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Journal Article

Author(s):
Graf, Gian Michele; Vaghi, Alessio

Publication date:
2007-01

Permanent link:
https://doi.org/10.3929/ethz-b-000158618

Originally published in:
Letters in Mathematical Physics 79(1), https://doi.org/10.1007/s11005-006-0120-4

Letters in Mathematical Physics (2007) 79:17–22
DOI 10.1007/s11005-006-0120-4

A Remark on the Estimate of a Determinant by Minami

GIAN MICHELE GRAF and ALESSIO VAGHI
Theoretische Physik, ETH-Hönggerberg, Zürich 8093, Switzerland.
e-mail: gmgraf@itp.phys.ethz.ch; vaghial@student.ethz.ch

Received: 20 April 2006 ; revised version: 9 September 2006
Published online: 8 December 2006

Abstract. In the context of the Anderson model, Minami proved a Wegner type bound on the expectation of 2×2 determinant of Green's functions. We generalize it so as to allow for a magnetic field, as well as to determinants of higher order.

Mathematics Subject Classification (2000). 81V99, 81Q10.

Keywords. random Schrödinger operators, Anderson model, eigenvalue statistics.

1. Introduction

Minami [2] considered the Anderson model

$$H = -\Delta + V$$

acting on $\ell^2(\mathbb{Z}^d)$, where Δ is the discrete Laplacian and $V = \{V_x\}_{x \in \mathbb{Z}^d}$ consists of independent, identically distributed real random variables, whose common density ρ is bounded. He showed that in the localization regime the eigenvalues of the Hamiltonian restricted to a finite box $\Lambda \subset \mathbb{Z}^d$ are Poisson distributed if appropriately rescaled in the limit as Λ grows large. More precisely, the eigenvalue statistics near an energy $E \in \mathbb{R}$ is described by the point process

$$\xi(\Lambda; E)(\mathrm{d}x) = \sum_{j=1}^{|\Lambda|} \delta_{|\Lambda|(E_j - E)}(\mathrm{d}x), \tag{1}$$

where E_j are the eigenvalues of the Hamiltonian H_Λ on $\ell^2(\Lambda)$ obtained by truncating H to Λ through Dirichlet boundary conditions. For E in the localization regime Minami showed that $\xi(\Lambda; E)$ converges in law weakly to the Poisson point process $\xi(E)$ of intensity $n(E)\mathrm{d}x$,

$$\xi(\Lambda; E)(\mathrm{d}x) \xrightarrow[w]{\text{law}} \xi(E)(\mathrm{d}x), \quad (\Lambda \uparrow \mathbb{Z}^d), \tag{2}$$

where n is the density of states.

The result and, up to small changes, its proof also apply when the kinetic energy $-\Delta$ is replaced by a more general operator $K = K^*$ with a rapid off-diagonal decay of its matrix elements $K(x, y)$ in the position basis ($x, y \in \mathbb{Z}^d$), as long as

$$K(x, y) = K(y, x)\,. \tag{3}$$

Use of this property is made in the proof of Lemma 2 in [2], where H, and hence its resolvent $G(z) = (H - z)^{-1}$, is assumed symmetric: $G(z;\, x, y) = G(z;\, y, x)$, cf. Equations (2.68) and (2.75).

In physical terms Equation (3) states invariance of K, and hence of H, under time reversal, i.e., under the antiunitary operator on $\ell^2(\mathbb{Z}^d)$ given by complex conjugation. In fact invariance means $K(x, y) = \overline{K(x, y)}$, which is equivalent to (3), because K is self-adjoint. In particular, that condition entails the absence of an external magnetic field, and it may thus be desirable to dispense with it. This is achieved in this note. But first we recall Minami's Lemma 2. Let $\operatorname{Im} G = (G - G^*)/2\mathrm{i}$. Then [2]

$$\mathbb{E}\left[\det\begin{pmatrix}(\operatorname{Im} G)(z;\, x, x) & (\operatorname{Im} G)(z;\, x, y)\\ (\operatorname{Im} G)(z;\, y, x) & (\operatorname{Im} G)(z;\, y, y)\end{pmatrix}\right] \leqslant \pi^2 \|\rho\|_\infty^2\,, \tag{4}$$

for $x \neq y$ and $\operatorname{Im} z > 0$, and the same applies to the Hamiltonian H_Λ with $\Lambda \ni x, y$ in place of H.

