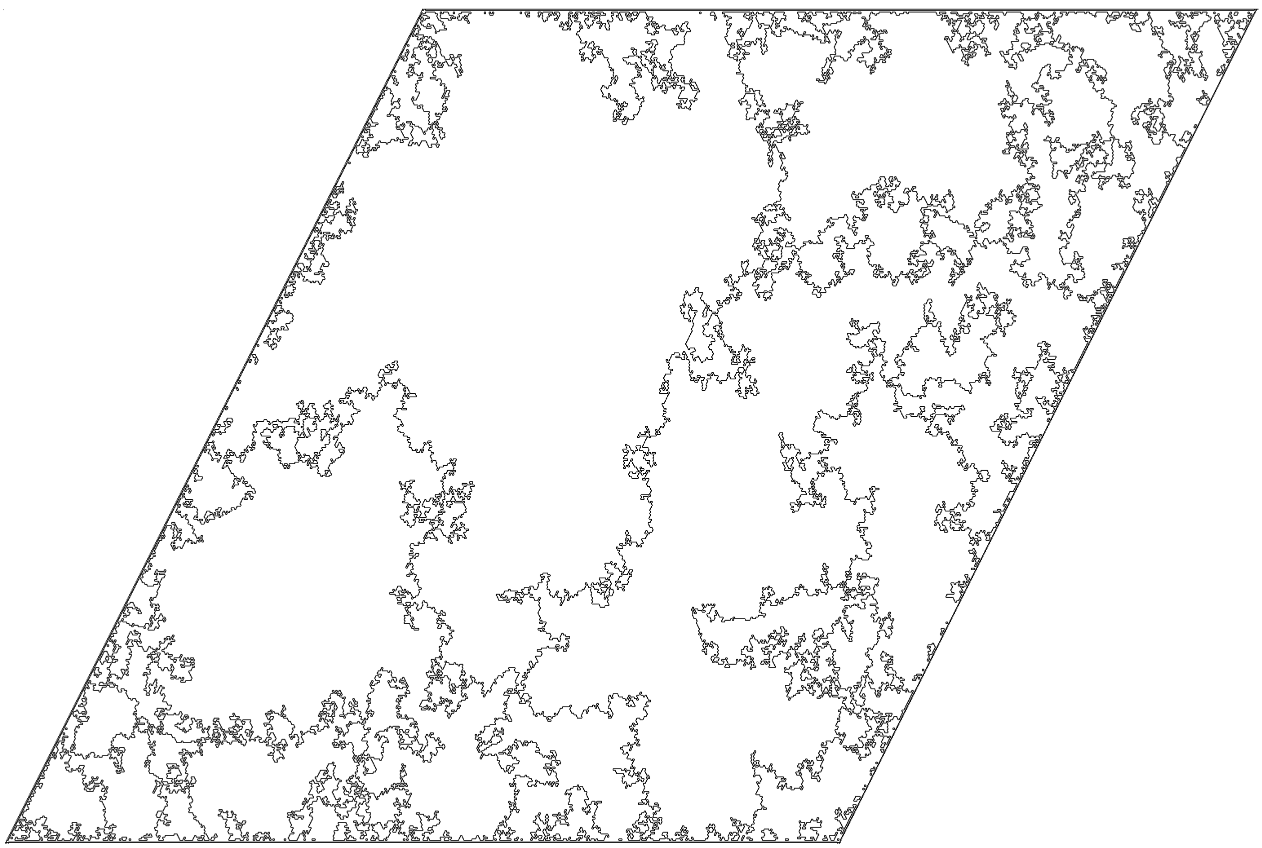
Exit sets of the continuum Gaussian free field in two dimensions and related questions



Author
Avelio Sepúlveda

Supervisor
Prof. Dr. Wendelin Werner

PhD Thesis submitted to ETH Zürich No. 24656
September 2017

DISS. ETH. No. 24656

Exit sets of the continuum Gaussian free field in two dimension and related questions

A thesis submitted to attain the degree of

DOCTOR OF SCIENCES of ETH Zürich

(Dr. sc. ETH Zürich)

presented by

Leonardo Avelio SEPÚLVEDA DONOSO

M. sc. ÉCOLE POLYTECHNIQUE

Engineer UNIVERSIDAD DE CHILE

Born July 13, 1990

Citizen of Chile

Accepted on the recommendation of:

Prof. Dr. Wendelin Werner	Examiner
Prof. Dr. Frederik Viklund	Co-Examiner

September 2017

Cada cuerpo tiene
su armonía y
su desarmonía.
En algunos casos
la suma de armonías
puede ser casi
empalagosa.
En otros
el conjunto
de desarmonías
produce algo mejor
que la belleza.

Teoría de conjuntos, Mario Benedetti

Al amor.

Abstract

The topic of this thesis is the study of geometric properties of the two-dimensional continuum Gaussian free field (GFF), which is the analogue of Brownian motion when time is replaced by a two-dimensional domain. This is part of the wider field of two-dimensional and conformally invariant geometry, which is currently a very active area of probability theory.

More specifically, the main theme of our work is the definition and properties of what we call *exit sets* of the GFF. These sets are the analogue of exit times of intervals by Brownian motion when one replace the one-dimensional time by a two-dimensional set. Because the GFF is not a continuous function, but only a generalised function, the definition and the study of these sets are somewhat challenging.

We discuss the definition and characterizations of these sets, study their size, the connectivity properties of their complement, some of their approximations via discrete structures, their relations to conformal loop ensembles and Schramm-Loewner evolutions, and how they can be used to construct some Liouville quantum gravity measures.

Résumé

Cette thèse porte sur l'étude de propriétés geométriques du champ libre Gaussien (GFF) en dimension deux (GFF), qui est l'analogue du mouvement Brownien lorsque l'axe temporel est remplacé par un domaine du plan. Cette thèse s'inscrit dans l'étude mathématique des object aléatoires invariants conforme dans le plan, qui constitue un sujet actuellement très actif en théorie des Probabilités.

Spécifiquement, le thème principal de ce travail est la définition et l'étude de ce que nous appelons les *ensembles de sorties* du GFF. Ces ensembles sont l'analogue pour le GFF de ce que sont les temps de sortie d'un intervalle pour le mouvement Brownien. Comme le GFF n'est pas une fonction continue, mais seulement une fonction généralisée, la définition même de ces ensembles est une question délicate.

Dans cette thèse, nous définissons, construisons, et caractérisons ces ensembles, nous étudiions leurs dimensions, des propriétés de connexité (de leur complémentaire), nous abordons la questions des approximations par des modèles discrets, nous discutons la relation entre ces ensembles avec les processus SLE et les ensembles de boucles CLE, et nous les utilisons pour donner une construction de mesures dites de gravité quantique de Liouville.

Acknowledgements

As a (Latin) American, the number of people to whom I have become indebted would fill a space larger than I could possibly fit in this chapter. So let me start by apologising to all people who have helped me in this endeavour and whose name is not written here.

I cannot start this section without recognising the enormous amount of effort put in by Wendelin Werner so that my PhD could arrive at this point. I would like to thank him, first, for accepting me as his student and for introducing me to the magic world of the Gaussian free field. It is really an honour to be guided by a man who is as capable in mathematics as he is wise in life, and who has guided me, not only in the intricate worlds of mathematical ideas and paper naming, but also in the (more) complex dimension of life itself. It has been a pleasure to work with a brilliant professor who is so generous at sharing his knowledge and intuition.

I thank Frederik Viklund for agreeing to be my co-examiner and for taking the time and effort to understand this work. I would also like to thank him for organising (and inviting me to) many challenging conferences in two-dimensional geometry.

As a Chilean, I would not be here if it weren't for all the courageous people who risked their lives to stop the dictatorship. To them, I will always be indebted: *Venceremos*.

It is also impossible not to recognise how (almost) all of this work was done together with my main collaborator Juhan Aru. He has illuminated me with his amazing creativity which can be seen in the way he approaches everything: maths, music, literature and life itself. I cannot thank him enough for all the hours of deep discussions. I am also profoundly grateful of my other collaborators:Ellen, for all the times we stopped to think about the dynamics, and for all the time spent correcting my (awful) English; and Titus, for the numerous discussions about Mathematics and names.

I owe a lot to my office friends and colleagues, without them life in Zürich would have been demoralising. Thank you: Ron, for the long conversations and the trivia Mondays; Mayra, for keeping the working place as Latin American as is Swissly possible; Angelo, for the introduction to the Swiss/Italian culture and for all the culinary moments you helped us enjoy; Xinyi, for your vast knowledge ranging from whisky and Brazilian music to geography and trekking; and Pablo, for teaching me how to survive in the ETH jungle. I would also like to thank all my Zuricher colleagues who gave taste to my life in Zürich: Antti, Vedran, Robin, Yukun, Wei, Yiling, Cyril, Emma, Martina, Adrien, Benjamin, Silvia, Quan, Daisuke, Gabriel, Gaultier, Bastien, Danijel, Pierre, Sara, and Brent (who also made the beautiful simulations of this thesis).

Je voudrais aussi remercier à mes amis Français, ceux qui m'ont introduit aux complexité du monde des universités français: Loïc, pour m'avoir accueilli avant que je ne parle français.

Raphaël, pour les journées d'étude et jeux. Fanny, pour m'avoir appris la vie à la française.

Me gustaría agradecer a todos mis profesores del colegio, en particular a aquellos que, con sus actos y enseñanzas, me permitieron ver la belleza (¿oculta?) de las matemáticas. Ricardo Mayorga, por mostrarme que las matemáticas son más que las operaciones básicas. Alejandro Candia,no sólo por abrirme las puertas de la física, sino también las de muchas escuelas. Gilda Valpuesta, por entender que no hay que exigir a los alumnos igual cantidad de resultados, pero si igual cantidad de dedicación. Manuel Carrillo, por permitir a sus alumnos disfrutar las Matemáticas y demostrar constantemente que es posible que alumnos de enseñanza media reflexionen a un nivel más alto.

No puedo olvidar de agradecer a todos mis amigos que, luego de haber conocido en el DIM, me acompañaron en este camino a las Europas. Les agradezco amigos: Pancho (por el sinnúmero de discusiones que nos ha permitido Zúrich), Charlie, Barbieri, Rodolfo, José Tomás, Nikolás, Nicolás y Frodo (por las infinitas conversaciones, que tuvimos y tendremos, sobre probabilidades y la vida).

Quise guardar para el final a mi familia. No hay ninguna duda que sin ella ninguna parte de este trabajo hubiera sido posible. Les agradezco profundamente: Marina, por haberme inculcado el amor al saber, por cada libro que libro que leímos juntos y por cada clase que compartimos; Héctor, por las innumerables discusiones de lo humano y lo divino, por tu capacidad de reflexión y por tu rigor intelectual; Daniela, por que me has mostrado que el amor puede ser más fuerte que las Matemáticas y por hacerme entender que la vida no es en blanco y negro; Carmen, por enseñarme que la edad no es un impedimento al momento de aprender; Naiomi, por que de ti he aprendido que la vida está llena de aventuras y que uno nunca debe temer a empezar una nueva.

Contents

Chapter 0

Introduction

The present thesis can be viewed as a tale about *exit sets* of the two-dimensional Gaussian free field (GFF). *Exit sets*, or rather their boundary, can be thought of as what exit times of intervals by standard Brownian motion (which is a random real-valued function) become when one replaces the time-set by a two-dimensional domain.

In the first section of this rather narrative introduction, we present informally the GFF, its first simple properties and the main question that will guide us throughout this work. In the second section, we describe, informally, some of our contributions on exit sets. In the third section, we describe some of the secondary yet important main casts in our story and briefly indicate what their role will be. In the fourth section, we very briefly explain the structure of the present thesis, and we then conclude with some comments on possible natural continuations of our work.

0.1 Gaussian free field

0.1.1 The canonical random surface

One of the primary goals of probability and statistical physics is to understand the propagation of randomness through different scales of a system, in other words, to answer the question: *how does microscopic randomness behave on a macroscopic scale?* The answer depends of course on the model and the meaning of "macroscopic". However, in general terms, there are only two possibilities: Randomness can either disappear at macroscopic scale or remain present.

The simplest way to exemplify these different scenarios is via the law of large numbers and the central limit theorem. What is the large scale behaviour of the trajectory of a random walk? If the expected value of individual steps is non-zero, then on a large scale, the law of large number says that the walk behaves ballistically. Its trajectory follows a deterministic straight line. On the other hand, the central limit theorem tells us that if the individual steps have zero expectation and bounded second moments, then the walk will converge in law to some variant of Brownian motion, which is indeed a random curve.

Just as in this particular example, the type of macroscopic randomness does not depend on the fine details of the microscopic scales, but only on the general type of model. This example also shows that continuous universal objects have properties and symmetries that are not present in the discrete models. For instance, the trace of d-dimensional Brownian motion is rotationally invariant and scale-invariant. In two dimensions, it is in fact conformally invariant: assume that $\varphi : D \subseteq \mathbb{C} \to D' \subseteq \mathbb{C}$ is a conformal function, and B is a Brownian motion started from $x \in D$ and stopped when it exists D. Then, the image of $\varphi \circ B$ has the law of the trace of a Brownian motion started at $\varphi(x)$ and stopped at the first moment it leaves D'.

It is natural to wonder whether a similar universal object associated with the fluctuation of random surfaces (continuous functions from $D \subseteq \mathbb{R}^d$ to $\mathbb{R}$ when $d \geq 2$ instead of $d = 1$) exists. As it turns out, the answer is yes: it is the continuum Gaussian free field (GFF), and its two-dimensional version (when $d = 2$) is the central object of study of this thesis. The definition of the GFF is straightforward: It is the centred Gaussian process with covariance function given by the (zero-boundary) Green's function in D. However, its study is delicate, because the Green's function blows up on the diagonal. This means that the GFF is not well-defined at a point and only makes sense as a random generalised function (or Schwartz distribution). One can, for instance, define its mean height on any non-empty disc and the collection of all these mean heights characterises this generalised function. For details, see Section 1.2.1 or [She07, Wer16].

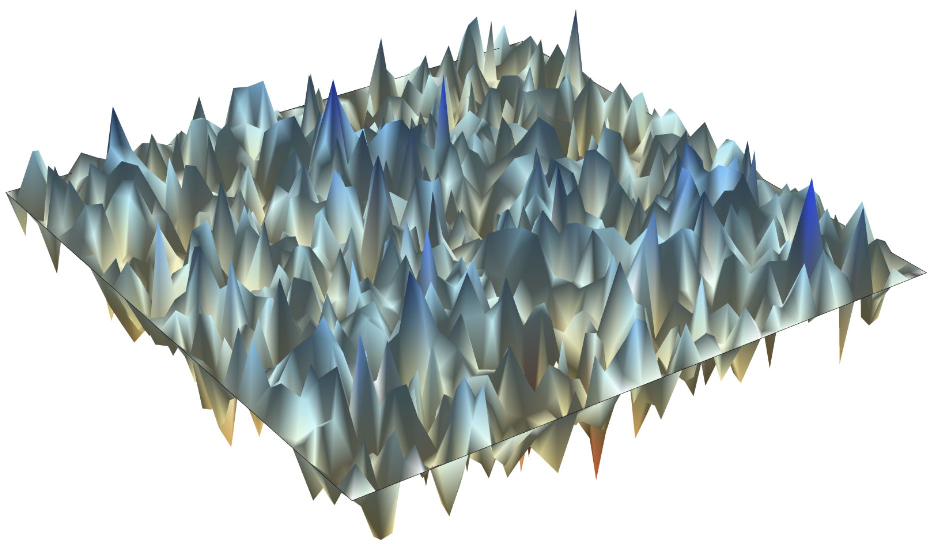


Figure 1: Simulation of a discrete two dimensional GFF. When the meshsize goes to zero, the spikes explode but the discrete GFF converges in law to a continuous GFF in a space of generalized functions.

The GFF is a central object in the context of Euclidean quantum field theory [Sim74, Gaw96]. It arises as the fluctuations of numerous models. It is the limit of the height function in the dimer model [Ken01] or many other random surfaces, the empirical measure of Coulomb gases [RV07, BBNY16], etc. As in the case of Brownian motion, the GFF in two-dimension has an important property: it is conformally invariant. That is to say, if $\varphi : D \subseteq \mathbb{C} \to D' \subseteq \mathbb{C}$ is a conformal map and Γ is a GFF in D, then $\Gamma \circ \varphi^{-1}$ has the law of a GFF in D'. This property explains why it has played a central role in many recent developments on two-dimensional conformally invariant structures. As we will recall in the third section of this introduction, it is closely related to SLE, CLE, Brownian loop-soups or Liouville quantum gravity.

0.1.2 Markov properties and local sets

Markov Property

One of the most important properties that allows the study of the Brownian motion is its Markov property. It is natural to wonder whether a similar statement holds for the GFF. Here, we find our first obstacle. On a general domain, it is not clear what "past" or "future" means, so one needs to go back to the one-dimensional Markov property. There, we can note that the Markov property naturally corresponds to the filtration indexed by all values in the past, i.e., the information corresponding to the values in intervals and not in points. This brings to light the fact that, in dimensions $d \geq 2$, the right objects to work with are not points but closed sets. We can now formulate the weak Markov property for the GFF.

Proposition 0.1 (Weak Markov property) *Let Γ be a GFF in $D \subseteq \mathbb{R}^d$ and $A \subseteq \overline{D}$ be a closed set. Then, there exist two independent fields Γ_A and Γ^A such that $\Gamma = \Gamma_A + \Gamma^A$, Γ^A has the law of a GFF in $D\backslash A$ and Γ_A restricted to $D\backslash A$ is equal to a harmonic function h_A.*

To upgrade this to the strong Markov property one can use the same strategy as for the Brownian motion. First, define $(\mathscr{F}_C : C \subseteq \overline{D}$ a closed set$)$ to be a filtration indexed by closed sets. In analogy with the one-dimensional case it is possible to make sense of a GFF in this filtration, i.e., an $\mathscr{F}_\cdot$-GFF. Then, we define stopping sets to be random sets A such that for all closed set C, $\{A \subseteq C\} \in \mathscr{F}_C$. With all these concepts, it is possible to prove the strong Markov property:

Proposition 0.2 (Strong Markov property) *Let $D \subseteq \mathbb{R}^d$ be an open set, $(\mathscr{F}_C : C \subseteq \overline{D}$ closed$)$ be a filtration indexed by closed sets, Γ be an $\mathscr{F}_\cdot$-GFF and A be an $\mathscr{F}_\cdot$-stopping set. Then, the set A induces a Markovian decomposition of Γ. In other words, there exist two fields Γ_A and Γ^A, such that $\Gamma = \Gamma_A + \Gamma^A$, and conditionally on A they are independent, Γ^A has the law of a GFF in $D\backslash A$ and Γ_A restricted to $D\backslash A$ is equal to a harmonic function h_A.*

The way we interpret this property is the following: Γ_A is the conditional value of the field, given the information that one can see in A. That is why "$\Gamma_A = \Gamma$" in A. Moreover, in $D\backslash A$, we have that $\Gamma_A = h_A$. We think of h_A as "the harmonic extension of the values of the field on ∂A".

Local sets

What we described in the last paragraphs is one natural setup to study Markovian fields, which was developed in the eighties for fairly general fields [Roz82]. On the other hand, the modern approach to the strong Markov property for the GFF is slightly different. Before discussing it, let us state a lemma relating stopping times to times that induce a Markovian decomposition in the case of Brownian motion:

Lemma 0.3 *Let (B, τ) be a coupling such that B is a Brownian motion and τ is a random* *time. The following are equivalent:*

(1) There exists a filtration $(\mathscr{F}_t)_{t\geqslant 0}$ such that B is an $\mathscr{F}_\cdot$-Brownian motion and τ is a $\mathscr{F}_\cdot$-stopping time.

(2) τ induces a Markovian decomposition of B. In other words, there exists two processes $\hat{B}$ and $\hat{B}'$ such that $B_t = \hat{B}_t + \hat{B}'_{t-\tau}\mathbf{1}_{\{t\geqslant\tau\}}$, and conditionally on τ, $(\hat{B}_t)_{t\geqslant 0}$ is independent of $(\hat{B}'_{t-\tau})_{t\geqslant 0}$, $\hat{B}'$ has the law of a Brownian motion and $\hat{B}$ is constant in (τ,∞).

Thus, in a certain sense, in the case of Brownian motion, inducing a Markovian decomposition is equivalent to being a stopping time.

Let us now present the approach initiated by Schramm-Sheffield [SS13] to understand the strong Markov property of the GFF. Informally speaking, instead of starting from (1), the idea is to build on (2). More precisely, the notion of *local sets* is defined as follows: let (Γ, A) be a coupling such that Γ is a GFF and A is a random set. We say that (Γ, A) is a local set coupling, or that A is a local set if A induces a Markovian decomposition of Γ. As in the case of Brownian motion, it is possible to show that being a local set is, in a certain sense, equivalent to being a stopping set.

In this thesis, we work mostly with local set techniques. There are three main reasons to do so. The first one is that, somewhat surprisingly, in many of the cases that we are interested in, it is much easier to prove that the set we are studying is local. The second one is that if (Γ, A) and (Γ', A') are local set couplings, there exists a useful way to create a coupling (Γ, A, A') such that A and A' are local for Γ and A and A' are conditionally independent given Γ. The final one is that this approach is better known to the community. (to our knowledge, they are no papers where the stopping set approach is developed in full detail; one can still read Chapter 1 of [Aru15a] to get a nice feeling about it).

0.1.3 Some questions

Despite working using the local set approach, our intuition will come mostly from the stopping time approach. In particular, the central questions studied in this thesis arise naturally in this setting:

Can we define "exit sets" of the GFF? And if we can, what are their properties?

By "exit sets" we mean the analogue of the exit times, i.e., the first time that a Brownian motion hits a given set of on or two values. For a one-dimensional Brownian motion there exist only two types of such stopping times: the two-sided one $\tau_{-a,b} := \inf\{t \geqslant 0 : B_t \in \{-a, b\}\}$ and the one-sided one $\tau_{-a} := \inf\{t \geqslant 0 : B_t = -a\}$ corresponding to the exit time of $(-a, b)$ and $(-a,\infty)$. Even though they look similar they have an inherent difference, $\tau_{-a,b}$ is "small" while τ_{-a} is "big". This can be quantified by noticing, for example, that $\tau_{-a,b}$ has exponential moments, while $\mathbb{E}\left[\sqrt{\tau_{-a}}\right] = \infty$. Furthermore, $(B_{t\wedge\tau_{-a,b}})_{t\geqslant 0}$ is a uniformly integrable martingale while $(B_{t\wedge\tau_{-a}})_{t\geqslant 0}$ is not uniformly integrable.

The natural way to construct "exit sets" of a particular level of a continuous field would be as follows: a point belongs to the set iff there is a path connecting it to the boundary, such that at no point of the path, the value of the field is equal to the given level. This definition only works in dimension 1. This is because as we have already mentioned, in dimension 2 or higher, the GFF is not a continuous function. This technical difficulty spices up our problems and induces the introduction of new tools and techniques.

0.2 Definition and main properties of "exit sets"

The objective of this section is to present some of our results regarding exit sets. We define exit sets through an axiomatic approach which allows one to define mathematical objects which are in some sense a priori ill-defined. The heuristic goes as follows: first, assume the object exists and study which properties it should satisfy. Collect enough properties such that one and only one object satisfies them. Finally, define the object as the unique one having all these properties.

0.2.1 Two-sided exit sets

Let us collect a list of properties that a two-sided "exit set" should satisfy. Let Γ be a GFF in a simply connected domain D, and assume A is its exit set of level $\{-a, b\}$, for $a, b > 0$, i.e., the analogue of $\tau_{-a,b}$. One expects that A should have the following properties:

(i) A is a local set of Γ.

(ii) h_A is constant in each connected component of $D\backslash A$ taking values in $\{-a, b\}$.

(iii) A is "small".

(iv) $A \cup \partial D$ has finitely many connected components.

Let us stress that the analog of $A_{-a,b}$ in the one-dimension Brownian motion case would be in fact the interval $[0, \tau_{-a,b}]$, and that conversely, the analog of $\tau_{-a,b}$ for the GFF would be in fact the boundary of $A_{-a,b}$. But, we will often just say that $A_{-a,b}$ is "the analog" of $\tau_{-a,b}$ (and similar statements for one-sided exit sets).

Let us stop for a while at the somewhat vague condition (iii). A priori, we would like to say that A is as small as possible, so that it is indeed the *first* exit set. However, as it turns out, it will be enough to ask that the set A is "small" in some general sense (this is a little bit like saying that the only Brownian stopping time τ with $B_\tau \in \{-a, b\}$ that has finite expectation is $\tau = \tau_{-a,b}$).

Understanding this general condition is the topic of the first chapter of this thesis. When $d \geq 2$, we will say that a local set A is "small" or "thin" if it carries no "mass" of the GFF itself. More precisely, if $h_A = \Gamma_A$. As we will discuss, it is not so trivial to turn this equality into a rigorous definition in a general setting, but when the harmonic function h_A is integrable (which is obviously the case when condition (ii) holds), this means that for all smooth function f, a.s. $(\Gamma_A, f) = \int_D h_A(x) f(x) dx$.

One may at first glance think that a local set is thin if only if its Lebesgue measure is 0. This is indeed the case when the set is deterministic, but this property does not hold for general local sets. We will construct local sets with 0 Lebesgue measure that are not thin. Furthermore, their Minkowski (a.k.a. box-counting) dimension is $\max\{d/2 + 1, d - 1\}$ (Proposition 1.8). On the other hand, it is true that small enough local sets are thin. More precisely, we will see that if A is a local set with Minkowski dimension strictly smaller than $\max\{d/2 + 1, d - 2\}$, then A is thin (Proposition 1.11).

Let us come back to the discussion about the two-sided "exit sets" in the case $d = 2$ (let us already note that for higher dimensions, we do not have many clues as to whether or not such exit sets exist at all). From now on in this introduction, we will assume that we are in the two-dimensional case.

Here, an interesting phenomenon arises corresponding to the existence of a height gap, as was first discovered in [SS09] in a slightly different setting. Let us explain how it appears in our framework (this corresponds to results of Chapter 2). Fix Γ a two-dimensional GFF, and define $2\lambda := \sqrt{\pi/2}$ that we call the height gap. Then, if $a + b < 2\lambda$ there is no local set A satisfying (i)-(iv). On the other hand, when $a + b \geqslant 2\lambda$, there exists a unique set such that (i)-(iv) holds. We denote this set $A_{-a,b}$ and we call it the two-valued set (TVS) of levels $-a$ and b (this statement will be first derived in the simply connected case in Section 2.6, and then generalised in Section 3.4.1).

It is also possible to define TVS for a GFF with non-zero boundary conditions. That is to say, if $u \in [-a, b]$ is a bounded harmonic function and Γ is a GFF we can define $A^u_{-a,b}$ the TVS of $\Gamma + u$ of level $-a$ and b. To do this, we just need to change condition (ii) to $h_A + u$ being constant in each connected component of $D\backslash A$ and taking values in $\{-a, b\}$.

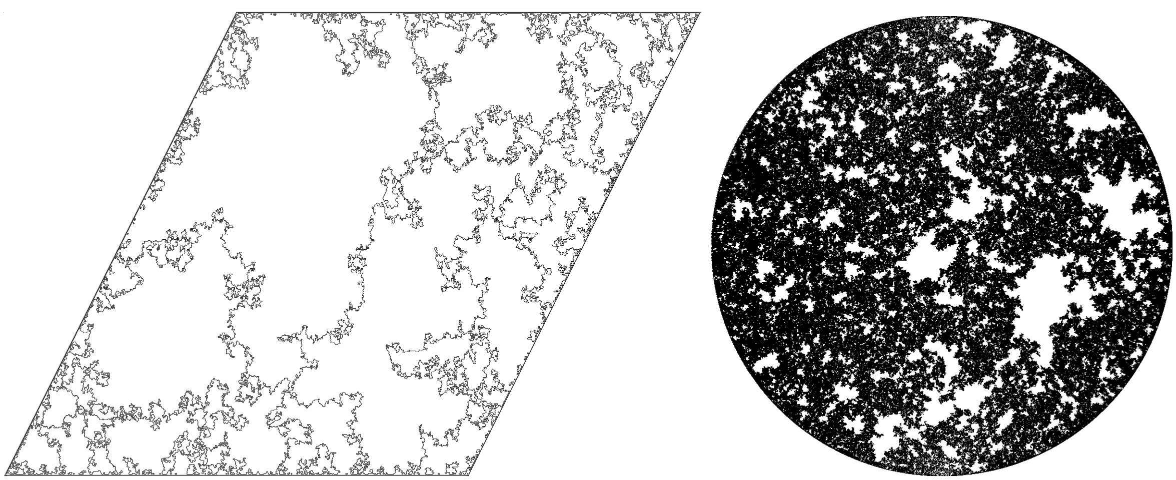


Figure 2: On the left a simulation of $A_{-\lambda,\lambda}$ by Brent Werness. On the right a simulation of $A_{-2\lambda,2\lambda}$ by David Wilson.

The TVS satisfies properties analogous to those of the two-sided exit times. Let us list a few. First, the TVS turns out to be much smaller than what condition (iii) required: Its Minkowski dimension is upper bounded by $2 - 2\lambda^2/((b + a)^2)$ (Proposition 2.23). Also, $A_{-a,b}$ is connected to the boundary (Proposition 2.4). Furthermore, these sets are monotone, in the sense that if $[a, b] \subseteq [-a', b']$, then $A_{-a,b} \subseteq A_{-a',b'}$. Additionally, if A is a set satisfying (i), (iii), (iv) and $|h_A| \leqslant a$, then $A \subseteq A_{-2\lambda-a,a+2\lambda}$ (Proposition 2.3) [1]. One motivation to study TVS is their very close relations to conformal loop ensembles (see discussion in Section 0.3.3).

It is interesting to note that contrary to the one-dimensional case, the geometry of the local sets is interesting on its own. One can study its dimension or the dimension of the boundary of a connected component of its complement (the dimension of $A_{-a,b}$ is $2 - 2\lambda^2/((b + a)^2)$

[1] It is possible to remove the additional height gap 2λ here in many cases. Sadly, we still have not found how to make it work in the general case

and that of the boundary of a connected component of $D\backslash A_{-a,b}$ is always 3/2). One can also ask other types questions that have no one-dimensional counterpart. For example: *can one recover the values of $h_{A_{-a,b}}$ just knowing the geometry of $A_{-a,b}$?*

Even though the last question is not essentially geometric its answer will come from the study of the geometry of $A_{-a,b}$. Let us define the loop graph of $A_{-a,b}$ as follows. The vertices are the connected components of $D\backslash A_{-a,b}$, and the edge set is constructed using a simple rule: there is an edge between O_1 and O_2 if $\overline{O}_1 \cap \overline{O}_2 \neq \emptyset$. It turns out that the connectivity properties of these graphs, as a function of a and b, undergo a phase transition when $a+b$ is equal to twice the height gap (Proposition 2.27). More precisely when $a+b \geqslant 4\lambda$, it is almost surely completely disconnected but when $2\lambda \leqslant a+b < 4\lambda$, the graph is connected and bipartite (because a component with label $-a$ can only be adjacent to components with label b and vice-versa). This property implies that in this last regime when one knows the label of one connected component of $D\backslash A_{-a,b}$, one can recover them all.

As we shall see, the answer to the question mentioned above about the measurability of the labels with respect to $A_{-a,b}$ is the following (Proposition 2.31):

- When $2\lambda \leqslant a+b < 4\lambda$ and $a \neq b$, it is possible to recover all the labels in a deterministic way by just observing $A_{-a,b}$. This is done by observing the Hausdorff dimension of the intersection between the boundary of the domain with the boundary of a component that intersects it.
- When $2\lambda \leqslant a+b < 4\lambda$ and $a = b$, then for obvious symmetry reasons, this is not possible. However, as soon as one knows the label of one component, one recovers them all, and for a given component, the conditional probability given $A_{-a,a}$ that its label is $-a$ is 1/2.
- When $a+b \geqslant 4\lambda$ then even if we know the value of $h_{A_{-a,b}}$ at a finite number of points, it is still not possible to recover deterministically the value of $h_{A_{-a,b}}$. To derive this last fact, one needs some other properties of two-valued sets.

0.2.2 One-sided exit set

We now turn to the exit set with only one level $-a$. This is the analogue of the first hitting time τ_{-a} of $-a$ by Brownian motion. A first way to define A_{-a} would be as the union (or the closure of the union) of $A_{-a,b}$ for all positive b. Instead, one can also follow the same axiomatic approach; we start by listing the basic properties that a one-sided "exit set" should satisfy. Let Γ be a GFF in $D \subseteq \mathbb{R}^d$ and assume A is its one-sided "exit set" of level $-a$. Then we expect that:

i) A is a local set of Γ.

ii) h_A is constant equal to $-a$.

iii) $\Gamma_A \geqslant -a$, in other words, $\Gamma_A + a$ is a positive measure.

iv) $A \cup \partial D$ has finitely many connected components.

To discuss existence and uniqueness, let us break up the cases according to the dimension of D. In dimension one there exists a unique set satisfying these properties: it is the one-sided exit time, τ_{-a}. The two-dimensional case is the subject of study of Chapter 3, and we discuss it briefly in the next paragraphs. The higher-dimensional case has not yet been worked out.

It seems possible to construct one-sided “exit sets” in dimension 3 to 5, but uniqueness still seems out of grasp. Some heuristics suggests that in dimension 7 and higher, there are new problems that arise when one tries to construct sets satisfying i)-iv).

Let us focus, now, on the two-dimensional case. There, we shall see that for each $a \geqslant 0$ there exists a unique set that we denote by A_{-a}, satisfying i)-iv) (Propositions 3.25 and 3.26). We call this set the first passage set (FPS) of level $-a$. Of course, it is also possible to define A_b, the FPS of level $b \geqslant 0$. We only need to change iii) to $\Gamma_A \leqslant b$.

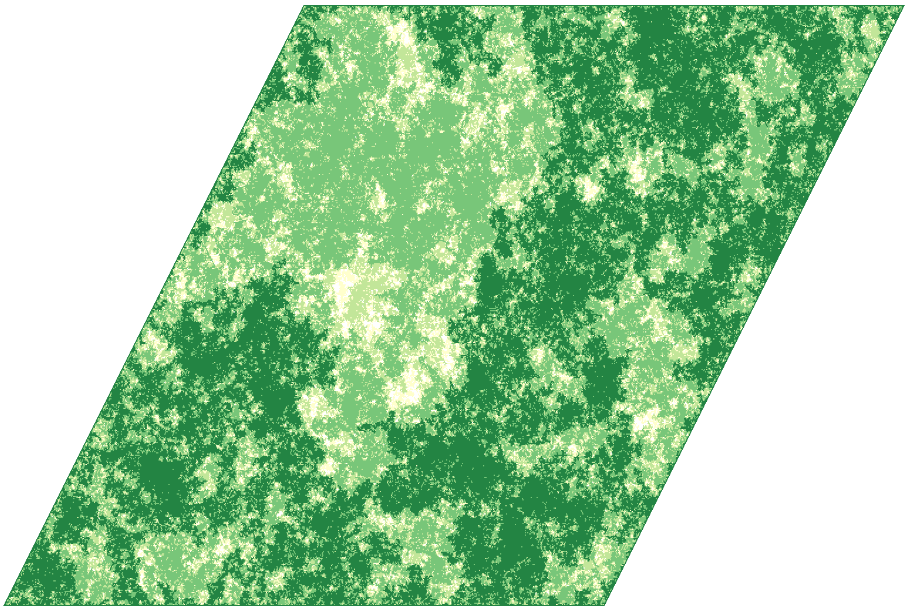


Figure 3: Simulation (from darker to lighter colours) of $A_{-\lambda}$, $A_{-2\lambda}$, $A_{-3\lambda}$ and $A_{-4\lambda}$ for the same GFF.

It turns out that the FPS is fairly “big” (even though it has zero Lebesgue measure). The set A_{-a} has Hausdorff dimension is 2 (Proposition 3.48), and it is not a thin set (Section 1.3.1). These properties are reminiscent of the non-uniform integrability of the exit time of a level by one-dimensional Brownian motion. Additionally, A_{-a} is connected to the boundary of the domain (this is a consequence of Proposition 3.25). Furthermore, the collection of FPS are also monotone in the sense that if $a \leqslant a'$, then $A_{-a} \subseteq A_{-a'}$ (Proposition 3.26).

Similarly to its two-sided counterpart, one can work out additional geometric properties of FPS. The dimension of the boundary of a connected component of $D \backslash A_{-a}$ turns out to be 3/2. One important geometric property of the FPS is related to the idea of level set percolation. To simplify the next statement, assume $D = \mathbb{H}$. In this case, there exists a path from 0 to ∞ in $\mathbb{H} \cap A_{-a}$ if and only if $a \geqslant 2\lambda$ (Remark 3.27). Remember that we interpret points in A_{-a} as the points of the GFF connected to the boundary by a path whose values are bigger or equal $-a$. Thus, in a certain sense this property may be read as: “the set $\{\Gamma \geqslant \alpha\}$ percolates if and only if $\alpha \leqslant -2\lambda$”.

In this framework, there are also interesting measurability questions. Even though it is obvious that $h_{A_{-a}} = -a$ is a measurable function of A_{-a}, this is not clear for $\Gamma_{A_{-a}} + a$, i.e., “the values of the GFF restricted to A_{-a}”. We shall derive the following a priori not trivial statement: This field $\Gamma_{A_{-a}} + a$ is in fact a measurable function of A_{-a} (Proposition 3.33). Our proof will make use of the connection between the GFF and Liouville measures that we will describe in the next section. On the other hand, we have not yet been able to describe $\Gamma_{A_{-a}}+a$ in terms of the Minkowski content of the set for a given gauge, even though heuristics strongly suggest that this should be the case.

To finish this section let us come back to the natural interplay between FPSs and TVSs. As in the one-dimensional case, we can recover the FPS from knowing certain TVSs, in fact, A_{-a} defined as above is indeed almost surely equal to $\overline{\bigcup_b A_{-a,b}}$ (Proposition 3.25). On the other hand, it is also simple to recover the TVS by just knowing two FPSs. To simplify the next statement assume that D is simply connected. Then, for all $a+b \geqslant 2\lambda$, $A_{-a,b} = A_{-a} \cap A_b$ (Proposition 3.35).

All these properties are consistent with the intuition that TVS and FPS are the natural two-dimensional analogues of exit times. This analogy will be even clearer via the metric graph GFFs approximation of the GFF using continuous functions that we will briefly describe in the next section of this introduction.

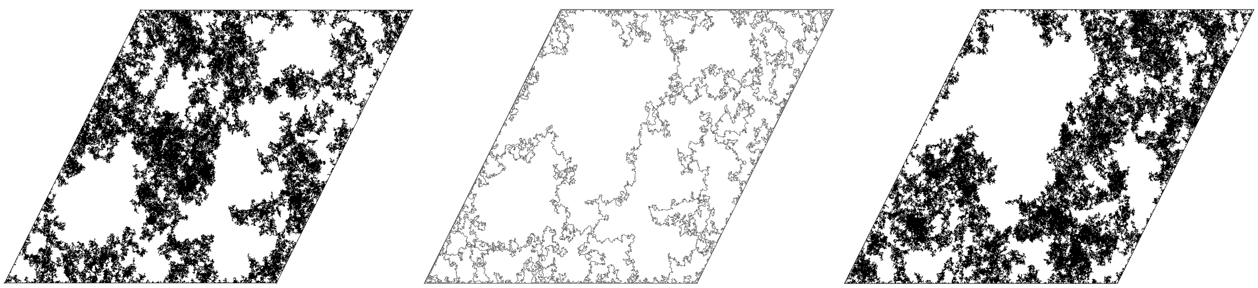


Figure 4: From left to right. Simulation of $A_{-\lambda}$, $A_{-\lambda,\lambda}$ and A_λ.

0.3 Relation with other conformally invariant objects.

In this section, we present the other main actors in our story. All of them are important enough to take the lead role in many plots, but in the current work, they will appear rather sporadically but will be however often pivotal.

0.3.1 Liouville measures

Liouville measures are the universal object corresponding to random "two-dimensional area measures". They can be constructed using a GFF. More precisely, if Γ is a two-dimensional GFF its corresponding Liouville measure with parameter $\gamma \in \mathbb{R}$ may be represented as $d\mu^\gamma :=$ "$\exp(\sqrt{2\pi}\gamma\Gamma)dx$ (the constant $\sqrt{2\pi}$ appears here for normalization issues). As Γ is not a function in the usual way, it needs to undergo a renormalization procedure to define its exponential. As a consequence of this procedure, $\mu^\gamma \neq 0$ only when $|\gamma| < 2$. It is interesting to know that in this regime, not only Liouville measures are a function of the GFF, but we can also recover the GFF just from the knowledge of one of its Liouville measures, with $0 < |\gamma| < 2$.

As we have already mentioned, one role that Liouville measures play in our story is as a tool to prove that $\Gamma_{A_{-a}} + a$ is a measurable function of the FPS A_{-a}.

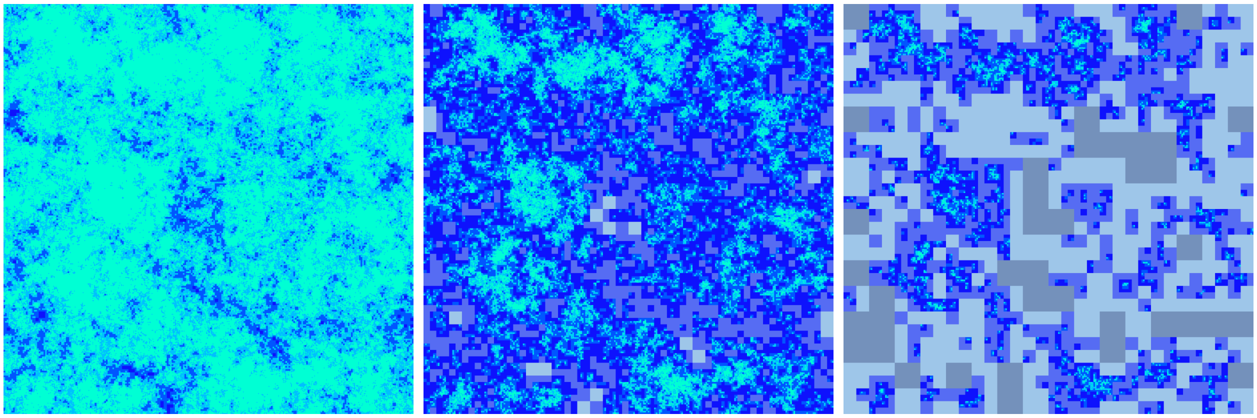


Figure 5: From left to right, simulation of Liouville measure with parameter 0.75, 1.25 and 1.75 by Jason Miller. Each square has roughly the same mass for the measure.

They will also in fact be our main focus of interest in Chapter 4: We will see that Liouville measures with positive γ can be obtained as a function of the FPS sequence $(A_b)_{b\geqslant 0}$. Furthermore, for each $b \geqslant 0$, one can construct a measure μ_b^γ depending only A_b, such that, $\mu_\gamma^b(A_b) = 0$ and $\mu_b^\gamma \to \mu^\gamma$ as $b \to \infty$ (Proposition 4.7). Of course this relationship does not only hold for FPS but also for TVS; in this case the approximated measure does not only depend on $A_{-a,b}$, but also on $h_{A_{-a,b}}$ (Proposition 4.10).

0.3.2 Schramm Loewner Evolution (SLE)

SLEs are the universal objects corresponding to "two-dimensional non-self-crossing paths". They were originally defined by Schramm [Sch00] via an axiomatic approach. He constructed them as candidates for scaling limits of paths naturally appearing in statistical physics models. His insight made it possible to reach a much better understanding of many statistical physics models.

In this thesis, we will use SLEs going between two points of the boundary. To do this, we do not distinguish the parametrization of the path. That is to say two path η_1 and η_2 are equivalent if there exists an increasing function σ such that $\eta_1 = \eta_2 \circ \sigma$.

As with all these models let us first characterize SLEs by their properties. Take $\kappa \geqslant 0$, $D \subseteq \mathbb{R}^2$ a simply connected domain and $a, b \in \partial D$. Suppose $(\eta(t))_{t\geqslant 0}$ has the law of an SLE$_\kappa$ in D going from a to b. Then, $\eta(\cdot)$ is a curve in $\overline{D}$, such that $\eta(0) = a$ and $\eta(\infty) = b$, and it satisfies:

(a) Conformal invariance: if $\varphi : D \mapsto D'$ is a conformal transformation, then $\varphi \circ \eta$ has the law of an SLE$_\kappa$ in D' going from $\varphi(a)$ to $\varphi(b)$.

(b) Markov property: Conditionally on $\eta([0, T])$, the law of $(\eta(t + T))_{t\geqslant 0}$ is that of an SLE$_\kappa$ going from $\eta(T)$ to b in the connected component of $D\backslash\eta([0, T])$ whose boundary contains b.

Schramm proved in [Sch00] that it is possible to parametrize, by a non-negative number κ, the family of non-self-crossing curves satisfying (a) and (b). There are many ways to distinguish between them. For example for $\kappa < 8$ the Hausdorff dimension of an SLE$_\kappa$ is $1 + k/8$ [Bef08].

The GFF has played an important auxiliary role in the development of the theory of SLE. This is because there is a natural way to couple SLE as a local set of the GFF, which was developed in [SS13, She05, Dub09, MS16a, MS16b, MS16c, MS17]. This natural coupling allowed, for example, to prove reversibility of non-simple SLEs and many SLE-types of curves.

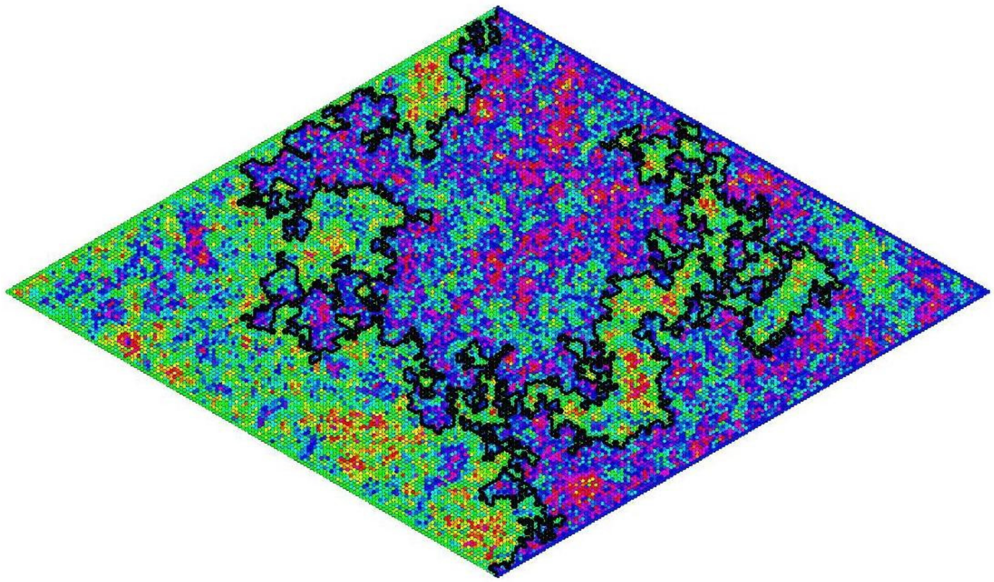


Figure 6: Coupling between the GFF and SLE_4 in a rhombus. The black curve corresponds to $A^{u_0}_{-\lambda,\lambda}$ (Simulation by Scott Sheffield).

In this thesis, we will use SLE-type curves as a tool to construct and understand exit sets. In fact, the only value of κ that will play a role here will be $\kappa = 4$. Let us briefly describe the techniques in this case. To simplify the next statements assume that $D = \mathbb{H}$, and u_0 is the bounded harmonic function with boundary values $-\lambda$ in $\mathbb{R}^-$ and λ in $\mathbb{R}^+$. Schramm and Sheffield showed, in [SS13], that $A^{u_0}_{-\lambda,\lambda}$ exists and that it has the law of the trace of an SLE_4. Additionally, they showed it is possible to discover this set progressively. They call this set the level line of the GFF. In fact, level lines for different boundary conditions have been studied in many cases, see for instance [MS16a, WW16, PW15]. Those techniques, together with the ideas of Chapter 2 enable us to show existence and uniqueness of TVS and FPS.

0.3.3 Conformal Loop Ensemble (CLE)

CLEs (here, we will only discuss the CLE carpets corresponding to CLE_κ for $\kappa \in (8/3, 4]$) are the universal conformally invariant objects associated with the joint collection of all simple "two-dimensional interfaces" that should correspond to the scaling limit of a number of discrete statistical physics models with non-intersecting loops, such as the $O(N)$-models. In [SW12], Sheffield and Werner constructed these CLEs using an axiomatic approach.

Let us now review some of the main properties of CLEs: Let $\kappa \in (8/3, 4]$, and K be a random closed set with the law of a CLE_κ in D. Then, K is a subset of $\overline{D}$, and for each $\varepsilon > 0$, there are only finitely many connected components of $D\backslash K$ with a diameter bigger than $\varepsilon > 0$. Additionally, for each O, O' distinct connected components of $D\backslash K$, ∂O is a simple curve, and $\overline{O} \cap \overline{O'} = \overline{O} \cap \partial D = \emptyset$. Finally, it satisfies:

a) Conformal invariance: if $\varphi : D \to D'$ is a conformal transformation, then $\varphi(K)$ has the law of a CLE_κ in D'.

b) Restriction property: for any closed set C, define $\widehat{C}$ the closed union of all the connected components of $D\backslash K$ intersecting C. Then, conditioned of $\widehat{C}$, the law of $K \cap (D\backslash\widehat{C})$ is that of a CLE_κ in each connected component of $D\backslash\widehat{C}$.

Sheffield and Werner proved in [SW12] that the family of laws satisfying a) and b) can be parametrized by a real number $\kappa \in [8/3, 4]$, each one having different Hausdorff dimension. They gave two different constructions: one using loop soups, and the other one using a specific SLE-type curve. In particular, the loops of a CLE_κ is an SLE_κ type loop.

In this thesis, the only CLE that will play a role is CLE_4, which is the one that is most naturally associated to the GFF (just as SLE_4 is the SLE that is most directly connected to the GFF). One can reformulate a result by Miller and Sheffield [MS11] by saying the law of $A_{-2\lambda,2\lambda}$ is that of a CLE_4 (we will also give a derivation of this result is Chapter 2 of this thesis). As we shall see, in this coupling, the CLE_4 is almost surely determined by the corresponding GFF. This will allow us also to understand the behaviour of the geometry of $A_{-a,b}$, resp. A_{-a}, in the regime where $a + b \geqslant 4\lambda$, resp. $a \geqslant 2\lambda$. On the other hand, as a by-product of the existence and uniqueness of TVS, this will provide an easy way to generalise CLE_4 to non-simply connected domains of CLE_4. Additionally, we provide various new level-line constructions of CLE_4 (see Remark 2.21).

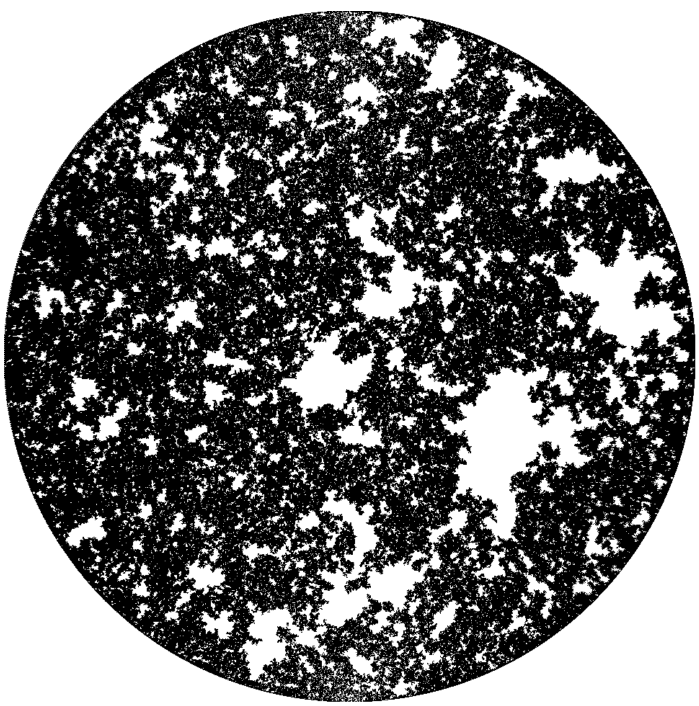


Figure 7: Simulation of CLE_4 by David Wilson.

0.3.4 Brownian loop-soup and Brownian excursions

A Brownian loop soup is a random collection of unrooted Brownian-loops contained in a domain D. Lawler and Werner introduced loop soups in [LW04], where they constructed them as a Poisson point process with intensity c times a natural Brownian loop measure.

The connectivity property of the loop soup undergoes a phase transition at $c = 1/2$ (this is sometimes also referred to $c = 1$ and depends on the choice of normalisation for the loop measure). More precisely, define the following equivalence class on the elements of a loop soup: $\ell \sim \ell'$ if there is a finite path of intersecting loops connecting them. Then, Sheffield and Werner proved in [SW12] that there is a unique equivalence class if and only if $c > 1/2$. Furthermore, when $c = 1/2$ the closed union of the outer boundary of the outermost clusters has the law of a CLE_4, i.e., the same law as $A_{-2\lambda,2\lambda}$. Additionally, in [QW17], Qian and Werner proved that there is a natural way to couple the GFF and the loop soup of parameter 1/2. In this coupling, the square of the GFF is equal to the occupation time of the loop soup and $A_{-2\lambda,2\lambda}$ is equal to the outer boundary of the outermost cluster of the loop soup.

Boundary-to-boundary excursions are a random collection of Brownian-excursions contained in a domain D. They can also be constructed as a Poisson point process of a given measure with intensity u. In this story, they are used to describe the law of FPS. In particular, sample a loop soup of parameter 1/2, and a boundary-to-boundary excursion of parameter u. Then, we can define the same equivalence relationship like the one used in the loop soup case. The FPS A_{-u} is equal in law to the closed union of the elements of the equivalence classes containing at least one excursion (Corollary 3.46). It is work in progress to see that in this case we also have a coupling with the same properties as that of [QW17].

Again, just as for SLE and CLE, the loop-soup that will play a role in the present thesis is the one that is most directly related to the GFF, i.e., the loop soup with intensity $c = 1/2$.

0.3.5 Metric graph GFF

The metric graph GFF is an approximation of the GFF using continuous functions. It was introduced by Lupu in [Lup16b], to have a better understanding of Le Jan's isomorphism between the square of the GFF and occupation time of the loop soup. To define it let us first describe the metric graph, or cable system, corresponding to a graph G. The metric graph associated to G, $\widetilde{G}$, is the metric space in which every edge of G is replaced by a copy of $[0, 1]$. The most useful property of this metric space is that its continuous functions satisfy the intermediate value property.

Before the introduction of metric graph GFF, the usual way to discretize the GFF was using the discrete GFF. That is to say, a centred Gaussian process defined on the vertices of a graph G with covariance given by the Green's function of the graph. The main problem with this approach is that functions on graphs have no way of satisfying the intermediate value property. This explains why it is not easy to understand the relationships between some functionals of the discrete GFF, like level lines, and their continuous counterparts.

Let us briefly describe how to sample, $\tilde{\phi}$, a metric graph GFF in $\widetilde{G}$. We can obtain the values on the vertices by sampling a discrete GFF on G. Then, the values of $\tilde{\phi}$ restricted to an edge are given by the sum of the linear interpolation of the values on the endpoints and a Brownian bridge independent of all the other randomness. This construction guarantees two important properties: $\tilde{\phi}$ is continuous and satisfies the Markov property. It is possible to see that in this setting Le Jan's isomorphism also holds, in fact, Lupu proved a stronger version of it in [Lup16a].

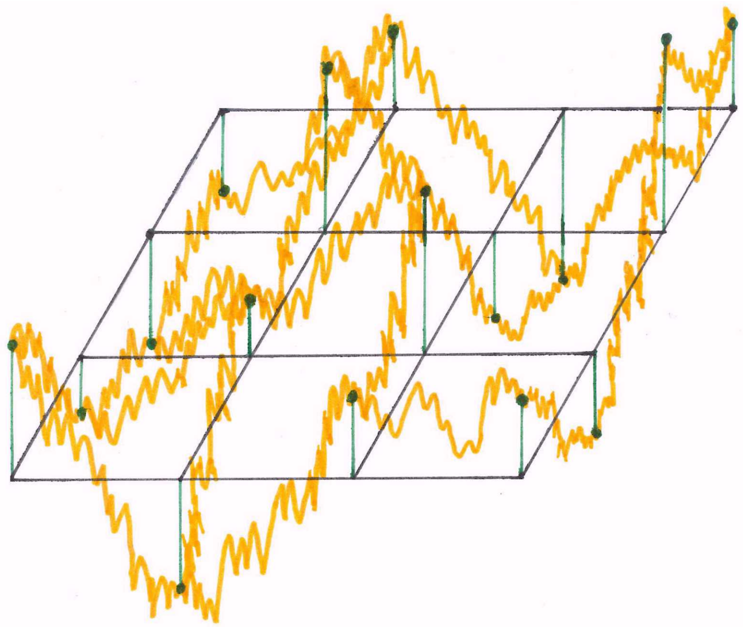


Figure 8: Artistic view of a metric GFF. Drawn by Titus Lupu.

The metric graph GFF is very helpful in understanding some functionals of the GFF and has been a useful ingredient in the proof of natural properties. In our story, it will provide a way to see why FPS are the right analogue of one-sided exit times: Let $(\widetilde{D}_n)_{n\in\mathbb{N}}$ be a sequence of metric graphs and $\tilde{\phi}_n$ a metric graph GFF in $\widetilde{D}_n$. Define $\tilde{A}_{-a}$ as the set of all points of $\widetilde{D}_n$ that can be connected to $\partial\widetilde{D}_n$ by a path in $\widetilde{D}_n$ where the values of $\tilde{\phi}_n$ are always bigger than $-a$. Then, if $\widetilde{D}_n \to D \subseteq \mathbb{R}^2$ in a suitable way, $(\tilde{\phi}_n, \tilde{A}_{-a})$ converges in law to (Γ, A_{-a}) (Proposition 3.40). This will allow us to prove the characterization of the distribution of the FPS given in the last section.

0.4 Structure of this thesis

This thesis is divided into four chapters. Each one corresponds to a separate paper or preprint, except Chapter 2 that consists of the concatenation of a paper with some work in progress. This explains some repetitions in the introductory/background parts of each chapter, and also why notations will sometimes vary from one chapter to the other. On the other hand, this gives to each chapter a more self-contained character. Here is a summary of each of them:

0.4.1 Chapter 1: Thin local sets

Based on [Sep17]. In this chapter, we introduce the concept of thin sets, and we study how small a local set of the GFF in dimension d has to be to ensure that this set is thin. We also show that this criterion is sharp by constructing small local sets that are not thin.

Note that this chapter is the only one where we also work with the GFF in a dimension greater than 2. In this first chapter, FPS make a first appearance as examples of non-thin local sets.

0.4.2 Chapter 2: Bounded-type local sets and TVS

Joint work with Juhan Aru and Wendelin Werner [ASW17] and ongoing work with Juhan Aru. The main focus of this chapter is the study of a certain class of local sets of the two-dimensional GFF in a simply-connected domain, and their relation to the conformal loop ensemble CLE_4. More specifically, we consider bounded-type thin local sets (BTLS), where bounded-type means that the harmonic function describing the mean value of the field away from the local set is bounded by some deterministic constant. We show that a local set is a BTLS if and only if it is contained in some nested version of the CLE_4 carpet, and prove that all BTLS are necessarily connected to the boundary of the domain.

This class of BTLS includes the two-valued sets (TVS) $A_{-a,b}$ that correspond to the exit times of intervals for Brownian motion. This is the chapter where we define and construct all these sets, and we show that a BTLS is necessarily contained in some TVS. We show that TVS (this includes the CLE_4 TVS $A_{-2\lambda,2\lambda}$) are in fact measurable functions of the GFF. Furthermore, we study geometric properties the TVS.

Then (and this corresponds to the ongoing work with Juhan Aru), we study connectivity properties of their complement, and we show that the answer depends on whether $a+b<4\lambda$ or not. We also obtain conditions that tell us when the harmonic function describing the mean value of the field away from the TVS is a measurable function of the TVS.

0.4.3 Chapter 3: First passage sets

Joint work with Juhan Aru and Titus Lupu [ALS17a]. This chapter is centred on the study of the FPS in finitely connected domains. We provide a continuum construction using level lines and study its properties. We further prove that the metric graph FPS converges towards the continuum FPS. This convergence allows us to show that the FPS can be represented as clusters of Brownian excursions and Brownian loops, and to revisit convergence results for level lines of the GFF.

0.4.4 Chapter 4: Liouville measure and local sets

Joint work with Juhan Aru and Ellen Powell [APS17]. In this chapter, we provide new constructions of the subcritical and critical Gaussian multiplicative chaos (GMC) measures corresponding to the 2D GFF. As a special case, we recover E. Aidekon's construction of random measures using nested conformally invariant loop ensembles, and thereby prove his conjecture that certain CLE_4 based limiting measures are equal in law to the GMC measures for the GFF. The constructions are based on the theory of local sets of the GFF and build a strong link between multiplicative cascades and GMC measures. This link allows us to directly adapt techniques used for multiplicative cascades to the study of GMC measures of the GFF. As a proof of principle, we do this for the so-called Seneta–Heyde rescaling of the critical GMC measure.

0.5 Future directions

Let us now briefly comment on current projects that we are working on or plan to work on in the near future, in the same vein. The presented projects will be ordered according to the likely order of completion (we end with those where we do not have many certainties on our angle of attack).

The results of the final sections of Chapter 2 are, in fact, part of an ongoing project with Juhan Aru. One goal of this project is to show how TVS' techniques allow us to understand basic conformal invariant metrics of CLE_4. We will show another coupling between $A_{-2\lambda,2\lambda-r}$ and a GFF Γ, in such a way that the graph distance from a loop to the boundary appears naturally in the labels of a loop. Then, we plan to show how, when one takes $r \to 0$, one recovers the coupling between labelled CLE_4 and the GFF first described in [WW16].

Similarly, Chapter 3 corresponds in fact to the first paper in a sequence of three, all joint with Juhan Aru and Titus Lupu. Motivated by [LW16], the goal is to describe the Lévy transform of the continuous GFF. That is to say to define, using to TVS and FPS, the "absolute value" of the GFF. In particular, we are interested in understanding how the coupling between the GFF and the loop soup looks under this transformation, and how certain metrics related to CLE_4 appear naturally. In the third paper in this series, we plan to carefully use the construction and convergences of "exit sets" to understand properties of CLE_4. In particular, we plan to compute the laws of various observables of CLE_4, such as the extremal distance between the loop containing 0 and the boundary.

Chapter 4 also has a natural continuation. In fact, it was originally thought as a step to prove that the derivative of the Liouville measure, when $\gamma = 2$, which is the critical Liouville measure. It seems that techniques from multiplicative cascades may come to an aid here, as for the case of Seneta-Heyde rescaling at critically. With Juhan Aru, Ellen Powell and Xin Sun, we are working on this problem and its applications to the construction of the critical measure in the mating of trees framework, introduced by Duplantier, Miller and Sheffield in [DMS14].

One natural question that comes to mind after the study of level set percolation for the continuum GFF is whether one can deduce analogous features for the discrete or metric GFF. In the discrete and metric setup, one is for instance interested to know whether 0 is connected to ∞ by a path where the discrete, or metric, GFF is bigger than a certain value α. There has been a lot of progress in the understanding of this question in transient graphs, in particular in $\mathbb{Z}^d$ for $d \geqslant 3$ [BLM87, RS13, Rod15, Szn16, AS16]. We are working with Pierre-Francois Rodriguez and Titus Lupu, to tackle this problem in the triangular lattice restricted to the half plane. There, the continuum GFF result suggests the following: there is percolation in the discrete, resp. metric, GFF, if and only if $\alpha \leqslant -3^{-1/4}\lambda$, resp. $\alpha \leqslant -3^{-1/4}2\lambda$. It seems that this would be the first case where the critical value is explicitly computed.

Another question we would like to understand is: If one looks at a natural dynamic whose invariant measure is the GFF, how do TVS, FPS and other fundamental local sets evolve under this dynamics? This question is delicate, and until now we do not have many solid results. To tackle this problem and its variants, we have been working on different projects with Juhan Aru, Christophe Garban, Ellen Powell, Hugo Vanneuville and Brent Werness. Most of the simulations done in this thesis are a by-product of trying to understand the discrete side of this question.

Finally, it is impossible not to mention the most obvious natural open question related to our story: what happens in higher dimensions? We have been thinking about it with Juhan Aru and Titus Lupu. As we have discussed in the case of the one-sided exit sets, we do have some guesses and heuristics about what should be going on. On the other hand, higher dimensions are more challenging because all the conformal geometry tools are lost. In particular, we have found no way to explore the GFF with bounded (or upper bounded) local sets. Note that there are still some available tools: for example, Le Jan's isomorphism still holds in dimension 3, and the metric graph does approximate the GFF in any dimension.

Chapter 1

Thin local sets

1.1 Introduction

The Gaussian Free Field (GFF) is the natural analogue of Brownian motion when the time-set is replaced by a d-dimensional open domain D. The GFF is a fundamental object in probability and statistical physics. In two dimensions its geometry is closely related to many other key objects such as stochastic Loewner evolutions [SS13, Dub09, MS16a], conformal loop ensembles [MS11, ASW17], Liouville quantum gravity [Aru15b, DS11], Quantum Loewner evolutions [MS16d, MS15] and loop soups [LJ11, Lup16a, QW15]; note that the relation to loop-soups is in fact not restricted to the two-dimensional GFF.

Unlike Brownian motion, when $d \geq 2$, the GFF is not a continuous function; it is only defined as a random generalized function from D into $\mathbb{R}$. However, the GFF shares many of the Brownian motion's properties and in particular its Markov property; loosely speaking, the spatial Markov property of the GFF is that for any deterministic closed set A the distribution of the GFF in the complement of A is equal to the sum of the harmonic extension of the values of the GFF on ∂A with an independent GFF in $D \setminus A$. Just as in the one-dimensional case, this Markov property can be upgraded into a strong Markov property, where the above decomposition holds also for some random sets A. Such multivariate Markov properties were first studied in the 70s and 80s [Roz82], and recently reinterpreted and applied in the two-dimensional imaginary geometry framework [MS16a, SS13]. These sets, called local sets in [SS13, MS16a], play roughly the same role, in the higher-dimensional setting, as stopping times; more precisely, the local set A is the analogue of the interval $[0, \tau]$ when τ is a one-dimensional stopping time. The notion of local sets makes sense and is natural for the GFF in any dimension, even if so far it has only been used when $d = 2$.

One way to formally describe local sets is to say that there exists a coupling (Γ, A, Γ_A) where Γ is a GFF in D, A is a random closed set and Γ_A is a random field with the following properties:

- Conditionally on (A, Γ_A), the distribution of $\Gamma - \Gamma_A$ is a GFF in $D \setminus A$.
- For every deterministic open set O, on the event where O and A are disjoint, the restriction of Γ_A to O is a harmonic function in O. More precisely, there exists a random harmonic function h_A in $D\backslash A$ such that for all smooth function f, $(\Gamma_A, f) = \int_{D\backslash A} h_A(x) f(x) dx$ on the event where the support of f is contained in $D\backslash A$.

The field Γ_A can be loosely speaking understood as being equal to the field Γ "within A" and as being equal in $D \setminus A$ to the harmonic extension h_A of the values of the field on ∂A.

In the present chapter, we investigate how small a local set has to be (for instance in terms of its fractal dimension) in order to ensure that, loosely speaking, Γ_A consists only of its harmonic part h_A, i.e, it carries no mass of the GFF on itself – we call such sets thin local sets: In the very special case where the harmonic function h_A is a.s. integrable on $D\setminus A$ (this for instance happens for the bounded-type thin local sets studied in the next chapters, particularly, in Chapter 2, where h_A is bounded), being thin means that for any compactly supported smooth function f, (Γ_A, f) is almost surely equal to $\int_{D\setminus A} h_A(z) f(z) dz$, even when the support of f intersects A. In the general case, where the local set is such that the function h_A is not integrable on $D \setminus A$, which should be thought of as the generic case (mind that h_A oscillates wildly when it approaches A, especially in higher dimensions – this is already the case when A is a deterministic non-polar set), there are various possible definitions that we will discuss, but we can sum it up in saying that thin local sets are the local sets for which for all given reasonable procedure to make sense of the not-absolutely-converging integral $\int f(z) h_A(z) dz$ turns out to be almost surely equal to (Γ_A, f).

Thin local sets are typically small. For instance, a deterministic set is a thin local set if and only it is of zero Lebesgue measure. But, as we shall see, when $d \geq 2$ there exist many (random) non-thin local sets that have zero Lebesgue measure. In some sense, this is due that one can explore GFF values in such a way to capture large values of the GFF while keeping the explored domain local and fairly small.

Let us briefly present our main results first when $d \in \{3, 4\}$, then $d = 2$ and then $d \geq 5$. For $d = 3, 4$, we have:

$(1)_d$ If A is a local set of the GFF with upper Minkowski dimension that is almost surely smaller than $(d/2) + 1$, then it is a thin local set.

$(2)_d$ There exist local sets of the GFF with upper Minkowski dimension that is almost surely not larger than $(d/2) + 1$ that are not thin local sets.

In other words, the dimensions $5/2$ and 3 play an important role for the size of local sets of the GFF in respective dimensions $d = 3$ and $d = 4$.

These statements also hold in the two-dimensional case, but the second one is rather void because $1 + (d/2) = 2$, so that one can just take A to be the entire domain $\overline{D}$, which is clearly not thin. We derive the following more refined result when $d = 2$:

$(1)_2$ If A is a local set of the two-dimensional GFF such that for some positive δ, the expected value of the area of the ε-neighborhood of A decays almost surely like $O(|\log \varepsilon|^{-1/2-\delta})$, then it is a thin local set.

$(2)_2$ There exist local sets of the two-dimensional GFF for which the expected value of the area of their ε-neighborhood decays almost surely like $O(|\log \varepsilon|^{-1/2})$ and that are not thin local sets.

When $d \geq 5$, another phenomenon related to the dimension of polar sets enters into the game. We shall prove that when $d \geq 5$,

$(1)_d$ If A is a local set of a d-dimensional GFF and has upper Minkowski dimension smaller than $\max\{d - 2, 1 + (d/2)\}$, then it is thin.

$(2)_d$ There exist local sets of the GFF with upper Minkowski dimension almost surely equal to $d-1$ that are not thin local sets

We believe that one can replace $d-1$ by $d-2$ in the statement $(2)_d$. The threshold $(d/2)+1$ would then be valid up to $d=6$, and for $d>6$, it should therefore be $d-2$.

Our proofs of statements of the type $(1)_d$ (i.e. "when the local set is small enough, then it is necessarily thin") are based on rather direct moment estimates: When $d \neq 2$, a first moment computation combined with a Borel-Cantelli argument suffices, and when $d=2$, we use a slightly more refined second moment computation.

It is somewhat more challenging to prove $(2)_d$, i.e. to construct well-chosen "fairly small" local sets and to prove that they are not thin, and this is arguably the main contribution of the present chapter. It is worthwhile noticing that in two-dimensions, it is possible to use the nested version of the Miller-Sheffield GFF-CLE_4 coupling to construct such a small yet non-thin local set, but when $d \geq 3$ other ideas are needed. Our strategy consists in relating a particular exploration of the GFF with a branching Brownian motion. This idea is reminiscent of the one that was for instance used in the two-dimensional case in [BDG01] to study the maximum of the discrete GFF. The constructed set may also be interpreted as a local set approximation of perfect thick points (in the sense of [HMP10], Section 3.2).

The structure of the chapter is the following: we first very briefly recall some basic properties of the continuous GFF, its local sets and we give a possible definition of thin local sets. Then, we use this definition to construct examples of local sets that prove the statements $(2)_d$. Thereafter, we prove the statements $(1)_d$ and conclude with some comments about the definitions of think local sets.

