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Rotation Quasimorphisms for Surfaces

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Zum Rumpumpelblitz und Dunnerchlapf isch das es Züügs!

Polderi

Abstract

Let S be an oriented hyperbolic surface with basepoint $x \in S$ and set $\Gamma = \pi_1(S, x)$. Let $\rho : \Gamma \to G = \mathrm{Isom}^+(\mathbb{D})$ be a holonomy representation into the group G of orientation preserving isometries of the Poincaré disc. The pullback $e_b^\Gamma \in \mathrm{H}_b^2(\Gamma)$ of the bounded Euler class $e_b \in \mathrm{H}_b^2(\mathrm{Homeo}^+(S^1))$ via the composition

$$\Gamma \xrightarrow{\rho} G \longrightarrow \mathrm{Homeo}^+(S^1)$$

is invariant under the action of the mapping class group $\mathcal{M}(S)$ of S. The latter group acts on the unit tangent bundle $\mathbb{T}^1 S$ of S by orientation preserving homeomorphisms and the pullback of e_b^Γ to the fundamental group $\pi_1(\mathbb{T}^1 S)$ is invariant under this lifted mapping class group action. We prove that this pulled back class is trivialised by a unique $\mathcal{M}(S)$-invariant homogeneous quasimorphism $\mathrm{Rot} : \pi_1(\mathbb{T}^1 S) \to \mathbb{R}$ which is moreover integral-valued and independent of the choice of hyperbolic metric on S.

For a closed regular curve c on S the integer $\mathrm{Rot}([c'])$ serves as a substitute for the classical rotation number for closed regular planar curves. In particular, we have the following analogon of Whitney's classification result for regular homotopy classes of planar curves: Two closed regular curves c_1, c_2 on S are regularly homotopic if and only if they are homotopic and, moreover, $\mathrm{Rot}([c_1']) = \mathrm{Rot}([c_2'])$. Chillingworth has introduced the related concept of winding numbers for curves and used it to prove an analogous result for non-compact surfaces. But in contrast to the quasimorphism Rot, the definition of the winding number functions involves the choice of a nowhere vanishing vector field on the surface and, for exactly this reason, is not well defined anymore for compact surfaces.

Zusammenfassung

Sei S eine orientierte hyperbolische Fläche, $x \in S$ ein Basispunkt und $\Gamma = \pi_1(S,x)$ deren Fundamentalgruppe. Wir betrachten eine Holonomiedarstellung $\rho : \Gamma \to G = \mathrm{Isom}^+(\mathbb{D})$ von Γ mit Werten in der Gruppe G der orientierungserhaltenden Homöomorphismen der hyperbolischen Ebene. Der Pullback $e_b^\Gamma \in \mathrm{H}_b^2(\Gamma)$ der beschränkten Eulerklasse $e_b \in \mathrm{H}_b^2(\mathrm{Homeo}^+(S^1))$ längs der Komposition

$$\Gamma \xrightarrow{\rho} G \longrightarrow \mathrm{Homeo}^+(S^1)$$

ist invariant unter der Operation der Abbildungsklassengruppe $\mathcal{M}(S)$ von S. Letztere operiert auf dem Einheitstangentialbündel $\mathbb{T}^1 S$ von S durch orientierungserhaltende Homöomorphismen, und der Pullback von e_b^Γ auf die Fundamentalgruppe $\pi_1(\mathbb{T}^1 S)$ ist invariant unter dieser Operation. Wir beweisen, dass diese Cohomologieklasse durch einen eindeutig bestimmten, $\mathcal{M}(S)$-invarianten homogenen Quasimorphismus $\mathrm{Rot} : \pi_1(\mathbb{T}^1 S) \to \mathbb{R}$ trivialisiert wird, welcher ausserdem auch noch ganzzahlig und unabhängig von der Wahl der hyperbolischen Metrik auf S ist.
Für reguläre geschlossene Kurven c auf S dient die Zahl $\mathrm{Rot}([c'])$ als Ersatz für die klassische Rotationszahl für ebene reguläre geschlossene Kurven. Insbesondere gilt folgendes Analogon von Whitneys Klassifikationsresultat für reguläre Homotopieklassen ebener Kurven: Zwei reguläre geschlossene Kurven c_1, c_2 auf S sind genau dann regulär homotop, wenn sie homotop sind und zusätzlich $\mathrm{Rot}([c_1']) = \mathrm{Rot}([c_2'])$ gilt. Chillingworth hat den verwandten Begriff der Windungszahl für Kurven eingeführt und mit dessen Hilfe ein analoges Resultat für nicht kompakte Flächen bewiesen. Im Gegensatz zum Quasimorphismus Rot fliesst in die Definition der Windungszahl allerdings die Wahl eines nirgends verschwindenden Vektorfeldes auf der Fläche ein. Und aus genau diesem Grund sind Windungszahlen im kompakten Fall nicht mehr wohldefiniert.

Contents

Introduction

Let S be an oriented surface of finite topological type such that the space $\mathrm{Hyp}(S)$ of complete hyperbolic metrics of finite volume is non-empty. We fix a universal covering $\tilde{S} \to S$, a base point $x \in S$ and set $\Gamma = \pi_1(S, x)$. Associated to every metric $h \in \mathrm{Hyp}(S)$ and every non-zero tangent vector of $\tilde{S}$ with base point lying above x there is a *holonomy representation*

$$\rho : \Gamma \to G$$

into the group $G = \mathrm{Isom}^+(\mathbb{D}) = \mathrm{PSU}(1,1)$ of orientation preserving isometries of the Poincaré disc. Observe that the induced action of G on the boundary $\partial\mathbb{D}$ gives an embedding $G \hookrightarrow \mathrm{Homeo}^+(S^1)$ into the group of orientation preserving homeomorphisms of the circle.

Consider the central $\mathbb{Z}$-extension

$$\langle t \rangle \rightarrowtail \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \twoheadrightarrow \mathrm{Homeo}^+(S^1)$$

where the group in the middle consists of orientation preserving homeomorphisms of the real line which commute with the integral translation $t : x \mapsto x + 1$. The cohomology class $e_{\mathbb{Z}} \in \mathrm{H}^2(\mathrm{Homeo}^+(S^1), \mathbb{Z})$ associated to this extension is called the *integral Euler class.* It is easy to represent $e_{\mathbb{Z}}$ by an explicit cocycle which only takes values in the set $\{0, 1\}$ and therefore also represents a bounded cohomology class $e_b \in \mathrm{H}^2_b(\mathrm{Homeo}^+(S^1))$, the *bounded Euler class.*

Consider the pullback $e_b^\Gamma = \rho^*(e_b) \in \mathrm{H}^2_b(\Gamma)$ of the bounded Euler class via a holonomy representation $\Gamma \xrightarrow{\rho} G \longrightarrow \mathrm{Homeo}^+(S^1)$. It is a non-trivial fact that e_b^Γ is independent of the choice of ρ and is moreover invariant under the outer action of the mapping class group $\mathcal{M}(S)$ of S on the fundamental group Γ.

Let's focus on the case of a compact surface S for the moment. Then we might as well consider the pullback $e_{\mathbb{Z}}^\Gamma = \rho^*(e_{\mathbb{Z}}) \in \mathrm{H}^2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$ of the integral Euler class via a holonomy representation (in the non-compact case Γ is a free group and so $\mathrm{H}^2(\Gamma, \mathbb{Z})$ is trivial). Again, as the notation suggests, $e_{\mathbb{Z}}^\Gamma$ is independent of the choice of ρ and is a $(2g - 2)$-fold multiple of a suitable generator in this group where g is the genus of the surface S. Now consider a central $\mathbb{Z}$-extension

$$\mathbb{Z} \rightarrowtail \overline{\Gamma} \overset{p}{\twoheadrightarrow} \Gamma$$

of Γ whose cohomology class in $\mathrm{H}^2(\Gamma, \mathbb{Z})$ equals $e_{\mathbb{Z}}^\Gamma$. Then the pullback $p^*(e_b^\Gamma)$ lies in the kernel of the comparison map

$$c : \mathrm{H}^2_b(\overline{\Gamma}) \to \mathrm{H}^2(\overline{\Gamma}, \mathbb{R})$$

from bounded to usual cohomology, and hence it is trivialised by a homogeneous quasimorphism $\varphi : \overline{\Gamma} \to \mathbb{R}$. As it turns out, the group $\overline{\Gamma}$ is isomorphic to the fundamental group $\pi_1(\mathbb{T}^1 S)$ of the unit tangent bundle of S (we omit basepoints from the notation of fundamental groups in what follows). There is an action of $\mathcal{M}(S)$ by orientation preserving homeomorphisms on $\mathbb{T}^1 S$ such that the induced outer action on $\pi_1(\mathbb{T}^1 S)$ lifts the outer action on $\pi_1(S)$. Indeed, this is true for all surfaces, not only compact ones. Clearly, the pullback $p^*(e_b^\Gamma) \in \mathrm{H}^2_b(\overline{\Gamma})$ is invariant under this lifted action since e_b^Γ is invariant.

Returning to the case of a general surface, it is natural to ask the following question motivated by the compact case: The pullback of the invariant class e_b^Γ to

$\mathrm{H}_b^2(\pi_1(\mathbb{T}^1 S))$ is invariant under the lifted mapping class group action and is trivialised by a homogeneous quasimorphism. *Is there an* invariant *homogeneous quasimorphism trivialising this class?* The first main objective of this thesis is to answer the above question affirmatively: There is a *unique* such quasimorphism $\mathrm{Rot} : \pi_1(\mathbb{T}^1 S) \to \mathbb{R}$ and it takes integral values.

We give two approaches to this problem. In Section 4 we restrict to the case of a compact surface and take a purely algebraic approach. The starting point of our consideration is the fact that the natural homomorphism $\mathcal{M}(S) \to \mathrm{Out}^+(\Gamma)$ is an isomorphism in the compact case by the Dehn-Nielsen-Baer theorem. The construction of the lifted mapping class group action by outer automorphisms on $\overline{\Gamma}$ reduces to a splitting problem and, ultimatively, to a cohomological vanishing result. We then derive the existence and uniqueness of an invariant trivialising quasimorphism using again cohomological methods.

Section 5 treats the case of a general surface using geometric methods. We first explicitely construct a lifted mapping class group action on the unit tangent bundle. Then we introduce simultaneous lifts of all holonomy representations and use these lifts to define Rot. As it turns out, the natural domain of definition of Rot is not a single fundamental group but rather the full fundamental groupoid $\pi_1(\mathbb{T}^1 S)$ or, at least, the subset of closed classes.

Our second main objective is to give a concrete description of the quasimorphism Rot in more classical terms. This is done in the final Section 6. We will see that Rot serves as an analogon of the notion of 'rotation number' of regular closed planar curves for regular closed curves on a hyperbolic surface. The rotation number of a regular closed planar curve c is defined as the number of times the tangent vector c' rotates in counter-clockwise direction as c is traversed once in positive direction. In order to give a similar definition for curves on more general surfaces one needs a reference frame with respect to which the rotation of the tangent vector c' can be measured. This has been carried out by Chillingworth and, indepenently, by Reinhart in the case of non-compact surfaces by choosing a nowhere vanishing vector field as reference frame. Fixing such a vector field, Chillingworth defines the notion of 'winding number' for curves and shows that it classifies regular homotopy classes: Two regular closed curves are regularly homotopic if and only if they are homotopic and have the same winding number. This generalises Whitney's classification result for regular homotopy classes of planar curves in terms of the rotation number.

For a regular closed curve c on a hyperbolic surface S we can interpret the homotopy class $[c']$ of its derivative as an element of $\pi_1(\mathbb{T}^1 S)$ and form the integer $\mathrm{Rot}([c'])$. In analogy to the above classification results we have: Two closed regular curves c_1, c_2 on S are regularly homotopic if and only if they are homotopic and, moreover, $\mathrm{Rot}([c_1']) = \mathrm{Rot}([c_2'])$. Hence we have an analogon of the rotation respectively winding number in the planar respectively non-compact setting which is intrinsically defined and does not dependent on the choice of a global reference frame. In particular, it is available for compact surfaces of genus $g \geq 2$ as well. We will also clarify the connection between the quasimorphism Rot and Chillingworth's winding number functions. In fact, Rot encodes the information of all winding number functions simultaneously.

Finally, we discuss the quasimorphism rot defined on the fundamental group $\pi_1(S)$ of a non-compact surface introduced by Calegari. It turns out that rot is a specific winding number function which clarifies the connection to Rot. We compare a formula describing rot obtained by Calegari with certain combinatorial formulas which we will derive for Rot.

1. Preliminaries

This section serves as a reference for some basic results that will be needed later on. At the same time we use the opportunity to fix notations and concepts.

1.1. **Coverings and Topological Groups.** Let $p : Y \to X$ be a covering map of connected spaces. Recall that a *covering transformation* is a homeomorphism $T : Y \to Y$ that descends to the identity on X. Either T is the identity or T has no fixed point. A covering p is called *regular* (or *Galois*) if the group of covering transformations acts transitively on each fibre of p. A covering p is *universal* if Y is simply connected. A topological space X admits a universal covering $\tilde{X} \to X$ if and only if it is *sufficiently connected* by which we mean that it is path-connected, locally path-connected and semi-locally simply connected.
Fix a base point $x \in X$ and choose a base point $y \in Y$ lying above x, i.e., $p(y) = x$. Every curve $c : I \to X$ starting at x has a unique lift to Y starting at y. If c_1 and c_2 are homotopic (rel {0,1}) and start at x then their lifts with starting point y are homotopic as well, in particular, they have the same end point. Hence if p is regular then for every class $\gamma \in \pi_1(X, x)$ and every point $y \in Y$ lying above x there is a unique covering transformation T^y_γ such that the lift of every curve representing γ to Y with starting point y has endpoint $T^y_\gamma(y)$. For fixed y the map $\gamma \mapsto T^y_\gamma$ from $\pi_1(X, x)$ to the group of covering transformations of p is a group homomorphism. When p is universal then it is an isomorphism identifying the group of covering transformations with the fundamental group $\pi_1(X, x)$.

There is a standard construction of a universal covering for sufficiently connected spaces. But since we will only need it for topological groups we restrict ourselves to this case where much more structure is available. Let G be a topological group with neutral element e. We assume that G is sufficiently connected and locally compact. For any two curves $u : I \to G$ and $v : I \to G$ we can form their pointwise product $t \mapsto u(t)v(t)$ which we denote by $u \cdot v$ for short. If we equip the space $\mathcal{G}$ of all curves in G with the compact-open topology (cf. Subsection 1.5) the binary operation $\cdot$ induces a structure of topological group on $\mathcal{G}$ with neutral element the class of the constant path $\tilde{e} : t \mapsto e$. Moreover, if u, u' are homotopic (rel $\{0, 1\}$) via the homotopy g and v, v' are homotopic (rel $\{0, 1\}$) via the homotopy h then

$$I \times I \to G, \qquad (t, s) \mapsto g(t, s)h(t, s)$$

is a homotopy (rel $\{0, 1\}$) from $u \cdot v$ to $u' \cdot v'$. From this it is easy to deduce that $\cdot$ descends to a topological group structure on the fundamental groupoid $\pi_1(G)$ of G equipped with the quotient topology. We point out that this group structure differs from the groupoid structure given by concatenation. A slightly different point of view is the following. The set $\mathcal{N}$ of all closed nullhomotopic paths in G based at e is a closed normal subgroup of $\mathcal{G}$ and we have $\pi_1(G) = \mathcal{G} / \mathcal{N}$.

The set $\tilde{G}$ of all classes of paths starting at e is a closed normal subgroup of $\pi_1(G)$. The map $p : \tilde{G} \to G$ sending a class of paths to their common endpoint in G is a continuous surjective group homomorphism with kernel the fundamental group $\pi_1(G, e)$. In fact we have:

Lemma 1.1. *The group $\tilde{G}$ is connected and simply connected and p is a universal covering map.*

For a topological group G there are two 'multiplications' available for homotopy classes (rel $\{0, 1\}$) of curves in G. On the one hand there is concatenation of composable paths which we denote by $*$ for the moment. Then there is the pointwise multiplication $\cdot$ discussed above. They are compatible in the following sense:

Lemma 1.2. *Let a, b, c, d be curves in G such that a and b are composable as well as c and d. Then*

$$(a * b) \cdot (c * d) \simeq (a \cdot c) * (b \cdot d) \quad (\text{rel } \{0, 1\}).$$

It is well known and easy to check that, as a consequence, the restrictions of $\cdot$ and $*$ to the fundamental group $\pi_1(G, e)$ agree and are both commutative.

1.2. **On Surfaces.** In this very short subsection we introduce some conventions concerning surfaces that will stay in effect throughout this thesis. By a *surface* S we always mean a connected, two-dimensional differentiable manifold (without boundary) which is orientable and *oriented*, i.e., equipped with a fixed orientation. Moreover, we focus on *topologically finite* surfaces, that is, we require the fundamental group $\pi_1(S)$ to be finitely generated. For such a surface we denote by $\mathrm{Hyp}(S)$ the set of complete hyperbolic metrics with *finite* volume. We always require that $\mathrm{Hyp}(S)$ is *non-empty*. In particular, if S is compact then its genus is ≥ 2.

1.3. **Definition of Unit Tangent Bundles.** Let M be a smooth manifold of dimension m. The *tangent bundle* of M is the union of all tangent spaces of M, formally we set

$$TM = \bigcup_{x \in M} \{x\} \times TM_x.$$

When there is no risk of confusion, a typical element $(x, v) \in TM$ will just be denoted by v and we will say that v is *based* at x or that x is the *basepoint* of v. It is well known that the differentials of the local charts in any smooth atlas of M give a smooth atlas of TM and that the basepoint map $TM \to M$ is an m-dimensional vector bundle.

Choose a riemannian metric g on M. This choice allows to define unit tangent spaces

$$T^{1,g}M_x = \{v \in TM_x \mid ||v||_g = 1\}$$

as well as the unit tangent bundle

$$T^{1,g}M = \bigcup_{x \in M} \{x\} \times T^{1,g}M_x \subset TM.$$

The unit tangent bundle is a codimension 1 submanifold of TM and is also an S^{m-1}-bundle over M via the basepoint map.

Although the definitions above depend on the choice of a metric g, the diffeomorphy type of $T^{1,g}M$ does not. Hence we are going to define a metric-independent version of the unit tangent bundle by 'semiprojectivising' the tangent spaces of M. For every $x \in M$ the group $\mathbb{R}^+$ acts on $TM_x \setminus \{0\}$ by scalar multiplication and the orbit space

$$\mathbb{T}^1 M_x = \big(TM_x \setminus \{0\}\big) / \mathbb{R}^+$$

can be interpreted as the space of rays in TM_x emanating from the origin. Similar as above the union

$$\mathbb{T}^1 M = \bigcup_{x \in M} \{x\} \times \mathbb{T}^1 M_x$$

has a natural structure of smooth manifold and of an S^{m-1}-bundle over M. For any metric g the canonical map

$$T^{1,g}M \to \mathbb{T}^1 M, \qquad (x, v) \mapsto (x, [v])$$

is clearly smooth and is a sphere bundle isomorphism.

1.4. **Coverings of Unit Tangent Bundles.** Let M be a smooth manifold and let $\Gamma < \mathrm{Diff}(M)$ act freely and properly discontinuously on M. Then the canonical map $p : M \to \Gamma\backslash M$ is a regular covering with group of covering transformations Γ. Equip the space $\Gamma\backslash M$ with the unique manifold structure such that p is a local diffeomorphism. Via pullback the projection p induces a bijection

$$\mathrm{Riem}(\Gamma\backslash M) \to \mathrm{Riem}(M)^\Gamma$$

between the corresponding spaces of (Γ-invariant) Riemannian metrics. Fix $g \in \mathrm{Riem}(\Gamma\backslash M)$ and set $\tilde{g} = p^*(g)$. Then p is a local isometry and hence the differential

$$dp : T^{1,\tilde{g}}(M) \to T^{1,g}(\Gamma\backslash M)$$

gives a local diffeomorphism between the unit tangent bundles. On the other hand, by definition of $\tilde{g}$, we have $\Gamma < \mathrm{Isom}(M, \tilde{g})$ and hence there is an action of Γ on $T^{1,\tilde{g}}(M)$ given by the differentials $\gamma \cdot v = d\gamma(v)$.

Lemma 1.3. *The map dp is a regular covering with Γ as group of covering transformations. Hence dp induces a diffeomorphism*

$$q : \Gamma\backslash T^1(M) \xrightarrow{\cong} T^1(\Gamma\backslash M).$$

Proof. Fix a point $(v, \overline{x}) \in T^1(\Gamma\backslash M)$ and choose an open neighbourhood U of $\overline{x}$ in $\Gamma\backslash M$ and a diffeomorphism $\psi : p^{-1}(U) \to \Gamma \times U$ such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "p"'] & & \Gamma \times U \arrow[dl, "\pi_2"] \\ & U & \end{tikzcd}$$

commutes. If we equip $\Gamma \times U$ with the pulled back metric $\pi_2^*(g)$ then ψ is isometric. Hence we can take the differentials of all maps in the above diagram and obtain again a commutative diagram

$$\begin{tikzcd} (dp)^{-1}(T^1(U)) \arrow[rr, "d\psi"] \arrow[dr, "dp"'] & & \Gamma \times T^1(U) \arrow[dl, "\pi_2"] \\ & T^1(U) & \end{tikzcd}$$

where, strictly speaking, the map on the right is the differential of π_2 but this is still just the projection to the second factor. As $T^1(U)$ is an open neighbourhood of $(v, \overline{x})$ in $T^1(\Gamma\backslash M)$ this shows that $dp : T^1(M) \to T^1(\Gamma\backslash M)$ is indeed a covering with fibre Γ. But Γ clearly acts transitively on the fibres of dp (via the differentials) and hence dp is a regular covering with Γ as group of covering transformations. $\square$

1.5. **The Compact-Open Topology.** We collect some general facts about compact-open topologies. Recall the definition of the *compact-open topology* on the set $C(X,Y)$ of continuous maps from X to Y: For $K \subset X$ and $U \subset Y$ define

$$D(K,U) = \{f \in C(X,Y) \mid f(K) \subset U\},$$

then the compact open topology has the familiy $\{D(K,U) \mid K \text{ compact}, U \text{ open}\}$ as a subbasis.

For a one-point space $X = *$ we can identify $C(*,Y)$ with Y and the compact open topology on the first space agrees with the topology on Y. A very useful result is the following:

Lemma 1.4. *Let X, Y, Z be topological spaces and let $f : X \times Y \to Z$ be a map. If f is continuous then so is the map $X \to C(Y,Z)$, $x \mapsto f(x,\cdot)$ where $C(Y,Z)$ is equipped with the compact-open topology. If Y is locally compact then the reverse implication is also true.*

For this reason we will assume that *all spaces in question are locally compact.* With this assumption the evaluation map

$$C(X,Y) \times X \to Y, \qquad (f,x) \mapsto f(x)$$

is continuous and, in fact, the compact open topology is the coarsest topology on $C(X,Y)$ such that it is continuous. Also the composition operation

$$\circ : C(Y,Z) \times C(X,Y) \to C(X,Z)$$

is continuous. Hence if X is locally compact then the group $\mathrm{Homeo}(X)$ equipped with the compact-open topology is a topological group.

Another important consequence of Lemma 1.4 is that, in the locally compact setting, there is a bijection between homotopies $H : I \times X \to Y$ between maps $f, g \in C(X,Y)$ and paths $h : I \to C(X,Y)$, $t \mapsto H(t,\cdot)$ connecting f and g. Hence the homotopy classes of maps in $C(X,Y)$ correspond to the path connected components of $C(X,Y)$:

$$[X,Y] = \pi_0\big(C(X,Y)\big).$$

Similar statements hold for homotopies with restrictions. For example a homotopy relative to $\{0,1\}$ between curves $w_i : I \to X$ having the same starting and end point is the same thing as a path $I \to C(I,X)$ which takes values in the subspace $\{f \in C(I,X) \mid f(0) = w_i(0), f(1) = w_i(1)\}$.

Finally, for later use we record:

Lemma 1.5. *If Y is totally disconnected, then so is $C(X,Y)$. In particular: For discrete X the space* $\mathrm{Homeo}(X)$ *is totally disconnected.*

Proof. Let $f, g \in C(X,Y)$ be different. Choose $x \in X$ with $f(x) \neq g(x)$ and choose open sets U, V in Y such that

$$U \cap V = \emptyset, \quad U \cup V = Y, \quad f(x) \in U, \quad g(x) \in V.$$

The open sets $U' = D(x,U)$ and $V' = D(x,V)$ in $C(X,Y)$ then give a similar decomposition, i.e.,

$$U' \cap V' = \emptyset, \quad U' \cup V' = C(X,Y), \quad f \in U', \quad g \in V'.$$

□

1.6. **Isometries of the Poincaré Disc.** We collect some basic facts from hyperbolic geometry for later reference. The *Poincaré disc* is the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ endowed with the Poincaré metric

$$h = \frac{4|dz|^2}{(1-|z|^2)^2}$$

of constant curvature -1. The group $\mathrm{SU}(1,1)$ consisting of all complex matrices of the form

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \qquad \text{with} \qquad |a|^2 - |b|^2 = 1$$

acts on $\mathbb{D}$ by fractional linear transformations, i.e., by

$$gz = \frac{az+b}{\bar{b}z+\bar{a}}.$$

This action is orientation preserving and isometric with kernel $\{\pm I\}$. Hence we get an action of the quotient $G = \mathrm{PSU}(1,1) = \mathrm{SU}(1,1)/\{\pm I\}$ on $\mathbb{D}$.

Lemma 1.6. *The map $G \to \mathrm{Isom}^+(\mathbb{D})$ to the group of orientation preserving isometries of $\mathbb{D}$ is an isomorphism.*

We use the notation

$$g = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$$

for a typical element of G where the square brackets indicate that we mean the residue class of the corresponding matrix in $\mathrm{SU}(1,1)$ and where we have the determinant condition $|a|^2 - |b|^2 = 1$. The unit element of G is denoted by e in the sequel. An element $g \neq e$ is called *elliptic*, *parabolic* or *hyperbolic* depending on whether $|\operatorname{tr}(g)|$ is less than, equal to or bigger than 2. An elliptic element has a unique fixed point on $\mathbb{D}$, a parabolic one has a unique fixed point on the boundary $\partial\mathbb{D}$ and a hyperbolic element has a pair of fixed points on the boundary $\partial\mathbb{D}$. We define the following three subgroups of G:

$$\begin{aligned} K &= \left\{ \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \,\middle|\, \zeta \in S^1 \right\} \\ A &= \left\{ \begin{bmatrix} \sqrt{1+t^2} & t \\ t & \sqrt{1+t^2} \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\} \\ N &= \left\{ \begin{bmatrix} 1+iy & -iy \\ iy & 1-iy \end{bmatrix} \,\middle|\, y \in \mathbb{R} \right\} \end{aligned}$$

Then every elliptic, parabolic respectively hyperbolic element is conjugated within G to an element in the group K, A respectively N. The subgroup K is maximal compact and isomorphic to $S^1/\{\pm 1\} \cong S^1$. It is the stabiliser of the point $0 \in \mathbb{D}$. The groups A and N are both isomorphic to $\mathbb{R}$ and their product $B = AN$ acts simply transitively on $\mathbb{D}$.

Since the action of G on $\mathbb{D}$ is isometric we have an induced action on the unit tangent bundle $T^1\mathbb{D}$ by the differentials, explicitely given by

$$g(z,v) = (gz, dg_z(v)) = \Big(\frac{az+b}{\bar{b}z+\bar{a}}, \frac{v}{(\bar{b}z+\bar{a})^2} \Big).$$

The above discussion shows:

Lemma 1.7. *The action of G on $T^1\mathbb{D}$ is simply transitive, hence the map*

$$G \xrightarrow{\cong} T^1\mathbb{D}, \qquad g \mapsto g(0,1)$$

is a homeomorphism. The left coset $gK \subset G$ is mapped homeomorphically to the fibre over the point $g(0,1)$.

2. Quasimorphisms

This section is about quasimorphisms on groups (and groupoids). We start by recalling the definition and basic properties of quasimorphisms, then discuss their connection to bounded cohomology. Next we compare the two seminorms on the space of homogeneous quasimorphisms given by the defect and by the pullback of the norm on H^2_b (the so-called Gromov norm). In Subsections 2.3 and 2.4 we present new results concerning these seminorms which make use of the heavy machinery introduced to the theory of bounded cohomology by Burger and Monod. In particular we use the amenable and doubly ergodic action on the poisson boundary of random walks on groups. In Subsection 2.5 we briefly review the translation quasimorphism, the rotation number and the Euler class which will play a crucial role in the rest of this work.

2.1. **Basic Properties.** Let Γ be a group with neutral element e. A *quasimorphism* on Γ is a map $\varphi : \Gamma \to \mathbb{R}$ that behaves like a homomorphism up to finite error. More precisely, we require that its *defect*

$$D(\varphi) = \sup_{x,y\in\Gamma} |\varphi(xy) - \varphi(x) - \varphi(y)|$$

is finite. An easy induction shows that for any n we have an inequality

$$\Big|\varphi(x_1\cdots x_n) - \sum_{k=1}^{n}\varphi(x_k)\Big| \leq (n-1)D(\varphi). \tag{1}$$

We shall say that two maps $\varphi_1, \varphi_2 : \Gamma \to \mathbb{R}$ are at finite distance if they are with respect to the sup-norm, i.e., if $||\varphi_1 - \varphi_2||_\infty$ is finite. Every map at finite distance of a quasimorphism is a quasimorphism as well.

We call a quasimorphism φ *homogeneous* if $\varphi(x^n) = n\varphi(x)$ for all integers n and all $x \in \Gamma$. To every quasimorphism φ we can associate its *homogenisation* $\overline{\varphi}$ defined by

$$\overline{\varphi}(x) = \lim_{n\to\infty}\frac{\varphi(x^n)}{n}.$$

Lemma 2.1. *The above limit exists for all $x \in \Gamma$. The map $\overline{\varphi}$ is a homogeneous quasimorphism at finite distance of φ, in fact $||\overline{\varphi} - \varphi||_\infty \leq D(\varphi)$.*

Proof. By (1) we have

$$\begin{aligned} |m\varphi(x^n) - n\varphi(x^m)| &\leq |m\varphi(x^n) - \varphi(x^{mn})| + |n\varphi(x^m) - \varphi(x^{mn})| \\ &\leq (m-1)D(\varphi) + (n-1)D(\varphi) \end{aligned}$$

for all $m, n \geq 1$. Dividing by mn gives

$$\Big|\frac{\varphi(x^n)}{n} - \frac{\varphi(x^m)}{m}\Big| \leq \Big(\frac{1}{n} + \frac{1}{m} - \frac{2}{mn}\Big)D(\varphi),$$

hence $\varphi(x^n)/n$ is a Cauchy sequence and the limit indeed exists. Specialising the first inequality in the proof to the case $m = 1$ gives

$$|\varphi(x^n) - n\varphi(x)| \leq (n-1)D(\varphi).$$

Dividing by n and passing to the limit $(n \to \infty)$ shows $|\overline{\varphi}(x) - \varphi(x)| \leq D(\varphi)$. Since this holds for all $x \in \Gamma$ we have $||\overline{\varphi} - \varphi||_\infty \leq D(\varphi)$. Finally, $\overline{\varphi}$ is homogeneous because

$$\overline{\varphi}(x^m) = \lim_{n\to\infty}\frac{\varphi(x^{mn})}{n} = m\lim_{n\to\infty}\frac{\varphi(x^{mn})}{mn} = m\overline{\varphi}(x).$$

$\square$

Denote by $Q(\Gamma)$ respectively $Q^h(\Gamma)$ the spaces of all quasimorphisms respectively homogeneous quasimorphisms on Γ. We call two quasimorphisms *equivalent* if they are at finite distance.

Corollary 2.2. *The homogenisation operator* $\overline{(\cdot)} : Q(\Gamma) \to Q(\Gamma)$ *is a projection with kernel* $l^\infty(\Gamma)$ *and image* $Q^h(\Gamma)$. *As a consequence, every equivalence class of quasimorphisms contains exactly one homogeneous one, and we have the decomposition*

$$Q(\Gamma) = l^\infty(\Gamma) \oplus Q^h(\Gamma)$$

with respect to which $\overline{(\cdot)}$ *is just the projection on the second factor.*

Proof. On the one hand, we obviously have $\overline{\varphi} = \varphi$ in case φ is already homogeneous, hence $\overline{(\cdot)}$ is a projection on its image $Q^h(\Gamma)$. On the other hand, $\overline{\varphi} = \overline{\psi}$ if and only if φ and ψ are equivalent. Indeed, the if part is clear and the only if part follows from Lemma 2.1 since

$$||\varphi - \psi||_\infty \leq ||\overline{\varphi} - \varphi||_\infty + ||\overline{\psi} - \psi||_\infty \leq D(\varphi) + D(\psi).$$

□

Next, we derive two results concerning the defect of a homogeneous quasimorphism. To do so, we need the following observation concerning commutators in groups.

Proposition 2.3. *The element* $x^{-2n}y^{-2n}(xy)^{2n}$ *can be written as a product of* n *commutators.*

Proof. See [2] Lemma 3.6. Actually this is a special case of a much more general result (compare [32] Lemma 2.1): Let $x_1, \ldots, x_{2n}$ be arbitrary elements and assume that z can be written as a product containing each x_i and each inverse x_i^{-1} exactly once as a factor, then z is a product of n commutators. □

Lemma 2.4. *Let* $\overline{\varphi}$ *denote the homogenisation of* φ. *Then* $D(\overline{\varphi}) \leq 2D(\varphi)$.

Proof. Define $\varphi'(x) = \frac{1}{2}\big(\varphi(x) - \varphi(x^{-1})\big)$ and observe that $||\varphi' - \varphi|| \leq D(\varphi)$. So φ' is a quasimorphism whose homogenisation agrees with $\overline{\varphi}$. Moreover we have $D(\varphi') \leq D(\varphi)$, hence for the proof we may replace φ by φ' and assume in what follows that $\varphi(x^{-1}) = -\varphi(x)$ for all $x \in \Gamma$ (i.e., we may assume that φ is *antisymmetric*).
We shall first derive bounds for the values of an antisymmetric quasimorphism on product of commutators. For a single commutator we have

$$|\varphi([x,y])| \leq \underbrace{|\varphi(x) + \varphi(y) + \varphi(x^{-1}) + \varphi(y^{-1})|}_{=0} + 3D(\varphi) = 3D(\varphi),$$

and if c is a product of n commutators then an induction shows that

$$|\varphi(c)| \leq (4n-1)D(\varphi). \tag{2}$$

Now by Proposition 2.3 and (2) we obtain

$$\begin{aligned} |\varphi((xy)^{2n}) - \varphi(x^{2n}) - \varphi(y^{2n})| &\leq |\varphi(x^{-2n}y^{-2n}(xy)^{2n})| + 2D(\varphi) \\ &\leq (4n-1)D(\varphi) + 2D(\varphi) = (4n+1)D(\varphi) \end{aligned}$$

for every positive integer n. Dividing this by $2n$ and passing to the limit ($n \to \infty$) gives

$$|\overline{\varphi}(xy) - \overline{\varphi}(x) - \overline{\varphi}(y)| \leq 2D(\varphi),$$

and the claim follows by taking the supremum over all $x, y \in \Gamma$. □

Lemma 2.5. *Every homogeneous quasimorphism is invariant under conjugation.*

Proof. Let φ be homogeneous. Then for all $x, y \in \Gamma$ and all positive n we have

$$n|\varphi(yxy^{-1}) - \varphi(x)| = |\varphi(yx^n y^{-1}) - \varphi(x^n)| \leq 2D(\varphi)$$

by (1). Dividing by n and passing to the limit ($n \to \infty$) gives the claim. $\square$

It is a remarkable fact that the defect of a homogeneous quasimorphism is encoded in its values on commutators.