Equation (4) should be compared with the Wegner bound $\mathbb{E}((\operatorname{Im} G)(z;\, x, x)) \leqslant \pi \|\rho\|_\infty$, which implies that the expected number of eigenvalues in an interval I is bounded by $\|\rho\|_\infty |I||\Lambda|$. Similarly (4) implies that the expected number of pairs of eigenvalues is bounded by $(\|\rho\|_\infty |I||\Lambda|)^2$. These bounds enter the proof of (2) as follows. The scaling of eigenvalues seen in (1) amounts to keeping $|I||\Lambda|$ constant. For large $|\Lambda|$, localization allows to approximate the point process $\xi(\Lambda; E)$ by $\sum_{k=1}^N \xi(C_k; E)$, where the box Λ has been broken into $N \gg 1$ cubes C_k, yet each of volume $|\Lambda|/N \gg 1$. Since the first bound is linear in $|\Lambda|$, the total number of scaled eigenvalues remains of order 1. However, the number of pairs of eigenvalues, both coming from a same cube, is $N \cdot \mathrm{O}(N^{-2}) = \mathrm{O}(N^{-1})$. Different eigenvalues thus effectively come from different cubes and are hence independent, leading to the Poisson distribution. For details, see [2].

Because of $G^*(z;\, x, y) = \overline{G(z;\, y, x)}$, the above matrix element

$$(\operatorname{Im} G)(z;\, x, y) = \frac{G(z;\, x, y) - \overline{G(z;\, y, x)}}{2\mathrm{i}} \tag{5}$$

agrees with $\operatorname{Im}(G(z;\, x, y))$ only if the symmetry (3) is assumed, which we shall not do here. Then the agreement is limited to $x = y$. For the sake of clarity we remark that it is the operator interpretation (5) of $\operatorname{Im} G$, and not the one in the sense of matrix elements, which makes (4) true and useful in the general case.

The core of the argument is contained in the following

LEMMA 1. Let $A=(a_{ij})_{i,j=1,2}$ with $\operatorname{Im} A>0$. Then

$$\int dv_1 dv_2 \det\big(\operatorname{Im}[\operatorname{diag}(v_1,v_2)-A]^{-1}\big)$$
$$=\pi^2 \frac{\det \operatorname{Im} A}{\sqrt{(\det \operatorname{Im} A)^2+\frac{1}{2}(\det \operatorname{Im} A)(|a_{12}|^2+|a_{21}|^2)+\frac{1}{16}(|a_{12}|^2-|a_{21}|^2)^2}}. \tag{6}$$

The right-hand side is trivially bounded by π^2, since $\det \operatorname{Im} A>0$. In [2], Equations (2.72) and (2.74), the equality was established in the special case $a_{12}=a_{21}$. It was applied to

$$-\big(A^{-1}\big)(u,v):=(\widehat{H}-z)^{-1}(u,v)\,, \quad (u,v=x,y),$$

where $\widehat{H}$ is H with V_x and V_y set equal to zero. With the so defined 2×2 matrix A the two matrices under “det Im” in (4) and on the left-hand side of (6) agree, a fact known as Krein’s formula. That A is actually well defined and satisfies $\operatorname{Im} A>0$ is seen from $\operatorname{Im}(z-\widehat{H})=\operatorname{Im} z>0$ and the following remarks [3], which apply to any complex $n\times n$ matrix C:

(i) $$\operatorname{Im} C>0 \Longleftrightarrow \operatorname{Im}(-C^{-1})>0\,. \tag{7}$$
Indeed, C is invertible, since otherwise $Cu=0$ for some $0\neq u\in\mathbb{C}^n$, implying $(u,(\operatorname{Im} C)u)=\operatorname{Im}(u,Cu)=0$, contrary to our assumption. Moreover, $\operatorname{Im}(-C^{-1})=C^{-1*}(\operatorname{Im} C)C^{-1}$. The converse implication is because $C\mapsto(-C)^{-1}$ is an involution.

(ii) $$\operatorname{Im} C>0 \Longrightarrow \operatorname{Im} \widehat{C}>0\,, \tag{8}$$
where $\widehat{C}$ is the restriction of C to a subspace, as a sesquilinear form. In fact, $\operatorname{Im}\widehat{C}=\widehat{\operatorname{Im} C}$.

A more qualitative understanding of the bound π^2 for (6) may be obtained from its generalization to $n\times n$ matrices:

LEMMA 2. Let $A=(a_{ij})_{i,j=1,\dots,n}$ with $\operatorname{Im} A>0$. Then

$$\int dv_1\cdots dv_n \det\big(\operatorname{Im}[\operatorname{diag}(v_1,\dots,v_n)-A]^{-1}\big)\leqslant\pi^n\,.$$

As a result, Equation (4) also generalizes to the corresponding determinant of order n.

