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FENCHEL-NIELSEN COORDINATES FOR MAXIMAL REPRESENTATIONS

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Zusammenfassung

Höhere Teichmüllertheorie befasst sich mit dem Studium von Räumen von Darstellungen der Fundamentalgruppe einer orientierbaren Fläche in gewisse Lie Gruppen. Ein Zweig der höheren Teichmüllertheorie ist das Studium von solchen Darstellungen in Hermitesche Liegruppen G mit maximaler Toledozahl.

In dieser Arbeit konstruieren wir Koordinaten für den Raum der Darstellungen mit maximaler Toledozahl in die symplektische Gruppe $\mathrm{Sp}(2n, \mathbb{R})$. Diese Koordinaten verallgemeinern Fenchel-Nielsen Koordinaten auf dem Teichmüllerraum. Dabei spielt die Zerlegung der zugrundeliegenden Fläche in Teilstücke (sogenannte Hosen) und das Studium der Darstellung der Fundamentalgruppe davon eine herausragende Rolle. Die resultierenden Koordinaten sind wie im klassischen Fall Längen- und Twistparameter. In unserem Fall sind dies Matrizen aus $\mathrm{GL}(n, \mathbb{R})$, die noch gewisse Relationen zu erfüllen haben.

Die Koordinaten nutzen wir für zwei Anwendungen: wir können zeigen, dass die Limeskurve zu einer maximalen Darstellung in gewissen Fällen auch für nicht-geschlossene Flächen stetig ist. Ausserdem zählen wir Zusammenhangskomponenten von maximalen Darstellungen für nicht geschlossene Flächen.

Darüberhinaus präsentieren wir die Konstruktion von Doppelverhältnissen für maximale Darstellungen. Dies ist eine gemeinsame Arbeit mit Tobias Hartnick. Wir konstruieren ein Doppelverhältnis auf Quadrupeln im Shilovrand eines beschränkten symmetrischen Gebiets, das durch gewisse Funktorialitätseigenschaften eindeutig charakterisiert ist. Mit Hilfe dieses Doppelverhältnis und der Limeskurve kann man einer maximale Darstellung ein striktes Doppelverhältnis im Sinne von Labourie zuordnen. Daraus ergeben sich gewisse Konsequenzen, wie zum Beispiel die Eigentlichkeit der Wirkung der Abbildungsklassengruppe auf dem Raum der maximalen Darstellungen.

Abstract

Higher Teichmüller theory is concerned with the study of spaces of representations of the fundamental group of an orientable surface into certain Lie groups. One branch of higher Teichmüller theory is the study of representations into Hermitian Lie group with maximal Toledo invariant.

In this text we give coordinates for the space of representations with maximal Toledo invariant into the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. These coordinates generalize Fenchel-Nielsen coordinates on Teichmüller space. The decomposition of the given surface into pieces (so called pairs of pants) and the study of the fundamental group of pairs of pants plays an important role. Our coordinates are, as in the classical case, length and twist parameters, which are here matrices from $\mathrm{GL}(n, \mathbb{R})$, which have to satisfy some relations.

We use these coordinates for two applications: we can show that the limit curve associated with a maximal representation is continuous for a class of maximal representations of the fundamental group of a non-closed surface. Furthermore we count connected components of spaces of maximal representations of non-closed surfaces.

Finally we present results obtained in joint work with Tobias Hartnick. We construct a cross ratio on quadruples on the Shilov boundary of a bounded symmetric domain of tube type which is uniquely characterized by their behavior under products, a functoriality condition and some normalization. We use this cross ratio and the limit curve for a maximal representation to associate a strict cross in the sense of Labourie to maximal representations. The cross ratio can be used to deduce some consequences, e.g. the properness of the action of the mapping class group on the space of maximal representations.

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Contents

Chapter 1

Introduction

1.1 Teichmüller Space and Representations of Fundamental Groups

Let Σ_g be a closed and oriented surface of genus $g \geq 2$. The space of marked conformal structures is the *Teichmüller space* $\mathcal{T}(\Sigma_g)$ and the study of this space is called *Teichmüller theory.*

The Teichmüller space has a number of other realizations, for example

- as the space of marked hyperbolic structures on Σ_g,
- as the space of quadratic holomorphic differentials (Teichmüller's theorem)
- a subset of the space of representations of the fundamental group $\pi_1(\Sigma_g)$ into $\mathrm{PSL}(2,\mathbb{R})$

For the correspondence with the first two structure see e.g. [42]. We explain the relation between hyperbolic structures and representations later in this section.

Since $\mathrm{PSL}(2,\mathbb{R})$ is a Hermitian Lie group as well as a real split Lie group, there are two natural ways for generalization of Teichmüller theory. Indeed, if G is a Hermitian Lie group or a real split Lie group, the space $\mathrm{Hom}(\pi_1(\Sigma_g), G)$ has connected components, which share a lot of properties with hyperbolizations. If G is a real split Lie group, then these components are the Hitchin components, which we briefly introduce in Section 2.5. If G is a Hermitian Lie group, the components are distinguished by the maximality of the Toledo invariant. This text is concerned with these representations. The goal is to give coordinates for several spaces of maximal representations.

We refer the reader to [1, 42, 51, 52] for an introduction to classical Teichmüller theory and [17] for an overview over higher Teichmüller theory and the relation between classical Teichmüller theory and representations of fundamental groups.

Maximal representations have been studied using various techniques. With methods from bounded cohomology one can obtain geometric results for maximal representations ([18], [13],[58],[15]). Higgs bundles techniques were used to get informations on the topology of the space of maximal representations ([34, 7, 31, 6, 32]) as well as to obtain results on the deformation behavior of maximal representations [8].

In the reminder of this section we will introduce some notations and present the relation between hyperbolic structures on a surface Σ and representations of $\pi_1(\Sigma)$. We give a short introduction to Fenchel-Nielsen Coordinates on $\mathcal{T}(\Sigma_g)$ in Section 1.2. In Section 1.3 we generalize these coordinates to coordinates on the space of maximal representations into $\mathrm{Sp}(2n,\mathbb{R})$. In Section 1.4 we present some applications for these coordinates. In Section 1.5 we show how to associate a cross ratio with maximal representations and deduce some consequences. Finally, Section 1.6 contains a guide for the reader.

Let us first fix some notation needed throughout this text: We denote by $\Sigma_{g,m,l}$ an oriented surface of genus g with m boundary components and l punctures and by $\Gamma_{g,m,l}$ its fundamental group. An oriented surface of genus g with m boundary component is denoted by $\Sigma_{g,m}$, its fundamental group by $\Gamma_{g,m}$. A surface of genus g without boundary is denoted by Σ_g, its fundamental group by Γ_g.

A *hyperbolic structure* on $\Sigma_{g,m,l}$ is a Riemannian metric of constant sectional curvature -1. We will always assume that the boundary components are geodesics. If the metric is complete, the punctures are "moved to infinity". A neighborhood of such a puncture at infinity is called a *cusp*.

Note that the fundamental group can not distinguish between punctures and boundary components. Therefore we mostly use the oriented surfaces $\Sigma_{g,m}$ of genus g and m boundary components and their fundamental groups $\Gamma_{g,m}$.

We fix the following standard presentation for $\Gamma_{g,m}$

$$\begin{aligned}\Gamma_{g,m} = \langle A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m | \\ [A_g, B_g] \ldots [A_1, B_1] C_m \ldots C_1 = e \rangle.\end{aligned}$$

The C_j correspond to loops around boundary components. We call the A_i, B_i and C_j the *standard generators*.

Let G be a topological group. We denote by $\mathrm{Hom}(\Gamma_{g,m}, G)$ the set of homomorphism from $\Gamma_{g,m}$ into G. The group G acts by pointwise conjugation on $\mathrm{Hom}(\Gamma_{g,m}, G)$. The quotient space with respect to this action is

$$\mathrm{Rep}(\Gamma_{g,m}, G) = \mathrm{Hom}(\Gamma_{g,m}, G)/G,$$

the *representation variety.* We equip Hom and Rep with a topology. If $m \neq 0$, then $\Gamma_{g,m}$ is a free group of degree $2g + m - 1$ and $\mathrm{Hom}(\Gamma_{g,m}, G)$ can be identified with G^{2g+m-1} and we can carry over the topology. If $m = 0$, then we have only one relation in Γ_g, hence $\mathrm{Hom}(\Gamma_g, G)$ is a quotient with respect to this relation and we can use the quotient topology. If G is algebraic, then so is $\mathrm{Hom}(\Gamma_{g,m}, G)$. We also take the quotient topology for $\mathrm{Rep}(\Gamma_{g,m}, G)$.

We now present the link between hyperbolic structures on oriented surfaces and representations of fundamental groups into $\mathrm{PSL}(2, \mathbb{R})$. For details see [57, Ch. 3.4]. A hyperbolic structure on a surface Σ gives rise to a $(\mathrm{PSL}(2, \mathbb{R}), \mathbb{D})$-structure, i.e. Σ locally looks like an open subset of $\mathbb{D}$ (local charts (U_i, φ_i)) and the transition maps between local charts $g_{i,j} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are locally constant and elements of $\mathrm{PSL}(2, \mathbb{R})$. A $(\mathrm{PSL}(2, \mathbb{R})), \mathbb{D})$-structure on a topological space induces a representation of the fundamental group $\pi_1(\Sigma)$ into $\mathrm{PSL}(2, \mathbb{R})$:

Let $[\gamma] \in \Gamma_{g,m,l}$, where $\gamma : [0, 1] \to \Sigma_g$ is a closed loop. There exist $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ such that $\gamma([t_{i-1}, t_i]) \subset U_i$ (where we possibly have to adapt the numeration of the U_i). Denote by $g_{i,i+1} \in \mathrm{PSL}(2, \mathbb{R})$ the transition map. Putting

$$\varrho([\gamma]) := g_{n-1,n} \cdots g_{1,2}$$

defines the *holonomy representation* $\varrho : \Gamma_{g,m,l} \to \mathrm{PSL}(2, \mathbb{R})$. The holonomy representation can be defined for every (G, X)-structure. For $(\mathrm{PSL}(2, \mathbb{R}), \mathbb{D})$-structures a holonomy representation is called a *hyperbolization*. They are faithful and have discrete image and they are characterized by this property. If $m = 0$ and the metric on $\Sigma_{g,0,l}$ is complete, one can identify $\tilde{\Sigma}_{g,0,l}$ with the hyperbolic disc $\mathbb{D}$ via the developing map $\tilde{\Sigma}_g \to \mathbb{D}$.

If $m \geq 1$ then one can identify $\tilde{\Sigma}_{g,m,l}$ with a subset of $\mathbb{D}$ with the following surface doubling construction. Let $\Sigma_{g,m,l}$ be a hyperbolic surface with boundary (i.e. $m \geq 1$). Then we can take a second copy of the same surface with the same hyperbolic structure and glue every boundary component of one copy with one boundary component of the same length of the other component. The result is the hyperbolic surface $\Sigma_{2g+m-1,0,2l}$ without boundary (see Figure 1.2). For this surface we get a representation of the fundamental group $\Gamma_{2g+m-1,0,2l}$ as described above and its universal cover can be identified with $\mathbb{D}$. We can restrict the representation of $\Gamma_{2g+m-1,0,2l}$ to a representation of $\Gamma_{g,m,l}$ into $\mathrm{PSL}(2,\mathbb{R})$. Furthermore we obtain an identification of the universal cover of $\Sigma_{g,m,l}$ with a subset of the hyperbolic disk. For more details on the doubling construction and the restriction of representation see Section 3.5.6.

Not all representations of $\Gamma_{g,m,l}$ into $\mathrm{PSL}(2,\mathbb{R})$ are holonomy representations, because they have to be faithful and have discrete image. It turns out, that holonomy representations ϱ into $\mathrm{PSL}(2,\mathbb{R})$ are also distinguished by a numerical invariant, the *Toledo invariant* T_ϱ , which we define in in Section 2.2.3. The Toledo invariant can be defined for every representation ϱ of the fundamental group of an oriented surface Σ of negative Euler characteristic $\chi(\Sigma)$ into a Hermitian Lie group G (which we define in Section 2.1.1). The Toledo invariant cannot take arbitrary values. By the Milnor-Wood inequality (see [18, Thm. 1] and Section 2.2):

$$|T_\varrho| \leq |\chi(\Sigma)| \operatorname{rk} \mathcal{X},$$

where $\operatorname{rk}\mathcal{X}$ is the rank of $\mathcal{X}$, the symmetric space associated with G.

If Σ has non-empty boundary, then $T_\bullet$ is surjective on the interval $[-|\chi(\Sigma)| \operatorname{rk} G, |\chi(\Sigma)| \operatorname{rk} G]$. If Σ is closed then $T_\bullet$ only takes finitely many values (see Theorem 2.2.10).

The following theorem classifies hyperbolizations in terms of the Toledo invariant.

Theorem 1.1.1. *(Goldman, [33]) A representation $\varrho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ is a hyperbolization if and only if $T_\varrho = 2g-2$.*

Since $2g-2 = |\chi(\Sigma_g)| \operatorname{rk}\mathbb{D}$, this motivates

Definition 1.1.2. Let G be a Hermitian Lie group. A representation $\varrho : \Gamma_{g,m} \to G$ is *maximal* if its Toledo invariant is maximal, i.e.

$$T_\varrho = |\chi(\Sigma)| \operatorname{rk} \mathcal{X}.$$

The space of maximal representations is denoted by $\mathrm{Rep}_{max}(\Gamma_{g,m}, G)$.

Theorem 1.1.1 says in particular that a representation into $\mathrm{PSL}(2,\mathbb{R})$ is a hyperbolization if and only if it is maximal.

Maximal representations share a lot of properties with hyperbolizations, whence $\mathrm{Rep}_{max}(\Gamma_{g,m}, G)$ is called *higher Teichmüller space.* In Section 2.2 below we define the Toledo invariant in full detail and present the most important properties.

1.2 Fenchel-Nielsen Coordinates for Teichmüller Space

Fenchel-Nielsen coordinates are coordinates on the Teichmüller space $\mathcal{T}(\Sigma_g)$. An introduction to these can be found in [42]. The basic idea for these coordinates is a decomposition of a given surface into smaller pieces, the investigation of hyperbolic structures on the smaller pieces and a careful study of the reconstruction of the given surface from the building blocks. In Section 1.3 we will generalize the ideas presented in this section to obtain coordinates for maximal representations.

The building block is the surface of genus 0 and 3 boundary components $\Sigma_{0,3}$. We call it *pair of pants* or Y-piece (Figure 1.2). A hyperbolic structure with geodesic boundaries on $\Sigma_{0,3}$ is uniquely determined by the length of the boundaries. Formally we have:

Proposition 1.2.1. *Let $\Sigma_{0,3}$ be a pair of pants and $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+$. Then there is a unique hyperbolic structure on $\Sigma_{0,3}$ such that the boundary length are λ_1, λ_2 and λ_3 respectively.*

For a geometric proof using hyperbolic geometry see [42, Thm. 4.3.1]. The proposition can also be deduced from the results in Section 1.3 (Corollary 3.1.5). The λ_i are called *length parameter.*

One can construct all surfaces with negative Euler characteristic $\chi(\Sigma_{g,m}) = 2-2g-m$ by gluing several copies of $\Sigma_{0,3}$, see for example a decomposition of Σ_2 in Figure 1.2.

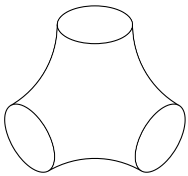


Figure 1.1: Pair of pants $\Sigma_{0,3}$

Indeed, to obtain $\Sigma_{g,m}$, we first glue $2g + m - 2 \geq 1$ pairs of pants such that the result is a surface of genus 0 and $2g + m$ boundary components ($\Sigma_{0,2g+m}$). Finally we close g handles as in Figure 1.4. The result is the surface $\Sigma_{g,m}$. This construction is explained in more detail in Section 3.5.1.

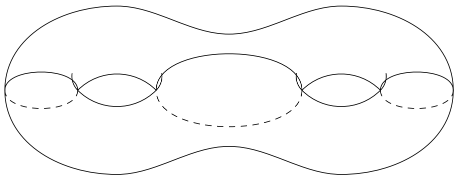


Figure 1.2: Σ_2 decomposed into pairs of pants.

The construction of a surface $\Sigma_{g,m}$ from pairs of pants fixes a set of simply closed curves $\{d_i\}$in $\Sigma_{g,m}$, namely the boundaries curves of the pairs of pants. The choice of a decomposition into pairs of pants (or of curves inducing such a decomposition) modulo deformation is called a *marking*. A surface with marking and hyperbolic structure has a *marked hyperbolic structure*.

We have seen that we can construct every surface of negative Euler characteristic from some copies of $\Sigma_{0,3}$, which we can equip with a hyperbolic metric. We can glue two boundary components of one or two hyperbolic surfaces if and only if the have the same length and obtain again a hyperbolic surface. This shows in particular that every surface with negative Euler characteristic can be equipped with a hyperbolic structure. For details for the gluing see Section 3.5.

A marked hyperbolic structure on a surface is determined by finitely many parameters. Given a decomposition $\{d_i\}$ of $\Sigma_{g,m}$ into pairs of pants and a hyperbolic structure on $\Sigma_{g,m}$. Any curve d_i is freely homotopic to a unique closed geodesic $\bar{d}_i$ with respect to the given hyperbolic metric. If we cut the surface along these closed geodesics, we get some pairs of pants with geodesic boundary. The hyperbolic structure on each pair of pants is uniquely determined by the length of the boundary geodesics, the *length parameter*. The gluing along geodesic boundary components of same length is unique up to rotation along the boundary, the *twist parameter*.

Thus length and twist parameter determine the marked hyperbolic structure uniquely and give *Fenchel-Nielsen coordinates* on Teichmüller space.

In particular we have:

Theorem 1.2.2. *(Fenchel-Nielsen coordinates) Let Σ_g be a closed surface of genus $g \geq 2$. Fix a marking on Σ_g. Then there are $3g-3$ length and $3g-3$ twist parameters, hence the Teichmüller space $\mathcal{T}(\Sigma_g)$ is homeomorphic to $\mathbb{R}^{6g-6}$.*

In summary, to obtain similar coordinates for representations, one has to understand representation of $\Gamma_{0,3}$ on one hand and the gluing on the other hand.

1.3 Coordinates for Maximal Representations

In this section we state the main results of this text, Theorem 1.3.1 and Theorem 1.3.10 below, which provide coordinates for the set of maximal representations $\Gamma_{0,3}$ into $\mathrm{Sp}(2n, \mathbb{R})$ on one hand and a the gluing construction for such representations on the other hand. In the last part, we give coordinates for maximal representations of the fundamental groups of $\Sigma_{1,1}$, $\Sigma_{1,2}$ and $\Sigma_{2,0}$. These surfaces can be obtained from one or two pairs of pants. Stating the coordinates in full detail is a little bit more involved, the most general statements can be found in Section 3.6.2.

1.3.1 Representations of $\Gamma_{0,3}$ into $\mathrm{Sp}(2n, \mathbb{R})$

To state the main theorem, denote by B the set of matrices in $\mathrm{GL}(n, \mathbb{R})$ whose eigenvalues have absolute value strictly less than 1 and define

$$R := \{(X_1, X_2, X_3) \in \bar{B}^3 | X_3 (X_2^\top)^{-1} X_1 \text{ is symmetric} \\ \text{and positive definite}\}.$$

Note that $\mathrm{O}(n)$ acts by diagonally by conjugation on R. Recall that $\Gamma_{0,3} = \{C_3, C_2, C_1 | C_3 C_2 C_1 = I\}$. Here and in the sequel we write elements of $\mathrm{Sp}(2n, \mathbb{R})$ as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are real $n \times n$-matrices, which have to satisfy some relations (see also Section 2.1.5).

The following theorem was inspired by [48, Ch. 10], where results of [30] are presented. We will explain the geometric idea behind it in Section 3.1 below.

Theorem 1.3.1. *Let* $\bar{f} : R \to \operatorname{Rep}(\Gamma_{0,3}, \operatorname{Sp}(2n, \mathbb{R}))$ *be the map which assigns to* $(X_1, X_2, X_3) \in R$ *the representation* $\varrho = \bar{f}(X_1, X_2, X_3)$ *of* $\Gamma_{0,3}$ *into* $\operatorname{Sp}(2n, \mathbb{R})$ *defined by*

$$\varrho(C_1) := c_1 = \begin{pmatrix} X_1 & 0 \\ X_1 + X_2^{-1} X_3^\top & (X_1^\top)^{-1} \end{pmatrix}$$

$$\varrho(C_2) := c_2 = \begin{pmatrix} -X_3^{-1} X_1^\top - X_2 - (X_2^\top)^{-1} & X_2 + X_3^{-1} X_1^\top \\ -X_3^{-1} X_1^\top - (X_2^\top)^{-1} & X_3^{-1} X_1^\top \end{pmatrix}$$

$$\varrho(C_3) := c_3 = \begin{pmatrix} (X_3^\top)^{-1} & -(X_3^\top)^{-1} - X_1^{-1} X_2^\top \\ 0 & X_3 \end{pmatrix}.$$

Then $\bar{f}$ *induces a homeomorphism*

$$f : R/\operatorname{O}(n) \to \operatorname{Rep}_{max}(\Gamma_{0,3}, \operatorname{Sp}(2n, \mathbb{R})).$$

The X_i can be seen as *generalized length parameters.*

Theorem 1.3.1 will we be proven in Section 3.3. For the proof we use the following theorem to identify for each $\varrho(C_i)$ a triples of points in the Shilov boundary $\check{S}$ (which will be introduced in Section 2.1.1 below) as well as its image under $\varrho(C_i)$. This yields equation for each $\varrho(C_i)$ which we can solve completely.

The calculations for these equations are preformed in T_Ω, the upper half plane model for the symmetric space associated with $\operatorname{Sp}(2n, \mathbb{R})$, respectively in a certain part of its boundary. The model T_Ω is defined and discussed in Section 2.1.1. In Remark 3.1.6 we describe the boundary and the action of the image of a maximal representation on it.

Theorem 1.3.2. *Let* G *be a Hermitian Lie group of tube type. Let* $\varrho : \Gamma_{0,3} \to G$ *be a representation and denote* $c_i := \varrho(C_i)$*. Assume that each* c_i *has a fixed point* $y_i \in \check{S}$*. Then we can express the Toledo invariant as follows:*

$$T_\varrho = \frac{1}{2}\big(\beta(y_1, y_2, y_3) + \beta(y_1, c_1 \cdot y_3, y_2)\big), \tag{1.1}$$

where β *denotes the Maslov index.*

We will define the Maslov index in Section 2.3.1 and prove the theorem in Section 3.2.

Theorem 1.3.2 yields that each representation as in Theorem 1.3.1 is maximal.

Before we turn to the twist parameters, we state some remarks:

Remark 1.3.3. In Theorem 1.3.1 $X_3(X_2^\top)^{-1}X_1$ was symmetric and positive definite. A direct calculation shows that $c_3c_2c_1 = I$ if and only if $X_3(X_2^\top)^{-1}X_1$ is symmetric. Its signature determines the Toledo invariant. Indeed with Formula (1.1) we get

$$T_\varrho = \frac{1}{2}(n + \operatorname{sgn} X_3(X_2^\top)^{-1}X_1).$$

This can be used to write down certain non-maximal representations explicitly, see Proposition 3.3.3 below.

The X_i describe the differential of the c_i in certain fixed points (Corollary 3.3.10).

Remark 1.3.4. Let now $n = 1$, i.e. we consider $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$. Then the X_i from Theorem 1.3.1 are just real numbers with the property: $X_1X_2X_3 > 0$. The real number X_i is the eigenvalue of c_i which has absolute value in $(0,1]$. These eigenvalues determine the translation length of c_i.

Recall that for a metric space (X,d) and an isometry g of X, the *translation length* of g is defined as

$$\tau_X(g) = \inf_{x\in X} d(x, gx).$$

For an isometry γ of $\mathbb{D}$ this translation length is non-zero if and only if γ is hyperbolic, i.e. g is conjugate to a matrix

$$\begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$$

with $x \neq \pm 1$. In this case the translation number is

$$\tau_{\mathbb{D}}(c_i) = 2\big|\log|X_i|\big|.$$

Since the translation length of c_i is equal to the length of the boundary geodesic at the boundary component C_i, the X_i determine these boundary length and vice versa. This is the relation between the length parameter in the Fenchel-Nielsen coordinates and our generalized length parameter.

In the sequel we need the following corollary, which gives us a standard form for certain elements of the image of a maximal representation. It is an immediate consequence of Theorem 1.3.1.

Corollary 1.3.5. *Let $\varrho : \Gamma_{g,m} \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation. Then there exists for each standard generator C_i matrices $X \in \mathrm{GL}(n, \mathbb{R})$ and S_1, S_2 and S_3 symmetric and positive definite, such that the image $\varrho(C_i)$ is conjugate to each of the following matrices:*

$$\begin{pmatrix} X & 0 \\ X + S_1 X & (X^\top)^{-1} \end{pmatrix}$$
$$\begin{pmatrix} -S_2X - X - (X^\top)^{-1} & X + S_2X \\ -S_2X - (X^\top)^{-1} & S_2X \end{pmatrix}$$
$$\begin{pmatrix} (X^\top)^{-1} & -(X^\top)^{-1} - S_3X \\ 0 & X \end{pmatrix}.$$

Remark 1.3.6. In Section 3.4 below we prove a result similar to Theorem 1.3.1 to parametrize representation of $\Gamma_{0,3}$ into a general Hermitian Lie group of tube type.

1.3.2 Gluing

To complete Fenchel-Nielsen coordinates for representations we need a gluing construction for representations. Here we briefly introduce the problem, all details and proofs can be found in Section 3.5. There are two gluing construction which we need to obtain a general surfaces from pairs of pants: gluing two surfaces (Figure 1.3) and closing handles (Figure 1.4).

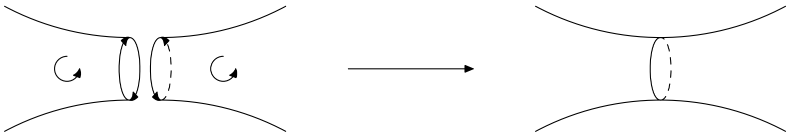


Figure 1.3: Gluing two surfaces.

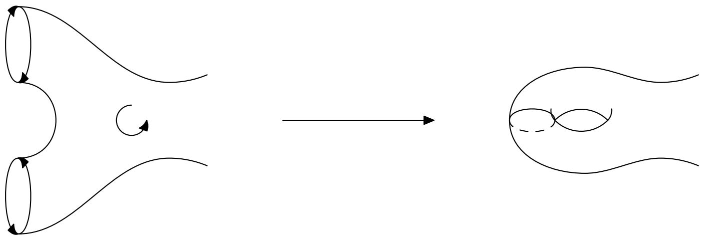


Figure 1.4: Closing a handle.

To obtain an oriented surface whose orientation is compatible with its building blocks, we have to respect the orientation of the building blocks.

Note that the loops generating the fundamental groups of the surfaces are oriented according to the boundary orientations. Denote the loops by C_1 and C_2. To write down the new fundamental group, we have to identify C_1 with C_2^{-1}. On the level of a representation we can glue representations ϱ_1 and ϱ_2 of the fundamental groups if and only if $\varrho_1(C_1) = \varrho_2(C_2)^{-1}$. Since we investigate representations up to conjugation, it is enough that $\varrho_1(C_1)$ and $\varrho_2(C_2)^{-1}$ are conjugate. We discuss this in detail in Section 3.5, where we also prove the following propositions.

(A) Gluing of two surfaces

Proposition 1.3.7. *Consider* $[\varrho_1'] \in \mathrm{Rep}(\Gamma_{g_1,n_1}, G)$ *and* $[\varrho_2'] \in \mathrm{Rep}(\Gamma_{g_2,n_2}, G)$ *with* $n_1 \geq 1$ *and* $n_2 \geq 1$. *Assume that there* *exists* $\varrho_1 \in [\varrho_1']$ *and* $\varrho_2 \in [\varrho_2']$ *such that* $\varrho_1(C_i) = g\varrho_2(\bar{C}_j)^{-1}g^{-1}$ *for some* $g \in G$. *Then there is a class of representations of* Γ *defined by*

$$\varrho := \varrho_1 * (g\varrho_2 g^{-1}) : \Gamma \to G$$

such that $[\varrho|_{\Gamma_{g_1,n_1}}] = [\varrho_1]$ *and* $[\varrho|_{\Gamma_{g_2,n_2}}] = [\varrho_2]$.

Remark 1.3.8. Note that ϱ is not unique. Let h be an element of the centralizer of $\varrho_2(\bar{C}_j)$. Then the representation $\varrho_h := \varrho_1 * (gh\varrho_2(gh)^{-1})$ also satisfies: $[\varrho_h|_{\Gamma_{g_1,n_1}}] = [\varrho_1]$ and $[\varrho_h|_{\Gamma_{g_2,n_2}}] = [\varrho_2]$.

(B) Closing handles

Proposition 1.3.9. *Let* $[\varrho'] \in \mathrm{Rep}(\Gamma_{g,n}, G)$ *with* $n \geq 2$. *Assume that there exists* $g \in G$ *such that* $\varrho'(C_i)^{-1} = g\varrho'(C_j)g^{-1}$. *Then there exists a representation* ϱ_f *of* Γ_f *such that* $\varrho_f|_\Gamma = \varrho$.

By Corollary 1.3.5 we can restrict the investigation of the gluing in the $\mathrm{Sp}(2n,\mathbb{R})$-case to the following case:

Theorem 1.3.10. *Let*

$$c = \begin{pmatrix} X & 0 \\ X + (X^\top)^{-1}S & (X^\top)^{-1} \end{pmatrix} \tag{1.2}$$

and

$$\bar{c} = \begin{pmatrix} (\bar{X}^\top)^{-1} & -(\bar{X}^\top)^{-1} - \bar{S}\bar{X} \\ 0 & \bar{X} \end{pmatrix} \tag{1.3}$$

be elements in $\mathrm{Sp}(2n,\mathbb{R})$ *with* X *and* $\bar{X}$ *invertible and* S *and* $\bar{S}$ *symmetric positive definite.*

(i) Suppose X *and* $\bar{X}$ *contracting. Then* $\bar{c}$ *and* c^{-1} *are conjugate in* $\mathrm{Sp}(2n,\mathbb{R})$ *if and only* $X^\top$ *and* $\bar{X}$ *are conjugate in* $\mathrm{GL}(n,\mathbb{R})$. *If* $\bar{X} = GX^\top G^{-1}$, *then* $\bar{c} = gc^{-1}g^{-1}$ *with*

$$g = g_1g_2g_3 = \begin{pmatrix} \bar{Y}GY^{-1} - (G^\top)^{-1} & -\bar{Y}G \\ GY^{-1} & -G \end{pmatrix},$$

where

$$Y = -\left(\sum_{i=0}^{\infty}(X^\top)^i(X^\top \cdot X + S)X_1^i\right)^{-1}$$

and

$$\bar{Y} = \sum_{i=0}^{\infty}(\bar{X}^\top)^i(I + \bar{X}^\top \bar{S}\bar{X})\bar{X}^i.$$

(ii) It X or $\bar{X}$ has an eigenvalue of absolute value 1, then $\bar{c}$ and c^{-1} are not conjugate in $\mathrm{Sp}(2n, \mathbb{R})$.

The matrix G from Theorem 1.3.9 (i) can be seen as a *generalized twist parameter.*

The matrices Y and $\bar{Y}$ are fixed points of c resp. $\bar{c}$ in the boundary of T_Ω.

Remark 1.3.11. Proposition 1.3.10 has a geometrical interpretation for $G = \mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$. It corresponds to the fact, that one can glues two hyperbolic surfaces along two geodesic boundaries if and only if they have the same length. Indeed, let Σ_1 and Σ_2 be surfaces with boundary components C_1 resp. C_2 and ϱ_1 and ϱ_2 hyperbolizations. Define $c := \varrho_1(C_1)$ and $\bar{c} := \varrho_2(C_2)$ and assume that they are in the form of Proposition 1.3.10. Then the lengths of the corresponding boundaries are equal to the translation length of c resp. $\bar{c}$. The translation length is uniquely determined by the eigenvalues of c and $\bar{c}$. Then one can glue along these boundary components if and only if X and $\bar{X}$ are equal and their absolute values is different from 1, i.e. c and $\bar{c}$ are hyperbolic and have the same translation length. But this is precisely the statement of Proposition 1.3.10.

1.3.3 Fenchel-Nielsen Coordinates for Maximal Representations into $\mathrm{Sp}(2n, \mathbb{R})$

Combining the results of the two previous sections we see, that we have a length parameter $X \in \mathrm{GL}(n, \mathbb{R})$ for every boundary component and a length parameter X and a twist parameter G for every boundary component of an embedded pair of pants.

Recall that $\Gamma_{1,1} = \langle A, B, C | [A,B]C = e \rangle$.

Proposition 1.3.12. *There exists a bijection between*

$$\left\{(X_1, X_2, G) \in \mathrm{GL}(n,\mathbb{R})^3 | X_1 \in B,\ (X_1, X_2, GX_1^\top G^{-1}) \in R\right\} / \mathrm{O}(n)$$

and $\mathrm{Rep}_{max}(\Gamma_{1,1}, \mathrm{Sp}(2n,\mathbb{R}))$ *induced by the map which assigns to* (X_1, X_2, G) *the representation* ϱ *defined by:*

$$\varrho(A) = \begin{pmatrix} X_1 & 0 \\ X_1 + (X_1^\top)^{-1}S & (X_1^\top)^{-1} \end{pmatrix},$$
$$\varrho(B) = \begin{pmatrix} \bar{Y}GY^{-1} - (G^\top)^{-1} & -\bar{Y}G \\ GY^{-1} & -G \end{pmatrix},$$
$$\varrho(C) = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

with

$$\begin{aligned}
C_1 =& -(X_2^\top)^{-1}X_1(S^\top)^{-1}X_1^\top - X_2 - (X_2^\top)^{-1} \\
C_2 =& X_2 + (X_2^\top)^{-1}X_1(S^\top)^{-1}X_1^\top \\
C_3 =& -(X_2^\top)^{-1}X_1(S^\top)^{-1}X_1^\top - (X_2^\top)^{-1} \\
C_4 =& (X_2^\top)^{-1}X_1(S^\top)^{-1}X_1^\top \\
S =& X_1^\top X_2^{-1}(G^\top)^{-1}X_1G^\top \\
Y =& -\left(\sum_{i=0}^{\infty}(X_1^\top)^i(X_1^\top \cdot X_1 + S)X_1^i\right)^{-1} \\
\bar{Y} =& (G^\top)^{-1}\left(\sum_{i=0}^{\infty}(X_1)^i(G^\top \cdot G + X_1G^\top S^{-1}GX^\top)(X_1^\top)^i\right)G^{-1}.
\end{aligned}$$

The matrices Y and $\bar{Y}$are fixed points in the boundary of T_Ω of $\varrho(A)$ respectively $\varrho(B)\varrho(A)^{-1}\varrho(B)^{-1}$.

Proposition 1.3.13. *There exists a bijection between*

$$\big\{(X_1, X_2, X_3, \bar{X}_1, \bar{X}_2, G) \in \mathrm{GL}(n,\mathbb{R})^6 | (X_1, X_2, X_3) \in R,$$
$$(\bar{X}_1, \bar{X}_2, GX_1^\top G^{-1}) \in R, X_1 \text{ contracting}\big\}/\sim$$

and $\mathrm{Rep}_{max}(\Gamma_{0,4}, \mathrm{Sp}(2n,\mathbb{R}))$*, where for* $k, l \in \mathrm{O}(n)$*,*

$$(X_1, X_2, X_3, \bar{X}_1, \bar{X}_2, G)$$

and

$$(kX_1k^{-1}, kX_2k^{-1}, kX_3k^{-1}, l\bar{X}_1l^{-1}, l\bar{X}_2l^{-1}, lGk^{-1})$$

are equivalent.

Proposition 1.3.14. *There exists a bijection between*

$$\big\{(X_1, X_2, X_3, G_3, G_2, G_1) \in \mathrm{GL}(n,\mathbb{R})^6 | (X_1, X_2, X_3) \in R,$$
$$(G_1X_3^\top G_1^\top, G_2X_2^\top G_2^{-1}, G_3X_1^\top G_3^{-1}) \in R,$$
$$X_i \text{ contracting}\big\}/\sim$$

and $\mathrm{Rep}_{max}(\Gamma_{2,0}, \mathrm{Sp}(2n,\mathbb{R}))$*, where for* $l, k \in \mathrm{O}(n)$*,*

$$(X_1, X_2, X_3, G_3, G_2, G_1)$$

and

$$(kX_1k^{-1}, kX_2k^{-1}, kX_3k^{-1}, lG_3k^{-1}, lG_2k^{-1}, lG_1k^{-1})$$

are equivalent.

1.4 Applications

We present three applications for our coordinates:

(i) We count connected components of $\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n,\mathbb{R}))$, if $m \geq 1$,

(ii) we show continuity of the limit curve for certain maximal representations of $\Gamma_{g,m}$ with $m \geq 1$,

(iii) we identify standard representations in the sense of [35].

We can also calculate the number of connected components of the space of maximal representations for surfaces with boundary, i.e. $\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))$ for $m \geq 1$ and we proof continuity of the limit curve of maximal representations of the fundamental group of a non-closed surface for which the standard generators have S-hyperbolic image (see Definition 1.4.3). Both results where known for closed surfaces. Connected components have been counted using Higgs-bundle techniques, which are quite far from our methods.

Theorem 1.4.1. *[31, 34] The space* $\mathrm{Rep}_{max}(\Gamma_g, \mathrm{Sp}(2n, \mathbb{R}))$ *has* $3 \cdot 2^{2g}$ *connected components if* $n \geq 3$ *and* $3 \cdot 2^{2g} + 2g - 4$ *connected components if* $n = 2$.

We use an invariant to find a lower bound for the connected components. For an upper bound of the number of connected components we use Theorem 3.6.9 and the gluing results from Section 1.3.2 to define paths in $\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))$ to join every representation with a standard representation. We obtain

Theorem 1.4.2. *Let* $m \geq 1$ *then* $\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))$ *has* 2^{2g+m-1} *connected components.*

For a proof see Section 4.2.

To distinguish the class of representations for which the limit curve is continuous, we need the following definition:

Definition 1.4.3. Let G be a Hermitian Lie group and $g \in G$. Then g is *Shilov-hyperbolic* (or S-hyperbolic) if it has a pair (X^+, X^-) of transversal fixed points in $\check{S}$, such that g contracts an open and dense subset of $\check{S}$ to X^+ and g^{-1} contracts an open and dense subset to X^-.

Remark 1.4.4. We will discuss S-hyperbolic elements in G in Section 3.3.3. An element $c \in \mathrm{Sp}(2n, \mathbb{R})$ as in Corollary 1.3.5 is S-hyperbolic if and only if X has no eigenvalue of absolute value 1.

For a maximal representation ϱ, one can glue along $\varrho(C_i)$ if and only if it is S-hyperbolic. In particular the generators $\varrho(A_i)$ and $\varrho(B_j)$ are automatically S-hyperbolic.

In Section 4.1 we show that the limit curve for any representation whose generators are S-hyperbolic is continuous:

Theorem 1.4.5. *Let $h : \Gamma_{g,m} \to \mathrm{PSL}(2, \mathbb{R})$ be a hyperbolization for a surface with geodesic boundaries. Denote by $\mathcal{L}$ its limit set in S^1. Let $\varrho : \Gamma_{g,m} \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation s.t. $\varrho(C_i)$ is S-hyperbolic for all i. Then there exists a monotone, ϱ-equivariant, continuous map*

$$\varphi : \mathcal{L} \to \check{S}.$$

For maximal representations of Γ_g into $\mathrm{Sp}(2n, \mathbb{R})$ the theorem is proved in [13].

Furthermore we will prove:

Proposition 1.4.6. *Let ϱ be maximal representation as in Theorem 1.4.5 . Then the associated limit curve is unique.*

The general parameters from Theorem 3.6.9 distinguish certain types of representations, which are *standard representations* in the sense of [35]:

Definition 1.4.7. A representation is a *product representation* if it is in the image of the diagonal map $i : \mathrm{SL}(2, \mathbb{R})^n \to \mathrm{Sp}(2n, \mathbb{R})$.

Let $\Delta : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})^n$ be the diagonal embedding. The centralizer of $i \circ \Delta(\mathrm{SL}(2, \mathbb{R})) \subset \mathrm{Sp}(2n, \mathbb{R})$ is precisely $\mathrm{O}(n)$. This gives an embedding φ_Δ of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(n)$ into $\mathrm{Sp}(2n, \mathbb{R})$. A representation is a *twisted diagonal representation* if its image is in $\varphi_\Delta(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(n))$.