Lemma 2.6. *If φ is homogeneous then*

$$D(\varphi) = \sup_{x,y\in\Gamma} |\varphi([x,y])|.$$

Proof. First notice that

$$|\varphi([x,y])| \leq |\varphi(x) + \varphi(yx^{-1}y^{-1})| + D(\varphi) = D(\varphi)$$

because φ is invariant under conjugation and antisymmetric. To prove the reversed inequality we can proceed similarly as in the proof of Lemma 2.4. Set

$$s = \sup_{x,y\in\Gamma} |\varphi([x,y])|.$$

For any positive integer n we obtain by Proposition 2.3:

$$\begin{aligned} 2n \cdot |\varphi(xy) - \varphi(x) - \varphi(y)| &= |\varphi((xy)^{2n}) - \varphi(x^{2n}) - \varphi(y^{2n})| \\ &\leq |\varphi(x^{-2n}y^{-2n}(xy)^{2n})| + 2D(\varphi) \\ &\leq (n-1)D(\varphi) + ns + 2D(\varphi) \\ &= (n+1)D(\varphi) + ns. \end{aligned}$$

Dividing this by $2n$ and passing to the limit ($n \to \infty$) gives

$$|\varphi(xy) - \varphi(x) - \varphi(y)| \leq \frac{1}{2}(D(\varphi) + s).$$

Taking the supremum over all $x, y \in \Gamma$ finally shows that $s \geq D(\varphi)$. $\square$

Homogeneous quasimorphisms have many more pleasant properties besides the ones described above. We list two of them for convenience, the easy proofs are all based on a variant of the limit argument in the proof of Lemma 2.6 and will be omitted. Let φ be a homogeneous quasimorphism on Γ.

Lemma 2.7. *If two elements x, y commute then $\varphi(xy) = \varphi(x) + \varphi(y)$. In particular, every homogeneous quasimorphism on an abelian group is a homomorphism.*

There is also an analogon to the homomorphism theorem of group theory:

Lemma 2.8. *Let $f : \Gamma \to \Lambda$ be a surjective homomorphism. Then there exists a quasimorphism ψ on Λ such that the diagram*

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi"] \arrow[d, "f"'] & \mathbb{R} \\ \Lambda \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

commutes if and only if φ vanishes identically on $\ker(f)$. *If this is the case then ψ is unique, homogeneous and $D(\varphi) = D(\psi)$.*

2.2. **Quasimorphisms and Bounded Cohomology.** In this subsection we discuss the link between quasimorphisms and the second bounded cohomology of a group. In analogy to usual group cohomology we consider the (inhomogeneous) bar resolution of Γ but restrict to bounded maps, i.e., consider the resolution

$$0 \longrightarrow l^\infty(\Gamma^0) \xrightarrow{d^0} l^\infty(\Gamma) \xrightarrow{d^1} l^\infty(\Gamma^2) \xrightarrow{d^2} l^\infty(\Gamma^3) \xrightarrow{d^3} \cdots$$

where the d^n are the usual inhomogeneous coboundary operators given by

$$\begin{aligned}(d^n\alpha)(x_0,\dots,x_n) &= \alpha(x_1,\dots,x_n)\\ &+ \sum_{k=1}^{n}(-1)^k\alpha(x_0,\dots,x_{k-2},x_{k-1}x_k,x_{k+1},\dots,x_n)\\ &+ (-1)^{n+1}\alpha(x_0,\dots,x_{n-1}).\end{aligned}$$

Then the bounded cohomology groups of Γ with real coefficients are defined by

$$\mathrm{H}^n_b(\Gamma,\mathbb{R}) = \ker(d^n)/\operatorname{Im}(d^{n-1}),$$

equipped with the quotient seminorm. We have chosen this very naive definition since, except for the next two subsections, it is the most appropriate one for our purposes. However, there is an extensive theory behind this cohomology. Originally introduced by Gromov, later on Burger and Monod developed a framework to define and study (continuous) bounded cohomology for arbitrary topological groups and a large class of coefficients. Their work includes in particular strong methods to compute $\mathrm{H}^\bullet_{cb}$ via a variety of resolutions of the coefficient space. A crucial role is played by the seminorm whose correct computation is guaranteed by additional assumptions on the resolution. For an introduction to bounded cohomology we refer the reader to [30] or [9]. There is also a modern approach to bounded cohomology via derived functors, see Bühler [7], giving the whole theory a more classical foundation. The essential difference in the derived functor approach is that the spaces $\mathrm{H}^\bullet_{cb}$ are only recovered up to isomorphism and not up to *isometry* of seminormed spaces. Hence the 'canonical' seminorm remains somewhat mysterious.
We will now focus on the second real bounded cohomology $\mathrm{H}^2_b(\Gamma) = \mathrm{H}^2_b(\Gamma,\mathbb{R})$ for discrete groups Γ since this is all we will need in the sequel. For these spaces Ivanov proved that the canonical seminorm is actually a *norm*, see [16].

Let φ be a quasimorphism on Γ. Then $d\varphi(x,y) = \varphi(x) - \varphi(xy) + \varphi(y)$ is an inhomogeneous 2-cocycle on Γ which is bounded since $||d\varphi||_\infty = D(\varphi) < \infty$ by definition. Hence it defines a class $[d\varphi] \in \mathrm{H}^2_b(\Gamma)$. The map $\varphi \mapsto [d\varphi]$ fits into a four-term exact sequence as follows.

Lemma 2.9. *The following diagram is commutative and has exact rows*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(\Gamma,\mathbb{R}) \oplus l^\infty(\Gamma) \arrow[r] \arrow[d, "\pi_1"] & Q(\Gamma) \arrow[r, "{[d\cdot]}"] \arrow[d, "\overline{(\cdot)}"] & \mathrm{H}^2_b(\Gamma,\mathbb{R}) \arrow[r, "c"] \arrow[d, equal] & \mathrm{H}^2(\Gamma,\mathbb{R}) \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{Hom}(\Gamma,\mathbb{R}) \arrow[r] & Q^h(\Gamma) \arrow[r, "{[d\cdot]}"] & \mathrm{H}^2_b(\Gamma,\mathbb{R}) \arrow[r, "c"] & \mathrm{H}^2(\Gamma,\mathbb{R}) \end{tikzcd}$$

where the so-called comparison map *c is induced by forgetting the boundedness of an inhomogeneous 2-cocycle.*

Proof. The commutativity of the diagram is clear by Corollary 2.2. We shall prove exactness of the upper row, the argument for the lower row being similar. The class of a 2-cocycle $\beta \in l^\infty(\Gamma^2)$ lies in the kernel of the comparison map if and only if there exists $\varphi : \Gamma \to \mathbb{R}$ with $\beta = d\varphi$. Since β is bounded, φ has to be a quasimorphism. Next, the class $[d\varphi] \in \mathrm{H}^2_b(\Gamma)$ is trivial if and only if there exists

$\alpha \in l^\infty(\Gamma)$ with $d(\varphi - \alpha) = 0$. Hence the quasimorphism φ is the sum of a bounded function and a homomorphism on Γ. $\square$

Denote by $|| \cdot ||_b$ the pullback of the canonical norm on $\mathrm{H}^2_b(\Gamma)$ to the space $Q(\Gamma)$. Then we have two seminorms on $Q(\Gamma)$ given by

$$\begin{aligned} D(\varphi) &= ||d\varphi||_\infty, \\ ||\varphi||_b &= \inf_{\alpha \in l^\infty(\Gamma)} ||d\varphi + d\alpha||_\infty. \end{aligned}$$

Corollary 2.10. *The restrictions of these two seminorms to $Q^h(\Gamma)$ are equivalent. More precisely, for all $\varphi \in Q^h(\Gamma)$ there is the estimate*

$$||\varphi||_b \leq D(\varphi) \leq 2||\varphi||_b.$$

Proof. The first inequality is immediate from the definitions. On the other hand, $\overline{\alpha}$ vanishes identically for any bounded function α and hence $\overline{\varphi + \alpha} = \varphi$. Now Lemma 2.4 implies

$$||d\varphi||_\infty \leq 2||d(\varphi + \alpha)||_\infty$$

for all α which gives the second inequality. $\square$

Proposition 2.11. *Let Γ be countable. Then $Q^h(\Gamma)$ is complete (with respect to either seminorm). Hence the quotient $Q^h(\Gamma)/\operatorname{Hom}(\Gamma, \mathbb{R})$ is a Banach space.*

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to the defect seminorm. We proceed in steps.
Step 1. We claim that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of homomorphisms $f_n : \Gamma \to \mathbb{R}$ such that the family of quasimorphisms $\psi_n = \varphi_n - f_n$ takes bounded values on any fixed element of Γ. More precisely, for every $x \in \Gamma$ there exists a constant C_x such that

$$|\psi_n(x)| \leq C_x \qquad \forall n \geq 1.$$

For the proof, consider the composition of homomorphisms $\alpha : \Gamma \to \Gamma_{ab} \to \Gamma_{ab} \otimes_\mathbb{Z} \mathbb{Q}$. Choose a set of elements $\{y_1, y_2, \ldots\}$ in Γ whose images $z_i = \alpha(y_i)$ form a basis of the $\mathbb{Q}$-vector space $\Gamma_{ab} \otimes_\mathbb{Z} \mathbb{Q}$. For $n \geq 1$ let $g_n : \Gamma_{ab} \otimes_\mathbb{Z} \mathbb{Q} \to \mathbb{R}$ be the unique $\mathbb{Q}$-linear map such that $g_n(z_i) = \varphi_n(y_i)$ for all i. We will prove that the homomorphisms $f_n = g_n \circ \alpha : \Gamma \to \mathbb{R}$ will do the job.
Fix an element $x \in \Gamma$. We claim that there exists a positive integer m, integers $a_1, a_2, \ldots$ almost all of which are zero and an element $z \in \Gamma'$ such that

$$x^m = \prod_i y_i^{a_i} \cdot z. \tag{3}$$

To show this, write $\alpha(x) = \sum_i r_i z_i$ as rational linear combination with $r_i = 0$ for almost all i. For an appropriate positive integer m' we therefore have $\alpha(x^{m'}) = \sum_i b_i z_i$ with all b_i integral. The image of the element $\prod_i y_i^{-b_i} \cdot x^{m'}$ in Γ_{ab} lies in the kernel of the map $\Gamma_{ab} \to \Gamma_{ab} \otimes_\mathbb{Z} \mathbb{Q}$ which is torsion. Hence there is a positive integer m'' such that $z = \prod_i y_i^{-m'' b_i} \cdot x^{m'm''} \in \Gamma'$. Now set $m = m'm''$ and $a_i = m'' b_i$.
Assume that z is a product of k commutators and that exactly s of the exponents a_i in (3) are non-zero. Then by (1) we have

$$m|(\varphi_n - f_n)(x)| \leq \sum_i a_i |\varphi_n(y_i) - f_n(y_i)| + |\varphi_n(z)| + |f_n(z)| + sD(\varphi_n).$$

Here all terms in the first sum vanish by the choice of f_n, moreover $f_n(z) = 0$ since $z \in \Gamma'$ and finally $|\varphi_n(z)| \leq (2k-1)D(\varphi_n)$ by Lemma 2.6. In summary we have

$$|\psi_n(x)| \leq \frac{2k + s - 1}{m} D(\varphi_n).$$

Since every Cauchy sequence is bounded, the defects $D(\varphi_n)$ are bounded independently of n and this proves the claim.
Step 2. There exists a subsequence $(\psi_{n_k})_{k\in\mathbb{N}}$ such that for all $x \in \Gamma$ the limit

$$\psi(x) = \lim_{k\to\infty} \psi_{n_k}(x)$$

exists. For the proof we write $\Gamma = \{x_1, x_2, \ldots\}$. By step 1 there exists a subsequence of the ψ_n such that the evaluation on the element x_1 converges. From this subsequence we can again extract a subsequence such that the evaluation on the element x_2 converges. Continuing this way and applying the usual diagonal trick we end up with a subsequence for which indeed all the above limits exist.
Step 3. We claim that the map ψ is a homogeneous quasimorphism and that $\lim_{n\to\infty} \varphi_n = \psi$ with respect to the defect seminorm. It is enough to prove the last statement. Indeed, by definition of ψ_n this is equivalent to $\psi_n \to \psi$. Since ψ_n is a Cauchy sequence as well it will suffice to prove that the subsequence ψ_{n_k} converges to ψ. Assume this is not the case, then there exists $\varepsilon > 0$ such that $D(\psi_{n_k} - \psi) > 2\varepsilon$ for infinitely many k. Choose N such that $D(\psi_{n_k} - \psi_{n_l}) < \varepsilon$ for all $k, l > N$. Hence there exist $k > N$ and $x, y \in \Gamma$ such that

$$\left|\big(\psi_{n_k}(xy) - \psi_{n_k}(x) - \psi_{n_k}(y)\big) - \big(\psi(xy) - \psi(x) - \psi(y)\big)\right| > 2\varepsilon.$$

So we have

$$\left|\big(\psi_{n_l}(xy) - \psi_{n_l}(x) - \psi_{n_l}(y)\big) - \big(\psi(xy) - \psi(x) - \psi(y)\big)\right| > \varepsilon$$

for *all* $l > N$. But this contradicts the existence of all limits in Step 2. $\square$

2.3. **Comparing the Two Seminorms.** In fact, all concrete examples of homogeneous quasimorphisms known to the author satisfy the equality $D(\varphi) = 2||\varphi||_b$. We conjecture that this is true in general, at least for countable groups:

Conjecture 2.12. *Let Γ be a countable group. Then every homogeneous quasimorphism φ on Γ satisfies the equality $D(\varphi) = 2||\varphi||_b$.*

Besides the fact that there seems not to be a known counterexample, we shall collect some additional evidence for the conjecture in this subsection. To do so, we need a result concerning the canonical norm on H^2_b.

Let Γ be a countable discrete group. Following [23], Section 0.3 we recall the construction of the Poisson boundaries of (right) random walks on Γ determined by certain symmetric probability measures μ on Γ. For this we always assume μ to be *non-degenerate*, i.e., Γ is generated by $\operatorname{supp}\mu$. (We point out that for the construction of Poisson boundaries on general locally compact groups μ has in addition to be *spread out*, i.e., that there exists a convolution power μ^{*n} which is non-singular with respect to the Haar measure on Γ. This last condition is automatically fullfilled in discrete groups because the Haar measure is the counting measure.)
Let $\Gamma^\infty = \prod_{k=0}^\infty \Gamma$ be the space of trajectories of the random walk and let $\mathbb{P}^\mu$ be the image of the product measure μ^∞ on the space $\prod_{k=1}^\infty \Gamma$ of increments by the map

$$\prod_{k=1}^\infty \Gamma \to \Gamma^\infty, \quad (x_1, x_2, \ldots) \mapsto (1, x_1, x_1x_2, \ldots).$$

Call a subset $A \subset \Gamma^\infty$ *stationary* (mod 0) if it is measurable and if it contains with almost every trajectory y also all trajectories y' which can be obtained from y by coordinate shifts and by replacing any finite number of coordinates, i.e., all y' such that $y'_{n+k} = y_n$ for all sufficiently large n and a fixed integer k. Denote by $\mathscr{A}$ the σ-algebra of classes of stationary sets (mod 0) and call it the *stationary σ-algebra*. The *Poisson boundary* (B, ν) of the random walk (Γ, μ) is the quotient space of the measure space $(\Gamma^\infty, \mathbb{P}^\mu)$ with respect to the measurable partition attached to the stationary σ-algebra $\mathscr{A}$. In particular, ν is the image of $\mathbb{P}^\mu$ by the canonical map $\Gamma^\infty \to B$. The action of Γ on Γ^∞ by component-wise left multiplication induces an action of Γ on $\mathscr{A}$ and hence on B. The measure ν is Γ-quasi-invariant.

Proposition 2.13. *Consider a surjective homomorphism $\pi : \Gamma_1 \to \Gamma_2$ between countable discrete groups. Let μ_1 be a symmetric non-degenerate probability measure on Γ_1 and let $\mu_2 = \pi\mu_1$ be its image by π. Denote by (B_i, ν_i) the Poisson boundary of (Γ_i, μ_i), $i = 1, 2$. Then π induces a Γ-equivariant measurable map $\pi_B : B_1 \to B_2$ such that $\nu_2 = \pi_B\nu_1$.*

Proof. First observe that μ_2 is also symmetric and non-degenerate because π is surjective. Consider the following diagram (ignore the dotted arrow for the moment):

$$\begin{tikzcd}
\left(\prod_{k=1}^\infty \Gamma_1, \mu_1^\infty\right) \arrow[r, "\alpha=\prod_{k=1}^\infty \pi"] \arrow[d] & \left(\prod_{k=1}^\infty \Gamma_2, \mu_2^\infty\right) \arrow[d] \\
(\Gamma_1^\infty, \mathbb{P}^{\mu_1}) \arrow[r, "\beta=\prod_{k=0}^\infty \pi"] \arrow[d] & (\Gamma_2^\infty, \mathbb{P}^{\mu_2}) \arrow[d] \\
(B_1, \nu_1) \arrow[r, dotted, "\pi_B"] & (B_2, \nu_2)
\end{tikzcd}$$

Obviously all maps are measurable and the top square commutes. Moreover the horizontal maps are equivariant homomorphisms. Observe first that $\mu_2^\infty = \alpha\mu_1^\infty$. This follows immediatley from Kolmogorov's consistency theorem (cf. [33], Theorem 5.1) as these two measures clearly agree on cylinder subsets of Γ_2^∞. By commutativity of the top square this implies that $\mathbb{P}^{\mu_2} = \beta\mathbb{P}^{\mu_1}$. As a consequence, for any stationary set $A \subset \Gamma_1^\infty$ the image $\beta(A)$ is contained in a stationary set in Γ_2^∞. Hence the map β descends to a measurable equivariant map π_B such that the lower square of the diagram commutes almost everywhere. By this commutativity and the definition of ν_i as the image of $\mathbb{P}^{\mu_i}$ we finally deduce $\nu_2 = \pi_B\nu_1$. □

Theorem 2.14. *Let $\pi : \Gamma_1 \to \Gamma_2$ be a surjective homomorphism of countable groups. Then the induced map $\mathrm{H}_b^2(\pi) : \mathrm{H}_b^2(\Gamma_2) \to \mathrm{H}_b^2(\Gamma_1)$ is an isometric injection.*

Proof. Let Γ be a countable discrete group and μ a symmetric non-degenerate probability measure on Γ. Then Γ acts measure class preserving on the Poisson boundary (B, ν) of Γ and this action is amenable and doubly ergodic, see [22], Theorem 3. We fix such a measure μ. By amenability of the Γ-action the complex

$$0 \to \mathbb{R} \xrightarrow{\varepsilon} L^\infty(B) \to L^\infty(B^2) \to L^\infty(B^3) \to \dots$$

is a strong augmented resolution of isometrically injective Γ-modules ([30], Lemma 2.3). The same is true for the subresolution

$$0 \to \mathbb{R} \xrightarrow{\varepsilon} L^\infty_{\mathrm{alt}}(B) \to L^\infty_{\mathrm{alt}}(B^2) \to L^\infty_{\mathrm{alt}}(B^3) \to \dots$$

of alternating functions because the obvious projection maps $L^\infty(B^n) \to L^\infty_{\mathrm{alt}}(B^n)$ have norm 1, compare [9], Lemma 2.12. Hence the complex of invariants

$$0 \to L^\infty_{\mathrm{alt}}(B)^\Gamma \to L^\infty_{\mathrm{alt}}(B^2)^\Gamma \to L^\infty_{\mathrm{alt}}(B^3)^\Gamma \to \dots$$

realises the bounded cohomology $\mathrm{H}_b^\bullet(\Gamma)$. Moreover, by the double ergodicity of the Γ-action all functions in $L^\infty(B^2)^\Gamma$ are constant and hence $L^\infty_{\mathrm{alt}}(B^2)^\Gamma = 0$. As a consequence we get an isometric identification

$$\mathrm{H}_b^2(\Gamma) = ZL^\infty_{\mathrm{alt}}(B^3)^\Gamma = \{c \in L^\infty_{\mathrm{alt}}(B^3, \nu^{\otimes 3})^\Gamma \mid dc = 0\}.$$

Choose a symmetric non-degenerate probability measure μ_1 on Γ_1 and let μ_2 and (B_i, ν_i) be as in Proposition 2.13. As π is surjective, μ_2 is a symmetric and non-degenerate probability measure on Γ_2. By the same proposition the maps $\pi_B^n : B_1^n \to B_2^n$ are measurable and equivariant for all positive integers n. In addition we have $\nu_2^{\otimes n} = \pi_B^n \nu_1^{\otimes n}$. The induced maps

$$\pi^* : L^\infty_{\mathrm{alt}}(B_2^n, \nu_2^{\otimes n}) \to L^\infty_{\mathrm{alt}}(B_1^n, \nu_1^{\otimes n}), \qquad f \mapsto f \circ \pi_B^n$$

are therefore equivariant and isometric as C is an essential bound for f if and only if it is one for $f \circ \pi_B^n$. Moreover, an easy check shows that these maps constitute a morphism of augmented resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\varepsilon"] \arrow[d, equal] & L^\infty_{\mathrm{alt}}(B_2) \arrow[r] \arrow[d, "\pi^*"] & L^\infty_{\mathrm{alt}}(B_2^2) \arrow[r] \arrow[d, "\pi^*"] & L^\infty_{\mathrm{alt}}(B_2^3) \arrow[r] \arrow[d, "\pi^*"] & \cdots \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\varepsilon"] & L^\infty_{\mathrm{alt}}(B_1) \arrow[r] & L^\infty_{\mathrm{alt}}(B_1^2) \arrow[r] & L^\infty_{\mathrm{alt}}(B_1^3) \arrow[r] & \cdots
\end{tikzcd}$$

By equivariance we hence get an isometry

$$ZL^\infty_{\mathrm{alt}}(B_2^3)^{\Gamma_2} \to ZL^\infty_{\mathrm{alt}}(B_1^3)^{\Gamma_1}$$

which realises the induced map $\mathrm{H}_b^2(\pi)$. □

Corollary 2.15. *If Conjecture 2.12 is true for a specific countable group, then it is also true for all its quotients. In particular, its validity for countable free groups would imply its validity in general.*

Proof. Let $\pi : \Gamma_1 \to \Gamma_2$ be a surjective homomorphism of countable groups and let φ be an arbitrary quasimorphism on Γ_2. On the one hand, we clearly have $D(\pi^*\varphi) = D(\varphi)$ by the surjectivity of π. On the other hand, $||\pi^*\varphi||_b = ||\varphi||_b$ by Theorem 2.14. Assume that Conjecture 2.12 is true for Γ_1. If φ is homogeneous then so is $\pi^*\varphi$, hence

$$D(\varphi) = D(\pi^*\varphi) = 2||\pi^*\varphi||_b = 2||\varphi||_b$$

where the second equality holds by assumption. □

A large family of homogeneous quasimorphisms on free groups was introduced by Brooks in [6]. And a slightly different construction was used by Rolli in [35] to produce an uncountable familiy of independent homogeneous quasimorphisms. It is worth noting that these essentially exhaust all explicitly known quasimorphisms on free groups and they all satisfy Conjecture 2.12 (cf. [35], Proposition 3.6).

2.4. **Harmonicity.** For $f \in l^\infty(\Gamma)$ and a probability measure μ on Γ we define the *convolution* $f * \mu$ by the formula

$$(f * \mu)(x) = \int_\Gamma f(xy^{-1})\, d\mu(y) = \sum_{x \in \Gamma} \mu(y) f(xy^{-1}).$$

Convolution with μ is a linear operator on $l^\infty(\Gamma)$ of norm 1 and the invariant functions are called *μ-harmonic.* More generally, one can define the convolution $f * \mu$ for unbounded functions by the same formula provided that the sum converges.

A function $f \in l^\infty(\Gamma^n)$ is called *μ-pluriharmonic* if it is harmonic with respect to every variable. We denote by $l^\infty_\mu(\Gamma)$ the space of μ-pluriharmonic functions.
Assume that μ is moreover non-degenerate. Then we can form the Poisson boundary (B, μ) of Γ with respect to μ on which Γ acts amenable and doubly ergodic. Recall the Poisson transform

$$\mathcal{P}^{(n)} : L^\infty(B^{n+1}, \nu^{\otimes(n+1)}) \to l^\infty_\mu(\Gamma^{n+1})$$

defined by

$$\mathcal{P}^{(n)} f(x) = \int_{B^{n+1}} f(x\xi)\, d\nu^{\otimes(n+1)}(\xi)$$

which is a Γ-equivariant isometric isomorphism which preserves alternating maps (cf. [9], Proposition 3.11).

Let Γ be finitely generated. Choose a word lengths metric d associated to a finite generating system of Γ. A probability measure μ on Γ is said to have *finite first moment* if the sum

$$\sum_{x\in\Gamma} d(1,x)\mu(x)$$

is finite (this definition does not depend on the choice of d as all such metrics are bi-Lipschitz equivalent). Let $\varphi : \Gamma \to \mathbb{R}$ be a quasimorphism and let μ have finite first moment. Then the convolution $\varphi * \mu$ makes sense as φ has at most linear growth with respect to d.

The following result supplements the discussion about harmonic quasimorphisms in [9] and shows that they are always defect minimising in their equivalence class.

Theorem 2.16. *Let Γ be a finitely generated group and let μ be a symmetric, non-degenerate probability measure on Γ with finite first moment. For every quasimorphism φ of Γ there is a unique antisymmetric μ-harmonic quasimorphism φ_μ at finite distance of φ. Moreover, we have*

$$D(\varphi_\mu) = ||\varphi_\mu||_b,$$

i.e., its defect is least possible among all quasimorphisms at finite distance of φ.

Proof. The existence of an antisymmetric μ-harmonic quasimorphism φ_μ at finite distance of φ was proven in [9], Corollary π.
We shall show uniqueness. To every quasimorphism $\varphi : \Gamma \to \mathbb{R}$ we can associate a function $\omega_\varphi : \Gamma^2 \to \mathbb{R}$ by setting $\omega_\varphi(x,y) = \varphi(x^{-1}y)$. One easily checks that ω_φ is Γ-invariant and moreover alternating if φ is antisymmetric. In addition it is μ-biharmonic if φ is μ-harmonic and antisymmetric. We prove harmonicity in the first variable, the argument for the second variable is similar but does not rely on φ being antisymmetric:

$$\begin{aligned}
(\omega_\varphi *_1 \mu)(x,y) &= \sum_{z\in\Gamma} \mu(z)\omega_\varphi(xz^{-1}, y) \\
&= \sum_{z\in\Gamma} \mu(z)\varphi(zx^{-1}y) \\
&= -\sum_{z\in\Gamma} \mu(z)\varphi(y^{-1}xz^{-1}) \\
&= -\varphi(y^{-1}x) = \varphi(x^{-1}y) = \omega_\varphi(x,y).
\end{aligned}$$

Assume now that there are two antisymmetric μ-harmonic quasimorphisms ψ_1, ψ_2 at finite distance of each other. Then we have

$$\omega_{\psi_1-\psi_2} \in l^\infty_{\mu,\mathrm{alt}}(\Gamma^2)^\Gamma = L^\infty_{\mathrm{alt}}(B^2, \nu^{\otimes 2})^\Gamma = 0$$

by the double ergodicity of the Poisson boundary. Hence $\omega_{\psi_1} = \omega_{\psi_2}$ and therefore $\psi_1 = \psi_2$.
Finally, we prove that φ_μ has minimal possible defect. By [9], Lemma 3.13 the inclusion of the subresolution

$$0 \longrightarrow \mathbb{R} \longrightarrow l^\infty_{\mu,\mathrm{alt}}(\Gamma) \longrightarrow l^\infty_{\mu,\mathrm{alt}}(\Gamma^2) \longrightarrow l^\infty_{\mu,\mathrm{alt}}(\Gamma^3) \longrightarrow \cdots$$

into the homogeneous standard resolution of Γ induces the identity at the level of cohomology. Moreover, we already noted that $l^\infty_{\mu,\mathrm{alt}}(\Gamma^2)^\Gamma = 0$, hence we get an isometric identification

$$H^2_b(\Gamma) \cong Zl^\infty_{\mu,\mathrm{alt}}(\Gamma^3)^\Gamma.$$

Now, by the above discussion, $d\omega_{\varphi_\mu}$ is a bounded alternating μ-triharmonic homogeneous 3-cocycle on Γ and hence we conclude

$$||d\omega_{\varphi_\mu}||_\infty = ||[d\omega_{\varphi_\mu}]||_{\mathrm{H}^2_b}.$$

On the other hand, there is an isometric isomorphism of chain complexes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathbb{R} \arrow[r, "d^i"] \arrow[d, "\alpha^0"] & l^\infty(\Gamma) \arrow[r, "d^i"] \arrow[d, "\alpha^1"] & l^\infty(\Gamma^2) \arrow[r, "d^i"] \arrow[d, "\alpha^2"] & L^\infty(\Gamma^3) \arrow[r, "d^i"] \arrow[d, "\alpha^3"] & \cdots \\
0 \arrow[r] & l^\infty(\Gamma)^\Gamma \arrow[r, "d^h"] & l^\infty(\Gamma^2)^\Gamma \arrow[r, "d^h"] & l^\infty(\Gamma^3)^\Gamma \arrow[r, "d^h"] & l^\infty(\Gamma^4)^\Gamma \arrow[r, "d^h"] & \cdots
\end{tikzcd}$$

where d^i respectively d^h are the (in-)homogeneous coboundary operators. The degree n map ist given by

$$\alpha^n f(x_0, \ldots, x_n) = f(x_0^{-1}x_1, x_1^{-1}x_2, \ldots, x_{n-1}^{-1}x_n).$$

A short calculation shows that $d^h\omega_{\varphi_\mu} = \alpha^2(d^i\varphi_\mu)$ and so we finally conclude that

$$D(\varphi_\mu) = ||d\varphi_\mu||_{H^2_b} = ||\varphi_\mu||_b.$$

□

2.5. **The Translation Number and the Euler Class.** In this subsection we are going to define the translation number quasimorphism and the rotation number introduced by Poincaré. For a detailed treatment of the objects in the title and their impact on the dynamics of individual circle homeomorphisms as well as more general group actions by homeomorphisms the reader is referred to [13], [14] or [10].

Denote by $\tilde{H} = \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ the group of orientation preserving homeomorphisms of the real line which commute with integral translations. In other words, its elements are the continuous, strictly monotonic increasing functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$. We will frequently use the following estimate valid for every such function f:

$$\big|\big(f(x) - x\big) - \big(f(y) - y\big)\big| \leq 1 \qquad \forall x, y \in \mathbb{R}. \tag{4}$$

Denote by $H = \mathrm{Homeo}^+(S^1)$ the group of orientation preserving homeomorphisms of the circle. Equipped with the compact-open topology both $\tilde{H}$ and H are topological groups. Interpreting $S^1 = \mathbb{R}/\mathbb{Z}$ as quotient of the real line there is a natural projection $p : \tilde{H} \to H$ which is clearly a continuous group homomorphism.

Lemma 2.17. *The group $\tilde{H}$ is contractible and $p : \tilde{H} \to H$ is a covering map realising $\tilde{H}$ as universal covering group of H. Moreover, there is a central $\mathbb{Z}$-extension*

$$\langle t \rangle \rightarrowtail \tilde{H} \xrightarrow{p} H \tag{5}$$

where $t : \mathbb{R} \to \mathbb{R}$ is the translation $t(x) = x + 1$.

Proof. Clearly, p is surjective and its kernel is generated by t. It therefore remains to prove that $\tilde{H}$ is contractible. But the continuous map

$$I \times \tilde{H} \to \tilde{H}, \qquad (s, f) \mapsto \big(x \mapsto sx + (1-s)f(x)\big)$$

is a homotopy between $\mathrm{id}_{\tilde{H}}$ and the constant map $f \mapsto \mathrm{id}_{\mathbb{R}}$. □

We now turn to the definition of the translation and the rotation number.

Lemma 2.18. *For $f \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ and $x \in \mathbb{R}$ the limit*

$$T(f) = \lim_{n\to\infty} \frac{f^n(x) - x}{n} \in \mathbb{R}$$

exists and is independent of the choice of x. The map $T : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ is a continuous, homogeneous quasimorphism with defect 1.

Proof. For $x \in \mathbb{R}$ define a function $T_x : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ by $T_x(f) = f(x) - x$. The inequality (4) exactly states that any two of these functions are at finite distance: $||T_x - T_y||_\infty \le 1$. For the same reason we have

$$|T_x(fg) - T_x(f) - T_x(g)| = |T_{g(x)}(f) - T_x(f)| \le 1$$

for all $f, g \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, hence each T_x is a quasimorphism. Everything except for the last claim now follows at once since the defining formula for T describes precisely the common homogenisation for the family of equivalent continuous quasimorphims T_x.
We finally prove that $D(T) = 1$. Observe that the quasimorphism $\lfloor T_0 \rfloor$ given by the integer part of T_0 is at finite distance of T_0 and that

$$\lfloor T_0 \rfloor(fg) - \lfloor T_0 \rfloor(f) - \lfloor T_0 \rfloor(g) \in \{0, 1\}$$

for all f, g. Hence the quasimorphism $\lfloor T_0 \rfloor - \frac{1}{2}$ has defect at most $\frac{1}{2}$ and has T as its homogenisation. The claim now follows from Lemma 2.4. □

The quasimorphism $T : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ is called *translation number.* For the definition of the rotation number function $R : \mathrm{Homeo}^+(S^1) \to \mathbb{R}/\mathbb{Z}$ consider $f \in H$ and choose an arbitrary lift $\tilde{f} \in \tilde{H}$ of f under p. By Lemma 2.17 and the definition of the translation number the image $R(f)$ of the real number $T(\tilde{f})$ in $\mathbb{R}/\mathbb{Z}$ is independent of the lift $\tilde{f}$ and is called the *rotation number of f.*

For later use we describe a method of computation for integral translation numbers. Since $\tilde{H}$ is the universal covering group of H, we can interpret elements of the former group as homotopy classes (rel $\{0, 1\}$) of curves starting at the identity in the latter, cf. Subsection 1.1. This point of view allows to compute translation numbers, whenever they are integral, as the degree of a certain self map of S^1:

Proposition 2.19. *Let the element $f \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ be represented by a path $w : I \to \mathrm{Homeo}^+(S^1)$ starting at the identity. The path*

$$I \xrightarrow{w} \mathrm{Homeo}^+(S^1) \xrightarrow{R} \mathbb{R}/\mathbb{Z}$$

is closed if and only if the translation number $T(f)$ is integral. If this is the case then the mapping degree of the induced map $S^1 \to S^1$ equals $T(f)$.

Proof. Consider the diagram

$$\begin{tikzcd} \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \arrow[r, "T"] \arrow[d, "p"'] & \mathbb{R} \arrow[d] \\ \mathrm{Homeo}^+(S^1) \arrow[r, "R"] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

where both vertical arrows are universal coverings. Since this diagram commutes the image $T(f)$ is represented by the path $R \circ w$, that is, $T(f)$ equals the endpoint of the unique lift of this path to $\mathbb{R}$ starting at 0. Hence $T(f)$ is integral if and only if $R \circ w$ is closed. The second claim is a consequence of the following general fact: Let $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be continuous and let $\tilde{g} : \mathbb{R} \to \mathbb{R}$ be the unique lift of g with $\tilde{g}(0) = 0$. Then we have $\deg(g) = \tilde{g}(1)$. □

We will finish this subsection by showing that the translation number trivialises the pullback via p of the cohomology class in $\mathrm{H}^2(H, \mathbb{Z})$ associated to the central extension (5). We refer to Section 4.1 for further details.

In what follows we identify $\mathbb{Z} \cong \langle t \rangle$ via the isomorphism $k \mapsto t^k$. Let $e_{\mathbb{Z}} \in \mathrm{H}^2(H, \mathbb{Z})$ be the *(integral) Euler class*, i.e., the cohomology class associated to the extension (5). The image of $e_{\mathbb{Z}}$ under the canonical homomorphism $\mathrm{H}^2(H, \mathbb{Z}) \to \mathrm{H}^2(H, \mathbb{R})$ is denoted by $e_{\mathbb{R}}$ and is referred to as the *real Euler class.* To construct a concrete cocycle representing $e_{\mathbb{Z}}$ respectively $e_{\mathbb{R}}$ we choose the section $\sigma : H \to \tilde{H}$ sending f to the unique lift $\tilde{f} \in \tilde{H}$ with $0 \leq \tilde{f}(0) < 1$. Then the corresponding cocycle is given by

$$\alpha(f, g) = \sigma(f)\sigma(g)\sigma(fg)^{-1}(0).$$

By definition of σ we have $\sigma(fg)(0) \in [0, 1[$ and similarly $\sigma(g)(0) \in [0, 1[$, hence $\sigma(f)\sigma(g)(0) \in [0, 2[$. From this it is immediate that α can only take the values 0 and 1, specifically

$$\alpha(f, g) = \lfloor \sigma(f)\sigma(g)(0) \rfloor \in \{0, 1\}. \tag{6}$$

In particular, α is bounded and hence defines a bounded cohomology class $e_b \in \mathrm{H}^2_b(H, \mathbb{R})$ as well, the *bounded Euler class.* The image of e_b under the comparison map is of course just $e_{\mathbb{R}}$. Since $p^*(e_{\mathbb{R}}) \in \mathrm{H}^2(\tilde{H}, \mathbb{R})$ vanishes according to Corollary 4.5 (a) there is a quasimorphism on $\tilde{H}$ trivialising $p^*(e_b) \in \mathrm{H}^2_b(\tilde{H}, \mathbb{R})$ by Lemma 2.9. In fact, this quasimorphism is just the translation number:

Lemma 2.20. *We have* $[dT] = p^*(e_b)$ *in* $\mathrm{H}^2_b(\tilde{H}, \mathbb{R})$.