1.2 Preliminaries

1.2.1 GFF and scaling

Introductions and basic results about the GFF can be found in [Aru15a, AS17a, SS13, She09, Wer16]. While the presentations in those references is in the two-dimensional setting, they can be extended without any difficulty to higher dimensions. Let us briefly remind some basic facts:

Throughout this chapter, we will use the function ϕ_d defined on $\mathbb{R}^d \setminus \{0\}$ by $\phi_d(x) = (1/2\pi) \times \log(1/\|x\|)$ when $d=2$ and by $\phi_d(x) = 1/(c_d\|x\|^{d-2})$ when $d \geq 3$, where c_d denote the $d-1$-dimensional surface of the unit sphere in $\mathbb{R}^d$.

Suppose that D is d-dimensional open domain with non-polar boundary (this boundary can be empty if $d \geq 3$), and consider the Green's function with Dirichlet boundary condition in D to be the unique function from $D \times D \setminus \{(x,x)\ x \in D\}$ to $\mathbb{R}_+$ that is harmonic in both variables, and such that for all given x in D, $G_D(x,y) \to 0$ as $y \to \partial D$ and $G_D(x,y) \sim \phi_d(x-y)$ as $y \to x$. Recall that when $D \subset \tilde{D}$, then $G_D(x,y) \leq G_{\tilde{D}}(x,y)$.

We can then define the space $\mathcal{H}^{-1}(D)$ of functions on D, such that

$$\iint_{D \times D} f(x) G_D(x,y) f(y) dx dy < \infty.$$

The GFF in D with zero boundary conditions is defined to be the centered Gaussian process $((\Gamma, f), f \in \mathcal{H}^{-1}(D))$ with covariance function

$$\mathbb{E}\left[(\Gamma, f)(\Gamma, g)\right] = \iint\limits_{D\times D} f(x)G_D(x,y)g(y)dxdy.$$

It is well-known that this process exists, and that it is possible to find a version of the GFF such that almost surely, for all $\varepsilon > 0$, Γ can be viewed as an element of the space $\mathcal{H}^{1/2-d/4-\varepsilon}$, the dual under the $\mathcal{L}^2$ product of the Sobolev space $\mathcal{H}^{d/4-1/2+\varepsilon}$ (see for instance Section 2.3 of [She09]).

The definition of the GFF immediately implies its scaling properties. If we define the domain $z_0 + rD := \{z_0 + rz \ : \ z \in D\}$, then

$$G_{z_0+rD}(z_0 + rx, z_0 + ry) = r^{2-d} G_D(x, y) \tag{1.1}$$

(in two dimensions, a stronger result holds, as the Green's function is conformally invariant), which yields the corresponding scaling properties for the GFF.

1.2.2 Local sets

We first very briefly review the definitions of local sets and some of their properties that are relevant for our purposes.

Denote the family of all closed subsets of D by $\mathcal{C}(D)$. Let Γ be a GFF in D and $C \in \mathcal{C}(D)$, one can decompose Γ into the sum of two independent processes Γ_C and Γ^C where almost surely, Γ_C restricted to $D \setminus C$ is a harmonic function, and where Γ^C is a GFF in $D\setminus C$ (this property is usually refered to as the spatial Markov property of the GFF). One can note that Γ_C and Γ^C are Gaussian processes that are also generalized functions, with respective covariance given by the Green's functions $G_D - G_{D\setminus C}$ and $G_{D\setminus C}$.

Let $(\mathscr{F}_C)_{C\in\mathcal{C}(D)}$ be a complete outside-continuous filtration indexed by $\mathcal{C}(D)$ i.e. $C \mapsto \mathscr{F}_C$ is non-decreasing, the σ-fields $\mathscr{F}_C$ are all complete with respect to the probability measure that we are working with, and for any decreasing sequence (C_n), one has $\mathscr{F}(\cap C_n) = \cap \mathscr{F}(C_n)$. We say that the GFF Γ is adapted with respect to this filtration if for all C, Γ_C is $\mathscr{F}_C$ measurable while Γ^C is independent of $\mathscr{F}_C$. We also say that a random set A is a local set in the filtration $(\mathscr{F}_C)$ if for all $C \in \mathcal{C}(D)$, the event $\{A \subset C\}$ is in $\mathscr{F}_C$. The filtration generated by a GFF Γ (or the "natural filtration" of Γ) is the smallest one for which each Γ_C is $\mathscr{F}_C$-measurable.

Let us list a couple of simple facts about local sets, whose properties are immediate consequences of the definition (see [AS17a]):

a) If A and B are local with respect to the filtration $(\mathscr{F}_C)$, then $A \cup B$ is also local.

b) If (A_n) is a family of local sets with respect to the filtration $(\mathscr{F}_C)$, then $\cap_n(\overline{\cup_{m\geq n} A_m})$ is also a local in the same filtration.

c) If A is a local set and Γ is a GFF adapted to $\mathscr{F}_\cdot$, then there exists a process Γ_A, such that it is a.s. harmonic in $D\setminus A$, and that conditionally on (A, Γ_A), $\Gamma^A := \Gamma - \Gamma_A$ is a GFF in $D\setminus A$.

In the literature, having a coupling (A, Γ) satisfying c) is usually used as a definition of local set (see for instance [SS13]). This property is equivalent to the existence of a filtration under which A is a local set and Γ is a GFF (see [AS17a]). The definition of local via filtration will be handy to show that the examples that we construct are indeed local sets.

Note that we can represent the restriction of Γ_A to $D\backslash A$ as a harmonic function h_A in $D \setminus A$, i.e., there exists a harmonic function h_A in the random domain $D \setminus A$ such that for all smooth function f with compact support in D, $(\Gamma_A, f) = \int h_A(z) f(z) dz$ almost surely on the event where the support of f is contained in $D \setminus A$.

Additionally, it holds that when A and B are local sets, a.s. for all z such that the connected component of $D\backslash A$ containing z is equal to the connected component of $D\backslash B$ containing z we have that $h_A(z) = h_B(z)$ (see [AS17a, She09, Wer16]).

Let us already point out that local sets have to be big enough in order to actually provide any information about the GFF:

Lemma 1.1 *Let Γ be a GFF on a domain D and A a local set. Then, $\Gamma_A = 0$ almost surely if only if A is almost surely polar for Brownian motion on D.*

Proof. Note that A is polar if and only if $G^D = G^{D\backslash A}$. Then for all smooth function f with bounded support,

$$\mathbb{E}\left[(\Gamma_A, f)^2\right] = \mathbb{E}\left[(\Gamma, f)^2\right] - \mathbb{E}\left[(\Gamma^A, f)^2\right],$$

Given that $G^{D\backslash A} \leqslant G^D$, we see that A is polar if and only if the right hand side is equal to 0 for all such f. □

Recall that Kakutani's Theorem (Theorem 8.2 in [MP10]) shows that being polar or not is in fact just a condition on the decay on the volume of small neighborhoods of A. In particular, we see that when $d \geq 3$, any local set with Minkowski dimension smaller than $d-2$ is polar for the BM, and it is therefore a local set with $\Gamma_A = 0$.

1.2.3 A first possible definition of thin local sets

We will discuss in more detail various possible definitions of thin local sets and questions related to those definitions in the last section of the chapter, but at this point, let us already give here a definition based on dyadic approximations. We will work with this definition in the coming sections.

Suppose that D is a fixed open domain in $\mathbb{R}^d$ for $d \geq 2$. For simplicity, we assume that D is a bounded set. For any $n \geq 0$, say that s is an open dyadic hyper-cube of side-length 2^{-n} (or just 2^{-n} dyadic hypercubes) if it is a translate of $(0, 2^{-n})^d$ by some element in $(2^{-n}\mathbb{Z})^d$. We call $\mathcal{S}_n$ the set of all non-empty intersections of open 2^{-n}-dyadic hypercubes with D and $\mathcal{T}_n$ the set of faces of elements of $\mathcal{S}_n$. If A is a closed set we define A_n to be the closure of the union of elements of $\mathcal{S}_n \cup \mathcal{T}_n$ intersecting A

With this definition, we can note that if A is a local set for the GFF Γ in D, A_n is also a local set. Note that $A_n \searrow A$ and for each $n \in \mathbb{N}$, A_n can take only finitely many possible values. This second fact makes it possible to define, for each smooth bounded function f in

D, random variables $(\Gamma_A, f1_{D\backslash A_n})$ and $(\Gamma_A, f\mathbf{1}_{D\backslash A_n})$. Indeed, one can simultaneously define $(\Gamma_A, f\mathbf{1}_u)$ for any possible value u of $D\backslash A_n$, and then $(\Gamma_A, f\mathbf{1}_{D\backslash A_n}) = \sum_n (\Gamma_A, f\mathbf{1}_u)\mathbf{1}_{\{D\backslash A_n=u\}}$.

Definition 1.2 *We now say that the local set A is thin if for any smooth bounded function f in D, the sequence of random variables $(\Gamma_A, f1_{D\backslash A_n})$ converges in probability to (Γ_A, f) as $n \to \infty$.*

The intuition behind this definition is that the limit of this sequence of random variables should be thought of as a way to make sense of $(\Gamma_A, f1_{D\backslash A})$, which then has to be the same as (Γ_A, f).

We leave it as an exercise to check that in the particular case of local sets where h_A is integrable, then this definition indeed coincides with the one given in the introduction. To do this, first one has to check that for all possible values of u of $D\backslash A_n$, $(\Gamma_A, f\mathbf{1}_u)$ is a.s. equal to $\int_u h_A(z)f(z)dz$, we will come back to this in Section 1.5

Finally, let us note already that the choice of working with dyadic approximations is somewhat arbitrary and the question whether changing this choice would change the definition is in fact not an easy one. Even if the examples that we will describe in the next section are clearly tailor-made for the particular definition, it is easy to adapt it to any other analogous choice. We will comment further on this in Section 1.5.

1.3 Examples of "small" non-thin local sets.

In the present section, we prove the statements $(2)_d$: We construct and describe the main features of a particular local set of the d-dimensional GFF in $d \geq 2$, which is not thin, yet rather small.

1.3.1 An example using CLE_4 in two dimensions

Before we construct our actual examples, let us first very quickly describe how it is possible to use the coupling of the two-dimensional GFF with the Conformal Loop Ensembles CLE_4 to construct a local set which implies the statement $(2)_d$ when $d = 2$. Due to the fact that such a relationship is only known in dimension 2, this construction can not be generalized to higher dimensions, but it will nevertheless help understanding some features of the example that we will provide in the next subsections. Since this CLE_4-based construction is not used in the main proofs of this Chapter, we review it in Section 2.4.

Let Γ be a GFF in a simply connected domain D. In Section 2.4, we will see that it is possible to define deterministically from Γ a local set A_1 of Minkowski dimension 15/8 such that the harmonic function h_{A_1} (that we denote by h_1) is constant and equal to $\pm 2\lambda$ in each connected component of $D \setminus A_1$, where here and throughout this section, 2λ is equal to the so-called height-gap $\sqrt{\pi/2}$ of the two-dimensional GFF. This set A_1 has the law of a CLE_4, and the coupling just described is usually called the natural coupling of CLE_4 with the GFF.

Furthermore this local set is thin (in the present case, the definition of thin is the one given in the introduction because h_1 is integrable) and conditionally on A_1, the sign of h_1 is chosen to be + or − independently in each connected component of $D \setminus A_1$.

We then define inductively, an increasing family A_n of local sets as follows: Suppose that for a given n, we have defined A_n in such a way that h_n is constant in each connected component of $D \setminus A_n$ and is equal to $2k\lambda$ for some integer $k \leq 1$. We then define A_{n+1} and h_{n+1} as follows:

- In the connected components of $D \setminus A_n$ where $h_n = 2\lambda$ we do nothing: these connected components are still in $D \setminus A_{n+1}$ and $h_{n+1} = 2\lambda$ there.
- In the other connected components, O, of $D \setminus A_n$, we construct, in O, the CLE_4 associated to the GFF Γ^{A_n} restricted to O. The connected components of $D \setminus A_{n+1} \cap O$ are defined to be the complement of this CLE_4, and the values of the harmonic function are $h_{n+1} = h_n \pm 2\lambda$.

We finally define our local set A to be the closure of $\cup_n A_n$.

It is then easy that A is a local set, that h_A is equal to 2λ in each of the connected components of the complement of A. It is also easy to see that the Lebesgue measure of A is almost surely equal to 0, and we leave it as a simple exercise to the reader who has read Section 2.6 to check that in fact, it satisfies $(2)_2$.

Since $h_A = 2\lambda$, the set A can not be thin. Indeed, for any smooth non-negative test function f, the integral $\int_{D\setminus A} h_A(z)f(z)dz$ would be almost surely non-negative, and it can therefore not be the same random variable as $(\Gamma, f) - (\Gamma^A, f)$ (unless $f = 0$).

1.3.2 Another example in two dimensions

The previous example relies on the CLE technology which is not available for the GFF in higher dimensions. In the present subsection we first describe another local set A of the two-dimensional GFF that has a simple generalization when $d \geq 3$. One main feature is reminiscent of the previous case: We discover the GFF in a self-similar fashion (but we use the boundary of dyadic squares instead of nested CLE_4), and explore the GFF until its mean value in the dyadic square that we are currently looking at is likely to be positive, in some sense that we will make precise

Notation. Choose the domain D to be the unit square $(0,1)^2$. As we are going to use nested dyadic squares, it is useful to use the following notation. We define $S^\emptyset$ to be equal to D, and when u is a finite sequence of n elements of $\{1,\dots,4\}$, then $S^{u1},\dots,S^{u4}$ are the four open dyadic subsquares of side-length 2^{-n-1} of S^u, where each one is a dyadic square of side-length 2^{-n}.We can for instance choose to associate the four indices respectively to the NW, NE, SW, SE subsquares. Thanks to this notation we can associate to each square a point in the tree $\{1,2,3,4\}^*$, and a genealogy.

Let us also define for each dyadic square S^u, the random variable $\gamma_n(S^u) := (\Gamma_{T_n}, 1_{S^u})$, where T_n is the union of elements in $\mathcal{T}_n$. This is the conditional expectation of $(\Gamma, 1_{S^u})$ in S^u, when one observes the GFF outside (i.e. on the boundary) of the ancestor of S^u with height n if $n \leqslant |u|$(the height of u), or the boundary of the childs of u with height n if $n > |u|$. It can also be viewed as (Γ, μ^u) where μ^u_n is a well-chosen measure supported on the boundary of the squares associated with S^u with height n.

We are going to discover progressively and simultaneously the GFF along the four segments from $(1/2, 0)$, $(1, 1/2)$, $(1/2, 1)$ and $(0, 1/2)$ to the middle point $(1/2, 1/2)$ (see the first image of figure Figure 1.1). When we have finished, then the unit square is divided into the four squares $S^1, \ldots, S^4$ of side-length 1/2. During this discovery, we can choose a modification of the conditional expectation of the random variable $(\Gamma, 1)$ (which is the mean value of Γ on $S^\emptyset$ given the discovered values of the GFF in the four segments we have discovered) so that it evolves like a continuous martingale. Thus, we can parametrize time in such a way that at time t, we have discovered four segments of length $l(t)$ so that this conditional expectation has the law of a Brownian motion $B = B^\emptyset$ at time t.

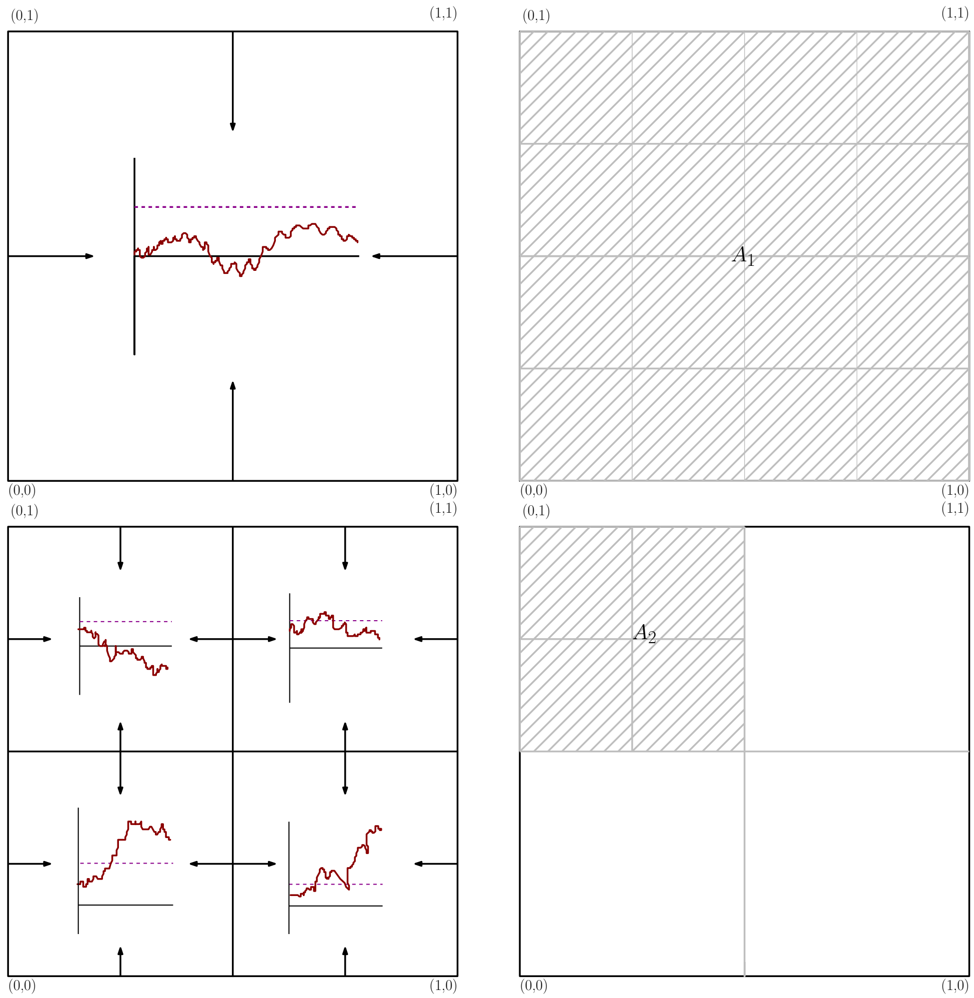


Figure 1.1: First two steps in the construction of A. In the left pictures we represent the Brownian motion associated to each point. In the right figure, the grey areas represents A_n.

Definition of A. If B hits 1 before time T, we define A^1 to be equal to the union of these four segments at the end-time T of this exploration, so that $U_1 := S^\emptyset \setminus A^1 = S^1 \cup \ldots \cup S^4$. If not we take $A^1 = D$. Note that $\mathbb{E}[(\Gamma, 1)| \sup_{t \leq T} B \geq 1] = 1$.

If the Brownian motion has not reached 1 before time T, we continue exploring, and we do this independently and simultaneously in all four squares $S^1, \ldots, S^4$ using the GFF Γ^{A^1} in each of them (note that Γ^{A^1} consists of four independent GFFs in the four squares). In each of these squares, we grow four boundary segments towards the center of the square, and we study the conditional expectation of $4(\Gamma^{A^1}, \mathbf{1}_{S^j})$ (the mean of the mass of Γ^{A^1} in S^j) given what one has discovered. By self-similarity, each of these four quantities evolve like four independent Brownian motions $B^1, \dots, B^4$ up to time T.

Now, in order to define A^2, if $A^1 \neq D$, then $A^2 = A^1$, if not we look, for each S^i, at whether the BM $W^i := B(t \wedge T) + B^i(T-t)\mathbf{1}_{\{t \geqslant 0\}}$ hits level 1 before time $2T$ or not. A^2 is made by the closed union of all the squares of size 2^{-1} where this BM did not hit the level 1 before time $2T$, with the boundaries of all the squares of the same size where this event happen (see Figure 1.1). In other words, for each $n \geq 1$:

- The sets A^n and ∂A^n are local sets made out of the union of 2^{-n} dyadic segments with elements of $\mathcal{S}_n$, and A^n is such that $(A^n)_n = A^n$. We say that a square $s \in \mathcal{S}_n$ is still active (meaning that we will continue exploring inside it) when $s \in A^n$. Furthermore, active squares also come equipped with a Brownian motion W^s stopped at time Tn. We call K_n the set of active squares in $\mathcal{S}_n$ and V_n the set of connected components of $D \backslash A^n$,i.e., the inactive components. Note that $V_n \subseteq \bigcup_{k=1}^n \mathcal{S}_n$.
- In order to construct A^{n+1} and to continue W, we proceed as follows: The components that were not active at step n remain inactive For $s \in K_n$ continuously grow the middle lines as done in the first step and define for $0 \leqslant t \leqslant n(T+1)$ and s^+ any direct descendent of s, $W^{s^+}(t) := W^s(t \wedge nt) + B^s(t - nT)\mathbf{1}_{\{t \geqslant nT\}}$, where B^s is the BM associated to the change of the conditional expectation of $2^n(\Gamma^{A^n}, 1_s)$ given the increasing procedure in s. We keep active those squares s^+ where its associated BM did not hit 1 before time $(n+1)T$, and we make s^+ inactive (i.e. $s^+ \in V_m$ for $m \geqslant n+1$) if its associated BM hit 1 before time $(n+1)T$. We define A^{n+1} as the closed union of all the active squares at time $(n+1)$ with the boundary of the inactive squares. We can also see it as A^n minus the squares s^+ that became inactive in this step.

Note that A^n is non-increasing, and that the family V_n is non-decreasing. We define A to be the intersection of all A^n. The complement of A is then just the union of the squares that stop being active at some point, more precisely, $D \backslash A$ is the disjoint union of the squares in $\cup_n V_n$. Thus, we have that $A_n = A^n$. Note that for a given dyadic square s, on the even that $s \in V_n$, the harmonic function h_A coincides with the harmonic function $h_{D \backslash T_n}$ on s (where T_n the union of all boundaries of 2^{-n}-dyadic squares) and that $(\Gamma_A, 1_s) = \gamma_n(s)$.

The set A is not large. The construction shows immeditaly that the probability that a given dyadic square s of side-length 2^{-n} is still active at step n is equal to the probability that a one-dimensional Brownian motion did not hit 1 before time $n \times T$, which decays like a constant times $1/\sqrt{n}$ as $n \to \infty$. From this, it follows readily that the size of A is indeed of the type required for $(2)_2$, i.e.:

Proposition 1.3 *The expected value of the area of the ε-neighborhood of A decays almost surely like $O(|\log \varepsilon|^{-1/2})$.*

Indeed, if $N_n = N_n(A)$ denotes the number of closed 2^{-n} dyadic squares that intersect A, then

$$\mathbb{E}[N_n] = \sum_{s \in \mathcal{S}_n} \mathbb{E}\left[\mathbf{1}_{\{s \subseteq A_n\}}\right] + C \sum_{j=1}^{n-1} \sum_{s \in \mathcal{S}_j} 2^{n-j} \mathbb{E}\left[\mathbf{1}_{\{s \subseteq A_j \backslash A_{j+1}\}}\right]$$
$$\leqslant 4^n \mathbb{P}(\text{BM does not hit one before } Tn) + C 2^n \sum_{j=1}^{n-1} j^{-3/2} 2^j \qquad \leqslant C \frac{4^n}{\sqrt{n}}$$

(mind that in N_n, we have to also count the squares that intersect the boundaries of squares that have stopped being active, which explains the sum in j).

A first moment estimate. Note that in order to define the set A, we have in fact associated a Branching Brownian motion (BBM) W to each GFF, where each BM splits into 4 independently evolving BM at each time which is a multiple of T. However, it should be emphasized that for a given dyadic square s of side-length 2^{-n}, the value of the corresponding Brownian motion at time nT is not equal to the expected mean height of the GFF in s given the exploration up to the n-th generation. Indeed, this mean height has clearly a higher value when s is towards the centre of S than when it is near its boundary, which is not mirrored by the Branching Brownian motion description. However, a key observation is that this difference is averaged out when one sums over all squares. For instance, it is easy to check by induction on n that

$$\sum_{s\in\mathcal{S}_n} \gamma_n(s) = \sum_{s\in\mathcal{S}_n} 4^{-n} W^s(nT),$$

if B^s denotes the Brownian motion that is following the branch of the BBM corresponding to s.

The variant of this result that will be useful for us is:

Lemma 1.4

$$\mathbb{E}\left[(\Gamma_A, \mathbf{1}_{D\backslash A_n})\right] = \mathbb{E}\left[\sum_{s\in V_n} Area(s)\right].$$

The left-hand side is equal to the probability that a Brownian motion started from 0 does hit 1 before time nT, which converges to 1. This shows already that $(\Gamma_A, \mathbf{1}_{D\backslash A_n})$ can not converge in L^1 to $(\Gamma, 1)$, which is a symmetric random variable with mean 0.

Proof. Note that $D\backslash A_n = \bigcup_{s\in V_n} s$ and that at time n, $\Gamma_{\partial A_n} = \Gamma_A$ in all elements of V_n and $\Gamma_{\partial A_n} = \Gamma_{T_n}$ in all of those in K_n. This implies that $\mathbb{E}\left[(\Gamma_A, \mathbf{1}_{D\backslash A_n})\right] = -\mathbb{E}\left[\sum_{s\in K_n} \gamma_n(s)\right]$. Then, it is enough to prove that

$$\mathbb{E}\left[\sum_{s\in K_n} \gamma_n(s)\right] = \mathbb{E}\left[\sum_{s\in K_n} W^s\right] = -\mathbb{E}\left[\sum_{s\in V_n} Area(s)\right].$$

The second equality just follows from the stopping time theorem. For the first equality we have to work harder. Let us note that for all $s' \in \mathcal{S}_m$ and $s \in \mathcal{S}_n$ with ancestor s', $W^s((m+1)T) - W^s(mT)$ is equal to $4^m(\gamma_{m+1}(s') - \gamma_m(s'))$ and that $\mathbb{E}\left[\mathbf{1}_{\{s\in K_n\}} \mid \Gamma_{T_{m+1}}\right]$ does not depend on s. Now, let us show that the increment of the harmonic function for $s \in K_n$ at level m can be computed using the Brownian motion,

$$\begin{aligned}
\sum_{s\in\mathcal{S}^n : s'\leqslant s} \mathbb{E}\left[(\gamma_{m+1} - \gamma_m)(s)\mathbf{1}_{\{s\in K_n\}}\right] &= \sum_{s\in\mathcal{S}^n : s'\leqslant s} \mathbb{E}\left[(\gamma_{m+1} - \gamma_m)(s)\mathbb{E}\left[\mathbf{1}_{\{s\in K_n\}} \mid \Gamma_{T_{m+1}}\right]\right] \\
&= \sum_{s\in\mathcal{S}^n : s'\leqslant s} 4^{m-n}\mathbb{E}\left[(\gamma_{m+1} - \gamma_m)(s')\mathbb{E}\left[\mathbf{1}_{\{s\in K_n\}} \mid \Gamma_{T_{m+1}}\right]\right] \\
&= 4^{-n} \sum_{s\in\mathcal{S}^n : s'\leqslant s} \mathbb{E}\left[(W^s((m+1)T) - W^s(mT))\mathbf{1}_{\{s\in K_n\}}\right].
\end{aligned}$$

We conclude by writing a $\sum_{s\in K_n}\gamma_n(s)$ as a telescopic sum. □

This set A is not thin. Our goal is now to derive the following fact, which combined with Proposition 1.3 proves the statement $(2)_2$:

Proposition 1.5 *This local set A is not thin.*

This is a direct consequence of the following claim:

Claim 1.6 *The sequence of random variables $(\Gamma_A, \mathbf{1}_{D\setminus A_n})$ is bounded in L^2.*

Indeed, if $(\Gamma_A, \mathbf{1}_{D\setminus A_n})$ would converge in probability towards $(\Gamma, 1)$, then it would converge also in L^1, and we have seen in the previous paragraph that this can not be the case.

Deriving Claim 1.6 requires some care. We have to bound covariances of the increments of the integral of the harmonic function in two squares, s and s', at each step of the process. In order to do that, we separate the increments according to whether or not they come from the conditional expected value of T_m with m bigger or equal, p, the height of $s\wedge s'$, the last common ancestor of s and s'. We realize that if we condition according to the values of the GFF in T_p there are many terms that become constant and allow us to go the increments of level p, instead of n.

In our proof, we use the following basic bound for centred Gaussian random variables X, Y: For any event E with positive probability,

$$\mathbb{E}\left[XY\mathbf{1}_E\right] \leqslant C\max\{\mathrm{Var}(X), \mathrm{Var}(Y)\}\mathbb{P}(E)\log(1/\mathbb{P}(E)) \tag{1.2}$$

(to prove it, note first that due to the fact that $2ab \leqslant a^2+b^2$, we can restrict ourselves to the case where $X = Y$, and by scaling it suffices to consider the case where X is a standard normal variable. The quantity $E[X^2 1_A]$ is maximal among all sets A with $P(A) = a$ for the set $A = \{X^2 > x\}$ where $P(X^2 > x) = a$, and the estimate then follows).

Proof of the claim. As in the beginning of Lemma 1.4, let us remember that $(\Gamma_A, \mathbf{1}_{D\setminus A_n}) = (\Gamma_A, 1) + \sum_{s\in K_n}\gamma_n(s)$. Given that $\mathrm{Var}(\Gamma_A, 1) \leqslant \mathrm{Var}(\Gamma, 1)$ it is just enough to bound

$$\mathbb{E}\left[\sum_{s,s'\in K_n}\gamma_n(s)\gamma(s')\right].$$

We will do this by writing $\gamma_n(s)$ and $\gamma_n(s')$ as the sum of the increments at each iteration step. Things are a little bit messier than for the first moment, because one has more terms to evaluate. For $s, s' \in \mathcal{S}_n$, we will have to consider the common ancestor $w = s\wedge s'$. In the following lines, we first fix $p \geq 2$ and w a 2^{-p}-daydic square.

For any $m, o \geqslant p$ conditionally on Γ_{T_p}, $(\gamma_{m+1}-\gamma_m)(s)\mathbf{1}_{\{s\in K_n\}}$ and $(\gamma_{o+1}-\gamma_o)(s')\mathbf{1}_{\{s\in K_n\}}$

are independent. Hence,

$$\begin{aligned}
&\sum_{p\leqslant m,o<n}\sum_{\substack{s,s'\in\mathcal{S}^n\\ s\wedge s'=w}}\mathbb{E}\left[(\gamma_{m+1}-\gamma_m)(s)(\gamma_{o+1}-\gamma_o)(s')\mathbf{1}_{\{s,s'\in K_n\}}\right]\\
=&\sum_{p\leqslant m,o<n}\sum_{\substack{s,s'\in\mathcal{S}^n\\ s\wedge s'=w}}\mathbb{E}\left[\mathbb{E}\left[(\gamma_{m+1}-\gamma_m)(s)\mathbf{1}_{\{s\in K_n\}}\mid\Gamma_{T_p}\right]\mathbb{E}\left[(\gamma_{o+1}-\gamma_o)(s')\mathbf{1}_{\{s'\in K_n\}}\mid\Gamma_{T_p}\right]\right]\\
=&\sum_{\substack{s,s'\in\mathcal{S}^n\\ s\wedge s'=w}}8^{-n}\mathbb{E}\left[\mathbb{E}\left[(W^s(nT)-W^s(pT))\mathbf{1}_{\{s\in K_n\}}\mid\Gamma_{T_p}\right]\mathbb{E}\left[(W^{s'}(nt)-W^{s'}(pt))\mathbf{1}_{\{s'\in K_n\}}\mid\Gamma_{T_p}\right]\right]\\
\leqslant&\sum_{\substack{s,s'\in\mathcal{S}^n\\ s\wedge s'=w}}8^{-n}\mathbb{E}\left[(W^s(pT)+1)^2\mathbf{1}_{\{w\in K_p\}}\right]\leqslant C8^{-p}\sqrt{p},
\end{aligned}$$

where for the third equality we used the same technique as in Lemma 1.4 and for the fourth and fifth we just use stopping time theorem for the BM B and for B_t^2-t.

It is also true that $\mathbb{P}(u\in K_n\mid T_p)$ is constant for all u with ancestor w and that conditionally on Γ_{T_p}, $\{s\in K_n\}$ is independent of $\{s'\in K_n\}$. This allows us to compute the following second term

$$\begin{aligned}
&\sum_{0\leqslant m,o<p}\sum_{\substack{s,s'\in\mathcal{S}^n\\ s\wedge s=w}}\mathbb{E}\left[(\gamma_{m+1}-\gamma_m)(s)(\gamma_{o+1}-\gamma_o)(s')\mathbf{1}_{\{s,s'\in K_n\}}\right]\\
=&\sum_{0\leqslant m,o<p}\sum_{\substack{s,s'\in\mathcal{S}^n\\ s\wedge s=w}}\mathbb{E}\left[(\gamma_{m+1}-\gamma_m)(s)\mathbb{P}\left[\mathbf{1}_{\{s\in K_n\}}\mid\Gamma_{T_p}\right](\gamma_{o+1}-\gamma_o)(s')\mathbb{P}\left[\mathbf{1}_{\{s'\in K_n\}}\mid\Gamma_{T_p}\right]\right]\\
=&\mathbb{E}\left[\gamma_p(w)^2\mathbf{1}_{\{w\in K_p\}}\right]\leqslant C8^{-p}\sqrt{p}\log(p),
\end{aligned}$$

where in the last step we have used (1.2) and the fact that the variance of $\gamma_p(w)$ is bounded by that of $(\Gamma,\mathbf{1}_w)$.

For the remaining term we need to bound the cross -product and using similar remarks as before we have that

$$\begin{aligned}
&\sum_{0\leqslant m<p\leqslant o<n}\sum_{\substack{s,s'\in\mathcal{S}^n\\ s\wedge s=w}}\mathbb{E}\left[(\gamma_{m+1}-\gamma_m)(s)(\gamma_{o+1}-\gamma_o)(s')\mathbf{1}_{\{s,s'\in K_n\}}\right]\\
=&\sum_{0\leqslant m<p\leqslant o<n}\sum_{\substack{s,s'\in\mathcal{S}^n\\ s\wedge s=w}}\mathbb{E}\left[(\gamma_{m+1}-\gamma_m)(s)\mathbb{P}\left[\mathbf{1}_{\{s\in K_n\}}\mid\Gamma_{T_p}\right]\mathbb{E}\left[(\gamma_{o+1}-\gamma_o)(s')\mathbf{1}_{\{s'\in K_n\}}\mid\Gamma_{T_p}\right]\right]\\
=&-\mathbb{E}\left[\gamma_p(w)(-W^w(pT)+1)c(W^w(pT),n-p)\mathbf{1}_{\{w\in K_p\}}\right]\leqslant C8^{-p}\sqrt{p}\log(p),
\end{aligned}$$

where $c(x,m)$ is the probability than a BM hits height $x+1$ before time mT.

Summing all the previous terms up, we get that

$$\mathbb{E}\left[\sum_{s,s'\in K_n}\gamma_n(s)\gamma_n(s')\right]\leqslant C'+C\sum_{p=2}^{\infty}4^{-p}\sqrt{p}\log(p)<\infty.$$

□

1.3.3 The example in higher dimensions

We now explain how to adapt the previous example to the higher-dimensional setting. The only slight difference is that in the two-dimensional case, we used the scale invariance of the GFF, while we will now use the scaling relation (1.1).

To adapt our example, let us define $D = S^{\emptyset} := (0,1)^d$. We use the d-dimensional dyadic hypercubes denoted now by S^u where u are finite sequences in $\{1,\dots,2^d\}$. When Γ is a GFF in D, we are now going to discover its values on all simultaneously growing all the $(d-1)$-dimensional mid-hyperplanes. Then, the iterative construction proceeds in almost exactly the same way, but with a notable difference. Due to the different scaling behaviour of the GFF, if the evolution of the conditional mean height during the first iteration evolves like a Brownian motion up to some time T, then the evolution during the second iteration is that of a Brownian motion during time $T \times 2^{d-2}$, and so on. In other words, the intervals between the branching times of the branching Brownian motion will grow exponentially, and the n-th branching time will be $T_n = T(2^{(d-2)n} - 1)/(2^{d-2} - 1)$ instead of nT.

Other than that, nothing in the previous discussion changes. Lemma 1.4 together with Claim 1.6 become readily:

Lemma 1.7 *For this A we have that* $\mathbb{E}\left[(\Gamma_A, \mathbf{1}_{D\setminus A_n})\right] = \mathbb{E}\left[\sum_{s\in V_n} \text{Volume}(s)\right]$ *and the second moment of* $(\Gamma_A, \mathbf{1}_{D\setminus A_n})$ *is uniformly bounded.*

Just as in the 2-dimensional case, this then implies that A is not-thin.

To upper bound the Minkowski dimension, the only difference is that the probability that a given dyadic hypercube of side-length 2^{-n} is active at the n-th iteration is now the probability that a Brownian motion does not hit level 1 before time T_n, which leads to the estimate on the size of A as in $(2)_d$. Indeed, if N_n denotes the number of closed dyadic hypercubes that intersect A,

$$\begin{aligned}\mathbb{E}\left[N_n\right] &\leqslant C \sum_{j=1}^{n} \sum_{s\in \mathcal{S}^j} 2^{(n-j)(d-1)} \mathbb{E}\left[\mathbf{1}_{\{s\subseteq A_j\}}\right] = C 2^{n(d-1)} \sum_{j=1}^{n} 2^j \mathbb{P}\left(\text{BM hits 1 after time } T_n\right) \\ &\leqslant C 2^{n(d-1)} \sum_{j=1}^{n} 2^{(-d/2+2)j} \leqslant C 2^{\max\{d-1, d/2+1\}n}.\end{aligned}$$

Thus, thanks the Markov inequality

$$\mathbb{P}\left[N_n \geqslant 2^{(\max\{d-1,d/2+1\}+\varepsilon)n}\right] \leqslant C 2^{-\varepsilon n},$$

and thanks to the Borel-Cantelli Lemma, we can conclude that the upper Minkowski dimension of A is almost surely bounded by $\max\{d-1, d/2+1\}$.

We conclude that $(2)_d$ holds for any $d \geq 3$.

Proposition 1.8 ($(2)_d$) *This local set A is not thin, and its upper Minkowski dimension is almost surely not larger than* $\max\{d-1, d/2+1\}$.

1.4 Small sets are thin (proof of $(1)_d$)

Let us briefly note that the definition of thin sets can be extended to non-local sets: we say that a set A is thin if for all f smooth bounded function D we have that $(\Gamma, f\mathbf{1}_{A_n}) \overset{\mathbb{P}}{\to} 0$ as $n \to \infty$. This definition is useful because a.s.

$$\sum_{s \in \mathcal{S}_n : s \not\subseteq D \setminus A_n} (\Gamma, f\mathbf{1}_s) = (\Gamma, f\mathbf{1}_{A_n}),$$

so that it is sufficient to bound the value of the GFF in hyper-cubes of size 2^{-n}.

The following proposition links both definitions.

Lemma 1.9 *Let Γ be a GFF on D and A a local set. A is thin in this last sense if only if A is a thin local set.*

Proof. It is enough to see that for all f smooth and bounded function :

$$(\Gamma, f\mathbf{1}_{A_n}) - ((\Gamma_A, f) - (\Gamma_A, f\mathbf{1}_{D\setminus A_n})) = (\Gamma^A, f\mathbf{1}_{A_n}) \overset{\mathbb{P}}{\to} 0 \qquad \text{as } n \to \infty,$$

□

This shows for instance that any deterministic closed set A with zero Lebesgue measure is a thin local set. Indeed, if $\|f\|_\infty < 1$, by dominated convergence,

$$\mathbb{E}\left[(\Gamma, f\mathbf{1}_{A_n})^2\right] = \iint\limits_{A_n \times A_n} f(x) G_D(x,y) f(y) dy dx \to 0$$

as $n \to \infty$.

1.4.1 The case $d \geq 3$

The idea of the proof will be just to get uniform bounds on the mean values of Γ on elements of $\mathcal{S}_n$ by second moment estimates and Borel-Cantelli arguments, and to then use Lemma 1.9 to conclude that if our sets are small enough then they are thin.

Let us recall that there exists an absolute constant C_d such that for any $s \in \mathcal{S}_n$ and any bounded function f,

$$\iint\limits_{s \times s} f(x) G(x,y) f(y) dx dy \leqslant C_d 2^{-(d+2)n}, \tag{1.3}$$

which readily implies the following:

Lemma 1.10 *Let $d \geqslant 3$, $D \subseteq \mathbb{R}^d$ be an open set and Γ a GFF in D. For any $\beta < d/2 + 1$ and any smooth bounded function f, almost surely,*

$$\lim_{n \to \infty} \sup_{s \in \mathcal{S}_n} |(\Gamma, f\mathbf{1}_s)| 2^{\beta n} = 0.$$

PROOF. For $n \in \mathbb{N}$ and $s \in \mathcal{S}_n$, $(\Gamma, f\mathbf{1}_s)$ is a centered Gaussian random variable with variance $\iint_{s\times s} f(x)G_D(x,y)f(y)dxdy$, so that by (1.3),

$$\sum_{n\in\mathbb{N}}\sum_{s\in\mathcal{S}_n}\mathbb{P}(|(\Gamma, f\mathbf{1}_s)| \geqslant \varepsilon 2^{-n\beta}) < \infty,$$

and we can conclude using the Borel-Cantelli Lemma. □

This lemma now enables to prove the statement $(1)_d$ for $d \geq 3$:

Proposition 1.11 ($(1)_d$) *Let $D \subseteq \mathbb{R}^d$ be an open set, Γ a GFF in D and A a local set of Γ. If the upper Minkowski dimension of A is almost surely strictly smaller than* $\max\{d-2, d/2+1\}$, *then A is a thin local set.*

PROOF. Let us first note that if the upper Minkowski dimension $\delta(A)$ of A is strictly smaller than $d-2$, then A is polar, so that Lemma 1.1 implies that $\Gamma_A = 0$, and that A is thin local set.

Let us now assume instead that $\delta(A) < d/2+1$ almost surely. The following argument will in fact not use the fact that A is a local set: Let us fix $f \in C_0^\infty(\mathbb{R}^d)$ define the following events:

$$\tilde{\Omega}_\beta := \left\{N_n = o(2^{n(d/2+1-\beta)}) \text{ as } n \to \infty\right\}.$$

Since $\delta(A) < d/2+1$, we know that $\cup_{\beta>0}\Omega$ holds almost surely. On the other hand, for each given β, if $\tilde{\Omega}_\beta$ holds, then by Lemma 1.10, we see that $(\Gamma, f\mathbf{1}_{A_n})$ tends to 0. Hence, we conclude that this convergence holds in fact almost surely and by Lemma 1.9 we conclude that A is thin. □

Note that with this proposition and its proof we can get some other basic properties of thin sets.

Corollary 1.12 *Let $D \subseteq \mathbb{R}^d$ be an open set, Γ a GFF on D and A, B thin local sets. If the upper Minkowski dimension of A is strictly smaller than $d/2+1$, then:*

1. *$A \cup B$ is also a thin set.*
2. *If A is a local set such that h_A is integrable (i.e., such that $\int_{D\backslash A}|h_A| < \infty$) and B has zero Lebesgue measure, then a.s. $B\backslash A$ is thin for $\Gamma^A := \Gamma - \Gamma_A$.*

PROOF. 1. Note that for any bounded smooth function f:

$$|(\Gamma, \mathbf{1}_{(A\cup B)_n})| \leqslant |(\Gamma, f\mathbf{1}_{B_n})| + |(\Gamma, f\mathbf{1}_{A_n\backslash B_n)})| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty,$$

where the second term goes to 0 because it can be written as a sum over elements of $\mathcal{S}_n$ and the amount of terms in that sum is smaller than the cardinal $\{s \in \mathcal{S}_n : s \subseteq A_n\}$, so the same argument used in the last proof to show that A is thin can be applied.

2. Let f be a bounded function and note that the fact that because h_A is integrable and B has 0 measure $\int_{D\backslash A} h_A(x)\mathbf{1}_{(B\backslash A)_n}f(x)dx$ goes to 0. Additionally $(\Gamma, f\mathbf{1}_{(B\backslash A)_n})$ because of the same reason as in the proof of the last fact.

□

In future work we plan to prove that when the upper Minkowski dimension of A is smaller than $d/2+1$, then h_A integrable on $D\backslash A$, which will allow to relax a little bit the conditions in this last corollary, see [AS17a].

Note that this does not answer the question whether the fact that B is thin implies that its Lebesgue measure is 0. Remark that such statements are non-trivial, due for instance to the fact that we cannot exclude at this point, the very unlikely fact that there exist thin local sets, with non-thin subsets.

1.4.2 The case $d = 2$

A useful fact in two dimensions is that in order to prove that a set is thin, it is not necessary to test against all smooth bounded functions:

Lemma 1.13 *Let A be a local set of a GFF Γ in a bounded domain $D \subseteq \mathbb{R}^2$. Then A is thin as soon as $\sum_{s\in\mathcal{S}_n: s\not\subseteq D\backslash A_n} |(\Gamma, 1_s)| \to 0$ in probability as $n\to\infty$.*

Proof. Let $s\in\mathcal{S}_n$ and $f\in C_0^\infty(\mathbb{R}^2)$. Define $\bar{f}_s := \int_s f(x)dx/\text{Leb}(s)$ as the mean of f in s. We have that

$$\mathbb{E}\left[\left(\bar{f}_s(\Gamma, \mathbf{1}_s) - (\Gamma, \mathbf{1}_s f)\right)\right] \leqslant C\|f\|_{C_0^1} n 2^{-6n}.$$

Using Borel Cantelli we have that the $\sup_{s\in\mathcal{S}_n}(\Gamma, 1_s(f-\bar{f}_s))$ is $O(n^2 2^{-3n})$. Due to the fact that the cardinal of $\mathcal{S}_n$ is of the order 4^n,

$$\sup_{B\subseteq\Gamma(D)}\left|\sum_{s\in\mathcal{S}_n: s\not\subseteq D\backslash A_n} \bar{f}_s(\Gamma, \mathbf{1}_s) - (\Gamma, \mathbf{1}_s f)\right| \leqslant \sum_{s\in\mathcal{S}_n: s\not\subseteq D\backslash A_n} |(\Gamma, \mathbf{1}_s(f-\bar{f}_s))| \overset{a.s.}{\to} 0 \quad \text{as } n\to\infty.$$

Thus, it is enough to show the convergence to 0 of $|\sum_{s\in\mathcal{S}_n: s\not\subseteq D\backslash A_n} \bar{f}_s(\Gamma, \mathbf{1}_s)|$, which clearly follows from the hypothesis. □

Now, we prove $(1)_2$, i.e., the following statement:

Proposition 1.14 $((1)_2)$ *Let $D\subseteq\mathbb{C}$ be a bounded open set, Γ a GFF on D and A random closed set. If there exists $\delta>0$ such that*

$$\mathbb{E}[N_n] = o(4^n/n^{1/2+\delta})$$

as $n\to\infty$, then A is a thin set.

Proof. Thanks to the scaling properties it is enough to work with a domain $D\subseteq B(0,1/4)$. Take Γ a GFF in D and $s\in\mathcal{S}_n$. Let us define B_s the event where $|(\Gamma, \mathbf{1}_s)|$ is bigger than the quantity

$$m_n := 4^{-n}\sqrt{n}\sqrt{\log n + 2\log\log n}\sqrt{\log 2}/\sqrt{2\pi},$$

and R_n the number of squares where this inequality holds. We also define

$$S_n = \sum_{s\in\mathcal{S}_n} |(\Gamma, \mathbf{1}_s)|\mathbf{1}_{B_s}.$$

Note S_n converges to 0 in L^1. Indeed, $\mathrm{Var}(\Gamma, \mathbf{1}_s) \leqslant 4^{-n}\sqrt{n}\sqrt{\log(2)}/\sqrt{2\pi}$ and:

$$\mathbb{E}\left[S_n\right] = \sum_{s \in \mathcal{S}_n} \mathbb{E}\left[|(\Gamma, \mathbf{1}_s)| \mathbf{1}_{B_s}\right] \leqslant C\sqrt{n} \exp\left(-\frac{\log(n) + 2\log\log(n)}{2}\right) \to 0 \quad \text{as } n \to \infty.$$

Now, let us note that if $|\{s \in \mathcal{S}_n : s \not\subseteq D \backslash A_n\}| \leqslant R_n$, then $\sum_{s \in \mathcal{S}_n : s \not\subseteq D\backslash A_n} |(\Gamma, \mathbf{1}_s)| \leqslant S_n$. Thus, we have that for all $\eta > 0$

$$\mathbb{P}\left(\sum_{s \in \mathcal{S}_n,\ s \not\subseteq D\backslash A_n} |(\Gamma, \mathbf{1}_s)| \geqslant \eta\right)$$
$$\leqslant \mathbb{P}\left(|\{s \in \mathcal{S}_n : s \not\subseteq D\backslash A_n\}| \geqslant \frac{4^n}{n^{1/2+\delta}}\right) + \mathbb{P}(S_n \geqslant \eta) + \mathbb{P}\left(R_n \leqslant \frac{4^n}{n^{1/2+\delta}}\right).$$

where the first term goes to 0 thanks to Markov inequality, and the second thanks to the fact that S_n converges to 0 in L^1. Thanks to Lemma 1.13, the only thing we need to finish the proof is to show that the last term converges to 0. In order to do that let us first prove that $\mathbb{E}(R_n) \geqslant C2^{2n}n^{-1/2-\delta/2}$ and $\mathrm{Var}(R_n) = O(2^{4n}n^{-4/3})$.

For $x, y \in D\backslash T_n$ take s_x the only element in $\mathcal{S}_n$ where it belongs and define

$$\alpha_x := \frac{m_n}{\sqrt{\mathrm{Var}((\Gamma, \mathbf{1}_{s_x}))}}$$
$$u_{x,y} := \frac{\mathrm{Cov}((\Gamma, \mathbf{1}_{s_x}), (\Gamma, \mathbf{1}_{s_y}))}{\sqrt{\mathrm{Var}((\Gamma, \mathbf{1}_{s_x}))}\sqrt{\mathrm{Var}((\Gamma, \mathbf{1}_{s_y}))}}.$$

We note that for all $(x, y) \in E^{(n)} := \{(x, y) \in D^2 : \ \min(|x - y|, d(y, \partial D), d(x, \partial D)) \geqslant 1/n\}$

$$\alpha_x \geqslant (1 - C\log(n)/n) \times \sqrt{\log n + 2\log\log(n)},$$
$$u_{x,y} \leqslant C\frac{\log(1/|x - y|)}{n}.$$

Using the lower bound for α_x and basic Gaussian estimates we can compute the lower bound for the first moment

$$\mathbb{E}\left[R_n\right] \geqslant C4^n(\log(n))^{-1/2} \exp\left(-\frac{\log n + \log\log N}{2}\right) \geqslant C\frac{4^n}{n^{1/2+\delta/2}}$$

The variance estimate is more involved and we use the lower bound for the correlation. First let us note that

$$4^{-2n}\mathrm{Var}[R_n] = \iint_{D \times D} [\mathbb{P}(\bar{\Gamma}_x \geqslant \alpha_x, \bar{\Gamma}_y \geqslant \alpha_y) - \mathbb{P}(\bar{\Gamma}_x \geqslant \alpha_x)\mathbb{P}(\bar{\Gamma}_y \geqslant \alpha_y)]dxdy,$$

where $\bar{\Gamma}_x = (\Gamma, \mathbf{1}_{s_x})/\mathrm{Var}(\Gamma, \mathbf{1}_{s_x})$, and $(\bar{\Gamma}_x, \bar{\Gamma}_y)$ has the law of the centred Gaussian vector whose coordinates have variance 1 and the covariance between them is $u_{x,y} \geqslant 0$. Note the fact that $\mathbb{P}(\bar{\Gamma}_x \geqslant \alpha_x) = O(n^{-1/2})$ and that the volume of $D \times D\backslash E^{(n)}$ is $O(n^{-1})$ implies that we only need to bound the term for $x, y \in E^{(n)}$. In this case we just note that by definition of Gaussian vector the term inside the integral is equal to (we denote $u_{x,y} = u$)

$$\frac{1}{2\pi}\int_{\alpha_x}\int_{\alpha_y}^{\infty} e^{-\frac{1}{2}(a^2+b^2)}\left(\frac{1}{\sqrt{1-u^2}}\exp\left(-\frac{u^2a^2+u^2b^2-2uab}{2(1-u^2)}\right)-1\right)dbda$$
$$\leqslant \frac{1}{2\pi}\int_{\alpha_x}^{\infty}\int_{\alpha_y}^{\infty} e^{-\frac{1}{2}(a^2+b^2)}\left(\frac{1}{\sqrt{1-u^2}}-1\right)dadb - \frac{C\log(|x-y|)}{n}\int_{\alpha_x}^{\infty}\int_{\alpha_y}^{\infty} abe^{-(a^2+b^2-4uab)/2}dbda$$

where to get the inequality we have used that for $x \geqslant 0$, $\exp(x)-1 \leqslant x\exp(x)$. A Taylor expansion shows that the first term is an $O(n^{-2})$. Note that the second term and is smaller than a constant times $n^{-1}\log(1/|x-y|)\mathbb{E}\left[XY\mathbf{1}_{\{X\geqslant \alpha_x, Y\geqslant \alpha_y\}}\right]$, where (X,Y) is a Gaussian vector such that each term has variance $1/(1-4u^2)$; using that $XY \leqslant X^2+Y^2$ it is easy to see that this term is an $O(n^{-4/3})$. Integrating over $(x,y)\in E^{(n)}$ we get the desired bound on the variance.

To conclude we note that $\mathbb{E}[R_n] \geqslant 4^n/n^{1/2+\delta}$ and

$$\mathbb{P}\left(R_n \leqslant \frac{4^n}{n^{1/2+\delta}}\right) \leqslant \mathbb{P}\left((\mathbb{E}[R_n]-R_n])^2 \geqslant \left(\mathbb{E}[R_n]-\frac{4^n}{n^{1/2+\delta}}\right)^2\right) = \mathrm{Var}[R_n]\left(\mathbb{E}[R_n]-\frac{4^n}{n^{1/2+\delta}}\right)^{-2};$$

the estimates on the mean and the variance of R_n show that this term is an $O(n^{-1/3+\delta})$ which concludes the proof. □

1.5 Some comments about the definitions of thin local sets

Let us now make some somewhat abstract comments about the definition of local sets. One general strategy in order to define local sets is to use some deterministic "enlargements" of the random sets A (see for instance [Wer16]). To the best of our knowledge, only dyadic-type enlargements have been used in earlier work, but this is a rather arbitrary choice. For our purposes here, it seems natural to consider also other possible deterministic enlargements – indeed, this a priori choice could be important, given that some property may hold for one approximation scheme, and not for the other.

Let us describe one possible class of discrete approximation schemes (DAS), for which the proofs of the present chapter can be adapted rather directly.

DAS when $d \geqslant 3$. Define a pre-DAS for a domain $D \subseteq \mathbb{R}^d$ to be a sequence $(\mathcal{A}_n)_{n\geq 0}$ of families of closed sets $\mathcal{A}_n = (\mathcal{B}_n, \mathcal{C}_n)$ for which there exists some (large) constant $C \in \mathbb{R}$ such that the following holds for any $n \in \mathbb{N}$:

1. For any two distinct c and c' in $\mathcal{C}_n$, the Lebesgue measure of $c \cap c'$ is zero.
2. For any c in $\mathcal{C}_n$ the diameter of c is upper bounded by $C2^{-n}$ and its volume is lower bounded by $2^{-nd}/C$.
3. $\mathrm{Leb}(\bigcup_{b\in\mathcal{B}_n} b) = 0$. And for all $E \subseteq \mathbb{R}^d$ compact, the cardinal of the elements of $\mathcal{B}_n$ that intersect E is finite

For a fixed pre-DAS $\mathcal{A}_n$, take $\mathcal{B}^n := \bigcup_{b\in\mathcal{B}_n} b$, the set of all points covered by elements of $\mathcal{B}$. For all closed set $A \subseteq \bar{D}$, define $\mathcal{A}\{A\}_n$ as the set of all elements of $\mathcal{C}_n$ that have a non empty intersection with $A\backslash\mathcal{B}^n$ and take $\mathcal{A}[A]_n$ the union of all sets in $\mathcal{A}\{A\}_n$ with all the set in $\mathcal{B}_n$ that have non-empty intersection with A. More formally,

$$\mathcal{A}\{A\}_n := \{c \in \mathcal{C}_n : c \cap A\backslash\mathcal{B}^n \neq \emptyset\},$$
$$\mathcal{A}[A]_n := \bigcup_{c\in\{A\}_n} c \cup \bigcup_{\substack{b\in\mathcal{B}_n,\\ b\cap A\neq\emptyset}} b.$$

We then say that a pre-DAS $\mathcal{A}_n$ is a DAS if for all closed set $A \subseteq \bar{D}$, $\mathcal{A}[A]_n \searrow A$.

In this context we understand $\mathcal{A}[A]_n$ as an approximation of A using a union of elements in $\mathcal{B}_n$ and $\mathcal{C}_n$, it should be understood that the elements of $\mathcal{C}_n$ are the only ones "giving mass" to $\mathcal{A}[A]_n$. $\mathcal{A}\{A\}_n$ represents all the set in $\mathcal{C}_n$ that where used to construct $\mathcal{A}[A]_n$.

Dyadic hyper-cubes provide an example of DAS – more precisely, when $\mathcal{C}_n$ are the closed dyadic hypercubes of side-length 2^{-n} intersected with D and $\mathcal{B}_n$ is empty. This is our canonical DAS and it is such that for all closed sets A the cardinal of $\mathcal{A}\{A\}_n$ is N_n.

Note that condition (2) implies that if A is bounded $|\mathcal{A}\{A\}_n| \leqslant CN_n$ and that there exists an absolute constant C_d such that for any $c \in \mathcal{C}_n$

$$\iint_{c\times c} f(x)G(x,y)f(y)dxdy \leqslant C_d 2^{-(d+2)n}. \tag{1.4}$$

DAS when $d = 2$. In two dimensions, we will modify slightly the definitions. A pre-DAS for a domain $D \subseteq \mathbb{R}^2$ is now a countable collection of families of closed sets $\mathcal{A}_n = (\mathcal{B}_n, \mathcal{C}_n)$ for which (1) and (3) holds and (2) is replaced another condition (2') that we now describe.

For each $c \in \mathcal{C}_n$, let us define

$$G_D(c_x) = \iint_{c_x\times c_x} G_D(x,y)dxdy, \qquad G_D(c_x,c_y) = \iint_{c_x\times c_y} G_D(x,y)dxdy.$$

We will say that (2') holds if for all c and c' in $\mathcal{C}_n$ that are at distance greater than $1/n$ from each other and from ∂D,

$$1 - \frac{C\log n}{n} \leqslant \frac{1}{\sqrt{2\pi}}\frac{2^{-2n}\sqrt{n}}{\sqrt{G_D(c_x)}} \leqslant 1 + \frac{C\log n}{n}, \tag{1.5}$$

$$u_{x,y} := \frac{G_D(c_x,c_y)}{G_D(c_x)G_D(c_y)} \leqslant C\frac{-\log(|x-y|)+1}{n}, \tag{1.6}$$

Note that this is the type of estimates that hold in the case of dyadic squares, and that we have used in our arguments.

We are now ready to give an alternative definition of thin local sets. This definition coincides with that of [Wer16] in the special case where h_A is integrable on $D \setminus A$ (so that working with DAS is not necessary). It is also similar to Lemma 3.10 [MS16a], where they ask Γ to be a.s. determined by the restriction of Γ to $D\backslash A$. On the other hand the first example presented in Section 1.3 is non-thin. However, it is proved in Proposition 3.26 that Γ is a function of the restriction of Γ to $D\backslash A$ (because A is measurable with respect to this restriction).

Definition 1.15 *Let Γ be a Gaussian Free Field on a domain D and $A \subseteq D$ a local set. We say that A is a thin set if for all f smooth and with bounded support in $\mathbb{R}^d$ ($C_0^\infty(\mathbb{R}^d)$) and for all DAS $\mathcal{A}_n$, the sequence $(\Gamma_A, f\mathbf{1}_{D\setminus \mathcal{A}[A]_n})$ converges in probability to (Γ_A, f) when $n \to \infty$.*

Note that $(\Gamma_A, f\mathbf{1}_{D\setminus \mathcal{A}[A]_n})$ is always well defined thanks to the fact that when the supp(f) is compact, $\mathcal{A}[A]_n$ can take only finitely many values. Also, as we have said before, if $\int_{D\setminus A} |h_A| < \infty$, then the limit of $(\Gamma_A, f\mathbf{1}_{D\setminus \mathcal{A}[A]_n})$ is a.s. equal to $\int_{D\setminus A} f(z)h_A(z)$ and this limit does not depend on the chosen DAS. The DAS framework is relevant in the case where the integral of $|h_A|$ on $D \setminus A$ diverges.

Additionally, when $\int_{D\setminus A} |h_A| < \infty$ it is actually enough to check the criteria for functions f in $C_0^\infty(D)$, because when we approximate one function in $C_0^\infty(\mathbb{R}^d)$ restricted to D by one in $C_0^\infty(D)$ both the left and right term of the definition converge to what they should.

Let us briefly note that the definition of thin sets can be extended to non-local sets: We say that a set A is thin if for all $f \in C_0^\infty(\mathbb{R}^d)$ and for all DAS $\mathcal{A}$.

$$\sum_{c\in\mathcal{A}\{A\}_n} (\Gamma, f\mathbf{1}_c) = (\Gamma, f\mathbf{1}_{\mathcal{A}[A]_n}) \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty.$$

it is easy to see that Lemma 1.9 also holds in this setup. This, together with the estimates (1.4), (1.5), and (1.6), allows us to prove that when a deterministic set A has 0 Lebesgue measure it is thin and the equivalents of Proposition 1.11 and 1.14. Note that (2') is necessary if we want to adapt the proof of Proposition 1.14. Note that if a set is a thin local set, then it is thin under the definition of Section 1.2.3. This implies that the sets A defined in Section 1.3 are not thin local sets.

We still do not understand many things about thin sets, for example, we still do not know how to prove that thin sets have 0 Lebesgue measure. On the other hand, it is possible to see that they have empty interior.

Lemma 1.16 *Let Γ be a GFF in D. If A is a thin local set of a Γ, then, a.s. A has empty interior.*

Proof. By contradiction we can assume that there exists an open set O such that with positive probability $O \subseteq A$. Define f_O a smooth function with bounded support inside O. Then, for n small enough, on the even $\{O \subseteq A\}$

$$\sum_{c\in\mathcal{A}\{A\}_n} (\Gamma, f_O\mathbf{1}_c) = (\Gamma, f_O) \neq \emptyset.$$

□

Let us conclude with the following general remarks: It is an open question whether thinness for one approximation scheme implies thinness for all of them. Another issue is that it does not allow to capture the fact that we are also asking our sets to be also local; our proofs that rely on the approximation schemes do not only to prove local thinness but also thinness. This is related to the fact that so far, the question whether the union of thin local sets is always a local set is also still open

Chapter 2

Bounded-type local sets, two-valued sets

2.1 Introduction

2.1.1 General introduction

Thanks to the recent works of Schramm, Sheffield, Dubédat, Miller and others (see [SS09, SS13, Dub09, She16, MS16a, MS16b, MS16c, MS17] and the references therein), it is known that many structures built using Schramm's SLE curves can be naturally coupled with the planar Gaussian Free Field (GFF). For instance, even though the GFF is not a continuous function, SLE_4 and the conformal loop ensemble CLE_4 can be viewed as "level lines" of the GFF. In these couplings, the notion of *local sets of the GFF* and their properties turn out to be instrumental. This general abstract concept appears already in the study of Markov random field in the 70s and 80s (see in particular [Roz82]) and can be viewed as the natural generalization of stopping times for multidimensional time. More precisely, if the random generalized function Γ is a (zero-boundary) GFF in a planar domain D, a random set A is said to be a local set for Γ if the conditional distribution of the GFF in $D \setminus A$ given A has the law of the sum of a (conditionally) independent GFF Γ^A in $D \setminus A$ with a random harmonic function h_A defined in $D \setminus A$. This harmonic function can be interpreted as the harmonic extension to $D \setminus A$ of the values of the GFF on ∂A.

In the present chapter, we are interested in a special type of local sets that we call *bounded-type thin local sets* (or BTLS – i.e., Beatles). Our definition of BTLS imposes three type of conditions, on top of being local sets:

1. There exists a constant K such that almost surely, $|h_A| \leq K$ in the complement of A.
2. The set A is a thin local set as defined in 1.15, which implies in particular that the harmonic function h_A carries itself all the information about the GFF also on A. More precisely, this means here (because we also have the first condition) that for any smooth test function f, the random variable (Γ, f) is almost surely equal to $(\int_{D\setminus A} h_A(x)f(x)dx) + (\Gamma^A, f)$.
3. Almost surely, each connected component of A that does not intersect ∂D has a neighbourhood that does intersect no other connected component of A.

If the set A is a BTLS with constant K in the first condition, we say that it is a K-BTLS.

The first two conditions are the key ones, and they appear at first glance antinomic (the first one tends to require the set A not to be too small, while the second requires it to be small), so that it is not obvious that such BTLS exist at all. Let us stress that Condition (1) is highly non-trivial: The GFF is only a generalized function and we loosely speaking require the GFF here to be bounded by a constant on ∂A. Notice for instance that a deterministic non-polar set is not a BTLS because the corresponding harmonic function is not bounded, so that a non-empty BTLS is necessarily random. As we shall see, the second condition which can be interpreted as a condition on the size of the set A (i.e., it can not be too large), can be compared in the previous analogy between local sets and stopping times to requiring stopping times to be uniformly integrable.

The third condition implies in particular that A has only countably many connected components that do not intersect ∂D. This is somewhat restrictive as there exist totally disconnected sets that are non-polar for the Brownian motion (for instance a Cantor set with Hausdorff dimension in $(0, 1)$), and that can therefore have a non-negligible boundary effect for the GFF. We however believe that this third condition is not essential and could be disposed of (i.e., all statements would still hold without this condition), and we comment on this at the end of the chapter.

We remark that the precise choice of definitions is not that important here. We will see, in fact, that the combination of these three conditions implies much stronger statements. For instance, the (upper) Minkowski dimension of such a K-BTLS A is necessarily bounded by some constant $d(K)$ that is smaller than 2 (and this is stronger than the second condition), and that $\partial D \cup A$ has almost surely only one connected component (which implies the third condition).

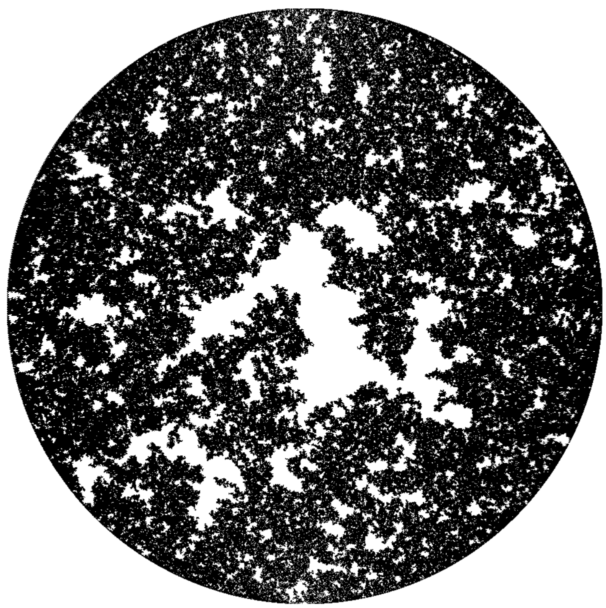
Non-trivial BTLS do exist, and the first examples are provided by SLE_4, CLE_4 and their variants (for their first natural coupling with the GFF) as shown in [Dub09, MS11, SS13]. In the present chapter, among other things, we prove that any BTLS A is contained in a nested version of CLE_4 and that $A \cup \partial D$ is necessarily connected. From our proofs it also follows that the CLE_4 and its various generalizations are deterministic functions of the GFF. Keeping the stopping time analogy in mind, one can compare these results with the following feature of one-dimensional Brownian motion B started from the origin: if T is a stopping time with respect to the filtration of B such $|B_T| \leq K$ almost surely, then either $T \leq \inf\{t \geq 0 : |B_t| = K\}$ almost surely, or T is not integrable. In other words, the CLE_4 and its nested versions and variants are the field analogues of one-dimensional exit times of intervals.

One important general property of local sets, shown in [SS13] and used extensively in [MS16a], is that when A and B are two local sets coupled with the same GFF, in such a way that A and B are conditionally independent given the GFF, then their union $A \cup B$ is also a local set. It seems quite natural to expect that it should be possible to describe the harmonic function $h_{A\cup B}$ simply in terms of h_A and h_B, but deriving a general result in this direction appears to be, somewhat surprisingly, tricky. The present approach provides a way to obtain results in this direction in the case of BTLS: we shall see that the union of two bounded-type thin local sets is always a BTLS.

2.1.2 An overview of results

We now state more precisely some of the results that we shall derive in this chapter. Throughout the present section, D will denote a simply connected domain with non-empty boundary (so that D is conformally equivalent to the unit disk).

Let us first briefly recall some features of the coupling between CLE_4 and GFF in such a domain D. Using a branching-tree variant of SLE_4 introduced by Sheffield in [She09], it is possible to define a certain random conformally invariant family of marked open sets (O_j, ε_j), where the O_j's form a disjoint family of open subsets of the upper half-plane, and the marks ε_j belong to $\{-1, 1\}$. The complement A of $\cup_j O_j$ is called a CLE_4 carpet (see Figure 2.1) and its Minkowski dimension is in fact almost surely equal to 15/8 (see [NW11, SSW09]). As pointed out by Miller and Sheffield [MS11], the set A can be coupled with the GFF as a BTLS in a way that h_A is constant and equal to $2\lambda\varepsilon_j$ (with $\lambda = \sqrt{\pi/8}$) in each O_j.



Simulation by David B. Wilson.

Figure 2.1: The set in black represents the CLE_4 carpet and the white holes are the open sets O_j.

By the property of local sets, conditionally on A, the field $\Gamma - h_A$ consists of independent Gaussian free fields Γ^j inside each O_j. We can then iterate the same construction independently for each of these GFFs Γ^j, using a new CLE_4 in each O_j. In this way, for each given $z \in \mathcal{H}$, if we define $O^1(z)$ to be the open set O_j that contains z (for each fixed z, this set almost surely exists) and set $\varepsilon^1(z) = \varepsilon_j$, we then get a second set $O^2(z) \subset O^1(z)$ corresponding to the nested CLE_4 in $O^1(z)$ and a new mark $\varepsilon^2(z)$. Iterating the procedure, we obtain for each given z, an almost surely decreasing sequence of open sets $O^n(z)$ and a sequence of marks $\varepsilon^n(z)$ in $\{-1, +1\}$. When $z' \in O^n(z)$, then $O^n(z') = O^n(z)$ and $\varepsilon^n(z') = \varepsilon^n(z)$, so that $\varepsilon^n(\cdot)$ can be viewed as a constant function in $O^n(z)$. Furthermore, for each $z \in D$, the sequence $\Upsilon_n(z) = \sum_{m=1}^{n} \varepsilon^m(z)$ is a simple random walk.

For each n, the complement A_n of the union of all $O^n(z)$ with $z \in \mathbb{Q}^2 \cap D$ is then a BTLS and the corresponding harmonic function h_n is just $2\lambda\Upsilon_n(\cdot)$. The set A_n is called the nested

CLE_4 of level n (we refer to it as $\mathrm{CLE}_{4,n}$ in the sequel). It is in fact easy to see that one can recover the GFF Γ from the knowledge of all these pairs (A_n, h_n) for $n \geq 1$ (because for each smooth test function f, the sequence $(\Gamma, f) - (h_n, f)$ converges to 0 in L^2).

For each integer $M \geq 1$, one can then define for each z in $\mathcal{H}$, the random variable $M(z)$ to be the smallest m at which $|\Upsilon_m(z)| = M$. As for each fixed z, the sequence $(\Upsilon_n(z))$ is just a simple random walk, $M(z)$ is almost surely finite. The complement A^M of the union of all these $O^{M(z)}(z)$ for z with rational coordinates defines a $2\lambda M$-BTLS: Indeed, the corresponding harmonic function is then constant in each connected component $O^{M(z)}(z)$ and takes values in $\{2M\lambda, -2M\lambda\}$, and we explain in Section 2.4 that the Minkowski dimension of A^M is almost surely bounded by $d = 2 - (1/(8M^2)) < 2$ which imply the thinness. We refer to this set A^M as a CLE_4^M (mind that M is in superscript, as opposed to $\mathrm{CLE}_{4,m}$, where we just iterated m times the CLE_4).