Corollary 1.4.8. *A maximal representation ϱ is conjugate to a product representation into* $\mathrm{Sp}(2n, \mathbb{R})$ *if and only if and only if there exist length and twist parameter which are diagonal.*
The representation ϱ is twisted diagonal, if and only if there exists length and twist parameter of the form $d \cdot k$ with d diagonal with one single eigenvalue and $k \in \mathrm{O}(n)$.

We prove this corollary in Section 3.6.2.

Corollary 1.4.9. *A maximal representation of $\Gamma_{g,m}$ such that all $\varrho(C_i)$ are S-hyperbolic is Anosov.*

We will prove Corollary 1.4.9 in Section 3.5.6.

For a definition of Anosov representation see Definition A.3.1.

1.5 Cross Ratios

Chapter 5 is joint work with Tobias Hartnick. It is concerned with three interrelated problems:

(i) the development of a functorial theory of generalized cross ratios on Shilov boundaries of bounded symmetric domains of tube type (following work of Clerc and Ørsted [25]);

(ii) estimates for the translation length of isometries of bounded symmetric domains of tube type, which have two transversal fixed points in the Shilov boundary, in terms of these cross ratios;

(iii) applications to maximal representations of surface groups into Hermitian Lie groups (as suggested by earlier work of Labourie [47] and Wienhard [58]).

Concerning (i) we recall that the classical four point cross ratio on $\mathbb{CP}^1$ is defined by the formula

$$[a : b : c : d] := \frac{(a-d)(c-b)}{(c-d)(a-b)};$$

its restriction to the circle classifies orbits of ordered quadruples under the action of $\mathrm{PSL}(2,\mathbb{R})$. For boundaries of more general symmetric spaces the space of invariant functions on 4-tuples will no longer be one-dimensional, hence it is not obvious how to extend the definition of the cross ratio to more general semisimple Lie groups. In fact, it is not even clear what would be the correct notion of boundary to be used in a general theory of cross ratios. Various inequivalent definitions of generalized cross ratios (in different degrees of generality) exist in the literature, see e.g. [56, 9, 43, 3] and [47, Subsec. 4.2.6].

A Hermitian symmetric space $\mathcal{D}$ is of tube type, if it is biholomorphic to $V \oplus i\Omega \subset V^{\mathbb{C}}$, where V is a real vector space, $V^{\mathbb{C}}$ its complexification and Ω an open cone (see Section 2.1.1). We will introduce a cross ratio on the Shilov boundary of a bounded symmetric domain $\mathcal{D}$ of tube type, which generalizes the classical cross ratio on S^1. Our basic idea is that a good generalization of the classical cross ratio should be functorial (in a sense to be made precise below) and well-behaved under products. If we demand these two properties then there is actually only one choice:

Theorem 1.5.1. *For every bounded symmetric domain $\mathcal{D}$ of tube type with Shilov boundary $\check{S}$ there exists a subset $\check{S}^{(4+)}$ of $\check{S}^4$ (defined in Definition 5.1.7 below) and a function $B_{\check{S}} : \check{S}^{(4+)} \to \mathbb{R}^{\times}$ called the* generalized cross ratio *of $\check{S}$, such that the family of functions $\{B_{\check{S}}\}$ is characterized uniquely by the following properties:*

(i) $B_{\check{S}}$ is invariant under the group of biholomorphic automorphisms of $\mathcal{D}$.

(ii) If $f : \mathcal{D}_1 \to \mathcal{D}_2$ is a balanced tight morphism (see Definition 5.1.14 below), then the corresponding generalized cross ratios

$B_{\check{S}_1}$, $B_{\check{S}_2}$ satisfy

$$B_{\check{S}_2}(\bar{f}(x), \bar{f}(y), \bar{f}(z), \bar{f}(t)) = B_{\check{S}_1}(x, y, z, t),$$

where $(x, y, z, t) \in \check{S}_1^{(4+)}$ and $\bar{f}$ is the boundary extension of f.

(iii) If $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ is a direct product of bounded symmetric domains of ranks r_1, r_2 with projections $p_j : \mathcal{D} \to \mathcal{D}_j$ and corresponding boundary extensions $\bar{p}_j : \check{S} \to \check{S}_j$ then

$$B_{\check{S}}(x, y, z, t)^{r_1+r_2} == \quad B_{\check{S}_1}(\bar{p}_1(x), \bar{p}_1(y), \bar{p}_1(z), \bar{p}_1(t))^{r_1} B_{\check{S}_2}(\bar{p}_2(x), \bar{p}_2(y), \bar{p}$$

(iv) B_{S^1} is the restriction of the classical four point cross ratio.

(Theorem 1.5.1 will be proved in Section 5.1.4 below.)

The proof of the theorem is constructive. Cross ratios for *irreducible* bounded symmetric domains of tube type have been constructed by Clerc and Ørsted in [25], and it is easy to modify their construction in such a way that it becomes functorial. The main difficulty is then to show that the extension of these generalized cross ratios to arbitrary bounded symmeric domains by means of (iii) is still functorial. In fact, as will be explained in more details in Section 5.1.2 below, this can only be achieved by restricting the class of admissible morphisms to exclude obvious pathologies.

One of the reasons for the importance the classical cross ratio in hyperbolic geometry is the fact that is can be used to define the hyperbolic metric. As a consequence, it can also be used to measure translation lengths (which we introduced in Remark 1.3.4) of hyperbolic isometries. In this case γ has an attractive fixed point $\gamma^+ \in S^1$ and a repellent fixed point $\gamma^- \in S^1$ and we have

$$\tau_{\mathbb{D}}(\gamma) = \tau_{\mathbb{D}}^{\infty}(\gamma) := \log[\gamma^- : \xi : \gamma^+ : \gamma.\xi], \tag{1.4}$$

where $\xi \in S^1 \setminus \{\gamma^\pm\}$ is an arbitrary auxiliary point. The right hand side of this equation is referred to as the *period* of γ. If g is an isometry of a bounded symmetric domain $\mathcal{D}$ of tube type which admits a pair of transverse fixed points $g^\pm$ in $\check{S}$, then we can use our generalized cross ratios we can define a period

$$\tau_{\mathcal{D}}^\infty(g, g^+, g^-) := \log B_{\check{S}}(g^-, \xi, g^+, g.\xi) \tag{1.5}$$

using a generic auxiliary point $\xi \in \check{S}$. Reordering $g^\pm$ if necessary we can always assume $\tau_{\mathcal{D}}^\infty(g, g^+, g^-) \geq 0$. Without any further assumptions we then find a constant $C_{\mathcal{D}}$ depending only on $\mathcal{D}$ such that (see Corollary 5.2.8 below)

$$\tau_{\mathcal{D}}(g) \geq C_{\mathcal{D}} \cdot \tau_{\mathcal{D}}^\infty(g, g^+, g^-). \tag{1.6}$$

Remarkably, no hyperbolicity assumptions on g are required for this inequality to hold. On the other hand, if g is hyperbolic with attractor g^+ and repellor g^-, then we also get the converse inequality

$$\tau_{\mathcal{D}}(g) \leq C'_{\mathcal{D}} \cdot \tau_{\mathcal{D}}^\infty(g, g^+, g^-). \tag{1.7}$$

for some constant $C'_{\mathcal{D}}$ depending only on $\mathcal{D}$. In fact, the hyperbolicity assumption can be weakened considerably to include e.g. products of unipotent isometries, but not dropped altogether; see Corollary 5.2.8 for details.

Our main application of Inequality (1.6) concerns maximal representations $\varrho : \Gamma_g \to G$. Let φ be the limit curve for the maximal representation. Then we may define Γ-invariant function on quadruples on the circle by the formula

$$b_\varrho(a, b, c, d) := B_{\check{S}}(\varphi(a), \varphi(b), \varphi(c), \varphi(d)).$$

We want a continuous cross ratio, so we ask φ to be continuous. This is true for closed surfaces as well as for a class of representations for surfaces with boundary (see Section 4.1). But we will not get any new informations for the latter as we will explain in Remark 1.5.7. Furthermore the definition of the period does not make any sense if $g \in G$ does not admit a transverse pair of fixed points. Therefore we will restrict ourselves to representations for closed surfaces.

However for representations ϱ of Γ_g this function turns out to be a strict weak cross ratio in the sense of Labourie [46], which we refer to as the *cross ratio of* ϱ.

By choosing a finite generating set S we can think of the group Γ as a metric space with word metric d_S. With respect to this metric the translation length of $\gamma \in \Gamma$ on Γ is given by the formula

$$l_S(\gamma) := \inf_{\eta \in \Gamma} \|\eta\gamma\eta^{-1}\|_S. \tag{1.8}$$

If we combine the estimate for b_ϱ arising from (1.6) with Labourie's equivalence theorem for strict weak cross ratios (see [47] and Theorem 5.3.4) then we obtain the following relation between l_S and translation length in $\mathcal{D}$:

Theorem 1.5.2. *Let Γ be the fundamental group of a closed oriented surface Σ, $\mathcal{D}$ a bounded symmetric domain and S a finite generating set S for Γ. Then for every maximal representation $\varrho : \Gamma \to G$ there exist $A, B > 0$ such that for all $\gamma \in \Gamma$,*

$$\tau_{\mathcal{D}}(\varrho(\gamma)) \geq A \cdot l_S(\gamma) - B.$$

(Theorem 1.5.2 will be proved in Theorem 5.3.10 below.)

In the language of [27] Theorem 1.5.2 says that maximal representations are *well-displacing*, where the constants A and B implicit in this statement depend on the maximal representation in question. This well-displacing property has a number of well-known consequences, which we list briefly. Firstly, given any finite generating set S of Γ we can define an associated word metric d_S on Γ. Then, using results from [27] we obtain:

Corollary 1.5.3. *For every $x \in \mathcal{D}$ and every finite generating set S of Γ the map*

$$(\Gamma, d_S) \to (\mathcal{D}, d_{\mathcal{D}}), \quad \gamma \mapsto \varrho(\gamma).x$$

is a quasi-isometric embedding.

Theorem 1.5.2, Corollary 1.5.3 and the Milnor-Švarc lemma imply:

Corollary 1.5.4. *There exists constants $C, D > 0$ such that for all $\gamma \in \Gamma$*

$$C^{-1}\tau_{\mathbb{D}}(\gamma) - D \leq \tau_{\mathcal{D}}(\varrho(\gamma)) \leq C\tau_{\mathbb{D}}(\gamma) + D$$

Another consequence of Theorem 1.5.2 concerns the mapping class group of Σ. Fix a bounded symmetric domain $\mathcal{D}$ of tube type and denote by G the corresponding automorphism group. The set $\mathrm{Hom}_{\max}(\Gamma, G)$ of maximal representations of Γ into G can be considered as a subset of G^S for any finite generating set S of Γ; this induces a locally compact topology on $\mathrm{Hom}_{\max}(\Gamma, G)$. We denote by $\mathrm{Rep}_{\max}(\Gamma, G)$ the quotient of $\mathrm{Hom}_{\max}(\Gamma, G)$ by the conjugation action of G, i.e. the moduli space of conjugacy classes of maximal representations of Γ into G. Combining Corollary 1.5.4 with results from [58] we obtain:

Corollary 1.5.5. *In the above situation the action of the mapping class group of Σ on $\mathrm{Rep}_{\max}(\Gamma, G)$ is proper.*

For classical simple groups Corollary 1.5.5 was proved by Wienhard [58] (see also [47] for the symplectic case).

As a final application we consider the energy functional of a maximal representation ϱ as introduced in [47]: we denote by $E_\varrho := (\tilde{\Sigma} \times \mathcal{D})/\Gamma$ the associated $\mathcal{D}$-bundle over Σ and by $\Gamma(E_\varrho)$ the space of smooth sections of E_ϱ. In this notation the energy of a complex structure J on Σ with respect to ϱ is given by (see [47, Sec. 5.1])

$$e_\varrho(J) := \inf\{\int_\Sigma \langle df \wedge df \circ J \rangle \mid f \in \Gamma(E_\varrho)\}.$$

Then e_ϱ descends to a functional e_ϱ on Teichmüller space $\mathcal{T}(\Sigma)$ called the *energy functional* of ϱ. In this context, our results imply:

Corollary 1.5.6. *For any maximal representation $\varrho : \Gamma \to G$ the associated energy functional $e_\varrho : \mathcal{T}(\Sigma) \to \mathbb{R}$ is proper.*

(The fact that Theorem 1.5.2 implies Corollaries 1.5.3-1.5.6 is well-known; see Subsection 5.3.5 below for precise references.)

Remark 1.5.7. Theorem 1.5.2, Corollary 1.5.3, Corollary 1.5.4 and Corollary 1.5.6 are also true for representations ϱ of $\Gamma_{g,m}$ which are S-hyperbolically generated. Corollary 1.5.6 is true if we replace $\mathrm{Rep}_{max}(\Gamma, G)$ by

$$\mathrm{Rep}^h_{max}(\Gamma, G) := \{\varrho \in \mathrm{Rep}_{max}(\Gamma, G) | \varrho \text{ is S-hyperbolically generated}\}.$$

This follows from the surface doubling construction described in Proposition 3.5.16. Indeed, let Σ_{2g+m-1} be a double of $\Sigma_{g,m}$ and $\tilde{\varrho} : \Gamma_{2g+m-1} \to G$ a double of ϱ. Then $\tilde{\varrho}$ is maximal and its restriction to $\Sigma_{g,m} \subset \Sigma_{2g+m-1}$ is equal to ϱ. The results above are true for $\tilde{\varrho}$, because it is a representation of a closed surface. Clearly they are also true for ϱ because it is a restriction of $\tilde{\varrho}$.

The representations for which we can show continuity of the limit curve are precisely the ones which are S-hyperbolically generated. For these representations we have a weak strict cross ratio and we can use this cross ratio to show the results directly.

Recall that we need a pair of transverse fixed points in $\check{S}$ for $g \in G$ to be able to define the period. In particular one cannot generalize the results above straight forward for representations which are not hyperbolically generated.

Therefore we restrict ourselves to representations for closed surfaces.

1.6 Guide for the reader

In Section 2.1 we introduce notations needed throughout this text. The first parts (Section 2.1.1 and Section 2.1.2) are concerned with bounded symmetric domains, the Shilov boundary, Jordan algebras and their relations. Thereafter we introduce and discuss boundary morphisms for Shilov boundaries (Section 2.1.3). Section 2.1.4 is concerned with the Cayley transform and we show that a certain action of the Levi factor of the stabilizer of a point in the Shilov boundary is linear (this is joint work with Tobias Hartnick). In Section 2.1.5 we present the group $\mathrm{Sp}(2n,\mathbb{R})$, its bounded symmetric domain and its Shilov boundary.

Section 2.2 contains definitions of (bounded) cohomology via homogeneous and inhomogeneous cocycles, because we need both to connect results from [18] and [23]. Furthermore we define the Toledo invariant and maximal representations and state some properties for both.

In Section 2.3.1 we define the Maslov index, discuss the relation with the Kähler class and introduce the Souriau index which we will need for the proof of Formula (1.1). We introduce the (generalized) rotation number from [18] and the limit curve.

Chapter 2.1 does not contain any new material. The reader may skip it and come back if necessary.

The geometric idea behind the parameters for maximal representations of $\Gamma_{0,3}$ into $\mathrm{Sp}(2n,\mathbb{R})$ is explained in Section 3.1.

The first results can be found in Section 3.2 where we prove Formula (1.1) and deduce some consequences for maximal representations.

In Section 3.3 we proof Theorem 1.3.1. Section 3.3.3 contains result on the dynamics of generators of maximal representations and their fixed points in the Shilov boundary which might be also interesting by their own.

In Section 3.4 we parametrize representations of $\Gamma_{0,3}$ into an arbitrary Hermitian Lie group of tube type.

Section 3.5 is dedicated to the study of the gluing construction which is used in Section 3.6 to state parameters for all maximal representations into $\mathrm{Sp}(2n, \mathbb{R})$.

In Sections 4.1 and 4.2 we show continuity for the limit curve in some cases and count connected components.

Section 5.1 is concerned with the construction of generalized cross ratios and the proof of their functorial characterization, i.e. Theorem 1.5.1. Along the way we establish various useful properties of cross ratios. The key step in the proof is Proposition 5.1.12 which establishes the desired functoriality.

Section 5.2 is devoted to the relation between generalized cross ratios and translation lengths. We first provide bounds for translation lengths of elements of the general linear group acting on the associated symmetric space. Using the linear representation of the Levi factor constructed in Section 2.1.4 we thereby obtain bounds for the translation length of special isometries of general bounded symmetric domains of tube type, which we are able to express in terms of the period of the isometry in question, see Corollary 5.2.8.

In Section 5.3 we associate with every maximal representation of the fundamental group of a closed, oriented surface of genus ≥ 2 a strict weak cross ratio in the sense of Labourie. Using Labourie's equivalence theorem for such cross ratios and the estimates from Section 5.2 we then establish the well-displacing property of maximal representations in Theorem 5.3.10. Finally, we indicate how to deduce Corollaries 1.5.3, 1.5.4, 1.5.5 and 1.5.6.

Appendix A.1 contains some matrix calculations. Appendix A.2 collects some Jordan theoretic facts used in our proof of the functoriality theorem. Appendix A.3 contains a definition of Anosov representation and establishes a certain uniqueness property of limit curves of maximal representations, which is a consequence of work of Burger, Iozzi and Wienhard in [15]. In the preparation of this appendix we profited from a manuscript on Anosov representations by Anna Wienhard and Olivier Guichard (which has now appeared as a part of [36]).

Chapter 2

Preliminaries

2.1 Basic Notions and Facts

In this section we collect definitions and facts used later in this text. The reader may skip this section and come back to it if necessary.

2.1.1 Hermitian Symmetric Spaces, Bounded Symmetric Domains and Tube Type Domains

Symmetric Spaces

The main references for this Subsection are [39] and [5].

Definition 2.1.1. Let (M, g) be a Riemannian manifold. It is called *locally symmetric space* if for any $x \in M$ there exists a local isometry s_x which is an involution and which has x as an isolated fixed point. If each s_x is a global isometry, M is a *(Riemannian) symmetric space.*

Example 2.1.2. The hyperbolic spaces H^n, the Euclidean spaces E^n and the spheres S^n are symmetric spaces.

The isometry s_x acts involutively on T_xM, hence the eigenvalues of $ds_x|_x$ can only be ± 1. But if one of the eigenvalues is 1, there would be a small piece of geodesic starting in x fixed by s_x which contradicts the assumption that x is an isolated fixed point, hence $ds_x|_x$ acts by multiplication with -1 on T_xM.

Riemannian symmetric spaces are geodesically complete. Indeed given a geodesic σ with $\sigma(0) = x$, then $s_x(\sigma(t)) = \sigma(-t)$ if $\sigma(t)$ is defined. In this case $s_{\sigma(t)}\sigma(0) = \sigma(2t)$. Since geodesics are always defined for small $t > 0$, this showns that geodesics are defined for all $t \in \mathbb{R}$. Then by the Hopf-Rinow Theorem, M is complete and we can join any two points x and y by a geodesic.

Let σ be a geodesic. Then the isometry $s_{\sigma(t/2)} \circ s_{\sigma(0)}$ is isotopic to the identity and it maps $\sigma(0)$ to $\sigma(t)$. Therefore the identity compontent G of the isometry group of M (which is in fact a finite dimensional Lie group [39, Lem. 3.2]) acts transitively on geodesics. Indeed, since M is geodesically complete, one can join any two points by a geodesic and G acts transitively on M. The stabilizer of a point in M is a compact subgroup K of G. A symmetric space is *irreducible* if it is not the product of two symmetric spaces. Every symmetric space M can be written as a product ([39, Ch. V.5]):

$$M = E \times M_1 \times \ldots \times M_m \times N_1 \times \ldots \times N_n,$$

where E is isometric to an Euclidean (flat) space, M_i is non-compact irreducible and N_j is compact irreducible. If $n = 0$, the space M is called *of non-compact type*, if $M = N_1 \times \ldots \times N_n$, it is *of compact type*.

Let G be a finite dimensional Lie group and K a maximal compact subgroup. Then G/K can be equipped with the structure of a symmetric space. It is called the *symmetric spaces associated with* G.

Definition 2.1.3. Let (M, g) be a (locally) Riemannian symmetric space. It is called (locally) *Hermitian symmetric space* if there exists an invariant complex structure on M, i.e. a complex structure

$$J : TM \to TM,$$

which commute with isometries.
A Lie group G is *Hermitian* if its associated symmetric space is Hermitian.

Hermitian symmetric spaces are automatically Kähler with Kähler form

$$\omega(X, Y) := g(JX, Y)$$

(for a proof of this fact see [13, Lemma 2.1]).

Bounded Symmetric Domains and Jordan Algebras

Hermitian symmetric spaces are closely related to bounded symmetric domains. This section contains material collected for [38] in joint work with Tobias Hartnick. We follow [29], see also [55] and [50].

Let W be finite-dimensional complex vector space. A connected open subset $\mathcal{D} \subset W$ is a *domain*. A bounded domain $\mathcal{D}$ is called *symmetric* if for every $z \in \mathcal{D}$ there exists a biholomorphic involutive automorphism s_z of $\mathcal{D}$ such that z is an isolated fixed point of s_z.
Recall that for any domain $\mathcal{D}$, the *Bergman space* $\mathcal{H}^2(\mathcal{D})$ is the space of holomorphic square integrable functions on W. If $\mathcal{D}$ is bounded then this space is infinite-dimensional, because it contains at least all polynomials. Thus a reproducing kernel $k_{\mathcal{D}} : \mathcal{D}^2 \to \mathbb{C}^\times$ can be defined by the formula

$$f(z) = \int_{\mathcal{D}} f(w) k_{\mathcal{D}}(w, z) dw \quad (f \in \mathcal{H}^2(\mathcal{D}), z \in \mathcal{D}).$$

(see e.g. [29, Chap. IX.2]). The kernel $k_{\mathcal{D}}$ is the *Bergmann kernel* of $\mathcal{D}$. Later we will consider also multiples of the Bergmann kernel and we will sometimes refer to the kernel introduced here as the *unnormalized* Bergmann kernel.

The kernel has an important transformation property. Let g be a biholomorphic transformation for $\mathcal{D}$. Then:

$$k_{\mathcal{D}}(gz, gw) = j(z, g)k_{\mathcal{D}}(z, w)\overline{j(w, g)}, \tag{2.1}$$

where $j(z, g)$ is the *automorphy factor*.

The tensor

$$g_{jk}(z) := \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log k_{\mathcal{D}}(z, z)$$

defines a Hermitian metric on $\mathcal{D}$, called the *Bergman metric*, for whichbiholomorphic transformations are isometries (see [29, Prop IX.2.6]). Similar to the kernel function we will sometimes refer to this metric as the *unnormalized* Bergman metric. Since the point involutions s_z are biholomorphic, they are also global isometries, hence a bounded symmetric domain is a Riemannian symmetric space. Since the point involutions are holomorphic, they commute with the complex structure, hence a bounded symmetric domain is in fact a Hermitian symmetric space.

Surprisingly there is a converse statement.

Theorem 2.1.4. *(Harish-Chandra embedding,[37]) Every Hermitian symmetric space is biholomorphic to a bounded symmetric domain.*

For a more detailed discussion and a proof see [39] or [59].

Let $\mathcal{D}$ and $\mathcal{D}'$ be bounded symmetric domains with involutions s_z and $s'_{z'}$, respectively. A holomorphic map $f : \mathcal{D} \mapsto \mathcal{D}'$ is a *morphism* if for any $z \in \mathcal{D}$ we have

$$f \circ s_z = s'_{f(z)} \circ f.$$

Equivalently, f is an *affine* holomorphic map with respect to the Bergman metric on $\mathcal{D}$. Given a bounded symmetric domain $\mathcal{D}$, we denote by $G(\mathcal{D})$ the group of all automorphisms of $\mathcal{D}$. Its identity component $G(\mathcal{D})^0$ is a finite-dimensional connected adjoint semisimple Lie group acting transitively on $\mathcal{D}$, and the stabilizer of each point is a maximal compact subgroup.

Example 2.1.5. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc. It is clearly a bounded domain in a complex vector space. By [29, Prop. X.4.5] the Bergmann kernel of $\mathbb{D}$ is

$$k_{\mathbb{D}}(z, w) = \frac{2}{\pi} \frac{1}{(1 - z\bar{w})^2}.$$

Example 2.1.6. The kernel for a polydisc $\mathbb{D}^r$ is

$$k_P(z, w) = \prod k_{\mathbb{D}}(z_i, w_i)$$

Example 2.1.7. Let $V := \mathrm{Sym}_n(\mathbb{R})$ be the vector space of real symmetric $n \times n$-matrices and $V^{\mathbb{C}}$ its compactification. Then

$$\mathcal{D} = \{X \in V^{\mathbb{C}} |\ I_n - X^* X \text{ is positive definite}\}$$

is a bounded symmetric domain with kernel function ([29, Thm. X.1.2]):

$$k_{\mathcal{D}}(z, w) = \frac{\Gamma_\Omega(2n)}{\pi^{n(n+1)/2}\Gamma_\Omega(n)} \det\left(\frac{z - \bar{w}}{i}\right)^{-2n},$$

where Γ_Ω is the generalized Γ-function defined in Corollary VII.1.3 in [29].

Tube Type Domains and Jordan Algebras

Recall that the unit disc and the upper half plane are two models for hyperbolic two space. A bounded symmetric domain can be seen as a disc model for a Hermitian symmetric space and Theorem 2.1.4 says that any Hermitian symmetric space has such a disc model. Now one can ask for a generalization of the upper half plane model.

Definition 2.1.8. A Hermitian symmetric space is of *tube type* if it is biholomorphic to $V \oplus i\Omega \subset V^{\mathbb{C}}$, where Ω is a proper, open cone in a real vector space V. A Hermitian Lie group is of *tube type* if its associated Hermitian symmetric space is of tube type.

Example 2.1.9. The Hermitian symmetric space associated with $\mathrm{Sp}(2n, \mathbb{R})$ is of tube type. The Hermitian symmetric space associated with $SU(p, q)$ is of tube type if and only if $p = q$.

Hermitian symmetric spaces of tube type admit nice descriptions using Jordan algebras. Our main reference is again [29].

Definition 2.1.10. A *Jordan algebra* is a finite dimensional algebra V with

(i) $xy = yx \quad \forall x, y \in V$ (Commutativity)

(ii) $x^2(xy) = x(x^2y) \quad \forall x, y \in V$.

Define the left multiplication L on V by

$$L(x)y := xy, \quad x, y \in V.$$

The Axiom (ii) for Jordan algebras can be rephrased as $[L(x), L(x^2)] = 0$ for all $x \in V$, i.e.the left multiplication with x and x^2 commute. One can use this property to show that a Jordan algebra is power associative ([29, Prop. II.1.2]), even if a Jordan algebra need not to be associative in general.

We introduce the *quadratic representation* for V:

$$P(x) := 2L(x)^2 - L(x^2)$$

and the *box operator*

$$x \Box y := L(xy) + [L(x), L(y)].$$

Definition 2.1.11. A real Jordan algebra V is called Euclidean if it is unital with unit e and if it admits an scalar product $(\cdot,\cdot)$ with

$$(xy, z) = (y, xz), \quad \forall x, y, z \in V$$

i.e. $L(x)$ is self-adjoint for all $x \in V$.

The *complexification* of a real Jordan algebra V, denoted by $V^{\mathbb{C}}$ is again a Jordan algebra.

Example 2.1.12. Let $V = Sym_n(\mathbb{R})$ be the set of symmetric $n \times n$-matrices. Let $\cdot$ be the usual matrix product. Then defining a product on $\mathrm{Sym}_n(\mathbb{R})$ via

$$xy := \frac{1}{2}(x \cdot y + y \cdot x),$$

gives V the structure of a Jordan algebra. Let

$$(X, Y) := \mathrm{tr}(X \cdot Y).$$

Then $(\cdot,\cdot)$ defines a scalar product which gives V the structure of an Euclidean Jordan algebra.

A consequence of the existence of the scalar product is the spectral theorem ([29, Ch.III]). Before we state it, we need the following definition.

Definition 2.1.13. An element c of V is called *idempotent* if $c^2 = c$. Two idempotents c_1 and c_2 are *orthogonal* if $c_1c_2 = 0$. An idempotent c is *primitive* if it is non-zero and cannot be written as the sum of two non-zero idempotents.
A collection $\{c_1, \ldots, c_r\}$ of pairwise orthogonal, primitive idempotents is called *Jordan frame* if

$$c_1 + \ldots + c_r = e.$$

Note that two orthogonal idempotents c_1 and c_2 are also orthogonal with respect to the scalar product:

$$(c_1, c_2) = (c_1^2, c_2) = (c_2, c_1c_2) = 0.$$

The cardinality r is the same for all Jordan frames. It is called the *rank* of V.

Now we can state the spectral theorem for Euclidean Jordan algebras:

Theorem 2.1.14. *([29, Thm. III.1.2]) Given an Euclidean Jordan algebra V of rank r and $x \in V$. Then there exists a Jordan frame $\{c_1, \ldots, c_r\}$ and $\lambda_i \in \mathbb{R}$ such that*

$$x = \sum_{i=1}^{\infty} \lambda_i c_i.$$

Definition 2.1.15. Let $x = \sum_{i=1}^{r} \lambda_i c_i$ for some Jordan frame $\{c_i\}$. The numbers λ_i are called *spectral values.* The *Jordan algebra determinant* and the *Jordan algebra trace* are defined as

$$\det_V x = \prod \lambda_i, \quad \operatorname{tr}_V x := \sum \lambda_i.$$

Example 2.1.16. Let $V = \mathrm{Sym}_n(\mathbb{R})$ be as in Example 2.1.12. The the spectral value decomposition corresponds to the fact that symmetric matrices are diagonalizable with real eigenvalues. In particular the matrices

$$c_i := \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix},$$

with 1 on the ith diagonal position and 0 everywhere else form a Jordan frame and all Jordan frames are conjugate under $O(n)$. The spectral theorem is a direct generalization of this fact. In fact the proof relies on this fact for $L(x)$. The Jordan algebra determinant as well as the Jordan algebra trace are the usual ones.

Another important notion is the open (symmetric) cone ([29, Ch.III]) associated with a Jordan algebra:

Definition 2.1.17. Let V be a Euclidean Jordan algebra. Define

$$\Omega := \{x \in V | L(x) \text{ positive definite}\}.$$

It is clear from the definition that Ω is proper open cone. It is in fact a symmetric cone (see [29, p.4]). In particular the group

$$G(\Omega) = \{g \in \mathrm{GL}(V) | g\Omega = \Omega\}$$

acts transitively on Ω. For other characterizations and more details see [29, Ch.III]. One important description is the following

$$\Omega = \{x \in V | \text{ All spectral values are strictly positive}\}.$$

The space

$$T_\Omega = V \oplus i\Omega = \{\sum \mu_i c_i | \operatorname{im} \mu_i > 0\}$$

is biholomorphic to a bounded symmetric domain in $V^{\mathbb{C}}$ via the *Cayley transform*

$$p : z \mapsto (z - ie)(z + ie)^{-1}.$$

It maps T_Ω biholomorphically to the bounded domain

$$\mathcal{D} := \big\{v = \sum \lambda_i c_i \in V^{\mathbb{C}} | \{c_i\} \text{ Jordan frame, } |\lambda_i| < 1\big\}. \tag{2.2}$$

The inverse of p is given by

$$c : z \mapsto i(e + z)(e - z)^{-1}.$$

See [29, p.190] for details.

We say that $\mathcal{D}$ is the bounded symmetric domain associated with V. The group of orientation preserving isometries of $\mathcal{D}$ is denoted by $G(\mathcal{D})$. The domain $\mathcal{D}$ is isomorphic to $G(\mathcal{D})/K$, where K is a stabilizer of a point in $\mathcal{D}$. The subgroup K is a maximal compact subgroup of $G(\mathcal{D})$.

Now we can state another spectral theorem:

Theorem 2.1.18. *Let V be a Euclidean Jordan algebra. Fix a Jordan frame $\{c_1, \ldots, c_R\}$ and let $z \in V^{\mathbb{C}}$. Then there exists $k \in K$ and $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r$, such that*

$$z = k \sum \lambda_i c_i.$$

Since T_Ω is biholomorphic to a bounded domain, we can define the Bergman metric on T_Ω as in Subsection 2.1.1. The map

$$j : \begin{cases} T_\Omega \to T_\Omega \\ z \mapsto -z^{-1}. \end{cases}$$

is an isometric involution having ie as a unique fixed points. We can use it to give T_Ω the structure of a Hermitian symmetric space ([29, Thm. X.1.1]).

Remark 2.1.19. We have seen in (2.2) how to relate a bounded symmetric domain $\mathcal{D}$ to a Euclidean Jordan algebra V. This relation is in fact functorial. For details see [2].

The group of biholomorphic transformations, denoted by $G(T_\Omega)$, admits a nice decomposition . Denote by N^+ the subgroup of $G(T_\Omega)$ which acts by addition with an element of V on T_Ω. By [29, Prop. IX.3.4] this is indeed an isometry. We denote by n_v^+ the element of N^+ with

$$n_v^+(x) = x + v.$$

Now we have

Proposition 2.1.20. *([29, Prop. X.5.5]) The group $G(T_\Omega)$ decomposes as*

$$G(T_\Omega) = N^+ G(\Omega) G(T_\Omega)_{ie},$$

where $G(T_\Omega)_{ie}$ is the stabilizer of $ie \in T_\Omega$.

Later we will need another subgroup $N^- := jN^+j^{-1}$. It acts as follows:

$$n_v^- : x \mapsto (x^{-1} - v)^{-1},$$

and it gives another decomposition of $G(T_\Omega)$:

$$G(T_\Omega) = N^- G(\Omega) G(T_\Omega)_{ie}$$

We observe that

$$(n_v^+)^{-1} = n_{-v}^+ \text{ and } (n_v^-)^{-1} = n_{-v}^-$$

Definition 2.1.21. Let V be a Jordan algebra. The *automorphism group* of V is

$$\mathrm{Aut}(V) := \{g \in \mathrm{GL}(V) | g(xy) = (gx)(gy)\ \forall x, y \in V\}.$$

Another important group is the structure group of a Jordan algebra.

Definition 2.1.22. Let V be a Jordan algebra and $\tau(x, y) := \operatorname{tr} L(xy)$. Then V is *semi-simple*, if τ is non-degenerate. The structure group $\mathrm{Str}(V)$ of a semi-simple Jordan algebra V is:

$$\mathrm{Str}(V) := \{g \in \mathrm{GL}(V) | P(gx) = gP(x)g^*\},$$

where g^* is the adjoint of g w.r.t. τ.

Proposition 2.1.23. *[29, Prop. VIII.2.4, Prop VIII.2.8] The group* $\mathrm{Aut}(V)$ *is a subgroup of* $\mathrm{Str}(V)$*. If V is simple Euclidean, then* $\mathrm{Str}(V) = G(\Omega) \otimes \{\pm 1\}$.

Finally we introduce the character χ (cf. [23, p.99]):

Definition 2.1.24. Let V be a real or complex Jordan algebra and $\mathrm{Str}(V)$ be the structure group of V. Then we define the character χ on $\mathrm{Str}(V)$ via:

$$\det_V(gx) = \chi(g) \det_V(x),$$

where $g \in \mathrm{Str}(V)$ and $x \in V$.

Lemma 2.1.25. *Let $\mathcal{D} \subset V^{\mathbb{C}}$ as in* (2.2). *Then $K \subset G(\mathcal{D})$ is contained in the structure group* $\mathrm{Str}(V^{\mathbb{C}})$.

Proof. By Lemma A.2.2 every $k \in K$ extends to a linear map $V^{\mathbb{C}} \to V^{\mathbb{C}}$, since it fixes 0. Since it also fixes the Shilov boundary it is contained in $G(\Sigma)$ in [29, X.3.], which contained in $\mathrm{Str}(V^{\mathbb{C}})$ by [29, Prop. X.3.1]. $\square$

2.1.2 The Shilov Boundary of a Bounded Symmetric Domain

For spaces of non-positive curvature there exists many notions of boundary. In this section we introduce a boundaries, the Bergmann-Shilov boundary for domains and the Shilov boundary for bounded domains. We follow [29, Ch. X].

Definition 2.1.26. Let D be a domain and $A(D)$ the space of continuous functions on $\bar{D}$ which are holomorphic on D. A *Bergman-Shilov* boundary is a closed set B in ∂D such that

$$\max_{z \in D} |f(z)| = \max_{z \in B} |f(z)|, \quad \forall f \in A(D).$$

The Shilov boundary $\check{S}_{\mathcal{D}}$ of a bounded domain $\mathcal{D}$ is the smallest closed subset in $\partial\mathcal{D}$ such that

$$\max_{z \in \mathcal{D}} |f(z)| = \max_{z \in \check{S}} |f(z)|, \quad \forall f \in A(\mathcal{D}).$$

We will usually write $\check{S}$ instead of $\check{S}_{\mathcal{D}}$.

Proposition 2.1.27. *[29, Prop. IX.5.5] V is the unique Bergman-Shilov boundary of T_Ω.*

The Shilov boundary behaves nicely with respect to products.

Proposition 2.1.28. *Let $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ a bounded symmetric domains with $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$. Then*

$$\check{S}_{\mathcal{D}} = \check{S}_{\mathcal{D}_1} \times \check{S}_{\mathcal{D}_2}.$$

Note that this is not true for the geodesic boundary! For any bounded symmetric domain, the action of its isometry group G extends to the topological boundary. This action has been studied in detail before, see [59, I.5.]. The closure of $\mathcal{D}$ in its ambient vector space decomposes into a finite number of G-orbits. It turns out that the Shilov boundary is precisely the unique closed orbit of G in $\bar{\mathcal{D}}$.

Proposition 2.1.29. *Let $\mathcal{D}$ be a bounded symmetric domain. Then the Shilov boundary $\check{S}$ is equal to the unique closed orbit in $\partial\mathcal{D}$, the stabilizer of a point in $\check{S}$ is a certain maximal parabolic subgroup Q of G.*

Let $\mathcal{D} \subset V^{\mathbb{C}}$ be a bounded symmetric domain in the complexification of an Euclidean Jordan algebra V. The Shilov boundary can be described in terms of the Jordan algebra structure. In fact the following are equivalent:

(i) $z \in \check{S}$,

(ii) $z = \sum \lambda_i c_i$, *where* $c_1, \ldots, c_r$ *is a Jordan frame and* $|\lambda_i| = 1$,

(iii) $z \in \overline{p(V)}$,

(iv) $\bar{z} = z^{-1}$.

Proof. The first part is proven in [59] Part I Section 6, the second part follows is [29, p.190, Prop.X.2.3, Thm.X.4.6]. □

Example 2.1.30. The unit disc $\mathbb{D} \subset \mathbb{C}$ is a bounded symmetric domain. Its topological compactification $\bar{\mathbb{D}}$ equals $\mathbb{D} \cup S^1$. The action of $Is(\mathbb{D})$ extends to S^1.

Example 2.1.31. By Proposition 2.1.28 the Shilov boundary of a polydisc $\mathbb{D}^r$ is equal to $(S^1)^r$.

Example 2.1.32. Let $\mathcal{D}$ be the bounded symmetric domain associated with $\mathrm{Sp}(2n, \mathbb{R})$ (cf. Example 2.1.7). Then

$$\check{S} = \{X \in V^{\mathbb{C}} | X^* X = I\}.$$

Another important concept is transversality of pairs of points in $\check{S}$. We define transversality here in terms of Jordan algebras:

Definition 2.1.33. Let V be a Jordan algebra. Then $x, y \in V$ are transversal (denoted by $x \pitchfork y$) if

$$\det_V(x - y) \neq 0.$$

In Proposition A.2.3 in the Appendix we show the equivalence of various notions of transversality.

Lemma 2.1.34. *Transversality in the topological closure $\bar{T}_\Omega \subset V^{\mathbb{C}}$ is invariant under the action of the isometry group $G(T_\Omega)$. Two points in $\bar{T}_\Omega$ are transverse if and only if $c(x)$ and $c(y)$ are transversal in $c(\bar{T}_\Omega) \subset \bar{\mathcal{D}}$.*

Proof. Follows from the proof of Proposition A.2.3. □

Since transversality is defined for all V respectively $V^{\mathbb{C}}$, in particular it is defined for $x, y \in \bar{\mathcal{D}}$ respectively $x, y \in \bar{T}_\Omega$.

Example 2.1.35. The set $p(V) \subset \partial\mathcal{D}$ is the set of points in $\check{S} \subset \bar{\mathcal{D}}$ transversal to e.