Proof. We set $s = \sigma \circ p : \tilde{H} \to \tilde{H}$, hence $s(\tilde{f}) = \tilde{f} - \lfloor \tilde{f}(0) \rfloor$. Then the pullback $\beta = p^*\alpha$ is given by

$$\beta(\tilde{f}, \tilde{g}) = \lfloor s(\tilde{f})s(\tilde{g})(0) \rfloor = \lfloor \tilde{f}\tilde{g}(0) \rfloor - \lfloor \tilde{f}(0) \rfloor - \lfloor \tilde{g}(0) \rfloor,$$

where we have used (6) for the first equality. In other words we have $\beta = d\lfloor T_0 \rfloor$, this completes the proof since T is the homogenisation of $\lfloor T_0 \rfloor$ (cf. proof of Lemma 2.18). □

Let $\mathbb{D}$ be the Poincaré disc and let $G = \mathrm{PSU}(1, 1)$ be the group of orientation preserving isometries of $\mathbb{D}$ acting by linear fractional transformations on $\mathbb{D}$, cf. Lemma 1.6. Since G also acts on the topological boundary $S^1 = \partial\mathbb{D} \subset \mathbb{C}$ we have a canonical embedding $G \hookrightarrow \mathrm{Homeo}^+(S^1)$. In particular, one can ask for the rotation number of elements $g \in G$. We give a formula which involves a lift of g to the two-fold cover $\mathrm{SU}(1, 1)$ of G:

Lemma 2.21. *Let $g \in \mathrm{SU}(1,1)$ be an element with upper left entry a. Then*

$$R(g) = \begin{cases} \operatorname{sign}\big(\operatorname{Im}(a)\big)\frac{1}{\pi}\arccos\big(\frac{1}{2}\operatorname{tr}(g)\big), & |\operatorname{tr}(g)| < 2, \\ 0, & |\operatorname{tr}(g)| \geq 2, \end{cases}$$

where in the first case $\operatorname{Im}(a) \neq 0$.

Proof. Every parabolic or hyperbolic element has a fixed point on the boundary $\partial\mathbb{D}$ and therefore vanishing rotation number. For an elliptic element of the form

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

with $\zeta = e^{i\varphi} \in S^1$ we have $gz = \zeta^2 z$ and hence g describes a rotation with angle 2φ. So g has rotation number $\frac{\varphi}{\pi}$. On the other hand, $\operatorname{tr}(g) = 2\cos\varphi$ and this leads to the above formula. The case of an arbitrary elliptic element follows at once from this because each such element is conjugated within $\mathrm{SU}(1,1)$ to one of the above diagonal form, and conjugation preserves the rotation number, preserves traces and also preserves the sign of the imaginary part of the upper left entry of every elliptic element. To see the latter, let

$$g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$$

be elliptic and write $a = x + iy$ and $b = z + iw$. Then by assumption on the trace we have $|x| < 1$ and the determinant condition gives $y^2 = z^2 + w^2 + (1 - x^2) > z^2 + w^2$. In particular y does not vanish. If we conjugate g with another element

$$h = \begin{pmatrix} c & d \\ \overline{d} & \overline{c} \end{pmatrix} \in \mathrm{SU}(1,1)$$

then the imaginary part of the upper left entry of the result is given by the expression

$$(|c|^2 + |d|^2)y + 2\operatorname{Im}(\overline{c}d)z - 2\operatorname{Re}(\overline{c}d)w.$$

Here the first term clearly has the same sign as y. Moreover,

$$\begin{aligned} \big(2|\operatorname{Im}(\overline{c}d)z| + 2|\operatorname{Re}(\overline{c}d)w|\big)^2 &\leq 4\big(\operatorname{Im}(\overline{c}d)^2 + \operatorname{Re}(\overline{c}d)^2\big)(z^2 + w^2) \\ &< 4|\overline{c}d|^2 y^2 \leq (|c|^2 + |d|^2)^2 y^2, \end{aligned}$$

where the first inequality follows from Cauchy-Schwarz, the second from the above remark and the last one from the arithmetic-geometric mean inequality. Hence the signs of the imaginary parts of the upper left entries in g and hgh^{-1} are the same. □

2.6. **Quasimorphisms on Groupoids.** The concept of quasimorphism naturally generalises from groups to groupoids. A *quasimorphism* on a groupoid G is a map $\varphi : G \to \mathbb{R}$ such that there exists a constant D with

$$|\varphi(xy) - \varphi(x) - \varphi(y)| \leq D$$

whenever xy is defined. The smallest such constant is called the *defect* of φ and denoted $D(\varphi)$. Virtually everything said in Subsection 2.1 applies to this more general setting with the obvious modifications. In particular, it makes sense to speak of *homogeneous* quasimorphisms by requiring that $\varphi(x^n) = n\varphi(x)$ for all integers n and all self-composable elements $x \in G$. Homogeneous quasimorphisms are invariant under conjugation whenever it is defined. Also we have $\varphi(xy) = \varphi(x) + \varphi(y)$ whenever xy and yx are defined and agree.

In the important special case where $G = \pi_1(X)$ is the fundamental groupoid of a topological space conjugation invariance implies that the restriction of φ to the subset of classes of closed curves depends only on the *free* homotopy class of a curve.

3. Holonomy Representations

In this section we consider the class of holonomy representations $\rho : \pi_1(S) \to \mathrm{PSU}(1,1)$ which are defined via metrics $h \in \mathrm{Hyp}(S)$ on a surface S. Then we discuss invariants of representations ρ that characterise the subset $\mathrm{Hol}(\pi_1(S)) \subset \mathrm{Hom}(\pi_1(S), \mathrm{PSU}(1,1))$ of holonomy representations inside the representation variety. These invariants are given by pulling back via ρ either the integral Euler class $e_{\mathbb{Z}} \in \mathrm{H}^2(\mathrm{Homeo}^+(S^1), \mathbb{Z})$ or the bounded Euler class $e_b \in \mathrm{H}^2_b(\mathrm{Homeo}^+(S^1))$. This section is taken from [8].

3.1. **Definitions.** Let S be an oriented surface. We fix once and for all a universal covering $p : \tilde{S} \to S$ of S and choose base points $x \in S$ and $\tilde{x} \in \tilde{S}$ lying above x. Set $\Gamma = \pi_1(S, x)$ and recall that for $\gamma \in \Gamma$ and $\tilde{y} \in \tilde{S}$ lying above x we denote by $T^{\tilde{y}}_{\gamma} \in \mathrm{Diff}^+(\tilde{S})$ the unique covering transformation of p such that the lift of any curve in γ to $\tilde{S}$ with starting point $\tilde{y}$ has endpoint $T^{\tilde{y}}_{\gamma}(\tilde{y})$. The map $T^{\tilde{x}}_{(-)}$ from Γ to the group of covering transformations of p is a group isomorphism. Hence, for what follows, we will identify the group of covering transformations with Γ.
Fix a metric $h \in \mathrm{Hyp}(S)$ and notice that the pullback $\tilde{h} = p^*(h)$ is a Γ-invariant complete hyperbolic metric on $\tilde{S}$ which, however, is not of finite volume anymore. If we equip $\tilde{S}$ with the metric $\tilde{h}$ all covering transformations are orientation preserving isometries, hence we obtain a homomorphism

$$T^{\tilde{x}}_{(-)} : \Gamma \to \mathrm{Isom}^+(\tilde{S}, \tilde{h})$$

into the group of orientation preserving isometries of $\tilde{S}$. By Cartan's theorem the universal cover $\tilde{S}$ is isometric to the Poincaré disc. Fixing such an isometry we can conjugate the above homomorphism into a representation

$$\rho_{h,x} : \Gamma \to G,$$

called *holonomy representation*, where $G = \mathrm{Isom}^+(\mathbb{D}) = \mathrm{PSU}(1,1)$. This of course depends on the choice of the lift $\tilde{x}$ and the choice of the isometry $f : (\tilde{S}, \tilde{h}) \to \mathbb{D}$. Changing $\tilde{x}$ amounts to pre-composition with a conjugation in Γ and changing f amounts to post-composition with a conjugation in G. To sum up, for each metric h we obtain a well-defined conjugacy class of representations $\Gamma \to G$.

However, we will need to be more specific about the involved choices. Therefore we give a modified definition of holonomy representations which keeps track of the input data. To start with, we shall pin down the isometry $\tilde{S} \to \mathbb{D}$. Let $\tilde{v}$ be a non-zero tangent vector of $\tilde{S}$ with base point $\tilde{x}$. Since $\mathrm{Isom}^+(\mathbb{D})$ acts simply transitively on the unit tangent bundle of $\mathbb{D}$ there is a *unique* orientation preserving isometry

$$f_{\tilde{h},\tilde{v}} : (\tilde{S}, \tilde{h}) \to \mathbb{D}$$

sending $\tilde{x}$ to $0 \in \mathbb{D}$ and $\tilde{v}$ to a positive multiple of the tangent vector $1 \in T\mathbb{D}_0$.
Define the *holonomy representation* associated to $\tilde{v}$ and h by

$$\rho_{h,\tilde{v}} : \Gamma \to G, \qquad \gamma \mapsto f_{\tilde{h},\tilde{v}} \circ T^{\tilde{x}}_{\gamma} \circ (f_{\tilde{h},\tilde{v}})^{-1}.$$

It is easy to see that, indeed, $\rho_{h,\tilde{v}}$ is a well-defined homomorphism with respect to which $f_{\tilde{h},\tilde{v}}$ is equivariant.

Lemma 3.1. *Let $\rho : \Gamma \to G$ be a holonomy representation. Then the image of ρ contains no elliptic elements.*

Proof. For $\gamma \neq 1$ the covering transformation $T^{\tilde{x}}_{\gamma}$ has no fixed point on $\tilde{S}$, hence the conjugated map $\rho(\gamma)$ has no fixed point on $\mathbb{D}$. $\square$

3.2. **Characterising Holonomy Representations via the Euler Class.** Let S be a surface as before. Remember that we always assume that $\mathrm{Hyp}(S)$ is non-empty, in particular, if S is compact then its genus is $g \geq 2$. We keep the notation $G = \mathrm{PSU}(1,1)$ and fix a base point $x \in S$. The goal of this subsection is to characterise the set $\mathrm{Hol}(\Gamma)$ of holonomy representations of the fundamental group $\Gamma = \pi_1(S,x)$ among all representations $\Gamma \to G$.

We first assume that S is compact. To every representation $\rho : \Gamma \to G$ we can associate a numerical invariant $e(\rho)$, called the *Euler number*, of ρ as follows. Consider the pullback $\rho^*(e_{\mathbb{Z}}) \in \mathrm{H}^2(\Gamma, \mathbb{Z})$ of the integral Euler class $e_{\mathbb{Z}} \in \mathrm{H}^2(\mathrm{Homeo}^+(S^1), \mathbb{Z})$ via the composition

$$\Gamma \xrightarrow{\ \rho\ } G \longrightarrow \mathrm{Homeo}^+(S^1)$$

(see Subsection 2.5 for the definition of $e_{\mathbb{Z}}$). Since S has genus $g \geq 2$ the classifying map $S \to B\Gamma$ is a homotopy equivalence and therefore induces an isomorphism $\mathrm{H}^2(S, \mathbb{Z}) \to \mathrm{H}^2(\Gamma, \mathbb{Z})$. Now we define $e(\rho)$ as the evaluation of $\rho^*(e_{\mathbb{Z}})$ on the image of the fundamental class $[S] \in \mathrm{H}^2(S, \mathbb{Z})$ under the above isomorphism. There is a famous bound on the Euler number of a representation known as Milnor-Wood inequality (cf. [29], [40]):

Theorem 3.2. *The Euler number $e(\rho)$ of every representation $\rho : \Gamma \to G$ satisfies*

$$|e(\rho)| \leq 2g - 2.$$

In fact, every integer between $-2g+2$ and $2g-2$ is realised as the Euler number of a suitable representation. For $\rho \in \mathrm{Hol}(\Gamma)$ the Euler number always takes the maximal possible value $e(\rho) = 2g - 2$, compare [8], Section 3.8. Goldman has shown in [15] that this property actually characterises holonomy representations:

Theorem 3.3. *A representation $\rho : \Gamma \to G$ belongs to $\mathrm{Hol}(\Gamma)$ if and only if $e(\rho) = 2g - 2$.*

Observe that for a non-compact surface S the fundamental group Γ is free and hence has trivial second cohomology. Therefore, the pullback $\rho^*(e_{\mathbb{Z}})$ vanishes and there is no analogon of the Euler number available for representations $\rho \in \mathrm{Hom}(\Gamma, G)$. But despite the vanishing of the usual second cohomology of Γ, the *bounded* cohomology $\mathrm{H}^2_b(\Gamma)$ is heavily non-trivial. Hence for arbitrary surfaces S we can try to use the pullback $\rho^*(e_b) \in \mathrm{H}^2_b(\Gamma)$ of the bounded Euler class as a characterising invariant for holonomy representations. This indeed works, see [8], Corollary 4.5:

Theorem 3.4. *The subset $\mathrm{Hol}(\Gamma) \subset \mathrm{Hom}(\Gamma, G)$ is characterised by the fact that for $\rho \in \mathrm{Hol}(\Gamma)$ the pullback $\rho^*(e_b)$ of the bounded Euler class equals a certain fixed class which depends only on S. In particular, the pullback $e^\Gamma_b = \rho^*(e_b) \in \mathrm{H}^2_b(\Gamma)$ is the same for all holonomy representations ρ.*

Corollary 3.5. *The class $e^\Gamma_b \in \mathrm{H}^2_b(\Gamma)$ is invariant under the action of the mapping class group $\mathcal{M}(S)$ of S.*

Proof. We begin with some computations concerning holonomy representations and start with a simple observation. For every $\psi \in \mathrm{Diff}^+(\tilde{S})$ and every non-zero tangent vector $\tilde{v}$ of $\tilde{S}$ based at $\tilde{x}$ we have

$$f_{\psi^*(\tilde{h}),\tilde{v}} = f_{\tilde{h},d\psi(\tilde{v})} \circ \psi. \tag{7}$$

Indeed, the map on the right hand side satisfies the defining properties of $f_{\psi^*(\tilde{h}),\tilde{v}}$. Now let $\varphi \in \mathrm{Diff}^+(S)$ be a diffeomorphism and choose a lift $\psi \in \mathrm{Diff}^+(\tilde{S})$. Then we obtain

$$\begin{aligned}
\rho_{\varphi^*(h),\tilde{v}}(\gamma) &= f_{\psi^*(\tilde{h}),\tilde{v}} \circ T_\gamma^{\tilde{x}} \circ \left(f_{\psi^*(\tilde{h}),\tilde{v}}\right)^{-1} \\
&= f_{\tilde{h},d\psi(\tilde{v})} \circ \psi T_\gamma^{\tilde{x}} \psi^{-1} \circ \left(f_{\tilde{h},d\psi(\tilde{v})}\right)^{-1} \\
&= f_{\tilde{h},d\psi(\tilde{v})} \circ T_{\varphi_*(\gamma)}^{\psi(\tilde{x})} \circ \left(f_{\tilde{h},d\psi(\tilde{v})}\right)^{-1} \\
&= \rho_{h,d\psi(\tilde{v})}(\varphi_*(\gamma)),
\end{aligned}$$

where we have used (7) in the second equality. Hence for every $\rho \in \mathrm{Hol}(\Gamma)$ and every $\varphi \in \mathrm{Diff}^+(S)$ the representation $\rho \circ \varphi_*$ is again contained in $\mathrm{Hol}(\Gamma)$. Therefore

$$(\varphi_*)^*(e_b^\Gamma) = (\varphi_*)^*(\rho^*(e_b)) = (\rho \circ \varphi_*)^*(e_b) = e_b^\Gamma$$

as desired. $\square$

4. The Algebraic Picture

In this section we describe the purely algebraic approach to our main result in the case of a compact surface S. From an algebraic point of view this is the most interesting case since the fundamental group $\Gamma = \pi_1(S, x)$ is non-free. A major role will be played by a certain central $\mathbb{Z}$-extension of Γ, denoted $\overline{\Gamma}_{2-2g}$ and isomorphic to the fundamental group of the unit tangent bundle of S. By the Dehn-Nielsen-Baer theorem the mapping class group $\mathcal{M}(S)$ can be identified with the group $\mathrm{Out}^+(\Gamma)$ of 'orientation preserving' outer automorphisms of Γ which is the starting point of our algebraic treatment. We prove that the natural homomorphism $\mathrm{Out}^+(\overline{\Gamma}_{2-2g}) \to \mathrm{Out}^+(\Gamma)$ splits and hence we obtain an action of $\mathrm{Out}^+(\Gamma)$ on the extension $\overline{\Gamma}_{2-2g}$ by outer automorphisms.
The pullback $\rho^*(e_b)$ to $\mathrm{H}^2_b(\Gamma)$ of the bounded Euler class via any holonomy representation is invariant under the mapping class group action. Its lift to $\mathrm{H}^2_b(\overline{\Gamma}_{2-2g})$ is invariant under the lifted mapping class group action and moreover lies in the kernel of the comparison map to the usual cohomology, hence is trivialised by a homogeneous quasimorphism. We prove that it is trivialised by a unique *invariant* homogeneous quasimorphism Rot which moreover takes integral values.

We start by recalling the connection between extensions of a group G by an abelian group A and the second cohomology $\mathrm{H}^2(G, A)$. Subsection 4.2 deals with lifting question in central extensions. We present a general result which describes, under certain assumptions, the (outer) automophism group of a central group extension of G by A in terms of the automorphism group of G and homomorphisms $G \to A$ (cf. Proposition 4.8). In Subsection 4.3 we specialise the discussion to the central $\mathbb{Z}$-extensions of a compact surface group Γ. In particular we determine their (outer) automorphism groups (Theorem 4.14). The next three subsections culminate in a description of the first and second cohomology group of $\mathrm{Out}^+(\Gamma)$ with (twisted) coefficients the dual of the abelianisation Γ_{ab}, see Theorem 4.30. For this we use a characterisation of the abelianisation of the Torelli group $\mathcal{I} \leq \mathcal{M}(S)$ obtained by Johnson as well as a result of Looijenga on the stable rational cohomology of mapping class groups. Finally we turn to the proof of the main results in Subsection 4.7 which will make use of most results obtained so far in this section.

4.1. **Abelian Extensions and H^2.** In this subsection we introduce the notion of abelian and central extensions of a group G by an abelian group A. Fixing a G-module structure on A the set of strong isomorphism classes of such extensions can be turned into a group which is naturally isomorphic to the second cohomology $\mathrm{H}^2(G, A)$. This is all classical, more information can be found in [25] for example.

An *abelian extension* of a group G by an abelian group A is an exact sequence E of the form

$$E: \qquad A \overset{i}{\rightarrowtail} \overline{G} \overset{p}{\twoheadrightarrow} G. \tag{8}$$

An abelian extension is *central* if moreover $i(A)$ is contained in the center of $\overline{G}$. We will write A additively but $\overline{G}$ and G multiplicatively. Since A is abelian the action of $\overline{G}$ on $i(A)$ by conjugation descends to an action of G. Identifying A with its image $i(A)$ we hence obtain a left action $\varphi_E : G \to \mathrm{Aut}(A)$ which turns A into a G-module. The extension E is central if and only if φ_E is trivial, i.e., if A is a trivial G-module.

The extension E is called *split* if there exists a homomorphism $\sigma : G \to \overline{G}$ with $p\sigma = \mathrm{id}_G$. We will call such maps σ *splittings* to distinguish them from arbitrary set-theoretical sections. An easy check shows that (8) splits if and only if $\overline{G} \cong A \rtimes_{\varphi_E} G$ is the semidirect product of A with G.
A *morphism of extensions* is a morphism of short exact sequences

(9)
$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, "h"] & \overline{H} \arrow[r, two heads] \arrow[d, "g"] & H \arrow[d, "f"] \\ A \arrow[r, tail] & \overline{G} \arrow[r, two heads] & G \end{tikzcd}$$

It is an isomorphism (i.e., it has a two-sided inverse) if and only if f, g, h are all isomorphisms. By the 5-lemma this is already the case if f and h are isomorphisms. An isomorphism of extensions of G by A of the special form

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & \overline{G}_1 \arrow[r, two heads] \arrow[d, "\cong"] & G \arrow[d, equal] \\ A \arrow[r, tail] & \overline{G}_2 \arrow[r, two heads] & G \end{tikzcd}$$

is called a *strong isomorphism.* Clearly, if E_1 and E_2 are strongly isomorphic then $\varphi_{E_1} = \varphi_{E_2}$, that is, conjugation induces the same G-module structure on A. Being strongly isomorphic is an equivalence relation and we will denote by $\mathcal{E}(G, A)$ the set of strong isomorphism classes of abelian extensions of G by the G-module A. We point out that the action of G on A is part of the input data, although this is not reflected in the notation (similar as in the notation of group (co-)homology).
Let E_1, E_2 be two abelian extensions of G by the G-module A. Form the pullback over p_1 and p_2 to get the lower right square and obtain d by the pullback property:

$$\begin{tikzcd} A \arrow[rr, "-\mathrm{id}"] \arrow[dr, "d"] \arrow[dd, "\mathrm{id}"'] & & A \arrow[d, tail, "i_2"] \\ & X \arrow[r] \arrow[d] & \overline{G}_2 \arrow[d, two heads, "p_2"] \\ A \arrow[r, tail, "i_1"] & \overline{G}_1 \arrow[r, two heads, "p_1"] & G \end{tikzcd}$$

Now $A \times A \rightarrowtail X \twoheadrightarrow G$ is exact and the conjugation action of G on $A \times A$ is the diagonal one. By setting $\overline{G} = \mathrm{coker}(d)$ we obtain another abelian extension

$$E_1 \cdot E_2 : \qquad A \overset{i}{\rightarrowtail} \overline{G} \overset{p}{\twoheadrightarrow} G$$

of G by A which is called the *Baer product* of E_1 and E_2 (cf. [1]). In fact, we have to be more careful since there are several choices involved in forming the above extension (everything is only defined up to unique isomorphism). To pin down the Baer product at least up to strong isomorphism we are going to make these choices now. Instead of working with universal properties we give explicit descriptions of the involved groups and maps. To start with, the pullback X is given by

$$X = \{(\overline{x}_1, \overline{x}_2) \in \overline{G}_1 \times \overline{G}_2 \mid p_1(\overline{x}_1) = p_2(\overline{x}_2)\}$$

and the map $d : A \to X$ by $a \mapsto (i_1(a), -i_2(a))$. Define

$$\begin{aligned} j : A \times A \to X, &\qquad (a_1, a_2) \mapsto (i_1(a_1), i_2(a_2)), \\ q : X \to G, &\qquad (\overline{x}_1, \overline{x}_2) \mapsto p_1(\overline{x}_1) = p_2(\overline{x}_2), \end{aligned}$$

then the upper part of the diagram

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, tail, "{(\mathrm{id}, -\mathrm{id})}"'] & A \arrow[d, tail, "d"] & \\
A \times A \arrow[r, tail, "j"] \arrow[d, two heads, "\Sigma"'] & X \arrow[r, two heads, "q"] \arrow[d, two heads] & G \arrow[d, equal] \\
A \arrow[r, tail, "i"] & \overline{G} \arrow[r, two heads, "p"] & G
\end{tikzcd}$$

commutes and the middle row is an abelian extension. Now we *define* the Baer product $E_1 \cdot E_2$ as the sequence of cokernels in the bottom row where Σ is the summation map. A routine verification shows:

Proposition 4.1. *The Baer product induces an abelian group structure on $\mathcal{E}(G, A)$ with the split extension as neutral element.*

Next we describe an isomorphism between $\mathcal{E}(G, A)$ and the second cohomology group $\mathrm{H}^2(G, A)$. Consider an extension E as in (8) and choose a set theoretical section $\sigma : G \to \overline{G}$ of p. For $x, y \in G$ the two elements $\sigma(x)\sigma(y)$ and $\sigma(xy)$ of $\overline{G}$ both lift the product xy, hence the map

$$\alpha : G^2 \to A, \qquad \alpha(x, y) = i^{-1}\big(\sigma(x)\sigma(y)\sigma(xy)^{-1}\big)$$

is well-defined. It is easy to see that

$$\sigma(x)\alpha(y, z)\sigma(x)^{-1}\alpha(x, yz) = \alpha(x, y)\alpha(xy, z)$$

for all $x, y, z \in G$ and hence α is an inhomogeneous 2-cocycle. Moreover, for a strong isomorphism

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, equal] & \overline{G} \arrow[r, two heads, "p"] \arrow[d, "g", "\cong"'] & G \arrow[d, equal] \\
A \arrow[r, tail, "i'"] & \overline{G}' \arrow[r, two heads, "p'"] & G
\end{tikzcd}$$

of extensions and sections σ, σ' of p, p' consider the map

$$\beta : G \to A, \qquad \beta(x) = i'^{-1}\big(\sigma'(x)g(\sigma(x))^{-1}\big).$$

A calculation shows that $\alpha' = \alpha + d\beta$ where α, α' are the cocycles computed using the two sections. Hence any strong isomorphism class of extensions gives rise to a well-defined class in $\mathrm{H}^2(G, A)$.

There is a more abstract way to describe the above construction using the 5-term exact sequence in group cohomology (cf. [38], 6.8.3.). Consider A as a $\overline{G}$-module via the map p. Then the 5-term exact sequence induced by the extension E takes the form

$$\mathrm{H}^1(G, A) \xrightarrow{p^*} \mathrm{H}^1(\overline{G}, A) \xrightarrow{i^*} \mathrm{Hom}(A, A)^G \xrightarrow{\delta} \mathrm{H}^2(G, A) \xrightarrow{p^*} \mathrm{H}^2(\overline{G}, A).$$

Now associate to E the class $[E] = \delta(\mathrm{id}_A) \in \mathrm{H}^2(G, A)$. By naturality of the 5-term sequence, strongly isomorphic extensions have the same class and hence we obtain a map $\mathcal{E}(G, A) \to \mathrm{H}^2(G, A)$ which agrees with the above construction.

Proposition 4.2. *The map $E \mapsto [E]$ induces a group isomorphism $\mathcal{E}(G, A) \to \mathrm{H}^2(G, A)$.*

The proof is somewhat tedious but straightforward, details can be found in [25] IV. Theorem 8.8. We content ourselves with describing the inverse map. Fix a class in

$\mathrm{H}^2(G, A)$ and choose a representing cocycle $\alpha : G^2 \to A$. Define a multiplication on the set $A \times G$ by

$$(a_1, x_1)(a_2, x_2) = (a_1 + a_2 + \alpha(x_1, x_2), x_1x_2).$$

One checks that this is a group structure, the associativity for example being equivalent to the cocycle identity for α. The resulting group $\overline{G}_\alpha$ fits in an abelian extension $A \rightarrowtail \overline{G}_\alpha \twoheadrightarrow G$ where the two homomorphisms are given by $i : A \to \overline{G}_\alpha$, $a \mapsto (a, 1)$ and $p : \overline{G}_\alpha \to G$, $(a, x) \mapsto x$. Moreover, if $\alpha' = \alpha + d\beta$ is another representative then the map

$$\overline{G}_\alpha \to \overline{G}_{\alpha'}, \qquad (a, x) \mapsto (a + \beta(x), x)$$

induces a strong isomorphism of extensions. Hence every class in $\mathrm{H}^2(G, A)$ gives rise to a well-defined element of $\mathcal{E}(G, A)$.

We note for later reference:

Lemma 4.3. *Assume that the extension E splits. Then the set of splittings is in bijection with the space of* 1*-cocycles $G \to A$ (where G acts on A by conjugation as usual). More precisely, the splittings form a one-dimensional affine space over the space of* 1*-cocycles. In particular we have:*

(a) *If the extension is central then the splittings are in bijection with* $\mathrm{Hom}(G, A)$.
(b) *If $H^1(G, A) = 0$ then all splittings are (as maps) conjugated under A, and the set of splittings is in bijection with A/A^G.*

Proof. Fix a splitting $s : G \to \overline{G}$. Then any section $\sigma : G \to \overline{G}$ is of the form $\sigma = (i \circ f) \cdot s$ with a well-defined map $f : G \to A$. A computation gives

$$\begin{aligned}
\sigma(x)\sigma(y) &= i(f(x))s(x)i(f(y))s(y) \\
&= i(f(x) + x \cdot f(y))s(x)s(y) \\
&= i\big(x \cdot f(y) - f(xy) + f(x)\big)i(f(xy))s(xy) \\
&= i\big(x \cdot f(y) - f(xy) + f(x)\big)\sigma(xy),
\end{aligned}$$

hence σ is a homomorphism if and only if f is a 1-cocycle.
From this (a) is clear as then $Z^1(G, A) = \mathrm{Hom}(G, A)$. In the situation of (b) there exists an element $a \in A$ such that $f(x) = a - xa$ and consequently

$$\sigma(x) = [a, x]s(x) = as(x)a^{-1}.$$

Therefore A acts transitively by conjugation on the set of splitting maps and all stabilisers are equal to A^G. $\square$

4.2. **Automorphisms of Central Extensions.** In this subsection we restrict our attention to *central extensions*

$$E : \qquad A \overset{i}{\rightarrowtail} \overline{G} \overset{p}{\twoheadrightarrow} G.$$

That is, we keep the standing assumption that $i(A)$ is a central subgroup of $\overline{G}$ or, what amounts to the same, that A is a trivial G-module. In the following we deal with lifting questions in central extensions and will finally describe, for suitable central extensions E, the (outer) automophism group of $\overline{G}$ in terms of the automorphism group of G and homomorphisms $G \to A$ (cf. Proposition 4.8).

Let E be as above, let H be a group and let $f : H \to G$ be a group homomorphism. Forming the pullback over f and p we obtain the diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & \widetilde{H} \arrow[r, two heads] \arrow[d] & H \arrow[d, "f"] \\ A \arrow[r, tail] & \overline{G} \arrow[r, two heads, "p"] & G \end{tikzcd}$$

where the top row is a central extension of H by A which we call the *pullback* of E under f and which we denote by f^*E. Every morphism $F \to E$ of central extensions as in (9) factors uniquely over the pullback f^*E in the following way:

$$\begin{tikzcd} F: & B \arrow[r, tail] \arrow[d, "h"] & \overline{H} \arrow[r, two heads] \arrow[d, "\overline{g}"] & H \arrow[d, equal] \\ f^*E: & A \arrow[r, tail] \arrow[d, equal] & \widetilde{H} \arrow[r, two heads] \arrow[d] & H \arrow[d, "f"] \\ E: & A \arrow[r, tail] & \overline{G} \arrow[r, two heads] & G \end{tikzcd}$$

It is easy to see that for a fixed homomorphism $f : H \to G$ the pullback construction induces a group homomorphism $f^*(-) : \mathcal{E}(G, A) \to \mathcal{E}(H, A)$. Considering A as trivial module for all groups we hence obtain a functor

$$\mathcal{E}(-, A) : \mathbf{Grp} \to \mathbf{Ab}^{\mathrm{op}}$$

and Proposition 4.2 can be strengthened as follows:

Theorem 4.4. *The map* $[-] : \mathcal{E}(-, A) \to \mathrm{H}^2(-, A)$ *is a natural isomorphism of functors* $\mathbf{Grp} \to \mathbf{Ab}^{\mathrm{op}}$.

As a consequence we have the following lifting result.

Corollary 4.5. *Let E be a central extension and let $f : H \to G$ be a group homomorphism as above.*

(a) *There exists a lift $\overline{f} : H \to \overline{G}$ if and only if $f^*([E]) = 0$ in $\mathrm{H}^2(H, A)$. In particular, the pullback $p^*([E]) \in \mathrm{H}^2(\overline{G}, A)$ vanishes.*

(b) *Assume that $\overline{f}$ is such a lift. Then the lifts are precisely the maps of the form $h \mapsto \overline{f}(h) \cdot i(g(h))$ for a homomorphism $g : H \to A$.*

Proof. By Theorem 4.4 the vanishing $f^*([E]) = 0$ is equivalent to the splitting of the pullback $f^*E = (A \rightarrowtail \widetilde{H} \twoheadrightarrow H)$. Assume first that $\overline{f}$ exists and consider the diagram

$$\begin{tikzcd} H \arrow[dr, dotted, "s"] \arrow[drr, bend left, "\mathrm{id}_H"] \arrow[ddr, bend right, "\overline{f}"'] & & \\ & \widetilde{H} \arrow[r, two heads] \arrow[d] & H \arrow[d, "f"] \\ & \overline{G} \arrow[r, two heads, "p"] & G \end{tikzcd}$$

By the pullback property of the square s exists and is a splitting of f^*E. On the other hand, if $s : H \to \widetilde{H}$ is a splitting then the composition $H \xrightarrow{s} \widetilde{H} \to \overline{G}$ is a lift of f. This proves (a). For the second part, assume that $\tilde{f}$ is a second lift. Then

$$p\big(\overline{f}(h)^{-1}\tilde{f}(h)\big) = 1$$

for every $h \in H$, hence there is a unique map $g : H \to A$ with $\overline{f}(h)^{-1}\tilde{f}(h) = i(g(h))$ for all $h \in H$. Now g is a homomorphism as $\overline{f}$ and $\tilde{f}$ are homomorphisms and A is central. $\square$

Let E again be as above. An endomorphism $\overline{\varphi} : \overline{G} \to \overline{G}$ descends to a well-defined endomorphism $\varphi : G \to G$ (and hence is a lift of a such) if and only if $\overline{\varphi}(i(A)) \subset i(A)$. In the other direction we have the following criterion:

Corollary 4.6. *An endomorphism φ of G lifts to an endomorphism of $\overline{G}$ if and only if $\varphi^*(\ker p^*) \leq \ker p^*$ where $p^* : \mathrm{H}^2(G, A) \to \mathrm{H}^2(\overline{G}, A)$ denotes the induced map on the level of cohomology.*

Proof. Assume that $\overline{\varphi}$ is a lift of φ. Then the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{H}^2(G, A) \arrow[r, "p^*"] \arrow[d, "\varphi^*"] & \mathrm{H}^2(\overline{G}, A) \arrow[d, "\overline{\varphi}^*"] \\
\mathrm{H}^2(G, A) \arrow[r, "p^*"] & \mathrm{H}^2(\overline{G}, A)
\end{tikzcd}$$

implies $\varphi^*(\ker p^*) \leq \ker p^*$. Conversely, the latter implies $\varphi^*([E]) \in \ker p^*$ by Corollary 4.5 (a). Hence $(\varphi \circ p)^*([E]) = 0$ and therefore $\varphi \circ p$ lifts to an endomorphism of $\overline{G}$ again by Corollary 4.5. $\square$

Denote by $\mathrm{End}_{\uparrow}(G)$ the set of endomorphisms of G which lift to endomorphisms of $\overline{G}$ and denote by $\mathrm{End}_A(\overline{G})$ the set of endomorphisms $\overline{\varphi}$ of $\overline{G}$ such that $\overline{\varphi}(i(A)) \subset i(A)$. Obviously, both sets are closed under composition. But in general the inverse of an automorphism in one of these sets will not automatically lie in the set again. For this to hold we have to impose further conditions. We shall need the following basic result.

Lemma 4.7. *Let $C \leq B$ be finitely generated abelian groups. If φ is an automorphism of B such that $\varphi(C) \leq C$, then actually $\varphi(C) = C$.*

Proof. It is well known that finitely generated abelian groups are hopfian, that is, every surjective endomorphism of such a group is an isomorphism. Consider the induced map $\overline{\varphi} : B/\varphi(C) \to B/\varphi(C)$. Then $\overline{\varphi}$ is surjective and hence is an isomorphism since $B/\varphi(C)$ is finitely generated. But clearly $C/\varphi(C) \leq \ker \overline{\varphi}$ and therefore $\varphi(C) = C$. $\square$

We turn to the main result of this subsection:

Proposition 4.8. *Let E be a central extension as above. Assume that:*

(i) *The center of $\overline{G}$ is finitely generated.*
(ii) *$\mathrm{H}^2(G, A)$ is finitely generated.*
(iii) *There is no non-trivial homomorphism $A/([\overline{G}, \overline{G}] \cap A) \to A$.*

Then the following holds:

(a) $\mathrm{Aut}_A(\overline{G}) = \mathrm{Aut}(\overline{G}) \cap \mathrm{End}_A(\overline{G})$ *is a subgroup of* $\mathrm{Aut}(\overline{G})$ *and* $\mathrm{Aut}_{\uparrow}(G) = \mathrm{Aut}(G) \cap \mathrm{End}_{\uparrow}(G)$ *is a subgroup of* $\mathrm{Aut}(G)$.
(b) *Every lift of every element in* $\mathrm{Aut}_{\uparrow}(G)$ *is an automorphism of $\overline{G}$ and there is an abelian extension*

$$\mathrm{Hom}(G, A) \stackrel{\Delta}{\rightarrowtail} \mathrm{Aut}_A(\overline{G}) \twoheadrightarrow \mathrm{Aut}_{\uparrow}(G)$$

where Δ sends $h : G \to A$ to the automorphism $\overline{x} \mapsto \overline{x} \cdot i(h(p(\overline{x})))$ of $\overline{G}$.