2. Proofs

Proof of Lemma 1. Following [2] we will use that

$$\int dx\,\frac{1}{|ax+b|^2}=\frac{\pi}{\operatorname{Im}(\bar{b}a)}\,, \quad (a,b\in\mathbb{C},\ \operatorname{Im}(\bar{b}a)>0) \tag{9}$$

and

$$\int \mathrm{d}x \, \frac{1}{ax^2+bx+c} = \frac{2\pi}{\sqrt{\Delta}}, \quad (a>0,\ b, c\in\mathbb{R},\ \Delta := 4ac-b^2>0)\,. \tag{10}$$

We observe that

$$\det \operatorname{Im} A = (\operatorname{Im} a_{11})(\operatorname{Im} a_{22}) - \frac{1}{4}|a_{12}-\overline{a_{21}}|^2\,, \tag{11}$$

and hence the right-hand side of (6), do not depend on $\operatorname{Re} a_{ii}$, $(i=1,2)$. Similarly the left hand side, by a shift of integration variables. We may thus assume $\operatorname{Re} a_{ii} = 0$. The matrix on the left-hand side of (6) is

$$\operatorname{Im}\big[\operatorname{diag}(v_1,v_2)-A\big]^{-1} = \big(A^*-\operatorname{diag}(v_1,v_2)\big)^{-1}(\operatorname{Im} A)\big(A-\operatorname{diag}(v_1,v_2)\big)^{-1}\,, \tag{12}$$

and its determinant equals

$$\det \operatorname{Im} A\cdot\big|\det\big(A-\operatorname{diag}(v_1,v_2)\big)\big|^{-2} = \det\operatorname{Im} A\cdot\big|(v_1-a_{11})(v_2-a_{22})-a_{12}a_{21}\big|^{-2}\,. \tag{13}$$

The v_2-integration of the second factor (13) is of the type (9) with $a=v_1-a_{11}$ and $b=(a_{11}-v_1)a_{22}-a_{12}a_{21}$. Then

$$\begin{aligned}\operatorname{Im}(\overline{b}a) &= (\operatorname{Im} a_{22})|v_1-a_{11}|^2+\operatorname{Im}(a_{12}a_{21})\big(v_1-\operatorname{Re} a_{11}\big)+\operatorname{Re}(a_{12}a_{21})(\operatorname{Im} a_{11})\\ &= (\operatorname{Im} a_{22})v_1^2+\operatorname{Im}(a_{12}a_{21})v_1+(\operatorname{Im} a_{22})(\operatorname{Im} a_{11})^2+\operatorname{Re}(a_{12}a_{21})(\operatorname{Im} a_{11})\,.\end{aligned} \tag{14}$$

By (10), the v_1-integral is obtained by computing the discriminant Δ of this quadratic function:

$$\int \mathrm{d}v_1\mathrm{d}v_2\,\big|\det\big(A-\operatorname{diag}(v_1,v_2)\big)\big|^{-2} = \frac{2\pi^2}{\sqrt{\Delta}}\,, \tag{15}$$

$$\begin{aligned}\Delta &= 4(\operatorname{Im} a_{11}\operatorname{Im} a_{22})^2+4(\operatorname{Im} a_{11}\operatorname{Im} a_{22})\operatorname{Re}(a_{12}a_{21})-\big(\operatorname{Im}(a_{12}a_{21})\big)^2\\ &= \big(2\operatorname{Im} a_{11}\operatorname{Im} a_{22}+\operatorname{Re}(a_{12}a_{21})\big)^2-|a_{12}a_{21}|^2\,.\end{aligned}$$

In doing so we tacitly assumed that Δ, and hence (14), are positive. This is indeed so, because $\Delta\leqslant 0$ would imply that $A-\operatorname{diag}(v_1,v_2)$ is singular for some $v_1,v_2\in\mathbb{R}$, which contradicts $\operatorname{Im} A>0$, cf. (7). It also follows because $\Delta/4$ equals the expression under the root in (6), a claim we need to show anyhow: from (11) and

$$|a_{12}-\overline{a_{21}}|^2 = |a_{12}|^2+|a_{21}|^2-2\operatorname{Re}(a_{12}a_{21})$$

we obtain

$$2\operatorname{Im} a_{11}\operatorname{Im} a_{22}+\operatorname{Re}(a_{12}a_{21}) = 2\det\operatorname{Im} A+\frac{1}{2}\big(|a_{12}|^2+|a_{21}|^2\big)$$

and hence

$$\Delta = 4(\det \operatorname{Im} A)^2 + 2\big(|a_{12}|^2 + |a_{21}|^2\big)(\det \operatorname{Im} A) + \frac{1}{4}\big(|a_{12}|^2 + |a_{21}|^2\big)^2 - |a_{12}a_{21}|^2.$$