A special case of our results is that this CLE_4^M is the only BTLS with harmonic function taking its values in $\{-2M\lambda, 2M\lambda\}$. More precisely:

Proposition 2.1 *Suppose that a CLE_4^M that we denote by C is coupled with a GFF in the way that we have just described. Suppose that A is a BTLS coupled to the same GFF Γ, such that the corresponding harmonic function h_A takes also its values in $\{-2M\lambda, 2M\lambda\}$. Then, A is almost surely equal to C.*

In particular, for $M = 1$ and taking A to be another CLE_4 coupled with the same GFF, this implies that any two CLE_4 that are coupled with the same GFF as local sets with harmonic function in $\{-2\lambda, 2\lambda\}$ are almost surely identical. This BTLS approach therefore provides a rather short proof of the fact that *the first layer CLE_4 and therefore also all nested $CLE_{4,m}$'s are deterministic functions of the GFF* (hence, that the information provided by the collection of nested labelled $\mathrm{CLE}_{4,m}$'s is equivalent to the information provided by the GFF itself). This fact is not new and is due to Miller and Sheffield [MS11], who have outlined the proof in private discussions, presentations and talks (and a paper in preparation). Our proof follows a somewhat different route than the one proposed by Miller and Sheffield, although the basic ingredients are similar (absolute continuity properties of the GFF, basic properties of local sets and the fact that SLE_4 itself is a deterministic function of the GFF).

Let us stress that the condition of being thin is important. Indeed, if we consider the union of a CLE_4 with, say, the component of its complement that contains the origin, we still obtain a local set for which the corresponding harmonic function is in $\{-2\lambda, 2\lambda\}$, yet it is almost surely clearly not contained in any of the CLE_4^M for $M \geq 1$.

We will study and characterize two-valued sets (TVS), all possible BTLS such that the harmonic function can take only two possible prescribed values. We will, in particular, derive the following facts:

Proposition 2.2 *Let us consider $-a < 0 < b$.*

1. *When $a + b < 2\lambda$, it is not possible to construct a BTLS A such that $h_A \in \{-a, b\}$ almost surely.*
2. *When $a + b \geq 2\lambda$, it is possible to construct a BTLS $A_{-a,b}$ coupled with a GFF Γ in such a way that $h_A \in \{-a, b\}$ almost surely. Moreover, the sets $A_{-a,b}$ are*
   - *Unique in the sense that if A' is another BTLS coupled with the same Γ, such that*

for all $z \in D$, $h_{A'}(z) \in \{-a, b\}$ almost surely, then $A' = A_{-a,b}$ almost surely.

- *Measurable functions of the GFF Γ that they are coupled with.*
- *Monotonic in the following sense: if $[a,b] \subset [a',b']$ and $-a < 0 < b$ with $b+a \geq 2\lambda$, then almost surely, $A_{-a,b} \subset A_{-a',b'}$.*

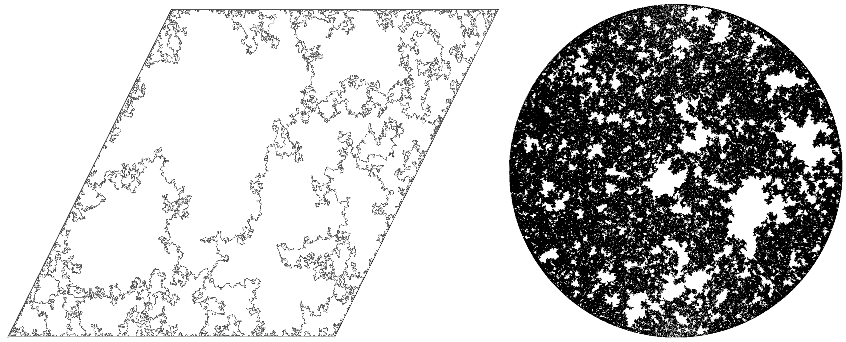


Figure 2.2: On the left a simulation of $A_{-\lambda,\lambda}$ done by Brent Werness. On the right a simulation of $A_{-2\lambda,2\lambda}$ done by David Wilson.

More information about the sets $A_{-a,b}$ and their properties (detailed construction) as well as some generalizations are also discussed. In particular:

- We present a new construction of CLE_4 only using $\text{SLE}_4(-1;-1)$ and $\text{SLE}_4(-1)$ processes.
- We will describe the Hausdorff dimension of the sets $A_{-a,b}$ as a function of a and b.
- We will also discuss connectivity properties of the complement of $A_{-a,b}$. More precisely, we say that two connected components are neighbors if their boundaries are not disjoint. Then, for this adjacency relation, it turns out that the graph of connected components is connected if and only if $a + b < 4\lambda$.
- We will also discuss when it is possible to recover the value of the harmonic function $h_{A_{-a,b}}$ by just looking at $A_{-a,b}$.

Note that these two-valued sets $A_{-a,b}$ will be instrumental in Chapters 3 and 4, and have been also turned out to be useful in the paper [QW17].

Another type of result that we derive using similar ideas as in the proof of Proposition 2.1 goes as follows:

Proposition 2.3 *If B is a $(2M\lambda)$-BTLS associated to the GFF Γ, then B is almost surely a subset of the CLE_4^{M+1} associated to Γ.*

Notice that one would expect to conclude that $B \subseteq \text{CLE}_4^M$ in Proposition 2.3 (and this would mean that CLE_4^M is maximal among all $2\lambda M$-BTLS), but this seems to require some more technical work that we do not discuss in the present chapter.

As a finite collection of BTLS is in fact a collection of $2\lambda M$-BTLS for some $M \in \mathbb{N}$, we see that their union is almost surely contained in CLE_4^{M+1}, and thus it is again a BTLS. This type of facts helps to derive the following result, that we already mentioned earlier in this introduction:

Proposition 2.4 *If A is a BTLS, then $A \cup \partial D$ is connected.*

The structure of the chapter is the following: We first recall some basic features about BTLS. Then, we discuss level lines of the GFF with non-constant boundary conditions and their boundary hitting behaviour. Thereafter, we recall features of the construction of the coupling of the GFF with CLE_4^M. This sets the stage for the proofs of the propositions involving CLE_4^M. We then finally turn to Proposition 2.2.

In the final sections, we study the dimensions of the sets $A_{-a,b}$ and of the intersections between the $A_{-a,b}$ loops and the boundary of the domain, we discuss the connectivity properties of the complement of $A_{-a,b}$ and the measurability of the harmonic function h with respect to the set $A_{-a,b}$.

2.2 Local sets and BTLS

In this section, we quickly browse through basic definitions and properties of the GFF and of bounded-type local sets. We only discuss items that are directly used in the current chapter. For a more general discussion of local sets, thin local sets (not necessarily of bounded type), we refer to Chapter 1.

Throughout this chapter, the set D denotes an open planar domain with a non-empty and non-polar boundary. In fact, we will always at least assume that the complement of D (a) has at most countably many connected components, (b) has only finitely many components that intersect each given compact subset of D, (c) has no connected component that is a singleton; this last condition (c) excludes for instance sets like $\mathbb{D} \setminus K$, where $K \subset [0,1]$ is the middle Cantor set). Recall that by a theorem of He and Schramm [HS95], such domains D are known to be conformally equivalent to circle domains (i.e. to $\mathbb{D} \setminus K$ or more conveniently for us to $\mathbb{H} \setminus K$, where K is a union of closed disjoint discs).

Recall that the (zero boundary) Gaussian Free Field (GFF) in a such a domain D can be viewed as a centered Gaussian process $\Gamma = ((\Gamma, f))$ (we also sometimes write Γ^D when the domain needs to be specified) indexed by the set of continuous functions f with compact support in D, with covariance given by

$$\mathbb{E}[(\Gamma, f)(\Gamma, g)] = \iint_{D\times D} f(x)G_D(x,y)g(y)dxdy$$

where G_D is the Green's function (with Dirichlet boundary conditions) in D, normalized such that $G_D(x,y) \sim (2\pi)^{-1}\log(1/|x-y|)$ as $x \to y$ for $y \in D$. For this choice of normalization of G (and therefore of the GFF), we set

$$\lambda = \sqrt{\pi/8}.$$

Sometimes, other normalizations are used in the literature: If $G_D(x,y) \sim c\log(1/|x-y|)$ as $x \to y$, then λ should be taken to be $(\pi/2) \times \sqrt{c}$. Note that it is in fact possible and useful to define the random variable (Γ, μ) for any fixed Borel measure μ, provided the energy $\iint \mu(dx)\mu(dy)G_D(x,y)$ is finite.

The covariance kernel of the GFF blows up on the diagonal, which makes it impossible to view Γ as a random function. However, the GFF has a version that lives in some space of generalized functions acting on some deterministic space of smooth functions f (see for example [Dub09]). This also justifies our notation (Γ, f). Let us stress that it is in general not possible to make sense of (Γ, f) for *random* functions that are correlated with the GFF, even when $f = 1_A$ is the indicator function of a random closed set A. Local sets form a class of random closed sets A, where this is (in a sense) possible. Here, by a random closed set we mean a random variable in the space of closed subsets of $\overline{D}$, endowed with the Hausdorff metric

Definition 2.5 (Local sets) *Consider a random triple (Γ, A, Γ_A), where Γ is a GFF in D, A is a random closed subset of $\overline{D}$ and Γ_A is a random distribution that can be viewed as a harmonic function, h_A, when restricted to compact subsets of $D \setminus A$. We say that A is a local set for Γ if conditionally on the couple (A, Γ_A), the field $\Gamma - \Gamma_A$ is a GFF in $D \setminus A$.*

We use the different notation h_A for the restriction of Γ_A to $D\backslash A$, in order to emphasize that in $D\backslash A$ the generalized function Γ_A is in fact a harmonic function and thus can be evaluated at any point $z \in D\backslash A$.

When A is a local set for Γ, we will define Γ^A to be equal to $\Gamma - \Gamma_A$. Note that the conditional distribution of Γ^A given (A, Γ_A) is in fact a function of A alone.

Notice that being a local set can also be seen as a property of the law of the couple (Γ, A), as if one knows this law and the fact that A is a local set of Γ, then one can recover h_A as the limit when $n \to \infty$ of the conditional expectation of Γ (outside of A) given A and the values of Γ in the smallest union of 2^{-n} dyadic squares that contains A. One can then recover Γ^A which is equal $\Gamma - h_A$ outside of A, and finally one reconstructs $\Gamma_A = \Gamma - \Gamma^A$ (including on A). This argument shows in particular that any local set can be coupled in a unique way with a given GFF: if two random triples (Γ, A, Γ_A) and (Γ, A, Γ'_A) are both local couplings, then Γ_A and Γ'_A are almost surely identical.

When A is a local set for Γ, we will denote by $\mathcal{F}_A$ the σ-field generated by (A, Γ_A). We will say that two local sets A and B that are coupled with the same Gaussian Free Field Γ are *conditionally independent local sets of* Γ if the sigma-fields $\mathcal{F}_A$ and $\mathcal{F}_B$ are conditionally independent given Γ.

Let us list the following two properties of local sets (see for instance [SS13] for derivations and further properties):

Lemma 2.6

1. *When A and B are conditionally independent local sets for the GFF Γ, then $A \cup B$ is also a local set for Γ.*
2. *When A and A' are conditionally independent local sets for Γ such that $A \subset A'$ almost surely, then for any smooth compactly supported test function f, $(\Gamma_A, f) = \mathbb{E}[(\Gamma_{A'}, f)|\mathcal{F}_A]$ almost surely.*

We now define local sets of bounded type:

Definition 2.7 (BTLS) *Consider a random relatively closed subset A of D (i.e. so that $D \setminus A$ is open) and Γ a GFF defined on the same probability space. Let $K \geqslant 0$, we say that A is a K-BTLS if the following four conditions are satisfied:*

1. *A is a local set of Γ.*
2. *Almost surely, $|h_A| \leq K$ in $D \setminus A$.*

3. *Almost surely, each connected component of A that does not intersect ∂D has a neighbourhood that does intersect no other connected component of A.*
4. *A is a thin local set in the sense defined in 1.15: for any smooth test function $f \in \mathcal{C}_0^\infty$, the random variable (Γ_A, f) is almost surely equal to $(\int_{D\setminus A} h_A(x)f(x)dx)$.*

If A is a K-BTLS for some K, we say that it is a BTLS.

Note also that it is in fact possible to remove all isolated points from a BTLS (of which there are at most countably many because of the third property) without changing the property of being BTLS. Indeed, the bounded harmonic function h_A can be extended to those points and a GFF does not see polar sets. We therefore replace the third property by:

(3') Almost surely, A contains no isolated points and each connected component of A that does not intersect ∂D has a neighbourhood that intersects no other connected component of A.

This reformulation is handy to keep our statements simple. In particular, this condition implies that for any $x \in A$ and for any neighbourhood J of x, $J \cap A$ is not polar. To see this, it is enough to notice that $D\setminus A$ is conformally equivalent to a circle domain described in the beginning of this section.

It is not hard to see that because the harmonic function is bounded, the condition (4) in the definition of BTLS could be replaced (without changing the definition) by the fact that if we define $[A]_n$ to be the union A_n of the 2^n-dyadic squares that intersect A, then for any compactly supported smooth test function f in D, the sequence of random variables $(\Gamma, f\mathbf{1}_{[A]_n})$ converges in probability to 0. From a Borel-Cantelli argument, one can moreover see that this equivalent condition is implied by the stronger condition (Proposition 1.14):

$(*)$ The expected volume of the ε-neighbourhood of A decays like $o(1/\log(1/\varepsilon))$ as $\varepsilon \to 0$.

In other words, if a set satisfies the first three conditions in our BTLS definition and $(*)$, then it is a BTLS.

Note that if A and B are two conditionally independent BTLS of Γ, then we know by Lemma 2.6 that $A \cup B$ is a local set, but not yet that it is a BTLS. In order to prove this, we will need to show it is thin and give an upper bound for the harmonic function $h_{A\cup B}$.

It is not hard to derive the following related fact (that will be used later in the chapter, at the end of the proof of the fact that the union of any two BTLS is a again a BTLS):

Lemma 2.8 *Let A and A' are two conditionally independent thin local sets of the same GFF Γ such that A satisfies the condition (3') of Definition 2.7, that A' is a K-BTLS and that $A \subseteq A'$ almost surely. Then A is a K-BTLS.*

Proof. We need to check that $|h_A| \leqslant K$. Let us choose a smooth non-negative test function ρ that is radially symmetric around the origin, of unit mass with support in the unit ball, and denote by ρ_ε^z the naturally shifted and scaled version of ρ, so that ρ_ε^z is radially symmetric around z, of unit mass and with support in the open ball $B(z,\varepsilon)$. The final statement of Lemma 2.6 shows that when $\varepsilon < d(z,\partial D)$, $\mathbb{E}[(\Gamma_{A'}, \rho_\varepsilon^z)|\mathcal{F}_A] = (\Gamma_A, \rho_\varepsilon^z)$ almost surely. As A' is a K-BTLS, we know that almost surely, $|(\Gamma_{A'}, \rho_\varepsilon^z)| \le K$, so that by Jensen's inequality, $|(\Gamma_A, \rho_\varepsilon^z)| \le K$ almost surely. But as Γ_A is equal to the harmonic function h_A in the complement of A, we have $h_A(z) = \Gamma_A(\rho_\varepsilon^z)$ as long as $d(z,A) > \varepsilon$. Thus we conclude that with full probability on the event that $d(z,A) > \varepsilon$, $|h_A(z)| \le K$. Since this holds almost surely for all

z with rational coordinate and every rational $\varepsilon < d(z, \partial D)$ simultaneously, we conclude that almost surely $|h_A| \leqslant K$ in $D \setminus A$. □

The following two tailor-made lemmas will be used in the proof of the fact that any BTLS is contained in some CLE_4^M:

Lemma 2.9 *Let A and B are two conditionally independent BTLS of the GFF Γ such that A almost surely satisfies condition $(*)$, and such that there exists $k \in \mathbb{R}$ such that a.s. for all $z \notin A \cup B$, $|h_A(z) + k| \geqslant |h_{A\cup B}(z) + k|$. Then $B \subseteq A$ almost surely.*

Proof of Lemma 2.9: First, let us briefly explain why the conditions on A and B imply that $A \cup B$ is a thin local set. From Lemma 2.6, we know that $A \cup B$ is local. To show that $A \cup B$ is thin, we use Corollary 1.12.

We denote by A_+ (resp. A_-) the set of points in $D \setminus A$ where $h_A + k$ is non-negative (resp. non-positive). Then, for any open set O,

$$\begin{aligned}\mathbb{E}\left[(\Gamma + k, \mathbf{1}_O)^2 \mathbf{1}_{\{O\subset A_+\}}\right] &= \mathbb{E}\left[\mathbb{E}\left[\left((h_A + k, \mathbf{1}_O) + (\Gamma^A, \mathbf{1}_O)\right)^2 |\mathcal{F}_A\right] \mathbf{1}_{\{O\subset A_+\}}\right] \\ &= \mathbb{E}\left[\left((h_A + k, \mathbf{1}_O)^2 + \int_{O\times O} G_{D\setminus A}(x,y)dxdy\right) \mathbf{1}_{\{O\subset A_+\}}\right].\end{aligned}$$

Similarly, by conditioning on $\mathcal{F}_{A\cup B}$ and using that $A \cup B$ is thin, one gets that this same quantity is equal also to

$$\mathbb{E}\left[\left((h_{A\cup B} + k, \mathbf{1}_O)^2 + \int_{(O\setminus B)\times(O\setminus B)} G_{D\setminus(A\cup B)}(x,y)dxdy\right) \mathbf{1}_{\{O\subset A_+\}}\right].$$

But by definition, when $O \subset A_+$, $(h_A + k, \mathbf{1}_O) \geq |(h_{A\cup B} + k, \mathbf{1}_O)|$ (because $h_A(x) + k \geq |h_{A\cup B}(x) + k|$ for all $x \in A_+$). Hence, using the fact that $G_{D\setminus(A\cup B)} \leq G_{D\setminus A}$, we conclude that for every open set, almost surely on the event $O \subset A_+$, $G_{D\setminus(A\cup B)} = G_{D\setminus A}$ on $O \times O$. The same statement holds for A_- instead of A_+. Therefore $B \setminus A$ is polar in $D \setminus A$ and condition (3') allows us to conclude. □

Let us now suppose that A is a local set of the GFF Γ in a bounded simply-connected domain D such that $D\setminus A$ is connected and $\partial D \cup A$ has only finitely many connected components. The following lemma says that if $|h_A| \leq C$ in the neighbourhood of all but finitely many prime ends of $D\setminus A$, then it is bounded by C in all of $D\setminus A$. To state this rigorously, it is convenient to note that by Koebe's circle domain theorem, one can use a conformal map ϕ to map $D\setminus A$ onto a circle domain $\tilde{O}$ (i.e., the unit disc with a finite number of disjoint closed discs removed). In this way, each prime end of $D\setminus A$ is in one-to-one correspondence with a boundary point of $\tilde{O}$. Define also the harmonic function $\tilde{h}_A := h_A \circ \phi^{-1}$ in $\tilde{O}$.

Lemma 2.10 *Let A be a local set of the GFF Γ as just described, and let $\tilde{O}$, $\tilde{h}_A$ be as above. Assume furthermore that there exist finitely many points $y_1, \ldots, y_n$ on $\partial\tilde{O}$ and a non-negative constant C such that for all $y \in \partial\tilde{O} \setminus \{y_1, \ldots, y_n\}$, one can find a positive $\varepsilon(y)$ such that $|\tilde{h}_A| \leq C$ in the $\varepsilon(y)$-neighbourhood of y in $\tilde{O}$. Then $|h_A|$ is in fact bounded by C in all of $D\setminus A$.*

PROOF. For some (random) small enough r_0, all connected components of $\partial\tilde{O}$ are at distance at least r_0 from each other. Let us now consider any ε smaller than $r_0/2$. By compactness of $\partial\tilde{O}$, one can cover $\partial\tilde{O} \setminus \cup_{j\le n} B(y_j,\varepsilon)$ by a union U of finitely many open balls of radius not larger than ε that are centered on points of $\partial\tilde{O}$ in such a way that $|\tilde{h}_A| \le C$ in all of $U \cap \tilde{O}$.

Let us now choose some $\tilde{z} \in \tilde{O}$ with $d(\tilde{z},\partial\tilde{O}) > \varepsilon$ and prove that $|\tilde{h}_A(\tilde{z})| \le C$. Define V to be the connected component of $\tilde{O} \setminus (U \cup \cup_{j\le n} B(y_j,\varepsilon))$ that contains $\tilde{z}$. The definition of U shows that except on the (possibly empty) part of ∂V that belongs to the boundary of the ε-balls around $y_1,\dots,y_n$, the function $|\tilde{h}_A|$ is bounded by C. Now, $\tilde{h}_A(\tilde{z})$ is the integral of the harmonic function $\tilde{h}_A$ with respect to the harmonic measure at $\tilde{z}$ on ∂V. Thus, in order to show that $|\tilde{h}_A(\tilde{z})| \le C$, it suffices to prove that the contribution J of the integral on $\partial V \cap \partial B(y_i,\varepsilon)$ goes to 0 as ε to 0 for all $i = 1,\dots,n$.

To justify this, we can first note that the density of the harmonic measure (with respect to the Lebesgue measure) on all these arcs is bounded by a positive constant independently of ε. On the other hand, it follows from the proof of Lemma 3.1 in [HMP10] that there exists a random constant C' such that almost surely, the absolute value of the circle average of Γ on the circle of radius r around x is bounded by $C'\log(2/r)$ for all $x \in D$ and $r \in (0,1]$ simultaneously. As for all $x \in D\backslash A$, $h_A(x)$ is equal to the average of $\Gamma - \Gamma^A$ on any circle of radius smaller than $d(x,\partial(D\backslash A))$ around x, we deduce that for some random constant C'' and for all $x \in (D\backslash A)$

$$|h_A(x)| \le C'' \log(2/d(x,\partial(D\backslash A))).$$

But now Beurling's estimate allows to compare $d(x,\partial(D\backslash A))$ with $d(\phi(x),\partial\tilde{O})$. We obtain that for some random positive C''' and for all $y \in \tilde{O}$ simultaneously

$$|\tilde{h}_A(y)| \le C''' \log(2/|d(y,\partial\tilde{O})|).$$

This in turn implies that J is almost surely $O(\varepsilon|\log\varepsilon|)$ as $\varepsilon \to 0$, which completes the proof. □

2.3 Absolute continuity, generalized level lines.

2.3.1 GFF absolute continuity

Let $D = \mathbb{H} \setminus K$, where K is a countable union of closed discs such that in any compact set of $\mathbb{H}$ there are only finitely many of them.

Let us recall first that, similarly to Brownian motion, the GFF can be viewed as the Gaussian measure associated to the Dirichlet space $\mathcal{H}_0^1$, which is the closure of the set of smooth functions of compact support in D with respect to the Dirichlet norm given by

$$(f,f)_\nabla = \int_D |\nabla f(x)|^2 dx.$$

The Dirichlet space is also the Cameron-Martin space for the GFF (see e.g. [Dub09, She16] for this classical fact):

Theorem 2.11 (Cameron-Martin for the GFF) *Let F be a function belonging to $\mathcal{H}_0^1(D)$ and Γ a GFF in D. Denote the law of Γ by $\mathbb{P}$ and the law of $\Gamma + F$ by $\tilde{\mathbb{P}}$. Then $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are* *mutually absolutely continuous and the Radon-Nikodym derivative* $d\tilde{\mathbb{P}}/d\mathbb{P}$ *at* γ *is a multiple of* $\exp((F,\gamma)_\nabla)$.

We are now going to use this in the framework of local sets of the GFF: Denote by S the interior of a finite union of closed dyadic squares in D with $I := \partial S \cap \mathbb{R} \neq \emptyset$, and a harmonic function H in D such that H extends continuously to an open neighbourhood I' of I in $\mathbb{R}$ in such a way that $H = 0$ on I' (we then say that the boundary value of H on I is zero). Using the Cameron-Martin Theorem it is not hard to see that the GFF Γ and $\Gamma + H$ are mutually absolutely continuous, when restricted to S, i.e. when restricted to all test functions f with support in S.

Indeed, let $\tilde{H}$ be the bounded harmonic function in $D\backslash\partial S$ that is equal to H on the boundary of S and to zero on the boundary of D. Note that $\tilde{H}$ belongs to $\mathcal{H}_0^1(D)$ and that $(\tilde{H},\Gamma)_\nabla$ depends only on the restriction of Γ to S. Thus, using the Cameron-Martin Theorem and the domain Markov property of the GFF we obtain (see [Dub09, She16]):

Lemma 2.12 *Let* Γ *be a (zero boundary) GFF in* D *and denote its law restricted to* S *by* $\mathbb{P}$. *Let also* $\tilde{\mathbb{P}}$ *be the law of* $\Gamma + H$, *restricted to* S. *Then* $\mathbb{P}$ *and* $\tilde{\mathbb{P}}$ *are mutually absolutely continuous with respect to each other. Moreover, the Radon-Nikodym derivative* $d\tilde{\mathbb{P}}/d\mathbb{P}$ *is a multiple of* $\exp((\tilde{H},\Gamma)_\nabla)$.

Furthermore, if D' *is another domain as above,* Γ' *is a GFF in* D' *and* $S \subset D \cap D'$ *is at positive distance of* $\partial D \triangle \partial D'$, *then the laws of* Γ *and* Γ' *restricted to* S *are mutually absolutely continuous.*

This absolute continuity property allows to change boundary conditions for local couplings away from the local sets:

Proposition 2.13 *Under the previous conditions, suppose that* A *is a BTLS for* Γ *an* D-*GFF such that* $A \subset S \subset D$ *almost surely. Define* $\tilde{\mathbb{P}} := Z\exp((\tilde{H},\Gamma)_\nabla)d\mathbb{P}$. *Then* A *is coupled as a local set with* $\tilde{\Gamma} := \Gamma - \tilde{H}$ *(that is a* $\tilde{\mathbb{P}}$ *zero-boundary GFF). The corresponding harmonic function* $\tilde{h}_A$ *is equal to the unique bounded harmonic function on* $D \setminus A$, *with boundary values equal to those of* $h_A - H$ *on* $\partial A \cup I$, *and to* 0 *on* $\partial D \setminus I$.

Proof. Notice that by the Cameron-Martin Theorem for the GFF, the law of Γ under $\tilde{\mathbb{P}}$ is that of $\tilde{\Gamma} + \tilde{H}$, where $\tilde{\Gamma}$ is a $\tilde{\mathbb{P}}$ GFF. Now we have to verify that with this change of variables A is still a local set. Note that conditionally on $\mathcal{F}_A$, we can write $\Gamma = h_A + \Gamma^A$, where Γ^A is a $\mathbb{P}$-GFF on $D \setminus A$. We have to show that we can write $\Gamma^A = \tilde{\Gamma}^A + \tilde{H} + \tilde{h}_A - h_A$, where $\tilde{\Gamma}^A$ is a $\tilde{\mathbb{P}}$-GFF on $D \setminus A$ and $\tilde{h}_A$ is as in the statement. Let M be a measurable function of the field Γ^A, then for some Z_A and Z'_A, measurable functions of A,

$$\begin{aligned}\tilde{\mathbb{E}}\left[M(\Gamma^A) \mid A, h_A\right] &= Z_A\mathbb{E}\left[M(\Gamma^A)\exp\left((\tilde{H},\Gamma)_\nabla\right) \mid \mathcal{F}_A\right]\\ &= Z'_A\mathbb{E}\left[M(\Gamma^A)\exp\left((\tilde{H},\Gamma^A)_\nabla\right) \mid \mathcal{F}_A\right]\\ &= Z'_A\mathbb{E}\left[M(\Gamma^A)\exp\left((\tilde{H}+\tilde{h}_A - h_A,\Gamma^A)_\nabla\right) \mid \mathcal{F}_A\right],\end{aligned}$$

where in the last equality we use that Γ^A is a GFF and that $\tilde{h}_A - h_A$ is harmonic in $D \setminus A$. But, we know that (under $\mathbb{P}$), conditionally on $\mathcal{F}_A$, Γ^A is just a GFF. Thus using again the Cameron-Martin Theorem, Lemma 2.6-(i) and the fact that $\tilde{h}_A - h_A$, $\tilde{H}$ are in $\mathcal{H}^1(D\backslash A)$ we can conclude. □

Corollary 2.14 *Let $D' \subseteq D$ be another domain with the same properties. Under the previous conditions, suppose that A is a BTLS for a GFF in D such that $A \subset S \subset D'$ almost surely. Then, there exist an absolutely continuous probability measure $\tilde{\mathbb{P}}$ under which A is coupled as a local set with a $\tilde{\mathbb{P}}$ zero-boundary GFF in D' and the corresponding harmonic function $\tilde{h}_A$ is equal to the unique bounded harmonic function on $D' \setminus A$, with boundary values equal to those of $h_A - H$ on $\partial A \cup I$, and to 0 on $\partial D' \setminus I$.*

Proof. It can be shown that A is a local set for $\Gamma^{D\setminus D'}$ such that its harmonic function $(h^{D\setminus D'})_A$ goes to 0 on $\overline{\partial} D \setminus I$ and $(h^{D\setminus D'})_A - h_A + h_K$ goes to 0 on $\partial A \cup I$ (see Lemma 3.9-1 of [SS13], or [Wer16]). We conclude using the fact that $h_{D\setminus D'}$ is independent of $h^{D\setminus D'}$ and Proposition 2.13. □

2.3.2 SLE$_4$ and the GFF with more general boundary conditions

SLE$_4$ as level lines of the GFF

Let us start by recalling some well-known features of the Schramm-Sheffield coupling of SLE$_4$ with the GFF in the upper half-plane $\mathcal{H}$: Consider the bounded harmonic function $F_0(z)$ in the upper-half plane with boundary values $-\lambda$ on $\mathbb{R}_-$ and $+\lambda$ on $\mathbb{R}_+$. There exists a unique law on random simple curves $(\eta(t), t \geqslant 0))$ (parametrized by half-plane capacity) in the closed upper half-plane from 0 to infinity that can be coupled with a GFF Γ so that the following property holds for all $t \geqslant 0$. (we state it in a somewhat strange way that will be easier to generalize):

(∗∗) The set $\eta[0,t]$ is a BTLS of the GFF Γ, with harmonic function h_t defined as follows: $h_t + F_0$ is the unique bounded harmonic function in $\mathbb{H} \setminus \eta[0, \min\{t, \tau\}]$ with boundary values $-\lambda$ on the left-hand side of η, $+\lambda$ on the right side of η, and with the same boundary values as F_0 on $\mathbb{R}_-$ and $\mathbb{R}_+$.

Furthermore, this curve is then a SLE$_4$ and when one couples SLE$_4$ with a GFF Γ in this way, then the SLE$_4$ process is in fact a measurable function of the GFF (see [Dub09, MS16a, SS13, She16, Wer16] for all these facts).

We are also going to work with piecewise boundary condition[1]. So, it is just needed to work with the result in [WW16]. We are now going to state Theorem 1.1.1 of [WW16]. Let u be a bounded harmonic function with piecewise constant boundary data such that $H(0^-) < \lambda$ and $H(0^+) > -\lambda$.

Lemma 2.15 (Existence of generalized level line targeted to ∞) *There exists a unique law on random simple curves $(\eta(t), t \geqslant 0)$ coupled with a GFF in a simply connected domain such that $(*)$ holds for the function H and possibly infinite stopping time τ that is defined as the first time when η hits a point $x \in \mathbb{R}$ such that $x \geqslant 0$ and $H(x^+) \leqslant -\lambda$ or $x \leqslant 0$ and $H(x^-) \geqslant \lambda$. We call η the generalized level line for the GFF $\Gamma + u$.*

For convenience we also use the notion of a $(-a, -a+2\lambda)$-level line of $\Gamma + u$: it is a generalized level line of $\Gamma + a - \lambda + u$ and has boundary conditions $-a, -a+2\lambda$ with respect to the field $\Gamma + u$. Moreover, it is known that when $u = 0$ this level line has the law of a SLE$_4(\rho_1, \rho_2)$ process [WW16].

[1] Here, and elsewhere this means piecewise constant that changes only finitely many times

Notice that as the level line is parametrized using half-plane capacity, it will accumulate at ∞ if not stopped earlier. In general, it is useful to know when a level line attains its target point (i.e. ∞ in the previous definition). The following lemma is a consequence of Proposition 2.13 and the fact that SLE_4 does not intersect the boundary. It says that a target point is reached with positive probability if one can start a level line from this point in the opposite direction.

Lemma 2.16 *Let η be a generalized level line of a GFF $\Phi + u$. If there exists a generalized level line of $-\Phi \circ \psi - u \circ \psi$ for $\psi(x) = -1/x$, then with positive probability we have that $\eta_\infty = \infty$.*

More general boundary conditions

Now we generalize this definition of level lines to the GFF with more general boundary conditions. By conformal invariance if we wish to define them for all domains D described at the beginning of Section 2.2 it is enough to consider the case where $D = \mathbb{H} \setminus K$, where K is a countable union of closed discs such that any compact subset of $\mathbb{H}$ intersects only finitely many of these discs.

Let H be a harmonic function on D with zero boundary conditions on some real neighbourhood I of the origin. For a random simple curve η in D we define time τ to the (possibly infinite) smallest positive time t at which $\liminf_{s \to t-} d(\eta_s, \partial D) = 0$. The generalized level line for the GFF in D with boundary conditions $F_0 + H$ up to the first time it hits the boundary is then defined as follows:

Lemma 2.17 *[Generalized level line] There is a unique law on random simple curves $(\eta(t), t \geqslant 0))$ in D parametrized by half-plane capacity (i.e., viewing $\eta((0,t])$ as a subset of $\mathbb{H}$ instead of $\mathbb{H} \setminus K$) with $\eta(0) = 0$, $\eta(0,\tau) \subset D$ that can be coupled with a GFF Γ so that $(**)$ holds for all $0 \leqslant t < \tau$, when one replaces F_0 by $F_0 + H$ and considers D instead of $\mathbb{H}$. Moreover, the curve η is measurable with respect to Γ. We call η the generalized level line of $\Gamma + F_0 + H$ in D.*

Proof. Let $\mathcal{S}$ denote the collection of regions S such that S is the interior finite union of closed dyadic squares in D with $0 \in \partial S \cap \partial D \subset I$ and with S simply connected. Note that is enough to show that for all $S \in \mathcal{S}$ there is at most one curve satisfying $(**)$, when one replaces F_0 by $F_0 + H$, until the time it exits S.

Suppose by contradiction that η_1 and η_2 are two different curves with this property and that with positive probability they do not agree. Thanks to Corollary 2.14, we can construct two local sets $\tilde{\eta}_1$ and $\tilde{\eta}_2$ coupled with the same GFF in a simply connected domain $D' \subset D$ that strictly contains S, such that when we apply ϕ, the conformal transformation from D' to $\mathbb{H}$, $(**)$ holds for $\phi(\tilde{\eta}_1)$ and $\phi(\tilde{\eta}_2)$ until the time they exit $\phi(S)$. The fact that they are different with positive probability contradicts the uniqueness of the Schramm-Sheffield coupling proved in [Dub09, SS13].

It remains to show the existence of this curve. Take S_n an increasing sequence of elements in $\mathcal{S}$ such that their union is D. We can define η until the time it exits S_n, by using the Schramm-Sheffield coupling and Proposition 2.14. The above argument shows that they are compatible and so we can define η until the first time it touches ∂D. The measurability is just a consequence of the existence and uniqueness. □

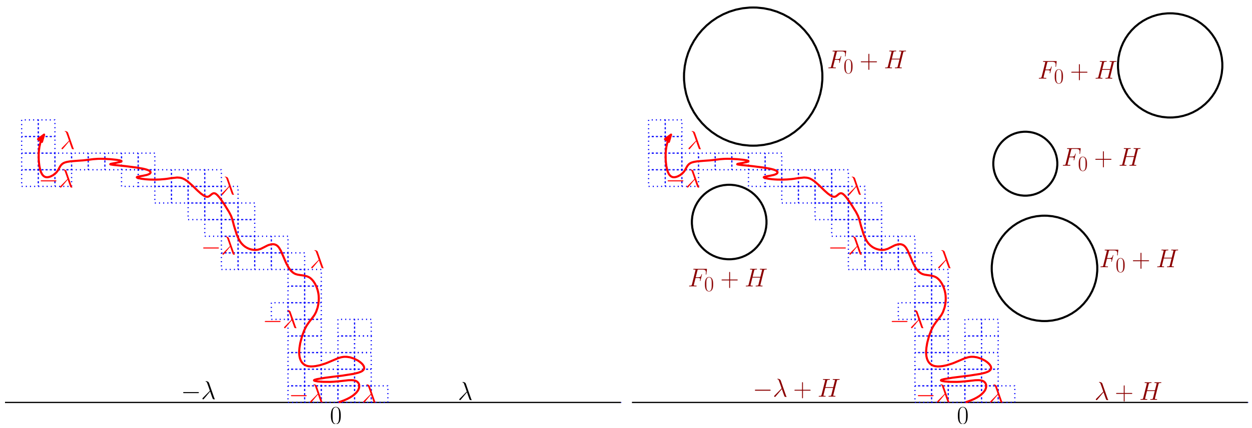


Figure 2.3: Defining generalized level lines by absolute continuity.

Note that for all stopping times τ of $\eta([0,t])$, such that τ is measurable with respect to the σ-algebra generated by η, we have that $(**)$ holds for $t = \tau$.

Boundary hitting of generalized level lines

In [PW15] Lemma 3.1, the authors show that a generalized level line in $\mathbb{H}$ corresponding to $F_0 + H_0$ with H_0 zero in a neighbourhood of the origin, bounded, $F_0 + H_0 \geqslant -\lambda$ on $(-c, 0)$ and $F_0 + H_0 \geqslant \lambda$ elsewhere cannot touch nor accumulate in a point of $\mathbb{R} \setminus [-c, 0]$. Combined with our previous considerations, we extend this to:

Lemma 2.18 (Boundary hitting of generalized level lines) *Let η be a generalized level line of $\Gamma + F_0 + H$ in D such that H is harmonic in D and has zero boundary values in a neighbourhood I of zero. Suppose $F_0 + H \geqslant \lambda$ in $J \cap \partial D$, with J some open set of D. Let τ denote the first time at which $\inf\{d(\eta_s, J \cap \partial D), s < t\} = 0$. Then, the probability that $\tau < \infty$ and that $\eta(t)$ converges to a point in J as $t \to \tau^-$ is equal to 0. This also holds if $F_0 + H \leqslant -\lambda$ in $J \cap \partial D$ for J an open set of D.*

Proof. Let S be the interior of a finite union of closed dyadic squares intersected with D such that $0 \in \partial S$, $x \in \partial S$, $\partial S \cap \partial D \subset I \cup J$ and $S \cap K = \emptyset$. From Lemma 2.17 we know that for any such S, a generalized level line η is measurable up to leaving S w.r.t. the GFF restricted to S.

We first consider the case $x < 0$ and $J \cap \partial D = (x - \varepsilon, x + \varepsilon)$ with $0 < \varepsilon < -x$. Define a harmonic function H_0 on $\mathbb{H}$ such that $H_0 = H$ on $(x - \varepsilon, x + \varepsilon)$, $H_0 = 0$ on $[x + \varepsilon, \infty)$ and $H_0 = 2\lambda$ on $(-\infty, x - \varepsilon]$. The laws of the GFFs $\Gamma + F_0 + H_0$ and $\Gamma + F_0 + H$ restricted to S are absolutely continuous by Lemma 2.12. Also, both of these generalized level lines are measurable w.r.t. the GFF until they exit S. Now $F_0 + H_0$ satisfies the conditions mentioned above and hence almost surely the generalized level line of $\Gamma + F_0 + H_0$ does not exit from S on $\partial S \cap (x - \varepsilon, x + \varepsilon)$ in finite time. By absolute continuity and measurability of both generalized level lines, the result follows for this J. The case $x > 0$ is treated similarly. By the union bound the result is true simultaneously for all $x_n \in \mathbb{Q}$ and $\varepsilon_n \in \mathbb{Q}$ with $\varepsilon_n < |x_n|$

and $F_0 + H \geqslant \lambda$ on $(x_n - \varepsilon_n, x + \varepsilon_n)$. Given that all $J \cap \partial D \subseteq \mathbb{R}$ satisfying the conditions of the theorem are written as the union of these intervals, we conclude.

To treat the case where x is on the boundary of some component D_1 of K, notice that it suffices to consider S that do not surround D_1 and that for any such S we can connect D_1 to $\mathbb{R}$ using some curve γ such that $\mathbb{H} \setminus (D_1 \cup \gamma)$ is simply connected and contains S. The previous argument and Corollary 2.14 then help to conclude. □

2.4 Review of the construction of CLE_4^M and its coupling with the GFF

In this section, we review the coupling of the GFF with the nested CLE_4 using Sheffield's SLE_4 exploration trees (i.e. the branching $\mathrm{SLE}_4(-2)$ process). As this section does not contain really new results, we try to be rather brief, and refer to [She09, Wer16] for details. Note that we will present an alternative construction of the CLE_4 and its coupling to the GFF in Section 2.6.

2.4.1 Radial symmetric $\mathrm{SLE}_4(-2)$

Let us first recall the definition of the radial (symmetric) $\mathrm{SLE}_4(-2)$ process targeted at 0 (for non-symmetric variants see [She09]). It is the radial Loewner chain of closed hulls $(K_t)_{t\geq 0}$ in the unit disc, with driving function ξ_t defined as follows:

- Start with a standard real-valued Brownian motion $(B_t, t \geq 0)$ with $B_0 = 0$.
- Then, define the continuous process $U_t := \int_0^t \cot(B_s) ds$.
- Finally, set $W_t = 2B_t + U_t$ and $\xi_t := \exp(iW_t)$.

We set $D_t = \mathbb{D} \setminus K_t$, and denote by f_t to be the conformal maps from D_t onto $\mathbb{D}$ normalized at the origin.

Notice that as the integral $\int_0^t |\cot(B_s)| ds$ is almost surely infinite, the definition of U_t can not be viewed as an usual absolutely converging integral. Yet, one can for instance define first U_t^ε as the integral of $\mathbf{1}_{\{d(B_s, \pi\mathbb{Z}) \geq \varepsilon\}} \cot(B_s)$ and show that these approximating processes U_t^ε converge in L^2 to a continuous processes U_t. See e.g. [WW13b] for a discussion of the chordal analogue.

We now describe some properties of the hulls generated by the radial $\mathrm{SLE}_4(-2)$. For more details, see [She09, Wer16, WW13b]:

1. The process $O_t := \exp(iU_t)$ describes the evolution of the marked point in the $\mathrm{SLE}_\kappa(\rho)$ framework. In particular, SLE coordinate change considerations (see [Dub07, SSW09]) show that during the time-intervals during which B_t is not in $\pi\mathbb{Z}$, the process behaves exactly like a chordal SLE_4 targeting the boundary prime end $f_t^{-1}(O_t)$ in the domain D_t.
2. It is therefore clear that during the open time-intervals at which B is not in $\pi\mathbb{Z}$, the Loewner chain is generated by a simple continuous curve. These excursions of B away from $\pi\mathbb{Z}$ correspond to the intervals during which the $\mathrm{SLE}_4(-2)$ traces "quasi-loops":

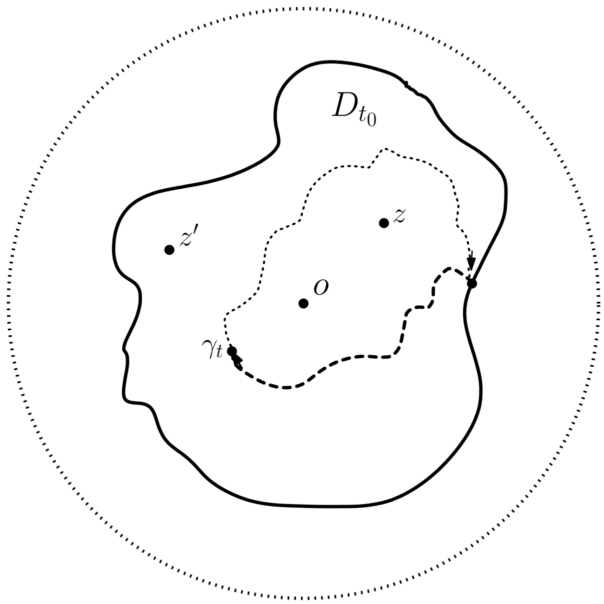


Figure 2.4: Sketch of D_{t_0}, the stretch $\gamma(t_0, t]$. The harmonic measure from 0 of the "top part" of $\gamma[t_0, t]$ corresponds to B_t. This loop corresponds to an excursion away from $\pi\mathbb{Z}$ by each of B, B^z and $B^{z'}$. It corresponds to an increase of π for B and for B^z, but not for $B^{z'}$. The end-time of this loop is $\sigma(z, z')$.

If (t_0, t_0') is an excursion interval away from $\pi\mathbb{Z}$, then $\gamma(t_0, t_0')$ is the image of a loop from ξ_{t_0} to itself in $\mathbb{D}$ via $f_{t_0}^{-1}$. When $B_{t_0} = B_{t_0'}$, then this quasi-loop does not surround the origin, when $B_{t_0'} = B_{t_0} \pm \pi$, it surrounds the origin clockwise or anti-clockwise. We stress that during any time-interval corresponding to an excursion of B away from $\pi\mathbb{Z}$ (e.g during (t_0, t_0')), the tip of the Loewner chain does not touch its past boundary and at the end of the excursion it accumulates at the prime end corresponding to the starting point.

3. It is known (via the loop-soup construction of CLE_4, see [SW12]) that these quasi-loops are in fact proper loops (i.e. that $f_{t_0}^{-1}(\xi_{t_0})$ is a proper boundary point). It can be also shown that the whole $\mathrm{SLE}_4(-2)$ is *generated* by a two-dimensional curve, see [MSW16], but we do not need this fact in the present chapter.

For any other $z \in \mathbb{D}$, one can define the radial $\mathrm{SLE}_4(-2)$ from 1 to z in $\mathbb{D}$ to be the conformal image of the radial $\mathrm{SLE}_4(-2)$ targeting the origin, by the Moebius transformation of the unit disc that maps 0 to z, and 1 onto itself. This is now a Loewner chain growing towards z, and it is naturally parametrized using the log-conformal radius seen from z.

Moreover the radial $\mathrm{SLE}_4(-2)$ processes for different target points can be coupled together in a nice way. This target-invariance feature was first observed in [Dub07, SSW09], and is closely related to the decompositions of $\mathrm{SLE}_4(-2)$ into SLE_4 "excursions" mentioned in (1). It says that for any z and z', the $\mathrm{SLE}_4(-2)$ targeting z and z' can be coupled in such a way that they coincide until the first time $\sigma(z, z')$ at which z' gets disconnected from z for the chain targeting z and evolve independently after that time. Note that in this coupling, the natural time-parametrizations of these two processes do not coincide. However, using the previous observation about the relation between the excursions of B and the intervals during which the $\mathrm{SLE}_4(-2)$ traces a simple curve, we see that up to time $\sigma(z, z')$ the excursion intervals away from $\pi\mathbb{Z}$ by the Brownian motions used to generate the $\mathrm{SLE}_4(-2)$ processes targeting z and z' correspond to each other. The time-change between the two Brownian motions can be calculated explicitly. For instance, if we define B_t^z to be the process obtained by time-changing the Brownian motion used in the construction that targets z via the log-conformal radius of D_t seen from 0 (instead of z), then as long as $t < \sigma(z, 0)$,

$$dB_t^z := 2\pi P^{f_t(z)} dB_t$$

where by P^z we denote the Poisson kernel in $\mathbb{D}$ seen from z and targeted at ξ_t (it follows from the Hadamard formula given below).

2.4.2 $\mathrm{SLE}_4(-2)$ branching tree and CLE_4

The target-invariance property of $\mathrm{SLE}_4(-2)$ leads to the construction of the radial $\mathrm{SLE}_4(-2)$ branching-tree. We next summarize its definition and its main properties:

- For a given countable dense collection $\mathcal{Z}$ of points in $\mathbb{D}$, it is possible to define on the same probability space a collection of radial $\mathrm{SLE}_4(-2)$ processes in such a way that for any z and z', they coincide (modulo time-change) up to the first time at which z and z' get disconnected from each other, and behave independently thereafter.
- For any z consider the first time τ_z at which the underlying driving function driving function exits the interval $(-\pi, \pi)$ and define $O(z)$ to be the complement of K_t at that time containing z. Define $O(z')$ similarly. The previous property shows that $z' \in O(z)$ if and only if $z \in O(z')$.
- The CLE_4 carpet is then defined to be the complement of $\cup_{z\in\mathcal{Z}} O(z)$. In this thesis, we write CLE_4 to denote this set (i.e. we call CLEs to be the random fractal sets, not the collection of loops).

Notice that from the construction, it is not obvious that the law of the obtained CLE_4 (and its nested versions) does not depend on the choice of the starting point (here 1) on $\partial\mathbb{D}$ (i.e. of the root of the exploration tree). However, Proposition 2.1 and Proposition 2.2 provide one possible proof of the fact that this choice of starting point does not matter (this is also explained in [WW13b], using the loop-soup construction of [SW12]). This justifies a posteriori that when one iterates the CLE_4 in order to construct the nested versions, one does in fact not need to specify where to continue the exploration.

On the other hand, from this construction it is easy to estimate the expected area of the ε-neighbourhood of CLE_4 and to see that the upper Minkowski dimension of CLE_4 is almost surely not greater than $15/8 = 2 - 1/8$. This follows from arguments in [SSW09]: One wants to estimate the probability that $d(z, \mathrm{CLE}_4) < r$. By conformal invariance it is enough to treat the case where z is the origin in the unit disc. But we can note that the conformal radius of D_t (in the $\mathrm{SLE}_4(-2)$) is comparable to $d(0, \partial D_t)$ by Koebe's 1/4-Theorem, and that the log-conformal radius of $\mathbb{D} \setminus \mathrm{CLE}_4$ from the origin is just minus the exit time of $(-\pi, \pi)$ by the Brownian motion B which is a well-understood random variable (see [PS78] for instance). Hence, the asymptotic behaviour as $r \to 0$ of the probability that $d(0, \mathrm{CLE}_4) < r$ can be estimated precisely.

One can also prove the somewhat stronger statement that the Hausdorff dimension of CLE_4 is in fact equal to $15/8$, by using second moment bounds (i.e., bounds on the probability that two points z and z' are close to CLE_4), see [NW11].

The nested $\mathrm{CLE}_{4,m}$ and CLE_4^M can then be defined by appropriately iterating independent CLE_4 carpets in the respective domains $O(z)$ (starting the explorations in the nested domains at the point where one did just close the loops). The definition of CLE_4^M is almost word for word the same, just replacing $\tau(z)$ by $\tau_M(z)$ which is the first time at which the underlying Brownian motion B exits the interval $(-M\pi, M\pi)$. A similar argument then shows that the upper Minkowski dimension of CLE_4^M is not greater than $2-(1/(8M^2))$ (the M^2 term is then just due to Brownian scaling – the exit time of $(-M\pi, M\pi)$ by Brownian motion is equal in distribution to M^2 times the exit time of $(-\pi,\pi)$).

2.4.3 Coupling with the GFF

To explain the coupling of CLE_4 with the GFF, we can first describe the coupling of a single radial $\mathrm{SLE}_4(-2)$ process with a GFF. The whole coupling then follows iteratively from the strong Markov property and from the branching tree procedure. The proof of the coupling of the radial $\mathrm{SLE}_4(-2)$ follows the steps of the coupling of the usual SLE_4 with the GFF, as explained for instance in [SS13, She16]. It is based on the following observations:

1. The Hadamard formula (see for instance [IK13]) gives the time-evolution of the Green's function under Loewner flow: for any two points x and y in $\mathbb{D}$, the Green's function $G_{D_t}(x,y)$ evolves until $\min\{\sigma(z,0),\sigma(w,0)\}$ according to

$$dG_{D_t}(z,w) = -2\pi P^{f_t(z)}P^{f_t(w)}dt,$$

where by P^z we denote the Poisson kernel in $\mathbb{D}$ seen from z and targeted at ξ_t as before.

2. The cross-variation $d\langle B^z, B^w\rangle_t$ between the two local martingales B^z and B^w is equal to $4\pi^2 P^{f_t(z)}P^{f_t(w)}dt$, so that

$$dG_{D_t}(z,w) = -\frac{1}{2\pi}d\langle B^z, B^w\rangle_t.$$

3. We can interpret $|B_t^z|/\pi$ (up to $\tau(z)$) as the harmonic measure in D_t seen from z, of the boundary arc between the tip of the curve and the force point U_t (the sign of B_t^z describes which of the two arcs one considers).

4. If we take $h_t(z)$ to be the harmonic extension to D_t of the function that has constant value $\mathrm{sign}(B_t)2\lambda$ on the boundary of ∂D_t between the tip and the force point, then the mean of $h_t(z)$ is zero and for $\lambda = \sqrt{\pi/8}$ we have:

$$dG_{D_t}(z,w) = -d\langle h_t^z, h_t^w\rangle.$$

Using these observations, one can first couple the GFF with the radial $\mathrm{SLE}_4(-2)$ up to the first time at which it surrounds the origin, exactly following the arguments of [She16]. This defines a BTLS with harmonic function in $\{-2\lambda, 0, 2\lambda\}$. For those domains where the harmonic functions are zero, one can then continue the $\mathrm{SLE}_4(-2)$ iteration targeting another well-chosen point in that domain and proceed.

This radial construction is arguably also the easiest one to explain that in fact, conditionally on the CLE_4, the labels ε_j are i.i.d. This is then just a consequence of the time-reversal of SLE_4, so that changing the orientation of the quasi-loop that traces the boundary of O_j corresponds to a measure-preserving transformation of the driving Brownian motion (see for instance [WW13b]). We come back to this in Section 2.9.

2.5 Comparisons of BTLS with CLE_4^M

We now derive Proposition 2.3, Proposition 2.1 and Proposition 2.4. In some sense, this section is the core of the chapter.

Throughout this section, D will denote a simply connected planar domain with non-empty boundary (so that the previous definition of CLE_4 makes sense).

2.5.1 Proof of Proposition 2.3

In this proof, (C, h_C) will denote the CLE_4^{M+1} and its corresponding harmonic function. Consider the triple $(\Gamma, (B, h_B), (C, h_C))$ where the two local sets (C, h_C) and (B, h_B) are conditionally independent given Γ and where B is a $(2\lambda M)$-BTLS. Recall from Section 2.4 that (C, h_C) can be constructed using a branching radial SLE_4 exploration tree denoted by SLE_4^r.

The two steps of the proof are then as follows:

- Given (B, h_B), we argue using Lemma 2.18 that when a branch of the SLE_4^r exploration tree is in the process of tracing a loop of C inside a connected component O of $D\backslash A$, then the loop it generates has to be contained in O.
- From this we deduce that for all $z \notin B \cup C$, one has $|h_{B\cup C}(z)| \leqslant 2\lambda(M+1)$. We then conclude using Lemma 2.9.

Let us now make it precise. For $z \in D$, we define by $O_B(z)$ (respectively $O_C(z)$) the connected component of $D \setminus B$ (resp. $D \setminus C$) containing z when $z \notin B$ (resp. $z \notin C$). Denote by ν^z the process obtained from the branch of the SLE_4^r tree that is directed at z and recall from Section 2.4 that $\nu^z([0,t])$ is a BTLS for all fixed t. Let K_t^z be the hull of ν_t^z with respect to the point z, i.e. the complement of the open component of $D \setminus \nu^z[0,t]$ containing z.

Claim 2.19 *Fix $t > 0$ and $z \in D$. Define $D_t(z) := O_B(z)\backslash K_t^z$ and let $E_{t,z}$ be event that $\nu^z(t)$ is in the middle of tracing a loop of C and that $\nu^z(t) \in O_B(z)$. Suppose that the probability of $E_{t,z}$ is strictly positive. Then, on the event $E_{t,z}$, the time $\tau = \sup\{t' : \nu^z[t,t'] \subset D_t(z)\}$ is finite and it is exactly the time at which the exploration closes the loop that it is tracing at time t. Thus, the starting and ending point of the loop correspond to at most two prime ends of $\partial D_\tau(z)$.*

Proof of Claim 2.19. Let us notice that $D_t(z)$ satisfies the condition of Lemma 2.17. Thus, conditioned on (B, h_B), $(\nu^z[0,t], h_{\nu^z[0,t]})$ and $E_{t,z}$ the process $\tilde{\nu}(s) := (\nu^z(t+s), s \geqslant 0)$ is a generalized level line in the domain D_t up to the time $\tau' := \sup\{s : \nu^z([t, t+s]) \subset D_t(z)\}$. On the event $E_{t,z}$, the path η_z is locally tracing a level line with heights $\pm(2\lambda(M+1))$ vs. $\pm(2\lambda M)$ at time t.

We know from Section 2.4 that almost surely the SLE_4^r exploration process touches itself

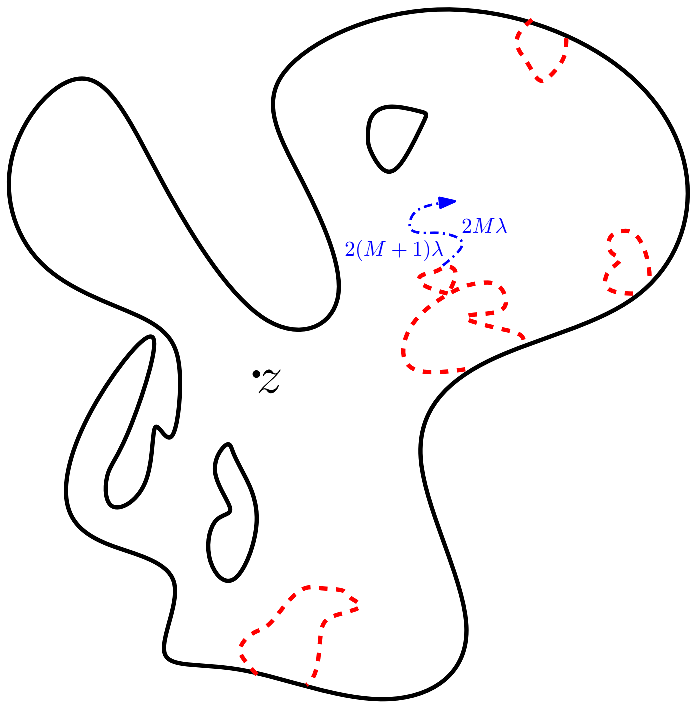


Figure 2.5: Sketch of the proof: in plain $\partial O_B(z)$; in dashed, the past of the radial exploration (used to define the CLE_4^{M+1}) that lies within $O_B(z)$. When one explores a loop along the dash-dotted interface: It cannot hit the dashed part before completing the CLE_4^{M+1} loop because of the almost sure properties of the radial $\mathrm{SLE}_4(-2)$ process. Also, it cannot touch the parts of $\partial O_B(z)$ that are away from the dashed part before completing the loop, because of the boundary hitting behaviour of the GFF level lines.

only when it closes a loop and stays at a positive distance of any other previously visited point.

It remains to show that $\tilde{\nu}_s$ does not touch any point of $\partial D_t \backslash K_t^z$. Take J any open set of D_t such that $d(J, K_t^z) \geqslant \varepsilon$. From Lemma 2.6, the boundary condition of $h_{B\cup\nu^z[0,t]}$ in $\partial J \cap \partial D_t$ are equal to those of h_B, thus their absolute value is not larger than $2M\lambda$. Lemma 2.18 now lets us conclude that $\tilde{\nu}(s)$ does not touch $\partial J \cap \partial D_t$ before τ'. The claim follows by taking the union over $\varepsilon > 0$. □

Claim 2.20 *Almost surely for all $z \notin B \cup C$, we have $|h_{B\cup C}(z)| \leqslant 2\lambda(M+1)$.*

Proof. Let $(z_n : n \in \mathbb{N})$ be a dense subset of D, such that for all $n \in \mathbb{N}$ the event $z_n \notin B$ has positive probability. It suffices to show that for all $n \in \mathbb{N}$ a.s. $|h_{B\cup C}(z_n)| \leqslant 2\lambda(M+1)$. On the event $z_n \notin B$ we have the following possibilities:

If E_{t,z_n} does not occur for any rational $t > 0$, then $O_B(z_n)$ is contained in $O_C(z_n)$. Thus, $|h_{B\cup C}(z_n)| = |h_B(z_n)| \leqslant 2\lambda(M+1)$.

If E_{t,z_n} occurs for some rational $t > 0$, from Claim 2.19 we deduce that either $O_C(z_n) \subset O_B(z_n)$ or the CLE_4^{M+1} loop surrounding z separates some components of B from the others, see Figure 2.6. Let us call $L(z_n)$ the connected component of the complement of $B \cup C$ that has this loop as part of its outer boundary. Importantly (see Lemma 3.11 of [SS13]), the boundary conditions of $h_{B\cup C}$ in $L(z_n)$ are given by those of h_B or h_C everywhere but at (at most) two prime ends corresponding to the beginning and the end of the relevant CLE_4^{M+1} loop. In this case the claim follows from Lemma 2.10 applied to the complement of $L(z_n)$ in D (note that because the CLE_4^{M+1} loop is at positive distance of ∂D and because of the BTLS condition for B, the complement of $L(z_n)$ can have only finitely many connected components). □

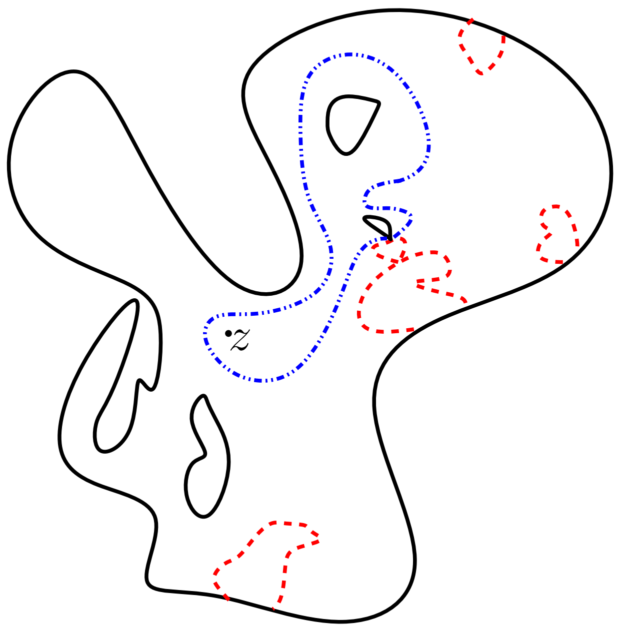


Figure 2.6: The continuous line represents ∂O_B, the dash-dotted line represents the recently closed loop and the dashed line represents the borders of the past loops traced by SLE_4^r.

Proposition 2.3 now follows from Lemma 2.9 applied to $A = C$ (noting that the set C satisfies Condition $(*)$).

2.5.2 Proof of Proposition 2.1

The proof of Proposition 2.1 goes along the same lines as that of Proposition 2.3. The difference lies in the fact that this time, C is a CLE_4^M and that $h_A \in \{-2\lambda M, 2\lambda M\}$. Thus, with the same notations as before, the boundary conditions on $\partial D_t(z) \setminus K_t^z$ are in $\{-2M\lambda, 2M\lambda\}$. As above, we conclude from Lemma 2.18 that the part of the radial $\mathrm{SLE}_4(-2)$ drawing $2\lambda(M-1)$ versus $2\lambda M$ level-line loop cannot exit $O_B(z)$ before completing that loop (one has to modify Figure 2.5, so that the dash-dotted interface η is now a $2M\lambda$ vs. $2(M-1)\lambda$ interface, and the continuous boundary data is $\pm 2M\lambda$).

Hence, it follows using the same argument as before that $|h_{A\cup C}| \leq 2\lambda M$, and then using Lemma 2.9 that $A \subseteq C$ almost surely. But this means that for all z, $|h_{A\cup C}(z)| = 2\lambda M = |h_A(z)|$ almost surely. Again, using Lemma 2.9, we see that $C \subseteq A$ almost surely.

2.5.3 Proof of Proposition 2.4

Suppose now that A is a K-BTLS such that with positive probability, there exists a connected component of A that is disconnected from ∂D. We choose some M such that $2\lambda(M-1) \geq K$.

We have just shown that almost surely, $A \subseteq C$ where C is the CLE_4^M coupled with the GFF.

On the other hand, with positive probability, $D \setminus A$ contains an annular open region that disconnects this connected component of A from ∂D. Hence, there exists a deterministic such annular region O such that with positive probability, O is in $D \setminus A$ and disconnects a connected component of A from ∂D (we call this event $E = E_O$).

Given A and h_A, the conditional distribution of Γ^A is a GFF. It follows from Lemma 2.12 that on the event E_O, the conditional distribution of Γ^A restricted to O is mutually absolutely continuous with respect to the conditional distribution of Γ itself restricted to O. It is possible to show, using Corollary 2.14 and the fact that with positive probability the radial $\text{SLE}_4(-2)$ makes a loop inside O, that with positive probability C does not intersect the interior part of the complement of O. But this contradicts the fact that $A \subseteq C$.

2.6 Two-valued sets

In this section, we describe the class of BTLS such that the harmonic function can only take two prescribed values and we determine for which values such a set does exist. These are what we will call the (thin) two-valued sets (TVS).

Some aspects of the following discussion are strongly related to the $\kappa = 4$ case of boundary conformal loop ensembles (and their nesting) as introduced and studied in [MSW16] (that was written up in parallel to the present chapter) for general κ.

2.6.1 A first special example

Let us first describe in some detail one specific example. Consider a (zero boundary) GFF in the unit disc and fix two boundary points, say $-i$ and i. Consider the level line of this GFF (i.e. for all $t \geqslant 0$, the curve that satisfies condition $(**)$ with $F_0 = 0$) . This is an $\text{SLE}_4(-1;-1)$ from $-i$ to i that is coupled with the GFF as a BTLS [MS16a]. It is known that this is a simple (boundary touching) continuous path η from $-i$ to i in the closed disc, and that its Minkowski dimension is almost surely equal to 3/2. It is measurable with respect to the GFF [MS16a] and thus we often say that we explore the GFF to find η.

The harmonic function h_η associated to this level line can be described as follows: First notice that the complement of the curve $\mathbb{D} \setminus \eta$ is a union of countably many connected components $(D_j^1)_{j\in J}$. Any component lies either to the right or to the left of the level line (if one views the level line as going from $-i$ to i). In each component D_j^1, the harmonic function has boundary conditions 0 on $\partial D_j^1 \cap \partial \mathbb{D}$. On $\partial D_j^1 \cap \eta$ the boundary condition is either λ or $-\lambda$, depending on whether D_j^1 is on the left or on the right of η.

As inside each component D_j^1 there is an independent GFF with these boundary conditions, we can iterate: Suppose for example that the D_j^1 lies to the right of η so that ∂D_j^1 is divided into two arcs, one of which is an excursion of η away from the ∂D, and the other one is a counter-clockwise arc from x_j to y_j of ∂D. We now explore the level line of this GFF from x_j to y_j with this boundary data (i.e. for all $t \geqslant 0$, the curve that satisfies condition $(**)$ with F_0 equal to the given boundary data). This level line has the law of an $\text{SLE}_4(-1)$ process from x_j to y_j and is again a simple curve. We proceed in a symmetric way in the connected components that lie above η. In this way, we obtain a new BTLS A_1, for which the harmonic function h_{A_1} is defined via the boundary conditions indicated in Figure 2.7.

The iteration then further proceeds by exploring additional level lines (which are usual $\mathrm{SLE}_4(-1)$ processes with just one marked point) in each of the remaining connected components which have a part of ∂D on their boundary. One then defines a second layer of loops and one proceeds iteratively. We then consider the closure $A_{-\lambda,\lambda}$ of the union of all the traced level lines.

Clearly, after any finite number of iterations in the previous construction, one has a λ-BTLS, and therefore a subset of the CLE_4^2 by our previous results. Hence, A is itself a subset of the CLE_4^2 and therefore a BTLS. It is also easy to see that a given point $z \in D$ is almost surely contained in a loop cut out after finitely many iterations, so that the harmonic function associated to A takes its values in $\{-\lambda, \lambda\}$.

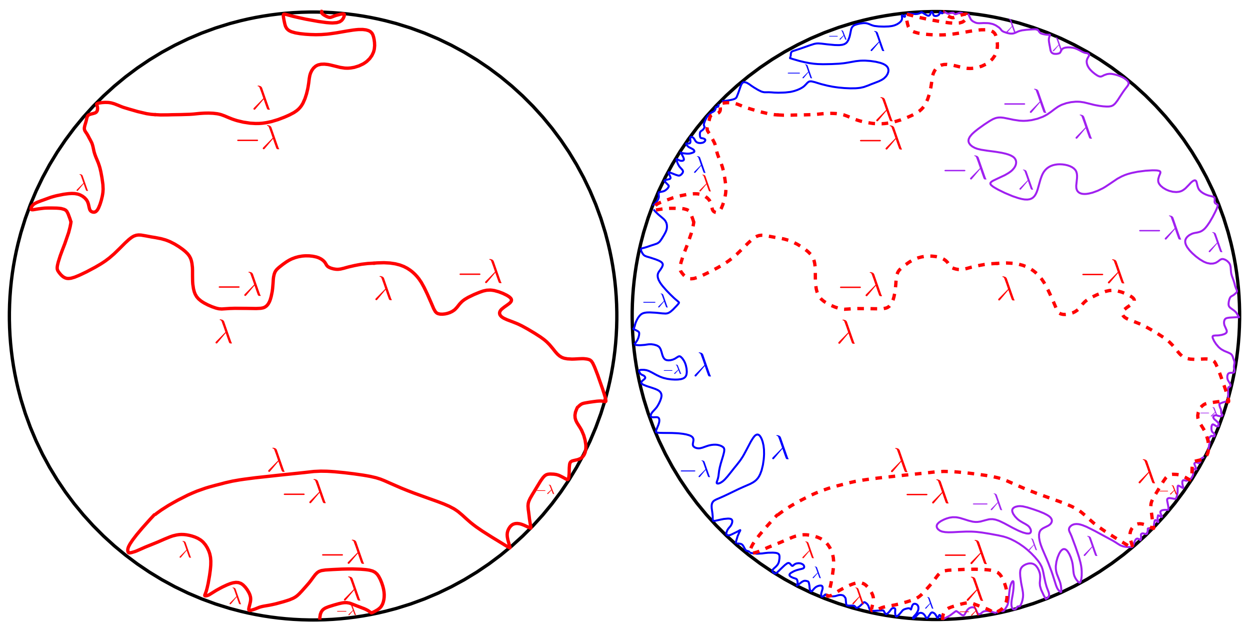


Figure 2.7: First iterations: The $\mathrm{SLE}_4(-1;-1)$ and its boundary conditions on the left. The $\mathrm{SLE}_4(-1)$ in the second layer on the right picture that creates loops with $\pm\lambda$ boundary conditions.

We can make the following observations about this set $A_{-\lambda,\lambda}$:

- As opposed to the CLE_4, the set $A_{-\lambda,\lambda}$ is just made out of the union of all SLE_4-type paths. For instance, each excursion of η away from $\partial\mathbb{D}$ is on the boundary between two connected components of $A_{-\lambda,\lambda}$ (one $+\lambda$ loop to its right and one $-\lambda$ loop to its left). This indicates that the Hausdorff dimension of $A_{-\lambda,\lambda}$ is almost surely $3/2$ (we will come back to this later).
- Because we have only used measurable sets in the construction $A_{-\lambda,\lambda}$, we know it is measurable function of the underlying GFF.
- In addition, it comes out that the set $A_{-\lambda,\lambda}$ is the only BTLS with boundary values in $\{-\lambda, \lambda\}$. In particular its law does not depend on the arbitrary choices of the start and end points in the previous layered construction. We prove it below in a more general context.

