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Factorization Homology as a Fully Extended Topological Field Theory

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Abstract

Given an E_n-algebra A we explicitly construct a fully extended n-dimensional topological field theory which is essentially given by factorization homology. Under the cobordism hypothesis, this is the fully extended n-TFT corresponding to the E_n-algebra A, considered as an object in a suitable Morita-(∞, n)-category Alg_n. We first give a precise definition of a fully extended n-dimensional topological field theory using complete n-fold Segal spaces as a model for (∞, n)-categories. This involves developing an n-fold Segal space Bord_n of n-dimensional bordisms and endowing it with a symmetric monoidal structure. Exploiting the equivalence between E_n-algebras and locally constant factorization algebras proven by Lurie we use locally constant factorization algebras on stratified spaces to construct an (∞, n)-category with E_n-algebras as objects, (pointed) bimodules as 1-morphisms, (pointed) bimodules between bimodules as 2-morphisms, etc. and endow it with a symmetric monoidal structure. Finally, given an E_n-algebra we construct a morphism of n-fold Segal spaces from Bord_n to Alg_n given by a suitable pushforward of the factorization algebra obtained by taking factorization homology. We show that this map respects the symmetric monoidal structure and thus is a fully extended n-TFT.

Zusammenfassung

Gegeben eine E_n-Algebra A konstruieren wir in expliziter Weise eine vollständig erweiterte n-dimensionale topologische Feldtheorie, die im Wesentlichen durch Faktorisierungshomologie gegeben ist. Unter Verwendung der Kobordismus-Hypothese entspricht diese der vollständig erweiterten n-TFT, die durch die E_n-Algebra A, als Objekt einer geeigneten Morita-(∞, n)-Kategorie Alg_n betrachtet, bestimmt ist. Als Modell für (∞, n)-Kategorien benutzen wir vollständige n-fache Segalräume und geben zunächst eine präzise Definition einer vollständig erweiterten n-dimensionalen topologischen Feldtheorie. Diese benötigt die Konstruktion eines n-fachen Segalraumes n-dimensionaler Bordismen Bord_n und einer symmetrisch monoidalen Struktur darauf. Motiviert durch die Äquivalenz zwischen E_n-Algebren und lokal konstanten Faktorisierungsalgebren, die von Lurie bewiesen wurde, verwenden wir lokal konstante Faktorisierungsalgebren auf stratifizierten Räumen um eine (∞, n)-Kategorie, deren Objekte E_n-Algebren, 1-Morphismen (punktierte) Bimoduln, 2-Morphismen (punktierte) Bimoduln zwischen Bimoduln, etc. sind, und eine symmetrisch monoidalen Struktur darauf zu definieren. Schließlich konstruieren wir, in Abhängigkeit einer E_n-Algebra, einen Morphismus n-facher Segalräume von Bord_n nach Alg_n, der durch einen gewissen Pushforward der Faktorisierungsalgebra, die mittels Faktorisierungshomologie erhalten wird, gegeben ist. Wir zeigen, dass diese Abbildung die symmetrisch monoidale Struktur respektiert und daher eine vollständig erweiterte n-TFT ist.

Résumé

Étant donné une algèbre E_n, nous construisons explicitement une théorie des champs topologiques pleinement étendue de dimension n, essentiellement donnée par l'homologie de factorisation. D'après l'Hypothèse du Cobordisme il s'agit de la n-TFT pleinement étendue qui correspond à l'algèbre E_n A, considérée comme un objet dans une (∞, n)-catégorie appropriée de Morita Alg_n. Nous donnons dans un premier temps une définition précise d'une théorie des champs topologiques pleinement étendue de dimension n en utilisant les espaces de Segal complets n-uples comme un modèle pour les (∞, n)-catégories. Pour cela nous construisons un espace de Segal complet n-uple $Bord_n$ de bordismes de dimension n et lui donnons une structure monoïdale symétrique. En exploitant ensuite l'équivalence, démontrée par Lurie, entre les algèbres E_n et les algèbres de factorisation localement constantes, nous utilisons des algèbres de factorisation localement constantes sur des espaces stratifiés pour construire une (∞, n)-catégorie ayant les algèbres E_n pour objets, les bimodules (pointés) pour 1-morphismes, les bimodules entre bimodules pour 2-morphismes, etc... lui donnons une structure monoïdale symétrique. Finalement, étant donné une algèbre E_n, nous construisons un morphisme entre espaces de Segal n-uples depuis Bord_n vers Alg_n, donné par un pushforward de l'algèbre de factorisation obtenue par l'homologie de factorisation. Nous montrons que cette construction préserve la structure monoïdale symétrique et donc ce morphisme est une n-TFT pleinement étendue.

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Introduction

Motivation

Topological field theories

Topological field theories (TFTs) arose as toy models for physical quantum field theories and have proven to be of mathematical interest, notably because they are a fruitful tool for studying topology. Inspired by Witten's paper [Wit82] relating supersymmetry and Morse theory and Segal's axioms of conformal field theories in [Seg04], they were first axiomatized by Atiyah in [Ati88]. An n-dimensional TFT is a symmetric monoidal functor from the category of bordisms, which has closed $(n-1)$-dimensional manifolds as objects and n-dimensional bordisms as morphisms, to any other symmetric monoidal category, which classically is taken to be the category of vector spaces or chain complexes. In particular it assigns topological invariants to closed n-dimensional manifolds, which has turned out to be very useful in the study of low-dimensional topology. Early results by Witten in [Wit89] showed that the Jones polynomial of knot theory arises from 3-dimensional Chern-Simons theory, which is a TFT. Interesting 4-dimensional examples are Donaldson invariants of 4-dimensional manifolds which arise from a twisted 4-dimensional supersymmetric gauge theory, [Wit88], and the related Seiberg-Witten invariants [Wit94, SW94a, SW94b].

A classification of 1- and 2-dimensional TFTs follows from classification theorems for 1- and 2-dimensional compact manifolds with boundary. In the 1-dimensional case, a 1-TFT is fully determined by its value at a point, which is a dualizable object in the target category and conversely, every dualizable object in the target gives rise to a 1-TFT. In the 2-dimensional case, a classification, given by the value at a circle, was proven by Abrams in [Abr96]. The question of a classification result for larger values of n appears naturally and raises the question of a suitable replacement of the classification of compact n-manifolds with boundary used in the low-dimensional cases. In [BD95], Baez and Dolan explain the need for higher categories of cobordisms for a classification of n-dimensional extended topological field theories. Here extended means that we need to be able to evaluate the n-TFT not only at n- and $(n-1)$-dimensional manifolds, but also at $(n-2)$-,..., 1-, and 0-dimensional manifolds. In light of the hope of computability of the invariants determined by an n-TFT, e.g. by a triangulation, it is natural to include this data. Furthermore, Baez and Dolan conjectured that, similarly to the 1-dimensional case, extended n-TFTs are fully determined by their value at a point, calling this the cobordism hypothesis. A proof of a classification theorem of extended TFTs for dimension 2 and in particular a definition of a suitable bicategory of 2-cobordisms was given in [SP09].

In his expository manuscript [Lur09c], Lurie suggested passing to (∞, n)-categories for a proof of the cobordism hypothesis in arbitrary dimension n and gave a detailed sketch of such a proof using a suitable (∞, n)-category of cobordisms, which, informally speaking, has zero-dimensional manifolds as objects, bordisms between objects as 1-morphisms, bordisms between bordisms as 2-morphisms, etc., and for $k > n$ there are only invertible k-morphisms. Finding an explicit model for such a higher category poses one of the difficulties in rigorously defining these n-dimensional TFTs, which are called "fully extended". His result shows that evaluation at a point gives a bijection, or more precisely an equivalence of ∞-groupoids, between (isomorphism classes of) fully extended n-TFTs with values in a target symmetric monoidal (∞, n)-category $\mathcal{C}$ and (isomorphism classes of) "fully dualizable" objects in $\mathcal{C}$. Thus any fully dualizable object in the target category determines a fully extended n-TFT. Full dualizability is a finiteness condition depending on the top dimension n which generalizes the condition of being a dualizable object in the 1-dimensional case.

Factorization homology and factorization algebras

Inspired by Segal's approach to conformal field theories in [Seg04] and Atiyah's axioms for TFTs mentioned above, there have been several approaches to describe (topological) quantum field theories in an axiomatic way. Factorization homology and factorization algebras are two such approaches which were developed and studied by many people, among them Beilinson-Drinfeld, Costello-Gwilliam, Francis, and Lurie.

Factorization homology, also called topological chiral homology, was first defined by Jacob Lurie in [Lur]. It is a homology theory for topological manifolds satisfying a generalization of the Eilenberg-Steenrod axioms for ordinary homology, see [Fra12, AFT12]. The construction depends on the data of an E_n-algebra in a suitable symmetric monoidal $(\infty, 1)$-category $\mathcal{S}$, which is an algebra in $\mathcal{S}$ for the operad E_n, which in turn is equivalent to the little cubes operad in dimension n. In the case $n = 1$, E_1-algebras are equivalent to associative algebras up to homotopy, i.e. A_∞-algebras, and in the case of $n = 2$, E_2-algebras in the category of categories are braided monoidal categories. In the special case that $\mathcal{S}$ is the $(\infty, 1)$-category of chain complexes, any commutative differential graded algebra A is in particular also an E_n-algebra and it was shown in [GTZ10] that in this case factorization homology recovers the (higher) Hochschild homology of A. Factorization homology for n-dimensional manifolds with boundary yields an n-TFT, as was shown by Ayala-Francis-Tanaka in [AFT12] and, with different techniques, by Horel in [Hor14b]. Moreover, Lurie stated in [Lur09c] that it should lead to a fully extended n-TFT.

Factorization algebras are algebraic structures encoding the structure of the observables of a quantum field theory (henceforth QFT), as was shown in [CG] for perturbative QFTs. One can think of them as a multiplicative, non-commutative version of cosheaves and they turn out to be a tool useful for describing well-known algebraic structures such as E_n-algebras ([Lur09c]) and bimodules between algebras ([Gin]). Factorization algebras and factorization homology are related in a local-to-global way: in [GTZ10] it was shown that considering factorization homology locally on a given manifold M yields a factorization algebra on M whose global sections are the factorization homology of M.

Overview of the thesis

Lurie's cobordism hypothesis gives a "recipe" for producing a fully extended n-TFT. Namely one first needs to find a suitable target, which is a symmetric monoidal (∞, n)-category, and then one needs to pick a fully dualizable object. However, this construction is not explicit in the sense that one might like to be able to compute the values of the n-TFT. The goal of this thesis was to, avoiding the use of the cobordism hypothesis, explicitly construct a family of examples of fully extended n-dimensional TFTs, which is essentially given by factorization homology with coefficients in a given E_n-algebra A. Under the cobordism hypothesis this fully extended n-TFT corresponds to the E_n-algebra A, which is a fully dualizable object in a suitable Morita-(∞, n)-category Alg_n. Informally it can be thought of as a higher category with E_n-algebras as objects, bimodules in E_{n-1}-algebras as 1-morphisms, bimodules between bimodules as 2-morphisms, etc. In fact, this (∞, n)-category is the truncation of an $(\infty, n+1)$-category $\widetilde{\mathrm{Alg}_n}$ whose $(n+1)$-morphisms are morphisms in $\mathcal{S}$. Our construction allows to compute the topological invariants given by the TFT by taking global sections of a factorization algebra, and the gluing condition (locality) of the factorization algebra allows this to be computed locally. This extends the excision property of factorization homology proved by Ayala, Francis, and Tanaka in [AFT12].

The first two chapters aim to give a precise definition of a fully extended n-dimensional topological field theory. In the third chapter we define the target category of E_n-algebras and the final chapter contains the construction of the fully extended n-TFT as a morphism of n-fold Segal spaces. We now give a more detailed overview of the chapters.

Symmetric monoidal complete n-fold Segal spaces

First, in chapter 1 we recall the necessary tools from higher category theory needed to define fully extended TFTs. We explain the model for (∞, n)-categories given by complete n-fold Segal spaces. Moreover, we give two possible definitions of symmetric monoidal structures on complete n-fold Segal spaces, once as a Γ-object in complete n-fold Segal spaces following [TV09] and once as a tower of suitable $(n+k)$-fold Segal spaces with one object, 1-morphism,..., $(k-1)$-morphism for $k \geqslant 0$ following the Stabilization Hypothesis.

Definition of a fully extended n-TFT

Chapter 2 deals with the symmetric monoidal (∞, n)-category of bordisms. Lurie gives a formal definition of this (∞, n)-category using complete n-fold Segal spaces, however, as we explain in section 2.4.2, this actually is not an n-fold Segal space. In our definition 2.3.1, we propose a stronger condition on elements in the levels of the Segal space and show that this indeed yields a n-fold Segal space PBord_n. Its completion Bord_n defines an (∞, n)-category of n-cobordisms and thus is a corrigendum to Lurie's n-fold simplicial space of bordisms from [Lur09c].

Instead of using manifolds with corners and gluing them, Lurie's idea was to conversely use embedded closed (not necessarily compact) manifolds and to specify points where they are cut into bordisms of which the embedded manifold is a composition. Whitney's embedding theorem ensures that every n-dimensional manifold M can be embedded into some large enough vector space and suitable versions for manifolds with boundary can be adapted to obtain an embedding theorem for bordisms, see 2.6.1. Moreover, the rough idea behind the definition of the levels of PBord_n is that the $(k_1,\dots,k_n)$-level of our n-fold Segal space PBord_n should be a classifying space for k_i-fold composable n-bordisms in the ith direction. Lurie's idea was to use the fact that the space of embeddings of M into $\mathbb{R}^\infty$ is contractible to justify the construction.

We base our construction of PBord_n on a simpler complete Segal space Int of closed intervals, which is defined in section 2.1. The closed intervals correspond to places where we are allowed to cut the manifold into the bordisms it composes. The fact that we prescribe closed intervals instead of just a point corresponds to fixing collars of the bordisms.

In section 2.2 we study a version of a time-dependent Morse lemma which serves as a motivation for our definition of the spatial structure of the levels of PBord_n. As we explain in 2.3.2, the spatial structure we define is almost obtained by taking differentiable simplices of the space of embeddings, but we add the data of a semi-group of diffeomorphisms between bordisms along a simplex. The time-dependent Morse lemma shows that this yields the same paths.

Section 2.3 is the central part of this chapter and consists of the construction of the complete n-fold Segal space Bord_n of cobordisms. We discuss variants of this construction in section 2.4 and compare our definition to Lurie's sketch.

In section 2.5 we endow Bord_n with a symmetric monoidal structure, both as a Γ-object and as a tower. The construction of the tower requires the construction of (∞,l)-categories of bordisms for arbitrary l.

In section 2.6 we show that its homotopy (bi)category is what one should expect, namely the homotopy category of its $(n-1)$-fold looping $L_{n-1}(\mathrm{Bord}_n)$ gives back the classical cobordism category $n\mathrm{Cob}$ and the homotopy bicategory of Bord_2 is Schommer-Pries' bicategory $2\mathrm{Cob}^{ext}$ from [SP09].

Finally, in section 2.7 we consider bordism categories with additional structure such as orientations, denoted by Bord_n^{or}, and framings, denoted by Bord_n^{fr}, which allows us to define fully extended n-dimensional topological field theories in section 2.8.

The target: E_n-algebras

In chapter 3 we define the target of our fully extended n-TFT, namely a symmetric monoidal Morita-(∞,n)-category $\mathrm{Alg}_n = \mathrm{Alg}_n(\mathcal{S})$ of E_n-algebras. By an E_n-algebra, we mean an E_n-algebra object in a suitable symmetric monoidal $(\infty,1)$-category $\mathcal{S}$. By suitable, we mean that it satisfies the following assumption.

Assumption 1. *Let $\mathcal{S}$ be a symmetric monoidal $(\infty,1)$-category which is $\otimes$-sifted cocomplete.*

Main examples we will be interested in are the category of chain complexes over a ring R, $\mathcal{S} = \mathrm{Ch}_R$, or the category of (Lagrangian) correspondences $\mathcal{S} = \mathrm{(Lag)Corr}$.

To define this as a complete n-fold Segal space, we exploit the equivalence of $(\infty,1)$-categories between E_n-algebras and locally constant factorization algebras on $\mathbb{R}^n \cong (0,1)^n$ (proven by Lurie in [Lur09c]) and define the objects of the n-fold Segal space to be locally constant factorization algebras on $(0,1)^n$. Furthermore, following the observation that the data of a factorization algebra on $(0,1)$ which is locally constant with respect to a stratification of the form $(0,1) \supset \{p\}$ for any $p \in (0,1)$ are equivalent to the data of a pointed (homotopy) bimodule, we model the "levels" of the n-fold Segal space as factorization algebras on $(0,1)^n$ which are locally constant with respect to certain stratifications.

As with Bord_n, we base the construction on a simpler complete Segal space Covers which we construct in section 3.1. The data given by Covers determine the stratification with respect to which the factorization algebras are locally constant.

Section 3.2 contains the main construction of the (∞, n)-category, i.e. the n-fold Segal space, Alg_n. In fact, it is the truncation of an $(\infty, n+1)$-category $\mathrm{Alg}_n^{(\infty,n+1)}$ given by an n-fold Segal object in Segal spaces. These Segal spaces, i.e. the levels, are $(\infty,1)$-categories of locally constant factorization algebras on $(0,1)$ which are locally constant with respect to a stratification of a particular form. The simplicial structure of Alg_n essentially comes from the simplicial structure of the Segal space Covers and is given by the pushforward of the factorization algebra along a suitable collapse-and-rescale map. With this definition composition in the homotopy category corresponds to sending two bimodules ${}_AM_B$ and ${}_BN_C$ to their tensor product $({}_AM_B) \otimes_B ({}_BN_C)$.

The fact that factorization algebras naturally lead to pointed objects has an important consequence. Namely, it implies that the n-fold Segal space Alg_n is complete. This is shown in section 3.2.9.

In section 3.3 we endow Alg_n with a symmetric monoidal structure, both as a Γ-object and as a tower.

Finally we show in section 3.4 that the homotopy category of Alg_1 is the Morita category, whose objects are (homotopy) algebras and whose morphisms are isomorphism classes of pointed (homotopy) bimodules.

Construction of the fully extended n-dimensional topological field theory

The final chapter, chapter 4 connects the two previous chapters. It contains the construction of the fully extended n-TFT as a morphism of n-fold Segal spaces.

The construction of the functor proceeds in two steps: we first define an auxillary symmetric monoidal complete n-fold Segal space Fact_n of factorization algebras on $(0,1)^n$ in section 4.2, which, like Bord_n is based on the Segal space Int. It translates the properties of PBord_n^{fr} via a map given by factorization homology with coefficients in a fixed E_n-algebra A,

$$\int_{(-)} A : \qquad \mathrm{Bord}_n^{fr} \qquad \longrightarrow \qquad \mathrm{Fact}_n,$$

$$\begin{array}{ccc} M & \overset{\iota}{\hookrightarrow} & V \times (0,1)^n \\ & {\scriptstyle\pi}\searrow & \downarrow \\ & & (0,1)^n \end{array} \qquad \longmapsto \qquad \pi_*\big(\textstyle\int_M A\big),$$

which is defined in section 4.3. However, this map is just a morphism of the underlying n-fold simplicial sets as it fails to extend to the spatial structure of the levels.

In a second step, in section 4.4, we define a map to an n-fold Segal space $\mathrm{FAlg}_n \supseteq \mathrm{Alg}_n$ of factorization algebras on $(0,1)^n$ which have certain locally constancy properties, but do not lead to bimodules,

$$\veebar : \mathrm{Fact}_n \longrightarrow \mathrm{FAlg}_n\,.$$

This map can be understood as "collapsing" parts of the factorization algebra and then rescaling. It arises from a map $\veebar$: Int $\to$ Covers of the simpler Segal spaces on which Fact_n and FAlg_n are based, which determines a collapse-and-rescale map $\varrho : (0,1)^n \to (0,1)^n$. Then the map $\veebar$ is given by the pushforward of the factorization algebra along ϱ.

One should think of this process as collapsing the part of the factorization algebra in which the factorization algebra might change along a path, or an even higher simplex in Bord_n^{fr}. The global sections of this part do not change, as the data of a higher simplex in Bord_n include diffeomorphisms between bordisms along this simplex. Following this argument we show in section 4.5 that the composition of the two constructed maps $\veebar \circ \int_{(-)} A$ is a morphism of n-fold Segal spaces and its image in fact lands in Alg_n,

$$\mathcal{FH}_n(A) : \mathrm{PBord}_n^{fr} \longrightarrow \mathrm{Alg}_n\,.$$

By the universal property of the completion, this map extends to a map of complete n-fold Segal spaces,

$$\mathcal{FH}_n(A) : \mathrm{Bord}_n^{fr} \longrightarrow \mathrm{Alg}_n\,.$$

To conclude that $\mathcal{FH}_n(A)$ is the desired fully extended topological field theory we show in 4.6 that it extends to the symmetric monoidal structure for both structures.

$$\begin{array}{ccccc}
\mathrm{PBord}_n^{fr} & \xrightarrow{\int_{(-)}} & \mathrm{Fact}_n & \xrightarrow{\veebar} & \mathrm{FAlg}_n \\
\uparrow & & \searrow & & \uparrow \\
\mathrm{Bord}_n & & \xrightarrow[\mathcal{FH}_n(A)]{} & & \mathrm{Alg}_n
\end{array}$$

Finally, our main theorem, which appears as corollaries 4.6.3 and 4.6.5 in chapter 4, summarizes the construction.

Theorem. *Let A be an E_n-algebra. Then the map*

$$\mathcal{FH}_n(A) : \mathrm{Bord}_n^{fr} \longrightarrow \mathrm{Alg}_n$$

is a fully extended topological field theory.

As can be seen in example 4.5.3, its value at a point is the given E_n-algebra A. So, by the cobordism hypothesis, this is the fully extended topological field theory with values in Alg_n exhibiting A as a fully dualizable object.

Guide to the reader

Parts of this thesis contain rather technical constructions of suitable (n-fold) Segal spaces, so let us explain which parts can be left aside on a first reading.

The first chapter mostly contains a recollection on complete n-fold Segal spaces as a model for (∞, n)-categories. The only original part in this section is that of the definition of a symmetric monoidal structure on an n-fold Segal space following the Delooping Hypothesis in subsection 1.6.2. We use the notion of k-hybrid n-fold Segal space, which is a suitable interpolation between complete n-fold Segal spaces and Segal n-categories. The second and third chapters are mostly independent of each other. In both, one can first brush over the rather technical constructions of the underlying simpler Segal spaces Int and Covers in sections 2.1 and 3.1 and go straight to the main constructions of the (∞, n)-categories Bord_n and Alg_n in sections 2.3 and 3.2. The forth chapter contains the heart of this thesis. The fully extended TFT is constructed within this chapter.

Warning. In chapter 1 we define an (∞, n)-category to be a complete n-fold Segal space. We try to be consistent with this definition throughout the thesis, but at times have to switch to different models for (∞, n)-categories, usually for $(\infty, 1)$-categories. We will state this explicitly where necessary.

Conventions. We will use the following conventions throughout this thesis.

- By *space*, we will mean a simplicial set. This is to distinguish the n simplicial "directions" of the n-fold Segal space from the simplicial set of the "levels", which we call spatial direction. The $(\infty, 1)$-category of spaces will be denoted by $\mathcal{S}pace$.
- We fix a diffeomorphism $(0,1) \overset{\chi}{\cong} \mathbb{R}$. This will endow $(0,1)$ with the structure of a vector space. Whenever we write "$(0,1) \cong \mathbb{R}$" we will mean this fixed diffeomorphism.
- To simplify notation, if we write $[a,b] \subseteq (0,1)$, we allow $a = 0$ or $b = 1$ and mean $[a,b] \cap (0,1)$.
- We denote $\{0,1,\ldots,n\}$ by $[n]$. They form the objects of the simplex category Δ whose morphisms are (weakly) order-preserving morphisms.

Chapter **1**

Preliminaries: symmetric monoidal (∞, n)-categories

A higher category, say, an n-category for $n \geqslant 0$, has not only objects and (1-)morphisms, but also k-morphisms between $(k-1)$-morphisms for $1 \leqslant k \leqslant n$. Strict higher categories can be rigorously defined, however, most higher categories which occur in nature are not strict. Thus, we need to weaken some axioms and coherence between the weakenings become rather involved to formulate explicitly. Things turn out to become somewhat easier when using a geometric definition, in particular when furthermore allowing to have k-morphisms for all $k \geqslant 1$, which for $k \geqslant n$ are invertible. Such a higher category is called an (∞, n)-category. There are several models for such (∞, n)-categories, e.g. Segal n-categories (cf. [HS98]), Θ_n-spaces (cf. [Rez10]), and complete n-fold Segal spaces, which all are equivalent in an appropriate sense (cf. [Toë05, BS11]). For our purposes, the latter model turns out to be well-suited and in this section we recall some basic facts about complete n-fold Segal spaces as higher categories. This is not at all exhaustive, and more details can be found in e.g. [BR13, Zha13].

1.1 The homotopy hypothesis and $(\infty, 0)$-categories

The basic hypothesis upon which higher category theory is based is the following

Hypothesis 1.1.1 (Homotopy hypothesis)**.** Topological spaces are models for ∞-groupoids, also referred to as $(\infty, 0)$-categories.

Given a topological space X, its points are thought of as objects of the $(\infty, 0)$-category, 1-morphisms as paths between points, 2-morphisms as homotopies between paths, 3-morphisms as homotopies between homotopies, and so forth. With this interpretation, it is clear that all n-morphisms are invertible up to homotopies, which are higher morphisms.

We take this hypothesis as the basic definition.

Definition 1.1.2. An $(\infty, 0)$-*category* is a space.

1.2 Complete Segal spaces as models for $(\infty, 1)$-categories

A good overview of different models for $(\infty, 1)$-categories can be found in [Ber10]. Additionally to the model we will discuss below in more detail, we would like to mention one particularly simple and quite rigid model, namely that of topologically enriched categories.

Definition 1.2.1. A *topological category* is a category enriched in topological spaces (or simplicial sets, depending on the purpose).

Topological categories are discussed and used in [Lur09a, TV05]. However, for our applications, complete Segal spaces, first introduced by Rezk in [Rez01] as models for $(\infty, 1)$-categories, turn out to be very well-suited. We recall the definition in this section.

1.2.1 Segal spaces

Definition 1.2.2. A *(1-fold) Segal space* is a simplicial space $X = X_\bullet$ which is level-wise fibrant and satisfies the *Segal condition*, i.e. for any $n, m \geqslant 0$,

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

induced by the maps $[m] \to [m+n]$, $(0 < \cdots < m) \mapsto (0 < \cdots < m)$, and $[n] \to [m+n]$, $(0 < \cdots < n) \mapsto (m < \cdots < m+n)$, is a homotopy pullback square. In other words,

$$X_{m+n} \longrightarrow X_m \overset{h}{\underset{X_0}{\times}} X_n,$$

is a weak equivalence.

Defining a *map of Segal spaces* to be a map of the underlying simplicial spaces gives a category of Segal spaces, $\mathbf{SSpaces} = \mathbf{SSpaces_1}$.

Remark 1.2.3. Following [Lur09c] we omit the Reedy fibrant condition which often appears in the literature. In particular, this condition would guarantees in particular that the canonical map

$$X_m \underset{X_0}{\times} X_n \longrightarrow X_m \overset{h}{\underset{X_0}{\times}} X_n$$

is a weak equivalence. This explains the different appearance of the Segal condition.

Remark 1.2.4. Rezk showed in [Rez01] that Reedy fibrant Segal spaces are the fibrant objects for a model structure on the category of simplicial spaces given by a suitable localization of the injective model structure. However, Horel showed in [Hor14a] that for a suitable localization of the projective model structure, fibrant objects are Segal spaces which are level-wise fibrant instead of Reedy fibrant. Moreover, these two models are Quillen equivalent. In the following text we will point out where these subtleties need to be taken into account. Moreover, we will sometimes mean this model category (or the ∞-category it represents) when writing **SSpaces**.

Example 1.2.5. Let $\mathcal{C}$ be a small topological category, i.e. a small category enriched over topological spaces. Then its nerve $N(\mathcal{C})$ is a Segal space.

Segal spaces as $(\infty, 1)$-categories

The above example motivates the following interpretation of Segal spaces as models for $(\infty,1)$-categories. If $X_\bullet$ is a Segal space then we view the set of 0-simplices of the space X_0 as the set of objects. For $x, y \in X_0$ we view

$$\mathrm{Hom}_X(x,y) = \{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\}$$

as the $(\infty,0)$-category, i.e. the space, of arrows from x to y. More generally, we view X_n as the $(\infty,0)$-category, i.e. the space, of n-tuples of composable arrows together with a composition. Note that given an n-tuple of composable arrows, there is a contractible space of compositions. Moreover, one can interpret paths in the space X_1 of 1-morphisms as 2-morphisms, which thus are invertible up to homotopies, which themselves are 3-morphisms, and so forth.

Definition 1.2.6. We will later refer to the spaces X_n as the *levels* of the Segal space.

1.2.2 The homotopy category of a Segal space

To a higher category one can intuitively associate an ordinary category, its *homotopy category*, having the same objects, with morphisms being 2-isomorphism classes of 1-morphisms. For Segal spaces, one can realize this idea as follows.

Definition 1.2.7. The *homotopy category* $h_1(X)$ of a Segal space $X = X_\bullet$ has as set of objects the set of vertices of the space X_0 and as morphisms between objects $x, y \in X_0$,

$$\begin{aligned}\mathrm{Hom}_{h_1(X)}(x,y) &= \pi_0\left(\mathrm{Hom}_X(x,y)\right)\\ &= \pi_0\left(\{x\} \underset{X_0}{\overset{h}{\times}} X_1 \underset{X_0}{\overset{h}{\times}} \{y\}\right).\end{aligned}$$

For $x, y, z \in X_0$, the following diagram induces the composition of morphisms, as weak equivalences induce bijections on π_0.

$$\begin{aligned}\left(\{x\} \underset{X_0}{\overset{h}{\times}} X_1 \underset{X_0}{\overset{h}{\times}} \{y\}\right) \times \left(\{y\} \underset{X_0}{\overset{h}{\times}} X_1 \underset{X_0}{\overset{h}{\times}} \{z\}\right) &\longrightarrow \{x\} \underset{X_0}{\overset{h}{\times}} X_1 \underset{X_0}{\overset{h}{\times}} X_1 \underset{X_0}{\overset{h}{\times}} \{z\}\\ &\overset{\simeq}{\longleftarrow} \{x\} \underset{X_0}{\overset{h}{\times}} X_2 \underset{X_0}{\overset{h}{\times}} \{z\}\\ &\longrightarrow \{x\} \underset{X_0}{\overset{h}{\times}} X_1 \underset{X_0}{\overset{h}{\times}} \{z\}.\end{aligned}$$

Example 1.2.8. Given a small (ordinary) category $\mathcal{C}$, the homotopy category of its nerve, viewed as a simplicial space with discrete levels, is equivalent to $\mathcal{C}$,

$$h_1(N(\mathcal{C})) \simeq \mathcal{C}.$$

1.2.3 Complete Segal spaces

In our interpretation of a Segal space $X = X_\bullet$ as an $(\infty,1)$-category above several Segal spaces give rise to the same $(\infty,1)$-category: we can replace X_0 by the ∞-groupoid, i.e. the space, obtained by discarding all non-invertible morphisms. To avoid this ambiguity, we impose an extra condition which ensures that the space X_0 is ∞-groupoid, or $(\infty,0)$-category, obtained by discarding all non-invertible morphisms.

Definition 1.2.9. An element $f \in X_1$ with source and target x and y, i.e. the two faces of f are x and y, is *invertible* if its image under

$$\{x\} \underset{X_0}{\times} X_1 \underset{X_0}{\times} \{y\} \longrightarrow \{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{y\} \longrightarrow \pi_0 \left(\{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{y\} \right) = \mathrm{Hom}_{\mathrm{h}_1(X)}(x, y)\,,$$

is an invertible morphism in $h_1(X)$.

Denote by X_1^{inv} the subspace of invertible arrows and observe that the map $X_0 \to X_1$ factors through X_1^{inv}, since the image of $x \in X_0$ under $X_0 \to X_1 \to \mathrm{Hom}_{\mathrm{h}_1(X)}(x, x)$ is id_x, which is invertible.

Definition 1.2.10. A Segal space $X_\bullet$ is *complete* if the map $X_0 \to X_1^{inv}$ is a weak equivalence. We denote the full subcategory of **SSpaces** whose objects are complete Segal spaces by **CSSpaces** = $\mathbf{CSSpaces_1}$.

Remark 1.2.11. Similarly to remark 1.2.4, Rezk showed in [Rez01] that complete Segal spaces are the fibrant objects for a model structure on the category of simplicial spaces, namely a suitable localization of **SSpaces**. Horel showed the analogous statement for the projective model structure in [Hor14b]. We will usually mean this model category when writing **CSSpaces**.

Complete Segal spaces are $(\infty, 1)$-categories

Rezk explained in [Rez01] that complete Segal spaces are a good model for $(\infty, 1)$-categories. This justifies the following definition.

Definition 1.2.12. An $(\infty, 1)$-*category* is a complete Segal space.

Remark 1.2.13. The completeness condition says that all invertible morphisms essentially are just identities up to the choice of a path. So strictly speaking, complete Segal spaces should be called *skeletal*, or, according to [Joy], *reduced* $(\infty, 1)$-categories.

Completion of Segal spaces

Rezk showed in [Rez01] that Segal spaces can always be completed. He showed that there is a completion functor which to every Segal space X associates a complete Segal space $\widehat{X}$ together with a map $i_X : X \to \widehat{X}$, which is a Dwyer-Kan equivalence, which in turn is defined below.

Even though the inclusion **CSSpaces** $\hookrightarrow$ **SSpaces** does not have a left adjoint, it does when passing to the homotopy categories (see remarks 1.2.4 and 1.2.11). Thus, $\widehat{X}$ is universal (in the homotopy category) among complete Segal spaces Y together with a map $X \to Y$.

Definition 1.2.14. A map $f : X \to Y$ of Segal spaces is a *Dwyer-Kan equivalence* if

1. the induced map $h_1(f) : h_1(X) \to h_1(Y)$ on homotopy categories is an equivalence of categories, and

2. for each pair of objects $x, y \in X_0$ the induced function on mapping spaces $\mathrm{Hom}_X(x, y) \to \mathrm{Hom}_Y(f(x), f(y))$ is a weak equivalence.

Relative categories and the classification diagram

In this section we recall a construction due to Rezk [Rez01] which produces a complete Segal space from a simplicial closed model category. More generally, Barwick and Kan proved in [BK11] that this construction also gives a complete Segal space for so-called partial model categories.

Definition 1.2.15. Let $(\mathcal{C}, \mathcal{W})$ be a relative category, i.e. a category $\mathcal{C}$ with a distinguished subcategory $\mathcal{W}$. Consider the simplicial object in categories $\mathcal{C}_\bullet$ given by $\mathcal{C}_n := \mathrm{Fun}\big([n], \mathcal{C}\big)$. It has a subobject $\mathcal{C}_\bullet^{\mathcal{W}}$, where $\mathcal{C}_n^{\mathcal{W}} \subset \mathcal{C}_n$ is the subcategory having the same objects and morphisms consisting only of those from $\mathcal{W}$. Taking its nerve we obtain a simplicial space $N(\mathcal{C}, \mathcal{W})_\bullet$ with

$$N(\mathcal{C}, \mathcal{W})_n = N(\mathcal{C}_n^{\mathcal{W}}).$$

It satisfies the Segal condition, but will not be level-wise fibrant unless we started with an ∞-groupoid. Its level-wise fibrant replacement is called the *relative/simplicial nerve* or the *classification diagram*, which, by abuse of notation, we again denote by $N(\mathcal{C}, \mathcal{W})$.

Example 1.2.16. Let C be a small category. Then it is straightforward to see that $N(\mathcal{C}, \mathrm{Iso}\,\mathcal{C})$ is a complete Segal space. Alternatively, if $\mathcal{C}$ has finite limits and colimits, it can be made into a closed model category in which the weak equivalences are the isomorphisms and all maps are fibrations and cofibrations. Then the above result also shows that the classification diagram is a complete Segal space, cf. [Rez01].

1.2.4 Segal categories

A second way to avoid the problem that in a Segal space and its homotopy category we do not use the topology on X_0 is to impose that X_0 is discrete. By this we obtain the notion of Segal categories, which are another model for $(\infty, 1)$-categories and briefly mention here. More details and references can be found in the above mentioned [Ber10].

Definition 1.2.17. A Segal (1-)category is a Segal space $X = X_\bullet$ such that X_0 is discrete.

Segal categories also are the fibrant objects of a model category which is Quillen equivalent to **CSSpaces**. For our purposes, complete Segal spaces turn out to be the more useful model.

1.3 Complete n-fold Segal spaces as models for (∞, n)-categories

As a model for (∞, n)-categories, we will use complete n-fold Segal spaces, which were first introduced by Barwick in his thesis and appeared prominently in Lurie's [Lur09c].

1.3.1 n-fold Segal spaces

An n-fold Segal space is an n-fold simplicial space with certain extra conditions.

Definition 1.3.1. An n-fold simplicial space $X_{\bullet,\dots,\bullet}$ is *essentially constant* if there is a weak homotopy equivalence of n-fold simplicial spaces $Y \to X$, where Y is constant.

Definition 1.3.2. An *n-fold Segal space* is an n-fold simplicial space $X = X_{\bullet,\dots,\bullet}$ such that

(i) For every $1 \leqslant i \leqslant n$, and every $k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n \geqslant 0$,

$$X_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n}$$

is a Segal space.

(ii) For every $1 \leqslant i \leqslant n$, and every $k_1, \dots, k_{i-1} \geqslant 0$,

$$X_{k_1,\dots,k_{i-1},0,\bullet,\dots,\bullet}$$

is essentially constant.

Defining a *map of n-fold Segal spaces* to be a map of the underlying n-fold simplicial spaces gives a category of n-fold Segal spaces, $\mathbf{SSpaces_n}$.

Remark 1.3.3. Alternatively, one can formulate the conditions iteratively. First, an *n-iterated Segal space* is a simplicial object $Y_\bullet$ in $(n-1)$-fold Segal spaces which satisfies the Segal condition. Then, an n-fold Segal space is an n-iterated Segal space such that Y_0 is essentially constant (as an $(n-1)$-fold Segal space). To get back the above definition, the ordering of the indices is crucial: $X_{k_1,\dots,k_n} = (Y_{k_1})_{k_2,\dots,k_n}$.

Remark 1.3.4. Similarly to remark 1.3.4 there is a model structure on the category of n-fold simplicial spaces obtained as a localization of the injective model structure whose fibrant objects are Reedy fibrant n-fold Segal spaces, see [Lur09b] or [BS11]. Similarly to the aforementioned [Hor14a], there is a model structure obtained by localizing the projective model structure and whose fibrant objects are complete Segal spaces which are level-wise fibrant instead of Reedy fibrant, see remark 1.5.6 in [Lur09b]. We will sometimes mean this model category (or the ∞-category it represents) when writing $\mathbf{SSpaces_n}$.

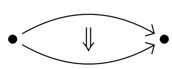
Interpretation as higher categories

An n-fold Segal space can be thought of as a higher category in the following way.

The first condition means that this is an n-fold category, i.e. there are n different "directions" in which we can "compose". An element of $X_{k_1,\dots,k_n}$ should be thought of as a composition consisting of k_i composed morphisms in the ith direction.

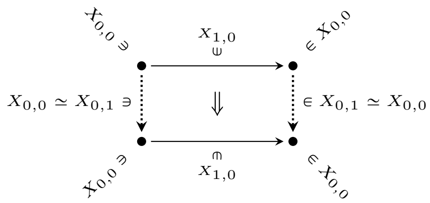
The second condition imposes that we indeed have a higher n-category, i.e. an n-morphism has as source and target two $(n-1)$-morphisms which themselves have the "same" (in the sense that they are homotopic) source and target.

For $n = 2$ one can think of this second condition as "fattening" the objects in a bicategory. A 2-morphism in a bicategory can be depicted as



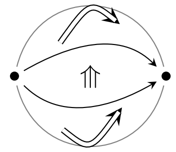
The top and bottom arrows are the source and target, which are 1-morphisms between the same objects.

In a 2-fold Segal space $X_{\bullet,\bullet}$, an element in $X_{1,1}$ can be depicted as

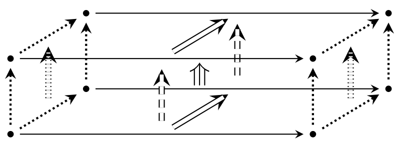


The images under the source and target maps in the first direction $X_{1,1} \rightrightarrows X_{1,0}$ are 1-morphisms which are depicted by the horizontal arrows. The images under the source and target maps in the second direction $X_{1,1} \rightrightarrows X_{0,1}$ are 1-morphisms, depicted by the dashed vertical arrows, which are essentially just identity maps, up to homotopy, since $X_{0,1} \simeq X_{0,0}$. Thus, here the source and target 1-morphisms (the horizontal ones) themselves do not have the same source and target anymore, but up to homotopy they do.

The same idea works with higher morphisms, in particular one can imagine the corresponding diagrams for $n = 3$. A 3-morphism in a tricategory can be depicted as



whereas a 3-morphism, i.e. an element in $X_{1,1,1}$ in a 3-fold Segal space X can be depicted as



Here the dotted arrows are those in $X_{0,1,1} \simeq X_{0,0,1} \simeq X_{0,0,0}$ and the dashed ones are those in $X_{1,0,1} \simeq X_{1,0,0}$.

Thus, we should think of the set of 0-simplices of the space $X_{0,\ldots,0}$ as the objects of our category, and elements of $X_{1,\ldots,1,0,\ldots,0}$ as i-morphisms, where $0 < i \leqslant n$ is the number of 1's. Pictorially, they are the i-th "horizontal" arrows. Moreover, the other "vertical" arrows are essentially just identities of lower morphisms. Similarly to before, paths in $X_{1,\ldots,1}$ should be thought of as $(n+1)$-morphisms, which therefore are invertible up to a homotopy, which itself is an $(n+2)$-morphism, and so forth.

1.3.2 Complete and hybrid n-fold Segal spaces

As with (1-fold) Segal spaces, we need to impose an extra condition to ensure that invertible k-morphisms are paths in the space of $(k-1)$-morphisms. Again, there are several ways to include its information.

Definition 1.3.5. Let X be an n-fold Segal space and $1 \leqslant i, j \leqslant n$. It is said to satisfy

CSS^i if for every $k_1, \ldots, k_{i-1} \geqslant 0$,

$$X_{k_1,\ldots,k_{i-1},\bullet,0,\ldots,0}$$

is a complete Segal space.

SC^j if for every $k_1, \ldots, k_{j-1} \geqslant 0$,

$$X_{k_1,\ldots,k_{j-1},0,\bullet,\ldots,\bullet}$$

is discrete, i.e. a discrete space viewed as a constant $(n-j+1)$-fold Segal space.

Definition 1.3.6. An n-fold Segal space is

1. *complete*, if for every $1 \leqslant i \leqslant n$, X satisfies (CSS^i).
2. a *Segal n-category* if for every $1 \leqslant j \leqslant n$, X satisfies (SC^j).
3. *m-hybrid* for $m \geqslant 0$ if condition (CSS^i) is satisfied for $i > m$ and condition (SC^j) is satisfied for $j \leqslant m$.

Denote the full subcategory of $\mathbf{SSpaces_n}$ of complete n-fold Segal spaces by $\mathbf{CSSpaces_n}$.

Remark 1.3.7. Note that an n-hybrid n-fold Segal space is a Segal n-category, while an n-fold Segal space is 0-hybrid if and only if it is complete.

For our purposes, the model of complete n-fold Segal spaces is well-suited, so we define

Definition 1.3.8. An (∞, n)-category is an n-fold complete Segal space.

Model structure and weak equivalences

As in remarks 1.2.4, 1.2.11, and 1.3.4, there is a model structure on the category of n-fold simplicial spaces $\mathbf{sSpaces_n}$ whose fibrant objects are complete n-fold Segal spaces, see [Lur09b] or [BS11]. Moreover, iterating the construction of Segal categories enriched in a suitable model category and the construction of complete Segal objects for so-called absolute distributors in [Lur09b] should give a model in which m-hybrid n-fold Segal spaces are the fibrant objects.

Since $\mathbf{SSpaces_n}$ and $\mathbf{CSSpaces_n}$ are localizations of $\mathbf{sSpaces_n}$, they inherit a subcategory of weak equivalences. One can prove that they are exactly the *Dwyer-Kan equivalences*, the analogous notion to definition 1.2.14 for $n = 1$. More details can be found e.g. in [Zha13].

Completion

In light of the iterative definition of an n-fold Segal space, i.e. viewing an n-fold Segal space as an $(n-1)$-fold Segal space, condition (CSS^i) above means that the ith iteration is a complete Segal space object. Thus, given an n-fold Segal space $X_{\bullet,\ldots,\bullet}$, one can apply the completion functor iteratively to obtain a complete n-fold Segal space $\widehat{X}_{\bullet,\ldots,\bullet}$, its *(n-fold)* *completion*. This yields a map $X \to \widehat{X}$, the *completion map*, which is universal among all maps (in the homotopy category) to complete n-fold Segal spaces. It is a left adjoint to the inclusion of the homotopy category of $\mathbf{CSSpaces_n}$ into the homotopy category of $\mathbf{SSpaces_n}$.

If an n-fold Segal space $X_{\bullet,\dots,\bullet}$ satisfies (SC^j) for $j \leqslant m$, we can apply the completion functor just to the last $(n-m)$ indices to obtain an m-hybrid n-fold Segal space $\widehat{X}^m_{\bullet,\dots,\bullet}$, its *m-hybrid completion*.

1.4 The homotopy bicategory of a 2-fold Segal space

To any higher category one can intuitively associate a bicategory having the same objects and 1-morphisms, and with 2-morphisms being 3-isomorphism classes of the original 2-morphisms.

Definition 1.4.1. The *homotopy bicategory* $\mathrm{h}_2(X)$ of a 2-fold Segal space $X = X_{\bullet,\bullet}$ is defined as follows: objects are the points of the space $X_{0,0}$ and

$$\mathrm{Hom}_{\mathrm{h}_2(X)}(x,y) = \mathrm{h}_1\big(\mathrm{Hom}_X(x,y)\big) = \mathrm{h}_1\left(\{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{y\}\right)$$

as Hom categories. Horizontal composition is defined as follows:

$$\begin{aligned}
\left(\{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{y\}\right) \times \left(\{y\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\}\right) &\longrightarrow \{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\} \\
&\overset{\sim}{\longleftarrow} \{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{2,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\} \\
&\longrightarrow \{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\}\,.
\end{aligned}$$

The second arrow happens to go in the wrong way but it is a weak equivalence. Therefore after taking h_1 it turns out to be an equivalence of categories, and thus to have an inverse (assuming the axiom of choice).

1.5 Constructions of n-fold Segal spaces

We describe several intuitive constructions of (∞, n)-categories in terms of (complete) n-fold Segal spaces.

1.5.1 Truncation

Given an (∞, n)-category, for $k \leqslant n$ its (∞, k)-truncation is the (∞, k)-category obtained by discarding the non-invertible m-morphisms for $k < m \leqslant n$.

In terms of n-fold Segal spaces, there is a functor of n-fold Segal spaces sending $X = X_{\bullet,\dots,\bullet}$ to its k-truncation, the k-fold Segal space

$$\tau_k X = X_{\underbrace{\bullet,\dots,\bullet}_{k \text{ times}},\underbrace{0,\dots,0}_{n-k \text{ times}}}\,.$$

Remark 1.5.1. Note that if X is m-hybrid then so is $\tau_k X$ by the definition of the conditions (CSS^i) and (SC^j).

Warning. Truncation does not behave well with completion, i.e. the truncation of the completion is not the completion of the truncation. However, we get a map in one direction.

$$\begin{array}{ccc} \tau_k(X) & \longrightarrow & \tau_k(\widehat{X}) \\ \big\downarrow & \nearrow & \\ \widehat{\tau_k(X)} & & \end{array}$$

In general, we do not expect this map to be an equivalence.

Thus in general one should always complete an n-fold Segal space before truncating it, as

$$X_{\underbrace{1,\ldots,1}_{k},0,\ldots,0} \hookrightarrow X_{\underbrace{1,\ldots,1}_{m},0,\ldots,0}$$

are the invertible m-morphisms for $k < m \leqslant n$ if and only if X satisfies (1.3.5) for $k < i \leqslant n$. For example, if $X = X_\bullet$ is a (1-fold) Segal space then X_0 is the underlying ∞-groupoid of invertible morphisms if and only if X is complete.

1.5.2 Extension

Any (∞, n)-category can be viewed as an $(\infty, n+1)$-category with only identities as $(n+1)$-morphisms.

In terms of iterated Segal spaces, any n-fold Segal space can be viewed as a constant simplicial object in n-fold Segal spaces, i.e. an $(n+1)$-fold Segal space which is constant in the first index. Explicitly, if $X_{\bullet,\ldots,\bullet}$ is an n-fold Segal space, then $\varepsilon(X)_{\bullet,\ldots,\bullet}$ is the constant $(n+1)$-fold Segal space such that for every $k \geqslant 0$,

$$\varepsilon(X)_{k,\bullet,\ldots,\bullet} = X_{\bullet,\ldots,\bullet}$$

with identities as face and degeneracy maps.

Lemma 1.5.2. *If X is complete, then $\varepsilon(X)$ is complete.*

Proof. Since X is complete, it satisfies (CSS^i) for $i > 1$. For $i = 0$, we have to show that $\varepsilon(X)_{\bullet,0,\ldots,0}$ is complete. This is satisfied because

$$(\varepsilon(X)_{1,0,\ldots,0})^{inv} = \varepsilon(X)_{1,0,\ldots,0} = X_{0,\ldots,0} = \varepsilon(X)_{0,0,\ldots,0},$$

since morphisms between two elements x, y in the homotopy category of $\varepsilon(X)_{\bullet,k_2,\ldots,k_n}$ are just connected components of the space of paths in $X_{k_2,\ldots,k_n}$, and thus are always invertible. □

We call ε the extension functor, which is left adjoint to τ_1. Moreover, the unit $\mathrm{id} \to \tau_1 \circ \varepsilon$ of the adjunction is the identity.

1.5.3 The higher category of morphisms and loopings

Given two objects x, y in an (∞, n)-category, morphisms from x to y should form an $(\infty, n-1)$-category. This can be realized for n-fold Segal spaces, which is one of the main advantages of this model for (∞, n)-categories.

Definition 1.5.3. Let $X = X_{\bullet,\cdots,\bullet}$ be an n-fold Segal space. As we have seen above one should think of objects as vertices of the space $X_{0,\ldots,0}$. Let $x, y \in X_{0,\ldots,0}$. The $(n-1)$-*fold Segal space of morphisms* from x to y is

$$\mathrm{Hom}_X(x,y)_{\bullet,\cdots,\bullet} = \{x\} \underset{X_{0,\bullet,\cdots,\bullet}}{\overset{h}{\times}} X_{1,\bullet,\cdots,\bullet} \underset{X_{0,\bullet,\cdots,\bullet}}{\overset{h}{\times}} \{y\}\,.$$

Remark 1.5.4. Note that if X is m-hybrid, then $\mathrm{Hom}_{X_\bullet}(x,y)$ is $(m-1)$-hybrid.

Example 1.5.5 (Compatibility with extension)**.** Let X be an $(\infty, 0)$-category, i.e. a space, viewed as an an $(\infty, 1)$-category, i.e. a constant (complete) Segal space $\varepsilon(X)_\bullet$, $\varepsilon(X)_k = X$. For any two objects $x, y \in \varepsilon(X)_0 = X$ the $(\infty, 0)$-category, i.e. the topological space, of morphisms from x to y is

$$\mathrm{Hom}_{\varepsilon(X)}(x,y) = \{x\} \underset{\varepsilon(X)_0}{\overset{h}{\times}} \varepsilon(X)_1 \underset{\varepsilon(X)_0}{\overset{h}{\times}} \{y\} = \{x\} \underset{X}{\overset{h}{\times}} \{y\} = \mathrm{Path}_X(x,y)\,,$$

the path space in X, which coincides with what one expects by the interpretation of paths, homotopies, homotopies between homotopies, etc. being higher invertible morphisms.

Definition 1.5.6. Let X be an n-fold Segal space, and $x \in X_0$ an object in X. Then the *looping of X at x* is the $(n-1)$-fold Segal space

$$L(X,x)_{\bullet,\ldots,\bullet} = \mathrm{Hom}_X(x,x)_{\bullet,\ldots,\bullet} = \{x\} \times^h_{X_{0,\bullet,\ldots,\bullet}} X_{1,\bullet,\ldots,\bullet} \times^h_{X_{0,\bullet,\ldots,\bullet}} \{x\}.$$

In the following, it will often be clear at which element we are looping, e.g. if there essentially only is one element, or at a unit for the monoidal structure. Then we omit the x from the notation and just write

$$LX = L(X) = L(X,x).$$

Note that even if there is not a unique unit, this will be independent of the choice of unit.

We can iterate this procedure as follows.

Definition 1.5.7. Let $L_0(X,x) = X$. For $1 \leqslant k \leqslant n$, let the *$k$-fold iterated looping* be the $(n-k)$-fold Segal space

$$L_k(X,x) = L(L_{k-1}(X,x),x),$$

where we view x as a trivial k-morphism via the degeneracy maps, i.e. an element in $L_{k-1}(X,x)_{0\ldots,0} \subset X_{1,\ldots,1,0,\ldots,0}$, with k 1's.

Remark 1.5.8. We remark that looping commutes with taking the ordinary or the m-hybrid completion, since completion is taken index per index.

1.6 Symmetric monoidal n-fold Segal spaces

1.6.1 Symmetric monoidal n-fold Segal spaces as a Γ-object

Following [Toe, TV09], we define a symmetric monoidal n-fold Segal space in analogy to so-called Γ-spaces.