We write

$$\check{S}^{(n)} := \{(z_1, \ldots, z_n) \in \check{S}^n \,|\, \forall i \neq j : \; z_i \pitchfork z_j\}$$

for the set of pairwise transverse n-tuples in $\check{S}$. Since the G-action preserves transversality, each $\check{S}^{(n)}$ is a union of G-orbits. For $n = 2$

we see from Proposition A.2.3 that $\check{S}^{(2)}$ is the unique G-orbit in $\check{S}^2$ of maximal dimension. This characterization can be used to identify $\check{S}^{(2)}$ in concrete examples.

Example 2.1.36. For example, if $G = \mathrm{Sp}(2n, \mathbb{R})$ the Shilov boundary $\check{S}$ is identified with the Lagrangian Grassmannian $L(\mathbb{R}^{2n})$ of $\mathbb{R}^{2n}$. Indeed the Lagrangian subspace spanned by the standard basis vectors $e_{n+1}, \ldots, e_{2n}$ is stabilized by the subgroup

$$\left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}) \right\}$$

and this is also the stabilizer of the point $0 \in V$. In both cases $\mathrm{Sp}(2n, \mathbb{R})$ acts transitively, hence one can identify $\check{S}$ and $L(\mathbb{R}^{2n})$. This identification give the following interpretation of the set $\check{S}^{(2)}$

$$L(\mathbb{R}^{2n})^{(2)} = \{(V, W) \in \mathcal{L}(\mathbb{R}^{2n})^2 \mid V \oplus W = \mathbb{R}^{2n}\},$$

since the right hand hand side is an open G-orbit.

Returning to the general case we recall that G-orbits in $\check{S}^{(3)}$ are classified by the generalized Maslov index $\mu_{\check{S}}$ of Clerc and Ørsted, see [25]. (For a complete classification of orbits in $\check{S}^3$ see [24].) Concerning G-orbits in $\check{S}^{(4)}$ we will confine ourselves with the following result. For the notion of Jordan frame used therein see [29, Ch. 4].

Let $\check{S}$ be the Shilov boundary of a tube type domain $\mathcal{D}$.

Proposition 2.1.37. *Let $(z_1, \ldots, z_4) \in \check{S}^{(4)}$, and suppose $\mu_{\check{S}}(z_i, z_j, z_k)$ is maximal for some $\{i, j, k\} \subset \{1, \ldots, 4\}$. Then $z_1, \ldots, z_4$ are contained in the boundary of a common maximal polydisc. More precisely, if $\mu_{\check{S}}(z_1, z_2, z_3)$ is maximal, then there exists $g \in G$ and a Jordan frame $(c_1, \ldots, c_r)$ such that*

$$g.(z_1, \ldots, z_4) = (\sum(-1) \cdot c_j, \sum(-i) \cdot c_j, \sum 1 \cdot c_j, \sum \lambda_j c_j).$$

Proof. Let $r := \mathrm{rk}(V)$. We may assume w.l.o.g. that $\mu_{\check{S}}(z_1, z_2, z_3)$ is maximal, i.e.

$$\mu_{\check{S}}(z_1, z_2, z_3) = r = \mu_{\check{S}}(-e, -ie, e).$$

Since the Maslov index classifies orbits of transverse triples we then find $g \in G$ with

$$g.(z_1, z_2, z_3) = (-e, -ie, e).$$

Let $z = g.z_4$. Then there exists a Jordan frame $(c_1, \ldots, c_r)$ and $\lambda_i \in \mathbb{C}$ with $|\lambda_i| = 1$ such that

$$z = \sum_{i=1}^{r} \lambda_i c_i,$$

see the proof of [29, Proposition X.2.3]. We deduce that

$$g.(z_1, \ldots, z_4) = (\sum (-1) \cdot c_j, \sum (-i) \cdot c_j, \sum 1 \cdot c_j, \sum \lambda_j c_j),$$

hence the quadruple $g.(z_1, \ldots, z_4)$ is contained in the Shilov boundary of the polydisc

$$\varphi_c : \mathbb{D}^r \to \mathcal{D}, \quad (\lambda_1, \ldots, \lambda_r) \mapsto \sum_{i=1}^{r} \lambda_i c_i$$

associated with the Jordan frame $c = (c_1, \ldots, c_r)$, and consequently $(z_1, \ldots, z_4)$ is contained in the Shilov boundary of the maximal polydisc $g^{-1} \circ \varphi_c$. $\square$

2.1.3 Boundary Morphisms

Let $\mathcal{D}$ be a bounded symmetric domain. Recall that we denote by $k_{\mathcal{D}}$ and $g_{\mathcal{D}}$ the Bergman kernel, respectively the Bergman metric of $\mathcal{D}$ and we write $G(\mathcal{D})$ for the group of biholomorphisms of $\mathcal{D}$. Recall that the action of $G(\mathcal{D})$ on $\mathcal{D}$ extends continuously to an action on $\check{S}_{\mathcal{D}}$. In the sequel we denote by $G_{\mathcal{D}}$ the identity component of $G(\mathcal{D})$ and by $\mathfrak{g}_{\mathcal{D}}$ its Lie algebra. We define $\hat{G}^{\mathbb{C}}_{\mathcal{D}}$ to be the simply-connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathcal{D}} \otimes \mathbb{C}$ and $\hat{G}$ to be the analytic subgroup of $\hat{G}^{\mathbb{C}}_{\mathcal{D}}$ with Lie algebra $\mathfrak{g}_{\mathcal{D}}$. We observe that $\hat{G}_{\mathcal{D}}$ is a finite covering of $G_{\mathcal{D}}$.

An affine holomorphic map between bounded symmetric domains will be referred to as a *morphism*; we recall that a holomorphic map is a morphism if and only if it commutes with the corresponding point involutions. Given any morphism $\beta : \mathcal{D}_1 \to \mathcal{D}_2$ of bounded symmetric domains, there exists a unique morphism $\widehat{\beta} : \hat{G}_{\mathcal{D}_1} \to \hat{G}_{\mathcal{D}_2}$ for which β is equivariant (see e.g. [2, Thm. V.1.9]); we refer to $\widehat{\beta}$ as the *equivariant lift* of β.

Definition 2.1.38. A morphism $\beta : \mathcal{D}_1 \to \mathcal{D}_2$ of bounded symmetric domains with respective Shilov boundaries $\check{S}_j := \check{S}_{\mathcal{D}_j}$ is called a *boundary morphism* if it admits a continuous extension $\bar{\beta} : \check{S}_1 \to \overline{\mathcal{D}_2}$ satisfying $\bar{\beta}(\check{S}_1) \subset \check{S}_2$.

Every bounded symmetric domain is isomorphic to the unit ball of a positive Hermitian Jordan triple system W with respect to the spectral norm (see e.g. [22]). If $\mathcal{D}$ is of tube type, then W can be chosen to be the complexification of a Euclidean Jordan algebra V, which we denote by $V^{\mathbb{C}} := V \otimes \mathbb{C}$; in this situation we write $\mathcal{D} = \mathcal{D}_V$ and abbreviate $G_V := G_{\mathcal{D}_V}$. We then define $K_V := \mathrm{stab}_0(G_V)$ and $Q_{\pm,V} := \mathrm{stab}_{\pm e}(G_V)$, where e_V denotes the unit element of the Jordan algebra V. We use the small gothic letters $\mathfrak{g}_V, \mathfrak{k}_V, \mathfrak{q}_{+,V}$ to denote the respective Lie algebras. The group K_V is a maximal compact subgroup of G_V and thus induces a Cartan decomposition $\mathfrak{g}_V = \mathfrak{k}_V \oplus \mathfrak{p}_V$, where $\mathfrak{p}_V$ is the Killing orthogonal complement of $\mathfrak{k}_V$ in $\mathfrak{g}_V$. In particular, $T_0\mathcal{D}_V \cong \mathfrak{p}_V$. The subgroups $Q_{\pm,V}$ are conjugate maximal parabolic subgroups of G_V. We refer to the parabolics in their conjugacy class as *Shilov parabolics*. Note that $Q_{+,V}$ and $Q_{-,V}$ share the same Levi factor $L(Q_{\pm,V}) = Q_{+,V} \cap Q_{-,V}$, which is the pointwise stabilizer of $\{\pm e_V\}$.

We will use the term *morphism* of Euclidean Jordan algebras as a shorthand for *unital* algebra homomorphism of Euclidean Jordan algebras. Every morphism $\alpha : V_1 \to V_2$ of Euclidean Jordan algebras in the above sense induces a morphism $\alpha^{\mathbb{C}} : V_1^{\mathbb{C}} \to V_2^{\mathbb{C}}$ which restricts to a morphism $\alpha^{\dagger} : \mathcal{D}_{V_1} \to \mathcal{D}_{V_2}$ of bounded symmetric domains. Every morphism of the form $\alpha^{\dagger}$ is automatically a boundary morphism; indeed this follows from the fact that the Shilov boundary $\check{S}_V := \check{S}_{\mathcal{D}_V}$ of $\mathcal{D}_V$ can be described as [29, Thm. X.4.6] (cf. Proposition 2.1.29)

$$\check{S}_V = \{z \in V^{\mathbb{C}} \mid z \text{ invertible}, z^{-1} = \bar{z}\}. \tag{2.3}$$

The following proposition implies that these are essentially the only boundary morphisms of bounded symmetric domains of tube type.

Proposition 2.1.39. *Let $\mathcal{D}_1, \mathcal{D}_2$ be bounded symmetric domains of tube type with respective Shilov boundaries $\check{S}_1$ and $\check{S}_2$, and $\beta : \mathcal{D}_1 \to \mathcal{D}_2$ be an injective morphism (i.e. affine holomorphic). Then the following are equivalent:*

(i) β is a boundary morphism.

(ii) There exist Euclidean Jordan algebras V_1, V_2, a unital Jordan algebra homomorphism $\alpha : V_1 \to V_2$ and isomorphisms $\mathcal{D}_j \cong \mathcal{D}_{V_j}$ intertwining β and $\alpha^{\dagger} : \mathcal{D}_{V_1} \to \mathcal{D}_{V_2}$.

(iii) β is a Kähler-tight map of Hermitian symmetric spaces.

(iv) The equivariant lift $\hat{\beta}$ of β is a tight homomorphism with respect to the Kähler class of $\hat{G}_{\mathcal{D}_2}$

The concepts of a tight map between symmetric spaces, respectively a tight homomorphism as referred to in the proposition are defined in [16], where the implications

$$\text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)}$$

are proved (see [16, Cor. 2.16 and Thm. 4.1]). As far as the implication (ii) $\Rightarrow$ (iii) is concerned, we learned the following argument from O. Guichard: We may assume $\mathcal{D}_j = \mathcal{D}_{V_j}$ and $\beta = \alpha^{\mathbb{C}}$ for some morphism $\alpha : V_1 \to V_2$ of Euclidean Jordan algebras. We then have embeddings of the Poincaré disc into $\mathcal{D}_j$ given by

$$\iota_j : \mathbb{D} \to \mathcal{D}_j, \quad \lambda \mapsto \lambda \cdot e_j,$$

where e_j is the unit element of V_j; these satisfy $\beta \circ \iota_1 = \iota_2$. Now the embeddings ι_1 and ι_2 are tight and positive; however, as proved in [16, Lemma 8.1], a morphism intertwining positive tight discs is itself tight. This implies (iii). Thus the only missing implication is (i) $\Rightarrow$ (ii); for this we provide a Jordan algebraic proof in the appendix (see Proposition A.2.1).

Remark 2.1.40. In the sequel we will often consider a fixed bounded symmetric domain $\mathcal{D}$. We then choose a Euclidean Jordan algebra V with $\mathcal{D}_V$ and denote by $G, K, Q_\pm, \Omega, T_\Omega, \check{S}$ respectively the objects denoted by $G_V, K_V, Q_{\pm,V}, \Omega_V, T_{\Omega_V}$ and $\check{S}_V$ in this section.

2.1.4 The Cayley transform and representations of Levi factors

To obtain a better understanding of the fine structure of G we observe that the Cayley transform $c : \mathcal{D}_V \to T_\Omega$ induces an isomorphism

$$\hat{c} : G \to G(T_\Omega)^0, \quad g \mapsto c \circ g \circ c^{-1}. \tag{2.4}$$

Denote by $\mathfrak{g}(T_\Omega)$ and $\mathfrak{g}(\Omega)$ the Lie algebras of $G(T_\Omega)$ and recall

$$G(\Omega) := \{g \in \mathrm{GL}(V) \mid g\Omega = \Omega\}.$$

and the decomposition $G(T_\Omega) = N^+ G(\Omega) G_{ie}$.

The Lie algebra $\mathfrak{g}(T_\Omega)$ admits a $\mathbb{Z}$-grading with $\mathfrak{g}(T_\Omega)_0 = \mathfrak{g}(\Omega)$, $\mathfrak{g}(T_\Omega)_{\pm 1} \cong V$ and $\mathfrak{g}(T_\Omega)_n = \{0\}$ for $|n| > 1$ (see e.g [49, Sec. 6]). The subgroups $N^\pm$ are the analytic subgroups of $G(T_\Omega)^0$ corresponding to $\mathfrak{g}(T_\Omega)_{\pm 1}$. Then $G(\Omega)$ normalizes $N^\pm$ and we can thus form the semidirect products $P^+ := N^+G(\Omega)$ and $P^- := N^-G(\Omega)$. (The reason for these sign conventions will become clear in Proposition 2.1.41.) It turns out that $P^\pm$ are maximal parabolic subgroups of $G(T_\Omega)^0$ and that P^- stabilizes $0 \in V$. Its unipotent radical is given by N^- and its Levi factor is given by $G(\Omega)$ (see [49, Sec. 7]). Now we have:

Proposition 2.1.41. *Let $\widehat{c} : G \to G(T_\Omega)^0$ be the isomorphism given by* (2.4)*. Then $\widehat{c}(Q_-) = P^-$ and $\widehat{c}(L(Q_-)) = G(\Omega)$.*

Proof. Since P^- stabilizes $0 \in V$ the group $\widehat{c}^{-1}(P^-)$ stabilizes $c^{-1}(0) = -e$.

Thus $\widehat{c}^{-1}(P^-) \subset Q_-$ is a subgroup, but being maximal parabolic itself we find $\widehat{c}^{-1}(P^-) = Q_-$. Passing to the corresponding Levi factors yields the second statement. □

For later reference we record the following consequences:

Corollary 2.1.42. *(i) The unipotent radical of a Shilov parabolic is abelian.*

(ii) The map $\widehat{c}$ provides a linear representation $\widehat{c} : L(Q_\pm) \to$ $\mathrm{GL}(V)$ for the Levi factor of the standard Shilov parabolics.

We will exploit the linear representation of $L(Q_\pm)$ in Section 5.2.1 below to estimate translation lengths.

2.1.5 $\mathrm{Sp}(2n, \mathbb{R})$

Let I_n be the $n \times n$-unit matrix and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The group $Sp(2n, \mathbb{R})$ consists of real $2n \times 2n$-matrices g with $g^\top J g = J$. It consists of four $n \times n$ matrices A, B, C and D such that

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The equation above give the following relations:

$$A^\top D - C^\top B = I, \quad A^\top C = C^\top A, \quad D^\top B = B^\top D. \tag{2.5}$$

A direct calculation shows that this set of equations is equivalent to the following:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^\top & -B^\top \\ -C^\top & A^\top \end{pmatrix}.$$

This is a generalisation to the formula for the inverse in $\mathrm{SL}(2, \mathbb{R})$.

The dimension of $\mathrm{Sp}(2n, \mathbb{R})$ over the real numbers is $2n^2 + n$

Let $V := \mathrm{Sym}_n(\mathbb{R})$ be the set of real symmetric $n \times n$-matrices. The product

$$x \cdot y := \frac{xy + yx}{2}$$

gives V the structure of a Jordan algebra (see Definition 2.1.10). It is a unital algebra with the unit matrix as the unit. In this context we will denote it by e. The set of positive definite matrices in V is an open cone and it is indeed the cone Ω. Let $V^{\mathbb{C}}$ be the complexification of V; it is a Jordan algebra as well and clearly $V^{\mathbb{C}} = \mathrm{Sym}_n(\mathbb{C})$.

By [29, Theorem X.1.1], we have $T_\Omega \subset D(p)$ and the set given by

$$\mathcal{D} := \{X \in V^{\mathbb{C}} |\; I_n - X^* X \text{ is positive definite}\}$$

is a bounded symmetric domain. Its topological closure is

$$\bar{\mathcal{D}} = \{X \in V^{\mathbb{C}} |\; I_n - X^* X \text{ is positive semi-definite}\}$$

The Shilov boundary of $\mathcal{D}$ is given by

$$\check{S} := \{X \in V^{\mathbb{C}} | \ X^*X = I_n\}.$$

It turns out that the group $G(T_\Omega)$ of biholomorphic transformations of T_Ω is $\mathrm{Sp}(2n, \mathbb{R})$.

The group $\mathrm{Sp}(2n, \mathbb{R})$ acts on T_Ω as follows:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \mapsto (AX + B)(CX + D)^{-1}.$$

Now we discuss the subgroups appearing in the decomposition of $G(T_\Omega)$ in Proposition 2.1.20.

First we calculate the stabilizer of $ie \in T_\Omega$.

$$\begin{aligned} & (A(ie) + B)(C(ie) + D)^{-1} = ie \\ \Leftrightarrow \ & iA + B = -C + iD \\ \Leftrightarrow \ & A = D, \ B = -C. \end{aligned}$$

The identification:

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB$$

identifies the stabilizer with $U(n)$.

Let

$$G(\Omega) := \{g \in GL(V) | g\Omega = \Omega\}.$$

It is contained in $G(T_\Omega)$, because it acts linearly with real matrices on $V^{\mathbb{C}}$. The group $G(\Omega)$ is isomorphic to $\mathrm{GL}(n, \mathbb{R})$ acting via

$$g : X \mapsto g^\top X g.$$

The action is transitive, because one can write any positive definite matrix g as $h^\top h$ (Cholesky decomposition). We have

$$G(\Omega) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix} \middle| A \in GL(n, \mathbb{R}) \right\}$$

Indeed, this follows directly of the definition of the action.

The vector space V itself acts by translation on T_Ω, hence it is also contained in $G(T_\Omega)$. We denote by the group of these transformations by N^+. Again by definition of the action we get:

$$N^+ = \left\{ \left(\begin{array}{cc} I_n & X \\ 0 & I_n \end{array} \right) \middle| X \in V \right\}$$

Lemma 2.1.43. *The subgroup of $G(T_\Omega)$ generated by $G(\Omega)$ and N^+ acts transitively on T_Ω.*

Proof. Let $x + iy \in T_\Omega$ and $g \in G(\Omega)$ such that $ge = y$. Then

$$x + iy = \left(\begin{array}{cc} I_n & x \\ 0 & I_n \end{array} \right) g(ie)$$

□

This shows that $T_\Omega = Sp(2n, \mathbb{R})/U(n)$.

The bounded symmetric domain $\mathcal{D}$ and the upper half plane T_Ω are biholomorphicaly equivalent under the Cayley transform.We can use this to calculate the action of $g \in Sp(2n, \mathbb{R})$ on $\mathcal{D}$:

$$\begin{aligned} p \circ g \circ c &= \left(\begin{array}{cc} I & -iI \\ I & iI \end{array} \right) \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \left(\begin{array}{cc} I & -iI \\ I & iI \end{array} \right) \\ &= \left(\begin{array}{cc} C - B + i(A + D) & C + B + i(A - D) \\ -C - D + i(A - D) & B - C + i(A + D) \end{array} \right) \end{aligned}$$

2.2 The Toledo Invariant

This section is concerned with the definition and properties of the Toledo invariant.

2.2.1 (Bounded) Cohomology

Later we will need methods from (bounded) group cohomology. Here we introduce two different definitions and shortly discuss the relation between them, because we have to combine results from different sources which are formulated in both pictures.

Let G be a group and $A = \mathbb{R}$, $\mathbb{C}$ or S^1. We define:

$$\begin{aligned} C^n(G,A) =& \{f : G^n \to A\} \\ C^n_b(G,A) =& \{f : G^n \to A | f \text{ bounded}\} \\ C^n_c(G,A) =& \{f : G^n \to A | f \text{ contiuous}\} \\ C^n_{cb}(G,A) =& \{f : G^n \to A | f \text{ bounded and continuous}\}. \end{aligned}$$

The Homogeneous Picture

Define the (usual) boundary operator $\delta_n : C^n_*(G,A) \to C^{n+1}_*(G,A)$:

$$(\delta_n f)(g_0, \dots, g_n) := \sum_{i=0}^{n} (-1)^i f(g_0, \dots, \hat{g}_i, \dots, g_n).$$

A direct calculation shows $\delta_{n+1} \circ \delta_n = 0$, hence $(C^n_*(G,A), \delta_n)$ is a complex. But its homology is zero. So we rather take the complex $(C^n_*(G,A)^G, \delta_n)$ of G-invariant functions, where now δ_n is the restriction of the boundary operator to $C^n(G,A)^G$. Define $B^n_*(G,A) := \operatorname{im} \delta_n \subset C^{n+1}_*(G,A)^G$ (space of homogeneous coboundaries) and $Z^n_*(G,A) := \ker \delta_{n+1} \subset C^{n+1}_*(G,A)^G$ (space of homogeneous cocycles) and

$$H^n_*(G,A) := Z^n_*(G,A)/B^n_*(G,A).$$

We call $H^n_b(G,A)$ the *bounded cohomology* of G and $H^n_{cb}(G,A)$ the *continuous bounded cohomology* of G.

The Inhomogeneous Picture

We can define the same cohomology theories using inhomogeneous cocycles. Define the boundary operator

$$(d_n f)(g_1, \ldots, g_{n+1}) = \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \ldots, g_n) + f(g_2, \ldots, g_{n+1})$$

Again this defines complexes $(C^n_*(G,A), d_n)$ with $B^n_*(G,A) := \operatorname{im} \delta_{n-1} \subset C^n_*(G,A)^G$ (space of inhomogeneous coboundaries) and $Z^n_*(G,A) := \ker \delta_n \subset C^n_*(G,A)^G$ (space of homogeneous cocycles)

$$H^n_*(G,A) := Z^n_*(G,A)/B^n_*(G,A).$$

Connection between these pictures

The definitions for H^n_* are equivalent. Here we present maps which maps homogeneous to inhomogeneous cocycles representing the same element in H^n and vice versa. Let f be an inhomogeneous n-cocycle, i.e. $\delta f = 0$, then

$$\tilde{f}(g_0, g_1, \ldots, g_n) := f(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n)$$

is an homogeneous cocycle. Its inverse is provided by

$$\bar{h}(g_1, \ldots, g_n) := h(e, g_1, g_1 g_2, g_1 g_2 g_3, \ldots, g_1 g_2 g_3 \cdots g_n),$$

where h is a homogeneous cocycle.

2.2.2 The Kähler cocycle and the Kähler class

We will now define the Kähler cocycle, the Maslov index and the Souriau index which we will need to define the Toledo invariant.

Let $\mathcal{D}$ be a bounded symmetric domain of rank r and G the unit component of its group of isometries. Equipped with the unnormalized Bergmann metric it has minimal holomorphic sectional curvature $-2/p$, where p is a constant (see [26, p. 273] resp. [28, III, prop. V.3.7]). We follow [26] and scale the metric with the constant $p/2$ to get a metric g with minimal sectional curvature -1. From now on we consider $\mathcal{D}$ equipped with this normalized metric and define the associated kernel function

$$k(z,w) := k_{\mathcal{D}}(z,w)^{\frac{2}{p}}.$$

(cf. [26, Ch.1]). Recall the Kähler form

$$\omega(X,Y) := g(JX,Y),$$

where g is the metric associated with ω.

Definition 2.2.1. We define the *Kähler cocycle.* Let z_1, z_2, z_3 in $\mathcal{D}$.

$$c(z_1,z_2,z_3) := \int_{T(z_1,z_2,z_3)} \omega.$$

where $T(x,y,z)$ is a triangle with geodesic sides spanned by x, y and z.

By Stoke's theorem c is independent of the triangle chosen.

This defines a map from $\mathcal{D}^3$ to $\mathbb{R}$ with the following properties

Proposition 2.2.2. *(i) c is G-invariant.*

(ii) $c(z_1,z_2,z_3) + c(z_1,z_3,z_4) = c(z_2,z_3,z_4) + c(z_1,z_2,z_4)$ (Cocycle property).

(iii) $|c| < r\pi$.

(iv) $\sup |\int_\Delta \omega| = r\pi$, where the supremum runs over all geodesic triangles Δ in $\mathcal{D}$.

Proof. The first properties comes from the G-invariance of ω and the second from Stokes theorem. The third property is Theorem 3.1 in [26] and the fourth follows from the remark thereafter. $\square$

For $g_1, g_2, g_3 \in G$ we define

$$c_G(g_1, g_2, g_3) := \frac{1}{2\pi} c(g_1 0, g_2 0, g_3 0),$$

which is a G-invariant homogeneous cocycle on G,

Definition 2.2.3. This defines cohomology classes $\kappa_G \in H^2_c(G, \mathbb{R})$ resp. $\kappa^b_G \in H^2_{cb}(G, \mathbb{R})$, the *Kähler class* resp. the *bounded Kähler class.*

2.2.3 Definition of the Toledo Invariant

For the most general definition of the Toledo invariant we follow [18].

The definition is easier for closed surfaces, so we begin with this case:
Let Σ be an orientable surface without boundary, Γ its fundamental group and G a group. Then $H_2(\Sigma, \mathbb{Z}) \simeq \mathbb{R}$. A generator of $H_2(\Sigma, \mathbb{R})$ is called *fundamental class*, it is denoted by $[\Sigma]$.

Let $\kappa \in H^2(G, \mathbb{R})$ and $\varrho : \Gamma \to G$ be a homomorphism. The pullback $\varrho^*(\kappa) \in H^2(\Gamma, \mathbb{R})$ can be considered as an element of $H^2(\Sigma, \mathbb{R})$. Therefore we can apply the natural pairing $H^2(\Sigma, \mathbb{R}) \times H_2(\Sigma, \mathbb{R}) \to \mathbb{R}$ and define:

$$T_{\varrho,\kappa} := \langle \varrho^*(\kappa), [\Sigma] \rangle \in \mathbb{R}.$$

For surfaces with boundary we have $H^2(\Sigma, \mathbb{R}) = 0$. But $H^2(\Sigma, \partial\Sigma, \mathbb{R}) \simeq \mathbb{R}$. Therefore we need a relative class which we can pair with $[\Sigma, \partial\Sigma]$. We have an element $\varrho^*(\kappa) \in H^2(\Gamma, \mathbb{R})$ and we need one in $H^2(\Sigma, \partial\Sigma, \mathbb{R})$. There is no way of doing this assignment in ordinary cohomology, because $H^2(\Sigma, \mathbb{R}) = 0$. We need bounded cohomology. Now let

$\kappa^b \in H_b^2(G, \mathbb{R})$. Its image under

$$H_{cb}^2(G, \mathbb{R}) \to H_b^2(\Gamma, \mathbb{R}) \simeq H_b^2(\Sigma, \mathbb{R}) \simeq H^2(\Sigma, \partial\Sigma, \mathbb{R})$$

can be paired with $[\Sigma, \partial\Sigma]$ to obtain a real number T_{ϱ,κ^b}.

Definition 2.2.4. Let $\varrho : \Gamma \to G$ be a representation. Then $T_\varrho := T_{\varrho,\kappa^b_G}$ is the *Toledo invariant* of ϱ.

Remark 2.2.5. It is important to notice that the Toledo invariant does not only depend on the fundamental group but also on the underlying surface. For example the surfaces $\Sigma_{0,3}$ and $\Sigma_{1,1}$ have fundamental groups

$$\Gamma_{0,3} = \langle C_1, C_2, C_3 | C_3C_2C_1 = e\rangle, \quad \Gamma_{1,1} = \langle A, B, C | [A, B]C = e\rangle$$

which are isomorphic as abstract groups, because both are free of rank 2. But clearly the Toledo invariant depends on the underlying surface.

Later we will need another form of the Toledo invariant which is also due to Burger, Iozzi and Wienhard [18, Ch.7] expressing the Toledo invariant in terms of a generalized rotation numbers.

Let $B \subset G$ be a closed subgroup. Consider the long exact sequence induced by the coefficient sequence:

$$0 \longrightarrow \mathrm{Hom}_c(B, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} \hat{H}_{cb}^2(B, \mathbb{Z}) \longrightarrow \hat{H}_{cb}^2(B, \mathbb{R}) \simeq H_{cb}^2(B, \mathbb{R}) \longrightarrow \dots, \tag{2.6}$$

where $\hat{H}^\bullet$ is the Borel cohomology as defined in [18, Sec. 2.3].

For $\kappa \in \hat{H}_{cb}^2(B, \mathbb{Z})$, we denote its image in $\hat{H}_{cb}^2(B, \mathbb{R})$ by $\kappa_\mathbb{R}$. If $\kappa_R = 0$, we have $\kappa = \delta(f_B)$ and f_B is unique. Note that for amenable groups B (e.g. compact or abelian groups) $H_{cb}^2(B, \mathbb{R}) = 0$. Therefore the following definition makes sense.

Definition 2.2.6. Let $\kappa \in H_{cb}^2(G, \mathbb{Z})$. Then the map

$$\mathrm{Rot}_\kappa : \begin{cases} G \to \mathbb{R}/\mathbb{Z} \\ g \mapsto f_{\overline{\langle g\rangle}}(g) \end{cases}$$

is called the *rotation number* of G.

From the definition there are some immediate consequences ([18, Lemma 7.2]):

Proposition 2.2.7. *(i) The rotation number is conjugation invariant,*

(ii) if $\kappa_{\mathbb{R}}|_B = 0$, then the rotation number is continuous homomorphism and $\delta(\mathrm{Rot}_\kappa|_B) = \kappa_B$,

(iii) if $\sigma : G_1 \to G_2$ is a continuous homomorphism and $\kappa_1 = \sigma^*(\kappa_2)$ then
$$\mathrm{Rot}_{\kappa_1}(g_1) = \mathrm{Rot}_{\kappa_2}(\sigma(g_1)).$$

Let now $\widetilde{\mathrm{Rot}}_\kappa : \tilde{G} \to \mathbb{R}$ be the unique lift of Rot_κ with $\widetilde{\mathrm{Rot}}_\kappa(e) = 0$. Then

Theorem 2.2.8. *[18, Thm. 12] Let $\kappa \in H^2_{cb}(G, \mathbb{Z})$. Let $m \geq 1$ and $\varrho : \Gamma_{g,m} \to G$ a representation. Choose a lift $\tilde{\varrho} : \Gamma_{g,m} \to \tilde{G}$. Then*

$$T_{\varrho,\kappa} = -\sum_{i=1}^{m} \widetilde{\mathrm{Rot}}_\kappa(\tilde{\varrho}(C_i)).$$

To state a the similar theorem for the closed surfaces we need the commutator map. Recall that $\tilde{G}$ is a central extension of G. Then the commutator map $G \times G \to \tilde{G}$ is defined

$$[g, h]^\sim := [\tilde{g}, \tilde{h}],$$

where $\tilde{g}$ and $\tilde{h}$ are arbitrary lifts.

Theorem 2.2.9. *(Theorem 8.3 in [18]) Let $\kappa \in H^2_c(G, \mathbb{Z})$ and $\varrho : \Gamma_g \to G$ be a representation. Then*

$$T_{\varrho,\kappa} = -\widetilde{\mathrm{Rot}}_\kappa \left(\prod_{j=1}^{g} [\varrho(A_i), \varrho(B_i)]^\sim \right).$$

In Section 2.3.4 we will express the (generalized) rotation number in terms of the Souriau index which enables a computation of the Toledo invariant using the $\varrho(\Gamma)$-action on $\check{S}$ rather than on its universal covering.

2.2.4 Properties of the Toledo Invariant and Maximal Representations

We collect here some important properties of the Toledo invariant.

Theorem 2.2.10. *([18, Thm 1, Prop. 3.2])*

(i) $|T_\varrho| \leq |\chi(\Sigma)| \operatorname{rk} \mathcal{X}$ *(Milnor-Wood inequality)*

(ii) $T_\bullet$ *is continuous*

(iii) If $\partial\Sigma = \emptyset$*, then the image of* $T_\bullet$ *is finite*

(iv) If $\partial\Sigma \neq \emptyset$*, then* $T_\bullet$ *is surjective on the interval*

$$\big[-|\chi(\Sigma)| \operatorname{rk} \mathcal{X}, |\chi(\Sigma)| \operatorname{rk} \mathcal{X}\big]$$

(v) Let Σ *be a surface divided by a separating loop* l *into two subsurfaces* Σ_1 *and* Σ_2*. Denote by* ϱ_1 *and* ϱ_2 *the restrictions of* ϱ *to* Σ_1 *resp.* Σ_2*. Then*

$$T_\varrho = T_{\varrho_1} + T_{\varrho_2}$$

(vi) Let Σ' *be a surface obtained by cutting a surface* Σ *along a non-separating loop. Let* $i : \Sigma' \to \Sigma$ *be the canonical map. Then*

$$T_{i^*\varrho} = T_\varrho.$$

Definition 2.2.11. A representation $\varrho : \pi_1(\Sigma) \to G$ with maximal Toledo invariant, i.e.

$$T_\varrho = |\chi(\Sigma)| \operatorname{rk} \mathcal{X}$$

is called *maximal*. We denote the set of maximal representations of Γ into G by $\operatorname{Rep}_{max}(\Gamma, G)$.

There are some ways to construct maximal representations:

Example 2.2.12. A representation $\varrho : \Gamma_{g,m} \to \mathrm{PSL}(2, \mathbb{R})$ is a hyperbolization for $\Sigma_{g,m}$ if and only it is maximal.

Example 2.2.13. Let $\varrho_i : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ be a family of maximal representations. Then $(\varrho_1, \ldots, \varrho_r) : \Gamma \to \mathrm{PSL}(2, \mathbb{R})^r$ is maximal.

Example 2.2.14. Let $\varrho : \Gamma \to G$ a maximal representation and $i : G \to H$ be tight (see [16] for a definition and properties). Then $i \circ \varrho$ is maximal as well.

Here is the structure theorem for maximal representations due to Burger, Iozzi and Wienhard:

Theorem 2.2.15. *([18, Thm.5]) Let $\mathbf{G}$ be a connected semisimple algebraic group defined over $\mathbb{R}$ such that $G = \mathbf{G}(\mathbb{R})^\circ$ is Hermitian. Let $\varrho : \Gamma \to G$ be a maximal representation Then*

(i) *ϱ is injective with discrete image,*

(ii) *the Zariski closure $\mathbf{H} < \mathbf{G}$ of the image of ϱ is reductive,*

(iii) *the reductive Lie group $H := \mathbf{H}(\mathbb{R})^\circ$ has compact centralizer in G and the symmetric space $\mathcal{Y}$ associated to H is Hermitian of tube type,*

(iv) *$\varrho(\Gamma)$ stabilizes a maximal tube type subdomain $\mathcal{T} \subset \mathcal{Y}$.*

Maximal representation are characterized by the existence of an equivariant boundary map with a certain monotonicity property:

Definition 2.2.16. A triple in $\mathbb{D}$ is called maximal if it positively oriented. A triple in $\mathbb{D}^r$ is maximal if all its components are maximal. A triple in $\check{S}$ is *maximal* if it is contained in the boundary of a maximal polydisc and maximal there. A map $S^1 \to \check{S}$ is *monotone* if it maps maximal triples to maximal triples.

Remark 2.2.17. In Section 2.3.1 we introduce the Maslov index β, which is a skew-symmetric, G-invariant map from $\check{S}^3$ to $\mathbb{R}$. It classifies orbits of transversal triples in $\check{S}^3$. A triple in $\check{S}$ is maximal as defined above if and only if its Maslov index is maximal.

For the following theorem fix a hyperbolization h of the surface $\Sigma_{g,m}$. It yields an action of $\Gamma_{g,m}$ on S^1.

Proposition 2.2.18. *A representation $\varrho : \Gamma_{g,m} \to G$ is maximal if and only if there exists a map $\varphi : S^1 \to \check{S}$ with the following properites:*

(i) φ maps transverse pairs to transverse pairs,

(ii) φ is ϱ-equivariant,

(iii) φ is monotone,

(iv) φ is left-continuous.

This is Theorem 8 in [18].

Remark 2.2.19. For closed surfaces the limit curve is continuous (see [13, 15]). We will show continuity for a class of representations for surfaces with boundary (Section 4.1).

2.3 Maslov Index and Rotation Numbers

2.3.1 The Maslov Index

Originally the Maslov index was defined for transverse triples of Lagrangian subspaces. Clerc and Ørstedt extended the definition from the space of Lagrangians to the Shilov boundary $\check{S}$ of a bounded symmetric domain of tube type. In [25] they defined it for triples of pairwise transverse points in $\check{S}$ using Jordan algebra techniques and show that is classifies orbits of such triples. Clerc extended this definition to arbitrary triples in the Shilov boundary of bounded symmetric domains of tube type ([20]) and later also to the non tube type case ([21]).

Since we only need the Maslov index for tube type domains, we will restrict ourselves to this case. Before recalling the definition for general triples from [20] we first present the definition of the Maslov index for triples of pairwise transverse points in $\check{S}$ following [25]. Fix a Jordan frame $\{c_i\}$. Let

$$\varepsilon_k := \sum_{i=1}^{k} c_i - \sum_{i=k+1}^{r} c_i \in \check{S}.$$

Proposition 2.3.1. *([25, Thm. 4.3]) There are $r+1$ orbits of pairwise transverse triples in S^3. Each ε_j represents one orbit.*

Now we define

Definition 2.3.2. Let (x_1, x_2, x_3) a triple of pairwise transverse points in $\check{S}$. The *Maslov index* β is

$$\beta(x_1, x_2, x_3) = 2k - r,$$

if (x_1, x_2, x_3) is in the orbit of $(-e, -i\varepsilon_k, e)$.

We will now sketch the definition of the Maslov index for general triples, which is due to Clerc in [20]. Let c be the Kähler cocycle defined in Section 2.2.2. We want to extend c to $\check{S}^3$ representing points in $\check{S}$ as endpoints of certain curves in $\mathcal{D}$.

Definition 2.3.3. Let $x \in \check{S}$. Let $\gamma : [0,1] \to \bar{\mathcal{D}}$ be a C^1- curve with $\gamma(0) = x$ and $\gamma(t) \in \mathcal{D}$ for $t \in (0,1]$. Then γ is a Γ-radial curve in x, if there exists g mapping $\mathcal{D}$ isometrically to T_Ω and $\gamma(0)$ to 0 such that

$$\frac{d}{dt}(g\gamma(t))|_0 \in i\Omega.$$

Theorem 2.3.4. *([20, Thm. 3.3]) Let* $(x_1, x_2, x_3) \in \check{S}^3$ *and* γ_1, γ_2 *and* γ_3 Γ*-radial curves in* x_1, x_2 *and* x_3 *respectively. Then*

$$\beta(x_1, x_2, x_3) := \lim_{t\to 0} \frac{1}{\pi} c(\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

exists. It is independent of the chosen Γ*-radial curves and it takes its values in* $[-r, r]$*. If* z_1, z_2 *and* z_3 *are pairwise transversal, then* β *defined here coincides with the Maslov index defined above.*

Definition 2.3.5. The function β defined above is the *Maslov index.* A triple $(x_1, x_2, x_3) \in \check{S}^3$ with $\beta(x_1, x_2, x_3) = r$ is *maximal*; it is *minimal* if $\beta(x_1, x_2, x_3) = -r$.

Proposition 2.3.6. *([20, Thm 3.5]) The Maslov index* β *has the following properties:*

(i) β *is* G*-invariant.*

(ii) $\beta(x_1, x_2, x_3) + \beta(x_1, x_3, x_4) = \beta(x_2, x_3, x_4) + \beta(x_1, x_2, x_4)$ *(Cocycle property).*

(iii) β *is skew-symmetric.*

(iv) On the set of transverse triples β *takes values in* $\{-r, -r+2, \ldots, r-2, r\}$ *and it classifies* G*-orbits of transverse triples.*

Remark 2.3.7. The Maslov index $\beta(x_1, x_2, x_3)$ can be seen as the area of an ideal triangle with vertices x_1, x_2 and x_3.