Remark 2.21 (A new construction of CLE_4) *Note that, similarly to the construction of CLE_4^M, we can iterate the construction of the set $A_{-\lambda,\lambda}$ to construct a BTLS with harmonic function that takes values in $\{-2\lambda, 2\lambda\}$. Indeed, for each z belonging to a given countable dense subset of D, one can iterate the construction in the component containing z until the boundary values are in $\{-2\lambda, 2\lambda\}$. At each step in the construction one has a $2\lambda-BTLS$ which is contained in CLE_4^2 and therefore, in the limit also, one still has a BTLS that is contained in the CLE_4^2. But from Proposition 2.1 it now follows that the obtained set is exactly the CLE_4. This therefore provides an alternative construction of CLE_4 (and of its iterated nested versions $CLE_{4,m}$ and CLE_4^M) that builds only on the coupling of the chordal $SLE_4(-1;-1)$ and $SLE_4(-1)$ process with the GFF. Notice that in this case the measurability of the CLE_4 just follows from that of the respective $SLE_4(-1;-1)$ and $SLE_4(-1)$ processes.*

As we point out in Section 2.7.1, there is also a direct way to see that this set $A_{-\lambda,\lambda}$ and its iterates are thin (without using the relation to CLE_4^M as we just did). Hence, we indeed obtain a stand-alone construction of CLE_4 and derivation of its properties. Interestingly, we do not know how to show that this construction gives the same law as CLE_4 without using the coupling with the GFF.

2.6.2 General sets $A_{-a,b}$ and proof of Proposition 2.2

We first construct a BTLS $A_{-a,b}$ such that h_A takes its values in $\{-a, b\}$, for all given pairs (a, b) such that $-a \leqslant 0 \leqslant b$ and $b + a \geq 2\lambda$. This generalizes our previous constructions of $A_{-\lambda,\lambda}$ and of CLE_4^M (that will be our $A_{-2M\lambda,2M\lambda}$). We then prove their uniqueness, the monotonicity of $A_{-a,b}$ with respect to a and b, and we show that there exist no BTLS with $h_A \in \{-a, b\}$ when $b + a < 2\lambda$ (unless a or b are equal to 0, in which case one can take the empty set).

Construction of $A_{-a,b}$ and measurability

We first the construct $A_{-a,b}$ for some ranges of values of a and b, and then describe the general case:

- *$a = 0$ or $b = 0$:* We set $A_{-a,b} = \emptyset$ and the corresponding harmonic function takes the value 0 everywhere.
- *$a = -n_1\lambda$ and $b = n_2\lambda$, where n_1 and n_2 are positive integers:* Note that similarly to the construction of CLE_4 in 2.21, we can iterate the construction of the set $A_{-\lambda,\lambda}$. Indeed, pick a countable number of dense $z \in D$ and iterate the construction in the component containing z until the boundary values are in $\{-n_1\lambda, n_2\lambda\}$.
- *$a + b = 2\lambda$:* Set $c := (b - a)/2 \in (-\lambda, \lambda)$ and repeat exactly the same construction as above, except that one now traces c-level lines i.e. $c - \lambda$ vs $c + \lambda$ interfaces iteratively instead of $-\lambda$ vs. λ interfaces. Exactly the same construction and the same arguments lead to the construction of a BTLS $A_{c-\lambda,c+\lambda}$ such that the corresponding harmonic function takes its values in $\{c - \lambda, c + \lambda\}$. These sets are called the boundary conformal loop ensembles for $\kappa = 4$ in [MSW16]. Notice that in these sets each interior boundary arc is shared by two components of the complement.
- *$a + b = n\lambda$ where $n \geqslant 3$ is an integer*: Define $c \in (-\lambda, \lambda)$ such that there exists two

non-negative integers n_1, n_2 with $a = c - n_1\lambda$ and $b = c + n_2\lambda$. Starting from $A_{c-\lambda,c+\lambda}$ we now iterate copies of $A_{-(n_1-1)\lambda,n_2\lambda}$ (resp. of $A_{-n_1\lambda,(n_2-1)\lambda}$) in the connected components of the complement of $A_{c-\lambda,c+\lambda}$ depending on the value of the harmonic function.

- *General case with $b + a > 2\lambda$*: We can assume that $b > \lambda$. Let $m \in \mathbb{N}$ such that $2m\lambda - a \in (b - 2\lambda, b]$. Define $A^1 := A_{-a,2m\lambda-a}$, and iteratively construct A^n in the following way:
  - If n is odd, then $D\backslash A^n$ is made of the closed union of loops with labels equal to either $-a$, b or $2m\lambda - a$. In every connected component, O, of $D\backslash A^n$ labelled $m\lambda - a$ we iterate $A_{b+a-4m\lambda,b+a-2m\lambda}$ of Γ^{A^n} restricted to O. Define A^{n+1} the closed union of A^n with the explored sets. Then all loops of A^{n+1} have labels $-a$, b or $b - 2m\lambda \in [-a, -a + 2\lambda)$.
  - If n is even, then $D\backslash A^n$ is made of the closed union of loops labelled either $-a$, b or $b - 2m\lambda$. In every connected component O of $D\backslash A^n$ labelled $b - 2m\lambda$ explore $A_{-a-b+2m\lambda,-a-b+4m\lambda}$ of Γ^{A^n} restricted to O. Define A^{n+1} the closed union of A^n with the newly explored sets. It is clear that all loops of A^{n+1} have label $-a$, b or $2m\lambda - a$.

  Then, $A_{-a,b} := \overline{\bigcup A^n}$.

We make the following observations about the $A_{-a,b}$ constructed above:

(i) In the construction we only need to use level lines whose boundary values are in $[-a, b]$.

(ii) For a fixed point $z \in D$ a.s. we only need a finite number of level lines to construct the loop of $A_{-a,b}$ surrounding z.

(iii) From the measurability of the level lines used in the construction, it follows that the sets $A_{-a,b}$ are measurable with respect to the underlying GFF.

Remark 2.22 *Note that $A_{-a,2\lambda-a}$ is constructed as an union of SLE$_4$-type paths. Moreover, each excursion of η away from $\partial\mathbb{D}$ is on the boundary between two connected components of Aa (one loop labelled $-a$ to its right and one $2\lambda - a$ loop to its left). Thus in particular the Hausdorff dimension of an ALE is almost surely equal to $3/2$. Additionally, each connected component O of Aa is such that $\overline{O} \cap \partial D \neq \emptyset$.*

Uniqueness

To show uniqueness, we follow loosely the strategy of the proof of Proposition 2.1.

Suppose that $\tilde{A}$ is another BTLS coupled with the same GFF, such that $h_{\tilde{A}}$ takes its values in $\{-a, b\}$ and such that conditionally on Γ, $A_{-a,b}$ and $\tilde{A}$ are independent. Consider some $z \notin \tilde{A}$ and denote by $O(z)$ the component of z in $D \setminus \tilde{A}$. Now, we claim that almost surely no level line in the construction of the component of z in $D \setminus A_{-a,b}$ can make an excursion inside of $O(z)$: Indeed, suppose that with positive probability a level lines does an excursion inside $O(z)$. On this event, using (ii) we can consider the first level line entering $O(z)$. Then, on the one hand this level line cannot exit $O(z)$ through the boundary of $\tilde{A}$, due to Lemma 2.18 and (i). On the other hand, it is also almost surely a simple path. Thus, it cannot exit $O(z)$ at all and we obtain a contradiction. As this holds for a countable dense family of z, we obtain that $A_{-a,b} \subseteq \tilde{A}$. We conclude using Lemma refmes with $k = (-a + b)/2$.

In particular, this implies that the arbitrary choices of points in the construction of $A_{-\lambda,\lambda}$ and also in the constructions of $A_{-a,b}$ do not matter.

As we, now, know that $A_{-a,b}$ is a measurable function of Γ, when there are several GFFs at hand, we sometimes write $A_{-a,b}(\Gamma)$ to be clear which GFF the set is coupled to. Sometimes, $\Gamma = \sum_O \Gamma^O$, where each O is a simply connected domain and Γ^O is an independent GFF in O. In those cases we write $A_{-a,b}(\Gamma, O)$ as the TVS of level $-a$ and b of the GFF Γ^O. Additionally, note that from the uniqueness statement we have that almost surely $A_{-a,b}(\Gamma) = A_{-b,a}(-\Gamma)$.

Monotonicity

Suppose $[-a, b] \subset [-a', b']$ and $-a < 0 < b$ with $b + a > 2\lambda$. Start with $A_{-a,b}(\Gamma)$. Inside the connected components O of $D\backslash A_{-a,b}$ labelled $-a$, resp. b, explore $A_{a-a',a+b'}(\Gamma^A, O)$, resp. $A_{-a'-b,b'-b}(\Gamma^A, O)$. We obtain a BTLS with boundary values in $\{-a', b'\}$. By uniqueness it follows that the obtained set is indeed equal to $A_{-a',b'}$ and by construction it contains $A_{-a,b}$.

There are no BTLS A with $h_A \in \{-a, b\}$ when a and b are non-zero and $a + b < 2\lambda$

First, one can discard the case where $-a$ and b have the same sign because the mean value of the field has to remain 0. When $-a < 0 < b$ and $b + a < 2\lambda$, suppose that A is a BTLS with $h_A \in \{-a, b\}$. Exploring $A_{-a-b,-a-b+2\lambda}$ in those connected components of the complement of A where the harmonic function $h_A = b$, we see that $A \subset A_{-a,-a+2\lambda}$. In particular, note that a connected component of $D \setminus A$ where the corresponding harmonic function h_A is equal to $-a$ remains a connected component of $D\backslash A_{-a,-a+2\lambda}$. Such a connected component has boundary value $-a$, but has no boundary arc that is shared with a component of $D \setminus A_{-a,-a+2\lambda}$ where the harmonic function is $-a + 2\lambda$ on the other side. This leads to a contradiction with an observation made above, because we know from the construction of $A_{-a,-a+2\lambda}$ that all interior boundary arcs are shared by two components of the complement of this set.

2.7 Dimensions

In this section, we will give results about the fractal dimension of the sets $A_{-a,b}$ and of subsets related to it.

2.7.1 Dimension of $A_{-a,b}$

Let us first derive the following fact:

Proposition 2.23 *For each given z, the random variable* $\log(\text{crad}(z, \mathbb{D})) - \log(\text{crad}(z, \mathbb{D} \setminus A_{-a,b}))$ *is distributed like a constant times the exit time from $(-a\pi/(2\lambda), b\pi/(2\lambda))$ by a one-dimensional Brownian motion B started from* 0.

In particular, for each given z, the probability that $d(z, A_{-a,b}) < r$ is (up to constants) comparable to r^s for $s = s_{-a,b} := 2\lambda^2/(b + a)^2$ as $r \to 0$.

PROOF. Let us first focus on the case of the set $A = A_{-\lambda,\lambda}$. Fix a point $z \in \mathbb{D}$ and note that the construction of the set A is obtained via a continuously increasing family of sets $(A_t, t \geq 0)$ that correspond to the concatenation of the various chordal $\mathrm{SLE}_4(-1;1)$ and $\mathrm{SLE}_4(-1)$ processes that one iterates. All these sets A_t are clearly local sets, and the value of the corresponding harmonic function $h_t(z)$ at z is always in $[-\lambda, \lambda]$ (as its boundary values are in $\{0, \lambda, -\lambda\}$). Furthermore, by definition it is a local martingale, and therefore a martingale. We know that it converges to either $+\lambda$ or $-\lambda$ as $t \to \infty$.

On the other hand, we know (see e.g.. [MS16a]) that when R_t is a continuously increasing family of local sets then $h_{R_t}(z)$ evolves like a $2\lambda/\pi$ times the standard Brownian motion when parametrized by the decrease of the log-conformal radius of $\mathbb{D}\backslash R_t$ seen from z. This therefore implies that $\log(\mathrm{CR}(z, \mathbb{D})) - \log(\mathrm{CR}(z, \mathbb{D} \setminus A))$ is distributed like the exit time of $(-\pi/2, \pi/2)$ by a one-dimensional Brownian motion. The tail estimate then follows from [SSW09].

Exactly the same argument can be applied to all $A_{-a,b}$'s that we have constructed – one just needs to note that in our iterative procedure, all the iterations are independent and the harmonic functions h_t always remain in $[-a, b]$. □

Note that this argument can be used to see that $A_{-\lambda,\lambda}$ is indeed a BTLS with upper Minkowski dimension almost surely not larger than 3/2 and that CLE_4 obtained by iterations of $A_{-\lambda,\lambda}$ is indeed a BTLS (as it satisfies (3)).

More generally, this result means that the so-called expected Minkowski dimension of $A_{-a,b}$ is equal to $2 - s_{-a,b}$. Using the GFF techniques (as in [MW17]), it should be then in principle be no problem to control second moment estimates, and to see that in fact, the Hausdorff dimension of $A_{-a,b}$ is almost surely equal to $2 - s_{-a,b}$.

2.7.2 Dimension of boundary-intersections of loops

Let us now study the boundaries of the connected components of the complement of $A_{-a,b}$.

Let A be a closed set with empty interior (as is the case for thin local sets, Lemma 1.16). We say that ℓ is a loop of A if ℓ is the boundary of a connected component of $D\backslash A$. In our case, these boundary components are indeed Jordan curves, motivating the name loop. Define $\mathrm{Loop}(A)$ as the set of loops of A, and note that $A = \overline{\bigcup_{\ell \in \mathrm{Loop}(A)} \ell}$. In this context, we say that a loop $\ell \in \mathrm{Loop}(A_{-a,b})$ is labelled $-a$, resp. b, if $h_{A_{-a,b}}$ restricted to the interior of ℓ is equal to $-a$, resp. b. We, sometimes, abuse notations and say that the inside of the loop is labelled $-a$ or b.

Let us already note that as SLE_4-type loops, these loops ℓ have Hausdorff dimension 3/2. We are now going to study their intersection with the boundary of the domain. A first result in this direction is the following:

Lemma 2.24 *Let $a, b, \delta \geqslant 0$ with $a + b \geqslant 2\lambda$. Then almost surely,*

1. *The loops of $A_{-a,b}$ with label $-a$ are loops of $A_{-a,b+\delta}$ with label $-a$.*
2. *A loop ℓ of $A_{-a,b}$ labelled $-a$ touches the boundary iff $a < 2\lambda$ and ℓ is a loop of $A_{-a,2\lambda-a}$ labelled $-a$.*

PROOF. For the first part, note that we can construct $A_{-a,b+\delta}$ in the following way.

- Explore $A_{-a,b}$.
- Explore $A_{-a-b,\delta}(\Gamma^{A_{-a,b}}, O)$ inside each connected component O of $D\backslash A_{-a,b}$ labelled b .

Defining A' as the closed union of $A_{-a,b}$ with the newly explored sets, we see that it is a BTLS where the harmonic functions is in $\{-a, b+\delta\}$. Thus, $A' = A_{-a,b+\delta}$ by uniqueness (Proposition 2.2). The claim follows because in the second step there were no explored set inside the loops labelled $-a$.

For the second part, let us first assume $a \geqslant 2\lambda$ and show that no loop with label $-a$ touches the boundary. This follows from the fact that $A_{-2\lambda,2\lambda}$ has the law of a CLE_4 (Section 2.4) and no loop of CLE_4 touches the boundary [SW12]. If $b \geqslant 2\lambda$, then $A_{-2\lambda,2\lambda} \subseteq A_{-a,b}$ by monotonicity of TVS and we see directly that there are no loops of $A_{-a,b}$ touching the boundary. If $b < 2\lambda$, then from part (1) it follows that all loops of $A_{-a,b}$ with label $-a$ are also loops of $A_{-a,2\lambda}$ with the same label and thus do not touch the boundary.

Now, let us study the case $a < 2\lambda$. By using the part (i) and the fact that all loops of $A_{-a,2\lambda-a}$ touch the boundary we get that all loops of $A_{-a,b}$ that are loops of $A_{-a,2\lambda-a}$ touch the boundary. We still need to show that these are the only ones. Suppose $b \geqslant -a+2\lambda$. Then one can construct $A_{-a,b}$ by first exploring $A_{-a,2\lambda-a}$ and then exploring $A_{-2\lambda,b+a-2\lambda}(\Gamma^{A_{-a,2\lambda-a}}, O)$ inside all connected components O of $D\backslash A_{-a,2\lambda-a}$ with the label $-a+2\lambda$. Thus, the loops with label $-a$ in $A_{-a,b}$ are of two types: either those with the label $-a$ in $A_{-a,2\lambda-a}$, or those with the label -2λ in $A_{-2\lambda,b+a-2\lambda}(\Gamma^{A_{-a,2\lambda-a}}, O)$. We conclude by noting that, thanks to the previous paragraph, the latter loops do not touch the boundary of the domain. □

To understand how 'often' the loops of $A_{-a,b}$ hit the boundary, we can see in [MW17, Sch17] that the Hausdorff dimension of the intersection of an $\text{SLE}_4(\rho)$ curve in the upper half-plane with the real line is almost surely equal to

$$1 - \frac{(\rho+2)^2}{4}$$

for $\rho \in (-2, 0)$ (note that the case $\kappa = 4$ is stated but not proved in [MW17] but that the techniques used there for $\kappa \neq 4$ should indeed provide the result). This implies readily the following fact about the dimension of the intersection of the loops labelled $-a$ in $A_{-a,2\lambda-a}$ with the boundary:

Corollary 2.25 *Let $0 < a < 2\lambda$, and Γ a GFF in $\mathbb{H}$. Then a.s. any loop of $A_{-a,2\lambda-a}$ labelled $-a$ either does not touch the boundary or it touches the boundary infinitely often. In the latter case the set of intersection points has Hausdorff dimension $1-(2-a/\lambda)^2/4$.*

Finally, recall that one can construct $A_{-a,b}$ for $b > 2\lambda - a$ by first exploring $A_{-a,2\lambda-a}$ and then exploring sets $A_{-2\lambda,b+a-2\lambda}$ inside the loops with the label $-a+2\lambda$. Thus from (2) of Lemma 2.24 and from the fact that $A_{-a,2\lambda-a} = \partial M$ where M is the union of all loops of $A_{-a,2\lambda-a}$ with the label $-a$ (see Remark 2.22), it follows that we can inversely reconstruct $A_{-a,2\lambda-a}$ from $A_{-a,b}$ by just observing its intersection with the boundary (if we suppose that this boundary is smooth):

Corollary 2.26 *Suppose $0 < a < 2\lambda$ and that $b > 2\lambda - a$ and that $a \neq b$. Let M be the union of all loops of $A_{-a,b}$ touching the boundary with dimension $1-(2-a/\lambda)^2/4$, then almost surely $\partial M = A_{-a,2\lambda-a}$.*

2.8 Connectivity properties

Let us now state and derive results on the connectivity properties of the loops of $A_{-a,b}$.

We will formulate these connectivity properties using two different graphs. We say that two loops of $A_{-a,b}$ are 'side-connected' if the intersection of these two loops contains a set that is homeomorphic to a segment; we say that the are 'point-connected' if their intersection is non-empty. Consider the graphs G_s, G_p whose vertex set are the loops of $A_{-a,b}$ and edge sets E_s, E_p consisting of pairs of loops that are either 'side-connected' or 'point-connected' respectively. Notice that by definition $E_s \subset E_p$.

We now state how the connectivity properties of the loops of $A_{-a,b}$ depend on $a+b$:

Proposition 2.27 *Let $A_{-a,b}$ with $a, b > 0$ and $a + b \geqslant 2\lambda$ be a two-valued set of level $-a$ and b. Then*

1. *If $a+b = 2\lambda$, the graph G_s is equal to G_p. Additionally, it is connected, i.e. one can pass from each loop to any other one in finite number of steps using 'side-connections'.*
2. *If $2\lambda < a+b < 4\lambda$, the edge set E_s is empty but the graph G_p is connected, i.e. one can pass from each loop to any other one in finite number of steps using 'point-connections'.*
3. *If $a+b \geqslant 4\lambda$, then E_p is empty, or in other words all loops are pairwise disjoint.*
4. *Moreover, in all cases, any two loops with the same label are neither side-nor point-connected, in particular G_p and G_s are bipartite.*

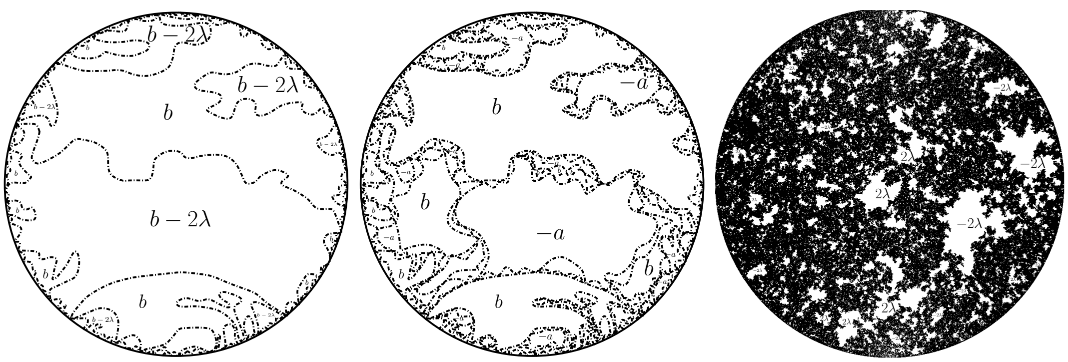


Figure 2.8: The three phases described in Proposition 2.27. The left picture represents $A_{b-2\lambda,b}$: two loops that intersect share a whole side. The middle pictures represents $A_{-a,b}$ with $a \in (2\lambda - b, 4\lambda - b)$: each loops intersects with infinitely many other loops, but no two loops share a side; also loops with the same label do not touch. The right picture is a simulation by D. Wilson of $A_{-2\lambda,2\lambda}$, in which case all loops are pairwise disjoint.

The rest of this section will be devoted to the proof of this proposition. We will prove parts (1), (2), (3) and (4) in that order:

Proof of (1): the ALE ($a+b = 2\lambda$)

We use the construction of the basic TVS given in Section 2.6.1; in particular recall the notation A^n from this Section - here n refers to the $n-$th layer of level lines in the construction.

By conformal invariance we may assume that we are working in $\mathbb{H}$ and that the first level line is started from 0 and targeted to ∞.

Let us start by showing that all loops of $A_{-a,2\lambda-a}$ that also belong to A^2 are connected via a finite path in G_s. We differentiate two types of loops: those which contain a segment joining $\mathbb{R}^-$ to $\mathbb{R}^+$, and those which touch either only $\mathbb{R}^+$ or $\mathbb{R}^-$. Notice that the loops of the second type are at a distance 1 from some loop of the first type. Thus, it suffices to prove that the loops of the first type are connected via a finite path only using loops of the first type, i.e., loops that will also belong to G_s. This, however, follows from the fact that the level line is a.s. non-self-crossing, continuous up to its target point, and attains its target point almost surely ([WW16]).

Now, notice that the rest follows inductively: indeed, any loop of $A_{-a,2\lambda-a}$ that was not present at A^n, but is present at A^{n+1} is side-connected to a loop that appears at level A^n. Thus, as any loop of $A_{-a,2\lambda-a}$ appears at A^N for some finite random N, we conclude that G_s is connected.

It also follows from the construction that $G_p(A^n) = G_s(A^n)$ and that any loops that share a segment have different labels.

Proof of (2): the connected phase ($2\lambda < a+b < 4\lambda$)

First notice that throughout this subsection it is sufficient to work in the case when $a < 2\lambda$ (otherwise we can consider $A_{-b,a}$).

From the construction of A^n (Section 2.6.2), it follows that in this phase two loops do not share sides. Indeed, we first construct $A_{-a,2\lambda-a}$ and then iterate TVS in loops with value $-a+2\lambda$; as no loop of any TVS shares a segment with the boundary, the claim follows.

To show that $A_{-a,b}$ is point-connected in this regime, it suffices to prove two things:

- All loops are point-connected to a loop with label $-a$ that touches the boundary.
- All loops labelled $-a$ that touch the boundary are point-connected between each other.

Claim 2.28 *Let $2\lambda \leqslant a+b < 4\lambda$. Then almost surely for every loop of $A_{-a,b}$ there exists a path in G_p connecting it to a loop that touches the boundary and has label $-a$.*

Proof. When $a+b = 2\lambda$, we are in the case of an ALE, and thus all loops are point-connected to the boundary. So suppose $2\lambda < a+b < 4\lambda$. To deal with this case, recall the very last construction of Section 2.6.2, where in this concrete case we have $m = 2$. Thus $A_{-a,b}$ can be constructed by starting from $A_{-a,2\lambda-a}$ and then iterating ALEs inside the loops that do not yet have value $-a$ or b. As all loops of any ALE touch the boundary, and we iterate at every step only in the loops that don't have value $-a$ or b, we see that any loop constructed at some finite step n intersects a loop constructed at step $n-1$. Thus, any loop constructed at step n is point-connected to a boundary-touching loop via a path of (side-length) n. As any loop is constructed at some finite random step N, the claim follows. □

We now show that any two boundary touching loops with label a are point-connected. Note that, thanks to Lemma 2.24 all boundary-touching loops of $A_{-a,b}$ with label $-a$ are also loops of $A_{-a,2\lambda-a}$ which is point-connected. Thus, it suffices to prove that any two loops with label $-a$ intersecting the boundary and at a distance 2 in $A_{-a,2\lambda-a}$ are at a finite distance in $G_p(A_{-a,b})$.

But (as in the previous claim) $A_{-a,b}$ can be constructed by first exploring $A_{-a,2\lambda-a}$ and then further exploring $A_{-2\lambda,b+a-2\lambda}$ inside the loops with label $-a+2\lambda$. Moreover, any two loops of $A_{-a,2\lambda-a}$ labelled $-a$, and that are at distance 2 in $A_{-a,2\lambda-a}$ are side-connected to a loop of label $-a+2\lambda$ of $A_{-a,2\lambda-a}$. In particular, to show that these two loops are at finite distance in $G_p(A_{-a,b})$, it suffices to prove the following claim:

Claim 2.29 *Let $2\lambda \leqslant a'+b' < 4\lambda$. Then, for any two fixed intervals of the boundary, there is almost surely a point-connected path in $G_p(A_{-a',b'})$ going from a loop touching one interval to another loop touching the other interval.*

For simplicity, assume that we work now in $\mathbb{D}$, we take again $a'=a$, $b'=b$ and fix two disjoint boundary arcs $I=[x,y]$ and $\tilde{I}=[\tilde{x},\tilde{y}]$, where the arcs are taken in a counter-clockwise sense.

Proof of the claim. The proof is again based on choosing a particular way of constructing $A_{-a,b}$: we first build loops that touch I and then iteratively build loops that are of larger and larger distance from I and go towards $\tilde{I}$. In this respect, consider $A^1 := \eta^1([0,\infty])$, where η^1 is a $(-a,-a+2\lambda)$ level line of from y to x. Either

- η_1 hits $\tilde{I}$: then from the construction of $A_{-a,b}$ in Section 2.6.2 we see that η_1 is part of the boundary of a loop of $A_{-a,b}$ labelled $-a$ touching both I and $\tilde{I}$ and we are done.
- η_1 closes at y without hitting $\tilde{I}$: in this case $\tilde{I}$ belongs to the boundary of a connected component of $D\backslash\eta_1$. By the fact that η_1 is non-crossing and ends at y, we know that the part of the level line boundary has boundary condition $-a+2\lambda$ on the side of $\tilde{I}$. Moreover, on the other side of this level line segment we can, using an additional level line, finish a loop labelled $-a$ that touches I.

In the latter case we continue the construction recursively: if A^n does not touch $\tilde{I}$, we continue the construction of $A_{-a,b}$ in O^n, the connected component of $D\backslash A^n$ that contains $\tilde{I}$. We now take I^n to be the subset $\partial O^n \cap A^n$. Notice that the extremal distance of I^n to I in $D\backslash A^n$ is larger than the extremal distance to $\tilde{I}$ as I. Denote $I^n=[x^n,y^n]$ in a counter-clockwise sense and depending on the parity of n, continue as follows:

- If n is odd, h_{A^n} is equal to $-a+2\lambda$ on I^n and zero on $\partial O^n\backslash A^n$. Hence, as $b-2\lambda < -a+2\lambda < b$, from Lemma 2.15 it follows that we can explore a $(b-2\lambda,b)$-level line η^{n+1} of $\Gamma^{A^n}+h_{A^n}$ restricted to O^n from x^n to y^n. We now have two scenarios as above: either the level line intersects $\tilde{I}$, then we stop as above and observe that there is now a loop with label b that joins $\tilde{I}$ and I^n. Otherwise we set $A^{n+1} := A^n \cup \eta^{n+1}([0,\infty])$ and note that the boundary values of $h_{A^{n+1}}$, restricted to closed connected component of $D\backslash A^{n+1}$ containing $\tilde{I}$, are equal to $b-2\lambda$ on the level line $\eta^{n+1}([0,\infty])$. Moreover, on the other side of this level line segment we can, using an additional level line, finish a loop labelled b that touches I^n.
- If n is even, h_{A^n} restricted to O has boundary values constant equal to $b-2\lambda$ on I_n and zero elsewhere on $\partial O^n\backslash A^n$. Thus, it is possible to construct a $(-a,-a+2\lambda)$ level line η^{n+1} of Γ^{A^n} from y^n to x^n. Again, if η^{n+1} intersects $\tilde{I}$ we stop. If not, we observe that the boundary values of $h_{A^{n+1}}$, restricted to closed connected component of $D\backslash A^{n+1}$ containing $\tilde{I}$, are equal to $-a+2\lambda$ in η^n. Moreover, on the other side of this level line

segment we can, using an additional level line, finish a loop labelled $-a$ that touches I^n.

We claim that this procedure stops at a finite (random) time N almost surely. Indeed, by using conformal invariance of generalised level lines, we can map O^n to the unit disk via ϕ^nsuch that I^n maps to a fixed interval. As the extremal distance between I^n and $\tilde{I}$ is decreasing, the $\phi^n(\tilde{I})$ is increasing in length. As on the other hand the boundary conditions are equal on I^n on even and odd steps separately, and equal to zero elsewhere on ∂O^n, we conclude that the probability of hitting $\tilde{I}$ before finishing at y is increasing separately in even and odd steps (for $n \geqslant 3$). As this probability is non-zero to begin with (as $\text{SLE}_4(\rho_1, \rho_2)$ process hits any interval that it potentially could hit wit positive probability), we see that N is stochastically dominated by a geometric random variable of positive parameter p.

As each I^n is joined to I via a path of n point-connected loops of labels $-a$ or b, it just remains to see that we can finish the construction of $A_{-a,b}$ (i.e. that these loops indeed belong to $A_{-a,b}$. Notice that after the part of the construction above, it then remains to construct $A_{-a,b}$ in simply connected components with possibly non-zero, but always piece-wise constant boundary conditions with values in $[-a, b]$. Thus, the claim follows from Proposition 3.22, which is just a slight generalisation of the construction in Section 2.6.2.

□

Proof of (3): the disconnected case ($a + b \geqslant 4\lambda$)

In the case of $a = b = 2\lambda$, the claim follows from the fact that $A_{-2\lambda,2\lambda}$ has the law of a CLE_4. Now consider $A_{-a,-a+4\lambda}$ with $a < 2\lambda$. We can construct it by first exploring $A_{-a,2\lambda-a}$. Then, inside connected components O of $D\backslash A_{-a,2\lambda-a}$ labelled $-a + 2\lambda$, we explore $A_{-2\lambda,2\lambda}(\Gamma^{A_{-a,2\lambda-a}}, O)$. The closed union of the explored sets gives precisely $A_{-a,-a+4\lambda}$.

We know that loops with the label $-a$ that also belong to $A_{-a,2\lambda-a}$ do not touch each other. But all other loops come from exploring $A_{-2\lambda,2\lambda}(\Gamma^{A_{-a,2\lambda-a}}, O)$ in the second step. These loops do not touch each other, nor the boundary of O, i.e. the loops with label $-a$.

For general $a + b \geqslant 4\lambda$, we conclude from the previous case and the monotonicity: any loop of $A_{-a,b}$ is contained in the interior of some loop of $A_{-a',-a'+4\lambda}$ for some $0 < a' < 4\lambda$.

Similarly, we can draw the following Corollary:

Corollary 2.30 *Suppose $a \geqslant 2\lambda$. Then the loops of $A_{-a,b}$ with the label $-a$ are pairwise disjoint. Similarly if $b \geqslant 2\lambda$, then all loops with the label b are pairwise disjoint.*

Proof. From Lemma 2.24, we know that loops with label $-a$ in $A_{-a,b}$ remain also loops with label $-a$ of $A_{-a,b+2\lambda}$. But we know that any two loops of the latter are disjoint. □

Proof of (4): loops with the same label do not touch

As this is trivially true in the case $a + b \geqslant 4\lambda$, we suppose $a + b < 4\lambda$.

First, note that for the ALE ($a+b = 2\lambda$) this follows from the construction using level lines. Indeed, the labels on the two sides of a level line are different, so two loops could only touch at the endpoints of $\text{SLE}_4(\rho_1, \rho_2)$ excursions constructing them. But any two $\text{SLE}_4(\rho_1, \rho_2)$

excursions away from the boundary are disjoint as they correspond to excursions of Bessel-type of processes with dimension strictly less than 2. Moreover, for the same reason none of the excursions also touches the starting point nor the endpoint of the process.

Suppose now that $2\lambda < a+b < 4\lambda$ and assume WLoG that $a < 2\lambda$. Let us first show that no two loops with label $-a$ touch each other. As in the proof of Lemma 2.24, we can construct $A_{-a,b}$ by (1) first exploring $A_{-a,-a+2\lambda}$ and then (2) exploring $A_{-2\lambda,b+a-2\lambda}$ inside the connected components of $D\backslash A_{-a,-a+2\lambda}$ with the label $-a+2\lambda$. Now, we know that no two loops labelled $-a$ from step (1) can touch each other by the previous paragraph. Also, by Lemma 2.24 no loop labelled$-a$ constructed in step (2) touches the loops with the label $-a$ of step (1). Finally, by Corollary 2.30 loops labelled $-a$ constructed in step (2) are also pairwise disjoint.

Consider now the loops with label b. If $b < 2\lambda$, then we can argue as just above. If $b \geqslant 2\lambda$, two loops with label b do not touch each other by Corollary 2.30.

2.9 Measurability of the labels

Note that conformal invariance of the GFF implies the conformal invariance of $A_{-a,b}$ and thus the fact that $\mathbb{P}(h_{A_{-a,b}}(z) = -a)$ does not depend on $z \in D$. Using the fact that $\mathbb{E}((\Gamma, 1)) = 0$, we therefore see that for all $z \in D$,

$$\mathbb{P}(h_{A_{-a,b}}(z) = -a) = \frac{b}{a+b} \quad \text{and} \quad \mathbb{P}(h_{A_{-a,b}}(z) = b) = \frac{a}{a+b}.$$

This already indicates some sort of asymmetry between the set of points with label $-a$ and the set of points with label b when $a \neq b$.

Let us note that the construction of CLE_4 recalled in Section 2.4 shows it is possible to first sample the CLE_4 $A_{-2\lambda,2\lambda}$ and then the labels ε_j using independent fair coins. The idea is that one can define the symmetric radial $\mathrm{SLE}_4(-2)$ using a Poisson point process of SLE_4 bubbles, which is invariant under resampling of the orientations of the bubbles, see e.g. [WW13b]. In other words, conditionally on $A_{-2\lambda,2\lambda}$ the value of $h_{-2\lambda,2\lambda}$ in each of the connected components is an i.i.d. family of fair coin-tosses. The labels are clearly not a deterministic function of $A_{-2\lambda,2\lambda}$.

Note that the conditional independence between the labels in different connected components feature is specific to CLE_4. For instance, for CLE_4^M the existence of a correlation between the heights in different domains is clear from the construction.

In the case of $A_{-\lambda,\lambda}$, the situation is actually quite different: Conditionally on $A_{-\lambda,\lambda}$, one single fair coin toss that decides the sign of the harmonic function at the origin is enough to determine the harmonic function in all the other connected components of the complement of $A_{-\lambda,\lambda}$. This follows from the fact that any two neighbouring components have necessarily different heights and from the connectivity properties stated in the previous section.

In the case of $A_{c-\lambda,c+\lambda}$ for $c \neq 0$, as it turns out, one does not even need to toss a coin, as the asymmetry (and conformal invariance) makes it in fact possible to detect almost surely the sign of the harmonic function at the origin. From this point of view, it is even quite intriguing that by iterating $A_{-\lambda,\lambda}$, one obtains a CLE_4 where the signs of the heights are independent in the different components. In fact it is possible to determine precisely to which extent the function h_A is a measurable functions of A when $A_{-a,b}$.

Let us now state the main result of this section:

Proposition 2.31 *Let $a, b > 0$ and consider the local set coupling $(\Gamma, A_{-a,b}, h_{A_{-a,b}})$:*

- *If $2\lambda \leqslant a+b < 4\lambda$ and $a \neq b$, then the labels of $A_{-a,b}$ are a measurable function of the set $A_{-a,b}$ (or in other words $h_{A_{-a,b}}$ is measurable w.r.t. $A_{-a,b}$).*
- *If $\lambda \leqslant a < 2\lambda$, the labels of $A_{-a,a}$ are a measurable function of the set $A_{-a,a}$ and the label of the loop surrounding* 0.
- *If $a+b \geqslant 4\lambda$, the labels of $A_{-a,b}$ cannot be recovered only knowing $A_{-a,b}$ and any finite number of labels.*

Again, the proof of these facts will be based on the iterative constructions of the TVSs. Another ingredient will be the results about the Hausdorff dimension of the $A_{-a,b}$ loops that allow to detect the label associated to a loop.

The rest of this section will be devoted to the proof of this proposition:

Connected case with $a \neq b$ $(2\lambda \leqslant a+b < 4\lambda)$

We may again assume that $a < 2\lambda$. From Corollary 2.25 we know the label of any loop touching the boundary with Hausdorff dimension $1-(2-a/\lambda)^2/4$ has the label $-a$. Moreover by Lemma 2.24 there is some loop ℓ_a with label a that touches the boundary. But now from Proposition 2.27 it follows that any loop ℓ is point-connected to ℓ_a. The label of ℓ is $-a$ if the graph distance in G_p between ℓ and ℓ_a is even, and b if it is odd.

Connected case with case with $a = b$ $(\lambda \leqslant a < 2\lambda)$

The previous proof fails as loops of label $\pm$ touch the boundary with the same Hausdorff dimension. However, as soon as we know the label of the loop surrounding 0 we can again similarly use the connectedness of G_p to deduce the claim.

Disconnected case $(a+b \geqslant 4\lambda)$

This case needs a bit more care. The idea is to use the fact (e.g see Section 2.4.3) that conditionally on the set $A_{-2\lambda,2\lambda}$, the labels are given by i.i.d. fair coin tosses.

In this respect, we show that there are two GFF Γ and $\tilde{\Gamma}$ such that a.s.

(a) $A_{-a,b}(\Gamma) = A_{-a,b}(\tilde{\Gamma})$.

(b) There are infinitely many loops of $A_{-a,b}(\Gamma)$, such that their loop under Γ is different than their loop under $\tilde{\Gamma}$.

(c) Any finite subset of labels has the same value for Γ and $\tilde{\Gamma}$ with positive probability.

Note that this implies the statement, as conditionally on $A_{-a,b}$ and any finite subset of labels, the (conditional) law of the rest of the labels is non-trivial.

First let us construct this coupling when $a = b \geqslant 2\lambda$. We first sample a GFF Γ and then explore $A_{-a,a}(\Gamma)$. We do this in two steps:

1. We explore $A_{-2\lambda,2\lambda}$.
2. We explore $A_{-a+2\lambda,a+2\lambda}(\Gamma^{A_{-2\lambda,2\lambda}}, O)$ in all connected components O of $D\backslash A_{-2\lambda,2\lambda}$ labelled -2λ, and we explore $A_{-a-2\lambda,a-2\lambda}(\Gamma^{A_{-2\lambda,2\lambda}}, O)$ in all connected components O with label 2λ .

Note that in this construction the law of the TVS being explored in each component O is the same as the law of $-\Gamma$ and Γ agree.

Let us now construct $\tilde{\Gamma}$. We start by constructing $A_{-a,a}(\tilde{\Gamma})$ and its labels. First, resample the labels of the loops of $A_{-2\lambda,2\lambda}(\Gamma)$ independently, by tossing an independent fair coin for each loop.

Now, inside the connected components of $D\backslash A_{-2\lambda,2\lambda}(\Gamma)$ where the new labels agree with the ones for $A_{-2\lambda,2\lambda}(\Gamma)$, we now just use the same set and labels as in (2). In other words, $A_{-a+2\lambda,a+2\lambda}(\Gamma^{A_{-2\lambda,2\lambda}}, O)$ or $A_{-a-2\lambda,a-2\lambda}(\Gamma^{A_{-2\lambda,2\lambda}}, O)$, depending on whether the label is -2λ or 2λ respectively, and its labels. In those connected components O where the sign changed we use again the same set, but change the sign of all the labels inside. We have thus constructed $\tilde{h}_{A_{-a,a}}$.

Finally, define $\tilde{\Gamma}^{A_{-a,a}}$ in some way, say by setting it equal to $\Gamma^{A_{-a,a}}$. Due to the equality in law noted at the beginning, $\tilde{\Gamma}$ has the law of a GFF. Additionally, it is clear to see that Γ and $\tilde{\Gamma}$ satisfy the desired properties.

For the general case $a \neq b$ assume WLoG that $a \leqslant b$ and define $m = (b-a)/2 > 0$. To have a realization of $A_{-a,b}$ we can first sample $A_{-a,m}$ and then explore $A_{-a-m,b-m} = A_{-(b+a)/2,(b+a)/2}$ inside the loops labelled m. We can now conclude as before by realizing the coupling for $A_{-(b+a)/2,(b+a)/2}$.

Law of the labels conditioned on $A_{-a,b}$ in the critical case $a + b = 4\lambda$

In the case $a + b = 4\lambda$ one can moreover precisely describe the law of the labels:

Proposition 2.32 *Let $0 < a \leqslant 2\lambda$. Then the law of the labels of $A_{-a,-a+4\lambda}$ given $A_{-a,-a+4\lambda}$ is the following:*

- *The loops touching the boundary are labelled $-a$.*
- *For each loop that does not touch the boundary we toss independent fair coins to decide whether the label is equal to $-a$ or $-a + 4\lambda$.*

Proof. Note that the result holds for $A_{-2\lambda,2\lambda}$ due to the fact that no loop touch the boundary. When $a < 2\lambda$, by Lemma 2.24, as $-a + 4\lambda \geqslant 2\lambda$, the loops that touch the boundary are labelled $-a$. The union of the loops touching the boundary is $A_{-a,2\lambda-a}$ by Remark 2.22. All the other loops are constructed by exploring $A_{-2\lambda,2\lambda}(\Gamma^{A_{-a,2\lambda-a}}, O)$ inside any connected component O of $D\backslash A_{-a,2\lambda-a}$ labelled $-a + 2\lambda$. But we know that the labels of $A_{-2\lambda,2\lambda}$ are given by independent fair coin tosses. □

Some comments

Let us conclude this chapter with some comments.

We will show in Chapter 3, that two valued local sets also exist when the given GFF has boundary values given by a piecewise constant function changing finitely many times or when the domain D is multiply connected.

The generalisation to piecewise constant boundary data is relatively simple and follows the same main ideas.

The case where D is n-connected requires a bit more of care as with positive probability there are connected components of $D\backslash A_{-a,b}$ that are not simply connected. This implies that there may be connected components of $D\backslash A_{-a,b}$ inside where $h_{A_{-a,b}}$ is not constant. Nevertheless, one can solve this issue by changing the definition of 'loops'. We say that ℓ is a loop of $A_{-a,b}$ if it is the connected component of a boundary of a connected component of $D\backslash A_{-a,b}$.

With this new definition, we can again associate a label to each loop and we can study the graph $G_p(A_{-a,b})$. When $2\lambda \leqslant a+b < 4\lambda$, the graph $G_p(A_{-a,b})$ may have up to $n+1$ connected components, each one corresponding to one connected component of the boundary. When, $a+b \geqslant 4\lambda$, $G_p(A_{-a,b})$ is again totally disconnected. To prove this for $A_{-2\lambda,2\lambda}$, one needs to use the $\text{SLE}_4(-2)$-type construction used in Section 2.4.3, the rest of the proof is the same.

In this case an interesting question is to know the probability that some boundaries are connected, i.e. in the case $a+b<4\lambda$ this translates to the question of when does $G_p(A_{-a,b})$ have less than $n+1$ connected components. This is known in the 1-connected case, where it follows from the construction and the behaviour of the derivative of $h_{A_{-a,b}}$ near the boundary (see [ALS17b]). In this case, the probability that a Brownian bridge of length equal to the extremal length between the two boundaries does not hit $-a$ or b. We are still working on the more general case.

As we have already mentioned in the introduction of this thesis, one motivation for the questions addressed in the final two sections of this chapter is that it provides an approach to the study of natural metrics between CLE_4 loops, which is a topic of some interest.

Chapter 3

First passage sets

3.1 Introduction

In the current chapter, our main focus will be the first passage set (FPS) A_{-a} of a two-dimensional GFF. Heuristically, this corresponds to the set of points in the domain that can be connected to the boundary by paths along which the height of the GFF is greater or equal to $-a$. In this regard, we first provide an axiomatic characterization of the continuum FPS, and a purely continuum construction using the methods and techniques of Chapter 2.

The present chapter will also illustrate how useful the metric graph GFF can turn out to be to study objects like first passage sets. Indeed, in the metric graph setting, first passage sets $\widetilde{A}_{-a}$ are well-defined: they are the points in the domain connected to the boundary via a path staying above $-a$ [LW16]. We will show that when the mesh of the lattice tends to 0, the metric FPS converges to its continuous counterparts. This will enable us to give a representation of the continuum FPS also in terms of clusters of Brownian boundary-to-boundary excursions and Brownian loops. These type of results will hold for non-simply connected domains and non-constant boundary conditions, which will enable us also to extend results of [QW15] about loop-soup cluster decompositions to such settings.

A consequence of our results is the fact that the right boundary of a certain metric graph FPS converges to the $\mathrm{SLE}_4(\rho)$ level lines (Corollary 3.54). The proof of this is much simpler than the proof of the convergence of level lines of the discrete GFF by Schramm and Sheffield [SS09, SS13], a similar result in a slightly different setup.

Let us, now, give an overview of the continuum FPS and its properties. To explain its definition, let us recall the local set coupling of a random set A with the Gaussian free field Φ in a domain D. It is a coupling (Φ, A) such that we Φ has a Markovian decomposition w.r.t. A, i.e., $\Phi = \Phi_A + \Phi^A$, where Φ_A is a random distribution that is harmonic on $D\backslash A$ and conditional on (A, Φ_A), Φ^A is a GFF on $D\backslash A$. We denote by h_A the harmonic function corresponding to Φ_A outside of A. The local set condition can be reworded as saying that conditional on A and Φ_A, the GFF Φ on $D\backslash A$ is given by $h_A + \Phi^A$.

Take Φ a zero-boundary two-dimensional GFF in an n-connected domain D and $a \geqslant 0$. The first passage set of level $-a$, denoted A_{-a}, is the only local set of Φ satisfying the following properties:

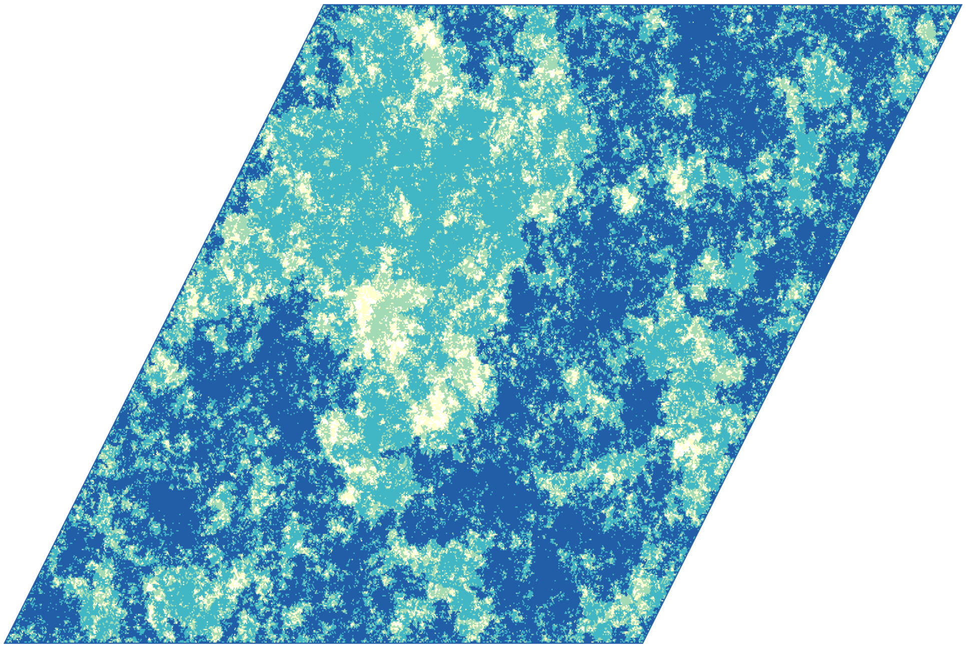


Figure 3.1: A simulation of four nested First passage sets. The first passage set $A_{-\lambda}$ with $\lambda = \sqrt{\pi/8}$ is in dark blue. The difference between $A_{-2\lambda}$ and A_λ is in lighter blue, difference between $A_{-2\lambda}$ and $A_{-3\lambda}$ in green and yellow depicts the missing part of $A_{-4\lambda}$. Image done by B. Werness.

- *Conditional on A_{-a}, the law of the restriction to $D\backslash A_{-a}$ is that of a GFF on $D\backslash A_{-a}$ with boundary condition $-a$, or in other words $h_{A_{-a}} = -a$.*
- *The GFF on A_{-a} is larger than $-a$, in the sense that for any positive test function f we have that $(\Phi_{A_{-a}} + a, f) \geqslant 0$ - that is to say $\Phi_{A_{-a}} + a$ is a positive measure.*

This definition is, in a sense, inspired by that of the two-level sets $A_{-a,b}$ given in Proposition 2.2. In that case, and when D is simply connected, we defined $A_{-a,b}$ the only thin local sets of the GFF such that $h_A \in \{-a, b\}$. Here, thin means that Φ_A contains no extra information on A, i.e. for any smooth f, we have that $(h_A, f) = (\Phi_A, f)$. In Proposition 3.22, we generalise this definition for n-connected domains and general boundary condition. Additionally, we show properties analogues to those of Proposition 2.2.

The general definition of FPS for general boundary conditions u is given in Definition 3.23. There, we also define the FPS in the other direction: $\mathbb{A}_b$ will heuristically correspond to the local set A such that $h_A = b$ and the GFF on A is smaller than b. As proved in Propositions 3.25 and 3.26 in the setting of more general boundary conditions, the first passage set

- *is unique in the sense that any other local set with the above conditions is a.s. equal to the FPS, and thus, it is a measurable with respect to the GFF it is coupled with;*
- *is monotone in the sense that for all $a \leqslant a'$ almost surely $A_{-a} \subset A_{-a'}$;*
- *as in the case of the Brownian motion can be constructed as a limit of two-valued local sets, $A_{-a} = \lim_{b\to\infty} A_{-a,b}$.*

In fact the relation to two-valued sets is even stronger: we will show that *the intersection of two FPS A_{-a} and $\mathbb{A}_b$ is precisely $A_{-a,b}$ in the simply connected case.*

One can also show that A_{-a} *has zero Lebesgue measure* but, contrary to $A_{-a,b}$, *its Hausdorff dimension is 2 and A_{-a} is not a thin local set*, i.e., Φ "charges" A_{-a}. Moreover and surprisingly (at least for us) $\Phi_{A_{-a}}$ *is measurable function of just the set A_{-a} itself* (Lemma 3.33). The argument uses the recent construction of Liouville quantum gravity measures via local sets, presented in Chapter 4, and the fact that the GFF is a measurable function of any of its Liouville quantum gravity measures [BSS14]. Finally, we are also able to give *simple continuum derivations to several identities in law for the FPS*, calculated in the metric graph setting in [LW16] (compare Proposition 3.6 and Propositions 3.28 and 3.29).

As we have already mentioned, the other main point of the chapter is the connection of the metric graph FPS of level $-a$, denoted by $\widetilde{A}_{-a}$, with the continuum FPS. We denote the metric graph GFF by $\tilde{\phi}$ and define the FPS as the set of points x in the metric graph for which there is a path from the boundary to x such that $\tilde{\phi} \geqslant -a$ on the whole path [LW16]. Given our knowledge on local sets, and the continuum characterisation of the FPS, it is not hard to see that when the mesh-size goes to zero, and $\tilde{\phi}$ converge in probability to Φ, then *the metric graph FPS converges in probability to its continuum counterpart* (Proposition 3.40). This convergence tells us that the continuum FPS corresponds to the heuristical definition presented at the beginning of the section.

Now, we also observe that in the metric graph setting, the FPS $\widetilde{A}_{-a}$ can be obtained from metric graph boundary-to-boundary excursions and loops. Indeed, the FPS is given by the closed union of all the loops that can be connected to an excursion using a path of pair-wise intersecting loops. With some extra work on the convergence of clusters of Brownian loops and excursions, we are then able to deduce that the same holds in the continuum (Corollary 3.46).

As a consequence, we obtain that certain level lines of the GFF are boundaries of clusters of Brownian loops and boundary-to-boundary excursions (Corollary 3.52). In the case of $\text{SLE}_4(\rho)$ we strengthen the results of [Wer03, WW13a], in the sense that the curves are coupled as level lines of a GFF. Additionally, our result works for a wider range of boundary conditions and without restriction to simply connected domains. In this way, the connection between the Brownian paths and the GFF, known since Symanzik [Sym69] and Dynkin [Dyn83, Dyn84a, Dyn84b] (isomorphism theorem), takes a more geometrical form. It does not only deal with moments and the (renormalized) square of the GFF but also with interfaces within the GFF. Another consequence is an explicit coupling between level lines of GFF-s with different boundary conditions such that the curves coincide with positive probability (Corollary 3.56). This coupling uses the isomorphism with Brownian loops and excursions.

The next sections of the chapter are as follows: Sections 3.2 and 3.3 contain preliminary material. In Section 3.2 we describe the metric graph setting: we introduce the GFF on the metric graph, the loops and excursions, Dynkin type isomorphisms and the FPS. In Section 3 we deal with the continuum preliminaries: Brownian loop soups and excursions, two-dimensional continuum GFF, local sets, and two-valued local sets of the GFF. In Section 3.4, we extend the theory of two-valued local sets, done in the simply-connected setting in Section 2.6 to the finitely-connected case. After that, we define and characterise the continuum FPS and prove several of its properties. In Section 3.5 we show that the metric graph FPS converges to the continuum FPS. We also demonstrate the representation of the FPS using Brownian loops and excursions and draw several consequences. We conclude with a brief discussion about ongoing work and open questions.

3.2 Preliminaries on the metric graph

In this section we first recall the definition of the metric graph and of the GFF on the metric graph , which corresponds to taking a discrete GFF on its vertices, and extending it using Brownian bridges of length 1 on all edges. Next, we browse through the definition of measures on loops and excursions on the metric GFF, and recall some aspects of the isomorphism theorems. In Subsection 2.4 we define the First passage set (FPS) of the metric graph introduced in [LW16] and bring out its representation using Brownian loops and excursions. Finally, we recall some calculations for metric graph FPS that we are able to reproduce in the continuum setting later on. The results in this section are either already in the literature or are slight extensions of already existing results. For example, we extend the isomorphism theorems on the metric graph to non-constant boundary conditions.

3.2.1 The Gaussian free field on metric graphs

We start from a finite connected undirected graph $\mathcal{G} = (V, E)$ with no multiple edges or self-loops. We interpret it is as an electrical network by equipping each edge $e = \{x, y\} \in E$ with a conductance $C(e) = C(x, y) > 0$. If $x, y \in V$, $x \sim y$ denotes that x and y are connected by an edge. A special subset of vertices $\partial\mathcal{G} \subset V$ will be considered as the boundary of the network. We assume that $\partial\mathcal{G}$ and $V\backslash\partial\mathcal{G}$ are non-empty. For $x \in V\backslash\partial\mathcal{G}$, we denote

$$C_{\text{tot}}(x) := \sum_{\substack{y\in V\\ y\sim x}} C(x, y).$$

Let $\Delta^{\mathcal{G}}$ be the discrete Laplacian:

$$(\Delta^{\mathcal{G}} f)(x) := \sum_{y\sim x} C(x, y)(f(y) - f(x)).$$

Let $\mathcal{E}_{\mathcal{G}}$ be the Dirichlet energy:

$$\mathcal{E}_{\mathcal{G}}(f, f) := -\sum_{x\in V}\sum_{y\sim x} f(x)(\Delta^{\mathcal{G}} f)(x) = \sum_{\{x,y\}\in E} C(x, y)(f(y) - f(x))^2.$$

Let ϕ be the discrete Gaussian free field (GFF) on $\mathcal{G}$, associated to the Dirichlet energy $\mathcal{E}_{\mathcal{G}}$, with boundary condition 0. That is to say, if we defined the Green's function $G_{\mathcal{G}}$ as the inverse of $-\Delta^{\mathcal{G}}$, with 0 boundary conditions on $\partial\mathcal{G}$, we have that ϕ is the only centred Gaussian process such that for any $f, g : V \mapsto \mathbb{R}$

$$\mathbb{E}\left[(\phi, f_1)(\phi, f_2)\right] = \sum_{x,y\in V\backslash\partial\mathcal{G}} f_1(x) G_{\mathcal{G}}(x, y) f_2(y).$$

We would sometimes be interested in a GFF with non-0 boundary conditions. For that we call $u : V \mapsto \mathbb{R}$ a boundary condition if it is harmonic function in $V\backslash\partial\mathcal{G}$ and when the context is clear we identify it with its restriction to $\partial\mathcal{G}$. Now note that $\phi + u$ is then the GFF with boundary condition u. Its expectation is u and its covariance is given by the Green's function.

Given an electrical network $\mathcal{G}$, we can associate to it a *metric graph*, also called *cable graph* or *cable system*, denoted $\widetilde{\mathcal{G}}$. Topologically, it is a simplicial complex of degree 1, where each edge is replaced by a continuous line segment. We also endow each such segment with a metric such that its length is equal to the resistance $C(x,y)^{-1}$, x and y being the endpoints. One should think of it as replacing a "discrete" resistor by a "continuous" electrical wire, where the resistance is proportional to the length.

Given a discrete GFF ϕ with boundary condition 0, we interpolate it to a function on $\widetilde{\mathcal{G}}$ by adding on each edge-line a conditional independent standard Brownian bridge. If the line joins the vertices x and y, the endvalues of the bridge would be $\phi(x)$ and $\phi(y)$, and its length $C(x,y)^{-1}$. By doing that we get a continuous function $\tilde{\phi}$ on $\widetilde{\mathcal{G}}$ (Figure 3.2). This is precisely the metric graph GFF with 0 boundary conditions. Consider the linear interpolation of u inside the edges, still denoted by u. $\tilde{\phi}+u$ is the metric graph GFF with boundary conditions u. The restriction of $\tilde{\phi}+u$ to the vertices is the discrete GFF $\phi+u$.

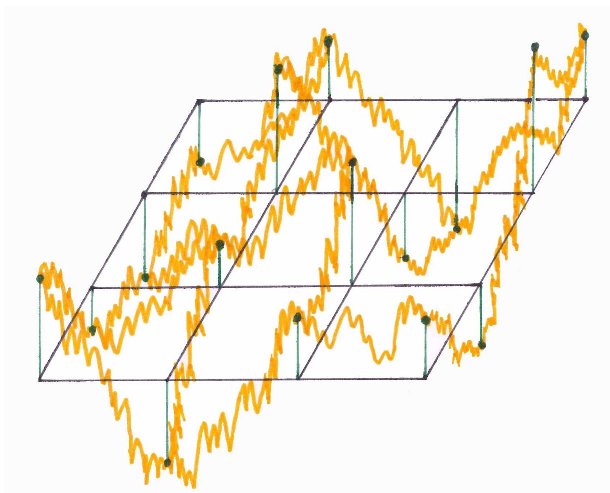


Figure 3.2: $\tilde{\phi}$ on a square lattice. Green dots represent the values of the discrete GFF. Orange curves are the Brownian bridges interpolating between these values.

The metric graph GFF satisfies the strong Markov property on $\widetilde{\mathcal{G}}$. More precisely, assume that A is a random compact subset of $\mathcal{G}$. We say that is *optional* for $\tilde{\phi}$ if for every O deterministic open subset of $\mathcal{G}$, the event $A \subseteq O$ is measurable with respect the restriction of $\tilde{\phi}$ to O. For simplicity we will also assume that a.s., A has finitely many connected components. Then $\widetilde{\mathcal{G}}\backslash A$ has finitely many connected components too, and the closure of each connected component is a metric graph, even if an edge of $\mathcal{G}$ is split among several connected components or partially covered by A.

Proposition 3.1 (Strong Markov property, [Lup16a]) *Let A be a random compact subset of $\widetilde{\mathcal{G}}$, with finitely many connected components and optional for the metric graph GFF $\tilde{\phi}$. Then we have a Markov decomposition*

$$\tilde{\phi} = \tilde{\phi}_A + \tilde{\phi}^A,$$

where, conditional on A, $\tilde{\phi}^A$ is a zero boundary metric graph GFF on $\widetilde{\mathcal{G}}\backslash A$ independent of $\tilde{\phi}_A$ (and by convention zero on A), and $\tilde{\phi}_A$ is on A the restriction of $\tilde{\phi}$ to A and on $\widetilde{\mathcal{G}}\backslash A$ equals a harmonic function $\tilde{h}_A$, whose boundary values are given by $\tilde{\phi}$ on $\partial\mathcal{G} \cup A$.

3.2.2 Measures on loops and excursions

Next, we introduce the measures on loops and boundary-to-boundary excursions which appear in isomorphism theorems in discrete and metric graph settings.

Consider the nearest neighbour Markov jump process on $\mathcal{G}$, with jump rates given by the conductances, and let $p_t^{\mathcal{G}}(x,y)$ and $\mathbb{P}_t^{\mathcal{G},x,y}$ be the associated transition probabilities and bridge probability measures respectively. Let $T_{\partial\mathcal{G}}$ be the first time the jump process hits the boundary $\mathcal{G}$. The *loop measure* on $\mathcal{G}$ is defined to be

$$\mu_{\text{loop}}^{\mathcal{G}}(\cdot) := \sum_{x\in V}\int_0^{+\infty} \mathbb{P}_t^{\mathcal{G},x,x}(\cdot, T_{\partial\mathcal{G}} > t) p_t^{\mathcal{G}}(x,x)\frac{dt}{t}. \tag{3.1}$$

$\mu_{\text{loop}}^{\mathcal{G}}$ is a measure on nearest neighbour paths in $V\backslash\partial\mathcal{G}$, parametrized by continuous time, which at the end return to the starting point. Note that it associates an infinite mass to trivial loops, which only stay at one given vertex. This measure was introduced by Le Jan in [LJ07, LJ10, LJ11]. If one restricts the measure to non-trivial loops and forgets the time-parametrisation, one gets the measure on random walk loops which appears in [LTF07, LL10].

Γ will denote the family of all finite paths parametrized by discrete time, which start and end in $\partial\mathcal{G}$, only visit $\partial\mathcal{G}$ at the start and at the end, and also visit $V\backslash\partial\mathcal{G}$. We see a path in Γ as the skeleton of an excursion from $\partial\mathcal{G}$ to itself. We introduce a measure $\nu_{\text{exc}}^{\mathcal{G}}$ on Γ as follows. The mass given to an admissible path $(x_0, x_1, \dots, x_n)$ is

$$\prod_{i=1}^{n} C(x_{i-1}, x_i)\prod_{i=1}^{n-1} C_{\text{tot}}(x_i)^{-1}.$$

Note that this measure is invariant under time-reversal. For $x, y \in \partial\mathcal{G}$, $\Gamma_{x,y}$ will denote the subset of Γ made of paths that start at x and end at y. We defined the kernel $H_{\mathcal{G}}(x,y)$ on $\partial\mathcal{G}\times\partial\mathcal{G}$ as

$$H_{\mathcal{G}}(x,y) := \nu_{\text{exc}}^{\mathcal{G}}(\Gamma_{x,y}).$$

It is symmetric. $H_{\mathcal{G}}$ is often referred to as the *discrete boundary Poisson kernel*, and this is the terminology we will use. $\mathbb{P}_{\text{exc}}^{\mathcal{G},x,y}$ will denote the probability measure on excursions from x to y parametrized by continuous time. The discrete-time skeleton of the excursion is distributed according to the probability measure $\mathbf{1}_{\Gamma_{x,y}}H_{\mathcal{G}}(x,y)^{-1}\nu_{\text{exc}}^{\mathcal{G}}$. The excursions under $\mathbb{P}_{x,y}^{\text{exc}}$ spend zero time at x and y, i.e. they immediately jump away from x and jump to y at the last moment. Conditional on the skeleton $(x_0, x_1, \dots, x_n)$, the holding time at x_i, $1 \leqslant i \leqslant n-1$, is distributed as an exponential r.v. with mean $C_{\text{tot}}(x_i)^{-1}$, and all the holding times are conditional independent. To a non negative boundary condition u on $\partial\mathcal{G}$ we will associate the measure

$$\mu_{\text{exc}}^{\mathcal{G},u} := \frac{1}{2}\sum_{(x,y)\in\partial\mathcal{G}\times\partial\mathcal{G}} u(x)u(y)H_{\mathcal{G}}(x,y)\mathbb{P}_{\text{exc}}^{\mathcal{G},x,y}. \tag{3.2}$$

Consider now the metric graph setting. We will consider on $\widetilde{\mathcal{G}}$ a diffusion we introduce now. For generalities on diffusion processes on metric graphs, see [BC84, EK01]. $(\widetilde{X}_t)_{t\geqslant 0}$ will be a Feller process on $\widetilde{\mathcal{G}}$. The domain of its infinitesimal generator $\Delta^{\widetilde{\mathcal{G}}}$ will contain all continuous functions which are $\mathcal{C}^2$ inside each edge and such that the second derivatives have limits at the vertices and which are the same for every adjacent edge. On such a function f,

$\Delta^{\widetilde{\mathcal{G}}}$ will act as $\Delta^{\widetilde{\mathcal{G}}} f = f''/4$, i.e. one takes the second derivative inside each edge. $\widetilde{X}$ behaves inside an edge like a one-dimensional Brownian motion. With our normalization of $\Delta^{\widetilde{\mathcal{G}}}$, it is not a standard Brownian motion, but with variance multiplied by 1/2. When $\widetilde{X}$ hits an edge of degree 1, it behaves like a reflected Brownian motion near this edge. When it hits an edge of degree 2, it behaves just like a Brownian motion, as we can always consider that the two lines associated to the two adjacent edges form a single line. When $\widetilde{X}$ hits a vertex of degree at least three, then it performs Brownian excursions inside each adjacent edge, until hitting an neighbouring vertex. Each adjacent edge will be visited infinitely many times immediately when starting from a vertex, and there is no notion of first visited edge. The rates of small excursions will be the same for each adjacent edge. See [Lup16a, EK01] for details.

Just as a one-dimensional Brownian motion, $(\widetilde{X}_t)_{t\geqslant 0}$ has local times. Denote $\tilde{m}$ the measure on $\widetilde{\mathcal{G}}$ such that its restriction to each edge-line is the Lebesgue measure. There is a family of local times $(L_t^x(\widetilde{X}))_{x\in\widetilde{\mathcal{G}},t\geqslant 0}$, adapted to the filtration of $(\widetilde{X}_t)_{t\geqslant 0}$ and jointly continuous in (x,t), such that for any f measurable bounded function on $\widetilde{\mathcal{G}}$,

$$\int_0^t f(\widetilde{X}_s)ds = \int_{\widetilde{\mathcal{G}}} f(x)L_t^x(\widetilde{X})d\tilde{m}(x).$$

On should note that in particular the local times are space-continuous at the vertices. See [Lup16a]. Consider the continuous additive functional

$$(t,(\widetilde{X})_{0\leqslant s\leqslant t}) \mapsto \sum_{x\in V} L_t^x(\widetilde{X}). \tag{3.3}$$

It is constant outside the times $\widetilde{X}$ spends at vertices. By performing a time change by the inverse of the CAF (3.3), one gets a continuous-time paths on the discrete network $\mathcal{G}$ which jumps to the nearest neighbours. It actually has the same law as the Markov jump process on $\mathcal{G}$ with the rates of jumps given by the conductances. See [Lup16a].

The process $(\widetilde{X}_t)_{t\geqslant 0}$ has transition densities and bridge probability measures, which we will denote $p_t^{\widetilde{\mathcal{G}}}(x,y)$ and $\mathbb{P}_t^{\widetilde{\mathcal{G}},x,y}$ respectively. $\widetilde{T}_{\partial\mathcal{G}}$ will denote the first time $(\widetilde{X}_t)_{t\geqslant 0}$ hits the boundary $\partial\mathcal{G}$. The loop measure on the metric graph $\widetilde{\mathcal{G}}$ is defined to be

$$\mu_{\text{loop}}^{\widetilde{\mathcal{G}}}(\cdot) := \int_{\widetilde{\mathcal{G}}}\int_0^{+\infty} \mathbb{P}_t^{\widetilde{\mathcal{G}},x,x}(\cdot, \widetilde{T}_{\partial\mathcal{G}} > t) p_t^{\widetilde{\mathcal{G}}}(x,x)\frac{dt}{t} d\tilde{m}(x).$$

It has infinite total mass. This definition is the exact analogue of the definition (3.1) of the measure on loops on discrete network $\mathcal{G}$. Under the measure $\mu_{\text{loop}}^{\widetilde{\mathcal{G}}}$, the loops do not hit the boundary $\widetilde{\mathcal{G}}$. One can almost recover $\mu_{\text{loop}}^{\mathcal{G}}$ from $\mu_{\text{loop}}^{\widetilde{\mathcal{G}}}$. Just as the process $(\widetilde{X}_t)_{t\geqslant 0}$ itself, the loops under $\mu_{\text{loop}}^{\widetilde{\mathcal{G}}}$ admit a continuous family of local times. One can consider the CAF (3.3) applied to a metric graph loop $\tilde{\gamma}$ that visits at least one vertex. By performing the time-change by the inverse of this CAF, one gets a nearest neighbour loop on the discrete network $\mathcal{G}$. The image by this map of the measure $\mu_{\text{loop}}^{\widetilde{\mathcal{G}}}$, restricted to the loops that visit at least one vertex, is $\mu_{\text{loop}}^{\mathcal{G}}$, up to a change of root (i.e. starting and endpoint) of the discrete loop. So, if one rather considers the unrooted loops and the measures projected on the quotients, then one obtains $\mu_{\text{loop}}^{\mathcal{G}}$ as the image of $\mu_{\text{loop}}^{\widetilde{\mathcal{G}}}$ by a change of time. Moreover, the holding times at vertices of discrete network loops are equal to the increments of local times at vertices of metric graph loops between two consecutive edge traversals. Note that $\mu^{\widetilde{\mathcal{G}}}_{\mathrm{loop}}$ also puts mass on the loops that do not visit any vertex. These loops do not matter for $\mu^{\mathcal{G}}_{\mathrm{loop}}$. See [FR14] for generalities on the covariance of the measure on loops by time change by an inverse of a CAF.

On the metric graph one also has the analogue of the measure $\mu^{\mathcal{G},u}_{\mathrm{exc}}$ on excursions from boundary to boundary defined by (3.2). Let $x \in \partial\mathcal{G}$ and let k be the degree of x. Let $\varepsilon > 0$ be smaller than the smallest length of an edge adjacent to x. $x_{1,\varepsilon}, \dots, x_{k,\varepsilon}$ will denote the points inside each of the adjacent edge to x which are located at distance ε from x. The measure on excursions from x to the boundary is obtained as the limit

$$\mu^{\widetilde{\mathcal{G}},x}_{\mathrm{exc}}(F(\tilde{\gamma})) = \lim_{\varepsilon\to 0} \varepsilon^{-1} \sum_{i=1}^{k} \mathbb{E}_{x_{i,\varepsilon}} \left[F((\widetilde{X}_t)_{0\leqslant t\leqslant \widetilde{T}_{\partial\mathcal{G}}}) \right],$$

where F is any measurable bounded functional on paths. If $y \in \partial\mathcal{G}$ is another boundary point, possibly the same, $\mu^{\widetilde{\mathcal{G}},x,y}_{\mathrm{exc}}$ will denote the restriction of $\mu^{\widetilde{\mathcal{G}},x}_{\mathrm{exc}}$ to excursions that end at y. $\mu^{\widetilde{\mathcal{G}},y,x}_{\mathrm{exc}}$ is the image of $\mu^{\widetilde{\mathcal{G}},x,y}_{\mathrm{exc}}$ by time-reversal. If $y \neq x$, $\mu^{\widetilde{\mathcal{G}},x,y}_{\mathrm{exc}}$ has a finite mass, which equals $H_{\mathcal{G}}(x,y)$. To the contrary, the mass of $\mu^{\widetilde{\mathcal{G}},x,x}_{\mathrm{exc}}$ is infinite. However, the restriction of $\mu^{\widetilde{\mathcal{G}},x,x}_{\mathrm{exc}}$ to excursions that visit $V\backslash\partial\mathcal{G}$ has a finite mass equal to $H_{\mathcal{G}}(x,x)$.