Definition 1.6.1. Segal's category Γ is the category whose objects are the finite sets

$$\langle m\rangle = \{0, \dots, m\},$$

for $m \geqslant 0$ which are pointed at 0. Morphisms are pointed functions, i.e. for $k, m \geqslant 0$, functions

$$f : \langle m\rangle \longrightarrow \langle k\rangle, \quad f(0) = 0.$$

For every $m \geqslant 0$, there are m canonical morphisms

$$\gamma_\beta : \langle m\rangle \longrightarrow \langle 1\rangle, \quad j \longmapsto \delta_{\beta j}$$

for $1 \leqslant \beta \leqslant m$, called the *Segal morphisms.*

Remark 1.6.2. Segal's category Γ is the skeleton of the category of finite pointed sets.

We would now like to define a symmetric monoidal (complete) n-fold Segal space to be an ∞-functor from Γ to the $(\infty, 1)$-category of n-fold complete Segal spaces which satisfies certain properties. Recall from remark 1.3.4 and section 1.3.2 that the $(\infty, 1)$-category of n-fold (complete) Segal spaces is presented by a model category $(\mathbf{C})\mathbf{SSpace_n}$. Using the strictification theorem of Toën-Vezzosi from [TV02] every such functor can be represented by a strict functor from Γ to $(\mathbf{C})\mathbf{SSpace_n}$. Moreover, the $(\infty, 1)$-category of $(\infty, 1)$-functors can be computed using the model category $((\mathbf{C})\mathbf{SSpace_n})^\Gamma$ of Γ-diagrams in $(\mathbf{C})\mathbf{SSpace_n}$. Thus the following definition suffices.

Definition 1.6.3. A *symmetric monoidal (complete) n-fold Segal space* is a (strict) functor from Γ to the (strict) category of complete n-fold Segal spaces $(\mathbf{C})\mathbf{SSpace_n}$,

$$A : \Gamma \longrightarrow (\mathbf{C})\mathbf{SSpace_n}$$

such that for every $m \geqslant 0$, the induced map

$$A\Big(\prod_{1\leqslant\beta\leqslant m} \gamma_\beta\Big) : A\langle m\rangle \longrightarrow (A\langle 1\rangle)^m$$

is an equivalence of n-fold (complete) Segal spaces.

The (complete) n-fold Segal space $X = A\langle 1\rangle$ is called the (complete) n-fold Segal space *underlying* A, and by abuse of language we will sometimes call a (complete) n-fold Segal space X symmetric monoidal, if there is a symmetric monoidal (complete) n-fold Segal space A such that $A\langle 1\rangle = X$.

Remark 1.6.4. Note that in particular, for $m = 0$, this implies that $A\langle 0\rangle$ is a point, viewed as a constant n-fold Segal space.

Definition 1.6.5. The $(\infty, 1)$-category, i.e. Segal space, of functors from Γ to $(\mathbf{C})\mathbf{SSpace_n}$, which as mentioned above can be computed using the model category of Γ-diagrams in $(\mathbf{C})\mathbf{SSpace_n}$, has a full sub-$(\infty, 1)$-category of symmetric monoidal (complete) n-fold Segal spaces. A 1-morphism in this category is called a *symmetric monoidal functor of* (∞, n)*-categories.*

Since the completion map $X \to \widehat{X}$ is a weak equivalence and preserves finite products of Segal spaces up to weak equivalence, we obtain the following

Lemma 1.6.6. *If* $A : \Gamma \longrightarrow \mathbf{SSpace_n}$ *is a symmetric monoidal* n*-fold Segal space, then*

$$\widehat{A} : \Gamma \longrightarrow \mathbf{CSSpace_n},$$
$$\langle m \rangle \longmapsto \widehat{A\langle m \rangle}$$

is a symmetric monoidal complete n*-fold Segal space.*

Remark 1.6.7. Similarly, one can define $\mathbf{E}_k$-monoidal (complete) n-fold Segal spaces as certain $(\infty, 1)$-functors from $\mathbf{E}_k$ to $(\mathbf{C})\mathbf{SSpace_n}$, where $\mathbf{E}_k$ denotes the $(\infty, 1)$-category obtained from the topological category of little-k-disks, see [Zha13] for more details on this definition.

Example 1.6.8. Let $A : \Gamma \longrightarrow \mathbf{SSpace_1}$ be a symmetric monoidal Segal space. Consider the product of maps $\gamma_1 \times \gamma_2$ and the map induced by the map $\gamma : \langle 2 \rangle \to \langle 1 \rangle; 1, 2 \mapsto 1$,

$$A\langle 1 \rangle \times A\langle 1 \rangle \xleftarrow[A(\gamma_1)\times A(\gamma_2)]{\simeq} A\langle 2 \rangle \xrightarrow{A(\gamma)} A\langle 1 \rangle.$$

Passing to the homotopy category, we obtain a map

$$h_1(A\langle 1 \rangle) \times h_1(A\langle 1 \rangle) \longrightarrow h_1(A\langle 1 \rangle).$$

Toën and Vezzosi showed in [TV09] that this is a symmetric monoidal structure on the category $h_1(A\langle 1 \rangle)$. Roughly speaking, this uses functoriality of A. Associativity uses the Segal space $A\langle 3 \rangle$, $A\langle 0 \rangle$ corresponds to the unit, and the map $c : \langle 2 \rangle \to \langle 2 \rangle; 1 \mapsto 2, 2 \mapsto 1$ induces the commutativity constraint.

Example 1.6.9. Truncations and extensions of symmetric monoidal (∞, n)-categories again are symmetric monoidal. Let A be a symmetric monoidal n-fold Segal space. Then we can define

$$\tau_k(A)\langle m \rangle = \tau_k(A\langle m \rangle), \quad \varepsilon(A)\langle m \rangle = \varepsilon(A\langle m \rangle).$$

Note that τ_k and ε are functors of n-fold Segal spaces which preserves weak equivalences. Thus, these assignments can be extended to functors $\tau_k(A)$ and $\varepsilon(A)$, and the images of $A\big(\prod_{1 \leqslant \beta \leqslant m} \gamma_\beta\big)$ are again weak equivalence.

Example 1.6.10. For every $m \geqslant 0$ there is a unique map $\langle 0 \rangle \to \langle m \rangle$, and since $A\langle 0 \rangle$ is the point as a constant (complete) n-fold Segal space, this induces, for every $m \geqslant 0$, a distinguished object $\mathbb{1}_{\langle m \rangle} \in A\langle m \rangle$. The looping of a symmetric monoidal n-fold Segal space A with respect this object also is symmetric monoidal, with

$$L(A)\langle m \rangle = L(A\langle m \rangle, \mathbb{1}_{\langle m \rangle}),$$

which extends to an appropriate functor similarly to in the previous example.

Example 1.6.11. Important examples come from the classification diagram construction. Let $\mathcal{C}$ be a small symmetric monoidal category and let $\mathcal{W} = \operatorname{Iso}\mathcal{C}$. As we saw in section 1.2.3, this gives a complete Segal space $\mathcal{C}_\bullet = N(\mathcal{C}, \mathcal{W})$. The symmetric monoidal structure of $\mathcal{C}$ endows $\mathcal{C}_\bullet$ with the structure of a symmetric monoidal complete Segal space:

First note that $\mathcal{W}^{\times m} = \operatorname{Iso}(\mathcal{C}^{\times m})$ for every m. On objects, let $A : \Gamma \longrightarrow \mathbf{CSSpace_1}$ be given by $A\langle m \rangle = N(\mathcal{C}^{\times m}, \mathcal{W}^{\times m})_\bullet$. We explain the image of the map $\langle 2 \rangle \to \langle 1 \rangle; 1, 2 \mapsto 1$, which should be a map $A\langle 2 \rangle \to A\langle 1 \rangle$. The image of an arbitrary map $\langle m \rangle \to \langle l \rangle$ can be defined analogously.

An l-simplex in $A\langle 2 \rangle_0 = N(\mathcal{C} \times \mathcal{C}, \mathcal{W} \times \mathcal{W})_0$ is a pair

$$C_0 \xrightarrow{w_1} \cdots \xrightarrow{w_l} C_l, \qquad D_0 \xrightarrow{w'_1} \cdots \xrightarrow{w'_l} D_l,$$

and is sent to

$$C_0 \otimes D_0 \xrightarrow{w''_1} \cdots \xrightarrow{w''_l} C_l \otimes D_l,$$

where $w''_i : C_{i-1} \otimes D_{i-1} \xrightarrow{w_i \otimes id_{D_{i-1}}} C_i \otimes D_{i-1} \xrightarrow{id_{C_i} \otimes w'_i} C_i \otimes D_i$ is in $\mathcal{W}$. More generally, an l-simplex in

$$A\langle 2 \rangle_k = N(\mathcal{C} \times \mathcal{C}, \mathcal{W} \times \mathcal{W})_k$$

is a pair of diagrams

$$\begin{array}{ccccccc}
C_{0,0} & \xrightarrow{f_{10}} & C_{1,0} & \xrightarrow{f_{20}} & \cdots & \xrightarrow{f_{k0}} & C_{k,0} \\
\downarrow{\scriptstyle w_{01}} & & \downarrow{\scriptstyle w_{11}} & & & & \downarrow{\scriptstyle w_{k1}} \\
C_{0,1} & \xrightarrow{f_{11}} & C_{1,1} & \xrightarrow{f_{21}} & \cdots & \xrightarrow{f_{k1}} & C_{k,1} \\
\downarrow{\scriptstyle w_{02}} & & \downarrow{\scriptstyle w_{21}} & & & & \downarrow{\scriptstyle w_{k2}} \\
\vdots & & \vdots & & & & \vdots \\
\downarrow{\scriptstyle w_{0l}} & & \downarrow{\scriptstyle w_{1l}} & & & & \downarrow{\scriptstyle w_{kl}} \\
C_{0,l} & \xrightarrow{f_{1l}} & C_{1,l} & \xrightarrow{f_{2l}} & \cdots & \xrightarrow{f_{k,l}} & C_{k,l}
\end{array}
\qquad
\begin{array}{ccccccc}
D_{0,0} & \xrightarrow{g_{10}} & D_{1,0} & \xrightarrow{g_{20}} & \cdots & \xrightarrow{g_{k0}} & D_{k,0} \\
\downarrow{\scriptstyle v_{01}} & & \downarrow{\scriptstyle v_{11}} & & & & \downarrow{\scriptstyle v_{k1}} \\
D_{0,1} & \xrightarrow{g_{11}} & D_{1,1} & \xrightarrow{g_{21}} & \cdots & \xrightarrow{g_{k1}} & D_{k,1} \\
\downarrow{\scriptstyle v_{02}} & & \downarrow{\scriptstyle v_{21}} & & & & \downarrow{\scriptstyle v_{k2}} \\
\vdots & & \vdots & & & & \vdots \\
\downarrow{\scriptstyle v_{0l}} & & \downarrow{\scriptstyle v_{1l}} & & & & \downarrow{\scriptstyle v_{kl}} \\
D_{0,l} & \xrightarrow{g_{1l}} & D_{1,l} & \xrightarrow{g_{2l}} & \cdots & \xrightarrow{g_{k,l}} & D_{k,l}
\end{array}$$

which is sent to the diagram

$$\begin{array}{ccccc}
C_{0,0} \otimes D_{0,0} & \xrightarrow{f_{10} \otimes g_{10}} & C_{1,0} \otimes D_{1,0} & \xrightarrow{f_{20} \otimes g_{20}} & \cdots \\
\downarrow & & \downarrow & & \\
C_{0,1} \otimes D_{0,1} & \xrightarrow{f_{11} \otimes g_{11}} & C_{1,1} \otimes D_{1,1} & \xrightarrow{f_{21} \otimes g_{21}} & \cdots \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & &
\end{array}$$

where the vertical maps are defined as for the objects.

Finally, we need to check that $A\big(\prod_{1 \leqslant \beta \leqslant m} \gamma_\beta\big)$ is a weak equivalence. This follows from the fact that

$$(A\langle m \rangle)_k = N(\mathcal{C}^{\times m}, \mathcal{W}^{\times m})_k = \big(N(\mathcal{C}, \mathcal{W})_k\big)^{\times m} = \big(A\langle 1 \rangle_k\big)^m.$$

Remark 1.6.12. If we start with a symmetric monoidal relative category $(\mathcal{C}, \mathcal{W})$ (a definition can e.g. be found in [Cam14]) such that all $N(\mathcal{C}^{\times m}, \mathcal{W}^{\times m})$ are (complete) Segal spaces, then the above construction for $(\mathcal{C}, \mathcal{W})$ yields a symmetric monoidal (complete) Segal space $N(\mathcal{C}, \mathcal{W})$.

1.6.2 Symmetric monoidal n-fold Segal spaces as a tower of $(n+i)$-fold Segal spaces

Our motivation for the following definition of a (k-)monoidal complete n-fold Segal space comes from the Delooping Hypothesis, which is inspired by the fact that a monoidal category can be seen as a bicategory with just one object. Similarly, a k-monoidal n-category should be a $(k+n)$-category (whatever that is) with only one object, one 1-morphism, one 2-morphism, and so on up to one $(k-1)$-morphism.

Hypothesis 1.6.13 (Delooping Hypothesis)**.** k-monoidal (∞, n)-categories can be identified with $(k-j)$-monoidal, $(j-1)$-simply connected $(\infty, n+j)$-categories for any $0 \leqslant j \leqslant k$, where $(j-1)$-simply connected means that any two parallel i-morphisms are equivalent for $i < j$. In particular, monoidal (∞, n)-categories can be identified with $(\infty, n+1)$-categories with (essentially) one object.

Monoidal n-fold complete Segal spaces

We use the last statement in the delooping hypothesis as the motivation for the following definition. However, first we need to explain what "having (essentially) one object" means.

Definition 1.6.14. An n-fold Segal space X is called *pointed* or *0-connected*, if

$$X_{0,\bullet,\ldots,\bullet},$$

is weakly equivalent to the point viewed as a constant n-fold Segal space.

Definition 1.6.15. A *monoidal* complete n-fold Segal space is a 1-hybrid $(n+1)$-fold Segal space $X^{(1)}$ which is pointed. We say that this endows the n-fold complete Segal space

$$X = L(X^{(1)}, *)$$

with a monoidal structure and that $X^{(1)}$ is a *delooping* of X.

Remark 1.6.16. Note that as $X^{(1)}$ is 1-hybrid, $X^{(1)}_{0,\bullet,\ldots,\bullet}$ is discrete. Thus, to be pointed implies that $X^{(1)}_{0,\bullet,\ldots,\bullet}$ is equal to the point viewed as a constant n-fold Segal space.

Without the completeness condition, we could define a monoidal n-fold Segal space as an $(n+1)$-fold Segal space $X^{(1)}$ which is pointed. Then $L(X^{(1)}, *) = \mathrm{Hom}_{X^{(1)}}(*, *)$ is independent of the choice of point $* \in X_{0,\ldots,0}$ and we can say that this endows the n-fold Segal space $X = L(X^{(1)}) = L(X^{(1)}, *)$ with a monoidal structure.

However, a complete Segal space will not have a contractible space as $X_{0,\ldots,0}$. Thus, we need to introduce a model for $(\infty, n+k)$-categories which can have a point as the set of objects, 1-morphisms, et cetera, which motivates our use of hybrid Segal spaces.

Remark 1.6.17. Let X be an m-hybrid n-fold Segal space with $m > 0$ which is pointed. Then $X_{0,\bullet,\ldots,\bullet} = *$, and the looping is

$$L(X)_{\bullet,\ldots,\bullet} = \{*\} \overset{h}{\underset{*}{\times}} X_{1,\bullet,\ldots,\bullet} \overset{h}{\underset{*}{\times}} \{*\} = X_{1,\bullet,\ldots,\bullet}.$$

A similar definition works for hybrid Segal spaces.

Definition 1.6.18. A *monoidal* m-hybrid n-fold Segal space is an $(m+1)$-hybrid $(n+1)$-fold Segal space $X^{(1)}$ which is pointed. We say that this endows the m-hybrid n-fold Segal space

$$X = L(X^{(1)})$$

with a monoidal structure and that $X^{(1)}$ is a *delooping* of X.

Example 1.6.19. Let $\mathcal{C}$ be a small monoidal category and let $\mathcal{W} = \mathrm{Iso}\,\mathcal{C}$. As we saw in section 1.2.3, this gives a complete Segal space $\mathcal{C}_\bullet = N(\mathcal{C}, \mathcal{W})$. The monoidal structure of $\mathcal{C}$ endows $\mathcal{C}_\bullet$ with the structure of a monoidal complete Segal space:

Let $\mathcal{C}_{m,n} = \mathcal{C}_n^{\otimes m}$ be the category which has objects of the form

$$C_{01} \otimes \cdots \otimes C_{0m} \xrightarrow{c_1} \cdots \xrightarrow{c_n} C_{n0} \otimes \cdots \otimes C_{nm}$$

and morphisms of the form

$$\begin{array}{ccc} C_{01} \otimes \cdots \otimes C_{0m} \xrightarrow{c_1} & \cdots & \xrightarrow{c_n} C_{n0} \otimes \cdots \otimes C_{nm} \\ \downarrow f^0 & & \downarrow f^n \\ D_{01} \otimes \cdots \otimes D_{0m} \xrightarrow{d_1} & \cdots & \xrightarrow{d_n} D_{n0} \otimes \cdots \otimes D_{nm}, \end{array}$$

where $c_1, \ldots, c_n, d_1, \ldots, d_n$, and $f^0, \ldots, f^n$ are morphisms in $\mathcal{C}$.

Consider its subcategory $\mathcal{C}^{\mathcal{W}}_{m,n} \subset \mathcal{C}_{m,n}$ which has the same objects, and vertical morphisms involving only the ones in $\mathcal{W} = \mathrm{Iso}\,\mathcal{C}$, i.e. $f^0, \ldots, f^n$ are morphisms in $\mathcal{W}$.

Now let

$$\mathcal{C}^{(1)}_{m,n} = N(\mathcal{C}^{\mathcal{W}}_{m,n}),$$

the (ordinary) nerve. By a direct verification one sees that the collection $\mathcal{C}^{(1)}_{\bullet,\bullet}$ is a 2-fold Segal space. Moreover,

1. $\mathcal{C}^{(1)}_{0,n} = N(\mathcal{C}_n^{\otimes 0}) = *$, so $\mathcal{C}^{(1)}_{0,\bullet}$ is discrete and equal to the point viewed as a constant Segal space, and
2. for every $m \geqslant 0$, $\mathcal{C}^{(1)}_{m,\bullet} = N(\mathcal{C}^{\mathcal{W}}_{m,\bullet}) = N((\mathcal{C}_\bullet^{\otimes m})^{\mathcal{W}})$, which is a complete Segal space.

Summarizing, $\mathcal{C}^{(1)}$ is a 1-hybrid 2-fold Segal space which is pointed and endows $L(\mathcal{C}^{(1)})_\bullet = \mathcal{C}_\bullet$ with the structure of a monoidal complete Segal space.

k-monoidal n-fold complete Segal spaces

To encode braided or symmetric monoidal structures, we can push this definition even further.

Definition 1.6.20. An n-fold Segal space X is called *j-connected* if for every $i < j$,

$$X_{\underbrace{1,\ldots,1,}_{i}\,0,\bullet,\ldots,\bullet}$$

is weakly equivalent to the point viewed as a constant n-fold Segal space.

Definition 1.6.21. A *k-monoidal* m-hybrid n-fold Segal space is an $(m+k)$-hybrid $(n+k)$-fold Segal space $X^{(k)}$ which is $(k-1)$-connected.

Remark 1.6.22. Note that as $X^{(k)}$ is $(m+k)$-hybrid, $X^{(k)}_{\underbrace{1,\ldots,1,}_{i}\,0,\bullet,\ldots,\bullet}$ is discrete. Thus, to be $(k-1)$-connected implies that $X^{(k)}_{\underbrace{1,\ldots,1,}_{i}\,0,\bullet,\ldots,\bullet}$ is equal to the point viewed as a constant $(n-i+1)$-fold Segal space.

By the following proposition this definition satisfies the delooping hypothesis. In practice this allows to define a k-monoidal n-fold complete Segal space step-by-step by defining a tower of monoidal i-hybrid $(n+i)$-fold Segal spaces for $0 \leqslant i < k$.

Proposition 1.6.23. *The data of a k-monoidal n-fold complete Segal space is the same as a tower of monoidal i-hybrid* $(n+i)$*-fold Segal spaces* $X^{(i+1)}$ *for* $0 \leqslant i < k$ *together with weak equivalences*

$$X^{(j)} \simeq L(X^{(j+1)})$$

for every $0 \leqslant j < k-1$.

Remark 1.6.24. We say that these equivalent data endow the complete n-fold Segal space

$$X = X^{(0)} \simeq L(X^{(1)})$$

with a k-monoidal structure. The $(n+i+1)$-fold Segal space $X^{(i+1)}$ is called an *i-fold delooping of* X.

Before we prove this proposition, we need some lemmas:

Lemma 1.6.25. *If* X *is a k-monoidal m-hybrid n-fold Segal space, and* $0 \leqslant l \leqslant k$*, then* X *is also an l-monoidal* $(m+k-l)$*-hybrid* $(n+k-l)$*-fold Segal space.*

Proof. Since X is a k-monoidal m-hybrid n-fold Segal space, X is a $(m+k)$-hybrid $(n+k)$-fold Segal space such that for every $0 \leqslant i < k$,

$$X_{\underbrace{1,\ldots,1,}_{i}\,0,\ldots,0} = *,$$

so in particular, this also holds for $0 \leqslant i < l$. □

Lemma 1.6.26. *Let* X *be a k-monoidal m-hybrid n-fold Segal space. Then* $\mathrm{Hom}_X(*,*)$ *is a* $(k-1)$*-monoidal* $(m-1)$*-hybrid n-fold Segal space.*

Proof. This follows from

$$(\mathrm{Hom}_X(*,*))_{\bullet,\ldots,\bullet} = \{*\} \times^h_{X_{0,\bullet,\ldots,\bullet}} X_{1,\bullet,\ldots,\bullet} \times^h_{X_{0,\bullet,\ldots,\bullet}} \{*\} = X_{1,\bullet,\ldots,\bullet},$$

since $X_{0,\bullet,\ldots,\bullet}$ is a point. □

Proof of Proposition 1.6.23. Let X be a k-monoidal n-fold complete Segal space. By Lemma 1.6.25 $X^{(k)} = X$ is a monoidal $(k-1)$-hybrid $(n+k-1)$-fold Segal space.

Now let $X^{(k-1)} = L(X^{(k)})$. By Lemmas 1.6.26 and 1.6.25, this is a monoidal $(k-2)$-hybrid $(n+k-2)$-fold Segal space.

Inductively, define $X^{(i)} = L(X^{(i+1)})$ for $1 \leqslant i \leqslant k-1$. Similarly to above, by Lemmas 1.6.26 and 1.6.25, this is a monoidal $(i-1)$-hybrid $(n+i-1)$-fold Segal space.

Conversely, assume we are given a tower $X^{(i)}$ as in the proposition. Since $X = X^{(k)}$ is a monoidal $(k-1)$-hybrid $(n+k-1)$-fold Segal space,

$$X_{0,\bullet,\dots,\bullet} = X^{(k)}_{0,\bullet,\dots,\bullet} = *. \tag{1.1}$$

Since $X^{(k-1)}$ is a monoidal $(k-2)$-hybrid $(n+k-2)$-fold Segal space and by (1.1),

$$\begin{aligned} X_{1,0,\bullet,\dots,\bullet} = X^{(k)}_{1,0,\bullet,\dots,\bullet} &= \{*\} \times^h_{X^{(k)}_{0,0,\bullet,\dots,\bullet}} X^{(k)}_{1,0,\bullet,\dots,\bullet} \times^h_{X^{(k)}_{0,0,\bullet,\dots,\bullet}} \{*\} \\ &= \left(\mathrm{Hom}^{(k)}_X(*,*)\right)_{0,\bullet,\dots,\bullet} \\ &\simeq X^{(k-1)}_{0,\bullet,\dots,\bullet} = *. \end{aligned} \tag{1.2}$$

Since $X^{(k)}$ is k-hybrid, $X_{1,0,\bullet,\dots,\bullet}$ is discrete and so $X_{1,0,\bullet,\dots,\bullet} = *$.

Inductively, for $0 \leqslant i < k$, since $X^{(k-i)}$ is a monoidal $(k-i-1)$-hybrid $(n+k-i-1)$-fold Segal space and by (1.1), (1.2),...

$$\begin{aligned} X_{\underbrace{1,\dots,1,}_{i}\,0,\bullet,\dots,\bullet} &= X^{(k)}_{\underbrace{1,\dots,1,}_{i}\,0,\bullet,\dots,\bullet} \\ &= \{*\} \times^h_{X^{(k)}_{0,\underbrace{1,\dots,1,}_{i-1}\,0,\bullet,\dots,\bullet}} X^{(k)}_{\underbrace{1,\dots,1,}_{i}\,0,\bullet,\dots,\bullet} \times^h_{X^{(k)}_{0,\underbrace{1,\dots,1,}_{i-1}\,0,\bullet,\dots,\bullet}} \{*\} \\ &= \left(\mathrm{Hom}^{(k)}_X(*,*)\right)_{\underbrace{1,\dots,1,}_{i-1}\,0,\bullet,\dots,\bullet} \\ &\simeq X^{(k-1)}_{\underbrace{1,\dots,1,}_{i-1}\,0,\bullet,\dots,\bullet} = \ldots \simeq X^{(k-i)}_{0,\bullet,\dots,\bullet} = *. \end{aligned}$$

Again, since $X^{(k)}$ is k-hybrid, $X_{\underbrace{1,\dots,1,}_{i}\,0,\bullet,\dots,\bullet}$ is discrete and so

$$X_{\underbrace{1,\dots,1,}_{i}\,0,\bullet,\dots,\bullet} = *.$$

□

Symmetric monoidal n-fold complete Segal spaces

The Stabilization Hypothesis, first formulated in [BD95], states that an n-category which is monoidal of a sufficiently high degree cannot be made "more monoidal", and thus it makes sense to call it symmetric monoidal, see e.g. [Sim98] for a proof for Tamsamani's weak n-categories.

Hypothesis 1.6.27 (Stabilization Hypothesis)**.** For $k \geqslant n+2$, a k-monoidal n-category is the same thing as an $(n+2)$-monoidal n-category.

In the world of (∞, n)-categories this statement must be false, otherwise, any E_n-algebra, which can be thought of as an n-monoidal, n-connected (∞, n)-category would already be symmetric monoidal, i.e. commutative.

However, in light of Proposition 1.6.23, we can require that a symmetric monoidal (∞, n)-category is k-monoidal for every $k \geqslant 0$ to encode that it is "monoidal enough".

Definition 1.6.28. A *symmetric monoidal structure* on a complete n-fold Segal space X is a tower of monoidal i-hybrid $(n+i)$-fold Segal spaces $X^{(i+1)}$ for $i \geqslant 0$ such that if we set $X = X^{(0)}$, for every $i \geqslant 0$,

$$X^{(i)} \simeq L(X^{(i+1)}).$$

Remark 1.6.29. This definition and definition 1.6.3 given in the previous section should be equivalent. Indeed, we should be able to switch between the definitions by setting

$$X\langle m\rangle_{\bullet,\dots,\bullet} = X^{(1)}_{m,\bullet,\dots,\bullet}.$$

Using the fact that $N(\Delta, \operatorname{Iso}\Delta)$ and $\mathbf{E}_1$ are equivalent as $(\infty, 1)$-categories and Dunn's additivity theorem ("$\mathbf{E}_n$ is $\mathbf{E}_1$ in $\mathbf{E}_1$ in $\dots$ in $\mathbf{E}_1$, n times"), k-monoidal (complete) n-fold Segal spaces in the sense of this section should equivalent to k-monoidal (complete) n-fold Segal spaces in the sense of remark 1.6.7.

Furthermore, since $N(\Gamma, \operatorname{Iso}\Gamma)$ and $\mathbf{E}_\infty$ are equivalent as $(\infty, 1)$-categories, this leads to the desired equivalence. There are certainly details to be worked out about to make these statements precise.

Chapter **2**

The (∞, n)-category of cobordisms

To rigorously define fully extended topological field theories we need a suitable (∞, n)-category of cobordisms, which, informally speaking, has zero-dimensional manifolds as objects, bordisms between objects as 1-morphisms, bordisms between bordisms as 2-morphisms, etc., and for $k > n$ there are only invertible k-morphisms. Finding an explicit model for such a higher category, i.e. defining a complete n-fold Segal space of bordisms, is the main goal of this chapter. We endow it with a symmetric monoidal structure and also consider bordism categories with additional structure, e.g. orientations and framings, which allows us, in section 2.8, to rigorously define fully extended topological field theories.

We build a rather explicit model suitable for our purposes, which will allow us to later define a functor to the desired target category. We will construct a variant of this construction, which is perhaps more conceptual and less adhoc, in subsequent publications.

2.1 The complete n-fold Segal space of closed intervals in $(0,1)$

In this section we define a Segal space $\mathrm{Int}_\bullet$ of closed intervals in $(0,1)$ which will form the basis of the n-fold Segal space of cobordisms. First we define the sets of vertices, i.e. of 0-simplices, of the levels. Then we define the spatial structure of the levels. Next we endow the collection of sets $(\mathrm{Int}_k)_k$ with a simplicial structure which we then extend to the l-simplices of the levels in a compatible way, giving the simplicial structure. Finally, we show that this construction yields a Segal space.

Definition 2.1.1. For an integer $k \geqslant 0$ let

$$\mathrm{Int}_k = \{I_0 \leqslant \cdots \leqslant I_k\}$$

be the set consisting of ordered $(k+1)$-tuples of intervals $I_j \subseteq (0,1)$ with left endpoints a_j and right endpoints b_j such that I_j has non-empty interior, is closed in $(0,1)$, and $a_0 = 0$, $b_k = 1$. By "ordered", i.e. $I_j \leqslant I_{j'}$, we mean that the endpoints are ordered, i.e. $a_j \leqslant a_{j'}$ and $b_j \leqslant b_{j'}$ for $j \leqslant j'$.

2.1.1 The spatial structure of the levels Int_k

The l-simplices of the space Int_k

The l-simplices do not just consist of the data of a smooth family of intervals, but we add an extra rescaling datum which records how the intervals are deformed into each other along the simplex. This is given by a suitable family of order-preserving diffeomorphisms.

Definition 2.1.2. Let $(I_0(s) \leqslant \cdots \leqslant I_k(s)) \in \mathrm{Int}_k$ be a smooth family of intervals over $|\Delta^l|$, i.e. denoting the left endpoints by $a_j(s)$ and the right endpoints by $b_j(s)$, the maps $|\Delta^l| \to \mathbb{R}$, $s \mapsto a_j(s), b_j(s)$ are smooth maps. A smooth family of strictly monotonically increasing diffeomorphisms

$$(\varphi_{s,t} : (0,1) \to (0,1))_{s,t \in |\Delta^l|}$$

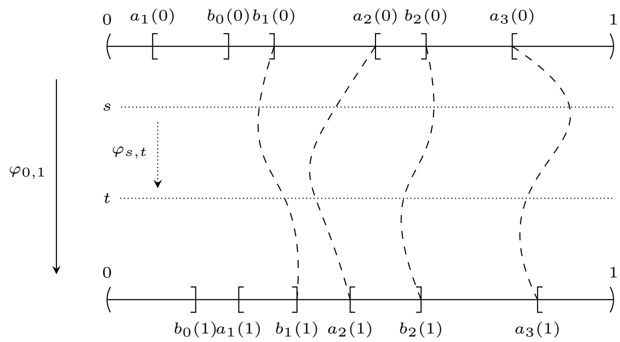
is said to *intertwine with the composed intervals* if the following condition is satisfied for every morphism $f : [m] \to [l]$ in the simplex category Δ.

Let $|f| : |\Delta^m| \to |\Delta^l|$ be the induced map between standard simplices. For every $0 \leqslant j < k$ such that

- either for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ is empty
- or for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ contains only one element,

we require that for every $s \in |f|(|\Delta^m|)$,

$$b_j(s) \overset{\varphi_{s,t}}{\longmapsto} b_j(t), \qquad a_{j+1}(s) \overset{\varphi_{s,t}}{\longmapsto} a_{j+1}(t);$$



Remark 2.1.3. Note that it is enough to check this condition for $m \leqslant l$.

Definition 2.1.4. An *l-simplex of* Int_k consists of

1. a smooth family of *underlying 0-simplices*, i.e. for every $s \in |\Delta^l|$,

$$(I_0(s) \leqslant \cdots \leqslant I_k(s)) \in \mathrm{Int}_k,$$

depending smoothly on s;

2. a *rescaling datum*, which is a smooth family of strictly monotonically increasing diffeomorphisms
$$(\varphi_{s,t} : (0,1) \to (0,1))_{s,t\in|\Delta^l|}$$
such that
   a) $\varphi_{s,s} = id$ for every $s \in |\Delta^l|$,
   b) $\varphi_{s,t} = \varphi_{t,s}^{-1}$ for every $s,t \in |\Delta^l|$, and
   c) $(\varphi_{s,t})_{s,t\in|\Delta^l|}$ intertwines with the composed intervals.

Remark 2.1.5. Note that in particular for $l = 0$ an l-simplex in this sense is an underlying 0-simplex together with $\varphi_{s,s} = id : (0,1) \to (0,1)$, so, by abuse of language we call both a 0-simplex.

Remark 2.1.6. The third condition will imply both that the simplicial structure is well defined (we can cut off intervals at the end) and that the spatial structure is well defined (the family of diffeomorphisms restricts well), as we will see in the next sections.

The space Int_k

The spatial structure arises similarly to that of the singular set of a topological space.

Fix $k \geqslant 0$ and let $f : [m] \to [l]$ be a morphism in the simplex category Δ, i.e. a (weakly) order-preserving map. Then let $|f| : |\Delta^m| \to |\Delta^l|$ be the induced map between standard simplices and let f^Δ be the map sending an l-simplex in Int_k given by
$$(I_0(s) \leqslant \cdots \leqslant I_k(s))_{s\in|\Delta^l|}, \quad \big(\varphi_{s,t} : (0,1) \to (0,1)\big)_{s,t\in|\Delta^l|},$$
to
$$I_0(|f|(s)) \leqslant \ldots \leqslant I_k(|f|(s))_{s\in|\Delta^m|}, \quad \big(\varphi_{|f|(s),|f|(t)} : (0,1) \longrightarrow (0,1)\big)_{s,t\in|\Delta^m|}.$$

Lemma 2.1.7. *This assignment gives a functor $\Delta^{op} \to \mathrm{Set}$ and thus Int_k is a space, i.e. a simplicial set.*

Proof. We need to verify that
$$I_0(|f|(s)) \leqslant \ldots \leqslant I_k(|f|(s))_{s\in|\Delta^m|}, \quad \big(\varphi_{|f|(s),|f|(t)} : (0,1) \longrightarrow (0,1)\big)_{s,t\in|\Delta^m|}$$
is an m-simplex in Int_k. To see this, we need to check that the smooth family of diffeomorphisms $(\varphi_{|f|(s),|f|(t)})_{s,t\in|\Delta^m|}$ intertwines with the composed intervals. Let $g : [m'] \to [m]$, and $|g| : |\Delta^{m'}| \to |\Delta^m|$. Let $0 \leqslant j < k$ such that one of the two conditions on $I_j(|f|(s)) \cap I_{j+1}(|f|(s))$ in definition 2.1.2 is satisfied for every $s \in |g|(|\Delta^{m'}|)$. Then $f \circ g : [m'] \to [l]$ and the same condition is satisfied on $I_j(\tilde{s}) \cap I_j(\tilde{s})$ for every $\tilde{s} \in |f \circ g|(|\Delta^{m'}|)$. Since $(\varphi_{s,t})_{s,t\in|\Delta^l|}$ intertwines with the composed intervals, for every $\tilde{s} \in |f \circ g|(|\Delta^{m'}|)$,
$$b_j(\tilde{s}) \xmapsto{\varphi_{s,t}} b_j(\tilde{t}), \qquad a_{j+1}(\tilde{s}) \xmapsto{\varphi_{s,t}} a_{j+1}(\tilde{t}),$$
so for every $s \in |g|(|\Delta^{m'}|)$, since $\tilde{s} = |f|(s) \in |f \circ g|(|\Delta^{m'}|)$,
$$b_j(|f|(s)) \xmapsto{\varphi_{|f|(s),|f|(t)}} b_j(|f|(t)), \qquad a_{j+1}(|f|(s)) \xmapsto{\varphi_{|f|(s),|f|(t)}} a_{j+1}(|f|(t)).$$

Functoriality follows from the functoriality of the geometric realization. □

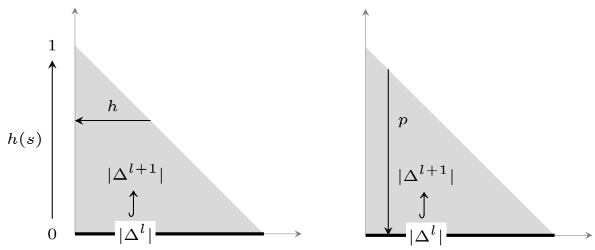
Notation 2.1.8. We denote the spatial face and degeneracy maps of Int_k by d_j^{Δ} and s_j^{Δ} for $0 \leqslant j \leqslant l$.

We will need the following lemma later for the Segal condition.

Proposition 2.1.9. *Each level* Int_k *is a contractible Kan complex. Moreover, the inclusion* $* \hookrightarrow \mathrm{Int}_\bullet$ *given by degeneracies, where* $*$ *is seen as a constant complete Segal space, is a weak equivalence of complete Segal spaces.*

Proof. We first prove the contractibility. For every $k \geqslant 0$, consider the composition of degeneracy maps, which is the inclusion of the point $((0,1) \leqslant \cdots \leqslant (0,1)) \in \mathrm{Int}_k$, which we will by abuse of notation again denote by $(0,1)$. Given an l-simplex, we need to find an $(l+1)$-simplex with one l-dimensional face the given l-simplex and additional vertex $(0,1)$ lying opposite that face.

Let $h : |\Delta^{l+1}| \to \mathbb{R}$ be the coordinate giving the distance from the given l-dimensional face and let $p : |\Delta^{l+1}| \to |\Delta^l|$ be the projection forgetting that coordinate,



For $s \in |\Delta^{l+1}|$ let

$$a_j(s) = (1 - h(s))a_j(p(s)), \qquad b_j(s) = (1 - h(s))b_j(p(s)) + h(s).$$

This is a smooth family of intervals and this construction commutes with the simplicial structure of $\mathrm{Int}_\bullet$.

To obtain the desired $(l+1)$-simplex we need to add a rescaling datum which restricts to the given rescaling datum on the given l-dimensional face. The condition for it to intertwine with the composed bordisms is non-trivial only on that face since every other face and the whole l-simplex itself contain the point $(0,1)$ and thus all pairs of intervals will start to intersect along the face or the whole l-simplex.

Thus it is enough to find a smooth family of order-preserving diffeomorphisms $\big(\varphi_{s,t} : (0,1) \to (0,1)\big)_{s,t \in |\Delta^{l+1}|}$ which restricts to the given one on $|\Delta^l| \hookrightarrow |\Delta^{l+1}|$ and satisfies $\varphi_{s,s} = id$ and $\varphi_{s,t} = \varphi_{t,s}^{-1}$, and in a way commuting with the simplicial structure. This follows from the fact that $\mathrm{Diff}^+(\mathbb{R})$ is contractible.

Then Kan condition follows similarly from the contractibility of $\mathbb{R}$ and $\mathrm{Diff}^+(\mathbb{R})$. $\square$

2.1.2 The simplicial set $\mathrm{Int}_\bullet$

In this subsection, the collection of sets Int_k is endowed with a simplicial structure by extending the assignment

$$[k] \longmapsto \mathrm{Int}_k$$

to a functor from Δ^{op}.

Let $f : [m] \to [k]$ be a morphism in Δ. Then, let

$$\begin{array}{rcl} \mathrm{Int}_k & \xrightarrow{f^*} & \mathrm{Int}_m, \\ I_0 \leqslant \cdots \leqslant I_k & \longmapsto & \rho_f(I_{f(0)} \leqslant \cdots \leqslant I_{f(m)}), \end{array}$$

where the rescaling map ρ_f is the unique affine transformation $\mathbb{R} \to \mathbb{R}$ sending $a_{f(0)}$ to 0 and $b_{f(m)}$ to 1.

Lemma 2.1.10. *The collection of sets* $(\mathrm{Int}_k)_k$ *is a simplicial set.*

Proof. Given two maps $[m] \xrightarrow{f} [k] \xrightarrow{g} [p]$, and $I_0 \leqslant \cdots \leqslant I_p$, the rescaling map $\rho_{g\circ f}$ and the composition of the rescaling maps $\rho_g \circ \rho_f$ both send $a_{g\circ f(0)}$ to 0 and $b_{g\circ f(m)}$ to 1 and, since affine transformations $\mathbb{R} \to \mathbb{R}$ are uniquely determined by the image of two points, this implies that they coincide. Thus, this gives a functor $\Delta^{op} \to \mathrm{Set}$. □

Notation 2.1.11. We denote the *(simplicial) face and degeneracy maps* by $d_j : \mathrm{Int}_k \to \mathrm{Int}_{k-1}$ and $s_j : \mathrm{Int}_k \to \mathrm{Int}_{k+1}$ for $0 \leqslant j \leqslant k$.

Explicitly, they are given by the following formulas. The jth degeneracy map is given by inserting the jth interval twice,

$$\begin{array}{rcl} \mathrm{Int}_k & \xrightarrow{s_j} & \mathrm{Int}_{k+1}, \\ I_0 \leqslant \cdots \leqslant I_k & \longmapsto & I_0 \leqslant \cdots \leqslant I_j \leqslant I_j \leqslant \cdots \leqslant I_k. \end{array}$$

The jth face map is given by deleting the jth interval and, for $j = 0, k$, by rescaling the rest linearly to $(0,1)$. For $j = 0$, the rescaling map is the affine map ρ_0 sending $(a_1, 1)$ to $(0,1)$, $\rho_0(x) = \frac{x-a_1}{1-a_1}$ and for $j = k$, it is the affine map $\rho_k : (0, b_{k-1}) \to (0,1)$, $\rho_k(x) = \frac{x}{b_{k-1}}$. Explicitly,

$$\begin{array}{rcl} \mathrm{Int}_k & \xrightarrow{d_j} & \mathrm{Int}_{k-1}, \\ I_0 \leqslant \cdots \leqslant I_k & \longmapsto & \begin{cases} I_0 \leqslant \cdots \leqslant \hat{I}_j \leqslant \cdots \leqslant I_k, & j \neq 0, k, \\ (0, \frac{b_1-a_1}{1-a_1}] \leqslant \cdots \leqslant [\frac{a_k-a_1}{1-a_1}, 1), & j = 0, \\ (0, \frac{b_0}{b_{k-1}}] \leqslant \cdots \leqslant [\frac{a_{k-1}}{b_{k-1}}, 1), & j = k. \end{cases} \end{array}$$

2.1.3 The Segal space $\mathrm{Int}_\bullet$

The simplicial space $\mathrm{Int}_\bullet$

We first extend the assignment $f \mapsto (f^* : \mathrm{Int}_k \to \mathrm{Int}_m)$ to l-simplices in a compatible way. Essentially, f^* arises from applying f^* to each of 0-simplices underlying the l-simplex.

Let $f : [m] \to [k]$ be a morphism in Δ. Recall that given $(I_0 \leqslant \cdots \leqslant I_k) \in \mathrm{Int}_k$ we have an affine rescaling map $\rho_f : \mathbb{R} \to \mathbb{R}$ which sends $a_{f(0)}$ to 0 and $b_{f(m)}$ to 1.

Given a smooth family $(I_0(s) \leqslant \cdots \leqslant I_k(s))_{s \in |\Delta^l|}$, denote by $\rho_f(s)$ the rescaling map associated to the sth underlying 0-simplex $(I_0(s) \leqslant \cdots \leqslant I_k(s))$. Moreover, denote by $D_j(s) = (a_{f(0)}(s), b_{f(m)}(s))$.

Let f^* send an l-simplex of Int_k

$$(I_0(s) \leqslant \cdots \leqslant I_k(s))_{s \in |\Delta^l|} \qquad (\varphi_{s,t})_{s,t \in |\Delta^l|}$$

to the following l-simplex of Int_m.

1. The underlying 0-simplices of the image are the images of the underlying 0-simplices under f^*, i.e. for $s \in |\Delta^l|$,

$$f^*\left(I_0(s) \leqslant \cdots \leqslant I_k(s)\right);$$

2. its rescaling datum is

$$f^*(\varphi_{s,t}) = \rho_f(t) \circ \varphi_{s,t}|_{D_j(s)} \circ \rho_f(s)^{-1} : (0,1)^n \to (0,1)^n.$$

Note that $(f^*(\varphi_{s,t}))_{s,t \in |\Delta^l|}$ again intertwines well with the composed intervals since it is a restriction of a rescaling datum.

Using the fact that the rescaling maps behave functorially, we obtain the following lemma.

Lemma 2.1.12. *The collection of spaces* $(\mathrm{Int}_k)_k$ *is a simplicial space.*

The complete Segal space $\mathrm{Int}_\bullet$

Proposition 2.1.13. $\mathrm{Int}_\bullet$ *is a complete Segal space.*

Proof. We have seen in lemma 2.1.9 that every Int_k is contractible. This ensures the Segal condition, namely that

$$\mathrm{Int}_k \xrightarrow{\simeq} \mathrm{Int}_1 \overset{h}{\underset{\mathrm{Int}_0}{\times}} \cdots \overset{h}{\underset{\mathrm{Int}_0}{\times}} \mathrm{Int}_1,$$

and completeness. □

Definition 2.1.14. Let

$$\mathrm{Int}^n_{\bullet,\ldots,\bullet} = (\mathrm{Int}_\bullet)^{\times n}.$$

Lemma 2.1.15. *The n-fold simplicial space* $\mathrm{Int}^n_{\bullet,\ldots,\bullet}$ *is a complete n-fold Segal space.*

Proof. The Segal condition and completeness follow from the Segal condition and completeness for $\mathrm{Int}_\bullet$. Since every Int_k is contractible by lemma 2.1.9, $(\mathrm{Int}_\bullet)^{\times n}$ satisfies essential constancy, so Int^n is an n-fold Segal space. □

2.2 A time-dependent Morse lemma

2.2.1 The classical Morse lemma

The following theorem is classical Morse lemma, as can be found e.g. in [Mil63].

Theorem 2.2.1 (Morse lemma). *Let f be a smooth proper real-valued function on a manifold M. Let $a < b$ and suppose that the interval $[a, b]$ contains no critical values of f. Then $M^a = f^{-1}((-\infty, a])$ is diffeomorphic to $M^b = f^{-1}((-\infty, b])$.*

We repeat the proof here since later on in this section we will adapt it to the situation we need.

Proof. Choose a metric on M, and consider the vector field

$$V = \frac{\nabla_y f}{|\nabla_y f|^2},$$

where ∇_y is the gradient on M. Since f has no critical value in $[a, b]$, V is defined in $f^{-1}((a - \epsilon, b + \epsilon))$, for suitable ϵ. Choose a smooth function $g : \mathbb{R} \to \mathbb{R}$ which is 1 on $(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$ and compactly supported in $(a - \epsilon, b + \epsilon)$. Extend g to a function $g : M \to \mathbb{R}$ by setting $g(y) = g(f(y))$. Then

$$\mathcal{V} = g \frac{\nabla_y f}{|\nabla_y f|^2}$$

is a compactly supported vector field on M and hence generates a 1-parameter group of diffeomorphisms

$$\psi_t : M \longrightarrow M.$$

Viewing $f - (a + t)$ as a function on $\mathbb{R} \times M$, $(t, y) \mapsto f(y) - (a + t)$, we find that in $f^{-1}((a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}))$,

$$\partial_t(f - (a + t)) = 1 = \frac{\nabla_y f}{|\nabla_y f|^2} \cdot (f - (a + t)) = V \cdot (f - (a + t)),$$

and so the flow preserves the set

$$\{(t, y) : f(y) = a + t\}.$$

Thus, the diffeomorphism ψ_{b-a} restricts to a diffeomorphism

$$\psi_{b-a}|_{M^a} : M^a \longrightarrow M^b.$$

□

2.2.2 The time-dependent Morse lemma

In Lemma 3.1 in [GWW] Gay, Wehrheim, and Woodward prove a time-dependent Morse lemma which shows that a smooth family of composed cobordisms in their (ordinary) category of (connected) cobordisms gives rise to a diffeomorphism which intertwines with the cobordisms. We adapt this lemma to a variant which will be suitable for our situation in the higher categorical setting.

Proposition 2.2.2. *Let M be a smooth manifold and let $(f_s : M \to (0,1))_{s\in[0,1]}$ be a smooth family of smooth functions which give rise to a smooth proper function $f : N = [0,1] \times M \to (0,1)$. Let $(I_0(s) \leqslant \cdots \leqslant I_k(s))_{s\in[0,1]}$ be a smooth family of closed intervals in $(0,1)$ such that for every $s \in [0,1]$, the function f_s has no critical value in $I_0(s) \cup \cdots \cup I_k(s)$. Then there is a rescaling datum $(\varphi_{s,t} : (0,1) \to (0,1))_{s,t\in[0,1]}$ which makes $(I_0(s) \leqslant \cdots \leqslant I_k(s))_{s\in[0,1]}$ into a 1-simplex in Int_k, and a smooth family of diffeomorphisms $(\psi_{s,t} : M \to M)_{s,t\in[0,1]}$ such that for*

$$\begin{aligned} t_j(s) \in I_j(s) : &\quad \varphi_{s,t}(t_j(s)) \in I_j(t), \text{ and} \\ t_l(s) \in I_l(s) : &\quad \varphi_{s,t}(t_l(s)) \in I_l(t), \end{aligned}$$

$\psi_{s,t}$ restricts to diffeomorphisms

$$\psi_{s,t}|_{f_s^{-1}([t_j,t_l])} : f_s^{-1}([t_j, t_l]) \longrightarrow f_t^{-1}([\varphi_{s,t}(t_j), \varphi_{s,t}(t_l)]).$$

Proof. The main strategy of the proof is the same as for the classical Morse lemma. Namely, we will construct a suitable vector field whose flow gives the desired diffeomorphisms.

Step 1: disjoint intervals

First assume that for all $0 \leqslant j \leqslant k$ and for every $s \in [0,1]$ we have

$$I_j(s) \cap I_{j+1}(s) = \varnothing.$$

Fix a metric on M. Denote the endpoints of the intervals by $a_j(s), b_j(s)$ as before, which yield smooth functions $a_j, b_j : [0,1] \to (0,1)$, and let

$$A_j = \bigcup_{s\in[0,1]} \{s\} \times f_s^{-1}(a_j(s)), \qquad B_j = \bigcup_{s\in[0,1]} \{s\} \times f_s^{-1}(b_j(s)).$$

Now for $0 \leqslant j \leqslant k$ consider the vector fields

$$V_j = \left(\partial_s, \partial_s(a_j(s) - f_s)\frac{\nabla_y f_s}{|\nabla_y f_s|^2}\right), \qquad W_j = \left(\partial_s, \partial_s(b_j(s) - f_s)\frac{\nabla_y f_s}{|\nabla_y f_s|^2}\right),$$

where ∇_y is the gradient on M. Since f_s has no critical value in $I_j(s)$, the vector fields V_j and W_j are defined on $f^{-1}(U_j)$, where U_j is a neighborhood of $\bigcup_{s\in[0,1]}\{s\} \times I_j(s)$. Moreover, viewing $a_j : (s,y) \mapsto a_j(s)$ as a function on N,

$$V_j(f - a_j) = \partial_s(f - a_j) + \partial_s(a_j - f)\frac{\nabla_y f}{|\nabla_y f|^2}(f - a_j) = \partial_s(f - a_j) + \partial_s(a_j - f) = 0,$$

So the vector field V_j is tangent to A_j and similarly, W_j is tangent to B_j.

We would now like to construct a vector field $\mathcal{V}$ on N which for every $0 \leqslant j \leqslant k$, at A_j restricts to V_j and at B_j restricts to W_j, and such that there exists a family of functions $\big(c_x : [0,1] \to (0,1)\big)_{x\in I_j(0)}$ such that

- $c_x(0) = x$, $c_x(s) \in I_j(s)$,

- the graphs of c_x for varying x partition $\bigcup_{s\in[0,1]}\{s\} \times [a_j(s), b_j(s)]$, and
- $\mathcal{V}$ is tangent to $C_x = \bigcup_{s\in[0,1]}\{s\} \times f_s^{-1}(c_x(s))$.

We will use c_x to define $\varphi_{0,s}(x) = c_x(s)$ and $\varphi_{s,t} = \varphi_{0,t} \circ \varphi_{0,s}^{-1}$. Moreover, the diffeomorphisms $\psi_{s,t}$ will arise as the flow along $\mathcal{V}$.

Fix smooth functions $g_j, h_j : [0,1] \times (0,1) \to \mathbb{R}$ which satisfy the following conditions:

1. g_j, h_j are compactly supported in U_j,
2. $g_j = 1$ in a neighborhood of graph $a_j = \{(s, a_j(s)) : s \in [0,1]\}$,
   $h_j = 1$ in a neighborhood of graph b_j
3. $g_j + h_j = 1$ in $\bigcup_{s\in[0,1]}\{s\} \times I_j(s)$, and the supports of the $g_j + h_j$ are disjoint.

By abuse of notation, extend the functions g_j, h_j to functions $g_j, h_j : N = [0,1] \times M \to \mathbb{R}$ by setting $g_j(s,y) := g_j(s, f_s(y))$. Then consider the following vector field on N:

$$\mathcal{V}_j = \left(\partial_s, (g_j\partial_s(a_j) + h_j\partial_s(b_j) - \partial_s(f))\,\frac{\nabla_y f}{|\nabla_y f|^2}\right)$$

This vector field is supported on the support of $g_j + h_j$ and thus extends to a vector field on N. Note that for $(s,y) \in A_j$, $\mathcal{V}_j(s,y) = V_j(s,y)$, and for $(s,y) \in B_j$, $\mathcal{V}_j(s,y) = W_j(s,y)$.