Example 2.3.8. For the unit disc $\mathbb{D}$ the Maslov index becomes particularly easy; it coincides with the orientation cocycle o. Let $(x_1, x_2, x_3) \in (S^1)^3$. Then:

$$o(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } (x_1, x_2, x_3) \text{ is positively oriented} \\ -1 & \text{if } (x_1, x_2, x_3) \text{ is negatively oriented} \\ 0 & \text{else.} \end{cases}$$

Example 2.3.9. Let $\mathcal{D} = \mathcal{P} = \mathbb{D}^r$. Then the Shilov boundary is equal to $(S^1)^r$ and we can define the Maslov index for a triple $x_1 = (\lambda_1, \cdots, \lambda_r)$, $x_2 = (\mu_1, \dots, \mu_r)$ and $x_3 = (\sigma_1, \dots, \sigma_r)$:

$$\beta(x_1, x_2, x_3) = \sum o(\lambda_i, \mu_i, \sigma_i).$$

Let $\{c_i\}$ be a Jordan frame in a bounded symmetric domain $\mathcal{D}$ associated with the Jordan algebra V. If

$$x_1 = \sum \lambda_i c_i, \quad x_2 = \sum \mu_i c_i, \quad x_3 = \sum \sigma_i c_i,$$

with $\lambda_i, \mu_i, \sigma_i \in S^1$, then

$$\beta(x_1, x_2, x_3) = \sum o(\lambda_i, \mu_i, \sigma_i).$$

Example 2.3.10. We denote by $\mathrm{Sp}(W)$ the group of linear transformations of W which leave ω invariant. We define the Maslov index for triples of Lagrangians. Let L_1, L_2, L_3 be Lagrangian subspaces. Then

$$q : \begin{cases} L_1 \oplus L_2 \oplus L_3 \to \mathbb{R} \\ (x_1, x_2, x_3) \mapsto \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1) \end{cases}$$

is a quadratic form on the vector space $L_1 \oplus L_2 \oplus L_3$. The Maslov index of (L_1, L_2, L_3) is defined to be the signature of this quadratic form. It is an integer which takes values between $-n$ and n. For triples of pairwise transverse triples, it only takes values $\{-n, -n+2, \dots, n-2, n\}$ and it classifies the $\mathrm{Sp}(W)$-orbits of transverse triples of Lagrangian subspaces. Furthermore it satisfies the cocycle property:

$$\beta(L_1, L_2, L_3) - \beta(L_1, L_2, L_4) + \beta(L_1, L_3, L_4) - \beta(L_2, L_3, L_4) = 0.$$

The space of Lagrangians of W forms a boundary of the Siegel upper half plane (cf. Example 2.1.36), which is a Hermitian symmetric space of tube type.

Example 2.3.11. For several interesting configuration of (x_1, x_2, x_3), the Maslov index has an explicit interpretation for $G = \mathrm{Sp}(2n, \mathbb{R})$. Recall that the Shilov boundary of the associated bounded symmetric domain has an open dense subset, which is mapped to V under the Cayley transform (see Section 2.1.1). For $G = \mathrm{Sp}(2n, \mathbb{R})$ the Jordan algebra V is precisely the set of symmetric real $n \times n$-matrices. Given a triple $(x_1, x_2, x_3) \in V^3$, assume that x_1 and x_3 as well as x_2 and x_3 are transversal . Since G acts transitively on transverse pairs there exists $\tilde{g} \in G$ mapping the ordered pair (x_1, x_3) to $(0, \infty)$. Furthermore, since $\tilde{g}x_2$ is a symmetric, it is diagonalisable via conjugating with an element k from $\mathrm{O}(n)$. But this conjugation can be expressed in terms of the action of $G(\Omega)$ on V and we can assume gx_2 is diagonal (see Section 2.1.5). Since the Maslov index is G-invariant, we get

$$\beta(x_1, x_2, x_3) = \beta(0, g.x_2, \infty).$$

But since 0, gx_2 and ∞ are codiagonal, they are in the same polydisc (see Example 2.1.6). In particular the Maslov index is the sum of the Maslov index of the orientation cocycles in the factors of the polydisc. This shows that the Maslov index $\beta(0, g.x_2, \infty)$ is the signature of the symmetric bilinear form defined by gx_2.

In particular β can be calculated using the signs of the eigenvalues of $g \cdot x_2$ and it takes values between $-n$ and n.

Remark 2.3.12. Definition 2.3.3 shows that the Maslov index is also defined for triples in V. For $V = \mathbb{R}$, the Maslov index $\beta_{\mathbb{R}}$ is the unique antisymmetric map with $\beta_{\mathbb{R}}(x_1, x_2, x_3) = 1$ if $x_1 < x_2 < x_3$ and $\beta_{\mathbb{R}}(x_1, x_2, x_3) = 0$ if $x_i = x_j$ for $i \neq j$.

Now let V be an arbitrary Euclidean Jordan algebra. If $x_1, x_2, x_3 \in V$, then there exists $g \in G(T_\Omega)$ and a Jordan frame $\{c_1, \ldots, c_r\}$ such that

$$gx_1 = \sum \lambda_i c_i, \quad gx_2 = \sum \mu_i c_i, \quad gx_3 = \sum \sigma_i c_i,$$

with $\lambda_i, \mu_i, \sigma_i \in \mathbb{R}$. Then

$$\beta(x_1, x_2, x_3) = \sum \beta(\lambda_i, \mu_i, \sigma_i).$$

2.3.2 The Kähler Class and the Maslov Index

Let c be the Kähler cocycle on $\mathcal{D}$. We define:

$$c_G(g_1, g_2, g_3) := \frac{1}{2\pi} c(g_1 0, g_2 0, g_3 0),$$

where $0 \in \mathcal{D}$ is an arbitrary base point. We use the same factor $\frac{1}{2\pi}$ as in [14, Ch. 4]. From the properties of c we see that c_G is a homogeneous cocycle for G. It defines a cohomology class κ_G, the *Kähler class* and a bounded cohomology class $\kappa_G^b \in H_{cb}^2(G, \mathbb{R})$.

The Maslov index defines a cohomology class κ_β via the homogeneous cocycle on G defined by:

$$c_{\beta,b}(g_1, g_2, g_3) := \beta(g_1 b, g_2 b, g_3 b).$$

Note that $c_{\beta,b}$ and $c_{\beta,b'}$ are maybe not equal, but they are cohomologous ([14, Prop 4.3]).

Proposition 2.3.13. *([14, Prop.4.3])*

$$\kappa_\beta = 2\kappa_G.$$

2.3.3 The Souriau Index

Let G be a Hermitian Lie group of tube type, $\mathcal{D}$ its associated bounded symmetric domain and $\check{S}$ its Shilov boundary, realized as a subset of the Jordan algebra $V^{\mathbb{C}}$. The universal covering $\check{S}$, denoted by $\check{R}$, is given by

$$\check{R} = \{(\sigma, \theta) | \sigma \in \check{S}, \theta \in \mathbb{R}, \det{}_{V^{\mathbb{C}}} \sigma = e^{ir\theta}\},$$

where $r = \operatorname{rk} G$ ([23, Thm 3.5]). Denote by $\tilde{G}$ the universal cover of G.

Definition 2.3.14. Let $\tilde{\sigma}_1 = (\sigma_1, \theta_1), \tilde{\sigma}_2 = (\sigma_2, \theta_2) \in \check{R}$. They are *transversal* ($\tilde{\sigma}_1 \pitchfork \tilde{\sigma}_2$) if σ_1 and σ_2 are transversal.

Let $\tilde{\sigma}_1 \pitchfork \tilde{\sigma}_2$ and $\tilde{g} \in \tilde{G}$ such that

$$\tilde{g}\tilde{\sigma}_1 = \left(\sum e^{i\theta_j} c_j, \theta\right), \quad \tilde{g}\tilde{\sigma}_2 = \left(\sum e^{i\varphi_j} c_j, \varphi\right).$$

Note that transversality is equivalent to $\sigma_i \neq \theta_i$ for all i. Define the *Souriau index* for transversal points:

$$m(\tilde{\sigma}_1, \tilde{\sigma}_2) = \frac{1}{\pi}\left[\sum \{\theta_j - \varphi_j + \pi\} - r(\theta - \varphi)\right],$$

where $\{x\}$ is the unique representative of $[x] \mod 2\pi$ in $(-\pi, \pi)$. If $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are not transversal then define

$$m(\tilde{\sigma}_1, \tilde{\sigma}_2) := \beta(\sigma_1, \sigma_2, \sigma_3) + m(\tilde{\sigma}_1, \tilde{\sigma}_3) + m(\tilde{\sigma}_3, \tilde{\sigma}_2),$$

where $\tilde{\sigma}_3 := (\sigma_3, \theta_3) \in \check{R}$ is transversal to $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ and β is the Maslov index defined above.

The following proposition collects properties of the Souriau index ([23, Prop. 5.3 and 5.4]):

Proposition 2.3.15. *The Souriau index is skew-symmetric, i.e.* $m(\tilde{\sigma}_1, \tilde{\sigma}_2) = -m(\tilde{\sigma}_2, \tilde{\sigma}_1)$ *and* $\tilde{G}$*-invariant.*

Remark 2.3.16. By [23, Thm. 6.1] we have the following relation between the Souriau-index and the Maslov index on the Shilov boundary: let $a, b, c \in \check{S}$ and $\tilde{a}, \tilde{b}, \tilde{c} \in \check{R}$ be arbitrary lifts. Then

$$\beta(a, b, c) = m(\tilde{a}, \tilde{b}) + m(\tilde{b}, \tilde{c}) + m(\tilde{c}, \tilde{a}). \tag{2.7}$$

Proposition 2.3.17. *Let* $\tilde{x}_1, \tilde{x}_2 \in \check{R}$ *two lifts of* $x \in \check{S}$ *and* $\tilde{y} \in \check{R}$. *Then*

$$m(\tilde{x}_1, \tilde{x}_2) = m(\tilde{x}_1, \tilde{y}) + m(\tilde{y}, \tilde{x}_2).$$

Proof. We use Formula 2.7. We have:

$$0 = \beta(x, x, y) = m(\tilde{x}_1, \tilde{x}_2) + m(\tilde{x}_2, \tilde{y}) + m(\tilde{y}, \tilde{x}_1)$$

and the statement follows from the skew-symmetry of m. $\square$

Immediate consequences from Proposition 2.3.17 and the G-invariance are:

Lemma 2.3.18. *Let $g \in G$ and $x \in \check{S}$ fixed by g. Let $\tilde{g} \in \tilde{G}$ and $\tilde{x} \in \check{R}$ be lifts. Then $m(\tilde{g}^n \tilde{x}, \tilde{x}) = n \cdot m(\tilde{g}\tilde{x}, \tilde{x})$.*

Lemma 2.3.19. *Let $x \in \check{S}$ and $\tilde{x} \in \check{R}$ a lift. Let $H < \tilde{G}$ be the lift of the stabilizer of x in G. Then*

$$m(\ \cdot \ \tilde{x}, \tilde{x}) : H \to \mathbb{R}$$

is a homogeneous quasimorphism.

Lemma 2.3.20. *Let $g \in G$, $y \in \check{S}$ a fixed point of g and $x \in \check{S}$ an arbitrary point. Let $\tilde{x}$ and $\tilde{y}$ be arbitrary lifts of x and y and $\tilde{g}$ a lift of g which fixes $\tilde{y} \in \check{R}$.*

Then

$$\beta(y, gx, x) = m(\tilde{g}\tilde{x}, \tilde{x}).$$

Proof. By Formula (2.7) and the assumptions we have

$$\beta(y, gx, x) = m(\tilde{y}, \tilde{g} \cdot \tilde{x}) + m(\tilde{g} \cdot \tilde{x}, \tilde{x}) + m(\tilde{x}, \tilde{y}).$$

By $\tilde{G}$-invariance and skew-symmetry we have $m(\tilde{y}, \tilde{g}\tilde{x}) + m(\tilde{x}, \tilde{y}) = 0$. □

Remark 2.3.21. Let $\sigma_1, \sigma_2 \in \check{S}$ transversal. Let $k \in K$ such that

$$k\sigma_1 = \sum e^{i\theta_j} c_j, \quad k\sigma_2 = \sum e^{i\varphi_j} c_j,$$

for some Jordan frame $\{c_j\}$. As in [23, Ch.5] we define

$$\Psi(\sigma_1, \sigma_2) := \sum \{\theta_j - \varphi_j + \pi\}.$$

If σ_1 and σ_2 are not transversal, there exists $\sigma_3 \in \check{S}$ transversal to both of them and we define

$$\Psi(\sigma_1, \sigma_2) := \pi\beta(\sigma_1, \sigma_2, \sigma_3) + \Psi(\sigma_3, \sigma_2) + \Psi(\sigma_1, \sigma_3).$$

In particular

$$\beta(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{\pi}\big[\Psi(\sigma_1, \sigma_2) + \Psi(\sigma_2, \sigma_3) + \Psi(\sigma_3, \sigma_2)\big]. \tag{2.8}$$

Ψ is invariant under K and skew-symmetric ([23, Prop.5.4]).

An important property of Ψ is [23, Formula (16)]:

$$e^{2i\Psi(\sigma,\tau)} = (\det \sigma)^2 (\det \tau)^{-2},$$

where det is the Jordan algebra determinant.

Proposition 2.3.22. *Fix $b \in \check{S}$. Then the map*

$$f : \begin{cases} K \to \mathbb{R}/\mathbb{Z} \\ k \mapsto \big[\frac{1}{\pi}\Psi(b, kb)\big] \end{cases}$$

is a homomorphism. It does not depend on b.

Proof. Let $k_1, k_2 \in K$. Then

$$e^{2i\Psi(b,k_1k_2b)} = (\det b)^2(\det k_1k_2b)^{-2} = \chi(k_1k_2)^{-2}(\det b)^2(\det b)^{-2}$$
$$=\chi(k_1)^{-2}(\det b)^2(\det b)^{-2}\chi(k_2)^{-2}(\det b)^2(\det b)^{-2} = e^{2i\big(\Psi(b,k_1b)+\Psi(b,k_2b)\big)},$$

where χ is the character on $\mathrm{Str}(V^{\mathbb{C}})$ introduced in Definition 2.1.24. Therefore f is a homomorphism.
Now we show independence of b: let $b, b' \in \check{S}$. Then there exists $l \in K$ such that $b' = lb$. Then $\Psi(b', kb') = \Psi(b, l^{-1}klb)$ for all $k \in K$. Since f is a homomorphism into the abelian group $\mathbb{R}/\mathbb{Z}$, $\Psi(b, l^{-1}klb) = \Psi(b, kb)$, for all k. Hence f does not depend on b. $\square$

2.3.4 The Rotation Number and the Souriau Index

Recall $\kappa_\beta \in H^(G, \mathbb{Z})$, defined in Section 2.3.2. We will express the rotation number $\mathrm{Rot}_{\kappa_\beta}$ in terms of the map Ψ defined in Section 2.3.3.

Proposition 2.3.23. *Fix $b \in \check{S}$. Given $g \in G$, let $g = g_e g_h g_u$ its refined Jordan decomposition. Let $k \in C(g_e) \cap K$, where $C(g_e)$ is the conjugacy class of g_e. Then*

$$\mathrm{Rot}_{\kappa_\beta}(g) = \left[\frac{1}{\pi}\Psi(b, kb)\right] \in \mathbb{R}/\mathbb{Z}.$$

For the refined Jordan decomposition see [44] or [4].

Proof. Recall that $\frac{1}{\pi}\Psi(b, kb)$ defines a homomorphism $K \to \mathbb{R}/\mathbb{Z}$ (Proposition 2.3.22). Define $B := \overline{\langle g \rangle}$. Given $h, h' \in B$ the refined Jordan decompositions are compatible, i.e. $h'_e, h'_u, h'_h, h_u, h_e, h_h$ commute pairwise. Since Rot is conjugation invariant ([18, Lem.7.2]), we can assume that the elliptical part k of g in the refined Jordan decomposition is in K. Under this assumptions this holds for all elements in B, because the Jordan decompositions are compatible.
We are searching for $f_B : B \to \mathbb{R}$ which defines a homomorphism $B \to \mathbb{R}/\mathbb{Z}$ and such that ∂f_b is a representative of κ_β. Let $g = g_u g_h k$ be the refined Jordan decomposition of g. Choose $b \in \check{S}$ such that $g_u g_h$ fixes b. Then for any $h \in B$, $h_u h_h$ fixes b, since h is the limit of powers of g and their refined Jordan decompositions are compatible.
Now we define

$$f_B : \begin{cases} B \to \mathbb{R} \\ h \mapsto \frac{1}{\pi}\Psi(b, hb). \end{cases}$$

Note that $\Psi(b, kb) = \Psi(b, h_e b)$ and therefore by Proposition 2.3.22 it defines a homomorphism $B \to \mathbb{R}/\mathbb{Z}$.
It remains to show $\delta f_B = \kappa_\beta|_B$. Let $g_1, g_2 \in B$ and denote by $k_1, k_2 \in K$ the respective elliptic parts of the refined Jordan decomposition:

$$\begin{aligned}\delta f_B(g_1, g_2) =& f_B(g_1) - f_B(g_1 g_2) + f_B(g_1)\\ =&\Psi(b, k_1 b) - \Psi(b, k_1 k_2 b) + \Psi(b, k_2)\\ =&\frac{1}{\pi}\left[\Psi(b, k_1 b) + \Psi(k_1 b, k_1 k_2 b) + \Psi(k_1 k_2 b, b)\right]\end{aligned}$$

$$=\beta(b, k_1 b, k_1 k_2 b) = \beta(b, g_1 b, g_1 g_2 b),$$

where we used K-invariance of Ψ and Formula (2.8). This finishes the proof, because $\beta(b, g_1 b, g_1 g_2 b)$ is an inhomogeneous cocycle defining κ_β (c.f. Section 2.2.1). □

Definition 2.3.24. Let G be a group and $f : G \to \mathbb{R}$ be a map. Then f is a *quasimorphism*, if there exists $C \in \mathbb{R}$ such that

$$|f(gh) - f(g) - f(h)| \leq C, \quad \forall g, h \in G.$$

A quasimorphism is homogeneous, if $f(g^n) = nf(g)$ for all $g \in G$ and $n \in \mathbb{N}$.

Proposition 2.3.25. *Let*

$$\tau(\tilde{g}) := \lim_{n\to\infty} \frac{m(\tilde{b}, \tilde{g}^n \tilde{b})}{n}.$$

be the homogenization of $\tilde{g} \mapsto m(\tilde{b}, \tilde{g}\tilde{b})$*. Then* τ *is a quasimorphism of* $\tilde{G}$ *and*

$$\tau = -\widetilde{\mathrm{Rot}}_{\kappa_\beta}$$

Proof. First note that τ does not depend on $\tilde{b}$ (cf. [23, Ch.10]). Since $\tau(e) = \widetilde{\mathrm{Rot}}_{\kappa_\beta}(e) = 0$ it is enough to show that

$$\tau(\tilde{g}) = -\operatorname{Rot}_{\kappa_\beta}(g) \mod \mathbb{Z}$$

for $g \in G$ and $\tilde{g}$ any lift, because the lifts of both sides are unique. First note that both sides are conjugation invariant and only depend on the elliptic part of the refined Jordan decomposition of g. Hence it is enough to show this equality for $g = k \in K$.
By Proposition 10.4 in [23] and Proposition 2.3.23 we have

$$e^{2i\pi\tau(\tilde{k})} = \chi(k)^2 = e^{-2i\pi\Psi(b,kb)/\pi} = e^{-2i\pi \operatorname{Rot}_{\kappa_\beta}}$$

for all $k \in K$ and all lifts $\tilde{k}$ of k. □

Corollary 2.3.26. *Let $g \in G$ and $x \in \check{S}$ a fixed point of g. Let $\tilde{g} \in \tilde{G}$ and $\tilde{x} \in \check{R}$ be lifts. Then*

$$\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{g}) = m(\tilde{g}\tilde{x}, \tilde{x})$$

and $\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{g}) = 0$ if $\tilde{g}$ has a fixed point in $\check{R}$.

Proof. We calculate

$$\tau(\tilde{g}) = \lim_{n\to\infty} \frac{m(\tilde{g}^n\tilde{x}, \tilde{x})}{n} = \lim_{n\to\infty} \frac{n \cdot m(\tilde{g}\tilde{x}, \tilde{x})}{n} = m(\tilde{g}\tilde{x}, \tilde{x}),$$

where we have used Lemma 2.3.18. □

Recall the classical *translation number* T on $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$. Let $g \in \mathrm{PSL}(2,\mathbb{R})$ and $\tilde{g} \in \widetilde{\mathrm{PSL}(2,\mathbb{R})}$ a lift and $\tilde{x} \in \mathbb{R}$. Then:

$$T(\tilde{g}) = \lim_{n\to\infty} \frac{\tilde{g}^n\tilde{x} - \tilde{x}}{n}.$$

Remark 2.3.27. The number $T(\tilde{g})$ is independent of $\tilde{x}$. If $g \in \mathrm{PSL}(2,\mathbb{R})$ has a fixed point $x \in S^1$ and $\tilde{g}$ and $\tilde{x}$ are lifts, then $T(\tilde{g}) = \tilde{g}\tilde{x} - \tilde{x}$.

Proposition 2.3.28. *Let $g \in \mathrm{PSL}(2,\mathbb{R})$ and $\tilde{g} \in \widetilde{\mathrm{PSL}(2,\mathbb{R})}$ a lift.*

$$\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{g}) = -T(\tilde{g}).$$

Proof. The map T satisfies $T(e) = 0$ and

$$-T(\tilde{g}) = \mathrm{Rot}_{\kappa_\beta} \mod \mathbb{Z}$$

(see Proposition 2.3.23 and Remark 2.3.21). Hence $-T(\tilde{g}) = \widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{g})$. □

We finish this subsection with a short lemma, which we will need later.

Lemma 2.3.29. *Let f be a homogeneous quasimorphism. Then $f(g) = -f(g^{-1})$.*

Proof. Since $f(e) = f(e^n) = nf(e)$ for all $n \in \mathbb{N}$, $f(e) = 0$. By the definition of quasimorphism we have:

$$n|f(g) + f(g^{-1})| = |f(g^n) + f(g^{-n}) - f(e)| \leq r$$

for all $n \in \mathbb{N}$, hence $f(g) + f(g^{-1}) = 0$. □

2.4 Limit Curves

One of the most important properties of maximal representations $\varrho : \Gamma \to G$ is the existence of a boundary curve from some boundary of Γ into $\check{S}$, the Shilov boundary of the Hermitian Lie group G. In this section we will shortly recall the construction of limit curve. Let $\mathcal{D}$ be the bounded symmetric domain associated with G and $\check{S}$ its Shilov boundary. The following results are due to Burger, Iozzi and Wienhard ([12, 14, 18] and the references therein).

The most general statement is

Theorem 2.4.1. *([14, Thm. 4.7]) Let $\mathbb{G}$ be a connected semisimple algebraic group defined over $\mathbb{R}$, $G = \mathbb{G}(\mathbb{R})^\circ$ and let $\mathbb{P}$ be a minimal parabolic subgroup defined over $\mathbb{R}$. Assume that the continuous homomorphism $\varrho : H \to G$ has Zariski dense image and let (B, ν) be a Poisson boundary for H such that the diagonal H-action on $(B \times B, \nu \times \nu)$ is ergodic.*
Then there exists a measurable H-equivariant map $\varphi : B \to G/P$. Moreover, any such map verifies that for almost every $(b_1, b_2) \in B^2$, the images $\varphi(b_1), \varphi(b_2)$ are transverse.

See [12, Ch.7] for a proof.

Remark 2.4.2. Such a Poisson boundary exists for any finitely generated group with finite generating set S and measure

$$\mu := \frac{1}{2|S|}\sum_{s\in S}(\delta_s + \delta_{s^{-1}})$$

([12, Prop. 6.1]).

Let $h : \Gamma_{g,m} \to \mathrm{PSL}(2,\mathbb{R})$ be a hyperbolization for $\Sigma_{g,m}$ with m cusps, i.e. the realization of $\Gamma_{g,m}$ as a lattice in $\mathrm{PSL}(2,\mathbb{R})$. This induces an action of $\Gamma_{g,m}$ on S^1. Later there will be $\varrho\circ h^{-1}$-equivariant maps from S^1 into $\check{S}$ for a given maximal representation $\Gamma_{g,m} \to G$ and we will shortly write ϱ-equivariant map.

Note that S^1 equipped with the round measure λ is a Poisson boundary for Γ_g for $(\varrho(\Gamma_g), \mu)$. Let $\varrho : \Gamma_{g,m} \to G$ be a maximal, then Theorem 2.4.1 gives a ϱ-equivariant map $\varphi : S^1 \to \check{S}$.

It is *monotone*, i.e. it maps positive/negative triples in S^1 to maximal/minimal triples in $\check{S}$.

This leads to the following characterization of maximality ([18, Thm. 8]):

Theorem 2.4.3. *The representation $\varrho : \Gamma \to G$ is maximal if and only if there exists a limit curve $\varphi : S^1 \to G$ which is left continuous, monotonous and ϱ-equivariant, where Γ acts on S^1 via h.*

Remark 2.4.4. For maximal representations of Γ_g, the limit curve φ is continuous ([13]).

The following theorem summarizes further properties of the limit curve:

Theorem 2.4.5. *([18, Thm. 5.1]) Let $\varrho : \Gamma_g \to G$ be a maximal representation with Zariski dense image. Then there are two Borel maps*

$$\varphi_\pm : S^1 \to \check{S}$$

with the following properties

(i) *φ_+ and φ_- are strictly ϱ-equivariant,*

(ii) *φ_- is left continuous and φ_+ is right continuous,*

(iii) *for every $x \neq y$, $\varphi_\epsilon(x)$ is transverse to $\varphi_\delta(y)$ for all $\epsilon, \delta \in \{+,-\}$,*

(iv) *for all $x, y, z \in S^1$*

$$\beta_{\check{S}}(\varphi_\epsilon(x), \varphi_\delta(y), \varphi_\eta(z)) = r_{\mathcal{X}}\beta(x, y, z),$$

for all $\epsilon, \delta, \eta \in \{+,-\}$.

Moreover φ_+ and φ_- are the unique maps satisfying (1) and (2).

In Appendix A.3 we prove:

Proposition 2.4.6. *Let $\varrho : \Gamma_g \to G$ be a maximal representation. Then the limit curve φ for ϱ is unique.*

Let $\varrho : \Gamma_{g,m} \to \mathrm{PSL}(2, \mathbb{R})$ be a hyperbolization. Denote by $\mathcal{L}$ the limit set of $\varrho(\Gamma_{g,m})$ in S^1 and by μ_{PS} its Patterson-Sullivan measure. Then $(\mathcal{L}, \mu_{PS})$ is a Poisson boundary for $\varrho(\Gamma_{g,m})$ and we can apply this combined with Theorem 2.4.1 to obtain results as Theorem 2.4.3 and Theorem 2.4.5 for non-closed surfaces (see Section 4.1).

2.5 Hitchin representations

Another branch of higher Teichmüller theory is the study of Hitchin representations.

Let G be a split real group. We define

$$\mathrm{Rep}^+(\Gamma_g, G) := \mathrm{Hom}^+(\Gamma_g, G)/G,$$

where $\mathrm{Hom}^+(\Gamma_g, G)$ is the set of reducible representations.

Hitchin identified in [40] connected components of $\operatorname{Rep}^+(\Gamma_g, G)$ which are homeomorphic to $\mathbb{R}^{(2g-2)\dim G}$. Their elements are characterized as follows: since G is a split real Lie group, there exists a unique irreducible representation $i : \mathrm{PSL}(2,\mathbb{R}) \to G$. A representation $\varrho : \Gamma_g \to G$ is called *Fuchsian* if $\varrho = i \circ \varrho'$, where $\varrho' : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ is a hyperbolisation.

Definition 2.5.1. A *Hitchin component* of $\operatorname{Rep}^+(\Gamma_g, G)$ is a component which contains Fuchsian representations.

Geometric properties of Hitchin representations have been studied by Labourie [47, 45, 46]. They share the following properties with maximal representations [47]

(i) They are faithful and have discrete image,

(ii) they admit limit curves from S^1 to some boundary of symmetric space associated with G,

(iii) they have the Anosov property.

See also Theorem in 6.1 [13], Proposition 2.2.18 and Theorem 2.2.15. The group $\mathrm{Sp}(2n,\mathbb{R})$ is the only Hermitian Lie group which is also split-real. Therefore one can define maximal representations as well as Hitchin representations for it. The Hitchin components form a proper subset of the maximal representations.

Chapter 3

Parameters for Representation Varieties

3.1 The Geometric Idea behind the Parameters

3.1.1 The $\mathrm{SL}(2,\mathbb{R})$-Case

In this section we explain the geometric idea behind Theorem 1.3.1 by discussing the case $G = \mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$.

The most important fact here is that Theorem 2.2.8 formulates in this situation as follows:

Theorem 3.1.1. . *Let* $\varrho : \Gamma_{g,m} \to \mathrm{PSL}(2,\mathbb{R})$ *be a representation.*

Choose a lift $\tilde{\varrho}:\Gamma_{g,m}\to\widetilde{\mathrm{PSL}(2,\mathbb{R})}$. *Then*

$$T_\varrho=\sum_{i=1}^m T(\tilde{\varrho}(C_i)).$$

Let now $\varrho:\Gamma_{0,3}\to\mathrm{Sp}(2,\mathbb{R})=\mathrm{SL}(2,\mathbb{R})$ be a hyperbolization (i.e. a maximal representation). For the sake of simplicity we assume that $c_i:=\varrho(C_i)$ is hyperbolic for all i, i.e. we assume the surface to have three geodesic boundary components and no cusps. Nevertheless the following discussion works along precisely the same lines if one or more of the generators are parabolic. Denote by c_i^+ the attractive and by c_i^- the repellent fixed point of c_i.

The discussion in the $\mathrm{SL}(2,\mathbb{R})$ case relies on four observations:
First observe that the fixed points of the hyperbolic isometries c_i are as in the relative position as indicated in Figure 3.1. Indeed, the c_i correspond to geodesic boundary components of $\Sigma_{0,3}$. Choosing a universal cover embedded in $\mathbb{D}$ shows that the axis and hence the endpoints of these axes are as in Figure 3.1. We can formalize this to:

$$o(c_1^\pm,c_2^\pm,c_3^\pm)=1,$$

where o is the orientation cocycle for triples of S^1 introduced in Section 2.3.1. This is a first constraint for the fixed points of the c_i.

The **second** observation concerns the position of $c_1\cdot c_3^+$, $c_2\cdot c_1^+$ and $c_3\cdot c_2^+$ relative to the $c_i^\pm$. We will show that maximality implies the relative positions are as in as in Figure 3.1. Therefore we get with the formula in Theorem 3.1.1:

$$T_\varrho=T(\tilde{c}_1)+T(\tilde{c}_2)+T(\tilde{c}_3),$$

where the $\tilde{c}_i\in\widetilde{\mathrm{SL}(2,\mathbb{R})}$ generate a lift of the representation ϱ into $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ and T is the translation number on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ (Definition 3.1.1). We can choose the lifts of, say, c_1 and c_2 such that they have fixed points in $\mathbb{R}=\tilde{S}^1$, because the c_i all have fixed points in S^1. In

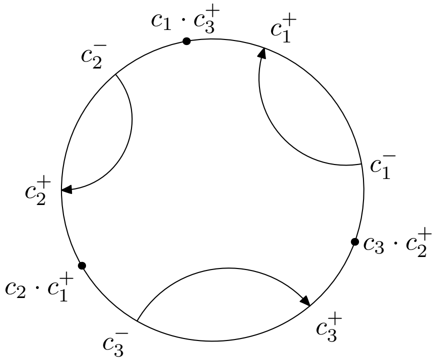


Figure 3.1: S^1

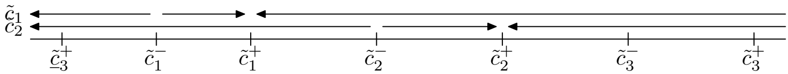


Figure 3.2: $\mathbb{R}$ with the dynamics of $\tilde{c}_1$ and $\tilde{c}_2$.

this case we have $T(\tilde{c}_1) = T(\tilde{c}_2) = 0$, hence $T_\varrho = T(\tilde{c}_3) = -T(\tilde{c}_2\tilde{c}_1)$. The second equality follow from the fact that the translation number T is a homogeneous quasimorphism (Proposition 2.3.25 and Proposition 2.3.28). By Remark 2.3.27 we have $T(\tilde{c}_2\tilde{c}_1) = \tilde{c}_2\tilde{c}_1\tilde{c}_3^+ - \tilde{c}_3^+$, so this reduces the calculation of T_ϱ to

$$T_\varrho = \tilde{c}_3^+ - \tilde{c}_2\tilde{c}_1\tilde{c}_3^+.$$

Now we can calculate of the relative position of $c_1 \cdot c_3^+$. In Figure 3.2 you see the universal cover $\mathbb{R}$ of S^1 as well as lifts of $c_i^\pm$. Recall

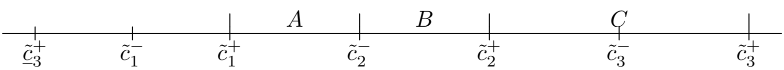


Figure 3.3: $\mathbb{R}$ with the intervals A, B and C.

that we have chosen $\tilde{c}_1$ and $\tilde{c}_2$ such that they fix all lifts of $c_1^\pm$ resp. $c_2^\pm$. The arrows indicate the direction of displacement for $\tilde{c}_1$ resp. $\tilde{c}_2$. The lifts of attractive fixed points remain attractive in a neighborhood, the lifts of repellent ones remain repellent.

Consider now Figure 3.3 and recall that by assumption $T_\varrho = \tilde{c}_3^+ - \tilde{c}_2\tilde{c}_1\tilde{c}_3^+ = 1$. Hence $\tilde{c}_2\tilde{c}_1\tilde{c}_3^+ = \underline{\tilde{c}}_3^+$. By definition of the action of $\tilde{c}_1$, we know that $\tilde{c}_1\tilde{c}_3^+$ is either in A, B or C. If it would be in C, then $\tilde{c}_2\tilde{c}_1\tilde{c}_3$ would be in C as well, and $T(\tilde{c}_2\tilde{c}_1) = 0$, which is a contradiction. For the same reason it cannot be in B. Hence it has to be in A. Hence $c_1 \cdot c_3^+$ is in the relative position indicated in Figure 3.1. The same argument can be used for the relative position of $c_3 \cdot c_2^+$ and $c_1 \cdot c_3^+$, but they can also be determined using properties of the orientation cocycle on S^1. One can summarize this discussion to:

$$o(c_1^+, c_1 \cdot c_3^+, c_2^+) = o(c_2^+, c_2 \cdot c_1^+, c_3^+) = o(c_3^+, c_3 \cdot c_2^+, c_1^+) = 1. \quad (3.1)$$

The first two equalities follow from the $\mathrm{PSL}(2,\mathbb{R})$-invariance and skew-symmetry of 0 and the relation $c_3c_2c_1 = e$.

Remark 3.1.2. In the discussion above we only used that the c_i^+ are fixed points of the c_i. It also works if we replace some c_i^+ by c_i^- and it also works for representations where one or more generators are parabolic.

The first and the second observation led to Theorem 1.3.2.

The **third** observation is the following calculation, which shows that for every maximal triple $f_1, f_2, f_3 \in S^1$ and every triple $z_1, z_2, z_3 \in S^1$ with

$$o(f_1, z_2, f_2) = o(f_2, z_3, f_3) = o(f_3, z_1, f_1) = 1, \quad (3.2)$$

there exists a maximal representation $\varrho : \Gamma_{0,3} \to \mathrm{SL}(2,\mathbb{R}) = \mathrm{Sp}(2,\mathbb{R})$, such that f_i is a fixed point of $\varrho(C_i) =: c_i$ and

$$z_1 = c_3 \cdot f_2 = c_1^{-1} \cdot f_2, \quad z_2 = c_1 \cdot f_3 = c_2^{-1} \cdot f_3, \quad z_3 = c_2 \cdot f_1 = c_3^{-1} \cdot f_1.$$

Note that if ϱ exists, it is automatically maximal by Theorem 1.3.2.

From now on we perform our calculations in the upper half plane model H^2 of hyperbolic 2-space, because Condition (3.2) as well as the calculation with matrices become particularly easy there. Indeed, the orientation cocycle carries over to the antisymmetric cocycle o on $\partial H^2 = \mathbb{R} \cup \{\infty\}$ with $o(x_1, x_2, x_3) = 1$ if $x_1 < x_2 < x_3$ and $o(x_1, x_2, x_3) = 0$ if $x_1 = x_2$ Recall that we investigate representations up to conjugation. Hence we can assume that $(f_1, f_2, f_3) = (0, 1, \infty)$, because $\mathrm{SL}(2, \mathbb{R})$ acts 3-transitively on maximal triples. Now our assumptions for the z_i become:

$$o(z_1, 0, 1) = o(0, z_2, 1) = o(1, z_3, \infty) = 1.$$

and by definition of o we can rephrase that to:

$$z_1 < 0 < z_2 < 1 < z_3. \tag{3.3}$$

Recall that the group $\mathrm{SL}(2, \mathbb{R})$ acts via Möbius transforms isometrically on H^2:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

This action extends to $\partial H^2 = \mathbb{R} \cup \{\infty\}$ with the same formula. For the point ∞ we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \infty \quad \text{iff} \quad cz + d = 0.$$

Now we come back to our desired representation. The isometries c_1, c_2 and c_3 are characterized by:

$$c_1 : \begin{cases} 0 \mapsto 0 \\ z_1 \mapsto 1 \\ \infty \mapsto z_2 \end{cases}, \quad c_2 : \begin{cases} 1 \mapsto 1 \\ z_2 \mapsto \infty \\ 0 \mapsto z_3 \end{cases}, \quad c_3 : \begin{cases} \infty \mapsto \infty \\ z_3 \mapsto 0 \\ 1 \mapsto z_1 \end{cases} \tag{3.4}$$

This characterization determines $c_i \in \mathrm{SL}(2,\mathbb{R})$ up to sign and we can calculate the c_i explicitly. We proceed with the explicit calculation of c_1: Assume

$$c_1 := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From (3.4) we get:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} 0 = \frac{b}{d} \stackrel{!}{=} 0$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z_1 = \frac{az_1 + b}{cz_1 + d} \stackrel{!}{=} 1$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c} \stackrel{!}{=} z_2$$

The first condition gives $b = 0$, the third gives $c = a/z_2$ and from the determinant condition we get $d = 1/a$. Together with the second equation this yields

$$a^2 = \frac{1}{-z_1(z_2^{-1} - 1)}$$

and the right-hand side is positive by assumption. Hence we can choose m_1, m_2 and m_3 such that

$$z_1 = -m_1^2,\ z_2 = (m_2^2 + 1)^{-1},\ z_3 = m_3^2 + 1. \tag{3.5}$$

This is possible by (3.3). Conversely for any triple of non-zero m_i, the z_i defined via these formulae are in the right position.

Therefore $a = \frac{1}{m_1 m_2}$ is a solution for any m_1 and m_2 chosen as above. Fixing such a triple (m_1, m_2, m_3), similar calculations as for c_1 give (cf. Section 3.3)

$$c_1 = \begin{pmatrix} \frac{1}{m_1 m_2} & 0 \\ \frac{m_2 + \frac{1}{m_2}}{m_1} & m_1 m_2 \end{pmatrix}$$

$$c_2 = \begin{pmatrix} -\frac{m_2}{m_3} - \frac{m_3}{m_2} - \frac{1}{m_2 m_3} & \frac{m_3 + \frac{1}{m_3}}{m_2} \\ -\frac{m_2 + \frac{1}{m_2}}{m_3} & \frac{1}{m_2 m_3} \end{pmatrix}$$
$$c_3 = \begin{pmatrix} \frac{m_1}{m_3} & -m_1(\frac{1}{m_3} + m_3) \\ 0 & \frac{m_3}{m_1} \end{pmatrix}$$

A direct computation shows that $c_3 c_2 c_1 = 1$. Hence there exists a representation with the desired properties 3.2. This finishes the third part.

Remark 3.1.3. In (3.5) we had to chose a sign for m_i. This sign does not affect the action of the c_i on $\bar{\mathbb{D}}$. This corresponds to the fact, that the center of $\mathrm{SL}(2,\mathbb{R})$ acts trivially on $\bar{\mathbb{D}}$. Furthermore this sign gives rise to different connected components of $\mathrm{Rep}_{max}(\Gamma_{0,3}, \mathrm{SL}(2,\mathbb{R}))$.

Recall from Remark 1.3.4 that eigenvalues of the c_i and the boundary length of the associated hyperbolic surfaces are related.