Proof. To prove (a) it is enough to show that both sets are closed under inversion. Consider an element $\overline{\varphi} \in \operatorname{Aut}_A(\overline{G})$. By (i) we can apply Lemma 4.7 to the groups $i(A) \leq Z(\overline{G})$ and the restriction $\overline{\varphi}|_{Z(\overline{G})}$ (which is automatically an automorphism) to conclude that $\overline{\varphi}(i(A)) = i(A)$. So $(\overline{\varphi})^{-1}$ is in $\operatorname{Aut}_A(\overline{G})$ as well. Next, consider $\varphi \in \operatorname{Aut}_\uparrow(G)$ and let $\psi \in \operatorname{Aut}(G)$ be its inverse. We shall use the lifting criterion from Corollary 4.6. By (ii) we can apply Lemma 4.7 to the groups $\ker p^* \leq \mathrm{H}^2(G, A)$ and the isomorphism φ^* to obtain $\varphi^*(\ker p^*) = \ker p^*$. This shows that $\psi^*(\ker p^*) = \ker p^*$ and hence ψ lifts as well.
To prove (b) we first look at elements in $\operatorname{End}_A(\overline{G})$ which lift the identity id_G. By Corollary 4.5 (b) these are precisely the maps of the form $\overline{x} \mapsto \overline{x} \cdot H(\overline{x})$ for a homomorphism $H : \overline{G} \to A$. Now H factors over the abelianisation of $\overline{G}$ and hence by (iii) the restriction $H|_A$ must be trivial. Therefore H factors over a homomorphism $h : G \to A$. Summarising, there is a bijection

$$\Delta : \operatorname{Hom}(G, A) \to \ker\left(\operatorname{End}_A(\overline{G}) \to \operatorname{End}_\uparrow(G)\right)$$

which sends h to the endomorphism $\overline{x} \mapsto \overline{x} \cdot i(h(p(\overline{x})))$ and a short calculation shows that Δ is actually a homomorphism. In particular we have $\Delta(h) \circ \Delta(-h) = \mathrm{id}_{\overline{G}}$ for all h and hence $\operatorname{im}(\Delta) \leq \operatorname{Aut}_A(\overline{G})$.
Finally we prove that every lift of every element in $\operatorname{Aut}_\uparrow(G)$ is bijective. As a consequence, the map $\operatorname{Aut}_A(\overline{G}) \to \operatorname{Aut}_\uparrow(G)$ is surjective and in combination with the above discussion about Δ this proves the exactness of the sequence

$$\operatorname{Hom}(G, A) \stackrel{\Delta}{\rightarrowtail} \operatorname{Aut}_A(\overline{G}) \twoheadrightarrow \operatorname{Aut}_\uparrow(G).$$

To do so we consider mutually inverse maps $\varphi, \psi \in \operatorname{Aut}_\uparrow(G)$ and arbitrary lifts $\overline{\varphi}$ and $\overline{\psi}$. By the previous paragraph there exists $h \in \operatorname{Hom}(G, A)$ such that $\overline{\varphi} \circ \overline{\psi} = \Delta(h)$. As the map on the right hand side is bijective, $\overline{\varphi}$ is surjective and $\overline{\psi}$ is injective. By symmetry, both maps are indeed bijective.

□

Since $\operatorname{Inn}(\overline{G}) \subseteq \operatorname{Aut}_A(\overline{G})$ and $\operatorname{Inn}(G) \subseteq \operatorname{Aut}_\uparrow(G)$ we can form the quotients

$$\begin{aligned} \operatorname{Out}_A(\overline{G}) &= \operatorname{Aut}_A(\overline{G}) / \operatorname{Inn}(\overline{G}), \\ \operatorname{Out}_\uparrow(G) &= \operatorname{Aut}_\uparrow(G) / \operatorname{Inn}(G) \end{aligned}$$

which are subgroups of the corresponding outer automorphisms groups.

Corollary 4.9. *Under the same assumptions as in Proposition 4.8 there is an abelian extension*

$$W \stackrel{\Delta}{\rightarrowtail} \operatorname{Out}_A(\overline{G}) \twoheadrightarrow \operatorname{Out}_\uparrow(G)$$

where W is the cokernel of the injective commutator map

$$p^{-1}(Z(G))/Z(\overline{G}) \xrightarrow{\overline{x} \mapsto [\overline{x}, -]} \operatorname{Hom}(G, A).$$

Proof. Since the sequence

$$1 \longrightarrow Z(\overline{G}) \longrightarrow p^{-1}(Z(G)) \xrightarrow{\mathrm{Int}} \operatorname{Inn}(\overline{G}) \longrightarrow \operatorname{Inn}(G) \longrightarrow 1$$

is exact the commutative diagram

$$\begin{tikzcd} p^{-1}(Z(G))/Z(\overline{G}) \arrow[r, tail] \arrow[d, "{[\overline{x}, -]}"] & \operatorname{Inn}(\overline{G}) \arrow[r, two heads] \arrow[d, tail] & \operatorname{Inn}(G) \arrow[d, tail] \\ \operatorname{Hom}(G, A) \arrow[r, tail, "\Delta"] & \operatorname{Aut}_A(\overline{G}) \arrow[r, two heads] & \operatorname{Aut}_\uparrow(G) \end{tikzcd}$$

has exact rows. Now apply the snake lemma. □

4.3. **Compact Surface Groups.** In this subsection we collect some basic facts about the fundamental group $\Gamma = \pi_1(S)$ of a closed orientable surface S of genus g and its central $\mathbb{Z}$-extensions. Then we apply the results of the last subsection to the situation at hand. In the beginning we closely follow [8], Section 3.3.

It is well known that Γ has the following standard presentation:

$$\Gamma = \Big\langle a_1, b_1, \dots, a_g, b_g \;\Big|\; \prod_{k=1}^{g} [a_k, b_k] = 1 \Big\rangle.$$

For any integer n we define the group

$$\overline{\Gamma}_n = \Big\langle a_1, b_1, \dots, a_g, b_g, c \;\Big|\; \prod_{k=1}^{g} [a_k, b_k] = c^n,\ c \text{ is central} \Big\rangle.$$

Lemma 4.10. *The group $\overline{\Gamma}_n$ fits into a central extension*

$$E_n: \qquad \mathbb{Z} \overset{i}{\rightarrowtail} \overline{\Gamma}_n \overset{p}{\twoheadrightarrow} \Gamma$$

where $i(k) = c^k$ and where p is the obvious projection. Moreover, $E_m \cdot E_n \cong E_{m+n}$ for all integers m, n.

Proof. The natural homomorphism $p : \overline{\Gamma}_n \to \Gamma$ induced by $a_k \mapsto a_k$, $b_k \mapsto b_k$ and $c \mapsto 1$ is surjective with kernel the central subgroup generated by c. To show that c has infinite order in $\overline{\Gamma}_n$ we can clearly assume $n \neq 0$. The homomorphism $\overline{\Gamma}_n \to H_3$ into the Heisenberg group given by

$$a_k \mapsto \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b_k \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, \qquad c \mapsto \begin{pmatrix} 1 & 0 & gn \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

maps c to an element of infinite order. Hence i is injective and E_n is indeed a central extension.
Next we compute the Baer product $E_m \cdot E_n$. The pullback $X \leq \overline{\Gamma}_m \times \overline{\Gamma}_n$ in the construction is generated by the elements

$$d = \big(c^{(m)}, 1\big), e = \big(1, c^{(n)}\big) \quad \text{and} \quad u_k = \big(a_k^{(m)}, a_k^{(n)}\big), v_k = \big(b_k^{(m)}, b_k^{(n)}\big), \quad 1 \leq k \leq g$$

and is actually given by the presentation

$$X = \Big\langle u_1, v_1, \dots, u_g, v_g, d, e \;\Big|\; \prod_{k=1}^{g} [u_k, v_k] = d^m e^n, d \text{ and } e \text{ are central} \Big\rangle.$$

Define the map $X \to \overline{\Gamma}_{m+n}$ by $u_k \mapsto a_k$, $v_k \mapsto b_k$ and $d \mapsto c$, $e \mapsto c$. Then the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] \arrow[d, tail, "{(\mathrm{id}, -\mathrm{id})}"'] & \mathbb{Z} \arrow[d, tail, "1 \mapsto de^{-1}"] & \\
\mathbb{Z} \times \mathbb{Z} \arrow[r, tail, "{(k,l) \mapsto d^k e^l}"] \arrow[d, two heads, "\Sigma"'] & X \arrow[r, two heads] \arrow[d, two heads] & \Gamma \arrow[d, equal] \\
\mathbb{Z} \arrow[r, tail, "i"] & \overline{\Gamma}_{m+n} \arrow[r, two heads, "p"] & \Gamma
\end{tikzcd}$$

commutes and hence indeed $E_m \cdot E_n \cong E_{m+n}$. $\square$

Lemma 4.11. *The class $[E_1]$ generates $\mathrm{H}^2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. ([8], Proposition 3.1.) We first prove that every central $\mathbb{Z}$-extension of Γ is strongly isomorphic to one of the extensions E_n. Let

$$\mathbb{Z} \rightarrowtail^{\iota} \Lambda \xrightarrow{\pi} \Gamma$$

be such an extension. Choose lifts α_k and β_k of a_k and b_k, these are determined up to central elements. Hence the product $l = \prod_{k=1}^{g}[\alpha_k, \beta_k]$ does not depend on the lifts and lies in the kernel of π. So there is an integer n with $\iota(n) = l$. The homomorphism $f : \overline{\Gamma}_n \to \Lambda$ defined by $a_k \mapsto \alpha_k$, $b_k \mapsto \beta_k$ and $c \mapsto \iota(1)$ fits into the commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, tail, "i"] \arrow[d, equal] & \overline{\Gamma}_n \arrow[r, two heads, "p"] \arrow[d, "f"] & \Gamma \arrow[d, equal] \\ \mathbb{Z} \arrow[r, tail, "\iota"] & \Lambda \arrow[r, two heads, "\pi"] & \Gamma \end{tikzcd}$$

and so gives a strong isomorphism of extensions by the 5-lemma. In combination with Lemma 4.10 this shows that $\mathcal{E}(\Gamma, \mathbb{Z})$ is generated by the class of E_1. But it is obvious that two extensions E_m, E_n are never strongly isomorphic for $m \neq n$. Hence the class of E_1 has infinite order. $\square$

We point out that this does not mean that the *groups* $\overline{\Gamma}_n$ are pairwise non-isomorphic. In fact $\overline{\Gamma}_n \cong \overline{\Gamma}_{-n}$ for all n. The computation of the abelianisations in the next lemma shows that these are the only non-trivial isomorphisms among the $\overline{\Gamma}_n$.

Lemma 4.12. *Let n be an integer.*

(a) *The center $Z(\overline{\Gamma}_n) = \langle c \rangle$ is isomorphic to $\mathbb{Z}$.*
(b) *We have $[\overline{\Gamma}_n, \overline{\Gamma}_n] \cap \langle c \rangle = \langle c^n \rangle$ and $(\overline{\Gamma}_n)_{ab} \cong \Gamma_{ab} \times (\mathbb{Z}/n\mathbb{Z})$.*

Proof. (a) The well known fact that Γ has trivial center implies $Z(\overline{\Gamma}_n) \leq \langle c \rangle$ while the other inclusion is trivial. (b) By definition, c^n is a product of commutators. On the other hand, there is a surjective homomorphism $\overline{\Gamma}_n \to \mathbb{Z}/n\mathbb{Z}$ given by $a_k \mapsto 0$, $b_k \mapsto 0$ and $c \mapsto 1$, hence $[\overline{\Gamma}_n, \overline{\Gamma}_n] \cap \langle c \rangle \leq \langle c^n \rangle$. So the diagram

$$\begin{tikzcd} n\mathbb{Z} \arrow[r, tail] \arrow[d] & {[\overline{\Gamma}_n, \overline{\Gamma}_n]} \arrow[r, two heads] \arrow[d] & {[\Gamma, \Gamma]} \arrow[d] \\ \mathbb{Z} \arrow[r, tail, "i"] & \overline{\Gamma}_n \arrow[r, two heads, "p"] & \Gamma \end{tikzcd}$$

has exact rows and the snake lemma gives the exact sequence

$$\mathbb{Z}/n\mathbb{Z} \rightarrowtail (\overline{\Gamma}_n)_{ab} \twoheadrightarrow \Gamma_{ab}.$$

The latter splits because Γ_{ab} is free abelian. $\square$

The extension E_{2-2g} has an important geometric meaning as it describes the long exact homotopy sequence of the unit tangent bundle $\mathbb{T}^1 S \to S$:

Lemma 4.13. *There is a central extension*

$$\pi_1(S^1) \rightarrowtail \pi_1(\mathbb{T}^1 S) \twoheadrightarrow \pi_1(S).$$

To be more precise, choose a standard curve system $a_1, b_1, \ldots, a_g, b_g$ on S such that the lift to the universal cover $\tilde{S}$ of $\prod_{i=1}^{g}[a_i, b_i]$ bounds a positively oriented disc and consider the isomorphism $\pi_1(S) \cong \Gamma$ induced by this curve system. Moreover identify $\pi_1(S^1) \cong \mathbb{Z}$ such that the composition $\mathbb{Z} \to \pi_1(S_1) \to \pi_1(\mathbb{T}^1 S)$ maps 1 to the class of the positively oriented fibre. Then the above extension is strongly isomorphic to the extension E_{2-2g}.

Proof. The long exact homotopy sequence for the S^1-bundle $\mathbb{T}^1S \to S$ gives an exact sequence

$$\pi_2(S) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(\mathbb{T}^1S) \longrightarrow \pi_1(S) \longrightarrow \pi_0(S^1).$$

Here the first term is trivial since the universal covering of S is contractible, and the last term is trivial since S^1 is path-connected. Clearly, the action of $\pi_1(S)$ on the fundamental group of the fibre is trivial, hence the extension is central.
It remains to determine the strong isomorphy type of this extension. For this we choose a point $x \in S$ which does not lie on any of the curves a_i, b_i and we choose a vector field X on S which vanishes only at the point x. Define lifts $\tilde{a}_i = X \circ a_i$ and $\tilde{b}_i = X \circ b_i$ and observe that the closed curve $\prod_{i=1}^{g}[\tilde{a}_i, \tilde{b}_i]$ is homotopic to $X \circ c$ where c is the boundary of a small positively oriented disc around the point x. But the later is homotopic to $i_X(x)$ times the positively oriented fibre of $\mathbb{T}^1S$ over the point x by definition of the index $i_X(x)$. Finally, by the Poincaré-Hopf theorem, we have $i_X(x) = \chi(S) = 2 - 2g$ which implies the claim by the proof of Lemma 4.11. $\square$

Set $H = \Gamma_{ab}$ and observe that H is a free abelian group of rank $2g$, freely generated by the images of the generators a_k, b_k of Γ. We will frequently use the notation $H^* = \mathrm{Hom}(H, \mathbb{Z})$ for the dual group of H. We now turn to the description of the (outer) automorphisms group of $\overline{\Gamma}_n$ in terms of H^* and the corresponding automorphism group of Γ:

Theorem 4.14. *Assume that $n \neq 0$. There are abelian extensions*

$$H^* \rightarrowtail^{\Delta} \mathrm{Aut}(\overline{\Gamma}_n) \twoheadrightarrow \mathrm{Aut}(\Gamma)$$

and

$$H^* \rightarrowtail^{\Delta} \mathrm{Out}(\overline{\Gamma}_n) \twoheadrightarrow \mathrm{Out}(\Gamma).$$

The map Δ sends a homomorphism $l : H \to \mathbb{Z}$ to the automorphism $z \mapsto z \cdot c^{l(q(z))}$ where $q : \overline{\Gamma}_n \to \Gamma \to H$ denotes the canonical projection.

Proof. We will derive the claim from Proposition 4.8 and start by verifying the three conditions in the statement there. We set $A = \langle c \rangle \cong \mathbb{Z}$ for notational clarity. By Lemma 4.12 (a) the center $Z(\overline{\Gamma}_n) = A$ is finitely generated and by (b) the group $A/([\overline{\Gamma}_n, \overline{\Gamma}_n] \cap A)$ is cyclic of order $n \neq 0$, hence does not admit any non-trivial homomorphisms to A. Finally, $\mathrm{H}^2(\Gamma, A) \cong \mathbb{Z}$ is finitely generated. Proposition 4.8 now gives the abelian extension

$$\mathrm{Hom}(\Gamma, A) \rightarrowtail^{\Delta} \mathrm{Aut}_A(\overline{\Gamma}_n) \twoheadrightarrow \mathrm{Aut}_{\uparrow}(\Gamma).$$

Since $A = Z(\overline{\Gamma}_n)$ is characteristic we have $\mathrm{Aut}_A(\overline{\Gamma}_n) = \mathrm{Aut}(\overline{\Gamma}_n)$. Moreover, every automorphism of Γ acts by multiplication with ± 1 on $\mathrm{H}^2(\Gamma, A) \cong \mathbb{Z}$ and therefore preserves $\ker(p^*)$, hence $\mathrm{Aut}_{\uparrow}(\Gamma) = \mathrm{Aut}(\Gamma)$. Finally, we observe that the identification $\mathbb{Z} \cong A$ given by $1 \mapsto c$ induces an identification $\mathrm{Hom}(\Gamma, A) \cong H^*$ and the description of the map Δ above follows from the description given in Proposition 4.8.
For the second extension we can apply Corollary 4.9 and observe that $p^{-1}(Z(\Gamma)) = A = Z(\overline{\Gamma}_n)$. $\square$

Note that for $n = 0$ the analogous statement fails. The automorphism of $\overline{\Gamma}_0$ induced by $a_k \mapsto a_k$, $b_k \mapsto b_k$ and $c \mapsto c^{-1}$ lifts the identity on Γ but does not lie in the image of Δ.

Lemma 4.15. *For $\varphi \in \mathrm{Aut}(\Gamma)$ the following conditions are equivalent:*

(i) *The induced isomorphism* φ^* *of* $\mathrm{H}^2(\Gamma, \mathbb{Z})$ *is the identity.*
(ii) *For every integer* $n \neq 0$ *and every lift* $\overline{\varphi} \in \mathrm{Aut}(\overline{\Gamma}_n)$ *of* φ *the restriction of* $\overline{\varphi}$ *to the center* $Z(\overline{\Gamma}_n)$ *is the identity.*

Proof. In the situation of (ii) we set $A = Z(\overline{\Gamma}_n) \cong \mathbb{Z}$ and consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[r, "\delta"] \arrow[d, "(\overline{\varphi}|_A)^*"'] & \mathrm{H}^2(\Gamma, A) \arrow[d, "\varphi^*"] \\
\mathrm{Hom}(A,A) \arrow[r, "\delta"] & \mathrm{H}^2(\Gamma, A)
\end{tikzcd}$$

where δ is the connecting homomorphism in the cohomological 5-term sequence associated to E_n. As $n \neq 0$ we have $\delta(\mathrm{id}_A) = [E_n] \neq 0$ by Lemma 4.11, in particular δ is non-trivial. Since the groups $\mathrm{Hom}(A, A)$ and $\mathrm{H}^2(\Gamma, A)$ are both infinite cyclic and since δ is non-trivial, the maps φ^* and $(\overline{\varphi}|_A)^*$ have to be multiplication with the same integer. Hence (i) and (ii) are equivalent. □

We denote by $\mathrm{Aut}^+(\Gamma)$ the group of automorphisms which satisfy the equivalent conditions in Lemma 4.15. For $n \neq 0$ we define $\mathrm{Aut}^+(\overline{\Gamma}_n)$ to be the preimage of $\mathrm{Aut}^+(\Gamma)$ under the projection $\mathrm{Aut}(\overline{\Gamma}_n) \to \mathrm{Aut}(\Gamma)$. For $n = 0$ we define $\mathrm{Aut}^+(\overline{\Gamma}_0)$ to be the group of automorphism $\overline{\varphi}$ which restrict to the identity on the center and which map to an element of $\mathrm{Aut}^+(\Gamma)$ under the canonical projection. It is easy to see that all the groups Aut^+ have index 2 in the corresponding full automorphism groups except for $|\mathrm{Aut}(\overline{\Gamma}_0) : \mathrm{Aut}^+(\overline{\Gamma}_0)| = 4$. Finally, since inner automorphisms are contained in Aut^+, it makes sense to form the quotients $\mathrm{Out}^+(\Gamma)$ and $\mathrm{Out}^+(\overline{\Gamma}_n)$.

Corollary 4.16. *If one replaces each* Aut *by* Aut^+ *and each* Out *by* Out^+ *in Theorem 4.14, one again obtains abelian extensions. This holds for all* n*, including the case* $n = 0$.

Proof. For $n \neq 0$ one only has to observe that the image of the map $\Delta : H^* \to \mathrm{Aut}(\overline{\Gamma}_n)$ actually lies in $\mathrm{Aut}^+(\overline{\Gamma}_n)$ since automorphisms of the form $\Delta(l)$ all descend to the identity on Γ. For $n = 0$ one can directly adapt the proof given in Theorem 4.14 with obvious small modifications. □

We denote the abelian extensions described in the above corollary by $\mathrm{Aut}^+(E_n)$ respectively $\mathrm{Out}^+(E_n)$.

Proposition 4.17. *The map* $\mathcal{E}(\Gamma, \mathbb{Z}) \to \mathcal{E}(\mathrm{Aut}^+(\Gamma), H^*)$ *induced by* $E_n \mapsto \mathrm{Aut}^+(E_n)$ *is a group homomorphism. The same holds if we replace* Aut^+ *by* Out^+.

Proof. Let m, n be integers. The Baer product $E_m \cdot E_n$ is given by the lower row of the diagram

(10)
$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] \arrow[d, tail, "(\mathrm{id}, -\mathrm{id})"'] & \mathbb{Z} \arrow[r] \arrow[d, tail] & 1 \arrow[d] \\
\mathbb{Z} \times \mathbb{Z} \arrow[r, tail, "i_m \times i_n"] \arrow[d, two heads, "\Sigma"'] & X \arrow[r, two heads] \arrow[d, two heads] & \Gamma \arrow[d, equal] \\
\mathbb{Z} \arrow[r, tail, "i"] & \overline{\Gamma} \arrow[r, two heads] & \Gamma
\end{tikzcd}$$

and is the sequence of cokernels of the upper part. Here

$$X = \{(\overline{x}_m, \overline{x}_n) \in \overline{\Gamma}_m \times \overline{\Gamma}_n \mid p_m(\overline{x}_m) = p_n(\overline{x}_n)\}.$$

Hence it is enough to prove that the group $\mathrm{Aut}^+(\overline{\Gamma})$ fits into a commutative diagram

(11)
$$\begin{tikzcd}
H^* \arrow[r, equal] \arrow[d, tail, "{(\mathrm{id},-\mathrm{id})}"'] & H^* \arrow[r] \arrow[d, tail] & 1 \arrow[d] \\
H^* \times H^* \arrow[r, tail, "\Delta_m \times \Delta_n"] \arrow[d, two heads, "\Sigma"'] & W \arrow[r, two heads] \arrow[d, "f"] & \mathrm{Aut}^+(\Gamma) \arrow[d, equal] \\
H^* \arrow[r, tail, "\Delta"] & \mathrm{Aut}^+(\overline{\Gamma}) \arrow[r, two heads] & \mathrm{Aut}^+(\Gamma)
\end{tikzcd}$$

with short exact columns and rows. The group W is given by

$$\begin{aligned} W \quad = \quad & \{(\overline{\varphi}_m, \overline{\varphi}_n) \in \mathrm{Aut}^+(\overline{\Gamma}_m) \times \mathrm{Aut}^+(\overline{\Gamma}_n) \mid \\ & \overline{\varphi}_m \text{ and } \overline{\varphi}_n \text{ descend to the same map in } \mathrm{Aut}^+(\Gamma)\} \end{aligned}$$

and the map $\Delta : H^* \to \mathrm{Aut}^+(\overline{\Gamma})$ by $\Delta(l) : \overline{x} \mapsto \overline{x} \cdot i(l(q(\overline{x}))$ where $q : \overline{\Gamma} \to H$ is the canonical projection. Note that the lower row is exact by Corollary 4.16. Hence it is sufficient to construct a homomorphism f such that the diagram commutes and such that the composition $H^* \to W \xrightarrow{f} \mathrm{Aut}^+(\overline{\Gamma})$ is trivial. The exactness of the middle column then follows from the nine lemma.
To define f, choose a pair $(\overline{\varphi}_m, \overline{\varphi}_n) \in W$ and denote by φ the common image of both maps in $\mathrm{Aut}^+(\Gamma)$. Since by assumption $\overline{\varphi}_m, \overline{\varphi}_n$ both restrict to the identity on the center there is an isomorphism of extensions

$$\begin{tikzcd}
\mathbb{Z} \times \mathbb{Z} \arrow[r, tail, "i_m \times i_n"] \arrow[d, equal] & X \arrow[r, two heads] \arrow[d, "\overline{\varphi}_m \times \overline{\varphi}_n"] & \Gamma \arrow[d, "\varphi"] \\
\mathbb{Z} \times \mathbb{Z} \arrow[r, tail, "i_m \times i_n"] & X \arrow[r, two heads] & \Gamma
\end{tikzcd}$$

which, by passing to the cokernel sequence in (10), induces an isomorphism

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, tail] \arrow[d, equal] & \overline{\Gamma} \arrow[r, two heads] \arrow[d, "\overline{\varphi}"] & \Gamma \arrow[d, "\varphi"] \\
\mathbb{Z} \arrow[r, tail] & \overline{\Gamma} \arrow[r, two heads] & \Gamma
\end{tikzcd}$$

We define $f\big((\overline{\varphi}_m, \overline{\varphi}_n)\big) = \overline{\varphi}$. Clearly, f is a homomorphism and the above diagram shows that it takes values in $\mathrm{Aut}^+(\overline{\Gamma})$. By the same reason, the lower right hand square in (11) commutes. A routine verification finally shows the commutativity of the lower left square in (11) as well as the triviality of the composition $H^* \to W \xrightarrow{f} \mathrm{Aut}^+(\overline{\Gamma})$.
The corresponding statement for $\mathrm{Out}^+(E_n)$ can be proved in the same way just by dividing out the inner automorphisms at the appropriate places. □

Corollary 4.18. *The cohomology class* $[\mathrm{Out}^+(E_n)]$ *lies in*

$$n \cdot \mathrm{H}^2\left(\mathrm{Out}^+(\Gamma), H^*\right).$$

We will show later (cf. Theorem 4.30) that the group $\mathrm{H}^2\left(\mathrm{Out}^+(\Gamma), H^*\right)$ is annihilated by $2g-2$. In combination with the previous corollary this shows that the extension $\mathrm{Out}^+(E_n)$ splits whenever n is divisible by $2g-2$.

4.4. **The Symplectic Group and the Torelli Group.** By the homotopy invariance of singular homology the mapping class group $\mathcal{M}(S)$ of a surface S acts on the integral homology $\mathrm{H}_1(S)$. This action clearly preserves the homological intersection form ω on $\mathrm{H}_1(S)$, hence it factors over the symplectic group $\mathrm{Sp}(\omega)$.
On the other hand, $\mathcal{M}(S)$ acts by outer automorphisms on $\Gamma = \pi_1(S)$ and the resulting homomorphism $\mathcal{M}(S) \to \mathrm{Out}^+(\Gamma)$ is an isomorphism by the Dehn-Nielsen-Baer theorem. Observe that $\mathrm{H}_1(S)$ is just the abelianisation of Γ by the Hurewitz theorem. Hence the action of the mapping class group on the homology of S corresponds to the natural homomorphism

$$\mathrm{Out}^+(\Gamma) \to \mathrm{Aut}(\Gamma_{ab}). \tag{12}$$

The latter is well-defined since the conjugation action of Γ on Γ_{ab} is trivial. Now, from a purely algebraic point of view, it is not so clear anymore why (12) should factor over a symplectic group. The first goal of this subsection is therefore to give an algebraic explanation of this phenomenon.

We keep the notation $H = \Gamma_{ab}$ for the abelianisation of Γ. For $1 \le i \le g$ we denote the image of a_i in H by x_i and the image of b_i by x_{i+g}, then these $2g$ elements form a $\mathbb{Z}$-base of the free abelian group H. Consider the non-degenerate integral symplectic form

$$\omega = \sum_{i=1}^{g} dx_i \wedge dx_{i+g} \in \wedge^2 H^*$$

on H with respect to which $\{x_1, \ldots, x_{2g}\}$ is a symplectic standard basis. We will prove that (12) indeed factors over the integral symplectic group $\mathrm{Sp}(\omega) \cong \mathrm{Sp}_{2g}(\mathbb{Z})$.

We take the opportunity to recall the construction of exterior powers of free abelian groups. Let H be free abelian with a free basis $\{x_1, \ldots, x_n\}$. For a positive integer m the *m-th exterior power* $\wedge^m H$ is defined as the quotient of the m-fold integral tensor power $\bigotimes^m H$ by the subgroup generated by all pure tensors $v_1 \otimes \cdots \otimes v_m$ where two of the elements v_i are the same. As usual, we denote the image in $\wedge^m H$ of a tensor $v_1 \otimes \cdots \otimes v_m$ by $v_1 \wedge \cdots \wedge v_m$. It is not difficult to see that $\wedge^m H$ is free abelian of rank $\binom{n}{m}$ and that a basis is given by the elements

$$x_{k_i} \wedge \ldots \wedge x_{k_m}, \qquad 1 \le k_1 < \ldots < k_m \le n.$$

In what follows, we shall take a more general point of view. Let F be a free group of rank n on the generators $a_1, \ldots, a_n$ and let $r \in F$. We consider the one-relator group

$$G = \langle a_1, \ldots, a_n \mid r \rangle,$$

i.e., the quotient $G = F/R$ where $R = \langle\langle r \rangle\rangle \le F$ is the normal subgroup generated by r. We make the *standing assumption* that $r \in F'$ lies in the commutator subgroup of F. As a consequence, the natural projection $F \to G$ descends to an isomorphism $F_{ab} \cong G_{ab}$ between the corresponding abelianisations. This common abelianisation will be denoted by H and is a free abelian group of rank n, freely generated by the images $\overline{a}_i$. We will now give an upper bound on the image of the homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(H)$. The proof uses a construction taken from Johnson [17] which will also play a role later on.

Theorem 4.19. *Write $r = [u_1, v_1] \cdots [u_m, v_m]$ as a product of commutators in F. Then the element*

$$\theta = \sum_{i=1}^{m} \overline{u}_i \wedge \overline{v}_i \in \wedge^2 H$$

depends only on r and not on the presentation as a product of commutators. Moreover, the image of the homorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(H)$ consists of automorphisms for which the induced automorphism of $\wedge^2 H$ preserves θ up to sign.

Proof. For a group Γ the *lower central series* is the sequence of characteristic subgroups defined recursively by $\Gamma^{(0)} = \Gamma$ and $\Gamma^{(k+1)} = [\Gamma, \Gamma^{(k)}]$. The factor group $\Gamma_k = \Gamma/\Gamma^{(k)}$ is the largest quotient which is nilpotent of class k. In particular, we have $\Gamma^{(1)} = \Gamma'$ and $\Gamma_1 = \Gamma_{ab}$. The functor $(-)_k : \mathbf{Grp} \to \mathbf{Nil}_k$ into the category of nilpotent groups of class k is left adjoint to the inclusion functor and hence is right exact. In the special case of the free group $F = \langle a_1, \ldots, a_n \rangle$, the quotients F_k are the free nilpotent groups of class k on n generators. They enjoy the following universal property: For any nilpotent group N of class k and for every n-tupel of elements $x_1, \ldots, x_n \in N$ there is a unique homomorphism $f : F_k \to N$ such that $f(\overline{a}_i) = x_i$.
Applying $(-)_2$ to the exact sequence $1 \to R \to F \to G \to 1$ we obtain the right-exact sequence

$$R_2 \to F_2 \to G_2 \to 1.$$

Since $r \in F'$ the element $\overline{r}$ is central in F_2 and the image of R_2 is given by the cyclic subgroup $\langle \overline{r} \rangle$. So we have a central extension

$$\langle \overline{r} \rangle \rightarrowtail F_2 \twoheadrightarrow G_2.$$

Lemma 4.20. *Every automorphism of G_2 lifts to an automorphism of F_2 which fixes the set $\{\overline{r}, \overline{r}^{-1}\}$.*

Proof. We denote by $\overline{a}_i$ respectively $\tilde{a}_i$ the images of the generators $a_i \in F$ in F_2 respectively G_2. Consider an automorphism φ of G_2, set $\tilde{b}_i = \varphi(\tilde{a}_i)$ and choose preimages $\overline{b}_i$ in F_2. By the universal property of F_2 there is a unique endomorphism $\psi : F_2 \to F_2$ such that $\psi(\overline{a}_i) = \overline{b}_i$. By construction, ψ lifts φ. Likewise, there is a lift ρ of the inverse φ^{-1}. Since the composition $\rho \circ \psi$ lifts the identity on G_2, there are integers k_i with $\rho \circ \psi(\overline{a}_i) = \overline{a}_i \overline{r}^{k_i}$ for $1 \le i \le n$. Because $\overline{r} \in F_2$ is central we can conclude that $\rho \circ \psi$ is the identity on $(F_2)'$, in particular we have

$$\rho \circ \psi(\overline{r}) = \overline{r}. \tag{13}$$

On the other hand, since both ψ and ρ descend to G_2, there are integers s, t such that $\psi(\overline{r}) = \overline{r}^s$ and $\rho(\overline{r}) = \overline{r}^t$. Now (13) implies $st = 1$ and hence $s = t = \pm 1$. In addition, this shows that ψ and ρ are surjective and therefore also injective because finitely generated nilpotent groups are hopfian. □

We set $N = F^{(1)}/F^{(2)}$. Note that N and H are both abelian and that we have a central extension

$$N \rightarrowtail F_2 \twoheadrightarrow H.$$

Lemma 4.21. *The commutator map on F_2 descends to an alternating bilinear and $\mathrm{Aut}(F_2)$-equivariant map $H \times H \to N$ which induces an equivariant isomorphism*

$$j : \wedge^2 H \to N.$$

Proof. ([17], Lemmas 1A,1B,1C) The commutator map $[\cdot, \cdot]$ on $F_2 \times F_2$ takes values in N and depends only on the residue class mod N of its arguments since N is central. Hence it descends to a well-defined map $b : H \times H \to N$ which is obviously equivariant and alternating. The bilinearity is a consequence of the well known commutator identities

$$[x, yz] = [x, y] \cdot y[x, z]y^{-1} \quad \text{and} \quad [xy, z] = x[y, z]x^{-1} \cdot [x, z].$$

The induced homomorphism $j : \wedge^2 H \to N$ is surjective since every element in N is a product of commutators. For the injectivity it is then enough to observe that the two groups $\wedge^2 H$ and N are both free abelian of the same rank $\binom{n}{2}$. This is clear for the first group, while the structure of N has been determined by Magnus, cf. [26], Theorems 5.11 and 5.12. □

Set $\theta = j^{-1}(\overline{r}) \in \wedge^2 H$. By construction of j this is the element defined in the statement of the theorem. The independence of the decomposition of w into commutators is now obvious. As a consequence of the $\mathrm{Aut}(F_2)$-equivariance of j the diagram

$$\begin{tikzcd}
\mathrm{Aut}(F_2) \arrow[r] \arrow[d] & \mathrm{Aut}(N) \arrow[d, "\mathrm{Int}(j^{-1})"] \\
\mathrm{Aut}(H) \arrow[r, "d"] & \mathrm{Aut}(\wedge^2 H)
\end{tikzcd}$$

commutes where d is given by the diagonal action. Hence we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Aut}(F_2)_{\{\pm\overline{r}\}} \arrow[r] \arrow[d, two heads] & \mathrm{Aut}(N)_{\{\pm\overline{r}\}} \arrow[r, "\mathrm{Int}(j^{-1})"] & \mathrm{Aut}(\wedge^2 H)_{\{\pm\theta\}} \arrow[d, tail] \\
\mathrm{Aut}(G) \arrow[r] & \mathrm{Aut}(G_2) \arrow[r] & \mathrm{Aut}(H) \arrow[r, "d"] & \mathrm{Aut}(\wedge^2 H)
\end{tikzcd}$$

where in the upper row we mean the corresponding set stabilisers and where the left vertical map is surjective by Lemma 4.20. This shows the desired invariance of θ up to sign. □

We now come back to the concrete case of the compact surface group Γ. Here F is the free group on the generators a_i, b_i for $1 \le i \le g$ and $r = \prod_{i=1}^{g} [a_i, b_i] \in F'$. The element θ is given by

$$\theta = \sum_{i=1}^{g} x_i \wedge x_{i+g} \in \wedge^2 H.$$

Denote the image of $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(H)$ by K. By Theorem 4.19 every element of K induces an automorphism of $\wedge^2 H$ which preserves θ up to sign. To make the link to the symplectic form ω we consider the map

$$H^* \to H^{**}, \qquad f \mapsto \big(g \mapsto (f \wedge g)(\theta)\big)$$

which is K-equivariant up to sign. Composition with the inverse of the $\mathrm{Aut}(H)$-equivariant evaluation map $H \to H^{**}$ gives a homomorphism $c : H^* \to H$ which is again K-equivariant up to sign. A short computation shows that $c(dx_i) = x_{i+g}$ and $c(dx_{i+g}) = -x_i$ for $1 \le i \le g$, in particular, c is an isomorphism. Now θ corresponds to the symplectic form ω via the K-equivariant isomorphism $c \otimes c : \wedge^2 H^* \to \wedge^2 H$, hence we can conclude that ω is K-invariant up to sign as well. Equivalently, we have

$$K \le \langle \sigma \rangle \ltimes \mathrm{Sp}(\omega) \tag{14}$$

where $\sigma \in \mathrm{Aut}(H)$ is the involution given by interchanging x_i and x_{i+g} for $1 \le i \le g$ and satisfying $(\sigma \otimes \sigma)^*(\omega) = -\omega$.