Now let $\mathcal{V}$ be the vector field on N constructed by combining the above vector fields as follows:

$$\mathcal{V} = \left(\partial_s, \sum_{0\leqslant j\leqslant k} (g_j\partial_s(a_j) + h_j\partial_s(b_j) - \partial_s(f))\,\frac{\nabla_y f_s}{|\nabla_y f_s|^2}\right).$$

Note that in $\bigcup_{s\in[0,1]}\{s\} \times f_s^{-1}(I_j(s))$, it restricts to $\mathcal{V}_j$.

In order for $\mathcal{V}$ to be tangent to C_x, the functions c_x must satisfy the following equation at points in C_x.

$$\begin{aligned} 0 &\overset{!}{=} \mathcal{V}_j(f - c_x) \\ &= \partial_s(f - c_x) + (g_j\partial_s(a_j) + h_j\partial_s(b_j) - \partial_s(f))\,\frac{\nabla f}{|\nabla f|^2}(f - c_x) \\ &= -\partial_s(c_x) + g_j\partial_s(a_j) + h_j\partial_s(b_j). \end{aligned}$$

This leads to the ordinary differential equation with smooth coefficients on $[0,1]$,

$$\begin{aligned} \partial_s(c_x)(s) &= g_j(s, c_x(s))\partial_s(a_j)(s) + h_j(s, c_x(s))\partial_s(b_j)(s), \\ c_x(0) &= x. \end{aligned}$$

By Picard-Lindelöf, it has a unique a priori local solution. To see that it extends to $s \in [0,1]$, consider the smooth function $F : N \to [0,1] \times (0,1), F(s,y) = (s, f(s,y)) = (s, f_s(y))$. Since f is proper, so is F. Moreover, $C_x = F^{-1}(\text{graph}\, c_x)$. For fixed x, we can show that C_x lies in a compact part of $N = [0,1] \times M$ similarly to the argument given in example 2.3.2, and thus the local solution exists for all $s \in [0,1]$.

We now define our rescaling data essentially by following the curve c_x. Explicitly, let $\varphi_{0,s} : (0,1) \to (0,1)$ be defined on $[a_j(0), b_j(0)]$ by sending x_0 to $c_{x_0}(s)$. Note that by construction, it sends $a_j(0), b_j(0)$ to $a_j(s), b_j(s)$. Since the solution c_x of the ODE varies smoothly with respect to the initial value x this map is a diffeomorphism. So we can define $\varphi_{s,t} : (0,1) \to (0,1)$ on $[a_j(s), b_j(s)]$ by sending $x_s = c_{x_0}(s)$ to $c_{x_0}(t)$. We extend $\varphi_{s,t}$ to a diffeomorphism in between these intervals in the following way. Let $\tilde{g}_j, \tilde{h}_j : [b_j(0), a_{j+1}(0)] \to \mathbb{R}$ be a partition of unity such that $\tilde{g}_j$ is strictly decreasing, $\tilde{g}_j(b_j(s)) = 1$, and $\tilde{h}_j(a_{j+1}(s)) = 1$. Then, for $x_0 \in [b_j(0), a_{j+1}(0)]$ set

$$c_{x_0}(s) = \tilde{g}_j(x_0) c_{b_j(0)}(s) + \tilde{h}_j(x_0) c_{a_{j+1}(0)}(s) \quad \text{and} \quad \varphi_{s,t}(c_{x_0}(s)) = c_{x_0}(t).$$

As mentioned above, we obtain the diffeomorphisms $\psi_{s,t}$ by flowing along the vector field $\mathcal{V}$. Since $\mathcal{V}$ is tangent to the sets $C_x = \bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}(c_x(s))$ for $x \in I_0(0) \cup \cdots \cup I_k(0)$, the flow preserves C_x, and $\bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}([b_j(s), a_{j+1}(s)])$ in between. Again, this implies that the flow exists for all $s \in [0,1]$. It is of the form $\Psi(t-s, (s,y)) = (t, \psi_{s,t}(y))$ for $0 \leqslant s \leqslant t \leqslant 1$, where $(\psi_{s,t})_{s,t \in [0,1]}$ is a family of diffeomorphisms and intertwines with the composed bordisms with respect to the rescaling data $\varphi_{s,t}$.

Step 2: common endpoints

Now consider the case that for $0 \leqslant j \leqslant k$ we have that either for every $s \in [0,1]$, $I_j(s) \cap I_{j+1}(s) = \varnothing$ as in the previous case or for every $s \in [0,1]$ we have

$$|I_j(s) \cap I_{j+1}(s)| = 1.$$

In this case, one can modify the above argument. We explain for the case of two intervals with one common endpoint, i.e. $b_j(s) = a_{j+1}(s)$.

Instead of choosing smooth functions $g_j, h_j, g_{j+1}, h_{j+1} : [0,1] \times (0,1) \to \mathbb{R}$ such that the supports of $g_j + h_j$ and $g_{j+1} + h_{j+1}$ are disjoint (which now is not possible), we fix three smooth functions $f_j, g_j, h_j : [0,1] \times (0,1) \to \mathbb{R}$ which satisfy the following conditions:

1. f_j, g_j, h_j are compactly supported in $U_j \cup U_{j+1}$,
2. $f_j = 1$ in a neighborhood of $\operatorname{graph} a_j = \{(s, a_j(s)) : s \in [0,1]\}$,
   $g_j = 1$ in a neighborhood of $\operatorname{graph} b_j = \operatorname{graph} a_{j+1}$,
   $h_j = 1$ in a neighborhood of $\operatorname{graph} b_{j+1}$,
3. $f_j + g_j + h_j = 1$ in $\bigcup_{s \in [0,1]} \{s\} \times (I_j(s) \cup I_{j+1}(s))$, and the support of the $f_j + g_j + h_j$ is disjoint to the sums associated to the other intervals.

Now continue the proof similarly to above.

Step 3: overlapping intervals

It remains to consider the case when for some $0 \leqslant j \leqslant k$ and some $s \in [0,1]$,

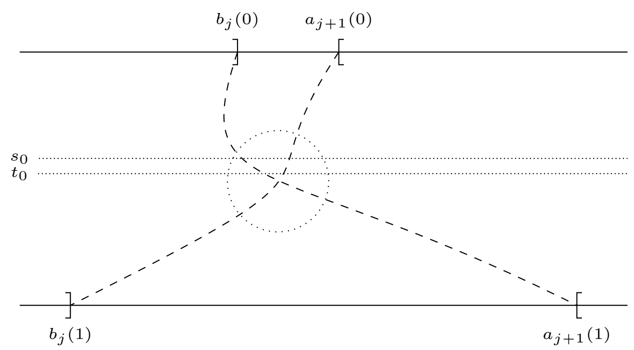
$$I_j(s) \cap I_{j+1}(s)$$

has non-empty interior.

Intervals always overlap. First, if $I_j(s) \cap I_{j+1}(s)$ has non-empty interior for every $s \in [0,1]$, then one can do the above construction with the intervals $I_j(s), I_{j+1}(s)$ replaced by the interval $I_j(s) \cup I_{j+1}(s)$.

Intervals do not always overlap. If $I_j(s) \cap I_{j+1}(s)$ sometimes has non-empty interior, but not for every $s \in [0,1]$, we can combine the cases treated so far.

We explain the process in the case that there is an $\tilde{s}$ such that for $s < \tilde{s}$, $I_j(s) \cap I_{j+1}(s) = \varnothing$ and for $s \geqslant \tilde{s}$, $I_j(s) \cap I_{j+1}(s) \neq \varnothing$. In this case, $\tilde{x} = b_j(\tilde{s}) = a_{j+1}(\tilde{s})$, which is a regular value of $f_{\tilde{s}}$. Since f is smooth, there is an open ball U_j centered at $(\tilde{s}, \tilde{x})$ in $[0,1] \times (0,1)$ such that for $(s,x) \in U$, x is a regular value of f_s. Let $\tilde{\tilde{s}} < \tilde{s}$ be such that for every $\tilde{\tilde{s}} \leqslant s \leqslant \tilde{s}$, the set $\{s\} \times [a_j(s), b_{j+1}(s)]$ is covered by $U \cup \big(\{s\} \times (I_j(s) \cup I_{j+1}(s))\big)$. Choose s_0 and t_0 such that $\tilde{\tilde{s}} \leqslant s_0 < t_0$.



In $[0, t_0]$, we are in the situation of disjoint intervals and can use the first construction to obtain $c_x^{(2)}(s)$ and $\mathcal{V}^{(2)}(s,y)$ for $s \leqslant t_0$.

In $[s_0, 1]$, we apply the construction from step 1 to the intervals $I_j(s)$ and $I_{j+1}(s)$ replaced by the interval $[a_j(s), b_{j+1}(s)]$ to obtain $c_x^{(2)}(s)$ and $\mathcal{V}^{(2)}(s,y)$ for $s \geqslant s_0$.

Now choose a partition of unity $G, H : [0,1] \to \mathbb{R}$ such that $G|_{[0,s_0]} = 1, H|_{[t_0,1]} = 1$, and G is strictly decreasing on $[s_0, t_0]$. For $s < t$ define

$$c_x(s) = G(s)c_x^{(1)}(s) + H(s)c_x^{(2)}(s), \qquad \mathcal{V}(s,y) = G(s)\mathcal{V}^{(1)}(s,y) + H(s)\mathcal{V}^{(2)}(s,y).$$

Then define $\varphi_{s,t}$ and $\psi_{s,t}$ as before. □

2.3 The (∞, n)-category of bordisms Bord_n

In this section we define an n-fold Segal space PBord_n in several steps. However, it will turn out not to be complete. By applying the completion functor we obtain a complete n-fold Segal space, the (∞, n)-category of bordisms Bord_n.

Let V be a finite dimensional vector space. We first define the levels relative to V with elements being certain submanifolds of the (finite dimensional) vector space $V \times (0,1)^n \cong V \times \mathbb{R}^n$. Then we let V vary, i.e. we take the limit over all finite dimensional vector spaces lying in some fixed infinite dimensional vector space, e.g. $\mathbb{R}^\infty$. The idea behind this process is that by Whitney's embedding theorem, every manifold can be embedded in some large enough vector space, so in the limit, we include representatives of every n-dimensional manifold. We use $V \times (0,1)^n$ instead of $V \times \mathbb{R}^n$ as in this case the spatial structure is easier to write down explicitly.

2.3.1 The level sets $(\mathrm{PBord}_n)_{k_1,\ldots,k_n}$

For $S \subseteq \{1,\ldots,n\}$ denote the projection from $(0,1)^n$ onto the coordinates indexed by S by $\pi_S : (0,1)^n \to (0,1)^S$.

Definition 2.3.1. Let V be a finite dimensional vector space. For every n-tuple $k_1,\ldots,k_n \geqslant 0$, let $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ be the collection of tuples $(M, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\ldots,n})$, satisfying the following conditions:

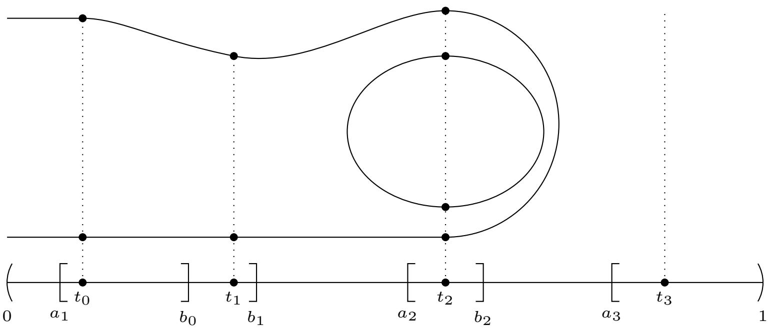
1. M is a closed n-dimensional submanifold of $V \times (0,1)^n$ and the composition $\pi : M \hookrightarrow V \times (0,1)^n \twoheadrightarrow (0,1)^n$ is a proper map.
2. For $1 \leqslant i \leqslant n$,
$$(I_0^i \leqslant \cdots \leqslant I_{k_i}^i) \in \mathrm{Int}_{k_i}.$$
3. For every $S \subseteq \{1,\ldots,n\}$, let $p_S : M \xrightarrow{\pi} (0,1)^n \xrightarrow{\pi_S} (0,1)^S$ be the composition of π with the projection π_S onto the S-coordinates. Then for every $1 \leqslant i \leqslant n$ and $0 \leqslant j_i \leqslant k_i$, at every $x \in p_{\{i\}}^{-1}(I_{j_i}^i)$, the map $p_{\{i,\ldots,n\}}$ is submersive.

Remark 2.3.2. For $k_1,\ldots,k_n \geqslant 0$, one should think of an element in $(\mathrm{PBord}_n)_{k_1,\ldots,k_n}$ as a collection of $k_1 \cdots k_n$ composed bordisms, with k_i composed bordisms with collars in the ith direction. They can be understood as follows.

- Condition 3 in particular implies that at every $x \in p_{\{n\}}^{-1}(I_j^n)$, the map $p_{\{n\}}$ is submersive, so if we choose $t_j^n \in I_j^n$, it is a regular value of $p_{\{n\}}$, and so $p_n^{-1}(t_j^n)$ is an $(n-1)$-dimensional manifold. The embedded manifold M should be thought of as a composition of n-bordisms and $p_n^{-1}(t_j^n)$ is one of the $(n-1)$-bordisms in the composition.
- At $x \in p_{\{n-1\}}^{-1}(I_j^{n-1})$, the map $p_{\{n-1,n\}}$ is submersive, so for $t_l^{n-1} \in I_l^{n-1}$, the preimage
$$p_{\{n-1,n\}}^{-1}\Big((t_l^{n-1}, t_j^n)\Big)$$
is an $(n-2)$-dimensional manifold, which should be thought of as one of the $(n-2)$-bordisms which are connected by the composition of n-bordisms M. Moreover, again since $p_{\{n-1,n\}}$ is submersive at $p_{\{n-1\}}^{-1}(I_l^{n-1})$, the preimage $p_{\{n-1\}}^{-1}(t_l^{n-1})$ is a trivial $(n-1)$-bordism between the $(n-2)$-bordisms it connects.
- Similarly, for $(t_{j_k}^k,\ldots,t_{j_n}^n) \in I_{j_k}^k \times \cdots \times I_{j_n}^n$, the preimage
$$p_{\{k,\ldots,n\}}^{-1}\Big((t_{j_k}^k,\ldots,t_{j_n}^n)\Big)$$
is a $(k-1)$-dimensional manifold, which should be thought of as one of the $(k-1)$-bordisms which is connected by the composition of n-bordisms M.

- Moreover, the following proposition shows that different choices of "cutting points" $t^i_j \in I^i_j$ lead to diffeomorphic bordisms. One should thus think of the n-bordisms we compose as $\pi^{-1}(\prod_{i=1}^n [b^i_j, a^i_{j+1}])$, and the preimages of the specified intervals as collars of the bordisms along which they are composed.

We will come back to this interpretation in section 2.6 when we compute homotopy (bi)categories.



Proposition 2.3.3. *For $1 \leqslant i \leqslant n$ let $u^i_j, v^i_j \in I^i_j$ and $u^i_{j+1}, v^i_{j+1} \in I^i_{j+1}$. Then there is a diffeomorphism*

$$p_{\{i\}}^{-1}([u^i_j, u^i_{j+1}]) \longrightarrow p_{\{i\}}^{-1}([v^i_j, v^i_{j+1}].$$

Proof. Since the map $p_{\{i\}}$ is submersive in I^i_j and I^i_{j+1}, we can apply the Morse lemma 2.2.1 to $p_{\{i\}}$ twice to obtain diffeomorphisms

$$p_{\{i\}}^{-1}([u^i_j, u^i_{j+1}]) \longrightarrow p_{\{i\}}^{-1}([v^i_j, u^i_{j+1}]) \longrightarrow p_{\{i\}}^{-1}([v^i_j, v^i_{j+1}]).$$

□

Applying the proposition successively for $i = 1, \ldots, n$ yields

Corollary 2.3.4. *Let $B_1, B_2 \subseteq (0,1)^n$ be products of closed intervals with endpoints lying in the same I^i_j's. Then there is a diffeomorphism*

$$\pi^{-1}(B_1) \longrightarrow \pi^{-1}(B_2).$$

2.3.2 The spaces $(\mathrm{PBord}_n)_{k_1 \ldots, k_n}$

The level sets $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ form the underlying set of 0-simplices of a space which we construct in this subsection.

The l-simplices of the space $(\mathrm{PBord}^V_n)_{k_1 \ldots, k_n}$

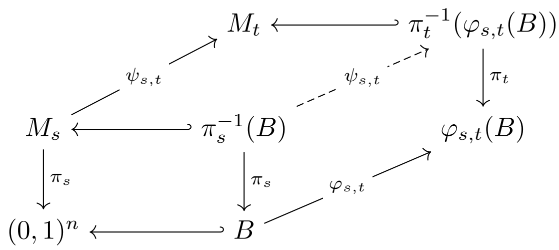
Let $|\Delta^l|$ denote the standard geometric l-simplex.

Definition 2.3.5. An l-simplex of $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ consists of the following data:

1. A smooth family of *underlying 0-simplices*, which is a smooth family of elements
$$\big(M_s \subseteq V \times (0,1)^n, (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))_{i=1,\ldots,n}\big) \in (\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$$
indexed by $s \in |\Delta^l|$. By this we mean that $\bigcup_{s\in|\Delta^l|}\{s\} \times M_s \subseteq |\Delta^l| \times V \times (0,1)^n$ is a smooth submanifold with corners, and that the endpoint maps a_j^i, b_j^i of the intervals are smooth;

2. For every $1 \leqslant i \leqslant n$, a *rescaling datum* $\big(\varphi_{s,t}^i : (0,1) \to (0,1)\big)_{s,t\in|\Delta^l|}$ which together with
$$\big(I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s)\big)_{s\in|\Delta^l|}$$
is an l-simplex in Int_{k_i};

3. A smooth family of *diffeomorphisms*
$$(\psi_{s,t} : M_s \longrightarrow M_t)_{s,t\in|\Delta^l|},$$
such that $\psi_{s,s} = id_{M_s}$ and $\psi_{s,t} = \psi_{t,s}^{-1}$, which *intertwine with the composed bordisms* with respect to the product of the rescaling data $\varphi_{s,t} = (\varphi_{s,t}^i)_{i=1}^n : (0,1)^n \to (0,1)^n$. By this we mean the following. Denoting by π_s the composition $M_s \hookrightarrow V \times (0,1)^n \twoheadrightarrow (0,1)^n$, for $1 \leqslant i \leqslant n$ and $0 \leqslant j_i, l_i \leqslant k_i$ let
$$\begin{array}{lll} t_{j_i}^i(s) \in I_{j_i}^i(s) \text{ such that} & \varphi_{s,t}(t_{j_i}^i(s)) \in I_{j_i}^i(t), & \text{and} \\ t_{l_i}^i(s) \in I_{l_i}^i(s) \text{ such that} & \varphi_{s,t}(t_{l_i}^i(s)) \in I_{l_i}^i(t). & \end{array}$$
Then $\psi_{s,t}$ restricts to a diffeomorphism
$$\pi_s^{-1}\left(\prod_{i=1}^n [t_{j_i}^i(s), t_{l_i}^i(s)]\right) \xrightarrow{\psi_{s,t}} \pi_s^{-1}\left(\prod_{i=1}^n [\varphi_{s,t}(t_{j_i}^i(s)), \varphi_{s,t}(t_{l_i}^i(s))]\right),$$
i.e. denoting $B = \prod_{i=1}^n [t_{j_i}^i(s), t_{l_i}^i(s)]$,



Remark 2.3.6. The condition that the diffeomorphisms $\psi_{s,t}$ intertwine with the composed bordisms in the elements of the family means that $\psi_{s,t}$ induces diffeomorphisms of the composed bordisms in the family and the rescaling data remembers to which choice of cutoffs the specified diffeomorphism restricts.

Remark 2.3.7. In the above definition we let the intervals vary as $s \in |\Delta^l|$ varies. In practice, when dealing with a fixed element of an l-simplex, we can assume that these intervals are fixed as s varies by choosing a fixed vertex $t_0 \in |\Delta^l|_0$ and composing each ι_s with $\varphi_{s,t_0} : (0,1)^n \to (0,1)^n$ and keeping the intervals constant at $I_j^i(t_0)$. This new path is connected by a homotopy to the original one.

The space $(\mathrm{PBord}_n)_{k_1,\ldots,k_n}$

We now lift the spatial structure of $\mathrm{Int}^n_{k_1,\ldots,k_n}$ to $(\mathrm{PBord}_n)_{k_1,\ldots,k_n}$.

Fix $k \geqslant 0$ and let $f : [m] \to [l]$ be a morphism in the simplex category Δ, i.e. a (weakly) order-preserving map. Then let $|f| : |\Delta^m| \to |\Delta^l|$ be the induced map between standard simplices.

Let f^Δ be the map sending an l-simplex in $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ to the m-simplex which consists of

1. for $s \in |\Delta^m|$,
$$M_{|f|(s)} \subseteq V \times (0,1)^n;$$
2. for $1 \leqslant i \leqslant n$, the m-simplex in Int_{k_i} obtained by applying f^Δ,
$$f^\Delta\Big(I^i_0(s) \leqslant \cdots \leqslant I^i_{k_i}(s), \varphi^i_{s,t}\Big);$$
3. for $s, t \in |\Delta^m|$,
$$\psi_{|f|(s),|f|(t)} : M_{|f|(s)} \longrightarrow M_{|f|(t)}.$$

Proposition 2.3.8. $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ *is a space. Moreover, it is a Kan complex, i.e. fibrant in the category of simplicial sets with Quillen model structure.*

Proof. The above assignment indeed is well-defined since the underlying assignment for the underlying intervals is well-defined and the conditions on $\psi_{|f|(s),|f|(t)}$ are a special case of those on $\psi_{s,t}$. Moreover, since this structure essentially comes from the spatial structure of Int_{k_i} and the simplicial structure of $N(\Delta)$, the assignment is functorial.

It remains to show that this space is a Kan complex. A morphism $\Lambda^l_k \to \mathrm{PBord}^V_n$ is the data of, for $s, t \in |\Lambda^l_k|$,

$$\big(M_s \subseteq V \times (0,1)^n, (I^i_0(s) \leqslant \cdots \leqslant I^i_{k_i}(s))_{i=1,\ldots,n}\big) \in (\mathrm{PBord}^V_n)_{k_1,\ldots,k_n},$$
$$\big(\varphi^i_{s,t} : (0,1) \to (0,1)\big), \text{and} \qquad \big(\psi_{s,t} : M_s \longrightarrow M_t\big).$$

We have seen in 2.1.9 in particular that every Int_k is a Kan complex. The proof is completed by the fact that the inclusions of spaces $\mathrm{Sub}^{\mathrm{sm}}(M, V \times (0,1)^n \hookrightarrow \mathrm{Sub}(M, V \times (0,1)^n$ and $\mathrm{Diff}^{\mathrm{sm},+}(M) \hookrightarrow \mathrm{Diff}^+(M)$ are weak equivalences (see e.g.), and both $\mathrm{Sub}(M, V \times (0,1)^n$ and $\mathrm{Diff}^+(M)$ are Kan complexes by an argument similar to that of showing that the singular complex of a manifold is a Kan complex. □

Notation 2.3.9. We denote the *spatial face and degeneracy maps* of $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ by d^Δ_j and s^Δ_j for $0 \leqslant j \leqslant l$.

So far the definition depends on the choice of the vector space V. However, in the bordism category we need to consider all (not necessarily compact) n-dimensional manifolds. By Whitney's embedding theorem any such manifold can be embedded into some $V \times (0,1)^n$ for some finite dimensional vector space V, so we need to allow big enough vector spaces.

Definition 2.3.10. Fix some (countably) infinite dimensional vector space, e.g. $\mathbb{R}^\infty$. Then

$$\mathrm{PBord}_n = \varinjlim_{V \subset \mathbb{R}^\infty} \mathrm{PBord}_n^V .$$

Example: Cutoff path

We now construct an example of a path which will be used several times later on. It shows that cutting off part of the collar of a bordism yields an element which is connected to the original one by a path.

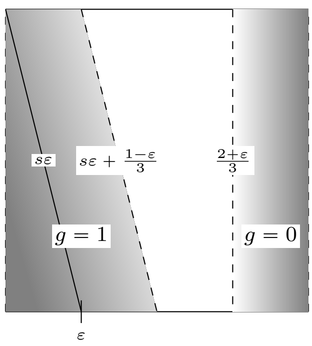
Let $(M) = (M \subseteq V \times (0,1)^n, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\dots,n}) \in (\mathrm{PBord}_n)_{k_1,\dots,k_n}$. We show that cutting off a short enough piece at an end of an element of $(\mathrm{PBord}_n)_{k_1,\dots,k_n}$ leads to an element which is connected by a path to the original one. Explicitly, for ε small enough, we show that there is a 1-simplex with underlying 0-simplices

$$\big(\iota_s : M_s \hookrightarrow V \times (0,1)^n, (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))_{i=1}^n\big) \in (\mathrm{PBord}_n^V)_{k_1,\dots,k_n}$$

such that $M_s = p_i^{-1}((s\varepsilon, 1))$ and $I_j^i(s) = \rho_s(I_j^i)$, where $\rho_s : (s\varepsilon, 1) \to (0,1)$ is the affine rescaling map $x \mapsto \frac{x - s\varepsilon}{1 - s\varepsilon}$, and

$$\iota_s : M_s \subseteq V \times (0,1)^{n-1} \times (s\varepsilon, 1) \xrightarrow{id \times \rho_s} V \times (0,1)^n.$$

Fix $1 \leqslant i \leqslant n$ and let $\varepsilon < b_0^i$. Let N be the manifold $[0,1] \times M \subseteq [0,1] \times V \times (0,1)^n$ endowed with the induced metric, and view p_i as a function on N by setting $p_i(s,y) = p_i(y)$. Choose a smooth cutoff function $g : [0,1] \times (0,1) \to \mathbb{R}$ such that $g = 1$ in a neighborhood U_ε of $\{(s,z) : z \in [s\varepsilon, s\varepsilon + \frac{1-\varepsilon}{3})\}$ and $g = 0$ on $U_1 = [0,1] \times (\frac{2+\varepsilon}{3}, 1)$ and extend g to N by setting $g(s,y) = g(s, p_i(y))$.



Consider the vector field on N given by

$$V = (\partial_s, \varepsilon g \frac{\nabla_y p_i}{|\nabla_y p_i|^2}),$$

where ∇_y denotes the gradient on M. Note that over U_ε, $V = (\partial_s, \varepsilon \frac{\nabla_y p_i}{|\nabla_y p_i|^2})$ and over U_1, $V = 0$. We now show that the flow along the vector field V exists for (s,y) such that $s\varepsilon < p_i(x) < 1$, For $\xi < \frac{1-\varepsilon}{3}$, $(s, s\varepsilon+\xi) \in U_\varepsilon$, and, defining $p_i - s\varepsilon + \xi$ to be the function $(s, y) \mapsto p_i(y) - s\varepsilon + \xi$ on N,

$$V \cdot (p_i - (s\varepsilon + \xi)) = -\varepsilon + \varepsilon \frac{\nabla_y p_i}{|\nabla_y p_i|^2}(p_i - (s\varepsilon + \xi)) = 0. \tag{2.1}$$

For $\alpha \neq i$ and $\xi_\alpha \in (0,1)$, since all components of V except for the ith are 0,

$$V \cdot (p_\alpha - \xi_\alpha) = 0, \tag{2.2}$$

where again we view $p_\alpha - \xi_\alpha$ as a function on N. Let

$$\vec{\xi} : [0,1] \to [0,1] \times (0,1)^n, \quad s \mapsto (s, \xi_1, \ldots, \xi_{i-1}, s\varepsilon + \xi, \xi_{i+1}, \ldots, \xi_n).$$

Equations 2.1 and 2.2 imply that the flow of V preserves the sets

$$\Xi_{\vec{\xi}} = \{(s, y) : \pi(y) = \vec{\xi}(s)\} = (id_{[0,1]} \times \pi)^{-1}(\operatorname{graph} \vec{\xi}).$$

The graph of $\vec{\xi}$ is closed and therefore compact as it is a closed subset of $[0,1] \times \{\xi_1\} \cdots \times [\xi, \varepsilon + \xi] \times \cdots \times \{\xi_n\}$. Since π is proper, $id \times \pi$ is proper, and thus $\Xi_{\vec{\xi}}$ is compact. Hence in

$$\{(s, y) : s\varepsilon < p_i(y) < \varepsilon + \frac{1-\varepsilon}{3}\}$$

the flow exists for all $s \in [0,1]$.

In U_1, the flow is of the form $\Psi(t - s, (s, y)) = (t, y)$ and so it also exists for $s \in [0,1]$.

For points $(s_0, y) \in N$ such that $p_i(y) \in [s_0\varepsilon + \frac{1-\varepsilon}{3}, \frac{2+\varepsilon}{3}]$, the flow preserves the set

$$\Xi_{\vec{\xi}} = (id_{[0,1]} \times \pi)^{-1}(\operatorname{graph} \vec{\xi}),$$

where $\vec{\xi} : s \mapsto (s, \xi_1, \ldots, \xi_{i-1}, \xi_i(s), \xi_{i+1}, \ldots, \xi_n)$, and $\vec{\xi}(s_0) = y$, and $\xi_i(s)$ is a solution at points in Ξ_ξ of the ordinary differential equation with smooth coefficients

$$\begin{aligned} 0 &\stackrel{!}{=} V \cdot (p_i - \xi_i) \\ &= -\partial_s \xi_i + \varepsilon g \partial_s \xi_i \frac{\nabla_y p_i}{|\nabla_y p_i|^2}(p_i - \xi_i) \\ &= -\partial_s \xi_i + \varepsilon g. \end{aligned}$$

By Picard-Lindelöf, this ordinary differential equation has a unique, a priori local, solution. Similarly, the flow exists locally. Furthermore, the preimage of the proper map $(id_{[0,1]} \times \pi)$ of the compact set $[0,1] \times [\frac{1-\varepsilon}{3}, \frac{2+\varepsilon}{3}]$ is compact. Since $\Xi_{\vec{\xi}}$ is a subset of this preimage, we are looking for solutions of the above differential equation on this compact manifold. By compactness, they exist globally and therefore the flow exists for all $s \in [0,1]$.

Piecing this together, the flow takes on the form

$$\Psi(t - s, y) = (t, \psi_{s,t}(y))$$

for $s\varepsilon < p_i(y)$ and exists for all $s \in [0,1]$. This gives the desired family of diffeomorphisms $\psi_{s,t} : p_i^{-1}((s\varepsilon, 1)) \to p_i^{-1}((t\varepsilon, 1))$. The rescaling data $\varphi_{s,t} : (0,1)^n \to (0,1)^n$ is the identity on coordinates except for the ith, where it is given by

$$\varphi^i_{s,t}(x_s) = \begin{cases} \rho_s(x_s + (t-s)\varepsilon), & \text{for } x_s < s\epsilon + \frac{1-\varepsilon}{3}, \\ \rho_s(x_s), & \text{for } x_s > \frac{2+\varepsilon}{3}, \\ \rho_s(\xi_i(t)), & \text{for } s\epsilon + \frac{1-\varepsilon}{3} \leqslant x_s \leqslant \frac{2+\varepsilon}{3}, \end{cases}$$

where ξ_i is the integral curve through x_s, which is the solution to the differential equation above.

Remark 2.3.11. In the above example we constructed a path from an element in $(\mathrm{PBord}_n^V)_{k_1,\dots,k_n}$ to its "cutoff", where we cut off the preimage of $p_i^{-1}((0,\varepsilon])$ for suitably small ε. Note that the same argument holds for cutting off the preimage of $p_i^{-1}([1-\delta,1))$ for suitably small δ. Moreover, we can iterate the process and cut off ε_i, δ_i strips in all i directions. Choosing $\varepsilon_i = \frac{b_0^i}{2}, \delta_i = \frac{a_{k_i}^i}{2}$ yields a path to its "cutoff" with underlying submanifold

$$cut(M) = \pi^{-1}\Big(\prod_{i=1}^{n}(\frac{b_0^i}{2}, \frac{a_{k_i}^i}{2})\Big).$$

The map $\pi : M \to (0,1)^n$ is proper, which implies that $\pi^{-1}(\prod_{i=1}^n[\frac{b_0^i}{2}, \frac{a_{k_i}^i}{2}]) \supset cut(M)$ is compact and thus bounded in the V-direction. Thus, any element in $(\mathrm{PBord}_n^V)_{k_1,\dots,k_n}$ is connected by a path to an element whose underlying submanifold is bounded in the V-direction.

Variants of the spatial structure

Following [Lur09c], one could define the spatial structure of $(\mathrm{PBord}_n^V)_{k_1,\dots,k_n}$ as follows to obtain classifying spaces of bordisms:

1. One could make $(\mathrm{PBord}_n^V)_{k_1,\dots,k_n}$ into a topological space (instead of a simplicial set) by endowing it with the following topology coming from the Whitney topology.

   On the set $\mathrm{Sub}(V \times (0,1)^n)$ of closed (not necessarily compact) submanifolds $M \subseteq V \times (0,1)^n$, a neighborhood basis at M is given by

   $$\{N \hookrightarrow V \times (0,1)^n : N \cap K = j(M) \cap K, j \in W\},$$

   where $K \subseteq V \times (0,1)^n$ is compact and $W \subseteq \mathrm{Emb}(M, V \times (0,1)^n)$ is a neighborhood of the inclusion $M \hookrightarrow V \times (0,1)^n$ in the Whitney C^∞-topology (see [Gal11]). Using the standard topology on $\mathbb{R}$ and the product topology gives a topology on

   $$\mathrm{Sub}(V \times \mathbb{R}^n) \times \bigcup_{i=1}^{n}\bigcup_{j=1}^{k_1}\{a_j^i, b_{j-1}^i \in [0,1] : a_j^i < b_j^i\}.$$

   We take the quotient topology of this topology with respect to the relation identifying elements $(M_0, I_j^i(0)\text{'s})$, $(M_1, I_j^i(1)\text{'s})$ if the preimages of the boxes $B(l) = [b_0^1(l), a_{k_1}^1(l)] \times \cdots \times [b_0^n(l), a_{k_n}^n(l)]$ for $l = 0,1$, respectively, under their composition with the projection to $(0,1)^n$ coincide, i.e.

   $$\pi_1^{-1}(B(0)) = \pi_2^{-1}(B(1)),$$

   where for $l = 0,1$, $\pi_l : M \hookrightarrow V \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$. Finally, $(\mathrm{PBord}_n^V)_{k_1,\dots,k_n}$ is a subspace thereof.

   However, the reason for our choice of using simplicial sets instead of spaces is that we eventually want to construct a fully extended topological field theory and the levels of our target which we construct in the next chapter will be naturally modelled as simplicial sets. Thus it is more natural to also model the levels of our source category, the bordism category, as simplicial sets. If one would rather have topological spaces as the spatial structure of the levels, one can apply geometric realization to the simplicial sets.

2. To model the levels of the bordism category as simplicial sets, we could start with the above version as a topological space and take singular or, even better, differentiable chains of this space to obtain a simplicial set. Then, the l-vertices would consist of smooth submanifolds

$$I : \Delta^l \times M \hookrightarrow \Delta^l \times V \times (0,1)^n,$$

where I commutes with the projections to Δ^l, such that $\forall s \in |\Delta^l|$,

$$(M_s = \mathrm{Im}(I(s,-)) \subseteq V \times (0,1)^n, (I_0^i(s) \leqslant \ldots \leqslant I_{k_i}^i(s))_{i=1,\ldots,n}) \in (\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}.$$

Note that as abstract manifolds, $M_s = M$, but as submanifolds, they are diffeomorphic images of the same abstract manifold along the path. Thus, there are diffeomorphisms

$$\psi_{s,t} : M_s \longrightarrow M_t$$

as in our definition. Moreover, for $l = 1$, proposition 2.3.12 below, which is a corollary of proposition 2.2.2, the time-dependent Morse lemma, implies that there exists such a family of diffeomorphisms and some rescaling data which intertwine. So paths in this simplicial set yield paths in ours.

It also implies that for $l > 0$, given any two fixed points $s, t \in |\Delta^l|$, we obtain a diffeomorphism $\psi_{s,t}$ and a rescaling function $\varphi_{s,t}$, by applying the lemma to any path between s and t and defining $\psi_{s,t} = \psi_{0,1}$ and $\varphi_{s,t} = \varphi_{0,1}$. Choosing the same (shape of) partitions of unity in this process we get a smooth family of such diffeomorphisms, so l-simplices in this simplicial set yield l-simplices in ours.

Moreover, the diffeomorphisms appearing are all isotopic to the identity and therefore arise as the flow of a (time-dependent) vector field [Thu74], which however may not be of the form of the ones we considered. We nevertheless believe that the two simplicial sets are weakly equivalent under the map simply forgetting the family of diffeomorphisms.

Proposition 2.3.12. *Consider a smooth one-parameter family of embeddings*

$$(I : [0,1] \times M \hookrightarrow [0,1] \times V \times (0,1)^n, [0,1] \ni s \mapsto (I_0^i(s) \leqslant \ldots \leqslant I_{k_i}^i(s))_{i=1,\ldots,n}),$$

which gives rise to

$$(M_s) = (M \xhookrightarrow{I(s,-)} V \times (0,1)^n, (I_0^i(s) \leqslant \ldots \leqslant I_{k_i}^i(s))_{i=1,\ldots,n})$$

in $(\mathrm{PBord}_n)_{k_1,\ldots,k_n}$. *Then there is a rescaling data* $(\varphi_{s,t} : (0,1)^n \to (0,1)^n)_{s,t\in[0,1]}$ *and a family of diffeomorphisms* $(\psi_{s,t} : M \to M)_{s,t\in[0,1]}$ *which intertwines with the rescaling data.*

Proof. For $1 \leqslant i \leqslant n$, let $0 \leqslant j_i \leqslant k_i - 1$. Let

$$\pi_s : M \xhookrightarrow{I(s,-)} V \times (0,1)^n \twoheadrightarrow (0,1)^n$$

and denote by $(p_i)_s : M \to (0,1)$ the composition of π_s with the projection to the ith coordinate. Note that by condition 3 in definition 2.3.1, the function $(p_i)_s$ does not have a critical point in $I_0^i(s) \cup \ldots \cup I_{k_i}^i(s)$.

We cannot quite apply the the time-dependent Morse lemma 2.2.2 to $(p_i)_s$, because we only have properness of the individual π_s, and moreover, this would ensure intertwining only in the ith direction. However, we can adapt the proof to our situation.

Choosing the metric on M coming from $I(0,-)$, and following the proof of the proposition 2.2.2, for each i we get a vector field

$$\begin{aligned}\mathcal{V}^i &= \left(\partial_s, \sum_{0\leqslant j\leqslant k} (g_j\partial_s(a_j) + h_j\partial_s(b_j) - \partial_s(p_i))\frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2}\right)\\ &=: \left(\partial_s, \Pi_i(s,y)\frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2}\right).\end{aligned}$$

We combine them to obtain a new vector field on $[0,1]\times M$,

$$\tilde{\mathcal{V}} = \left(\partial_s, \sum_{i=1}^{n} \Pi_i(s,y)\frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2}\right).$$

The projections $(p_i)_0$ and $(p_j)_0$ are orthogonal with respect to the metric on M induced by the embedding $I(0,-)$, and moreover, $(p_i)_s$, $(p_j)_s$ stay orthogonal along the path, because the change of metric on M induced by the change of embedding respects orthogonality on $(0,1)^n$. This implies that

$$\frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2}p_j = \delta_{ij},$$

and so $\tilde{\mathcal{V}}$ still is tangent to the respective C^i_x in each direction and thus its flow, if it exists globally, will give rise to the desired diffeomorphisms and rescaling data.

The global existence follows from the special form of the vector field. Given a point $(t, y_t) \in N$, the flow will preserve a set of the form

$$\{(s,y) : \pi_s(y_s) = (c^1_{x_0}(s), \ldots, c^n_{x_0}(s)) = (\xi_1(s), \ldots, \xi_n(s))\},$$

where the right hand side is in the notation of example 2.3.2, and $\vec{c}_{x_0}(t) = \vec{\xi}(t) = y_t$. Similarly to in the example, one can show that this set lies in a compact part of N and thus the flow exists globally. □

2.3.3 The n-fold simplicial set $(\mathrm{PBord}_n)_{\bullet,\cdots,\bullet}$

In the next two subsections we will make the collection of spaces $(\mathrm{PBord}_n)_{\bullet,\ldots,\bullet}$ into an n-fold simplicial space by lifting the simplicial structure of $\mathrm{Int}^{\times n}_{\bullet,\ldots,\bullet}$. In this section we define the structure on 0-simplices, which makes $(\mathrm{PBord}_n)_{\bullet,\ldots,\bullet}$ into an n-fold simplicial set. In the next subsection we extend the structure to l-vertices of the levels to obtain an n-fold simplicial space $(\mathrm{PBord}_n)_{\bullet,\ldots,\bullet}$.

Fixing $1 \leqslant i \leqslant n$, we first need to extend the assignment

$$[k_i] \longmapsto (\mathrm{PBord}_n)_{k_1,\ldots,k_n}$$

to a functor from Δ^{op}. Let $f : [m_i] \to [k_i]$ be a morphism in the simplex category Δ, i.e. a (weakly) order-preserving map. Then we need to define the map

$$(\mathrm{PBord}_n)_{k_1,\ldots,k_i,\ldots k_n} \xrightarrow{f^*} (\mathrm{PBord}_n)_{k_1,\ldots,m_i,\ldots k_n}.$$

Notation 2.3.13. Recall that the map $f^* : \mathrm{Int}_{k_i} \to \mathrm{Int}_{m_i}$ is defined using an affine rescaling map $\rho_f : \mathbb{R} \to \mathbb{R}$ which sends $a^i_{f(0)}$ to 0 and $b^i_{f(m)}$ to 1 and thus restricts to a diffeomorphism $\rho_f : D_f = (a^i_{f(0)}, b^i_{f(m)}) \to (0,1)$. By abuse of notation, we again denote by ρ_f the map

$$\rho_f : V \times \prod_{\alpha \neq i} (0,1) \times (a^i_{f(0)}, b^i_{f(m)}) \to V \times (0,1)^n,$$

which is ρ_f in the ith component of $(0,1)^n$ and the identity otherwise.

Definition 2.3.14. Let $f : [m_i] \to [k_i]$ be a morphism in Δ. Then

$$(\mathrm{PBord}_n)_{k_1,\ldots,k_i,\ldots k_n} \xrightarrow{f^*} (\mathrm{PBord}_n)_{k_1,\ldots,m_i,\ldots k_n}.$$

applies f^* to the ith tuple of intervals and perhaps cuts the manifold and rescales. Explicitly, it sends an element

$$(M) := (\iota : M \hookrightarrow V \times (0,1)^n, (I_0^\alpha \leqslant \cdots \leqslant I_{k_\alpha}^\alpha)_{\alpha=1}^n)$$

to

$$\left(\rho_f \circ \iota|_{p_i^{-1}(D_f)} : p_i^{-1}(D_f) \hookrightarrow V \times (0,1)^n, (I_0^\alpha \leqslant \cdots \leqslant I_{k_i}^\alpha)_{\alpha \neq i}, f^*(I_0^i \leqslant \cdots \leqslant I_{k_i}^i)\right).$$

Remark 2.3.15. In the following, we will omit explicitly writing out the restriction of ι to $p_i^{-1}(D_f)$ for readability.

Notation 2.3.16. We denote the *(simplicial) face and degeneracy maps* by $d^i_j : (\mathrm{PBord}_n)_{k_1,\ldots,k_n} \to (\mathrm{PBord}_n)_{k_1,\ldots,k_i-1,\ldots,k_n}$ and $s^i_j : (\mathrm{PBord}_n)_{k_1,\ldots,k_n} \to (\mathrm{PBord}_n)_{k_1,\ldots,k_i+1,\ldots,k_n}$ for $0 \leqslant j \leqslant k_i$.

Proposition 2.3.17. $(\mathrm{PBord}_n)_{\bullet,\ldots,\bullet}$ *is an n-fold simplicial set.*

Proof. This follows from the fact that $\mathrm{Int}_\bullet$ is a simplicial set and rescalings behave functorially. □

Remark 2.3.18. Recall from remark 2.3.2 that for $k_1, \ldots, k_n \geqslant 0$, one should think of an element in the set $(\mathrm{PBord}_n)_{k_1,\ldots,k_n}$ as a collection of $k_1 \cdots k_n$ composed bordisms with k_i composed bordisms with collars in the ith direction. These composed collared bordisms are the images under the maps

$$D(j_1, \ldots, j_k) : (\mathrm{PBord}_n)_{k_1,\ldots,k_n} \longrightarrow (\mathrm{PBord}_n)_{1,\ldots,1}$$

for $(1 \leqslant j_i \leqslant k_i)_{1 \leqslant i \leqslant n}$ arising as compositions of face maps, i.e. $D(j_1, \ldots, j_k)$ is the map determined by the maps

$$[1] \to [k_i], \quad (0 < 1) \mapsto (j_i - 1 < j_i)$$

in the category Δ of finite ordered sets. This should be thought of as sending an element to the $(j_1, \ldots, j_k)$-th collared bordism in the composition.

2.3.4 The full structure of $(\mathrm{PBord}_n)_{\bullet,\cdots,\bullet}$ as an n-fold simplicial space

In this subsection, we show that the maps defined in the previous paragraph are compatible with the structure of the levels as simplicial sets, i.e. for a morphism $f : [m_i] \to [k_i]$ in the simplex category Δ, we will define compatible maps f^* for l-simplices of the simplicial set $(\mathrm{PBord}_n)_{k_1,\ldots,k_n}$. They will be defined similarly as on vertices, namely by applying the map f^* to each underlying 0-simplex and by perhaps restricting the rescaling data and the diffeomorphisms. For the face and degeneracy maps, this will amount to the following.

- Degeneracy maps arise from the degeneracy maps of $\mathrm{Int}^n_{\bullet,\ldots,\bullet}$ by repeating one of the families of intervals $I^i_j(s)$.

Fix $1 \leqslant i \leqslant n$.

- For $0 < j < k_i$ the jth face map d^i_j arises from the face map of $\mathrm{Int}^n_{\bullet,\ldots,\bullet}$ by deleting the jth family of intervals $I^i_j(s)$ in the ith direction.
- Face maps for $j = 0, k_i$ require cutting and rescaling:

Notation 2.3.19. Recall that for a morphism f of the simplex category Δ, we have a rescaling map $\rho_f : \mathbb{R} \to \mathbb{R}$ which restricts to a diffeomorphism $\rho_f : D_f \to (0,1)$, with $D_f = (a_{f(0)}, b_{f(m)})$. By abuse of notation, we also denote by ρ_f the diffeomorphism $\rho_f : \prod_{\alpha\neq i}(0,1) \times D_f \to (0,1)^n$ which is ρ_f in the ith coordinate and the identity otherwise. Moreover, denote by $\rho_f(s)$ be the analog of the map ρ_f associated to the sth underlying 0-simplex $(I^i_0(s) \leqslant \cdots \leqslant I^i_{k_i}(s)) \in \mathrm{Int}_{k_i}$.

Definition 2.3.20. Let $f : [m_i] \to [k_i]$ be a morphism in the simplex category Δ. Consider an l-simplex of $(\mathrm{PBord}_n)_{k_1,\ldots,k_n}$ consisting of

$$\left(\iota_s : M_s \hookrightarrow V \times (0,1)^n, (I^i_0(s) \leqslant \cdots \leqslant I^i_{k_i}(s))^n_{i=1}\right)_{s\in|\Delta^l|},$$

$$(\varphi_{s,t} : (0,1)^n \longrightarrow (0,1)^n)_{s,t\in|\Delta^l|}, \quad \text{and} \quad (\psi_{s,t} : M_s \longrightarrow M_t)_{s,t\in|\Delta^l|}.$$

Let f^* send it to the l-simplex of $(\mathrm{Bord}_n)_{k_1,\ldots,k_i-1,\ldots,k_n}$ consisting of the following data.

1. The underlying 0-simplices of the image are the images of the underlying 0-simplices under f^*, i.e. for $s \in |\Delta^l|$,

$$\begin{aligned} f^*\left(M_s \subseteq V \times (0,1)^n, (I^i_0(s) \leqslant \cdots \leqslant I^i_{k_i}(s))\right) = \\ = \Big(\rho_f(s) \circ \iota_s|_{N_s} : N_s \hookrightarrow V \times (0,1)^n, \\ (I^\alpha_0(t) \leqslant \cdots \leqslant I^\alpha_{k_\alpha}(t))_{\alpha\neq i}, f^*(I^i_0(t) \leqslant \cdots \leqslant I^i_{k_i}(t))\Big), \end{aligned}$$

   where $N_s = (p_s)_i^{-1}(D_f(s))$.

2. The underlying l-simplex in Int_{k_i} is sent to its image under f^*, i.e. its rescaling data is $f^*(\varphi_{s,t})$. Recall from section 2.1.3 that this is

$$f^*(\varphi_{s,t}) = \rho_f(s) \circ \varphi_{s,t}|_{D_f(s)} \circ \rho_f(s)^{-1} : (0,1)^n \to (0,1)^n.$$

3. Since the diffeomorphisms $\psi_{s,t}$ intertwine with the composed bordisms with respect to the rescaling data $\varphi_{s,t}$, for every $s, t \in \Delta^l$ we have diffeomorphisms
$$\psi_{s,t}|_{N_s} : N_s \to N_t,$$
which intertwine with the (new) composed bordisms with respect to with the (new) rescaling data.

Proposition 2.3.21. *The spatial and simplicial structures of* $(\mathrm{PBord}_n)_{\bullet,\dots,\bullet}$ *are compatible, i.e. for* $g : [l] \to [p]$*,* $f_\alpha : [m_\alpha] \to [k_\alpha]$*, and* $f_\beta : [m_\beta] \to [k_\beta]$*, for* $1 \leqslant \alpha < \beta \leqslant n$*, the induced maps*
$$g^\Delta, f_\alpha^*, \text{and}\, f_\beta^*$$
commute. We thus obtain an n-fold simplicial space $(\mathrm{PBord}_n)_{\bullet,\cdots,\bullet}$.

Proof. By construction, g^Δ commutes with the simplicial structure. Moreover, the maps f_α^*, f_β^* commute since they modify different parts of the structure. ☐

2.3.5 The complete n-fold Segal space Bord_n

Proposition 2.3.22. $(\mathrm{PBord}_n)_{\bullet,\dots,\bullet}$ *is an n-fold Segal space.*

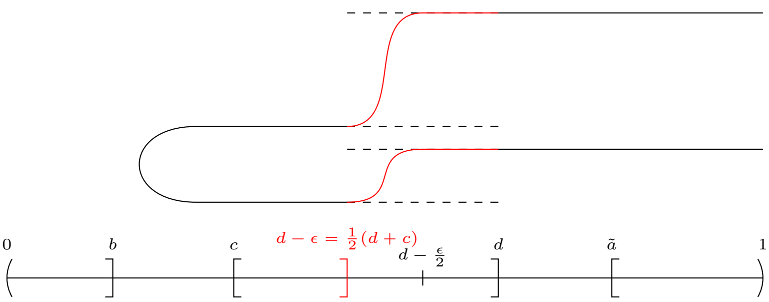
Proof. We need to prove the following conditions:

1. *The Segal condition is satisfied.* For clarity, we explain the Segal condition in the following case. The general proof works similarly. We will show that
$$(\mathrm{PBord}_n)_{k_1,\dots,2,\dots,k_n} \xrightarrow{\sim} (\mathrm{PBord}_n)_{k_1,\dots,1,\dots,k_n} \overset{h}{\underset{(\mathrm{PBord}_n)_{k_1,\dots,0,\dots,k_n}}{\times}} (\mathrm{PBord}_n)_{k_1,\dots,1,\dots,k_n}.$$
We will omit the indices and corresponding intervals for $\alpha \neq i$ for clarity. Our goal is to construct a map *glue* such that $glue \circ (d_0 \times d_2) \sim id$, $(d_0 \times d_2) \circ glue \sim id$,
$$(\mathrm{PBord}_n)_1 \overset{h}{\underset{(\mathrm{PBord}_n)_0}{\times}} (\mathrm{PBord}_n)_1 \underset{d_0 \times d_2}{\overset{glue}{\rightleftarrows}} (\mathrm{PBord}_n)_2$$
Since every level set $(\mathrm{PBord}_n)_{k_1,\dots,k_n}$ is a Kan complex by proposition 2.3.8, i.e. fibrant, the homotopy fiber product consists of triples consisting of two points and a path between them. Choose such an element given by
$$\begin{matrix} (M) = (\iota : M \hookrightarrow V \times (0,1)^n, (0,b] \leqslant [a,1)), \\ (\tilde M) = (\tilde\iota : \tilde M \hookrightarrow V \times (0,1)^n, (0,\tilde b] \leqslant [\tilde a,1)) \end{matrix} \in (\mathrm{PBord}_n)_1 \overset{h}{\underset{(\mathrm{PBord}_n)_0}{\times}} (\mathrm{PBord}_n)_1.$$
We will construct their image under *glue*, which is an element in $(\mathrm{PBord}_n)_2$, essentially by gluing them.

We saw in example 2.3.2 that cutting off a short enough piece at an end of an element of $(\mathrm{PBord}_n)_1$ leads to an element which is connected by a path to the original one, i.e. $(\iota : M \hookrightarrow V \times (0,1)^n, (0,b] \leqslant [a,1)) \sim (\iota : p_i^{-1}((0,1-\epsilon)) \hookrightarrow V \times (0,1)^n, (0,b] \leqslant [a,1-\epsilon))$, composed with suitable rescalings, for $0 < \epsilon < a$. So if the source of our glued element is such a "cutoff", there is a path to the original (M).

Since we started with an element in the homotopy fiber product, there is a path between the target of the first, $N = t((M))$, and the source of the second, $\tilde{N} = s((\tilde{M}))$. Composing this path with the inverse of the path connecting $t((M))$ and its cutoff as described above gives a path between the "cutoff" and $(\tilde{N})$.