Given u a non-negative boundary condition on $\partial\mathcal{G}$, we define the following measure on excursions from boundary to boundary on the metric graph:

$$\mu^{\widetilde{\mathcal{G}},u}_{\mathrm{exc}} = \frac{1}{2} \sum_{(x,y)\in\partial\mathcal{G}\times\partial\mathcal{G}} u(x)u(y)\mu^{\widetilde{\mathcal{G}},x,y}_{\mathrm{exc}}.$$

If one restricts $\mu^{\widetilde{\mathcal{G}},u}_{\mathrm{exc}}$ to excursions that visit $V\backslash\partial\mathcal{G}$ and performs on these excursions the time-change by the inverse of the CAF (3.3), one gets a measure on discrete-space continuous-time boundary-to-boundary excursions which is exactly $\mu^{\mathcal{G},u}_{\mathrm{exc}}$. Particular cases of above metric graph excursion measures were used in [Lup15].

Next we state a Markov property for the metric graph excursion measure $\mu^{\widetilde{\mathcal{G}},x}_{\mathrm{exc}}$. Let K be a compact connected subset of $\widetilde{\mathcal{G}}$. The boundary ∂K of K will be by definition the union of the topological boundary of K as a subset of $\widetilde{\mathcal{G}}$ and $\partial\mathcal{G} \cap K$. K is a metric graph itself. Its set of vertices is $(V \cap K) \cup \partial K$. If an edge of $\mathcal{G}$ is entirely contained inside K, it will be an edge of K and it will have the same conductance. K can also contain one or two disjoint subsegments of an edge of $\mathcal{G}$. Each subsegment is a (different) edge for K, and the corresponding conductances are given by the inverses of the lengths of subsegments. So K is naturally endowed with a boundary Poisson kernel $(H_K(x,y))_{x,y\in\partial K}$ and boundary-to-boundary excursion measures $(\mu^{K,x,y}_{\mathrm{exc}})_{x,y\in\partial K}$. Note that these objects depend only on K and ∂K, and not on how K is embedded in $\widetilde{\mathcal{G}}$.

Proposition 3.2 *Let $x \in \partial\mathcal{G}$, and K a compact connected subset of the metric graph $\widetilde{\mathcal{G}}$ which contains x and such that $\widetilde{\mathcal{G}}\backslash K \neq \emptyset$. Denote by $\gamma_1 \circ \gamma_2$ the concatenation of paths γ_1 and γ_2, where γ_1 comes first. For any F bounded measurable functional on paths, we have*

$$\mu^{\widetilde{\mathcal{G}},x}_{\mathrm{exc}}(F(\gamma), \gamma \text{ visits } \widetilde{\mathcal{G}}\backslash K) = \sum_{y\in\partial K\backslash\partial\mathcal{G}} \mu^{K,x,y}_{\mathrm{exc}} \otimes \mathbb{E}_y \left[F(\gamma_1 \circ (\widetilde{X}_t)_{0\leqslant t\leqslant \widetilde{T}_{\partial\mathcal{G}}}) \right],$$

where $\mathbb{E}_y$ stands for the metric graph Brownian motion $\widetilde{X}$ inside $\widetilde{\mathcal{G}}$, started from y.

3.2.3 Isomorphism theorems

The continuous time random walk *loop-soup* $\mathcal{L}_\alpha^{\mathcal{G}}$ is a Poisson point process (PPP) of intensity $\alpha\mu_{\text{loop}}^{\mathcal{G}}$, $\alpha > 0$. We view it as a random countable collection of loops. We will also consider PPP-s of boundary-to-boundary excursions $\Xi_u^{\mathcal{G}}$, of intensity $\mu_{\text{exc}}^{\mathcal{G},u}$, where $u : \partial\mathcal{G} \to \mathbb{R}_+$ is a non-negative boundary condition.

The *occupation field* of a path $(\gamma(t))_{0\leqslant t\leqslant t_\gamma}$ in $\mathcal{G}$, parametrized by continuous time, is

$$L^x(\gamma) = \int_0^{t_\gamma} \mathbf{1}_{\gamma(t)=x} dt.$$

The occupation field of a loop-soup $\mathcal{L}_\alpha^{\mathcal{G}}$ is

$$L^x(\mathcal{L}_\alpha^{\mathcal{G}}) = \sum_{\gamma\in\mathcal{L}_\alpha^{\mathcal{G}}} L^x(\gamma).$$

Same definition for the occupation field of $\Xi_u^{\mathcal{G}}$. At the intensity parameter $\alpha = 1/2$, these occupation fields are relate to the square of GFF:

Proposition 3.3 *Let $u : \partial\mathcal{G} \to \mathbb{R}_+$ be a non-negative boundary condition. Take $\mathcal{L}_{1/2}^{\mathcal{G}}$ and $\Xi_u^{\mathcal{G}}$ independent. Then the sum of occupation fields*

$$\left(L^x(\mathcal{L}_{1/2}^{\mathcal{G}}) + L^x(\Xi_u^{\mathcal{G}})\right)_{x\in V\setminus\partial\mathcal{G}}$$

is distributed like

$$\left(\frac{1}{2}(\phi + u)^2(x)\right)_{x\in V\setminus\partial\mathcal{G}},$$

where $\phi + u$ is the GFF with boundary condition u.

Proof. If $u \equiv 0$, there are no excursions we are in the setting of Le Jan's isomorphism for loop-soups ([LJ07, LJ11]). If u is constant and strictly positive, then the proposition follows by combining Le Jan's isomorphism and the generalized second Ray-Knight theorem ([MR06, Szn12b]). Indeed, then one can consider the whole boundary $\partial\mathcal{G}$ as a single vertex, and the boundary to boundary excursions as excursions outside this vertex.

The case of u non-constant can be reduced to the previous one. We first assume that u is strictly positive on $\partial\mathcal{G}$. The general case can be obtained by taking the limit. We define new conductances on the edges:

$$\widehat{C}(x, y) := C(x, y)u(x)u(y),$$

where x and y are neighbours in $\mathcal{G}$. Let $\hat{\phi}$ be the 0 boundary GFF associated to the new conductances $\widehat{C}$. We claim that

$$(\hat{\phi}(x))_{x\in V} \overset{(d)}{=} (u(x)^{-1}\phi(x))_{x\in V}.$$

To check the identity in law one has to check the identity of energy functions:

$$\begin{aligned}\mathcal{E}_{\mathcal{G}}(uf,uf) &= -\sum_{x\in V}\sum_{y\sim x} u(x)f(x)C(x,y)(u(y)f(y)-u(x)f(x))\\ &= -\sum_{x\in V}\sum_{y\sim x} u(x)f(x)C(x,y)u(y)(f(y)-f(x)) + \sum_{x\in V}\sum_{y\sim x} C(x,y)(u(y)-u(x))u(x)f(x)^2\\ &= -\sum_{x\in V}\sum_{y\sim x} f(x)\widehat{C}(x,y)(f(y)-f(x)) + \sum_{x\in \partial\mathcal{G}}\sum_{y\sim x} C(x,y)(u(y)-u(x))u(x)f(x)^2\\ &= \widehat{\mathcal{E}}(f,f)+0.\end{aligned}$$

From the second to the third line we used that u is harmonic.

Now we can apply the case of constant boundary conditions to $\frac{1}{2}(\hat{\phi}+1)^2$. We get that it is distributed like the occupation field of a loop-soup of parameter $\alpha = 1/2$ and an independent Poissonian family of excursions from $\hat{x}$ to $\hat{x}$, both associated to the jump rates $\widehat{C}(x,y)$. If on these paths we perform the time change

$$dt = u(x)^{-2}ds, \tag{3.4}$$

we get $\mathcal{L}^{\mathcal{G}}_{1/2}$ and $\Xi^{\mathcal{G}}_u$. The time change (3.4) multiplies the occupation field by u^2, which exactly transforms $(\hat{\phi}+1)^2$ into $(\phi+u)^2$. □

Note that the coupling $(L(\mathcal{L}^{\mathcal{G}}_{1/2}), L(\mathcal{L}^{\mathcal{G}}_{1/2}) + L(\Xi^{\mathcal{G}}_u))$ is not the same as $(\frac{1}{2}\phi^2, \frac{1}{2}(\phi+u)^2)$.

The proposition above belongs to a family of results, called *isomorphism theorems*, that relate the square of a discrete Gaussian free field on a network to the occupation times of a collection of trajectories obtained from a symmetric Markov jump process on the network, such as excursions, loops and interlacements. These are the generalized first and the second Ray-Knight theorems, Dynkin's isomorphism, Eisenbaum's isomorphism, Sznitman's isomorphism for random interlacements and Le Jan's isomorphism for loop-soups ([MR06, Szn12b, Kni63, Ray63, Dyn83, Dyn84a, Dyn84b, Eis95, LJ07, LJ11, Szn12a]).

On a metric graph, the isomorphism given by Proposition 3.3 still holds. But in this setting one has a stronger version of it, which takes in account the sign of the GFF. Consider a PPP of loops (loop-soup) $\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2}$ on the metric graph $\widetilde{\mathcal{G}}$, of intensity $\frac{1}{2}\mu^{\widetilde{\mathcal{G}}}_{\text{loop}}$, and an independent PPP of metric graph excursions from boundary to boundary, $\Xi^{\widetilde{\mathcal{G}}}_u$, of intensity $\mu^{\widetilde{\mathcal{G}},u}_{\text{exc}}$. For $x \in \widetilde{\mathcal{G}}$, $L^x(\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2})$ is defined as the sum over the loops of the local time at x accumulated by the loops. The occupation field $L^x(\Xi^{\widetilde{\mathcal{G}}}_u)$ is defined similarly. $L^x(\Xi^{\widetilde{\mathcal{G}}}_u)$ is a locally finite sum, except at the boundary points $\partial\mathcal{G}$, but there it converges to $\frac{1}{2}u^2$. Indeed, for this limit only matter the excursions that do not visit $V\backslash\partial\mathcal{G}$, but then we are in the case of excursions of a one-dimensional Brownian motion. To the contrary, $L^x(\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2})$ is a.s. an infinite sum at a fixed point $x \in \widetilde{\mathcal{G}}\backslash\partial\mathcal{G}$. However $x \mapsto L^x(\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2})$ admits a continuous version ([Lup16a]), and we will only consider it. We will also consider the clusters formed by $\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2} \cup \Xi^{\widetilde{\mathcal{G}}}_u$. Two trajectories (loops or excursions) belong to the same *cluster* if there is a finite chain of trajectories which connects the two, such that any two consecutive elements of the chain intersect each other. The zero set of $L^x(\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2}) + L^x(\Xi^{\widetilde{\mathcal{G}}}_u)$, which is non-empty with positive probability, is exactly the set of points not visited by any loop or excursion. The connected components of the positive set of $L^x(\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2}) + L^x(\Xi^{\widetilde{\mathcal{G}}}_u)$ are exactly the clusters of $\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2} \cup \Xi^{\widetilde{\mathcal{G}}}_u$, i.e. all the trajectories inside such a connected component belong to the same cluster. In [Lup16a] it is proved only for clusters of loops, but one can easily generalize it to the case with excursions. Also note that on the metric graph with positive probability the clusters of loops and excursions are strictly larger than the ones on the discrete network, i.e. they connect more vertices. We state next isomorphism without proof as it can be deduced from Proposition 3.3 following the method of [Lup16a].

Proposition 3.4 *Let u be a non-negative boundary condition and $\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2}$ and $\Xi^{\widetilde{\mathcal{G}}}_u$ be as previously. Let $\sigma(x)$ be a random sign function with values in $\{-1,1\}$, defined on the set*

$$\{x \in \widetilde{\mathcal{G}} | L^x(\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2}) + L^x(\Xi^{\widetilde{\mathcal{G}}}_u) > 0\},$$

such that

- *$\sigma(x)$ is constant on the connected components of its domain,*
- *conditional on $(\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2}, \Xi^{\widetilde{\mathcal{G}}}_u)$, the value of $\sigma(x)$ does not depend on the values of σ on other connected components,*
- *$\sigma(x)$ equals 1 if the cluster of x contains at least one excursion,*
- *if the cluster of x contains no excursion (or equivalently does not intersect $\partial\mathcal{G}$), then conditional on $(\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2}, \Xi^{\widetilde{\mathcal{G}}}_u)$, $\sigma(x)$ equals -1 or 1 with probability 1/2 each.*

The definition of σ will be extended to $\widetilde{\mathcal{G}}$ by letting σ to equal 0 on $\{x \in \widetilde{\mathcal{G}} | L^x(\mathcal{L}^{\widetilde{\mathcal{G}}}_{1/2}) + L^x(\Xi^{\widetilde{\mathcal{G}}}_u) = 0\}$. Then the field

$$\left(\sigma(x)\sqrt{2}\left(L^x(\mathcal{L}^{\mathcal{G}}_{1/2}) + L^x(\Xi^{\mathcal{G}}_u)\right)^{1/2}\right)_{x\in\widetilde{\mathcal{G}}}$$

is distributed like $\tilde{\phi} + u$, the metric graph GFF with boundary condition u.

3.2.4 First passage sets of the GFF on a metric graph

There is a natural notion of *first passage sets* for the metric graph GFF $\tilde{\phi} + u$, which are analogues of first passage bridges for the one-dimensional Brownian motion. Let $a \in \mathbb{R}$. Define

$$\widetilde{A}^u_{-a} = \widetilde{A}^u_{-a}(\tilde{\phi}) := \{x \in \widetilde{\mathcal{G}} | \exists\gamma \text{ continuous path from } x \text{ to } \partial\mathcal{G} \text{ such that } \tilde{\phi} \geqslant -a \text{ on } \gamma\}.$$

We report to Figure 3.8 for a picture of a first passage set on metric graph. $\widetilde{A}^u_{-a}$ is a compact optional set and $\tilde{\phi} + u$ equals $-a$ on $\partial\widetilde{A}^u_{-a} \backslash \partial\mathcal{G}$. Moreover, each connected component of $\widetilde{A}^u_{-a}$ intersects $\partial\mathcal{G}$. $\partial\widetilde{A}^u_{-a}$ is the first passage set of level $-a$. These first passage sets were introduced in [LW16]. From Proposition 3.4 we obtain a representation of the FPS using Brownian loops and excursions:

Proposition 3.5 *If $a = 0$ and the boundary condition u is non-negative, then in the coupling of Proposition 3.4, $\widetilde{A}^u_0$ is the union of topological closures of clusters of loops and excursions that contain at least an excursions (i.e. are connected to $\partial\mathcal{G}$).*

Several identities in law related to the first passage sets appear in [LW16]. In this chapter, we will see how to prove the continuum analogues of these identities - see Propositions 3.28 and 3.29.

Let us first introduce the quantities involved. Let K_1 and K_2 be two compact subsets of $\widetilde{\mathcal{G}}$ and for simplicity we assume that each has finitely many connected components. We also require that $\partial\mathcal{G} \subset K_1 \cup K_2$. We are interested in the quantity $R^{\text{eff}}(K_1, K_2)$, the *effective resistance* between K_1 and K_2 in the electrical sense. If $K_1 \cap K_2 \neq \emptyset$, then $R^{\text{eff}}(K_1, K_2) = 0$. Otherwise let $O_1, \dots, O_n$ be the connected components of $\widetilde{\mathcal{G}} \backslash (K_1 \cup K_2)$. $\overline{O}_i$ will denote the topological closure of O_i. We will endow $\overline{O}_i$ with a boundary $\partial\overline{O}_i = \overline{O}_i \cap (K_1 \cup K_2)$. Then

$$R^{\text{eff}}(K_1, K_2) = \left(\sum_{i=1}^{n} \sum_{x_i \in \partial\overline{O}_i \cap K_1} \sum_{y_i \in \partial\overline{O}_i \cap K_2} H_{\overline{O}_i}(x_i, y_i) \right)^{-1} > 0.$$

Moreover, if K_1 is a compact subset of $\widetilde{\mathcal{G}}$ containing $\partial\mathcal{G}$ and $K_2 = \{x\}$ is just an interior point in the complement of K_1, then the effective resistance $R^{\text{eff}}(K_1, \{x\})$ is a value of a Green's function in the complement of K_1. The Green's function on $\widetilde{\mathcal{G}} \backslash K_1$ with condition 0 on K_1 is

$$G_{\widetilde{\mathcal{G}} \backslash K_1}(x_1, x_2) = G_{\widetilde{\mathcal{G}} \backslash K_1}(x_2, x_1) = \mathbb{E}_{x_1}\left[L^{x_2}_{\widetilde{T}_K}(\widetilde{X}) \right],$$

where $\widetilde{T}_{K_1}$ is the first hitting time of K_1. The restriction of $G_{\widetilde{\mathcal{G}} \backslash K_1}$ to vertices is the discrete Green's function. See [Lup16a] for the expression of the interpolation to $(\widetilde{\mathcal{G}} \backslash K_1)^2$. We have that

$$R^{\text{eff}}(K_1, \{x\}) = G_{\widetilde{\mathcal{G}} \backslash K_1}(x, x).$$

Now we are ready to state the proposition:

Proposition 3.6 (Lupu-Werner, [LW16]) *Take $-a \leqslant \inf_{\partial\mathcal{G}} u$. We can then measure the distance of the FPS to boundary components and interior points as follows*

1. *Let $\mathcal{B} \subset \partial\mathcal{G}$ such that $\mathcal{B}$ and $\partial\mathcal{G} \backslash \mathcal{B}$ non-empty. Assume that the boundary condition u is constant equal to $u_e < -a$ on $\partial\mathcal{G} \backslash \mathcal{B}$ and that $-a \leqslant \inf_{\mathcal{B}} u$. Then*

$$R^{\text{eff}}(\mathcal{B}, \partial\mathcal{G} \backslash \mathcal{B}) - R^{\text{eff}}(\widetilde{A}^u_{-a} \backslash (\partial\mathcal{G} \backslash \mathcal{B}), \partial\mathcal{G} \backslash \mathcal{B})$$

is distributed like the first exit time of level $-a$ by a Brownian bridge of length $R^{\text{eff}}(\mathcal{B}, \partial\mathcal{G} \backslash \mathcal{B})$, starting at u_s and ending at u_e, where

$$u_s = R^{\text{eff}}(\mathcal{B}, \partial\mathcal{G} \backslash \mathcal{B}) \sum_{x \in \mathcal{B}} \sum_{y \in \partial\mathcal{G} \backslash \mathcal{B}} H_{\mathcal{G}}(x, y) u(x).$$

2. *Let $x \in \widetilde{\mathcal{G}} \backslash \partial\mathcal{G}$. Then*

$$R^{\text{eff}}(\partial\mathcal{G}, \{x\}) - R^{\text{eff}}(\widetilde{A}^u_{-a}, \{x\})$$

is distributed like the first exit time of level $-a$ by a Brownian motion started from $u(x)$ and with finite life-time $R^{\text{eff}}(\partial\mathcal{G}, \{x\})$. The event when the Brownian motion does not hit $-a$ on $[0, R^{\text{eff}}(\partial\mathcal{G}, \{x\})]$ corresponds to $x \in \widetilde{A}^u_{-a}$.

3.3 Continuum preliminaries

In this section we discuss about the continuum counterpart of the objects defined in the last section. First, we go over the basic notion of distance in complex analysis - the extremal length. Then, we discuss Brownian loop and excursion measures, and the continuum two-dimensional GFF. Finally, we introduce local sets and bounded type local sets.

Most of the results in this section are either already in the literature or slight extensions, e.g. as in Subsection 3.3.5 where we generalize bounded type local sets of Chapter 2 to more general boundary conditions. The only new result is Proposition 3.14, where we show how certain observables evolve when a local set growths according to their extremal length to a fixed boundary component.

We denote by $D \subseteq \mathbb{C}$ an open planar domain with a non-empty and non-polar boundary. By conformal invariance, we can always assume that D is a subset of the unit disk $\mathbb{D}$. The most general case that we work with are domains D such that the complement of D has at most finitely many connected component and no complement being a singleton. Recall that by the Riemann mapping for multiple-connected domains [Koe22], such domains D are known to be conformally equivalent to a circle domain (i.e. to $\mathbb{D}\backslash K$, where K is a finite union of closed disjoint disks, disjoint also from $\partial\mathbb{D}$).

3.3.1 Boundary Poisson kernel and extremal length

In multiply-connected domains, the natural way to measure distances between the components is the so-called *extremal length* (or *extremal distance*) and its reciprocal *conformal modulus*. Both of the quantities are conformally invariant and extremal length is the analogue of the effective resistance on electrical networks [Duf62]. We introduce it shortly here and refer to [Ahl10], Section 4 for more details.

If $\rho(z)|dz|$ is a metric on D conformal equivalent to the Euclidean metric, we will denote by

$$\mathrm{Lenght}_\rho(\gamma) := \int_\gamma \rho(z)|dz|$$

the ρ-length of a path γ, and by

$$\mathrm{Area}_\rho(D) := \int_D \rho(z)^2 dz$$

the ρ-area of D. The extremal length between $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined as

$$\mathrm{EL}(\mathcal{B}_1, \mathcal{B}_2) = \sup_\rho \inf_{\substack{\gamma \text{ connecting} \\ \mathcal{B}_1 \text{ and } \mathcal{B}_2}} \frac{\mathrm{Lenght}_\rho(\gamma)^2}{\mathrm{Area}_\rho(D)}.$$

The conformal modulus $\mathrm{M}(\mathcal{B}_1, \mathcal{B}_2)$ is then defined as $\mathrm{EL}(\mathcal{B}_1, \mathcal{B}_2)^{-1}$. We state here also a theorem about explicit formula for the extremal distance .

Proposition 3.7 (Theorem 4-5 of [Ahl10]) *Let D be finitely connected, $\mathcal{B}_1$ and $\mathcal{B}_2$ be unions of finitely many boundary arcs, such that $d(\mathcal{B}_1, \mathcal{B}_2) > 0$. Then $\mathrm{EL}(\mathcal{B}_1, \mathcal{B}_2)$ is given by the Dirichlet energy $\int_D |\nabla \bar{u}|^2$ of the harmonic function $\bar{u}$ equal to 0 on $\mathcal{B}_1$, 1 on $\mathcal{B}_2$, and having* *zero normal derivative on* $\partial D\backslash(\mathcal{B}_1 \cup \mathcal{B}_2)$*. If* $\mathcal{B}_1$*, resp.* $\mathcal{B}_2$*, has piecewise smooth boundary* $\int_D |\nabla \bar{u}|^2$ *is equal to* $-\int_{\mathcal{B}_1} \partial_n \bar{u}$*, resp.* $\int_{\mathcal{B}2} \partial_n \bar{u}$*, where* ∂_n *is the outward derivative.*

One can also express extremal length in terms of the Poisson kernel, which we introduce next. Most of it is covered in [Law08], Section 5.2.

To begin with, let $D \neq \mathbb{C}$ be an open domain with locally analytic boundary. In this case the boundary Poisson kernel is defined as

$$H_D(x, y) = \partial_{n_x} \partial_{n_y} G_D(x, y), \; x, y \in \partial D,$$

where ∂_{n_x} respectively ∂_{n_y} are the normal derivatives at x respectively y. If D and D' are domains with locally analytic boundaries and f is a conformal transformation from D to D', then

$$H_{D'}(f(x), f(y)) = |f'(x)||f'(y)|H_D(x, y). \tag{3.5}$$

One can see the boundary Poisson kernel as a measure on $\partial D \times \partial D$ rather than a function, by setting

$$H_D(dx, dy) = H_D(x, y)dxdy,$$

where on the right-hand side dx and dy denote the length measure on ∂D. This measure is conformal invariant by (3.5). Also note that it has infinite total mass due to diagonal divergence.

Now consider any D, which is an open bounded domain of $\mathbb{C}$, such that $\mathbb{C}\backslash D$ has finitely many connected components, none reduced to a point. Such a domain is conformally equivalent to a circle domain D' (i.e. to $\mathbb{D}\backslash K$ where K is a finite union of closed disjoint disks) [Koe20, Koe22, HS95] Thus, we can define the boundary Poisson kernel for D as a measure $H_D(dx, dy)$ as the image of the boundary Poisson kernel of D'. This is true even in the case ∂D has locally infinite length, e.g. in the cases where the boundary "looks like" an SLE_4 curve.

Finally, notice that from the definition using the Green's function and Theorem 3.7 we see that the extremal length introduced above can be expressed using the boundary Poisson kernel. Indeed, let $\mathcal{B}$ be a union of finitely many boundary components. Then

$$M(\mathcal{B}, \partial D\backslash B) = \mathrm{EL}(\mathcal{B}, \partial D\backslash B)^{-1} = \iint_{\mathcal{B}\times\partial D\backslash B} H_D(dx, dy).$$

In general, if $\mathcal{B}_1, \mathcal{B}_2 \subseteq \partial D$ are disjoint,

$$M(\mathcal{B}_1, \mathcal{B}_2) = \mathrm{EL}(\mathcal{B}_1, \mathcal{B}_2)^{-1} \geqslant \iint_{\mathcal{B}_1\times\mathcal{B}_2} H_D(dx, dy).$$

This can be seen for example using Theorem 3.7.

3.3.2 Brownian loop and excursion measures

Next, we discuss Brownian loop and excursion measures in the continuum. Consider a non-standard Brownian motion $(B_t)_{t\geqslant 0}$ on $\mathbb{C}$, such that its infinitesimal generator is the Laplacian Δ, so that $\mathbb{E}\left[\|B_t\|^2\right] = 4t$. The reason we use a non-standard Brownian motion comes from the fact that the isomorphisms with the continuum GFF have nicer forms. We will denote $\mathbb{P}_t^{z,w}$ the bridge probability measures corresponding to $(B_t)_{t\geqslant 0}$. Given D an open subset of $\mathbb{C}$, we will denote

$$T_{\partial D} := \inf\{t \geqslant 0 | B_t \notin D\} \in (0, +\infty].$$

The *Brownian loop measure* on D is defined as

$$\mu_{\text{loop}}^D(\cdot) = \int_D \int_0^{+\infty} \mathbb{P}_t^{z,z}(\cdot, T_{\partial D} > t) \frac{1}{4\pi t} \frac{dt}{t} dz,$$

where dx denotes the Lebesgue measure on $\mathbb{C}$. This is a measure on rooted loops, but it is natural to consider unrooted loops, where one "forgets" the position of the start and endpoint. This Brownian loop measure was introduced in [LW04], see also [Law08], Section 5.6 [1] From the definition follows that the Brownian loop measure satisfies a *restriction property*: if $D' \subset D$,

$$d\mu_{\text{loop}}^{D'}(\gamma) = \mathbf{1}_{\gamma \text{ contained in } D'} d\mu_{\text{loop}}^D(\gamma).$$

It also satisfies a conformal invariance property. The image of μ_{loop}^D by a conformal transformation of D is μ_{loop}^D up to a change of root and time reparametrization. In particular the measure on the range of the loop is conformal invariant. For $\mu_{\text{loop}}^{\mathbb{C}}$, there is also invariance by polar inversions (up to change of root and reparametrization). A *Brownian loop-soup* in D with intensity parameter $\alpha > 0$ is a Poisson point process of intensity measure $\alpha\mu_{\text{loop}}^D$, which we will denote by $\mathcal{L}_\alpha^D$.

Now we get to the excursion measure. Given $x \neq y \in \partial D$, $\mathbb{P}_{\text{exc}}^{D,x,y}$ will denote the probability measure on the boundary-to-boundary Brownian excursion in D from x to y, associated to the non-standard Brownian motion of generator Δ. Let u be a non-negative bounded Borel-measurable function on ∂D. We define the boundary-to-boundary excursion measure associated to u as

$$\mu_{\text{exc}}^{D,u} = \frac{1}{2} \iint_{\partial D \times \partial D} u(x)u(y) \mathbb{P}_{\text{exc}}^{D,x,y} H_D(dx, dy).$$

These excursion measure are analogous to the one on metric graphs defined in Section 3.2.2. In the particular case of D simply connected and u positive constant on a boundary arc and zero elsewhere, the measure $\mu_{\text{exc}}^{D,u}$ appears in the construction of restriction measures ([LSW03] and [Wer05], Section 4.3). Next we state without proof some fundamental properties of these excursion measures that follow just from properties of boundary Poisson kernel and 2D Brownian motion.

Proposition 3.8 *Let D be a domain as above and u a bounded non-negative condition. The boundary-to-boundary excursion measure $\mu_{\text{exc}}^{D,u}$ satisfies the following properties:*

1. *Conformal invariance: [Proposition 5.27 of [Law08]] Let D' be a domain conformally equivalent to D and f a conformal transformation from D to D'. Then $\mu_{\text{exc}}^{D',u}$ is the image of $\mu_{\text{exc}}^{D,u}$ by f, up to a change of time $ds = |f'(\gamma(t))|^{-2} dt$.*

2. *Markov property: Let $\mathcal{B}$ be a compact subsets of ∂D and assume that u is supported on $\mathcal{B}$. Let K be a compact subset of $\overline{D}$, at positive distance from $\mathcal{B}$. We assume that K*

[1] In [Law08], Section 5.6 the authors rather consider the loop measure associated to a standard Brownian motion. This is just a matter of a change of time $ds = dt/\sqrt{2}$.

has finitely many connected components. For any F bounded measurable functional on paths, we have

$$\mu_{\text{exc}}^{D,u}(F(\gamma), \gamma \text{ visits } K) = \int \mathbf{1}_{\gamma_1(0)\in\mathcal{B}} \mathbb{E}_{\gamma_1(t_{\gamma_1})} \left[u(B_{T_{\partial D}}) F(\gamma_1 \circ (B_t)_{0\leqslant t\leqslant T_{\partial D}})\right] d\mu_{\text{exc}}^{D\setminus K, \mathbf{1}_K + u\mathbf{1}_{\mathcal{B}}}(\gamma_1),$$

where $\gamma_1(t_{\gamma_1})$ is the endpoint of the path γ_1 and $\circ$ denotes the concatenation of paths.

The Markov property above is analogous to the Markov property on metric graphs given by Proposition 3.2.

3.3.3 The continuum GFF

In this section, we quickly browse through basic definitions and properties of the continuous GFF.

Recall that the (zero boundary) Gaussian Free Field (GFF) in a domain D can be viewed as a centred Gaussian process Φ (we also sometimes write Φ^D when we the domain needs to be specified) indexed by the set of continuous functions with compact support in D, with covariance given by

$$\mathbb{E}[(\Phi, f_1)(\Phi, f_2)] = \iint\limits_{D\times D} f_1(z) G_D(z, w) f_2(w) dz dw,$$

where G_D is the Green's function (with Dirichlet boundary conditions) in D, normalized such that $G_D(z, w) \sim (2\pi)^{-1} \log(1/|z - w|)$ as $z \to w$. It is often useful to write

$$G_D(z, w) = (2\pi)^{-1} \log(1/|z - w|) + g_D(z, w), \tag{3.6}$$

where $g_D(z, \cdot)$ is the harmonic function with boundary values given by $(2\pi)^{-1} \log(|z - x|)$ for $x \in \partial D$. In simply connected domains, $g_D(z, z)$ is the log conformal radius:

$$g_D(z, z) = \log(\text{CR}(z, D)).$$

For this choice of normalization of G (and therefore of the GFF), we set

$$\lambda = \sqrt{\pi/8}.$$

Sometimes, other normalizations are used in the literature: If $G_D(z, w) \sim c \log(1/|z - w|)$ as $z \to w$, then λ should be taken to be $(\pi/2) \times \sqrt{c}$. Note that it is in fact possible and useful to define the random variable (Φ, μ) for any fixed Borel measure μ, provided the energy $\iint \mu(dz)\mu(dw) G_D(z, w)$ is finite.

The covariance kernel of the GFF blows up on the diagonal, which makes it impossible to view Φ as a random function. It can, however, be shown that the GFF has a version that lives in some space of generalized functions, what justifies the notation (Φ, f) for Φ acting on functions f (see for example [Dub09]).

In this chapter, Φ always denotes the zero boundary GFF. We also consider GFF-s with non-zero Dirichlet boundary conditions - they are given by $\Phi + u$ where u is some harmonic function that is piecewise constant changing finitely many times on ∂D.

3.3.4 Local sets: Definitions and basic properties

In this section, we only discuss items that are directly used in the current chapter. For a more general discussion of local sets we refer to [SS13, Wer16], and for thin local sets (not necessarily of bounded type), we refer to Chapter 1.

Even though, it is not possible to make sense of (Φ, f) when $f = \mathbf{1}_A$ is the indicator function of an arbitrary random set A, local sets form a class of random sets where this is (in a sense) possible:

Definition 3.9 (Local sets) *Consider a random triple (Φ, A, Φ_A), where Φ is a GFF in D, A is a random closed subset of $\overline{D}$ and Φ_A a random distribution that can be viewed as a harmonic function when restricted to $D\backslash A$. We say that A is a local set for Φ if conditionally on (A, Φ_A), $\Phi^A := \Phi - \Phi_A$ is a GFF in $D\backslash A$.*

Throughout this chapter, we use the notation $h_A : D \to \mathbb{R}$ for the function that is equal to Φ_A on $D\backslash A$ and 0 on A.

Let us list a few properties of local sets (see for instance [SS13, AS17a] for derivations and further properties):

Lemma 3.10

1. *Any local set can be coupled in a unique way with a given GFF: Let be $(\Phi, A, \Phi_A, \widehat{\Phi}_A)$, where (Φ, A, Φ_A) and $(\Phi, A, \widehat{\Phi}_A)$ satisfy the conditions of this definition. Then, a.s. $\Phi_A = \widehat{\Phi}_A$. Thus, being a local set is a property of the coupling (Φ, A), as Φ_A is a measurable function of (Φ, A).*
2. *When A and B are local sets coupled with the same GFF Φ, and that (A, Φ_A) and (B, Φ_B) are conditionally independent given Φ, then $A \cup B$ is also a local set coupled with Φ. Additionally, $B\backslash A$ is a local set of Φ^A with $(\Phi^A)_{B\backslash A} = \Phi_{B\cup A} - \Phi_A$.*
3. *Let (Φ, A_n) be such that or all $n \in \mathbb{N}$ (Φ, A_n) is a local set coupling, the sets A_n are increasing in n and there exists $k \in \mathbb{N}$ such that the cardinal of connected components of $A_n \cup \partial D$ is bounded by k. Then, $\overline{\bigcup A_n}$ is also a local set and $\Phi_{A_n} \to \Phi_{\overline{\bigcup A_n}}$ in probability as $n \to \infty$.*
4. *Let (Φ, A_n) be such that or all $n \in \mathbb{N}$ (Φ, A_n) is a local set coupling and the sets A_n are decreasing in n. Then, $\bigcap A_n$ is also a local set and $\Phi_{A_n} \to \Phi_{\bigcap_n A_n}$ a.s. as $n \to \infty$.*

The property (3) follows from the fact that under the conditions on A_n Beurling estimate ensures that $G_{D\backslash A_n} \to G_{D\backslash A}$ as $n \to \infty$ (the proof is basically the same as that of Theorem 3.42).Property (4) just follows from inverse martingale theorem.

Often one is interested in a growing sequence of local sets, which we call local set processes.

Definition 3.11 (Local set process) *We say that a coupling $(\Phi, (\eta_t)_{t\geqslant 0})$ is a local set process if Φ is a GFF in D, $\eta_0 \subseteq \partial D$, and η_t is an increasing continuous family of local sets such that for all stopping time τ of the filtration $\mathscr{F}_t := \sigma(\eta_s : s \leqslant t)$, (Φ, η_τ) is a local set.*

Local processes can be naturally parametrized from the viewpoint of any interior point z: the expected height $h_{\eta_t}(z)$ then becomes a Brownian motion. More precisely, we have that:

Proposition 3.12 (Proposition 6.5 of [MS16a]) *For any $z \in D$ if $(\eta_t)_{t\geqslant 0}$ is parametrized* *such that* $(G_D - G_{D\backslash\eta_t})(z,z) = t$*, then* $(h_{\eta_t}(z))_{t\geqslant 0}$ *has the law of a Brownian motion.*

Remark 3.13 *Notice that whereas* G_D *diverges on the diagonal, the difference of Green's functions can be given a canonical sense, using* (3.6)*. In fact when* D*, and* $D\backslash\eta_t$ *are simply connected domains, it is a difference of logarithms of conformal radii:*

$$(G_D - G_{D\backslash\eta_t})(z,z) = \log(\mathrm{CR}(z,D)) - \log(\mathrm{CR}(z,D\backslash\eta_t)).$$

One can ask whether it is possible to similarly parametrize local set processes seen from the boundary. As the boundary values of the GFF away from η_t do not change, one has to rather consider the normal derivative. In order to obtain a conformally invariant quantity, notice that if $\mathcal{B} \subseteq \partial D$, then by Green's identities the quantity $\int_{\mathcal{B}} \partial_n h$ can be given a conformally invariant meaning: $\int_{\mathcal{B}} \partial_n h = \int_D \nabla h \nabla \bar{u}$ where $\bar{u}$ is the harmonic extension of the function that takes the value 1 on $\mathcal{B}$ and 0 on $\partial D\backslash\mathcal{B}$. This motivates the following proposition:

Proposition 3.14 *Let* D *be finitely connected with all boundary components larger than a point and let* (Φ, η_t) *be a local set process with* Φ *a GFF in* D*. Take* $\mathcal{B} \subseteq \partial D$ *be a union of finitely many boundary components. Then, if* η_t *is parametrized by its conformal modulus, i.e. such that*

$$t = \mathrm{M}(\mathcal{B}, (\partial D \cup \eta_t)\backslash\mathcal{B}) - \mathrm{M}(\mathcal{B}, \partial D\backslash\mathcal{B})$$

then $W_t := \int_{\mathcal{B}} \partial_n h_{\eta_t}$ *has the law of a standard Brownian motion started from 0.*

Equivalently, when parametrized by the extremal length

$$t = \mathrm{EL}(\mathcal{B}, \partial D\backslash\mathcal{B}) - \mathrm{EL}(\mathcal{B}, (\partial D \cup \eta_t)\backslash\mathcal{B}),$$

the process

$$\widehat{W_t} := \mathrm{EL}(\mathcal{B}, (\partial D \cup \eta_t)\backslash\mathcal{B}) \int_{\mathcal{B}} \partial_n h_{\eta_t},$$

has the law of a Brownian bridge from 0 to 0 with length $\mathrm{EL}(\mathcal{B}, \partial D\backslash\mathcal{B})$*.*

In fact this proposition can be slightly generalized to also consider cases where $\mathcal{B}$ is not a whole boundary component. This has interesting consequences in terms of measuring distances to the boundary of say the SLE_4 curves, but it will appear soon elsewhere.

Proof. As the conformal modulus is the reciprocal of the extremal length, a simple calculation shows that it is enough to prove the first claim.

Using the conformal invariance both of the quantity W_t, the Gaussian free field and the extremal length, it suffices to work in a circle domain and consider the case where $\mathcal{B}$ is equal to the union of circles. In fact, for simplicity, we will only prove the case where $\mathcal{B}$ is equal to a circle of radius 1. We can moreover assume that the circles of radius $1+\varepsilon$, denoted by $\mathcal{B}_\varepsilon$ are contained in D for all $\varepsilon > 0$ small enough. In this case $\varepsilon^{-1}\int_{\mathcal{B}_\varepsilon} h_{\eta_t}$ converges a.s. to h_{η_t}.

Now write $h_{\eta_t} = \Phi - \Phi^{\eta_t}$. Now note that the variance of any approximation of this random variables have

$$\sigma_\varepsilon^2 := \mathbb{E}\left[(\varepsilon^{-1}\int_{\mathcal{B}_\varepsilon} \Phi)^2\right] = \varepsilon^{-2} \iint\limits_{\mathcal{B}_\varepsilon \times \mathcal{B}_\varepsilon} G_D(z,w)dzdw,$$

$$\sigma_{\varepsilon,t}^2 := \mathbb{E}\left[(\varepsilon^{-1}\int_{\mathcal{B}_\varepsilon} \Phi^{\eta_t})^2 \mid \eta_t\right] = \varepsilon^{-2} \iint\limits_{\mathcal{B}_\varepsilon \times \mathcal{B}_\varepsilon} G_{D\backslash\eta_t}(z,w)dzdw.$$

As $\varepsilon \to 0$, both terms individually diverge. However, their difference converges to

$$\iint\limits_{\mathcal{B}\times\mathcal{B}} \partial_{n_y}\partial_{n_x}(G_D(x,y) - G_{D\backslash\eta_t}(x,y))dxdy$$

Moreover, this convergence is uniform in all sets η_t that are at extremal distance of at least δ to $\mathcal{B}$ for some $\delta > 0$,

Now, by Theorem 3.7 and the fact that $\partial_{n_x} G_D(x,y)dx$ is the Poisson kernel and we have that this term is equal to $\mathrm{M}(\mathcal{B}, (\partial D \cup \eta_t)\backslash\mathcal{B}) - \mathrm{M}(\mathcal{B}, \partial D\backslash\mathcal{B})$. To conclude, let us first note that a.s. $\varepsilon^{-1}\int_{\mathcal{B}} h_{\eta_t}(z) \to \int_{\mathcal{B}} \partial_n h_{\eta_t}(z)dz$, and that for all $\lambda \in \mathbb{R}$, by first conditioning on η_t,

$$e^{-\frac{\lambda^2}{2}(\sigma_\varepsilon^2 - \sigma_{\varepsilon,t}^2)}\mathbb{E}\left[\exp\left(\lambda i \varepsilon^{-1}\int_{\mathcal{B}} h_{\eta_t}(z)dz\right) \mid \eta_t\right] = e^{-\sigma_\varepsilon^2}\mathbb{E}\left[(\varepsilon^{-1}\int_{\mathcal{B}_\varepsilon}\Phi)^2 \mid \eta_t\right].$$

We conclude using by bounding $\sigma_\varepsilon^2 - \sigma_{\varepsilon,t}^2$ by $\mathrm{M}(\mathcal{B}, \partial D\backslash\mathcal{B}) - \mathrm{M}(\mathcal{B}, (\partial D\cup\eta_t)\backslash\mathcal{B}) \pm \delta$ and taking expected value. □

3.3.5 Bounded type local sets

Next we discuss a specific type of local sets introduced in Chapter 2: the bounded type local sets (BTLS). The word bounded here reflects the fact that the harmonic function h_A is bounded.

In fact, in Chapter 2 it is also required the local sets to be thin. Here, by a thin local set we mean the following condition:

- For any smooth test function $f \in \mathcal{C}_0^\infty$, the random variable (Φ, f) is almost surely equal to $(\int_{D\backslash A} h_A(z)f(z)dz) + (\Phi^A, f)$.

This definition assumes that h_A belongs to $\mathbb{L}^1(D\backslash A)$ which is the case in this chapter. For the general definition see Section 1.15. In order to say that the union of two thin sets is thin, it is more convenient to use a stronger condition. Indeed, it is not hard to show that (see Proposition 1.14):

- If h_A is $\mathbb{L}^1(D\backslash A)$ and for any compact set $K \subseteq D$, the Minkowski dimension of $A \cap K$ is strictly smaller than 2 then A is thin.

We can now define the bounded type local sets:

Definition 3.15 (BTLS) *Consider a closed subset A of $\overline{D}$ and Φ a GFF in D defined on the same probability space. Let $K \in \mathbb{R}^+$, we say that A is a K-BTLS for Φ if the following four conditions are satisfied:*

1. *A is a thin local set of Φ.*
2. *Almost surely, $|h_A| \leq K$ in $D\backslash A$.*
3. *Almost surely, each connected component of A that does not intersect ∂D has a neighborhood that does intersect no other connected component of A.*

If A is a K-BTLS for some K, we say that it is a BTLS.

3.3.5.1 Generalized level lines

One of the simplest family of BTLS are the generalized level lines, first described in [SS13], that correspond to $\mathrm{SLE}_4(\rho)$ processes.

Let $D := \mathbb{H}\backslash \bigcup_{k=1}^n C_k$, where $(C_k)_{1\leqslant k\leqslant n}$ is a finite family of disjoint closed disks, be a circle domain in the upper half plane. Further, let u be a harmonic function in D. We say that η_t, a curve parametrized by half plane capacity, is the generalized level line for the GFF $\Phi + u$ in D up to a stopping time τ if for all $t \geqslant 0$:

(∗∗) The set $\eta[0, t \wedge \tau]$ is a BTLS of the GFF Φ, with harmonic function $h_t := h_{\eta[0,t\wedge\tau]}$ satisfying the following properties: $h_t + u$ is a harmonic function in $D\backslash\eta[0, \min(t, \tau)]$ with boundary values $-\lambda$ on the left-hand side of η, $+\lambda$ on the right side of η, and with the same boundary values as u on ∂D.

The first example of level lines comes from [SS13]: Let u_0 be the unique bounded harmonic function in $\mathbb{H}$ with boundary condition $-\lambda$ in $\mathbb{R}-$ and λ in $\mathbb{R}^+$. Then it is shown in [SS13] that there exists a unique η satisfying (∗∗) for $\tau = \infty$, and its law is that of an SLE_4.

Several subsequent papers [SS13, MS16a, WW16, PW15] have studied more general boundary data in simply-connected case and also level lines in a non-simply connected setting (see Lemma 2.17). The following Lemma is a slight variant of the latter, stopping the level lines when we either hit another component, or we cannot continue it any longer. It is an easy consequence of Theorem 1.1.3 of [WW16] and Corollary 2.14.

Let u be a bounded harmonic function with piecewise constant boundary data [2] such that $u(0^-) < \lambda$ and $u(0^+) > -\lambda$.

Lemma 3.16 (Existence of generalized level line targeted to ∞) *There exists a unique law on random simple curves $(\eta(t), t \geqslant 0)$ coupled with the GFF such that (∗∗) holds for the function u and possibly infinite stopping time τ that is defined as the first time when η hits or accumulates at a point $x \in \partial D\backslash\mathbb{R}$ or hits a point $x \in \mathbb{R}$ such that $x \geqslant 0$ and $u(x^+) \leqslant -\lambda$ or $x \leqslant 0$ and $u(x^-) \geqslant \lambda$. We call η the generalized level line for the GFF $\Phi + u$.*

Notice that as the level line is parametrized using half-plane capacity, it will accumulate at ∞ if not stopped earlier. It is in general useful to know when a level line actually attains its target point (i.e. ∞ in the previous definition). The following lemma says that a target point is reached with positive probability if one can start a level line from this point in the opposite direction. It follows from Theorem 1.1.6 of [WW16], absolute continuity of the GFF and the fact that SLE_4 in the unit disk from $-i$ to i stays inside a macroscopic tube containing neighbourhoods around i and $-i$ with positive probability:

Lemma 3.17 *Let η be a generalized level line of a GFF $\Phi + u$. If there exists a generalized level line of $-\Phi \circ \psi - u \circ \psi$ for $\psi(x) = -1/x$, then with positive probability we have that $\eta_\infty = \infty$.*

Furthermore, we also need to understand when the level lines can touch the boundary. Let us recall Lemma 2.18:

Lemma 3.18 (Boundary hitting of generalized level lines) *Let η be a generalised level line*

[2] Here and elsewhere this means piecewise constant that changes only finitely many times

of $\Phi + u$ in D with u as above. Suppose $u \geqslant \lambda$ in some open neighbourhood J of $x \in \partial D$. Let τ denote the first time at which $\inf_{s<t} d(\eta_s, J) = 0$. Then the probability that $\tau < \infty$ and that η_t converges to a point in J as $t \to \tau^-$ is equal to 0. This also holds if $u \leqslant -\lambda$ in some open neighbourhood J of $x \in \partial D$.

We formulate a simple but convenient corollary of this result. It roughly says that if in any connected component of the complement of a local set A, the boundary conditions do not allow the generalized level line to touch the boundary, then the level line cannot enter inside this component. This observation was key in Chapter 2, and it follows just from the facts that 1) a generalized level line does not hit itself 2) it has to exit such a component in finite time. For the convenience of the reader we will present the argument here too:

Lemma 3.19 *Let η be a generalized level line of a GFF $\Phi + u$ in D as above and A a BTLS of Φ conditionally independent of η. Take $z \in D$ and define $O(z)$ the connected component of $D \backslash A$ containing z. On the event where on any connected component of $\partial O(z)$ the boundary values of $(h_A + u)\mid_{O(z)}$ are either everywhere $\geqslant \lambda$ or everywhere $\leqslant -\lambda$, we have that a.s. $\eta([0,\infty]) \cap O(z) = \emptyset$.*

Notice that we allow for the situation where on some boundary components the value is $\geqslant \lambda$ and on some it is $\leqslant -\lambda$. One of the key ingredients is the Lemma 3.11 of [SS09] that says that for a union of conditionally independent local sets A, B the boundary values on $A \cup B$ do not change at any point that is of positive distance of the boundary of $\partial A \cap \partial B$.

Proof. Define as E^z the event where on any connected component of $\partial O(z)$ the boundary values of $(h_A + u)\mid_{O(z)}$ are either everywhere $\geqslant \lambda$ or everywhere $\leqslant -\lambda$. Suppose for contradiction that on the event E^z, $\eta([0,\infty]) \cap (O(z)) \neq \emptyset$ with positive probability.

Take $\varepsilon > 0$ and $\tau := \tau(\varepsilon)$ the first time such that $\eta(\tau) \in O(z)$ and is at distance $\varepsilon > 0$ of $\partial O(z)$. Note that under our assumption for small enough ε, the event $\{\tau < \infty\} \cap E^z$ has non-zero probability. One can verify that $\bar\eta(t) := \eta(t + \tau)$ is a generalized level line of $\Phi^{A\cup\eta([0,\tau])} + (\Phi_{A\cup\eta([0,\tau])} + u)$. As the generalized level-line is simple continuous curve, it stays at positive distance of $\partial O(z) \cap \eta([0,\tau])$. Additionally, from Lemma 3.11 of [SS09] it follows that the boundary values of $(\Phi_{A\cup\eta_{\tau_\varepsilon}} + u)$ are $\geqslant \lambda$ or $\leqslant -\lambda$ around any point on $\partial O(z)$ that is at positive distance from $\eta([0,\tau])$. Thus from 3.18 it follows that $\bar\eta$ cannot exit $O(z)$ nor accumulate on its boundary. On the other hand from Lemma 3.16, we have that $\bar\eta(\infty)$ ends at (or accumulates at) a point on ∂D that is different from any of its previously visited points. Thus, we obtain a contradiction. □

A particularly useful corollary is the following: no level line η can enter a connected component of $D\backslash A$ inside which h_A is smaller than the boundary values of h_η.

3.3.5.2 Two-valued local sets in simply connected domains

In Section 2.6 two-valued local sets of the zero boundary GFF were introduced in the simply connected case, which we assume to be $\mathbb{D}$ for convenience. Two-valued local sets are thin local sets A such that the harmonic function h_A takes precisely two values. More precisely, take $a, b > 0$, and consider thin local sets $A_{-a,b}$ coupled with the GFF such that $h_{A_{-a,b}}$ is constant in each connected component of $\mathbb{D}\backslash A_{-a,b}$ and for all $z \in \mathbb{D}\backslash A_{-a,b}$, $h_{A_{-a,b}} \in \{-a, b\}$.

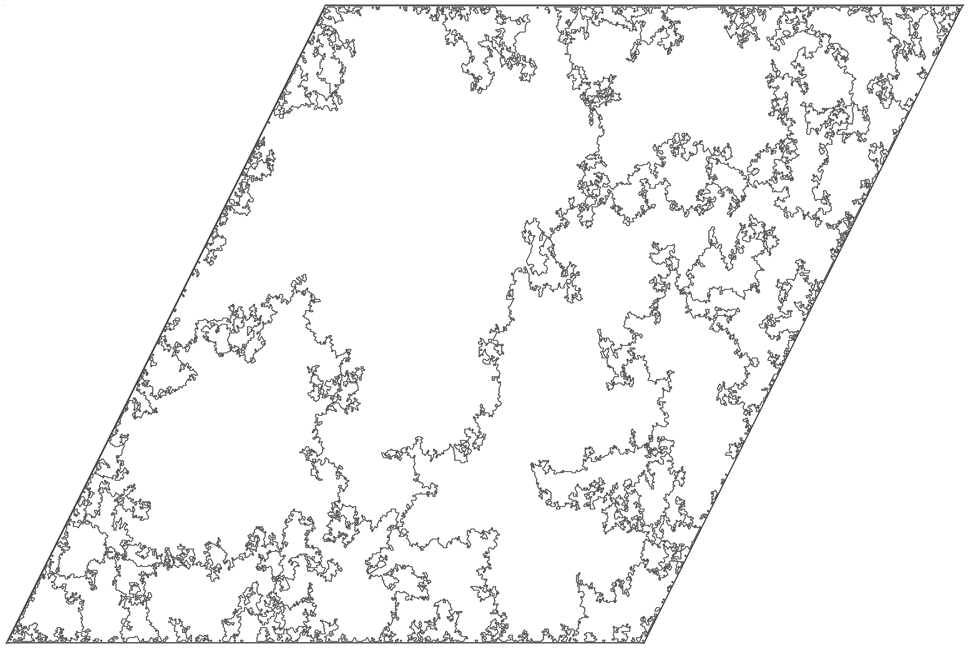


Figure 3.3: Simulation of $A_{-\lambda,\lambda}$ done by B. Werness.

It is somewhat more convenient to assume that the two-valued local sets and first passage sets introduced later also by convention contain the boundary.

In Section 2.6 we dealt with the construction, measurability, uniqueness and monotonicity of two-valued local sets in the case of the zero boundary GFF in simply connected domains. Here we state a slight generalization of this main theorem for more general boundary values.

In this respect, let u be a bounded harmonic function with piecewise constant boundary values. Take $a, b > 0$ and define $u^{-a,b}$ to be the part of the boundary where the values of u are outside of $[-a, b]$. As long as $u^{-a,b}$ is empty, the harmonic function h_A still takes only two values $-a$ and b. Otherwise, we also allow for components where some of the boundary data for h_A (corresponding to $u^{-a,b}$) is not equal to $-a$ or b. More precisely, the complement of the two-valued set $A^u_{-a,b}$ has two types of components O:

1. Those where $\partial O \cap \partial \mathbb{D}$ is a totally disconnected set. In these components $h_{A^u_{-a,b}} + u$ takes the constant value, $-a$ or b.
2. Those where $\partial O \cap \partial \mathbb{D} \subset u^{-a,b}$. In these components $h_{A^u_{-a,b}} + u$ takes boundary values u on the part $\partial O \cap \partial u^{-a,b}$ and has either constant boundary value $-a$ or b on the rest of ∂O, in such a way that $h_{A^u_{-a,b}} + u$ is a bounded harmonic function that is either greater or equal to b or smaller or equal to $-a$ throughout the whole component.

The next proposition basically says that all the properties of the zero-boundary case generalize to the general boundary:

Proposition 3.20 *Consider a bounded harmonic function u as above. If $|a + b| \geqslant 2\lambda$ and $[\min(u), \max(u)] \cap (-a, b) \neq \emptyset$, then it is possible to construct $A^u_{-a,b} \neq \emptyset$ coupled with a GFF Φ . Moreover, the sets $A^u_{-a,b}$ are*

- *Unique in the sense that if A' is another BTLS coupled with the same Φ, such that a.s. it satisfies the conditions above, then $A' = A^u_{-a,b}$ almost surely.*

- *Measurable functions of the GFF Φ that they are coupled with.*
- *Monotonic in the following sense: if $[-a,b] \subset [-a',b']$ with $b+a \geq 2\lambda$, then almost surely, $A^u_{-a,b} \subset A^u_{-a',b'}$.*

The proof is a simple extension of the arguments in Section 2.6. We will often refer to those arguments in the following proof, and only explain the small modifications in greater detail.

Proof. Construction: We know from Section 2.6.2 that the condition $|a+b| \geqslant 2\lambda$ is necessary. Also, if $[\min(u), \max(u)] \cap (-a,b) \neq \emptyset$ does not hold, then the empty set satisfies our conditions. Thus, suppose that $|a+b| \geqslant 2\lambda$ and $[\min(u), \max(u)] \cap (-a,b) \neq \emptyset$. Notice that as soon as we have constructed the basic sets with $|b+a| = 2\lambda$, the rest of the proof follows exactly as in the construction in Section 2.6.2 - one just iterates inside the components. Moreover in this basic case, one can only concentrate on $A_{-\lambda,\lambda}$ as for any other (a,b) with $b = -a + 2\lambda$ it is enough to construct $A^{u+a-\lambda}_{-\lambda,\lambda}$.

So let us now see how to build $A^u_{-\lambda,\lambda}$. To do this, partition the boundary $\partial\mathbb{D} = \bigcup_{k=1}^n \mathcal{B}_k$ such that each $\mathcal{B}_k$ is a finite segment, throughout each $\mathcal{B}_k$ the function u is either larger or equal to λ, smaller or equal to $-\lambda$, or is contained in $(-\lambda,\lambda)$, and n is as small as possible. Call n the *boundary partition size*. Notice that n is finite by our assumption. We will now show the existence by induction on n.

In fact the heart of the proof is in the case $n = 2$, so we will start from this. If u is, say, larger than λ on $\mathcal{B}_1$ and smaller than $-\lambda$ on $\mathcal{B}_2$, then by Lemma 3.16 we can draw a generalized level line from one point in $\partial\mathcal{B}_1$ to the other one, by Lemmas 3.18 and 3.17 it almost surely finishes at the other point of $\partial\mathcal{B}_1$ and decomposes the domain into components satisfying (2).

So suppose u is larger than λ on $\mathcal{B}_1$ but in $(-\lambda,\lambda)$ on $\mathcal{B}_2$. Then we can similarly start a generalized level line from one point in $\partial\mathcal{B}_1$ targeted to the other one. Again we know that it finishes there almost surely. It will decompose the domain into one piece that satisfies the condition (2) and possibly infinitely many simply connected pieces that have a boundary partition size equal to 2. We can iterate the level line in each of these components. Now for any $z \in \mathbb{D} \cap \mathbb{Q}^2$, denoting the local set process arising from the construction and continuing always in the connected component containing z by A^z_t, we have (say from Proposition 3.12) that $h_{A_t}(z)$ is a martingale. We claim that from this it follows that any z is in a component satisfying (1) or (2) above after drawing a finite number of generalized level lines. Indeed, fix some $z \in \mathbb{D} \cap \mathbb{Q}^2$; then any level line iterated in a component containing z that stays on the same side of z than the previous level line will have a larger harmonic measure than the previous one; as the sign of the level line facing z changes, we see that $h_{A_t}(z)$ changes by a bounded amount. This can happen only a finite number of times and thus the claim. Hence we have shown the construction in the case $n = 2$.

Now, if $n = 1$, then the only possible case is that u takes values in $(-\lambda,\lambda)$. In this case the generalized level lines can be started and ended at all points of the boundary. In choosing any two different points on the boundary and drawing a level line, we will decompose D into simply-connected components such that their boundary partition size equal to 2 (See Figure 3.4).

For $n \geqslant 3$, we must have at least two $\mathcal{B}_k$, say $\mathcal{B}_1$ and $\mathcal{B}_2$ (not necessarily adjacent) such

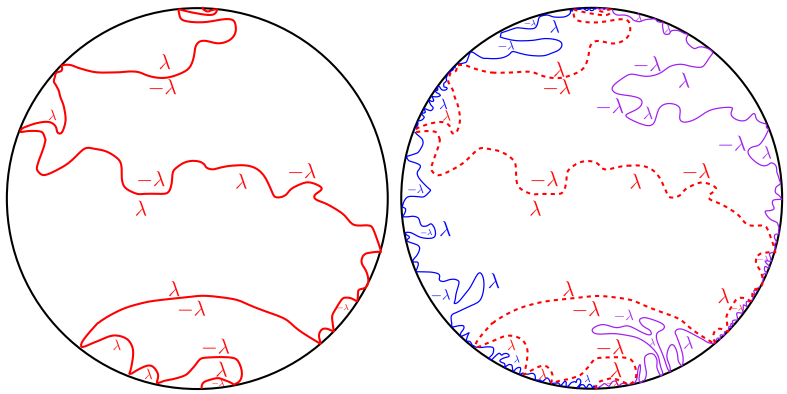


Figure 3.4: An illustration of the first two steps in the construction of $A^u_{-\lambda,\lambda}$. Does it look similar to Figure 3.3?

that $|u| \geqslant \lambda$ on them. We then start our generalized level line from a possible starting point in $\partial\mathcal{B}_1$ towards a possible target point (see Lemma 3.17) in $\partial\mathcal{B}_2$. By Lemmas 3.16, 3.17, 3.18 it stops at either at its target point or at a point between two $\mathcal{B}_i$, $\mathcal{B}_j$ such that on both $|u| \geqslant \lambda$. One can verify that in each of these cases, in each component cut out the boundary partition size is strictly smaller than n. □

Let us now make the following remarks

(i) In the construction we only need to use level lines of $\Phi + u$ whose boundary values are in $[-a, b]$.

(ii) For a fixed point $z \in \mathbb{D}$ a.s. we only need a finite number of level lines to construct the connected component of $\mathbb{D}\backslash A^u_{-a,b}$ containing z.

(iii) As none of the level lines is started inside $u^{-a,b}$ nor can touch $u^{-a,b}$, any connected component of $u^{-a,b}$ belongs entirely to the boundary of a single connected component of $\mathbb{D}\backslash A^u_{-a,b}$. In particular each connected component O of the complement of $A_{-a,b}$ with $\partial O \cap \partial D \neq \emptyset$ has only finitely many intersection points and by Lemma 2.10 we can assign them any values, in particular those that u already takes on $\partial\mathbb{D}$.

Proof. Uniqueness, measurability and monotonicity: follow exactly as in the zero-boundary case, i.e. one follows the proof of the Proposition 2.2. The only small difference is in the uniqueness part: The first step is to show that $A^u_{-a,b} \subset A'$. To do this one uses the construction and Lemma 3.19 as in the zero boundary case. Now, to show the opposite inclusion, notice that by conditions (1) and (2) we have that in any connected component O of $\mathbb{D}\backslash A^u_{-a,b}$ the boundary values of $h_{A^u_{-a,b}} + u$ are either bigger or equal b or smaller or equal $-a$. In particular

$$|h_{A^u_{-a,b}} + u + a/2 - b/2| \geqslant |h_{A'} + u + a/2 - b/2.|$$

Thus, given that $A' = A' \cup A^u_{-a,b}$ we can just use Lemma 2.9, where instead of k we use $u + a/2 - b/2$ (the proof is exactly the same in this case).

□

Remark 3.21 *In fact in the monotonicity statement, one could also include the changes in the harmonic function: if u, u' are two bounded harmonic function with piecewise constant* *boundary data, take* $\delta = \sup_H |u - u'|$ *if* $[-a,b] \subset [-a'+\delta, b'-\delta]$ *with* $b+a \geq 2\lambda$*, then almost surely,* $A^u_{-a,b} \subset A^{u'}_{-a',b'}$*. The proof just follows from the construction and Lemma 3.19.*

3.4 First passage sets of the 2D continuum GFF

The aim of this section is to define the first passage sets of the 2D continuum GFF, prove its characterization and properties. In order to do so, we first need to extend the definition of two-valued local set to n-connected domains.

3.4.1 Two-valued local sets in n-connected domains

Recall that the two-valued sets $A_{-a,b}$ correspond heuristically to sets that are connected to the zero valued boundary via a path on which the GFF takes values only in $[-a,b]$. If there are several boundary components, then each component gives rise to its own exploration - it gives the set of points that are connected via described paths to this boundary component. Such an exploration may or may not connect to a another boundary component. Thus we cannot expect the harmonic functions h_A of two valued sets $A = A_{-a,b}$ take only values in n-connected domains even for zero boundary condition: when two boundary components do not connect via the exploration, we have to allow for non-simply connected components of the complement of A that have different boundary values on each component.

Let u be a bounded harmonic function with piece-wise constant boundary values. Recall that by we $u^{-a,b}$ denoted the part of ∂D where the values of u are outside of $[-a,b]$.The two-valued set $A^u_{-a,b}$ in n-connected domains is then a BTLS such that in each connected component O of $D \backslash A^u_{-a,b}$ the bounded harmonic function $h_{A^u_{-a,b}}$ satisfies the following conditions:

(♫) On every boundary component of $\partial O \backslash u^{-a,b}$ the harmonic function $h_{A^u_{-a,b}} + u$ takes constant value a or $-b$, and in $\partial O \cap u^{-a,b}$ it takes the value u. Additionally, we require that in every connected component of ∂O either $h_{A^u_{-a,b}} + u \leqslant -a$ or $h_{A^u_{-a,b}} + u \geqslant b$ holds.

Note that in particular when $-a \leqslant \inf u \leqslant \sup u \leqslant b$, the condition (♫) simplifies to $h_{A^u_{-a,b}} + u$ takes constant values $-a$ or b in every connected component of ∂O. We now announce the fundamental proposition for two-valued sets in n-connected domains:

Proposition 3.22 *Let D be an n-connected domain and u be a bounded harmonic function that has piecewise constant boundary values. Consider $-a < 0 < b$, with $a+b \geqslant 2\lambda$. Then, $A^u_{-a,b}$ exist and are non-empty if $[\min(u), \max(u)] \cap (-a,b) \neq \emptyset$. Moreover, $A^u_{-a,b}$ satisfy the following properties:*

- *They are unique in the sense that if A' is another BTLS coupled with the same Φ, such A' satisfies (♫) almost surely, $A^u_{-a,b} = A'$.*
- *They are all measurable functions of the GFF Φ that they are coupled with.*
- *They are monotonic in the following sense: if $[a,b] \subset [a',b']$ and $-a < 0 < b$ with $b+a \geq 2\lambda$, then almost surely, $A^u_{-a,b} \subset A^u_{-a',b'}$.*

- *For each $A^u_{-a,b}$ there are at most n connected components of $D \backslash A^u_{-a,b}$ who are not simply connected.*

We will now prove the existence and measurability. Uniqueness will be a consequence of Lemma 3.35 and monotonicity follows from uniqueness as in the proof of Proposition 2.2. Until then by $A^u_{-a,b}$ we always mean the set constructed just below.

The proof is in its spirit very similar to the proof of Proposition 3.20, that itself is modelled after Section 2.6.1 - we just need a few extra arguments to treat the multiply-connected setting.

Proof. Construction Again we can assume that we are in the non-trivial case, in other words that $[\min(u), \max(u)] \cap (-a, b) \neq \emptyset$. As in the proof of Proposition 3.20 it suffices to construct $A^u_{-\lambda,\lambda}$.

This time we need a double induction. Let N be the number of boundary components and as in proof of Proposition 3.20. We take the minimal partition of any boundary component $\mathcal{B}_i$ as $\mathcal{B}_i = \bigcup_{k=1}^{n_i} \mathcal{B}_i^k$, such that each $\mathcal{B}_i^k$ is a finite segment, throughout each $\mathcal{B}_i^k$ the function u is either bigger or equal to λ, smaller or equal to $-\lambda$, or contained in $(-\lambda, \lambda)$. Recall that we called n_i boundary partition size of $\mathcal{B}_i$. We will now use induction on pairs $(N, \max_{i \leqslant N} n_i)$.

The case $(1, n)$ is given by Proposition 3.20. The key case is $(N, 2)$, so let us prove this by inducting on the number of boundary components N.

On any $\mathcal{B}_i$ satisfying $|u| \geqslant \lambda$, draw a generalized level line starting from one point of ∂B_i^1 to the other. If it hits some other boundary component, we have reduced the number of boundary components in each of the domains cut out and we can use induction hypothesis. Otherwise by Lemmas 3.17, 3.18 it ends at the other point of $\partial \mathcal{B}_i^1$ and reduces the boundary partition size of this boundary component to 1. Hence we can suppose that the only boundary components with boundary partition size equal to 2 have one part with $u \in (-\lambda, \lambda)$.

Now pick any such component, say, $\mathcal{B}_1$ and suppose u is larger than λ on $\mathcal{B}_1^1$. Then we can start a generalized level line from one points on $\partial \mathcal{B}_1^1$ towards the other one. If the generalized level line hits some other component or cuts the domain into subdomains with strictly less than m boundary components, we can use induction hypothesis. Otherwise we have finished all components O such that $\partial O \cap u^{-a,b}$ is non-zero. It now remains to see that all 'inner components' are also finished in finite time. This follows similarly to the proof of Proposition 3.20 by using the fact that $h_{A_t}(z)$ is a bounded martingale and converges almost surely.

Suppose now (N, n) satisfy $N \geqslant 2$, $n \geqslant 3$. Then we can similarly to the proof of Proposition 3.20 pick a generalized level line on some boundary component with boundary partition size bigger than 2 such that by drawing it we either reduce the boundary partition size to 2 for any subdomain with $N \geqslant 2$, or reduce the number of boundary components in each subdomain. Using a finite number of such lines we have reduced to either $(N, 2)$, or $(N', 3)$ with N' smaller than 3.

It remains to treat the case $(N, 1)$, if all components satisfy $|u| \geqslant \lambda$, we are done. Otherwise, in any component with $u \in (-\lambda, \lambda)$ we can start a level line from any point for some short amount of time. This will either reduce the setting to $(N, 3)$, $(N, 2)$ or reduce the number of boundary components.

□

Examining closely the proof the following holds:

(i) In the construction we only need to use level lines whose boundary values are in $[-a, b]$.

(ii) For a fixed point $z \in D$ a.s. we only need a finite number of level lines to construct the connected component of $D \backslash A^u_{-a,b}$ containing z.

(iii) As none of the level lines is started inside $u^{-a,b}$ nor can touch $u^{-a,b}$, any connected component of $u^{-a,b}$ belongs entirely to the boundary of a single connected component of $D \backslash A^u_{-a,b}$. In particular each connected component O of the complement of $A_{-a,b}$ with $\partial O \cap \partial D \neq \emptyset$ has only finitely many intersection points and by Lemma 2.10 we can assign them any values, in particular those that u already takes on ∂D.

(iv) Due to the construction $D \backslash A^u_{-a,b}$ has at most n non-simply connected components.

Proof. Measurability of the sets $A^u_{-a,b}$ with respect to the GFF just follows from the measurability of the level lines used in the construction and the measurability result of Proposition 3.20. □

3.4.2 First passage sets of the 2D continuum GFF

We are now ready to define continuum version of the First passage set (FPS). We first state an axiomatic definition of the continuum FPS inspired by the definition of the FPS in metric graphs (Section 3.2.4): i.e. the FPS stopped at value $-a$ is given by all points in D that can be connected to the boundary via a path on which the values of the GFF do not decrease below $-a$. From this description it is clear that the continuum analogue has to satisfy a Markov property: what remains outside of it, is just a $-a$ boundary GFF. Hence it is a local set, denoted A_{-a}. Its harmonic function has to satisfy $h_{A_{-a}}$ as we stop at value $-a$. Finally the question is how to translate the property for the values, as the GFF is not defined pointwise. The right way is to ask the distribution $\Phi_{A_{-a}} + a$ to be a positive measure.

The set-up is again as follows: D is a finitely-connected domain where no component is a single point and u is a bounded harmonic function with piecewise constant boundary conditions. Here is the definition for general boundary values:

Definition 3.23 *Let $a \in \mathbb{R}$ and Φ be a GFF in the multiple-connected domain D. We define the first passage set of (Φ, A^u_{-a}) of level $-a$ and boundary condition u as the local set of Φ such that $\partial D \subseteq A^u_{-a}$, with the following properties:*

1. *Inside each connected component O of $D \backslash A^u_{-a}$, the harmonic function $h_{A^u_{-a}} + u$ is equal to $-a$ on $\partial A^u_{-a} \backslash \partial D$ and equal to u on $\partial D \backslash A^u_{-a}$ in such a way that $h_{A^u_{-a}} + u \leqslant -a$.*
2. *$\Phi_{A^u_{-a}} + u - h_{A^u_{-a}} \geqslant 0$, i.e., for any smooth positive test function f we have $(\Phi_{A^u_{-a}} + u - h_{A^u_{-a}}, f) \geqslant 0$.*
3. *Additionally, for any connected component O of the complement of A^u_{-a}, for any $\varepsilon > 0$ and $z \in \partial O$ and for all sufficiently small open ball U_z around z, we have that a.s. $h_{A^u_{-a}}(z) + u(z) \geqslant \min\{-a, \inf_{w \in U_z \cap \overline{O}} u(w)\} - \varepsilon$.*

Notice that if $u \geqslant -a$, then the conditions (1) and (2) correspond more precisely to the heuristic and are equivalent to

(1)' $h_{A^u_{-a}} + u = a$ in $D \backslash A^u_{-a}$.