The eigenvalues of the c_i are:

$$c_1 : \; m_1 m_2, \; \frac{1}{m_1 m_2}, \; c_2 : \; -\frac{m_2}{m_3}, \; -\frac{m_3}{m_2}, \; c_3 : \; \frac{m_1}{m_3}, \; \frac{m_3}{m_1}.$$

It turns out that they determine the representation uniquely:

We put $x_1 := 1/m_1 m_2$, $x_2 := m_3/m_2$ and $x_3 := m_3/m_1$ and get

$$c_1 = \begin{pmatrix} x_1 & 0 \\ x_1 + x_2^{-1} x_3 & x_1^{-1} \end{pmatrix}$$
$$c_2 = \begin{pmatrix} -x_3^{-1} x_1 - x_2 - x_2^{-1} & x_2 + x_3^{-1} x_1 \\ -x_3^{-1} x_1 - x_2^{-1} & x_3^{-1} x_1 \end{pmatrix}$$
$$c_3 = \begin{pmatrix} x_3^{-1} & -x_3^{-1} - x_1^{-1} x_2 \\ 0 & x_3 \end{pmatrix}$$

These matrices are precisely in the form of the matrices in Theorem 1.3.1 and the x_i are coordinates for the representation.

The c_i have by construction at least one fixed points in S^1. A direct calculation shows that each c_i has a fixed point f_i,

$$f_1 = \frac{\left(-1+x_1^2\right)x_2}{x_1(x_1x_2+x_3)},\ f_2 = \frac{x_1x_2+x_2^2x_3}{x_1x_2+x_3},\ f_3 = \frac{-x_1-x_2x_3}{x_1\left(-1+x_3^2\right)}.$$

which may or may not coincide with the given one.

We summarize: any maximal representation $\varrho : \Gamma_{0,3} \to \mathrm{SL}(2,\mathbb{R})$ is determined by the eigenvalues of its generators and every triple of non-zero numbers may appear as triple of such generators. But we want to remark that different triples may define conjugated representations.

The **forth** and last observation concerns fixed points of the generators and their dynamical properties.

Each x_i controls the dynamic of c_i in its fixed points. The point f_1 is equal to 0 if and only if $|x_1| = 1$. The fixed point 0 is attractive if and only if $|x_1| < 1$, it is repellent if and only if $|x_1| > 1$. Analog results hold for the other c_i.

Above we have constructed parameters (x_1, x_2, x_3) for maximal representations into $\mathrm{SL}(2,\mathbb{R})$. But they depend on the choice of the fixed points (f_1, f_2, f_3). For example the triple $(f_1, f_2, f_3) = (c_1^+, c_2^+, c_3^+)$ gives other parameters than $(f_1, f_2, f_3) = (c_1^-, c_2^-, c_3^-)$. We want to use the dynamical characterization of fixed points to obtain unique coordinates for a representation. We choose for the calculation of x_i the attractive fixed point if the corresponding generator is hyperbolic and the unique fixed point if it is parabolic. This yields a triple (x_1, x_2, x_3) with $|x_i| < 1$ or $x_i \in B$.

This proves Proposition 1.2.1 and

Proposition 3.1.4.

$$\mathrm{Rep}_{max}(\Gamma_{0,3}, \mathrm{SL}(2,\mathbb{R})) = \{(x_1, x_2, x_3) | x_i \in (0,1] \cup [-1,0),\ x_1x_2x_3 > 0\}.$$

Corollary 3.1.5. $\mathrm{Rep}_{max}(\Gamma_{0,3}, \mathrm{PSL}(2,\mathbb{R})) = \{(x_1, x_2, x_3) | x_i \in (0,1]\}$.

Together with Remark 1.3.4 this proves Proposition 1.2.1

3.1.2 The $\mathrm{Sp}(2n, \mathbb{R})$-case

Now we generalize this discussion for maximal representations $\Gamma_{0,3}$ to $\mathrm{Sp}(2n, \mathbb{R})$ and follow the four steps from the $\mathrm{SL}(2, \mathbb{R})$-case.

The main tool for the generalization are the limit curves (Prop. 2.2.18). Let $\varrho : \Gamma_{0,3} \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation. Its limit curve $\varphi : S^1 \to \check{S}$ is ϱ-equivariant and maps maximal triples to maximal triples. Therefore every generator $c := \varrho(C_i)$ of ϱ has a fixed point $f_i = \varphi(c_i^+)$ in $\check{S}$. Combining (3.1) with the fact that limit curves map maximal triples to maximal triples we get:

$$\beta(f_1, f_2, f_3) = \beta(f_1, c_1 f_3, f_2) = \beta(f_2, c_2 f_1, f_3) = \beta(f_3, c_3 f_2, f_1) = n$$

It remains to show that any such configuration can come from a maximal representation. So let (f_1, f_2, f_3) be a maximal triple and triple (Z_1, Z_2, Z_3) with

$$\beta(f_1, Z_3, f_2) = \beta(f_2, Z_1, f_3) = \beta(f_3, Z_2, f_1) = n. \tag{3.6}$$

We may ask if there is a maximal representation $\varrho : \Gamma_{0,3} \to \mathrm{Sp}(2n, \mathbb{R})$ such that f_i are fixed points for $c_i := \varrho(C_i)$ and the $Z_j = c_j f_i$? Again the answer is yes. Note that by Formula (1.1) such a representation is automatically maximal.

We need a suitable model for our calculations on the Shilov boundary.

Remark 3.1.6. As in the $\mathrm{SL}(2, \mathbb{R})$-case we choose a suitable model and perform direct calculation with matrices. Since the symmetric space associated with $\mathrm{Sp}(2n, \mathbb{R})$ is of tube type, we can turn to the tube model $T_\Omega = V \oplus i\Omega$, where $V = \mathrm{Sym}(n, \mathbb{R})$ and $\Omega = \mathrm{Sym}(n, \mathbb{R})^+$, the set of symmetric positive matrices. This is a generalized upper half plane model which is only available for tube type groups.

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$ and $X \in T_\Omega$. Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} X = (AX + B)(CX + D)^{-1}.$$

The action of $\mathrm{Sp}(2n, \mathbb{R})$ on its bounded symmetric domain $\mathcal{D}$ extends to an action on the Shilov boundary. The Cayley transform, which maps T_Ω to $\mathcal{D}$, maps $V \subset \bar{T}_\Omega$ to an open and dense subset of $\check{S}$. It is precisely the set of points transversal to a certain point in $\check{S}$ (see Section 2.1.4). So it is clear that $\mathrm{Sp}(2n, \mathbb{R})$ does not act on V, since the Shilov boundary is a $\mathrm{Sp}(2n, \mathbb{R})$-orbit. But nevertheless there exists a $\varrho(\Gamma)$ invariant subset of $\check{S}$, namely the limit curve. Furthermore, by Theorem 2.2.18 (i) any two points in the image of the limit curve are transversal. So we can conjugate the representation such that the Cayley transform maps $\varphi(S^1) - \{\mathrm{pt}\}$ into V. It remains to add one single point ∞ to V to obtain a $\varrho(\Gamma)$-invariant set.

For the calculation with ∞ we have the following formulas:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \infty = AC^{-1}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} X = \infty \quad \text{iff} \quad CX + D = 0.$$

See also Remark 3.2.6.

Remark 3.1.7. A direct calculation shows that the stabilizer of the maximal triple $(0, e, \infty)$ is

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{O}(n) \right\}$$

Since all maximal triples are in the same G-orbit the stabilizers of maximal triples are conjugate to $\mathrm{O}(n)$.

Since $\mathrm{Sp}(2n,\mathbb{R})$ acts three-transitively on maximal triples on the Shilov boundary, we can assume without loss of generality $(f_1, f_2, f_3) = (0, 1, \infty)$ and summarize (3.6)

$$c_1 : \begin{cases} 0 \mapsto 0 \\ Z_1 \mapsto 1 \\ \infty \mapsto Z_2 \end{cases}, \ c_2 : \begin{cases} 1 \mapsto 1 \\ Z_2 \mapsto \infty \\ 0 \mapsto Z_3 \end{cases}, \ c_3 : \begin{cases} \infty \mapsto \infty \\ Z_3 \mapsto 0 \\ 1 \mapsto Z_1 \end{cases}$$

So we have a system of three equations for each c_i and these equation determine c_i up to multiplication with an element of G stabilizing a maximal triple. Therefore we have more solutions than in the $\mathrm{SL}(2,\mathbb{R})$-case. We solve these equations completely in Section 3.3.2 where we prove Proposition 3.3.3.

But as in the $\mathrm{SL}(2,\mathbb{R})$ case we can use a dynamical criterion to find a canonical fixed point for every generator. We explain the idea using a representation $\varrho : \Gamma_{0,3} \to \mathrm{PSL}(2,\mathbb{R})^r$. In this case the Shilov boundary is $(S^1)^r$ and the Maslov index is the sum of the orientation cocycles from each component. As usual we denote the generators by c_1, c_2 and c_3. We now investigate c_1, the investigation for the other generators works along the same lines. We write $c_1 = (a_1, \ldots, a_r)$, where $a_i \in \mathrm{SL}(2,\mathbb{R})$.

Each a_i is either parabolic or hyperbolic, hence it has one or two fixed points on S^1. If a_i is hyperbolic, we denote the repellent fixed point of a_i by a_i^-, the attractive fixed point by a_i^+. If it is parabolic we denote the unique fixed point by a^0. We assume without loss of generality that $a_1, \ldots, a_k$ are hyperbolic and $a_{k+1}, \ldots, a_n$ are parabolic. Hence any fixed point of c_1 has the form $(a_1^{\epsilon_1}, \ldots, a_k^{\epsilon_1}, a_{k+1}^0, \ldots, a_n^0)$, where $\epsilon_i \in \{+, -\}$. Observe that among these we have two canonical fixed points, namely:

$X^+ := (a_1^+, \ldots, a_+, a_{k+1}^0, \ldots, a_r^0)$ and $X^- := (a_1^-, \ldots, a_k^-, a_{k+1}^0, \ldots, a_r^0)$.

Both are characterized by the dynamical properties of c_1: in X^+ the action is non-expanding, in X^- the action is non-contracting (see below for the precise definitions). Proposition 3.3.4 is a generalization of this fact for $\mathrm{Sp}(2n,\mathbb{R})$.

3.2 A Formula for the Toledo Invariant

Recall Thorem 1.3.2:

Theorem 3.2.1. *Let G be a Hermitian Lie group of tube type. Let $\varrho : \Gamma_{0,3} \to G$ and denote $c_i := \varrho(C_i)$. Assume that each c_i has a fixed point $y_i \in \check{S}$. Then we can express the Toledo invariant as follows:*

$$T_{\varrho,\kappa_\beta} = \beta(y_1, y_2, y_3) + \beta(y_1, c_1 \cdot y_3, y_2). \tag{3.7}$$

Remark 3.2.2. The Maslov index is antisymmetric and G-invariant, hence

$$\beta(y_1, c_1 \cdot y_3, y_2) = \beta(y_2, c_2 \cdot y_1, y_3) = \beta(y_3, c_3 \cdot y_2, y_1), \tag{3.8}$$

i.e. we can also express the Toledo invariant in terms of $c_2 y_1$ or $c_3 y_2$ and the fixed points y_1, y_2 and y_3.

Remark 3.2.3. Recall that the Maslov index can be seen as the area of an ideal triangle in $\bar{D}$ with vertices in $\check{S}$ (cf. Theorem 2.3.4). In Section 1.1 we defined the Toledo invariant for closed surfaces as an integral over Σ_g, which can be identified with a $4g$-gon in hyperbolic 2-space.

Proof of Proposition 3.2.1. We will calculate the Toledo invariant using the following formula from [18, Thm.12]

$$T_{\varrho,\kappa_\beta} = -\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_1) - \widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_2) - \widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_3), \tag{3.9}$$

where the $\tilde{c}_i$ generate some lift of the representation ϱ to $\tilde{G}$, the universal covering of G, and $\widetilde{\mathrm{Rot}}$ is the generalized rotation number introduced in Section 2.3.4. Since this formula is independent of the chosen lift, we can choose a lift which makes the calculation easier.

We choose $\tilde{c}_1$ and $\tilde{c}_2$ such that both have fixed points in $\check{R}$, which is possible since c_1 and c_2 have fixed points in $\check{S}$. Then $\tilde{c}_3 := (\tilde{c}_2 \tilde{c}_1)^{-1}$ is a lift of c_3 and by definition $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$ generate a representation of $\Gamma_{0,3}$ into $\tilde{G}$.

Since $\tilde{c}_1$ and $\tilde{c}_2$ have fixed points in $\check{R}$, we have $\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_1) = \widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_2) = 0$ (Corollary 2.3.26). Therefore it suffices to calculate $\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_3)$. Since $\widetilde{\mathrm{Rot}}$ is a homogeneous quasimorphism ([18, Thm.11]), we have $\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}) = -\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}^{-1})$ (see [18] and Prop 2.3.29), hence

$$\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_3) = \widetilde{\mathrm{Rot}}_{\kappa_\beta}((\tilde{c}_2\tilde{c}_1)^{-1}) = -\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_2\tilde{c}_1).$$

By Corollary 2.3.26, $\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_2\tilde{c}_1) = m(\tilde{y}_3, \tilde{c}_2\tilde{c}_1\tilde{y}_3)$, since $\tilde{y}_3$ is the lift of a fixed point of c_2c_1. Furthermore (Lemma 2.3.17)

$$m(\tilde{c}_2\tilde{c}_1\tilde{y}_3, \tilde{y}_3) = m(\tilde{c}_1\tilde{y}_3, \tilde{y}_3) + m(\tilde{c}_2\tilde{c}_1\tilde{y}_3, \tilde{c}_1\tilde{y}_3)$$

We can summarize the discussion above to

$$\begin{aligned} T_\varrho =& -\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_1) - \widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_2) - \widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_3) = -\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_3) \\ =&\widetilde{\mathrm{Rot}}_{\kappa_\beta}(\tilde{c}_2\tilde{c}_1) = m(\tilde{c}_2\tilde{c}_1\tilde{y}_3, \tilde{y}_3) = m(\tilde{c}_1\tilde{y}_3, \tilde{y}_3) + m(\tilde{c}_2(\tilde{c}_1)\tilde{y}_3, \tilde{c}_1\tilde{y}_3) \end{aligned}$$

Now we will express the right hand side in terms of the Maslov index. By Lemma 2.3.20 we get

$$m(\tilde{c}_1\tilde{y}_3, \tilde{y}_3)+m(\tilde{c}_2(\tilde{c}_1\tilde{y}_3), \tilde{c}_1\tilde{y}_3) = \beta(y_2, c_2(c_1y_3), c_1y_3)+\beta(y_1, c_1{\cdot}y_3, y_3)$$

and

$$\begin{aligned} &\beta(y_2, c_2(c_1y_3), c_1y_3) + \beta(y_1, c_1 \cdot y_3, y_3) \\ =&\beta(c_1 \cdot y_3, y_2, y_3) + \beta(y_1, c_1 \cdot y_3, y_3) \\ =&\beta(y_1, c_1 \cdot y_3, y_2) + \beta(y_1, y_2, y_3). \end{aligned}$$

In the first step we used G-invariance, the fact that $c_2c_1 = c_3^{-1}$ and the anti-symmetry and in the second step the cocycle property. This finishes the proof. ☐

Remark 3.2.4. The proof of Proposition 3.2.1 relies on the fact that $m(\tilde{x}, \tilde{g}\tilde{x})$ behaves like a homogeneous quasimorphism if $\tilde{x}$ is the lift of a fixed point of g. Therefore we cannot expect a similar formula for representations where one or more generators do not have a fixed point. In particular the right hand side of the formula only takes a finite number of values, but since $\Gamma_{0,3}$ is free, the Toledo invariant is surjective on the interval $[-r, r]$, where r is the rank of G ([18, Thm.1]).

Corollary 3.2.5. *Let ϱ be a maximal representation of $\Gamma_{0,3}$ into a Hermitian Lie group G of tube type. Then*

(i) each $c_i := \varrho(C_i)$ has a fixed point $y_i \in \check{S}$,

(ii) $\beta(y_1, y_2, y_3) = r$,

(iii) $\beta(y_1, c_1 \cdot y_3, y_2) = r$.

Conversely if ϱ satisfies (i)-(iii), then ϱ is maximal.

Proof. Let $\varrho : \Gamma_{0,3} \to G$ be a maximal representation. Then (i) follows from [18, Lemma 8.8]. Properties (ii) and (iii) as well as the converse follow immediately from Formula 1.1. □

Remark 3.2.6. An important consequence of Proposition 3.2.5 (ii) is the fact that given a fixed point y_i of c_i, the fixed points of all c_j with $j \neq i$ are transverse to y_i. In particular, if we calculate in T_Ω as in Section 3.1.2, we can assume that $y_i = \infty$. Then every fixed point of c_j, $i \neq j$ is contained in V.

3.3 Parameters for $\mathrm{Rep}_{max}(\Gamma_{0,3}, \mathrm{Sp}(2n, \mathbb{R}))$

3.3.1 Proof of Theorem 1.3.1

First we introduce and recall some terminology:

Definition 3.3.1. Let $G \in \mathrm{GL}(k, \mathbb{R})$. Then we denote by $\sigma(G)$ the spectrum of G and G is

(i) *contracting* if $\sigma(G) \subset \mathbb{D}$,

(ii) *expanding* if $\sigma(G) \subset \mathbb{C} - \bar{\mathbb{D}}$,

(iii) *non-expanding* if $\sigma(G) \subset \bar{\mathbb{D}}$,

(iv) *non-contracting* if $\sigma(G) \subset \mathbb{C} - \mathbb{D}$.

Let $c \in \mathrm{Sp}(2n, \mathbb{R})$ and $X \in \bar{T}_\Omega$ a fixed point for c. Then we say that c is

(i) *contracting* in X if $dc|_X$ is contracting,

(ii) *expanding* in X if $dc|_X$ is expanding,

(iii) *non-expanding* in X if $dc|_X$ is non-expanding,

(iv) *non-contracting* in X if $dc|_X$ is non-contracting

Define

$$B := \{G \in \mathrm{GL}(n, \mathbb{R}) | G \text{ contracting.}\}$$

and

$$R := \{(X_1, X_2, X_3) \in \bar{B}^3 | X_3 (X_2^\top)^{-1} X_1 \text{ is symmetric and positive definite}\}.$$

Note that

$$\bar{B} = \{X \in \mathrm{GL}(n, \mathbb{R}) | \sigma(X) \subset \bar{\mathbb{D}}\}.$$

and that $O(n)$ acts by component wise conjugation on R. Recall $\Gamma_{0,3} = \{C_3, C_2, C_1 | C_3 C_2 C_1 = e\}$.

Recall Theorem 1.3.1:

Theorem 3.3.2. *Let* $\bar{f} : R \to \mathrm{Rep}(\Gamma_{0,3}, \mathrm{Sp}(2n, \mathbb{R}))$ *be the map which assigns to* $(X_1, X_2, X_3) \in R$ *the representation* $\varrho = \bar{f}(X_1, X_2, X_3)$ *of* $\Gamma_{0,3}$ *into* $\mathrm{Sp}(2n, \mathbb{R})$ *defined by*

$$\varrho(C_1) := c_1 = \begin{pmatrix} X_1 & 0 \\ X_1 + X_2^{-1} X_3^\top & (X_1^\top)^{-1} \end{pmatrix}$$

$$\varrho(C_2) := c_2 = \begin{pmatrix} -X_3^{-1}X_1^\top - X_2 - (X_2^\top)^{-1} & X_2 + X_3^{-1}X_1^\top \\ -X_3^{-1}X_1^\top - (X_2^\top)^{-1} & X_3^{-1}X_1^\top \end{pmatrix}$$

$$\varrho(C_3) := c_3 = \begin{pmatrix} (X_3^\top)^{-1} & -(X_3^\top)^{-1} - X_1^{-1}X_2^\top \\ 0 & X_3 \end{pmatrix}.$$

$$f : R/\operatorname{O}(n) \to \operatorname{Rep}_{max}(\Gamma_{0,3}, \operatorname{Sp}(2n, \mathbb{R})).$$

For the proof of Theorem 3.3.2 we need two propositions which we will prove in Section 3.3.2 resp. Section 3.3.3.

Proposition 3.3.3. *Let $\varrho : \Gamma_{0,3} \to \operatorname{Sp}(2n, \mathbb{R}))$ be a representation such that $c_1 := \varrho(C_1)$ fixes 0, $c_2 := \varrho(C_2)$ fixes e and $\varrho(C_3)$ fixes ∞. Then there exist $X_1, X_2, X_3 \in \operatorname{GL}(n, \mathbb{R})$ with $X_3(X_2^\top)^{-1}X_1$ symmetric such that*

$$c_1 = \begin{pmatrix} X_1 & 0 \\ X_1 + X_2^{-1}X_3^\top & (X_1^\top)^{-1} \end{pmatrix}$$

$$c_2 = \begin{pmatrix} -X_3^{-1}X_1^\top - X_2 - (X_2^\top)^{-1} & X_2 + X_3^{-1}X_1^\top \\ -X_3^{-1}X_1^\top - (X_2^\top)^{-1} & X_3^{-1}X_1^\top \end{pmatrix}$$

$$c_3 = \begin{pmatrix} (X_3^\top)^{-1} & -(X_3^\top)^{-1} - X_1^{-1}X_2^\top \\ 0 & X_3 \end{pmatrix},$$

and

$$T_\varrho = \frac{1}{2}(n + \operatorname{sgn} X_3(X_2^\top)^{-1}X_1). \tag{3.10}$$

Proposition 3.3.4. *Let*

$$c = \begin{pmatrix} A & 0 \\ A + (A^\top)^{-1}S & (A^\top)^{-1} \end{pmatrix} \in \operatorname{Sp}(2n, \mathbb{R}), \tag{3.11}$$

where A is invertible and S symmetric and positive definite.

(i) If $\sigma(A) \subset S^1$, then 0 is the unique fixed point of c in V.

(ii) If $\sigma(A) \not\subset S^1$, then c has a unique *fixed point Y in which the action is non-expanding. If g maps 0 to Y then $gcg^{-1} =$*

$\begin{pmatrix} A' & 0 \\ C' & (A'^{\top})^{-1} \end{pmatrix}$ *where* A' *has no eigenvalue of absolute strictly value bigger that* 1 *and* C' *is some* $n \times n$*-matrix.*

In case (i) we call 0, in case (ii) we call Y *the canonical fixed point of* c*.*

Proof of Theorem 3.3.2. First we show that the map f is well-defined. A direct calculation shows that for c_1, c_2 and c_3 as above, the product $c_3c_2c_1$ is equal to the identity if and only if $X_3(X_2^{\top})^{-1}X_1$ is symmetric. It remains to show that $\bar{f}(X_1, X_2, X_3)$ does not depend on the representative of the equivalence class in $R/O(n)$. Let $(X_1, X_2, X_3) \in R$ and $k \in O(n)$ and c_1, c_2 and c_3 be the generators of $\bar{f}(X_1, X_2, X_3)$. Then the generators of $\bar{f}(kX_1k^{-1}, kX_2k^{-1}, kX_3k^{-1})$ are lc_1l^{-1}, lc_2l^{-1} and lc_3l^{-1} where

$$l = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix},$$

hence $\bar{f}(X_1, X_2, X_3)$ and $\bar{f}(kX_1k^{-1}, kX_2k^{-1}, kX_3k^{-1})$ are in the same conjugacy class in $\mathrm{Rep}(\Gamma_{0,3}, \mathrm{Sp}(2n, \mathbb{R}))$ and f is well-defined. Furthermore $\bar{f}(X_1, X_2, X_3)$ is maximal by Formula (3.10) in Proposition 3.3.3.

To show that f is bijective, we construct an inverse map. The main ingredients are Proposition 3.3.3 and Proposition 3.3.4. Let ϱ' be a maximal representation and define $c_i' := \varrho'(C_i)$.

Every c_i' is conjugate to some c as in Corollary 1.3.5. Therefore we can apply Proposition 3.3.4 and each c_i' has a canonical fixed point X_i^+ (in which it acts non-expandingly). Since the triple (X_1^+, X_2^+, X_3^+) is maximal (Cor. 3.2.5), there exists $h \in \mathrm{Sp}(2n, \mathbb{R})$ which maps this triple to $(0, e, \infty)$. Therefore the images of C_1, C_2 resp. C_3 under the representation $\varrho := h\varrho'h^{-1}$ fix 0, e resp. ∞. We can apply Proposition 3.3.3 and get (X_1, X_2, X_3) with $X_3(X_2^{\top})^{-1}X_1$ symmetric positive definite, such that $\varrho = \bar{f}(X_1, X_2, X_3)$. By construction the $\varrho(C_i)$ are non-expanding in 0, e and ∞, respectively, hence

$(X_1, X_2, X_3) \in R$. Since h is unique up to left-multiplication with $k \in \mathrm{stab}_G((0, e, \infty)) = O(n)$, the triple (X_1, X_2, X_3) is unique up to conjugation by an element in $O(n)$. This provides a map inverse to $\bar{f}$. $\square$

3.3.2 Parameters for Maximal Representations

In this section we proof Proposition 3.3.3. We do all calculation in the tube model T_Ω, in particular we use the boundary $V \subset \bar{T}_\Omega$.

We define

$$\tilde{R}_n := \{(X_1, X_2, X_3) \in \mathrm{GL}(n, \mathbb{R})^3 | X_3 (X_2^\top)^{-1} X_1 \text{ symmetric} \\ \text{and positive definite}\}$$

Remark 3.3.5. Note that $(X_1, X_2, X_3) \in \tilde{R}_n$ if and only if

$$(\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3) \in \tilde{R}$$

for $(\lambda_1, \lambda_2, \lambda_3) \in \tilde{R}_1$.

Proof of Proposition 3.3.3. The proof relies on the information for the position of the points $c_1\infty$, $c_2 0$ and $c_3 e$ in V expressed in terms of the Maslov index in Corollary 3.2.5.

Define Z_1, Z_2 and Z_3 in $\check{S}$ by

$$Z_1 := c_3 e = c_1^{-1} e, \quad Z_2 := c_1 \infty = c_2^{-1} \infty, \quad Z_3 := c_2 0 = c_3^{-1} 0,$$

where the second equalities hold because $c_3 c_2 c_1 = id$ and each of 0, e and ∞ is a fixed points of one of the c_i.

We can summarize that to the following conditions for the c_i:

$$c_1 : \begin{cases} 0 \mapsto 0 \\ Z_1 \mapsto e \\ \infty \mapsto Z_2 \end{cases} \qquad c_2 : \begin{cases} e \mapsto e \\ Z_2 \mapsto \infty \\ 0 \mapsto Z_3 \end{cases} \qquad c_3 : \begin{cases} \infty \mapsto \infty \\ Z_3 \mapsto 0 \\ e \mapsto Z_1 \end{cases} \tag{3.12}$$

These conditions gives a system of three equations for each c_i, which determine each c_i up to an element of a stabilizer of a maximal triple in $\check{S}$. We will calculate all solutions of these equations.

By (3.8) we have

$$\beta(0, Z_2, e) = \beta(Z_1, 0, \infty) = \beta(e, Z_3, \infty) \tag{3.13}$$

and by Example 2.3.11 this is the case if and only if $-Z_1$, $Z_2^{-1} - I_n$ and $Z_3 - I_n$ have the same signatures. If Z is a symmetric matrix of signature l, then there exists an invertible matrix M, such that $Z = MI_{p,q}M^\top$ with $p - q = l$ and $I_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$.

The matrix M is unique up to right multiplication with $k \in O(p,q)$. Note that $k \in O(p,q)$ if and only if $kI_{p,q}k^\top = I_{p,q}$ or equivalently $I_{p,q}kI_{p,q} = (k^\top)^{-1}$. So for the rest of the proof we fix M_i, s.t.

$$-Z_1 = M_1 I_{p,q} M_1^\top,\ Z_2^{-1} - I_n = M_2 I_{p,q} M_2^\top,\ Z_3 - I_n = M_3 I_{p,q} M_3^\top, \tag{3.14}$$

Now we solve the systems of equations. Write the c_i in block form as in Section 2.1.5 and recall that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is symplectic if and only if

$$A^\top D - C^\top B = I, \quad A^\top C = C^\top A, \quad D^\top B = B^\top D$$

and the action of $\mathrm{Sp}(2n, \mathbb{R})$ on V as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} X = (AX + B)(CX + D)^{-1}.$$

Lets start with c_1:

$$c_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{cases} 0 \mapsto 0 \\ \infty \mapsto Z_2 \\ Z_1 \mapsto e \end{cases}$$

- From the first condition we get $B = 0$.
- Together with the condition that g is a symplectic matrix we have $D = (A^\top)^{-1}$.
- The second condition is equivalent to $AC^{-1} = Z_2$. Hence $C = Z_2^{-1}A$.
- The last condition is $AZ_1(CZ_1 + D)^{-1} = e$, therefore $D = AZ_1 - CZ_1$.

Combining the last two results we get

$$(A^\top)^{-1} = D = AZ_1 - Z_2^{-1}AZ_1 = \underbrace{(Z_2^{-1} - I_n)}_{=M_2 I_{p,q} M_2^\top} A \underbrace{(-Z_1)}_{=M_1 I_{p,q} M_1^\top}. \quad (3.15)$$

Therefore $A := (M_2^\top)^{-1} k_1 M_1^{-1}$ is a solution of this equation for all $k_1 \in O(p,q)$ and these are the only solutions. Indeed, since $D = (A^\top)^{-1}$ we can write Equation (3.15) as

$$M_2^{-1} D (M_1^\top)^{-1} = M_2^{-1}(A^\top)^{-1}(M_1^\top)^{-1} = I_{p,q} M_2^\top A M_1 I_{p,q},$$

which is equivalent to

$$((M_2^\top A M_1)^\top)^{-1} = I_{p,q} M_2^\top A M_1 I_{p,q},$$

hence $M_2^\top A M_1 \in O(p,q)$. For C we calculate:

$$\begin{aligned} C =& (Z_2^{-1} - I)A + A = M_2 I_{p,q} M_2^\top (M_2^\top)^{-1} k_1 M_1^{-1} + (M_2^\top)^{-1} k_1 M_1^{-1} \\ =& M_2 I_{p,q} k_1 M_1^{-1} + (M_2^\top)^{-1} k_1 M_1^{-1}. \end{aligned}$$

We summarize this to:

$$c_1 = \begin{pmatrix} (M_2^\top)^{-1} k_1 M_1^{-1} & 0 \\ M_2 I_{p,q} k_1 M_1^{-1} + (M_2^\top)^{-1} k_1 M_1^{-1} & M_2 (k_1^\top)^{-1} M_1^\top \end{pmatrix}$$

Here is the calculation of c_2:

$$c_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{cases} I_n \mapsto I_n \\ Z_2 \mapsto \infty \\ 0 \mapsto Z_3 \end{cases}$$

- From the first condition we get $A + B = C + D$.
- The second gives $CZ_2 + D = 0$, hence $C = -DZ_2^{-1}$.
- The third is equivalent to $BD^{-1} = Z_3$, therefore $B = Z_3 D$.
- Furthermore the matrix has to be symplectic, hence $A^\top D - C^\top B = I_n$. This is equivalent to $A = (D^\top)^{-1} + (D^\top)^{-1} B^\top C$.

Combining the first and the last of these items, we get $(D^\top)^{-1} = C + D - B - (D^\top)^{-1} B^\top C$. Together with the other items this yields

$$(D^{-1})^\top = \underbrace{(Z_3 - I_n)}_{M_3 I_{p,q} M_3^\top} D \underbrace{(Z_2^{-1} - I_n)}_{M_2 I_{p,q} M_2^\top}$$

and we see that $D = (M_3^\top)^{-1} k_2 M_2^{-1}$ is a solution for any $k_2 \in O(p,q)$, as above. We get

$$c_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$\begin{aligned} A =& -(M_3^\top)^{-1} I_{p,q} (k_2^\top)^{-1} M_2^\top - M_3 I_{p,q} k_2 M_2^{-1} - (M_3^\top)^{-1} k_2 M_2^{-1} \\ B =& M_3 I_{p,q} k_2 M_2^{-1} + (M_3^\top)^{-1} k_2 M_2^{-1} \\ C =& -(M_3^\top)^{-1} k_2 I_{p,q} M_2^\top - (M_3^\top)^{-1} k_2 M_2^{-1} \\ D =& (M_3^\top)^{-1} k_2 M_2^{-1} \end{aligned}$$

Now we calculate c_3, it is similar to the case above:

$$c_3 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{cases} \infty \mapsto \infty \\ Z_3 \mapsto 0 \\ e \mapsto Z_1 \end{cases}$$

Note that by the construction of the configuration the Z_i are all invertible.

- From the first condition we get $C = 0$.
- Together with the condition that g is a symplectic matrix we have $D = (A^\top)^{-1}$.
- The second condition is equivalent to $AZ_3 + B = 0$. Hence $B = -AZ_3$.
- The last condition says $(A + B)D^{-1} = Z_1$.

This gives

$$A \underbrace{(Z_3 - I_n)}_{M_3 I_{p,q} M_3^\top} A^\top = \underbrace{-Z_1}_{M_1 I_{p,q} s M_1^\top} .$$

and $A := M_1 k_3 M_3^{-1}$ is a solution for all $k_3 \in O(p, q)$, as in the calculation for c_1 and we get:

$$c_3 = \begin{pmatrix} M_1 k_3 M_3^{-1} & -M_1 k_3 I_{p,q} M_3^\top - M_1 k_3 M_3^{-1} \\ 0 & (M_1^\top)^{-1} (k_3^\top)^{-1} M_3^\top \end{pmatrix}$$

Calculating $c_3^{-1} c_1^{-1}$ and comparing this matrix with c_2 shows that the product $c_3 c_2 c_1$ is the identity if and only if $k_3 (k_2^\top)^{-1} k_1 = 1$.

Note that the sixtuples $(M_1, M_2, M_3, k_1, k_2, k_3)$ as above define precisely the same representation as $(M_1 k_1^{-1}, M_2, M_3 (k_2^\top)^{-1}, e, e, e)$. Hence the k_i do not give new parameters. We choose $k_1 = k_3 = I_{p,q}$ and $k_2 = \mathrm{id}$.

We get:

$$\begin{aligned}
c_1 &= \begin{pmatrix} (M_2^\top)^{-1} I_{p,q} M_1^{-1} & 0 \\ M_2 I_{p,q} I_{p,q} M_1^{-1} + (M_2^\top)^{-1} I_{p,q} M_1^{-1} & M_2 I_{p,q} M_1^\top \end{pmatrix} \\
c_2 &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\
c_3 &= \begin{pmatrix} M_1 I_{p,q} M_3^{-1} & -M_1 I_{p,q} I_{p,q} M_3^\top - M_1 I_{p,q} M_3^{-1} \\ 0 & (M_1^\top)^{-1} I_{p,q} M_3^\top \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
A =& - (M_3^\top)^{-1} I_{p,q} M_2^\top - M_3 I_{p,q} M_2^{-1} - (M_3^\top)^{-1} M_2^{-1} \\
B =& M_3 I_{p,q} M_2^{-1} + (M_3^\top)^{-1} M_2^{-1} \\
C =& - (M_3^\top)^{-1} I_{p,q} M_2^\top - (M_3^\top)^{-1} M_2^{-1} \\
D =& (M_3^\top)^{-1} 2 M_2^{-1}
\end{aligned}$$

Now define

$$\begin{aligned}
X_1 :=& (M_2^\top)^{-1} I_{p,q} M_1^{-1} \\
X_2 :=& M_3 I_{p,q} I_{p,q} M_2^{-1} \\
X_3 :=& (M_1^\top)^{-1} I_{p,q} M_3^\top .
\end{aligned}$$

Observe $X_3 (X_2^\top)^{-1} X_1 = (M_1^{-1})^\top I_{p,q} M_1^{-1}$ is a symmetric matrix of signature $p - q$.

We use Formula (1.1) to deduce the Formula 3.10. By construction the generators have $y_1 = 0$, $y_2 = e$ and $y_3 = \infty$ as fixed points, hence $\beta(y_1, y_2, y_3) = n$. For the second term in Formula (1.1) we calculate

$$\beta(y_3, c_3 \cdot y_2, y_1) = \beta(c_3 \cdot y_2, y_1, y_3) = \beta(c_3 \cdot e, 0, \infty). \tag{3.16}$$

By Example 2.3.11, $-\beta(c_3 \cdot e, 0, \infty)$ is the signature of the symmetric matrix $c_3 \cdot e$ and we calculate

$$c_3 \cdot e = (X_3^\top)^{-1} - (X_3^\top)^{-1} - X_1^{-1} X_2^\top X_3^{-1} = -X_1^{-1} X_2^\top X_3^{-1}.$$

Hence $\beta(c_3 \cdot e, 0, \infty) = \operatorname{sgn} X_1^{-1} X_2^\top X_3^{-1} = \operatorname{sgn} X_3 (X_2^\top)^{-1} X_1$. This finishes the proof. □

The matrices of the form $X_3(X_2^\top)^{-1}X_1$ have a geometrical interpretation:

$$\begin{aligned} X_3(X_2^\top)^{-1}X_1 &= -Z_1^{-1} = (M_1 I_{p,q} M_1^\top)^{-1} \\ X_1(X_3^\top)^{-1}X_2 &= (Z_2^{-1} - I_n)^{-1} = (M_2 I_{p,q} M_2^\top)^{-1} \\ X_2(X_1^\top)^{-1}X_3 &= Z_3 - I_n = M_3 I_{p,q} M_3^\top . \end{aligned}$$

3.3.3 Fixed Points of Generators of Maximal Representations

From Proposition 3.3.3 we know that image of a standard generator, $\varrho(C_i)$, under a maximal representation $\varrho : \Gamma_{0,3} \to \mathrm{Sp}(2n, \mathbb{R})$ is conjugate to

$$c = \begin{pmatrix} A & 0 \\ A + (A^\top)^{-1} S & (A^\top)^{-1} \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}), \tag{3.17}$$

with A invertible and S symmetric definite. Throughout this section we will assume that c has this form. It has at least one fixed point in $\check{S}$, but maybe more. We will show that c has a unique fixed point in $\check{S}$, in which it acts non-expandingly. The same proof can be used to show that c has a unique fixed point in which it acts non-contractingly. The fixed non-attracting fixed point and the non-repellent fixed point are transversal if and only if A has no eigenvalue of absolute value 1. We use the non-expanding fixed point as the *canonical fixed point*.

Remark 3.3.6. All fixed points of c are in V. Indeed, c can appear as the image of a standard generator under a maximal representation of $\Gamma_{0,3}$, say $c = \varrho(C_1)$. We can assume that a fixed point of, say, $\varrho(C_3)$ is equal to ∞. By Formula (1.1) every fixed point of c is transverse to ∞, hence is contained in V.

Remark 3.3.7. In the sequel we sometimes write C for $A+(A^\top)^{-1}S$.

Proof of Proposition 3.3.4

We prove the theorem for S-hyperbolic c in Section 3.3.3 and for S-parabolic c in Section 3.3.3 and use this to write down the desired fixed point explicitly. Note that we use here that there are $\mathrm{Sp}(2n,\mathbb{R})$ contains copies of $\mathrm{Sp}(2k,\mathbb{R})$ for $k \leq n$. This statement does not hold in an analogous form for other Hermitian Lie groups (e.g. the exceptional one), whence this proof can not be generalized one-to-one.

Proof of Proposition 3.3.4. (i) follows immediately from Proposition 3.3.16.

For (ii) we have to combine methods from the last two subsections. As in the proof of Proposition 3.3.13 we can assume that A is of block form $\begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix}$ where the eigenvalues of A_2 all have absolute value 1 and the absolute values of the eigenvalues of A_1 are different from 1. If A_1 is a $k \times k$ matrix, we denote by C_1 the upper left $k \times k$ block in the $n \times n$-matrix $A+(A^\top)^{-1}S$. Then $C_1 = A_1 + (A_1^\top)^{-1}S_1$, where S_1 is the upper left $k \times k$ block of the right size of S; it is automatically symmetric positive definite. Then

$$c_1 = \begin{pmatrix} A_1 & 0 \\ C_1 & (A_1^\top)^{-1} \end{pmatrix} \in \mathrm{Sp}(2k,\mathbb{R})$$

is by construction hyperbolic. Hence it has a unique fixed point Y_1 in $V_k = Sym(k,\mathbb{R})$ in which the action of c_1 is contracting. By Lemma 3.3.9 $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix} \in V$ is a fixed point of c and c acts non-expandingly in X.