Let us now assume that we have rescaled the embeddings and intervals such that they fit into $(0,d)$ respectively $(c,1)$, and moreover, $(a,1)$ and $(0,b)$ are sent to (c,d). Now we glue the embeddings along $(d-\epsilon, d)$ for $\epsilon = \frac{1}{2}(d-c)$ using a partition of unity subordinate to the cover $\{(0, d-\frac{\epsilon}{2}), (d-\epsilon, 1)\}$. This gives a new embedded manifold $\tilde{\tilde{\iota}} : \tilde{\tilde{M}} \hookrightarrow V \times (0,1)^n$ and together with the intervals $(0,b] \leqslant [c, \frac{1}{2}(d-c)] \leqslant [\tilde{a}, 1)$ they form an element in $(\mathrm{PBord}_n)_2$.



This construction extends to l-simplices: by construction, this procedure depends smoothly on the parameter for a smooth family of manifolds and intervals. It remains to see that given an l-simplex in the homotopy fiber product we can obtain a family of diffeomorphisms intertwining with the composed bordisms. This follows because the data of a path between l-simplices includes the data of diffeomorphisms which we can use to patch together the given diffeomorphisms of the source and target.

2. *For every i and every $k_1, \ldots, k_{i-1}$, the $(n-i)$-fold Segal space* $(\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,\bullet,\cdots,\bullet}$ *is essentially constant.*

We show that the degeneracy inclusion map

$$(\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,0,\ldots,0} \longrightarrow (\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$$

admits a deformation retraction and thus is a weak equivalence.

For $s \in [0,1]$, consider the map γ_s sending an element in $(\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$ represented by

$$(M) := \left(M \subseteq V \times (0,1)^n, \left(I_0^\beta \leqslant \cdots \leqslant I_{k_\beta}^\beta\right)_{1\leqslant\beta<i}, (0,1), (I_0^\alpha \leqslant \cdots \leqslant I_{k_\alpha}^\alpha)_{i<\alpha\leqslant n}\right)$$

to

$$(M)_s := \Big(M \subseteq V \times (0,1)^n, (I_0^\beta \leqslant \cdots \leqslant I_{k_\beta}^\beta)_{1\leqslant\beta<i}, (0,1),$$
$$(I_0^\alpha(s) \leqslant \cdots \leqslant I_{k_\alpha}^\alpha(s))_{i<\alpha\leqslant n}\Big),$$

where for $\alpha > i$, $a_j^\alpha(s) = (1-s)a_j^\alpha$ and $b_j^\alpha(s) = (1-s)b_j^\alpha + s$. Note that for $s = 0$, $I_0^\alpha(0) = I_0^\alpha$, $I_j^\alpha(0) = I_j^\alpha$ and for $s = 1$, $I_j^\alpha(1) = (0,1)$.

The maps γ_s form a homotopy between the degeneracy inclusion and the identity on $(\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$ provided that every γ_s indeed maps to this space. It suffices to check condition (3) in definition 2.3.1 for $(M)_s$. Since $(M) \in (\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$, this reduces to checking

> *For every* $i < \alpha \leqslant n$ *and* $0 \leqslant j \leqslant k_\alpha$*, at every* $x \in p_{\{\alpha\}}^{-1}(I_j^\alpha(s))$*, the map* $p_{\{\alpha,\ldots,n\}}$ *is submersive.*

Condition (3) on (M) for i implies that $p_{\{i,\ldots,n\}}$ is a submersion in $p_{\{i\}}^{-1}((0,1)) = M \supset p_{\{\alpha\}}^{-1}(I_j^\alpha(s))$, so $p_{\{\alpha,\ldots,n\}}$ is submersive there as well.

Again, this construction extends to l-simplices since it only involves moving the intervals.

□

Remark 2.3.23. An interesting property of PBord$_n$ is that it also satisfies the strict Segal condition,

$$(\mathrm{PBord}_n)_k \xrightarrow{\simeq} (\mathrm{PBord}_n)_1 \underset{(\mathrm{PBord}_n)_0}{\times} \cdots \underset{(\mathrm{PBord}_n)_0}{\times} (\mathrm{PBord}_n)_1,$$

where as above, we omit all indices except for the ith. This follows from the fact that we can glue the embedded manifolds along open sets. In fact, the maps

$$(\mathrm{PBord}_n)_k \longrightarrow (\mathrm{PBord}_n)_0$$

are fibrations and therefore the homotopy fiber product and the fiber product are weakly equivalent.

The last condition necessary to be a good model for the (∞, n)-category of bordisms is completeness, which PBord$_n$ in general does not satisfy. However, we can pass to its completion Bord$_n$.

Definition 2.3.24. The (∞, n)*-category of cobordisms* Bord$_n$ is the n-fold completion $\widehat{\mathrm{PBord}_n}$ of PBord$_n$, which is a complete n-fold Segal space.

Remark 2.3.25. For $n \geqslant 6$, PBord$_n$ is not complete, see the full explanation in [Lur09c], 2.2.8. For $n = 1$ and $n = 2$, by the classification theorems of one- and two-dimensional manifolds, PBord$_n$ is complete, and therefore Bord$_n$ = PBord$_n$.

2.4 Variants of Bord$_n$ and comparison with Lurie's definition

2.4.1 Bounded submanifolds, cutting points, and $\mathbb{R}$ as a parameter space

Bounded submanifolds

Recall from 2.3.11 that for every element in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$, we constructed a path to its cutoff, whose underlying submanifold

$$cut(M) = \pi^{-1}\Big(\prod_{i=1}^n (\frac{b_0^i}{2}, \frac{a_{k_i}^i}{2})\Big)$$

is bounded in the V-direction. This construction extends to l-simplices and yields a map of n-fold simplicial spaces

$$cut : \mathrm{PBord}_n \longrightarrow \mathrm{PBord}_n,$$

sending an element to its "cutoff". Its image lands in $\mathrm{PBord}_n^{bd} \subseteq \mathrm{PBord}_n$, the sub n-fold Segal space of elements for which the underlying submanifold is bounded in the V-direction. Moreover, it induces a strong homotopy equivalence between PBord_n^{bd} and PBord_n.

Cutting points

Another variant of an n-fold Segal space of cobordisms can be obtained by replacing the intervals I_j^i in definition 2.3.1 of PBord_n by specified "cutting points" $t_j^i \in (0,1)$, which correspond to where we cut our composition of bordisms. Equivalently, we can say that in this case the intervals are replaced by intervals consisting of just one point, i.e. $a_j^i = b_j^i =: t_j^i$. The levels of this n-fold Segal space PBord_n^t can be made into spaces as we did for PBord_n, but we now need to impose the extra condition that elements of the levels are connected by a path if they coincide inside the "box" of t's, i.e. over $[t_0^1, t_{k_1}^1] \times \cdots \times [t_0^n, t_{k_n}^n]$. However, for PBord_n^t the Segal condition is more difficult to prove, as in this case we do specify the collar along which we glue. Since the space of collars is contractible, sending an interval $I = [a,b] \cap (0,1)$ to its midpoint $t = \frac{1}{2}(a+b)$ induces a level-wise weak equivalence from PBord_n to PBord_n^t.

$\mathbb{R}$ as a parameter space

There also is a version of PBord_n replacing the closed intervals $I_j^i \subseteq (0,1)$ by closed intervals in $\mathbb{R}$. We impose conditions on elements in this n-fold Segal space PBord_n^∞ which are analogous to (1)–(3) in definition 2.3.1 of PBord_n. This amounts to using the identification $(0,1) \overset{\chi}{\cong} \mathbb{R}$. However, in this case the face and degeneracy maps d_j^i, s_j^i for $j = 0, k_i$ are more complicated to write down since they require the use of rescaling maps $\rho_0 : (a_1^i, \infty) \to \mathbb{R}$, respectively $\rho_{k_i} : (-\infty, b_{k_i-1}^i) \to \mathbb{R}$. In this case, sending an interval to its midpoint as above leads to an variant with cutting points and $\mathbb{R}$ as a parameter space $\mathrm{PBord}_n^{t,\infty}$.

2.4.2 Comparison with Lurie's definition of cobordisms

In [Lur09c], Lurie defined the n-fold Segal space of cobordisms as follows:

Definition 2.4.1. Let V be a finite dimensional vector space. For every n-tuple $k_1, \ldots, k_n \geqslant 0$, let $(\mathrm{PBord}_n^{V,L})_{k_1,\ldots,k_n}$ be the collection of tuples $(M, (t_0^i \leqslant \ldots \leqslant t_{k_i}^i)_{i=1,\ldots n})$, where

1. M is a closed n-dimensional submanifold of $V \times \mathbb{R}^n$ and the composition $\pi : M \hookrightarrow V \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is a proper map.

2. For $1 \leqslant i \leqslant n$,
$$t_0^i \leqslant \cdots \leqslant t_{k_i}^i$$
is an ordered (k_i+1)-tuple of elements in $\mathbb{R}$.

$\tilde{3}$. For every $S \subseteq \{1, \ldots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \leqslant j_i \leqslant k_i$, the composition $p_S : M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S$ does not have $(t_{j_i})_{i \in S}$ as a critical value.

$\tilde{4}$. For every $x \in M$ such that $p_{\{i\}}(x) \in \{t_0^i, \ldots, t_{k_i}^i\}$, the map $p_{\{i+1,\ldots,n\}}$ is submersive at x.

It is endowed with a topology coming from the Whitney topology similar to what we described in remark 2.3.2, which we will not repeat here. Similarly to before, we define

$$\mathrm{PBord}_n^L = \varinjlim_{V \subset \mathbb{R}^\infty} \mathrm{PBord}_n^{V,L}$$

Comparing this definition with definition 2.3.1 and $\mathrm{PBord}_n^{t,\infty}$ from above, note that our condition (3) on $\mathrm{PBord}_n^{t,\infty}$ is replaced by the two strictly weaker conditions ($\tilde{3}$) and ($\tilde{4}$) on PBord_n^L, which are implied by (3):

Lemma 2.4.2. *Let M be a closed n-dimensional manifold and $\pi : M \to \mathbb{R}^n$. Moreover, for $1 \leqslant i \leqslant n$ let $(t_0^i \leqslant \ldots \leqslant t_{k_i}^i)$ be an ordered $(k_i + 1)$-tuple of elements in $\mathbb{R}$. Denote for $S \subseteq \{1, \ldots, n\}$ the composition $M \xrightarrow{\pi} \mathbb{R}^n \twoheadrightarrow \mathbb{R}^S$ by p_S. Assume that condition (3) from definition 2.3.1 holds, i.e. for every $1 \leqslant i \leqslant n$ and $0 \leqslant j_i \leqslant k_i$ for $x \in M$ such that $p_{\{i\}}(x) = t_{j_i}^i$ the map $p_{\{i,\ldots,n\}}$ is submersive at x. Then,*

$\tilde{3}$. *For every $S \subseteq \{1, \ldots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \leqslant j_i \leqslant k_i$, the composition $p_S : M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S$ does not have $(t_{j_i})_{i \in S}$ as a critical value.*

$\tilde{4}$. *For every $x \in M$ such that $p_{\{i\}}(x) \in \{t_0^i, \ldots, t_{k_i}^i\}$, the map $p_{\{i+1,\ldots,n\}}$ is submersive at x.*

Proof. Let $i_0 = \inf S$. Consider the following diagram

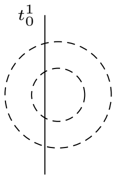
$$\begin{array}{ccc} M & \xrightarrow{p_{\{i_0,\ldots,n\}}} & \mathbb{R}^{\{i_0,\ldots,n\}} \\ & \searrow^{p_S} & \downarrow{\scriptstyle proj} \\ & & \mathbb{R}^S \end{array}$$

For $\tilde{3}$. let $x \in p_S^{-1}((t_{j_i})_{i \in S})$. Then $p_{\{i_0\}}(x) = t_{j_{i_0}}^{i_0}$, so by assumption the map $p_{\{i_0,\ldots,n\}}$ is submersive at x. Since *proj* is submersive, $p_S = proj \circ p_{\{i_0,\ldots,n\}}$ also is submersive at x.

For $\tilde{4}$. note that $p_{\{i,\ldots,n\}}$ being submersive at x implies that $p_{\{i+1,\ldots,n\}}$ is submersive at x. □

However, Lurie's n-fold simplicial space PBord_n^L is not an n-fold Segal space as we will see in the example below. Thus, our $\mathrm{PBord}_n^{t,\infty}$ is a corrigendum of Lurie's PBord_n^L from [Lur09c].

Example 2.4.3.



Consider the 2 dimensional torus T in $\mathbb{R} \times \mathbb{R}^2$, and consider the tuple $(T \hookrightarrow \mathbb{R} \times \mathbb{R}^2, t_0^1, t_0^2 \leqslant \ldots \leqslant t_{k_2}^2)$, where t_0^1 is indicated in the picture of the projection plane $\mathbb{R}^2$ below. Then, because of condition $(\tilde{3})$, $t_0^2 \leqslant \ldots \leqslant t_{k_2}^2$ can be chosen everywhere such that any (t_0^1, t_j^2) is not a point where the vertical $(t_0^1$-)line intersects the two circles in the picture. Thus, the space of these choices is not contractible. However, it satisfies the conditions (1), (2), $(\tilde{3})$, and $(\tilde{4})$ in the definition of $(\mathrm{PBord}_2^L)_{0,k_2}$, so $(\mathrm{PBord}_2^L)_{0,\bullet}$ is not essentially constant.

2.4.3 The n-fold category Bord_n^{uple}

Condition $(\tilde{3})$ in definition 2.4.1 ensures that the fibers $p_S^{-1}\left((t_{j_i})_{i \in S}\right)$ are $(n - |S|)$-dimensional smooth manifolds, i.e. that a k-morphisms, which is an k-dimensional cobordism, indeed goes from a $(k-1)$-dimensional cobordism to another one.

Our condition (3) ensures in addition the globularity condition, i.e. essential constancy, namely that we have an "n-category" instead of an "n-fold category" (unfortunately the use of "fold" here does not match up with the conventions for "fold" and "uple" for Segal spaces). This difference for $n = 2$ is the same as the difference between a "bicategory" and a "double category".

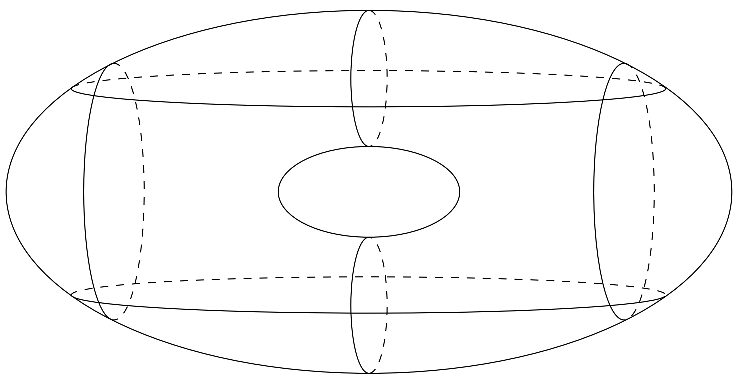
Consider the following interval version of condition $(\tilde{3})$

$\tilde{3}$. For every $S \subseteq \{1, \ldots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \leqslant j_i \leqslant k_i$, the composition $p_S : M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S$ does not have any critical value in $(I_{j_i})_{i \in S}$.

Relaxing our condition (3) in definition 2.3.1 to this $(\tilde{3})$ gives an n-uple Segal space PBord_n^{uple}. Completing gives a complete n-uple Segal space Bord_n^{uple}.

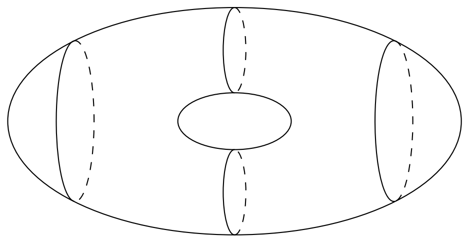
Example 2.4.4 (The torus as a composition)**.** The difference between the n-fold and the n-uple Segal spaces can be seen when decomposing the torus, viewed as a 2-morphism in the respective n-(fold) categories. We will omit drawing the intervals outside of the torus and just draw the "cutting lines", which should be understood as actually extending to small closed intervals around them.

The torus as a 2-morphism in Bord_2^{uple} can be decomposed simultaneously in both directions. One possible decomposition into in some sense elementary pieces is the following:

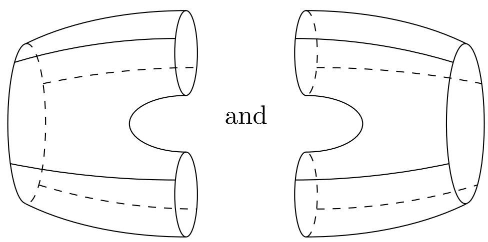


However, similar to the argument in example 2.4.3, this decomposition is not a valid decomposition in Bord_2, as condition 3 in definition 2.3.1 is violated.

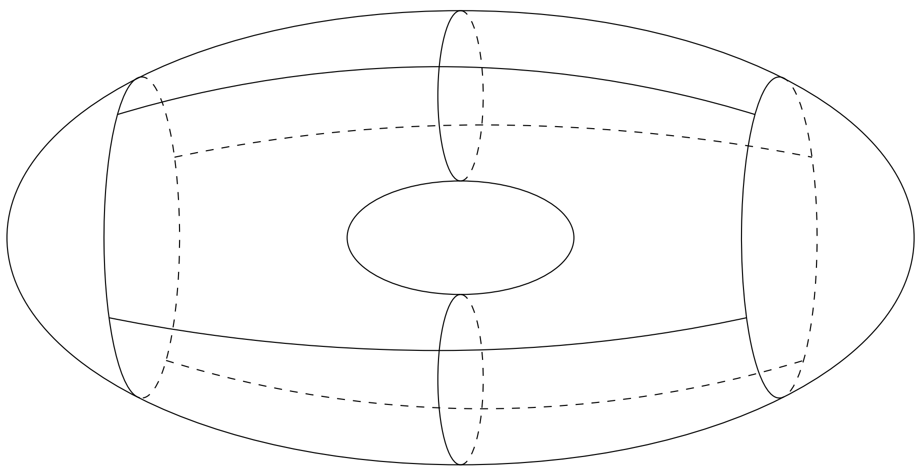
The torus as a 2-morphism in Bord_2 can only be decomposed "successively", so we first decompose it in the first direction, i.e. the first coordinate, e.g. as



which is an element in $(\mathrm{Bord}_2)_{4,1}$ and then decompose the two middle pieces, the images under the face maps $d_1, d_2 : (\mathrm{Bord}_2)_{4,1} \rightrightarrows (\mathrm{Bord}_2)_{1,1}$, as



Altogether a possible decomposition of the torus into elementary pieces in Bord_2 is



This of course also is a valid decomposition in the 2-fold category Bord_2^{uple}.

2.5 The symmetric monoidal structure on Bord_n

The (∞, n)-category Bord_n is symmetric monoidal with its symmetric monoidal structure essentially arising from taking disjoint unions. In this section we endow Bord_n with a symmetric monoidal structure in two ways. In section 2.5.1 the symmetric monoidal structure arises from a Γ-object. In section 2.5.2 a symmetric monoidal structure is defined using a tower of monoidal i-hybrid $(n+i)$-fold Segal spaces.

2.5.1 The symmetric monoidal structure arising as a Γ-object

We construct a sequence of n-fold Segal spaces $(\mathrm{Bord}_n^V[m])_{\bullet,\dots,\bullet}$ which form a Γ-object which endows Bord_n with a symmetric monoidal structure as defined in section 1.6.1.

Definition 2.5.1. Let V be a finite dimensional vector space. For every $k_1, \dots, k_n$, let $(\mathrm{PBord}_n^V[m])_{k_1,\dots,k_n}$ be the collection of tuples

$$(M_1, \dots, M_m, (I_0^i \leqslant \dots \leqslant I_{k_i}^i)_{i=1,\dots,n}),$$

where $M_1, \dots M_m$ are disjoint n-dimensional submanifolds of $V \times (0,1)^n$ and each $(M_\beta, (I_0^i \leqslant \dots \leqslant I_{k_i}^i)_{i=1,\dots n})$ is an element of $(\mathrm{PBord}_n^V)_{k_1,\dots,k_n}$. It can be made into a simplicial set similarly to PBord_n^V. Moreover, similarly to the definition of Bord_n, we take the limit over all $V \subset \mathbb{R}^\infty$ and complete to get an n-fold complete Segal space $\mathrm{Bord}_n[m]$.

Proposition 2.5.2. *The assignment*

$$\begin{aligned} \Gamma &\longrightarrow \mathbf{SSpace_n}, \\ [m] &\longmapsto \mathrm{Bord}_n[m] \end{aligned}$$

extends to a functor and endows Bord_n *with a symmetric monoidal structure.*

Proof. By lemma 1.6.6 it is enough to show that the functor sending $[m]$ to $\mathrm{PBord}_n[m]$ and a morphism $f : [m] \to [k]$ to the morphism

$$\begin{aligned} \mathrm{PBord}_n[m] &\longrightarrow \mathrm{PBord}_n[k], \\ (M_1, \dots, M_m, I's) &\longmapsto (\coprod_{\beta \in f^{-1}(1)} M_\beta, \dots, \coprod_{\beta \in f^{-1}(k)} M_\beta, I's), \end{aligned}$$

is a functor $\Gamma \to \mathbf{SSpace_n}$ with the property that

$$\prod_{1 \leqslant \beta \leqslant n} \gamma_\beta : \mathrm{PBord}_n[m] \longrightarrow (\mathrm{PBord}_n[1])^m$$

is an equivalence of n-fold Segal spaces.

The map $\prod_{1 \leqslant \beta \leqslant n} \gamma_\beta$ is an inclusion of n-fold Segal spaces and we show that level-wise it is a weak equivalence of spaces. Let $\Big((M_1), \dots, (M_n)\Big) \in (\mathrm{PBord}_n[1])^m$. We construct a path to an element in the image of $\prod_{1 \leqslant i \leqslant n} \gamma_\beta$ which induces a strong homotopy equivalence between the above spaces. First, there is a path to an element for which all (M_α) have the same specified intervals by composing all except one with a suitable smooth rescaling. Secondly, there is a path with parameter $s \in [0,1]$ given by composing the embedding $M_\alpha \hookrightarrow V \times (0,1)^n$ with the embedding into $\mathbb{R} \times V \times (0,1)^n$ given by the map $V \to \mathbb{R} \times V$, $v \mapsto (s\alpha, v)$. □

2.5.2 The monoidal structure and the tower

Our goal for this section is to endow Bord_n with a symmetric monoidal structure arising from a tower of monoidal l-hybrid $(n+l)$-fold Segal spaces $\mathrm{Bord}_n^{(l)}$ for $l \geqslant 0$.

The $(\infty, n+l)$-category of n-bordisms for $l \geqslant -n$

We now define an $(n+l)$-fold Segal space whose $(n+l)$-morphisms are n-bordisms for $l \geqslant -n$.

Definition 2.5.3. Let V be a finite dimensional vector space and let $n \geqslant 0, l \geqslant -n$. For every n-tuple $k_1, \ldots, k_{n+l} \geqslant 0$, we let $(\mathrm{PBord}_n^{l,V})_{k_1,\ldots,k_{n+l}}$ be the collection of tuples $(M \hookrightarrow V \times (0,1)^{n+l}, (I_0^i \leqslant \ldots \leqslant I_{k_i}^i)_{i=1,\ldots,n+l})$ satisfying conditions analogous to (1)-(3) in definition 2.3.1, i.e.

1. M is a closed n-dimensional submanifold of $V \times (0,1)^{n+l}$, and $I_j^i \subseteq (0,1)$ are closed intervals in $(0,1)$ with endpoints $a_j^i < b_j^i$, $a_0^i = 0$, $b_{k_i}^i = 1$, and $I_j^i \leqslant I_l^i$ iff $a_j^i \leqslant a_l^i$, $b_j^i \leqslant b_l^i$,
2. the composition $\pi : M \hookrightarrow V \times (0,1)^{n+l} \twoheadrightarrow (0,1)^{n+l}$ is a proper map,
3. for every $S \subseteq \{1, \ldots, n+l\}$ let p_S be the composition $p_S : M \xrightarrow{\pi} (0,1)^{n+l} \to (0,1)^S$. Then for every $1 \leqslant i \leqslant n+l$ and $0 \leqslant j_i \leqslant k_i$, at every $x \in p_{\{i\}}^{-1}(I_{j_i}^i)$, the map $p_{\{i,\ldots,n+l\}}$ is submersive.

We make $(\mathrm{PBord}_n^{l,V})_{k_1,\ldots,k_{n+l}}$ into a space similarly to $(\mathrm{PBord}_n^{V})_{k_1,\ldots,k_n}$, and again we take the limit over all finite dimensional vector spaces in a given infinite dimensional vector space, say $\mathbb{R}^\infty$:

$$\mathrm{PBord}_n^l = \lim_{V \subset \mathbb{R}^\infty} \mathrm{PBord}_n^{l,V}.$$

Proposition 2.5.4. $(\mathrm{PBord}_n^l)_{\bullet,\cdots,\bullet}$ *is an* $(n+l)$*-fold Segal space.*

Proof. The proof is completely analogous to the proof of Proposition 2.3.22. □

Definition 2.5.5. For $l \leqslant 0$ let Bord_n^l be the $(n+l)$-fold completion of PBord_n^l, the $(\infty, n+l)$-category of n-bordisms.

Remark 2.5.6. For $l > 0$, the underlying submanifold of objects of PBord_n^l, i.e. elements in $(\mathrm{PBord}_n^l)_{0,\ldots,0}$, are n-dimensional manifolds M which have a submersion onto $(0,1)^{n+l}$. This implies that $M = \varnothing$. Thus, the only object is $(\varnothing, (0,1), \ldots, (0,1))$. Similarly, $(\mathrm{PBord}_n^l)_{0,k_2,\ldots,k_{n+l}}$ has only one element, which is the image of compositions of the degeneracy maps. Thus, $(\mathrm{PBord}_n^l)_{0,\bullet,\ldots,\bullet}$ is the point viewed as a constant $(n-1)$-fold Segal space. Similarly, $(\mathrm{PBord}_n^l)_{1,\ldots,1,0,\bullet,\ldots,\bullet}$, with $(l-1)$ 1's, is the point viewed as a constant $(n-l)$-fold Segal space. Thus for $l > 0$ it makes sense and is more useful to use the l-hybrid completion of PBord_n^l.

Definition 2.5.7. For $l > 0$ let $\mathrm{Bord}_n^{(l)}$ be the l-hybrid completion of PBord_n^l.

Loopings of PBord_n^l

In any PBord_n^l, there is the distinguished object $\varnothing = (\varnothing, (0,1))$ in PBord_n^l, the unit for the monoidal structure. Recall from definition 1.5.7 the k-fold iterated loopings of PBord_n^l for $k \leqslant n+l$,

$$L_k(\mathrm{PBord}_n^l) = L(L_{k-1}(\mathrm{PBord}_n^l, \varnothing), \varnothing), \quad L_k(\mathrm{Bord}_n^l) = L(L_{k-1}(\mathrm{Bord}_n^l, \varnothing), \varnothing).$$

Proposition 2.5.8. *For $n + l \geqslant k \geqslant 0$, there are weak equivalences*

$$L_k(\mathrm{PBord}_n^l) \underset{\ell}{\overset{u}{\rightleftarrows}} \mathrm{PBord}_n^{l-k}.$$

Proof. We show that $L(\mathrm{PBord}_n^l) = \mathrm{Hom}_{\mathrm{PBord}_n^l}(\varnothing, \varnothing) \simeq \mathrm{PBord}_n^{l-1}$. The statement for general k follows by induction.

We define a map

$$u : L(\mathrm{PBord}_n^l) \xrightarrow{\simeq} \mathrm{PBord}_n^{l-1}$$

by sending an element in $\mathrm{Hom}_{\mathrm{PBord}_n^l}(\varnothing, \varnothing)_{k_2,\dots,k_{n+l}}$,

$$(M_l) = \big(M \subseteq V \times (0,1)^{n+l}, (0, b_0^1] \leqslant [a_1^1, 1), (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=2}^{n+l}\big) \in (\mathrm{PBord}_n^l)_{1,k_2,\dots,k_{n+l}}$$

to

$$(M_{l-1}) = (M \subseteq (\underbrace{V \times (0,1)}_{=\tilde{V}}) \times (0,1)^{n+l-1}, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=2,\dots,n+l}),$$

so it “forgets” the first specified intervals. First of all, we need to check that this map is well-defined, that is, that $(M_{l-1}) \in (\mathrm{PBord}_n^{l-1})_{k_2,\dots,k_{n+l}}$. Note that in the above, we view $\tilde{V} = V \times (0,1)$ as a vector space using the identification $(0,1) \overset{\chi}{\cong} \mathbb{R}$. The condition we need to check is the second one, i.e. we need to check that $M \hookrightarrow \tilde{V} \times (0,1)^{n+l-1} \twoheadrightarrow (0,1)^{n+l-1}$ is proper. We know that $M \to (0,1)^{n+l}$ is proper, and moreover, since $p_1^{-1}((0, b_0^1)) = p_1^{-1}((a_1^1, 1)) = \varnothing$, we know that M is bounded in the direction of the first coordinate, since $M = p_1^{-1}([b_0^1, a_1^1])$. Together this implies the statement. Note that the map u we just constructed actually is defined by a system of maps

$$u_V : L(\mathrm{PBord}_n^{l,V}) \longrightarrow \mathrm{PBord}_n^{l,\tilde{V}},$$

where $\tilde{V} = V \oplus \langle v \rangle$.

To construct a map in the other direction we will also need to change the vector space V, but this time we need to ”delete” a direction. To make this procedure precise, we fix the following notations. In the definition of $\mathrm{PBord}_n^{l,V}$ we let V vary within a fixed countably infinite dimensional space. Choose $\mathbb{R}^\infty$ with a countable basis consisting of vectors $v_1, v_2, \dots$ In taking the limit is enough to consider the finite dimensional subspaces V_d spanned by the first d vectors $v_1, \dots, v_d$. Then the map u we constructed above was defined as an inductive system of maps

$$\begin{aligned} u_d : L(\mathrm{PBord}_n^{l,V_d}) \quad &\longrightarrow \quad \mathrm{PBord}_n^{l,V_{d+1}}, \\ \Big(M \subseteq V_d \times (0,1) \times (0,1)^{n+l-1}\Big) \quad &\longmapsto \quad \Big(M \subseteq (\underbrace{\langle v_1 \rangle}_{\cong (0,1)} \oplus \underbrace{\langle v_2, \dots, v_{d+1} \rangle}_{\cong V_d}) \times (0,1)^{n+l-1}\Big), \end{aligned}$$

where we use the canonical morphisms $(0,1) \cong \mathbb{R} \cong \langle v_1 \rangle$ and $\langle v_2, \dots, v_{d+1} \rangle \to V_d$, $v_\beta \mapsto v_{\beta-1}$.

In remark 2.3.11, we constructed a path from an element in PBord_n to its “cutoff”, whose underlying submanifold is $\pi^{-1}(\prod_{i=1}^n (\frac{b_0^i}{2}, \frac{a_{k_i}^i}{2}))$, which is bounded in the V-direction. We saw in section 2.4.1 that this map gives rise to a strong homotopy equivalence

$$cut : \mathrm{PBord}_n \longrightarrow \mathrm{PBord}_n^{bd}.$$

Similarly, we obtain equivalences of n-fold Segal spaces

$$cut : \mathrm{PBord}_n^{l-1} \longrightarrow \mathrm{PBord}_n^{l-1,bd}, \qquad cut : L(\mathrm{PBord}_n^{l}) \longrightarrow L(\mathrm{PBord}_n^{l,bd}).$$

Note that u_d restricts to a map between the bounded versions,

$$u_d^{bd} : L(\mathrm{PBord}_n^{l,V_d,bd}) \longrightarrow \mathrm{PBord}_n^{l,V_{d+1},bd}$$

It suffices to show that this map induces a strong homotopy equivalence, with homotopy inverse given by the following inductive system of maps

$$\ell_d^{bd} : \mathrm{PBord}_n^{l-1,V_{d+1},bd} \longrightarrow L(\mathrm{PBord}_n^{l,V_d}).$$

Start with an element $(M_{l-1}) = \big(M \subseteq V_{d+1} \times (0,1)^{n+l-1}, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=2,\dots,n+l}\big) \in \mathrm{PBord}_n^{l-1,V_{d+1},bd}$. Since it is bounded in the V-direction, there are A, B such that

$$B < \pi_{v_1}(M) < A,$$

where $\pi_{v_1} : M \subseteq \big(\langle v_1\rangle \oplus \langle v_2,\dots,v_{d+1}\rangle\big) \times (0,1)^{n+l-1} \twoheadrightarrow \langle v_1\rangle = \mathbb{R}v_1$. Let $\tilde{B}$ be the supremum of such B and let $\tilde{A}$ be the infimum of such A. Let $\tilde{\tilde{B}} = \frac{\tilde{B}}{2}, \tilde{\tilde{A}} = \frac{\tilde{A}+1}{2}$. Now let $b, a \in (0,1) \cong \mathbb{R}$ correspond to $\tilde{\tilde{B}}, \tilde{\tilde{A}}, B$. Finally, we send (M_{l-1}) to

$$(M_l) = \Big(M \subseteq \underbrace{\langle v_2,\dots,v_{d+1}\rangle}_{\cong V_d} \times \underbrace{(0,1)}_{\cong\langle v_1\rangle} \times (0,1)^{n+l-1}, (0,b] \leqslant [a,1), (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=2}^{n+l}\Big).$$

By construction,

$$\ell_d^{bd} \circ u_d \sim id, \qquad u_d \circ \ell_d^{bd} = id,$$

where $\ell_d^{bd} \circ u_d$ just changes the first two intervals $I_0^1 \leqslant I_1^1$ and thus is homotopy equivalent to the identity. □

Definition 2.5.9. The map ℓ in the proof is called the *looping* and u the *delooping map*.

Recall from remark 1.5.8 that looping commutes with completion. Taking the appropriate completions, we obtain the following corollary.

Corollary 2.5.10. *Let* $k \geqslant 0$.

1. *If* $l - k > 0$,
$$L_k(\mathrm{Bord}_n^{(l)}) \simeq \mathrm{Bord}_n^{(l-k)}. \tag{2.3}$$
2. *If* $k \geqslant l > 0$ *and* $n + l - k \geqslant 0$,
$$L_k(\mathrm{Bord}_n^{(l)}) \simeq \mathrm{Bord}_n^{l-k}. \tag{2.4}$$
3. *If* $l \leqslant 0$ *and For* $n + l \geqslant k \geqslant 0$, $n + l - k \geqslant 0$,
$$L_k(\mathrm{Bord}_n^{l}) \simeq \mathrm{Bord}_n^{l-k}. \tag{2.5}$$

The tower and the symmetric monoidal structure

Recall from definition 2.5.7 that $\mathrm{Bord}_n^{(l)}$ is the l-hybrid completion of PBord_n^l. By remark 2.5.6 and (2.3) in corollary 2.5.10, proposition 2.5.8 has an immediate corollary.

Corollary 2.5.11. *The $(n+l)$-fold Segal spaces $\mathrm{Bord}_n^{(l)}$ are l-hybrid and endow Bord_n with the structure of a symmetric monoidal n-fold Segal space.*

2.6 The homotopy (bi)category

2.6.1 The homotopy category $h_1(L_{n-1}(\mathrm{Bord}_n))$

The symmetric monoidal structure on $h_1(L_{n-1}(\mathrm{Bord}_n))$

The $(n-1)$-fold looping $L_{n-1}(\mathrm{Bord}_n) \simeq \mathrm{Bord}_n^{-(n-1)}$ is a $(\infty,1)$-category with a symmetric monoidal structure defined in two ways similarly to that of Bord_1. Both induce a symmetric monoidal structure on the homotopy category $h_1(L_{n-1}(\mathrm{Bord}_n)) \simeq h_1(\mathrm{Bord}_n^{-(n+1)})$.

...coming from a Γ-object We can either obtain the symmetric monoidal structure as a Γ-object on $L_{n-1}(\mathrm{Bord}_n) \simeq \mathrm{Bord}_n^{-(n-1)}$ by iterating the construction of the symmetric monoidal structure on the looping from example 1.6.10 or by constructing a functor from an assignment $[m] \mapsto \mathrm{Bord}_n^{-(n-1)}[m]$. In the second case, $\mathrm{Bord}_n^{-(n-1)}[m]$ arises, similarly to $\mathrm{Bord}_n[m]$, from the spaces $(\mathrm{PBord}_n^{V,-(n-1)}[m])_{k_1,\ldots,k_n}$, which as a set is the collection of tuples

$$(M_1, \ldots, M_m, (I_0 \leqslant \ldots \leqslant I_k)),$$

where $M_1, \ldots M_m$ are disjoint n-dimensional submanifolds of $V \times (0,1)^n$ and each $(M_\beta, (I_0 \leqslant \ldots \leqslant I_k)) \in (\mathrm{PBord}_n^{V,-(n-1)})_{k_1,\ldots,k_n}$.

We saw in example 1.6.8 that a Γ-object endows the homotopy category of its underlying Segal space with a symmetric monoidal structure. Explicitly, in the second case, it comes from the following maps.

$$\begin{array}{ccccc} \mathrm{Bord}_n^{-(n+1)}[1] \times \mathrm{Bord}_n^{-(n+1)}[1] & \xleftarrow[\gamma_1\times\gamma_2]{\simeq} & \mathrm{Bord}_n^{-(n+1)}[2] & \xrightarrow{\gamma} & \mathrm{Bord}_n^{-(n+1)}[1], \\ (M_1, I\text{'s}), (M_2, I\text{'s}) & \longleftarrow & (M_1, M_2, I\text{'s}) & \longmapsto & (M_1 \amalg M_2, I\text{'s}) \end{array}$$

...coming from a tower The understand the symmetric monoidal structure on $h_1(L_{n-1}(\mathrm{Bord}_n))$ coming from a symmetric monoidal structure as a tower, we use that $L_{n-1}(\mathrm{Bord}_n) \simeq \mathrm{Bord}_n^{-(n-1)}$ and that $\mathrm{Bord}_n^{-(n-1)}$ has a symmetric monoidal structure coming from the collection of l-hybrid $(l+1)$-fold Segal spaces given by the l-hybrid completion of PBord_n^{l-n+1}, the completion in the last index. This symmetric monoidal structure induces one on the homotopy category $h_1(\mathrm{Bord}_n^{-(n-1)}) \simeq h_1(L_{n-1}(\mathrm{Bord}_n))$, which we will explain explicitly. Since completion is a Dold-Kan equivalence, see 1.2.3, it is enough to understand the symmetric monoidal structure on $h_1(\mathrm{PBord}_n^{-(n-1)})$.

Essentially, the monoidal structure is given by composition in $\mathrm{PBord}_n^{1-(n-1)}$, the next layer of the tower $\mathrm{PBord}_n^{2-(n-1)}$ gives a braiding and the higher layers show that it is symmetric monoidal. Consider the diagram

$$(\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet} \times (\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet} \xleftarrow[d_0^1\times d_2^1]{\simeq} (\mathrm{PBord}_n^{1-(n-1)})_{2,\bullet} \xrightarrow{s_1^1} (\mathrm{PBord}_1^{n-(n-1)})_{1,\bullet}.$$

Using the fact from remark 1.6.17 that $L(\mathrm{PBord}_n^{1-(n-1)})_\bullet = (\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet}$, we find that $(\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet} \simeq (\mathrm{PBord}_n^{-(n-1)})_\bullet$, which induces a map

$$h_1(\mathrm{PBord}_n^{-(n-1)}) \times h_1(\mathrm{PBord}_n^{-(n-1)}) \longrightarrow h_1(\mathrm{PBord}_n^{-(n-1)}).$$

This is a monoidal structure on $h_1(\mathrm{PBord}_n^{-(n-1)})$. We can explicitly construct this map. Consider two objects or 1-morphisms (M) and (N) in $(\mathrm{PBord}_n^{-(n-1)})_k$ for $k = 0$ or $k = 1$,

$$(M) = (M \subseteq V \times (0,1), I_0 \leqslant \cdots \leqslant I_k), \quad (N) = (N \subseteq \tilde{V} \times (0,1), \tilde{I}_0 \leqslant \cdots \leqslant \tilde{I}_k).$$

Without loss of generality $V = \tilde{V} = V_d$, and $(M), (N) \in (\mathrm{Bord}_1^{bd})_k$.

Under the map $\ell_d^{bd} : \mathrm{Bord}_1^{bd} \to L(\mathrm{Bord}_1^{1,bd})$ from proposition 2.5.8, (M) and (N) are sent to

$$(M_1) = (M \subseteq V_{d-1} \times (0,1)^2, (0,b] \leqslant [a,1), I_0 \leqslant \cdots \leqslant I_k),$$
$$(N_1) = (N \subseteq \tilde{V}_{d-1} \times (0,1)^2, (0,\tilde{b}] \leqslant [\tilde{a},1), \tilde{I}_0 \leqslant \cdots \leqslant \tilde{I}_k).$$

In the proof of the Segal condition for PBord_n proposition 2.3.22 we explicitely constructed a homotopy inverse *glue* to $d_0^1 \times d_2^1$. Similarly one can obtain such a homotopy inverse for PBord_n^l, which applied to (M_1) and (N_1) gives

$$\left(M \amalg N \hookrightarrow \tilde{V}_{d-1} \times (0,1)^2, (0,b_0^1] \leqslant [a_1^1, b_1^1] \leqslant [a_2^1, 1), \tilde{I}_0 \leqslant \cdots \leqslant \tilde{I}_k\right),$$

since $d_1^1((M_1)) = d_0^1((N_1)) = \varnothing$. The third face map sends it to

$$\left(M \amalg N \hookrightarrow \tilde{V}_{d-1} \times (0,1)^2, (0,b_0^1] \leqslant [a_2^1, 1), \tilde{I}_0 \leqslant \cdots \leqslant \tilde{I}_k\right)$$

which by $u_d^{bd} : L(\mathrm{Bord}_1^{1,bd}) \to \mathrm{Bord}_1^{bd}$ is sent to

$$\left(M \amalg N \hookrightarrow \tilde{V}_d \times (0,1), \tilde{I}_0 \leqslant \cdots \leqslant \tilde{I}_k\right).$$

The homotopy category and nCob

The homotopy category of Bord_1 turns out to be what we expect, namely 1Cob. We can show even more, namely that our higher categories of cobordisms also give back the ordinary categories of n-cobordisms, as we see in the following proposition.

Proposition 2.6.1. *There is an equivalence of symmetric monoidal categories between the homotopy category of the $(n-1)$-fold looping of Bord_n and the category of n-cobordisms,*

$$h_1(L_{n-1}(\mathrm{Bord}_n)) \simeq n\mathrm{Cob}.$$

Proof. We first show that there is an equivalence of categories $h_1(L_{n-1}(\mathrm{Bord}_n)) \simeq n\mathrm{Cob}$ and then show that it respects the symmetric monoidal structures.

Rezk's completion functor is a Dwyer-Kan equivalence of Segal spaces, and thus by definition induces an equivalence of the homotopy categories. Moreover, completion commutes with looping, so it is enough to show that

$$h_1(L_{n-1}(\mathrm{PBord}_n)) \simeq n\mathrm{Cob}.$$

We define a functor

$$F : h_1(L_{n-1}(\mathrm{PBord}_n)) \longrightarrow n\mathrm{Cob}$$

and show that it is essentially surjective and fully faithful.

Definition of the functor By definition,

$$(L_{n-1}(\mathrm{PBord}_n))_k = \{\varnothing\} \underset{(L_{n-2}(\mathrm{PBord}_n))_{0,k}}{\overset{h}{\times}} (L_{n-2}(\mathrm{PBord}_n))_{1,k} \underset{(L_{n-2}(\mathrm{PBord}_n))_{0,k}}{\overset{h}{\times}} \{\varnothing\},$$

and, iterating this process, we find that an element in $L_{n-1}(\mathrm{PBord}_n)_k$ is an element (M) of $(\mathrm{PBord}_n)_{1,\dots,1,k}$ such that for every $i \neq n$, $d_j^i((M))$ has $\varnothing$ as its underlying manifold, i.e. in every direction except for the nth direction, the source and target both have $\varnothing$ as its underlying manifold.

So an object in $h_1(L_{n-1}(\mathrm{PBord}_n))$ is an element $(M) \in (\mathrm{PBord}_n)_{1,\dots,1,0}$ such that for $i \neq n$, the underlying manifold of$d_j^i((M))$ is $\varnothing$. We let the functor F send (M) to

$$\pi^{-1}([b_0^1, a_1^1] \times \cdots \times [b_0^{n-1}, a_1^{n-1}] \times \{\frac{1}{2}\}).$$

Since $\frac{1}{2}$ is a regular value of $p_{\{n\}}$, $F((M))$ is an $(n-1)$-dimensional manifold, and since π is proper, it is compact. Moreover its boundary is empty. This follows from

$$F((M)) \hookrightarrow V \times [b_0^1, a_1^1] \times \cdots [b_0^{n-1}, a_1^{n-1}] \times \{\frac{1}{2}\}$$

which implies that

$$\partial F((M)) = F((M)) \cap \partial \left(V \times [b_0^1, a_1^1] \times \cdots [b_0^{n-1}, a_1^{n-1}] \times \{\frac{1}{2}\} \right)$$

and since for every $i \neq n$, the underlying manifold of $d_j^i((M))$ is $\varnothing$.

So, as an abstract manifold, $F((M))$ is a closed compact $(n-1)$-dimensional manifold, i.e. an object in $n\mathrm{Cob}$.

Similarly, the functor F sends a morphism in $h_1(L_{n-1}(\mathrm{PBord}_n))$, which is an element in $\pi_0(L_{n-1}(\mathrm{PBord}_n)_1)$ which is represented by an element $(M) \in (\mathrm{PBord}_n)_{1,\dots,1,1}$ such that for $i \neq n$, the underlying manifold of $d_j^i((M))$ is $\varnothing$, to the isomorphism class of

$$\bar{M} = \pi^{-1}([b_0^1, a_1^1] \times \cdots \times [b_0^{n-1}, a_1^{n-1}] \times [b_0^n, a_1^n]).$$

This is an n-dimensional manifold with boundary

$$\pi^{-1}([b_0^1, a_1^1] \times \cdots \times [b_0^{n-1}, a_1^{n-1}] \times \{b_0^n\}) \amalg \pi^{-1}([b_0^1, a_1^1] \times \cdots \times [b_0^{n-1}, a_1^{n-1}] \times \{a_1^n\}).$$

This is well-defined, since a path in $L_{n-1}(\mathrm{PBord}_n)_1$ by definition gives diffeomorphism $\psi_{0,1} : M_0 \to M_1$ which intertwines with the composed bordisms and thus restricts to diffeomorphisms of the images defined above.

The functor is an equivalence of categories Whitney's embedding theorem shows that F is essentially surjective. Moreover, it is injective on morphisms: Let $\iota_0 : M_0 \hookrightarrow V \times (0,1)^n$ and $\iota_0 : M_1 \hookrightarrow V \times (0,1)^n$ be representatives of two 1-morphisms which have diffeomorphic images. This means that there is a diffeomorphism $\psi : \bar{M}_0 \to \bar{M}_1$, which can be extended to their collars, i.e. we get a diffeomorphism $\psi : M_0 \to M_1$. Since $\mathrm{Emb}(M_1, \mathbb{R}^\infty \times (0,1)^n)$ is contractible, the quotient $\mathrm{Emb}(M_1, \mathbb{R}^\infty \times (0,1)^n)/\mathrm{Diff}(M_1)$ is path-connected, so there is a path of embedded submanifolds $\tilde{\iota}_s : M_1 \hookrightarrow \mathbb{R}^\infty \times (0,1)^n$

such that $\tilde{\iota}_1 = \iota_1$ is the given one and $\tilde{\iota}_0 = \iota_0 \circ \psi$. Note that $\tilde{\iota}_0$ and ι_0 give the same submanifold. By lemma 2.3.12, this family ι_s determines a rescaling data and a family of diffeomorphisms $\psi_{s,t}$ which intertwine and thus a path in PBord_n, which by construction lies in $L_{n-1}(\mathrm{Bord}_n)$. It remains to show that F is full.

In the case $n = 1, 2$ this is easy to show, as we have a classification theorem for 1- and 2-dimensional manifolds with boundary. In the 1-dimensional case it is enough to show that an open line, the circle and the half-circle, once as a bordism from 2 points to the empty set and once vice versa, lie in the image of the map, which is straightforward. In the two dimensional case, the pair-of-pants decomposition tells us how to embed the manifold.

For general n we first embed the manifold with boundary into $\mathbb{R}^+ \times \mathbb{R}^{2n}$ using a variant of Whitney's embedding theorem for manifolds with boundary, cf. [Lau00]. Then the boundary of the halfspace is $\partial(\mathbb{R}^+ \times \mathbb{R}^{2n}) = \mathbb{R}^{2n}$. We want to transform this embedding into an embedding into $(0,1) \times \mathbb{R}^{2n}$ such that the incoming boundary is sent into $\{\epsilon\} \times \mathbb{R}^{2n}$ and the outgoing boundary is sent into $\{1-\epsilon\} \times \mathbb{R}^{2n}$.

We first show that the boundary components can be separated by a hyperplane in $\mathbb{R}^{2n}$. The boundary components are compact so they can be embedded into balls B^{2n}. By perhaps first applying a suitable "stretching" transformation, one can assume that these balls do not intersect. Now, since $2n > 1$, $\pi_0(\mathrm{Conf}(B^{2n}, \mathbb{R}^{2n})) = *$, there is a transformation to a configuration in which the boundary components are separated by a hyperplane, without loss of generality given by the equation $\{x_1 = 0\} \subset \mathbb{R}^{2n}$.

Consider the (holomorphic) logarithm function on $(\mathbb{R}^+ \times \mathbb{R})\backslash\{(0,0)\} \cong \mathbb{H}\backslash 0 \subseteq \mathbb{C}$ with branch cut $-i\mathbb{R}^+$. It is a homeomorphism to $\{(x,y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant \pi\}$. We can apply $\log \times id_{\mathbb{R}^{2n-1}}$ to $(\mathbb{R}^+ \times \mathbb{R}_{x_1}) \times \mathbb{R}^{2n-1}$ and, composing this with a suitable rescaling, obtain an embedding into $(\epsilon, 1-\epsilon) \times \mathbb{R}^{2n}$. Now choose a collaring of the bordism to extend the embedding to $(0,1) \times \mathbb{R}^{2n}$.

The functor is a symmetric monoidal equivalence Explicitly analyzing the two symmetric monoidal structures on $h_1(\mathrm{Bord}_n^{-(n-1)})$, one sees that they both send two elements (represented by)

$$(M) = (M \subseteq V_d \times (0,1), I_0 \leqslant \cdots \leqslant I_k), \quad (N) = (N \subseteq V_d \times (0,1), \tilde{I}_0 \leqslant \cdots \leqslant \tilde{I}_k)$$

for $k = 0$ or $k = 1$ to an embedding of $M \amalg N$ into V_{d+1}, which sends M and N to different heights in the extra $(d+1)$st direction.

In the case of the structure coming from a Γ-object, one can similarly to in the previous paragraph define an equivalence of categories

$$F[m] : \mathrm{Bord}_n^{-(n-1)}[m] \longrightarrow n\mathrm{Cob}^m .$$

Then one can easily check that the following diagram commutes.

$$\begin{array}{ccccc}
\mathrm{Bord}_n^{-(n-1)}[1] \times \mathrm{Bord}_n^{-(n-1)}[1] & \xleftarrow{\simeq} & \mathrm{Bord}_n^{-(n-1)}[2] & \longrightarrow & \mathrm{Bord}_n^{-(n-1)}[1] \\
\downarrow{\scriptstyle F \times F} & & \downarrow{\scriptstyle F[2]} & & \downarrow{\scriptstyle F} \\
n\mathrm{Cob} \times n\mathrm{Cob} & = & n\mathrm{Cob} \times n\mathrm{Cob} & \xrightarrow{\amalg} & n\mathrm{Cob}
\end{array}$$

For the case of the structure coming from a tower, we explicitly saw that the symmetric structure on $h_1(\mathrm{Bord}_n^{-(n-1)})$ sends two objects or 1-morphisms determined by

$$(M) = (M \subseteq V \times (0,1), I_0 \leqslant \cdots \leqslant I_k), \quad (N) = (N \subseteq \tilde{V} \times (0,1), \tilde{I_0} \leqslant \cdots \leqslant \tilde{I_k})$$

to

$$(M \amalg N) = \left(M \amalg N \hookrightarrow \tilde{V}_d \times (0,1), \tilde{I_0} \leqslant \cdots \leqslant \tilde{I_k} \right),$$

where the embedding of M is changed by a rescaling. This change of rescaling is precisely such that under the functor F the element $(M \amalg N)$ is sent to $F((M)) \amalg F((N))$. □

2.6.2 The homotopy bicategory $h_2(\mathrm{Bord}_2)$ and comparison with $2\mathrm{Cob}^{ext}$

C. Schommer-Pries defined a symmetric monoidal bicategory $n\mathrm{Cob}^{ext}$ of n-dimensional cobordisms in his thesis [SP09]. In this section we show that the homotopy bicategory of our $(\infty, 2)$-category of 2-dimensional bordisms is symmetric monoidally equivalent to this bicategory.

The bicategory $2\mathrm{Cob}^{ext}$

We first briefly recall the definition of $2\mathrm{Cob}^{ext}$.