Lemma 4.22. *For $\varphi \in \mathrm{Aut}(\Gamma)$ the following conditions are equivalent:*

(i) $\varphi \in \mathrm{Aut}^+(\Gamma)$.
(ii) *The image of φ in $\mathrm{Aut}(H)$ lies in $\mathrm{Sp}(\omega)$.*
(iii) *The image of φ in $\mathrm{Aut}(H)$ has determinant 1.*

Proof. The group

$$H_\omega = \left\langle x_1, \dots, x_{2g}, y \;\middle|\; [x_i, x_j] = y^{\omega(x_i,x_j)},\ y \text{ is central} \right\rangle$$

fits into a central extension

$$C: \qquad \mathbb{Z} \overset{i}{\rightarrowtail} H_\omega \twoheadrightarrow H$$

where $i(1) = y$. Since $(\varphi \otimes \varphi)^*(\omega) = \pm\omega$ a similar argument as in the proof of Lemma 4.11 shows that the pullback φ^*C is strongly isomorphic to $\pm C$, hence $\varphi^*([C]) = \pm[C] \in \mathrm{H}^2(H,\mathbb{Z})$. Moreover, a check show that $q^*([C]) = [E_g] \in \mathrm{H}^2(\Gamma,\mathbb{Z})$ where $q : \Gamma \to H$ is the canonical projection. Now consider the commutative diagram

$$\begin{tikzcd} \langle [C] \rangle \arrow[r, "q^*"] \arrow[d, "\varphi^*"'] & \mathrm{H}^2(\Gamma,\mathbb{Z}) \arrow[d, "\varphi^*"] \\ \langle [C] \rangle \arrow[r, "q^*"] & \mathrm{H}^2(\Gamma,\mathbb{Z}) \end{tikzcd}$$

where in the left column we mean the subgroup of $\mathrm{H}^2(H,\mathbb{Z})$ generated by $[C]$. Since $q^* \neq 0$ the two vertical maps are multiplication with the same integer ±1, hence (i) and (ii) are equivalent. The equivalence of (ii) and (iii) is a consequence of (14), the well known fact that all elements of $\mathrm{Sp}(\omega)$ have determinant 1 and $\det(\sigma) = -1$. $\square$

In summary we have shown so far that the image of the natural map $\mathrm{Out}^+(\Gamma) \to \mathrm{Aut}(H)$ is contained in $\mathrm{Sp}(\omega)$. In the second half of this subsection we are going to prove that it actually equals $\mathrm{Sp}(\omega)$. To do so, we first take a closer look at the symplectic group and use the opportunity to introduce some notation.

With respect to the base $\{x_1, \dots, x_{2g}\}$ of H the group $\mathrm{Sp} = \mathrm{Sp}(\omega)$ is realised as the integral symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ consisting of integral square matrices φ of size $2g$ satisfying the equation

$$\varphi^T J \varphi = J, \qquad \text{where} \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Decomposing φ into four square blocks we have the equivalent conditions

$$\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}) \qquad \Longleftrightarrow \qquad \begin{cases} A^T C = C^T A \\ B^T D = D^T B \\ A^T D - C^T B = I \end{cases}$$

We will introduce some special elements of Sp which will be useful later on. For $1 \le i \le g$ let E_i be the square matrix of size g with all entries zero except the (i,i)-entry which is 1. Set

$$\lambda_i = \begin{pmatrix} I & E_i \\ 0 & I \end{pmatrix} \quad \text{and} \quad \mu_i = \begin{pmatrix} I & 0 \\ E_i & I \end{pmatrix}.$$

For different indices $1 \le i, j \le g$ define F_{ij} to be the square matrix of size g with all entries zero except for the entries at the spots (i,i) and (j,j) which are 1 and the entries at the spots (i,j) and (j,i) which are -1. Set

$$\nu_{ij} = \begin{pmatrix} I & 0 \\ F_{ij} & I \end{pmatrix}.$$

Finally, for a permutation $\pi \in S_g$ let P_π be the permutation matrix associated to π, i.e., $(P_\pi)_{ij} = \delta_{\pi(i)j}$, and set

$$\theta_\pi = \begin{pmatrix} P_\pi & 0 \\ 0 & P_\pi \end{pmatrix}.$$

Among the various relations between these elements we shall only use the following easy to verify ones:

$$\begin{aligned} \theta_\pi \lambda_i \theta_\pi^{-1} &= \lambda_{\pi(i)}, \\ \theta_\pi \mu_i \theta_\pi^{-1} &= \mu_{\pi(i)}, \\ \theta_\pi \nu_{ij} \theta_\pi^{-1} &= \nu_{\pi(i)\pi(j)}. \end{aligned}$$

Proposition 4.23. *The group* Sp *is generated by the four elements* $\lambda_1, \mu_1, \nu_{12}$ *and* θ_π *where* $\pi = (12\dots g)$ *is a cycle of length* g.

Proof. By [3], Theorem 1 the group Sp is generated by λ_i and μ_i for $1 \le i \le g$ together with $\nu_{12}, \dots, \nu_{(g-1)g}$. By the above relations these are all conjugated under the subgroup $\langle \theta_\pi \rangle$ to one of the three elements $\lambda_1, \mu_1, \nu_{12}$. $\square$

Theorem 4.24. *The image of the map* $\mathrm{Out}^+(\Gamma) \to \mathrm{Aut}^+(H)$ *equals* Sp.

Proof. It suffices to prove the inclusion $\supseteq$. By Proposition 4.23 it is enough to show that the four elements $\mu_1, \lambda_1, \nu_{12}$ and θ_π lift to $\mathrm{Aut}(\Gamma)$. For this we shall lift them to automorphisms of the free group F which preserve the normal subgroup R generated by $r = \prod_{i=1}^g [a_i, b_i]$. These then descend to the desired automorphisms of Γ.
Let $\tilde{\lambda}_1$ be the endomorphism of F which maps b_1 to $b_1 a_1$ and fixes all other generators. Then clearly $\tilde{\lambda}_1$ is invertible and fixes r, hence lifts λ_1. Similarly, the automorphism $\tilde{\mu}_1$ of F which maps a_1 to $a_1 b_1$ and fixes all other generators is a lift of μ_1. Next, let $\tilde{\nu}_{12}$ be defined by

$$\begin{cases} a_1 \mapsto a_1 a_2 b_2^{-1} a_2^{-1} b_1, \\ b_1 \mapsto b_1^{-1} a_2 b_2 a_2^{-1} b_1 a_2 b_2^{-1} a_2^{-1} b_1, \\ a_2 \mapsto b_1^{-1} a_2 b_2, \end{cases}$$

and by fixing all other generators. A calculation again shows that $\tilde{\nu}_{12}$ fixes r and lifts ν_{12}. Moreover, one verifies that an inverse is given by

$$\begin{cases} a_1 \mapsto a_1 b_1^{-1} a_2 b_2 a_2^{-1}, \\ b_1 \mapsto a_2 b_2^{-1} a_2^{-1} b_1 a_2 b_2 a_2^{-1}, \\ a_2 \mapsto a_2 b_2^{-1} a_2^{-1} b_1 a_2. \end{cases}$$

Finally, a lift $\tilde{\theta}_\pi$ of θ_π is given by $a_i \mapsto a_{i+1}$ and $b_i \mapsto b_{i+1}$ where all indices are to be taken (mod g). This is clearly an isomorphism and since $\tilde{\theta}_\pi(r) = [a_1, b_1]^{-1} r [a_1, b_1]$ it preserves R. $\square$

Hence there are natural surjections $\mathrm{Aut}^+(\Gamma) \to \mathrm{Sp}$ and $\mathrm{Out}^+(\Gamma) \to \mathrm{Sp}$. The kernel of the latter homomorphism is called the *Torelli group* and denoted $\mathcal{I}$. Geometrically, $\mathcal{I}$ is the subgroup of the mapping class group consisting of those classes which act trivially on the first homology $\mathrm{H}_1(S)$ of the surface S.

4.5. **The Abelianisation of the Torelli Group.** For the cohomological computations in the next subsection we need a description of the abelianisation $\mathcal{I}_{ab}$ of the Torelli group. The first complete description has been given by Johnson in a sequence of three papers on the Torelli group ([19], [20], [21]). Johnson mainly considered the case of a surface with one boundary component where things are slightly easier to handle. He then derived the corresponding results for closed surfaces from this. We are going to present his work for the case of closed surfaces only and will allways assume that the genus g of the surface is at least 3.

Johnson constructed two different homomorphisms σ and τ from $\mathcal{I}$ into abelian groups. In [18] he defines σ as a combination of a family of $\mathbb{Z}_2$-valued homomorphisms constructed by Birman and Craggs based on the Rochlin invariant in [4]. Its image can be described as a certain space of boolean polynomials over $\mathbb{Z}_2$ (see below). On the other hand, the definition of τ is purely algebraic in nature and is related to the arguments we have used in the proof of Theorem 4.19. Its image is a quotient of the third exterior power of H. Nowadays τ is known as *Johnson homomorphism.* These two homomorphisms are not independent but share certain information (mod 2) in a sense made precise below. Hovewer, it turns out that they combine all the information about $\mathcal{I}_{ab}$ (compare Theorem 4.29).

Before we can describe the two homomorphisms σ and τ we need some preparations. We will frequently be dealing with the second and third exterior power of H and of its (mod 2)-reduction $\overline{H} = H/2H$. To simplify notation we use abbreviations like

$$x_{ijk} = x_i \wedge x_j \wedge x_k, \qquad \text{for } i, j, k \in \{1, \dots, 2g\} \text{ pairwise different}$$

in $\wedge^3 H$ and analogous ones for elements in $\wedge^2 H$. The set $\{x_{ijk} \mid i < j < k\}$ is then a $\mathbb{Z}$-basis of $\wedge^3 H$. Recall the element

$$\theta = \sum_{i=1}^{g} x_i \wedge x_{i+g} \in \left(\wedge^2 H \right)^{\mathrm{Sp}}$$

which is the dual of the invariant symplectic form ω on H. Since θ is invariant the map

$$\beta : H \to \wedge^3 H, \qquad v \mapsto \theta \wedge v$$

is an Sp-equivariant homomorphism.

Lemma 4.25. *The cokernel V of β is free abelian.*

Proof. We use the decomposition

$$\wedge^3 H = \bigoplus_{i=1}^{g} K_i \oplus \bigoplus_{i=1}^{g} L_i \oplus W$$

into a sum of $2g + 1$ subspaces where K_i is generated by the elements $x_{ij(j+n)}$ for $1 \le j \le g$ and $j \ne i$, L_i is generated by the elements $x_{(i+g)j(j+g)}$ for $1 \le j \le g$ and $j \ne i$ and W is generated by the set of elements x_{jkl} where no two indices differ by a multiple of g. Since $\beta(x_i) = \sum_{j\ne i}^{g} x_{ij(j+g)}$ the image of β decomposes into the direct sum of subspaces $\mathbb{Z}a_i \le K_i$ and $\mathbb{Z}b_i \le L_i$ with

$$a_i = \sum_{j \ne i}^{g} x_{ij(j+g)} \qquad \text{and} \qquad b_i = \sum_{j \ne i}^{g} x_{(i+g)j(j+g)}.$$

The elements a_i and b_i are obviously primitive, hence the quotients $K_i/\mathbb{Z}a_i$ and $L_i/\mathbb{Z}b_i$ are free abelian. This finishes the proof. $\square$

Lemma 4.26. *The map* $\gamma : \wedge^3 H \to H$*,*

$$u \wedge v \wedge w \mapsto \omega(u,v)w + \omega(v,w)u + \omega(w,u)v$$

is a surjective, Sp*-equivariant homomorphism. The composition* $\gamma \circ \beta : H \to H$ *is multiplication with* $g-1$*.*

Proof. Again, the equivariance follows from the invariance of ω. Since

$$\gamma(x_{i(i+n)j}) = x_j$$

for $1 \leq i \leq g$ and $j \neq i, i+g$, γ is surjective. Finally, a simple calculation proves $\gamma(\beta(x_i)) = (g-1)x_i$ for all i, hence $\gamma \circ \beta = (g-1)\,\mathrm{id}_H$. □

We now turn to the description of the Johnson homomorphism τ and closely follow [17], Section 3. We will refer to the proof of Theorem 4.19 and the notations introduced therein. Set $M = \Gamma^{(1)}/\Gamma^{(2)}$, then there is a commutative diagram

$$\begin{tikzcd}
\langle \overline{r} \rangle \arrow[r, equal] \arrow[d, tail] & \langle \overline{r} \rangle \arrow[d, tail] & \\
N \arrow[r, tail] \arrow[d, two heads] & F_2 \arrow[r, two heads] \arrow[d, two heads] & H \arrow[d, equal] \\
M \arrow[r, tail] & \Gamma_2 \arrow[r, two heads] & H
\end{tikzcd}$$

Let $\varphi \in \mathcal{I}$, i.e., φ acts trivially on H. Let $v \in H$ and choose a lift $\tilde{v} \in \Gamma_2$. Then the element $\delta(\varphi)(v) = \varphi(\tilde{v})\tilde{v}^{-1}$ is independent of the choice of the lift $\tilde{v}$ and projects to $0 \in H$. Hence it lies in M and we obtain a map $\delta(\varphi) : H \to M$. For two elements $v, w \in H$ with lifts $\tilde{v}, \tilde{w}$ we have

$$\delta(\varphi)(vw) = \varphi(\tilde{v}\tilde{w})(\tilde{v}\tilde{w})^{-1} = \varphi(\tilde{v})\delta(\varphi)(w)\tilde{v}^{-1} = \delta(\varphi)(v)\delta(\varphi)(w)$$

since $\delta(\varphi)(w) \in M$ is central in Γ_2. Therefore, $\delta(\varphi)$ is a homomorphism.

Lemma 4.27. *The map*

$$\delta : \mathcal{I} \to \mathrm{Hom}(H, M)$$

is an Sp*-equivariant homomorphism (where the action on* $\mathcal{I}$ *is given by conjugation and the action on* $\mathrm{Hom}(H,M)$ *is given by* $(\varphi \cdot f)(v) = \varphi\big(f(\varphi^{-1}(v))\big)$ *as usual).*

Proof. For $\varphi, \psi \in \mathcal{I}$ we have

$$\delta(\varphi\psi)(v) = \varphi(\psi(\tilde{v}))\tilde{v}^{-1} = \varphi\big(\delta(\psi)(v)\big) \cdot \delta(\psi)(v) = \delta(\varphi)(v) \cdot \delta(\psi)(v),$$

where in the last equality we have used that $\delta(\psi)(v) \in M$ and that $\mathcal{I}$ acts trivially on M. So δ is indeed a homomorphism. Moreover, for $\varphi \in \mathrm{Aut}^+(\Gamma)$ and $\psi \in \mathcal{I}$ we obtain

$$\delta(\varphi\psi\varphi^{-1})(v) = \varphi(\psi(\varphi^{-1}(\tilde{v})))\tilde{v}^{-1} = \varphi\big(\psi(\varphi^{-1}(\tilde{v}))\varphi^{-1}(\tilde{v})^{-1}\big) = \varphi\big(\delta(\psi)(\varphi^{-1}v)\big)$$

and hence δ is Sp-equivariant. □

To give the final form of the Johnson homomorphism we first notice that the isomorphism j of Lemma 4.21 induces an isomorphism $k : \wedge^2 H/\langle\theta\rangle \to M$. Using this we obtain equivariantly

$$\mathrm{Hom}(H,M) \cong M \otimes H^* \cong M \otimes H \cong \big(\wedge^2 H/\langle\theta\rangle\big) \otimes H$$

where the second isomorphism is given by the self-duality of H induced by the symplectic form ω and the third by k^{-1}. The last space now canonically projects onto $(\wedge^3 H)/(\theta \wedge H) = V$, the cokernel of the map β defined above. Indeed, the diagram

$$\begin{tikzcd}
\langle\theta\rangle \otimes H \arrow[r] \arrow[d, "\cong"] & (\wedge^2 H) \otimes H \arrow[r] \arrow[d] & \big(\wedge^2 H/\langle\theta\rangle\big) \otimes H \arrow[d] \\
H \arrow[r, "\beta"] & \wedge^3 H \arrow[r] & V
\end{tikzcd}$$

commutes and the right column consists of the cokernels of the maps on the left. Summing up the above discussion, δ induces an Sp-equivariant homomorphism

$$\tau : \mathcal{I}_{ab} \to V.$$

Johnson proved that τ is surjective (cf. [17], Section 4).

Next, we turn to the homomorphism σ. We will not actually define it since this would lead us too far away, the interested reader is referred to [18]. Instead, we content ourselves with the description of its image. All that follows is taken from the before mentioned reference. Denote by $\overline{H} = H/2H \cong H \otimes \mathbb{Z}_2$ the reduction of $H \pmod 2$. To avoid cumbersome notation we shall abstain from distinguishing between objects related to H and their reduction in $\overline{H}$. This will not cause any problems since we are exclusively working over $\mathbb{Z}_2$ for the moment. In particular, we denote the reduction $\overline{H} \wedge \overline{H} \to \mathbb{Z}_2$ of the symplectic form ω again by ω. The induced action of Sp on $\overline{H}$ and on all related objects introduced below naturally factors over the finite simple group $\mathrm{Sp}_{2g}(\mathbb{Z}_2)$.

A *quadratic form* on $\overline{H}$ is a map $q : \overline{H} \to \mathbb{Z}_2$ such that

$$q(u+v) = q(u) + q(v) + \omega(u,v)$$

for all $u, v \in \overline{H}$. The set Ω of quadratic forms on $\overline{H}$ is an affine space over the $2g$-dimensional $\mathbb{Z}_2$-vector space $\overline{H}^* = \mathrm{Hom}(\overline{H}, \mathbb{Z}_2)$ and carries the induced action of Sp given by $(\varphi q)(v) = q(\varphi^{-1}(v))$. Denote by L the vector space of affine linear maps $f : \Omega \to \mathbb{Z}_2$ and let B be the subalgebra of $\mathrm{Map}(\Omega, \mathbb{Z}_2)$ generated by L. In other words, B is the algebra of all functions on Ω which can be written as a polynomial expression in affine linear maps. Since $f^2 = f$ for all $f \in B$ one may interpret B as an algebra of 'boolean polynomials'. The symplectic group Sp acts linearly on B via $(\varphi f)(q) = f(\varphi^{-1} q)$.

Consider the evaluation map $\alpha : \overline{H} \to L$ given by $\alpha(v)(q) = q(v)$. It is easy to see that α is equivariant, however, α is *not* a homomorphism. Indeed, we have

$$\alpha(v+w) = \alpha(v) + \alpha(w) + \omega(v,w) \tag{15}$$

for all $v, w \in \overline{H}$. Setting $e_i = \alpha(x_i) \in L$, a basis of L is given by $\{1, e_1, \ldots, e_{2g}\}$ where 1 denotes the constant function on Ω. The boolean property of B then implies that a basis of B as a $\mathbb{Z}_2$-vector space is given by all monomials of the form $e_{i_1} \cdots e_{i_k}$ with $i_1 < \ldots < i_k$.

There is a filtration $B_0 < B_1 < \ldots$ of B where B_n is the subspace of functions which can be written as a polynomial of degree $\leq n$. The Sp-action preserves this filtration and therefore descends to actions on the quotients B_n/B_{n-1}. However, the action does not preserve homogeneous elements as a consequence of the fact that α is not a homomorphism. For example we have

$$\begin{aligned}
\mu_1(e_1) &= \mu_1 \alpha(x_1) = \alpha(\mu_1(x_1)) = \alpha(x_1 + x_{1+g}) \\
&= \alpha(x_1) + \alpha(x_{1+g}) + \omega(x_1, x_{1+g}) \\
&= e_1 + e_{i+g} + 1
\end{aligned}$$

where in the second line we have used (15). The quotients B_n/B_{n-1} have a familiar structure. Using the explicit basis of B given above it is not difficult to prove:

Lemma 4.28. *The map α extends to an Sp-equivariant isomorphism of graded algebras*

$$\bigoplus_{n=0}^{\infty} \wedge^n \overline{H} \to \bigoplus_{n=0}^{\infty} B_n/B_{n-1}.$$

Consider the element

$$\mathrm{Arf} = e_1 e_{1+g} + \ldots + e_g e_{2g} \in (B_2)^{\mathrm{Sp}},$$

the *Arf-invariant*. It is indeed Sp-invariant and corresponds to the (mod 2)-reduction of θ under the isomorphism $B_2/B_1 \cong \wedge^2 \overline{H}$. Johnson now constructs σ as a surjective equivariant homomorphism

$$\sigma : \mathcal{I}_{ab} \to B_3/(B_1 \,\mathrm{Arf}).$$

Notice that the target spaces of both τ and σ naturally surject on the space

$$\overline{V} = (\wedge^3 \overline{H})/(\theta \wedge \overline{H}) \cong V \otimes \mathbb{Z}_2,$$

by reduction (mod 2) respectively by Lemma 4.28. The abelianisation $\mathcal{I}_{ab}$ of the Torelli group is now characterised in terms of σ and τ as follows:

Theorem 4.29. *Assume that $g \geq 3$. The diagram*

$$\begin{tikzcd} \mathcal{I}_{ab} \arrow[r, "\tau"] \arrow[d, "\sigma"'] & V \arrow[d, "(\mathrm{mod}\ 2)"] \\ B_3/(B_1 \,\mathrm{Arf}) \arrow[r, "s"] & \overline{V} \end{tikzcd}$$

commutes, where s is induced by the isomorphism $B_3/B_2 \cong \wedge^3 \overline{H}$. It is in fact a pullback diagram in the category of $\mathbb{Z}[\mathrm{Sp}]$-modules, in particular there is an exact sequence

$$B_2/\langle \mathrm{Arf} \rangle \rightarrowtail \mathcal{I}_{ab} \twoheadrightarrow V$$

where the group on the left is finite 2-torsion and the group on the right is free abelian of rank $\binom{2g}{3} - 2g$.

Proof. For the first part, see [21], Theorem 6. The last statement follows directly from the fact that the inclusion $B_2/\langle \mathrm{Arf} \rangle \to B_3/(B_1 \,\mathrm{Arf})$ is the kernel of the homomorphism s. $\square$

The assumption $g \geq 3$ on the genus of S is crucial in the previous theorem. Indeed, for $g = 1$ the Torelli group is trivial and for $g = 2$ Mess has shown in [28] that it is free of infinite rank. As a consequence, $\mathcal{I}_{ab}$ is free abelian of infinite rank in this case.

4.6. **On the Cohomology of the Mapping Class Group.** The goal of this subsection is to give upper bounds on the size of the first and second cohomology of $\mathrm{Out}^+(\Gamma)$ with coefficients in H^* equipped with the natural action. Our result is the following:

Theorem 4.30. *Let $\mathrm{Out}^+(\Gamma)$ act on H in the natural way.*

(a) *Assume that $g \geq 3$. The group $\mathrm{H}_1(\mathrm{Out}^+(\Gamma), H)$ is annihilated by $2g-2$. Moreover,*

$$\mathrm{H}^1\big(\mathrm{Out}^+(\Gamma), \mathrm{Hom}(H, A)\big) = 0$$

for every torsion-free abelian group A equipped with the trivial action.

(b) *For $g \geq 6$ the group* $\mathrm{H}^2(\mathrm{Out}^+(\Gamma), H^*)$ *is annihilated by* $2g-2$.

The main work is devoted to the proof that $\mathrm{H}_1(\mathrm{Out}^+(\Gamma), H)$ is annihilated by $2g-2$. This is already known, actually Morita has even shown that $\mathrm{H}_1(\mathrm{Out}^+(\Gamma), H) \cong \mathbb{Z}_{2g-2}$, see [31], Corollary 5.4. However, we take an alternative approach which makes use of the short exact sequence $\mathcal{I} \rightarrowtail \mathrm{Out}^+(\Gamma) \twoheadrightarrow \mathrm{Sp}$ and the explicit description of $\mathcal{I}_{ab}$ in Theorem 4.29. For (b) we will combine this calculation with a vanishing result of Looijenga for the rational cohomology group $\mathrm{H}^2(\mathrm{Out}^+(\Gamma), H^* \otimes \mathbb{Q})$ which holds for $g \geq 6$.

We start by recalling some tools from group cohomology which will be needed in the sequel. First, we recall the 5-term exact sequences of low degree terms in the classical Lyndon/Hochschild-Serre spectral sequence:

Proposition 4.31. *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of groups and let M be a G-module. The there are exact sequences*

$$\mathrm{H}_2(G, M) \to \mathrm{H}_2(Q, M_N) \to \mathrm{H}_1(N, M)_Q \to \mathrm{H}_1(G, M) \to \mathrm{H}_1(Q, M_N) \to 0,$$

respectively

$$0 \to \mathrm{H}^1(Q, M^N) \to \mathrm{H}^1(G, M) \to \mathrm{H}^1(N, M)^Q \to \mathrm{H}^2(Q, M^N) \to \mathrm{H}^2(G, M).$$

Proof. See [38], 6.8.3. □

Next, we give a version of the universal coefficient theorem which also applies to coefficients with non-trivial action.

Proposition 4.32. *Let G be a group, M an arbitrary G-module and A a trivial G-module. Then for every $n \geq 0$ there is a short exact sequence*

$$0 \to \mathrm{H}_n(G, M) \otimes_{\mathbb{Z}} A \to \mathrm{H}_n(G, M \otimes_{\mathbb{Z}} A) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}_{n-1}(G, M), A) \to 0.$$

Similarly, there is a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{n-1}(G, M), A) \to \mathrm{H}^n(G, \mathrm{Hom}_{\mathbb{Z}}(M, A)) \to \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{H}_n(G, M), A\big) \to 0.$$

Proof. To prove the second statement, consider the right exact functors

$$\mathbb{Z}G - \mathbf{Mod} \xrightarrow{C} \mathbf{Ab} \xrightarrow{D} \mathbf{Ab}^{\mathrm{op}}$$

between abelian categories where

$$C = \mathbb{Z} \otimes_{\mathbb{Z}G} - = (-)_G \qquad \text{and} \qquad D = \mathrm{Hom}_{\mathbb{Z}}(-, A).$$

The value of the composite functor DC on a left $\mathbb{Z}G$-module M is given by

$$\begin{aligned} DC(M) &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} M, A) \\ &= \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Hom}_{\mathbb{Z}}(M, A)) \\ &= \mathrm{Hom}_{\mathbb{Z}}(M, A)^G \end{aligned}$$

Note that both $\mathbb{Z}G - \mathbf{Mod}$ and $\mathbf{Ab}$ have enough projectives and that moreover C sends projective objects to D-acyclic ones. To see the latter, consider a projective $\mathbb{Z}G$-left module M and choose a complement N such that $M \oplus N$ is free. Then

$$(\mathbb{Z} \otimes_{\mathbb{Z}G} M) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}G} N) = \mathbb{Z} \otimes_{\mathbb{Z}G} (M \oplus N)$$

is a free abelian group, so the same holds for $\mathbb{Z} \otimes_{\mathbb{Z}G} M$. In particular, the latter group is D-acyclic. Hence we can invoke the Grothendieck spectral sequence

$$E^2_{pq} = (L_p G)(L_q F)(M) \Rightarrow L_{p+q}(GF)(M).$$

Now, because $\mathbb{Z}$ has cohomological dimension 1, only the first two columns in the E^2-sheet of that spectral sequence are non-zero and the sequence collapses to give the desired result.

One can derive the first sequence in a similar fashion using the two right exact functors

$$\mathbb{Z}G - \mathbf{Mod} \xrightarrow{\mathbb{Z}\otimes_{\mathbb{Z}G}-} \mathbf{Ab} \xrightarrow{A\otimes_{\mathbb{Z}}-} \mathbf{Ab}\,.$$

□

We now turn to the proof of Theorem 4.30 and start with an analysis of the homology group $\mathrm{H}_1(\mathrm{Out}^+(\Gamma), H)$. As $\mathcal{I} \rightarrowtail \mathrm{Out}^+(\Gamma) \twoheadrightarrow \mathrm{Sp}$ is short exact we obtain an exact sequence

$$\mathrm{H}_1(\mathcal{I}, H)_{\mathrm{Sp}} \longrightarrow \mathrm{H}_1(\mathrm{Out}^+(\Gamma), H) \longrightarrow \mathrm{H}_1(\mathrm{Sp}, H_{\mathcal{I}})$$

by Proposition 4.31. Since the action of $\mathcal{I}$ on H is trivial by definition we have the following description of the two outer groups:

- By Proposition 4.32 we have

$$\mathrm{H}_1(\mathcal{I}, H)_{\mathrm{Sp}} \cong \big(\mathrm{H}_1(\mathcal{I}, \mathbb{Z}) \otimes H\big)_{\mathrm{Sp}} = (\mathcal{I}_{ab} \otimes H)_{\mathrm{Sp}}$$

  (the Tor-term clearly vanishes).
- $\mathrm{H}_1(\mathrm{Sp}, H_{\mathcal{I}}) = \mathrm{H}_1(\mathrm{Sp}, H)$.

Our claim will follow as soon as we prove that the first group is cyclic of order $g-1$ and the second group is annihilated by 2. The latter is rather easy to see and is a consequence of the following general fact:

Proposition 4.33. *Let G be a group and let M be a G-module. Assume that there exists a central element in G which acts by multiplication with -1 on M. Then $\mathrm{H}_1(G, M)$ is annihilated by 2.*

Proof. Let $z \in Z(G)$ act by multiplication with -1 on M. We shall work with the inhomogeneous bar-resolution and consider a 1-cycle $c = \sum_i m_i[g_i] \in Z_1(G, M)$, i.e.,

$$\sum_i g_i m_i - m_i = 0. \tag{16}$$

Subtracting the two equations

$$\begin{aligned} \partial\big(m_i[g_i|z]\big) &= g_i(m_i)[z] - m_i[g_i z] + m_i[g_i] \\ \partial\big(m_i[z|g_i]\big) &= z(m_i)[g_i] - m_i[z g_i] + m_i[z] \end{aligned}$$

and using that $g_i z = z g_i$ as well as $z(m_i) = -m_i$ we obtain

$$2m_i[g_i] = \partial\big(m_i[g_i|z]\big) - \partial\big(m_i[z|g_i]\big) - (g_i m_i - m_i).$$

Summing over i and using (16) shows that $2c$ is a boundary, as desired. □

We can apply this to the situation of Sp acting on H. The negative of the identity matrix is central in Sp and indeed acts by multiplication with -1 on H, hence $\mathrm{H}_1(\mathrm{Sp}, H)$ is annihilated by 2. Actually, a little more work shows that $\mathrm{H}_1(\mathrm{Sp}, H)$ is isomorphic to $\mathbb{Z}_2$, a generator being given by the class of the inhomogeneous 1-cycle

$$x_1[\nu_{12}] - x_1[\mu_1] + x_2[\mu_2].$$

We now turn to the more difficult task to compute $(\mathcal{I}_{ab} \otimes H)_{\mathrm{Sp}}$ where we keep the standing assumption $g \geq 3$. Recall that by Theorem 4.29 the group $\mathcal{I}_{ab}$ is given by a pullback construction and in particular it fits into an exact sequence

$$B_2/\langle \mathrm{Arf} \rangle \rightarrowtail \mathcal{I}_{ab} \twoheadrightarrow V.$$

We will proceed in several steps.

Lemma 4.34. *We have*

$$\begin{array}{rcl} H_{\mathrm{Sp}} & = & 0, \\ (H\otimes H)_{\mathrm{Sp}} & \cong & \mathbb{Z}, \\ (\wedge^2 H\otimes H)_{\mathrm{Sp}} & = & 0, \\ (\wedge^3 H\otimes H)_{\mathrm{Sp}} & \cong & \mathbb{Z}. \end{array}$$

The isomorphism in the second line is induced by the symplectic Form $\omega : H\times H \to \mathbb{Z}$ and the one in the fourth by the map $\omega\circ(\gamma\times \mathrm{id}) : \wedge^3 H\times H \to \mathbb{Z}$.

Proof. The isomorphism in the second line is a consequence of the one in the fourth since the equivariant homomorphism $\gamma\otimes \mathrm{id} : \wedge^3 H\otimes H \to H\otimes H$ is surjective and $(-)_{\mathrm{Sp}}$ is right-exact. Similarly, the vanishing of the first group follows from the vanishing of the third.
We prove that the third group is trivial. For $i \le g$ and $j \notin \{i, i+g\}$ we have the equations

$$\begin{array}{rcll} (\mu_i - 1)(x_{ij}\otimes x_l) & = & x_{(i+g)j}\otimes x_l, & l\neq i \\ (\lambda_i - 1)(x_{(i+g)j}\otimes x_l) & = & x_{ij}\otimes x_l, & l\neq i+g \end{array}$$

Using these and the fact that the wedge product is antisymmetric we can conclude that the coinvariants are generated by the elements

$$x_{i(i+g)}\otimes x_l, \qquad x_{i(i+g)}\otimes x_{l+g}$$

where $i, l \le g$. Now the equations

$$\begin{array}{rcl} (\mu_l - 1)(x_{i(i+g)}\otimes x_l) & = & x_{i(i+g)}\otimes x_{l+g}, \\ (\lambda_l - 1)(x_{i(i+g)}\otimes x_{l+g}) & = & x_{i(i+g)}\otimes x_l \end{array}$$

which are also valid for $l = i$ show the vanishing of $(\wedge^2 H\otimes H)_{\mathrm{Sp}}$.
We now turn to the fourth group, the argument in fact being very similar. For $i \le g$ and $j, k \notin \{i, i+g\}$ we have

$$\begin{array}{rcll} (\mu_i - 1)(x_{ijk}\otimes x_l) & = & x_{(i+g)jk}\otimes x_l, & l\neq i \\ (\lambda_i - 1)(x_{(i+g)jk}\otimes x_l) & = & x_{ijk}\otimes x_l, & l\neq i+g \end{array}$$

Hence the coinvariants are generated by the images of the elements

$$x_{i(i+g)j}\otimes x_{j+g}, \qquad x_{i(i+g)(j+g)}\otimes x_j$$

for $i, j \le g$ and $i \neq j$. Since Sp clearly acts transitively on the set of such elements up to sign, their images in $(\wedge^3 H\otimes H)_{\mathrm{Sp}}$ are all the same up to sign. Hence the latter group is cyclic and generated by each of them. On the other hand, the equivariant bilinear map $\omega\circ(\gamma\times\mathrm{id})$ descends to an equivariant homomorphism $\wedge^3 H\otimes H \to \mathbb{Z}$. This in turn factors over the coinvariants since the action on $\mathbb{Z}$ is trivial. The claim now follows because the image of each element $x_{i(i+g)j}\otimes x_{j+g}$ in $\mathbb{Z}$ equals 1. □

Proposition 4.35. *We have*

$$(V\otimes H)_{\mathrm{Sp}} \cong \mathbb{Z}_{g-1}.$$

Proof. Because coinvariants are right-exact we have

$$(V\otimes H)_{\mathrm{Sp}} = \operatorname{coker}\Big((H\otimes H)_{\mathrm{Sp}} \xrightarrow{\beta\otimes\mathrm{id}} \big(\wedge^3 H\otimes H\big)_{\mathrm{Sp}}\Big).$$

Consider the following diagram where the vertical isomorphisms are given by Lemma 4.34:

$$\begin{tikzcd}
\big(H\otimes H\big)_{\mathrm{Sp}} \arrow[r, "\beta\otimes\mathrm{id}"] \arrow[d, "\cong"] & \big(\wedge^3 H\otimes H\big)_{\mathrm{Sp}} \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "\cong"] & \big(H\otimes H\big)_{\mathrm{Sp}} \arrow[d, "\cong"] \\
\mathbb{Z} \arrow[r] \arrow[rr, bend right, "\cdot(g-1)"'] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z}
\end{tikzcd}$$

By Lemma 4.26 the composition in the upper row equals multiplication with $(g-1)$. Hence the same holds for the composition in the lower row. But since the right map in the lower row is surjective, the first one is actually multiplication by $\pm(g-1)$ and this gives the claim. $\square$

Proposition 4.36. *The space*

$$(B_2\otimes H)_{\mathrm{Sp}}$$

is generated by the image of the element $e_{1+g}\otimes x_1$.