Since the last two terms equal $(|a_{12}|^2 - |a_{21}|^2)^2/4$, we establish the claim and, by Equations (13) and (15), the lemma. □

Proof of Lemma 2. By induction in n. It may start with $n=0$, in which case the determinant is 1 by natural convention, or with $n=1$, where the claim, i.e.,

$$\int \mathrm{d}v \; \operatorname{Im}\Big(\frac{1}{v-a}\Big) \leqslant \pi\,, \quad (\operatorname{Im} a > 0)\,,$$

is easily seen to hold as an equality. We maintain the induction step

$$\int \mathrm{d}v_n \; \det\big(\operatorname{Im}[\operatorname{diag}(v_1,\ldots,v_n) - A]^{-1}\big) \leqslant \pi \det\big(\operatorname{Im}[\operatorname{diag}(v_1,\ldots,v_{n-1}) - B]^{-1}\big)$$

for some $(n-1)\times(n-1)$ matrix B with $\operatorname{Im} B > 0$. This is actually a special case of

$$\int \mathrm{d}v \; \det\big(\operatorname{Im}[\operatorname{diag}(0,\ldots,0,v) - A]^{-1}\big) \leqslant \pi \det \operatorname{Im}(-B)^{-1}\,, \tag{16}$$

where B is the Schur complement of a_{nn}, given as

$$B = \widehat{A} - a_{nn}^{-1}(a_V \otimes a_H) \tag{17}$$

in terms of the $(n-1,1)$-block decomposition of an $n\times n$ matrix:

$$C = \begin{pmatrix} \widehat{C} & c_V \\ c_H & c_{nn} \end{pmatrix}.$$

By a computation similar to (12) the integrand in (16) is

$$\begin{aligned} \frac{\det \operatorname{Im} A}{\big|\det\big(A - \operatorname{diag}(0,\ldots,0,v)\big)\big|^2} &= \frac{\det \operatorname{Im} A}{\big|\det A - v \det \widehat{A}\big|^2} \\ &= \frac{\det \operatorname{Im} A}{|\det A|^2 \big|1 - v(A^{-1})_{nn}\big|^2} = \frac{\det \operatorname{Im}(-A^{-1})}{\big|1 - v(A^{-1})_{nn}\big|^2}\,. \end{aligned}$$

In the first line we used that $v \in \mathbb{R}$ and that the determinant is linear in the last row; in the second that

$$(C^{-1})_{nn} \cdot \det C = \det \widehat{C}\,. \tag{18}$$

By (9) the integral is π times

$$\frac{\det \operatorname{Im}(-A^{-1})}{\operatorname{Im}(-A^{-1})_{nn}} = \frac{\det \operatorname{Im}(-A^{-1})}{\big(\operatorname{Im}(-A^{-1})\big)_{nn}} \leqslant \det\big[\widehat{\operatorname{Im}(-A^{-1})}\big] = \det \operatorname{Im}\big[-\widehat{A^{-1}}\big]\,,$$

where the estimate is by applying

$$\det C \leqslant c_{nn} \cdot \det \widehat{C}\,, \quad (C > 0)$$

to $C = \mathrm{Im}(-A^{-1})$, cf. (7). This inequality is by Cauchy's for the sesquilinear form C: letting $\delta_n = (0, \dots, 0, 1)$,

$$1 = (C^{-1}\delta_n, C\delta_n)^2 \leqslant (C^{-1}\delta_n, CC^{-1}\delta_n) \cdot (\delta_n, C\delta_n) = (C^{-1})_{nn} \cdot c_{nn}\,,$$

cf. (18). Finally, $\widehat{A^{-1}}$ may be computed by means of the Schur (or Feshbach) formula [1, p. 18]: $\widehat{A^{-1}} = B^{-1}$ with B as in (17). Note that the left-hand side is invertible because of $\mathrm{Im}(-\widehat{A^{-1}}) > 0$, cf. (7) and (8), and that it is the inverse of a matrix with positive imaginary part. □

Note added in proof: After completion of this work, we learnt about a different proof (J. Bellissard et al., in preparation) of Lemma 2.

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