Now it remains to show that this is the unique fixed point with this property. After conjugating c with $g = \begin{pmatrix} 1 & -Y \\ 0 & 1 \end{pmatrix}$, we can assume that 0 is a non-repellent fixed point. Let $\bar{Y}$ be another fixed point. Again, after eventual conjugation with an isometry $h \in O(n)$ (which stabilizes 0) we can assume that $\bar{Y} = \begin{pmatrix} \bar{Y}_1 & 0 \\ 0 & 0 \end{pmatrix}$, with $\bar{Y}_1$ invertible. By Lemma 3.3.9 (ii)

$$hgc_1(hg)^{-1} = \begin{pmatrix} \bar{A} & 0 \\ \bar{C} & (\bar{A}^\top)^{-1} \end{pmatrix},$$

with

$$\bar{A} = \begin{pmatrix} \bar{A}_1 & \bar{A}_2 \\ 0 & \bar{A}_4 \end{pmatrix} \text{ and } \bar{C} = \begin{pmatrix} \bar{C}_1 & \bar{C}_2 \\ \bar{C}_3 & \bar{C}_4 \end{pmatrix}$$

and $\bar{Y}_1$ is a fixed point of $\bar{c}_1 = \begin{pmatrix} \bar{A}_1 & 0 \\ \bar{C}_1 & (\bar{A}_1^\top)^{-1} \end{pmatrix}$. Since $\bar{Y}_1$ is invertible we can calculate $d\bar{c}_1|_{\bar{Y}_1}$ as in the proof of Lemma 3.3.14 and we get

$$d\bar{c}_1|_{\bar{Y}_1} : v \mapsto (\bar{Y}_1(A_1^\top)^{-1}\bar{Y}_1^{-1})c(\bar{Y}_1(A^\top)^{-1}\bar{Y}_1^{-1})^\top$$

and the action of $\bar{c}_1$ in $\bar{Y}_1$ is expanding. Hence c has at least one expanding direction in any fixed point different from 0 and 0 is the only non-repellent fixed point. This finishes the proof. □

In Section 3.3.3 and 3.3.3 we introduce and discuss hyperbolic respectively parabolic isometries and we prove Proposition 3.3.4 for these two cases before we conclude the general case in 3.3.3.

Now we recall shortly some facts used later, fix some terminology and show how to construct fixed points for c.

Recall that the equation for a fixed point $Y \in V$ is

$$Y(A + (A^\top)^{-1}S)Y + Y(A^\top)^{-1} - AY = 0. \tag{3.18}$$

Remark 3.3.8. Later we will sometimes assume that certain matrices $Y \in Sym(n, \mathbb{R})$ have the special form $\begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}$ with $Y_1 \in$

$\mathrm{Sym}(k,\mathbb{R})$ diagonal and invertible. This is allowed since every element of V is a symmetric matrix. Hence there exists $k \in \mathrm{O}(n)$ such that kYk^{-1} has this form. Furthermore $l := \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$ and if Y is a fixed point for $g = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, then kYk^{-1} is a fixed point for

$$lgl^{-1} = \begin{pmatrix} kAk^{-1} & 0 \\ kCk & kDk^{-1} \end{pmatrix}.$$

Clearly the spectrum of A is equal to the spectrum of kAk^{-1}.

We can use a block structure of A to construct fixed points of c in $\check{S}$.

Lemma 3.3.9. *(i) Let* $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R})$, *such that* A_1 *is a* $k \times k$*-matrix. Write* $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$, *where* S_1 *has the same size as* A_1. *Now let* $i \in \{1,4\}$ *and define*

$$d_i = \begin{pmatrix} A_i & 0 \\ A_i + (A_i^\top)^{-1} S_i & (A_i^\top)^{-1} \end{pmatrix} \in \mathrm{Sp}(2k,\mathbb{R}).$$

Let $Y_1 \in \mathrm{Sym}_k(\mathbb{R})$ *be a fixed point for* d_1. *Then* $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}$ *is a fixed point for* c.

(ii) Conversely if $\begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}$ *is a fixed point for* c *with* Y_1 *invertible, then* $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$, *where* A_1 *has the same size as* Y_1 *and* Y_1 *is a fixed point for* c_1 *defined as in (i).*

(iii) Let Y *be a fixed point of* c. *Then the differential of* c *in* Y *is*

$$dc|_Y(v) : v \mapsto (-YC + A)v(CY + D)^{-1}.$$

Corollary 3.3.10. *Using the notation from Theorem 1.3.1 we have:*

$$dc_1|_0(v) = X_1 v X_1^\top, \quad dc_2|_e(v) = X_2 v X_2^\top, \quad dc_3|_\infty(v) = X_3 v X_3^\top.$$

Remark 3.3.11. The matrices d_1 and d_4 can appear as images of standard generators of a maximal representations into $\mathrm{Sp}(2k,\mathbb{R})$ resp. $\mathrm{Sp}(2(n-k),\mathbb{R})$ (see Proposition 3.3.3).

Proof of Lemma 3.3.9. (i) For statement (i) note that S_1 is positive definite symmetric because S is. The verification that Y is a fixed point point of c is straight forward. Indeed inserting $\begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}$ in the fixed point equation (3.18) gives

$$\begin{aligned} &\begin{pmatrix} Y_1 A_1 Y_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Y_1((A^\top)^{-1}S)_1 Y_1 & 0 \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} Y_1(A^\top)^{-1}_1 & Y_1((A^\top)^{-1})_2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} A_1 Y_1 & 0 \\ A_3 Y_1 & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

Since we assume in (i) that $A_3 = 0$ and that Y_1 is a fixed point for c_1, this equality is true.

(ii) follows from the same equation. The matrix A_3 is equal to 0 since Y_1 is invertible.

(iii) We calculate the differential of c at some point $Y \in V$. We know:

$$c : Y \mapsto AY(CY + D)^{-1}.$$

First we calculate a power series for the map $v \mapsto (C(Y+v)+D)^{-1}$ for small $v \in V$. We abbreviate $M := CY + D$.

$$\begin{aligned} (C(Y+v)+D)^{-1} =& (M + Cv)^{-1} = (1 + M^{-1}Cv)^{-1}M^{-1} \\ =& \sum_{i=0}^{\infty} (-M^{-1}Cv)^i M^{-1}, \end{aligned}$$

where we where allowed to use the geometric series for matrices since we asked v to be small.

Therefore we get

$$\begin{aligned}&A(Y+v)(C(Y+v)+D)^{-1}\\=&AY(C(Y+v)+D)^{-1}+Av(C(Y+v)+D)^{-1}\\=&\sum_{i=0}^{\infty}AY(-M^{-1}Cv)^{i}M^{-1}+\sum_{i=0}^{\infty}Av(-M^{-1}Cv)^{i}M^{-1}\end{aligned}$$

and the differential in the point Y is:

$$dc|_Y(v)=(-AY(CY+D)^{-1}C+A)v(CY+D)^{-1}.$$

If Y is a fixed point of c this is:

$$dc|_Y(v)=(-YC+A)v(CY+D)^{-1}.$$

□

Proof of Proposition 3.3.4 for S-hyperbolic Isometries

We recall

Definition 3.3.12. Let G be a Hermitian Lie group and $g \in G$. Then g is *Shilov-hyperbolic* (or S-hyperbolic) if it has a pair (g^+, g^-) of transversal fixed points in $\check{S}$, such that g contracts an open and dense subset of $\check{S}$ to g^+ and g^{-1} contracts an open and dense subset to g^-. Note that the fixed points c^+ and c^- are uniquely determined.

Proposition 3.3.13. *Let c be as in* (3.17) *with $\sigma(A) \cap S^1 = \emptyset$. Then c is S-hyperbolic.*

Before we give the general proof, we need a proof in a special case:

Lemma 3.3.14. *Let c be as in* (3.17).

(i) *Assume that A only has eigenvalues of absolute value strictly smaller than* 1. *Then c has a unique fixed Y point transversal to* 0. *It satisfies $\beta(Y, 0, \infty) = n$. The action of c in Y is expanding and the differential $dc|_Y$ acts as on $T_Y V$ as*

$$dc|_Y : v \mapsto (Y(A^\top)^{-1}Y^{-1})c(Y(A^\top)^{-1}Y^{-1})^\top.$$

(ii) *If A only has eigenvalues of absolute value strictly bigger that* 1*, then c has a unique fixed point Y transversal to* 0. *It satisfies $\beta(0, Y, \infty) = n$. The action of c in Y is contracting and the differential $dc|_Y$ acts as on $T_Y V$ as*

$$dc|_Y : v \mapsto (Y(A^\top)^{-1}Y^{-1})v(Y(A^\top)^{-1}Y^{-1})^\top.$$

Furthermore in both cases Y is invertible and depends continuously on c.

Proof. (i) We are searching for a fixed point Y transversal to 0, hence we search for an invertible one. We can reformulate the fixed point equation 3.18 to $A^\top Y^{-1}A - Y^{-1} = \bar{S}$, where $\bar{S} := A^\top A + S$ is positive definite symmetric.

One verifies easily that

$$Y^{-1} = -\sum_{i=0}^{\infty}(A^\top)^i \bar{S} A^i,$$

is a solution, which is clearly negative definite. The sum converges since A is contracting. Furthermore it is unique because the equation for Y^{-1} is a linear matrix equation [41, Ch.4.3] which has a unique solution if and only if for any eigenvalues λ and μ of A, $\lambda\mu \neq 1$. Here this is clearly true by assumption. In particular Y depends continuously on c.

From Lemma 3.3.9 we know that if Y is a fixed point of c, then:

$$dc|_Y(v) = (-YC + A)v(CY + D)^{-1}.$$

For $Y = 0$ we have $dc|_0(v) = AvA^\top$ and for general Y we get, using the fixed point formula,

$$A - YC = Y(A^\top)^{-1}Y^{-1} \text{ and } (CY + D)^{-1} = Y^{-1}A^{-1}Y, \tag{3.19}$$

hence

$$dc|_Y(v) = (Y(A^\top)^{-1}Y^{-1})v(Y(A^\top)^{-1}Y^{-1})^\top.$$

Therefore c is expanding in Y.

(ii) Analogously.

□

Proof of Proposition 3.3.13. By Lemma A.1.1 we can assume that

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix},$$

such that the eigenvalues of the $k \times k$-matrix A_1 have absolute value strictly bigger than 1 and the absolute values of A_4 have absolute value strictly less than 1. We use Lemma 3.3.14 and Lemma 3.3.9 to construct the desired fixed points X^+ and X^-. By Lemma 3.3.14

$$\begin{pmatrix} A_1 & 0 \\ A_1 + (A_1^\top)^{-1}S_1 & (A_1^\top)^{-1} \end{pmatrix} \in \mathrm{Sp}(2k, \mathbb{R})$$

and

$$\begin{pmatrix} A_4 & 0 \\ A_4 + (A_4^\top)^{-1}S_4 & (A_4^\top)^{-1} \end{pmatrix} \in \mathrm{Sp}(2(n-k), \mathbb{R})$$

have fixed points Y_1 resp. Y_4 transversal to 0 in their respective Shilov boundaries. By Lemma 3.3.9

$$y_1 := \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad y_4 := \begin{pmatrix} 0 & 0 \\ 0 & Y_4 \end{pmatrix}$$

are fixed points for c, and since $Y_1 \in \mathrm{Sym}_k(\mathbb{R})$ (with k as above) and $Y_4 \in \mathrm{Sym}_{n-k}(\mathbb{R})$ are invertible, $y_1 - y_4$ is invertible, hence y_1 and y_4 are transversal.

By Lemma 3.3.9 (iii) we know that

$$dc|_{y_1} : v \mapsto (-y_1 C + A)v(Cy_1 + D)^{-1}.$$

A straight forward calculation shows that

$$-y_1 C + A = \begin{pmatrix} A_1 - Y_1 C_1 & Y_1 C_2 \\ 0 & A_4. \end{pmatrix}$$

Since Y_1 is invertible and a fixed point for c_1 we get from (3.18):

$$A_1 - Y_1 C_1 = Y_1^{-1}(A_1^\top)^{-1} Y_1,$$

hence

$$-y_1 C + A = \begin{pmatrix} Y_1^{-1}(A_1^\top)^{-1} Y_1 & Y_1 C_2 \\ 0 & A_4. \end{pmatrix}$$

The eigenvalues are the eigenvalues of $(A_1^\top)^{-1}$ and A_4, hence it is contracting

The same calculation shows

$$(Cy_1 + D)^{-1} = (-y_1 C + A)^\top.$$

Hence c acts contracting in y_1 Along the same lines one can show that y_4 is a repellent fixed point for c.

Let g be an isometry which maps $(0, \infty)$ to (y_4, y_1). Then $gcg^{-1} = \begin{pmatrix} \bar{A} & 0 \\ 0 & (\bar{A}^\top)^{-1} \end{pmatrix}$ where $\bar{A}$ is a conjugate to $-y_1 C + A$, hence contracting. Therefore gcg^{-1} contracts V to 0 and V can be seen as an open and dense subset of the Shilov boundary $\check{S}$ of the bounded symmetric space associated with $\mathrm{Sp}(2n, \mathbb{R})$. $\square$

Proof of Proposition 3.3.4 for S-parabolic Isometries

Definition 3.3.15. Let G be a Hermitian Lie group. Then $g \in G$ is *Shilov-parabolic* or S-parabolic if g has a unique fixed point in $\check{S}$.

Proposition 3.3.16. *Let c be as in 3.17. Assume $\sigma(A) \subset S^1$. Then c is S-parabolic.*

Proof. Let $Y \in V$ be a fixed point. Since Y is a symmetric matrix we can assume without loss of generality that

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}$$

such that Y_1 is a square matrix and diagonal invertible. Now we can apply Lemma 3.3.9 (ii). Therefore A decomposes into a block form and the eigenvalues of A_1 also have absolute value 1. Multiplying both sides of the fixed point equation (3.18) for Y_1 from the left with $A_1^\top Y_1^{-1}$ and from the right with Y_1^{-1} (which we are allowed to, since Y_1 was chosen to be invertible), we get

$$A_1^\top Y_1^{-1} A_1 - Y_1^{-1} = A_1^\top A_1 + S_1.$$

The right hand side is positive definite, but for the left hand we can choose an eigenvector v to a (possibly complex) eigenvalue λ with $|\lambda| = 1$. Since A_1 is a real matrix, we have $A_1^\top = A^*$ and hence

$$\begin{aligned} v^* A_1^\top Y_1^{-1} A_1 v - v^* Y_1^{-1} v =& v^* A_1^* Y_1^{-1} A_1 v - v^* Y_1^{-1} v \\ =& \underbrace{\bar{\lambda}\lambda}_{=1} v^* Y_1^{-1} v - v^* Y_1^{-1} v = 0, \end{aligned}$$

which is a contradiction. Hence 0 is the only fixed point of c. ☐

3.4 Parameters for $\mathrm{Rep}(\Gamma_{0,3}, G)$

3.4.1 Motivation

Remark 3.4.1. As a first hint for a generalization for general groups of tube type we decompose c_1, c_2 and c_3 from Section 3.3.2 according to the decomposition of $G(T_\Omega)$ from Proposition 2.1.20 as:

$$
\begin{aligned}
c_1 &= \begin{pmatrix} (M_2^\top)^{-1}M_1^{-1} & 0 \\ (M_2 + (M_2^\top)^{-1})M_1^{-1} & M_2M_1^\top \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} 1 & 0 \\ M_2M_2^\top + 1 & 1 \end{pmatrix}}_{\in N^-} \underbrace{\begin{pmatrix} (M_2^\top)^{-1}M_1^{-1} & 0 \\ 0 & M_2M_1^\top \end{pmatrix}}_{\in G(\Omega)} \\
c_2 &= \underbrace{\begin{pmatrix} 1 & 1 + M_3M_3^\top \\ 0 & 1 \end{pmatrix}}_{\in N^+} \underbrace{\begin{pmatrix} M_3M_2^{-1} & 0 \\ 0 & (M_3^\top)^{-1}M_2^\top \end{pmatrix}}_{\in G(\Omega)} \cdot \\
&\quad \cdot \underbrace{\begin{pmatrix} M_2M_2^\top & 0 \\ -1 - (M_2M_2^\top)^{-1} & (M_2^\top)^{-1}M_2^{-1} \end{pmatrix}}_{\in N^- G(\Omega)} \\
c_3 &= \begin{pmatrix} M_1M_3^{-1} & -M_1(M_3^\top + M_3^{-1}) \\ 0 & (M_1^\top)^{-1}M_3^\top \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} M_1M_3^{-1} & 0 \\ 0 & (M_1^\top)^{-1}M_3^\top \end{pmatrix}}_{\in G(\Omega)} \underbrace{\begin{pmatrix} 1 & -1 - M_3M_3^\top \\ 0 & 1 \end{pmatrix}}_{\in N^+}
\end{aligned}
$$

or

$$
c_1 = n^-_{-Z_2^{-1}}\theta(g_2)g_1^{-1}, \quad c_2 = n^+_{Z_3}g_3\theta(g_2)^{-1}n^-_{Z_2^{-1}}, \quad c_3 = g_1g_3^{-1}n^+_{-Z_3},
$$

where

$$g_i = \begin{pmatrix} M_i & 0 \\ 0 & (M_i^\top)^{-1}. \end{pmatrix} \in G(\Omega),\ n^-_{Z_2^{-1}} = \begin{pmatrix} 1 & 0 \\ Z_2^{-1} & 1 \end{pmatrix} \in N^-$$

and

$$n^+_{Z_3} = \begin{pmatrix} 1 & Z_3 \\ 0 & 1 \end{pmatrix} \in N^+$$

Note that the third matrix in the decomposition of c_2 only depends on M_2.

3.4.2 Generalization

In this section we will generalize the results from Section 3.3 to representations from $\Gamma_{0,3}$ to arbitrary Hermitian Lie groups G of tube type. The main result is 3.4.4.

Recall that the symmetric space associated with G is biholomorphic to a tube domain $V \oplus i\Omega$, where V is an Euclidean Jordan algebras and Ω is the symmetric cone in V. The group $G(\Omega) \subset GL(V)$, which fixes Ω acts transitively on Ω. It is closed under taking adjoints with respect to the Euclidean structure of V. We denote the adjoint of $g \in G(\Omega)$ by g^*. From [29, Thm. III.5.3] we have with $\theta(g) := (g^*)^{-1}$ for all $x \in \Omega$:

$$(gx)^{-1} = \theta(g)x^{-1}.$$

If $x = ge$ for $g \in G(\Omega)$, then $x^{-1} = \theta(g)e$. In particular the properties symmetric and positive definite are defined in $G(\Omega)$.

Before we can formulate the generalization for the results from previous section, we show that the symmetric cone Ω in a simple Euclidean Jordan algebra and the cone of symmetric positive definite matrices in $G(\Omega)$ are $G(\Omega)$-equivariantly identifiable.

Proposition 3.4.2. *Let $\Omega \subset V$ the symmetric cone in a simple Euclidean Jordan algebra. Let $K := G(\Omega) \cap O(V)$. Then the cone C of symmetric positive definite matrices in $G(\Omega)$ is the homogeneous cone $G(\Omega)/K \simeq \Omega$.*

Proof. The map $g \mapsto gg^*$ provides a map from $G(\Omega)$ to C and the stabilizer of e is equal to K. Surjectivity for this map follows from Proposition 3.4.3 (ii). □

Proposition 3.4.3. *(i) Let V be an Euclidean Jordan algebra with scalar product $\langle \cdot, \cdot \rangle$. Fix a Jordan frame $\{c_1, \dots, c_r\}$. Define $K := G(\Omega) \cap O(V)$ and*

$$A := \{P(a) | a = \sum \lambda_i c_i,\ \lambda_i > 0\}.$$

Let $g \in G(\Omega)$. Then there exists $k, h \in K$ and $a \in A$ such that $g = kah$.

(ii) Let $s \in G(\Omega) \subset GL(V)$ be positive definite and symmetric with respect to the given scalar product. Then $s = gg^*$ for some $g \in G(\Omega)$.

Proof. (i) Let $g \in G(\Omega)$. Let $x := g.e$. Then $x = k \sum \lambda_i c_i$ ([29, Ch. IV.2]), hence $g.e = ka.e$, with $a \in A$ and $k \in K$. In particular $g = kah$ with $h \in K$, since $(ka)^{-1}g \in G(\Omega)$ stabilizes e.

(ii) We can assume that all $a \in A$ are diagonal matrices. By (i) we can write $s = kP(a)h$ and since the property symmetric positive definite is invariant under conjugation with $h \in K$, we can assume $s = P(a)k$.

Since $P(a)$ is symmetric ([29, Prop. VII.2.4]) and by construction positive define, k only has positive eigenvalues. Indeed let λ be an eigenvalue and v be a non-zero eigenvector, we have

$$\langle v, P(a)kv \rangle = \lambda \underbrace{\langle v, P(a)v \rangle}_{>0} > 0.$$

But since k is orthogonal, the eigenvalue has to be 1. Hence $k = id$ and $s = P(a)$.

Define $\bar{a} := \sum \sqrt{\lambda_i} c_i$. It is an element of Ω and it satisfies $\bar{a}^2 = a$. and there exists a matrix $\sqrt{a}$ with positive eigenvalues and $\sqrt{a}^2 = a$. Furthermore $P(\bar{a})P(\bar{a})^\top = P(\bar{a})^2 = P(\bar{a}^2) = P(a) = s$ ([29, Ch. II.3]), where we used $P(a) = P(a)^\top \in G(\Omega)$.

□

Let $\varrho : \Gamma_{0,3} \to G$ be a maximal representation and define $c_i := \varrho(C_i)$. From the discussion in Section 3.3 and Formula 1.1 we know that the c_i have fixed points $y_i \in \check{S}$. As in the previous section we can conjugate ϱ such that the fixed points are 0, e and ∞ respectively.

Recall that in the last section we used the points in V:

$$Z_1 := c_3 e = c_1^{-1} e, \quad Z_2 := c_1 \infty = c_2^{-1} \infty, \quad Z_3 := c_2 0 = c_3^{-1} 0, \tag{3.20}$$

and by Corollary 3.2.5 they satisfy

$$\beta(Z_1, 0, \infty) = \beta(0, Z_2, e) = \beta(e, Z_3, \infty) = r, \tag{3.21}$$

We will express these conditions in terms of the spectral values of Z_i (see Theorem 2.1.14) and relate these points to points in Ω, the open symmetric cone in V.

As in Example 2.1.6 we use the fact that we can calculate the Maslov index easily if the three points are in a common polydisc. Recall that

$$\Omega = \left\{ x \in V | x = \sum \lambda_i c_i, \ \lambda_i > 0 \text{ for some Jordan frame } (c_i) \right\}.$$

Now we can characterise the properties above as follows:

- $\beta(Z_1, 0, \infty) = r$ if and only if all spectral values of Z_i are strictliy negative if and only if $-Z_1 \in \Omega$,
- $\beta(0, Z_2, e) = r$ if and only if all spectral values are in $(0, 1)$ if and only if $Z_2^{-1} - e \in \Omega$,

- $\beta(e, Z_3, \infty) = r$ if and only if all spectral values are strictly bigger than 1 if and only if $Z_3 - e \in \Omega$.

Since $G(\Omega)$ acts transitively on Ω all conditions are fulfilled if and only if there exists g_1, g_2 and g_3 in $G(\Omega)$ with

$$Z_1 = -g_1 e, \quad Z_2 = (g_2 e + e)^{-1}, \quad Z_3 = g_3 e + e. \tag{3.22}$$

Note that the g_i are unique up to an element in the stabilizer of e.

We use these observations to generalise Proposition 3.3.3:

Proposition 3.4.4. *Let g_1, g_2 and g_3 in $G(\Omega)$ and $Z_i \in V$ as in* (3.22)*. Then the representations of $\Gamma_{0,3}$ into G defined by*

$$c_1 = n^-_{-Z_2^{-1}} \theta(g_2) g_1^{-1}, \quad c_2 = n^+_{Z_3} g_3 \theta(g_2)^{-1} n^-_{Z_2^{-1}}, \quad c_3 = g_1 g_3^{-1} n^+_{-Z_3}$$

is maximal. Conversely the generators $c_i = \varrho(C_i)$ of any maximal representation $\varrho : \Gamma_{0,3} \to G$, such that c_1 fixes 0, c_2 fixed e and c_3 fixes ∞, are of this form.

Remark 3.4.5. The idea for these formulae for c_i comes from Remark 3.4.1.

Proof. It is immediately clear that $c_3 c_2 c_1 = e$, for c_1, c_2 and c_3 constructed as above, hence they define a representation ϱ of $\Gamma_{0,3}$ and by construction of the points Z_i and Formula (1.1) ϱ is maximal.

Conversely, let $\varrho : \Gamma_{0,3} \to G$ be a maximal representation. As usual we define $c_i := \varrho(C_i)$. Defining the $Z_i \in V$ as above, we can summarize properties of c_i as in Section 3.3:

$$c_1 : \begin{cases} Z_1 \mapsto e \\ 0 \mapsto 0 \\ \infty \mapsto Z_2 \end{cases} \qquad c_2 : \begin{cases} e \mapsto e \\ 0 \mapsto Z_3 \\ Z_2 \mapsto \infty \end{cases} \qquad c_3 : \begin{cases} e \mapsto Z_1 \\ Z_3 \mapsto 0 \\ \infty \mapsto \infty. \end{cases}$$

These properties determine c_i up to an element of the stabilizer of a maximal triple. Choose $\tilde{g}_1$, $\tilde{g}_2$ and $\tilde{g}_3$ in $G(\Omega)$ such that

$$Z_1 = -\tilde{g}_1 e, \quad Z_2 = (\tilde{g}_2 e + e)^{-1}, \quad Z_3 = \tilde{g}_3 e + e. \tag{3.23}$$

Note that the $\tilde{g}_i$ exist because $G(\Omega)$ acts transitively on Ω, but they are not unique. We define

$$\tilde{c}_1 := n^-_{-Z_2^{-1}}\theta(\tilde{g}_2)\tilde{g}_1^{-1}, \quad \tilde{c}_2 := n^+_{Z_3}\tilde{g}_3\theta(\tilde{g}_2)^{-1}n^-_{Z_2^{-1}}, \quad \tilde{c}_3 := \tilde{g}_1\tilde{g}_3^{-1}n^+_{-Z_3}.$$

A direct computation shows

$$\begin{aligned}
\tilde{c}_1(Z_1, 0, \infty) =& c_1(Z_1, 0, \infty) = (e, 0, Z_2)\\
\tilde{c}_2(e, 0, Z_2) =& c_2(e, 0, Z_2) = (e, Z_3, \infty)\\
\tilde{c}_3(e, Z_2, \infty) =& c_3(e, Z_3, \infty) = (Z_1, 0, \infty).
\end{aligned}$$

Hence $\tilde{c}_i$ and c_i coincide up to multiplication with an element of G stabilizing a maximal triple. Using $\tilde{c}_1 = \big(n^-_{-Z_2^{-1}}\theta(\tilde{g}_2)\big)\tilde{g}_1^{-1}$, we observe

$$\begin{aligned}
&(n^-_{-Z_2^{-1}}\theta(\tilde{g}_2))^{-1}c_1(Z_1, 0, \infty)\\
=&\theta(\tilde{g}_2)^{-1}(n^-_{-Z_2})^{-1}c_1(Z_1, 0, \infty) = \theta(\tilde{g}_2)^{-1}(n^-_{-Z_2})^{-1}(e, 0, Z_2)\\
=&\theta(\tilde{g}_2)^{-1}((e - Z_2^{-1})^{-1}, 0, \infty) = (-e, 0, \infty)
\end{aligned}$$

and

$$\tilde{g}_1^{-1}(Z_1, 0, \infty) = (-e, 0, \infty).$$

This shows that $\theta(\tilde{g}_2)^{-1}(n^-_{-Z_2})^{-1}c_1$ and $\tilde{g}_1$ only differ by an element k_1 of the stabilizer of $(-e, 0, \infty)$. In particular $c_1 = n^-_{Z_2}\tilde{g}_2 k_1\tilde{g}_1^{-1}$. Along the same lines one can show that there exists k_2 and k_3 in the stabilizer of $(-e, 0, \infty)$ such that

$$c_2 = n^+_{Z_3}\tilde{g}_3 k_2\tilde{g}_2^{-1}n^-_{-Z_2}$$

and

$$c_3 = \tilde{g}_1 k_3\tilde{g}_3^{-1}n^+_{-Z_3}.$$

A direct calculation shows that $c_3c_2c_1 = 1$ if and only if $k_3k_2k_1 = 1$. Now defining

$$g_1 := \tilde{g}_1 k_1^{-1}, \quad g_2 := \tilde{g}_2, \quad \tilde{g}_3 := g_3 k_2$$

finishes the proof. $\square$

Remark 3.4.6. The fixed point discussion from the previous section is harder to generalize. We used subspaces of the Shilov boundary associated with $Sp(2n,\mathbb{R})$ which are Shilov boundaries associated with $Sp(2m,\mathbb{R})$ with $m < n$. Such subspaces do not exist for all types of Hermitian Lie groups of tube type, e.g. the exceptional one. However we shortly indicate how to obtain the derivative of c_1 in a fixed point $Y \in V$. Recall that the stabilizer of 0 in $G(T_\Omega)$ is equal to $N^-G(\Omega)$. Now let $c_1 = n_v^+ g$ as above and $Y \in V$ a fixed point. Then

$$n_Y^+ c n_Y^+ = n_{\tilde{v}}^- \tilde{g}$$

for some $\tilde{v} \in V$ and $\tilde{g} \in G(\Omega)$ and

$$\tilde{g} = n_{-\tilde{v}}^- n_Y^+ c n_Y^+$$

and $\tilde{v}$ can be obtained by evaluating both sides at the point ∞, since $n_{\tilde{v}}^-(\infty) = \tilde{v}^{-1}$. A direct calculation shows that the differential of $n_{\tilde{v}}^- \tilde{g}$ in 0 is equal to $\tilde{g}$.

3.5 Gluing

So far we only considered representations of $\Gamma_{0,3}$. To obtain parameters for more general representations, we will discuss gluing constructions for surfaces (Section 3.5.1), the effect on fundamental groups and their representations (Section 3.5.2) as well as gluing for conjugacy classes of representations (Section 3.5.3). In Section 3.5.4 we apply the gluing criteria obtained in these sections to representations into $\mathrm{Sp}(2n,\mathbb{R})$. Finally we introduce the gluing graph in Section 3.5.5 which is a tool to encode the gluing for a surface $\Sigma_{g,m}$.

3.5.1 Gluing constructions for Surfaces

In this section we recall gluing constructions for oriented surfaces.

We consider all surfaces oriented and all boundary components carry the induced orientation. We have two operations to get new surfaces from given ones:

(A) Gluing two surfaces
Let Σ_l and Σ_r two surfaces each of which has at least one boundary component. Using an orientation *reversing* homeomorphism $f : C_l \to C_r$ between boundary components[1] C_l of Σ_l and C_r of Σ_r, we can glue the two surfaces along C_r resp. C_l and we obtain a new surface

$$\Sigma_f := \Sigma_l \amalg_f \Sigma_r = (\Sigma_l \amalg \Sigma_r)/\{x = f(x)\ \forall x \in C_l\}.$$

The Euler characteristic of the new surface is:

$$\chi(\Sigma_f) = \chi(\Sigma_l) + \chi(\Sigma_r).$$

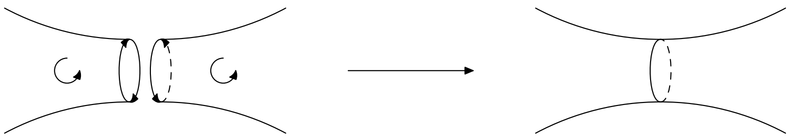


Figure 3.4: Gluing two surfaces.

(B) Closing handles
Let Σ be a surface which has at least two boundary components and let $f : C_1 \to C_2$ be an orientation *reversing* homeomorphism between two different boundary components. Gluing along these boundary components gives a new surface

$$\Sigma_f = \Sigma/\{x = f(x)\ \forall x \in C_1\}$$

[1] l for left, r for right.

The genus of Σ_f is the genus of Σ plus one, the number of boundary components has shrunk by two. For the Euler characteristic we get:

$$\chi(\Sigma_f) = \chi(\Sigma).$$

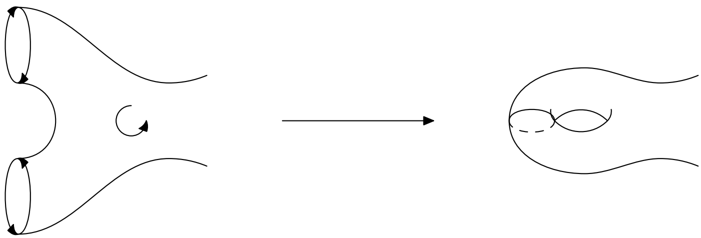


Figure 3.5: Closing a handle.

3.5.2 Gluing for Fundamental Groups and Representations

We express the fundamental group Γ_f of Σ_f in terms of the fundamental groups Σ_l and Σ_r or Σ respecively.

(A) Gluing two surfaces.

Proposition 3.5.1. *Let (Σ_l, x_l) and (Σ_r, x_r) be pointed oriented surfaces with fundamental groups*

$$\pi_1(\Sigma_l, x_l) = \Gamma_{g_1,n_1} = \langle A_1, B_1, \dots, A_{g_1}, B_{g_1}, C_1, \dots, C_{n_1} | \\ C_{n_1} \dots C_1 [A_{g_1}, B_{g_1}] \dots [A_1, B_1] = e \rangle.$$

and

$$\pi_1(\Sigma_r, x_r) = \Gamma_{g_2,n_2} = \langle \bar{A}_1, \bar{B}_1, \dots, \bar{A}_{g_2}, \bar{B}_{g_2}, \bar{C}_1, \dots, \bar{C}_{n_2} | \\ \bar{C}_{n_2} \dots \bar{C}_1 [\bar{A}_{g_2}, \bar{B}_{g_2}] \dots [\bar{A}_1, \bar{B}_1] = e \rangle.$$

Assume that x_l is on the boundary component of Σ_l labeled by C_i and x_r is on the boundary component of Σ_r labeled by $\bar{C}_j$. Let c_l be a representative of C_i, c_r be a representative for $\bar{C}_j$ and $f : c_l \to c_r$ a orientation reversing homeomorphism with $f(x_l) = x_r$.
Then the fundamental group of Σ_f is generated by

$$\begin{aligned}\{&A_1, B_1, \ldots, A_{g_1}, B_{g_1}, C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_{n_1},\\ &\bar{A}_1, \bar{B}_1, \ldots, \bar{A}_{g_2}, \bar{B}_{g_2}, \bar{C}_1, \ldots, \bar{C}_{j-1}, \bar{C}_{j+1}, \ldots, \bar{C}_{n_2}\}\end{aligned}$$

and the only relation is

$$\begin{aligned}&C_{n_1} \cdots C_{i+1}\bar{C}_{j-1} \cdots \bar{C}_1[\bar{A}_{g_2}, \bar{B}_{g_2}] \cdots [\bar{A}_1, \bar{B}_1]\cdot\\ &\cdot\bar{C}_{n_2} \cdots \bar{C}_{j+1}C_{i-1} \cdots C_1[A_{g_1}, B_{g_1}] \cdots [A_1, B_1] = e.\end{aligned}$$

In particular $\pi_1(\Sigma_f)$ is the amalgam of $\pi_1(\Sigma_1)$ and $\pi(\Sigma_2)$ along C_i.

Proof. By assumption we have $[f(c_l)] = \bar{C}_j^{-1}$, because f is orientation reversing. Since we identify c_l and $f(c_l)$, we get

$$C_i = \bar{C}_j^{-1} = \bar{C}_{j-1} \cdots \bar{C}_1[\bar{A}_{g_2}, \bar{B}_{g_2}] \cdots [\bar{A}_1, \bar{B}_1]\bar{C}_{n_2} \cdots \bar{C}_{j+1}.$$

Inserting this into the relation of $\pi_1(\Gamma_l)$ gives the desired result. □

Example 3.5.2. Let Σ and $\bar{\Sigma}$ be two surfaces homeomorphic to $\Sigma_{0,3}$. Let $\langle C_1, C_2, C_3 | C_3C_2C_1 = e\rangle$ and $\langle \bar{C}_1, \bar{C}_2, \bar{C}_3 | \bar{C}_3\bar{C}_2\bar{C}_1 = e\rangle$ be their fundamental groups. Given a homeomorphism between boundary components as in Proposition 3.5.1 we can glue them along boundary components and get a surface homeomorphic to $\Sigma_{0,4}$. Assume $\bar{C}_1 = C_3^{-1} (= C_2C_1)$ and we finally get the fundamental group of $\Sigma_{0,4}$ in the presentation

$$\langle C_1.C_2.\bar{C}_2, \bar{C}_1 | \bar{C}_3\bar{C}_2C_2C_1 = e\rangle.$$

Given representations of Γ_{g_1,m_1} and Γ_{g_2,m_2} with $m_1, m_2 \geq 1$ into a group G. Under a certain condition they can be extended to a representation of $\Gamma_{g_1+g_2,m_1+m_2-2}$.

Proposition 3.5.3. *Let Σ_{g_1,n_1}, Σ_{g_2,n_2}, Γ_{n_1,g_1} and Γ_{g_2,n_2} as above. Let $\varrho_1 : \Gamma_{g_1,n_1} \to G$ and $\varrho_2 : \Gamma_{g_2,n_2} \to G$ be representations. Assume that there exists C_i and $\bar{C}_j$ such that $\varrho_1(C_i) = \varrho_2(\bar{C}_j)^{-1}$. Then there exists a unique representation*

$$\varrho_f : \pi_1(\Sigma_f) \to G$$

such that $\varrho_f|_{\Gamma_{g_1,n_1}} = \varrho_1$ and $\varrho_f|_{\Gamma_{g_2,n_2}} = \varrho_2$.

We say that we *glue* ϱ_1 and ϱ_2.

Proof. We define:

$$\begin{aligned}
\varrho_f(A_l) &:= \varrho_1(A_l) \\
\varrho_f(B_l) &:= \varrho_1(B_l) \\
\varrho_f(\bar{A}_k) &:= \varrho_2(\bar{A}_k) \\
\varrho_f(\bar{B}_k) &:= \varrho_2(\bar{B}_k) \\
\varrho_f(C_l) &:= \varrho_1(C_l) \\
\varrho_f(\bar{C}_k) &:= \varrho_2(\bar{C}_k).
\end{aligned}$$

This is well defined since $\varrho_1(C_i) = \varrho_2(\bar{C}_j)^{-1}$. □

(B) Closing handles.
As above we have:

Proposition 3.5.4. *Recall*

$$\pi_1(\Sigma, x) = \Gamma_{g,n} = \langle A_1, B_1, \ldots, A_{g_1}, B_{g_1}, C_1, \ldots, C_n | \\ C_n \ldots C_1[A_g, B_g] \ldots [A_1, B_1] = e \rangle.$$

We assume that the base point for Γ is on C_i and we glue along the boundary curves C_i and C_j. Then

$$\pi_1(\Sigma_f) = \langle \Gamma, t | tC_i t^{-1} = C_j^{-1} \rangle.$$

Remark 3.5.5. We can consider Γ as a subset of $\pi_1(\Sigma_f)$.

Example 3.5.6. Given $\Sigma_{0,3}$ and f, say, an orientation reversing homeomorphism $C_3 \to C_2$ as in Proposition 3.5.4. Then we get a new loop T and we have the presentation of $\pi_1(\Sigma_f)$

$$\langle C_1, C_3, T | C_3 T C_3^{-1} T^{-1} C_1 = e \rangle.$$

We have an statement analog to Proposition 3.5.3.

Proposition 3.5.7. *Let ϱ be a representation of Γ into some group G and assume there exists $g \in G$ such that $\varrho(C_i)^{-1} = g\varrho(C_j)g^{-1}$. Then there exists a unique representation*

$$\varrho_f : \pi_1(\Sigma_f) \to G$$

with $\varrho_f(t) = g$ and $\varrho_j|_{\pi_1(\Sigma)} = \varrho$.

3.5.3 Gluing in $\mathrm{Rep}(\Gamma, G)$

Recall that G acts on $\mathrm{Hom}(\Gamma, G)$ by conjugation and we defined the quotient

$$\mathrm{Rep}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)/G.$$

We denote by $[\varrho]$ the equivalence class generated by ϱ in $\mathrm{Rep}(\Gamma, G)$. In this section we extend the result from the previous section to $\mathrm{Rep}_{max}(\Gamma_{g,m}, G)$.

As above we have to discuss two cases

(A) Gluing of two surfaces

Proposition 3.5.8. *Consider $[\varrho_1'] \in \mathrm{Rep}(\Gamma_{g_1,n_1}, G)$ and $[\varrho_2'] \in \mathrm{Rep}(\Gamma_{g_2,n_2}, G)$ with $n_1 \geq 1$ and $n_2 \geq 1$. Assume there exists $\varrho_1 \in [\varrho_1']$ and $\varrho_2 \in [\varrho_2']$ such that $\varrho_1(C_i) = g\varrho_2(\bar{C}_j)^{-1}g^{-1}$ for some $g \in G$. Then there is a class of representations of Γ defined by*

$$\varrho := \varrho_1 * (g\varrho_2 g^{-1}) : \Gamma \to G$$

such that $[\varrho|_{\Gamma_{g_1,n_1}}] = [\varrho_1]$ and $[\varrho|_{\Gamma_{g_2,n_2}}] = [\varrho_2]$.