Proof. We consider the filtration $0<B_0<B_1<B_2$ with successive quotients $\mathbb{Z}_2$, $\overline{H}$ and $\wedge^2\overline{H}$. Hence there is an exact sequence

$$(\mathbb{Z}_2\otimes H)_{\mathrm{Sp}} \longrightarrow (B_1\otimes H)_{\mathrm{Sp}} \longrightarrow (\overline{H}\otimes H)_{\mathrm{Sp}} \longrightarrow 0.$$

According the Lemma 4.34 the first space is trivial and the last one is generated by the image of the element $\overline{x}_{1+g}\otimes x_1$. Hence $(B_1\otimes H)_{\mathrm{Sp}}$ is generated by the image of the element $e_{1+g}\otimes x_1$. Now consider the exact sequence

$$(B_1\otimes H)_{\mathrm{Sp}} \longrightarrow (B_2\otimes H)_{\mathrm{Sp}} \longrightarrow (\wedge^2\overline{H}\otimes H)_{\mathrm{Sp}} \longrightarrow 0.$$

Again by Lemma 4.34 the last space is trivial and hence the claim follows. $\square$

Proposition 4.37. *We have*

$$(\mathcal{I}_{ab}\otimes H)_{\mathrm{Sp}} \cong \mathbb{Z}_{g-1}.$$

Proof. Tensoring the exact sequence

$$0 \longrightarrow B_2/\langle \mathrm{Arf}\rangle \longrightarrow \mathcal{I}_{ab} \longrightarrow V \longrightarrow 0$$

with H and taking coinvariants we obtain the exact sequence

$$\big((B_2/\langle \mathrm{Arf}\rangle)\otimes H\big)_{\mathrm{Sp}} \longrightarrow \big(\mathcal{I}_{ab}\otimes H\big)_{\mathrm{Sp}} \longrightarrow \big(V\otimes H\big)_{\mathrm{Sp}} \longrightarrow 0.$$

By Proposition 4.35 the group on the right is isomorphic to $\mathbb{Z}_{g-1}$. The key point of the proof is to show that the first map is trivial which will imply the claim. To do so we use the right-exactness of the coinvariants once more to put ourselves in a slightly more transparent setting. Define X via the pull back diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \wedge^3 H \arrow[d] \\
B_3 \arrow[r, "r"] & \wedge^3\overline{H}
\end{tikzcd}$$

so X has the following explicit description:

$$X=\{(f,z)\in B_3\times(\wedge^3 H)\mid r(f)=z \pmod 2\}.$$

Since the inclusion $i: B_2\to B_3$ is the kernel of r the sequence

$$0 \longrightarrow B_2 \xrightarrow{(i,0)} X \longrightarrow \wedge^3 H \longrightarrow 0$$

is exact and we obtain a commutative diagram

$$\begin{tikzcd}
\big(B_2 \otimes H\big)_{\mathrm{Sp}} \arrow[r, "(i,0)\otimes\mathrm{id}"] \arrow[d] & \big(X \otimes H\big)_{\mathrm{Sp}} \arrow[d] \\
\big((B_2/\langle \mathrm{Arf}\rangle) \otimes H\big)_{\mathrm{Sp}} \arrow[r] & \big(\mathcal{I}_{ab} \otimes H\big)_{\mathrm{Sp}}
\end{tikzcd}$$

where the vertical maps are the canonical projections. It will be enough to prove that the top horizontal map is trivial. A rather tedious calculation shows that

$$\begin{aligned}
\big((i,0)\otimes \mathrm{id}\big)(e_{1+g}\otimes x_1) &= (\nu_{12}-1)\big((e_1 e_{1+g} e_{2+g}, x_{1(1+g)(2+g)})\big)\otimes x_1\big) \\
&- (\mu_1 - 1)\big(((1+e_1)e_{1+g}e_{2+g}, x_{1(1+g)(2+g)})\big)\otimes x_1\big) \\
&+ (\mu_2 - 1)\big(((1+e_1)e_{1+g}e_{2+g}, x_{1(1+g)(2+g)})\big)\otimes x_2\big) \\
&+ (\mu_2 - 1)\big((e_2 e_{1+g}, 0)\otimes x_1\big)
\end{aligned}$$

and hence the image of the element $e_{1+g}\otimes x_1$ in $(X\otimes H)_{\mathrm{Sp}}$ vanishes. On the other hand, $(B_2\otimes H)_{\mathrm{Sp}}$ is generated by the image of this element according to Proposition 4.36. This finishes the proof. $\square$

So far we have shown that the group $\mathrm{H}_1(\mathrm{Out}^+(\Gamma), H)$ is indeed annihilated by $2g-2$. The fact that it is torsion now easily implies the vanishing of the cohomology groups $\mathrm{H}^1\big(\mathrm{Out}^+(\Gamma), \mathrm{Hom}(H,A)\big)$ for A torsion-free. Indeed, by the universal coefficient theorem (Proposition 4.32) we have

$$\mathrm{H}^1\big(\mathrm{Out}^+(\Gamma), \mathrm{Hom}(H,A)\big) \cong \mathrm{Hom}\,\big(\mathrm{H}_1(\mathrm{Out}^+(\Gamma), H), A\big) = 0$$

(the Ext-term vanishes since H has trivial coinvariants by Lemma 4.34).

Finally, we prove Theorem 4.30 (b). The missing main ingredient is a deep result of Looijenga who has computed the *stable* cohomology ring of the mapping class group for a certain class of rational coefficients in terms of the stable cohomology ring with trivial rational coefficients. We will not discuss the stability results on the cohomology of mapping class groups here but instead refer to [24] for an overview and to the references therein. The main result in [24] is essentially the following. Let U be a finite dimensional rational algebraic representation of the group $\mathrm{Sp}_{2g}(\mathbb{Q})$. Then U also naturally carries an action of the mapping class group of a closed surface S of genus g via the composition $\mathrm{Out}^+(\Gamma) \twoheadrightarrow \mathrm{Sp}(\omega) \to \mathrm{Sp}_{2g}(\mathbb{Q})$. The cohomology of the mapping class group with coefficients in U can then be explicitely computed within the stable range in terms of the rational cohomology and certain combinatorial data associated with the representation U. In the case of the rational representation $U = H^*_{\mathbb{Q}} = H^*\otimes\mathbb{Q}$ this gives:

Theorem 4.38. *For $g \geq 6$ the cohomology group $\mathrm{H}^2(\mathrm{Out}^+(\Gamma), H^*_{\mathbb{Q}})$ vanishes.*

Proof. In the notation of [24], Theorem 1 the tensor product of graduated algebras on the left hand side has the form $b\cdot \mathrm{H}^\bullet(\Gamma_\infty, \mathbb{Q})[c]$ where b is of degree 3 and c is of degree 2. In particular, the homogeneous part of degree 2 is trivial. $\square$

Now, observe that there is an equivariant isomorphism

$$H^*_{\mathbb{Q}} = \mathrm{Hom}(H,\mathbb{Z})\otimes\mathbb{Q} \xrightarrow{\cong} \mathrm{Hom}(H,\mathbb{Q}),$$

hence by the universal coefficient theorem we have

$$0 = \mathrm{H}^2\big(\mathrm{Out}^+(\Gamma), \mathrm{Hom}(H,\mathbb{Q})\big) \cong \mathrm{Hom}\,\big(\mathrm{H}_2(\mathrm{Out}^+(\Gamma), H), \mathbb{Q}\big),$$

the Ext-term being trivial. So $\mathrm{H}_2(\mathrm{Out}^+(\Gamma), H)$ does not admit any non-trivial homomorphisms to $\mathbb{Q}$ and therefore has to be torsion since $\mathbb{Q}$ is injective as a group. Another application of the universal coefficient theorem gives

$$\mathrm{H}^2\big(\mathrm{Out}^+(\Gamma), H^*\big) \cong \mathrm{Ext}^1_{\mathbb{Z}}\left(\mathrm{H}_1(\mathrm{Out}^+(\Gamma), H), \mathbb{Z}\right) \cong \mathrm{H}_1(\mathrm{Out}^+(\Gamma), H),$$

where the second isomorphism holds because $\mathrm{H}_1(\mathrm{Out}^+(\Gamma), H)$ is finitely generated and torsion. This finishes the proof of Theorem 4.30.

4.7. **Trivialising the Bounded Euler Class.** Finally, we will put the cohomological computations of the last subsections to good use. We start with a splitting result.

Theorem 4.39. *Assume that $g \geq 6$. The extension*

$$H^* \rightarrowtail \mathrm{Out}^+(\overline{\Gamma}_{2-2g}) \twoheadrightarrow \mathrm{Out}^+(\Gamma)$$

splits and any two splitting maps are conjugated.

Proof. On the one hand, the cohomology class

$$[\mathrm{Out}^+(E_{2-2g})] \in \mathrm{H}^2(\mathrm{Out}^+(\Gamma), H^*)$$

of the above extension is a $(2g-2)$-fold multiple by Corollary 4.18. On the other hand, the latter group is annihilated by $2g-2$ according to Theorem 4.30 (b). Hence $[\mathrm{Out}^+(E_{2-2g})] = 0$ and the extension splits. The second claim is a consequence of Lemma 4.3 and the vanishing of the first cohomology $\mathrm{H}^1(\mathrm{Out}^+(\Gamma), H^*) = 0$. □

Now consider a holonomy representation $\rho : \Gamma \to \mathrm{PSU}(1,1) \to \mathrm{Homeo}^+(S^1)$. The pullback $e_b^\Gamma \in \mathrm{H}_b^2(\Gamma)$ via ρ of the bounded Euler class $e_b \in \mathrm{H}_b^2(\mathrm{Homeo}^+(S^1))$ is independent of the choice of ρ by Theorem 3.4 and is invariant under the action of the mapping class group $\mathrm{Out}^+(\Gamma)$ by Corollary 3.5.
Moreover, by Theorem 3.3 the pullback $e_{\mathbb{Z}}^\Gamma \in \mathrm{H}^2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$ of the integral Euler class $e_{\mathbb{Z}} \in \mathrm{H}^2(\mathrm{Homeo}^+(S^1), \mathbb{Z})$ via ρ equals $2g-2$ times a generator. Hence the pullback of $e_{\mathbb{Z}}^\Gamma$ to $\mathrm{H}^2(\overline{\Gamma}_{2-2g}, \mathbb{Z})$ vanishes by Corollary 4.5 (a) and therefore the pullback α of e_b^Γ lies in the kernel of the comparison map:

$$\alpha \in \ker\left(\mathrm{H}_b^2(\overline{\Gamma}_{2-2g}) \xrightarrow{c} \mathrm{H}^2(\overline{\Gamma}_{2-2g}, \mathbb{R})\right).$$

By Theorem 4.39 the canonical action of the mapping class group $\mathrm{Out}^+(\Gamma)$ by outer automorphisms on Γ lifts to an action on $\overline{\Gamma}_{2-2g}$ by outer automorphisms. Fixing such a lift we have actions of $\mathrm{Out}^+(\Gamma)$ on the cohomology group $\mathrm{H}_b^2(\overline{\Gamma}_{2-2g}, \mathbb{R})$ as well as on the space $Q^h(\overline{\Gamma}_{2-2g})$ of homogeneous quasimorphisms. Since e_b^Γ is invariant the same holds for α:

$$\alpha \in \mathrm{H}_b^2(\overline{\Gamma}_{2-2g})^{\mathrm{Out}^+(\Gamma)}.$$

Hence α is invariant and trivialised by a homogeneous quasimorphism by Lemma 2.9. The question now arises whether there exists an *invariant* homogeneous quasimorphism which trivialises α. In general this will not be the case, however, we have:

Lemma 4.40. *Let Γ be a group and let G be a group which acts on Γ (by outer automorphisms). Consider a G-invariant bounded cohomology class α in the kernel of the comparison map $\mathrm{H}_b^2(\Gamma) \to \mathrm{H}^2(\Gamma)$.*

(a) *If $\mathrm{H}^1(G, \mathrm{Hom}(\Gamma, \mathbb{R})) = 0$ then there exists a G-invariant homogeneous quasimorphism φ_G which trivialises α, i.e., $\alpha = [d\varphi_G]$.*

(b) *If $A \leq R$ is a subgroup,* $\mathrm{H}^1(G, \mathrm{Hom}(\Gamma, A)) = 0$ *and if α is trivialised by a homogeneous quasimorphism taking values in A, then φ_G can be chosen to have values in A as well.*

If moreover $\mathrm{Hom}(\Gamma, A)^G = 0$ *then φ_G is unique.*

Proof. Observe that (a) is a consequence of (b) by Lemma 2.9. Hence we assume that $\varphi \in Q^h(\Gamma)$ is A-valued and trivialises α. For every $g \in G$ the map $g\varphi$ is an A-valued homogeneous quasimorphism as well. In addition we have

$$[d(g\varphi - \varphi)] = g\alpha - \alpha = 0$$

since α is assumed to be G-invariant. Hence $g\varphi - \varphi$ is a homomorphism by Lemma 2.9 and we obtain a map

$$u : G \to \mathrm{Hom}(\Gamma, A), \qquad g \mapsto g\varphi - \varphi.$$

Obviously, u satisfies the cocycle identity and therefore defines an element in the cohomology group $\mathrm{H}^1(G, \mathrm{Hom}(\Gamma, A))$. By assumption, the latter is trivial and so there exists $f \in \mathrm{Hom}(\Gamma, A)$ with $u(g) = gf - f$ for all $g \in G$. Now the map $\varphi_G = \varphi - f$ satisfies all conditions: It is a homogeneous A-valued quasimorphism which trivialises α and which is moreover G-invariant by the choice of f.
For the last claim we consider two such G-invariant quasimorphisms. Their difference is then a G-invariant homomorphism $\Gamma \to A$, hence trivial by assumption. $\square$

As a direct consequence of the cohomological computations in the last subsection we obtain:

Theorem 4.41. *Assume that $g \geq 3$, then*

$$\big(\mathrm{H}_b^2(\Gamma)\big)^{\mathrm{Out}^+(\Gamma)} = \mathbb{R}e_b^\Gamma \oplus \big(Q^h(\Gamma)\big)^{\mathrm{Out}^+(\Gamma)}.$$

Proof. We have

$$\mathrm{H}^1(\mathrm{Out}^+(\Gamma), \mathrm{Hom}(\Gamma, \mathbb{R})) = 0, \qquad (\Gamma_{ab})_{\mathrm{Out}^+(\Gamma)} = 0$$

by Theorem 4.30 respectively by Lemma 4.34. Hence by Lemma 4.40 there is an exact sequence

$$0 \longrightarrow \big(Q^h(\Gamma)\big)^{\mathrm{Out}^+(\Gamma)} \longrightarrow \big(\mathrm{H}_b^2(\Gamma)\big)^{\mathrm{Out}^+(\Gamma)} \xrightarrow{c} \big(\mathrm{H}^2(\Gamma, \mathbb{R})\big)^{\mathrm{Out}^+(\Gamma)}.$$

By the very definition of $\mathrm{Out}^+(\Gamma)$ its action on $\mathrm{H}^2(\Gamma, \mathbb{R}) \cong \mathbb{R}$ is trivial. On the other hand the image of e_b^Γ under the comparison map is non-trivial by Theorem 3.3. Now the claim follows since the above sequence of real vector spaces splits. $\square$

Theorem 4.42. *Assume that $g \geq 6$. The invariant class $\alpha \in \mathrm{H}_b^2(\overline{\Gamma}_{2-2g})$ is trivialised by a unique homogeneous quasimorphism* Rot *which is invariant under the lifted mapping class group action. Moreover,* Rot *is integral-valued.*

Proof. By Corollary 4.5 there exists a homomorphism σ such that the diagram

$$\begin{tikzcd}
\overline{\Gamma}_{2-2g} \arrow[rr, dotted, "\sigma"] \arrow[d] & & \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \arrow[d, "q"] \\
\Gamma \arrow[r, "\rho"] & \mathrm{PSU}(1,1) \arrow[r] & \mathrm{Homeo}^+(S^1)
\end{tikzcd}$$

commutes. On the one hand, the pullback $q^*(e_b) \in \mathrm{H}_b^2(\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}))$ is trivialised by the translation quasimorphism T by Lemma 2.20 and hence the quasimorphism $T \circ \sigma \in Q^h(\overline{\Gamma}_{2-2g})$ trivialises α. On the other hand, every element in the image of ρ is parabolic or hyperbolic by Lemma 3.1 and therefore has vanishing rotation number. Consequently, $T \circ \sigma$ is integral valued.

Now we can apply Lemma 4.40 (b) to the invariant class α and the value group $A = \mathbb{Z}$. Observe that $\mathrm{Hom}(\overline{\Gamma}_{2-2g}, \mathbb{Z}) = H^*$ by Lemma 4.12 (b) and hence

$$\mathrm{H}^1(\mathrm{Out}^+(\Gamma), \mathrm{Hom}(\overline{\Gamma}_{2-2g}, \mathbb{Z})) = \mathrm{H}^1(\mathrm{Out}^+(\Gamma), H^*) = 0$$

by Theorem 4.30 (a). Moreover, every $\mathrm{Out}^+(\Gamma)$-invariant homomorphism $\overline{\Gamma}_{2-2g} \to \mathbb{Z}$ factors over $H_{\mathrm{Out}^+(\Gamma)}$ which is trivial by Lemma 4.34. Therefore, there exists a unique invariant homogeneous quasimorphism $\mathrm{Rot} : \overline{\Gamma}_{2g-2} \to \mathbb{R}$ trivialising α and it is integer-valued. □

By Lemma 4.13 the group $\overline{\Gamma}_{2-2g}$ is isomorphic to the fundamental group of the unit tangent bundle $\mathbb{T}^1 S$. In the next section we will actually construct Rot geometrically on the unit tangent bundle of an arbitrary surface S.

5. The Geometric Picture

In this section we parallel the results of the last one. However, there are two essential differences. On the one hand, we use geometric methods here instead of algebraic ones. On the other hand, the constructions will work in full generality, i.e., for arbitrary not necessarily compact hyperbolic surfaces. We start by constructing an action of the mapping class group $\mathcal{M}(S)$ of a surface S by orientation preserving homeomorphisms on the unit tangent bundle of S. This is well known, we follow [37]. Then we define the quasimorphism Rot_h in dependence of a metric $h \in \mathrm{Hyp}(S)$. As it turns out, the natural domain of Rot_h is not an individual fundamental group of the unit tangent bundle of S but rather the whole fundamental groupoid $\pi_1(\mathbb{T}^1 S)$. We prefer the (metric independent) semi-projectivised version of the unit tangent bundle for several reasons. First, in this interpretation the quasimorphisms Rot_h form a continuous family in h, and second, the restriction of all Rot_h to the set of closed classes is integer-valued and hence independent of the metric h.

The first subsection deals with an algebraic invariant characterising homotopy classes of self maps of a space Y which depends on a chosen covering space of Y. This mainly serves as a technical preparation for Subsection 5.2 where we recall the construction of the lifted action of the mapping class group on the unit tangent bundle of a surface. In the remaining three subsections we first construct a common lift of all holonomy representations of all fundamental groups of S associated to a fixed metric h. Then we use this lift to pull back the translation quasimorphism to obtain the quasimorphism Rot_h. Finally, we prove various nice properties of the family $(\mathrm{Rot}_h)_{h \in \mathrm{Hyp}(S)}$.

5.1. **Coverings and Homotopies.** Associated to a covering $X \to Y$, we introduce an endomorphism-valued invariant which classifies the homotopy classes of continuous self maps of Y which lift to X. In this subsection *all spaces are assumed to be locally compact.*

Consider a regular covering $p : X \to Y$ of connected spaces with group of covering transformations Γ. We denote by $C(X)$ respectively $C(Y)$ the space of continuous self maps of X respectively Y equipped with the compact-open topology.

Denote by $N_{C(X)}(\Gamma)$ the set of all $\psi \in C(X)$ satisfying the following property: For all $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ such that $\psi\gamma = \gamma'\psi$. The element γ' is uniquely determined by ψ and γ since a covering transformation is determined by the value at one single point. We denote it by $a_\psi(\gamma)$. It is easy to see that $N_{C(X)}(\Gamma) \leq C(X)$ is a submonoid, that the map $a_\psi : \Gamma \to \Gamma$ is an endomorphism and that

$$a_{(-)} : N_{C(X)}(\Gamma) \to \mathrm{End}(\Gamma)$$

is a homomorphism. Observe that $N_{C(X)}(\Gamma) \cap \mathrm{Homeo}(X) = N_{\mathrm{Homeo}(X)}(\Gamma)$ is the usual normaliser of the subgroup Γ and that for $\psi \in N_{\mathrm{Homeo}(X)}(\Gamma)$ the automorphism a_ψ of Γ is induced by the conjugation with ψ.

Assume that $\varphi \in C(Y)$ lifts to a continuous map $X \to X$. Then every lift lies in $N_{C(X)}(\Gamma)$ and any two such lifts differ only by left multiplication with an element of Γ. Conversly, every element in $N_{C(X)}(\Gamma)$ descends to a continuous map $Y \to Y$. In the special case where X is simply connected, every element of $C(Y)$ lifts and the following sequence is exact:

$$1 \longrightarrow \Gamma \longrightarrow N_{C(X)}(\Gamma) \longrightarrow C(Y) \longrightarrow 1.$$

Now consider a tower of regular coverings $Z \xrightarrow{p'} X \xrightarrow{p} Y$ of connected spaces and let Γ respectively Γ' be the groups of covering transformations of p respectively p'. Denote by Λ the group of covering transformations of $p \circ p'$. There is a short exact sequence

$$1 \longrightarrow \Gamma' \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow 1$$

where the second map is given by descent. Denote by

$$\begin{aligned} A_{(-)} &: N_{C(Z)}(\Lambda) \to \mathrm{End}(\Lambda), \\ a_{(-)} &: N_{C(X)}(\Gamma) \to \mathrm{End}(\Gamma) \end{aligned}$$

the corresponding maps defined above. Observe that Γ' acts by left multiplication on $N_{C(Z)}(\Lambda)$ and by (pointwise) conjugation on $\mathrm{End}(\Lambda)$. With respect to these actions the map $A_{(-)}$ is Γ'-equivariant.

Lemma 5.1. *For every $\psi \in N_{C(Z)}(\Lambda)$ and every $\gamma' \in \Gamma'$ we have $A_\psi(\gamma') = \mathrm{id}_Z$. Hence A_ψ descends to a well-defined endomorphism of Γ. The following diagram commutes and has exact columns:*

$$\begin{tikzcd} \Gamma' \arrow[r, equal] \arrow[d] & \Gamma' \arrow[d, "\mathrm{Int}"] \\ N_{C(Z)}(\Lambda) \arrow[r, "A_{(-)}"] \arrow[d] & \mathrm{End}(\Lambda) \arrow[d] \\ N_{C(X)}(\Gamma) \arrow[r, "a_{(-)}"] & \mathrm{End}(\Gamma) \end{tikzcd}$$

Proof. All of this is routine verification. □

We keep the notations introduced above.

Proposition 5.2. *Let $p : X \to Y$ be a regular covering with group of covering transformations Γ. Let $\tilde{X} \to X$ be a universal covering of X and let Γ' be its group of covering transformations. Assume that there exists a Γ'-invariant metric on $\tilde{X}$ such that $\tilde{X}$ is* CAT(0)*. Then the non-empty fibres of the map*

$$a_{(-)} : N_{C(X)}(\Gamma) \to \mathrm{End}(\Gamma)$$

are precisely the path-connected components of $N_{C(X)}(\Gamma)$.

Proof. We first prove that every path-component is mapped to a single point. For this none of the assumptions on $\tilde{X}$ is needed. Notice that $\mathrm{End}(\Gamma)$ is totally disconnected when equipped with the compact-open topology since Γ is discrete (cf. Lemma 1.5). Hence it is enough to prove that $a_{(-)}$ is continuous. For this we have to check the continuity of the map $N_{C(X)}(\Gamma) \times \Gamma \to \Gamma$, $(\psi, \gamma) \mapsto a_\psi(\gamma)$. But this is obvious since Γ is discrete and since, for fixed γ, the map $a_{(-)}(\gamma)$ is clearly locally constant.
It remains to show that the fibres of $a_{(-)}$ are path-connected. Assume that $a_\psi = a_{\psi'}$. By Lemma 5.1 we can choose lifts $\theta, \theta' \in N_{C(\tilde{X})}(\Lambda)$ such that $A_\theta = A_{\theta'} = A$. For $\tilde{x} \in \tilde{X}$ let $\sigma_{\tilde{x}} : I \to \tilde{X}$ be the unique constant speed geodesic segment connecting $\theta(\tilde{x})$ to $\theta'(\tilde{x})$. Because $\tilde{X}$ was assumed to be CAT(0), the map

$$H : I \times \tilde{X} \to \tilde{X}, \qquad t \mapsto \sigma_{\tilde{x}}(t)$$

is continuous (cf. [5], Proposition 1.4. (1)). Moreover, observe that for $\lambda \in \Lambda$ we have two constant speed geodesic segments $\sigma_{\lambda(\tilde{x})}$ and $A(\lambda)\sigma_{\tilde{x}}$ with equal starting and end point. Therefore, $\sigma_{\lambda(\tilde{x})} = A(\lambda)\sigma_{\tilde{x}}$ and

$$H(t, \lambda(\tilde{x})) = \sigma_{\lambda(\tilde{x})}(t) = A(\lambda)\sigma_{\tilde{x}}(t) = A(\lambda)H(t, \tilde{x}).$$

Hence $t \mapsto H(t, \cdot)$ is a continuous path $I \to N_{C(\tilde{X})}(\Lambda)$ connecting θ to θ' which descends to a continuous path $I \to N_{C(X)}(\Gamma)$ connecting ψ to ψ'. This finishes the proof. $\square$

Theorem 5.3. *We keep the notations and assumptions of Proposition 5.2. Consider two elements $\varphi_1, \varphi_2 \in C(Y)$ which both lift to X. Then the following are equivalent:*

(i) $\varphi_1 \simeq \varphi_2$.
(ii) a_{ψ_1} *and* a_{ψ_2} *are Γ-conjugate for some (hence all) lifts ψ_i of φ_i, $i = 1, 2$.*

Proof. The remark in the brackets in (ii) is a consequence of Lemma 5.1. Assume that (i) holds. Choose a lift ψ_1 of φ_1, then the homotopy between $\varphi_1 \simeq \varphi_2$ lifts to a path $I \to N_{C(X)}(\Gamma)$ connecting ψ_1 to a lift ψ_2 of φ_2. By Proposition 5.2 we conclude $a_{\psi_1} = a_{\psi_2}$. On the other hand, if (ii) holds we can assume that actually $a_{\psi_1} = a_{\psi_2}$. Then again by Proposition 5.2 there is a path $I \to N_{C(X)}(\Gamma)$ connecting ψ_1 to ψ_2. This path descends to a homotopy $\varphi_1 \simeq \varphi_2$. $\square$

5.2. **The Action of the Mapping Class Group on T^1S.** Let S be a surface equipped with a fixed metric $h \in \mathrm{Hyp}(S)$. We shall construct an alternative model of the unit tangent bundle $T^1S = T^{1,h}S$. Then we use this model to define an action of the mapping class group $\mathcal{M}(S)$ by orientation preserving homeomorphisms on T^1S. This construction is well known, our exposition follows [37]. Using the methods of the last subsection we finally prove that this action lifts the 'outer action' of $\mathcal{M}(S)$ on S.

We start by recalling some facts concerning boundaries at infinity, for a detailed exposition the reader is referred to [5], Chapter III.H 3. The boundary at infinity $\partial\mathbb{D}$ of the Poincaré disc $\mathbb{D}$ is defined as the space of equivalence classes of quasi-geodesic rays modulo finite distance. It is homeomorphic to S^1 and the union $\mathbb{D} \cup \partial\mathbb{D}$ carries a natural topology turning it into a compactification of $\mathbb{D}$ which is homeomorphic to $\overline{\mathbb{D}}$, the closed disc embedded in $\mathbb{C}$. Denote by $G = \mathrm{Isom}^+(\mathbb{D}) = \mathrm{PSU}(1,1)$ the group of orientation preserving isometries of $\mathbb{D}$.
We set

$$\partial\mathbb{D}^{(3)} = \{(x, y, z) \in \partial\mathbb{D}^3 \mid x, y, z \text{ are pairwise different}\}$$

and define a map $\partial\mathbb{D}^{(3)} \to T^1\mathbb{D}$ as follows. To a triple of pairwise different points (x, y, z) we construct the unique non-oriented geodesic γ with endpoints x, y and then the unique geodesic δ which is orthogonal to γ and with endpoint $\delta^+ = z$. Then the image of (x, y, z) is the unit tangent vector based at $\gamma \cap \delta$ pointing in the direction of δ. Of course, this construction gives the same result if we swap x and y, hence we obtain a map

$$f : (\partial\mathbb{D}^{(3)})/\mathbb{Z}_2 \longrightarrow T^1\mathbb{D} \tag{17}$$

where $\mathbb{Z}_2$ acts on $\partial\mathbb{D}^{(3)}$ by interchanging the first two entries. Clearly, f is a G-equivariant homeomorphism where G acts diagonally on the left hand side. Fix a universal covering $p : \tilde{S} \to S$ and an isometric identification of $\tilde{S}$ with $\mathbb{D}$. Denote (the image of) the group of covering transformations of p by $\Gamma < G$. As f is G-equivariant we can divide out the Γ-action in (17) and obtain a homeomorphism

$$f_S : \Gamma\backslash(\partial\mathbb{D}^{(3)})/\mathbb{Z}_2 \longrightarrow T^1S$$

by Lemma 1.3.

We are now going to construct an action of the mapping class group $\mathcal{M}(S)$ by homeomorphisms on T^1S. For this we start with an arbitrary element $\varphi \in \mathrm{Homeo}^+(S)$ and choose a lift $\psi \in N_{\mathrm{Homeo}^+(\mathbb{D})}(\Gamma)$ of φ to $\mathbb{D}$. Then ψ is 'type preserving' and hence induces a homeomorphism $\partial\psi$ on $\partial\mathbb{D}$ that continuously extends ψ. The map

$$\partial\psi^{(3)} \in N_{\mathrm{Homeo}^+((\partial\mathbb{D})^{(3)})}(\Gamma)$$

is obviously $\mathbb{Z}_2$-equivariant, hence it descends to a homeomorphism

$$\Psi \in N_{\mathrm{Homeo}^+((\partial\mathbb{D})^{(3)}/\mathbb{Z}_2)}(\Gamma)$$

which, in turn, descends to a homeomorphism $\Phi \in \mathrm{Homeo}^+(\Gamma\backslash(\partial X)^{(3)}/\mathbb{Z}_2)$. As the notation suggests, Φ is independent of the choice of the lift ψ. Actually, it only depends on the homotopy class of φ. Indeed, let $\varphi_1 \simeq \varphi_2$ be homotopic and choose lifts ψ_1 and ψ_2. By Theorem 5.3 applied to the covering $\mathbb{D} \to S$ there exists $\gamma \in \Gamma$ such that $a_{\psi_1} = \gamma a_{\psi_2}\gamma^{-1}$. But then we have $\partial\psi_1 = \gamma\partial\psi_2$ and hence $\Psi_1 = \gamma\Psi_2$, so finally indeed $\Phi_1 = \Phi_2$. At last, we can conjugate Φ with the homeomorphism f_S to obtain $\varphi^{\#} \in \mathrm{Homeo}^+(T^1S)$. Clearly, we have:

Theorem 5.4. *The above construction yields a well-defined homomorphism*

$$(-)^{\#} : \mathcal{M}(S) \longrightarrow \mathrm{Homeo}^+\left(T^1S\right).$$

The homomorphism $(-)^{\#}$ lifts the outer action of $\mathcal{M}(S)$ on the surface S in the following sense:

Theorem 5.5. *Let $\varphi \in \mathrm{Diff}^+(S)$ be a diffeomorphism. Then the two homeomorphisms $\varphi^{\#}$ and $d\varphi$ of T^1S are homotopic.*

Proof. We start by recalling the involved coverings. First we have $\mathbb{D} \to S$ with group of covering transformations Γ. Then there is a commutative diagram

$$\begin{tikzcd}
(\partial\mathbb{D})^{(3)}/\mathbb{Z}_2 \arrow[r, "f"] \arrow[d] & T^1\mathbb{D} \arrow[d] \\
\Gamma\backslash(\partial\mathbb{D})^{(3)}/\mathbb{Z}_2 \arrow[r, "f_S"] & T^1S
\end{tikzcd}$$

where f and f_S are homeomorphisms. The two vertical maps are isomorphic regular coverings with group of covering transformations Γ. Here Γ acts on $(\partial\mathbb{D})^{(3)}/\mathbb{Z}_2$ via the maps $\partial\gamma$ and on $T^1\mathbb{D}$ via the differentials $d\gamma$. Notice that all coverings satisfy the assumptions of Theorem 5.3. For $\mathbb{D} \to S$ this is clear as $\mathbb{D}$ is simply connected and the Poincaré metric is Γ-invariant with sectional curvature -1, hence is CAT(0). Similarly, the universal covering of $T^1\mathbb{D}$ can be identified with $\mathbb{D} \times \mathbb{R}$ with the product metric Poincaré $\times$ Euclidian which has non-positive sectional curvature. We denote the corresponding invariants by

$$\begin{aligned}
a_{(-)} &: N_{\mathrm{Homeo}^+(\mathbb{D})}(\Gamma) \to \mathrm{Aut}(\Gamma) \\
b_{(-)} &: N_{\mathrm{Homeo}^+(T^1\mathbb{D})}(\Gamma) \to \mathrm{Aut}(\Gamma) \\
c_{(-)} &: N_{\mathrm{Homeo}^+((\partial\mathbb{D})^{(3)}/\mathbb{Z}_2)}(\Gamma) \to \mathrm{Aut}(\Gamma)
\end{aligned}$$

To start with the actual proof, choose a lift $\psi \in N_{\mathrm{Diff}^+(\mathbb{D})}(\Gamma)$ of φ. On the one hand, for every $\gamma \in \Gamma$ we have $\psi\gamma = a_\psi(\gamma)\psi$. Looking at the induced maps on

$\partial\mathbb{D}$ we obtain $\partial\psi\partial\gamma = \partial a_\psi(\gamma)\partial\psi$ and hence clearly $c_\Psi = a_\psi$. On the other hand, we obtain similarly $d\psi d\gamma = da_\psi(\gamma)d\psi$ and therefore $b_{d\psi} = a_\psi$. Finally, we have $b_{f\Psi f^{-1}} = c_\Psi$. So we can conclude that

$$b_{f\Psi f^{-1}} = b_{d\psi}$$

and by Theorem 5.3 the induced maps $[\varphi]^\#$ and $d\varphi$ on T^1S are homotopic. $\square$

5.3. **Lifting the Holonomy Representations.** We fix a metric $h \in \mathrm{Hyp}(S)$ on the surface S for this subsection. We are going to construct lifts of the various holonomy representations to the fundamental groups $\pi_1(\mathbb{T}^1S,[v])$ of the unit tangent bundle of S. We fix a universal covering $p : \tilde{S} \to S$ and we denote by $q : \mathbb{T}^1S \to S$ the basepoint map.