(2)' $\Phi_{A^u_{-a}} + u + a \geqslant 0$.

Moreover, in this case the technical condition (3) is not necessary. This condition roughly says that nothing odd can happen at boundary values that we have not determined: those on the intersection ∂A^u_{-a} and ∂D. This condition enters in the case $u < -a$: we do not necessarily want to prescribe the value of the harmonic function at the intersection of ∂D and ∂A^u_{-a} when we are taking the limit of the metric graph FPS. Notice that in contrast we did prescribe the values at intersection points for two-valued sets.

Remark 3.24 *One could similarly define excursions sets in the other direction, i.e. stopping the sets from above. We denote these sets by V_b. In this case the definition goes the same way except that (2) should now be replaced by $\Phi_{V^u_b} \leqslant h_{V^u_b}$. Let us also remark that in Section 4 the sets V_b are called A_b.*

We start from the existence of the FPS. One way to prove the existence of FPS is to consider the scaling limit of the metric graph FPS when the mesh size goes to zero. However, here we provide a purely continuum construction using the two-valued sets $A_{-a,b}$.

Proposition 3.25 *Denote by $A^u_{-a,n}$ the two-valued local sets coupled with the GFF Φ in the domain D. Then for every $a \geqslant 0$ the local set $A^u_{-a} := \overline{\cup_{n\in\mathbb{N}} A^u_{-a,n}}$ is a FPS of level $-a$.*

Proof. Let A^u_{-a} be as in the statement. Then A^u_{-a} is the closed union of nested measurable local sets so it is a measurable local set: it is a local set by Lemma 3.10 and measurable as a limit of measurable functions.

We first prove the condition (1) of the Definition 3.23. Take a countable dense set in D, $(z_i)_{i\in\mathbb{N}}$, and note that almost surely for all $i \in \mathbb{N}$, $z_i \notin A^u_{-a}$. Consider $n > \sup u$. It suffices to show that for any z_i, there will be some finite n such that the component of the complement of $A^u_{-a,n}$ containing z_i does not take the value n. Indeed, in this case by the definition of two-valued sets above, it would take a value as described in (1).

Now, $h_{A^u_{-a,n}}(z_i)$ is a martingale n and is lower bounded and thus has to converge almost surely. It can, however, only converge when for some n it belongs to the component of the complement of $A^u_{-a,n}$ not taking the value n. Hence we deduce the condition (1).

The condition (3) just comes from the fact that the value at the intersection points is prescribed by the definition of two-valued sets and it satisfies the appropriate condition.

It remains to prove (2), i.e. that $\Phi_{A^u_{-a}} + u - h_{A^u_{-a}} \geqslant 0$. Note that for all positive $f \in \mathcal{C}_0^\infty$ and all $n \in \mathbb{N}$, we have that $(\Phi_{A^u_{-a,n}} + u - h_{A^u_{-a}}, f) \geqslant 0$. Thus, we conclude using Lemma 3.10 (iii).

□

Let us make the following observation about the construction above:

(i) In the construction we only need to use generalized level lines whose boundary values are in $[-a, \infty)$, moreover these generalized level lines never hit themselves.

(ii) For a fixed point $z \in D$, it will belong to a component of the complement of $A^u_{-a,n}$ with value n only for a finite number of n. Thus, we need only a finite number of level lines to construct the loop of A^u_{-a} surrounding z.

We now want to use these remarks and the techniques of Chapter 2 to prove the uniqueness of the FPS:

Proposition 3.26 (Uniqueness and Monotonicity of the FPS) *Let (Φ, A') be a FPS of level $-a$ for the GFF with boundary condition u. Then $A' = A^u_{-a}$ a.s. Additionally, if $a \leqslant a'$ and $u \leqslant u'$, then $A^u_{-a} \subseteq A^{u'}_{-a'}$*

Proof. First let us prove that if A is a local set such that almost surely $\Phi_A \geqslant 0$, then a.s. A is a polar set. Given this condition, we have that $(\Phi_A, 1) \geqslant 0$ and due to the Markov property we know that $\mathbb{E}[(\Phi_A, 1))] = 0$, thus a.s. $(\Phi_A, 1) = 0$. Additionally, we know that $G_D \geqslant G_{D\backslash A}$, using again the strong Markov property we get that

$$\iint_{D\times D} G_D(z,w)dzdw = \mathbb{E}\left[(\Phi,1)^2\right] = \mathbb{E}\left[(\Phi_G^{D\backslash A},1)^2\right] = \iint_{D\backslash A\times D\backslash A} G_{D\backslash A}(z,w)dzdw.$$

Hence almost everywhere $G_{D\backslash A} = G_D$, and thus A is a polar set.

We now prove the uniqueness. If $A^u_{-a} \subseteq A'$, we claim that then $A'\backslash A^u_{-a}$ is a polar set. Consider $B := A'\backslash A^u_{-a}$. From Lemma 3.10, B is a local set of the zero boundary GFF $\Phi^{A^u_{-a}}$. Moreover, one can check that from our conditions on the FPS, it follows that $h_{A^u_{-a}}+u \leqslant h_{A'}+u$ and hence $(\Phi^{A^u_{-a}})_B \geqslant 0$. Thus, by the previous argument B is polar.

Now, it suffices to prove that a.s. $A^u_{-a} \subseteq A'$. We prove the monotonicity using arguments similar to those of Section 2.6. Take A' an FPS for $u' \geqslant u$ and $a' \geqslant a$. Suppose by contradiction, A^u_{-a} is not contained in A'. Then choosing a countable dense set in D, $(z_i)_{i\in\mathbb{N}}$, there must be some z_i such that with positive probability during the construction of A^u_{-a} a generalized level line enters the component $O(z_i)$ of the complement of A' containing z_i. Thus, there should be some finite $n \in \mathbb{N}$ such that with positive probability, η, the n^{th}-level line pointed towards z_i, is the first one to enter $O(z_i)$ and $\eta \cap O(z_i) \neq \emptyset$. This is, however, in contradiction with Lemma 3.19 and the remark just after the proof: indeed, the boundary values of $h_{A'}$ inside $O(z)$ are equal to $-a' - u' \leqslant -a - u$ and by the remarks above the boundary values of η are in $[-a, \infty)$. Thus the uniqueness follows and thus monotonicity just follows from the construction given in Lemma 3.25. □

Remark 3.27 *When $a < 2\lambda$ one can use Lemma the construction of $A_{-a,-a+2\lambda}$ to prove the following: take x, $y \in \partial D$. A.s. there exists one connected component O of $D\backslash A_{-a}$ labelled $-a$ such that x and y belong, each one, to the boundary of different connected components of $D\backslash O$. In the case where D is simply connected one can use Proposition 2.24 and Proposition 5.3 of [MSW16] to prove that the connectivity properties of FPS undergoes a phase transition. That is to say, fix two points $x, y \in \partial D$. Then, there exists a path in $D \cap A_{-a}$ connecting x and y iff $a \geqslant 2\lambda$.*

3.4.3 Distance to interior points and boundary for A^u_{-a} and $A^u_{-a,b}$

We will now state and prove the continuum analogue of Proposition 3.6, that gave an exact description of the law of the distance of A^u_{-a} and $A^u_{-a,b}$ to interior points and boundary points in the setting of metric graphs.

In this section it is convenient to work straight away in a circle domain, i.e. in $D = \mathbb{D}\backslash K$, where K is a finite union of closed disjoint disks not touching $\partial\mathbb{D}$. Notice that by conformal invariance this is not a restriction (see the discussion in beginning of Section 3.3).

In fact the distance of $A_{-a,b}$ to interior points was already calculated in Proposition 2.23. Its extension to general boundary conditions, multiply-connected setting and to FPS A_{-a}^u follows exactly the same way, this is - by applying Proposition 3.12 to the level line construction of the set. Thus we will skip the proofs of these parts.

Proposition 3.28 *Let $a, b > 0$ such that $a + b \geqslant 2\lambda$ and D as above. Moreover, let u be a bounded harmonic function with piecewise constant boundary data changing finitely many times. Let $z \in D$ and take W_t to be a Brownian motion started from $u(z)$ and with life-time $g_D(z, z)$.*

1. *If $u \subseteq [-a, b]$, then*
$$G_D(z,z) - G_{D\backslash A^u_{-a,b}}(z,z)$$
is distributed like the first hitting time of the set $\{-a, b\}$ by W_t.
2. *If $u \geqslant -a$, then*
$$G_D(z,z) - G_{D\backslash A^u_{-a}}(z,z)$$
is distributed like the one-sided hitting time of the level $-a$ by W_t.

The second part of this proposition corresponds to the second part of Proposition 3.6. Also, one should notice that in the simply connected case $G_D(z,z) - G_{D\backslash A^u_{-a,b}}(z,z)$ corresponds to the change in the log of conformal radius at point z.

Now let us find the distance to boundary components, i.e. the analogue of the first part of Proposition 3.6. Recall that until the uniqueness of two-valued sets is proved, we consider $A^u_{-a,b}$ just to the the one given by the level line construction given in Sub-section 4.1.

Proposition 3.29 *Let $a, b > 0$ such that $a + b \geqslant 2\lambda$, D a multiple connected domain, and $\mathcal{B}$ a union of connected components of ∂D. Moreover, let u be a bounded harmonic function with piecewise constant boundary data changing finitely many times such that u on $\partial D \backslash \mathcal{B}$ is a constant equal to $u_e \leqslant -a$. Let $\widehat{W}_t$ be a Brownian bridge with starting point:*
$$u_s := \mathrm{EL}(\mathcal{B}, \partial D\backslash\mathcal{B}) \iint_{\mathcal{B}\times\partial D\backslash\mathcal{B}} u(x) H_D(dx, dy)$$
endpoint u_e, and with length $\mathrm{EL}(\mathcal{B}, \partial D\backslash\mathcal{B})$.

- *If $u \in [-a, b]$ on $\mathcal{B}$, then*
$$\mathrm{EL}(\mathcal{B}, \partial D\backslash\mathcal{B}) - \mathrm{EL}(\mathcal{B} \cup (A^u_{-a,b} \cap D), \partial D\backslash\mathcal{B})$$
has the law of the first hitting time of $\{-a, b\}$ by $\widehat{W}_t$.
- *If $u \geqslant -a$ on $\mathcal{B}$, then*
$$\mathrm{EL}(\mathcal{B}, \partial D\backslash\mathcal{B}) - \mathrm{EL}(\mathcal{B} \cup (A^u_{-a}\backslash\partial D), \partial D\backslash\mathcal{B})$$
has the law of the first hitting time of $-a$ by $\widehat{W}_t$.

The proof is very similar in spirit to that of Proposition 2.23.

Proof. We just need to prove the first part of the proposition, the latter part then follows from Proposition 3.25. From the construction of two-valued sets in the proof of Proposition

3.22, we know that the two-valued set can be constructed by using only level lines with boundary values in $[-a, b]$ that do not touch $\partial D \backslash \mathcal{B}$. We can parametrize the part of the construction that always continues in the connected component containing $\partial D \backslash \mathcal{B}$ on its boundary using its extremal length to $\partial D \backslash \mathcal{B}$. We denote the resulting local set process by $(A_t)_{0 \leqslant t \leqslant \tau}$. Here τ is the first time that this component is ready. In other words $\tau := \mathrm{EL}(\mathcal{B}, \partial D \backslash \mathcal{B}) - \mathrm{EL}(\mathcal{B} \cup (A^u_{-a,b} \backslash \partial D), \partial D \backslash \mathcal{B}))$ and moreover τ is the first time that satisfies the following property:

- ⌚ Restricted to the connected component of O of $A_\tau \backslash D$ such that $\partial D \backslash \mathcal{B} \subseteq \partial O$, $h_{A_\tau} + u$ is the bounded harmonic function with boundary values either $-a$ or b in $\partial O \backslash \mathcal{B}$ and $u_e \in \mathcal{B}$.

Using Proposition 3.14 for the underlying GFF Φ, we deduce that

$$\widehat{W}_t := \mathrm{EL}(\mathcal{B} \cup (A_t \backslash \partial D), \partial D \backslash \mathcal{B}) \left(- \int_{\partial D \backslash \mathcal{B}} \partial_n (h_{A_t} + u) \right) + u_e$$

is a Brownian bridge from u_s to u_e and of length $\mathrm{EL}(\mathcal{B}, \partial D \backslash \mathcal{B})$. From an explicit calculation using Green's identities and Theorem 3.1 we see that:

$$\mathrm{EL}(\mathcal{B}, \partial D \backslash \mathcal{B}) \iint_{\mathcal{B} \times \partial D \backslash \mathcal{B}} u(x) H_D(dx, dy) = -\,\mathrm{EL}(\mathcal{B}, \partial D \backslash \mathcal{B}) \int_{\partial D \backslash B} \partial_n u(x) dx + u_e.$$

Moreover, similarly we obtain that ⌚ is equivalent to $\widehat{W}_\tau \in \{-a, b\}$. Indeed, assuming for example that we get the boundary values $-a$ on $\partial O \backslash \mathcal{B}$, we can calculate using Theorem 3.7 that

$$\widehat{W}_\tau = (a + u_e)\, \mathrm{EL}(\mathcal{B} \cup (A_\tau \backslash \partial D), \partial D \backslash \mathcal{B}) \int_{\partial D \backslash x \mathcal{B}} \partial_n \bar{u}(x) + u_e = -a.$$

Here $\bar{u}$ is the bounded harmonic function with values 0 in $\partial D \backslash \mathcal{B}$ and 1 in $\mathcal{B} \cup A_\tau$. The same calculation yields that for all times smaller than τ, $\widehat{W}_\tau \in (-a, b)$, from where we conclude. □

Let us point out the following corollary

Corollary 3.30 *Let A^u_{-a} be an FPS with boundary condition u of Φ, where u is a bounded harmonic function with piecewise constant boundary data. Then $A^u_{-a} \cap D$ is at positive distance of any connected component of ∂D where $u \leqslant -a$.*

Notice that if only a part of the boundary satisfies $u \leqslant -a$, then we also know that the FPS stays at a positive distance of any point on this interval. Indeed, this follows from the level line construction: we know from the proof of Proposition 3.22 and the remarks following the proof that for $A^u_{-a,b}$ any connected set of $u^{-a,b}$ is entirely part of the boundary of a component of $D \backslash A_{-a,b}$; on the other hand we also know that any component with $h_{A^u_{-a,b}} + u \leqslant -a$ in the complement of $A^u_{-a,b}$, will also be a component of the complement of A^u_{-a}. Putting this together we conclude:

Corollary 3.31 *Let A^u_{-a} be an FPS with boundary condition u of Φ, where u is a bounded harmonic function with piecewise constant boundary data. Then $A^u_{-a} \cap D$ is at positive distance of any connected $J \subseteq \partial D$ where there is an open neighbourhood J_ε such that $u(x) \leqslant -a$ for all $x \in J_\varepsilon \cap \partial D$.*

3.4.4 Level lines as boundaries of FPS

Now, let us see that level line can be seen as the boundary of certain FPS. Let D be finitely connected domain and $\partial_{\text{ext}} D$ be the outermost connected component of ∂D, that is to say the one that separates D from infinity. We consider two boundary points $x_0 \neq y_0 \in \partial_{\text{ext}} D$ that split $\partial_{\text{ext}} D$ in two boundary arcs, $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $y_0, \mathcal{B}_2, x_0$ come in clockwise order. Assume that u is a harmonic function such that on the boundary it is piecewise constant, it is smaller than or equal to $-\lambda$ on $\mathcal{B}_2$, $\inf_{\mathcal{B}_1} u > -\lambda$ and $\inf_{\partial D \backslash \partial_{\text{ext}} D} u \geqslant \lambda$. Note that thanks to Lemma 3.16 there is a generalized level line η of $\Phi + u$ starting at y_0 and targeted at x_0.

Corollary 3.32 *Let u be as above and Φ be a GFF in D. Then, there exists a connected component O of $D \backslash A^u_{-\lambda}$ such that it contains $\mathcal{B}_2$ on its boundary as well as the generalized level line of $\Phi + u$ from y_0 to x_0.*

Proof. This basically just follows from the uniqueness of the FPS, i.e. Proposition 3.26.

First, construct η, the generalized level line of $\phi + u$ from y_0 to x_0. By Lemma 3.18 it intersects $\mathcal{B}_2$ only at y_0 and x_0 and it does not intersect $\partial D \backslash \partial B(0,1)$. Denote by O the connected component of $D \backslash \eta([0,\infty])$ containing $\mathcal{B}_2$ on its boundary: i.e. when we denote by $\partial_{\text{ext}} O$ the outermost connected component of its boundary, then it holds that $\partial_{\text{ext}} O = \mathcal{B}_2 \cup \eta([0,\infty])$. To construct the FPS of level $-\lambda$ in D it remains to construct the FPS of level $-\lambda$ inside each connected component of $D \backslash \eta([0,\infty])$.

If all of the boundary components of $\partial D \backslash \partial_{\text{ext}} D$ are outside of O, then we can see that O remains fixed in the remaining construction and thus by the uniqueness of FPS is a component of the complement of $A^u_{-\lambda}$.

If some of them are inside O, we have to make sure that the FPS of level $-\lambda$ construced inside O does not touch $\mathcal{B}_2 \cup \eta([0,\infty])$. To do this, notice that the boundary values of $\partial_{\text{ext}} O = \mathcal{B}_2 \cup \eta([0,\infty])$ are bounded from above by $-\lambda$. Thus by Corollary the FPS of level $-\lambda$ inside O stays at positive distance of $\partial_{\text{ext}} O = \mathcal{B}_2 \cup \eta([0,\infty])$.

□

3.4.5 Measurability of $\Phi_{A^u_{-a}}$

Corollary 3.31 implies that in the coupling $(\Phi, A^u_{-\lambda})$, the harmonic extension $h_{A^u_{-a}}$ is a measurable function only of the set A^u_{-a}. In a sense, this just follows from condition (1) of the definition. A bit surprisingly, it comes out that $\Phi_{A^u_{-a}}$ is also measurable function of A^u_{-a}. Notice that this is not true in the case of the graph metric FPS. The proof is short and uses the construction of the Liouville quantum gravity (LQG) measure via the local sets of the GFF in Chapter 4. For an introduction to the LQG see [DS11, Ber15a]. Here we just recall that the subcritical Liouville measure μ_γ corresponding to the parameter $\gamma \in (0,2)$ is the limit in probability of $d\mu^\varepsilon_\gamma(z) := \varepsilon^{\gamma^2/2} \exp(\gamma \Phi_\varepsilon(z)) dz$ where by $\Phi_\varepsilon(z)$ we mean the circle average of the GFF Φ on the circle of radius ε around the point z.

Proposition 3.33 *Let Φ be a GFF in D. We have that $\Phi_{A^u_{-a}}$ is a measurable function of the set A^u_{-a}.*

In fact it is clearer to prove the claim for the sets V^u_b. The ones that will be used in

Chapter 4 (see Remark 3.24) and then we can take γ positive.

PROOF. To prove the measurability of $\Phi_{V_b^u}$, take $0 < \gamma < 2$ and define μ_γ the $\gamma-$LQG measure corresponding to Φ. The proof is based on three measurability statements:

1. $\Phi_{V_b^u}$ is a measurable function of Φ: this follows from Lemma 3.25.
2. μ_γ is a measurable function of V_b^u, $\Phi^{V_b^u}$ and $h_{V_{-a}^u}$: this follows basically from the proof of Proposition 4.7.
3. Φ is a measurable function of μ_γ for a fixed $\gamma < 2$: this is Theorem 1.1 of [BSS14].

Indeed, in the proof of Proposition 4.7 (see also Section 4.5) it is shown that almost surely as measures on D

$$e^{\gamma u}\mu_\gamma^D(\Phi) = \exp\left(\gamma\left(u + h_{V_b^u}\right)\right)\mu_\gamma^{D\setminus V_b^u}(\Phi^{V_b^u}).$$

Thus, if F is a bounded measurable function, we have that

$$F(\Phi_{V_b^u}) = \mathbb{E}\left[F(\Phi_{V_b^u}) \mid \Phi\right] = \mathbb{E}\left[F(\Phi_{V_b^u}) \mid \mu_\gamma^D(\Phi)\right] = \mathbb{E}\left[F(\Phi_{V_b^u}) \mid V_b^u, \Phi^{V_b^u}, h_{V_b^u}\right].$$

Note that $h_{V_b^u}$ is a measurable function of V_b^u. Additionally, when A is a local set, then from the definition it follows that Φ^A and Φ_A are conditionally independent given A. Hence

$$\mathbb{E}\left[F(\Phi_{V_b^u}) \mid V_b^u, \Phi^{V_b^u}, h_{V_{-a}^u}\right] = \mathbb{E}\left[F(\Phi_{V_b^u}) \mid V_b^u, \Phi^{V_b^u}\right] = \mathbb{E}\left[F(\Phi_{V_b^u}) \mid V_b^u\right]$$

This proves the proposition. □

Remark 3.34 *Using the fact that the GFF can be seen as the derivative in γ at $\gamma = 0$ of the Liouville measure, one can deduce an explicit expression for the measure on V_b^u:*

$$\lim_{\gamma\to 0^+}\gamma\int_{D\setminus V_b^u}|\log(d(z,V_b^u))|d(z,V_b^u)^{\frac{\gamma^2}{2}}dz.$$

This has an interpretation as a generalized Minkowski content with gauge $\sqrt{|\log\varepsilon|}$. To see this take $\gamma = \sqrt{|\log\varepsilon|}^{-1}$ and observe that points at distance $d = \varepsilon$ from the boundary then maximize the integrand, which is then of constant order. Work in progress suggests that the measure $\Phi_{V_b^u} + u - h_{V_b^u}$ is up to a bounded multiplicative constants equal to the Minkowski content.

3.4.6 Uniqueness and monotonicity of $A_{-a,b}^u$

In this subsection we will prove that $A_{-a,b}^u$ is the only BTLS satisfying (♫). The monotonicity then follows as in the proof of Proposition 2.2.

To prove the uniqueness of $A_{-a,b}^u$, we show that it can represented as a simple function of A_{-a}^u and V_b^u, and then use the uniqueness of the FPS.

We say that a set $A \subseteq D$ is connected to the boundary if every connected component of $A \cup \partial D$ contains at least one boundary component of ∂D.

Proposition 3.35 *Let Φ be a GFF in an n-connected domain D. Then almost surely the union of the components of $A_{-a}^u \cap V_b^u$ that are connected to boundary is equal to $A_{-a,b}^u$. In particular, if D is simply connected we have that $A_{-a,b}^u = A_{-a}^u \cap V_b^u$*

Proof. From the monotonicity of two-valued sets (Proposition 3.22) and the construction of the FPS (Lemma 3.25) we see that for any $n \geqslant a \vee b$, we have that $A^u_{-a,b} \subseteq A^u_{-a,n} \subseteq A^u_{-a}$ and $A^u_{-a,b} \subseteq A^u_{-n,b} \subseteq V^u_b$. Hence $A^u_{-a,b} \subseteq A^u_{-a} \cap V^u_b$. Moreover, as by construction $A^u_{-a,b}$ is connected to the boundary, we deduce that $A^u_{-a,b}$ is contained on the union of connected components of $A^u_{-a} \cap V^u_b$ that are connected to the boundary.

We will now prove the converse inclusion. To do this, it suffices to show that for any connected component of the complement of $A_{-a,b}$, it is contained in either the complement of A^u_{-a}, in the complement of V^u_b, or its intersection with $A^u_{-a} \cap V^u_b$ is not connected to the boundary.

First, let us consider components of the complement of $A_{-a,b}$ where $h_{A^u_{-a,b}} + u$ is less or equal $-a$. By construction of the FPS (Lemma 3.25), we see that such a component is also a connected component of $D \backslash A^u_{-a}$. Similarly, if $h_{A^u_{-a,b}} + u$ is greater ot equal to b in a component, then this component is a component of the complement of $D \backslash V^u_b$.

Thus it remains to deal with components O of $D \backslash A^u_{-a,b}$ which have more than one boundary component, and where u takes values that less than or equal to $-a$ on some of the boundary components, and values that are greater than or equal to b on all the other boundary components. Pick one such component O.

Now, by the uniqueness of the FPS (Proposition 3.26) we know that in order to construct the part of FPS A^u_{-a} intersecting O, we can first take $A^u_{-a,b}$, and then construct the FPS of level $-a$ in O. But now, by Corollary 3.30 we know that an FPS of level $-a$ inside O cannot touch any part of the boundary where u is smaller than or equal to $-a$. But similarly we see that the part of FPS V^u_b intersecting O, cannot touch the boundary of O where u is larger than or equal to b. Hence the intersection of $A^u_{-a} \cap V^u_b \cap O$ cannot touch the boundary of O and hence is not connected to the boundary of D. The proposition follows.

□

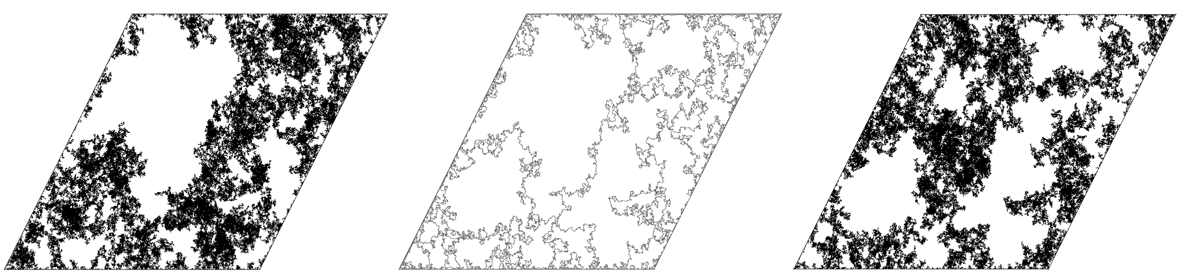


Figure 3.5: Graphical evidence of Lemma 3.35: On the left $A_{-\lambda}$, on the right V_λ and in the middle $A_{-\lambda,\lambda}$. Simulation done by B. Werness.

We are now ready to prove the uniqueness of two-valued sets $A^u_{-a,b}$ for general boundary data in n-connected domains. See Proposition 3.22 for the setting and precise statement.

Uniqueness of two-valued sets. In the proof of Proposition 3.22, we showed the existence of a two-valued set. Denote this set by $A^u_{-a,b}$. Suppose A' is another two-valued set coupled with the same GFF, i.e. it satisfies the condition (♫) given just before Proposition 3.22.

First, notice that A' has to be connected to the boundary: indeed, suppose for contradiction that there is a component B of A' that is not connected to the boundary. Consider the component O of $D\backslash B$ that has D on part of its boundary, and let $\mathcal{B} = \partial O \cap B$. WLOG suppose that the boundary conditions on $\mathcal{B}$ are smaller or equal to $-a$. Then as in the last paragraph of proof of Lemma 3.35, we see that the FPS of height $-a$ will also contain this component, and moreover it will also not be connected to the boundary. However, from the construction (Proposition 3.25) and uniqueness of the FPS (Proposition 3.26) we know that the FPS is connected to the boundary.

Now, inspecting the proof of Lemma 3.35, we conclude that exactly the same proof gives that almost surely the union of the components of $A^u_{-a} \cap V^u_b$ that are connected to boundary is equal to A'. But we know from Lemma 3.35 that it is equal to $A^u_{-a,b}$ and the claim follows.

□

3.5 Relating the first passage sets on metric graphs and in continuum

In this section we show that the metric graph FPS converges to the continuum FPS and draw several consequences. After detailing the set-up, we prove the convergence in probability of the metric graph FPS towards its continuum counterpart. This follows rather simply from general theory of local sets and the continuum characterization of the FPS. Next, we extend the results of [Lup15] on the convergence of loop soup clusters to derive the equivalent of Lemma 3.5 in the continuum, i.e. we show that first passage sets can be represented using Brownian loops and excursions. Finally, we derive some corollaries involving level lines of the GFF. For example, we prove the convergence of certain interfaces of the metric graph GFF towards generalized level lines, and thus in particular towards $\mathrm{SLE}_4(\rho)$ processes.

3.5.1 Set-up and basic convergence results

In this section we review some basic convergence results for random closed sets, random fields and path measures. Most of the content is standard, but slightly reworded and reinterpreted. For simplicity we restrict ourselves to $\widetilde{\mathbb{Z}}^2_n$, the metric graph induced by vertices $(2^{-n}\mathbb{Z})^2$ and with unit conductances on every edge. However, we believe that the convergence results we obtain may be extended without problem to isoradial graphs. We consider our metric graph $\widetilde{\mathbb{Z}}^2_n$ to be naturally embedded in $\mathbb{C}$, and when we mention distances and diameters for objects living on metric graphs, we always consider the Euclidean distance inherited from $\mathbb{C}$.

3.5.1.1 Topologies and convergences on sets and functions

We work with multiply-connected bounded domains D. For us, domains are by definition connected. We approximate them with metric graph domains obtained as intersections of $\widetilde{\mathbb{Z}}^2_n$ with open subsets of $\mathbb{C}$, i.e. by $\widetilde{D}_n = \widetilde{\mathbb{Z}}^2_n \cap D_n$, where $D_n \to D$ in an appropriate sense detailed below. We say that such an approximation satisfies the condition $\circledast$ if

- $\circledast$ There exists $C, C' > 0$ such that $D_n \subseteq [-C, C]^2$, and the amounts of connected components of $\mathbb{C}\backslash D_n$ is less or equal to C'.

We say that a sequence of metric graph pointed domains $(\widetilde{D}_n, z_n)$ satisfy $\circledast$ if D_n satisfies $\circledast$. Sometimes we also need to work with open sets that are non connected, but have at most countably many connected components (e.g. the complement of a CLE_4 carpet). The same condition makes sense in this case too.

We use the following topologies for open and closed sets:

- For open domains D^z with a marked point $z \in D^z$, we work with the Carathéodory topology: (D_n, z_n) converges to (D^z, z) in the sense of Carathéodory if

  1. $z_n \to z$,
  2. $D^z \subseteq \cup_{N\in\mathbb{N}} \cap_{n\geqslant N} D_n$,
  3. for any $x \in \partial D$ there are $x_n \in \partial D_n$ with $x_n \to x$.

  Notice that in this wording we do not have to assume simply-connectedness and indeed, the Carathéodory topology generalizes nicely to multiply-connected setting (e.g. see [Com13] and references within in this case).

- For closed sets, we work with the Hausdorff distance: the distance between two sets A, B is the infimum over $\varepsilon > 0$ such that $A \subset B + B(0,\varepsilon)$ and $B \subset A + B(0,\varepsilon)$, where $B(0,\varepsilon)$ is the unit disk of radius ε and we consider the Minkowski sum. Note that the ensemble of closed sets such that its complement is contained in $[-C,C]^2$ is compact for the Hausdorff topology.

- For open sets D that may not be connected, it is convenient to consider the Hausdorff distance on their complements. In this case, $D_n \to D$ implies the Carathéodory convergence for any pointed connected component (D^z, z) of D, see for example Theorem 1 of [Com13].

We are interested also in the convergence of functions on $\widetilde{D}_n = \widetilde{\mathbb{Z}}^2_n \cap D_n$ to (generalized) functions on D. In fact, it is more convenient to look at functions to be defined on a fixed domain $[-C,C]^2$ containing D and D_n. Thus we extend the functions on $\widetilde{D}_n$ to the whole of $[-C,C]^2$ by extending it harmonically outside of $\widetilde{D}_n$, in particular inside the square faces delimited by $\widetilde{D}_n$. In the case of the metric graph GFF $\tilde{\phi}$, its extension is still a Gaussian process. We silently use these extensions everywhere when talking about the convergences and often omit the word 'extension' for readability. If we want to be explicit, we use the decoration $\widehat{\ }$ for extensions, in particular $\widehat{G}_{\widetilde{D}_n}$ will denote the Green's function of the metric graph GFF defined on $\widetilde{D}_n$ and extended to $[-C,C]^2$.

Both harmonic functions and GFF-s can be considered on any open set. If Φ is a GFF in D, then we can write $\Phi = \sum_{D^z} \Phi^{D^z}$ where the sum runs on the connected components D^z of D and where Φ^{D^z} is a GFF in D^z independent of all the others. We consider the following topologies:

- For convergence of (the extensions of) harmonic function we use the uniform norm on compact subsets of D. We avoid ∂D because we want to allow for a finite number of jumps on ∂D.

- For GFF-s on metric graphs, it is convenient to work (with their extensions) in the Sobolev space $H^{-1-\varepsilon}([-C,C]^2)$. For background on Sobolev spaces we refer the reader to e.g. [AF03].

We will shortly see that these convergences are well-behaved. A key ingredient, that we will also use later, is the weak Beurling estimate (see for e.g. Proposition 2.11 of [CS11] for the discrete case and Proposition 3.73 of [Law08] for the continuum case):

Lemma 3.36 (Beurling estimate) *There exists $\beta > 0$ such that for all $K \subseteq \widetilde{\mathbb{Z}}_n^2$ with C connected components all of them with size at least δ, and for all $z \in \widetilde{\mathbb{Z}}_n^2 \backslash K$ and $\varepsilon \leqslant \delta/2$*

$$\mathbb{P}^x(\widetilde{X} \text{ hits } B(z,\varepsilon) \text{ before hitting } K) \leqslant const(C,\delta)\left(\frac{d(z,K)}{\varepsilon}\right)^\beta,$$

where $\widetilde{X}$ is a metric graph Brownian motion started at z. The same estimate holds in the continuum, i.e. if we replace $\widetilde{\mathbb{Z}}_n^2$ by $\mathbb{C}$ and consider the two-dimensional Brownian motion.

The following lemma is basically contained in [CS11] Proposition 3.3 and Corollary 3.11. Although the statements there include more stringent conditions (like boundaries being given by Jordan curves and simply-connectedness), one can verify that this is not really used in the proofs. For similar statements one can also see Proposition 3.5 and Lemma A.1 in [BL14].

Lemma 3.37 *Suppose $(D_n)_{n\in\mathbb{N}}$ are open sets that satisfy condition $\otimes$, and $D_n \to D$ in the sense that their complements converge in the Hausdorff topology. Then, we have that:*

1. *If H is a bounded function on $[-C,C]^2$ with at most a finite number of discontinuity points on ∂D and continuous elsewhere, then the (extensions of the) harmonic functions defined on $\widetilde{D}_n$ by boundary values given by H converge to the unique harmonic function with boundary values H.*
2. *For continuous bounded f defined on $Q_C = [-C,C]^2$ we have that,*

$$\lim_{n\to+\infty} \iint_{Q_C\times Q_c} f(z)\widehat{G}_{\widetilde{D}_n}(z,w)f(w)dzdw = \iint_{Q_C\times Q_C} f(z)G_D(z,w)f(w)dzdw,$$

   where $\widehat{G}_{\widetilde{D}_n}$ is the Green's function of the extension of the metric graph GFF.

Similarly, for any connected component D^z of D containing z, if (D_n, z_n) converge towards (D,z) in the Carathéodory sense, then the statements also hold.

Proof. As commented on above, the proofs are basically contained in [CS11]. Hence we will only sketch the steps with appropriate references.

1. Pre-compactness in the uniform norm on compacts of D, and harmonicity outside of ∂D both follow from the proof of Proposition 3.1 in [CS11]. In particular, we know that each subsequential limit is a bounded harmonic function. To determine the boundary values one finally uses Beurling estimate as in the proof of Proposition 3.3 in [CS11].
2. The convergence of the (extension of the) discrete Green's function on $[-C,C]^2 \cap \mathbb{Z}_n$ to the continuum Green's function on $[-C,C]^2$ is well-known and can be explicitly shown via eigenfunction expansion of the Green's function. The convergence of $G_{[-C,C]^2\cap\widetilde{\mathbb{Z}}_n}$ and $\widehat{G}_{[-C,C]^2\cap\widetilde{\mathbb{Z}}_n}$ follows.

   For the general case, note that function $\widehat{G}_{\widetilde{D}_n}(z,\cdot) - G_{[-C,C]^2\cap\widetilde{\mathbb{Z}}_n^2}(z,\cdot)$ is harmonic outside far of ∂D_n with uniformly bounded boundary values. Thus it converges by (1). To deduce the convergence of the integral one finally uses the known fact that $\widehat{G}_{[-C,C]^2\cap\widetilde{\mathbb{Z}}_n^2}(z,w) \leqslant c(\log(\|z-w\|)+1)$ and dominated convergence. For more details see e.g. Proposition 3.5 of [BL14].

□

We can now deduce a useful corollary for the convergence of the metric GFF-s:

Corollary 3.38 *Suppose $(D_n)_{n\in\mathbb{N}}$ are open sets that satisfy condition $\otimes$, $D_n \to D$ in the sense that their complements converge in the Hausdorff topology. Then the (extensions of the) metric graph GFF-s on $\widetilde{D}_n$ converge to a GFF on D. Moreover, for any connected component D^z of D containing z, if (D_n, z_n) converge towards (D, z) in the Carathéodory sense, then the restrictions of $\tilde{\phi}$ to D converge to a zero boundary GFF on D.*

Proof. Lemma 3.37 (2) guarantees finite-dimensional convergence, thus it remains to prove tightness. The norm of the Sobolev space $H^{-1}(D)$ is given by (e.g. see [Dub09], Section 4.2.)

$$\|f\|_{H^{-1}}^2 = \iint_{D\times D} f(z)G_D(z,w)f(w)dzdw.$$

But using Lemma 3.37 (2) and denoting $Q_C = [-C,C]^2$ we can explicitly calculate to see that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|\tilde{\phi}_n\|_{H^{-1}(Q_C)}^2\right] = \sup_{n\in\mathbb{N}} \iint_{Q_C\times Q_C} \widehat{G}_{\widetilde{D}_n}(z,w)G_D(z,w)dzdw < \infty.$$

Hence by the Sobolev embedding theorem, we have that $(\tilde{\phi}_n)_{n\in\mathbb{N}}$ is tight in $H^{-1-\varepsilon}([-C,C]^2)$ for any $\varepsilon > 0$ and the convergence follows.

The latter part follows similarly. □

3.5.1.2 Topologies and convergences on loops and excursions

Now let $\mathcal{L}_\alpha^D$ and $\mathcal{L}_\alpha^{\widetilde{D}_n}$, be respectively a continuum and a metric graph loop-soup. Moreover for u positive on ∂D and u_n positive on $\partial\widetilde{D}_n$, let Ξ_u^D and $\Xi_{u_n}^{\widetilde{D}_n}$ be respectively independent PPP of boundary-to-boundary Brownian excursions of intensity $\mu_{\text{exc}}^{D,u}$ and boundary-to-boundary metric graph excursions of intensity $\mu_{\text{exc}}^{\widetilde{D}_n,u_n}$. We use the following topologies when we work with for the measures of paths, i.e. excursions and loops:

- It suffices to consider paths as closed subsets in $\overline{D}$, i.e. to work with Hausdorff distance.
- For sets of paths $(\Gamma_n)_{n\in\mathbb{N}}$, we consider for each ε the subset Γ_n^ε of Γ_n, consisting of paths that have diameter larger than ε. We say that $(\Gamma_n)_{n\in\mathbb{N}}$ converges to Γ if for all dyadic $\varepsilon > 0$, we have that Γ_n^ε is finite, each element of Γ_n^ε converges to some element in $\Gamma_n^{\varepsilon'}$ for some $\varepsilon' \leqslant \varepsilon$, and conversely each element in Γ^ε is a limit of a path in some Γ_n^ε.

The following Lemma says that these convergences also behave nicely:

Lemma 3.39 *Suppose $(D_n)_{n\in\mathbb{N}}$ are open sets that satisfy condition $\otimes$, $D_n \to D$ in the sense that their complements converge in the Hausdorff topology. Moreover, let u be a positive harmonic function in $[-C,C]^2\backslash\partial D$ defined by piecewise constant boundary values on ∂D and u_n harmonic functions on $\widetilde{D}_n$ converging to u. Then, we have that for all $\varepsilon \geqslant 0$:*

1. $\mu_{\text{loop}}^{\widetilde{D}_n}\mathbf{1}_{\text{Diam}(\gamma)\geqslant\varepsilon} \to \mu_{\text{loop}}^{D}\mathbf{1}_{\text{Diam}(\gamma)\geqslant\varepsilon}$. *In other words, for all $\alpha \geqslant 0$ it exists a coupling such that $\mathcal{L}_\alpha^{\widetilde{D}_n} \to \mathcal{L}_\alpha^D$.*

2. $\mu_{\text{exc}}^{\widetilde{D}_n,u_n}\mathbf{1}_{\text{Diam}(\gamma)\geqslant\varepsilon} \to \mu_{\text{exc}}^{\widetilde{D},u}\mathbf{1}_{\text{Diam}(\gamma)\geqslant\varepsilon}$. *In other words, there exists a coupling such that* $\Xi_{u_n}^{\widetilde{D}_n} \to \Xi_u^D$.

Similarly, for any connected component D^z of D containing z, if (D_n, z_n) converge towards (D, z) in the Carathéodory sense, then the statements also hold.

Proof. (1) The equivalent statement for random walk loop-soups on $\mathbb{Z}_n^2 \cap D^z$ for a domain D^z is given in Corollary 5.4 of [LTF07]. The proof for the metric graph loop-soups in that context is exactly the same. As remarked just after the proof, the ideas extend to our non-simply connected case with finitely many boundary components. Moreover, one can verify that one can also approximate D^z using $\mathbb{Z}_n^2 \cap D_n$ where $(D_n, z_n) \to (D^z, z)$ in the sense of Carathéodory. As the convergence of $D_n \to D$ in the sense that the complements converge in the Hausdorff metric implies the Carathéodory convergence for all components, and we have only countably many components, the claim follows.

(2) Essentially the proof follows the steps of [LTF07]: we need to first show convergence of excursions with diameter larger than ε that visit some compact set inside D, and then to show that there are no excursions of diameter ε that stay δ close to the boundary.

For the first part it suffices to show that for any closed square $Q \subseteq D$ with rational endpoints, we have weak convergence $\mathbf{1}_{\gamma\cap Q\neq\emptyset}\mu_{\text{exc}}^{\widetilde{D}_n,u_n} \to \mathbf{1}_{\gamma\cap Q\neq\emptyset}\mu_{\text{exc}}^{D,u}$. This follows from the Markov property for the metric graph excursions (Proposition 3.2) and the Brownian excursion measure (Proposition 3.8). Indeed, we can decompose the excursions in D (or $\widetilde{D}_n$) at their first hitting time at Q into an excursion from ∂D (or $\partial\widetilde{D}_n$) to ∂Q and a Brownian motion (continuum 2D or on metric graph) started on ∂Q and stopped at its first hitting time of ∂D (or $\partial\widetilde{D}_n$). The convergence of the second part just follows from the convergence of random walks to Brownian motion inside compacts of D and the Beurling estimate for the convergence of the actual hitting point. For the excursion from ∂D (or $\partial\widetilde{D}_n$) to ∂Q, we can decompose it further into an excursion from $\partial Q'$ to ∂Q, where Q' is some closed square with rational endpoints containing Q in its interior, and a time-reversed Brownian motion (continuum 2D or on metric graph) from $\partial Q'$ to the boundary of ∂D (or $\partial\widetilde{D}_n$). The convergence of both pieces is now clear.

Finally, we need to show that for all $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\mu_{\text{exc}}^{\widetilde{D}_n,u_n}\left(\text{Diam}(\gamma)\geqslant 2\varepsilon, \sup_{x\in\gamma} d(x,\partial D)\leqslant\delta\right) = 0.$$

To do this we can again use the Markov decomposition. We cover the boundary of $\widetilde{D}_n$, for all n, with open disks $(B(z_i,\varepsilon))_{i\in I}$. The minimal number of disks needed depends on ε, but is uniformly upper bounded in n. Any excursion that is at least 2ε in diameter and has one endpoint in $B(z_i,\varepsilon)$, has to hit $\partial B(z_i,\varepsilon)$. But then it can be decomposed into an excursion from $\partial\widetilde{D}_n$ to $\partial B(z_i,\varepsilon)$ and a metric graph BM from $\partial B(z_i,\varepsilon)$ to $\partial\widetilde{D}_n$. The probability that the latter goes ε far without getting δ far from ∂D can be bounded by Beurling estimate (Lemma 3.36) and goes to 0 as $\delta \to 0$ uniformly in sufficiently large n. □

3.5.2 Convergence of first passage sets

In this subsection we prove that the discrete FPS converge to the continuum FPS. Recall that by convention the FPS always contains the boundary of the domain.

Proposition 3.40 *Let $(\widetilde{D}_n)_{n\in\mathbb{N}}$ satisfy condition $\circledast$ and suppose that the complements of D_n converge in the Hausdorff topology to the complement of some open set D. Moreover suppose that $(u_n)_{n\in\mathbb{N}}$ is a sequence of bounded harmonic functions in $\widetilde{D}_n$ such that $u_n \to u$, a bounded harmonic function with piecewise constant boundary values. Denote further for any $z \in D$ by D^z the connected component of z in D. Then for any D^z, the FPS restricted to this component converges in law: $(\tilde{\phi}_n^{D^z}, (\widetilde{A}_{-a}^{u_n} \cap D^z) \cup \partial D^z) \Rightarrow (\Phi^{D^z}, A_{-a}^u)$ as $n \to \infty$, where A_{-a}^u is the FPS in the component D^z.*

Furthermore, if we couple $(\tilde{\phi}_n)_{n\in\mathbb{N}}$ and Φ^D such that $\tilde{\phi}_n^D \to \Phi^D$ in probability as generalized functions, then $(\tilde{\phi}_n^{D^z}, (\widetilde{A}_{-a}^{u_n} \cap D^z) \cup \partial D^z) \to (\Phi^{D^z}, A_{-a}^u)$ in probability.

Remark 3.41 *The convergence of the open sets $D_n \to D$ in the sense that their complements converge implies, for any $z \in D$ and any $z_n \to z$, the Carathéodory convergence of (D_n, z_n) to (D^z, z). Yet it does not imply that ∂D_n converge to ∂D^z in the Hausdorff metric, hence the need to treat the boundary separately.*

The proof follows from two simple lemmas in the spirit of the proof of Theorem 1.2 of [SS13]. The first one says that the metric graph local sets converge towards continuum local sets. The second one is a general lemma, which in our case will imply that, due to the uniqueness of the FPS, the convergence in law of the pair (GFF, FPS) can be promoted to a convergence in probability.

Lemma 3.42 (Convergence of metric local sets) *Let $(\widetilde{D}_n)_{n\in\mathbb{N}}$ satisfy condition $\circledast$ and suppose that the complements of D_n converge to the complement of an open set D in the Hausdorff topology. Moreover, let $(\tilde{\phi}_n, A_n)$ be such that A_n is optional for ϕ_n and that for some $c > 0$, the sets A_n have almost surely less than c components none of which reduces to a point.*

Then $(\tilde{\phi}_n, A_n, (\tilde{\phi}_n)_{A_n})$ is tight and any sub-sequential limit (Φ, A, Φ_A) is a local set coupling. Additionally, for any connected component D^z of D we have that $(\tilde{\phi}_n^{D^z}, (A_n \cap D^z))$ converges to a local set coupling in D^z and the harmonic function is given just by the restriction of Φ_A to D^z.

Proof. Let us first argue tightness. By Lemma 3.38 we know that the GFF-s converge in law. Moreover, the space of closed subsets of the closure of a bounded domain is compact for the Hausdorff distance. Hence the sequence A_n is tight. By conditioning on A_n and using Jensen's inequality, we can uniformly bound the expected value of the $H^{-1-2\varepsilon}([-C, C]^2)$ norm of $(\tilde{\phi}_n)^{A_n}$ and obtain tightness of $(\tilde{\phi}_n)^{A_n}$. Finally, by the Markov decomposition $\tilde{\phi} - (\tilde{\phi}_n)_{A_n} = \tilde{\phi}^{A_n}$ and the triangle inequality, we see that also $(\tilde{\phi}_n)_{A_n}$ are tight in $H^{-1-\varepsilon}([-C, C]^2)$. Thus we have tightness of the quadruple $(\tilde{\phi}_n, A_n, (\tilde{\phi}_n)_{A_n}, (\tilde{\phi}_n)^{A_n})$. Now pick a subsequence (that we denote the same way) such that this quadruple converges in law to $(\Phi, A, \Phi^1, \Phi^2)$. From the joint convergence we then have that for any bounded continuous functionals f_1 and f_2

$$\lim_{n\to+\infty} \mathbb{E}\left[f_1((\tilde{\phi}_n)^{A_n}) f_2((\tilde{\phi}_n)_{A_n}, A_n)\right] = \mathbb{E}\left[f_1(\Phi^1) f_2(\Phi^2, A)\right].$$

On the other hand, conditionally on $(A_n, (\widetilde{\phi}_n)_{A_n})$, the law of $(\widetilde{\phi}_n^{A_n})$ is that of a metric graph GFF in $\widetilde{D}_n \backslash A_n$. By Lemma 3.38, it follows that $\mathbb{E}[f_1((\tilde{\phi}_n)^{A_n})|A_n] \to \mathbb{E}[f_1(\Phi^A)|A]$ in law. Thus, by bounded convergence, we have that

$$\lim_{n\to+\infty} \mathbb{E}\left[\mathbb{E}^{A_n}[f_1((\tilde{\phi}_n)^{A_n})] f_2((\tilde{\phi}_n)_{A_n}, A_n)\right] = \mathbb{E}\left[f_1(\Phi^A) f_2(\Phi^2, A)\right] = \mathbb{E}\left[\mathbb{E}[f_1(\Phi^1)|A] f_2(\Phi^2, A)\right],$$

giving the claim.

It remains to show that Φ^1 is almost surely harmonic in $D\backslash A$: indeed, then from Lemma 3.10, it would follow that A is local and $\Phi^1 = \Phi_A$ and $\Phi^2 = \Phi^A$. In this respect, let Δ_n be the discrete Laplacian. From Lemma 2.2 of [CS11], it follows that for any smooth function f, inside any compact set where derivatives of f remain bounded we have that $\Delta_n f(u) = \Delta f(u) + O(2^{-n})$. But from integration by parts it follows that if f is a smooth function with compact support in $D\backslash A$, then $((\tilde{\phi}_n)_{A_n}, \Delta_n f) = 0$ for sufficiently large n. Hence $(\Phi^1, \Delta f) = 0$ almost surely and thus Φ^1 is harmonic in $D\backslash A$.

The final claim just follows from Lemma 3.38 and the simple fact that if A is a local set for Φ in a non-connected domain D, then for any component of D, D^z, we have that $A \cap D^z$ is a local set of Φ^D

□

The next lemma shows how to promote convergence in law to convergence in probability. See Lemma 4.5 in [SS09], and Lemma 31 in [Sha16] for earlier appearances in the context of GFF and level lines, and of Gaussian multiplicative chaos respectively. We give a slight rewording of the latter proof adapted to our setting.

Lemma 3.43 *Let $(X_n, Y_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence of random variables in a metric space, living all of them in the same probability space. Suppose we know that*

1. *$(X_n, Y_n) \Rightarrow (X_\infty, Y_\infty)$*
2. *$X_n \to X_\infty$ in probability.*
3. *There exists a measurable function F such that $F(X_\infty) = Y_\infty$.*

Then $(X_n, Y_n) \to (X_\infty, F(X_\infty))$ in probability.

Proof. Denote $M_n := (X_n, Y_n, X_\infty, F(X_\infty))$. Because, each coordinate is tight we have that up to a subsequence $M_n \Rightarrow (\bar{X}_\infty, F(\bar{X}_\infty), X_\infty, F(X_\infty))$. Thus, any linear combination of them will also converge in law. Note that by (2), $(X_n, X_\infty) \to (X_\infty, X_\infty)$, so $\bar{X}_\infty = X_\infty$. This fact implies that a.s. $\bar{Y}_\infty = F(X_\infty)$, thus $Y_n - F(X_\infty)$ converges in law, and therefore in probability, to 0. □

We have now all the tools to prove the convergence.

Proof of Proposition 3.40. When $\min_{\partial \tilde{D}_n} u_n \geqslant -a$, we know that $(\tilde{\phi}_n)_{A_n} + u_n$ is constantly equal to $-a$ on $\tilde{D}_n \backslash A_n$ and the claim follows directly from the two Lemmas above.

When $\min_{\partial \tilde{D}_n} u_n < -a$, we can again use the two Lemmas above to obtain the convergence to a local set (A, Φ_A) in probability. Moreover, it is easy to see that the conditions (1) and (2) in the Definition 3.23 hold for A, as these properties hold for all approximations and pass to the limit. Thus, it just remains to argue for (3). This condition however follows from the Beurling estimate. Pick some component O of the complement of A and any z on its boundary. We can then choose a small enough ball U_z^1 around z such that the boundary conditions only change once in this neighbourhood. By the Beurling estimate (Lemma 3.36) we can further choose a even smaller ball U_z such that the Brownian motion started inside $U_z \cap O$ exits O through $U_z^1 \cap \partial O$ with a probability larger than $1 - \frac{\varepsilon}{4 \max |u|}$. By the convergence of $A_n \to A$ in probability and Beurling estimate again, we can choose n_0 large enough so that for all $n \geqslant n_0$ the metric graph Brownian motion started inside $U_z \cap (\widetilde{\mathbb{Z}}_n^2 \backslash A_n)$ exits $\widetilde{\mathbb{Z}}_n^2 \backslash A_n$ through $U_z^1 \cap (\widetilde{\mathbb{Z}}_n^2 \backslash A_n)$ with probability larger than $1 - \frac{\varepsilon}{2 \max |u|}$ and $u_n - u \leqslant \varepsilon/2$ uniformly over $\widetilde{D}_n \cap D$. A final use of Beurling estimate then implies that for any $z_n \in U_z \cap (\widetilde{\mathbb{Z}}_n^2 \backslash A_n)$, we have that $\tilde{h}_{A_n}(z_n) + u_n(z_n) \geqslant \min\{-a, \inf_{w \in U_z \cap \overline{O}} u(w)\} - \varepsilon$, where h_{A_n} is the metric graph harmonic function outside of A_n as in Proposition 3.1.The claim follows.

□

3.5.3 Convergence of clusters of loops and excursions

In this subsection we assume that u is non-negative. Let $\mathcal{L}_\alpha^D$ and $\mathcal{L}_\alpha^{\widetilde{D}_n}$ denote respectively a continuum and metric graph loop-soups of intensity $\alpha \in (0, 1/2]$. Similarly, let Ξ_u^D and $\Xi_{u_n}^{\widetilde{D}_n}$ denote PPP of boundary-to-boundary excursions in the continuum of intensity $\mu_{\text{exc}}^{D,u}$ and in the metric graph setting of intensity $\mu_{\text{exc}}^{\widetilde{D}_n, u_n}$ respectively. We sample the loop-soups and PPP of excursions independently and are interested in the clusters of $\mathcal{L}_\alpha^D \cup \Xi_u^D$ and $\mathcal{L}_\alpha^{\widetilde{D}_n} \cup \Xi_{u_n}^{\widetilde{D}_n}$ that contain at least one excursion. By definition two paths belong to the same cluster if they are joined by a finite chain of paths along which two consecutive ones intersect. We denote by $\mathcal{A} = \mathcal{A}(\mathcal{L}_\alpha^D, \Xi_u^D)$ and $\widetilde{\mathcal{A}}_n = \widetilde{\mathcal{A}}_n(\mathcal{L}_\alpha^{\widetilde{D}_n}, \Xi_{u_n}^{\widetilde{D}_n})$ the closed union of such clusters.

The main content of this subsection shows that metric graph clusters converge to their continuum counterparts:

Proposition 3.44 *Suppose $(\widetilde{D}_n, z_n)$ satisfy the condition $\circledast$ and converge to (D, z) in the Carathéodory sense. Moreover suppose that u is a non-negative bounded harmonic function and $u_n \to u$ uniformly on compact subsets of D. We also assume that whenever $u = 0$ on a part of the boundary $\mathcal{B}$, then for any sequence of metric graph boundary points $x_n \to x \in \overline{\mathcal{B}}$ we have that $u_n(x_n) = 0$ as well, for n large enough. Then the sequence of compact sets $(\overline{\widetilde{\mathcal{A}}_n \cap D})_{n \geqslant 0}$ converges in law for the Hausdorff metric towards $\mathcal{A}$.*

Let us explain the additional condition on the convergence of u_n. We want to avoid the following situation. Assume $\mathcal{B}$ is an arc of the boundary ∂D and u equals 0 on $\mathcal{B}$. Then $\mathcal{A}$ does not intersect $\mathcal{B}$. However one could approximate u by u_n small but positive on $\mathcal{B}_n \subseteq \partial \widetilde{D}_n$ approaching $\mathcal{B}$. Then almost surely $\mathcal{B}_n \subset \widetilde{\mathcal{A}}_n$ and the limit of $\widetilde{\mathcal{A}}_n$ would contain $\mathcal{B}$.

The core of our proof is the following lemma, saying that there are no loop-soup clusters that at the same time stay at a positive distance from the boundary, but also come microscopically close to it.

Lemma 3.45 *Let $\alpha \in (0, 1/2]$. Suppose that $(\widetilde{\Omega}_n, w_n)_{n \in \mathbb{N}}$ satisfy $\circledast$ and $(\widetilde{\Omega}_n, w_n) \to (\Omega, w)$ in the Carathéodory sense. Then, for all $\delta > 0$,*

$$\lim_{\zeta \to 0} \sup_{n \in \mathbb{N}} \mathbb{P} \left(\textit{There is } \mathcal{C} \textit{ cluster of } \mathcal{L}_\alpha^{\widetilde{\Omega}_n} \textit{ s.t. } d(\mathcal{C}, \partial \widetilde{\Omega}_n) \leqslant \zeta, \sup_{z \in \mathcal{C}} d(z, \partial \Omega) \geqslant \delta \right) = 0.$$

Note that the above lemma is not implied by the convergence result proved by Lupu in [Lup15]. However, it could have been proved using the same strategy as in [Lup15]. In this Chapter we will have a slightly different approach, relying on the convergence of first passage sets. We will first show how the proposition follows from this lemma, and then prove the lemma.

Proof of Proposition 3.44. From Lemma 3.37 we know that

$$\{\gamma \in \mathcal{L}_\alpha^{\widetilde{D}_n} | \gamma \cap D \neq \emptyset\} \Rightarrow \mathcal{L}_\alpha^D, \qquad \{\gamma \in \Xi_{u_n}^{\widetilde{D}_n} | \gamma \cap D \neq \emptyset\} \Rightarrow \Xi_u^D,$$

as $n \to \infty$. Also $(\widetilde{\mathcal{A}}_n)_{n\in\mathbb{N}}$ is a sequence of random closed sets and thus is tight. Thus we can extract a subsequence (which we denote in the same way) along which

$$(\{\gamma \in \mathcal{L}_\alpha^{\widetilde{D}_n} | \gamma \cap D \neq \emptyset\}, \{\gamma \in \Xi_{u_n}^{\widetilde{D}_n} | \gamma \cap D \neq \emptyset\}, \overline{\widetilde{\mathcal{A}}_n \cap D})_{n\in\mathbb{N}}$$

converges in law to a triple $(\mathcal{L}_\alpha^D, \Xi_u^D, \mathbf{A})$. By using Skorokhod's representation theorem, we may assume that this convergence is almost sure. Then, as $\mathcal{A}$ is a measurable function of $\mathcal{L}_\alpha^D$ and Ξ_u^D, it remains to show that $\mathcal{A} = \mathbf{A}$ almost surely.

Let us first show that $\mathcal{A} \subseteq \mathbf{A}$. To do this we consider loops and excursions with cutoff on the diameter and the clusters formed by these loops and excursions. More precisely, respectively in the continuum and on the metric graph, let $\mathcal{A}^\varepsilon$ and $\widetilde{\mathcal{A}}^\varepsilon$ denote the union of clusters, that are formed of loops and excursions that have diameter greater or equal to $\varepsilon > 0$, and that contain at least one excursion. Here by diameter we always mean the diameter for the Euclidean distance on $\mathbb{C}$, even for paths living on metric graphs. Note that both $\mathcal{A}^\varepsilon$ and $\widetilde{\mathcal{A}}^\varepsilon$ consist a.s. of finitely many path, and are in particular compact, since a.s. there are finitely many loops and excursions of diameter larger than some value. Now, in our coupling almost surely metric graph loops converge to continuum Brownian loops, metric graph excursions to Brownian excursions, and moreover by (Lemma 2.7 in [Lup16b]) their intersection relations also converge. Hence we have that $\overline{\widetilde{\mathcal{A}}_n^\varepsilon \cap D} \overset{a.s.}{\to} \mathcal{A}^\varepsilon$. On the other hand $\widetilde{\mathcal{A}}_n^\varepsilon \subseteq \widetilde{\mathcal{A}}_n$ and $\mathcal{A}^\varepsilon \to \mathcal{A}$ as $\varepsilon \to 0$. We conclude that $\mathcal{A} \subseteq \mathbf{A}$ almost surely.

Let us now show that $\mathbf{A} \subseteq \mathcal{A}$. First notice that there exists a deterministic sequence $\varepsilon(n) \searrow 0$ such that for a deterministic subsequence we denote the same way $\overline{\widetilde{\mathcal{A}}_n^{\varepsilon(n)} \cap D} \overset{a.s.}{\to} \mathcal{A}$. Indeed, as both $\overline{\widetilde{\mathcal{A}}_n^\varepsilon \cap D} \overset{a.s.}{\to} \mathcal{A}^\varepsilon$ as $n \to \infty$, and $\mathcal{A}^\varepsilon \overset{a.s.}{\to} \mathcal{A}$ as $\varepsilon \to 0$ in the Hausdorff distance, we can apply a diagonal argument to choose the sequence $\varepsilon(n)$.

Now, fix a dense sequence of distinct points $(w_i)_{i\geqslant 0}$ in D. Let $\widetilde{O}_n(w_i)$ and $\widetilde{O}_n^{\varepsilon(n)}(w_i)$, denote the connected components containing w_i of $\widetilde{D}_n\backslash\widetilde{\mathcal{A}}_n$ and $\widetilde{D}_n\backslash\widetilde{\mathcal{A}}_n^{\varepsilon(n)}$ respectively. By connected component of w_i on a metric graph we mean the connected component that either contains w_i or contains the dyadic square surrounding w_i. For any fixed w_i it is defined only with certain probability that converges to 1 as $n \to +\infty$. Further, define $O(w_i)$ as the connected component of w_i in $D\backslash\mathcal{A}$ and for any $\delta > 0$ let $\Theta_\delta(w_i)$ be the connected component of w_i in $D\backslash\overline{(\mathcal{A} + B(0,\delta))}$. As the condition on the boundary convergence of $u_n \to u$ guarantees that $\mathbf{A} \cap \partial D = \mathcal{A} \cap \partial D$, it remains to prove that $\mathbf{A} \cap D \subseteq \mathcal{A} \cap D$. To do this it suffices to show that for all w_i and $\delta > 0$

$$\lim_{n\to+\infty} \mathbb{P}(\Theta_\delta(w_i) \subseteq \widetilde{O}_n(w_i)) = 1. \tag{3.7}$$

For any fixed w_i, we will apply Lemma 3.45 to $\Omega = O(w_i)$ and $\widetilde{\Omega}_n = \widetilde{O}_n^{\varepsilon(n)}(w_i)$. Note that $\mathbb{C}\backslash O(w_i)$ has at most as many connected components as $\mathbb{C}\backslash D$. Moreover, from Theorem 1 of [Com13] we know that the Hausdorff convergence of $\widetilde{\mathcal{A}}_n^{\varepsilon(n)}$ to $\mathcal{A}$ implies the Carathéodory convergence of $(\widetilde{O}_n^{\varepsilon(n)}(w_i), w_i) \to (O(w_i), w_i)$. Finally, conditioned on $\widetilde{\mathcal{A}}_n^{\varepsilon(n)}$, the law of $\mathcal{L}_\alpha^{\widetilde{O}_n^{\varepsilon(n)}(w_i)}$

(i.e. the law of the metric graph loops of $\mathcal{L}_{1/2}^{\widetilde{D}_n}$ that are contained inside $\widetilde{O}_n^{\varepsilon(n)}(w_i)$), is that of a metric graph loop soup in $\widetilde{O}_n^{\varepsilon(n)}(w_i)$. Hence Lemma 3.45 implies that

$$\lim_{n\to+\infty} \mathbb{P}\left(\text{There is } \mathcal{C} \text{ cluster of } \mathcal{L}_\alpha^{\widetilde{O}_n^{\varepsilon(n)}(w_i)} \text{ s.t. } d(\mathcal{C}, \widetilde{\mathcal{A}}_n^{\varepsilon(n)}(w_i)) \leqslant 2\varepsilon(n), \sup_{z\in\mathcal{C}} d(z, \partial O(w_i)) \geqslant \delta\right) = 0. \tag{3.8}$$

The metric graph loops that intersect but are not contained in $\widetilde{\mathcal{A}}_n^{\varepsilon(n)}$ are by construction all of diameter smaller than $\varepsilon(n)$. Thus, the only way for $\widetilde{\mathcal{A}}_n$ to have points δ-far from $\widetilde{\mathcal{A}}_n^{\varepsilon(n)}$ is the event in (3.8) to be satisfied. We conclude that, with probability converging to 1, we have $\widetilde{\mathcal{A}}_n \cap \Theta_\delta(w_i) = \emptyset$. Hence we obtain (3.7) and conclude the proof of the proposition. □

Now, we present a short proof of the lemma using the already proved convergence of FPS. The idea is to add Brownian excursions to the loop soup to get an FPS. Then, when the event of having a macroscopic cluster close to the boundary occurs, we use bounds on the FPS and the fact that Brownian excursions intersect any cluster that goes from microscopically close to the boundary to a macroscopic distance, to conclude.

Proof of Lemma 3.45. Notice that by monotonicity of the clusters in α, it suffices to prove the claim for $\alpha = 1/2$. By Lemma 3.37 we can couple $\mathcal{L}_{1/2}^{\widetilde{\mathbb{Z}}_n^2}$ in such a way that $\mathcal{L}_{1/2}^{\widetilde{\mathbb{Z}}_n^2} \overset{a.s.}{\to} \mathcal{L}_{1/2}^{\mathbb{C}}$. We also add PPP-s of excursions $\Xi_n^{\widetilde{\Omega}_n}$ and Ξ_u^{Ω} for some constant $u > 0$ to be chosen later. We do it in such a way that $\Xi_n^{\widetilde{\Omega}_n}$ independent of $\mathcal{L}_{1/2}^{\widetilde{\mathbb{Z}}_n^2}$, Ξ_u^{Ω} independent of $\mathcal{L}_{1/2}^{\mathbb{C}}$, and

$$\{\gamma \in \Xi_n^{\widetilde{\Omega}_n} | \gamma \cap \Omega \neq \emptyset\} \overset{a.s.}{\to} \Xi_u^{\Omega}.$$

Now, let us define

$$E^{n,\zeta} = \left\{\text{There is } \mathcal{C} \text{ cluster of } \mathcal{L}_\alpha^{\widetilde{\Omega}_n} \text{ s.t. } d(\mathcal{C}, \partial\widetilde{\Omega}_n) \leqslant \zeta, \sup_{z\in\mathcal{C}} d(z, \partial\Omega) \geqslant \delta\right\}.$$

Then, by the representation of a metric graph first passage set $(\widetilde{A}_0^u)_n$ inside $\widetilde{\Omega}_n$ by loops and excursions (Corollary 3.5), we can bound $E^{n,\zeta} \subset E_1^{n,u} \cup E_2^{n,\zeta,u}$, where

$$E_1^{n,u} = \left\{\sup_{z\in(\widetilde{A}_0^u)_n} d(z, \partial\Omega) \geqslant \delta/2\right\}$$

$$E_2^{n,\zeta,u} = \left\{\text{There is } \mathcal{C} \text{ cluster of } \mathcal{L}_\alpha^{\widetilde{\Omega}_n} \text{ s.t. } d(\mathcal{C}, \partial\widetilde{\Omega}_n) \leqslant \zeta, \sup_{z\in\mathcal{C}} d(z, \partial\Omega) \geqslant \delta, \text{ but } \Xi_u^{\widetilde{\Omega}_n} \cap \mathcal{C} = \emptyset\right\}.$$

Now by Proposition 3.40 for any constant and positive boundary condition u, we have $((\widetilde{A}_0^u)_n \cap \Omega) \cup \partial\Omega \Rightarrow A_0^u$ in law in Hausdorff topology. On the other hand by convergence of nested local sets (Lemma 3.10), monotonicity of FPS (Lemma 3.26) and the fact that $A_0^0 = \partial\Omega$, we know that $\mathbb{P}\left(\sup_{z\in A_0^u} d(z, \partial\Omega) \geqslant \delta\right) \to 0$ as $u \to 0$. Thus we get

$$\lim_{u\to 0} \limsup_{n\to+\infty} \mathbb{P}(E_1^{n,u}) = 0.$$

So we can chose u such that $\mathbb{P}(E_1^{n,u})$ is arbitrarily small, uniformly in n large.

It remains to show that, for any fixed value of u,

$$\lim_{\zeta\to 0}\limsup_{n\to+\infty}\mathbb{P}(E_2^{n,\zeta,u})=0.$$

On one hand, as the excursion measure has infinite mass on the diagonal, it follows that for any $x\in\partial\Omega$, there is is a.s. a Brownian excursion in Ξ_u^Ω disconnecting x from $\Omega\backslash B(x,\delta/2)$ in Ω. Hence, any connected set joining x to a point at distance δ from $\partial\Omega$ has to intersect this excursion. On the other hand, we know that $\{\gamma\in\Xi_u^{\widetilde{\Omega}_n}|\gamma\cap\Omega\neq\emptyset\}$ converges in law to Ξ_u^Ω. The lemma follows.

□

3.5.4 Representation of the continuum FPS with Brownian loops and excursions

From Proposition 3.5 we know that a FPS on a metric graph is represented as closure of clusters of metric graph loops and excursions. By using the convergence of the metric graph FPS to the continuum FPS (Proposition 3.40) and the convergence of clusters of metric graph loops and excursions to their continuum counterparts (Proposition 3.44), we obtain a similar representation in continuum:

Corollary 3.46 (FPS = clusters with excursions) *Let u be a non-negative harmonic function with piecewise constant boundary values that change a finite number of times. Then the set $\mathcal{A}(\mathcal{L}_{1/2}^D,\Xi_u^D)\cup\partial D$ corresponding to the closure of clusters containing excursions, and the first passage set A_0^u, have the same law.*

Suppose now that we are in the setting of Proposition 3.40. We know that $(\widetilde{A}_{-a}^{u_n}\cap D)\cup\partial D$ converges in law to A_{-a}^u. However, by convention A_{-a}^u is defined to contain ∂D, and Proposition 3.40 does not guarantee that there is no part of $\widetilde{A}_{-a}^{u_n}$ that intersects D but at the limit converges to a non-trivial part on ∂D. This can be addressed using Corollary 3.46.

Corollary 3.47 *Suppose we are in the setting of Proposition 3.40. Let $\mathcal{B}$ denote*

$$\mathcal{B}=\{x\in\partial D|u(x)\leqslant -a\}.$$

Assume that for any sequence of metric graph boundary points $x_n\in\partial\widetilde{D}_n$ converging to a $x\in\overline{\mathcal{B}}$, we have that $u_n(x_n)\leqslant -a$ for n large enough. Then Then $\overline{(\widetilde{A}_{-a}^{u_n}\backslash\partial\widetilde{D}_n)\cap D}$ converges in law to $\overline{A_{-a}^u\backslash\partial D}$.

Proof. It is clear that a subsequential limit of $\overline{(\widetilde{A}_{-a}^{u_n}\backslash\partial\widetilde{D}_n)\cap D}$ is at least as large as $\overline{A_{-a}^u\backslash\partial D}$. It is also contained in A_{-a}^u. What we need to show is that it does not contain unwanted points on ∂D.