Proof. Follows immediately from Proposition 3.5.3. □

Remark 3.5.9. Note that ϱ is not unique. Let h be an element of the centralizer of $\varrho_2(\bar{C}_j)$. Then the representation $\varrho_h := \varrho_1 * (gh\varrho_2(gh)^{-1})$ also satisfies: $[\varrho_h|_{\Gamma_{g_1,n_1}}] = [\varrho_1]$ and $[\varrho_h|_{\Gamma_{g_2,n_2}}] = [\varrho_2]$.

(B) Closing handles

Proposition 3.5.10. *Let $[\varrho'] \in \mathrm{Rep}(\Gamma_{g,n}, G)$ with $n \geq 2$. Assume that there exists $g \in G$ such that $\varrho'(C_i)^{-1} = g\varrho'(C_j)g^{-1}$. Then there exists a representation ϱ_f of Γ_f such that $\varrho_f|_\Gamma = \varrho$.*

3.5.4 Gluing in $\mathrm{Sp}(2n, \mathbb{R})$

In the last section we have seen that we can glue along two generators c and $\bar{c}$ of maximal representations if and only if $\bar{c}$ and c^{-1} are conjugate in the target group of the maximal representation. In the next proposition we show that this is possible if and only if c and $\bar{c}$ are hyperbolic.

Theorem 3.5.11. *Let*

$$c = \begin{pmatrix} X & 0 \\ X + (X^\top)^{-1}S & (X^\top)^{-1} \end{pmatrix} \tag{3.24}$$

and

$$\bar{c} = \begin{pmatrix} (\bar{X}^\top)^{-1} & -(\bar{X}^\top)^{-1} - \bar{S}\bar{X} \\ 0 & \bar{X} \end{pmatrix} \tag{3.25}$$

be elements in $\mathrm{Sp}(2n, \mathbb{R})$ with X and $\bar{X}$ invertible and S and $\bar{S}$ symmetric positive definite.

(i) Suppose X and $\bar{X}$ contracting. Then $\bar{c}$ and c^{-1} are conjugate in $\mathrm{Sp}(2n, \mathbb{R})$ if and only $X^\top$ and $\bar{X}$ are conjugate in $\mathrm{GL}(n, \mathbb{R})$. If $\bar{X} = GX^\top G^{-1}$, then $\bar{c} = gc^{-1}g^{-1}$ with

$$g = g_1 g_2 g_3 = \begin{pmatrix} \bar{Y}GY^{-1} - (G^\top)^{-1} & -\bar{Y}G \\ GY^{-1} & -G \end{pmatrix},$$

where

$$Y = -\left(\sum_{i=0}^{\infty}(X^\top)^i(X^\top \cdot X + S)X_1^i\right)^{-1}$$

and

$$\bar{Y} = \sum_{i=0}^{\infty}(\bar{X}^\top)^i(I + \bar{X}^\top \bar{S}\bar{X})\bar{X}^i.$$

(ii) *It X or $\bar{X}$ has an eigenvalue of absolute value* 1*, then $\bar{c}$ and c^{-1} are not conjugate in* $\mathrm{Sp}(2n, \mathbb{R})$*.*

Remark 3.5.12. The length parameters from Theorem 3.3.2 are only unique up to conjugation with an element of $\mathrm{O}(n)$. For the gluing of two representations we have to choose representatives from these equivalence classes and glue them. The conjugation class of the resulting representation does not depend on this choice. Indeed, replace X and S by kXk^{-1} and kSk^{-1} resp. $\bar{X}$ and $\bar{S}$ by $\bar{k}\bar{X}\bar{k}^{-1}$ and $\bar{k}\bar{S}\bar{k}^{-1}$ and the G is replaced by $\bar{k}Gk^{-1}$, and the resulting representation from both sets of parameters are conjugate.

Proof of Proposition 3.5.11. (i) First note that since c and $\bar{c}$ are hyperbolic, they have fixed points Y resp. $\bar{Y}$ which are transversal to 0 resp. ∞. We are searching for g with $\bar{c} = gc^{-1}g^{-1}$. We want to write $g = g_1g_2g_3$, where the g_i have the following properties: g_3 maps the transverse pair $(0, Y)$ to $(0, \infty)$, g_2 fixes $(0, \infty)$ and g_1 maps the transverse pair $(0, \infty)$ to $(\infty, \bar{Y})$. We choose

$$g_1 := \begin{pmatrix} \bar{Y} & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$g_3 := \begin{pmatrix} Y^{-1} & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$g_1^{-1}\bar{c}g_1 = \begin{pmatrix} \bar{X} & \\ & (\bar{X}^\top)^{-1} \end{pmatrix}$$

and

$$g_3 c^{-1} g_3^{-1} = \begin{pmatrix} X^\top & \\ & X^{-1} \end{pmatrix}$$

By assumption $\bar{X}$ as well as $X^\top$ are contracting. Hence, if there exists g_2 such that $g_1^{-1}\bar{c}g_1 = g_2 g_3 c^{-1} g_3^{-1} g_2^{-1}$, then g_2 has to stabilize the pair $(0, \infty)$, i.e.

$$g_2 = \begin{pmatrix} G & \\ & (G^\top)^{-1} \end{pmatrix}$$

In particular there has to be a $G \in \mathrm{GL}(n, \mathbb{R})$ such that $\bar{X} = GX^\top G^{-1}$. Then $g = g_1 g_2 g_3$.

(ii) From Corollary 1.3.5 we know that we can assume

$$c = \begin{pmatrix} X & 0 \\ M & (X^\top)^{-1} \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} \bar{X} & 0 \\ \bar{M} & (\bar{X}^\top)^{-1} \end{pmatrix},$$

As explained in the second part of the proof of Theorem 3.3.2 we can assume that X^{-1} and $\bar{X}$ are non-expanding. Then 0 is the unique fixed point for c^{-1} and $\bar{c}$ where the differential is non-expanding. Hence if there exists $g \in \mathrm{Sp}(2n, \mathbb{R})$ with $g\bar{c}g^{-1} = c^{-1}$, then g has to fix 0. Assume

$$g = \begin{pmatrix} A & 0 \\ X & (A^\top)^{-1} \end{pmatrix}.$$

Assume that $g\bar{c}g^{-1} = c^{-1}$. Then $A\bar{X}A^{-1} = X^{-1}$. This is a first condition for c^{-1} and $\bar{c}$ to be conjugate. If $\bar{X}$ and X^{-1} are not conjugate, we are done.

Now assume that $A\bar{X}A^{-1} = X^{-1}$. Then we can write

$$g = \begin{pmatrix} A & 0 \\ C & (A^\top)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix}.$$

Define $\bar{C} := CA^{-1}$ and $M' := (A^\top)^{-1}\bar{M}A^{-1}$. Recall $A\bar{X}A^{-1} = X^{-1}$ and $(X^\top)^{-1}M'$ is symmetric and positive definite.

We can summarize that to the equation

$$\begin{aligned} g\bar{c}g^{-1} &= \begin{pmatrix} 1 & 0 \\ \bar{C} & 1 \end{pmatrix} \begin{pmatrix} X^{-1} & 0 \\ M' & X^\top \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -M' & 1 \end{pmatrix} \\ &= \begin{pmatrix} X^{-1} & 0 \\ \bar{C}X^{-1} + M' - X^\top \bar{C} & \bar{X}^\top \end{pmatrix} \stackrel{!}{=} c^{-1} = \begin{pmatrix} X^{-1} & 0 \\ -M^\top & X^\top \end{pmatrix}. \end{aligned}$$

In particular: $\bar{C}X^{-1} + M' - X^\top \bar{C} = -M^\top$, which is equivalent to

$$(X^\top)^{-1}\bar{C}X^{-1} - \bar{C} + (X^\top)^{-1}M' = -(X^\top)^{-1}M^\top.$$

Note that by construction $(X^\top)^{-1}\bar{M}$ is positive definite, hence $-(X^\top)^{-1}M^\top$ is negative definite. Let λ be an eigenvalue of X^{-1} with $|\lambda| = 1$ and let v be a non-zero eigenvector for λ. Such an eigenvalue exists by assumption. Then

$$\begin{aligned} v^*\big((X^*)^{-1}C'X^{-1} - C' + (X^*)^{-1}M'\big)v =& v^*(X^*)^{-1}M'v \\ =& -v^*(X^*)^{-1}M^*v, \end{aligned}$$

which is a contradiction since the left hand side is strictly positive and the right hand side is strictly negative. Therefore c^{-1} and $\bar{c}$ cannot be conjugate.

□

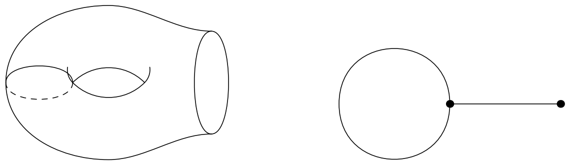
3.5.5 The Gluing Graph

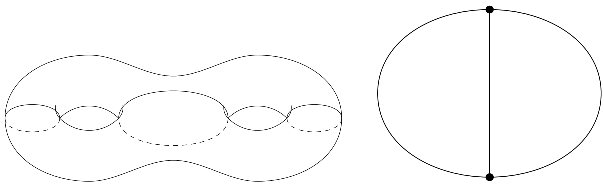
To be able to state coordinates for more general surfaces with need to encode the gluing involving several pairs of pants and handles in a clear way.

Let $\Sigma_{g,m}$ be the topological surface with genus g and $m \geq 1$ boundary components and $\chi(\Sigma_{g,m}) < 0$. It can be build using $2g - 2 + m$ pairs of pants (see Chapter 3.5.1).

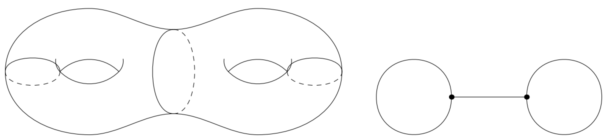
This gluing can be visualized in a *gluing graph*. Given $\Sigma_{g,m}$ with a decomposition into pairs of pants. We construct the gluing graph for this decomposition as follows: we represent any pair of pants and any boundary component by a vertex. We add an edge between two pairs of pants with a common boundary component for each common boundary component. Furthermore we join every pair of pants with the vertices associated with its boundary components. Note that these graphs are connected.

Here are some examples:





Clearly the graph depends on the decomposition into pairs of pants.



Lemma 3.5.13. *Let let n_3 be the number of threevalent vertices and n_1 the number of univalent vertices in the graph. Then the genus of the associated surface is*

$$g = \frac{n_3 - n_1}{2} + 1.$$

The Euler characteristic of the graph is equal to $1 - g$.

Definition 3.5.14. Let $\Sigma_{g,m}$ be a surface with negative Euler characteristic. Then we call the decomposition into pairs of pant as in Figure 3.5.14 with graph *standard decomposition.* We denote the

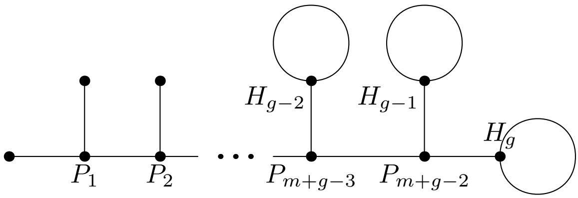


Figure 3.6: Standard graph

handle by H_i and the pairs of pants by P_i. This graph is the *standard graph.*

3.5.6 Surface Doubling

Definition 3.5.15. Given a surface $\Sigma_1 = \Sigma_{g,m}$ with at least one boundary component. Then we can take a second copy Σ_2 of this surface and glue each boundary component of Σ_1 to one boundary component Σ_2. The result is a closed surface of genus $2g + m - 1$. This construction is the *surface doubling.*

Given a surface with hyperbolic structure and geodesic boundary components, we can perform the surface doubling as well. The double carries a hyperbolic structure which is unique up to a twist along the former boundary curves.

From Proposition 3.5.11 and the fact that every matrix in $\mathrm{GL}(n, \mathbb{R})$ is conjugate to its transverse (Lemma A.1.5), we get

Proposition 3.5.16. *We can perform this doubling construction for a representation into* $\mathrm{Sp}(2n, \mathbb{R})$ *if and only if each generator of the boundary components is S-hyperbolic.*

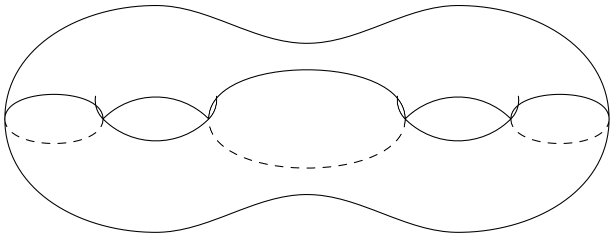


Figure 3.7: Surface doubling for $\Sigma_{0,3}$

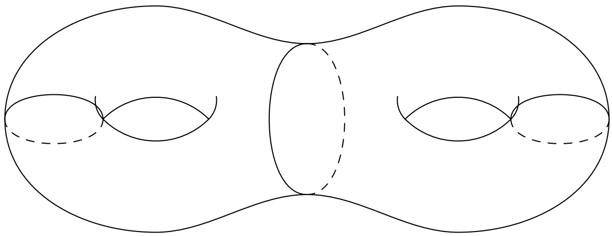


Figure 3.8: Surface doubling for $\Sigma_{1,1}$

Remark 3.5.17. The double of a representation is not unique.

Lemma 3.5.18. *Let $\varrho : \Gamma_{g,m} \to G$ a representation into some group. Then there exists a canonical restriction ϱ_{H_i} and ϱ_{P_i} to the fundamental group for any handle H_i and any pair of pants P_j as in the standard graph.*

Proof. Recall that $[A_g, B_g] \dots [A_1, B_1] C_m \cdots C_1 = e$.

(i) Restriction to H_i

$$\varrho_{H_i}(A) := \varrho(A_i)$$
$$\varrho_{H_i}(B) := \varrho(B_i)$$

$$\varrho_{H_i}(C) := [\varrho(A_i), \varrho(B_i)]^{-1}$$

(ii) Restriction to P_k

(a) $k \in \{1, \ldots, m-1\}$

$$\begin{aligned}
\varrho_{P_k}(C_1) &:= \varrho(C_k, \ldots, C_1)\\
\varrho_{P_k}(C_2) &:= \varrho(C_{k+1})\\
\varrho_{P_k}(C_3) &:= \varrho([A_g, B_g] \ldots [A_1, B_1] C_m \cdots C_{k+2})
\end{aligned}$$

(b) $k \in \{m, \ldots, m+g-2\}$

$$\begin{aligned}
\varrho_{P_k}(C_1) &:= \varrho([A_{k-m}, B_{k-m}] \ldots [A_1, B_1] C_m \ldots C_1)\\
\varrho_{P_k}(C_2) &:= \varrho([A_{k-m+1}, B_{k-m+1}])\\
\varrho_{P_k}(C_3) &:= \varrho([A_g, B_g] \ldots [A_{k-m+2}, B_{k-m+2}])
\end{aligned}$$

□

From Proposition 3.5.16 we can deduce Corollary 1.4.9:

Proof. (Proof of Corollary 1.4.9) Since the generators are S-hyperbolic, there exists a double of ϱ, which is a maximal representation of the fundamental group of the closed surface Σ_{2g+m-1}. From [13] we know that such a representation is Anosov. Hence the same holds for ϱ. □

3.6 More Parameters

We use the gluing graph introduced in the previous section to state parameters for $\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))$. In Section 3.6.1 we give parameters for representations of $\Gamma_{1,1}$, $\Gamma_{0,4}$, $\Gamma_{1,2}$ and $\Gamma_{2,0}$, since their underlying surfaces can be obtained from one or two pairs of pants, which makes the statements slightly easier.

Recall

$$B = \{X \in \mathrm{GL}(n, \mathbb{R}) | X \text{ contracting}\}.$$

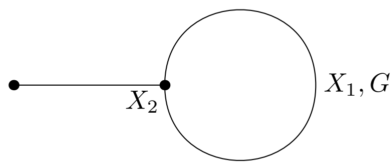
and

$$R := \{(X_1, X_2, X_3) \in \bar{B}^3 | X_3(X_2^\top)^{-1}X_1 \text{ is symmetric} \\ \text{and positive definite}\}.$$

3.6.1 Parameters for Surfaces obtained from one or two Pairs of Pants

Recall $\Gamma_{1,1} = \langle A, B, C | [A, B]C = e\rangle$ (see also Example 3.5.6).

We label the gluing graph for $\Sigma_{1,1}$ as follows:



Proposition 3.6.1. *There exists a bijection between*

$$\left\{(X_1, X_2, G) \in \mathrm{GL}(n, \mathbb{R})^3 | X_1 \in B,\ (X_1, X_2, GX_1^\top G^{-1}) \in R\right\} / \mathrm{O}(n),$$

and $\mathrm{Rep}_{max}(\Gamma_{1,1}, \mathrm{Sp}(2n, \mathbb{R}))$.

Proof. Let $\varrho : \Gamma_{1,1} \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation. Then we can define $\varrho' : \Gamma_{0,3} \to \mathrm{Sp}(2n, \mathbb{R})$ by

$$\varrho'(C_1) := \varrho(A), \quad \varrho'(C_2) := \varrho(C), \quad \varrho'(C_3) := \varrho(BA^{-1}B^{-1}).$$

By Theorem 2.2.10 ϱ' is a maximal representation of $\Gamma_{0,3}$.

We can assume that the $\varrho'(C_i)$ are as in Theorem 3.3.2 for some triple $(X_1, X_2, X_3) \in R$. Then $\varrho'(C_1)$ has the form of c and $\varrho'(C_3)$ has the form of $\bar{c}$ in Proposition 3.5.11 and by construction $\varrho'(C_1)^{-1}$ and $\varrho'(C_3)$ are conjugate and by the same proposition they are both hyperbolic. Since $\varrho'(C_1)^{-1}$ and $\varrho'(C_3)$ are conjugate, there exists $G \in \mathrm{GL}(n, \mathbb{R})$ with $X_3 = GX_1^\top G^{-1}$ and

$$\varrho(B) = \begin{pmatrix} Y_3GY_1^{-1} - (G^\top)^{-1} & Y_3G \\ GY_1^{-1} & -G \end{pmatrix},$$

where Y_1 is the fixed point of $\varrho'(C_1)$ transversal to 0 and Y_3 is the fixed point of $\varrho'(C_3)$ transversal to ∞. By Remark 3.5.12 this triple (X_1, X_2, G) is unique up to conjugation with an element from $\mathrm{O}(n)$.

We can construct a maximal representation of $\Gamma_{0,3}$ for any triple X_1, X_2 and G with X_1 contracting and $(X_1, X_2, (GX_1G^{-1})^\top) \in R$ and close the handle according to Proposition 3.5.10 resp. Proposition 3.5.11. This provides an inverse map to the construction given above. □

Corollary 3.6.2. *There exists a surjective map from* $B \times \mathrm{GL}(n, \mathbb{R}) \times \Omega$ *onto* $\mathrm{Rep}_{max}(\Gamma_{1,1}, \mathrm{Sp}(2n, \mathbb{R}))$.

Remark 3.6.3. Note that the triple $(X_1, X_2, GX_1 \top G^{-1})$ is an element of $\tilde{R}$ if and only if $GX_1^\top G^{-1}(X_2^\top)^{-1}X_1$ is symmetric positive definite, which is the case if and only if

$$(X_1^\top)^{-1}GX_1^\top G^{-1}(X_2^\top)^{-1} = [(X_1^\top)^{-1}, G](X_2^\top)^{-1}$$

is symmetric positive definite.

Proposition 3.6.4. *There exists a bijection between*

$$\big\{(X_1, X_2, X_3, \bar{X}_1, \bar{X}_2, G) \in \mathrm{GL}(n, \mathbb{R})^6 | (X_1, X_2, X_3) \in R,$$
$$(\bar{X}_1, \bar{X}_2, GX_1^\top G^{-1}) \in R, X_1 \text{ contracting}\big\}/\sim$$

and $\mathrm{Rep}_{max}(\Gamma_{0,4}, \mathrm{Sp}(2n, \mathbb{R}))$ *where for* $k, l \in \mathrm{O}(n)$

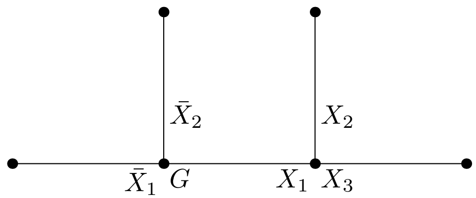
$$(X_1, X_2, X_3, \bar{X}_1, \bar{X}_2, G)$$

and

$$(kX_1k^{-1}, kX_2k^{-1}, kX_3k^{-1}, l\bar{X}_1l^{-1}, l\bar{X}_2l^{-1}, lGk^{-1})$$

are equivalent.

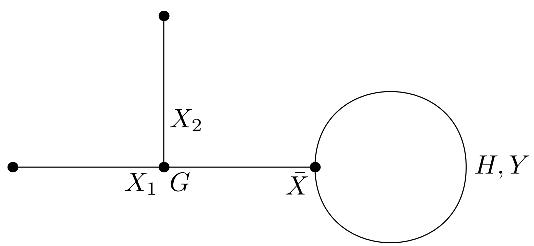
Proof. The graph



is a gluing graph for $\Sigma_{0,4}$. We denote the left pair of pants by P_1, the right one by P_2. The restriction of a maximal representation ϱ to the fundamental groups of P_1 and P_2 yield two maximal representations ϱ_1 and ϱ_2 of $\Gamma_{0,3}$. They admit parameters (X_1, X_2, X_3) respectively $(\bar{X}_1, \bar{X}_2, \bar{X}_3)$ (Theorem 3.3.2) such that X_1 is contracting and by Proposition 3.5.11 there exists $G \in \mathrm{GL}(n, \mathbb{R})$ such that $\bar{X}_3 = GX_1^\top G^{-1}$. The matrices $G, X_1, X_2, X_3, \bar{X}_1, \bar{X}_2$ determine the representation uniquely. By Remark 3.5.12, these parameters are unique up to the two $\mathrm{O}(n)$ actions. $\square$

Corollary 3.6.5. *There exists a surjective map from* $B \times \mathrm{GL}(n, \mathbb{R})^3 \times \Omega^2$ *onto* $\mathrm{Rep}_{max}(\Gamma_{0,4}, \mathrm{Sp}(2n, \mathbb{R}))$.

We label the gluing graph for $\Sigma_{1,2}$ as follows



As above we show:

Proposition 3.6.6. *There exists a bijection between*

$$\big\{(X_1, X_2, G, Y, \bar{X}, H) \in \mathrm{GL}(n,\mathbb{R})^6 | (X_1, X_2, G\bar{X}^\top G^{-1}) \in R,$$
$$(Y, \bar{X}, HY^\top H^{-1}) \in R, Y, \bar{X} \text{ contracting}\big\}/\sim$$

and $\mathrm{Rep}_{max}(\Gamma_{1,2}, \mathrm{Sp}(2n,\mathbb{R}))$*, where for* $k, l \in \mathrm{O}(n)$

$$(X_1, X_2, G, Y, \bar{X}, H)$$

and

$$(kX_1k^{-1}, kX_2k^{-1}, kGl^{-1}, lYl^{-1}, l\bar{X}l^{-1}, lHl^{-1})$$

are equivalent.

Corollary 3.6.7. *There exists a surjective map from* $B^2 \times \mathrm{GL}(n,\mathbb{R})^2 \times \Omega^2$ *onto* $\mathrm{Rep}_{max}(\Gamma_{1,2}, \mathrm{Sp}(2n,\mathbb{R}))$*.*

Proposition 3.6.8. *There exists a bijection between*

$$\big\{(X_1, X_2, X_3, G_3, G_2, G_1) \in \mathrm{GL}(n,\mathbb{R})^6 | (X_1, X_2, X_3) \in R,$$
$$(G_1 X_3^\top G_1^\top, G_2 X_2^\top G_2^{-1}, G_3 X_1^\top G_3^{-1}) \in R, X_i \text{ contracting}\big\}/\sim$$

and $\mathrm{Rep}_{max}(\Gamma_{2,0}, \mathrm{Sp}(2n,\mathbb{R}))$*, where for* $l, k \in \mathrm{O}(n)$

$$(X_1, X_2, X_3, G_3, G_2, G_1)$$

and

$$(kX_1k^{-1}, kX_2k^{-1}, kX_3k^{-1}, lG_3k^{-1}, lG_2k^{-1}, lG_1k^{-1})$$

are equivalent.

3.6.2 General parameters

In this section we state the most general theorem for Fenchel-Nielsen coordinates for maximal representations of $\Gamma_{g,m}$ into $\mathrm{Sp}(2n,\mathbb{R})$.

The strategy to obtain these coordinates is the same as for the examples above.

(i) Choose a decomposition of the underlying surface into pairs of pants and handles and write down the corresponding gluing graph,

(ii) Theorem 1.3.1 gives us coordinates for representations of $\Gamma_{0,3}$ and Proposition 1.3.12 gives coordinates for representations of $\Gamma_{1,1}$,

(iii) From Proposition 3.5.11 and Remark 3.5.12 we know in which cases we can glue representations and how we get twist parameters.

Theorem 3.6.9. *Let $\varrho : \Gamma_{g,m} \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation. Then there exist length and twist parameters as in the following gluing graph:*

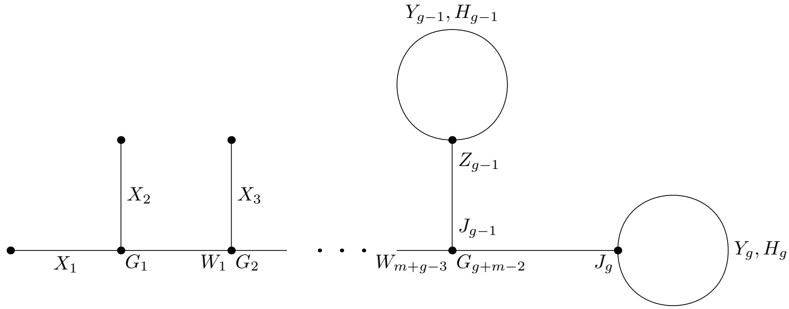


Figure 3.9: General gluing graph

where the $W_\bullet, X_\bullet, Y_\bullet, Z_\bullet$ are length parameter and the $G_\bullet, H_\bullet, J_\bullet$ are twist parameters subject to the usual relations and identifications.

Conversely any representations defined with these parameters is maximal.

Remark 3.6.10. Theorem 3.6.9 will be used in Section 4.2 to define paths in $\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))$.

Having these general coordinates at hand we can finally prove Corollary 1.4.8.

Proof. (Proof of Corollary 1.4.8) Let $\varrho = (\varrho_1, \ldots, \varrho_r)$ be a product representation. We can assume that the representations ϱ_i are all in the form given by Theorem 3.3.2 for $\mathrm{Sp}(2, \mathbb{R}) = SL(2, \mathbb{R})$. Then for any generator $c_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ of ϱ the blocks A_i, B_i, C_i and D_i are diagonal and the c_i have by construction the right fixed points and the right dynamics. In particular all length and twist parameters are diagonal.

The converse follows from the explicit formulae in Theorem 1.3.1 and Theorem 1.3.10.

The second statement follows immediately from the definition. $\square$

Chapter 4

Applications

In this part we use our parameters to show continuity of the limit curve for representations of the fundamental groups of surfaces with boundary (Section 4.1). Furthermore, we count connected components of $\mathrm{Rep}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))$ with $m \geq 1$ in Section 4.2.

4.1 Continuity of Limit Curves

In this section we prove Theorem 1.4.5. It was known before that limit curves are continuous for representations of Γ_g into $\mathrm{Sp}(2n, \mathbb{R})$ ([13]) and for maximal representations into general Hermitian Lie groups of tube type ([15]). In this section we use our parameters to extend this result to another class of maximal representations.

The proof of Theorem 1.4.5 is a modification of the proof in [13], inspired by [15]. It can be found at the end of this section. We prepare the proof by reformulating some results from [13] and [18].

First we need an existence statement over some maps from $\mathcal{L}$ to $\check{S}$. It is a modified version of [18, Thm 5.1].

Theorem 4.1.1. *Let h be as in Theorem 1.4.5 and $\varrho : \Gamma_{g,m} \to G$ a maximal representation with Zariski dense image. Then there are two Borel maps*

$$\varphi_{\pm} : \mathcal{L} \to \check{S}$$

with the following properties

(i) φ_+ *and* φ_- *are strictly* ϱ*-equivariant,*

(ii) φ_- *is left continuous and* φ_+ *is right continuous,*

(iii) for every $x \neq y$, $\varphi_\epsilon(x)$ *is transverse to* $\varphi_\delta(y)$ *for all* $\epsilon, \delta \in \{+,-\}$,

(iv) for all $x, y, z \in \mathcal{L}$

$$\beta_{\check{S}}(\varphi_\epsilon(x), \varphi_\delta(y), \varphi_\eta(z)) = r_{\mathcal{X}} \beta(x, y, z),$$

for all $\epsilon, \delta, \eta \in \{+,-\}$.

Moreover φ_+ *and* φ_- *are the unique maps satisfying (1) and (2).*

Proof. Existence of the limit curve follows from Theorem 2.4.1 and the fact that $\mathcal{L}$ with the Patterson-Sullivan measure λ is a Poisson boundary for $\Gamma_{g,m}$.

Property (iv) is the only part where we need our assumptions for the generators. Since Lemma 5.7 in [18] is not true for arbitrary representations for surfaces with boundary, we have to replace this lemma by Lemma 4.1.2). Property (iv) follows from Lemma 5.6 in [18] and Lemma 4.1.2. All other properties can be proved as in [18]. □

Recall the interval

$$((x, y)) = \{z \in \mathcal{L} | \beta(x, z, y) = 1\}$$

and the *essential graph* $\mathrm{EssGr}(\varphi)$, which is the support of the push-forward measure of λ into $\mathcal{L} \times \check{S}$ under the map

$$x \mapsto (x, \varphi(x)).$$

We use the notation of Theorem 1.4.5.

Lemma 4.1.2. *Let ϱ be as in Theorem 1.4.5 and $(x_1, f_1), (x_2, f_2) \in \mathrm{EssGr}(\varphi)$ with $x_1 \neq x_2$. Then $f_1 \pitchfork f_2$.*

Proof. The limit set $\mathcal{L}$ is a Cantor set in S^1. If $((x_1, x_2))$ and $((x_2, x_1))$ both are non-empty, one can apply the proof of Lemma 5.7 in [18] and we are done. If, say, $((x_1, x_2))$ is empty, then x_1 and x_2 are fixed points of a hyperbolic isometry γ, whose axis projects to a boundary of the hyperbolic surface. We can assume $x_1 = \gamma^+$ is the attractive fixed point for γ. We show that f_1 is the unique attractive fixed point for $\varrho(\gamma) \in \check{S}$.
Indeed, since γ represents a boundary of the surface, there exists $g \in h(\Gamma_{0,3})$ such that $(x_1, x_2) = g(c_i^+, c_i^-)$ for some generator c_i of $h(\Gamma_{0,3})$. Therefore $\gamma = g c_i g^{-1}$, hence $\varrho(\gamma) = \varrho(g)\varrho(c_i)\varrho(g)^{-1}$ is hyperbolic. Hence it contracts an open and dense subset of $\check{S}$, which contains almost all of $\varphi(\mathcal{L})$ to $\varrho(\gamma)^+$, it also contracts a set of positive measure in $\mathrm{EssGr}(\varphi)$ to $(x_1, \varrho(\gamma)^+)$, hence the latter is in $\mathrm{EssGr}(\varphi)$. Since the set $F_{x_1} = \{f \in \check{S} | (x_1, f) \in \mathrm{EssGr}(\varphi)\}$ contains precisely the right and the left limit of points in the fiber in the essential graph over x_1 ([13, Lem.8.6]). Therefore this fiber only contains $(x_1, \varrho(\gamma)^+)$. Working with γ^{-1} in stead of γ shows that the fiber over x_2 only contains $(x_2, \varrho(\gamma)^-) = (x_2, f_2)$. Hence, by assumption f_1 and f_2 are transversal. $\square$

The set of positive triples in $\mathcal{L}$, denoted by $\mathcal{L}^{(3)}$, is invariant under $h(\Gamma)$. Define

$$E^\varrho := \Gamma \backslash (\mathcal{L}^{(3)} \times V).$$

and denote by p the canonical projection from E^ϱ to $h(\Gamma) \backslash \mathcal{L}^{(3)}$.

Let $\varphi_\pm$ be the limit curves from Theorem 4.1.1. Recall $\check{S}$ can be identified with $L(V)$, the space of Lagrangian in V and $\varphi_-(x) \oplus \varphi_+(y) = V$ for all $x \neq y$ in S^1 (Theorem 4.1.1 (iii)). This induces a splitting of bundles

$$E^\varrho = E^\varrho_- \oplus E^\varrho_+.$$

As explained in [13, Sec. 8.2], there are metrics $\|\cdot\|^+_u$ and $\|\cdot\|^-_u$ on E^ϱ for $u \in \mathcal{L}^{(3)}$.

Note that $\mathcal{L}^{(3)}$ is not invariant under the geodesic flow g_t. But for $u = (u_-, u_o, u_+) \in \mathcal{L}^{(3)} \subset (S^1)^{(3)}$ we can define the set

$$T_{[u]} = \{t \in \mathbb{R} | [g_t u] \in h(\Gamma)\backslash\mathcal{L}^{(3)}\}.$$

and for all $\xi \in E^\varrho$ the flow $g_t\xi$ is defined for all $t \in T_{p(\xi)}$

Lemma 4.1.3. *(i) For every $\xi \in E^\varrho_+$*

$$\lim_{\substack{t\to\infty \\ t\in T_{p(\xi)}}} \|g^\varrho_t \xi\|^\pm = 0 \textit{ monotonically and}$$

$$\lim_{\substack{t\to-\infty \\ t\in T_{p(\xi)}}} \|g^\varrho_t \xi\|^\pm = \infty \textit{ monotonically}$$

(ii) For every $\xi \in E^\varrho_-$

$$\lim_{\substack{t\to-\infty \\ t\in T_{p(\xi)}}} \|g^\varrho_t \xi\|^\pm = 0 \textit{ monotonically and}$$

$$\lim_{\substack{t\to\infty \\ t\in T_{p(\xi)}}} \|g^\varrho_t \xi\|^\pm = \infty \textit{ monotonically}$$

The proof works as the proof of Lemma 8.8 in [18].

Lemma 4.1.4. *There exists a continuous metric $\|\cdot\|$ on E^ϱ which is equivalent to $\|\cdot\|^+$ and $\|\cdot\|^-$.*

It is clear that $\Gamma\backslash\mathcal{L}^{(3)}$ is compact and from Lemma 4.1.3 we get: for any $C \subset \mathcal{L}^{(3)}$ compact, the set of metrics

$$\{\|\cdot\|^\pm_u | u \in C\}$$

is bounded.

Lemma 4.1.5. *Let $E^\varrho = E^\varrho_- \oplus E^\varrho_+$ and $\|\cdot\|$ the continuous metric as in 4.1.4.*

(i) For every $\xi \in E^\varrho_+$

$$\lim_{\substack{t\to-\infty\\ t\in T_{p(\xi)}}} \|g_t^\varrho \xi\| = \infty \text{ and } \lim_{\substack{t\to\infty\\ t\in T_{p(\xi)}}} \|g_t^\varrho \xi\| = 0$$

(ii) For every $\xi \in E^\varrho_-$

$$\lim_{\substack{t\to\infty\\ t\in T_{p(\xi)}}} \|g_t^\varrho \xi\| = \infty \text{ and } \lim_{\substack{t\to-\infty\\ t\in T_{p(\xi)}}} \|g_t^\varrho \xi\| = 0$$

Proof. From the proof of Lemma 8.8 in [13] we get $\|g_t\xi\|^- \to \infty$ if $t \to -\infty$ for $\xi \in E^\varrho_+$ and $\|g_t\xi\|^+ \to \infty$ if $t \to \infty$ for $\xi \in E^\varrho_-$. Lemma 4.1.5 follows from the equivalence of $\|\cdot\|^\pm$ to the continuous metric $\|\cdot\|$. □

Now we have all ingredients to prove continuity of limit curves in our special case.

Proof of Theorem 1.4.5. Throughout this proof we omit the statement $t \in T_{[u]}$ for expressions as $\|g_t\xi\|_u$.

We have the following characterization of the splitting

$$E^\varrho_\pm = \{\xi \in E^\varrho : \lim_{t\to\pm} \|g_t^\varrho \xi\| = 0\}.$$

We want to show continuity of the splitting, which implies continuity of the limit curves. Let u_m be a converging sequence in $T^1\Sigma$ with limit u, and let $F \subset E^\varrho(u)$ be any accumulation point of

$$\{E^\varrho_+(u_m) : m \geq 1\}.$$

Let $\{m_k\}$ be any subsequence such that $\lim_{k\to\infty} E_+^\varrho(u_{m_k}) = F$. For every $\xi \in F$ take $\xi_k \in E_+^\varrho$ such that $\xi_k \to \xi$. We want to show $\xi \in E_+^\varrho$. Let $\xi = \xi^+ \oplus \xi^-$ with $\xi^+ \in E_+^\varrho$ and $\xi^- \in E_-^\varrho$. Every functions $\|g_t^\varrho \xi_k\| : [0,\infty) \to \mathbb{R}$ is bounded above by $A^2\|\xi_k\|$, where A is the constant coming from the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|^\pm$. But since $\|\xi_k\|$ converges to $\|\xi\|$ the functions are uniformly bounded and the limit $\|g_t^\varrho\xi\|$ is bounded as well. But if $\xi^- \neq 0$, by Lemma 4.1.5, $\|g_t^\varrho\xi\|_{t\to\infty} \to \infty$, which is a contradiction to boundedness of $\|g_t^\varrho\xi\|$. □

Remark 4.1.6. An adaption of this argument for representations, where $\varrho(C_i)$ are parabolic fails, because in this case one has to choose $h(\Gamma)$ such that it does not act cocompactly on the hyperbolic plane. Hence one can not construct the continuous metric as above. Nevertheless it seems quite plausible that the limit curve is continuous in this case, since the images of the generators C_i only have one fixed point in $\check{S}$ (Proposition 3.3.4). Hence the limit curve is automatically continuous in these points.

Proposition 4.1.7. *Let ϱ be maximal representation which is S-hyperbolically generated. Then the associated limit curve is unique.*

Proof. Let $\tilde{\varrho}$ be a double of ϱ. Then by Lemma A.3.2, every element $\varrho(\gamma) \in \tilde{\varrho}(\Gamma)$ contracts an open and dense subset of to $\tilde{\varphi}(\gamma)$, which is equal to $\varrho(\gamma)$ if $\varrho(\gamma) \in \varrho(\Gamma)$. Hence the limit curve is unique. □

4.2 Connected components

We have seen in Section 1, that connected components of spaces of maximal representations of fundamental groups of closed surfaces have been counted using Higgs bundle techniques. Olivier Guichard and Anna Wienhard gave in [35] one example for a representation in each connected component of $\mathrm{Rep}_{max}(\Gamma_g, \mathrm{Sp}(2n,\mathbb{R}))$.

The parameters from Theorem 3.3.2 allow us to count the connected components of $\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))$ for surfaces with boundary $\Sigma_{g,m}$ with $m \geq 1$. First we show that $\#\pi_0(\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))) \geq 2^{2g+m-1}$ (Proposition 4.2.5). Then we show, that one can deform every representation of $\Gamma_{g,m}$ with $m \geq 1$ into some standard representation (Proposition 4.2.6). This gives an upper bound and shows that $\#\pi_0(\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))) = 2^{2g+m-1}$.

Before we start, we collect some fact needed later in this section.

Lemma 4.2.1. *The set*

$$B = \{X \in \mathrm{GL}(n, \mathbb{R}) | X \text{ contracting}\}$$

has two connected components distinguished by the sign of the determinant.

Recall

$$\begin{aligned}
\tilde{R}_n =& \{(X_1, X_2, X_3) \in \mathrm{GL}(n, \mathbb{R})^3 | X_3 (X_2^\top)^{-1} X_1 \text{ symmetric} \\
& \text{and positive definite}\}, \\
R =& \{(X_1, X_2, X_3) \in \bar{B}^3 | X_3 (X_2^\top)^{-1} X_1 \text{ symmetric and positive definite}\} \\
R^* =& \{(X_1, X_2, X_3) \in B^3 | X_3 (X_2^\top)^{-1} X_1 \text{ symmetric and positive definite}\}
\end{aligned}$$

and

Lemma 4.2.2. *Note that $(X_1, X_2, X_3) \in \tilde{R}_n$ if and only if*

$$(\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3) \in \tilde{R}_n$$

for $(\lambda_1, \lambda_2, \lambda_3) \in \tilde{R}_1$.