Recall from Subsection 1.3 that there is a canonical isomorphism

$$T^{1,\tilde{h}}\tilde{S} \to \mathbb{T}^1\tilde{S}, \qquad (\tilde{x},\tilde{v}) \mapsto (\tilde{x},[\tilde{v}])$$

of S^1-bundles over $\tilde{S}$. We will denote by $B_{\tilde{h}} : \mathbb{T}^1\tilde{S} \xrightarrow{\cong} T^{1,\tilde{h}}\tilde{S}$ the inverse of this isomorphism. We also recall from Subsection 3.1 that for every non-zero tangent vector $\tilde{v}$ of $\tilde{S}$ with base point $\tilde{x}$ there exists a unique orientation preserving isometry

$$f_{\tilde{h},\tilde{v}} : (\tilde{S},\tilde{h}) \to \mathbb{D}$$

sending $\tilde{x}$ to $0 \in \mathbb{D}$ and $\tilde{v}$ to a positive multiple of the tangent vector $1 \in T\mathbb{D}_0$.

Let $c : I \to \mathbb{T}^1S$ be a curve. Choose a lift $\tilde{c} : I \to \mathbb{T}^1\tilde{S}$ and set $\tilde{v} = B_{\tilde{h}}(\tilde{c}(0))$. Define a curve $w_c : I \to G$ by the composition

$$I \xrightarrow{\tilde{c}} \mathbb{T}^1\tilde{S} \xrightarrow{B_{\tilde{h}}} T^{1,\tilde{h}}\tilde{S} \xrightarrow{df_{\tilde{h},\tilde{v}}} T^1\mathbb{D} \xrightarrow{\cong} G$$

where the homeomorphism on the right is given by the simply transitive action of G on $T^1\mathbb{D}$, cf. Lemma 1.7. Observe that, by definition of the isometry $f_{\tilde{h},\tilde{v}}$, the path w_c starts at the neutral element $e \in G$. We need the following technical result:

Lemma 5.6. (a) *The path w_c does not depend on the choice of the lift $\tilde{c}$.*
(b) *If c and c' are homotopic relative $\{0,1\}$ then so are the paths w_c and $w_{c'}$.*
(c) *Let c_1 and c_2 be two composable curves, i.e., $c_1(1) = c_2(0)$. Then*

$$w_{c_1c_2} \simeq w_{c_1} \cdot w_{c_2} \quad (\mathrm{rel}\ \{0,1\})$$

where on the right hand side we mean pointwise multiplication in G.

Proof. (a) Consider two lifts $\tilde{c}_1$, $\tilde{c}_2$ of c to $\tilde{S}$. For $i = 1,2$ set $\tilde{v}_i = B_{\tilde{h}}(\tilde{c}_i(0))$, let $\tilde{x}_i$ be the base point of $\tilde{v}_i$ and let $x \in S$ lie below both $\tilde{x}_i$. According to Lemma 1.3 there exists $\gamma \in \pi_1(S,x)$ such that $\tilde{v}_2 = dT^{\tilde{x}_1}_\gamma(\tilde{v}_1)$. This implies $\tilde{c}_2 = [dT^{\tilde{x}_1}_\gamma] \circ \tilde{c}_1$ by the uniqueness of lifts. Moreover, we have the equality

$$f_{\tilde{h},\tilde{v}_1} = f_{\tilde{h},\tilde{v}_1} \circ T^{\tilde{x}_1}_\gamma$$

since the composition on the right hand side satisfies the defining property of $f_{\tilde{h},\tilde{v}_1}$. These two observations imply that the diagram

$$\begin{tikzcd}
I \arrow[r, "\tilde{c}_1"] \arrow[d, equal] & \mathbb{T}^1\tilde{S} \arrow[r, "B_{\tilde{h}}"] \arrow[d, "{[dT^{\tilde{x}_1}_\gamma]}"] & T^{1,\tilde{h}}\tilde{S} \arrow[r, "df_{\tilde{h},\tilde{v}_1}"] \arrow[d, "dT^{\tilde{x}_1}_\gamma"] & T^1\mathbb{D} \arrow[r, "\cong"] \arrow[d, equal] & G \arrow[d, equal] \\
I \arrow[r, "\tilde{c}_2"] & \mathbb{T}^1\tilde{S} \arrow[r, "B_{\tilde{h}}"] & T^{1,\tilde{h}}\tilde{S} \arrow[r, "df_{\tilde{h},\tilde{v}_2}"] & T^1\mathbb{D} \arrow[r, "\cong"] & G
\end{tikzcd}$$

commutes and this implies the claim. To see (b) just note that we can lift a homotopy between c and c' to obtain homotopic lifts (rel $\{0,1\}$) of these curves to $\mathbb{T}^1\tilde{S}$. Finally, we prove (c). Choose composable lifts $\tilde{c}_1$ and $\tilde{c}_2$ and set $\tilde{v}_i = B_{\tilde{h}}(\tilde{c}_i(0))$. Let $g \in G$ be the unique element such that

$$df_{\tilde{h},\tilde{v}_1}(\tilde{v}_2) = g(1).$$

Then the diagram

$$\begin{tikzcd}
I \arrow[r, "\tilde{c}_2"] \arrow[d, equal] & \mathbb{T}^1\tilde{S} \arrow[r, "B_{\tilde{h}}"] \arrow[d, equal] & T^{1,\tilde{h}}\tilde{S} \arrow[r, "df_{\tilde{h},\tilde{v}_2}"] \arrow[d, equal] & T^1\mathbb{D} \arrow[r, "\cong"] \arrow[d, "g"] & G \arrow[d, "g\cdot"] \\
I \arrow[r, "\tilde{c}_2"] & \mathbb{T}^1\tilde{S} \arrow[r, "B_{\tilde{h}}"] & T^{1,\tilde{h}}\tilde{S} \arrow[r, "df_{\tilde{h},\tilde{v}_1}"] & T^1\mathbb{D} \arrow[r, "\cong"] & G
\end{tikzcd}$$

commutes and hence $w_{c_1c_2}$ is the concatenation of the paths w_{c_1} and $g\cdot w_{c_2}$. Observe that $g = w_{c_1}(1)$, so by Lemma 1.2 we have

$$w_{c_1c_2} = w_{c_1} * \big(w_{c_1}(1)\cdot w_{c_2}\big) = \big(w_{c_1} * (w_{c_1}(1)\big)\cdot w_{c_2} \simeq w_{c_1}\cdot w_{c_2} \quad (\text{rel } \{0,1\})$$

where $w_{c_1}(1)$ denotes the constant path. This finishes the proof. □

Corollary 5.7. *Interpreting the homotopy class* (rel $\{0,1\}$) *of the curve w_c as an element of the universal covering group $\tilde{G}$ of G, the above construction induces a well-defined map*

$$\sigma_h : \pi_1(\mathbb{T}^1S) \to \tilde{G}, \qquad [c] \mapsto [w_c]$$

on the fundamental groupoid of $\mathbb{T}^1S$. This map is a homomorphism.

Proof. Indeed, by Lemma 5.6 a) and b) the map σ_h is well-defined and by c) it is a homomorphism. □

Proposition 5.8. *The homomorphism σ_h simultaneously lifts all holonomy representations of all fundamental groups of S. More precisely, for any non-zero tangent vector $v \in TS$ with base point x there is a commutative diagram*

$$\begin{tikzcd}
\pi_1(\mathbb{T}^1S,[v]) \arrow[r, "\sigma_h"] \arrow[d, "q_*"'] & \tilde{G} \arrow[d] \\
\pi_1(S,x) \arrow[r, "\rho_{h,v}"] & G
\end{tikzcd}$$

Here the right vertical map is the universal covering, which in our interpretation of $\tilde{G}$ is given by mapping a homotopy class of curves in G to the common endpoint of these curves.

Proof. Let $c : I \to \mathbb{T}^1S$ be a closed curve based at $[v]$ and let $\tilde{c}$ be a lift to $\mathbb{T}^1\tilde{S}$ whose starting vector is based at $\tilde{x}$. Set $\tilde{v} = B_{\tilde{h}}(\tilde{c}(0))$. Let b denote the composition

$$I \xrightarrow{\tilde{c}} \mathbb{T}^1\tilde{S} \xrightarrow{B_{\tilde{h}}} T^{1,\tilde{h}}\tilde{S}$$

and let $g = w_c(1) \in G$ denote the endpoint of the curve w_c. Finally, denote by $\gamma = [\bar{c}] \in \pi_1(S,x)$ the class of the curve $\bar{c}$ on S induced by c. Now, on the one hand, we have

$$b(1) = dT_\gamma^{\tilde{x}}\big(b(0)\big)$$

by definition of γ and Lemma 1.3. On the other hand,

$$df_{\tilde{h},\tilde{v}} \circ b(1) = g\big(df_{\tilde{h},\tilde{v}} \circ b(0)\big).$$

Therefore, the two orientation preserving isometries $T^{1,\tilde{h}}\tilde{S} \to T^1\mathbb{D}$ given by $df_{\tilde{h},\tilde{v}} \circ dT^{\tilde{x}}_{\gamma}$ and $g \circ df_{\tilde{h},\tilde{v}}$ agree at the point $b(0)$. Hence they have to be the same and we can conclude $\rho_{h,v}(\gamma) = g$. □

5.4. **The Quasimorphism** Rot**.** We keep a metric $h \in \mathrm{Hyp}(S)$ fixed for the moment. Let Rot_h be the pullback of the translation quasimorphism under the lifted holonomy representation σ_h, i.e., we define Rot_h as the composition

$$\pi_1(\mathbb{T}^1S) \xrightarrow{\sigma_h} \tilde{G} \longrightarrow \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \xrightarrow{T} \mathbb{R}$$

where the second homomorphism is the lift of the canonical inclusion $G \hookrightarrow \mathrm{Homeo}^+(S^1)$ described in Subsection 2.5.

Theorem 5.9. *The map* Rot_h *is a homogeneous quasimorphism on the fundamental groupoid* $\pi_1(\mathbb{T}^1S)$ *with defect* 1 *whose values on classes of closed curves is integral. Its restrictions to the various fundamental groups trivialise the lifts of the corresponding bounded Euler classes. More precisely, for any non-zero tangent vector* $v \in TS$ *with base point* x *we have*

$$(q_*)^*(e_b^{\pi_1(S,x)}) = [d(\mathrm{Rot}_h|_{\pi_1(\mathbb{T}^1S,[v])})] \in \mathrm{H}^2_b\left(\pi_1(\mathbb{T}^1S,[v])\right),$$

where $e_b^{\pi_1(S,x)} \in \mathrm{H}^2_b\left(\pi_1(S,x)\right)$ *is the pullback via* $\rho_{h,v}$ *(or any other holonomy representation of* $\pi_1(S,x)$*) of the bounded Euler class* $e_b \in \mathrm{H}^2_b\left(\mathrm{Homeo}^+(S^1)\right)$ *and where* $q : \mathbb{T}^1S \to S$ *is the basepoint map.*

Proof. Since T is a homogeneous quasimorphism with defect 1, Rot_h is homogeneous as well and has defect ≤ 1. Next, consider the diagram

$$\begin{tikzcd}
\pi_1(\mathbb{T}^1S,[v]) \arrow[r, "\sigma_h"] \arrow[d, "q_*"'] & \tilde{G} \arrow[r] \arrow[d] & \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \arrow[d] \\
\pi_1(S,x) \arrow[r, "\rho_{h,v}"] & G \arrow[r] & \mathrm{Homeo}^+(S^1)
\end{tikzcd}$$

which commutes according to Proposition 5.8. Let $\gamma \in \pi_1(\mathbb{T}^1S,[v])$ be the class of a closed curve. Its image in $\pi_1(S,x)$ is mapped to a parabolic or hyperbolic element in G by Lemma 3.1 and therefore has rotation number 0. Hence the image $\sigma_h(\gamma)$ has integral translation number $\mathrm{Rot}_h(\gamma)$.
For the claim concerning the Euler classes we again refer to the above diagram. The pullback of $e_b \in \mathrm{H}^2\left(\mathrm{Homeo}^+(S^1)\right)$ along the bottom and left side of the big rectangle equals $(q_*)^*(e_b^{\pi_1(S,x)})$. Using Lemma 2.20 on the other hand, the pullback along the right and upper side equals

$$(\sigma_h)^*([dT]) = [d(\sigma_h^*(T))] = [d\,\mathrm{Rot}_h].$$

Finally, we can conclude that the defect of Rot_h is indeed 1. Since otherwise the restriction of Rot_h to any fundamental group $\pi_1(\mathbb{T}^1S,[v])$ would be a homomorphism by the above integrality statement. But then the pullback $(q_*)^*(e_b^{\pi_1(S,x)})$ is trivial which is absurd. □

Next, we deal with the dependence of Rot_h from the metric h.

Proposition 5.10. *For every* $\gamma \in \pi_1(\mathbb{T}^1S)$ *the map*

$$\mathrm{Hyp}(S) \to \mathbb{R}, \qquad h \mapsto \mathrm{Rot}_h(\gamma)$$

is continuous. The value of Rot_h *on closed classes is independent of the metric* h*.*

Proof. Unraveling the definition of Rot_h and of σ_h the continuity claim reduces to the continuity of the maps

$$\mathrm{Hyp}(S) \to T^1\mathbb{D}, \qquad h \mapsto df_{\tilde{h},B_{\tilde{h}}([\tilde{v}])} \circ B_{\tilde{h}}([\tilde{v}])$$

for every fixed non-zero tangent vector $\tilde{v} \in T\tilde{S}$. But $B_{\tilde{h}}$ as well as $f_{\tilde{h},\tilde{v}}$ depends continuously on h (and $\tilde{v}$). The second claim follows form the fact that $\mathrm{Hyp}(S)$ is connected and all Rot_h take integral values on classes of closed curves by Theorem 5.9. $\square$

Hence if $\gamma \in \pi_1(\mathbb{T}^1S)$ is closed we will frequently suppress the metric h and just write $\mathrm{Rot}(\gamma)$.

Proposition 5.11. *The value* $\mathrm{Rot}([c])$ *for a closed curve* c *depends only on the free homotopy class of* c*.*

Proof. Fix a metric $h \in \mathrm{Hyp}(S)$. Since Rot_h is homogeneous it is invariant under conjugation. But two closed classes $[c]$ and $[d]$ are conjugated in $\pi_1(\mathbb{T}^1S)$ if and only if c and d are freely homotopic. $\square$

5.5. **Invariance of** Rot **Under the Mapping Class Group Action.** In this subsection we study the transformation behaviour of Rot. The essential technical result is the following.

Proposition 5.12. *For every* $\varphi \in \mathrm{Diff}^+(S)$ *we have* $\mathrm{Rot}_h \circ (d\varphi)_* = \mathrm{Rot}_{\varphi^*(h)}$.

Proof. We have to show the equality $\mathrm{Rot}_h([d\varphi \circ c]) = \mathrm{Rot}_{\varphi^*(h)}([c])$ for every curve $c : I \to \mathbb{T}^1S$. In fact we prove that $\sigma_h([d\varphi \circ c]) = \sigma_{\varphi^*(h)}([c])$ and start with a simple observation. For every $\psi \in \mathrm{Diff}^+(\tilde{S})$ and every non-zero tangent vector $\tilde{v}$ of $\tilde{S}$ we have

$$f_{\psi^*(\tilde{h}),\tilde{v}} = f_{\tilde{h},d\psi(\tilde{v})} \circ \psi. \tag{18}$$

Indeed, the map on the right hand side satisfies the defining properties of $f_{\psi^*(\tilde{h}),\tilde{v}}$. Choose a lift $\tilde{c} : I \to \mathbb{T}^1\tilde{S}$ and set $\tilde{v} = B_{\tilde{h}}(\tilde{c}(0))$. Choose a lift $\psi \in \mathrm{Diff}^+(\tilde{S})$ of φ and observe that $d\psi \circ \tilde{c}$ is a lift of the curve $d\varphi \circ c$. In the diagram

$$\begin{tikzcd}
I \arrow[r, "\tilde{c}"] \arrow[d, equal] & \mathbb{T}^1\tilde{S} \arrow[r, "B_{\psi^*(\tilde{h})}"] \arrow[d, "{[d\psi]}"] & T^{1,\psi^*(\tilde{h})}\tilde{S} \arrow[r, "df_{\psi^*(\tilde{h}),\tilde{v}}"] \arrow[d, "d\psi"'] & T^1\mathbb{D} \arrow[r, "\cong"] \arrow[d, equal] & G \arrow[d, equal] \\
I \arrow[r, "d\psi\circ\tilde{c}"] & \mathbb{T}^1\tilde{S} \arrow[r, "B_{\tilde{h}}"] & T^{1,\tilde{h}}\tilde{S} \arrow[r, "df_{\tilde{h},d\psi(\tilde{v})}"] & T^1\mathbb{D} \arrow[r, "\cong"] & G
\end{tikzcd}$$

the first two squares obviously commute and the third does so because of (18). Now the composition of the maps in the first row represents $\sigma_{\varphi^*(h)}([c])$, while the composition in the second row represents $\sigma_h([d\varphi \circ c])$. This proves the claim. $\square$

As a consequence we obtain:

Theorem 5.13. *The restriction of* Rot *to the set of closed classes is invariant under the lifted action of the mapping class group* $\#(-) : \mathcal{M}(S) \to \mathrm{Homeo}^+(\mathbb{T}^1S)$.

Proof. We fix a metric $h \in \mathrm{Hyp}(S)$. Let c be a closed curve in $\mathbb{T}^1 S$ and let $\varphi \in \mathrm{Diff}^+(S)$. Then we have

$$\mathrm{Rot}([\#\varphi \circ c]) = \mathrm{Rot}_h([\#\varphi \circ c]) = \mathrm{Rot}_h([d\varphi \circ c]) = \mathrm{Rot}_{\varphi^*(h)}([c]) = \mathrm{Rot}([c])$$

where the second equality holds by Theorem 5.5 and Proposition 5.11 and the third by Proposition 5.12. $\square$

6. A Link to Winding Numbers

In this final section we are going to describe the geometric meaning of the quasimorphism Rot. For a regular closed curve in the plane one can define its ‘rotation number’ as the number of times the tangent vector $c'(t)$ rotates in counter-clockwise direction as c is traversed once in positive direction. In [39] Whitney has shown that two regular closed plane curves are regularly homotopic if and only if they have the same rotation number. In [12] Chillingworth introduced a concept of winding number on general surfaces which serves as a kind of substitute for the rotation number in the planar case (see also Reinhart [34] who does essentially the same thing). Since there is no canonical global directional reference frame on a general surface S he chooses a nowhere vanishing vector field X on S and then measures the number of times the tangent vector $c'(t)$ rotates in counter-clockwise direction with respect to the vector field X. In the non-compact case this leads to a satisfying generalisation of Whitney’s classification result for regular homotopy classes of curves: Fixing X, two (not nullhomotopic) regular closed curves on S are regularly homotopic if and only if they are homotopic and have the same winding number with respect to X. However, when S is compact there is no nowhere vanishing vector field on S anymore and one merely ends up with an invariant taking values in $\mathbb{Z}_{|\chi(S)|}$. We will prove that Rot is a natural analogon of the planar rotation number for arbitrary hyperbolic surfaces making any global reference frame superfluous. We prove an analogon of Whitney’s theorem in full generality, see Theorem 6.13. We will also clarify the connection of Rot to the various winding number functions in Theorem 6.14. An explicit combinatorial formula for Rot will be derived which is heavily inspired by a formula for winding numbers in [12]. In the final subsection we discuss the connection between Rot and a certain quasimorphism rot defined on the fundamental group of a non-compact surface introduced by Calegari in [11].

6.1. **A Family of Retractions on** PSU(1, 1).

Let $G = \mathrm{PSU}(1,1)$. We use the notation introduced in subsection 1.6 and start by recalling the Iwasawa decomposition for reductive groups in the special case of G. We have already defined the three subgroups

$$\begin{aligned} K &= \left\{ \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \,\middle|\, \zeta \in S^1 \right\} \\ A &= \left\{ \begin{bmatrix} \sqrt{1+t^2} & t \\ t & \sqrt{1+t^2} \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\} \\ N &= \left\{ \begin{bmatrix} 1+iy & -iy \\ iy & 1-iy \end{bmatrix} \,\middle|\, y \in \mathbb{R} \right\} \end{aligned}$$

and the essence of the decomposition theorem is that the multiplication map $K\times A\times N \to G$ is a diffeomorphism. The subgroup K is maximal compact and isomorphic to S^1. Whenever we write $K \cong S^1$ we mean the isomorphism given by

$$K \to S^1, \qquad \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \mapsto \zeta^2.$$

The groups A and N are both isomorphic to $\mathbb{R}$ and their product $B = AN$ is a Borel subgroup of G complementing K. An element $g \in G$ belongs to B if and only if $a + b \in \mathbb{R}$.

Let $\mathcal{P}$ be the set of parabolic elements and let $\mathcal{H}$ be the set of hyperbolic elements in G. We also set $\mathcal{C} = \mathcal{P} \cup \mathcal{H} \cup \{e\}$ for convenience.

Lemma 6.1. *There is a continuous map $u : \mathcal{C} \to K$ with the following properties:*

(i) *$g \in u(g)Bu(g)^{-1}$ for all $g \in \mathcal{C}$.*

(ii) *$u(kgk^{-1}) = ku(g)$ for all $g \in \mathcal{H} \cup \mathcal{P}$ and all $k \in K$.*

Proof. Let g be as above. We need to find

$$u(g) = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \in K$$

such that

$$\begin{bmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{bmatrix} \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix} \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \in B.$$

This is the case if and only if $a + \zeta^{-2}b \in \mathbb{R}$ or, equivalently, $\mathrm{Im}(\zeta^{-2}b) = -\mathrm{Im}(a)$. Now, since $p \in \mathcal{C}$ we have $\mathrm{Re}(a) \geq 1$ and hence by the determinant condition

$$|b|^2 = |a|^2 - 1 = \mathrm{Im}(a)^2 + (\mathrm{Re}(a)^2 - 1) \geq \mathrm{Im}(a)^2.$$

Therefore, there is allways a $\mu \in S^1$ such that $\mathrm{Im}(\mu^{-1}b) = -\mathrm{Im}(a)$ and we may define $u(g)$ as the pre-image of μ under the isomorphism $K \cong S^1$. The problem here is that the solution μ is not unique for $g \in \mathcal{H}$. To ensure that u is continuous and satisfies (ii) we proceed slightly different. Set

$$T = \left\{ \begin{bmatrix} a & z \\ z & \overline{a} \end{bmatrix} \;\middle|\; z \in \mathbb{R},\ \mathrm{Re}(a) \geq 1,\ |a| = z + 1,\ a \neq 1 \right\} \subset \mathcal{P} \cup \mathcal{H}$$

and observe that the restriction of the conjugation map

$$K \times T \to \mathcal{P} \cup \mathcal{H}, \qquad (k, g) \mapsto kgk^{-1}$$

is a homeomorphism. For $g \in T$ we proceed as before and choose the unique solution to the equation $\mathrm{Im}(\mu^{-1}z) = -\mathrm{Im}(a)$ with $\mathrm{Re}(\mu) \geq 0$ and observe that it depends continuously on g. Let $u(g)$ be the pre-image of μ under the isomorphism $K \cong S^1$. Next, for $k \in K$ and $g \in T$ we set

$$u(kgk^{-1}) = ku(g).$$

By the above observation this defines a continuous map on $\mathcal{H} \cup \mathcal{P}$ which clearly satisfies (i) and (ii). Finally, u extends continuously to $g = e$ by $u(e) = e$. □

If we conjugate the equality $G = BK$ with $k \in K$ we obtain another decomposition $G = kBk^{-1} \cdot K$ into two subgroups with trivial intersection. Actually, the multiplication map $kBk^{-1} \times K \to G$ is still a homeomorphism. Let $g \in \mathcal{C}$ and define $r_g : G \to K$ by the composition

$$G \xrightarrow{\cong} u(g)Bu(g)^{-1} \times K \xrightarrow{\mathrm{pr}_2} K.$$

We collect the essential properties of the maps r_g:

Lemma 6.2. (a) *For $g \in \mathcal{C}$ the map $r_g : G \to K$ is a deformation retraction and is K-equivariant with respect to the action by right multiplication on G and K. In particular, r_g restricts to orientation preserving homeomorphisms $hK \to K$ on all left cosets of K in G.*

(b) *For $k \in K$ and $g \in \mathcal{P} \cup \mathcal{H}$ the diagram*

$$\begin{tikzcd} G \arrow[r, "r_g"] \arrow[d, "k\cdot"'] & K \arrow[d, "k\cdot"] \\ G \arrow[r, "r_{kgk^{-1}}"] & K \end{tikzcd}$$

commutes.

(c) *The set $\mathcal{C}$ is path connected and the map*

$$\mathcal{C} \times G \xrightarrow{(g,h)\mapsto r_g(h)} K$$

is continuous.

Proof. (a) Clearly, r_g restricts to the identity on K, and since B is contractible it is a deformation retraction. The equivariance statement also follows immediately from the definition.
(b) Let $h \in G$ and assume that $r_g(h) = l$, i.e., there exists $b \in B$ such that $h = u(g)bu(g)^{-1} \cdot l$. Then we have

$$\begin{aligned} kh &= ku(g)bu(g)^{-1} \cdot l \\ &= (ku(g))b(ku(g))^{-1} \cdot kl \\ &= u(kgk^{-1})bu(kgk^{-1})^{-1} \cdot kl, \end{aligned}$$

where we have used that $u(kgk^{-1}) = ku(g)$. Hence $r_{kgk^{-1}}(kh) = kr_g(h)$.
(c) By Lemma 6.1 the set $\mathcal{C}$ is the union of all conjugates of the subgroup B. But since B is path connected and since the neutral element lies in all those conjugates, $\mathcal{C}$ is path connected as well.
Denote by $m : B \times K \to G$ the homeomorphism induced by the multiplication on G. For fixed $g \in \mathcal{C}$ the retraction r_g can be written as the composition

$$G \xrightarrow{\mathrm{Int}(u(g)^{-1})} G \xrightarrow{m^{-1}} B \times K \xrightarrow{\mathrm{Int}(u(g))\times\mathrm{Int}(u(g))} G \times K \xrightarrow{\mathrm{pr}_2} K$$

and together with the continuity of u this implies the continuity of the given map. $\square$

The main use of the retractions r_g lies in the computation of translation numbers:

Proposition 6.3. *Let $w : I \to G$ be a path starting at the neutral element and such that $g = w(1) \in \mathcal{C}$. Then the path*

$$I \xrightarrow{w} G \xrightarrow{r_g} K \xrightarrow{\cong} S^1$$

is closed and the degree of the induced map $S^1 \to S^1$ equals the translation number of the element $\tilde{g} = [w] \in \tilde{G}$.

Proof. We begin with a technical Lemma:

Lemma 6.4. *Let $g \in G$ and let $r_1, r_2 : G \to K$ be two retractions such that $r_1(g) = r_2(g) = e$. There exists a constant c such that the following holds: Let $w : I \to G$ be a path with $w(0) = e$ and $w(1) = g$ and for $i = 1, 2$ denote by $\deg_i(w)$ the degree of the map $S^1 \to S^1$ induced by the closed path*

$$I \xrightarrow{w} G \xrightarrow{r_i} K \xrightarrow{\cong} S^1.$$

Then $\deg_2(w) = \deg_1(w) + c$.

Proof. If $r : G \to K$ is a retraction then r and the inclusion $K \hookrightarrow G$ induce mutually inverse isomorphisms between $H_1(G)$ and $H_1(K)$. In particular, the map $r_* : H_1(G) \to H_1(K)$ is independent of the retraction r.
Let w, w' be two such paths. Then $w'w^{-1}$ is closed and its homology class in $H_1(G)$ is mapped to the same element in $H_1(K)$ by r_1 and r_2 since both are retractions. This gives the equality

$$[r_1 \circ w'] - [r_1 \circ w] = [r_1 \circ (w'w^{-1})] = [r_2 \circ (w'w^{-1})] = [r_2 \circ w'] - [r_2 \circ w]$$

in $H_1(K)$. But for a suitable generator a of $H_1(K) \cong \mathbb{Z}$ we have $[r_i \circ w] = \deg_i(w)a$ and similar for w', hence

$$\deg_2(w) - \deg_1(w) = \deg_2(w') - \deg_1(w').$$

□

Let r be the map

$$G \xrightarrow{R} S^1 \xrightarrow{\cong} K$$

induced by the rotation number. Then r is a retraction and moreover $r(g) = e$ whenever $g \in \mathcal{C}$. We claim that for $g \in \mathcal{C}$ and all paths $w : I \to G$ starting at e and ending at g we have

$$\deg_r(w) = \deg_{r_g}(w). \tag{19}$$

By Lemma 6.4 it is enough to find a suitable path w for each g such that we have equality in (19). Since $\mathcal{C}$ is path-connected we can choose $w : I \to \mathcal{C}$ starting at e and ending at g. Then, on the one hand, the composition $r \circ w$ is constant and therefore $\deg_r(w) = 0$. On the other hand, the composition

$$I \times I \xrightarrow{(s,t)\mapsto(w(s),w(st))} \mathcal{C} \times G \xrightarrow{(g,h)\mapsto r_g(h)} K$$

descends to a continuous map $f : I \times S^1 \to S^1$ giving a homotopy between a constant path and $r_g \circ w$. Hence $\deg_{r_g}(w) = 0$ as well.

Now by Proposition 2.19 $\deg_r(w)$ equals the translation number of the element $\tilde{g} \in \tilde{G}$ represented by the curve w. This finishes the proof.

□

6.2. **A Homological Interpretation of** Rot**.** Let $c : I \to S$ be a closed curve based at x. Denote by $q : \mathbb{T}^1 S \to S$ the basepoint map and let

$$\mathcal{V}_c = \{Y : I \to \mathbb{T}^1 S \mid c = q \circ Y \text{ and } Y(0) = Y(1)\}$$

be the space of closed curves lifting c. Interpreting c as being defined on S^1 we can form an S^1-bundle $E_c \to S^1$ by the pullback

$$\begin{array}{ccc} E_c & \longrightarrow & \mathbb{T}^1 S \\ {\scriptstyle q_c}\downarrow & & \downarrow{\scriptstyle q} \\ S^1 & \xrightarrow{c} & S \end{array}$$

Note that, since S is oriented, the pullback bundle has an induced orientation. In particular, it is trivial and the total space E_c is homeomorphic to a torus. Consider a trivialisation of E_c, i.e., an orientation preserving bundle isomorphism $E_c \to S^1 \times S^1$. Each such map is of the form (q_c, ψ) where $\psi : E_c \to S_1$ is continuous and restricts to an orientation preserving homeomorphism to S^1 on each fibre of E_c. We call such a map ψ *admissible.*

The elements of $\mathcal{V}_c$ are in bijection to the sections of the pullback bundle E_c. For $Y \in \mathcal{V}_c$ and an admissible map ψ we define $\deg_\psi(Y)$ as the degree of the map

$$S_1 \xrightarrow{Y} E_c \xrightarrow{\psi} S^1.$$

Intuitively, $\deg_\psi(Y)$ counts how often the curve Y winds around the fibres of E_c with respect to the reference system given by ψ.

Lemma 6.5. *Let ψ_1 and ψ_2 be two admissible maps. Then there exists a constant $z \in \mathbb{Z}$ such that for all $Y \in \mathcal{V}_c$ we have*

$$\deg_{\psi_2}(Y) = \deg_{\psi_1}(Y) + z.$$

Proof. For $i = 1, 2$ set $\varphi_i = (q_c, \psi_i) : E_c \to S^1 \times S^1$. We take the homology classes $a = [S^1 \times \{1\}]$, $b = [\{1\} \times S^1]$ of these oriented loops as a base of $H_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since φ_1, φ_2 are orientation preserving homeomorphisms there is an isomorphism $\tau = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that the diagram

$$\begin{tikzcd} H_1(E_c) \arrow[r, "(\varphi_1)_*"] \arrow[d, equal] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "\tau"'] \\ H_1(E_c) \arrow[r, "(\varphi_2)_*"] & \mathbb{Z} \oplus \mathbb{Z} \end{tikzcd}$$

commutes. Moreover, the homology class of the oriented fibre $(E_c)_1$ over 1 is mapped to b by both ψ_1 and ψ_2, hence we have $y = 0$ and $w = 1$. Now for every $Y \in \mathcal{V}_c$ we obtain the commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "Y_*"] \arrow[d, equal] & H_1(E_c) \arrow[r, "(\varphi_1)_*"] \arrow[d, equal] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)"'] \\ \mathbb{Z} \arrow[r, "Y_*"] & H_1(E_c) \arrow[r, "(\varphi_2)_*"] & \mathbb{Z} \oplus \mathbb{Z} \end{tikzcd}$$

where the two compositions in the top and bottom row send the element $1 \in \mathbb{Z}$ to the pair $\big(1, \deg_{\psi_1}(Y)\big)$ respectively $\big(1, \deg_{\psi_2}(Y)\big)$. This gives the claim. $\square$

We are now going to construct a specific admissible map which will provide a link to the quasimorphism Rot. Choose a lift $\tilde{c} : I \to \tilde{S}$ based at $\tilde{x}$ and define an S^1-bundle $E_{\tilde{c}}$ by the pullback

$$\begin{tikzcd} E_{\tilde{c}} \arrow[r] \arrow[d] & \mathbb{T}^1 \tilde{S} \arrow[d, "\tilde{q}"'] \\ I \arrow[r, "\tilde{c}"] & \tilde{S} \end{tikzcd}$$

Observe that, by the pullback property of E_c, there is a unique bundle morphism $\alpha : E_{\tilde{c}} \to E_c$ such that the diagrams

$$\begin{tikzcd} E_{\tilde{c}} \arrow[r, "\alpha"] \arrow[d] & E_c \arrow[d] \\ I \arrow[r] & S^1 \end{tikzcd} \qquad \begin{tikzcd} E_{\tilde{c}} \arrow[r, "\alpha"] \arrow[d] & E_c \arrow[d] \\ \mathbb{T}^1 \tilde{S} \arrow[r] & \mathbb{T}^1 S \end{tikzcd}$$

both commute. The map α is identifying, i.e., E_c carries the quotient topology with respect to α. We fix a metric $h \in \mathrm{Hyp}(S)$ and a unit tangent vector $\tilde{v} \in T^{1,\tilde{h}} \tilde{S}$ based at $\tilde{x}$. Denote by $\gamma = [c] \in \pi_1(S, x)$ the homotopy class of the curve c and observe that $T_\gamma^{\tilde{x}} \tilde{x} = \tilde{c}(1)$ by definition. Consider the map $\tilde{\psi}$ given by the composition

$$E_{\tilde{c}} \longrightarrow \mathbb{T}^1 \tilde{S} \xrightarrow{\cong} T^{1,\tilde{h}} \tilde{S} \xrightarrow{df_{\tilde{h},\tilde{v}}} T^1 \mathbb{D} \xrightarrow{\cong} G \xrightarrow{r_{\rho_{h,\tilde{v}}(\gamma)}} K \xrightarrow{\cong} S^1$$

where, as usual, $f_{\tilde{h},\tilde{v}}$ denotes the unique orientation preserving isometry $\tilde{S} \to \mathbb{D}$ sending $\tilde{v}$ to $1 \in T^1 \mathbb{D}_0$ and $r_{\rho_{h,\tilde{v}}(\gamma)}$ denotes the retraction $G \to K$ associated to the element $\rho_{h,\tilde{v}}(\gamma) \in \mathcal{C}$.

Then $\tilde{\psi}$ has the following properties:

(i) It is continuous.

(ii) Its restriction to each fibre of $E_{\tilde{c}}$ is an orientation preserving homeomorphism to S^1.

(iii) The restrictions to the fibres over 0 and 1 'are the same'. More precisely, $\tilde{\psi}$ factors through α.

For (ii) observe that each fibre of $E_{\tilde{c}}$ is mapped homeomorphically and orientation preservingly to a fibre of $T^1\mathbb{D}$ and hence to a left coset of K in G. The claim now follows from Lemma 6.2(a). To see (iii) note that a pair of points in $E_{\tilde{c}}$ that is identified under α is mapped by the canonical map $E_{\tilde{c}} \to \mathbb{T}^1\tilde{S}$ to a pair of the form $\tilde{u}$, $dT_\gamma^{\tilde{x}}\tilde{u}$ for some $\tilde{u}$ based at $\tilde{x}$. Their images in G are then of the form k, $\rho_{h,\tilde{v}}(\gamma)k$ for some $k \in K$. But since $r_{\rho_{h,\tilde{v}}(\gamma)}(\rho_{h,\tilde{v}}(\gamma)) = e$ and since $r_{\rho_{h,\tilde{v}}(\gamma)}$ is K-right-equivariant by Lemma 6.2(a) their images in S^1 are indeed the same.