Definition 2.6.2. The bicategory $2\mathrm{Cob}^{ext}$ has

- 0-dimensional manifolds as objects,
- 1-morphisms are 1-bordisms between objects, and
- 2-morphisms are isomorphism classes of 2-bordisms between 1-morphisms,

where

1. a *1-bordism* between two 0-dimensional manifolds Y_0, Y_1 is a smooth compact 1-dimensional manifold with boundary W with a decomposition and isomorphism

$$\partial W = \partial_{in} W \amalg \partial_{out} W \cong Y_0 \amalg Y_1;$$

2. a *2-bordism* between two 1-bordisms W_0, W_1 between objects Y_0, Y_1 is a compact 2-dimensional $<2>$-manifold S equipped with
   - a decomposition and isomorphism
   $$\partial_0 S = \partial_{0,in} S \amalg \partial_{0,out} S \xrightarrow{\sim} W_0 \amalg W_1,$$
   - a decomposition and isomorphism
   $$\partial_1 S = \partial_{1,in} S \amalg \partial_{1,out} S \xrightarrow{\sim} Y_0 \times [0,1] \amalg Y_1 \times [0,1].$$

   Recall that a $< 2 >$-manifold is a manifold with faces X with a pair of faces $(\partial_0 X, \partial_1 X)$ such that

   $$\partial_0 X \cup \partial_1 X = \partial X, \qquad \partial_0 X \cap \partial_1 X \text{ is a face.}$$

3. Two 2-bordisms S, S' are *isomorphic* if there is a diffeomorphism $h : S \to S'$ compatible with the boundary data.

Vertical and horizontal compositions of 2-morphisms are defined by choosing collars and gluing. This is well-defined because 2-morphisms are isomorphism classes of 2-bordisms, and thus the composition doesn't depend on the choice of the collar. However, composition of 1-morphisms requires the use of a choice of a collar, which requires the axiom of choice, and then composition is defined by the unique gluing. However, this gluing is associative only up to non-canonical isomorphism of 1-bordisms which gives a canonical isomorphism class of 2-bordisms realizing the associativity of horizontal composition in the axioms of a bicategory.

It is symmetric monoidal, with symmetric monoidal structure given by taking disjoint unions. For the exact details we refer to the above mentioned thesis [SP09].

The symmetric monoidal structure on $h_2(\mathrm{Bord}_2)$

The symmetric monoidal structure on Bord_2 arising as a Γ-object gives us

$$\mathrm{Bord}_2[1] \times \mathrm{Bord}_2[1] \xleftarrow{\simeq} \mathrm{Bord}_2[2] \longrightarrow \mathrm{Bord}_2[1]$$

which induces

$$h_2(\mathrm{Bord}_2) \times h_2(\mathrm{Bord}_2) \longrightarrow h_2(\mathrm{Bord}_2).$$

This makes $h_2(\mathrm{Bord}_2)$ into a symmetric monoidal bicategory, where the associativity follows from the equivalence $\mathrm{Bord}_2[3] \xrightarrow{\sim} \mathrm{Bord}_2[1]^{\times 3}$.

The homotopy bicategory and $2\mathrm{Cob}^{ext}$

In this section we show that our $(\infty, 2)$ category of 2-cobordisms indeed gives back the bicategory $2\mathrm{Cob}^{ext}$ as its homotopy bicategory.

Proposition 2.6.3. *There is an equivalence of symmetric monoidal bicategories between* $\mathrm{h}_2(\mathrm{Bord}_2)$ *and* $2\mathrm{Cob}^{ext}$.

Proof. By Whitehead's theorem for symmetric monoidal bicategories, see [SP09], theorem 2.21, it is enough to find a functor F which is

1. essentially surjective on objects, i.e. F induces an isomorphism $\pi_0(\mathrm{h}_2(\mathrm{Bord}_2)) \cong \pi_0(2\mathrm{Cob}^{ext})$,
2. essentially full on 1-morphisms, i.e. for every $x, y \in \mathrm{Ob}\,\mathrm{h}_2(\mathrm{Bord}_2)$, the induced functor $F_{x,y} : \mathrm{h}_2(\mathrm{Bord}_2)(x, y) \to 2\mathrm{Cob}^{ext}(Fx, Fy)$ is essentially surjective, and
3. fully-faithful on 2-morphisms, i.e. for every $x, y \in \mathrm{Ob}\,\mathrm{h}_2(\mathrm{Bord}_2)$, the induced functor $F_{x,y} : \mathrm{h}_2(\mathrm{Bord}_2)(x, y) \to 2\mathrm{Cob}^{ext}(Fx, Fy)$ is fully-faithful.

First of all, recall from remark 2.3.25 that for $n = 2$, PBord_2 is a complete 2-fold Segal space, so $\mathrm{Bord}_2 = \mathrm{PBord}_2$.

Definition of the functor Let

$$F : \mathrm{h}_2(\mathrm{Bord}_2) \longrightarrow 2\mathrm{Cob}^{ext}$$

be the functor defined as follows:

On objects,

$$(M \subseteq V \times \mathbb{R}^2, (0,1), (0,1)) \in (\mathrm{Bord}_2)_{0,0} \quad \stackrel{F}{\longmapsto} \quad \pi^{-1}\left((\frac{1}{2}, \frac{1}{2})\right),$$

where the image is thought of as an abstract manifold. This is well-defined, because as π is proper and $(\frac{1}{2}, \frac{1}{2})$ is a regular value of π, the preimage $\pi^{-1}\left((\frac{1}{2}, \frac{1}{2})\right)$ is compact and 0-dimensional, so it is a finite disjoint union of points. Note that because of condition (3) in the definition of $\mathrm{Bord}_2 = \mathrm{PBord}_2$, we could have taken the fiber over any other point in $(0,1)^2$ and would have gotten a diffeomorphic image.

On 1-morphisms,

$$(M \subseteq V \times \mathbb{R}^2, (0, b_0^1] \leqslant [a_1^1, 1), (0,1)) \in (\mathrm{Bord}_2)_{1,0} \stackrel{F}{\longmapsto} \pi^{-1}\left([b_0^1, a_1^1] \times \{\frac{1}{2}\}\right).$$

The point $\frac{1}{2}$ is a regular value of the projection map $p_2 : M \hookrightarrow V \times (0,1)^2 \twoheadrightarrow (0,1)$, so $\pi^{-1}\left([b_0^1, a_1^1] \times \{\frac{1}{2}\}\right)$ is a 1-dimensional manifold with boundary. Moreover, the decomposition of the boundary of the image is given by

$$\pi^{-1}\left((b_0^1, \frac{1}{2})\right) \amalg \pi^{-1}\left((a_1^1, \frac{1}{2})\right).$$

Note that again, we could have taken the preimage $\pi^{-1}([c,d] \times \{t\})$ for any $t \in [0,1]$, $c \in (0, b_0^1]$, and $d \in [a_1^1, 1)$ and would have gotten a diffeomorphic image.

On 2-morphisms, the functor F comes from the assignment

$$(M \subseteq V \times \mathbb{R}^2, (0, b_0^1] \leqslant [a_1^1, 1), (0, b_0^2] \leqslant [a_1^2, 1)) \stackrel{F}{\longmapsto} \pi^{-1}\left([b_0^1, a_1^1] \times [b_0^2, a_1^2]\right) =: S.$$

As π is proper, S is a compact 2-dimensional manifold with corners and moreover has the structure of a $\langle 2 \rangle$-manifold coming from the decomposition of the boundary coming from the inverse images under π of the sides of the rectangle $[b_0^1, a_1^1] \times [b_0^2, a_1^2]$,

$$\partial_0 S = \pi^{-1}\left([b_0^1, a_1^1] \times \{b_0^2\}\right) \amalg \pi^{-1}\left([b_0^1, a_1^1] \times \{a_1^2\}\right),$$

and

$$\partial_1 S = \pi^{-1}\left(\{b_0^1\} \times [b_0^2, a_1^2]\right) \amalg \pi^{-1}\left(\{a_1^1\} \times [b_0^2, a_1^2]\right).$$

By condition (3) in definition 2.3.1,

$$\pi^{-1}\left(\{b_0^1\} \times [b_0^2, a_1^2]\right) \cong \pi^{-1}\left((b_0^1, b_0^2)\right) \times [b_0^2, a_1^2]$$

and

$$\pi^{-1}\left(\{a_0^1\} \times [b_0^2, a_1^2]\right) \cong \pi^{-1}\left((a_0^1, b_0^2)\right) \times [b_0^2, a_1^2].$$

This makes S into a 2-bordism between the images under F of the source and target of our 2-bordism.

This assignment descends to 2-morphisms which are elements in $\pi_0((\mathrm{Bord}_2)_{1,1})$, as any path in $(\mathrm{Bord}_2)_{1,1}$ by definition induces a diffeomorphism $\psi_{0,1} : M_0 \to M_1$ which intertwines with the composed bordisms and thus induces an isomorphism of the images under F defined above.

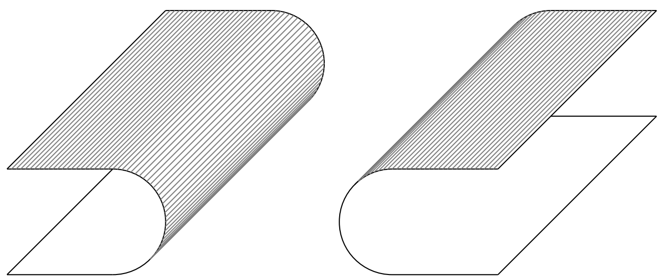
The functor is an equivalence of bicategories We check (1)-(3) of Whitehead's theorem.

For (1), the point is the image of the plane $(M = (0,1)^2 \overset{id}{\hookrightarrow} (0,1)^2, (0,1), (0,1))$. For k points, we can take k disjoint parallel planes in $(0,1) \times (0,1)^2$ which intersect $V = \mathbb{R}$ in k different points, e.g. $0, \dots, k-1$ and the intervals $I_0^1 = I_0^2 = (0,1)$.

For (2), we use the classification of 1-dimensional manifolds with boundary. Any connected component can be cut into pieces diffeomorphic to straight lines and left and right half circles. These all lie in the image of F in a very simple way, e.g. a straight line is the image of

$$\left(M = (0,1)^2 \overset{id}{\hookrightarrow} (0,1)^2, (0, \frac{1}{3}] \leqslant [\frac{2}{3}, 1), (0,1)\right),$$

and the right and left half circles are the images of the following embeddings $(0,1)^2 \hookrightarrow \mathbb{R} \times (0,1)^2$ with suitable choices of intervals.



By gluing these preimages in a suitable way, we get an element whose image is diffeomorphic to the connected component we started with.

For (3), to show that it is full on 2-morphisms, we use the classification theorem 3.33 of Schommer-Pries in [SP09]. He gives a set of generating 2-morphisms of $2\mathrm{Cob}^{ext}$ for which one easily sees that they all are the image of an element in $(\mathrm{Bord}_2)_{1,1}$. Moreover, the preimages can be glued. For faithfullness, a similar argument as in the proof of proposition 2.6.1 works: we use the fact that $\mathrm{Emb}(M, \mathbb{R}^\infty \times (0,1)^n)$ is contractible, so $\mathrm{Emb}(M, \mathbb{R}^\infty \times (0,1)^n)/\mathrm{Diff}(M)$ is path connected. Using lemma 2.3.12, an isomorphism of 2-bordisms will give rise to a path in $(\mathrm{Bord}_2)_{1,1}$.

The functor is a symmetric monoidal equivalence Similarly to in the previous subsection, the equivalence of bicategories

$$F : h_2(\mathrm{Bord}_2) \xrightarrow{\simeq} 2\mathrm{Cob}^{ext}$$

respects the symmetric monoidal structures. This can been seen by explicitly writing out the symmetric monoidal structure for $h_2(\mathrm{Bord}_2)$. □

Remark 2.6.4. In [SP09], Schommer-Pries also defined a bicategory $n\mathrm{Cob}^{ext}$ with objects being $(n-2)$-dimensional manifolds, 1-morphisms being $(n-1)$-cobordisms, and 2-morphisms being equivalence classes of 2-bordisms, which are suitable n-dimensional $\langle 2 \rangle$-manifolds. A similar argument should show that $h_2(L_{n-2}(\mathrm{Bord}_n)) \simeq n\mathrm{Cob}^{ext}$. However, one would need a suitable embedding theorem for cobordisms between cobordisms.

One should be able to adapt the embedding theorem for $\langle 2\rangle$-manifolds from [Lau00], similarly to how we adapted the embedding theorem for manifolds with boundary.

2.7 Cobordisms with additional structure: orientations and framings

In the study of fully extended topological field theories, one is particularly interested in manifolds with extra structure, especially that of a framing. In this section we explain how to define the (∞, n)-category of structured n-bordisms, in particular for the structure of an orientation or a framing.

2.7.1 Structured manifolds

We first need to recall the definition of structured manifolds and the topology on their morphism spaces making them into a topological category. In the next subsection we will see that the simplicial set of chains on these topological spaces essentially will give rise to the spatial structure of the levels of the n-fold Segal space of structured bordisms similarly to the construction in section 2.3.2.

Throughout this subsection, let M be an n-dimensional (smooth) manifold.

Definition 2.7.1. Let X be a topological space and $E \to X$ a topological n-dimensional vector bundle which corresponds to a (homotopy class of) map(s) $e : X \to B\mathrm{GL}(\mathbb{R}^n)$ from X to the classifying space of the topological group $\mathrm{GL}(\mathbb{R}^n)$. More generally, we could also consider a map $e : X \to B\mathrm{Homeo}(\mathbb{R}^n)$ to the classifying space of the topological group of homeomorphisms of $\mathbb{R}^n$, but for our purposes vector bundles are enough. An (X, E)-*structure or, equivalently, an* (X, e)*-structure on an* n*-dimensional manifold* M consists of the following data:

1. a map $f : M \to X$, and
2. an isomorphism of vector bundles
$$triv : TM \cong f^*(E).$$

Denote the set of (X, E)-structured n-dimensional manifolds by $\mathrm{Man}_n^{(X,E)}$.

An interesting class of such structures arises from topological groups with a morphism to $O(n)$.

Definition 2.7.2. Let G be a topological group together with a continuous homomorphism $e : G \to O(n)$, which induces $e : BG \to B\mathrm{GL}(\mathbb{R}^n)$. As usual, let $BG = EG/G$ be the classifying space of G, where EG is total space of its universal bundle, which is a weakly contractible space on which G acts freely. Then consider the vector bundle $E = (\mathbb{R}^n \times EG)/G$ on BG. A (BG, E)-structure or, equivalently, a (BG, e)-structure on an n-dimensional manifold M is called a *G-structure on M*. The set of G-structured n-dimensional manifolds is denoted by Man_n^G.

For us, the most important examples will be the following three examples.

Example 2.7.3. If G is the trivial group, $X = BG = *$ and E is trivial. Then a G-structure on M is a trivialization of TM, i.e. a framing.

Example 2.7.4. Let $G = O(n)$ and $e = id_{O(n)}$. Then, since the inclusion $O(n) \to \mathrm{Diff}(\mathbb{R}^n)$ is a deformation retract, an $O(n)$-structured manifold is just smooth manifolds.

Example 2.7.5. Let $G = SO(n)$ and $e : SO(n) \to O(n)$ is the inclusion. Then an $SO(n)$-structured manifold is an oriented manifold.

Definition 2.7.6. Let M and N be (X,E)-structured manifolds. Then let the space of morphisms from M to N be

$$\mathrm{Map}^{(X,E)}(M,N) = \mathrm{Emb}(M,N) \underset{\mathrm{Map}_{/B\mathrm{Homeo}(\mathbb{R}^n)}(M,N)}{\overset{h}{\times}} \mathrm{Map}_{/X}(M,N).$$

Taking (singular or differentiable) chains leads to a space, i.e. a simplicial set of morphisms from M to N. Thus we get a topological (or simplicial) category $\mathcal{M}an_n^{(X,E)}$ of (X,E)-structured manifolds. Disjoint union gives $\mathcal{M}an_n^{(X,E)}$ a symmetric monoidal structure.

Remark 2.7.7. For $G = O(n)$ we recover $\mathrm{Emb}(M,N)$, and for $G = SO(n)$, the space of orientations on a manifold is discrete, so an element in $\mathrm{Map}^{SO(n)}(M,N)$ is an orientation preserving map.

If G is the trivial group we saw above that a G-structure is a framing. In this case, the above homotopy fiber product reduces to

$$\mathrm{Map}^{(X,E)}(M,N) = \mathrm{Emb}(M,N) \underset{\mathrm{Map}_{GL(d)}(\mathrm{Fr}(TM),\mathrm{Fr}(TN))}{\overset{h}{\times}} \mathrm{Map}(M,N).$$

Thus, a framed embedding is a pair (f,h), where $f : M \to N$ lies in $\mathrm{Emb}(M,N)$ and h is a homotopy between between the trivialization of TM induced by the framing of M and that induced by the pullback of the framing on N.

2.7.2 The (∞,n)-category of structured cobordisms

Fix a type of structure given by the pair (X,E). In this subsection we define the n-fold (complete) Segal space of (X,E)-structured cobordisms $\mathrm{Bord}_n^{(X,E)}$.

Compared to definition 2.3.1 we add an (X,E)-structure to the data of an element in a level set.

Definition 2.7.8. Let V be a finite dimensional vector space. For every n-tuple $k_1,\ldots,k_n \geqslant 0$, let $\big(\mathrm{PBord}_n^{(X,E),V}\big)_{k_1,\ldots,k_n}$ be the collection of tuples $(M, f, triv, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\ldots,n})$, where

1. $(M, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1}^n)$ is an element in the set $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$, and
2. $(f, triv)$ is an (X,E)-structure on the (abstract) manifold M.

Remark 2.7.9. Note that there is a forgetful map

$$U : \big(\mathrm{PBord}_n^{(X,E),V}\big)_{k_1,\ldots,k_n} \to (\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$$

forgetting the (X,E)-structure.

Definition 2.7.10. An l-simplex of $\big(\mathrm{PBord}_n^{(X,E),V}\big)_{k_1,\ldots,k_n}$ consists of the following data:

1. A family of elements
$$(M_s, f_s, triv_s) = \big(M_s \subseteq V \times (0,1)^n, f_s, triv_s, (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))_{i=1,\ldots,n}\big)$$
in $\big(\mathrm{PBord}_n^{(X,E),V}\big)_{k_1,\ldots,k_n}$ indexed by $s \in |\Delta^l|$, which are called the *underlying* (X,E)*-structured 0-simplices*;

2. For every $1 \leqslant i \leqslant k_i$,
$$\big(I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s)\big)_{s\in|\Delta^l|}$$
is an l-simplex in Int_{k_i} with *rescaling datum* $\varphi_{s,t}^i : (0,1) \to (0,1)$;

3. A family of elements in $\mathcal{M}an_n^{(X,E)}(M_s, M_t)$ with underlying *diffeomorphisms*
$$\psi_{s,t} : M_s \longrightarrow M_t,$$
indexed by $s, t \in |\Delta^l|$;

such that the triple
$$U(M_s, f_s, triv_s), \quad (\varphi_{s,t})_{s,t\in|\Delta^l|}, \quad (\psi_{s,t})_{s,t\in|\Delta^l|}$$
is an l-simplex in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$.

Similarly as for PBord_n the levels can be given a spatial structure with the above l-simplices and then the collection of levels can be made into a complete n-fold Segal space $\mathrm{Bord}_n^{(X,E)}$.

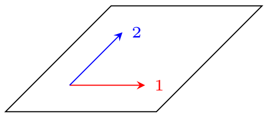
Moreover, $\mathrm{Bord}_n^{(X,E)}$ has a symmetric monoidal structure given by (X,E)-structured versions of the Γ-object and of the tower giving Bord_n a symmetric monoidal structure.

2.7.3 Example: Objects in Bord_2^{fr} are 2-dualizable

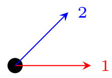
In dimension one, a framing is the same as an orientation. Thus the first interesting case is the two-dimensional one. In this case, the existence of a framing is a rather strong condition. However, we will see that nevertheless, any object in Bord_2^{fr} is 2-dualizable. Being 2-dualizable means that it is dualizable with evaluation and coevaluation maps themselves have adjoints, see [Lur09c].

Consider an object in Bord_2^{fr}, which, since in this case $\mathrm{Bord}_2^{fr} = \mathrm{PBord}_2^{fr}$ by remark 2.3.25, is an element of the form
$$\big(M \subseteq V \times (0,1)^2, F, (0,1), (0,1)\big),$$
where F is a framing of M. By the submersivity condition 3 in the definition 2.3.1 of PBord_2, M is a disjoint union of manifolds which are diffeomorphic to $(0,1)^2$. Thus, it suffices to consider an element of the form
$$\big((0,1)^2 \subseteq (0,1)^2, F, (0,1), (0,1)\big),$$

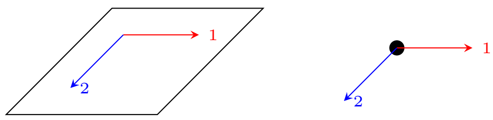
where F is a framing of $(0,1)^2$. Depict this element by

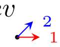


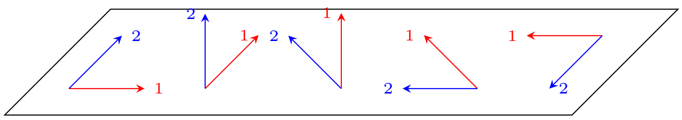
One should think of this as a point together with a 2-framing,

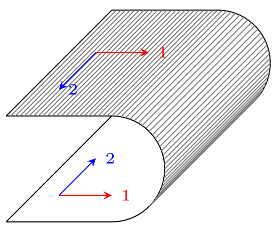


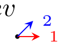
We claim that its dual is the same underlying unstructured manifold together with the opposite framing



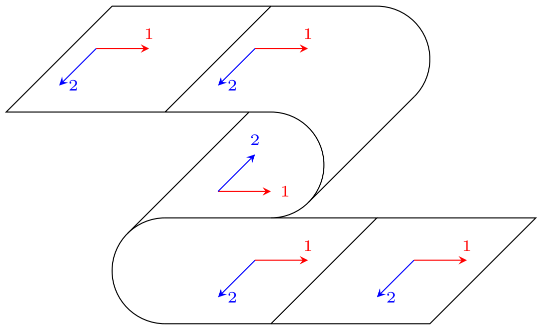
An evaluation 1-morphism *ev*  between them is given by the element in $(\mathrm{Bord}_2^{fr})_{1,0}$ which is a strip, i.e. $(0,1)^2$, with the framing given by slowly rotating the framing by 180°, and is embedded into $\mathbb{R} \times (0,1)^2$ by folding it over once as depicted further down.



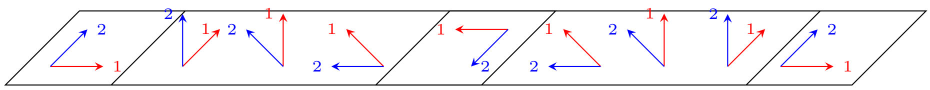


A coevaluation *coev*  is given similarly by rotating the framing along the strip in the other direction, by -180°.

The composition



is connected by a path to the flat strip with the following framing given by pulling at the ends of the strip to flatten it.

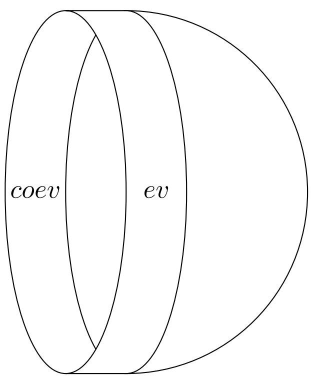


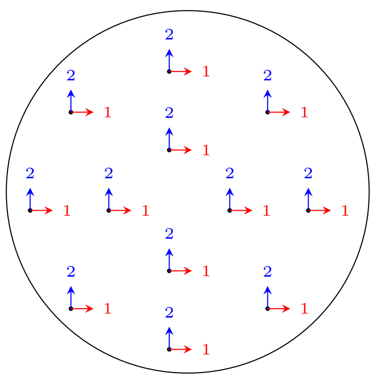
This strip is homotopic to the same strip with the trivial framing. Thus the composition is connected by a path to the identity and thus is the identity in the homotopy category. Similarly,

$$\left(ev \otimes id\right) \circ \left(id \otimes coev\right) \simeq id\,.$$

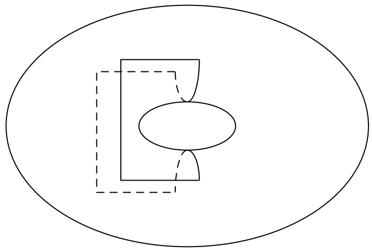
In the above construction, we used ev and $coev$ which arose from strips with framing rotating by $\pm 180°$. A similar argument holds if you use for the evaluation any strip with the framing rotating by $\alpha\pi$ for any odd integer α and for the coevaluation rotation by $\beta\pi$ for any odd β. Denoting these by $ev(\alpha)$ and $coev(\beta)$, they will be adjoints to each other if $\alpha + \beta = 2$.

The counit of the adjunction is given by the cap with the framing coming from the trivial framing on the (flat) disk.

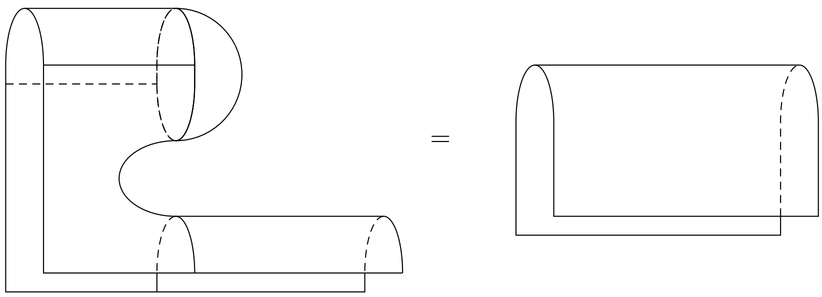




Similarly, the unit of the adjunction is given by a saddle with the framing coming from the one of the torus which turns by 2π along each of the fundamental loops.



Then the following 2-bordism also is framed and exhibits the adjunction.



2.8 Fully extended topological field theories

Now that we have a good definition of a symmetric monoidal (∞, n)-category of bordisms modelled as a symmetric monoidal complete n-fold Segal space, we can define fully extended topological field theories à la Lurie.

2.8.1 Definition

Definition 2.8.1. A *fully extended unoriented n-dimensional topological field theory* is a symmetric monoidal functor of (∞, n)-categories with source $Bord_n$.

Remark 2.8.2. Consider a fully extended unoriented n-dimensional topological field theory

$$Z : \mathrm{Bord}_n \longrightarrow \mathcal{C},$$

where $\mathcal{C}$ is a symmetric monoidal complete n-fold Segal space. We have seen in section 2.6 that $h_1(L_{n-1}(\mathrm{Bord}_n)) \simeq n\mathrm{Cob}$. The Z induces a symmetric monoidal functor

$$n\mathrm{Cob} \simeq h_1(L_{n-1}(\mathrm{Bord}_n)) \longrightarrow h_1(L_{n-1}(\mathcal{C}, Z(*))),$$

i.e. an ordinary n-dimensional topological field theory. The converse for $n > 1$ is not always true and poses interesting questions whether a theory can be "extended down".

Similarly, a fully extended unoriented 2TFT with target $\mathcal{C}$ yields an extended 2TFT

$$2\mathrm{Cob}^{ext} \simeq h_2(\mathrm{Bord}_2) \longrightarrow h_2(\mathcal{C}).$$

Additional structure Recall from the previous section that there are variants of Bord_n which require that the underlying manifolds of their elements to be endowed with some additional structure, e.g. an orientation or a framing. These variants lead to the following definitions.

Definition 2.8.3. Fix a type of structure given by the pair (X, E). A *fully extended n-dimensional (X, E)-topological field theory* is a symmetric monoidal functor of (∞, n)-categories with source $Bord_n^{(X,E)}$.

In particular, the most interesting cases are the following:

Definition 2.8.4. A *fully extended n-dimensional framed topological field theory* is a symmetric monoidal functor of (∞, n)-categories with source $Bord_n^{fr}$.

Definition 2.8.5. A *fully extended n-dimensional oriented topological field theory* is a symmetric monoidal functor of (∞, n)-categories with source $Bord_n^{or}$.

Remark 2.8.6. We will sometimes be imprecise when specifying the type of fully extended TFT. From now on, if we do not specify explicitly that it is unoriented or oriented, we will usually mean that it is framed.

2.8.2 n-TFT yields k-TFT

We will see that every fully extended n-dimensional (unoriented, oriented, framed) TFT yields a fully extended k-dimensional (unoriented, oriented, framed) TFT for any $k \leqslant n$ by truncation from subsection 1.5.1.

Note that for $k < n$, we have a map of k-fold Segal spaces

$$\mathrm{PBord}_k \longrightarrow \tau_k(\mathrm{PBord}_n) = (\mathrm{PBord}_n)_{\underbrace{\bullet,\ldots,\bullet}_{k \text{ times}},\underbrace{0,\ldots,0}_{n-k \text{ times}}}$$

induced by sending $\left(M \hookrightarrow V \times (0,1)^k, (I_j^i\text{'s})_{i=1}^k\right) \in \mathrm{PBord}_k$ to

$$\left(M \times (0,1)^{n-k} \hookrightarrow V \times (0,1)^n, (I_j^i\text{'s})_{i=1}^k, (0,1), \ldots, (0,1)\right).$$

The completion map $\mathrm{PBord}_n \to \mathrm{Bord}_n$ induces a map on the truncations. Precomposition with the above map yields a map of (in general non-complete) n-fold Segal spaces

$$\mathrm{PBord}_k \longrightarrow \tau_k(\mathrm{PBord}_n) \longrightarrow \tau_k(\mathrm{Bord}_n).$$

Recall from 1.5.1 that since $\tau_k(\mathrm{Bord}_n)$ is complete, by the universal property of the completion we obtain a map $\mathrm{Bord}_k \to \tau_k(\mathrm{Bord}_n)$. This ensures that any fully extended n-dimensional (unoriented, oriented, framed) TFT with values in a complete n-fold Segal space $\mathcal{C}$, $\mathrm{Bord}_n \to \mathcal{C}$ leads to a k-dimensional (unoriented, oriented, framed) TFT given by the composition

$$\mathrm{Bord}_k \longrightarrow \tau_k(\mathrm{Bord}_n) \longrightarrow \tau_k(\mathcal{C})$$

with values in the complete k-fold Segal space $\tau_k(\mathcal{C})$.

2.8.3 Cobordism Hypothesis à la Baez-Dolan-Lurie and outlook

In his seminal paper [Lur09c], Lurie gave a detailed sketch of proof of the *Cobordism Hypothesis*, which in its simplest form says that a fully extended framed TFT is fully determined by its value at the object given by a point which will be denoted by $*$. Conversely, any object in the target category which satisfies a suitable finiteness condition can be obtained in this way. The finiteness condition in question is called *fully dualizability*, which we will not explain here. For a full definition, we refer to [Lur09c].

Theorem 2.8.7 (Cobordism Hypothesis, [Lur09c] Theorem 1.4.9)**.** *Let C be a symmetric monoidal (∞, n)-category. The evaluation functor $Z \mapsto Z(*)$ determines a bijection between (isomorphism classes of) symmetric monoidal functors* $\mathrm{Bord}_n^{fr} \to \mathcal{C}$ *and (isomorphism classes of) fully dualizable objects of $\mathcal{C}$.*

Thus to construct a fully extended n-dimensional framed TFT, it suffices to find a fully dualizable object in the target $\mathcal{C}$, and the cobordisms hypothesis does the rest for us. However, fully dualizability is a condition which in general is not completely straightforward to check. Moreover, even though the proof of the cobordism hypothesis tells you that the (∞, n)-category Bord_n of cobordisms is freely generated by the point, it does not give you a simple algorithm with which one can compute all values of the fully extended n-TFT.

Our goal in this thesis is precisely this, namely, for a very special fully extended TFT, to explicitly construct it without invoking the cobordism hypothesis. In the next chapter we will construct our target, a symmetric monoidal (∞, n)-category Alg_n of E_n-algebras, and in the last chapter we will, given any object A in Alg_n, build a fully extended n-TFT by defining a strict functor of n-fold Segal spaces

$$\mathcal{FH}_n(A) : \mathrm{Bord}_n^{fr} \longrightarrow \mathrm{Alg}_n,$$

whose evaluation at the point is A. By the cobordism hypothesis, this in particular shows that any object in Alg_n is fully dualizable.

Chapter 3

The Morita (∞, n)-category of E_n-algebras

In this chapter, we define the target category for our fully extended n-dimensional topological field theory, which is a symmetric monoidal Morita (∞, n)-category $\mathrm{Alg}_n = \mathrm{Alg}_n(\mathcal{S})$ of E_n-algebras. By an E_n-algebra, we mean an E_n-algebra object in a suitable symmetric monoidal $(\infty, 1)$-category $\mathcal{S}$. In [Lur], Lurie proved that there is an equivalence of $(\infty, 1)$-categories between E_n-algebras and locally constant factorization algebras on $(0,1)^n \overset{\chi}{\cong} \mathbb{R}^n$, see theorem 3.2.21. We will use this equivalence to define the objects of our (∞, n)-category of E_n-algebras as a suitable space of locally constant factorization algebras on $(0,1)^n$. As (higher) morphisms we essentially use factorization algebras which are locally constant with respect to a certain stratification to model the Morita category of E_n-algebras as a complete n-fold Segal space $\mathrm{Alg}_n = \mathrm{Alg}_n(\mathcal{S})$. Informally speaking, Alg_n is the (∞, n)-category with E_n-algebras as objects, pointed (A, B)-bimodules in E_{n-1}-algebras as 1-morphisms in $\mathrm{Hom}(A, B)$, and so on.

For the interpretation of our n-fold Segal space as the Morita (∞, n)-category of E_n-algebras we need the following assumption on $\mathcal{S}$.

Assumption 1. *Let $\mathcal{S}$ be a symmetric monoidal $(\infty, 1)$-category which is $\otimes$-sifted cocomplete.*

3.1 The complete n-fold Segal space of closed covers in $(0,1)$

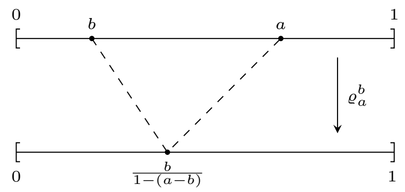
In this section, we construct a (1-)fold Segal space $\mathrm{Covers}_\bullet$ of covers of $(0,1)$ by closed intervals, which we will later enhance by suitable spaces of factorization algebras to give the desired complete n-fold Segal space of E_n-algebras. Before we begin with its construction, we introduce a family of collapse-and-rescale maps ϱ_a^b which will be used to define the simplicial structure.

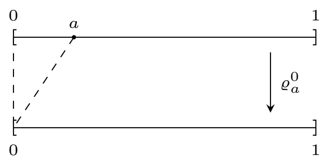
3.1.1 Collapse-and-rescale maps

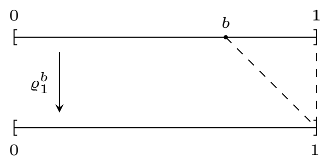
We first define collapse-and-rescale maps $\varrho_a^b : [0,1] \to [0,1]$ which delete the interval $(b, a]$ and rescale the rest back to $[0,1]$.

Definition 3.1.1. Let $0 \leqslant b, a \leqslant 1$ such that $(b, a) \neq (0, 1)$. If $a \leqslant b$, let $\varrho_a^b = id_{[0,1]}$. If $b < a$, let $\varrho_a^b : [0,1] \to [0,1]$,

$$\varrho_a^b(x) = \begin{cases} \frac{x}{1-(a-b)}, & x \leqslant b, \\ \frac{b}{1-(a-b)}, & b \leqslant x \leqslant a, \\ \frac{x-(a-b)}{1-(a-b)}, & a \leqslant x. \end{cases}$$







To simplify notation, we define the following composition of collapse-and-rescaling maps.

Definition 3.1.2. Let $0 \leqslant d, c, b, a \leqslant 1$. Then let

$$\varrho_c^d * \varrho_a^b = \varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b.$$

Remark 3.1.3. Note that if $(b, a) \subseteq (d, c)$, $\varrho_c^d * \varrho_a^b = \varrho_c^d$.

The following lemma shows that if the intervals (d, c) and (b, a) are disjoint, the composition of the respective collapse-and-rescale maps is independent of order in which we delete and rescale and so is determined by the data of the intervals which are collapsed.

Lemma 3.1.4. *Let $0 \leqslant d, c, b, a \leqslant 1$ such that $(d, c) \neq (0, 1) \neq (b, a)$. Furthermore, let $(d, c) \cap (b, a) = \varnothing$. Then*

$$\varrho_c^d * \varrho_a^b = \varrho_a^b * \varrho_c^d.$$

Moreover, if $b = c$ or $a = d$, the above composition is equal to $\varrho_{\max(b,a)}^{\min(d,c)}$.

Proof. Note that ϱ_a^b and ϱ_c^d are monotonically increasing and piecewise linear functions. We first consider the cases in which one of the functions in the composition is the identity.

1. If $d \geqslant c$, $\varrho_a^b(d) \geqslant \varrho_a^b(c)$ and so $\varrho_c^d = id = \varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)}$. Thus,

$$\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b = \varrho_a^b = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d.$$

If $b = c$, $\varrho_{\max(b,a)}^{\min(d,c)} = \varrho_{\max(b,a)}^{c} = \varrho_{\max(b,a)}^{b} = \varrho_a^b$, since if $\max(b, a) \neq a$, $a \leqslant b$, and $\varrho_a^b = id = \varrho_b^b$.

2. If $b \geqslant a$, similarly, $\varrho_a^b = id = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)}$ and
$$\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b = \varrho_c^d = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d.$$
If $b = c$, $\varrho_{\max(b,a)}^{\min(d,c)} = \varrho_b^{\min(d,c)} = \varrho_c^{\min(d,c)} = \varrho_c^d$, since if $\min(d,c) \neq d$, $c \leqslant d$, and $\varrho_c^d = id = \varrho_c^c$.

Since ϱ_a^b and ϱ_c^d are piecewise linear functions their composition again is piecewise linear. Thus in the remaining case it suffices to compute their value at the "break points". The computation of the composition in between the break points is essentially the same so we include it as well.

3. In the remaining case we can assume wlog that $c \leqslant b$ and thus $d < c \leqslant b < a$. This implies that
$$\varrho_a^b(d) = \tfrac{d}{1-(a-b)}, \quad \varrho_a^b(c) = \tfrac{c}{1-(a-b)},$$
$$\varrho_c^d(b) = \tfrac{b-(c-d)}{1-(c-b)}, \quad \varrho_c^d(b) = \tfrac{b-(c-d)}{1-(c-b)}.$$
If $x \leqslant d$,
$$\begin{aligned}\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) &= \frac{x}{1-(a-b)} \frac{1}{1-\frac{c-d}{1-(a-b)}} \\ &= \frac{x}{1-(a-b)-(c-d)} \\ &= \frac{x}{1-(c-d)} \frac{1}{1-\left(\frac{a-(c-d)}{1-(c-d)} - \frac{b-(c-d)}{1-(c-d)}\right)} \\ &= \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x).\end{aligned}$$
If $d \leqslant x \leqslant c$,
$$\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) = \frac{d}{1-(a-b)-(c-d)} = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x).$$
If $c \leqslant x \leqslant b$,
$$\begin{aligned}\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) &= \frac{\frac{x}{1-(a-b)} - \frac{c-d}{1-(a-b)}}{1-\frac{c-d}{1-(a-b)}} \\ &= \frac{x-(c-d)}{1-(a-b)-(c-d)} \\ &= \frac{\frac{x-(c-d)}{1-(c-d)}}{1-\left(\frac{a-(c-d)}{1-(c-d)} - \frac{b-(c-d)}{1-(c-d)}\right)} \\ &= \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x).\end{aligned}$$

If $b \leqslant x \leqslant a$,

$$\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) = \frac{b-(c-d)}{1-(a-b)-(c-d)} = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x).$$

If $a \leqslant x$,

$$\begin{aligned}
\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) &= \frac{\frac{x-(a-b)}{1-(a-b)} - \frac{c-d}{1-(a-b)}}{1-\frac{c-d}{1-(a-b)}} \\
&= \frac{x-(a-b)-(c-d)}{1-(a-b)-(c-d)} \\
&= \frac{\frac{x-(c-d)}{1-(c-d)} - \left(\frac{a-(c-d)}{1-(c-d)} - \frac{b-(c-d)}{1-(c-d)}\right)}{1-\left(\frac{a-(c-d)}{1-(c-d)} - \frac{b-(c-d)}{1-(c-d)}\right)} \\
&= \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x).
\end{aligned}$$

If $b = c$,

$$\frac{d}{1-(a-b)-(c-d)} = \frac{d}{1-(a-d)} = \frac{b-(c-d)}{1-(a-b)-(c-d)},$$

so the composition reduces to

$$\varrho_a^d = \varrho_{\max(b,a)}^{\min(d,c)}.$$

□

In the following, since the intervals we consider lie in $(0,1)$ we often use the restriction of ϱ_a^b to the domain $D(\varrho_a^b)$, which is defined as follows.

Definition 3.1.5. Let $0 \leqslant b, a \leqslant 1$ such that $(b,a) \neq (0,1)$. Let

$$D(\varrho_a^b) = \begin{cases} (0,1), & 0 \leqslant b, a \leqslant 1, \\ (a,1), & b = 0, \\ (0,b), & a = 1. \end{cases}$$

We might like to restrict to an even smaller domain to get a partial inverse.

Definition 3.1.6. The restriction of the collapse-and-rescale map ϱ_a^b to $D_a^b = (0,b] \cup (a,1) \subset D(\varrho_a^b)$,

$$\varrho_a^b|_{(0,b]\cup(a,1)} : D_a^b \longrightarrow (0,1),$$

is injective. We call D_a^b its *domain of injectivity.* In the following, let $(\varrho_a^b)^{-1}$ be the inverse of this restriction, $(\varrho_a^b|_{D_a^b})^{-1} : (0,1) \to D_a^b$.

3.1.2 The level sets Covers_k

We first define 0-simplices of the levels Covers_k as sets.

Definition 3.1.7. For an integer $k \geqslant 0$ let

$$\mathrm{Covers}_k = \{I_0 \leqslant \cdots \leqslant I_k\}$$

be the set consisting of ordered $(k+1)$-tuples of intervals $I_j \subseteq (0,1)$ such that I_j has non-empty interior, is closed in $(0,1)$ and $\bigcup_{j=0}^{k} I_j = (0,1)$. As in the definition of Int_k in 2.1, by "ordered" we mean that the left endpoints, denoted by a_j, and the right endpoints, denoted by b_j, are ordered.

Remark 3.1.8. Note that the condition that the intervals form a cover, $\bigcup_{j=0}^{k} I_j = (0,1)$, implies that $a_0 = 0$ and $b_k = 1$.

3.1.3 The spatial structure of Covers_k

The l-simplices of the space Covers_k

Similarly to the definition of Int_k, the l-simplices of Covers_k do not just consist of the data of a continuous family of covers. We add a rescaling datum which remembers how the intervals were deformed.

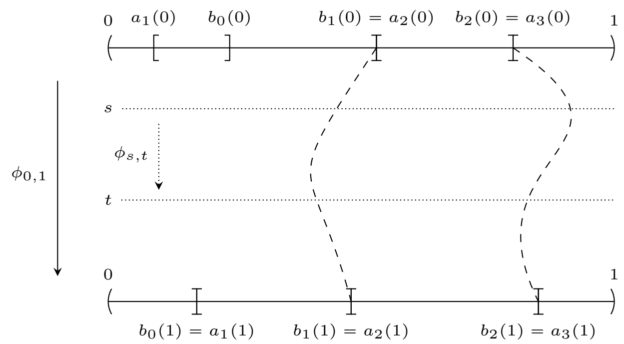
Definition 3.1.9. Let $(I_0(s) \leqslant \cdots \leqslant I_k(s)) \in \mathrm{Covers}_k$ be a continuous family of covers by closed intervals over $|\Delta^l|$, i.e. denoting the left endpoints by $a_j(s)$ and the right endpoints by $b_j(s)$, the maps $|\Delta^l| \to \mathbb{R}$, $s \mapsto a_j(s), b_j(s)$ are continuous maps. A continuous family of orientation-preserving homeomorphisms

$$(\phi_{s,t} : (0,1) \to (0,1))_{s,t \in |\Delta^l|}$$

is said to *intertwine with the composed covers* if the following condition is satisfied for every morphism $f : [m] \to [l]$ in the simplex category Δ.

Let $|f| : |\Delta^m| \to |\Delta^l|$ be the induced map between standard simplices. The homeomorphism $\phi_{s,t}$ should send the common endpoint at s to the corresponding endpoint at t for intervals which do not overlap along $|f|(\Delta^m|)$. Explicitly, this means that for every $0 \leqslant j < k$ such that for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ consists of exactly one element, namely $b_j(s) = a_{j+1}(s)$, we require that for every $s \in |f|(|\Delta^m|)$,

$$b_j(s) = a_{j+1}(s) \overset{\phi_{s,t}}{\longmapsto} b_j(t) = a_{j+1}(t),$$



Remark 3.1.10. Note that as with Int_k, it is enough to check this condition for $m \leqslant l$.

Definition 3.1.11. An l-simplex of Covers_k consists of

1. a continuous family of *underlying 0-simplices*, i.e. for every $s \in |\Delta^l|$,
$$(I_1(s) \leqslant \cdots \leqslant I_k(s)) \in \mathrm{Covers}_k,$$
depending continuously on s;
2. a *rescaling datum*, which is a collection of strictly monotonically increasing homeomorphisms
$$(\phi_{s,t} : (0,1) \to (0,1))_{s,t \in |\Delta^l|}$$
such that
   a) $\phi_{s,s} = id$ for every $s \in |\Delta^l|$,
   b) $\phi_{s,t} = \phi_{t,s}^{-1}$ for every $s, t \in |\Delta^l|$, and
   c) $(\phi_{s,t})_{s,t \in |\Delta^l|}$ intertwines with the composed covers.

Remark 3.1.12. Note that in particular for $l = 0$ an l-simplex in this sense is an underlying 0-simplex together with $\phi_{s,s} = id : (0,1) \to (0,1)$, so, by abuse of language we call both a 0-simplex.

The space Covers_k

The spatial structure arises similarly to that on Int_k.

Fix $k \geqslant 0$ and let $f : [m] \to [l]$ be a morphism in the simplex category Δ and $|f| : |\Delta^m| \to |\Delta^l|$ the induced map between standard simplices. Then let f^* be the map sending an l-simplex in Covers_k given by

$$(I_0(s) \leqslant \cdots \leqslant I_k(s))_{s \in |\Delta^l|}, \qquad \big(\phi_{s,t} : (0,1) \to (0,1)\big)_{s,t \in |\Delta^l|}$$

to

$$I_0(|f|(s)) \leqslant \ldots \leqslant I_k(|f|(s))_{s \in |\Delta^m|}, \qquad \big(\phi_{|f|(s),|f|(t)} : (0,1) \longrightarrow (0,1)\big)_{s \in |\Delta^m|}.$$

Analogous to lemma 2.1.7 we have

Lemma 3.1.13. *This gives a functor* $\Delta^{op} \to \mathrm{Set}$ *and thus* Covers_k *is a space, i.e. a simplicial set.*

Notation 3.1.14. We denote the spatial face and degeneracy maps by δ_j^{Δ} and σ_j^{Δ} for $0 \leqslant j \leqslant l$.

We will need the following lemma later for the Segal condition. Its proof is similar to that of lemma 2.1.9

Lemma 3.1.15. *Each level* Covers_k *is contractible.*

3.1.4 The simplicial set $\mathrm{Covers}_\bullet$

In this section, we make the collection of sets $\mathrm{Covers}_\bullet$ (ignoring the spatial structure we just constructed) into a simplicial set by defining degeneracy and face maps, which use the family of collapse-and-rescale maps $\varrho_a^b : (0,1) \to (0,1)$ defined in subsection 3.1.1.

Definition 3.1.16. The *jth degeneracy map* is given by inserting the jth interval twice,

$$\begin{array}{rcl} \mathrm{Covers}_k & \xrightarrow{\sigma_j} & \mathrm{Covers}_{k+1}, \\ I_0 \leqslant \cdots \leqslant I_k & \longmapsto & I_0 \leqslant \cdots \leqslant I_j \leqslant I_j \leqslant \cdots \leqslant I_k. \end{array}$$

The *jth face map* is given by deleting the jth interval, collapsing what now is not covered, and rescaling the rest linearly to $(0,1)$. Explicitly,

$$\begin{array}{rcl} \mathrm{Covers}_k & \xrightarrow{\delta_j} & \mathrm{Covers}_{k-1}, \\ I_0 \leqslant \cdots \leqslant I_k & \longmapsto & \varrho_{a_{j+1}}^{b_{j-1}}(I_0) \cap (0,1) \leqslant \cdots \leqslant \widehat{\varrho_{a_{j+1}}^{b_{j-1}}(I_j)} \leqslant \cdots \leqslant \varrho_{a_{j+1}}^{b_{j-1}}(I_k) \cap (0,1), \end{array}$$

where $\varrho_{a_{j+1}}^{b_{j-1}}$ is the collapse-and-rescale map associated to b_{j-1}, a_{j+1} from the previous section.

Proposition 3.1.17. $\mathrm{Covers}_\bullet$ *is a simplicial set.*

Proof. We need to show that the simplicial relations are satisfied. Two conditions are obviously fulfilled, namely $\sigma_l\sigma_j = \sigma_{j+1}\sigma_l$ for $l \leqslant j$ and

$$\delta_l\sigma_j = \begin{cases} id, & l = j, j+1, \\ \sigma_{j-1}\delta_l, & l < j, \\ \sigma_j\delta_{l-1} & l > j+1. \end{cases}$$

It remains to check that

$$\delta_j\delta_l = \delta_{l-1}\delta_j \quad \text{for } j < l.$$

Let $I_0 \leqslant \cdots \leqslant I_k$ be an element in Covers_k. Since the same intervals are deleted in both compositions, it is enough to show that the compositions of the respective collapse-and-rescale maps coincide on both sides. This follows from lemma 3.1.4 with

$$d = b_{j-1}, \quad c = a_{j+1}, \quad b = b_{l-1}, \quad a = a_{l+1},$$

given that $(b_{j-1}, a_{j+1}) \cap (b_{l-1}, a_{l+1}) = \varnothing$, which requires that

$$a_{j+1} = c \leqslant b = b_{l-1}.$$

Assume the opposite, that is, that $b_{l-1} \leqslant a_{j+1}$. By definition, $a_{j+1} < b_{j+1} \leqslant b_\alpha$ for $\alpha > j$, so this implies that $l-1 \leqslant j$. Since we need to check the identity for $j < l$, this implies that $l = j+1$. The intervals $(I_j)_j$ must form a cover of $(0,1)$, so $b_{l-1} \geqslant a_l = a_{j+1}$ and therefore $a_{j+1} = b_{l-1}$. So in any case

$$a_{j+1} = c \leqslant b = b_{l-1}.$$

□

3.1.5 The Segal space $\mathrm{Covers}_\bullet$

Face and degeneracy maps on l-simplices

We first need to extend the (simplicial) face and degeneracy maps δ_j, σ_j to l-simplices in a compatible way. They essentially arise from applying the face and degeneracy maps δ_j, σ_j to each of the 0-simplices underlying the l-simplex.

Notation 3.1.18. Let

$$\Big((I_0(s) \leqslant \cdots \leqslant I_k(s))_{s\in|\Delta^l|}, \qquad (\phi_{s,t})_{s,t\in|\Delta^l|}\Big)$$

be an l-simplex of Covers_k. For $s \in |\Delta^l|$, denote by $\varrho^{b_{j-1}}_{a_{j+1}}(s) = \varrho^{b_{j-1}(s)}_{a_{j+1}(s)}$ the collapse-and-rescale map associated to the sth underlying 0-simplex $(I_0(s) \leqslant \cdots \leqslant I_k(s))$ of the above l-simplex, and by $D^b_a(s) = (0, b_{j-1}(s)] \cup (a_{j+1}(s), 1)$ its domain of injectivity.

Degeneracy maps on l-simplices For $0 \leqslant j \leqslant k$ the *jth degeneracy map* σ_j sends an l-simplex of Covers_k

$$\Big((I_0(s) \leqslant \cdots \leqslant I_k(s))_{s\in|\Delta^l|}, \qquad (\phi_{s,t})_{s,t\in|\Delta^l|}\Big)$$

to the l-simplex of Covers_{k+1} given by

$$\Big(\sigma_j(I_0(s) \leqslant \cdots \leqslant I_k(s))_{s\in|\Delta^l|}, \qquad (\phi_{s,t})_{s,t\in|\Delta^l|}\Big)$$

This is well-defined, since the conditions on the $\phi_{s,t}$ stays the same.

Face maps on l-simplices For $0 \leqslant j \leqslant k$ the *jth face map* δ_j sends an l-simplex of Covers_k

$$\Big((I_0(s) \leqslant \cdots \leqslant I_k(s))_{s\in|\Delta^l|}, \qquad (\phi_{s,t})_{s,t\in|\Delta^l|}\Big)$$

to the following l-simplex of Covers_{k-1}.

1. The underlying 0-simplices of the image are the images of the underlying 0-simplices under δ_j, i.e. for $s \in |\Delta^l|$,
$$\delta_j\left(I_0(s) \leqslant \cdots \leqslant I_k(s)\right);$$
2. Its rescaling datum is
$$\delta_j(\phi_{s,t}) = \varrho^{b_{j-1}}_{a_{j+1}}(t) \circ \phi_{s,t}|_{D^b_a(s)} \circ \varrho^{b_{j-1}}_{a_{j+1}}(s)^{-1} : (0,1)^n \to (0,1)^n.$$

The complete Segal space $\mathrm{Covers}_\bullet$

Proposition 3.1.19. $\mathrm{Covers}_\bullet$ *is a complete Segal space.*

Proof. That the simplicial and spatial face and degeneracy maps commute follows directly from the definition. Furthermore, we have seen in lemma 3.1.15 that every space Covers_k is contractible. This ensures the Segal condition, namely that

$$\mathrm{Covers}_k \xrightarrow{\simeq} \mathrm{Covers}_1 \overset{h}{\underset{\mathrm{Covers}_0}{\times}} \cdots \overset{h}{\underset{\mathrm{Covers}_0}{\times}} \mathrm{Covers}_1,$$

and completeness. □

Definition 3.1.20. Let

$$\mathrm{Covers}^n_{\bullet,\dots,\bullet} = (\mathrm{Covers}_\bullet)^{\times n}.$$

Lemma 3.1.21. *The n-fold simplicial space* $\mathrm{Covers}^n_{\bullet,\dots,\bullet}$ *is a complete n-fold Segal space.*

Proof. The Segal condition and completeness follow from the Segal condition and completeness for $\mathrm{Covers}_\bullet$. Since every Covers_k is contractible by lemma 3.1.15, $(\mathrm{Covers}_\bullet)^{\times n}$ satisfies essential constancy, so Covers^n is an n-fold Segal space. □

3.2 The Morita (∞, n)-category of E_n-algebras Alg_n

This section contains the main construction of the complete n-fold Segal space $\mathrm{Alg}_n(\mathcal{S})$. To shorten notation, we will abbreviate $\mathrm{Alg}_n = \mathrm{Alg}_n(\mathcal{S})$ throughout. We first recall the definition of an E_n-algebra.