First assume that $-a\leqslant\inf u$. Then A_{-a}^u has the same law as A_0^{u+a} and $\overline{A_0^{u+a}\backslash\partial D}$ has same law as $\mathcal{A}(\mathcal{L}_{1/2}^D,\Xi_{u+a}^D)$. Similarly $\overline{(\widetilde{A}_{-a}^{u_n}\backslash\partial\widetilde{D}_n)\cap D}$ has the law of $\overline{(\widetilde{A}_0^{u_n+a}\backslash\partial\widetilde{D}_n)\cap D}$ that has the law of $\overline{\widetilde{\mathcal{A}}(\mathcal{L}_{1/2}^{\widetilde{D}_n},\Xi_{u_n+a}^{\widetilde{D}_n})\cap D}$. Thus the claim follows from Proposition 3.44.

For the general case, consider the boundary condition $u^*:=u\vee(-a)$ and $u_n^*=u_n\vee(-a)$ on D and $\widetilde{D}_n$ respectively. Notice that then u_n^*,u^* still satisfy the assumption in the statement.

Furthermore, by monotonicity of the FPS (Lemma 3.26) we have that $A_{-a}^{u} \subseteq A_{-a}^{u^*}$. Similarly, on the metric graph $\widetilde{A}_{-a}^{u_n} \subseteq \widetilde{A}_{-a}^{u_n^*}$. We now apply the previous case to $\widetilde{A}_{-a}^{u_n^*}$ and conclude using the assumption and the fact that $\overline{A_{-a}^{u} \backslash \partial D} = \overline{A_{-a}^{u^*} \backslash \partial D} = \overline{\partial D \backslash \mathcal{B}}$. □

We derive some more consequences of Corollary 3.46. Firstly, the non-degenerate first passage sets are have almost surely Hausdorff dimension equal to 2.

Corollary 3.48 (Hausdorff dimension of FPS) *Let u be harmonic piecewise constant on ∂D. Suppose that $\{z \in \partial D | u(x) > -a\}$ is non-empty. Then A_{-a}^{u} has almost surely Hausdorff dimension 2.*

Notice that if $u \leqslant -a$ then $A_{-a}^{u} = \partial D$ almost surely.

Proof. First consider the case $u \geqslant -a$ on D. Then A_{-a}^{u} has the law of A_0^{u+a} and by Corollary 3.46 the first passage set is obtainable from clusters of Brownian loops and excursion. Since the trace of a planar Brownian motion has Hausdorff dimension 2, so has A_{-a}^{u}.

Now we do not assume that $u \geqslant -a$ everywhere on D. Then first sample $\mathbb{V}_{-a}^{u}$. Then $D \backslash \mathbb{V}_{-a}^{u}$ has almost surely a connected component O on which $u_O > -a$, where u_O is the harmonic function with boundary condition u on $\partial O \cap \partial D$ and $-a$ on $\partial O \cap \partial \mathbb{V}_{-a}^{u}$. Then the first passage set of level $-a$ inside this component O, A_{-a}^{O,u_O}, is of Hausdorff dimension 2. Since $A_{-a}^{O,u_O} \subseteq A_{-a}^{u}$, so is A_{-a}^{u}. □

Next, we see that A_0^u satisfies an Harris-FKG inequality. This follows from the general Harris-FKG inequality for Poisson point processes (Lemma 2.1 in [Jan84])

Corollary 3.49 (Harris-FKG) *The boundary condition u is non-negative. Let F_1 and F_2 be two bounded measurable functionals on compact sets. We assume that F_1 and F_2 are increasing, that is to say if $K \subseteq K'$, $F_i(K) \subseteq F_i(K')$. Then*

$$\mathbb{E}[F_1(A_0^u)F_2(A_0^u)] \geqslant \mathbb{E}[F_1(A_0^u)]\mathbb{E}[F_2(A_0^u)].$$

Remark 3.50 *One could also obtain a Harris-FKG inequality for A_{-a}^{u} from a Harris-FKG inequality for the GFF Φ. Then one does not need the constraint $u \geqslant -a$. First, note that A_{-a}^{u} is an non-decreasing function of Φ: if $f \in H_0^1(D)$, $f \geqslant 0$, then $A_{-a}^{u}(\Phi) \subseteq A_{-a}^{u}(\Phi + f)$ a.s. This can be proven similarly to the monotonicity part in Proposition 3.26. Further, Φ satisfies itself a Harris-FKG inequality: if F_1 and F_2 are functionals such that $F_i(\Phi + f) \geqslant F_i(\Phi)$ a.s. for $f \in H_0^1(D)$, $f \geqslant 0$, then $\mathbb{E}[F_1(\Phi)F_2(\Phi)] \geqslant 0$. See [Pit82] for the Harris-FKG property for finite-dimensional Gaussian vectors with covariance matrix having non-negative entries.*

Next, we obtain a geometric description of the outermost clusters in a Brownian loop-soup $\mathcal{L}_{1/2}^{D}$ when we condition on their outer boundary. More precisely, let D now be simply connected. Then the outer boundaries of outermost clusters (not surrounded by others) in a Brownian loop-soup $\mathcal{L}_{1/2}^{D}$ are distributed like a conformal loop ensembles CLE_4 ([SW12]). It is also shown there that conditioned on one of these boundaries Υ, the Brownian loops in the interior surrounded by Υ ($\text{Int}(\Upsilon)$) that do not touch Υ are distributed like a Brownian loop-soup $\mathcal{L}_{1/2}^{\text{Int}(\Upsilon)}$ inside $\text{Int}(\Upsilon)$. Moreover, Qian and Werner showed in [QW15] that conditioned on Υ, the loops that intersect Υ are independent from those that do not intersect it, and they have the law of a PPP of Brownian excursions from Υ to Υ inside $\text{Int}(\Upsilon)$ with intensity

$\mu_{\text{exc}}^{\text{Int}(\Upsilon),2\lambda}$. Combining this with Corollary 3.46, we can give a geometric description of the the whole outermost cluster:

Corollary 3.51 (Cluster of $\mathcal{L}_{1/2}^D = A_0^{2\lambda} = A_{-2\lambda}$) *Let the domain D be simply connected. Conditioned on the outer boundary Υ of a Brownian loop-soup cluster in $\mathcal{L}_{1/2}^D$, the topological closure of the cluster itself is distributed like a first passage set $A_0^{2\lambda} = A_{-2\lambda}$ inside* $\text{Int}(\Upsilon)$, *the interior surrounded by Υ.*

3.5.5 Representation of level lines and convergence to level lines

In [WW13a] the authors show that in simply connected domains $\text{SLE}_\kappa(\rho)$ curve with $\kappa \in (8/3, 4]$ can be obtained as "envelopes" of clusters of Brownian excursions from boundary to boundary and Brownian loops inside the domain. We will first show how to extend this to our generalized level lines in multiply connected domains, and then use this to prove that certain interfaces of the metric graph GFF converge to these generalized level lines.

So let D be finitely connected and let $\partial_{\text{ext}} D$ be the outermost connected component of ∂D, that is to say the one that separates D from infinity. We consider two boundary points $x_0 \neq y_0 \in \partial_{\text{ext}} D$ that split $\partial_{\text{ext}} D$ in two boundary arcs, $\mathcal{B}_1$ and $\mathcal{B}_2$. Assume that u is a harmonic function such that on the boundary it is piecewise constant, equal to $-\lambda$ on $\mathcal{B}_2$, $\inf_{\mathcal{B}_1} u > -\lambda$ and $\inf_{\partial D \setminus \partial_{\text{ext}} D} u \geqslant \lambda$.

Then by Lemma 3.16 there is a generalized level line η of $\Phi + u$ starting at y_0 and targeted at x_0. Moreover by Lemmas 3.17 and 3.18, we know that it almost surely ends at x_0. Denote the component of $D \setminus \eta$ that contains $\mathcal{B}_2$ on its boundary by D_2. Set $D_1 = D \setminus (D_2 \cup \eta)$.

Consider on the other hand an independent PPP-s of loops $\mathcal{L}_{1/2}^D$ and boundary-to-boundary excursions $\Xi_{u+\lambda}^D$. By definition there are no excursions hitting $\mathcal{B}_2 \setminus \{x_0, y_0\}$ in $\Xi_{u+\lambda}^D$. Let $\mathscr{D}_2$ be the unique connected component of $D \setminus \mathcal{A}(\mathcal{L}_{1/2}^D, \Xi_{u+\lambda}^D)$ such that $\mathcal{B}_2 \subset \partial \mathscr{D}_2$ and let $\partial_2 \mathcal{A}(\mathcal{L}_{1/2}^D, \Xi_{u+\lambda}^D) = \partial \mathscr{D}_2 \cap \mathcal{A}(\mathcal{L}_{1/2}^D, \Xi_{u+\lambda}^D)$. It is also a path in D joining x_0 and y_0 like the generalized level line η. The following corollary says that these two paths agree (see Figure 3.6 for an illustration):

Corollary 3.52 (Level line = envelope of Brownian excursions and loops) *Let D be finitely connected and u, η and $\partial_2 \mathcal{A}(\mathcal{L}_{1/2}^D, \Xi_{u+\lambda}^D)$ as above. Then the generalized level line η has same law as $\partial_2 \mathcal{A}(\mathcal{L}_{1/2}^D, \Xi_{u+\lambda}^D)$*

Proof. It suffices to show that D_2 has same law as $\mathscr{D}_2$. Consider the first passage set $A_{-\lambda}^u$. Under the same GFF, it is equal to $A_0^{u+\lambda}$ which in turn has same law as $\mathcal{A}(\mathcal{L}_{1/2}^D, \Xi_{u+\lambda}^D)$. Thus $\partial_2 \mathcal{A}(\mathcal{L}_{1/2}^D, \Xi_{u+\lambda}^D)$ has the same law as the boundary of the connected component of $D \setminus A_{-\lambda}^u$ that has $\mathcal{B}_2$ on its boundary.

Note that D_2 is connected. We can construct $A_{-\lambda}^u$ as follows. First, sample η and then construct in D_1 the first passage set $A_{-\lambda}^{D_1, u_1}$. Here u_1 is the harmonic function that has boundary condition λ on η and u on $\partial D_1 \setminus \eta$. Then one can check that $A_{-\lambda}^{D_1, u_1} \cup \mathcal{B}_2$ satisfies the definition of FPS, and thus by uniqueness $A_{-\lambda}^u = A_{-\lambda}^{D_1, u_1} \cup \mathcal{B}_2$. In this construction, the connected component of $D \setminus A_{-\lambda}^u$ that has $\mathcal{B}_2$ on its boundary is D_2. □

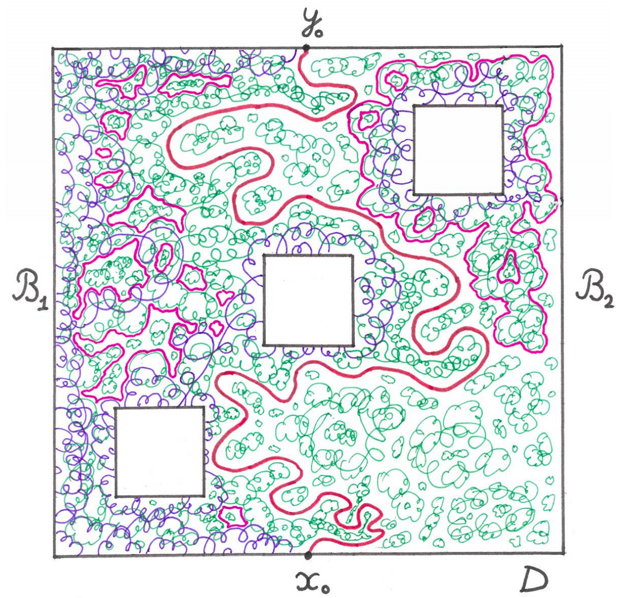


Figure 3.6: Artistic view of the level line (in red) as envelope of Brownian excursions (in blue) and loops (in green). Magenta contours outline some other boundary components of $A^u_{-\lambda}$.

Remark 3.53 *More generally, other level lines, or families of multiple level lines, can be obtained as boundaries of clusters of Brownian loops and excursions, as long as these level lines are boundaries of a same first passage set. For instance, in a simply connected domain, one can get in this way multiple commuting SLE*$_4$ *curves, which correspond to alternating boundary conditions* 0, 2λ *(Figure 3.7). In [PW17] appears an expression for probabilities of different pairings.*

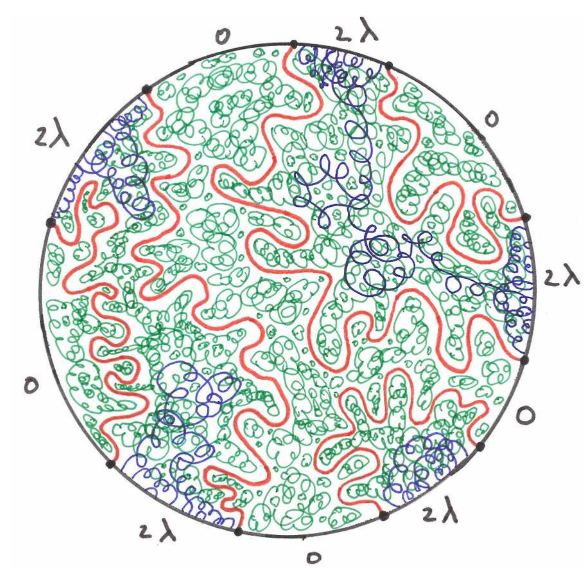


Figure 3.7: Multiple (here 5) commuting SLE_4 as boundaries of clusters of Brownian loops (green) and excursions (blue).

Next we show that certain interfaces of the metric graph GFF converge in law to level lines of the continuum GFF. Let D, x_0, y_0, $\mathcal{B}_1$, $\mathcal{B}_2$, u, η be as previously. Consider $\widetilde{D}_n$

open subset of $\widetilde{\mathbb{Z}}_n^2$ such that we have Hausdorff convergence of $\widetilde{D}_n \cup \partial \widetilde{D}_n$ to $\overline{D}$ and that of $\widetilde{\mathbb{Z}}_n^2 \backslash \widetilde{D}_n$ to $\mathbb{C} \backslash D$. Let $\partial_{\text{ext}} \widetilde{D}_n$ be the boundary of the only unbounded connected component of $\widetilde{\mathbb{Z}}_n^2 \backslash \widetilde{D}_n$. We assume that $\mathcal{B}_{1,n} \cup \mathcal{B}_{2,n}$ is a partition of $\partial_{\text{ext}} \widetilde{D}_n$, such that $\mathcal{B}_{i,n}$ converges to $\mathcal{B}_i$, and moreover $\mathcal{B}_{1,n}$ and $\mathcal{B}_{2,n}$ are separated by exactly two $2^{-n} \times 2^{-n}$ dyadic squares, of which one contains x_0 and the other y_0 (see Figure 3.8). Let u_n be harmonic on $\widetilde{D}_n$ such that u_n is constant $-\lambda$ on $\mathcal{B}_{2,n}$, $\inf_{\mathcal{B}_{1,n}} > -\lambda$, $\inf_{\partial \widetilde{D}_n \backslash \partial_{\text{ext}} \widetilde{D}_n} \geqslant \lambda$ and u_n converges to u uniformly on compact subsets of D. We have seen that with this boundary conditions, the metric graph first passage set $\widetilde{A}_{-\lambda}^{u_n}$ contains the boundary $\mathcal{B}_{2,n}$ only by convention, i.e. it satisfies

$$\overline{\widetilde{A}_{-\lambda}^{u_n} \backslash \partial \widetilde{D}_n} = \partial \widetilde{D}_n \backslash \mathcal{B}_{2,n}.$$

Let $\partial_2 \widetilde{A}_{-\lambda}^{u_n}$ be all the points in $\partial \widetilde{A}_{-\lambda}^{u_n}$ that are connected in $A_{-\lambda}^{u_n}$ to $\mathcal{B}_{1,n}$ and in $\widetilde{D}_n \backslash A_{-\lambda}^{u_n}$ to $\mathcal{B}_{2,n}$. A.s. $\partial_2 \widetilde{A}_{-\lambda}^{u_n}$ contains no vertices and the edges it intersects define a path from x_0 to y_0 in the dual lattice of $(2^{-n}\mathbb{Z})^2$ (in red on Figure 3.8).

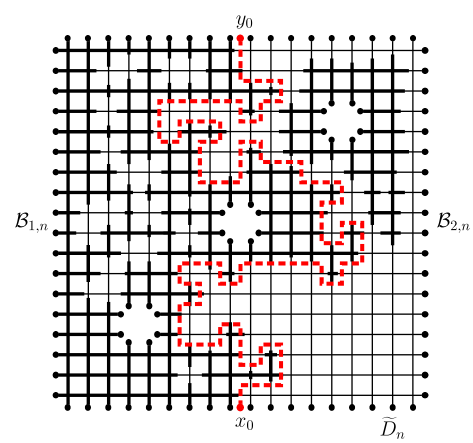


Figure 3.8: In thick black lines the first passage set $\widetilde{A}_{-\lambda}^{u_n}$ on the metric graph $\widetilde{D}_n$. Black dots represent $\partial \widetilde{D}_n$. The red interface converges in law to a level line of the continuum GFF.

Corollary 3.54 (Convergence to level lines from metric graph) *With the notations above, $\partial_2 \widetilde{A}_{-\lambda}^{u_n}$ converges in law for the Hausdorff topology to the level line η of the continuum GFF. In particular, if the domain D is simply connected and the boundary condition u is constant equal to $b > -\lambda$ on $\mathcal{B}_1$, $\partial_2 \widetilde{A}_{-\lambda}^{u_n}$ converges in law to the trace of an SLE$_4(\rho)$ process, with $\rho = b/\lambda - 1$.*

Proof. $\partial_2 \widetilde{A}_{-\lambda}^{u_n}$ is a boundary "component" of $\widetilde{A}_{-\lambda}^{u_n}$ and η that of $A_{-\lambda}^{u}$. The convergence of $\widetilde{A}_{-\lambda}^{u_n}$ to $A_{-\lambda}^{u}$ in the Hausdorff topology implies that the limit of $\partial_2 \widetilde{A}_{-\lambda}^{u_n}$ contains η and does not intersect D_2, i.e. the $\mathcal{B}_2$ side of η (right on Figures 3.6 and 3.8). Yet this convergence does not exclude that in the limit there are bubbles attached to η on its $\mathcal{B}_1$ side (left on Figures 3.6 and 3.8). To address this issue, we are going to use the representation of the level line η as the boundary of clusters of loops and excursions, and some results from [vdBCL16] that state that the clusters of a Brownian loop-soup are "well connected", that is to say that, if we remove the microscopic Brownian loops up to some scale, the outer boundaries of clusters do not change too much.

From Corollary 3.52 we have the representation $\eta = \partial_2 \mathcal{A}(\mathcal{L}^D_{1/2}, \Xi^D_{u+\lambda})$. Consider further metric graph loop soup $\mathcal{L}^{\widetilde{D}_n}_{1/2}$, metric graph PPP of excursions $\Xi^{\widetilde{D}_n}_{u_n+\lambda}$ and the union of clusters containing at least one excursion $\widetilde{\mathcal{A}}_n = \widetilde{\mathcal{A}}_n(\mathcal{L}^{\widetilde{D}_n}_{1/2}, \Xi^{\widetilde{D}_n}_{u_n+\lambda})$. Using Lemma 3.37 we can couple everything on the same probability space so that the metric graph PPP and unions of clusters converge to their continuum counterparts.

Now define $\partial_2 \widetilde{\mathcal{A}}_n$ to be the set of points on $\partial \widetilde{\mathcal{A}}_n$ that are connected in $\widetilde{\mathcal{A}}_n$ to $\mathcal{B}_{1,n}$ and in $\widetilde{D}_n \backslash \widetilde{\mathcal{A}}_n$ to $\mathcal{B}_{2,n}$. As before, it has the same law as $\partial_2 \widetilde{A}^{u_n}_{-\lambda}$.

As in the proof of Proposition 3.44 we also consider clusters of loops and excursions that have diameter larger than ε, denoted by $\mathcal{A}^\varepsilon = \mathcal{A}^\varepsilon(\mathcal{L}^D_{1/2}, \Xi^D_{u+\lambda})$ and $\widetilde{\mathcal{A}}^\varepsilon_n = \widetilde{\mathcal{A}}^\varepsilon_n(\mathcal{L}^{\widetilde{D}_n}_{1/2}, \Xi^{\widetilde{D}_n}_{u_n+\lambda})$ in the continuum and on the metric graph respectively. Define $\partial_2 \mathcal{A}^\varepsilon$ and $\partial_2 \widetilde{\mathcal{A}}^\varepsilon_n$ as above.

From Corollary 5.3 in [vdBCL16] it follows that for fixed $\varepsilon > 0$, $\partial_2 \widetilde{\mathcal{A}}^\varepsilon_n$ converges as $n \to +\infty$ in Hausdorff topology to $\partial_2 \mathcal{A}^\varepsilon$. Thus, as $\partial_2 \mathcal{A}^\varepsilon$ is on the $\mathcal{B}_1$ side of η, we obtain that $\partial_2 \widetilde{\mathcal{A}}_n$ is asymptotically "squeezed" between $\partial_2 \mathcal{A}^\varepsilon$ and η.

But now Theorem 4.1 in [vdBCL16] implies that as $\varepsilon \to 0$, $\partial_2 \mathcal{A}^\varepsilon$ converges to $\partial_2 \mathcal{A} = \eta$ and hence the claim follows. □

Remark 3.55 *Using absolute continuity of level lines, one can extend the convergence result above to the case where the boundary condition is not constantly equal to* $-\lambda$ *on* $\mathcal{B}_2$*, but is less or equal to* $-\lambda$ *on* $\mathcal{B}'_2$ *and equal to* $-\lambda$ *on* $\mathcal{B}_2 \backslash \mathcal{B}'_2$*, where* $\mathcal{B}'_2 \subset \mathcal{B}_2$ *and* $d(\mathcal{B}'_2, \{x_0, y_0\}) > 0$*.*

Finally, we will show how the representation of level lines as boundaries provides an explicit coupling of level lines for the GFF-s with different boundary conditions. Moreover, we also give an exact formula for the conditional probability that the two level lines agree in this coupling, conditioned on one of the level lines. In fact, in the non-boundary touching case, the existence of a coupling where level lines of two GFF-s with different boundary conditions agree with positive probability follows already from Corollary 2.14. Here we provide an explicit such coupling with exact formulas.

Let D, x_0, y_0, $\mathcal{B}_1$, $\mathcal{B}_2$, u, η be as previously. Moreover let u^* be another harmonic function that on the boundary is piecewise constant and changes value only finitely many times. Suppose $u^* \geqslant u$ and let

$$\mathcal{B}_3 = \{x \in \partial D | u^*(x) > u(x)\}.$$

Let Φ^* be a GFF. Then we can define η^*, a generalized level line of $\Phi^* + u^*$ from y_0 to x_0.

Corollary 3.56 (Coupling of level lines with different boundary conditions) *Assume that* $d(\mathcal{B}_3, \mathcal{B}_2) > 0$*. Then there is a coupling of random curves* η *and* η^* *such that the event* $\eta = \eta^*$ *has positive probability. The conditional probability of this event given* η *is*

$$\mathbb{P}(\eta^* = \eta | \eta) = \mathbf{1}_{\eta \cap \mathcal{B}_3 = \emptyset} \exp\left(-\mathcal{M}(u, u^*, \eta)\right),$$

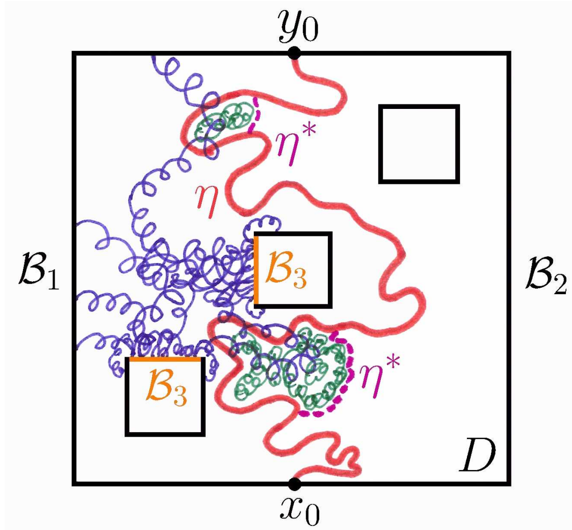


Figure 3.9: Coupling level lines by adding additional excursions. $\mathcal{B}_3$ is in orange. In blue are the excursions of $\delta\Xi$. Each has at least an endpoint in $\mathcal{B}_3$. In green are the clusters of $\mathcal{L}^D_{1/2}$ right to η that are intersected by $\delta\Xi$.

where

$$\mathcal{M}(u,u^*,\eta) = \frac{1}{2}\sum_{i=1,2}\iiint_{\substack{\mathcal{B}_3\cap\partial D_i\\ \times\eta\times\mathcal{B}_3}}(u^*-u)(x_1)H_{D_i}(dx_1,dx_2)\mu^D_{\text{harm}}(x_2,dx_3)(u^*-u)(x_3)$$
$$\sum_{i=1,2}\iiint_{\substack{\mathcal{B}_3\cap\partial D_i\\ \times\eta\times\partial D\setminus\mathcal{B}_2}}(u^*-u)(x_1)H_{D_i}(dx_1,dx_2)\mu^D_{\text{harm}}(x_2,dx_3)(u+\lambda)(x_3),$$

where $H_{D_i}(dx_1,dx_2)$ is the boundary Poisson kernel on $\partial D_i\times\partial D_i$ and $\mu^D_{\text{harm}}(x_2,dx_3)$ is the harmonic measure on ∂D seen from x_2.

Remark 3.57 *A crude lower bound for above probability is given by*

$$\mathbf{1}_{\eta\cap\mathcal{B}_3=\emptyset}\exp\left(-\frac{1}{2}\sum_{i=1,2}\sup(u^*-u)\sup(u+\lambda)M(\mathcal{B}_3\cap\partial D_i,\eta)\right)\sum_{i=1,2}\sup(u^*-u)^2M(\mathcal{B}_3\cap\partial D_i,\eta),$$

where the modulus $M(\mathcal{B}_3\cap\partial D_i,\eta)$ is taken inside D_i.

Proof. If we have used the same zero-boundary GFF Φ, then the generalized 0-level lines from x_0 to y_0 of $\Phi+u$ and $\Phi+u^*$ would have been a.s. different (unless $u^*=u$). To construct the coupling we rather apply Corollary 3.52 as follows. Consider an independent loop soup $\mathcal{L}^D_{1/2}$, PPP of excursions $\Xi^D_{u+\lambda}$ and another PPP of excursions with intensity $\mu^{D,u^*+\lambda}_{\text{exc}}-\mu^{D,u+\lambda}_{\text{exc}}$, $\delta\Xi$. Set $\Xi^D_{u^*+\lambda}=\Xi^D_{u+\lambda}\cup\delta\Xi$. We now construct η as the envelope of $\mathcal{L}^D_{1/2}\cup\Xi^D_{u+\lambda}$, and η^* as the one of $\mathcal{L}^D_{1/2}\cup\Xi^D_{u^*+\lambda}$ (Figure 3.9). In this construction,

$$\mathbb{P}(\eta^*=\eta|\eta)=\mathbb{P}(\forall\gamma\in\delta\Xi,\gamma\cap\eta=\emptyset|\eta)=\mathbf{1}_{\eta\cap\mathcal{B}_3=\emptyset}\exp\left(-(\mu^{D,u^*+\lambda}_{\text{exc}}-\mu^{D,u+\lambda}_{\text{exc}})(\{\gamma|\gamma\cap\eta\neq\emptyset\})\right),$$

which exactly gives the right expression. □

3.6 Future directions

In this section, we briefly explain how the set-up of this chapter provides a platform to approach several other questions and allows, not only to describe the GFF but also to calculate the law of certain observables of CLE_4 explicitly.

Let us note that it is possible to construct a zero-boundary GFF in a simply connected domain D as follows.

1. Sample a CLE_4, then, independently inside each CLE_4 loop set the boundary condition to 2λ or -2λ with probability $1/2$.
2. Inside each CLE_4 loop, sample a first passage set $A_0^{2\lambda}(= A_{-2\lambda})$ or $V_0^{-2\lambda}(= A_{-2\lambda}(-\Phi))$ according to the boundary conditions given in the last step. This deterministically induces $\Phi_{A_0^{2\lambda}}$ or $\Phi_{V_0^{-2\lambda}}$ respectively.
3. Inside each hole of these first passage sets one has a zero-boundary GFF independent of all the others. Iterate this construction inside each one of this.

Thanks to the Miller-Sheffield CLE_4 coupling it is possible to see that the sum of the measures $\Phi_{A_0^{2\lambda}}$ and $\Phi_{V_0^{2\lambda}}$ sample through this procedures converges to a GFF. Because all the sampled sets are in fact measurable functions of the GFF obtained, we have decomposed the whole two-dimensional GFF by alternating CLE_4-s and $2\lambda-$FPS-s.

As shown in Corollary 3.51, each of the FPS $A_0^{2\lambda}$, resp. $V_0^{-2\lambda}$, inside a CLE_4 loop, is the topological closure of a cluster in a Brownian loop-soup of parameter 1/2. So we have the following picture: given a Brownian loop-soup $\mathcal{L}_{1/2}^D$ of parameter 1/2, one tosses for each cluster $\mathcal{C}$ of Brownian loops an independent and uniform sign $\sigma_\mathcal{C}$; the geometry of the cluster $\mathcal{C}$ induces a positive measure $\mu_\mathcal{C}$ supported on it by the function $A_{-2\lambda} \mapsto (\Phi_{A_{-2\lambda}} + 2\lambda)$; then one sums these measures (with disjoint supports), signed by $\sigma_\mathcal{C}$, $\sum_{\mathcal{C} \text{ cluster of } \mathcal{L}_{1/2}^D} \sigma_\mathcal{C}\mu_\mathcal{C}$, and obtains in law a zero-boundary GFF. This construction should work not only on simply connected domains, but also on finitely connected ones, and more generally on bordered Riemann surfaces. In this construction, the clusters of a Brownian loop-soup of parameter 1/2 appear as same-sign components of the GFF, and one can call them *excursion sets*, by analogy with the excursions away from 0 of a one-dimensional Brownian motion. Thus, in some sense, we obtain a *excursion decomposition* of the continuum GFF in dimension 2.

In fact, one would hope to obtain, as in the case of the one-dimensional Brownian motion, a deterministic construction of the two-dimensional GFF from the Brownian loop-soup. In the one-dimensional case it goes as follows: take the occupation time of the loop-soup, its square-root gives the absolute value of the Brownian motion. Next, consider the local time at zero of this Brownian motion and subtract it from the absolute value. By Lévy's theorem, this has the law of a Brownian motion. This construction has an analogue on metric graphs [LW16]. The local time is replaced by the local time distance to the boundary. By convergence, this would induce a similar coupling in the continuum. The clusters of $\mathcal{L}_{1/2}^D$ would correspond to positive excursions of Φ above some negative level, and the value of Φ on the cluster would be again given just by the geometry of the cluster. In a simply connected domain, the "local time distances" between the boundary ∂D and the outermost clusters would be the time labels of CLE_4 loops in a conformal invariant growth mechanism of CLE_4 (Section 4 in [WW13b]) and the overall coupling between the CLE_4 and the GFF Φ would be the one of Theorem 1.2.2 in [WW16] (the "second coupling", see also [AS17b]). However, we still lack the understanding why the time labels of the CLE_4 loops are measurable functions of the CLE_4 loop ensemble, or in other words why the Lévy transformation remains measurable in the continuum. Understanding this and showing the convergence from metric graph to continuum would also allow us to prove several of the conjectures of [LW16].

Next, our techniques also allow us to do some precise calculations, still in progress: for example, we can calculate the law of the extremal length between the CLE_4 loop surrounding zero and the boundary. Moreover, we can also calculate certain joint laws between different nested loops. This will be a topic of [ALS17b].

Finally, there is a question about the positive measure on the trace of the FPS. We already have an explicit formula for constructing the measure $\Phi_{A_{-a}}$ from A_{-a} (see Remark 3.34) and we hope to show that it is up to multiplicative constants equal to a Minkowski content w.r.t. the right gauge. This would provide the existence of a non-trivial Minkowski measure on clusters of Brownian loops and excursions, which would be of interest on its own. It would be interesting to see whether this measure could also be seen a Hausdorff measure with respect to possibly a different gauge.

Chapter 4

Liouville measure via level sets

4.1 Introduction

Gaussian multiplicative chaos (GMC) theory, initiated by Kahane in the 80s [Kah85] as a generalization of multiplicative cascades, aims to give a meaning to "exp(Γ)" for rough Gaussian fields Γ. In a simpler setting it was already used in the 70s to model the exponential interaction of bosonic fields [HK71], and over the past ten years it has gained importance as a key component in constructing probabilistic models of so-called Liouville quantum gravity in 2D [DS11, DKRV16].

One of the important cases of GMC theory is when the underlying Gaussian field is equal to $\gamma\Gamma$, for Γ a 2D Gaussian free field (GFF) [DS11] and $\gamma > 0$ a parameter. It is then possible to define random measures with area element "$\exp(\gamma\Gamma) dx \wedge dy$". These measures are sometimes also called Liouville measures [DS11] and we will do so for convenience in this chapter. Due to the recent work of many authors [RV07, DS11, Ber15b, Sha16] one can say that we have a rather thorough understanding of Liouville measures in the so-called subcritical regime ($\gamma < 2$). The critical regime ($\gamma = 2$) is trickier, but several constructions are also known [DRSV14a, DRSV14b, JS15, Pow17].

Usually, in order to construct the GMC measure, one first approximates the underlying field using either a truncated series expansion or smooth mollifiers, then takes the exponential of the approximated Gaussian field, renormalizes it and shows that the limit exists in the space of measures. In [Aïd15] the author proposed a different way to construct measures of multiplicative nature using nested conformally invariant loop ensembles, inspired by multiplicative cascades. He conjectured that in the subcritical and critical regime, and in the case where these loop ensembles correspond to certain same-height contour lines of the underlying GFF, the limiting measure should have the law of the Liouville measure. In this chapter we confirm his conjecture. This is done by providing new constructions of the subcritical and critical Liouville measures using a certain family of so called local sets of the GFF [SS13, ASW17] and reinterpreting his construction as a special case of this general setting. Some of our local-set based constructions correspond to simple multiplicative cascades, and others in some sense to stopping lines constructions of the multiplicative cascade measures [Kyp00]. To our knowledge we provide a first "non-Gaussian" approximation of Liouville measures that is both local and conformally invariant. We also remark that our construction strongly uses the Markov property of the GFF and hence does not easily generalize to other log-correlated fields.

One simple, but important, consequence of our results is the simultaneous construction of a GFF in a simply connected domain and its associated Liouville measure using nested CLE_4 and a collection of independent coin tosses. Start with a height function $h_0 = 0$ on $\mathbb{D}$ and sample a CLE_4 in $\mathbb{D}$. Inside each connected component of its complement add either $\pm\pi$ to h_0 using independent fair coins. Call the resulting function h_1. Now repeat this procedure independently in each connected component: sample an independent CLE_4, toss coins and add $\pm\pi$ to h_1 to obtain h_2. Iterate. Then, it is a consequence of Section 2.4,that these piecewise constant fields h_n converge to a GFF Γ. It is also possible to show that the nested CLE_4 used in this construction is a measurable function of Γ. Proposition 4.10 of the current chapter implies that one can construct the Liouville measures associated to Γ by just taking the limit of measures

$$M_n^\gamma(dz) = e^{\gamma h_n(z)}\,\mathrm{CR}(z;\mathbb{D}\setminus A^n)^{\frac{\gamma^2}{2}}\,dz.$$

Here $\mathrm{CR}(z;\mathbb{D}\setminus A^n)$ is the conformal radius of the point z inside the n-th level loop.

Observe that the above approximation is different from taking naively the exponential of h_n and normalizing it pointwise by its expectation. In fact, it is not hard to see that in this setting the latter naive procedure that is used for mollifier and truncated series approximations would not give the Liouville measure.

In the critical case, and keeping to the above concrete approximation of the GFF, regularized Liouville measures can be given by the so-called derivative approximations:

$$D_n(dz) = \int_{\mathcal{O}} \left(-h_n(z) + 2\log \mathrm{CR}^{-1}(z,\mathbb{D}\setminus A^n)\right) e^{2h_n(z)}\,\mathrm{CR}(z;\mathbb{D}\setminus A^n)^2\,dz.$$

As the name suggests, they correspond to the derivative of the above measure M_n^γ w.r.t. to γ, taken at the critical parameter $\gamma = 2$. We show that these approximate signed measures converge to a positive measure that agrees (up to a constant factor 2) with the limiting measure of [Aïd15] described in Section 4.3.3, and also to the critical Liouville measure constructed in [DRSV14b, Pow17].

The connection between multiplicative cascades and the Liouville measure established by our construction makes it possible to directly adapt many techniques developed in the realm of branching random walks and multiplicative cascades to the study of the Liouville measure. For example, this allows us to prove a “Seneta–Heyde” rescaling result in the critical regime by following very closely the proof for the branching random walk in [AS14] and doing minimal extra work. Finally, our proofs are robust enough to study the Liouville measure in non-simply connected domains and also to study the boundary Liouville measure.

The rest of the chapter is structured as follows. We start with preliminaries on the GFF, its local sets and Liouville measure. Then, we treat the subcritical regime and discuss generalizations to non-simply connected domains and to the boundary Liouville measure. Finally, we handle the critical case: we first show that our construction agrees with both a construction by E. Aidekon (up to a constant factor 2) and a mollifier construction of the critical Liouville measure; then, we consider the case of Seneta-Heyde scaling.

4.2 Preliminaries on the Gaussian free field and its local sets

Let $D \subseteq \mathbb{R}^2$ denote a bounded, open and simply connected planar domain. By conformal invariance, we can always assume that D is equal to $\mathbb{D}$, the unit disk. Recall that the Gaussian Free Field (GFF) in D can be viewed as a centered Gaussian process Γ, indexed by the set of continuous functions in D, with covariance given by

$$\mathbb{E}\left[(\Gamma, f)(\Gamma, g)\right] = \iint_{D\times D} f(x)G_D(x,y)g(y)dxdy. \tag{4.1}$$

Here G_D is the Dirichlet Green's function in D, normalized such that $G_D(x,y) \sim \log(1/|x-y|)$ as $x \to y$ for all $y \in D$. Note that this is the ONLY chapter of this thesis where we use this normalization.

Let us denote by ρ_z^ε the uniform measure on the circle of radius ε around z. Then for all $z \in D$ and all $\varepsilon > 0$, one can define $\Gamma_\varepsilon := (\Gamma, \rho_z^\varepsilon)$. We remark that this concrete choice of mollifying the free field is of no real importance, but is just a bit more convenient in the write-up of the critical case.

An explicit calculation then shows that:

$$\mathbb{E}\left[\varepsilon^{\frac{\gamma^2}{2}} \exp\left(\gamma(\Gamma, \rho_z^\varepsilon)\right)\right] \begin{cases} = \mathrm{CR}(z;D)^{\gamma^2/2} & \text{if } d(z,\partial D) \geqslant \varepsilon, \\ \leqslant 1 & \text{if } d(z,\partial D) < \varepsilon, \end{cases} \tag{4.2}$$

where $\mathrm{CR}(z;D)$ is the conformal radius of z in the simply-connected domain D.

The Gaussian free field satisfies a spatial Markov property, and in fact it also satisfies a strong spatial Markov property. To formalise this, the concept of local sets was introduced in [SS13]. They can be thought as the generalisation of stopping times to a higher dimension.

Definition 4.1 (Local sets) *Consider a random triple (Γ, A, Γ_A), where Γ is a GFF in D, A is a random closed subset of $\overline{D}$ and Γ_A a random distribution that can be viewed as a harmonic function, h_A, when restricted to $D \setminus A$. We say that A is a local set for Γ if conditionally on A and Γ_A, $\Gamma^A := \Gamma - \Gamma_A$ is a GFF in $D \setminus A$.*

Here, by a random closed set we mean a probability measure on the space of closed subsets of $\overline{D}$, endowed with the Hausdorff metric and its corresponding Borel $\sigma-$algebra. For simplicity, we will only work with local sets A that are measurable functions of Γ and such that $A \cup \partial D$ is connected. In particular, this implies that all connected components of $D\backslash A$ are simply-connected. We define $\mathscr{F}_A = \sigma(A) \vee \sigma(\Gamma_A)$.

Other than the Markov property apparent from the definition, we will use the following simple properties of local sets. See for instance [SS13, Wer16] for further properties.

Lemma 4.2 *Let $(A^n)_{n\in\mathbb{N}}$ be an increasing sequence of local sets measurable w.r.t. Γ. Then*

1. *$\mathscr{F}_{A^n} \subset \mathscr{F}_{A^{n+1}}$,*
2. *$\overline{\bigcup A^n}$ is also a local set and $\Gamma_{A_N} \to \Gamma_{\overline{\bigcup A^n}}$ in probability as $N \to \infty$,*
3. *if $\overline{\bigcup A^n} = \overline{D}$, then the join of the σ-algebras $\mathscr{F}_{A^n}$ is equal to $\sigma(\Gamma)$. Moreover, $\Gamma_n := \Gamma_{A^n}$ then converges to Γ in probability in the space of distributions.*

The property (1) follows from the fact that our local sets are measurable w.r.t. Γ and the characterization of local sets found in [SS13]. Properties (2) and (3) follow from the fact that when $A^n \cup \partial D$ is connected we have that $G_{D\backslash A^n} \to G_{D\backslash A}$.

In other words, one can approximate the Gaussian free field by taking an increasing sequence of measurable local sets $(A^n)_{n\in\mathbb{N}}$ and for each n defining $\Gamma_n := \Gamma_{A_n}$. In some sense these give very intrinsic approximations to the GFF. For example, one could intuitively think that A^n are the sets that discover the part of the surface described by the GFF that is linked to the boundary and has height between $-n$ and n.

4.2.1 Two useful families of local sets

One useful family of local sets are the so-called two-valued local sets introduced in Chapter 2 and denoted by $A_{-a,b}$. For fixed $a, b > 0$, $A_{-a,b}$ is a local set of the GFF such that: the value of h_A inside each connected component of $D \setminus A$ is constant with value either $-a$ or b; and that is thin in the sense that for all f smooth we have $(\Gamma_A, f) = \int_{D\setminus A} f(z)h_A(z)\,dz$. The prime example of such a set is CLE_4 coupled with the Gaussian free field as $A_{-2\lambda,2\lambda}$, where λ is an explicit constant equal to $\lambda = \pi/2$ in our case. In analogy with stopping times, they correspond to exit times of Brownian motion from the interval $[-a, b]$. We recall the main properties of two-valued sets:

Proposition 4.3 *Let us consider $-a < 0 < b$.*

1. *When $a + b < 2\lambda$, there are no local sets of Γ with the characteristics of $A_{-a,b}$.*
2. *When $a + b \geq 2\lambda$, it is possible to construct $A_{-a,b}$ coupled with a GFF Γ. Moreover, the sets $A_{-a,b}$ are*
   - *Unique in the sense that if A' is another local set coupled with the same Γ, such that for all $z \in D$, $h_{A'}(z) \in \{-a, b\}$ almost surely and A' is thin in the sense above, then $A' = A_{-a,b}$ almost surely.*
   - *Measurable functions of the GFF Γ that they are coupled with.*
   - *Monotonic in the following sense: if $[a, b] \subset [a', b']$ and $-a < 0 < b$ with $b+a \geq 2\lambda$, then almost surely, $A_{-a,b} \subset A_{-a',b'}$.*
   - *$A_{-a,b}$ has almost surely Lebesgue measure 0.*
   - *For any z, $\log \text{CR}(z; D\backslash A_{-a,b}) - \log \text{CR}(z; D)$ has the distribution of the hitting time of $\{-a, b\}$ by a standard Brownian motion.*

Another nice class of local sets are those that only take one value in the complement of A. We call them first passage sets and denote them by A_a (if they only take the value a). These correspond to one-sided hitting times of the Brownian motion: hence the name. They are of interest in describing the geometry of the Gaussian free field and are treated in more detail in Chapter 3. Here, we only provide one working definition and refer to Chapter 3 for a more intrinsic definition, uniqueness and other properties not needed in the current chapter.

Definition 4.4 (First passage set) *Take $a \geqslant 0$. We say that A_a is the first passage set (FPS) of a GFF Γ, with height a, if it is given by $\overline{\bigcup_n A_{-n,a}}$.*

We need a few properties of these sets. The first follows from the definition, the second and third from Proposition 2.23:

- We have that $\Gamma_{A_a} = a - \nu_a$, where ν_a is a positive measure supported on A_a;
- A_a has zero Lebesgue measure;
- For any $a_n \to \infty$ we have that $\overline{\bigcup A_{a_n}} = D$.

Note that because the circle-average of the GFF $(\Gamma, \rho_z^\varepsilon)$ is a.s. well-defined for all $z \in D$, $\varepsilon > 0$ simultaneously, it also means that $(\nu_a, \rho_z^\varepsilon)$ is a.s. well-defined and positive for all z, ε as above.

We have already seen in Proposition 3.26 these three properties characterize A_a uniquely. However, in this chapter we only need a weaker uniqueness statement that is a consequence of the following lemma:

Lemma 4.5 *Denote $A^1 = A_{-a,a}$ with $a \geqslant \lambda$ and define iteratively A^n by exploring copies of $A_{-a,a}$ in each connected component of the complement of A^{n-1}. Then, almost surely for a dense countable set $z \in D$ the following holds: for $k \in \mathbb{N}$, let n_z be the first iteration when $h_{A^{n_z}}(z) = ak$, the connected component $D \backslash A^{n_z}$ containing z is equal to the connected component of $D \backslash A_{ak}$ containing z.*

Proof. The proof follows from the uniqueness of two-valued sets $A_{-a,b}$. Indeed, construct sets B^n by taking $B^1 = A^1$ and then repeating the construction of A^i only in the components where the value of h_{B^n} is not yet ak. Thus, by construction $B^n \subset A^n$. Moreover, for any z up to and including the first iteration where $\Gamma_{B^k}(z) = ak$, the connected component of the complement of A^n and B^n containing z coincide.

Now, note that for a fixed $z \in D$, n_z is almost surely finite. Thus it suffices to prove that for all $n \in \mathbb{N}$, the set B^n is contained in $A_{-\lceil an \rceil, ak}$ and that all connected components of $D \backslash B^n$ where h_{B^n} takes the value ak are connected components of $D \backslash A_{-\lceil an \rceil, ak}$ where $h_{A_{-\lceil an \rceil, ak}}$ is equal to ak. To see this, first note that $h_{B^n} \in \{-an, -a(n-1), \dots, ak\}$. In particular, in each connected component where $h_{B^n} = c \notin \{-\lceil an \rceil, ak\}$ we can construct the two-valued sets $A_{-\lceil an \rceil - c, ak - c}$. This gives us a local set $\tilde{B}$ s.t. $h_{\tilde{B}}$ takes only values in $\{-\lceil an \rceil, k\}$. It is also possible to see that $\tilde{B}$ is thin, by noting that inside each compact set its Minkowski dimension is smaller than 2 (Proposition 1.14 and comment after Definition 1.15). Then, by uniqueness of the two-valued sets, Lemma 4.3, $\tilde{B}$ is equal to $A_{\lceil an \rceil, k}$. To finish, notice that we the connected components of $D \backslash B^n$ where h_{B^n} took the value ak are also connected components of $\tilde{B}$ with the same value. □

In particular, from this lemma it follows that we can also construct A_a in a different way: denote $A^1 = A_{-a,a}$ and define A^2 by iterating independent copies of $A_{-a,a}$ in each component of the complement of $D \setminus A^1$ where $h_{A_1} \neq a$. Repeat this procedure again in all components of the complement for which the value still differs from a. This iteration gives an increasing sequence of local sets A^n, whose limit is equal to A_a. For a concrete example, one could take $A_{-2\lambda,2\lambda}$ to be equal to CLE_4 in its coupling with the GFF, and the above procedure would yield $A_{2\lambda}$. In fact the sets $(A_{2\lambda n})_{n\in\mathbb{N}}$ are exactly the sets that the author [Aïd15] proposes as a basis for the construction of the Liouville measure.

4.3 Overview of the Liouville measure and loop constructions of Aïdekon

There are many ways to define the Liouville measure in the subcritical case, the differences amounting to how one approximates the underlying GFF. We will first describe the approximations using circle averages in the subcritical case. Then we will discuss the critical regime, and finally present the nested-loop based constructions from [Aïd15] that are conjectured to give the Liouville measure. From now on we will set $D = \mathbb{D}$ for simplicity.

4.3.1 Subcritical regime

Let us recall that we denote $\Gamma_\varepsilon(z) = (\Gamma, \rho_z^\varepsilon)$ the ε-circle average of the GFF around the point z as before. It is known that $\Gamma_\varepsilon(z)$ is a continuous Gaussian processes that converge to Γ a.s. in the space of distributions as $\varepsilon \to 0$. Thus, one can define approximate Liouville measures

$$\mu_\varepsilon^\gamma(dz) := \varepsilon^{\frac{\gamma^2}{2}} \exp\left(\gamma \Gamma_\varepsilon(z)\right) dz.$$

In the subcritical regime we have the following result [DS11, Ber15b]:

Theorem 4.6 *For $\gamma < 2$ the measures μ_ε^γ converge to a non-trivial measure μ^γ weakly in probability. Moreover, for any fixed Borel set $\mathcal{O} \subset \mathbb{D}$ we have that $\mu_\varepsilon^\gamma(\mathcal{O})$ converges in L^1 to $\mu(O)$.*

In fact it is known that the measure is also unique, in the sense that the same limit can be obtained using any sufficiently nice mollifier instead of the circle average. We will show that the approximations using local sets give the same measure.

4.3.2 Critical regime

It is known that for $\gamma \geqslant 2$, the measures μ_γ^ε converge to zero [RV07]. To define the critical measures an additional renormalization is therefore required. One way to do it is to use the so-called derivative martingale, originating from studies on branching random walks. Define

$$\nu_\varepsilon(dz) := \left.\frac{\partial}{\partial \gamma}\right|_{\gamma=2} \mu_\varepsilon^\gamma(dz) = (-\Gamma_\varepsilon(z) + 2\log(1/\varepsilon))\varepsilon^2 \exp\left(2\Gamma_\varepsilon(z)\right) dz$$

It has been recently shown in [Pow17, Theorem 1.1] that ν_ε converges weakly in probability to a non-trivial limiting measure μ_2' as $\varepsilon \to 0$. Moreover, μ_2' coincides with the critical Liouville measure defined in [DRSV14a, DRSV14b]. We will again show that the approximations using local sets converge towards same measure.

Another way to define the critical measure is to use the so-called Seneta-Heyde renormalization [AS14, DRSV14b]. In the case of the circle-average process the approximating measures would be defined as:

$$\bar{\nu_\varepsilon}(dz) := \sqrt{\log 1/\varepsilon}\, \mu_\varepsilon^2(dz).$$

It has been shown [HRV15, JS15] that $\bar{\nu_\varepsilon}$ converges in probability to $\sqrt{\frac{2}{\pi}}\mu_2'$ as $\varepsilon \to 0$. We will prove an analogous result in our setting.

4.3.3 Measures constructed using nested loops

In [Aïd15] the author proposes a construction of measures, analogous to the Liouville measure, using nested conformally-invariant loop ensembles. We will now describe it in a concrete context that is related to this chapter.

Consider a CLE_4, and inside each loop toss an independent fair coin. Keep the loops with heads on top, and sample new CLE_4 loops in the others. Also toss new independent coins inside these loops. Keep track of all the coin tosses for each loop and repeat the procedure inside each loop where the number of heads is not yet larger than the number of tails. Define the resulting set as $\tilde{A}^1$. Now define $\tilde{A}^k$ iteratively by sampling an independent copy of $\tilde{A}^1$ inside each connected component of $\mathbb{D} \setminus \tilde{A}^{k-1}$.

For any Borelian $\mathcal{O} \subset \mathbb{D}$ we can now define

$$\tilde{M}_k^\gamma(\mathcal{O}) = \frac{1}{\mathbb{E}\mathrm{CR}(0, \mathbb{D} \setminus \tilde{A}^1)^{\gamma^2/2^k}} \int_{\mathcal{O}\cap\mathbb{D}\setminus\tilde{A}^k} \mathrm{CR}(z, \mathbb{D} \setminus \tilde{A}^k)^{\frac{\gamma^2}{2}} dz \tag{4.3}$$

It is shown in [Aïd15] that for $\gamma < 2$ the measures defined by $\tilde{M}_k^\gamma$ converge weakly almost surely to a non-trivial measure $\tilde{M}^\gamma$. It is also conjectured there that the limiting measures coincide with the Liouville measures μ^γ. We will prove this statement below.

It is further proved in [Aïd15] that for $\gamma \geqslant 2$, these measures converge almost surely to zero. In the critical case, however, one can again define a derivative martingale $\tilde{D}_k^\gamma$ by taking a derivative with respect to $-\gamma$. In other words one sets:

$$\tilde{D}_k^\gamma(\mathcal{O}) = -2\frac{\partial}{\partial\gamma}\tilde{M}_k^\gamma(\mathcal{O})$$

(we include the factor 2 here to be consistent with the definition in [Aïd15]). It is shown in [Aïd15] that the measures $\tilde{D}_k := \tilde{D}_k^2$ converge to a non-trivial positive measure $\tilde{D}_\infty$. In this chapter, we prove that $\tilde{D}_\infty = 2\mu_2'$.

4.4 Local set approximations of the subcritical Liouville measure

In this section we prove that one can approximate the Liouville measure of a GFF in a simply connected domain using increasing sequences of local sets $(A^n)_{n\in\mathbb{N}}$ with $\overline{\bigcup A^n} = \mathbb{D}$. In particular, the measure constructed in [Aïd15] will fit in our framework and thus it agrees with the Liouville measure. In fact, for simplicity, we first present the proof of convergence in this specific case.

First, recall that we denote by h_A the harmonic function given by the restriction of Γ_A to $\mathbb{D} \setminus A$. For any local set A with Lebesgue measure 0 and bounded h_A, we define for any Borelian set $\mathcal{O} \subseteq D$:

$$M^\gamma(\mathcal{O}, A) := \int_{\mathcal{O}} e^{\gamma h_A}\, \mathrm{CR}(z; \mathbb{D} \setminus A)^{\gamma^2/2}\, dz.$$

Notice that as h_A is bounded, we can define it arbitrarily on the 0 Lebesgue measure set A.

Proposition 4.7 *Fix $\gamma \in [0,2)$. For $a>0$, let A_a be the a-FPS of Γ and μ^γ be the Liouville measure defined by Γ. Then for each Borelian set $\mathcal{O} \subset \mathbb{D}$,*

$$M_a^\gamma(\mathcal{O}) := M^\gamma(\mathcal{O}, A_a) = e^{\gamma a} \int_{\mathcal{O}} CR(z; D\backslash A)^{\gamma^2/2} dz$$

is a martingale with respect to $\mathscr{F}_{A_a}$ and converges a.s. to $\mu^\gamma(\mathcal{O})$ as $a \to \infty$. Thus, a.s. the measures M_a^γ converge weakly to μ^γ.

Before the proof, we make two remarks. First, we make the connection between our martingale and the martingales of [Aïd15]:

Remark 4.8 *As a consequence of Lemma 4.5, the fact that $A_{-2\lambda,2\lambda}$ has the law of CLE$_4$ and the fact that the value of its corresponding harmonic function is independent in each connected component of $\mathbb{D}\backslash A_{-2\lambda,2\lambda}$, we see that $\tilde{A}^1$ of Section 4.3.3 is equal in law to $A_{2\lambda}$. Furthermore, the sequence $(\tilde{A}^k)_{k\in\mathbb{N}}$ has the same law has the sequence $(A_{2\lambda k})_{k\in\mathbb{N}}$.*

Now, by the iterative construction and conformal invariance the random variables

$$\log \mathrm{CR}(0, \mathbb{D} \setminus \tilde{A}^i) - \log \mathrm{CR}(0, \mathbb{D} \setminus \tilde{A}^{i-1})$$

with $A^0 = \emptyset$ are i.i.d. Thus, $\mathbb{E}\mathrm{CR}(0, \mathbb{D} \setminus \tilde{A}^1)^{\frac{\gamma^2}{2}k} = \mathbb{E}\mathrm{CR}(0, \mathbb{D} \setminus \tilde{A}^k)^{\frac{\gamma^2}{2}}$.

Moreover, it is a consequence of Lemma 2.23 that $-\log \mathrm{CR}(0, \mathbb{D}\backslash\tilde{A}^k)$ corresponds precisely to the hitting time of $k\pi$ by a standard Brownian motion started from zero. In our case, when $2\lambda = \pi$, we therefore see that

$$e^{\gamma 2\lambda k} = \mathbb{E}\left[\mathrm{CR}(0, \mathbb{D} \setminus \tilde{A}^1)^{\frac{\gamma^2}{2}}\right]^{-k}.$$

Furthermore, since $Leb(A_{2\lambda}) = 0$ implies that $M_a^\gamma(\mathcal{O} \cap A_{2\lambda}) = 0$, we have that $M_{2\lambda k}^\gamma$ agrees with the measure $\tilde{M}_k^\gamma$ defined in (4.3). Hence Proposition 4.7 confirms that the limit of $\tilde{M}_k^\gamma$ corresponds to the Liouville measure.

Remark 4.9 *Second, in order to avoid repetition, we recall here as a remark the standard argument showing that the almost sure weak convergence of measures is implied by the almost sure convergence of $M_a^\gamma(\mathcal{O})$ over all boxes $\mathcal{O}$ with dyadic coordinates. This follows from two observations: first, the sub-space of Radon measures on $\mathbb{D}$ with bounded mass is compact and second, the boxes $\mathcal{O}$ with dyadic coordinates generate the Borel σ-algebra. Notice that we do not show that we have strong convergence of measures, i.e. we do not know whether almost surely $\mu^\gamma(\mathcal{O})$ is the limit of $M_a^\gamma(\mathcal{O})$ for all Borelian $\mathcal{O}$.*

Proof of Proposition 4.7. By Remark 4.9, it suffices to prove the convergence statement for $M_a^\gamma(\mathcal{O})$. When $\gamma \in [0,2)$, we know that $\mu_\varepsilon^\gamma(\mathcal{O}) \to \mu^\gamma(\mathcal{O})$, in $\mathcal{L}^1$ as $\varepsilon \to 0$, where μ_ε^γ is as in Theorem 4.6. Thus,

$$\mathbb{E}\left[\mu^\gamma(\mathcal{O}) \mid \mathscr{F}_{A_a}\right] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\mu_\varepsilon^\gamma(\mathcal{O}) \mid \mathscr{F}_{A_a}\right].$$

The key is to argue that

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\mu_\varepsilon^\gamma(\mathcal{O}) \mid \mathscr{F}_{A_a}\right] = M_a^\gamma(\mathcal{O}). \tag{4.4}$$

Then $M_a^\gamma(\mathcal{O}) = \mathbb{E}\left[\mu^\gamma(\mathcal{O}) \mid \mathscr{F}_{A_a}\right]$ and we can conclude using the martingale convergence theorem and the fact that $\overline{\bigcup A_a} = \mathbb{D}$.

To prove (4.4), define A_a^ε as the ε-enlargement of A_a. By writing $\Gamma = \Gamma_{A_a} + \Gamma^{A_a}$ and using that $(\Gamma_{A_a}, \rho_\varepsilon^z) = a$ for any $z \in \mathbb{D}\backslash A_a^\varepsilon$, we have

$$\mathbb{E}\left[\int_{\mathcal{O}\backslash A_a^\varepsilon} \mathrm{e}^{\gamma(\Gamma,\rho_\varepsilon^z)}\, \varepsilon^{\gamma^2/2}\, dz \middle| \mathscr{F}_{A_a}\right] = \int_{\mathcal{O}\backslash A_a^\varepsilon} \mathrm{e}^{\gamma a}\, \varepsilon^{\gamma^2/2} \mathbb{E}\left[\mathrm{e}^{(\Gamma^{A_a},\rho_\varepsilon^z)} \middle| \mathscr{F}_{A_a}\right] dz$$

Using (4.2) we recognize that the right hand side is just $M_a^\gamma(\mathcal{O}\backslash A_a^\varepsilon)$.

But now for any fixed a, as $\mathrm{CR}(z,\mathbb{D}) \leqslant 1$ and A_a has zero Lebesgue measure, we have that $M_a^\gamma(\mathcal{O} \cap A_a^\varepsilon) = o_\varepsilon(1)$. On the other hand, from the fact that $(\Gamma_{A_a}, \rho_\varepsilon^z) \leqslant a$ for any z, and (4.2), it follows that

$$\mathbb{E}\left[\int_{\mathcal{O}\cap A_a^\varepsilon} \mathrm{e}^{\gamma(\Gamma,\rho_\varepsilon^z)}\, \varepsilon^{\gamma^2/2}\, dz \middle| \mathscr{F}_{A_a}\right] \leqslant Leb(A_a^\varepsilon) e^{\gamma a}.$$

Thus, we conclude (4.4) and the proof. □

We now state a more general version of this result, which says that one can construct the Liouville measure using a variety of local set approximations. The proof is a simple adaptation of the proof above. We say that a generalized function T on $\mathbb{D}$, for which the circle-average process $T_\varepsilon(z) := (T, \rho_z^\varepsilon)$ can be defined, is bounded from above by K if for all $z \in D$ and $\varepsilon > 0$, we have that $T_\varepsilon(z) \leqslant K$.

Proposition 4.10 *Fix $\gamma \in [0,2)$ and let $(A^n)_{n\in\mathbb{N}}$ be an increasing sequence of local sets for a GFF Γ with $\overline{\bigcup_{n\in\mathbb{N}} A^n} = \overline{\mathbb{D}}$. Suppose that almost surely for all $n \in \mathbb{N}$, we have that $Leb(A^n) = 0$ and that Γ_{A^n} is bounded from above by K_n for some sequence of finite K_n. Then for any Borel $\mathcal{O} \subset \mathbb{D}$, $M_n^\gamma(\mathcal{O})$ defined by*

$$M_n^\gamma(\mathcal{O}) = \int_{\mathcal{O}} e^{\gamma h_{A^n}(z)}\, \mathrm{CR}(z; \mathbb{D} \setminus A^n)^{\gamma^2/2}\, dz$$

is a martingale with respect to $\{\mathcal{F}_{A^n}\}_{n>0}$ and

$$\lim_{n\to\infty} M_n^\gamma(\mathcal{O}) = \mu^\gamma(\mathcal{O}) \ a.s.$$

where μ^γ is the Liouville measure defined by Γ. Thus, almost surely the measures M_n^γ converge weakly to μ^γ.

Let us mention two natural sequences of local sets for which this proposition applies. The first is when we take $a_n, b_n \nearrow \infty$ and study the sequence $(A_{-a_n,b_n})_{n\in\mathbb{N}}$. The second is when we take the sequence $(A_{-a,b}^n)_{n\in\mathbb{N}}$ for some $a, b > 0$, where $A_{-a,b}^n$ is defined by iteration [1]. Note that in the case where $a = b = 2\lambda$, we recover the result described in the introduction for the iterated CLE_4.

Observe that whereas our martingale agrees with the one given in [Aïd15] for the case of first-passage sets, for any cases where h_{A^n} can take more than one value, the martingales are in fact different. Yet, we can still identify the limit of the martingale $\tilde{M}_n^\gamma(\mathcal{O})$ of [Aïd15], corresponding to an iterated CLE_4 (i.e. $(\mathrm{CLE}_4^n)_{n\in\mathbb{N}}$.) In this case Aidekon's martingale converges in distribution to $\eta^\gamma(\mathcal{O}) := \mathbb{E}\left[\mu^\gamma(\mathcal{O})|\mathscr{F}_\infty\right]$, where μ^γ is the Liouville measure and

[1] We set $A_{-a,b}^1 = A_{-a,b}$ and define $A_{-a,b}^n$ by sampling the $A_{-a,b}$ of $\Gamma^{A_{-a,b}^{n-1}}$ inside each connected component of $D\backslash A_{-a,b}^{n-1}$

$\mathscr{F}_\infty$ is the σ-algebra containing only the geometric information from all iterations of the CLE_4. This σ-algebra is strictly smaller than $\mathcal{F}_{A^n_{-2\lambda,2\lambda}}$, which also contains information on the labels of CLE_4 in its coupling with the GFF. It is not hard to see that η^γ is not equal to μ^γ.

4.5 Generalizations

In this section, we describe some other situations where an equivalent of Proposition 4.10 can be proven using the same techniques as the proof of Proposition 4.7. In the following we do not present any new methods, but focus instead on announcing the propositions in context, so that they may be used in other works. We also make explicit the places where the results are already, or may in the future, be used.

4.5.1 Non-simply connected domains and general boundary conditions.

Here we consider the case when Γ is a GFF in an n-connected domain $D \subseteq \mathbb{D}$ (for more context see Chapter 3). First, let us note that in this set-up (4.2) becomes

$$\mathbb{E}\left[\varepsilon^{\frac{\gamma^2}{2}} \exp\left(\gamma(\Gamma, \rho_z^\varepsilon)\right)\right] \begin{cases} = e^{-\frac{\gamma^2}{2}\tilde{G}_D(z,z)} & \text{if } d(z,\partial D) \geqslant \varepsilon, \\ \leqslant 1 & \text{if } d(z,\partial D) < \varepsilon, \end{cases}$$

where we write $G_D(z,w) = -\log|z-w| + \tilde{G}_D(z,w)$, i.e. for any $z \in D$, $\tilde{G}_D(z,\cdot)$, is the bounded harmonic function that has boundary conditions $\log(|z-w|)$ for $w \in \partial D$. Additionally, if we work with local sets A such that all connected components of $A \cup \partial D$ contain an element of ∂D, then Lemma 4.2 will hold. All local sets we refer to here are assumed to satisfy this condition. These facts and assumptions are enough to prove the following proposition:

Proposition 4.11 *Fix $\gamma \in [0,2)$ and let $(A^n)_{n\in\mathbb{N}}$ be an increasing sequence of local sets for a GFF Γ with $\overline{\bigcup_{n\in\mathbb{N}} A^n} = \mathbb{D}$. Suppose that almost surely for all $n \in \mathbb{N}$, we have that $Leb(A^n) = 0$ and that Γ_{A^n} is bounded from above by K_n for some sequence of finite K_n. Then for any Borel $\mathcal{O} \subset D$, $M_n^\gamma(\mathcal{O})$ defined by*

$$M_n^\gamma := \int_{\mathcal{O}} e^{\gamma h_{A_n}(z) - \frac{\gamma^2}{2}\tilde{G}_{D\setminus A_n}(z,z)}\, dz, \qquad \mathcal{O} \subset \mathbb{D}$$

is a martingale with respect to $\{\mathcal{F}_{A^n}\}_{n>0}$ and

$$\lim_{n\to\infty} M_n^\gamma(\mathcal{O}) = \mu^\gamma(\mathcal{O}) \text{ a.s.}$$

where μ^γ is the Liouville measure defined by Γ. Thus, almost surely the measures M_n^γ converge weakly to μ^γ.

The equivalent of the sets $A_{-a,b}$ and A_a are defined in n-connected domains in Section 3.4 and it is easy to see that their iterated versions satisfy the hypothesis of Proposition 4.11. In particular, the above construction allowed us to prove that the measure Γ_{A_a} is a measurable function of A_a.

4.5.2 Dirichlet-Neumann GFF

In this section we take Γ to be a GFF with Dirichlet-Neumann boundary conditions in $\mathbb{D}^+ = \mathbb{D} \cap \mathcal{H}$. That is, Γ satisfies (4.1), with G_D replaced by $G_{\mathbb{D}^+}$: the Green's function in $\mathbb{D}^+$ with Dirichlet boundary conditions on $\partial\mathbb{D}$ and Neumann boundary conditions on $[-1,1]$. To be more specific, we set $G_{\mathbb{D}^+}(x,y) = G_{\mathbb{D}}(x,y) + G_{\mathbb{D}}(x,\bar{y})$, with $G_{\mathbb{D}}$ as in Section 4.2. Then $G_{\mathbb{D}^+}(x,y) \sim \log(1/|x-y|)$ as $x \to y$ in the interior of $\mathbb{D}^+$ and $G_{\mathbb{D}^+}(x,y) \sim 2\log(1/|x-y|)$ when $y \in (0,1)$.

Let A be a closed subset of $\bar{\mathbb{D}}^+$. Suppose that Γ is a Dirichlet-Neumann GFF in $\mathbb{D}^+\backslash A$ with Neumann boundary conditions on $[-1,1]\backslash A$ and Dirichlet boundary conditions on the rest of the boundary. Let $z \in [-1,1]$ and define ϱ_z^ε to be the uniform measure on $\partial B(z,\varepsilon) \cap \mathbb{D}^+$. Then, in this set-up (4.2) becomes

$$\mathbb{E}\left[\varepsilon^{\gamma^2/4} \exp\left(\frac{\gamma}{2}(\Gamma, \varrho_x^\varepsilon)\right)\right] \begin{cases} = \mathrm{CR}(x; \mathbb{D}\backslash\breve{A})^{\gamma^2/4} & \text{if } d(z, \partial(\mathbb{D}\backslash\breve{A})) \geqslant \varepsilon, \\ \leqslant 1 & \text{if } d(z, \partial(\mathbb{D}\backslash\breve{A})) < \varepsilon. \end{cases} \tag{4.5}$$

Here we set $\breve{A} := A \cup \bar{A}$ for $\bar{A} = \{z \in \mathbb{C} : \bar{z} \in A\}$.

There is also a notion of local sets for this Dirichlet-Neumann GFF. We say that (Γ, A, Γ_A) describes a local set coupling if, conditionally on (A, Γ_A), $\Gamma^A := \Gamma - \Gamma_A$ is a GFF with Neumman boundary conditions on $[-1,1]\backslash A$ and Dirichlet on the rest. For connected local sets such $\partial\mathbb{D}^+ \cup A$ is connected, Lemma 4.2 still holds (by the same proof given for the 0-boundary GFF).

We are interested in the boundary Liouville measure on $[-1,1]$. Take $\gamma < 2$, $\varepsilon > 0$ and a Borel set $\mathcal{O} \subseteq [-1,1]$. We define the approximate boundary Liouville measures as follows:

$$\upsilon_\varepsilon^\gamma(\mathcal{O}) := \varepsilon^{\gamma^2/4} \int_{\mathcal{O}} \exp\left(\frac{\gamma}{2}(\Gamma, \varrho_x^\varepsilon)\right) dx$$

where here dx is the Lebesgue density on $[-1,1]$. It is known (see [DS11, Ber15b]) that $\upsilon_\varepsilon^\gamma \to \upsilon^\gamma$ in $\mathcal{L}^1$ as $\varepsilon \to 0$. Moreover, it is also easy to see that υ^γ is a measurable function of $\mathscr{F}_{[-1,1]}$ - this just comes from the fact that the Dirichlet GFF contains no information on the boundary. Thus, we have all the necessary conditions to deduce the following Proposition using exactly the same proof as in Section 4.4.

Proposition 4.12 *Fix $\gamma \in [0,2)$ and let $(A^n)_{n\in\mathbb{N}}$ be an increasing sequence of local sets for a GFF Γ with $\overline{\bigcup_{n\in\mathbb{N}} A^n} \supseteq [-1,1]$. Suppose that almost surely for all $n \in \mathbb{N}$, we have that $Leb_{[-1,1]}(A_n) = 0$ and that Γ_{A^n} restricted to A^n is bounded from above by K_n for some sequence of finite K_n. Then for any Borel $\mathcal{O} \subset [-1,1]$, $M_n^\gamma(\mathcal{O})$ defined by*

$$M_n^\gamma(\mathcal{O}) := \int_{\mathcal{O}} e^{\frac{\gamma}{2} h_{A_n}(z)} \mathrm{CR}(z; \mathbb{D}\backslash\breve{A}_n)^{\frac{\gamma^2}{4}} dz$$

is a martingale with respect to $\{\mathcal{F}_{A^n}\}_{n>0}$ and

$$\lim_{n\to\infty} M_n^\gamma(\mathcal{O}) = \upsilon^\gamma(\mathcal{O}) \; a.s.$$

where where μ^γ is the boundary Liouville measure defined by Γ. Thus, almost surely the measures M_a^γ converge weakly to υ^γ.

It has recently been proven in [QW17] that sets satisfying the above hypothesis do exist, and that they can be used to couple the Dirichlet GFF with the Neumman GFF. Let us describe some concrete examples of these sets. If Γ is a Dirichlet-Neumman GFF, then in [QW17] it is shown that there exists a (measurable) thin local set $\tilde{A}(\Gamma)$ of the GFF such that:

- $\tilde{A}(\Gamma)$ has the law of the trace of an $\mathrm{SLE}_4(0;-1)$ going from -1 to 1
- $h_{\tilde{A}(\Gamma)}$ is equal to 0 in the only connected component of $\mathbb{D}^+\backslash\tilde{A}(\Gamma)$ whose boundary intersects $\partial\mathbb{D}\cap\mathcal{H}$
- in the other connected components, $h_{\tilde{A}(\Gamma)}$ is equal to $\pm 2\lambda$, where conditionally on $\tilde{A}(\Gamma)$ the sign is chosen independently in each component.