Hence $\tilde{\psi}$ descends to a map $\psi : E_c \to S^1$ which is admissible by the above observations. The main point is now the following:

Proposition 6.6. *For every $Y \in \mathcal{V}_c$ we have* $\deg_\psi(Y) = \mathrm{Rot}(Y)$.

Proof. Let $\tilde{Y}$ be the lift of Y starting at the vector $\tilde{u}$ based at $\tilde{x}$ and consider the commutative diagram

$$\begin{tikzcd}
I \arrow[r, "\tilde{Y}"] \arrow[d] & E_{\tilde{c}} \arrow[r] \arrow[d, "\alpha"] & \mathbb{T}^1\tilde{S} \arrow[r, "B_{\tilde{h}}"] & T^{1,\tilde{h}}\tilde{S} \arrow[r, "df_{\tilde{h},\tilde{v}}"] & T^1\mathbb{D} \arrow[r, "\cong"] & G \arrow[r, "r_{\rho_{h,\tilde{v}}(\gamma)}"] & K \arrow[r, "\cong"] & S^1 \arrow[d, equal] \\
S^1 \arrow[r, "Y"] & E_c \arrow[rrrrrr, "\psi"] & & & & & & S^1
\end{tikzcd}$$

On the one hand, the degree of the map in the bottom row equals $\deg_\psi(Y)$ by definition. On the other hand, since $\tilde{v}$ and $\tilde{u}$ have the same base point, there exists $k \in K$ with $f_{\tilde{h},\tilde{v}} = k \circ f_{\tilde{h},\tilde{u}}$ and therefore the composition in the upper row equals

$$I \xrightarrow{w_Y} G \xrightarrow{k\cdot} G \xrightarrow{r_{\rho_{h,\tilde{v}}(\gamma)}} K \xrightarrow{\cong} S^1. \tag{20}$$

But since $\rho_{h,\tilde{v}}(\gamma) = k\rho_{h,\tilde{u}}(\gamma)k^{-1}$ the diagram

$$\begin{tikzcd}
G \arrow[r, "r_{\rho_{h,\tilde{u}}(\gamma)}"] \arrow[d, "k\cdot"'] & K \arrow[d, "k\cdot"] \\
G \arrow[r, "r_{\rho_{h,\tilde{v}}(\gamma)}"] & K
\end{tikzcd}$$

commutes according to Lemma 6.2 (b) and hence the map $S^1 \to S^1$ induced by (20) differs from the map induced by

$$I \xrightarrow{w_Y} G \xrightarrow{r_{\rho_{h,\tilde{u}}(\gamma)}} K \xrightarrow{\cong} S^1 \tag{21}$$

only by post-composition with a rotation of S^1. In particular, they have the same mapping degree. But for (21) this degree equals $\mathrm{Rot}(Y)$ by Proposition 6.3 and the definition of Rot. $\square$

We can summarise the above discussion as follows:

Theorem 6.7. *Let $c : I \to S$ be a closed curve and let $\psi : E_c \to S^1$ be admissible. Then there exists a constant a_ψ depending only on ψ such that for all $Y \in \mathcal{V}_c$ we have*

$$\mathrm{Rot}(Y) = \deg_\psi(Y) + a_\psi$$

6.3. **Regular Curves and Homotopies.** We call a curve $c : I \to S$ *regular* if it is continuously differentiable and has nowhere vanishing tangent vector, i.e., $c'(t) \neq 0$ for all $t \in I$. Such a curve is called *regular closed* if in addition $c(0) = c(1)$ and $c'(0) = c'(1)$. A *regular homotopy* is a homotopy by regular curves. More precisely, it is a continuous map $f : I \times I \to S$, $(s,t) \mapsto f_s(t)$ such that the partial derivative $\partial f/\partial t$ exists and is continuous on $I \times I$. For a regular curve c we can form the derivative $c' : I \to \mathbb{T}^1 S$ which is well-defined and continuous by assumption. If c is regular closed then c' is closed as well.
Let v be a non-zero tangent vector based at $x \in S$, and consider the set $\pi_1^{\text{reg}}(S,v)$ of regular homotopy classes (rel $\{0,1\}$) of regular closed curves on S such that $c'(0) = c'(1) = v$. By the above discussion there is a natural map

$$\pi_1^{\text{reg}}(S,v) \to \pi_1(\mathbb{T}^1 S, [v])$$

induced by taking derivatives. Smale proved that it is actually a bijection, the hard part of course being the injectivity of the map, see [36], Theorem A. For later reference we state this in the following form:

Theorem 6.8. *Let c_1, c_2 be two regular closed curves on S. If c_1', c_2' are (freely) homotopic as curves in $\mathbb{T}^1 S$ then c_1, c_2 are (freely) regularly homotopic.*

Let $c : S^1 \to S$ be a closed curve. A *loop* of c is a restriction of c to an arc $z_1 z_2$ of positive length of S^1 such that $c(z_1) = c(z_2)$, in other words, it is a closed part of c. Note that this definition includes the curve c itself as a loop. We call a closed curve *direct* if it contains no nullhomotopic loop. In particular, a direct curve is never nullhomotopic.
We are going to derive equivalent conditions for a closed curve c to be direct. To do so, let c be based at $x \in S$ and denote by $\gamma \neq 1$ the class of c in $\pi_1(S,x)$. Consider a covering $p : R \to S$ and a point $y \in R$ lying above x such that the image of the homomorphism $p_* : \pi_1(R,y) \to \pi_1(S,x)$ is the subgroup generated by γ. Such a pair (R,y) exists and is unique up to unique isomorphism of pointed coverings. For brevity we will say that the pair (R,y) is *adapted to c*. The lift of c to R with starting point y is closed by construction.

Proposition 6.9. *Let c be as before. Then the following are equivalent:*

(i) *c is direct.*
(ii) *The (closed) lift of c to R with starting point y is simple closed, where (R,y) is adapted to c.*

Proof. We only give a sketch, for details the reader is referred to [12], Lemma 2.4. If c contains a nullhomotopic loop then so does its lift to R by the homotopy lifting property. For the reverse implication observe that R is homeomorphic to a cylinder (cf. proof of Proposition 6.10) and the lift of c to R represents a generator of $\pi_1(R) \cong \mathbb{Z}$. But such a curve on a cylinder is either simple or contains a nullhomotopic loop. In the second case this loop projects to a nullhomotopic loop of c. □

Next, we are going to prove that if two direct regular closed curves are homotopic then they are actually regularly homotopic. This has been shown in [12], Theorem 5.5 and Theorem 6.2 using Smales result on regular homotopies (Theorem 6.8). We present a proof which is independent of the latter.

Proposition 6.10. *If two direct regular closed curves are (freely) homotopic then they are (freely) regularly homotopic.*

Proof. We only treat the case of free homotopies since this is all we will need. The case of based homotopies follows by a slight modification of our argument. For

$i = 0, 1$ let c_i be direct closed and be based at x_i and let h_s be a free homotopy between c_0 and c_1. Choose a pair (R, y_0) adapted to c_0. Let a be the curve given by $d(t) = h_t(0)$ and denote by b its lift to R starting at y_0. Then clearly the endpoint y_1 of b lies above x_1 and we claim that the pair (R, y_1) is adapted to c_1. Indeed, on the one hand we have

$$\pi_1(R, y_1) = [b]^{-1}\pi_1(R, y_0)[b]$$

within the fundamental groupoid of R. On the other hand, the curve $a^{-1}c_0a$ is homotopic (rel $\{0, 1\}$) to c_1 and hence

$$p_*(\pi_1(R, y_1)) = [a]^{-1}p_*(\pi_1(R, y_0))[a] = [a]^{-1}\langle[c_0]\rangle[a] = \langle[c_1]\rangle.$$

By Proposition 6.9 the two lifts of c_i to R with starting point y_i $(i = 0, 1)$ are simple closed, not nullhomotopic and are moreover homotopic since h also lifts.

We will now show that these two lifts are regularly homotopic which finishes the proof. Since the fundamental group of S is torsion-free the fundamental group of R is isomorphic to $\mathbb{Z}$. Moreover, R is orientable and hence, by the classification of surfaces, has to be a cylinder. So we can choose a diffeomorphism from R to the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$. The images of the two curves under this diffeomorphism are two regular Jordan curves with the same non-zero winding number around the origin. Hence they have the same orientation and both contain the origin in their interior. We may now regularly homotope one of the curves by a radial dilatation around the origin such that it contains the other curve in its interior. By the differentiable version of the Schoenflies theorem there is a C^1-diffeomorphism of the punctured plane which maps the two curves into standard circles around the origin, one containing the other. We may assume that they are given by $t \mapsto e^{2\pi it}$ and $t \mapsto 2e^{2\pi it}$, then a regular homotopy between them is given by the map

$$(s, t) \mapsto (1 + s)e^{2\pi it}.$$

$\square$

Proposition 6.11. *Let $c : I \to S$ by a direct regular closed curve. Then we have* $\mathrm{Rot}(c') = 0$.

Proof. Choose a metric $h \in \mathrm{Hyp}(S)$. We distinguish two cases:

(i) The free homotopy class of c contains a shortest rectifiable curve. Clearly, this curve has to be a closed geodesic an cannot contain any nullhomotopic loops. In particular, every regular parametrisation $g : I \to S$ of this geodesic is direct. Now, since c is not nullhomotopic, we can choose such a parametrised geodesic g freely homotopic to c and conclude from Proposition 6.10 that these two curves are actually regularly homotopic. In particular we have $c' \simeq g'$ in $\mathbb{T}^1S$ and hence $\mathrm{Rot}(c') = \mathrm{Rot}(g')$. It remains to show that $\mathrm{Rot}(g') = 0$. Let $\tilde{g}$ be a lift of g to $\tilde{S}$ and set $\tilde{v} = \tilde{g}'(0)$. then the curve $f_{h,\tilde{v}} \circ \tilde{g}$ is a regularly parametrised geodesic in $\mathbb{D}$ starting at 0 and pointing into the direction of the vector $1 \in T^1\mathbb{D}_0$. Hence its image is contained in $\mathbb{D} \cap \mathbb{R}$ and so the curve $w_{g'}$ takes values in the subgroup $A \leq G$. But the rotation number vanishes on A and therefore $\mathrm{Rot}(g') = 0$ by Proposition 2.19.

(ii) The free homotopy class of c contains curves of arbitrary small length. Then c has to wind around a single cusp of S. For each $\varepsilon > 0$ we can choose a direct regular closed curve g_ε of length $\leq \varepsilon$ winding around that cusp. By the same argument as before we have $\mathrm{Rot}(c') = \mathrm{Rot}(g'_\varepsilon)$. Let $\tilde{g}_\varepsilon$ be a lift of g_ε to $\tilde{S}$ and set $\tilde{v} = \tilde{g}'_\varepsilon(0)$. Then the curve $f_{h,\tilde{v}} \circ \tilde{g}_\varepsilon$ is not quite a geodesic but almost in the following sense: Its lenght tends to 0 as $\varepsilon \to 0$ and the tangent vectors stay in a sector around $1 \in (T^1\mathbb{D})_z$ whose opening angle also tends to 0 as $\varepsilon \to 0$. Hence by Proposition 2.19 and continuity reasons we again conclude that $\mathrm{Rot}(c') = 0$.

□

Theorem 6.12. *Let $X_0, X_1 : S^1 \to \mathbb{T}^1 S$ be two closed curves and set $c_i = q \circ X_i$ for $i = 0, 1$. Then the following are equivalent:*

(i) *X_0 and X_1 are (freely) homotopic.*

(ii) *c_0 and c_1 are (freely) homotopic and* $\mathrm{Rot}(X_0) = \mathrm{Rot}(X_1)$.

Proof. The implication (i) $\Rightarrow$ (ii) is clear. For the reverse implication we make use of the pullback bundles introduced in Subsection 6.2. Let $h : I \times S^1 \to \mathbb{T}^1 S$, $(s, t) \mapsto h_s(t)$ be a (free) homotopy between c_0 and c_1. Define an S^1-bundle E_h by the pullback diagram

$$\begin{tikzcd} E_h \arrow[r] \arrow[d, "q_h"'] & \mathbb{T}^1 S \arrow[d, "q"'] \\ I \times S^1 \arrow[r, "h"] & S \end{tikzcd}$$

Then, since E_h is orientable and $I \times S^1$ is homotopy equivalent to S^1, E_h is the trivial bundle and hence we may choose a continuous map $\psi : E_h \to S^1$ such that $(q_c, \psi) : E_h \to (I \times S^1) \times S^1$ is an orientation preserving bundle isomorphism (where on the right hand side the bundle map is projection to the first factor). For each $s \in I$ the bundle E_{h_s} belonging to the curve h_s fits into the pullback diagram

$$\begin{tikzcd} E_{h_s} \arrow[r, "i_s"] \arrow[d, "q_{h_s}"'] & E_h \arrow[d, "q_h"'] \\ S^1 \arrow[r, "\{s\}\times\mathrm{id}"] & I \times S^1 \end{tikzcd}$$

and the composition $\psi_s = \psi \circ i_s : E_{h_s} \to S^1$ is an admissible map. Now, according to Theorem 6.7, there exists a constant a_s depending only on s such that $\deg_{\psi_s}(Z) = \mathrm{Rot}(Z) + a_s$ for all closed lifts Z to $\mathbb{T}^1 S$ of the curve h_s. But clearly a_s is locally constant in s and hence $a_0 = a_1$.
By assumption we have $\mathrm{Rot}(X_1) = \mathrm{Rot}(X_2)$ and therefore $\deg_{\psi_0}(X_0) = \deg_{\psi_1}(X_1)$. This means by definition that the two maps

$$S^1 \xrightarrow{X_i} E_{h_i} \xrightarrow{\psi_i} S^1$$

$(i = 0, 1)$ have the same mapping degree and hence we may choose a homotopy $k : I \times S^1 \to S^1$ between them. The continuous map

$$I \times S^1 \xrightarrow{\mathrm{id} \times k} (I \times S^1) \times S^1 \xrightarrow{(q_h, \psi)^{-1}} E_h \longrightarrow \mathbb{T}^1 S$$

is then indeed a homotopy between X_0 and X_1. □

Combining the above result with Theorem 6.8 we obtain:

Theorem 6.13. *Let c_0, c_1 be two homotopic regular closed curves in S. Then they are regularly homotopic if and only if* $\mathrm{Rot}(c_0') = \mathrm{Rot}(c_1')$.

Hence Rot is a full invariant for the regular homotopy classes of homotopic regular closed curves. Analogous results were proven for plane curves in [39] where Rot is replaced by the classical rotation number and for non-compact surfaces in [12], Theorem 3.1 where Rot is replaced by a winding number function w_X depending on a non-vanishing vector field X on S. We will define winding numbers in the next subsection and shall see that Rot and w_X are actually closely related, so the two results are essentially equivalent. However, we would like to point out that Theorem 6.13 also holds for compact surfaces.

6.4. **Winding Numbers.** In [12] Chillingworth introduces a concept of winding number on surfaces which generalises the traditional rotation number of curves in the plane. We would also like to mention Reinhart [34] who gives an essentially equivalent definition. For a beautiful treatment of rotation numbers of regular plane curves including a classification result analogous to Theorem 6.13 we refer the reader to Whitney [39]. Intuitively, the rotation number of a regular closed plane curve c measures how often the (non-zero) tangent vector $c'(t)$ rotates in counter-clockwise direction when c is traversed once. Note that in the plane every constant non-zero vector field serves as a global reference frame to which the tangent vector $c'(t)$ can be compared. On a general orientable surface S this canonical global reference frame has to be replaced by a non-zero vector field X on S. If S is compact and $\chi(S) \neq 0$, no such vector field exists. Therefore we assume for the moment that S is non-compact.

For a non-vanishing vector field X on S and for a regular closed curve $c : S^1 \to S$ Chillingworth defines the *winding number* $w_X(c)$ as follows. As in subsection 6.2 we define an S^1-bundle $E_c \to S^1$ by the pullback

$$\begin{tikzcd} E_c \arrow[r] \arrow[d, "q_c"'] & \mathbb{T}^1 S \arrow[d, "q"] \\ S^1 \arrow[r, "c"] & S \end{tikzcd}$$

Then we can consider the two curves c' and $X \circ c$ as sections of $E_c \to S_1$ and set

$$w_X(c) = \deg_\psi(c') - \deg_\psi(X \circ c)$$

for an admissible map $\psi : E_c \to S^1$. By Lemma 6.5 the value of $w_X(c)$ is independent of the choice of ψ. He shows that if c_1, c_2 are homotopic direct regular closed curves which are not nullhomotopic, then $w_X(c_1) = w_X(c_2)$ (cf. [12], Theorem 2.7). This allows to define winding numbers for non-trivial homotopy classes of closed curves in S by using particularly simple representatives: For a homotopy class $\gamma \neq 1$ define its *winding number* $w_X(\gamma)$ to be $w_X(c)$ for an arbitrary direct regular closed representative curve c.

The case of a compact surface S is slightly more involved. Assume that $\chi(S) < 0$ and let X be a vector field on S which vanishes only at the point $p \in S$. For a regular closed curve c on $S \setminus \{p\}$ we can define the winding number $w_X(c)$ in the same way as above. However, even if c_1 and c_2 are regular closed curves in $S \setminus \{p\}$ which are regularly homotopic we need not have $w_X(c_1) = w_X(c_2)$. Indeed, if a curve is homotoped to traverse the point p then $w_X(c)$ changes by the index $i_X(p)$ which equals $\chi(S)$ by the Poincaré-Hopf theorem. Hence we obtain a $\mathbb{Z}_{|\chi(S)|}$-valued winding number for non-trivial homotopy classes of closed curves by setting $w_X(\gamma) \equiv w_X(c) \pmod{|\chi(S)|}$ for an arbitrary direct regular closed representative curve c in $S \setminus \{p\}$.

In the meantime it should come as no surprise that the quasimorphism Rot is related to winding numbers. And indeed, Rot encodes the information of all winding number functions simultaneously:

Theorem 6.14. *Let S be non-compact and let X be a non-vanishing vector field on S. Then*

$$w_X = -\operatorname{Rot} \circ X_*$$

as maps $\pi_1(S) \to \mathbb{Z}$ where we interpret $X : S \to \mathbb{T}^1 S$ as section of the unit tangent bundle. In case S is compact and X is a vector field on S which vanishes only at the point $p \in S$ the composition

$$\pi_1(S \setminus \{p\}) \xrightarrow{X_*} \pi_1\big(\mathbb{T}^1(S \setminus \{p\})\big) \longrightarrow \pi_1(\mathbb{T}^1 S) \xrightarrow{-\operatorname{Rot}} \mathbb{Z} \longrightarrow \mathbb{Z}_{|\chi(S)|}$$

induces a well-defined map $\pi_1(S) \to \mathbb{Z}_{|\chi(S)|}$ which agrees with w_X.

Proof. We start with the non-compact case. Let c be a direct regular closed curve in S and choose an admissible map $\psi : E_c \to S^1$. Then by Theorem 6.7 there exists a constant a such that $\deg_\psi(Y) = \operatorname{Rot}(Y) + a$ for all $Y \in \mathcal{V}_c$. Subtracting the corresponding equalities for $Y = c'$ and $Y = X \circ c$ gives

$$\deg_\psi(c') - \deg_\psi(X \circ c) = \operatorname{Rot}(c') - \operatorname{Rot}(X \circ c) = -\operatorname{Rot}(X \circ c)$$

by Proposition 6.11 since c is direct. But the left hand side equals $w_X([c])$ by definition which gives the claim.
In the compact case let z be the boundary of a positively oriented disc in S containing p in its interior. Then the kernel of the homomorphism $\pi_1(S \setminus \{p\}) \to \pi_1(S)$ induced by the inclusion is the normal (i.e. conjugation invariant) subgroupoid generated by the class $[z]$. On the other hand, the composition $X \circ z : S^1 \to \mathbb{T}^1 S$ is homotopic to $i_X(p)$ times the oriented fibre of $\mathbb{T}^1 S$ over p where $i_X(p)$ is the index of the vector field X at p. Now the class of this fibre is central in $\pi_1(\mathbb{T}^1 S)$ and mapped to 1 by Rot, moreover we have $i_X(p) = \chi(S)$ by the Poincaré-Hopf theorem. Hence the composition in the statement of the theorem is indeed constant on the residue classes of the normal subgroupoid generated by $[z]$ and therefore descends to a well-defined map $\pi_1(S) \to \mathbb{Z}_{|\chi(S)|}$. This map agrees with w_X by the same argument as in the non-compact case treated before. ☐

6.5. **An Explicit Formula for** Rot. In section 7 of [12] Chillingworth gives an explicit combinatorial formula for his winding number functions. As it turns out, with some minor modifications his argument can be used to derive an analogous formula for the quasimorphism Rot. We are going to present a sketch of the proof and refer the reader to [12] for a more detailed exposition.

We assume that S has genus g and has p cusps and we fix a base point $x \in S$. Consider a *canonical curve system* for S, i.e., a collection of curves

$$a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_p$$

based at x with the following properties:

(i) The curves are all simple and disjoint outside x.
(ii) Each curve c_i winds around a single cusp and cuts out a punctured disc from S whose interior is disjoint from all other curves.

(iii) Cutting along all curves dissects S into the disjoint union of the punctured discs from (ii) and a disc whose (oriented) boundary runs

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, \dots, a_g^{-1}, b_g^{-1}, c_1, \cdots, c_p$$

where a_i^{-1} means a_i backwards etc.

One verifies that these curves and their inverses leave the base point x in the following order with respect to the *negative* orientation:

$$\mathcal{O}: \quad a_1, b_1^{-1}, a_1^{-1}, b_1, a_2, b_2^{-1}, \dots, a_g^{-1}, b_g, c_1, c_1^{-1}, c_2, c_2^{-1}, \cdots, c_p, c_p^{-1}.$$

Moreover, they (or more precisely: their homotopy classes) generate the fundamental group $\pi_1(S)$ and satisfy the standard relation

$$\prod_{i=1}^{g}[a_i, b_i] \cdot \prod_{i=1}^{p} c_i = 1.$$

In addition to the above properties we may also require the canonical curve system to satisfy:

(iv) All curves are regular. For a fixed non-zero tangent vector v based at x their derivative at the starting point equals v while their derivative at the endpoint equals $-v$. Hence each curve 'changes direction by 180 degrees at x', in particular none of them is regular closed.

In a next step we modify these curves and their inverses in such a way that they become regular closed and their derivatives are all based at the vector v. This can be done by adding a small half turn based at x at the end of every curve and making sure that the resulting curve is regularly homotopic to a simple curve. Hence we add a positive oriented half turn to the curves a_i^{-1}, b_i and c_i^{-1} and a negative oriented one to their respective inverses. In order to distinguish these modified curves from the original ones notationally, we will decorate them with a tilde. Notice that with this definition $\widetilde{a_i^{-1}}$ is *not* technically the inverse of $\tilde{a}_i$ but is only so up to choice of a basepoint, i.e., up to a free regular homotopy. However, this implies that its derivative is freely homotopic to the inverse of the derivative $(\tilde{a}_i)'$ in $\mathbb{T}^1 S$ and this is all we will need in the sequel. Similar remarks apply to the other curves.

Denote by $A_i^{\pm 1}$ the (class of the) derivative of $\widetilde{a_i^{\pm 1}}$ in $\pi_1(\mathbb{T}^1 S, v)$, similarly for the other curves. Then by construction $A_i^{\pm 1}$ lifts $a_i^{\pm 1}$ to the unit tangent bundle. It satisfies $\mathrm{Rot}(A_i^{\pm 1}) = 0$ by Proposition 6.11 since $\widetilde{a_i^{\pm 1}}$ is regularly homotopic to a simple curve. Moreover, it is the unique such lift up to homotopy by Theorem 6.12. The same holds for the other curves.

Let Z be the (class of the) positively oriented fibre over x. By construction the curves A_i, B_i, C_i together with Z generate $\pi_1(\mathbb{T}^1 S, v)$ and a calculation shows that they satisfy the relation

$$\prod_{i=1}^{g}[A_i, B_i] \cdot \prod_{i=1}^{p} C_i = Z^{\chi(S)}. \tag{22}$$

Since Z is central, every element of $\pi_1(\mathbb{T}^1 S, v)$ can be written in the form $Z^r \cdot W$ where W is a word in the generators $A_i^{\pm 1}$, $B_i^{\pm 1}$ and $C_i^{\pm 1}$. Denote by w the word obtained by replacing all capital letters in W by the corresponding small ones and denote by $\tilde{w}$ the word obtained by decorating all letters with a tilde. We keep the *standing assumption* that no cyclic subword of w represents the trivial element of $\pi_1(S, x)$, in particular w is cyclically reduced.

In what follows, we will not distinguish between a word and the curve it describes. The curves a_i, b_i, c_i on S are simple and disjoint except for the basepoint x. Therefore every loop of w is, up to change of basepoint, given by a cyclic subword of w. By assumption on w such a loop is not nullhomotopic and we can conclude that w is direct. Choose a pointed covering (R, y) adapted to the class of w as described in subsection 6.3. Then the lift of w to R with starting point y is simple closed. By construction of the modified curves the lift of $\tilde{w}$ to R with starting point y is simple except for small half turns at every transition between two letters. Let xy be a cyclic subword of length 2 of w and consider the half turn between the corresponding curves in the lift of $\tilde{w}$ to R. This turn might be harmless in the sense that it can be regularly homotoped such that the curve becomes locally simple there, or it might not be. In the latter case the half turn has 'the wrong orientation' and changing this orientation makes it harmless. Technically, after taking the derivative $\tilde{w}'$, such a change of orientation amounts to multiplication with $Z^{\pm 1}$ up to homotopy. A closer analysis of the possible cases shows that all half turns become harmless after the following modifications:

(i) If $x \in U$ and $y > x^{-1}$ in the ordering $\mathcal{O}$ then change the orientation from negative to positive (multiply $\tilde{w}'$ by Z).
(ii) If $x \in U^{-1}$ and $y < x^{-1}$ in the ordering $\mathcal{O}$ then change the orientation from positive to negative (multiply $\tilde{w}'$ by Z^{-1}).

Here U denotes the set of letters a_i, b_i^{-1}, c_i and U^{-1} denotes the set of letters a_i^{-1}, b_i, c_i^{-1}. Hence after these modifications, the resulting curve on R is regularly homotopic to a simple one and, consequently, the corresponding modified curve $\hat{w}$ on S is regularly homotopic to a direct one. By Proposition 6.11 we have $\mathrm{Rot}(\hat{w}') = 0$ and this leads to the following formula:

Theorem 6.15. *Consider an element $Z^r \cdot W \in \pi_1(\mathbb{T}^1 S, v)$ where W is a word in the letters $A_i^{\pm 1}$, $B_i^{\pm 1}$ and $C_i^{\pm 1}$. Let w be the corresponding word where all capital letters are replaced by the corresponding small ones. Assume that no cyclic subword of w represents the trivial element of $\pi_1(S, x)$. Then*

$$\begin{aligned}\mathrm{Rot}(Z^r W) \quad = \quad & r \,-\, \#\big(xy \text{ cyclic subword of } w \text{ with } x \in U \text{ and } y > x^{-1}\big)\\ & \quad +\, \#\big(xy \text{ cyclic subword of } w \text{ with } x \in U^{-1} \text{ and } y < x^{-1}\big).\end{aligned}$$

An elementary combinatorial argument shows that we can reformulate this in the following form:

Theorem 6.16. *In the situation of the last theorem we have*

$$\begin{aligned}\mathrm{Rot}(Z^r W) \quad &= \quad r \,+\, \#\big(xy \text{ cyclic subword of } w \text{ with } x \neq y, y < x^{-1}\big)\\ &\qquad -\, \#\big(\text{letters of } w \text{ belonging to } U\big)\\ &= \quad r \,-\, \#\big(xy \text{ cyclic subword of } w \text{ with } x \neq y, y > x^{-1}\big)\\ &\qquad +\, \#\big(\text{letters of } w \text{ belonging to } U^{-1}\big)\end{aligned}$$

In combination with Theorem 6.14 this gives back Chillingworth's formula for the winding number functions.

6.6. **Calegari's** rot**.** In subsection 4.2 of [11] Calegari introduces a function closely related to the quasimorphism Rot (in fact, rather to the winding number functions) on a non-compact hyperbolic surface S. He observes that for a holonomy representation $\rho : \pi_1(S) \to \mathrm{PSU}(1,1)$ the pullback of the cohomology class associated to the central extension

$$\mathbb{Z} \rightarrowtail \widetilde{\mathrm{PSU}(1,1)} \twoheadrightarrow \mathrm{PSU}(1,1)$$

is trivial since $\mathrm{H}^2(\pi_1(S), \mathbb{R}) = 0$. Hence ρ lifts to a homomorphism $\tilde{\rho} : \pi_1(S) \to \widetilde{\mathrm{PSU}(1,1)}$ and we can consider the pullback rot via $\tilde{\rho}$ of the translation quasimorphism on $\widetilde{\mathrm{PSU}(1,1)} \leq \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. In other words, he studies the maps

$$\text{rot} = \text{Rot} \circ s \tag{23}$$

where $s : \pi_1(S) \to \pi_1(\mathbb{T}^1 S)$ is a splitting (in fact this is only true up to sign since another sign convention is used in [11]). According to Theorem 6.14 these maps are precisely the winding number functions for S. The above definition of course depends on s, however the restriction of rot to the commutator subgroup $\pi_1(S)'$ is independent of the splitting since $\pi_1(\mathbb{T}^1 S)$ is a central extension of $\pi_1(S)$. Reclaiming the notation from the last subsection, the fundamental group of S is free on the set of generators

$$a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_{p-1} \tag{24}$$

(recall that $p \geq 1$ by assumption). Hence there is a unique splitting map s such that $\text{Rot} \circ s$ takes the value 0 on each of these generators. For this specific choice Calegari denotes the resulting quasimorphism on $\pi_1(S)$ by $\text{rot}_{g,p}$.
The curves A_i etc. from the last subsection are constructed in such a way that the map s sends each of the generators in (24) to the curve in $\mathbb{T}^1 S$ given by the corresponding capital letter. As a consequence of equality (22) we obtain $\text{Rot}(s(c_p)) = -\chi(S) \neq 0$ which shows that the definition of $\text{rot}_{g,p}$ involves a heavy break of symmetry.

Calegari gives a formula for $\text{rot}_{g,p}$ in terms of homogeneous counting functions which we will describe next. Consider a free group on a given set of generators. For a fixed reduced word v in these generators (and their inverses) and a reduced word w we denote by $C_v(w)$ the number of copies of v inside the word w. We will only consider words v consisting of one letter or two distinct letters in which cases distinct copies of v inside w are always disjoint. As antisymmetrisations of the counting functions we obtain the counting quasimorphisms

$$H_v(w) = C_v(w) - C_{v^{-1}}(w)$$

and the homogenisations of these functions will be benoted by $\overline{C}_v$ respectively $\overline{H}_v$. By [11], Theorem 4.76 $\text{rot}_{g,p}$ can be expressed as a rational linear combination of certain homogeneous counting functions on $\pi_1(S)$ considered as a free group on the set G of generators given by (24). More precisely, there is a formula of type

$$\text{rot}_{g,p} = \frac{1}{2\,\text{rank}(\pi_1(S))} \sum_{(x,y)} n_{xy} \overline{C}_{xy} \tag{25}$$

where the sum ranges over the set of pairs $\{(x,y) \in G^2 \mid x \neq y, y^{-1}\}$ and where all n_{xy} are non-zero integers. We point out that the denominators in (25) grow linearly with the topological complexity of S.

The results of the last subsection can be directly applied to the current situation to obtain similar formulas. For example Theorem 6.16 can be rephrased as

$$\begin{aligned}\mathrm{Rot} &= \overline{H}_Z + \sum_{x\neq y, y<x^{-1}} \overline{C}_{XY} - \sum_{x\in U} \overline{C}_X \\ &= \overline{H}_Z - \sum_{x\neq y, y>x^{-1}} \overline{C}_{XY} + \sum_{x\in U^{-1}} \overline{C}_X.\end{aligned}$$

Some care has to be taken here since this is not a valid identity of functions on the fundamental group $\pi_1(\mathbb{T}^1 S)$, in fact the above counting functions are not even well-defined there. But applying both sides to a word satisfying the conditions given in the description of Theorem 6.15 results in a correct equality of integers. Taking the arithmetic mean of the two expressions above one obtains the alternative form

$$\mathrm{Rot} = \overline{H}_Z + \frac{1}{2}\Big(\sum_{x\neq y, x<y^{-1}} \overline{H}_{XY} - \sum_{x\in U} \overline{H}_X \Big).$$

Following Calegari's convention we can consider $\pi_1(S)$ as a free group on the set G of generators (24) and upon ignoring all terms involving the letters $C_p^{\pm 1}$ the above formula gives

$$\mathrm{rot}_{g,p} = \frac{1}{2}\Big(\sum_{\substack{(x,y)\in G^{(2)} \\ x<y^{-1}}} \overline{H}_{xy} - \sum_{x\in U\cap G} \overline{H}_x \Big), \tag{26}$$

this time as an actual equality of functions. Observe that the second sum in the bracket is a homomorphism which vanishes on the commutator subgroup $\pi_1(S)'$.

A slightly different expression can be derived from Theorem 6.15. To group the occuring terms consider the four sets

$$\begin{aligned} S_1 &= \{(x,y)\in U\times U \mid y > x^{-1}\}, \\ S_2 &= \{(x,y)\in U\times U^{-1} \mid y > x^{-1}\}, \\ S_3 &= \{(x,y)\in U^{-1}\times U \mid y < x^{-1}\}, \\ S_4 &= \{(x,y)\in U^{-1}\times U^{-1} \mid y < x^{-1}\}. \end{aligned}$$

Then the map $(x,y)\mapsto (y^{-1},x^{-1})$ is a bijection between S_1 and S_4. Similarly, the map $(x,y)\mapsto(y,x)$ is a bijection between S_2 and S_3 as an easy verification shows. Hence Theorem 6.15 implies

$$\mathrm{Rot} = \overline{H}_Z - \sum_{(x,y)\in S_1} \overline{H}_{XY} - \sum_{(x,y)\in S_2} (\overline{C}_{XY} - \overline{C}_{YX}),$$

which in turn leads to the formula

$$\mathrm{rot}_{g,p} = -\sum_{S_1\cap G^{(2)}} \overline{H}_{xy} - \sum_{S_2\cap G^{(2)}} (\overline{C}_{xy} - \overline{C}_{yx}). \tag{27}$$

Note that in contrast to (25) all coefficients are integral here.

To give a concrete example, we consider the once-punctured torus where $g = p = 1$. In this case (25) gives after collecting terms

$$\mathrm{rot}_{1,1} = \frac{1}{4}\Big(\overline{H}_{a_1 b_1} + \overline{H}_{b_1 a_1^{-1}} + \overline{H}_{b_1^{-1} a_1} + \overline{H}_{a_1^{-1} b_1^{-1}} \Big)$$

while (27) reads

$$\mathrm{rot}_{1,1} = \overline{C}_{b_1 a_1} - \overline{C}_{a_1 b_1}.$$

Although these expressions look quite different, they are indeed equivalent (up to sign), due to identities of the form

$$\overline{C}_{xy} + \overline{C}_{xy^{-1}} = \overline{C}_{yx} + \overline{C}_{y^{-1}x}, \qquad x \neq y, y^{-1}$$

valid for a free group of rank 2. Similar remarks apply to the thrice-punctured sphere where $g = 0, p = 3$ which is the only other surface with free fundamental group of rank 2. For more complicated surfaces however, the relation between the formulas (25) and (27) remains somewhat mysterious.

Finally, we briefly discuss another geometric interpretation of the function rot given in [11]. Let S again be non-compact and consider an element a in the commutator subgroup $\pi_1(S)'$. For such elements the value of rot as defined in (23) is independent of the choice of the section s. Let γ be the unique closed geodesic in the free homotopy class of a, then the complement $S \setminus \gamma$ consists of finitely many connected regions R_i with piecewise geodesic boundaries. Since, by assumption on a, the curve γ is homologically trivial it is the boundary of an integral linear combination $\sum_i n_i R_r$ which is unique as S is non-compact. The value $\mathrm{rot}(a)$ is linked to the hyperbolic areas of the regions R_i as follows (cf. Lemma 4.68):

$$\sum_i n_i \,\mathrm{area}(R_i) = -2\pi \,\mathrm{rot}(a).$$

A similar relation was established earlier by McIntyre and Cairns in [27] for winding number functions on compact surfaces and not necessarily homologically trivial curves.

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