquium. *Rev. Mod. Phys.*, 82:3045, 2010.

[12] G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, and T. Esslinger. Studying band-topology with ultracold fermions in an optical lattice: Experimental realisation of the haldane model. *Mitt. SPG/Commun. SPS*, 47:22, 2015.

[13] C. L. Kane and E. J. Mele. Z_2 topological order and the quantum spin Hall effect. *Phys. Rev. Lett.*, 95:146802, 2005.

[14] J. Klinovaja, P. Stano, and D. Loss. Exotic states at the edge: Majorana fermions and parafermions. *Mitt. SPG/Commun. SPS*, 43:31, 2014.

[15] M. Kohmoto. Topological invariant and the quantization of the Hall conductance. *Ann. Phys.*, 160:343, 1985.

[16] J. M. Kosterlitz and D. J. Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *J. Phys. C*, 6:1181, 1973.

[17] R. B. Laughlin. Quantized Hall conductivity in 2 dimensions. *Phys. Rev. B*, 23:5632, 1981.

[18] P. Martinoli and C. Leemann. Two dimensional josephson junction arrays. *J. Low Temp. Phys.*, 118:699, 2000.

[19] O. A. McBryan and T. Spencer. On the decay of correlations in SO(n)-symmetric ferromagnets. *Comm. Math. Phys.*, 53:299, 1977.

[20] N. Mermin and H. Wagner. Absence of ferromagnetism or antiferromagnetism in one- or 2-dimensional isotropic Heisenberg models. *Phys. Rev. Lett.*, 17:1133, 1966.

[21] B. Nienhuis and K. Schoutens. Physics Nobel Prizes 2016: Topology in condensed matter physics. *Europhys. News*, 47(5-6):16, 2016.

[22] C. P. Scheller, B. Braunecker, D. Loss, and D. M. Zumbühl. Spontaneous helical order of electron and nuclear spins in a Luttinger liquid. *Mitt. SPG/Commun. SPS*, 44:23, 2014.

[23] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig. Classification of topological insulators and superconductors in three spatial dimensions. *Phys. Rev. B*, 78:195125, 2008.

[24] R. Süsstrunk and S. D. Huber. Topological mechanics: Topology's route to applications? *Mitt. SPG/Commun. SPS*, 50:26, 2016.

[25] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs. Quantized Hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.*, 49:405, 1982.