There are two interesting sequences of local sets we can construct using this basic building-block. The first one is the boundary equivalent of $(A^n_{-2\lambda,2\lambda})_{n\in\mathbb{N}}$, and the second is the boundary equivalent of $(A_{2\lambda n})_{n\in\mathbb{N}}$. The first one is also described in [QW17, Section 3]. The construction goes as follows: choose $A^1=\tilde{A}(\Gamma)$ and construct A^n by induction. In the connected components O of $\mathbb{D}\backslash A^n$ that contain an interval of $\mathbb{R}$, we have that Γ^{A_n} restricted to O is a Dirichlet-Neumman GFF (with Neumann boundary condition on $\mathbb{R}\cap\partial O$). Thus, by conformal invariance we can explore the set $\tilde{A}(\Gamma\mid_O)$ in each such component O. We define A^{n+1} to be the closed union of A^n with $\tilde{A}(\Gamma\mid_O)$ over all explored components O. Note that $h_{A^n}\in\{2\lambda k\}$ where k ranges between $-n$ and n. It is also not hard to see that A^n is thin (it follows from the fact that $h_A\in\mathcal{L}^1(\mathbb{D}\backslash A)$ and that for any compact set $K\subseteq\mathbb{D}^+$ the Minkowski dimension of $A^n\cap K$ is a.s. equal to $3/2$, see Proposition 1.11). Thus, we deduce that $\Gamma_{A^n}\leqslant 2\lambda n$. Additionally, note that by adjusting [MS11, Lemma 6.4], we obtain from the construction of A^1 that for any $z\in(-1,1)$ the law of $2(\log(\mathrm{CR}^{-1}(z,\mathbb{D}\backslash\breve{A}^1))-\log(\mathrm{CR}^{-1}(z,\mathbb{D}))$ is equal to the first time that a BM exits $[-2\lambda,2\lambda]$. It follows that for all $n\in\mathbb{N}$, $Leb_{\mathbb{R}}(A_n\cap[-1,1])=0$ and also $\overline{\bigcup_{n\in\mathbb{N}}A^n}\supseteq[-1,1]$. Hence we see that the sequence $(A^n)_{n\in\mathbb{N}}$ satisfies the conditions of Proposition 4.12.

For the second sequence of local sets, take $B^1=\tilde{A}(\Gamma)$ and define B^{n+1} to be the closed union of B^n with all $\tilde{A}(\Gamma\mid_O)$ such that O is a connected component of $\mathbb{D}\backslash B^n$, $h_{B^n}\mid_O\leqslant 2\lambda$ and ∂O contains an interval of $\mathbb{R}$. Denote $A^1(\Gamma)$ the closed union of all the B^n. Due to the fact that B^n are BTLS with $h_{B^n}\leqslant 2\lambda$ on $[-1,1]$, we have that Γ_{A^1} restricted to $[-1,1]$ is smaller than or equal to 2λ. Additionally, note that $2(\log(\mathrm{CR}^{-1}(z,\mathbb{D}\backslash\breve{A}^1))-\log(\mathrm{CR}^{-1}(z,\mathbb{D}))$ is distributed as the first time a BM hits 2λ. Now, we iterate to define $A^n(\Gamma)$ as the closed union of $A^{n-1}(\Gamma)$ and $A^1(\Gamma\mid_O)$, where O ranges over all connected components of $\mathbb{D}^+\backslash A^{(n-1)}$ containing an interval of $\mathbb{R}$. The sequence $(A^n)_{n\in\mathbb{N}}$ satisfies the condition of Proposition 4.12. Note that in this case the martingale simplifies and contains only information on the geometry of the sets A^n:

$$M_n^\gamma := e^{\gamma 2\lambda n}\int_{\mathcal{O}} CR(z;\mathbb{D}\backslash\breve{A}^n)^{\gamma^2/4}dz.$$

The fact that this martingale is a measurable function of A^n allows us to use the same techniques as in Lemma 3.33 to prove that the measure $2\lambda n-\Gamma_{A^n}$ on $\mathbb{R}$ is a measurable function of A^n.

It is also explained in [QW17] that the sets A^n we have just constructed, and the definition of the boundary Liouville measure using them, might help to reinterpret an SLE-type of conformal welding first studied in [She16].

4.6 Critical and supercritical regimes

In this section it is technically simpler to restrict ourselves to the simply connected case and study a special family of sequences of local sets, though the results hold in a more general setting. Namely, we assume that our sets A^n are formed by an iterative procedure. That is, $A^1 = A(\Gamma)$ is some measurable local set coupled with the GFF Γ, and A^{n+1} is formed from A^n by, in each component O of $\mathbb{D} \setminus A^n$, exploring $A(\Gamma^{A^n})$. Notice, for example, that the iterated CLE_4 coupling described in the introduction is covered by this hypothesis, as are the couplings with A_{an} for any $a > 0$.

We first show that the martingales defined in Section 4.4 converge to zero for $\gamma \geqslant 2$. Then, in the critical case $\gamma = 2$, we define a derivative martingale and show it converges to the same measure as the critical measure μ_2' from [DRSV14a, DRSV14b, Pow17], and 1/2 times the critical measure $\tilde{D}_\infty$ from [Aïd15]. Finally, we show that for $A^n = A_{an}$ we can also construct the critical measure using the Seneta-Heyde rescaling (analogous to the main theorem of [AS14].) More precisely, for all Borelian $\mathcal{O} \subset \mathbb{D}$, we have that $\sqrt{an}M^2(\mathcal{O}, A^n)$ converges in probability to $\frac{4}{\sqrt{\pi}}\mu_2'(\mathcal{O})$ as $n \to \infty$.

4.6.1 (Super)critical regime.

Lemma 4.13 *Set $\gamma \geqslant 2$ and assume that A^n is formed by iteration as above and that $\overline{\bigcup_n A_n} = \mathbb{D}$. Assume further that A is such that h_A is constant in each connected component of $\mathbb{D} \setminus A$ almost surely. Then $M_n^\gamma \to 0$ almost surely.*

Remark 4.14 *Due to the iterative nature of the construction, the condition $\overline{\bigcup_n A_n} = \mathbb{D}$, i.e. that in the limit the iterated sets cover the whole domain, is implied for example by a simple requirement on the local set $A = A_1$ - it suffices to have that there exists $\varepsilon, \delta > 0$ such that for any point $z \in \mathbb{D}$, the probability that $\mathrm{CR}(z; \mathbb{D} \backslash A_1) < (1-\varepsilon)\,\mathrm{CR}(z; \mathbb{D})$ is bigger than δ.*

In [Aïd15], Aïdekon also considers the critical and supercritical cases for his iterated loop measures. In particular, from his results one can read out that, with the notation of Proposition 4.7, for any $a > 0$ and $\gamma \geqslant 2$, we have $M_{an}^\gamma \to 0$ almost surely as $n \to \infty$.

The proof follows from a classical technique stemming from the literature on branching random walks [Lyo97], but is based on the local set coupling with the GFF.

Proof. From (4.2) and a direct calculation we see that $M_n^\gamma(\mathbb{D})/M_0^\gamma(\mathbb{D})$ is a mean one martingale, where $M_0^\gamma(\mathbb{D}) = \int_{\mathbb{D}} \mathrm{CR}(z, \mathbb{D})^{\gamma^2/2}\, dz$. Let us define a new probability measure $\hat{\mathbb{P}}$ via the change of measure

$$\left.\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathscr{F}_{A^n}} = \frac{M_n^\gamma(\mathbb{D})}{M_0^\gamma(\mathbb{D})}. \tag{4.6}$$

It is well known, see for example [Dur10], that in order to show that $M_n^\gamma(\mathbb{D}) \to 0$ almost surely under $\mathbb{P}$, it suffices to prove that $\limsup_n M_n^\gamma(\mathbb{D}) = +\infty$ a.s. under $\hat{\mathbb{P}}$.

To show this we actually consider a change of measure on an enlarged probability space. Define a measure $\mathbb{P}^*$ on $(\Gamma, (A^n)_n, Z)$ by sampling $(\Gamma, (A^n)_n)$ from $\mathbb{P}$ and then independently, sampling a random variable $Z \in \mathbb{D}$ with law proportional to Lebesgue measure. Note that under $\mathbb{P}^*$ the process

$$\xi_n = \mathrm{e}^{\gamma h_{A^n}(Z)+\gamma^2/2\log \mathrm{CR}(Z,\mathbb{D}\setminus A^n)}$$

is a martingale with respect to the filtration $\mathscr{F}^*_{A^n} = \mathscr{F}_{A^n} \vee \sigma(Z)$. Thus we can define a new probability measure $\hat{\mathbb{P}}^*$ by

$$\left.\frac{d\hat{\mathbb{P}}^*}{d\mathbb{P}^*}\right|_{\mathscr{F}^*_{A^n}} := \frac{\xi_n}{\mathbb{E}[\xi_0]} \tag{4.7}$$

Then if $\hat{\mathbb{P}}$ is the restriction of $\hat{\mathbb{P}}^*$ to $\mathscr{F}_{A^n}$, $\hat{\mathbb{P}}$ and $\mathbb{P}$ satisfy (4.6). Therefore it suffices to prove that under $\hat{\mathbb{P}}^*$ and conditionally on Z, we have $\limsup_n M_n^\gamma(\mathbb{D}) = +\infty$ almost surely. By the Köebe-(1/4) Theorem and [Aïd15, Lemma 2.4] we only need to prove that under this law almost surely

$$\limsup_n \mathrm{e}^{\gamma h_{A^n}(Z)+(\gamma^2/2+2)\log \mathrm{CR}(Z,\mathbb{D}\setminus A^n)} = +\infty.$$

However we can calculate, using (4.2) that

$$\gamma h_{A^n}(Z) + (\gamma^2/2+2)\log \mathrm{CR}(Z, \mathbb{D}\setminus A^n)$$

is a random walk with non-negative mean (started from $(\gamma^2/2+2)\log \mathrm{CR}(Z,\mathbb{D})$) under this law. This allows us to conclude.

□

4.6.2 The derivative martingale in the critical regime

We now show the convergence of the derivative martingale (when $\gamma = 2$, defined below) for the particular case of iterated $A_{-a,a}$, $a \geqslant \lambda$. For any Borel set $\mathcal{O} \subseteq \mathbb{D}$ and local set A, we define

$$D^\gamma(\mathcal{O}, A) := \int_{\mathcal{O}} \left(-h_A(z) + \gamma \log \mathrm{CR}^{-1}(z, \mathbb{D}\setminus A)\right) e^{\gamma h_A(z)}\, \mathrm{CR}(z; \mathbb{D}\setminus A)^{\gamma^2/2}\, dz.$$

The rest of this section is devoted to proving the following proposition.

Proposition 4.15 *Assume that A^n is formed by iterating $A_{-a,a}$ n times, for $a \geqslant \lambda$. Then for any Borel $\mathcal{O} \subset \mathbb{D}$ we have that $\hat{D}_n(\mathcal{O}) := D^2(\mathcal{O}, A^n)$ is a martingale and converges almost surely to a finite, positive limit $\hat{D}_\infty(\mathcal{O})$ as $n \to \infty$. In particular the signed measures $\hat{D}_n(\mathcal{O})$ converge weakly to a limiting measure that is independent of the choice of $a > 0$ and agrees with the critical measure μ_2' defined in [DRSV14a, DRSV14b], and $1/2$ times the critical measure $\tilde{D}_\infty$ defined in Theorem 1.3 of [Aïd15].*

Before the proof, let us first comment on the case where the set we are going to iterate is A_a; so n iterations gives A_{an}. In the case $a = 2\lambda$, observe that twice the derivative martingale $2D^2(\mathcal{O}, A_{2\lambda n})$ is equal to $\tilde{D}_n$ defined in (1.3) of [Aïd15] (see Remark 4.8). Thus, we know that when we iterate $A_{2\lambda}$, its associated sequence of measures converges to a limit $\tilde{D}_\infty$. In fact it follows from [Aïd15], that for all dyadic $a \geqslant 0$, $D_n(\mathcal{O}) := 2D^2(\mathcal{O}, A_{an})$ converges to the same limit. Doob's maximal inequality then implies that there exists a modification of $2D^2(\mathcal{O}, A_t)$ that also converges to $\tilde{D}_\infty$ as $t \to \infty$.

These martingales are not uniformly integrable (U.I.) and thus, our previous techniques do not apply directly. However, we will discuss how to pass through certain U.I. martingales to get convergence in the case of $D_n(\mathcal{O})$. We will then use this case to show convergence for $\hat{D}_n(\mathcal{O})$. We remark that these U.I. martingales, given in the proof below, are similar but not exactly the same as the analogous U.I. martingales introduced in [DRSV14a, Aïd15].

We make $D_n(\mathcal{O})$ uniformly integrable via localization. To do this, let us introduce the following stopping times:

$$\tau_\beta := \inf_n \left\{ n \in \mathbb{N} : \inf_{z \in \mathbb{D} \setminus A_{an}} -h_{A_{an}}(z) + 2 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_{an}) \leq -\beta \right\}.$$

Notice that $A_{a(n\wedge\tau_\beta)}$ is then also a local set of Γ and we can define $D_n^\beta(\mathcal{O}) := 2D^2(\mathcal{O}, A_{a(n\wedge\tau_\beta)})$. As always, we include the factor 2 for comparison with martingales in [Aïd15]. Then $D_n^\beta(\mathcal{O})$ is a martingale, due to the fact that it is the derivative with respect to $-\gamma$ of the martingale $2M^\gamma(\mathcal{O}, A_a^{n\wedge\tau_\beta})$.

4.6.2.1 $D_n^\beta(\mathcal{O})$ is uniformly integrable

Let us first show the following claim:

Claim 4.16 *The martingale $D_n^\beta(\mathcal{O})$ is uniformly integrable for all $\beta \geqslant 0$ and so converges almost surely and in $\mathcal{L}^1$ to some limit $L(\mathcal{O}, \beta)$ as $n \to \infty$.*

Proof. Indeed, for $\eta > 0$, let $E_\eta(n, z)$ be the event that

$$\{-am + 2 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_a^m) \geq -\eta \text{ for all } m \leqslant n\}.$$

Then Proposition 3.2. of [Aïd15] implies that for all $\eta > 0$,

$$\bar{D}_n^\eta(\mathcal{O}) := \int_{\mathcal{O}} h_1 \left(-2an + 4 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_a^n) + 2\eta\right) 1_{E_\eta(n,z)} e^{2an} \, \mathrm{CR}(z; \mathbb{D} \setminus A_a^n)^2 \, dz \qquad (4.8)$$

is a U.I. martingale. Here $h_1(u)$ is a so-called renewal function, that satisfies $h_1(u) \geqslant Ru$ for some $R > 0$. We conclude that the stopped martingale $\bar{D}_{n\wedge\tau_\beta}^\eta(\mathcal{O})$ is also U.I. Given that

$$-h_{A_a^{n\wedge\tau_\beta}}(z) + \gamma \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_a^{n\wedge\tau_\beta}) \geqslant -a - \beta$$

we can bound

$$|D_n^\beta(\mathcal{O})| \leqslant R^{-1} |\bar{D}_{n\wedge\tau_\beta}^{2\beta+2a}|.$$

The claim follows. □

4.6.2.2 Comparison with Aïdekon's limit

We first show that our UI martingales converge to Aïdekon's limit, and then use this to treat the case where A^n is formed by iterating $A_{-a,a}$.

Claim 4.17 *The martingales $D_n^\beta(\mathcal{O})$ converge to $\tilde{D}_\infty(\mathcal{O})$ as first $n \to \infty$ and then $\beta \to \infty$.*

Proof. From the definition, $D_n^\beta(\mathcal{O})$ and $2D^2(\mathcal{O}, A_{an})$ are equal on the event $\{\tau_\beta = \infty\}$. Additionally, $\mathbb{P}(\tau_\beta = \infty)$ is equal to $1 - o(\beta)$, due to the fact that almost surely

$$\inf_{z \in \mathbb{D}} \inf_{n \in \mathbb{N}} \left(-2an + 4 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_{an})\right) > -\infty. \tag{4.9}$$

This is proved [Aïd15] after the statement of Proposition 3.2 (see also Remark 4.8 of the current chapter).

In particular, as $2D^2(\mathcal{O}, A_{an})$ tends to $\tilde{D}_\infty(\mathcal{O})$ by [Aïd15] (together with the comments after Proposition 4.15) we see that $\lim_{\beta\to\infty} L(\mathcal{O}, \beta)$ is also equal to $\tilde{D}_\infty(\mathcal{O})$ in this case. □

Now, let us come back to the case where A^n is formed by iterating $A_{-a,a}$. To do this, define $\hat{D}_n^\beta(\mathcal{O}) := D^2(\mathcal{O}, A^{n \wedge \hat{\tau}_\beta})$, for

$$\hat{\tau}_\beta := \inf_n \left\{ n \in \mathbb{N} : \inf_{z \in \mathbb{D} \setminus A^n} -h_{A^n}(z) + 2 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A^n) \le -\beta \right\}.$$

First, observe that when both A^n, the iterated $A_{-a,a}$, and A_{an} are coupled with the same GFF as local sets, Lemma 4.5 implies that a.s. $\{\hat{\tau}_\beta = \infty\} = \{\tau_\beta = \infty\}$ and that for all $n \leqslant m$, $A^{n\wedge\hat{\tau}_\beta} \subseteq A_{a(m\wedge\tau_\beta)}$. Thus, as long as the limit on the lefthand side exists, we have

$$\lim_{\beta\to\infty} \lim_{n\to\infty} \hat{D}_n^\beta(\mathcal{O}) = \lim_{n\to\infty} \hat{D}_n(\mathcal{O}).$$

So, it suffices to argue that

$$2\hat{D}_n^\beta(\mathcal{O}) \mathbf{1}_{\{\hat{\tau}_\beta = \infty\}} \to L(\mathcal{O}, \beta) \mathbf{1}_{\{\hat{\tau}_\beta = \infty\}}, \qquad \text{as } n \to \infty.$$

To see this we will use the strategy of the proof of Proposition 4.7.

Namely, consider a local set A with zero Lebesgue measure, and such that (Γ_A, f) is bounded from above by $K \geqslant 0$ (in the sense explained before Proposition 4.10). Then from an explicit calculation similar to the key claim of Proposition 4.7 we have that for any $\gamma > 0$, a.s. and in $\mathcal{L}^1(\Omega)$ as $\varepsilon \to 0$,

$$\mathbb{E}\left[\int_{\mathcal{O}} (\gamma \log(1/\varepsilon) - \Gamma_\varepsilon(z)) e^{\gamma \Gamma_\varepsilon(z)} \varepsilon^{\frac{\gamma^2}{2}} dz \mid \mathscr{F}_A\right] \to D^\gamma(\mathcal{O}, A).$$

Now, for $n \leqslant m$ let $\hat{\mathscr{F}}_n$ and $\mathscr{G}_m$ be the sigma-algebras corresponding to the local sets $A^{n\wedge\hat{\tau}_\beta}$ and $A_{a(m\wedge\tau_\beta)}$ respectively. Noting that a.s. $A^{n\wedge\hat{\tau}_\beta} \subseteq A_{a(m\wedge\tau_\beta)}$,

$$\begin{aligned}\frac{1}{2}\mathbb{E}\left[D_m^\beta(\mathcal{O}) \mid \hat{\mathscr{F}}_n\right] &= \mathbb{E}\left[\lim_{\varepsilon\to 0} \mathbb{E}\left[\int_{\mathcal{O}} (\gamma \log(1/\varepsilon) - \Gamma_\varepsilon(z)) e^{\gamma \Gamma_\varepsilon(z)} \varepsilon^{\frac{\gamma^2}{2}} dz \mid \mathscr{G}_m\right] \mid \hat{\mathscr{F}}_n\right] \\ &= \lim_{\varepsilon\to 0} \mathbb{E}\left[\int_{\mathcal{O}} (\gamma \log(1/\varepsilon) - \Gamma_\varepsilon(z)) e^{\gamma \Gamma_\varepsilon(z)} \varepsilon^{\frac{\gamma^2}{2}} dz \mid \hat{\mathscr{F}}_n\right] = \hat{D}_n^\beta(\mathcal{O}).\end{aligned}$$

Due to the fact that $D_m^\beta(\mathcal{O}) \to L(O, \beta)$ in L^1 we have that $\mathbb{E}\left[L(\mathcal{O}, \beta) | \hat{\mathscr{F}}_n\right] = 2\hat{D}_n^\beta(\mathcal{O})$ and so $2\hat{D}_n^\beta(\mathcal{O}) \to \mathbb{E}\left[L(\mathcal{O}, \beta) | \hat{\mathscr{F}}_\infty\right]$. However, the event $\{\hat{\tau}_\beta = \infty\}$ is $\hat{\mathscr{F}}_\infty$-measurable and on this event the limit of $A^{n\wedge\hat{\tau}_\beta}$ is $\mathbb{D}$ (by Lemma 4.3 and Remark 4.14). Similarly to Lemma 4.2, it then follows that $F(\Gamma) 1_{\{\hat{\tau}_\beta = \infty\}}$ is $\hat{\mathscr{F}}_\infty$ measurable for any measurable function of Γ. Thus, we have that $2\hat{D}_n^\beta(\mathcal{O}) \mathbf{1}_{\{\tau_\beta = \infty\}} \to L(\mathcal{O}, \beta) \mathbf{1}_{\{\tau_\beta = \infty\}}$, as required.

4.6.2.3 Comparison with mollified measures.

It remains to prove the latter claim of the proposition, i.e. to show that the limiting measure $\hat{D}_\infty = \frac{1}{2}\tilde{D}_\infty$ is equal to the measure μ_2' from [DRSV14b, Pow17], described in Section 4.3.3. We again mollify our measures using the circle average, and choose a sequence $\varepsilon_k \to 0$ such that $\nu_\varepsilon \to \mu_2'$ a.s. Whenever we write $\varepsilon \to 0$, it means that we are converging to 0 via $(\varepsilon_k)_{k\in\mathbb{N}}$. We set, for fixed $\mathcal{O} \subset \mathbb{D}$,

$$\nu_\varepsilon^\beta(\mathcal{O}) = \int_{\mathcal{O}} (-\Gamma_\varepsilon(z) + 2\log(1/\varepsilon))\mathbf{1}_{\{T_\beta(z)\leqslant\varepsilon\}}\,\mathrm{e}^{2\Gamma_\varepsilon(z)-2\log(1/\varepsilon)}$$

where $T_\beta(z) = \sup\{\varepsilon \leqslant \varepsilon_0 : 2\Gamma_\varepsilon(z) - 2\log(1/\varepsilon) \leqslant -\beta\}$ and ε_0 is some fixed starting point such that $\mathrm{dist}(z, \partial\mathbb{D}) > \varepsilon_0$ for all $z \in \mathcal{O}$. It is shown in [Pow17, Proposition 3.6] that $\nu_\varepsilon^\beta(\mathcal{O})$ is uniformly integrable for fixed $\beta \geqslant 0$. Additionally, define

$$C_\beta := \{-\Gamma_\varepsilon(z) + 2\log(1/\varepsilon) + \beta > 0 \text{ for all } z \in \mathbb{D}, 0 < \varepsilon \leqslant d(z, \partial\mathbb{D})\}$$

then $\mathbb{P}(C_\beta) = 1 - o(1)$ thanks to [HRV15, Theorem 6.3].

The strategy is to prove, for A_n the n-FPS of the GFF, that

$$\lim_{\beta\to\infty}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\mathbb{E}\left[\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathscr{F}_{A_n}\right]\mathbf{1}_{\{\tau_\beta=\infty\}} \tag{4.10}$$

is equal to both $\mu_2'(\mathcal{O})$ and $\frac{1}{2}\tilde{D}_\infty(\mathcal{O})$ almost surely.

Let us first show that (4.10) is equal to $\mu_2'(\mathcal{O})$. Observe that since $\nu_\varepsilon^\beta(\mathcal{O})$ is uniformly integrable, we have by Fatou's and reverse Fatou's lemma that, if the limit in ε exists (we will show that is does in the next step)

$$\mathbb{E}\left[\liminf_{\varepsilon\to 0}\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathscr{F}_{A_n}\right] \leqslant \lim_{\varepsilon\to 0}\mathbb{E}[\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathscr{F}_{A_n}] \leqslant \mathbb{E}\left[\limsup_{\varepsilon\to 0}\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathscr{F}_{A_n}\right].$$

Taking the limit as $n, \beta \to \infty$ we obtain that

$$\lim_{\beta\to\infty}\liminf_{\varepsilon\to 0}\nu_\varepsilon^\beta(\mathcal{O}) \leqslant \lim_{\beta\to\infty}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\mathbb{E}[\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathscr{F}_{A_n}] \leqslant \lim_{\beta\to\infty}\limsup_{\varepsilon\to 0}\nu_\varepsilon^\beta(\mathcal{O}).$$

However, since $\nu_\varepsilon^\beta(\mathcal{O}) = \nu_\varepsilon(\mathcal{O})$ on the event C_β, and almost surely $\mathbf{1}_{C_\beta} \uparrow 1$ as $\beta \to \infty$, the right and left hand sides of the above two expressions are equal to $\mu_2'(\mathcal{O})$. Since also almost surely $\mathbf{1}_{\{\tau_\beta=\infty\}} \to 1$ as $\beta \to \infty$, we deduce that (4.10) is equal to $\mu_2'(\mathcal{O})$.

We now show that (4.10) is equal to $\frac{1}{2}\tilde{D}_\infty(\mathcal{O})$. Write $\mathbb{E}[\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathscr{F}_{A_n}] := E^1(n,\beta,\varepsilon) + E^2(n,\beta,\varepsilon)$ where

$$E^1(n,\beta,\varepsilon) := \int_{\mathcal{O}\backslash A_n^\varepsilon}\mathbb{E}^{A_n}\left[(\Gamma_\varepsilon(z) + 2\log(1/\varepsilon))\mathbf{1}_{\{T_\beta(z)\leqslant\varepsilon\}}\,\mathrm{e}^{2\Gamma_\varepsilon(z)-2\log(1/\varepsilon)}\right]dz;$$

$$E^2(n,\beta,\varepsilon) := \int_{\mathcal{O}\cap A_n^\varepsilon}\mathbb{E}^{A_n}\left[(\Gamma_\varepsilon(z) + 2\log(1/\varepsilon))\mathbf{1}_{\{T_\beta(z)\leqslant\varepsilon\}}\,\mathrm{e}^{2\Gamma_\varepsilon(z)-2\log(1/\varepsilon)}\right]dz;$$

and $\mathbb{E}^{A_n}$ is the regular conditional expectation w.r.t. $\mathscr{F}_{A_n}$. Here we used that $\Gamma_\varepsilon(z) = (\Gamma_{A_n}, \rho_z^\varepsilon) + \Gamma_\varepsilon^{A_n}(z)$, where conditionally on $\mathscr{F}_{A_n}$ (i.e. under $\mathbb{P}^{A_n}$) Γ^{A_n} is a GFF in $\mathbb{D}\backslash A_n$. This implies that $\lim_{\varepsilon\to 0} E^2(n,\beta,\varepsilon) = 0$ almost surely. To see why, note that $(\Gamma_{A_n}, \rho_z^\varepsilon) \leqslant n$ and that when $\varepsilon \geqslant d(z, A_n)$, the variance of $\Gamma_\varepsilon^{A_n}(z)$ is uniformly bounded (independently of z and ε). This implies that the integrand is of order $\varepsilon^2 \log(1/\varepsilon)$ uniformly in z.

To deal with E^1, observe that if $\varepsilon \leqslant d(A_n, z)$ then $(\Gamma_{A_n}, \rho_z^\varepsilon) = n$. Additionally, due to the Markov property of the GFF and an explicit computation, we have that conditionally on $\mathscr{F}_{A_n}$, i.e., under the probability $\mathbb{P}^{A_n}$,

$$\left(-n - \Gamma_\delta^{A_n}(z) + 2\log(1/\delta)\right) \mathbf{1}_{\{T_\beta(z)\leqslant\delta\}}\, \mathrm{e}^{2n+2\Gamma_\delta^{A_n}(z)-2\log(1/\delta)}$$

is a (reverse) martingale for $0 < \delta \leqslant \delta_n(z) := d(z, \partial\mathbb{D} \setminus A_n)$. Thus, we have that $E^1(n,\beta,\varepsilon)$ is equal to

$$\int_{\mathcal{O}\setminus A_n^\varepsilon} \mathbb{E}^{A_n}\left[(-n - \Gamma_{\delta_n(z)}^{A_n}(z) + 2\log(1/\delta_n(z)))\mathbf{1}_{\{T_\beta(z)\leqslant\delta_n(z)\}}\, \mathrm{e}^{2n+2\Gamma_{\delta_n(z)}^{A_n}(z)-2\log(1/\delta_n(z))}\right] dz.$$

Since the integrand does not depend on ε, taking the limit in ε simply yields the integral over the whole of $\mathcal{O} \setminus A_n$.

Now, we rewrite $\lim_{\varepsilon\to 0} E^1(n,\beta,\varepsilon)$ as a difference between

$$\int_{\mathcal{O}\setminus A_n} \mathbb{E}^{A_n}\left[(-n - \Gamma_{\delta_n(z)}^{A_n}(z) + 2\log(1/\delta_n(z)))\, \mathrm{e}^{2n+2\Gamma_{\delta_n(z)}^{A_n}(z)-2\log(1/\delta_n(z))}\right] dz$$

and

$$\int_{\mathcal{O}\setminus A_n} \mathbb{E}^{A_n}\left[(-n - \Gamma_{\delta_n(z)}^{A_n}(z) + 2\log(1/\delta_n(z)))\mathbf{1}_{\{T_\beta(z)>\delta_n(z)\}}\, \mathrm{e}^{2n+2\Gamma_{\delta_n(z)}^{A_n}(z)-2\log(1/\delta_n(z))}\right] dz$$

Notice that the first of these terms is equal to $D_n(\mathcal{O})/2$. Let us further rewrite the second term. First, we use Girsanov's theorem. Since $\Gamma_{\delta_n(z)}^{A_n}$ is a normal random variable with mean 0 and variance $\log(1/\delta_n(z)) - \log(1/\operatorname{CR}(z, \mathbb{D} \setminus A_n))$, we see that this term is equal to

$$\int_{\mathcal{O}\setminus A_n} \mathrm{e}^{2n-2\log \operatorname{CR}^{-1}(z,\mathbb{D}\setminus A_n)}\, \tilde{\mathbb{E}}_z^{A_n}\left[\left(-n - \Gamma_{\delta_n(z)}^{A_n} + 2\log\left(1/\delta_n(z)\right)\right) \mathbf{1}_{\{T_\beta(z)>\delta_n(z)\}}\right] dz$$

where $\tilde{\mathbb{P}}_z^{A_n}$ is the measure under which the process $(\Gamma_\delta^{A_n}(z))_\delta$ has the same covariance structure as under $\mathbb{P}^{A_n}$ but with means shifted by $0 \leqslant \operatorname{Cov}(\Gamma_\delta^{A_n}(z), \Gamma_{\delta_n(z)}^{A_n}(z)) \leqslant 2$.

Next, we further decompose this as a sum of $E^3(n,\beta)$ and $E^4(n,\beta)$ with

$$E^3(n,\beta) := \int_{\mathcal{O}\setminus A_n} \mathrm{e}^{2n-2\log \operatorname{CR}^{-1}(z,\mathbb{D}\setminus A_n)}\; \tilde{\mathbb{E}}_z^{A_n}\left[\left(-\Gamma_{\delta_n(z)}^{A_n} + 2\log\left(\frac{\operatorname{CR}(z,\mathbb{D}\setminus A_n)}{\delta_n(z)}\right)\right) \mathbf{1}_{\{T_\beta(z)>\delta_n(z)\}}\right] dz$$

and

$$E^4(n,\beta) := \int_{\mathcal{O}\setminus A_n} \mathrm{e}^{2n-2\log \operatorname{CR}^{-1}(z,\mathbb{D}\setminus A_n)} \left(-n + 2\log \operatorname{CR}^{-1}(z, \mathbb{D}\setminus A_n)\right) \tilde{\mathbb{P}}_z^{A_n}\left(T_\beta(z) > \delta_n(z)\right) dz.$$

To bound E^3, we notice that $\Gamma_{\delta_n(z)}^n(z)$ has bounded variance under $\tilde{\mathbb{E}}_z^{A_n}$, and that the quantity $(\operatorname{CR}(z, \mathbb{D} \setminus A_n))/\delta_n(z)$ is uniformly bounded by the Koebe's 1/4-Theorem. This means that the whole expression is less than some constant times $M_n(\mathcal{O})$, which we know converges to 0 a.s. as $n \to \infty$.

For E^4 first note that the law of $\Gamma_\delta^{A_n}$ under $\tilde{\mathbb{P}}_z^{A_n}$ is equal to its law under $\mathbb{P}^{A_n}$ up to a shift that is uniformly bounded by 2. Thus we have that $\tilde{\mathbb{P}}_z^{A_n}(T_\beta(z) > \delta_n(z)) \leqslant \mathbb{P}^{A_n}(C^c_{\beta-2})$. Additionally, on the event $\{\tau_\beta = \infty\}$ we also have for all $z \in \mathcal{O}$

$$-n + 2\log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_n) + \beta \geqslant 0.$$

This implies that

$$|E^4(n,\beta)|\mathbf{1}_{\{\tau_\beta=\infty\}} \leqslant (|D_n(\mathcal{O})|/2 + \beta M_n(\mathcal{O}))\mathbb{E}[C^c_{\beta-2}|\mathscr{F}_{A_n}]\mathbf{1}_{\{\tau_\beta=\infty\}}.$$

But the limit of the RHS as $n \to \infty$ is equal to $\frac{1}{2}\tilde{D}_\infty(\mathcal{O})\mathbf{1}_{C^c_{\beta-2}}\mathbf{1}_{\{\tau_\beta=\infty\}}$. As this tends to 0 as $\beta \to \infty$ we conclude.

4.6.3 Seneta-Heyde rescaling.

Finally, we show that one can also perform a so-called Seneta-Heyde rescaling for the construction of the critical Liouville measure using local sets. While this result itself is of interest, one of the other main objectives of this section is a proof to demonstrate how simple it is in this framework to transfer techniques and methods from multiplicative cascades and branching random walks to the study of the Liouville measure. We plan to make further use of this in a follow-up paper. The proof in this section follows very closely that of [AS14], so we only give an outline, point to concrete analogies, and highlight some minor differences. It might be helpful to have the article [AS14] on the side, although we aimed to make the section readable on its own too.

Proposition 4.18 (Seneta-Heyde Rescaling) *For all $a \geqslant 0$, and Borelian $\mathcal{O} \subset \mathbb{D}$, we have that $\sqrt{an}M^2(\mathcal{O}, A_{an}) \to \frac{4}{\sqrt{\pi}}\mu_2'(O)$ in probability as $n \to \infty$. In particular the measures $\sqrt{an}M_n^2$ converge weakly in probability to $\frac{4}{\sqrt{\pi}}\mu_2'$.*

Again, by Remark 4.9, it suffices to prove the convergence statement for $\sqrt{an}M^2(\mathcal{O}, A_{an})$. For simplicity, we work in the case $a = 1$ and define $M_n := M^2(O, A_n)$. Before proving this proposition we need to define carefully a certain family $\hat{\mathbb{Q}}^\eta$ of *rooted measures*. Recall that if (Γ, Z) has the law $\hat{\mathbb{P}}^*(d\Gamma, dz)$ defined in (4.7), then the process

$$S_n := -2n + 4\log \mathrm{CR}^{-1}(Z, \mathbb{D} \setminus A_n)$$

is a random walk with mean-zero increments under the conditional law $\hat{\mathbb{P}}^*(d\Gamma|Z)$. Recall also the definition of $\bar{D}_n^\eta$ given in (4.8):

$$\bar{D}_n^\eta(\mathcal{O}) := \int_\mathcal{O} h_1\left(-2n + 4\log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_n) + 2\eta\right) 1_{E_\eta(n,z)} e^{2n}\,\mathrm{CR}(z; \mathbb{D} \setminus A_n)^2\, dz.$$

We already showed that $\bar{D}_n^\eta$ is a positive martingale with respect to $(\mathscr{F}_{A_n})_n$ and our initial probability measure $\mathbb{P}$. Hence we can define a new probability measure $\mathbb{Q}^\eta$ by setting it, when restricted to $\mathscr{F}_{A_n}$, to have Radon-Nikodym derivative $\bar{D}_n^\eta(\mathcal{O})/\bar{D}_0^\eta(\mathcal{O})$ with respect to $\mathbb{P}$.

Again we extend this to a *rooted* measure on the field Γ plus a distinguished point Z by setting $\hat{\mathbb{Q}}^\eta(d\Gamma, dz)$ restricted to $\mathscr{F}^*_{A_n} = \mathscr{F}_{A_n} \vee \sigma(Z)$ to be

$$h_1(-2n + 4\log \mathrm{CR}^{-1}(z, \mathbb{D}\setminus A_n) + 2\eta)\, \mathrm{e}^{2n-2\log \mathrm{CR}^{-1}(z,\mathbb{D}\setminus A_n)}\, \mathbf{1}_{E_\eta(z,n)} \frac{\mathbf{1}_{\mathcal{O}}(z)}{\bar{D}^\eta_0}\, dz\, \mathbb{P}[d\Gamma].$$

We make the following observations:

1. The marginal law of Z under $\hat{\mathbb{Q}}^\eta$ is proportional to
$$h_1(4\log \mathrm{CR}^{-1}(z,\mathbb{D}) + 2\eta)\, \mathrm{CR}(z,\mathbb{D})^2 \mathbf{1}_{\mathcal{O}}(z) dz.$$
2. The marginal law of the field Γ under $\hat{\mathbb{Q}}^\eta$ is given by $\mathbb{Q}^\eta$.
3. Write $\hat{\mathbb{Q}}^\eta_z = \hat{\mathbb{Q}}^\eta[\cdot \mid Z = z]$ for the law of Γ given the point $Z = z$. The law of the sequence $(A_n)_n$ under this measure can be described as follows. First sample A_1 with law weighted by
$$\frac{h_1(-2 + 4\log \mathrm{CR}^{-1}(z,\mathbb{D}\setminus A_1) + 2\eta)}{h_1(\eta + 4\log \mathrm{CR}^{-1}(z,\mathbb{D}))} \mathbf{1}_{E_\eta(z,1)}\, \mathrm{e}^{2-2\log \mathrm{CR}^{-1}(z,\mathbb{D}\setminus A_1)}.$$
Then given A_k for any $k \geqslant 1$, construct an independent copy of $(A_n)_n$ inside each component of $\mathbb{D}\setminus A_k$ that does *not* contain the point z. Inside the component containing z, let us call this $\mathcal{B}_k$, construct the components of $A_{k+1} \cap \mathcal{B}_k$ by weighting their laws by
$$\frac{h_1(-2(k+1) - 4\log \mathrm{CR}(z,\mathbb{D}\setminus A_{k+1}) + 2\eta)}{h_1(-2k - 4\log \mathrm{CR}(z,\mathbb{D}\setminus A_k) + 2\eta)} \mathbf{1}_{E_\eta(z,k+1)}\, \mathrm{e}^{2+2\log \mathrm{CR}(z,\mathbb{D}\setminus A_{k+1}) - 2\log \mathrm{CR}(z,\mathbb{D}\setminus A_k)}.$$
This defines the law of the sets A_n, and hence also by iteration the law of Γ.

It follows directly from the above construction that the law of $S_n = -2n + 4\log \mathrm{CR}(z,\mathbb{D}\setminus A_n)$ under $\hat{\mathbb{Q}}^\eta_z$ has the same as its law under $\hat{\mathbb{P}}^*[\cdot \mid Z = z]$, but conditioned to stay above -2η.

Now, we note some useful properties of the renewal function see for example [AS14, Section 2]):.

- First, recall from Claim 4.16 that $h_1(u) \geqslant Ru$ for all $u \geqslant 0$ for some positive R.
- By the renewal theorem, $c_0 := \lim_{u\to\infty}(h_1(u)/u)$ exists and lies in $(0,\infty)$.
- Let $\theta = 2/(\sqrt{\pi}c_0)$ (in the case of A_{an}, $\theta = 2/(\sqrt{\pi a}c_0)$). Then
$$\hat{\mathbb{P}}^*\left[\min_{1\leqslant i\leqslant n} S_i \geqslant -u \mid z\right] \sim \frac{\theta h_1(S_0 + u)}{\sqrt{n}} \tag{4.11}$$
as $n \to \infty$, for any $u \geqslant 0$. Moreover, the above holds uniformly in $u \in [0, b_n]$ for any sequence $b_n \in \mathbb{R}_+$ such that $\lim_{n\to\infty} b_n/\sqrt{n} = 0$.

Now set
$$\bar{M}^\eta_n(\mathcal{O}) := \int_{\mathcal{O}} e^{2n}\, \mathrm{CR}(x;\mathbb{D}\setminus A_n)^2 \mathbf{1}_{E_\eta(x,n)}\, dx.$$

Using the fact that $\bar{D}^\eta_n$ converges a.s. to a positive measure and that $\bar{D}^\eta_n = \bar{D}_n$ for η large enough, one can show, following exactly [AS14, Proof of Theorem 1.1], that in order to prove the Proposition 4.18 it suffices to establish the next claim:

Claim 4.19 *For any $\eta > 0$, $\sqrt{n}\bar{M}_n^\eta / \bar{D}_n^\eta \to \theta$ in $\mathbb{Q}^\eta$ probability as $n \to \infty$.*

Proof. The overall strategy follows very closely [AS14] and is to control the first and second moments of $\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta}$ as $n \to \infty$:

$$\hat{\mathbb{Q}}^\eta \left[\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta} \right] = \frac{\theta}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}} \right) \quad \text{and} \quad \hat{\mathbb{Q}}^\eta \left[\left(\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta} \right)^2 \right] \leqslant \frac{\theta^2}{n} + o\left(\frac{1}{n} \right) \tag{4.12}$$

These estimates prove the result by Jensen and Chebyshev's inequalities. The key observation lies in rewriting the moments using the rooted measure. Indeed, we can write for (Γ, Z) distributed under $\hat{\mathbb{Q}}^\eta$

$$\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta} = \hat{\mathbb{Q}}^\eta \left[\frac{1}{h_1(-2n + 4 \log \mathrm{CR}(Z, \mathbb{D} \setminus A_n) + 2\eta)} \mid \mathscr{F}_{A_n} \right] \tag{4.13}$$

and thus we have $\hat{\mathbb{Q}}^\eta[\bar{M}_n^\eta / \bar{D}_n^\eta] = \int_z \hat{\mathbb{Q}}_z^\eta \left[1/h_1(-2n + 4 \log \mathrm{CR}(z, \mathbb{D} \setminus A_n)) \right] d\hat{\mathbb{Q}}^\eta[dz]$. The first moment estimate (4.12) then follows easily using estimates on the renewal function, as in [AS14, proof of Proposition 4.1, Equation (4.1)].

We now move on to the second moment claim. Using random walk estimates and Jensen inequality, exactly as in [AS14, Lemmas 4.3-4.4], and (4.13), one can see that

$$\hat{\mathbb{Q}}_z^\eta \left[\left(\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta} \right)^2 \right] = \hat{\mathbb{Q}}_z^\eta \left[\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta} \frac{1}{h_1(-2n + 4 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_n) + 2\eta)} \right] = O\left(\frac{1}{n} \right). \tag{4.14}$$

Thus, to prove the second moment bound, it suffices to find a sequence of events E_n with $\hat{\mathbb{Q}}^\eta(E_n) \to 1$ as $n \to \infty$ so that

$$\hat{\mathbb{Q}}_z^\eta \left[\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta} \frac{\mathbf{1}_{E_n}}{h_1(-2n + 4 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_n) + 2\eta)} \right] \leqslant \frac{\theta^2}{n} + o\left(\frac{1}{n} \right) \tag{4.15}$$

holds uniformly in z. To do this, we pick a sequence $k_n \to \infty$ such that $k_n/\sqrt{n} \to 0$ and $k_n/(\log n)^6 \to \infty$ as $n \to \infty$ (the reason for this choice will become clear later). We then decompose $\bar{M}_n^\eta$ and $\bar{D}_n^\eta$ by writing

$$\bar{M}_n^\eta = \bar{M}_n^{\eta,[0,k_n]} + \bar{M}_n^{\eta,[k_n,n]}; \quad \bar{D}_n^\eta = \bar{D}_n^{\eta,[k_n,n]} + \bar{D}_n^{\eta,[0,k_n]}$$

where the superscript $[0, k_n]$ refers to the integral over $\mathcal{B}_{k_n}$ and the superscript $[k_n, n]$ refers to the integral over $\mathcal{O} \setminus \mathcal{B}_{k_n}$, where $\mathcal{B}_{k_n}$ is the connected component of $D \setminus A_n$ containing z.

We now define our sequence of events E_n by setting $E_n = E_n^1 \cap E_n^2$, where

$$E_n^1 := \{ \bar{D}_n^{\eta,[k_n,n]} \leqslant 1/n^2 \}; \quad E_n^2 = \{ S_{k_n} \in [k_n^{1/3}, k_n] \}.$$

Since under $\hat{\mathbb{Q}}_z^\eta$, S_n is a centered random walk conditioned to stay above -2η, it is clear at least that $\hat{\mathbb{Q}}^\eta [E_n^2] \to 1$ as $n \to \infty$. Putting aside the issue of whether or not $\hat{\mathbb{Q}}^\eta [E_n^1] \to 1$ for the moment, the next step is to bound (4.15) above by

$$\hat{\mathbb{Q}}_z^\eta \left[\frac{\bar{M}_n^{\eta,[k_n,n]}}{\bar{D}_n^\eta} \frac{\mathbf{1}_{E_n^1}}{h_1(-2n + 4 \log \mathrm{CR}^{-1}(z, \mathcal{B}_n) + 2\eta)} \right]$$
$$+ \hat{\mathbb{Q}}_z^\eta \left[\frac{\bar{M}_n^{\eta,[0,k_n]}}{\bar{D}_n^{\eta,[0,k_n]}} \frac{\mathbf{1}_{E_n^2}}{h_1(-2n + 4 \log \mathrm{CR}^{-1}(z, \mathcal{B}_n) + 2\eta)} \right].$$

Then, using that $R\bar{M}_n^{\eta,[k_n,n]} \leqslant \bar{D}_n^{\eta,[k_n,n]} \leqslant 1/n^2$, as in [AS14, Proof of Lemma 4.5], it can be deduced that the first term is $o(1/n)$. For the second term, we use that the two products in the expectation are conditionally independent given $\mathscr{F}^*_{A_{k_n}}$. We then have, by (4.11) and the assumption that $k_n/\sqrt{n} \to 0$, that

$$\mathbf{1}_{E_n^2}\hat{\mathbb{Q}}_z^\eta\left[h_1^{-1}(-2n + 4\log \mathrm{CR}^{-1}(z, \mathcal{B}_n) + 2\eta) \mid \mathscr{F}^*_{A_{k_n}}\right] = \theta/\sqrt{n} + o(1/\sqrt{n})$$

uniformly in ω and z. Since E_n^2 is $\mathscr{F}^*_{A_{k_n}}$ measurable, it therefore remains to prove that

$$\hat{\mathbb{Q}}_z^\eta\left[(\bar{M}_n^{\eta,[0,k_n]}/\bar{D}_n^{\eta,[0,k_n]})\mathbf{1}_{E_n^2}\right] \leqslant \theta/\sqrt{n} + o(1/\sqrt{n}). \tag{4.16}$$

This is a consequence of our first moment estimate and the fact that $\bar{M}_n^{\eta,[0,k_n]}/\bar{D}_n^{\eta,[0,k_n]}$ is comparable to $\bar{M}_n^\eta/\bar{D}_n^\eta$ on the event E_n^1. The details of this claim are exactly as in [AS14, Lemma 4.5].

Thus, to finish the prove the proposition, it remains to establish that $\hat{\mathbb{Q}}^\eta\left[E_n^1\right] \to 1$ as $n \to \infty$. In fact we need to prove a stronger slightly stronger statement to also deduce (4.16) as described above:

Lemma 4.20 *Suppose that $k_n/\sqrt{n} \to 0$ and $k_n/(\log n)^6 \to \infty$ as $n \to \infty$. Then there exists a deterministic sequence $p_n \nearrow 1$ such that $\mathbf{1}_{E_n^2}\hat{\mathbb{Q}}_z^\eta\left[E_n^1 \mid \mathscr{F}^*_{A_{k_n}}\right] \geqslant p_n$.*

It is only in the proof of this lemma that we need to do a bit of extra work. The extra work comes from the fact that, unlike in the case of multiplicative cascades, in our setting the sets at the $n-$th level have different shapes and sizes.

Proof of Lemma 4.20. Define further events E_n^3 and E_n^4 by setting

$$E_n^3 = \cap_{k_n\leqslant j\leqslant n}\{S_j \geqslant k_n^{1/6}\}; \quad E_n^4 = \cap_{k_n\leqslant j\leqslant n}\{\sup_{w\in\mathcal{B}_j}|z-w| \leqslant j^c\,\mathrm{CR}(z,\mathcal{B}_j)\}$$

where c is some fixed constant to be chosen just below (as in [Aïd15, Lemma 3.5]). We argue that:

(i) $\mathbf{1}_{E_n^2}\hat{\mathbb{Q}}_z^\eta\left[E_n^3 \mid \mathscr{F}^*_{A_{k_n}}\right] \geqslant p_n$, where $p_n \to 1$ is deterministic;

(ii) $\mathbf{1}_{E_n^2}\hat{\mathbb{Q}}_z^\eta\left[E_n^4 \mid \mathscr{F}^*_{A_{k_n}}\right] \geqslant q_n$, where $q_n \to 1$ is deterministic; and finally

(iii) $\hat{\mathbb{Q}}_z^\eta[\bar{D}_n^{\eta,[k_n,n]}\mathbf{1}_{E_n^3\cap E_n^4} \mid \mathscr{F}^*_{A_{k_n}}] \leqslant r_n$ where $r_n = o(1/n^2)$ is deterministic.

This proves the lemma by conditional Markov's inequality. For (i), one uses the fact that under the given conditional law, $(S_j - S_{k_n};\ j \geqslant S_{k_n})$ is a centered random walk conditioned to stay above $-S_{k_n} + 2\eta$. The details rely on estimates for the renewal function, and are as in [AS14, Proof of Lemma 4.7]. Claim (ii) follows from the proof of [Aïd15, Lemma 3.5]. This proof shows that, uniformly in z,

$$\hat{\mathbb{Q}}_z^\eta\left[\sup_{w\in\mathcal{B}_j}|z-w| > j^c\,\mathrm{CR}(z,\mathcal{B}_j) \mid \mathscr{F}^*_{A_{k_n}}\right] \leqslant c'\sqrt{j-k_n}\,j^{-c'c},$$

for some positive constant c' that does not depend on c (note the right-hand side is deterministic.) Choosing c large enough gives (ii). For (iii), we condition further on all the *brother loops* of the point z (that is, for each $k_n \leqslant j \leqslant n-1$, the components of $D \setminus A_j$ contained in

$\mathcal{B}_j$ but not $\mathcal{B}_{j+1}$). After applying this conditioning, and using the description of $(A_n)_n$ given after the statement of Proposition 4.18, we see that $\hat{\mathbb{Q}}_z^\eta \left[\bar{D}_n^{\eta,[k_n,n]} \mathbf{1}_{E_n^3 \cap E_n^4} \mid \mathscr{F}^*_{A_{k_n}}\right]$ is equal to

$$\sum_{j=k_n}^{n-1} s_j \, + \, \hat{\mathbb{Q}}_z^\eta \left[\int_{\mathcal{B}_n} h_1(-2n + 4\log \mathrm{CR}^{-1}(w,\mathcal{B}_n) + 2\eta)\, \mathrm{e}^{2n - 2\log \mathrm{CR}^{-1}(w,\mathcal{B}_n)}\, \mathbf{1}_{E_n^3 \cap E_n^4}\, dw \mid \mathscr{F}^*_{A_{k_n}}\right],$$

where s_j is

$$\begin{aligned}
&\hat{\mathbb{Q}}_z^\eta \left[\mathbf{1}_{E_n^3 \cap E_n^4} \int_{\mathcal{B}_j \setminus \mathcal{B}_{j+1}} h_1(4\log \mathrm{CR}^{-1}(w, \mathbb{D}\setminus A_{j+1}) + 2\eta - 2(j+1))\, \mathrm{e}^{2(j+1) - 2\log \mathrm{CR}^{-1}(w,\mathbb{D}\setminus A_{j+1})}\, dw \mid \mathscr{F}^*_{A_{k_n}}\right] \\
&\leqslant C \hat{\mathbb{Q}}_z^\eta \left[\mathbf{1}_{E_n^3 \cap E_n^4} \int_{\mathcal{B}_j \setminus \mathcal{B}_{j+1}} h_1(4\log \mathrm{CR}^{-1}(w,\mathcal{B}_j) + 2\eta - 2j)\, \mathrm{e}^{2j - 2\log \mathrm{CR}^{-1}(w,\mathcal{B}_j)}\, dw \mid \mathscr{F}^*_{A_{k_n}}\right],
\end{aligned}$$

for some constant $C \in \mathbb{R}$. The inequality here follows because $\mathrm{CR}(w, \mathbb{D}\setminus A_j)$ is decreasing in j and h_1 is bounded on either side by a linear function. Now, note that on the event $E_n^3 \cap E_n^4$, thanks to Koebe's theorem, $2j - 2\log CR^{-1}(w,\mathcal{B}_j)$ is smaller than $S_j + 2c\log(j) + 2\log \mathrm{CR}^{-1}(z,\mathcal{B}_j)$, and the area of each B_j is also $O(\mathrm{CR}(z,\mathcal{B}_j)^2)$. This means that each s_j is $O(\exp(-\sqrt[6]{k_n}/2) n^{4c+1})$, and the assumption that $k_n/(\log n)^6 \to \infty$ therefore implies (iii). □

□

Bibliography

[AF03] Robert A Adams and John JF Fournier. *Sobolev spaces*, volume 140. Academic press, 2003.

[Ahl10] Lars Valerian Ahlfors. *Conformal invariants: topics in geometric function theory*, volume 371. American Mathematical Society, 2010.

[Aïd15] Ellie Aïdékon. The extremal process in nested conformal loops. *Preprint available on the webpage of the author*, 2015.

[ALS17a] Juhan Aru, Titus Lupu, and Avelio Sepúlveda. First passage sets of the 2D continuum Gaussian free field. *arXiv preprint arXiv:1706.07737*, 2017.

[ALS17b] Juhan Aru, Titus Lupu, and Avelio Sepúlveda. On extremal distances of CLE_4. *In preparation*, 2017.

[APS17] Juhan Aru, Ellen Powell, and Avelio Sepúlveda. Approximating Liouville measure using local sets of the Gaussian free field. *arXiv preprint arXiv:1701.05872*, 2017.

[Aru15a] Juhan Aru. *Géométrie du champ libre Gaussien en relation avec les processus SLE et la formule KPZ*. PhD thesis, Lyon, École normale supérieure, 2015.

[Aru15b] Juhan Aru. KPZ relation does not hold for the level lines and SLE_κ flow lines of the Gaussian free field. *Probability Theory and Related Fields*, 163(3-4):465–526, 2015.

[AS14] Elie Aidekon and Zhan Shi. The Seneta–Heyde scaling for the branching random walk. *The Annals of Probability*, 42(3):959–993, 2014.

[AS16] Angelo Abächerli and Alain-Sol Sznitman. Level-set percolation for the Gaussian free field on a transient tree. *arXiv preprint arXiv:1606.02411*, 2016.

[AS17a] Juhan Aru and Avelio Sepúlveda. Survey in preparation. 2017.

[AS17b] Juhan Aru and Avelio Sepúlveda. Two-valued local sets of the Gaussian free field: Their connectivity, labels and induced metrics. *In preparation*, 2017.

[ASW17] Juhan Aru, Avelio Sepúlveda, and Wendelin Werner. On bounded-type thin local sets of the two-dimensional Gaussian free field. *Journal of the Institute of Mathematics of Jussieu*, pages 1–28, 2017.

[BBNY16] Roland Bauerschmidt, Paul Bourgade, Miika Nikula, and Horng-Tzer Yau. The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem. *arXiv preprint arXiv:1609.08582*, 2016.

[BC84] Jesse Randall Baxter and Rafael van Severen Chacon. The equivalence of diffusions on networks to Brownian motion. In *Contemporary Mathematics*, volume 26, pages 33–48. American Mathematical Society, 1984.

[BDG01] Erwin Bolthausen, Jean-Dominique Deuschel, and Giambattista Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic. *The Annals of Probability*, 29(4):1670–1692, 2001.

[Bef08] Vincent Beffara. The dimension of the SLE curves. *The Annals of Probability*, pages 1421–1452, 2008.

[Ber15a] Nathanaël Berestycki. An elementary approach to Gaussian multiplicative chaos. *arXiv preprint arXiv:1506.09113*, 2015.

[Ber15b] Nathanaël Berestycki. Introduction to the Gaussian free field and Liouville quantum gravity. *Lecture notes, available on author's webpage*, 2015.

[BL14] Marek Biskup and Oren Louidor. Conformal symmetries in the extremal process of two-dimensional discrete Gaussian free field. *arXiv preprint arXiv:1410.4676*, 2014.

[BLM87] Jean Bricmont, Joel L Lebowitz, and Christian Maes. Percolation in strongly correlated systems: the massless Gaussian field. *Journal of statistical physics*, 48(5):1249–1268, 1987.

[BSS14] Nathanaël Berestycki, Scott Sheffield, and Xin Sun. Equivalence of Liouville measure and Gaussian free field. *arXiv preprint arXiv:1410.5407*, 2014.

[Com13] Mark Comerford. The Carathéodory topology for multiply connected domains I. *Open Mathematics*, 11(2):322–340, 2013.

[CS11] Dmitry Chelkak and Stanislav Smirnov. Discrete complex analysis on isoradial graphs. *Advances in Mathematics*, 228(3):1590–1630, 2011.

[DKRV16] François David, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Liouville quantum gravity on the Riemann sphere. *Communications in Mathematical Physics*, 342(3):869–907, 2016.

[DMS14] Bertrand Duplantier, Jason Miller, and Scott Sheffield. Liouville quantum gravity as a mating of trees. *arXiv preprint arXiv:1409.7055*, 2014.

[DRSV14a] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Critical Gaussian multiplicative chaos: convergence of the derivative martingale. *The Annals of Probability*, 42(5):1769–1808, 2014.

[DRSV14b] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Renormalization of critical Gaussian multiplicative chaos and KPZ relation. *Communications in Mathematical Physics*, 330(1):283–330, 2014.

[DS11] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Inventiones mathematicae*, 185(2):333–393, 2011.

[Dub07] Julien Dubédat. Commutation relations for Schramm-Loewner evolutions. *Communications on Pure and Applied Mathematics*, 60(12):1792–1847, 2007.

[Dub09] Julien Dubédat. SLE and the free field: partition functions and couplings. *Journal of the American Mathematical Society*, 22(4):995–1054, 2009.

[Duf62] Richard Duffin. The extremal lenght of a network. *Journal of Mathematical*

Analysis and Applications, 5(2):200–215, 1962.

[Dur10] Rick Durrett. *Probability: theory and examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010.

[Dyn83] Eugene Dynkin. Markov processes as a tool in field theory. *Journal of Functional Analysis*, 50(2):167–187, 1983.

[Dyn84a] Eugene Dynkin. Gaussian and non-Gaussian random fields associated with Markov processes. *Journal of Functional Analysis*, 55(3):344–376, 1984.

[Dyn84b] Eugene Dynkin. Local times and quantum fields. In *Seminar on Stochastic Processes, Gainesville 1983*, volume 7 of *Progress in Probability and Statistics*, pages 69–84. Birkhauser, 1984.

[Eis95] Nathalie Eisenbaum. Une version sans conditionnement du théorème d'isomorphisme de Dynkin. In *Séminaire de Probabilités XXIX*, volume 1613 of *Lecture Notes in Mathematics*, pages 266–289. Springer-Verlag, Berlin, 1995.

[EK01] Nathanaël Enriquez and Yuri Kifer. Markov chains on graphs and Brownian motion. *Journal of Theoretical Probability*, 14(2):495–510, 2001.

[FR14] Patrick Fitzsimmons and Jay Rosen. Markovian loop soups: permanental processes and isomorphism theorems. *Electronic Journal of Probability*, 19(60):1–30, 2014.

[Gaw96] Krzysztof Gawedzki. Lectures on conformal field theory. *Nuclear Physics B*, 328:733–752, 1996.

[HK71] Raphael Høegh-Krohn. A general class of quantum fields without cut-offs in two space-time dimensions. *Communications in Mathematical Physics*, 21(3):244–255, 1971.

[HMP10] Xiaoyu Hu, Jason Miller, and Yuval Peres. Thick points of the gaussian free field. *The Annals of Probability*, 38(2):896–926, 2010.

[HRV15] Yichao Huang, Rémi Rhodes, and Vincent Vargas. Liouville quantum gravity on the unit disk. *arXiv preprint arXiv:1502.04343*, 2015.

[HS95] Zheng-Xu He and Oded Schramm. The inverse Riemann mapping theorem for relative circle domains. *Pacific Journal of Mathematics*, 171(1):157–165, 1995.

[IK13] Konstantin Izyurov and Kalle Kytölä. Hadamard's formula and couplings of SLEs with free field. *Probability Theory and Related Fields*, pages 1–35, 2013.

[Jan84] Svante Janson. Bounds on the distribution of extremal values of a scanning process. *Stochastic Processes and their Applications*, 18:313–328, 1984.

[JS15] Janne Junnila and Eero Saksman. The uniqueness of the gaussian multiplicative chaos revisited. *arXiv preprint arXiv:1506.05099*, 2015.

[Kah85] Jean-Pierre Kahane. Sur le chaos multiplicatif. *Annales des Sciences Mathématiques du Québec*, 9(2):105–150, 1985.

[Ken01] Richard Kenyon. Dominos and the Gaussian free field. *Annals of Probability*, pages 1128–1137, 2001.

[Kni63] Frank Knight. Random walks and a sojourn density process of Brownian motion. *Transactions of the American Mathematical Society*, 109(1):56–86, 1963.

[Koe20] Paul Koebe. Abhandlungen zur Theorie der konformen Abbildung. *Mathematische Zeitschrift*, 7(1):235–301, 1920.

[Koe22] Paul Koebe. Über die Konforme Abbildung Endlich-und Unendlich-Vielfach Zusammenhängender Symmetrischer Bereiche. *Acta Mathematica*, 43(1):263–287, 1922.

[Kyp00] Andreas E Kyprianou. Martingale convergence and the stopped branching random walk. *Probability theory and related fields*, 116(3):405–419, 2000.

[Law08] Gregory Lawler. *Conformally invariant processes in the plane*. Number 114. American Mathematical Society, 2008.

[LJ07] Yves Le Jan. Markov loops, determinants and Gaussian fields. *arXiv preprint arXiv:1012.4797*, 2007.

[LJ10] Yves Le Jan. Markov loops and renormalization. *The Annals of Probability*, 38(3):1280–1319, 2010.

[LJ11] Yves Le Jan. Markov paths, loops and fields. In *2008 St-Flour summer school, Lecture Notes in Mathematics*, volume 2026. Springer, 2011.

[LL10] Gregory Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123. Cambridge University Press, 2010.

[LSW03] Gregory Lawler, Oded Schramm, and Wendelin Werner. Conformal restriction: the chordal case. *Journal of the American Mathematical Society*, 16(4):917–955, 2003.

[LTF07] Gregory Lawler and José Trujillo Ferreras. Random walk loop soup. *Transactions of the American Mathematical Society*, 359(2):767–787, 2007.

[Lup15] Titus Lupu. Convergence of the two-dimensional random walk loop soup clusters to CLE. *arXiv preprint arXiv:1502.06827*, 2015. To appear in *Journal of European Mathematical Society.*

[Lup16a] Titus Lupu. From loop clusters and random interlacements to the free field. *The Annals of Probability*, 44(3):2117–2146, 2016.

[Lup16b] Titus Lupu. Loop percolation on discrete half-plane. *Electronic Communications in Probability*, 21(30), 2016.

[LW04] Gregory Lawler and Wendelin Werner. The Brownian loop soup. *Probability Theory and Related Fields*, 128(4):565–588, 2004.

[LW16] Titus Lupu and Wendelin Werner. The random pseudo-metric on a graph defined via the zero-set of the Gaussian free field on its metric graph. *arXiv preprint arXiv:1607.06424*, 2016. To appear in Probability Theory and Related Fields.

[Lyo97] Russell Lyons. A simple path to biggins' martingale convergence for branching random walk. In *Classical and modern branching processes*, pages 217–221. Springer, 1997.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*, volume 30. Cambridge University Press, 2010.

[MR06] Michael B. Marcus and Jay Rosen. *Markov processes, Gaussian processes and local times*, volume 100. Cambridge University Press, 2006.

[MS11] Jason Miller and Scott Sheffield. The GFF and CLE(4), 2011. Slides, talks and private communications.

[MS15] Jason Miller and Scott Sheffield. Liouville quantum gravity and the Brownian map I: The QLE (8/3, 0) metric. *arXiv preprint arXiv:1507.00719*, 2015.

[MS16a] Jason Miller and Scott Sheffield. Imaginary geometry I: interacting sles. *Probability Theory and Related Fields*, 164(3-4):553–705, 2016.

[MS16b] Jason Miller and Scott Sheffield. Imaginary geometry II: reversibility of $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (0,4)$. *The Annals of Probability*, 44(3):1647–1722, 2016.

[MS16c] Jason Miller and Scott Sheffield. Imaginary geometry III: reversibility of $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (4,8)$. *Annals of Mathematics*, 184(2):455–486, 2016.

[MS16d] Jason Miller and Scott Sheffield. Quantum Loewner evolution. *Duke Mathematical Journal*, 165(17):3241–3378, 2016.

[MS17] Jason Miller and Scott Sheffield. Imaginary geometry iv: interior rays, whole-plane reversibility, and space-filling trees. *Probability Theory and Related Fields*, pages 1–141, 2017.

[MSW16] Jason Miller, Scott Sheffield, and Wendelin Werner. CLE percolations. *arXiv preprint arXiv:1602.03884*, 2016.

[MW17] Jason Miller and Hao Wu. Intersections of SLE paths: the double and cut point dimension of SLE. *Probab. Theory Related Fields*, 167(1-2):45–105, 2017.

[NW11] Şerban Nacu and Wendelin Werner. Random soups, carpets and fractal dimensions. *Journal of the London Mathematical Society*, 83(3):789–809, 2011.

[Pit82] Loren Pitt. Positively correlated normal variables are associated. *The Annals of Probability*, 10(2):496–499, 1982.

[Pow17] Ellen Powell. Critical Gaussian chaos: convergence and uniqueness in the derivative normalisation. *arXiv preprint arXiv:1704.06058*, 2017.

[PS78] Sidney C. Port and Charles J. Stone. *Brownian motion and classical potential theory*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. Probability and Mathematical Statistics.

[PW15] Ellen Powell and Hao Wu. Level lines of the Gaussian free field with general boundary data. *arXiv preprint arXiv:1509.02462*, 2015. To appear in *Annales de l'Institut Henri Poincaré*.

[PW17] Eveliina Peltola and Hao Wu. Global and local multiple SLEs for $\kappa \leqslant 4$ and connection probabilities for level lines of GFF. *arXiv preprint arXiv:1703.00898*, 2017.

[QW15] Wei Qian and Wendelin Werner. Decomposition of Brownian loop-soup clusters. *arXiv preprint arXiv:1509.01180*, 2015.

[QW17] Wei Qian and Wendelin Werner. Coupling the Gaussian free fields with free and with zero boundary conditions via common level lines. *arXiv preprint arXiv:1703.04350*, 2017.

[Ray63] Daniel Ray. Sojourn times of diffusion processes. *Illinois Journal of mathematics*, 7(4):615–630, 1963.

[Rod15] Pierre-François Rodríguez. *On level set percolation for the Gaussian free field.* PhD thesis, 2015.

[Roz82] Yu A Rozanov. *Markov Random Fields.* Springer, 1982.

[RS13] Pierre-François Rodriguez and Alain-Sol Sznitman. Phase transition and level-set percolation for the Gaussian free field. *Comm. Math. Phys.*, 320(2):571–601, 2013.

[RV07] Brian Rider and Bálint Virág. The noise in the circular law and the gaussian free field. *International Mathematics Research Notices*, rnm006:1–32, 2007.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel Journal of Mathematics*, 118(1):221–288, 2000.

[Sch17] Lukas Schoug. A multifractal $\mathrm{SLE}_\kappa(\rho)$ boundary spectrum. *In preparation*, 2017.

[Sep17] Avelio Sepúlveda. On thin local sets of the Gaussian free field. *arXiv preprint arXiv:1702.03164*, 2017.

[Sha16] Alexander Shamov. On Gaussian multiplicative chaos. *Journal of Functional Analysis*, 270(9):3224–3261, 2016.

[She05] Scott Sheffield. Local sets of the Gaussian free field: Slides and audio, 2005.

[She07] Scott Sheffield. Gaussian free fields for mathematicians. *Probability theory and related fields*, 139(3):521–541, 2007.

[She09] Scott Sheffield. Exploration trees and conformal loop ensembles. *Duke Mathematical Journal*, 147(1):79–129, 2009.

[She16] Scott Sheffield. Conformal weldings of random surfaces: Sle and the quantum gravity zipper. *The Annals of Probability*, 44(5):3474–3545, 2016.

[Sim74] Barry Simon. *The $P(\Phi)_2$ Euclidean (Quantum) Field Theory.* Princeton Series in Physics. Princeton University Press, 1974.

[SS09] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field. *Acta Mathematica*, 202(1):21, 2009.

[SS13] Oded Schramm and Scott Sheffield. A contour line of the continuum Gaussian free field. *Probability Theory and Related Fields*, 157(1-2):47–80, 2013.

[SSW09] Oded Schramm, Scott Sheffield, and David B Wilson. Conformal radii for conformal loop ensembles. *Communications in Mathematical Physics*, 288(1):43–53, 2009.

[SW12] Scott Sheffield and Wendelin Werner. Conformal loop ensembles: the Markovian characterization and the loop-soup construction. *Annals of Mathematics*, 176(3):1827–1917, 2012.

[Sym69] Kurt Symanzik. Euclidean quantum field theory. In *Scuola intenazionale di Fisica "Enrico Fermi". XLV Corso*, pages 152–223. Academic Press, 1969.

[Szn12a] Alain-Sol Sznitman. An isomorphism theorem for random interlacements. *Electronic Commununications in Probability*, 17(9):1–9, 2012.

[Szn12b] Alain-Sol Sznitman. *Topics in occupation times and Gaussian free field*. Zurich lectures in advanced mathematics. European Mathemtical Society, 2012.

[Szn16] Alain-Sol Sznitman. Coupling and an application to level-set percolation of the gaussian free field. *Electronic Journal of Probability*, 21, 2016.

[vdBCL16] Tim van de Brug, Federico Camia, and Marcin Lis. Random walk loop soups and conformal loop ensembles. *Probability Theory and Related Fields*, 166(1):553–584, 2016.

[Wer03] Wendelin Werner. SLEs as boundaries of clusters of Brownian loops. *Comptes Rendus de l'Académie des Sciences de Paris*, 337:481–486, 2003.

[Wer05] Wendelin Werner. Conformal restriction and related questions. *Probability Surveys*, 2:145–190, 2005.

[Wer16] Wendelin Werner. Topics on the GFF and CLE(4), 2016.

[WW13a] Wendelin Werner and Hao Wu. From CLE(κ) to SLE(κ, ρ)'s. *Electronic Journal of Probability*, 18:1–20, 2013.

[WW13b] Wendelin Werner and Hao Wu. On conformally invariant CLE explorations. *Communications in Mathematical Physics*, 320:637–661, 2013.

[WW16] Menglu Wang and Hao Wu. Level lines of Gaussian free field I: zero-boundary GFF. *Stochastic Processes and their Applications*, 2016.