3.2.1 Structured disks and E_n-algebras

As in section 2.7.1, let X be a topological space and $E \to X$ a topological n-dimensional vector bundle which corresponds to a (homotopy class of) map(s) $e : X \to B\mathrm{GL}(\mathbb{R}^n)$ from X to the classifying space of the topological group $\mathrm{GL}(\mathbb{R}^n)$.

Definition 3.2.1. The symmetric monoidal topological category $\mathcal{D}isk_n^{(X,E)}$ of (X, E)-*structured disks* is the full topological subcategory of $\mathcal{M}an_n^{(X,E)}$ whose objects are disjoint unions of (X, E)-structured n-dimensional Euclidean disks $\mathbb{R}^n$.

Example 3.2.2. Recall from section 2.7.1 that interesting examples of (X, E)-structures arise from a topological group G together with a continuous homomorphism $e : G \to O(n)$ by setting $X = BG$ and $e : BG \to B\mathrm{GL}(\mathbb{R}^n)$. In this case, we refer to (BG, e)-structured disks as G-structured disks and use the notation $\mathcal{D}isk_n^G = \mathcal{D}isk_n^{(BG,e)}$.

Definition 3.2.3. Let $\mathcal{S}$ be a symmetric monoidal $(\infty, 1)$-category. The $(\infty, 1)$-category $\mathcal{D}isk_n^{(X,E)}$-$\mathrm{Alg}(\mathcal{S})$ of $\mathcal{D}isk_n^{(X,E)}$-*algebras* is the $(\infty, 1)$-category of symmetric monoidal functors $\mathrm{Fun}^\otimes(\mathcal{D}isk_n^{(X,E)}, \mathcal{S})$.

Remark 3.2.4. Recall from section 1.2 that topological categories are a model for $(\infty, 1)$-categories. By perhaps changing to a different, suitable, model of $(\infty, 1)$-categories, the above definition makes sense.

The most common examples are the following three special cases.

Example 3.2.5. If G is the trivial group, then $X = BG = *$, and the topological category $\mathcal{D}isk_n^G$ is denoted by $\mathcal{D}isk_n^{fr}$. Using the fixed diffeomorphism $\chi : (0, 1) \cong \mathbb{R}$ it is equivalent to the topological category Cube_n whose objects are disjoint unions of $(0, 1)^n$ and whose spaces of morphisms are the spaces of embeddings $\coprod_I (0, 1)^n \to \coprod_J (0, 1)^n$ which are rectilinear on every connected component. As Cube_n-algebras are equivalent to E_n-algebras, the category $\mathcal{D}isk_n^{fr}$-$\mathrm{Alg}(\mathcal{S})$ is equivalent to the usual category of E_n-algebras in $\mathcal{S}$.

Remark 3.2.6. Note that morphisms in the category $\mathcal{D}isk_n^{fr}$-$\mathrm{Alg}(\mathcal{S})$ are morphisms of E_n-algebras, i.e. natural transformations of functors. In the Morita category we will construct in this section morphisms will be bimodules of E_n-algebras.

Example 3.2.7. If $G = O(n)$, the topological category $\mathcal{D}isk_n^G$ is denoted by $\mathcal{D}isk_n^{un}$. We call $\mathcal{D}isk_n^{un}$-algebras *unoriented E_n-algebras*. Similarly to in the previous example $\mathcal{D}isk_n^{un}$ is equivalent to the topological category Cube_n^{un} whose objects are disjoint unions of $(0,1)^n$ and whose spaces of morphisms are the spaces $\mathrm{Cube}_n(\coprod_I (0,1)^n, \coprod_J (0,1)^n) \ltimes O(n)^{\times J}$.

Example 3.2.8. If $G = SO(n)$ and $X = BG$, the topological category $\mathcal{D}isk_n^{(X,E)}$ is denoted by $\mathcal{D}isk_n^{or}$. We call $\mathcal{D}isk_n^{(X,E)}$-algebras *oriented E_n-algebras*. Again similarly to above $\mathcal{D}isk_n^{or}$ is equivalent to the topological category Cube_n^{or} whose objects are disjoint unions of $(0,1)^n$ and whose spaces of morphisms are the spaces $\mathrm{Cube}_n(\coprod_I (0,1)^n, \coprod_J (0,1)^n) \ltimes SO(n)^{\times J}$.

3.2.2 Factorization algebras

We recall the definition of factorization algebras and basic facts. Factorization algebras were first introduced by Beilinson-Drinfeld in the algebro-geometric context for curves in [BD04]. Inspired by this, Lurie first coined the term factorizing cosheaf in [Lur]. They were further studied and developped in [CG], [GTZ10], and others. We mainly follow the conventions in [Gin]. The other main reference is [CG]. Note that all our factorization algebras will be non-lax homotopy (i.e. derived) factorization algebras in the language of [CG]. For simplicity, let us assume that $\mathcal{S}$ is given by a relative category $(\mathcal{S}, \mathcal{W})$.

Definition 3.2.9. Let X be a topological space. A *prefactorization algebra on X with values in $\mathcal{S}$* is a functor

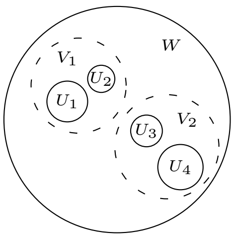
$$\mathcal{F} : \mathrm{Open}(X) \longrightarrow \mathcal{S}$$

together with structure maps

$$f_{U_1 \amalg \ldots \amalg U_n \subseteq V} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V)$$

for every finite disjoint union of open sets U_i (independent of the ordering of the U_i) included in another open set V such that

1. for $U \subseteq V$,
$$\mathcal{F}(U \subseteq V) = f_{U \subseteq V},$$
2. the following coherence (or associativity) condition is satisfied: for any finite collection of pairwise disjoint open subsets $(V_j)_{j \in J}$ lying in an open set W and for every $j \in j$, a finite collection of open subsets $(U_{i,j})_{i \in J_i}$ lying in V_j, the following diagram induced by the structure maps commutes:



$$\rightsquigarrow \quad \begin{array}{ccc} \bigotimes_{j\in J} \bigotimes_{i \in J_i} \mathcal{F}(U_{i,j}) & \longrightarrow & \bigotimes_{j \in J} \mathcal{F}(V_j) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(W) & \end{array}$$

Remark 3.2.10. The inclusion of the empty set induces map $\mathcal{F}(\varnothing) \to \mathcal{F}(U)$ for every open set U. Thus prefactorization algebras are pointed objects.

Remark 3.2.11. Prefactorization algebras on X are algebras over the colored operad with open sets in X as colors and

$$\mathcal{P}re\mathcal{F}act_X(U_1, \ldots, U_n; V) = \begin{cases} \{*\} & \text{if } U_1 \amalg \ldots \amalg U_n \subseteq V; \\ \varnothing & \text{otherwise.} \end{cases}$$

The ∞-operad associated to this operad was denoted by $N(\mathrm{Disk}(X))$ in [Lur].

Definition 3.2.12. Let $\mathscr{U}$ be an open cover of U. Then the *Čech complex* of $\mathscr{U}$ with values in a prefactorization algebra $\mathcal{F}$ on U is the simplicial object in $\mathcal{S}$ constructed as follows:

Let $P\mathscr{U}$ be the set of finite pairwise disjoint open subsets $\{U_1, \ldots, U_n : U_i \in \mathscr{U}, n \geqslant 0\}$. For $i \geqslant 0$, let

$$\check{C}^i(\mathscr{U}, \mathcal{F}) = \bigoplus_{\alpha \in P\mathscr{U}^{i+1}} \bigotimes_{U_j \in \alpha_j} \mathcal{F}\left(\bigcap_{j=0}^{i} U_j\right).$$

To define the face maps for $0 \leqslant j_0 \leqslant i$

$$\partial_{j_0} : \check{C}^{i+1}(\mathscr{U}, \mathcal{F}) \longrightarrow \check{C}^i(\mathscr{U}, \mathcal{F}),$$

consider the structure maps of the inclusions of open sets

$$\bigcap_{j=0}^{i} U_j \hookrightarrow \bigcap_{\substack{k=0 \\ k \neq j_0}}^{i} U_j.$$

Then the j_0th face map is given by the direct sum of their tensor products

$$\bigotimes_{U_j \in \alpha_j} \mathcal{F}\left(\bigcap_{j=0}^{i} U_j\right) \longrightarrow \bigotimes_{\substack{U_k \in \alpha_k, \\ k \neq j_0}} \mathcal{F}\left(\bigcap_{\substack{k=0 \\ k \neq j_0}}^{i} U_j\right).$$

Degeneracy maps are given by repeating one of the α_j's. Summarizing we obtain a simplicial diagram is $\mathcal{S}$,

$$\check{C}^\bullet(\mathscr{U}, \mathcal{F}) = \left(\bigoplus_{\alpha \in P\mathscr{U}} \bigotimes_{U \in \alpha} \mathcal{F}(U) \leftleftarrows \bigoplus_{\alpha \in P\mathscr{U}^2} \bigotimes_{U_j \in \alpha_j} \mathcal{F}(U_0 \cap U_1) \; \substack{\leftarrow \\ \leftarrow \\ \leftarrow} \; \cdots \right).$$

Remark 3.2.13. Note that the structure maps associated to the inclusions $\bigcap_{j=0}^{i} U_j \subseteq U$ give maps $\mathcal{F}(\bigcap_{j=0}^{i} U_j) \to \mathcal{F}(U)$ and thus $\check{C}^i(\mathscr{U}, \mathcal{F}) \longrightarrow \mathcal{F}(U)$ which commute with the simplicial maps by coherence. We obtain a map of simplicial objects in $\mathcal{S}$

$$\check{C}^\bullet(\mathscr{U}, \mathcal{F}) \longrightarrow \mathcal{F}(U).$$

Definition 3.2.14. An open cover $\mathscr{U}$ of U is said to be *factorizing* if for every finite set of pairwise distinct points $\{x_1, \ldots, x_n\} \subseteq U$ there is a finite subset $U_1, \ldots, U_k \subseteq \mathscr{U}$ of pairwise disjoint open sets such that

$$\{x_1, \ldots, x_n\} \subseteq U_1 \amalg \ldots \amalg U_k.$$

A *(homotopy) factorization algebra on X with values in $\mathcal{S}$* is prefactorization algebra on X with values in $\mathcal{S}$ for which the following *gluing condition* holds: for every open $U \subseteq X$ and every factorizing cover $\mathscr{U}$ of U the map

$$\check{C}^\bullet(\mathscr{U}, \mathcal{F}) \longrightarrow \mathcal{F}(U)$$

is a weak equivalence.

Remark 3.2.15 (Unitality)**.** If $\mathcal{F}$ is a factorization algebra on X with values in $\mathcal{S}$, then $\mathcal{F}(\varnothing) \simeq \mathbb{1}$ the monoidal unit in $\mathcal{S}$: Consider the Čech complex for the empty cover $\mathscr{U} = \varnothing$ of $U = \varnothing$. There is exactly one $\alpha \in P\mathscr{U}$, namely $\alpha = \varnothing$ and thus every factor $\bigotimes_{U_j \in \alpha_j} \mathcal{F}\left(\bigcap_{j=0}^{i} U_j\right) = \bigotimes_{\varnothing} = \mathbb{1}$. Thus the gluing condition requires the map

$$(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbb{1} \rightrightarrows \mathbb{1}) \longrightarrow \mathcal{F}(\varnothing)$$

to be a weak equivalence.

Remark 3.2.16 (Comparison with [CG])**.** Let $U = U_1 \amalg \cdots \amalg U_n$. Then $\mathscr{U} = \{U_1, \cdots, U_n\}$ is a factorizing open cover. Then gluing condition implies that the structure map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \xrightarrow{\simeq} \mathcal{F}(U). \tag{3.1}$$

Thus we do not consider lax factorization algebras in the sense of [CG].

Moreover, they require a gluing condition for covers with a stronger property, namely being a *Weiss cover*, named after Michael Weiss. However, every factorizing open cover $\mathscr{U}$ generates a Weiss open cover $\mathscr{V} = \{U_1 \amalg \cdots \amalg U_n : U_i \in \mathscr{U}\}$. Then the gluing condition for Weiss covers together with (3.1) is equivalent to the gluing property in our sense.

Definition 3.2.17. A *morphism of prefactorization algebras* $\mathcal{F} \to \mathcal{G}$ is a collection of maps

$$\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

for every open set U which commutes with the structure maps. Since $\mathcal{S}$ was a symmetric monoidal $(\infty, 1)$-category this gives a symmetric monoidal $(\infty, 1)$-category of prefactorization algebras $\mathcal{PFact}_X(\mathcal{S})$: e.g. if we start with a symmetric monoidal relative category $(\mathcal{S}, \mathcal{W})$, then define weak equivalences of prefactorization algebras to be the morphisms with values in $\mathcal{W}$. The symmetric monoidal structure is defined level-wise.

The category $\mathcal{F}act_X(\mathcal{S})$ of factorization algebras on X with values in $\mathcal{S}$ is the symmetric monoidal full sub-$(\infty, 1)$-category of the $(\infty, 1)$-category of prefactorization algebras on X with values in $\mathcal{S}$.

3.2.3 Stratifications and locally constant factorization algebras

The full definition of locally constant factorization algebras on a (stratified) space can be found in [Gin]. In this paper, we will only deal with stratifications of a very special type, so we recall the definition in an easier setting suitable for the factorization algebras appearing in this thesis here.

Definition 3.2.18. Let X be an n-dimensional manifold. By a *stratification of X* we mean a filtration

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X,$$

where X_α is an α-dimensional closed submanifold of $X_{\alpha+1}$. The connected components of $X_\alpha \backslash X_{\alpha-1}$ are called the *dimension α-strata of X*. An open disk D in X is said to have *index α*, if $D \cap X_\alpha \neq \varnothing$ and $D \subset X \backslash X_{\alpha-1}$. We say that a disk D is a *good neighborhood at X_α* if α is the index of D and $D \cap (X_\alpha \backslash X_{\alpha-1})$ is connected.

Definition 3.2.19. Let $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$ be a stratification of an n-dimensional manifold X. A factorization algebra $\mathcal{F}$ on X is called *locally constant with respect to the stratification* if for any inclusion of disks $U \hookrightarrow V$ such that both U and V are good neighborhoods at X_α for the same index $\alpha \in \{0, \ldots, n\}$, the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a weak equivalence.

A factorization algebra $\mathcal{F}$ on X is called *locally constant* if it is locally constant with respect to the stratification given by $X_\alpha = \varnothing$ for every $\alpha \neq n$, i.e.

$$\varnothing \subset X.$$

Notation 3.2.20. We denote by $\mathcal{F}act_X^{lc}$ the full sub-$(\infty,1)$-category of $\mathcal{F}act_X(\mathcal{S})$ whose objects are locally constant factorization algebras.

E_n-algebras as locally constant factorization algebras We will base our construction on factorization algebras which are locally constant with respect to certain stratifications. That our objects, which will be locally constant factorization algebras on $(0,1)^n$, indeed are E_n-algebras as defined in the previous section follows from the following theorem due to Lurie.

Theorem 3.2.21 (Lurie, [Lur], Theorem 5.3.4.10)**.** *There is an equivalence of $(\infty,1)$-categories*

$$\mathcal{D}isk_n^{fr}\text{-}Alg(\mathcal{S}) \xrightarrow{\simeq} \mathcal{F}act_{\mathbb{R}^n}^{lc}.$$

Remark 3.2.22. In fact, the equivalence in the proof is given by factorization homology, which we will construct in the next section, in section 4.1. That is, the image of an E_n-algebra A is its factorization homology $\int_{\mathbb{R}^n} A$.

The choice of diffeomorphism $\chi : (0,1) \cong \mathbb{R}$ yields the following corollary, see also [Gin], Remark 23, or [Cal].

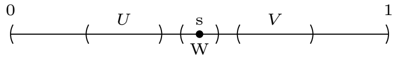
Corollary 3.2.23. *There is an equivalence of $(\infty,1)$-categories*

$$E_n\text{-}Alg(\mathcal{S}) \xrightarrow{\simeq} \mathcal{F}act_{(0,1)^n}^{lc}.$$

Bimodules as locally constant factorization algebras

Our second motivation for using factorization algebras is the following. For more details, see [Gin].

Let A, B be associative algebras in $\mathcal{S}$, M a pointed (A,B)-bimodule, with pointing $\mathbb{1} \xrightarrow{m} M$. Then the following assignment extends to a factorization algebra $\mathcal{F}_M$ on $(0,1)$: Let $0 < s < 1$. For open intervals U, V, and W in $(0,1)$ as in the picture



we set

$$U \longmapsto \mathcal{F}_M(U) = A, \qquad V \longmapsto \mathcal{F}_M(V) = B,$$

$$p \in W \longmapsto \mathcal{F}_M(W) = M.$$

The structure maps of the factorization algebra are given by the bimodule structure and by

$$A \otimes B \cong A \otimes \mathbb{1} \otimes B \xrightarrow{m} M.$$

This special case comes from the fact that factorization algebras naturally are pointed, as we can always include the empty set into any other open set. The inclusion $\varnothing \subseteq W$ induces a map

$$\mathbb{1} \longrightarrow M.$$

In the case where $\mathcal{S} = Ch_k$ is the $(\infty, 1)$-category of chain complexes over a field k, the pointing is a map $k \to M$ which is determined by the image of $1 \in k$,

$$1 \longmapsto m \in M.$$

In this case the structure map of $U \amalg V \subset (0,1)$ is given by

$$A \otimes B \longrightarrow M, \quad (a, b) \longmapsto amb.$$

The factorization algebra $\mathcal{F}_M$ defined by a bimodule M as above is locally constant with respect to the stratification

$$\varnothing \subset \{s\} \subset (0,1).$$

Conversely, any factorization algebra $\mathcal{F}$ which is locally constant with respect to a stratification of the above form determines a homotopy bimodule M over homotopy algebras A, B as we show in the following lemma.

Lemma 3.2.24. *Let $0 < s < 1$ and let $\mathcal{F}$ be a factorization algebra on $(0,1)$ which is locally constant with respect to the stratification*

$$\varnothing \subset \{s\} \subset (0,1).$$

Then $M = \mathcal{F}((0,1))$ is, up to homotopy, a pointed (A, B)-bimodule for the (E_1-)algebras $A = \mathcal{F}((0,s))$ and $B = \mathcal{F}((s,1))$ and pointing $\mathbb{1} \to M$ induced by the structure map for the inclusion $\varnothing \subset (0,1)$.

Proof. Since $U \subset (0,s)$ and $V \subset (s,1)$ are weak equivalences, the structure map of the factorization algebra associated to the inclusion of open sets $U \amalg V \subset (0,1)$ as in the picture above induces the homotopy bimodule structure. □

Corollary 3.2.25. *The data of a homotopy bimodule over E_1-algebras is the same as the data of a factorization algebra on $(0,1)$ which is locally constant with respect to a stratification of the form*

$$\varnothing \subset \{s\} \subset (0,1)$$

for some $0 < s < 1$.

Locally constant factorization algebras on products

We will need the following theorem later on, which is proposition 18 and corollary 6 in [Gin].

Theorem 3.2.26. *Let X, Y be stratified manifolds with finitely many dimension α-strata for every α.*

1. *The pushforward along the projection $pr_1 : X \times Y \to X$ induces a functor*
$$\underline{pr_1}_* : \mathcal{PFact}_{X\times Y} \longrightarrow \mathcal{PFact}_X(\mathcal{PFact}_Y),$$
which restricts to a functor
$$\underline{pr_1}_* : \mathcal{Fact}_{X\times Y} \longrightarrow \mathcal{Fact}_X(\mathcal{Fact}_Y).$$
2. *Consider the stratification on the product $X \times Y$ given by*
$$(X \times Y)_k := \bigcup_{i+j=k} X_i \times Y_j \subset X \times Y.$$
The functor from 1 induces a functor
$$\underline{pr_1}_* : \mathcal{Fact}^{lc}_{X\times Y} \longrightarrow \mathcal{Fact}^{lc}_X(\mathcal{Fact}^{lc}_Y)$$
between the subcategories of factorization algebras which are locally constant with respect to the stratifications of the respective spaces.

3.2.4 The level sets $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$

For $S \subseteq \{1, \ldots, n\}$ we denote the projection from $(0,1)^n$ onto the coordinates indexed by S by $\pi_S : (0,1)^n \to (0,1)^S$ and for $1 \leqslant i \leqslant n$, we abbreviate $\pi_{\{i\}}$ to π_i.

Definition 3.2.27. For every $k_1, \ldots, k_n \geqslant 0$, let $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ be the collection of tuples
$$(\mathcal{F}, (I^i_0 \leqslant \cdots \leqslant I^i_{k_i})_{i=1,\ldots,n}),$$
satisfying the following conditions:

1. $\mathcal{F}$ is a factorization algebra on $(0,1)^n$.
2. For $1 \leqslant i \leqslant n$,
$$(I^i_0 \leqslant \cdots \leqslant I^i_{k_i}) \in \mathrm{Covers}_{k_i}.$$
3. $\mathcal{F}$ is locally constant with respect to the stratification defined inductively by
$$X_n = (0,1)^n \quad \text{and} \quad X_{n-i} = X_{n-i+1} \cap Y_i$$
for $1 \leqslant i \leqslant n$, where, denoting by $(I^i_j)^\circ = (a^i_j, b^i_j)$ the interior of the interval I^i_j,
$$Y_i = \pi_i^{-1}\big((0,1)\backslash \bigcup_{j=0}^{k_i} (a^i_j, b^i_j)\big) = (0,1)^n \backslash \bigcup_{j=0}^{k_i} \pi_i^{-1}\big((I^i_j)^\circ\big).$$

Remark 3.2.28. Given an element in $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$, let $0 < s_1^i \leqslant \ldots \leqslant s_{l_i}^i < 1$ be the points such that

$$S^i = \{s_1^i, \ldots, s_{l_i}^i\} = (0,1) \setminus \bigcup_{j=0}^{k_i} (a_j^i, b_j^i).$$

Then $Y_i = \pi_i^{-1}(S^i)$ is a disjoint union of parallel hyperplanes and

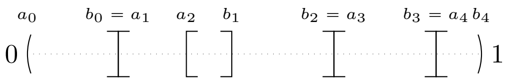
$$\begin{aligned} X_{n-i} &= Y_n \cap \cdots \cap Y_i \\ &= \bigcup_{(1 \leqslant j_\alpha \leqslant k_\alpha)_{\alpha=1}^i} \pi_{\{1,\ldots,i\}}^{-1}(s_{j_1}^1, \ldots, s_{j_i}^i) \\ &= S^1 \times \cdots \times S^i \times (0,1)^{\{i+1,\ldots,n\}}. \end{aligned}$$

The stratification has the form

$$(0,1)^n \supset \bigcup_{1 \leqslant j \leqslant k_1} \pi_1^{-1}(s_j^1) \supset \bigcup_{\substack{1 \leqslant j_1 \leqslant k_1 \\ 1 \leqslant j_2 \leqslant k_2}} \pi_{\{1,2\}}^{-1}(s_{j_1}^1, s_{j_2}^2) \supset \cdots \supset \bigcup_{(1 \leqslant j_i \leqslant k_i)_{i=1}^n} \pi^{-1}(s_{j_1}^1, \ldots, s_{j_n}^n).$$

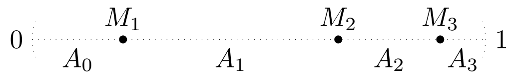
Remark 3.2.29. In fact, the data of the points in S^i is the essential one in the sense that they are the information of $(I_0^i \leqslant \cdots \leqslant I_{k_i}^i) \in \mathrm{Covers}_{k_i}$ we use. It might thus seem more natural to basing our construction on a Segal space of points instead of $\mathrm{Covers}_\bullet$. However, the points alone do not form a simplicial space because degeneracy maps cannot be defined. The extra information coming from the fact that points come from endpoints of intervals allows to define the missing structure.

Example 3.2.30. For $n = 1$, objects, which are elements in $(\mathrm{Alg}_1)_0$, are locally constant factorization algebras on $(0,1) \cong \mathbb{R}$, which in turn by the above mentioned equivalence 3.2.21 are E_1-algebras. Morphisms, i.e. elements in $\mathrm{Map}(A, B) = \{A\} \times^h_{(\mathrm{Alg}_1)_0} (\mathrm{Alg}_1)_1 \times^h_{(\mathrm{Alg}_1)_0} \{B\}$, are pointed homotopy (A, B)-bimodules as we have seen in lemma 3.2.24. For example, an element in $(\mathrm{Alg}_1)_4$ could have a cover of the form

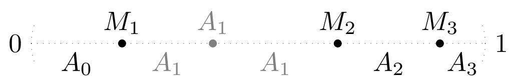


and therefore factorization algebras $\mathcal{F}$ which are locally constant with respect to a stratification of the following form

Since $\mathcal{F}|_{(0,s_1)}$ is locally constant on $(0, s_1) \simeq (0,1)$ it equivalent to the data of an E_1-algebra A_0. Similarly, $\mathcal{F}$ determines E_1-algebras $A_1, \ldots, A_3$. Moreover, the restriction $\mathcal{F}|_{(0,s_2)}$ determines a pointed homotopy (A_0, A_1)-bimodule M_1 and similarly, $\mathcal{F}$ determines bimodules M_2, M_3:

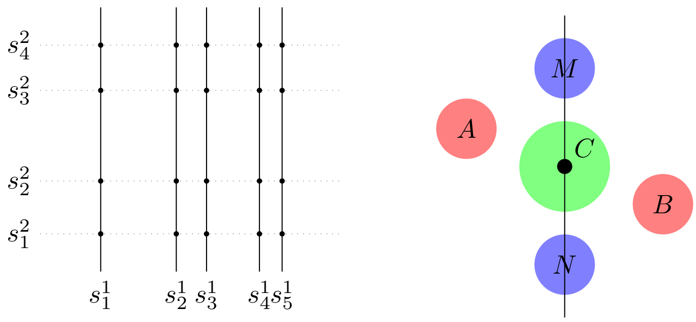


One may think of the overlapping intervals as also giving a point of the stratification, but one which is "degenerate", and thus gives a "degenerate" bimodule, by which we mean an E_1-algebra viewed as a bimodule over itself.

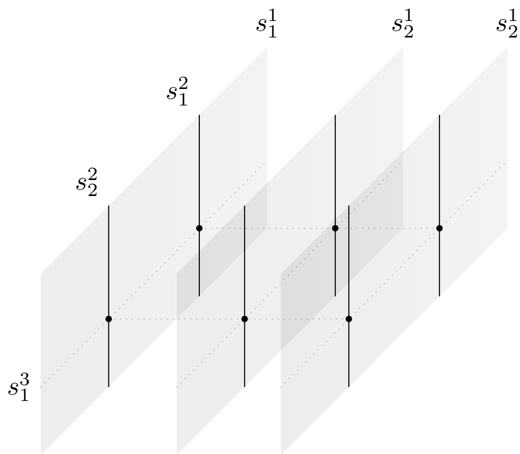


Remark 3.2.31. One should be a bit careful with the interpretation of the degenerate points of the stratification, as this data does not behave well with respect to the simplicial structure. As we explained above, this is the reason we do not use this as a definition, but keep track of the intervals instead.

Example 3.2.32. For $n = 2$, stratifications which appear in the definition of Alg_2 give pictures as in the left picture below. A 2-morphism, i.e. an element in $(\mathrm{Alg}_2)_{1,1}$, leads to a bimodule C between bimodules M and N of E_2-algebras A and B which are the images of open disks as in the right picture below.



For $n = 3$, stratifications which appear in the definition of Alg_3 give pictures of the following type:



3.2.5 The spaces $(\mathrm{Alg}_n)_{k_1\dots,k_n}$

The level sets $(\mathrm{Alg}_n)_{k_1,\dots,k_n}$ form the underlying set of 0-simplices of a space which we construct in this section.

The space of factorization algebras

We first need suitable spaces of factorization algebras.

Recall from 3.2.17 that the category $\mathcal{F}act_X(\mathcal{S})$ of factorization algebras on X with values in $\mathcal{S}$ is a symmetric monoidal $(\infty,1)$-category. Thus it has a space of objects, which is the space of factorization algebras we are interested in.

If we start with a symmetric monoidal relative category $(\mathcal{S},\mathcal{W})$, then $\mathcal{F}act_X(\mathcal{S})$ again is a symmetric monoidal relative category. The classification diagram as explained in section 1.2.3 of $\mathcal{F}act_X(\mathcal{S})$ with its level-wise weak equivalences gives a symmetric monoidal complete Segal space $N(\mathcal{F}act_X(\mathcal{S}),\mathcal{W})$ of factorization algebras.

The objects of this Segal space form a space, i.e. a simplicial set, of factorization algebras. Explicitly, if we begin with a relative category, this space of factorization algebras is the nerve of the category of factorization algebras with weak equivalences as morphisms, i.e. a k-simplex is a sequence

$$\mathcal{F}_0 \xrightarrow{w_1} \mathcal{F}_1 \xrightarrow{w_2} \cdots \xrightarrow{w_k} \mathcal{F}_k.$$

However, it will not be a Kan complex unless $\mathcal{S}$ was an ∞-groupoid. Thus, we use a modification of this definition which seems to be close to a Kan fibrant replacement of the above construction. We will give a more conceptual construction using families of factorization algebras in a subsequent paper.

The spatial structure of $(\mathrm{Alg}_n)_{k_1\dots,k_n}$

Definition 3.2.33. An l-simplex of $(\mathrm{Alg}_n)_{k_1,\dots,k_n}$ consists of the following data:

1. A family of *underlying 0-simplices*, which is a collection of elements

$$\big(\mathcal{F}_s, (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))_{i=1}^n\big) \in (\mathrm{Alg}_n)_{k_1,\dots,k_n}$$

   indexed by $s \in |\Delta^l|$;

2. For every $1 \leqslant i \leqslant n$, a *rescaling datum* $\big(\phi_{s,t}^i : (0,1) \to (0,1)\big)_{s,t\in|\Delta^l|}$ making

$$\big(I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s)\big)_{s\in|\Delta^l|}$$

   into an l-simplex in Covers_{k_i};

3. A collection of *weak equivalences*

$$(\phi_{s,t})_*\mathcal{F}_s \xrightarrow{w_{s,t}} \mathcal{F}_t$$

   for $s,t \in |\Delta^l|$, where $\phi_{s,t} = (\phi_{s,t}^i)_{i=1}^n : (0,1)^n \to (0,1)^n$ is the product of the rescaling data.

Remark 3.2.34. This space can be thought of as a "rescaled" version of the space of factorization algebras explained above, namely, we compare factorization algebras after rescaling using the diffeomorphisms $\phi_{s,t}$. When the context is clear we will sometimes omit writing the pushforwards along the rescaling maps and simply write $\mathcal{F}_s \to \mathcal{F}_t$.

Similarly to the construction of PBord_n, the spatial structure on $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ extends the one on $\mathrm{Covers}^n_{k_1,\ldots,k_n}$.

Fix $k \geqslant 0$ and let $f : [m] \to [l]$ be a morphism in the simplex category Δ, i.e. a (weakly) order-preserving map. Then let $|f| : |\Delta^m| \to |\Delta^l|$ be the induced map between standard simplices.

Let f^Δ be the map sending an l-simplex in $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ to the m-simplex which consists of

1. for $s \in |\Delta^m|$,
$$\mathcal{F}_{|f|(s)};$$
2. for $1 \leqslant i \leqslant n$, the m-simplex in Covers_{k_i} obtained by applying f^Δ,
$$f^\Delta\Big(I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s), \phi^i_{s,t}\Big);$$
3. for $s, t \in |\Delta^m|$,
$$w_{|f|(s),|f|(t)} : \mathcal{F}_{|f|(s)} \longrightarrow \mathcal{F}_{|f|(t)}.$$

Proposition 3.2.35. *$(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ is a space. Moreover, it is a Kan complex, i.e. fibrant in the category of simplicial sets with Quillen model structure.*

Proof. The spatial structure essentially comes from the spatial structure of $\mathrm{Covers}^n_{k_1,\ldots,k_n}$ and the simplicial structure of $N(\Delta)$, so the assignments above are well-defined and behave functorially.

The proof of the Horn filling condition is similar to that of the singular set of a topological space. A morphism $\Lambda^l_k \to \mathrm{Alg}_n$ is the data of, for $s, t \in |\Lambda^l_k|$,
$$\big(\mathcal{F}_s, (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))_{i=1,\ldots,n}\big) \in (\mathrm{Alg}_n)_{k_1,\ldots,k_n},$$
$$\big(\phi^i_{s,t} : (0,1) \to (0,1)\big), \text{and} \qquad \big(w_{s,t} : \mathcal{F}_s \longrightarrow \mathcal{F}_t\big).$$
Now choose a retraction $p : |\Delta^l| \to |\Lambda^l_k|$. Then for $\tilde{s}, \tilde{t} \in |\Delta^l|$, consider
$$\big(\mathcal{F}_{p(\tilde{s})}, (I_0^i(p(\tilde{s})) \leqslant \cdots \leqslant I_{k_i}^i(p(\tilde{s})))_{i=1,\ldots,n}\big) \in (\mathrm{Alg}_n)_{k_1,\ldots,k_n},$$
$$\big(\phi^i_{p(\tilde{s}),p(\tilde{t})} : (0,1) \to (0,1)\big), \text{and} \qquad \big(w_{p(\tilde{s}),p(\tilde{t})} : \mathcal{F}_{p(\tilde{s})} \longrightarrow \mathcal{F}_{p(\tilde{t})}\big).$$
This defines an l-simplex in $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ whose restriction to the kth horn is the given one. □

Notation 3.2.36. We denote the *spatial face and degeneracy maps* of $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ by δ^Δ_j and σ^Δ_j for $0 \leqslant j \leqslant l$.

3.2.6 The n-fold simplicial set Alg_n

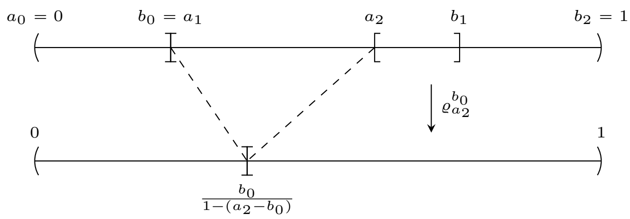
In the next two sections, we make the collection of spaces $(\mathrm{Alg}_n)_{\bullet,\dots,\bullet}$ into an n-fold simplicial space by defining suitable face and degeneracy maps. They essentially arise from the face and degeneracy maps of the n-fold simplicial set $\mathrm{Covers}^n_{\bullet,\dots,\bullet}$ of covers of $(0,1)^n$ by products of closed intervals. In this section we define faces and degeneracies on 0-simplices, which makes $(\mathrm{Alg}_n)_{\bullet,\dots,\bullet}$ into an n-fold simplicial set, ignoring the spatial structure of the levels. We will lift the n-fold simplicial set to an n-fold simplicial space using the spatial structure of the levels in the next section.

Before giving the full definition of the face and degeneracy maps of the n-fold simplicial set Alg_n, we first demonstrate them for $n = 1$.

Example 3.2.37. For $n = 1$, elements in $(\mathrm{Alg}_1)_1$ consist of a factorization algebra $\mathcal{F}$ on $(0,1)$ and two intervals $(0,b]$ and $[a,1)$ such that $a \leqslant b$. The source and target maps $(\mathrm{Alg}_1)_1 \rightrightarrows (\mathrm{Alg}_1)_0$ are given by restricting the factorization algebra which then is rescaled back to $(0,1)$. Explicitly, the source map pushes forward the restriction of the factorization algebra $\mathcal{F}$ to $(0,b)$ by the collapse-and-rescale map ϱ_0^b to $(0,1)$, which is the unique affine bijection $(0,b) \to (0,1)$. Similarly the target map pushes forward the restriction of the factorization algebra $\mathcal{F}$ to $(a,1)$ by the collapse-and-rescale map ϱ_a^0 to $(0,1)$. We saw in example 3.2.30 that elements in $(\mathrm{Alg}_1)_1$ can be viewed as pairs (A,B) of E_1-algebras and a pointed homotopy (A,B)-bimodule M. The source and target maps $(\mathrm{Alg}_1)_1 \rightrightarrows (\mathrm{Alg}_1)_0$ map M to the source A, respectively the target B.

The degeneracy map $(\mathrm{Alg}_1)_0 \to (\mathrm{Alg}_1)_1$ sends a pair $(\mathcal{F},(0,1))$ consisting of a locally constant factorization algebra F on $(0,1)$ to the element $(\mathcal{F},(0,1) \leqslant (0,1))$. In the language of algebras and bimodules, it sends an E_1-algebra A to itself, now viewed as an (A,A)-bimodule.

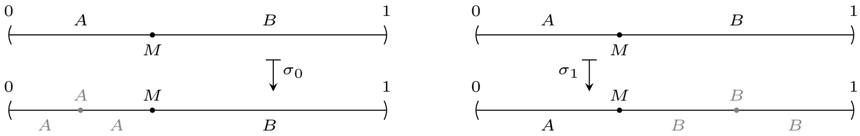
Two of the face maps, $\delta_0, \delta_2 : (\mathrm{Alg}_1)_2 \rightrightarrows (\mathrm{Alg}_1)_1$ are defined similarly, by "forgetting" part of the data, i.e. by restricting the factorization algebra and rescaling. In the language of modules, the map δ_2, which corresponds to the "source map", sends an element consisting of a triple (A,B,C) of E_1-algebras and a pair $({}_AM_B, {}_BN_C)$ of bimodules to (A,B) and ${}_AM_B$. The "target map" δ_0 sends the same element to (B,C) and ${}_BM_C$. The third map δ_1, which corresponds to composition, sends an element $(\mathcal{F}, I_0 \leqslant I_1 \leqslant I_2)$ to the pushforward along the collapse-and-rescale map $\varrho_{a_2}^{b_0} : (0,1) \to (0,1)$, illustrated in the following picture for the case $b_0 = a_1$.



If $b_0 \geqslant a_2$, then $\varrho_{a_2}^{b_0} = id$. Moreover, either $A = B$ and ${}_AM_B = {}_BB_B$, or $B = C$ and ${}_BN_C = B$ (or both). In the first case δ_1 sends $({}_BB_B, {}_BM_C)$ to just ${}_BM_C$. In the second case δ_1 sends $({}_AM_B, {}_BB_B)$ to just ${}_AM_B$.

If $b_0 < a_2$, the gluing axiom of factorization algebras implies that the homotopy bimodule associated to image under δ_1 of the pair of homotopy bimodules $({}_AM_B, {}_BN_C)$ is the tensor product $({}_AM_B) \otimes_B ({}_BN_C)$, i.e. composition sends an element consisting of E_1-algebras A, B, C and bimodules ${}_AM_B$ and ${}_BN_C$ to A, C and the bimodule $({}_AM_B) \otimes_B ({}_BN_C)$.

The two degeneracy maps $\sigma_0, \sigma_1 : (\mathrm{Alg}_1)_1 \rightrightarrows (\mathrm{Alg}_1)_2$ send $(\mathcal{F}, I_0 \leqslant I_1)$ to $\sigma_0(\mathcal{F}, I_0 \leqslant I_1) = (\mathcal{F}, I_0 \leqslant I_0 \leqslant I_1)$, $\sigma_1(\mathcal{F}, I_0 \leqslant I_1) = (\mathcal{F}, I_0 \leqslant I_1 \leqslant I_1)$. In the language of modules, they send an (A, B)-bimodule ${}_AM_B$ to the pairs $({}_AA_A, {}_AM_B)$ respectively $({}_AM_B, {}_BB_B)$.



Notation 3.2.38. Before we start defining the face and degeneracy maps, recall that we used collapse-and-rescale maps ϱ_a^b to define the simplicial structure on $\mathrm{Covers}_\bullet$. More precisely, the jth face map was defined using $\varrho_{a_{j+1}}^{b_{j-1}}$. For simplicity of notation, we will denote this map by ϱ_j in the following and its domain of injectivity by D_j.

Since $1 \leqslant i \leqslant n$ will be fixed throughout the constructions, by abuse of notation, we also denote by ϱ_j the map $\varrho_{a_{j+1}^i}^{b_{j-1}^i}$ used for the jth face map in the ith direction of the n-fold simplicial structure of $\mathrm{Covers}^n_{\bullet,\dots,\bullet}$ and its domain of injectivity by $D_j = (0, b_{j-1}^i) \cup (a_{j+1}^i, 1)$.

By even more abuse of notation we again denote by ϱ_j the map

$$\varrho_j : (0,1)^n \to (0,1)^n,$$

which is ϱ_j in the ith coordinate and the identity otherwise. By ϱ_j^{-1} we mean the inverse of

$$\varrho_j|_{\pi_i^{-1}(D_j)} : \pi_i^{-1}(D_j) = \prod_{\alpha \neq i} (0,1) \times D_j \to (0,1)^n.$$

Degeneracy maps Fix $1 \leqslant i \leqslant n$. For $0 \leqslant j \leqslant k_i$ the *jth degeneracy map*

$$\sigma_j^i : (\mathrm{Alg}_n)_{k_1,\dots,k_n} \to (\mathrm{Alg}_n)_{k_1,\dots,k_i+1,\dots,k_n}$$

applies the jth degeneracy map of $\mathrm{Covers}_\bullet$ to the ith tuple of intervals, i.e. it repeats the jth specified interval in the ith direction, I_j^i,

$$\begin{aligned}(\mathcal{F}, (I_0^\alpha \leqslant \dots \leqslant I_{k_\alpha}^\alpha)_{\alpha=1}^n) \quad \longmapsto \quad & \big(\mathcal{F}, (I_0^\alpha \leqslant \dots \leqslant I_{k_\alpha}^\alpha)_{\alpha \neq i}, \sigma_j(I_1^i \leqslant \dots \leqslant I_{k_i}^i)\big) = \\ & \big(\mathcal{F}, (I_0^\alpha \leqslant \dots \leqslant I_{k_\alpha}^\alpha)_{\alpha \neq i}, I_1^i \leqslant \dots \leqslant I_j^i \leqslant I_j^i \leqslant \dots \leqslant I_{k_i}^i\big).\end{aligned}$$

Since this does not change the stratification with respect to which $\mathcal{F}$ must be locally constant this map is well-defined.

Face maps Fix $1 \leqslant i \leqslant n$. For $0 \leqslant j \leqslant k_i$ the *jth face map*

$$\delta_j^i : (\mathrm{Alg}_n)_{k_1,\dots,k_n} \to (\mathrm{Alg}_n)_{k_1,\dots,k_i-1,\dots,k_n}$$

applies the jth face map δ_j of $\mathrm{Covers}_\bullet$ to the ith tuple of intervals, which forgets the jth interval I_j^i and applies the collapse-and-rescale map ϱ_j to the other intervals, and pushes the factorization algebra, restricted to $\pi_i^{-1}(D_j)$, forward along the map ϱ_j. Explicitly, $(\mathcal{F}, (I_0^\alpha \leqslant \cdots \leqslant I_{k_\alpha}^\alpha)_{\alpha=1}^n)$ is sent to

$$\left((\varrho_j)_* \mathcal{F}|_{\pi_i^{-1}(D_j)}, (I_0^\alpha \leqslant \cdots \leqslant I_{k_\alpha}^\alpha)_{\alpha \neq i}, \delta_j(I_1^i \leqslant \cdots \leqslant I_{k_i}^i)\right).$$

This is well-defined since the restriction of the factorization algebra and the stratification with respect to which it must be locally constant are rescaled by the same rescaling map.

Remark 3.2.39. In the following, we will omit explicitly writing out the restriction of $\mathcal{F}$ to $\pi_i^{-1}(D_j)$ for readability.

Proposition 3.2.40. *The face and degeneracy maps defined above define an n-fold simplicial set* $(\mathrm{Alg}_n)_{\bullet,\ldots,\bullet}$.

Proof. This follows from the fact that $\mathrm{Covers}_\bullet$ is a simplicial set and pushforward of factorization algebras is a functor. ☐

3.2.7 The full structure of Alg_n as an n-fold simplicial space

In this section we "extend" the simplicial face and degeneracy maps δ_j^i, σ_j^i to the l-simplices of $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ in a way that they commute with the spatial structure of $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$. This gives Alg_n the structure of an n-fold simplicial space.

Degeneracy maps on l-simplices. Fix $1 \leqslant i \leqslant n$. For $0 \leqslant j \leqslant k_i$ the *jth degeneracy map* σ_j^i sends an l-simplex of $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ to the l-simplex of $(\mathrm{Alg}_n)_{k_1,\ldots,k_i+1,\ldots,k_n}$ defined by applying the degeneracy map σ_j^i to each underlying 0-simplex,

$$\sigma_j^i\left(\mathcal{F}_s, (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))\right) \in (\mathrm{Alg}_n)_{k_1,\ldots,k_i+1,\ldots,k_n},$$

and keeping the same rescaling data $\phi_{s,t}$ and weak equivalences $(\phi_{s,t})_* \mathcal{F}_s \xrightarrow{w_{s,t}} \mathcal{F}_t$.

Face maps on l-simplices. Fix $1 \leqslant i \leqslant n$.

For $0 \leqslant j \leqslant k_i$ the *jth face map* δ_j^i sends an l-simplex of $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ consisting of

$$\left(\mathcal{F}_s, (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))\right)_{s \in |\Delta^l|}$$

$$\left(\phi_{s,t} : (0,1)^n \longrightarrow (0,1)^n\right)_{s,t \in |\Delta^l|}, \quad \text{and} \quad \left((\phi_{s,t})_* \mathcal{F}_s \xrightarrow{w_{s,t}} \mathcal{F}_t\right)_{s,t \in |\Delta^l|}$$

to the l-simplex of $(\mathrm{Alg}_n)_{k_1,\ldots,k_i-1,\ldots,k_n}$ consisting of the following data.

Denote by $\varrho_j(s)$ be the analog of the above map ϱ_j associated to the sth underlying 0-simplex $(I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s)) \in \mathrm{Covers}_{k_i}$.

1. The underlying 0-simplices of the image are the images of the underlying 0-simplices under δ_j^i, i.e. for $s \in |\Delta^l|$,

$$\delta_j^i\left(\mathcal{F}_s, (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))\right) = \left(\varrho_j(s)_* \mathcal{F}_s|_{\ldots}, (I_0^\alpha(s) \leqslant \cdots \leqslant I_{k_\alpha}^\alpha(s))_{\alpha \neq i}, \delta_j(I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))\right),$$

where we omit writing down the precise restriction domain from now on. It can be checked easily that they match up where needed.

2. The underlying l-simplex in Covers_{k_i} is sent to its image under δ^i_j, i.e. its rescaling data is $\delta^i_j(\phi_{s,t})$. Recall from section 3.1.5 that this is the map
$$\delta^i_j(\phi_{s,t}) = \varrho_j(t) \circ \phi_{s,t}|_{\dots} \circ \varrho_j(s)^{-1} : (0,1)^n \to (0,1)^n.$$

3. Pushforward along $\varrho_j(s)$ is an endofunctor of the category of factorization algebras on $(0,1)^n$ which preserves weak equivalences, so for every $s \in |\Delta^l|$ we have the following weak equivalences
$$\delta^i_j(\phi_{s,t})_*\Big(\varrho_j(s)_*\mathcal{F}_s|_{\dots}\Big) = \varrho_j(t)_*(\phi_{s,t}|_{\dots})_*\mathcal{F}_s|_{\dots} \xrightarrow{\varrho_j(t)_* w_{s,t}} \varrho_j(t)_*\mathcal{F}_t|_{\dots}.$$

Proposition 3.2.41. $(\mathrm{Alg}_n)_{\bullet,\cdots,\bullet}$ *is an n-fold simplicial space.*

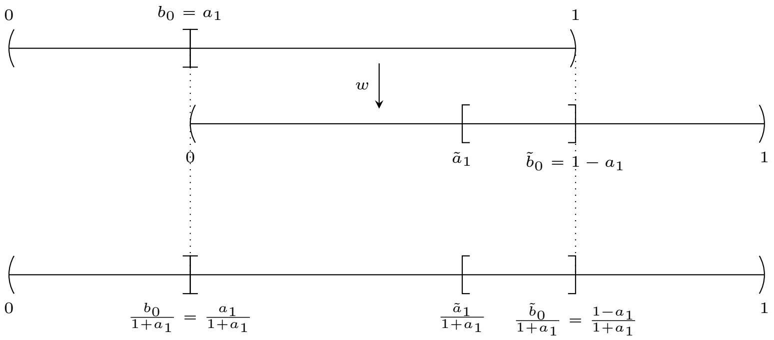
Proof. The degeneracy and face maps σ^i_j, δ^i_j defined above satisfy the simplicial relations since we showed in lemma 3.1.21 that $\mathrm{Covers}^n_{\bullet,\dots,\bullet}$ is an n-fold Segal space, in particular, we proved that the rescaling maps commute in the appropriate way. Moreover, they commute with each other since they modify different parts of the structure and they commute with the spatial structure maps f^Δ of the spaces $(\mathrm{Alg}_n)_{k_1,\dots,k_n}$. □

3.2.8 The n-fold Segal space Alg_n

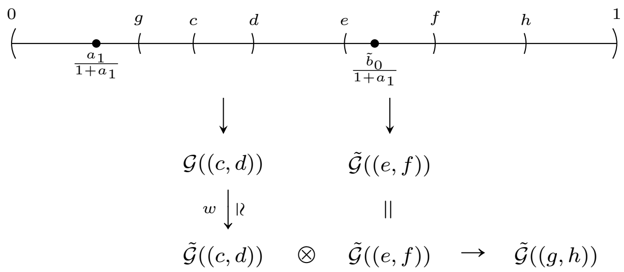
Proposition 3.2.42. $(\mathrm{Alg}_n)_{\bullet,\dots,\bullet}$ *is an n-fold Segal space.*

Proof. We need to prove the following conditions:

1. *The Segal condition is satisfied.* For clarity, we explain the Segal condition in the following case. The general proof works similarly. We will show that
$$(\mathrm{Alg}_n)_{k_1,\dots,2,\dots k_n} \xrightarrow{\sim} (\mathrm{Alg}_n)_{k_1,\dots,1,\dots,k_n} \overset{h}{\underset{(\mathrm{Alg}_n)_{k_1,\dots,0,\dots,k_n}}{\times}} (\mathrm{Alg}_n)_{k_1,\dots,1,\dots,k_n}.$$
To simplify notation, we omit the indices and the specified points in all directions except for the ith one, as this procedure only depends on this specified direction. We construct a map
$$(\mathrm{Alg}_n)_1 \overset{h}{\underset{(\mathrm{Alg}_n)_0}{\times}} (\mathrm{Alg}_n)_1 \xrightarrow{glue} (\mathrm{Alg}_n)_2$$
which is a deformation retraction, i.e. $glue \circ (\delta_0 \times \delta_2) = id$, $(\delta_0 \times \delta_2) \circ glue \sim id$.
Since every level set $(\mathrm{Alg}_n)_{k_1,\dots,k_n}$ is a Kan complex by proposition 3.2.35, i.e. fibrant, the homotopy fiber product consists of triples consisting of two points and a path between them. Thus an element in $(\mathrm{Alg}_n)_1 \times^h_{(\mathrm{Alg}_n)_0} (\mathrm{Alg}_n)_1$ consists of two factorization algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ on $(0,1)^n$, specified intervals $(0, b_0] \leqslant [a_1, 1)$, $(0, \tilde{b}_0] \leqslant [\tilde{a}_1, 1)$ in the i-th direction, and a path between their target and source. The path in particular gives a weak equivalence $\delta_1(\tilde{\mathcal{G}}) \xrightarrow{w} \delta_0(\mathcal{G})$. Here again, we omit the rescaling in the notation. We glue them to an element in $(\mathrm{Alg}_n)_2$ in the following way. By first applying a piecewise linear rescaling, we can assume that $1 - a_1 = \tilde{b}_0$.



Send the above data to the factorization algebra $\mathcal{F}$ on $(0,1)^n$ defined by $\mathcal{G}$ on $(0, \frac{\tilde{b}_0}{1+a_1}) \times \prod_{\alpha \neq i}(0,1)$ and $\tilde{\mathcal{G}}$ on $(\frac{a_1}{1+a_1}, 1) \times \prod_{\alpha \neq i}(0,1)$ using rescaling maps which, again, we will omit for clarity of notation. It remains to "glue" them together using the weak equivalence w. On an interval (a,b) such that $\frac{a_1}{1+a_1} < a < \frac{\tilde{b}_0}{1+a_1} < b$, $\mathcal{F}((a,b)) := \mathcal{G}((a,b))$. Moreover, we define the factorization algebra structure by



Note that this way the factorization algebra is defined on a factorizing cover and can be extended by the gluing condition.

This construction extends to the spatial structure since weak equivalences are defined locally and thus can be glued along open sets. By construction, $glue \circ (\delta_0 \times \delta_2) = id$. Moreover, the weak equivalence w coming from the path gives $(\delta_0 \times \delta_2) \circ glue \sim id$.

2. *For every i and every $k_1, \ldots, k_{i-1}$, $(\mathrm{Alg}_n)_{k_1,\ldots,k_{i-1},0,\bullet,\cdots,\bullet}$ is essentially constant.*

   An element in $(\mathrm{Alg}_n)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$ is of the form

$$(\mathcal{F}, I_0^1 \leqslant \ldots \leqslant I_{k_1}^1, \ldots, I_0^{i-1} \leqslant \ldots \leqslant I_{k_{i-1}}^{i-1}, (0,1), \\ I_0^{i+1} \leqslant \ldots \leqslant I_{k_{i+1}}^{i+1}, \ldots, I_0^n \leqslant \ldots \leqslant I_{k_n}^n),$$

   so by definition the stratification with respect to which $\mathcal{F}$ is locally constant reduces to

$$(0,1)^n = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_{i+1} \supseteq X_i = X_{i+1} \cap \pi_{n-i}^{-1}\Big((0,1) \backslash (0,1)\Big) = \varnothing.$$

   Since the stratification only depends on the first $i-1$ tuples of intervals we can freely move the remaining intervals I_j^α for $\alpha > i$ and still have a well-defined element in $(\mathrm{Alg}_n)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$. In particular, we can move them to $I_0^\alpha = \cdots = I_{k_\alpha}^\alpha = (0,1)$, which is in the image of the composition of degeneracy maps S. We can chose the endpoints to move linearly (by setting $a_j^\alpha(t) = (1-t)a_j^\alpha$ and $b_j^\alpha(t) = (1-t)b_j^\alpha + t$), so this construction extends to a homotopy. Hence

$$S : (\mathrm{Alg}_n)_{k_1,\dots,k_{i-1},0,\dots,0} \xrightarrow{\simeq} (\mathrm{Alg}_n)_{k_1,\dots,k_{i-1},0,k_{i+1},\dots,k_n}$$

is a weak equivalence.

□

Remark 3.2.43. One can alternatively show the Segal condition by showing that the source and target maps $s, t : (\mathrm{Alg}_n)_{k_1,\dots,1,\dots k_n} \to (\mathrm{Alg}_n)_{k_1,\dots,0,\dots k_n}$ are Serre fibrations and then showing that

$$(\mathrm{Alg}_n)_{k_1,\dots,k_i,\dots k_n} \xrightarrow{\sim} (\mathrm{Alg}_n)_{k_1,\dots,1,\dots,k_n} \underset{(\mathrm{Alg}_n)_{k_1,\dots,0,\dots,k_n}}{\times} \cdots \underset{(\mathrm{Alg}_n)_{k_1,\dots,0,\dots,k_n}}{\times} (\mathrm{Alg}_n)_{k_1,\dots,1,\dots,k_n}.$$

One can show the homotopy lifting property for cubes I^k for s, t explicitly by constructing a lift. This construction is similar to the construction of the map *glue* above. The second "strict Segal" condition follows from the fact that factorization algebras satisfy a descent condition, see e.g. 4.3.5 in [Gin].

3.2.9 Completeness of Alg_n and the Morita (∞, n)-category of E_n-algebras

Factorization algebras with values in a symmetric monoidal relative category with all coproducts $\mathcal{S}$ are pointed in the sense that given a factorization algebra $\mathcal{F}$, for any open set U the inclusion of the empty set $\varnothing \hookrightarrow U$ gives a map $\mathbb{1} \to \mathcal{F}(U)$, where $\mathbb{1}$ is the unit for the monoidal product of the symmetric monoidal structure of $\mathcal{S}$. In this subsection we show that this pointing ensures completeness of Alg_n.

We will first explain the argument for $n = 1$ using the language of algebras and bimodules following corollary 3.2.25, and then give the general argument.

Proposition 3.2.44. *The Segal space $\mathrm{Alg}_1(\mathcal{S})$ is complete, i.e.*

$$s_0 : (\mathrm{Alg}_1(\mathcal{S}))_0 \longrightarrow (\mathrm{Alg}_1(\mathcal{S}))_1^{inv}$$

is a weak equivalence.

Proof. An element in $(\mathrm{Alg}_1)_1^{inv}$ is a pointed (A, B)-bimodule $\mathbb{1} \xrightarrow{m} M$ such that there is a pointed (B, A)-bimodule $\mathbb{1} \xrightarrow{n} N$ and weak equivalences

$$A \xrightarrow[m \otimes n]{\simeq} M \otimes_B N, \quad \text{and} \quad B \xrightarrow[n \otimes m]{\simeq} N \otimes_A M,$$

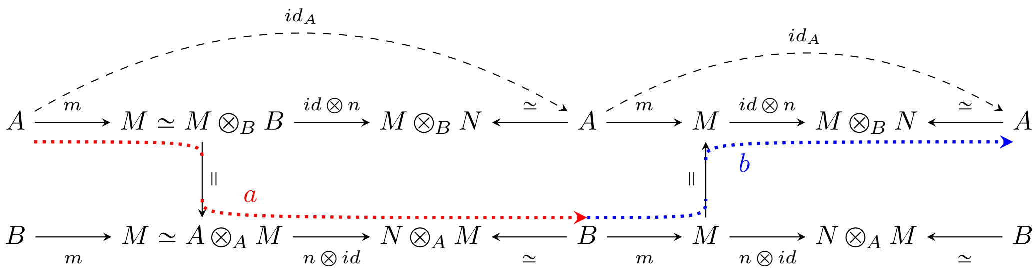
of (A, A), respectively (B, B)-bimodules. We need to show that $A \simeq B \simeq M$. This implies that there is a path from ${}_AM_B$ to ${}_AA_A$. This construction extends to a homotopy, since a weak equivalence from ${}_AM_B$ to a different bimodule ${}_CN_D$ includes the data of a weak equivalence from A to C.

First note that

$$\begin{array}{ccccccc} A & \longrightarrow & M & \xrightarrow{id_M \otimes n} & M \otimes_B N & \xleftarrow{\simeq} & A, \\ 1 & \longmapsto & m & \longmapsto & m \otimes n & \longleftarrow\!\shortmid & 1 \end{array}$$

are maps of (A, A)-bimodules, and induce the identity $A \to A$ in the homotopy category $h_1\mathcal{S}$ of $\mathcal{S}$.

Consider all following maps in $h_1\mathcal{S}$, in particular the maps $a : A \to B, b : B \to A$ given by the following diagram:



Their composition is equal to the composition of the dashed arrows, which are identities, so $b \circ a = id_A$. Similarly, $a \circ b = id_B$, so A and B are weakly equivalent. Moreover, $A \to M \to M \otimes_B N \simeq A$ is the identity, so $A \to M$ is a split monomorphism and $M \to A$ is a split epimorphism. Similarly, $A \to N$ is a split monomorphism.

Since tensoring with an object preserves split monomorphisms, $M \to M \otimes_B N \simeq A$ is a monomorphism, and thus an isomorphism (all in $h_1\mathcal{S}$). Similarly for N. □

Proposition 3.2.45. *The n-fold Segal space* $\mathrm{Alg}_n(\mathcal{S})$ *is complete.*

Proof. The statement for general n follows from the statement for $n = 1$, which is proposition 3.2.44.

Let $\underline{n} = \{1, \dots, n\}$. Factorization algebras on $(0,1)^{\underline{n}\setminus\{i\}}$ form a relative category $\mathcal{S}$ satisfying the assumptions 1. Elements in $(\mathrm{Alg}_n)_{k_1,\dots,1,\dots,k_n}$ are modules in $\mathcal{S}$ over E_1-algebra objects in $\mathcal{S}$, so we can apply proposition 3.2.44 which proves the statement. □

Definition 3.2.46. The *Morita (∞, n)-category of E_n-algebras* is the complete n-fold Segal space Alg_n.

3.3 The symmetric monoidal structure on Alg_n

3.3.1 The symmetric monoidal structure arising as a Γ-object

Similarly to Bord_n we can endow Alg_n with a symmetric monoidal structure arising as a Γ-object. It essentially comes from the fact that factorization algebras have a symmetric monoidal structure as a relative category.

Definition 3.3.1. For every $k_1, \ldots, k_n$, let $(\mathrm{Alg}_n[m])_{k_1,\ldots,k_n}$ be the collection of tuples

$$(\mathcal{F}_1, \ldots, \mathcal{F}_m, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\ldots,n}),$$

where for every $1 \leqslant \beta \leqslant m$, $(\mathcal{F}_\beta, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\ldots,n}) \in (\mathrm{Alg}_n)_{k_1,\ldots,k_n}$. Similary to Alg_n this can be made into a complete n-fold Segal space.

Proposition 3.3.2. *The assignment*

$$\begin{aligned} \Gamma &\longrightarrow \mathbf{SSpace_n}, \\ [m] &\longmapsto \mathrm{Alg}_n[m] \end{aligned}$$

extends to a functor and endows Alg_n with a symmetric monoidal structure.

Proof. The functor sends a morphism $f : [m] \to [k]$ to

$$\begin{aligned} \mathrm{Alg}_n[m] &\longrightarrow \mathrm{Alg}_n[k], \\ (\mathcal{F}_1, \ldots, \mathcal{F}_m, (I_j^i)_{i,j}) &\longmapsto (\bigotimes_{\beta \in f^{-1}(1)} \mathcal{F}_\beta, \ldots, \bigotimes_{\beta \in f^{-1}(k)} \mathcal{F}_\beta, (I_j^i)_{i,j}). \end{aligned}$$

Here the tensor product is the tensor product of factorization algebras with values in the given symmetric monoidal category (defined level-wise). This is well-defined as every $\mathcal{F}_\beta$, and therefore also the tensor product of several $\mathcal{F}_\beta$'s are locally constant with respect to the same stratification.

To show that

$$\prod_{1 \leqslant \beta \leqslant n} \gamma_\beta : \mathrm{Alg}_n[m] \longrightarrow (\mathrm{Alg}[1])^m$$

is an equivalence of n-fold complete Segal spaces we need to show that for any element in the right hand side we can rescale the intervals $(I_j^i)_{i,j}$ by a smooth family of rescaling maps $\phi_{s,t}^i$ so that they coincide. This follows from the fact that rescaling $(0,1)^n$ by some suitable rescaling data $\phi_{s,t}$ leads to weak equivalences of factorization algebras given by pushforward along $\phi_{s,t}$. This rescaling yields a path in the right hand space to an element in the image of $\prod_{1 \leqslant \beta \leqslant n} \gamma_\beta$ and the collection of these paths form a homotopy. ☐

3.3.2 The monoidal structure and the tower

Our goal for this section is to endow Alg_n with a symmetric monoidal structure arising from a tower of monoidal l-hybrid $(n+l)$-fold Segal spaces $\mathrm{Alg}_n^{(l)}$ for $l \geqslant 0$.

The deloopings $\mathrm{Alg}_n^{(l)}$

Our construction of the (∞, n)-category of E_n-algebras $\mathrm{Alg}_n(\mathcal{S})$ relies on a symmetric monoidal $(\infty, 1)$-category $\mathcal{S}$. Independent of which model for symmetric monoidal $(\infty, 1)$-categories we choose there is a distinguished object in $\mathcal{S}$, the unit $\mathbb{1}$ for the symmetric monoidal structure. This object naturally is an E_n-algebra, the constant factorization algebra on $\mathbb{R}^n$ with value $\mathbb{1}$, which determines an object $(\mathbb{1}, (0,1), \ldots, (0,1))$ in $(\mathrm{Alg}_n)_{0,\ldots,0}$.

The first layer of the tower

Definition 3.3.3. Let $\mathrm{Alg}_n^{(1)}$ be the fiber of Alg_{n+1} over $\mathbb{1}_0 = \mathbb{1}$ in the first direction, i.e. $(\mathrm{Alg}_n^{(1)})_{k_1,\ldots,k_{n+1}}$ is the fiber over $\mathbb{1}^{k_1+1} \in \big((\mathrm{Alg}_{n+1})_{0,k_2,\ldots,k_{n+1}}\big)^{k_1+1}$ of the map

$$(\mathrm{Alg}_{n+1})_{k_1,\ldots,k_{n+1}} \longrightarrow \big((\mathrm{Alg}_{n+1})_{0,k_2,\ldots,k_{n+1}}\big)^{k_1+1},$$

which is the product of the (k_1+1) different possible compositions of face maps

$$(\mathrm{Alg}_{n+1})_{k_1,\ldots,k_{n+1}} \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} (\mathrm{Alg}_{n+1})_{0,k_2,\ldots,k_{n+1}}.$$

Proposition 3.3.4. $\mathrm{Alg}_n^{(1)}$ *is a monoidal complete n-fold Segal space.*

Proof. By construction the $(n+1)$-fold Segal space $\mathrm{Alg}_n^{(1)}$ is 1-hybrid and pointed. □

Remark 3.3.5. It may seem unnatural to take the actual fiber here instead of a homotopy fiber. This is needed as we need hybridness which requires certain spaces to be equal to a point and not just contractible. As explained in remark 3.2.43, the maps s,t : $(\mathrm{Alg}_n)_{k_1,\ldots,1,\ldots k_n} \to (\mathrm{Alg}_n)_{k_1,\ldots,0,\ldots k_n}$ are fibrations. Thus, in this case, the homotopy fiber and the fiber actually coincide.

The higher layers Similarly, we define the higher layers of the tower.

Assume that we have defined $\mathrm{Alg}_n^{(0)} = \mathrm{Alg}_n, \mathrm{Alg}_n^{(1)}, \ldots, \mathrm{Alg}_n^{(l-1)}$ for every n such that $\mathrm{Alg}_n^{(k)}$ is a k-hybrid $(n+k)$-fold Segal space which is $(j-1)$-connected for every $0 < j \leqslant k$. Note that, via the degeneracy maps, $\mathbb{1}$ can be viewed as a trivial l-morphism in any $\mathrm{Alg}_n^{(k)}$ for any $1 \leqslant l \leqslant n+k$, i.e. an element

$$\mathbb{1}_l = (\mathbb{1}, (0,1) \leqslant (0,1), \ldots, (0,1) \leqslant (0,1), (0,1), \ldots, (0,1)) \in (\mathrm{Alg}_n^{(k)})_{\underbrace{1,\ldots,1,}_{l}\,0,\ldots,0}.$$

Definition 3.3.6. Let $\mathrm{Alg}_n^{(l)}$ be the fiber of $\mathrm{Alg}_{n+1}^{(l-1)}$ over $\mathbb{1}_{l-1}$, i.e. $(\mathrm{Alg}_n^{(l)})_{k_1,\ldots,k_{n+l}}$ is the fiber over $\mathbb{1}_{l-1} \in (\mathrm{Alg}_{n+l})_{1,\ldots,1,0,k_{l+1},\ldots,k_{l+n}}$ of the product of all different possible compositions of face maps

$$(\mathrm{Alg}_{n+1}^{(l-1)})_{k_1,\ldots,k_{n+l}} \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \big((\mathrm{Alg}_{n+1}^{(l-1)})_{1,\ldots,1,0,k_{l+1},\ldots,k_{l+n}}.$$

Proposition 3.3.7. $\mathrm{Alg}_n^{(l)}$ *is a k-monoidal complete n-fold Segal space.*

Proof. Again by construction the $(n+l)$-fold Segal space $\mathrm{Alg}_n^{(l)}$ is l-hybrid and $(j-1)$-connected for every $0 < j \leqslant l$. □

The tower and the symmetric monoidal structure

The monoidal complete n-fold Segal space $\mathrm{Alg}_n^{(1)}$ turns out to be a delooping of Alg_n. The following proposition shows that the collection of the l-monoidal complete n-fold Segal spaces $\left(\mathrm{Alg}_n^{(l)}\right)_l$ forms a tower which gives Alg_n a symmetric monoidal structure.

Proposition 3.3.8. *For* $n, l \geqslant 0$*, there are weak equivalences*

$$L(\mathrm{Alg}_n^{(l)}, \mathbb{1}) \underset{\ell}{\overset{u}{\rightleftarrows}} \mathrm{Alg}_n^{(l-1)},$$

defined as follows.

1. *The map* u *sends an element* $(\mathcal{F}) = \big(\mathcal{F}, (0, b_0^1] \leqslant [a_1^1, 1), (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=2}^{n+l}\big)$ $\in L(\mathrm{Alg}_n^{(l)})$ *to*
$$(u(\mathcal{F})) = (u(\mathcal{F}), (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=2}^{n+l}),$$
where $u(\mathcal{F}) = (\pi_{\{2,\ldots,n+1\}})_*\mathcal{F}$ *is the pushforward of* $\mathcal{F}$ *along the projection*
$$\pi_{\{2,\ldots,n+1\}} : (0,1)^{\{1,\ldots,n+1\}} \to (0,1)^{\{2,\ldots,n+1\}}.$$

2. *The map* ℓ *sends an element* $(\mathcal{G}) = (\mathcal{G}, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=2}^{n+l}) \in \mathrm{Alg}_n^{(l-1)}$ *to*
$$(\ell(\mathcal{G})) = \big(\ell(\mathcal{G}), (0, \tfrac{1}{2}] \leqslant [\tfrac{1}{2}, 1), (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=2}^{n+l}\big),$$
where $\ell(\mathcal{G}) = \iota_*(\mathcal{G})$ *is the pushforward of* $\mathcal{G}$ *along the inclusion*
$$\iota : (0,1)^{n+l-1} \to (0,1)^{n+l}, (x_2, \ldots, x_{n+l}) \mapsto (\tfrac{1}{2}, x_2, \ldots, x_{n+l}).$$

The map ℓ *is called the* looping *and* u *the* delooping map.

The main step in the proof of proposition 3.3.8 is the following observation, for which we need factorization algebras which are supported on a subspace.

Definition 3.3.9. Let M be a topological space and $N \subseteq M$ be a closed subspace. Then a prefactorization algebra on M is said to be *supported on* N, if

$$\mathcal{F}|_{M \setminus N} = \mathbb{1}.$$

Lemma 3.3.10. *Let* X *and* Y *be manifolds with the stratifications* $X \supset \{s\} \supset \varnothing$ *for* $s \in X$ *and* $Y = Y_n \supset Y_{n-1} \supset Y_0 \supset Y_{-1} = \varnothing$*. Consider the stratification on* $X \times Y$ *given by*
$$X \times Y \supset \{s\} \times Y \supset \{s\} \times Y_{n-1} \supset \cdots \supset \{s\} \times Y_1 \supset \{s\} \times Y_0 \supset \varnothing.$$
Denoting by $\mathcal{F}act_{X\times Y}^{lc,\{s\}\times Y}$ *factorization algebras on* $X \times Y$ *which are locally constant with respect to this stratification and are supported on* $\{s\} \times Y$*, there is a one-to-one correspondence*

$$\begin{aligned} \mathcal{F}act_{X\times Y}^{lc,\{s\}\times Y} &\longrightarrow \mathcal{F}act_Y^{lc} \\ \mathcal{F} &\longmapsto (pr_2)_*\mathcal{F} \\ \iota_*\mathcal{G} &\longleftarrow\!\shortmid\; \mathcal{G}, \end{aligned}$$

where $\iota : Y \to X \times Y$ *sends* $y \mapsto (s, y)$.

Proof. First we need to show that these maps are well-defined.

Note that the stratification on $X \times Y$ given above is coarser than the one from theorem 3.2.26. Thus, if $\mathcal{F}$ is locally constant with respect to this coarser stratification, it will also be locally constant with respect to the finer stratification from 3.2.26 and thus by the theorem the above map factors through $\mathcal{F}act_X^{lc}(\mathcal{F}act_Y^{lc})$ followed by the global section functor to $\mathcal{F}act_Y^{lc}$,

$$\mathcal{F}act_{X\times Y}^{lc,\{s\}\times Y} \subseteq \mathcal{F}act_{X\times Y}^{lc,fine} \longrightarrow \mathcal{F}act_X^{lc}(\mathcal{F}act_Y^{lc}) \longrightarrow \mathcal{F}act_Y^{lc}.$$

Conversely, let $\mathcal{G} \in \mathcal{F}act_Y^{lc}$. If $U \subseteq (X \times Y)\backslash(\{s\} \times Y)$, then $\iota^{-1}(U) = \varnothing$, so $\iota_*\mathcal{G}(U) = \mathbb{1}$ and $\iota_*\mathcal{G}$ is supported on $\{s\} \times Y$. To see that $\iota_*\mathcal{G}$ is locally constant with respect to the stratification above, it is enough to check this on a factorizing basis, , so it is enough to check them on products of open sets.

Let $U \times V \subset U' \times V'$ be an inclusion of disks such that both $U \times V$ and $U' \times V'$ are good neighborhoods at α for the same index $\alpha \in \{0, \ldots, n+1\}$.

If $\alpha = n+1$, then $U \times V \subset U' \times V' \subset (X \times Y)\backslash(\{s\} \times Y)$ and by the above,

$$\iota_*\mathcal{G}(U \times V) = \mathcal{G}(\varnothing) = \mathbb{1} = \mathcal{G}(\varnothing) = \iota_*\mathcal{G}(U' \times V').$$

If $0 \leqslant \alpha \leqslant n$, then $(U \times V) \cap (\{s\} \times Y) \neq \varnothing$ and $(U' \times V') \cap \{s\} \times Y \neq \varnothing$, so we get that both $\iota^{-1}(U \times V) = V$ and $\iota^{-1}(U' \times V') = V'$ are disks of index α for the same α, so since $\mathcal{G}$ is locally constant with respect to the stratification on Y,

$$\iota_*\mathcal{G}(U \times V) = \mathcal{G}(V) \simeq \mathcal{G}(V') = \iota_*\mathcal{G}(U' \times V').$$

To see that these maps form a bijection, since $pr_2 \circ \iota = id_Y$, it is enough to check that $\iota_* \circ (\mathrm{pr}_2)_*\mathcal{F} = \mathcal{F}$. Again it is enough to check this on a factorizing basis, which we will choose to consists of open sets of the form $U \times V$, where U is a disk. Then,

$$\iota_*(pr_2)_*(\mathcal{F})(U \times V) = \mathcal{F}(pr_2^{-1}(\iota^{-1}(U \times V))) = \begin{cases} \mathcal{F}(X \times V), & \text{if } s \in U, \\ \mathcal{F}(\varnothing) = \mathbb{1}, & \text{otherwise.} \end{cases}$$

Since $\mathcal{F}$ is locally constant with respect to the above stratification and is supported on $\{s\} \times Y$, it agrees with the above. □

Proof of Proposition 3.3.8. Setting $X = (0,1)$, $Y = (0,1)^{\{2,\ldots,n+l\}}$, and choosing $s = \frac{1}{2}$ in lemma 3.3.10,

$$u(\mathcal{F}) = (\pi_{\{2,\ldots,n+l\}})_*\mathcal{F} = (pr_2)_*(\mathcal{F}),$$

and the maps u and ℓ are well-defined. Moreover, by definition, $u \circ \ell = id$. It remains to show that $\ell \circ u \simeq id$.

Given an element $(\mathcal{F}) \in (L(\mathrm{Alg}_n^{(l)}))_{1,k_2,\ldots,k_{n+l}}$, note that the associated stratification on $X = (0,1)$ is given by $(0,1)\backslash\big((0,b_0^1) \cup (a_1^1,1)\big)$ either is empty or is equal to a point $s = b_0^1 = a_1^1$. This data is lost when applying u. By the lemma above, the factorization algebra is recovered under $\ell \circ u$ except for the data of s, which in the definition of ℓ we chose to be $s = \frac{1}{2}$. However, a homotopy from $\ell \circ u$ to the identity is given by the following construction. Let $\xi \in [0,1]$. Send an element $(\mathcal{F})$ to its pushforward along f_ξ, which is the (restriction to $(0,1)$ of the) unique piecewise affine map $\mathbb{R} \to \mathbb{R}$ such that

$$0 \mapsto 0, \quad s \mapsto s\xi + (1-s)\frac{1}{2}, \quad 1 \mapsto 1.$$

□

Corollary 3.3.11. *The l-monoidal complete n-fold Segal spaces $\mathrm{Alg}_n^{(l)}$ endow Alg_n with a symmetric monoidal structure.*

3.4 The homotopy category of Alg_1 and the Morita category

The idea behind our construction of Alg_1 was to model an $(\infty,1)$-category of algebras and pointed bimodules between them. Indeed, the homotopy category of Alg_1 turns out to be what we expect.

Definition 3.4.1. Let Mor be the category whose objects are algebras and whose morphisms from an algebra A to an algebra B are equivalences classes of (A,B)-bimodules ${}_AM_B$, where ${}_AM_B$ is equivalent to ${}_{A'}M'_{B'}$ iff $A \simeq A'$, $B \simeq B'$, $M \simeq M'$.

Remark 3.4.2. Keep in mind that we are considering algebra and bimodule objects in some symmetric monoidal relative category $\mathcal{S}$, e.g. $\mathcal{S} = \mathrm{Ch}_k$. If we choose $\mathcal{S} = \mathrm{Vect}_k$ with isomorphisms as weak equivalences, we get the classical category of algebras and bimodules. If we want to specify which relative category the algebra and module objects take values in, we write $\mathrm{Mor}(\mathcal{S})$. The symmetric monoidal structure comes from the one on $\mathcal{S}$, which sends (A, A') to their tensor product $A \otimes A'$ in $\mathcal{S}$ and $({}_AM_B, {}_CN_D)$ to ${}_{A\otimes C}M \otimes N_{B\otimes D}$.

Proposition 3.4.3. *There is an equivalence of symmetric monoidal categories*

$$h_1(\mathrm{Alg}_1) \simeq \mathrm{Mor}\,.$$

Proof. We have seen in examples 3.2.30 and 3.2.37 using 3.2.21 that objects of Alg_1, and thus also of $h_1(\mathrm{Alg}_1)$ are equivalent to (homotopy) algebras. A (1-)morphisms in Alg_1 from A to B is a factorization algebra $\mathcal{F}$ on $\mathbb{R}$ which gives the data of an (A,B)-bimodule ${}_AM_B$. The extra information it encodes is a choice of intervals $(0,b] \leqslant [a,1)$ which corresponds to choosing where on $(0,1)$ the module is located. The space of this extra information is the space of $s \in (0,1)$ and thus contractible. Moreover, paths from ${}_AM_B$ to ${}_{A'}M'_{B'}$ by definition give weak equivalences $A \simeq A'$, $B \simeq B'$, $M \simeq M'$. Thus, a connected component of the space of (1-)morphisms in Alg_1 from A to B is an equivalence class of (A,B)-bimodules M. Summarizing, there is an equivalence of categories

$$F : h_1(\mathrm{Alg}_1) \longrightarrow \mathrm{Mor},$$

which sends an object $(\mathcal{F}, (0,1)) \in (\mathrm{Alg}_1)_0$ to $\mathcal{F}((0,1))$ and a 1-morphism represented by $(\mathcal{F}, (0,b] \leqslant [a,1))$ to the $\Big(\mathcal{F}((0,b)), \mathcal{F}((a,1))\Big)$-bimodule $\mathcal{F}((0,1))$.

We saw in example 1.6.8 that the symmetric monoidal structure on Alg_1 induces one on the ordinary category $h_1(\mathrm{Alg}_1)$ coming from the diagram

$$\mathrm{Alg}_1[1] \times \mathrm{Alg}_1[1] \xleftarrow[\gamma_1\times\gamma_2]{\simeq} \mathrm{Alg}_1[2] \xrightarrow{\gamma} \mathrm{Alg}_1[1],$$

where the first arrow is an equivalence of complete Segal spaces. By the definition of the map γ, it is clear that the equivalence of categories F respects the monoidal structure. □

3.5 Variants and extensions of Alg_n

3.5.1 The $(\infty, n+1)$-category of E_n-algebras

As mentioned above, starting with a symmetric monoidal $(\infty, 1)$-category $\mathcal{S}$ with all products, factorization algebras on any space X with values in $\mathcal{S}$ again form a symmetric monoidal $(\infty, 1)$-category. Thus, Alg_n can be extended to an n-fold complete Segal object in $(\infty, 1)$-categories, and from this one can extract an $(\infty, n+1)$-category $\mathrm{Alg}_n^{(\infty,n+1)}$, which is moreover symmetric monoidal. It can also be explicitly constructed as a complete $(n+1)$-fold Segal spaces using morphisms of stratified factorization algebras, which are weak equivalences outside of the lowest dimensional stratum, as $(n+1)$-morphisms. In the full $(\infty, n+1)$-category $\mathrm{Alg}_n^{(\infty,n+1)}$ every object is n-dualizable, but there are much fewer fully dualizable objects.

In the case of $n = 1$ we saw in the previous section that the homotopy category of Alg_1 is just the Morita category Mor of algebras and equivalence classes of bimodules. This equivalence can be extended to an equivalence of the homotopy bicategory of the $(\infty, 2)$-category $\mathrm{Alg}_1^{(\infty,2)}$ with the full bicategory of algebras, bimodules, and intertwiners, which one might want to call the full "Morita bicategory Mor^2 of E_1-algebras".

3.5.2 An unpointed version

Note that in our construction we use factorization algebras and weak equivalences to model objects, 1-morphisms, and 2-morphisms. As we discussed in section 3.2.9, factorization algebras are pointed, with pointing coming from the monoidal unit $\mathbb{1}$ of the underlying category $\mathcal{S}$. This pointing leads to pointed bimodules and intertwiners.

For applications one might be interested in an unpointed version to obtain a category with unpointed bimodules as morphisms, which leads to the usual Morita category. For such a construction an unpointed version of factorization algebras which are locally constant with respect to the same stratifications is needed. Such "unpointed factorization algebras" can be defined using an operad similar to the one used in the definition of factorization algebras, see remark 3.2.11, but allowing only certain inclusions of empty sets. However, there is no reason for such an unpointed version of an (∞, n)-category of E_n-algebras to be complete.

3.5.3 The n-fold category Alg_n^{uple}

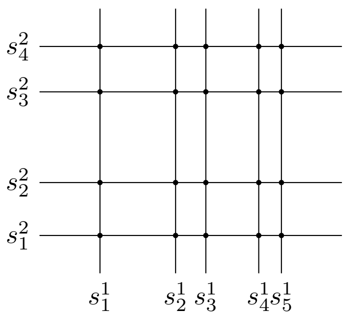
Similarly to the n-fold category version of the bordism category Bord_n^{uple} from section 2.4.3, there also is an n-fold category Alg_n^{uple}. The stratifications with respect to which factorization algebras are constant in condition (2) in definition 3.2.27 are designed such that Alg_n satisfies the essentially constancy condition of an n-fold Segal space. Relaxing this condition to allowing more general stratifications of the form

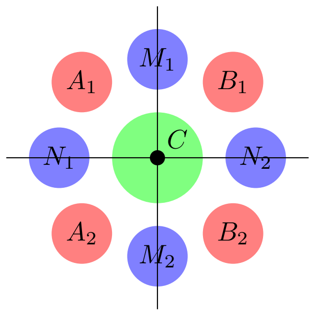
$$(0,1)^n \supset \bigcup_{1\leqslant i\leqslant n}\ \bigcup_{1\leqslant j_i\leqslant k_i} \pi_i^{-1}(s^i_{j_i}) \supset \bigcup_{1\leqslant i_1,i_2\leqslant n}\ \bigcup_{\substack{1\leqslant j_1\leqslant k_{i_1}\\ 1\leqslant j_2\leqslant k_{i_2}}} \pi^{-1}_{\{i_1,i_2\}}(s^{i_1}_{j_1}, s^{i_2}_{j_2}) \supset \cdots$$
$$\cdots \supset \bigcup_{(1\leqslant j_i\leqslant k_i)_{i=1}^n} \pi^{-1}(s^1_{j_1},\ldots,s^n_{j_n})$$

gives a complete n-uple Segal space Alg_n^{uple}.

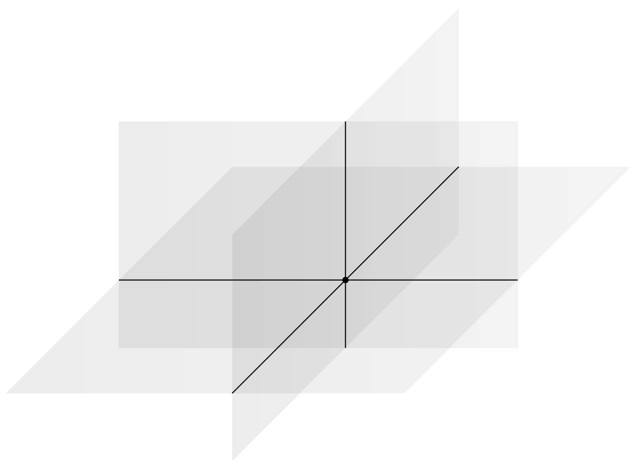
For $n=2$, stratifications which appear in the definition of $\mathrm{Alg}_2^{uple}(\mathcal{S})$ give pictures as in the left picture below. Now the interpretation is slightly different: the images of open disks as in the right picture below now give:

- E_2-algebras A_1, A_2, B_1, and B_2
- an (A_1,B_1)-bimodule M_1, an (A_2,B_2)-bimodule M_2, (A_1,A_2)-bimodule N_1, and a (B_1,B_2)-bimodule N_2
- a pointed element C in $\mathcal{S}$ which is an (M_1,M_2)-bimodule and an (N_1,N_2)-bimodule.





For $n=3$, stratifications which appear in the definition of Alg_3^{uple} give pictures of the following type:



Chapter **4**

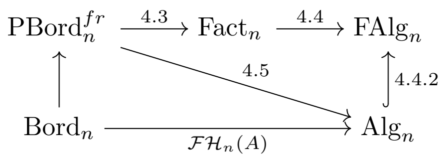
Factorization homology as a fully extended topological field theory

Recall that the main task of this thesis is the following. Given any E_n-algebra A, i.e. any object in $\mathrm{Alg}_n = \mathrm{Alg}_n(\mathcal{S})$, we would like to define a map of symmetric monoidal n-fold Segal spaces

$$\mathcal{FH}_n(A) : \mathrm{Bord}_n^{fr} \longrightarrow \mathrm{Alg}_n$$

essentially given by taking factorization homology of A. As complete n-fold Segal spaces are models for (∞, n)-categories, this defines a fully extended topological field theory with values in $\mathcal{C} = \mathrm{Alg}_n(\mathcal{S})$.

This chapter deals with the construction of this functor. For better overview we split the construction in two steps. First we construct a map, which is just a map of n-fold simplicial sets, to an auxiliary complete n-fold Segal space of factorization algebras which is essentially given by factorization homology. Then we construct a map which can be understood as "collapsing" and then "rescaling" a factorization algebra. Their composition yields the desired map of n-fold Segal spaces. The construction can be summarized in the following diagram. We indicate in which section the individual maps are constructed.



This map extends to the symmetric monoidal structures and yields the desired fully extended topological field theory.

4.1 Factorization Homology

Inspired by an algebro-geometric version by Beilinson and Drinfeld in [BD04] and a similar construction by Salvatore in [Sal01], Lurie introduced factorization homology in [Lur] calling it topological chiral homology. It has been studied, amongst others, in [Fra12, Fra13, AFT12, GTZ10, GTZ12, Hor14b]. We briefly recall the definition and the most important properties we will use in this chapter.

Again, as in sections 2.7.1 and 3.2.1, let X be a topological space and $E \to X$ a topological n-dimensional vector bundle which corresponds to a (homotopy class of) map(s) $e : X \to B\mathrm{GL}(\mathbb{R}^n)$ from X to the classifying space of the topological group $\mathrm{GL}(\mathbb{R}^n)$.

Recall from definition 3.2.3 that a $\mathcal{D}isk_n^{(X,E)}$-algebra in $\mathcal{S}$ is a symmetric monoidal (covariant) functor

$$A : \mathcal{D}isk_n^{(X,E)} \longrightarrow \mathcal{S}.$$

Now consider an (X, E)-structured n-dimensional manifold M. Since $\mathcal{D}isk_n^{(X,E)} \subseteq \mathcal{M}an_n^{(X,E)}$, it yields a contravariant functor

$$\begin{aligned} \underline{M} : (\mathcal{D}isk_n^{(X,E)})^{op} &\longrightarrow \mathcal{S}pace, \\ \coprod_I \mathbb{R}^n &\longmapsto \mathrm{Emb}^{(X,E)}(\coprod_I \mathbb{R}^n, M). \end{aligned}$$

Definition 4.1.1. Let the *factorization homology of M with coefficients in A* be the homotopy coend of the functor $\underline{M} \times A \longrightarrow \mathcal{S}pace \times \mathcal{S} \xrightarrow{\otimes} \mathcal{S}$ and denote it by

$$\int_M A = \underline{M} \otimes_{\mathcal{D}isk_n^{(X,E)}} A.$$

Remark 4.1.2. By [AFT14], assumption 1 ensures the existence of factorization homology.

Remark 4.1.3. Equivalently, the homotopy coend can be computed by the following homotopy colimit,

$$\int_M A = \underset{\coprod_{i=1}^n U_i \to M}{\mathrm{hocolim}} \bigotimes_{i=1}^n A(U_i),$$

i.e. the homotopy colimit is taken over

$$(\mathcal{D}isk_n^{(X,E)})_{/M} = \mathcal{D}isk_n^{(X,E)} \otimes_{\mathcal{M}an_n^{(X,E)}} (\mathcal{M}an_n^{(X,E)})_{/M}.$$

Example 4.1.4. If $M = \mathbb{R}^n$, then $\int_{\mathbb{R}^n} A \simeq A$.

In [GTZ10] it was proven that if we consider this construction locally on M for $X = BG$, where G is the trivial group, we obtain a locally constant factorization algebra on M.

Theorem 4.1.5 ([GTZ10], Proposition 13)**.** *Given an E_n-algebra A, i.e. a $\mathcal{D}isk_n^{fr}$-algebra, the rule*

$$U \mapsto \int_U A$$

for open subsets $U \subseteq M$ with the induced framing extends to a locally constant factorization algebra on M.

Remark 4.1.6. By abuse of notation, we will denote this factorization algebra by $\int_M A$, i.e. for an open subset $U \subseteq M$,

$$\Big(\int_M A\Big)(U) = \int_U A = \underline{U} \otimes_{\mathcal{D}isk_n^{fr}} A \in \mathcal{S}.$$

4.2 The auxiliary (∞, n)-category Fact_n

The main idea for our functor is that, given an E_n-algebra A, we define a map , also called $\int_{(-)} A$, which should be given by first taking factorization homology to obtain a factorization algebra on the manifold M and then pushing it forward to obtain a factorization algebra on $(0,1)^n$, i.e.

$$\begin{array}{ccc} M & \xhookrightarrow{\iota} & V \times (0,1)^n \\ & \overset{\pi}{\searrow} & \downarrow \\ & & (0,1)^n \end{array} \quad \xmapsto{\int_{(-)} A} \quad \pi_*(\int_M A).$$

We define an auxiliary complete n-fold Segal space Fact_n by translating the properties 1.-3. in the definition of PBord_n to conditions on the factorization algebra. We will show that this is the correct translation in section 4.3.

Similarly as to in the definition of Alg_n, for $S \subseteq \{1, \dots, n\}$, denote by $\pi_S : \mathbb{R}^n \to \mathbb{R}^S$ the projection onto the coordinates indexed by S.

Definition 4.2.1. Let elements in $(\mathrm{Fact}_n)_{k_1,\dots,k_n}$ be pairs

$$(\mathcal{F}, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\dots,n}),$$

satisfying the following conditions:

1. $\mathcal{F}$ is a factorization algebra on $(0,1)^n$.
2. For $1 \leqslant i \leqslant n$,
$$(I_0^i \leqslant \cdots \leqslant I_{k_i}^i) \in \mathrm{Int}_{k_i}.$$
3. For $1 \leqslant i \leqslant n$, the factorization algebra $\mathcal{F}$ is an E_{n-i+1}-algebra in factorization algebras on $(0,1)^{\{1,\dots,i-1\}}$ in a neighborhood of $\pi_i^{-1}(I_0^i \cup \ldots \cup I_{k_i}^i) \subset (0,1)^n$.

Remark 4.2.2. In condition 3 we first use theorem 3.2.26 to view $\mathcal{F}$ as a factorization algebra on $(0,1)^{\{i,\dots,n\}}$ in the $(\infty,1)$-category of factorization algebras on $(0,1)^{\{1,\dots,i-1\}}$ and then require that this factorization algebra on $(0,1)^{\{i,\dots,n\}}$ is locally constant. This is translated to saying that it is an E_{n-i+1}-algebra by using theorem 3.2.21.

4.2.1 The spaces $(\mathrm{Fact}_n)_{k_1,\dots,k_n}$

The spatial structure of $(\mathrm{Fact}_n)_{k_1,\dots,k_n}$ is a mixture of that on Bord_n, essentially coming from the one on the spaces Int_{k_i}, and that on Alg_n.

Definition 4.2.3. An l-simplex in $(\mathrm{Fact}_n)_{k_1,\dots,k_n}$ is given by the data of

1. *underlying 0-simplices*, i.e. for every $s \in |\Delta^l|$,
$$\big(\mathcal{F}_s, (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))\big) \in (\mathrm{Fact}_n)_{k_1,\dots,k_n};$$
2. for every $1 \leqslant i \leqslant k_i$,
$$\big(I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s)\big)_{s \in |\Delta^l|}$$
is an l-simplex in Int_{k_i} with rescaling datum $\varphi_{s,t}^i : (0,1) \to (0,1)$;

3. for every $s, t \in |\Delta^l|$, weak equivalences

$$(\varphi_{s,t})_* \mathcal{F}_s \xrightarrow{w_{s,t}} \mathcal{F}_t,$$

where $\varphi_{s,t} = (\varphi^i_{s,t})^n_{i=1} : (0,1)^n \to (0,1)^n$ is the product of the rescaling data.

The spatial face and degeneracy maps δ^Δ_l, σ^Δ_l arise from the face and degeneracy maps of (Δ^l) similarly to those of PBord_n, and we obtain a space $(\mathrm{Fact}_n)_{k_1,\dots,k_n}$.

4.2.2 The n-fold Segal space Fact_n

We now define face and degeneracy maps on the 0-simplices of the levels of $\mathrm{Fact}_{\bullet,\dots,\bullet}$ essentially coming from those of the n-fold Segal space $(\mathrm{Int})^n_{\bullet,\dots,\bullet} = \mathrm{Int}_\bullet \times \cdots \times \mathrm{Int}_\bullet$. They are similar to those of Alg_n, but use the rescaling maps ρ_j coming from $\mathrm{Int}_\bullet$ instead of the collapse-and-rescale maps ϱ^b_a coming from $\mathrm{Covers}_\bullet$. Recall that for $j = 0$ or$j = k$, in the usual notation they are the linear rescaling maps

$$\rho_0 : D_0 = (a_1, 1) \to (0,1), \quad x \mapsto \tfrac{x-a_1}{1-a_1}, \quad \rho_k : D_k = (0, b_{k-1}) \to (0,1), \quad x \mapsto \tfrac{x}{b_{k-1}}.$$

Since $1 \leqslant i \leqslant n$ will be fixed throughout the following constructions, by abuse of notation we define

$$\rho_j : \pi_i^{-1}(D_j) = \prod_{\alpha \neq i} (0,1) \times D_j \to (0,1)^n,$$

which is ρ_j in the ith coordinate and the identity otherwise.

Degeneracy maps on 0-simplices Fix $1 \leqslant i \leqslant n$. For $0 \leqslant j \leqslant k_i$ the *jth degeneracy map*

$$s^i_j : (\mathrm{Fact}_n)_{k_1,\dots,k_n} \to (\mathrm{Fact}_n)_{k_1,\dots,k_i+1,\dots,k_n}$$

applies the jth degeneracy map of $\mathrm{Int}_\bullet$ to the ith tuple of intervals, i.e. it repeats the jth interval in the ith direction,

$$\begin{aligned}(\mathcal{F}, (I^\alpha_0 \leqslant \cdots \leqslant I^\alpha_{k_\alpha})^n_{\alpha=1}) \quad \longmapsto \quad & \big(\mathcal{F}, (I^\alpha_0 \leqslant \cdots \leqslant I^\alpha_{k_\alpha})_{\alpha \neq i}, s_j(I^i_0 \leqslant \cdots \leqslant I^i_{k_i})\big) = \\ & \big(\mathcal{F}, (I^\alpha_0 \leqslant \cdots \leqslant I^\alpha_{k_\alpha})_{\alpha \neq i}, I^i_1 \leqslant \cdots \leqslant I^i_j \leqslant I^i_j \leqslant \cdots \leqslant I^i_{k_i}\big).\end{aligned}$$

Face maps on 0-simplices Fix $1 \leqslant i \leqslant n$. For $0 \leqslant j \leqslant k_i$ the *jth face map*

$$d^i_j : (\mathrm{Fact}_n)_{k_1,\dots,k_n} \to (\mathrm{Fact}_n)_{k_1,\dots,k_i-1,\dots,k_n}$$

applies the jth face map of $\mathrm{Int}_\bullet$ to the ith tuple of intervals, which forgets the jth interval, and, if necessary, rescales them and pushes the factorization algebra forward along the rescaling map ρ_j. Explicitly, for $j \neq 0, k_i$, the 0-simplex $(\mathcal{F}) = (\mathcal{F}, (I^\alpha_0 \leqslant \cdots \leqslant I^\alpha_{k_\alpha})^n_{\alpha=1})$ is sent to

$$\begin{aligned}&\big(\mathcal{F}, (I^\alpha_0 \leqslant \cdots \leqslant I^\alpha_{k_i})_{\alpha \neq i}, d_j(I^i_0 \leqslant \cdots \leqslant I^i_{k_i})\big) = \\ &\qquad = \big(\mathcal{F}, (I^\alpha_0 \leqslant \cdots \leqslant I^\alpha_{k_i})_{\alpha \neq i}, I^i_0 \leqslant \cdots \leqslant I^i_{j-1} \leqslant I^i_{j+1} \leqslant \cdots \leqslant I^i_{k_i}\big).\end{aligned}$$

For $j = 0$ or $j = k_i$, the 0-simplex $(\mathcal{F})$ is sent to

$$((\rho_j)_* \mathcal{F}|_{\pi_i^{-1}(D_j)}, (I^\alpha_0 \leqslant \cdots \leqslant I^\alpha_{k_i})_{\alpha \neq i}, d_j(I^i_0 \leqslant \cdots \leqslant I^i_{k_i})).$$

The full structure as an n-fold Segal space Face and degeneracy maps on l-simplices are defined analogous to for Alg_n, by which we obtain an n-fold simplicial space Fact_n.

Proposition 4.2.4. *$(\mathrm{Fact}_n)_{\bullet,\dots,\bullet}$ is an n-fold Segal space.*

Proof. The proof of the Segal condition works similarly as for Alg_n and essentially follows from the fact that paths of objects arise from weak equivalences and rescaling, which we can use to glue.

It remains to check that for every i and every $k_1, \dots, k_{i-1}$, the $(n-i)$-fold Segal space $(\mathrm{Fact}_n)_{k_1,\dots,k_{i-1},0,\bullet,\dots,\bullet}$ is essentially constant.

We claim that the composition of degeneracy maps

$$(\mathrm{Fact}_n)_{k_1,\dots,k_{i-1},0,\dots,0} \hookrightarrow (\mathrm{Fact}_n)_{k_1,\dots,k_{i-1},0,k_{i+1},\dots,k_n}$$

is a deformation retract.

For $s \in [0,1]$, consider the path γ_s in $(\mathrm{Fact}_n)_{k_1,\dots,k_{i-1},0,k_{i+1},\dots,k_n}$ sending an element represented by

$$(\mathcal{F}) := (\mathcal{F}, \Big(I_0^\beta \leqslant \dots \leqslant I_{k_\beta}^\beta)_{1 \leqslant \beta < i}, (0,1), (I_0^\alpha \leqslant \dots \leqslant I_{k_\alpha}^\alpha)_{i < \alpha \leqslant n}\Big)$$

to

$$(\mathcal{F})_s := \Big(\mathcal{F}, (I_0^\beta \leqslant \dots \leqslant I_{k_\beta}^\beta)_{1 \leqslant \beta < i}, (0,1), (I_0^\alpha(s) \leqslant \dots \leqslant I_{k_\alpha}^\alpha(s))_{i < \alpha \leqslant n}\Big),$$

where for $\alpha > i$, $a_j^\alpha(s) = (1-s)a_j^\alpha$ and $b_j^\alpha(s) = (1-s)b_j^\alpha + s$. Note that for $s = 0$, $I_0^\alpha(0) = I_0^\alpha$, $I_j^\alpha(0) = I_j^\alpha$ and for $s = 1$, $I_j^\alpha(1) = (0,1)$.

The collection of paths γ_s form a deformation retraction provided that each path is well-defined, i.e. indeed maps to $(\mathrm{Fact}_n)_{k_1,\dots,k_{i-1},0,k_{i+1},\dots,k_n}$. It suffices to check condition (3) in definition 4.2.1 for $(\mathcal{F})_s$. Since $(\mathcal{F}) \in (\mathrm{Fact}_n)_{k_1,\dots,k_{i-1},0,k_{i+1},\dots,k_n}$, this reduces to checking

For every $i < \alpha \leqslant n$, $\mathcal{F}$ is an $\mathrm{E}_{n-\alpha+1}$-algebra in factorization algebras on $(0,1)^{\{1,\dots,\alpha-1\}}$ in a neighborhood of $\pi_\alpha^{-1}(I_0^\alpha(s) \cup \ldots \cup I_{k_\alpha}^\alpha(s)) \subseteq (0,1)^n$.

Condition (3) on $(\mathcal{F})$ for i implies that in particular, $\mathcal{F}$ is an E_{n-i+1}-algebra in factorization algebras on $(0,1)^{\{1,\dots,i-1\}}$ in (a neighborhood of) $\pi_i^{-1}((0,1)) = (0,1)^n$.

This in turn implies that for every $\alpha > i$, $\mathcal{F}$ is an $E_{n-\alpha+1}$-algebra in factorization algebras on $(0,1)^{\{1,\dots,\alpha-1\}}$ in a neighborhood of $\pi_\alpha^{-1}((0,1)) = (0,1)^n \supseteq \pi_\alpha^{-1}(I_0^\alpha(s) \cup \ldots \cup I_{k_j}^\alpha(s))$. ☐

4.2.3 Completeness of Fact_n

We now show that the auxiliary n-fold Segal space of factorization algebras Fact_n always is complete, and thus is an (∞, n)-category.

Proposition 4.2.5. *The n-fold Segal space Fact_n is complete.*

Proof. We need to show that for any $k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n$, the degeneracy map

$$(\mathrm{Fact}_n)_{k_1,\dots,k_{i-1},0,k_{i+1},\dots,k_n} \xrightarrow{s_0} (\mathrm{Fact}_n)^{inv}_{k_1,\dots,k_{i-1},1,k_{i+1},\dots,k_n}$$

is a weak equivalence.

For any element in the right hand side

$$(\mathcal{F}) = \big(\mathcal{F}, I_0^i \leqslant I_1^i, (I_0^\alpha \leqslant \cdots \leqslant I_{k_\alpha}^\alpha)_{\alpha \neq i}\big)$$

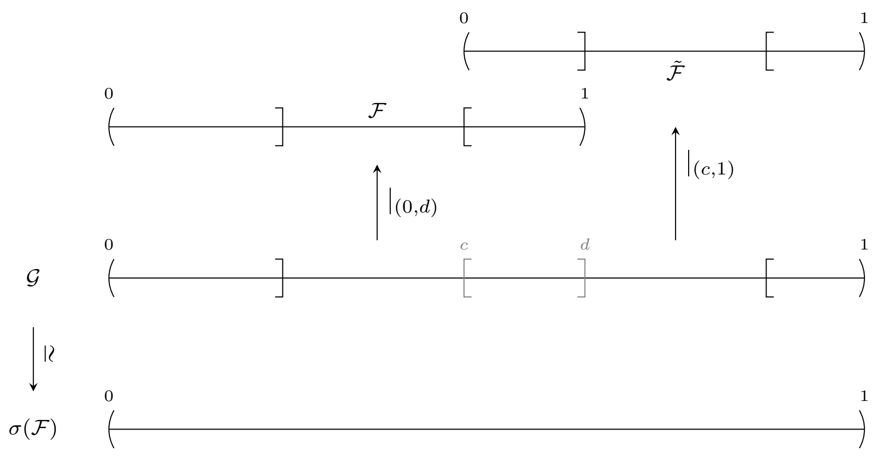
there is another element

$$(\tilde{\mathcal{F}}) = \big(\tilde{\mathcal{F}}, \tilde{I}_0^i \leqslant \tilde{I}_1^i, (\tilde{I}_0^\alpha \leqslant \cdots \leqslant \tilde{I}_{k_\alpha}^\alpha)_{\alpha \neq i}\big)$$

which, in the homotopy category, is an inverse of $(\mathcal{F})$. The composition in the homotopy category is represented by an element

$$(\mathcal{G}) = \big(\mathcal{G}, \tilde{\tilde{I}}_0^i \leqslant \tilde{\tilde{I}}_1^i, (\tilde{\tilde{I}}_0^\alpha \leqslant \cdots \leqslant \tilde{\tilde{I}}_{k_\alpha}^\alpha)_{\alpha \neq i}\big)$$

which, for some $0 \leqslant c \leqslant d \leqslant 1$, where the pair (c, d) is not equal to $(1, 0)$, on $\pi_i^{-1}((0, d))$ restricts to (the rescaled) $\mathcal{F}$ and on $\pi_i^{-1}((c, 1))$ restricts to (the rescaled) $\tilde{\mathcal{F}}$. Moreover, there is a path to the E_{n-i+1}-algebra in factorization algebras on $(0, 1)^{\{1,\dots,i-1\}}$ which is the source $d_1(\mathcal{F})$ of $\mathcal{F}$ which in turn is weakly equivalent to the target $d_0(\tilde{\mathcal{F}})$ of $\tilde{F}$, i.e. there is a weak equivalence $\mathcal{G} \longrightarrow d_1(\mathcal{F})$.



As a factorization algebra on $(0, 1)^{\{i\}}$ with values in factorization algebras on $(0, 1)^{\underline{n} \setminus i}$, $d_1(\mathcal{F})$ is locally constant and and therefore weakly equivalent to its restrictions to $(0, d)$ and $(c, 1)$. Since $\mathcal{G} \simeq d_1(\mathcal{F})$ and $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are its restrictions to $(0, d)$ and $(c, 1)$,

$$\mathcal{F} \simeq d_1(\mathcal{F}).$$

This construction yields a deformation retraction. □

4.2.4 The symmetric monoidal structure on Fact_n

Fact_n has a symmetric monoidal structure defined by a Γ-object which arises similarly to the structure of Alg_n as a Γ-object.

Definition 4.2.6. For every $k_1, \ldots, k_n$ and $m \geqslant 0$, let $(\mathrm{Fact}_n[m])_{k_1,\ldots,k_n}$ be the collection of tuples

$$(\mathcal{F}_1, \ldots, \mathcal{F}_m, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\ldots,n}),$$

where for every $1 \leqslant \beta \leqslant m$, $(\mathcal{F}_\beta, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\ldots,n}) \in (\mathrm{Fact}_n)_{k_1,\ldots,k_n}$. Similarly to Fact_n this can be made into a complete n-fold Segal space.

Proposition 4.2.7. *The assignment*

$$\begin{aligned} \Gamma &\longrightarrow \mathbf{SSpace_n}, \\ [m] &\longmapsto \mathrm{Fact}_n[m] \end{aligned}$$

extends to a functor and endows Fact_n *with a symmetric monoidal structure.*

Proof. Just as for Alg_n, a morphism $f : [m] \to [k]$ is sent to the functor

$$\begin{aligned} \mathrm{Fact}_n[m] &\longrightarrow \mathrm{Fact}_n[k], \\ (\mathcal{F}_1, \ldots, \mathcal{F}_m, I's) &\longmapsto (\bigotimes_{\beta \in f^{-1}(1)} \mathcal{F}_\beta, \ldots, \bigotimes_{\beta \in f^{-1}(k)} \mathcal{F}_\beta, I's). \end{aligned}$$

□

Remark 4.2.8. It is not quite as straightforward to write down the symmetric monoidal structure as a tower and we will not need it later on.

4.3 The map of n-fold simplicial sets $\int_{(-)} A$

In this section, given a fixed E_n-algebra A, we define a map of n-fold simplicial sets

$$\int_{(-)} A : \mathrm{PBord}_n^{fr} \longrightarrow \mathrm{Fact}_n$$

from the framed bordism category to the auxiliary category of factorization algebras. This map essentially translates the properties of the bordisms to factorization algebras on $(0,1)$. It thus in a certain sense encodes the geometry of the embedded manifold. It will not, however, be a map of n-fold Segal spaces as it does not extend to the simplicial structure of the "levels", as we explain below in problem 4.3.5.

We will use the following proposition to show that the third condition on factorization algebras in Fact_n is the exact translation via the map $\pi_*(\int_{(-)} A)$ of the third condition on elements in PBord_n^{fr}.

Proposition 4.3.1. *Let $\mathcal{G}$ be a locally constant factorization algebra on a smooth manifold M. Let X be a smooth manifold and Y a topological space. Consider $f : M \to X \times Y$ such that the composition with the projection, $\pi_X \circ f : M \to X$, is submersive at $(\pi_X \circ f)^{-1}(t)$ for some $t \in X$. Then $f_*\mathcal{G}$ is a locally constant factorization algebra on X with values in factorization algebras on Y in a neighborhood of $\pi_X^{-1}(t)$. If $X = \mathbb{R}^s$, $f_*\mathcal{G}$ is an E_s-algebra in factorization algebras on Y in this neighborhood.*

Proof. $\mathcal{F} := f_*\mathcal{G}$ is a factorization algebra on $X \times Y$, so by theorem 3.2.26, (1), the image of the functor induced by the pushforward along the projection is a factorization algebra $\tilde{\mathcal{F}} = \underline{pr_1}_*\mathcal{F}$ on X with values in factorization algebras on Y. Note that we have $\tilde{\mathcal{F}} : U \mapsto \mathcal{F}_U$ for $U \subset X$, where

$$\mathcal{F}_U : W \mapsto \mathcal{F}(U \times W) \text{ for } W \subset Y.$$

We need to show that $\tilde{\mathcal{F}}$ is locally constant in a neighborhood of t. Take $V \subset U \subset X$ two sufficiently small open sets containing t such that $U \simeq V$. Then the structure map $\mathcal{F}_V \to \mathcal{F}_U$ is a weak equivalence if for every open set $W \subset Y$, the map $\mathcal{F}_V(W) \to \mathcal{F}_U(W)$ is a weak equivalence. Consider

$$\mathcal{F}_U(W) = \mathcal{F}(U \times W) = f_*\mathcal{G}(U \times W) = \mathcal{G}(f^{-1}(U \times W)),$$

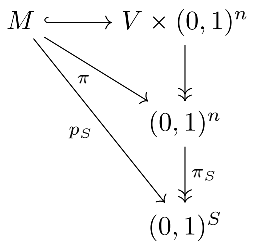
$$\mathcal{F}_V(W) = \mathcal{F}(V \times W) = f_*\mathcal{G}(V \times W) = \mathcal{G}(f^{-1}(V \times W)).$$

Since $\mathcal{G}$ is locally constant, it is enough to show that the inclusion $f^{-1}(V \times W) \subset f^{-1}(U \times W)$ is a weak equivalence. Since

$$f^{-1}(V \times W) = (\pi_X \circ f)^{-1}(V) \cap (\pi_Y \circ f)^{-1}(W), \quad \text{and}$$
$$f^{-1}(U \times W) = (\pi_X \circ f)^{-1}(U) \cap (\pi_Y \circ f)^{-1}(W),$$

it is enough to show that $(\pi_X \circ f)^{-1}(V) \hookrightarrow (\pi_X \circ f)^{-1}(U)$ is a weak equivalence. This holds because we assumed that $V \simeq U$ and that $\pi_X \circ f$ is a submersion at $(\pi_X \circ f)^{-1}(t)$, so locally a projection map. □

Now recall from definition 2.3.1 that for an element (M) in PBord_n^{fr} we used the following notation, where $S \subseteq \{1, \ldots, n\}$:



Applying the above proposition to $\mathcal{G} = \pi_*(\int_M A)$, $f = \pi$, $X = (0,1)^S$, $Y = (0,1)^{\underline{n}\setminus S}$, we obtain the desired property.

Corollary 4.3.2. *Let A be an E_n-algebra and let M be an n-dimensional framed manifold. For $S \subseteq \{1, \ldots, n\}$, let $p_S : M \to (0,1)^S$ be submersive at $x \in p_S^{-1}((t^\alpha)_{\alpha \in S})$. Then $\mathcal{F} := \pi_*(\int_M A)$ is an $\mathrm{E}_{|S|}$-algebra in factorization algebras on $(0,1)^{\underline{n}\setminus S}$ in a neighborhood of $\pi_S^{-1}((t^\alpha)_{\alpha\in S})$.*

By the corollary, the following is well-defined.

Definition 4.3.3. Let A be an E_n-algebra. Let

$$\int_{(-)} A : \mathrm{PBord}_n^{fr} \longrightarrow \mathrm{Fact}_n$$

send

$$\big(M \hookrightarrow V \times (0,1)^n, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\ldots,n}\big) \in (\mathrm{PBord}_n)_{k_1,\ldots,k_n}$$

to

$$\left(\pi_*(\int_M A), (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\ldots,n}\right) \in (\mathrm{Fact}_n)_{k_1,\ldots,k_n},$$

where, as in the previous sections, $\pi : M \hookrightarrow V \times (0,1)^n \twoheadrightarrow (0,1)^n$.

Proposition 4.3.4. $\int_{(-)} A$ *is a well-defined map of n-fold simplicial sets.*

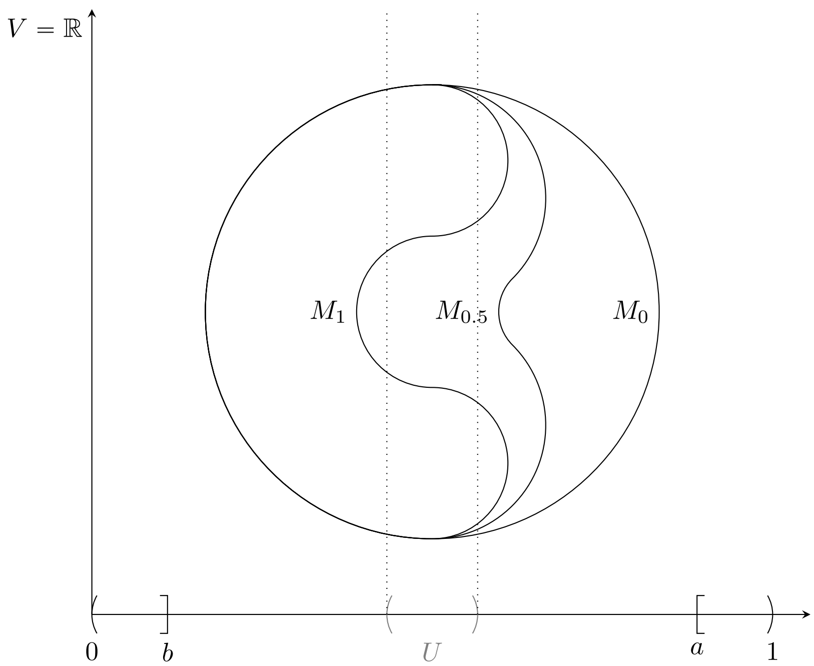
Proof. By the above proposition, $(\int_{(-)} A)\,((M))$ is an element in $(\mathrm{Fact}_n)_{k_1,\ldots,k_n}$. Moreover, $\int_{(-)} A$ commutes with the face and degeneracy maps d_j^i, s_j^i and δ_j^i, σ_j^i of the n-fold simplicial sets $(\mathrm{PBord}_n^{fr})_{\bullet,\ldots,\bullet}$ and $(\mathrm{Fact}_n)_{\bullet,\ldots,\bullet}$ by construction. □

Problem 4.3.5. $\int_{(-)} A$ does not extend to a map between l-simplices of the levels, i.e. $(\int_{(-)} A)_{k_1,\ldots,k_n}$ is not a map of simplicial sets

$$(\int_{(-)} A)_{k_1,\ldots,k_n} : (\mathrm{PBord}_n)_{k_1,\ldots,k_n} \longrightarrow (\mathrm{Fact}_n)_{k_1,\ldots,k_n},$$

as can be seen in the following example.

Consider the following 1-simplex in $(\mathrm{Bord}_1)_1$, which is given by a smooth deformation of the standard embedding of the circle, $[0,1] \times S^1 \hookrightarrow [0,1] \times \mathbb{R} \times (0,1)$, and the pair of intervals $(0,b] \leqslant [a,1)$.



The factorization algebra $\mathcal{F}_1 = (\pi_1)_*(\int_{S^1} A)$ associated to $s = 1$ is not weakly equivalent to that associated to $s = 0$ (even after any rescaling of $(0,1)$), as its value on the open set U as given in the picture is

$$\mathcal{F}_1(U) = A^{\otimes 2} \otimes (A^{op})^{\otimes 2},$$

but $\mathcal{F}_0 = (\pi_0)_*(\int_{S^1} A)$ on intervals takes on values $\mathbb{1}, A, A^{op}$, or $A \otimes A^{op}$.

4.4 Collapsing the factorization algebra and FAlg_n

In this section, we explain how to "collapse" a factorization algebra in Fact_n. We define a map of n-fold simplicial sets

$$\underline{\mathbb{V}} : \mathrm{Fact}_n \longrightarrow \mathrm{FAlg}_n$$

to an n-fold Segal space $\mathrm{FAlg}_n \supseteq \mathrm{Alg}_n$ of factorization algebras on $(0,1)^n$, which have certain locally constancy properties, but do not lead to bimodules.

We first define a collapse-and-rescale map $\veebar : \mathrm{Int}_\bullet \to \mathrm{Covers}_\bullet$ given by applying a collapse-and-rescale map $\varrho_{\underline{a}}^{\underline{b}} : (0,1) \to (0,1)$ to a tuple of intervals with endpoints $\underline{a}, \underline{b}$. This map is lifted to a map $\underline{\mathbb{V}} : \mathrm{Fact}_n \to \mathrm{FAlg}_n$ by pushing forward the factorization algebra along the product of the collapse-and-rescale maps.

4.4.1 The collapse-and-rescale map $\veebar : \mathrm{Int}_\bullet \to \mathrm{Covers}_\bullet$

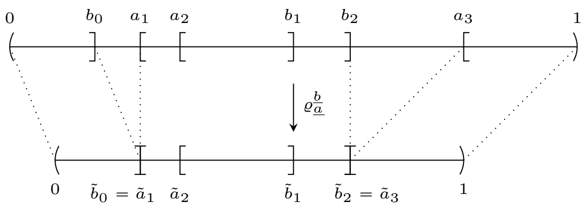
... on the levels

Informally speaking, we first collapse the complement of all intervals and then rescale the rest to $(0,1)$. We saw in lemma 3.1.4 that the collapse-and-rescale maps ϱ_a^b commute in a suitable way. This ensures that we can define the collapse-and-rescale map $\varrho_{\underline{a}}^{\underline{b}}$ by a successive application of ϱ_a^b's.

Definition 4.4.1. Let $I_0, \ldots, I_k$ be closed intervals in $(0,1)$ with non-empty interior and endpoints $\underline{a} = (a_0, \ldots, a_k)$, $\underline{b} = (b_0, \ldots, b_k)$. Then, let

$$\varrho_{\underline{a}}^{\underline{b}} = \varrho_{a_1}^{b_0} * \varrho_{a_2}^{b_1} \cdots * \varrho_{a_k}^{b_{k-1}}.$$

Note that since by definition (a_α, b_α) is non-empty, $(b_{\alpha-1}, a_\alpha) \cap (b_\alpha, a_{\alpha+1}) = \varnothing$. So we can apply lemma 3.1.4 and the map $\varrho_{\underline{a}}^{\underline{b}}$ is independent of the order of maps $\varrho_{a_{\alpha+1}}^{b_\alpha}$. In the following, we will apply this to $(I_0 \leqslant \cdots \leqslant I_k) \in \mathrm{Int}_k$.



Notation 4.4.2. For $I_0, \ldots, I_k$ as above let $\{j_1, \ldots, j_l\} \subseteq \{0, \ldots, k-1\}$ be the indices for which $b_{j_\beta} < a_{j_\beta+1}$, i.e. $\varrho_{a_{j_\beta+1}}^{b_{j_\beta}} \neq id$. Then, similarly as we saw for ϱ_a^b,

$$\varrho_{\underline{a}}^{\underline{b}}|_{D_{\underline{a}}^{\underline{b}}},$$

for $D^b_{\underline{a}} = (0, b_{j_1}] \cup (a_{j_1+1}, b_{j_2}] \cup \cdots \cup (a_{j_l+1}, 1)$ is bijective. We denote its inverse by

$$(\varrho^b_{\underline{a}})^{-1} = (\varrho^b_{\underline{a}}|_{D^b_{\underline{a}}})^{-1} : (0,1) \longrightarrow D^b_{\underline{a}}.$$

... as a map of complete Segal spaces

Proposition 4.4.3. *The map*

$$\mathrm{Int}_k \xrightarrow{\underline{\nabla}_k} \mathrm{Covers}_k,$$
$$(I_0 \leqslant \cdots \leqslant I_k) \longmapsto (\varrho^b_{\underline{a}}(I_0) \leqslant \cdots \leqslant \varrho^b_{\underline{a}}(I_k)),$$

extends to a map of complete Segal spaces.

Proof. We first need to show that the map $\underline{\nabla}_k$ extends to a map of spaces $\underline{\nabla}_k : \mathrm{Int}_k \to \mathrm{Covers}_k$, i.e. we need to define it on l-simplices and show that it commutes with the spatial face and degeneracy maps s^Δ_l, d^Δ_l of Int_k and $\sigma^\Delta_l, \delta^\Delta_l$ of Covers_k. Finally we need to show that all $\underline{\nabla}_k$ together form a map of simplicial spaces, i.e. they commutes with the simplicial face and degeneracy maps s_j, d_j of $\mathrm{Int}_\bullet$ and σ_j, δ_j of $\mathrm{Covers}_\bullet$.

... on l-simplices Consider an l-simplex in Int_k consisting of underlying 0-simplices $(I_1(s) \leqslant \cdots \leqslant I_k(s))_{s\in|\Delta^l|}$ and a rescaling datum $(\varphi_{s,t} : (0,1) \to (0,1))_{s,t\in|\Delta^l|}$. It is sent to the l-simplex in Covers_k defined as follows:

1. for $s \in |\Delta^l|$, the sth underlying 0-simplex of the image is
$$\Big(\varrho^b_{\underline{a}}(I_0(s)) \leqslant \cdots \leqslant \varrho^b_{\underline{a}}(I_k(s))\Big) \in \mathrm{Covers}_k;$$

2. for $s, t \in |\Delta^l|$, the rescaling datum is
$$\phi_{s,t} = \varrho^{b(t)}_{\underline{a}(t)} \circ \varphi_{s,t}|_{D^b_{\underline{a}}(s)} \circ (\varrho^{b(s)}_{\underline{a}(s)})^{-1} : (0,1) \to (0,1).$$

... commutes with the spatial degeneracy and face maps The map $\underline{\nabla}_k$ commutes with spatial degeneracy and face maps since these come from the degeneracy and face maps of the simplicial set $(\Delta^l)_l$.

... commutes with the simplicial degeneracy and face maps This essentially follows from the behaviour of the collapse-and-rescale maps ϱ^b_a, which is summarized in the following lemma. □

Lemma 4.4.4. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{Int}_{k+1} & \xrightarrow{\underline{\nabla}_{k+1}} & \mathrm{Covers}_{k+1} \\
\uparrow{\scriptstyle d_j} & & {\scriptstyle \delta_j}\uparrow \\
\mathrm{Int}_k & \xrightarrow{\underline{\nabla}_k} & \mathrm{Covers}_k \\
\downarrow{\scriptstyle s_j} & & {\scriptstyle \sigma_j}\downarrow \\
\mathrm{Int}_{k-1} & \xrightarrow{\underline{\nabla}_{k-1}} & \mathrm{Covers}_{k-1}
\end{array}$$

Proof. The collapse-and-rescaling maps on the top and in the middle coincide, since $\varrho_{a_j}^{b_j} = id$ and therefore

$$\varrho_{d_j(\underline{a})}^{d_j(\underline{b})} = \cdots * \varrho_{a_j}^{b_{j-1}} * \varrho_{a_j}^{b_j} * \varrho_{a_{j+1}}^{b_j} = \varrho_{\underline{a}}^{\underline{b}}.$$

Thus the top diagram commutes.

For the lower diagram, we need to compare the composition of the (collapse-and-)rescaling maps.

$$\begin{array}{ccc} \mathrm{Int}_k & \xrightarrow{\varrho_{\underline{a}}^{\underline{b}}} & \mathrm{Covers}_k \\ {\scriptstyle id \text{ or } \rho_j}\downarrow & & \downarrow {\scriptstyle \varrho_{a_{j+1}}^{b_{j-1}}} \\ \mathrm{Int}_{k-1} & \xrightarrow{\varrho_{\underline{a}}^{\underline{b}}} & \mathrm{Covers}_{k-1} \end{array}$$

The upper right composition $\sigma_j \circ \veebar_k$ has as rescaling map $\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{\underline{a}}^{\underline{b}}$. Using lemma 3.1.4 and remark 3.1.3 we obtain

$$\begin{aligned}
\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{\underline{a}}^{\underline{b}} &= \varrho_{a_{j+1}}^{b_{j-1}} * (\varrho_{a_1}^{b_0} * \varrho_{a_2}^{b_1} \cdots * \varrho_{a_k}^{b_{k-1}}) \\
&\overset{3.1.4}{=} \varrho_{a_{j+1}}^{b_{j-1}} * (\varrho_{a_j}^{b_{j-1}} * \varrho_{a_{j+1}}^{b_j} * \mathop{\circledast}_{\alpha \neq j-1,j} \varrho_{a_{\alpha+1}}^{b_\alpha}) \\
&= (\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_j}^{b_{j-1}}) * \varrho_{a_{j+1}}^{b_j} * \mathop{\circledast}_{\alpha \neq j-1,j} \varrho_{a_{\alpha+1}}^{b_\alpha} \\
&\overset{3.1.3}{=} (\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+1}}^{b_j}) * \mathop{\circledast}_{\alpha \neq j-1,j} \varrho_{a_{\alpha+1}}^{b_\alpha} \\
&\overset{3.1.3}{=} \varrho_{a_{j+1}}^{b_{j-1}} * \mathop{\circledast}_{\alpha \neq j-1,j} \varrho_{a_{\alpha+1}}^{b_\alpha} \\
&\overset{3.1.4}{=} \varrho_{a_1}^{b_0} * \cdots * \varrho_{a_{j-1}}^{b_{j-2}} * \varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+2}}^{b_{j+1}} * \cdots * \varrho_{a_k}^{b_{k-1}} \\
&= \varrho_{\widehat{\underline{a}}^j}^{\widehat{\underline{b}}^j},
\end{aligned}$$

where $\widehat{\underline{a}}^j = (a_0, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k)$ and $\widehat{\underline{b}}^j = (b_0, \ldots, b_{j-1}, b_{j+1}, \ldots, b_k)$.

For $j \neq 0, k$, we have that $s_j(\underline{a}) = \widehat{\underline{a}}^j$ and $s_j(\underline{b}) = \widehat{\underline{b}}^j$ and thus the lower left composition $\veebar_{k-1} \circ s_j$ has as rescaling map

$$\varrho_{s_j(\underline{a})}^{s_j(\underline{b})} = \varrho_{\widehat{\underline{a}}^j}^{\widehat{\underline{b}}^j}.$$

For $j = 0$ or $j = k$ we have that $s_j(\underline{a}) = \rho_j(\widehat{\underline{a}}^j) = \varrho_{a_{j+1}}^{b_{j-1}}(\widehat{\underline{a}}^j)$ and $s_j(\underline{b}) = \rho_j(\widehat{\underline{b}}^j) = \varrho_{a_{j+1}}^{b_{j-1}}(\widehat{\underline{b}}^j)$, and thus the lower left composition $\veebar_{k-1} \circ \sigma_j$ has as rescaling map

$$\begin{aligned}
\varrho_{s_j(\underline{a})}^{s_j(\underline{b})} &= \varrho_{\widehat{\underline{a}}^j}^{\widehat{\underline{b}}^j} * \varrho_{a_{j+1}}^{b_{j-1}} \\
&= (\varrho_{a_1}^{b_0} * \cdots * \varrho_{a_{j-1}}^{b_{j-2}} * \varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+2}}^{b_{j+1}} * \cdots * \varrho_{a_k}^{b_{k-1}}) * \varrho_{a_{j+1}}^{b_{j-1}} \\
&= \varrho_{a_1}^{b_0} * \cdots * \varrho_{a_{j-1}}^{b_{j-2}} * \varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+2}}^{b_{j+1}} * \cdots * \varrho_{a_k}^{b_{k-1}} \\
&= \varrho_{\widehat{\underline{a}}^j}^{\widehat{\underline{b}}^j},
\end{aligned}$$

since similarly to above, by lemma 3.1.4 we can first reorder the terms in the parentheses, use $\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+1}}^{b_{j-1}} = \varrho_{a_{j+1}}^{b_{j-1}}$ by remark 3.1.3, and then reorder again. □

4.4.2 The "faux" Alg_n, the n-fold Segal space FAlg_n

Recall that in definition 3.2.27, given $(I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\dots,n} \in (\mathrm{Covers}^n)_{k_1,\dots,k_n}$, we inductively defined a stratification of $(0,1)^n$ by

$$X_n = (0,1)^n, \quad X_{n-i} = X_{n-i+1} \cap Y_i,$$

for $1 \leqslant i \leqslant n$, where

$$Y_i = \pi_i^{-1}\big(S^i\big) \quad \text{for } S^i = (0,1)\backslash \bigcup_{j=0}^{k_i} (a_j^i, b_j^i) = (0,1)\backslash \bigcup_{j=0}^{k_i} (I_j^i)^\circ,$$

and $(I_j^i)^\circ = (a_j^i, b_j^i)$ is the interior of the interval I_j^i. Note that the set $X\backslash Y_i = \bigcup_{j=0}^{k_i}(a_j^i, b_j^i) \times (0,1)^{\underline{n}\backslash i}$ is a disjoint union of products of the form

$$(0, s_1^i) \times (0,1)^{\underline{n}\backslash i}, \quad (s_j^i, s_{j+1}^i) \times (0,1)^{\underline{n}\backslash i}, \quad \text{or } (s_{l_i}^i, 1) \times (0,1)^{\underline{n}\backslash i},$$

where $\underline{n} = \{1, \dots, n\}$.

We now define a "faux" n-fold Segal space FAlg_n of E_n-algebras, whose objects are E_n-algebras, but the morphisms do not behave like modules.

Definition 4.4.5. For every $k_1, \dots, k_n \geqslant 0$, let $(\mathrm{FAlg}_n)_{k_1,\dots,k_n}$ be the collection of tuples

$$(\mathcal{F}, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\dots,n}),$$

satisfying the following conditions:

1. $\mathcal{F}$ is a factorization algebra on $(0,1)^n$.
2. For $1 \leqslant i \leqslant n$,
$$(I_0^i \leqslant \cdots \leqslant I_{k_i}^i) \in \mathrm{Covers}_{k_i},$$
3. For $1 \leqslant i \leqslant n$, on every connected component of $X\backslash Y_i$, the factorization algebra $\mathcal{F}$ is an E_{n-i+1}-algebra in factorization algebras on $(0,1)^{\{1,\dots,i-1\}}$.

We make the collection $(\mathrm{FAlg}_n)_{\bullet,\dots,\bullet}$ into an n-fold Segal space similarly to $(\mathrm{Alg}_n)_{\bullet,\dots,\bullet}$.

Remark 4.4.6. Similarly to definition 4.2.1 we use theorem 3.2.26 to formulate the condition on the factorization algebra $\mathcal{F}$.

Example 4.4.7. For $n = 1$, $(\mathrm{FAlg}_1)_k$ consists of elements of the form

$$(\mathcal{F}, I_0 \leqslant \ldots \leqslant I_k),$$

where $\mathcal{F}$ is a factorization algebra on $(0,1)$ and is locally constant everywhere except at the points $S = \{s_1, \dots, s_l\} = (0,1)\backslash(I_0 \cup \ldots \cup I_k)$. In particular, $(\mathrm{FAlg}_1)_0 = (\mathrm{Alg}_1)_0$ and consists of locally constant factorization algebras on $(0,1)$, i.e. E_1-algebras. However, for $k > 1$, $(\mathrm{Alg}_1)_k$ is the proper subset of $(\mathrm{FAlg}_1)_k$ of elements which furthermore satisfy the condition that if U, V are intervals containing the same point s_j, $\mathcal{F}(U) \simeq \mathcal{F}(V)$.

Proposition 4.4.8. *There is an inclusion of n-fold Segal spaces*

$$\mathrm{Alg}_n \subset \mathrm{FAlg}_n.$$

Proof. Recall from definition 3.2.27 that

$$X_{n-\alpha} = S^1 \times \cdots S^\alpha \times (0,1)^{\{\alpha+1,\dots,n\}}.$$

Thus, the stratification induces a stratification on $X \backslash Y_i$ of the form

$$\begin{aligned}(X\backslash Y_i) \cap X_{n-\alpha} &= S^1 \times \cdots \times S^\alpha \times (0,1)^{\{\alpha+1,\dots,i-1\}} \times \big((0,1)\backslash S^i\big) \times (0,1)^{\{i+1,\dots,n\}} \\ &= \tilde{X}_{n-\alpha} \times (0,1)^{\{i+1,\dots,n\}},\end{aligned}$$

where

$$\tilde{X}_{n-\alpha} = S^1 \times \cdots \times S^\alpha \times (0,1)^{\{\alpha+1,\dots,i-1\}} \times \big((0,1)\backslash S^i\big).$$

for $0 \leqslant \alpha < i$.

Let $(\mathcal{F}, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{1\leqslant i \leqslant n}) \in (\mathrm{Alg}_n)_{k_1,\dots k_n}$. The restriction $\mathcal{F}|_{X\backslash Y_i}$ is locally constant with respect to the stratification $(X\backslash Y_i) \cap X_{n-\alpha}$. Thus, as a factorization algebra on $(0,1)^{\{i+1,\dots,n\}}$ with values in factorization algebras on $(0,1)^{\{1,\dots,i\}}$ it is locally constant. □

4.4.3 The collapsing map $\underline{\nabla}$: $\mathrm{Fact}_n \to \mathrm{FAlg}_n$

We can now lift the collapsing map ∇ : Int → Covers to a collapsing map $\underline{\nabla}$: $\mathrm{Fact}_n \to \mathrm{FAlg}_n$.

Notation 4.4.9. Let $(I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\dots,n}) \in \mathrm{Int}^n_{k_1,\dots,k_n}$. For $1 \leqslant i \leqslant n$ denote the collapse-and-rescale map associated to $(I_0^i \leqslant \cdots \leqslant I_{k_i}^i) \in \mathrm{Int}_{k_i}$ by $\varrho_{\underline{a}^i}^{b^i}$, and denote their product by

$$\varrho_{\underline{\underline{a}}}^{\overline{b}} = (\varrho_{\underline{a}^1}^{b^1}, \dots, \varrho_{\underline{a}^n}^{b^n}) : (0,1)^n \longrightarrow (0,1)^n.$$

Note that

$$\varrho_{\underline{\underline{a}}}^{\overline{b}} = \varrho_{\underline{a}^1}^{b^1} \circ \dots \circ \varrho_{\underline{a}^n}^{b^n},$$

where as before we again denote by $\varrho_{\underline{a}^i}^{b^i}$ the map $(0,1)^n \longrightarrow (0,1)^n$ which is $\varrho_{\underline{a}^i}^{b^i}$ in the ith coordinate and the identity otherwise, and the order in the above composition does not matter.

Proposition 4.4.10.

$$\begin{aligned}(\mathrm{Fact}_n)_{k_1,\dots k_n} &\xrightarrow{\underline{\nabla}} (\mathrm{FAlg}_n)_{k_1,\dots k_n} \\ (\mathcal{F}, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{1\leqslant i\leqslant n}) &\longmapsto \Big((\varrho_{\underline{\underline{a}}}^{\overline{b}})_*(\mathcal{F}), \big(\nabla(I_0^i \leqslant \cdots \leqslant I_{k_i}^i)\big)_{1\leqslant i\leqslant n}\Big)\end{aligned}$$

is a map of n-fold Segal spaces.

Proof. As we have seen in lemma 4.4.4 that the (collapse-and-)rescaling maps behave well with respect to face and degeneracy maps of the simplicial space, it is enough to show that $\underline{\nabla}$ indeed maps to FAlg_n.

We need to check the third condition in definition 4.4.5, i.e. that for $1 \leqslant i \leqslant n$, on

$$X\backslash Y_i = \pi_i^{-1}\Big(\bigcup_{j=0}^{k_i} \varrho_{\underline{a}^i}^{b^i}(I_j^i)^\circ\Big)$$

$(\varrho_{\underline{a}}^{\overline{b}})_*\mathcal{F}$ is an E_{n-i+1}-algebra in factorization algebras on $(0,1)^{\{1,\dots,i-1\}}$.

For this it is enough to show that for every $0 \leqslant j \leqslant k_i$, we have

$$\Big((\varrho_{\underline{a}}^{\overline{b}})_*\mathcal{F}\Big)|_{\pi_i^{-1}\left(\varrho_{\underline{a}^i}^{b^i}(I_j^i)^\circ\right)} = (\varrho_{\underline{a}}^{\overline{b}})_*\Big(\mathcal{F}|_{\pi_i^{-1}(I_j^i)^\circ}\Big)$$

is an E_{n-i+1}-algebra in factorization algebras on $(0,1)^{\{1,\dots,i-1\}}$.

Since $(\mathcal{F}, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{1\leqslant i\leqslant n}) \in (\mathrm{Fact}_n)_{k_1,\dots k_n}$, $\mathcal{F}|_{\pi_i^{-1}(I_j^i)^\circ}$ is an E_{n-i+1}-algebra in factorization algebras on $(0,1)^{\{1,\dots,i-1\}}$, so the following lemma finishes the proof. □

Lemma 4.4.11. *Let $\mathcal{G}$ be a locally constant factorization algebra on $(0,1)$ and let ϱ_a^b be a collapse-and-rescaling map. Then $(\varrho_a^b)_*\mathcal{G}$ is locally constant on $(0,1)$.*

Proof. This follows from the fact that preimages of intervals under ϱ_a^b again are intervals. □

4.5 The functor of (∞, n)-categories $\mathcal{FH}_n$

We now show that, given an E_n-algebra A, the composition $\underline{\nabla} \circ \int_{(-)} A$ lands in Alg_n and thus yields a map

$$\mathcal{FH}_n = \mathcal{FH}_n(A) : \mathrm{PBord}_n^{fr} \longrightarrow \mathrm{Alg}_n\,.$$

Proposition 4.5.1. *Let $(M) = \big(M \hookrightarrow V \times (0,1)^n, (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{1\leqslant i\leqslant n}\big) \in (\mathrm{PBord}_n)_{k_1,\dots k_n}$. Then*

$$\Big(\underline{\nabla} \circ \int_{(-)} A\Big)((M)) \in (\mathrm{Alg}_n)_{k_1,\dots,k_n},$$

Proof. Let $\pi : M \hookrightarrow V \times (0,1)^n \twoheadrightarrow (0,1)^n$ and as usual denote the endpoints of the interval I_j^i by a_j^i, b_j^i. We need to show that the underlying factorization algebra of $\mathcal{FH}_n((M))$, which is

$$\mathcal{F}_{(M)} = (\varrho_{\underline{a}}^{\overline{b}})_*\pi_* \int_M A,$$

is locally constant with respect to the stratification associated to the intervals $\big(\varrho_{\underline{a}^i}^{b^i}(I_0^i) \leqslant \cdots \leqslant \varrho_{\underline{a}^i}^{b^i}(I_{k_i}^i)\big)_{i=1}^n$.

Let $V \subseteq U$ be good neighborhoods at $X_{n-\alpha} = S^1 \times \cdots \times S^\alpha \times (0,1)^{\{\alpha+1,\dots,n\}}$ from definition 3.2.27 respectively remark 3.2.28. We can assume that they are boxes, i.e. products of intervals

$$U = U^1 \times \cdots \times U^n, \qquad V = V^1 \times \cdots \times V^n$$

and meet exactly one connected component

$$(s_j^i)_{\beta=1}^\alpha \times (0,1)^{\{\alpha+1,\dots,n\}}$$

of $X_{n-\alpha}$. We need to show that the structure map $\mathcal{F}_{(M)}(V) \to \mathcal{F}_{(M)}(U)$ is a weak equivalence. By definition,

$$\mathcal{F}_{(M)}(V) = (\int_M A)(\pi^{-1}(\tilde{V})) \quad \text{and} \quad \mathcal{F}_{(M)}(U) = (\int_M A)(\pi^{-1}(\tilde{U})),$$

where $\tilde{V} = (\varrho_{\underline{a}}^{\overline{b}})^{-1}(V)$ and $\tilde{U} = (\varrho_{\underline{a}}^{\overline{b}})^{-1}(U)$. Since $\int_M A$ is locally constant, it is enough to show that the inclusion $\pi^{-1}(\tilde{V}) \hookrightarrow \pi^{-1}(\tilde{U})$ is a weak equivalence.

The open sets $\tilde{V}$ and $\tilde{U}$ are boxes of open intervals,

$$\tilde{V} = (e^1, f^1) \times \cdots \times (e^n, f^n) \quad \text{and} \quad \tilde{U} = (c^1, d^1) \times \cdots (c^n, d^n)$$

where $(c^i, d^i) = (\varrho_{\underline{a}^i}^{b^i})^{-1}(U^i)$ and $(e^i, f^i) = (\varrho_{\underline{a}^i}^{b^i})^{-1}(V^i)$. The endpoints c^i and e^i, respectively d^i and f^i, either lie in the same closed specified interval $I_{j_i}^i$ or in ones connected by a chain of overlapping intervals. An argument similar to that in corollary 2.3.4 or, if an endpoint is 0 or 1, example 2.3.2 gives a diffeomorphism

$$\pi^{-1}(\tilde{U}) \longrightarrow \pi^{-1}(\tilde{V}).$$

□

Definition 4.5.2. Let

$$\mathcal{FH}_n = \mathcal{FH}_n(A) = \underline{\nabla} \circ \int_{(-)} A : \mathrm{PBord}_n^{fr} \longrightarrow \mathrm{Alg}_n,$$

$$\mathcal{FH}_n\big((M)\big) = (\mathcal{F}_M, \big(\underline{\nabla}\,(I_0^i \leqslant \ldots \leqslant I_{k_i}^i)\big)_{i=1}^n),$$

where $\mathcal{F}_{(M)} = (\varrho_{\underline{a}}^{\overline{b}})_* \pi_* \int_M A$. By the universal property of the completion it extends to a map of complete n-fold Segal spaces

$$\mathcal{FH}_n = \mathcal{FH}_n(A) : \mathrm{Bord}_n^{fr} \longrightarrow \mathrm{Alg}_n\,.$$

Example 4.5.3. [The value at a point] A point viewed as an object $*$ in the bordism category is represented by the trivial bordism

$$(*) = \big(M = (0,1)^n \hookrightarrow (0,1)^n, \big((0,1)\big)_{i=1}^n\big)\,.$$

We have seen in example 4.1.4 that $\int_{(0,1)^n} A \simeq \int_{\mathbb{R}^n} A \simeq A$ as a factorization algebra on $M = (0,1)^n$. Then $\pi : M = (0,1)^n \to (0,1)^n$ is the identity map and $\varrho_{\underline{a}}^{\overline{b}} = id$. So,

$$\mathcal{FH}_n(A)(*) = \big(A, \big((0,1)\big)_{i=1}^n\big) \in (\mathrm{Alg}_n)_{0,\ldots,0}.$$

4.6 The fully extended topological field theory $\mathcal{FH}_n$

To obtain a fully extended topological field theory the functor $\mathcal{FH}_n(A)$ needs to be symmetric monoidal. In this section we extend it to a symmetric monoidal functor, both by defining a natural transformation of Γ-objects and by defining compatible functors between the layers of the towers.

4.6.1 Symmetric monoidality via Γ-objects

We extend the map $\mathcal{FH}_n(A)$ to a natural transformation between functors $\Gamma \to \mathbf{SSpace_n}$, $[m] \mapsto \mathrm{PBord}_n^{fr}[m], \mathrm{Alg}_n[m]$.

Proposition 4.6.1. *For every object $[m] \in \Gamma$, let $\mathcal{FH}_n[m] = \mathcal{FH}_n(A)[m]$ be the map of n-fold Segal spaces*

$$\begin{aligned} \mathrm{PBord}_n^{fr}[m] &\longrightarrow \mathrm{Alg}_n[m], \\ \big(M_1, \ldots, M_m, (I_0^i \leqslant \ldots \leqslant I_{k_i}^i)_{i=1}^n)\big) &\longmapsto \big(\mathcal{F}_{(M_1)}, \ldots, \mathcal{F}_{(M_m)}, \text{\Large$\curlyvee$}(I_0^i \leqslant \ldots \leqslant I_{k_i}^i)_{i=1}^n\big). \end{aligned}$$

This assignment endows the functor $\mathcal{FH}_n(A)$ of (∞, n)-categories with a symmetric monoidal structure.

Proof. The map $\mathcal{FH}_n[m]$ is well-defined since the the image of the left-hand element under the inclusion $\mathrm{PBord}_n[m] \subseteq (\mathrm{PBord}_n[1])^m$ is the collection of

$$(M_\beta, (I_0^i \leqslant \ldots \leqslant I_{k_i}^i)_{i=1}^n) \in (\mathrm{PBord}_n^{fr,V})_{k_1,\ldots,k_n}$$

which under $\mathcal{FH}_n(A)$ are sent to elements in $(\mathrm{Alg}_n)_{k_1,\ldots,k_n}$ with underlying factorization algebras $\mathcal{F}_{(M_\beta)}$ and the same underlying element in $(\mathrm{Covers}^n)_{k_1,\ldots,k_n}$

$$\big(\curlyvee(I_0^i \leqslant \ldots \leqslant I_{k_i}^i)\big)_{i=1}^n.$$

Thus the collection of the images lies in the image of the inclusion $\mathrm{Alg}_n[m] \subseteq (\mathrm{Alg}[1])^m$. The map $\mathcal{FH}_n[m]$ is a map of n-fold Segal spaces by the same argument as for $\mathcal{FH}_n$.

To see that this assignment defines a natural transformation, let $f : [m] \to [k]$, and $1 \leqslant \alpha \leqslant k$. Let $\pi = \pi[1] \amalg \cdots \amalg \pi[m] : M_1 \amalg \cdots \amalg M_m \to (0,1)^n$. By the following lemma we have

$$\pi_* \int_{\coprod_{\beta \in f^{-1}(\alpha)} M_\beta} A = \bigotimes_{\beta \in f^{-1}(\alpha)} \pi[\beta]_* \int_{M_\beta} A,$$

and thus the following diagram commutes.

$$\begin{array}{ccc} \mathrm{PBord}_n^{fr}[m] & \xrightarrow{\mathcal{FH}_n[m]} & \mathrm{Alg}_n[m] \\ \big\downarrow f & & \big\downarrow f \\ \mathrm{PBord}_n^{fr}[k] & \xrightarrow{\mathcal{FH}_n[k]} & \mathrm{Alg}_n[k] \end{array}$$

□

Lemma 4.6.2. *Let $f : X \to Z$, $g : Y \to Z$ be continuous maps of topological spaces and let $\mathcal{F}$ be a factorization algebra on $X \amalg Y$. Then*

$$(f \amalg g)_* \mathcal{F} = f_* \mathcal{F}|_X \otimes g_* \mathcal{F}|_Y.$$

Proof. Let $U \subset Z$ be open. Then $(f \amalg g)^{-1}(U) = f^{-1}(U) \amalg g^{-1}(U)$ and by the gluing property of $\mathcal{F}$, we have

$$\mathcal{F}(f^{-1}(U) \amalg g^{-1}(U)) = \mathcal{F}(f^{-1}(U)) \otimes \mathcal{F}(g^{-1}(U)).$$

□

As a corollary of proposition 4.6.1 we obtain our main result.

Corollary 4.6.3. *Let A be an E_n-algebra. Then the map*

$$\mathcal{FH}_n(A) : \mathrm{Bord}_n^{fr} \longrightarrow \mathrm{Alg}_n$$

is a fully extended topological field theory.

4.6.2 Symmetric monoidality via the tower

In this section we extend the map to the layers of the tower in a compatible way.

On the lth layer the extension $\mathcal{FH}_n^{(l)}$ is the composition of maps $\int_{(-)} A$ and $\mathbb{V}^{(l)}$ analogous to those for $l = 0$. For simplicity, instead of defining the layers for the auxiliary spaces Fact_n and FAlg_n we define $\mathcal{FH}_n^{(l)}$ directly.

Proposition 4.6.4. *For every $l \geqslant 0$, the assignment*

$$\begin{gathered} \mathrm{PBord}_n^{fr,l} \longrightarrow \mathrm{Alg}_n^{(l)}, \\ \left(\pi^{(l)} : M \to (0,1)^{n+l}, (I_0^i \leqslant \ldots \leqslant I_{k_i}^i)_{i=1}^{n+l}\right) \longmapsto \left(\mathcal{F}_{M^{(l)}}, \mathbb{V}(I_0^i \leqslant \ldots \leqslant I_{k_i}^i)_{i=1}^{n+l}\right), \end{gathered}$$

where $\mathcal{F}_{M^{(l)}} = (\varrho^{\overline{b}}_{\underline{a}})_*(\pi^{(l)})_* \int_M A$, is a map of n-fold Segal spaces $\mathcal{FH}_n^{(l)} = \mathcal{FH}_n^{(l)}(A)$. It commutes with the looping and delooping maps u and ℓ from propositions 2.5.8 and 3.3.8.

Proof. We need to check:

1. *$\mathcal{FH}_n^{(l)}$ is well-defined, i.e. its image indeed lies in $\mathrm{Alg}_n^{(l)}$.*

   Similarly to propositions 4.4.10 and 4.5.1 one can show that $\mathcal{F}_{M^{(l)}}$ is locally constant with respect to the stratification associated to $\mathbb{V}(I_0^i \leqslant \ldots \leqslant I_{k_i}^i)_{i=1}^{n+l}$, and thus $\mathcal{FH}_n^{(l)}$ maps to Alg_{n+l}. Moreover, as noted in remark 2.5.6, $(\mathrm{PBord}_n^{fr,l})_{1,\ldots,1,0,\bullet,\ldots,\bullet}$, with $(l-1)$ 1's, is the point viewed as a constant $(n-l)$-fold Segal space. This implies that $\mathcal{FH}_n^{(l)}$ indeed maps to $\mathrm{Alg}_n^{(l)} \subset \mathrm{Alg}_{n+l}$.

2. *$\mathcal{FH}_n^{(l)}$ commutes with the looping and delooping maps u, ℓ from propositions 2.5.8 and 3.3.8, i.e. the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{PBord}_n^{fr,l} & \xrightarrow{\mathcal{FH}_n^{(l)}} & \mathrm{Alg}_n^{(l)} \\ u \uparrow \big\downarrow \ell & & u \uparrow \big\downarrow \ell \\ L(\mathrm{PBord}_n^{fr,l+1}) & \xrightarrow{\mathcal{FH}_n^{(l+1)}} & L(\mathrm{Alg}_n^{(l+1)}) \end{array}$$

   It is straightforward to see from the constructions of u that the diagram for u commutes. The commutativity for ℓ follows from the properties of the collapse-and-rescale maps.

□

By the universal property of the (l-hybrid) completion, we obtain maps

$$\mathcal{FH}_n^{(l)} : \mathrm{Bord}_n^{fr,(l)} \longrightarrow \mathrm{Alg}_n^{(l)} .$$

Corollary 4.6.5. *The maps $\mathcal{FH}_n^{(l)}$ endow the functor $\mathcal{FH}_n(A) : \mathrm{Bord}_n^{fr} \to \mathrm{Alg}_n$ of (∞, n)-categories with a symmetric monoidal structure. Thus, given an E_n-algebra A, the map*

$$\mathcal{FH}_n(A) : \mathrm{Bord}_n^{fr} \longrightarrow \mathrm{Alg}_n$$

is a fully extended topological field theory.

4.7 Variants

4.7.1 Geometric structures

As in section 2.7.1, let X be a topological space and $E \to X$ a topological n-dimensional vector bundle which corresponds to a (homotopy class of) map(s) $e : X \to B\mathrm{GL}(\mathbb{R}^n)$ from X to the classifying space of the topological group $\mathrm{GL}(\mathbb{R}^n)$.

Given a $\mathcal{D}isk_n^{(X,E)}$-algebra, one might ask if we can obtain a fully extended n-dimensional (X, E)-topological field theory by a similar procedure. Indeed, given a $\mathcal{D}isk_n^{(X,E)}$-algebra A, in 4.1.1 we defined factorization homology for (X, E)-structured n-dimensional manifolds with coefficients in A. Analyzing its proof one sees that theorem 4.1.5 also holds in this case. Thus, following the same steps as in section 4.3 we obtain a map of n-fold simplicial sets

$$\int_{(-)} A : \mathrm{PBord}_n^{(X,E)} \longrightarrow \mathrm{Fact}_n .$$

The rest of the construction remains the same and we obtain the following result.

Theorem 4.7.1. *Let A be a $\mathcal{D}isk_n^{(X,E)}$-algebra. By the universal property of the completion, the composition $\mathcal{FH}_n(A) = \underline{\mathrm{V}} \circ \int_{(-)} A : \mathrm{PBord}_n^{(X,E)} \to \mathrm{Alg}_n$ extends to an (X, E)-structured fully extended n-dimensional topological field theory*

$$\mathcal{FH}_n(A) : \mathrm{Bord}_n^{(X,E)} \longrightarrow \mathrm{Alg}_n .$$

4.7.2 A variant for n-fold categories

Recall from sections 2.4.3 and 3.5.3 the n-fold categories of bordisms and E_n-algebras, the complete n-uple Segal spaces Bord_n^{uple} and Alg_n^{uple}. Similarly, one can define a complete n-uple Segal spaces Fact_n^{uple} and FAlg_n^{uple} by relaxing condition (3) in definitions 4.2.1 and 4.4.5. Given an E_n-algebra, or, more generally as in the previous section, a $\mathcal{D}isk_n^{(X,E)}$-algebra A, the maps $\int_{(-)} A$ and $\underline{\mathrm{V}}$ give maps

$$\int_{(-)} A : \mathrm{PBord}_n^{(X,E),uple} \longrightarrow \mathrm{Fact}_n^{uple}$$

and

$$\underline{\mathrm{V}} : \mathrm{Fact}_n^{uple} \longrightarrow \mathrm{FAlg}_n^{uple} .$$

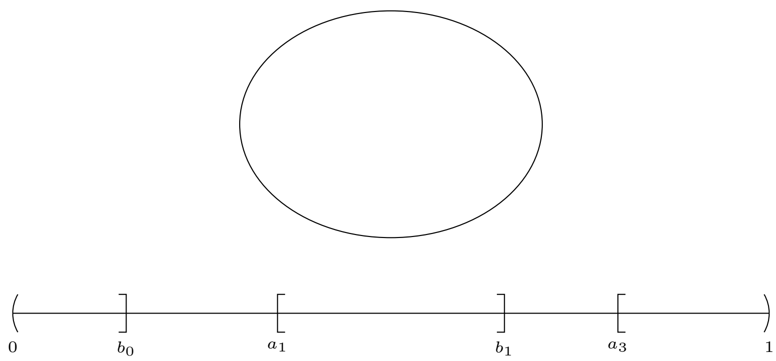
Altogether, we obtain a fully extended "n-fold" TFT

$$\mathcal{FH}_n(A) : \mathrm{Bord}_n^{(X,E),uple} \longrightarrow \mathrm{Alg}_n^{uple} .$$

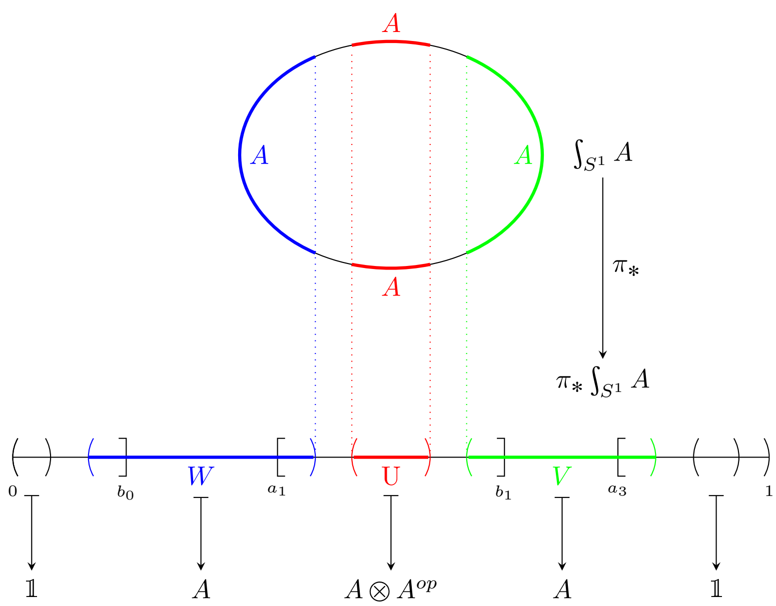
4.8 The simplest example: $n = 1$

Let A be an algebra. Then $\mathcal{FH}_1(A)$ is the field theory we expect: It sends S^1 considered as a 1-morphism from $\varnothing$ to $\varnothing$ to $\mathrm{HC}(A) = A \otimes_{A \otimes A^{op}} A$, the Hochschild chains of A. Note that we work in the derived setting, so tensor products are derived tensor products.

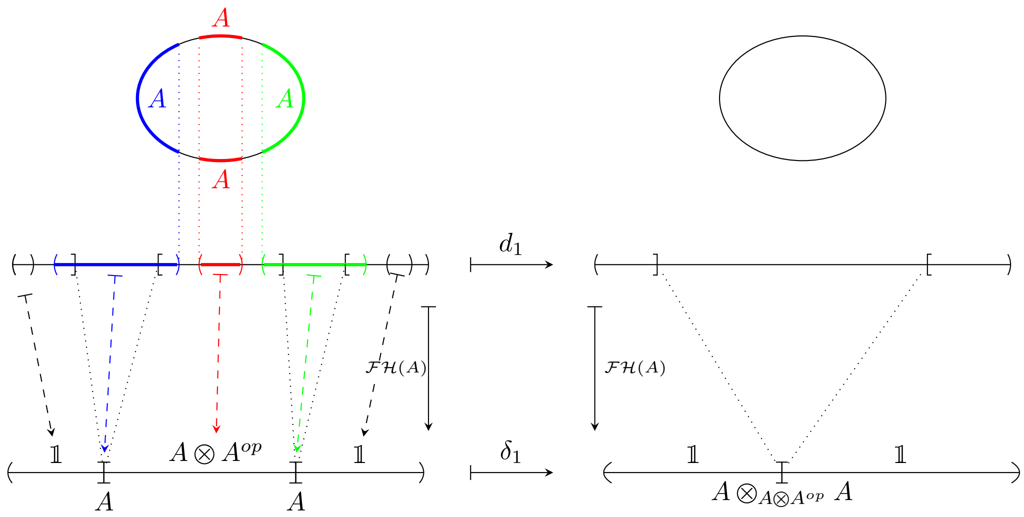
Consider the following element in $(\mathrm{Bord}_1)_2$, a pair of composable morphisms:



Taking factorization homology and pushing it to the base leads to the following factorization algebra on $(0, 1)$:



Thus pushing forward again along the collapse-and-rescale map $\rho_{\frac{b}{a}}$ gives the following diagram



A
A
A
A
d_1
δ_1
$\mathcal{FH}(A)$
$\mathcal{FH}(A)$
$\mathbb{1}$
$A \otimes A^{op}$
$\mathbb{1}$
A
A
$\mathbb{1}$
$\mathbb{1}$
$A \otimes_{A \otimes A^{op}} A$

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