Proposition 4.2.3. *The sets $\tilde{R}_n$, R and R^* have four connected components distinguished by the signs of the determinants of the X_i.*

Proof. We begin with $\tilde{R}$. It can be identified with $\mathrm{GL}(n, \mathbb{R})^2 \times \Omega$, where Ω is the set of symmetric and positive definite matrices in $\mathrm{GL}(n, \mathbb{R})$. Indeed, the map from $\tilde{R}$ to $\mathrm{GL}(n, \mathbb{R})^2 \times \Omega$

$$(X_1, X_2, X_3) \mapsto (X_1, X_2, X_3 (X_2^\top)^{-1} X_1)$$

is a homoeomorphism between these spaces. Since $\mathrm{GL}(n,\mathbb{R})$ has two connected components distinguished by the signs of the determinants of X_1 and X_2, $\tilde{R}$ has four connected components (Ω is connected). Now we proof the proposition for R. The main ingredient for the proof is Lemma 4.2.2.
In any connected component of $\tilde{R}$ there is at least one connected component of R. Indeed, let $(X_1, X_2, X_3) \in \tilde{R}$ arbitrary, then there exists λ_1, λ_2 and λ_3 in $(0,1]$ such that $(\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3) \in R$. Hence $|\pi_0(R)| \geq |\pi_0(\tilde{R})|$.
Now we show equality. Let (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) be triples in R, such that there is a path $s = (s_1, s_2, s_3)$ joining them in $\tilde{R}$. By Lemma 4.2.2 there exists for any t a triple $(\lambda_1(t), \lambda_2(t), \lambda_3(t)) \in (0,1]^3$ such that $(\lambda_1(t)s_1(t), \lambda_2(t)s_2(t), \lambda_3(t)s_3(t)) \in R$ for all t. Since the image of s is compact, there exists λ_1, λ_2 and λ_3 such that the path $\tilde{s}(t) := (\lambda_1 s_1(t), \lambda_2 s_2(t), \lambda_3 s_3(t))$ is in R. It joins $(\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3)$ and $(\lambda_1 Y_1, \lambda_2 Y_2, \lambda_3 Y_3)$ in R. Furthermore by construction there is a path joining (X_1, X_2, X_3) and $(\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3)$ as well as a path joining $(\lambda_1 Y_1, \lambda_2 Y_2, \lambda_3 Y_3)$ and (X_1, X_2, X_3). Hence (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are in the same connected component of R and $|\pi_0(R)| = |\pi_0(\tilde{R})|$.

The proof for R^* goes along the same lines. □

We use the notions from Theorem 3.6.9 resp. Figure 3.9.

Proposition 4.2.4. *The signs of the determinants of length parameters X_i and Y_j and twist parameters H_k distinguish connected components.*

Proof. By Lemma 1.4.9 representations ϱ with $\varrho(C_i)$ S-hyperbolic for all i are Anosov. Since they are dense in the representation variety $\mathrm{Rep}(\Gamma_{g,m}, \mathrm{Sp}(2n,\mathbb{R})$, it is enough to prove the proposition for this case. For Anosov representations we can apply Lemma 4.11 in [35], which expresses the first Stiefel-Whitney class of a certain bundle in terms of the representation ϱ. We use the notation from [35]. For this lemma we need the interpretation of the Shilov boundary as the space of Lagrangian subspaces (see Example 2.1.36). We can assume $\xi(t_\gamma^s) = \langle e_{n+1}, \dots, e_{2n}\rangle$ and

$$\varrho(\gamma) = \begin{pmatrix} X & 0 \\ Y & (X^\top)^{-1} \end{pmatrix}.$$

The matrix $\varrho(\gamma)$ acts on the last n components of the vectors in $\xi(t_\gamma^s)$ by multiplication with $(X^\top)^{-1}$. Therefore we get by [35, Lemma 4.11]

$$sw_1(\varrho)([\gamma]) = \operatorname{sgn}(\det \varrho(\gamma)|_{\xi(t_\gamma^s)}) = \operatorname{sgn}(\det (X^\top)^{-1}) = \operatorname{sgn}(\det X)$$

For the twist parameter, we also have to consider elements of $\mathrm{Sp}(2n, \mathbb{R})$ of the form

$$\varrho(\bar\gamma) = \begin{pmatrix} Y_3 G Y_1^{-1} - (G^\top)^{-1} & Y_3 G \\ G Y_1^{-1} & -G \end{pmatrix}.$$

We want to calculate $sw_1(\varrho)(\bar\gamma)$ as above. Hence we investigate the differential of $\varrho(\bar\gamma)$ in a fixed point. We know that $\varrho(\bar\gamma)$ is S-hyperbolic and that it has a pair of transversal fixed points Y and Y' with

$$\beta(Y_1, Y, 0) = \beta(e, Y', \infty) = n.$$

In particular $Y - Y_1$ is positive definite. By construction Y_1 is negative definite, hence the sign of its determinant only depends on n. To obtain the derivative of d in Y we calculate

$$\begin{pmatrix} 1 & -Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y_3 G Y_1^{-1} - (G^\top)^{-1} & Y_3 G \\ G Y_1^{-1} & -G \end{pmatrix} \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} (Y_3 G Y_1^{-1} - (G^\top)^{-1}) - Y G Y_1^{-1} & 0 \\ G Y_1^{-1} & G Y_1^{-1} Y - G \end{pmatrix}$$

Hence the same argument as above applies.

It is enough to consider the length parameters X_i and Y_j and twist parameters H_k because they already determine the Stiefel-Whitney classes. Indeed, $sw_1(\varrho)(A_i)$, $sw_1(\varrho)(B_i)$ and $sw_1(\varrho)(C_j)$ are uniquely determined by them. $\square$

Define:

$$X_\pm := \begin{pmatrix} \pm\frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad G_\pm := \begin{pmatrix} \pm 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Proposition 4.2.5. $\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n, \mathbb{R}))$ *has at least* 2^{2g+m-1} *connected components if* $m \geq 1$.

Proof. We can explicitly write down parameters for 2^{2g+m-1} representations which are by Proposition 4.2.4 in different connected components. The length parameters of all of these representations are either X_+ or X_-. The twist parameter are either G_+ or G_-.

We have a complete freedom of choice for the twist parameters H_k between G_+ or G_- as well as for all length parameter X_i and Y_j, except X_1, betweeen X_+ or X_-. This leaves only one choice for all other length parameter if we want them to be either X_+ or X_-. By Proposition 4.2.4 they all lie in different connected compontens. ☐

The representations associated with these parameters are twisted diagonal representations as defined in [35].

Proposition 4.2.6. *Let* $\varrho : \Gamma_{g,m} \to \mathrm{Sp}(2n, \mathbb{R})$ *be a maximal representation and we label its parameters as in Figure 3.9. It can be deformed into a representation* $\bar{\varrho}$ *with length parameters*

$$\bar{L}_i := X_{\mathrm{sgn}\,\det(L_i)}$$

where $L_i \in \{W_i, X_i, Y_i, Z_i\}$ *is a length parameter from Figure 3.9 and*

$$\bar{T}_j := G_{\mathrm{sgn}\,\det(T_j)},$$

where $T_j \in \{G_j, H_j, J_j\}$ *is a twist parameter and* $\mathrm{sgn}\det \in \{+, -\}$

Together with Proposition 4.2.5 we get:

Theorem 4.2.7. $\mathrm{Rep}_{max}(\Gamma_{g,m}, \mathrm{Sp}(2n,\mathbb{R}))$ *has* 2^{2g+m-1} *connected components if* $m \geq 1$.

Before we can prove Proposition 4.2.6 we need two lemmas, which are special cases.

Lemma 4.2.8. *Let* $\varrho : \Gamma_{0,3} \to \mathrm{Sp}(2n,\mathbb{R})$ *be a maximal representation with three contracting parameters* X_1, X_2 *and* X_3. *Let* $X_3(t) : [0,1] \to B$ *and* $X_2(t) : [0,1] \to B$ *be two path starting in* X_1 *resp.* X_2. *Then there exists a path* $X_1(t) : [0,1] \to B$ *such that* $X_3(t)\big(X_2(t)^\top\big)^{-1} X_1(t)$ *is symmetric and positive definite for any* t, *i.e. the path* $(X_1(t), X_2(t), X_3(t))$ *gives parameters for maximal representations of* $\Gamma_{0,3}$ *into* $\mathrm{Sp}(2n,\mathbb{R})$ *for any* t, *such that all* $X_i(t)$ *are contracting. Hence it defines a path in* $\mathrm{Rep}_{max}(\Gamma_{0,3}, \mathrm{Sp}(2n,\mathbb{R}))$.

Proof. Since X_1, X_2 and X_3 are parameters for a representation, we know that $S := X_3(X_2^\top)^{-1}X_1$ is symmetric and positive definite. Defining $\tilde{X}_1(t) := (X_3(t)(X_2(t)^\top)^{-1})^{-1}S$, we get a path with $\tilde{X}_1(0) = X_1$ such that $\tilde{X}_1(t)$, $X_2(t)$ and $X_3(t)$ are parameters for a maximal representation for any $t \in [0,1]$. To fix the issue that $\tilde{X}_1(t)$ might be non-contracting for some t, we choose a curve $\lambda : [0,1] \to R_{>0}$ such that $\lambda(0) = 1$ and $1/\lambda(t)$ is bigger than the absolute value of the biggest eigenvalue of $\tilde{X}_1(t)$ for all t. Now putting $X_1(t) := \lambda(t)\tilde{X}_1(t)$ finishes the proof. □

Corollary 4.2.9. *Every maximal representation of* $\Gamma_{0,3}$ *into* $\mathrm{Sp}(2n,\mathbb{R})$ *with parameters* (X_1, X_2, X_3) *can be deformed into the representation with parameters* $(X_{\mathrm{sgn}\det X_1}, X_{\mathrm{sgn}\det X_2}, X_{\mathrm{sgn}\det X_3})$.

Lemma 4.2.10. *Let* $\varrho : \Gamma_{1,1} \to \mathrm{Sp}(2n,\mathbb{R})$ *be a maximal representation with parameters* (X_1, X_2, G) *as in Proposition 3.6.1. Then* ϱ *can be deformed into a maximal representation* $\bar{\varrho}$ *with parameters*

$$\bar{X}_1 = X_{\mathrm{sgn}\det X_1},\ \bar{X}_2 = X_{\mathrm{sgn}\det X_2},\ G = G_{\mathrm{sgn}\det G}.$$

Proof. Let $X_1(t)$ and $G(t)$ be paths in $\mathrm{GL}(n,\mathbb{R})$ joining X_1 and $X_\pm$ resp. G and $G_\pm$. Choose a path $S(t)$ in the symmetric positive definite matrices joining $GX_2^\top G^{-1}(X_2^\top)^{-1}X_1$ and $\frac{1}{2}I$ such that $X_1(t) := (G(t)X(t)_2^\top G(t)^{-1}(X(t)_2^\top)-1)^{-1}S(t)$ is contracting for all t. This is possible since one can scale $S(t)$ as in the proof of Proposition 4.2.8 such that the eigenvalues of $X_1(t)$ are small enough. These paths defines a path in $\mathrm{Rep}_{max}(\Gamma_{1,1},\mathrm{Sp}(2n,\mathbb{R}))$ joining ϱ with the desired representation. □

Proof of Proposition 4.2.6. We prove Proposition 4.2.6 by recurrence. By Lemma 4.2.8 and Lemma 4.2.10 it is true for representations of $\Gamma_{0,3}$ and $\Gamma_{1,1}$. So we assume now that it is true for $\Gamma_{g,m}$ with $m \geq 1$.

(i) The case $\Gamma_{g,m+1}$
Let $\varrho : \Gamma_{g,m+1} \to \mathrm{Sp}(2n,\mathbb{R})$ be a maximal representation. First note that we obtain $\Sigma_{g,m+1}$ by gluing $\Sigma_{g,m}$ and $\Sigma_{0,3}$. Let (X_1, X_2, X_3) be parameters of the restriction of ϱ to $\pi_1(\Sigma_{0,3})$.

By recurrence assumption we can deform the parameters of $\varrho|_{\Gamma_{g,m}}$ as requested. This produces a path $X(t)$ for the boundary component of $\Sigma_{g,m}$ along which we can glue $\Sigma_{g,m}$ and $\Sigma_{0,3}$. Joining the twist parameter for this boundary component with G_+ resp. G_- defines a path from X_3 to X_+ resp. X_-. By Lemma 4.2.8 this path, together with a path $X_2(t)$ joining X_2 and X_+ resp. X_- produces a path which joins ϱ with the desired representation.

(ii) The case $\Gamma_{g+1,m}$
This case works analogously by gluing a surface $\Sigma_{1,2}$ to $\Sigma_{g,m}$.

□

Chapter 5

Cross Ratios

This chapter is joint work with Tobias Hartnick.

5.1 Construction of functorial cross ratios

5.1.1 Definition and basic properties of generalized cross ratios

Let V be a simple Euclidean Jordan algebra and denote by $\mathcal{D} = \mathcal{D}_V$ the associated irreducible bounded symmetric domain of tube type. Since $\mathcal{D}$ is simply-connected, the rational powers of a continuous function $k : \mathcal{D} \times \mathcal{D} \to \mathbb{C}^\times$ with $k(0,0) = 1$ are well-defined. Indeed, given integers $p, q \in \mathbb{Z}$, $q > 0$ we define $k^{p/q}$ as the unique continuous function satisfying

$$(k^{p/q})^q = k^p, \quad k(0,0) = 1. \tag{5.1}$$

There are two natural kernel functions on $\mathcal{D}$, which are closely related. On the one hand, there is the Bergman kernel $k_{\mathcal{D}}$ of $\mathcal{D}$, on the other hand there is the kernel $k_{\det} := \det \circ K$ associated with the *canonical automorphy kernel* $K : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathrm{End}(V^{\mathbb{C}})$ of V (see [25] and also [29, 55] for background). By [29, Prop. X.4.5] there exists a constant $C = C(V)$ such that

$$k_{\mathcal{D}_V} = C \cdot k_{\det}^{-1}. \tag{5.2}$$

This implies that the *weighted Bergman cross ratio* of weight α as given by

$$B_{\mathcal{D}}^{(\alpha)} : \mathcal{D}^4 \to \mathbb{C}^\times, \quad (x, y, z, t) \mapsto \frac{k_{\mathcal{D}}^{\alpha}(t, x) k_{\mathcal{D}}^{\alpha}(y, z)}{k_{\mathcal{D}}^{\alpha}(t, z) k_{\mathcal{D}}^{\alpha}(y, x)}. \tag{5.3}$$

can be expressed in terms of the canonical automorphy kernel as

$$B_{\mathcal{D}_V}^{(\alpha)}(x, y, z, t) = \frac{k_{\det}^{-\alpha}(t, x) k_{\det}^{-\alpha}(y, z)}{k_{\det}^{-\alpha}(t, z) k_{\det}^{-\alpha}(y, x)}. \tag{5.4}$$

Weighted Bergman cross ratios on bounded symmetric domains of tube type were introduced by Clerc and Ørsted in [25]. For simple V the normalization in their paper corresponds to the choice $\alpha = -\frac{\mathrm{rk}(V)}{\dim V}$ (see [29, Prop.III.4.3]). Here we will choose a different normalization.

Definition 5.1.1. Let $\mathcal{D}$ be an irreducible bounded symmetric domain. Then the *normalized cross ratio* $B_{\mathcal{D}}$ is defined as the weighted Bergman cross ratio $B_{\mathcal{D}}^{(-\frac{1}{2\dim V})}$ of weight $\alpha := -\frac{1}{2\dim V}$. Moreover, the *normalized kernel function* is defined to be $k_V := k_{\det}^{-\alpha}$.

For later reference we record that, by definition,

$$B_{\mathcal{D}_V}(x, y, z, t) = \frac{k_V(t, x) k_V(y, z)}{k_V(t, z) k_V(y, x)}. \tag{5.5}$$

Moreover, our kernel k_V and the kernel function denoted k in [25] are related by the formula

$$k_V^{2 \cdot \mathrm{rk}(V)} = k. \tag{5.6}$$

There are two reasons for our different normalization: In the rank one case, our cross ratio is the square root of the cross ratio of Clerc and Ørsted. In view of [25, Lemma 5.4] this implies that the continuous extension of our cross ratio to the circle yields the classical cross ratio rather than its square, as demanded by Property (iv) in Theorem 1.5.1 (see Example 5.1.8 below). Our reason for taking an additional rth root in the higher rank case is as follows: Suppose $\alpha : V_1 \to V_2$ is a morphism of Euclidean Jordan algebras of respective ranks r_1, r_2; then the corresponding kernels $k_{(1)}, k_{(2)}$ in the sense of [25] have the following equivariance property [25, Prop. 6.2]:

$$k_{(2)}(\alpha^{\mathbb{C}}(z), \alpha^{\mathbb{C}}(w)) = k_{(1)}(z, w)^{\frac{r_2}{r_1}} \quad (z, w \in \mathcal{D}_{V_1}).$$

Our normalization is chosen in such a way that this translates into the following *invariance* property:

Lemma 5.1.2 (Clerc-Ørsted)**.** *Let $\alpha : V_1 \to V_2$ be a morphism of simple Euclidean Jordan algebras. Then for all $x, y, z, t \in \mathcal{D}_{V_1}$ we have*

$$k_{V_2}(\alpha^{\mathbb{C}}(z), \alpha^{\mathbb{C}}(w)) = k_{V_1}(z, w),$$

and hence

$$B_{V_2}(\alpha^{\mathbb{C}}(x), \alpha^{\mathbb{C}}(y), \alpha^{\mathbb{C}}(z), \alpha^{\mathbb{C}}(t)) = B_{V_1}(x, y, z, t).$$

This is a first instance of the functoriality property of Theorem 1.5.1. In order to obtain a similar functoriality property for general cross ratios, we choose the following normalization in the reducible case:

Definition 5.1.3. Let V be a Euclidean Jordan algebra and $V = V_1 \oplus \cdots \oplus V_m$ a decomposition of V into simple ideals. Then the *normalized kernel function* k_V is defined by the formula

$$k_V(z, w)^{\operatorname{rk} V} = \prod_{j=1}^{m} k_{V_j}(z_i, w_i)^{\operatorname{rk} V_j}, \quad k_V(0, 0) = 1,$$

where the kernels k_{V_j} are defined as in (5.1.1), and the *normalized cross ratio* of $\mathcal{D} = \mathcal{D}_V$ is defined by (5.5).

Remark 5.1.4. The definition of weighted Bergman cross ratios makes sense for arbitrary domains which are biholomorphic to a bounded domain (so that the Bergman kernel is well-defined). In this generality we do, however, not see any preferable normalization. For bounded symmetric domains, which are not of tube type, the normalization can be chosen as follows: Define $B_{\mathcal{D}}$ by the formula

$$B_{\mathcal{D}} := B_{\mathcal{D}}^{(-\frac{1}{2\dim_{\mathbb{C}}\mathcal{D}})}$$

if $\mathcal{D}$ is irreducible, and extend to the non-irreducible case as above. With these definitions, large parts of the theory carry over to the non-tube type case, since the automorphy kernel is defined for any Jordan triple system. However, since all our applications are concerned with the tube type case, and the notation in the non-tube type case is considerably more complicated, we do not develop the theory in this larger generality.

Our normalized kernel functions extend continuously to the Shilov boundary. More precisely:

Proposition 5.1.5. *Let V be a Euclidean Jordan algebra. Then the normalized kernel k_V extends continuously to the Shilov boundary and for $z, w \in \check{S}$ we have*

$$k_V(z, w) \neq 0 \Leftrightarrow z \pitchfork w.$$

Let us first assume that V is simple. Then $k_{\det}$ extends continuously to the Shilov boundary and the extension satisfies $k_{\det}(z, w) \neq 0$ iff $z \pitchfork w$, see Proposition A.2.3 in the appendix. It follows that we can extend k_V to a continuous nowhere-vanishing function on $\check{S}^{(2)}$.

Lemma 5.1.6. *Let X be a topological space, $f : X \to \mathbb{C}$ be a continuous function and denote $X' := f^{-1}(\mathbb{C} \setminus \{0\})$. Let $\widetilde{f} : X' \to \mathbb{C} \setminus \{0\}$ be any continuous function with $\widetilde{f}^n = f|_{X'}$. Then $\widetilde{f}$ extends continuously by 0 to all of X.*

Proof. We have to show that the extension $\widetilde{f}$ to all of X by 0 is continuous. For this let $x_k \in X'$ with $x_k \to x$, where $x \in X \setminus X'$. Then $f(x_k) \to f(x) = 0$ by continuity of f, hence $\widetilde{f}(x_k)^n \to 0$. This, however, implies already $\widetilde{f}(x_k) \to 0 = \widetilde{f}(x)$, which yields continuity of the extended function. $\square$

Applying the lemma we deduce that for simple V the kernel k_V extends continuously to all of $\check{S}$ with zero set given by the complement of $\check{S}^{(2)}$. Another application of the lemma then reduces the general case to the irreducible one, thereby finishing the proof of Proposition 5.1.5.

As an immediate consequence of Proposition 5.1.5 and (5.5) we deduce that $B_{\mathcal{D}}$ extends continuously to a function

$$B_{\check{S}} : \check{S}^{(2)} \times \check{S}^{(2)} = \{(x, y, z, t) \in \check{S}^4 \,|\, x \pitchfork y, z \pitchfork t\} \to \mathbb{C},$$

which is nonzero on $\check{S}^{(4)} \subset \check{S}^{(2)} \times \check{S}^{(2)}$. It turns out, however, that the present domain for $B_{\check{S}}$ is too large for our purposes: Neither is the extended cross ratio real-valued on $\check{S}^{(2)} \times \check{S}^{(2)}$, nor can we show functoriality for these domains. It turns out, a posteriori, that the following domain is ideally suited for our purposes:

Definition 5.1.7. Let $\mathcal{D}$ be a bounded symmetric domain and $\check{S}$ the associated Shilov boundary. A quadruple $(x, y, z, t) \in \check{S}^{(4)}$ is called *extremal* if any triple $(a, b, c) \in \check{S}^3$ of pairwise distinct points with $a, b, c \in \{x, y, z, t\}$ has either maximal or minimal Maslov index. (Such a triple is then called *maximal* or *minimal* accordingly.) We denote the set of extremal quadruples in $\check{S}^4$ by $\check{S}^{(4+)}$. Then the *generalized cross ratio* of the Shilov boundary $\check{S}$ is the function

$$B_{\check{S}} : \check{S}^{(4+)} \to \mathbb{C}^\times, \quad (x, y, z, t) \mapsto \frac{k_V(t, x) k_V(y, z)}{k_V(t, z) k_V(y, x)}. \tag{5.7}$$

The term *generalized* refers to the following example:

Example 5.1.8. Let $V = (\mathbb{R}, \cdot)$ so that $V^{\mathbb{C}} = \mathbb{R}^{\mathbb{C}} = \mathbb{C}$ and $\mathcal{D}_V = \mathbb{D}$ is the Poincaré disc. Then for $x, w, z \in \mathbb{C}$ we have $K(z,w)x = (1 - z\bar{w})^2 x$, in particular $k_{\mathbb{R}}(z,w) = 1 - z\bar{w}$ and thus

$$B_{\mathbb{D}}(a,b,c,d) = \frac{(1-d\bar{a})(1-b\bar{c})}{(1-d\bar{c})(1-b\bar{a})}.$$

We deduce that

$$B_{S^1}(a,b,c,d) = \frac{(1-d\bar{a})(1-b\bar{c})}{(1-d\bar{c})(1-b\bar{a})} = \frac{(a-d)(c-b)}{(c-d)(a-b)} = [a:b:c:d].$$

Remark 5.1.9. A similar computation for then rank r polydisc $\mathcal{D} = \mathbb{D}^r$ (or Lemma 5.1.15 below) shows that

$$B_{(S^1)^r}(a,b,c,d) = \left(\prod_{i=1}^{r} \frac{(a_i - d_i)(c_i - b_i)}{(c_i - d_i)(a_i - b_i)} \right)^{1/r}.$$

In particular, the cross ratio is invariant under the diagonal embedding of $\mathbb{D}$ into $\mathbb{D}^r$. This sort of functoriality explains our normalization in the reducible case.

The following proposition summarizes the basic properties of our construction:

Proposition 5.1.10. *Let $\mathcal{D}$ be a bounded symmetric domain of tube type with Shilov boundary $\check{S}$.*

(i) The normalized cross ratio $B_{\mathcal{D}}$ and the generalized cross ratio $B_{\check{S}}$ are invariant under $G(\mathcal{D})$.

(ii) Suppose $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ is the product of two bounded symmetric domains $\mathcal{D}_1, \mathcal{D}_2$ of respective ranks r_1, r_2 with corresponding Shilov boundaries $\check{S}, \check{S}_1, \check{S}_2$ and $\check{S}$ is identified with $\check{S}_1 \times \check{S}_2$. Then

$$B_{\check{S}}(x,y,z,t)^{r_1+r_2} = B_{\check{S}_1}(x_1,y_1,z_1,t_1)^{r_1} B_{\check{S}_2}(x_2,y_2,z_2,t_2)^{r_2}.$$

Indeed, the second statement holds by definition and can be used to reduce the proof of the first statement to showing invariance of $B_{\mathcal{D}}$ for irreducible bounded symmetric domains. In this case we can appeal to the following general fact:

Proposition 5.1.11. *Let $\mathcal{C}$ be complex domains biholomorphic to a bounded domain and let $c : \mathcal{D} \to \mathcal{C}$ be a biholomorphism. Then for all $(x, y, z, t) \in \mathcal{D}^4$ and for every $\alpha \in \mathbb{Q}$ we have*

$$B_{\mathcal{D}}^{(\alpha)}(x, y, z, t) = B_{\mathcal{C}}^{(\alpha)}(c(x), c(y), c(z), c(t)).$$

Proof. Since the equality is invariant under taking rational powers, it suffices to prove the proposition for $\alpha = 1$. According to [29, Prop. IX.2.4] the Bergman kernels on $\mathcal{D}$ and $\mathcal{C}$ are related by the formula,

$$k_{\mathcal{D}}(z, w) = k_{\mathcal{C}}(c(z), c(w)) \det{}_{\mathbb{C}}(J_c(z))\overline{\det{}_{\mathbb{C}}(J_c(w))},$$

where J_c denotes the complex Jacobian of c. Now write out the definition of the Bergman cross ratio an cancel the Jacobian terms to obtain

$$B_{\mathcal{D}}^{(1)}(x, y, z, t) \quad = \quad B_{\mathcal{C}}^{(1)}(c(x), c(y), c(z), c(t)).$$

□

5.1.2 Balanced morphisms and functoriality

We now aim to establish the functoriality property of generalized cross ratios, which motivated (and a posteriori justifies) our normalizations. Indeed, we will prove the following proposition; the notion of a balanced tight morphism will be defined in Definition 5.1.14 below (see also Example 5.1.17).

Proposition 5.1.12. *Let $\mathcal{D}_1, \mathcal{D}_2$ be bounded symmetric domains of tube type with respective Shilov boundaries $\check{S}_1, \check{S}_2$, let $\beta : \mathcal{D}_1 \to \mathcal{D}_2$ be* *a balanced tight morphism and $\bar{\beta} : \check{S}_1 \to \check{S}_2$ its boundary extension. Then for all $(x, y, z, t) \in \check{S}^{(4+)}$ we have*

$$B_{\check{S}_2}(\bar{\beta}(x), \ldots, \bar{\beta}(t)) = B_{\check{S}_1}(x, \ldots, t).$$

In the case of *irreducible* bounded symmetric domains the proposition is true for arbitrary morphisms. Indeed, this follows by continuous extension from Lemma 5.1.2. However, in the general case one cannot expect such an unconditional result, as the following generic counterexample shows:

Example 5.1.13. Consider the Jordan algebra embedding $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ given by $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_1, \lambda_2)$. Then

$$\begin{aligned} k_{\mathbb{R}^2}(\lambda, \mu) &= (1 - \lambda_1\overline{\mu_1})^{\frac{1}{2}}(1 - \lambda_2\overline{\mu_2})^{\frac{1}{2}} \\ &\neq (1 - \lambda_1\overline{\mu_1})^{\frac{2}{3}}(1 - \lambda_2\overline{\mu_2})^{\frac{1}{3}} = k_{\mathbb{R}^3}(\alpha^{\mathbb{C}}(\lambda), \alpha^{\mathbb{C}}(\mu)). \end{aligned}$$

We will ask for functoriality under all morphisms except for those which on some polydisc look like the one in Example 5.1.13. This leads to the definition of a balanced morphism. Denote by tr_V the Jordan algebra trace of V and remind the reader that [29, Thm. III.1.2] for any Jordan frame $(c_1, \ldots, c_r)$ of V we have

$$x = \sum_{j=1}^{r} \lambda_j c_j \Rightarrow \mathrm{tr}_V(x) = \sum_{j=1}^{r} \lambda_j.$$

Now we define:

Definition 5.1.14. A Jordan algebra homomorphism $\alpha : V \to W$ is called *balanced* if for all $v \in V$

$$\frac{1}{\mathrm{rk}\, V} \mathrm{tr}_V(v) = \frac{1}{\mathrm{rk}\, W} \mathrm{tr}_W(\alpha(v)).$$

A tight morphism $\beta : \mathcal{D}_1 \to \mathcal{D}_2$ is called *balanced* if there exists Jordan algebras V, W and isomorphisms $\mathcal{D}_1 \cong \mathcal{D}_V$ and $\mathcal{D}_2 \cong \mathcal{D}_W$ intertwining β with the complexification of a balanced morphism of Euclidean Jordan algebras.

The notion is clearly invariant under complexification. Note that nonzero idempotents have positive trace and thus go to nonzero idempotents under balanced morphisms; this shows that every balanced Jordan algebra homomorphism is injective. Moreover, we have the following characterization of balanced Jordan algebra homomorphisms: Let $(c_1, \ldots, c_r)$ be a Jordan frame in V and $\alpha : V \to W$ a Jordan algebra homomorphism. Then $\alpha(c_1), \ldots, \alpha(c_r)$ is a family of idempotents with $\alpha(c_i)\alpha(c_j) = 0$ and $\sum \alpha(c_i) = e$. We thus find a Jordan frame $(c_{11}, \ldots, c_{1l_1}, \ldots, c_{r1}, \ldots, c_{rl_r})$ of W such that

$$\alpha(c_j) = \sum_{k=1}^{l_j} c_{jk}.$$

We have $\mathrm{tr}_W(\alpha(c_j)) = l_j$ and thus α is balanced if and only if

$$l_1 = \cdots = l_r.$$

Conversely, if the latter condition is true for any Jordan frame $(c_1, \ldots, c_r)$ of V, then α is balanced. Note that we obtain in particular

$$\mathrm{rk}\, W = l_j \cdot \mathrm{rk}\, V \quad (j = 1, \ldots, r),$$

so that $\mathrm{rk}\, W$ is divisible by $\mathrm{rk}\, V$. The morphism in Example 5.1.13 clearly violates this condition, and thus is not balanced. Heaving clarified our notion of morphism, we now turn to the proof of Proposition 5.1.12. The following lemma adapts results from [25] to our setting:

Lemma 5.1.15. *If $(c_1, \ldots, c_r)$ is a Jordan frame in a Euclidean Jordan algebra W of rank r and $\lambda_j \in \mathbb{D}$, $\mu_j \in \mathbb{D}$, then*

$$k_W\left(\sum_{j=1}^{r} \lambda_j c_j, \sum_{j=1}^{r} \mu_j c_j\right) = \prod_{j=1}^{r} (1 - \lambda_j \overline{\mu_j})^{\frac{1}{r}}. \tag{5.8}$$

Proof. If W is simple, then [25, Lemma 5.4] applies directly and in view of (5.6) yields the explicit formula

$$k_W^{2r}\left(\sum_{j=1}^r \lambda_j c_j, \sum_{j=1}^r \mu_j c_j\right) = k\left(\sum_{j=1}^r \lambda_j c_j, \sum_{j=1}^r \mu_j c_j\right) = \prod_{j=1}^r (1-\lambda_j\overline{\mu_j})^2\,,$$

which implies (5.8). For the general case, consider a decomposition $W = W_1 \oplus \cdots \oplus W_n$ into simple ideals. Let $r_l := \mathrm{rk}(W_l)$ and $(c_{l1}, \ldots, c_{lr_l})$ be a Jordan frame for W_l. Then $(c_{11}, \ldots, c_{nr_n})$ is a Jordan frame for W and, in fact, any Jordan frame for W is of this form (as follows e.g. from [29, Prop. X.3.2]). Let

$$\begin{aligned} z &:= \sum_{l=1}^n \sum_{j=1}^{r_l} \lambda_{lj} c_{lj}, \quad & w &:= \sum_{l=1}^n \sum_{j=1}^{r_l} \mu_{lj} c_{lj} \\ z_l &= \sum_{j=1}^{r_l} \lambda_{lj} c_{jl}, \quad & w_l &:= \sum_{j=1}^{r_l} \mu_{lj} c_{lj}. \end{aligned}$$

By the simple case we have

$$k_{W_l}(z_l, w_l)^{\mathrm{rk}\, W_l} = \prod_{j=1}^{r_l} (1-\lambda_{lj}\overline{\mu_j}),$$

hence

$$k_W(z,w)^{\mathrm{rk}\, W} = \prod_{l=1}^n (k_{W_l}(z_l, w_l))^{\mathrm{rk}\, W_l} = \prod_{l=1}^n \prod_{j=1}^{r_l} (1-\lambda_{lj}\overline{\mu_j}),$$

which is (5.8). □

For general bounded symmetric domains it is not true that the normalized kernel function itself is functorial; the following proposition provides a substitute, which is sufficient for our purposes. Let us call two elements $v_1, v_2 \in V$ *co-diagonalizable* if there exists a Jordan frame $(c_1, \ldots, c_r)$ of V and elements $\lambda_j \in \mathbb{D}$, $\mu_j \in \mathbb{D}$ such that

$$v_1 = \sum_{j=1}^{r} \lambda_j c_j \in \mathcal{D}_V, \quad v_2 = \sum_{j=1}^{r} \mu_j c_j \in \mathcal{D}_V.$$

By [29, X.2.2] x and y are co-diagonalizable if and only if $[L(x), L(y)] = 0$. Then we have:

Proposition 5.1.16. *Let $\alpha : V \to W$ be an injective homomorphism of Euclidean Jordan algebras. If α is balanced, then for every pair of co-diagonalizable elements $v_1, v_2 \in \mathcal{D}$ we have*

$$k_W(\alpha^{\mathbb{C}}(v_1), \alpha^{\mathbb{C}}(v_2)) = k_V(v_1, v_2). \tag{5.9}$$

Conversely, if (5.9) *holds for all co-diagonalizable $v_1, v_2 \in \mathcal{D}_V$, then α is balanced.*

Proof. If α is balanced, then $r_V := \mathrm{rk}(V)$ and $r_W := \mathrm{rk}(W)$ are related by $r_W = m_\alpha r_V$ for some constant multiplicity m_α. Given a Jordan frame $(c_1, \ldots, c_r)$ in V and elements

$$v_1 = \sum_{j=1}^{r} \lambda_j c_j \in \mathcal{D}_V, \quad v_2 = \sum_{j=1}^{r} \mu_j c_j \in \mathcal{D}_V$$

with $\lambda_j \in \mathbb{D}$, $\mu_j \in \mathbb{D}$ we have

$$\alpha^{\mathbb{C}}(v_1) = \sum_{j=1}^{r} \lambda_j \alpha(c_j), \quad \alpha^{\mathbb{C}}(v_2) = \sum_{j=1}^{r} \mu_j \alpha(c_j).$$

Now each $\alpha(c_j)$ decomposes as

$$\alpha(c_j) = d_{j1} + \cdots + d_{j\mu_\alpha},$$

where the d_{jl} are primitive idempotents. Now we obtain

$$k_V(v_1, v_2)^{r_V} = \prod_{j=1}^{r} (1 - \lambda_j \overline{\mu_j}),$$

whence

$$k_V(v_1,v_2)^{r_W} = \left(\prod_{j=1}^{r}(1-\lambda_j\overline{\mu_j})\right)^{m_\alpha} = \prod_{j=1}^{r}(1-\lambda_j\overline{\mu_j})^{m_\alpha}.$$

Similarly,

$$k_W(\alpha^{\mathbb{C}}(v_1),\alpha^{\mathbb{C}}(v_2))^{r_W} = \prod_{j=1}^{r}\prod_{l=1}^{m_\alpha}(1-\lambda_j\overline{\mu_j}) = \prod_{j=1}^{r}(1-\lambda_j\overline{\mu_j})^{m_\alpha}.$$

As $k_V(0,0) = k_W(\alpha^{\mathbb{C}}(0),\alpha^{\mathbb{C}}(0))^{r_W}$, this implies (5.9). On the other hand, if α is not balanced and v_1, v_2 are as above, then the multiplicity function m_α is non-constant and thus

$$\begin{aligned} k_V(v_1,v_2) = \prod_{j=1}^{r}(1-\lambda_j\overline{\mu_j})^{\frac{1}{r_V}} &\neq \prod_{j=1}^{r}(1-\lambda_j\overline{\mu_j})^{\frac{m_\alpha(c_j)}{r_W}} \\ &= k_W(\alpha^{\mathbb{C}}(v_1),\alpha^{\mathbb{C}}(v_2)). \end{aligned}$$

$\square$

Now we can finally prove Proposition 5.1.12:

Proposition 5.1.12. In view of Proposition 2.1.39 we may assume that $\mathcal{D}_1 = \mathcal{D}_V$ and $\mathcal{D}_2 = \mathcal{D}_W$ for Euclidean Jordan algebras V, W and $\bar{\beta} = \alpha^{\mathbb{C}}|_{\check{S}_1}$ for a balanced morphism $\alpha : V \to W$. Let $\hat{\beta} : \hat{G}_V \to \hat{G}_W$ be the equivariant lift of β so that $\alpha^{\mathbb{C}}(g.w) = \hat{\beta}(g).\alpha^{\mathbb{C}}(w)$ for all $g \in \hat{G}_V$ and $v \in \check{S}_1$. Observe that the actions of $\hat{G}_V$ and $\hat{G}_W$ factor through G_V and G_W; hence for every $g \in G_V$ there exists $h \in G_W$ such that for all $v \in \check{S}_V$,

$$\alpha^{\mathbb{C}}(gv) = h\alpha^{\mathbb{C}}(v). \tag{5.10}$$

Now assume $(x,y,z,t) \in \check{S}^{(4+)}$; then by Proposition 2.1.37 we find $g \in G_V$ such that gx, gy, gz, gt are diagonalized by a common Jordan frame $(c_1, \ldots, c_r)$. Let $h \in G_W$ be an element such that (5.10) holds for all $v \in \check{S}_1$. Using Proposition 5.1.10 and Proposition 5.1.16 we now obtain

$$\begin{aligned}
B_{\check{S}_1}(x,y,z,t) &= B_{\check{S}_1}(gx,gy,gz,gt)\\
&= \frac{k_V(gt,gx)k_V(gy,gz)}{k_V(gt,gz)k_V(gy,gx)}\\
&= \frac{k_W(\alpha^{\mathbb{C}}(gt),\alpha^{\mathbb{C}}(gx))k_W(\alpha^{\mathbb{C}}(gy),\alpha^{\mathbb{C}}(gz))}{k_W(\alpha^{\mathbb{C}}(gt),\alpha^{\mathbb{C}}(gz))k_W(\alpha^{\mathbb{C}}(gy),\alpha^{\mathbb{C}}(gx))}\\
&= \frac{k_W(h\alpha^{\mathbb{C}}(t),h\alpha^{\mathbb{C}}(x))k_W(h\alpha^{\mathbb{C}}(y),h\alpha^{\mathbb{C}}(z))}{k_W(h\alpha^{\mathbb{C}}(t),h\alpha^{\mathbb{C}}(z))k_W(h\alpha^{\mathbb{C}}(y),h\alpha^{\mathbb{C}}(x))}\\
&= B_{\check{S}_2}(h\alpha^{\mathbb{C}}(x),h\alpha^{\mathbb{C}}(y),h\alpha^{\mathbb{C}}(z),h\alpha^{\mathbb{C}}(t))\\
&= B_{\check{S}_2}(\bar{\beta}(x),\bar{\beta}(y),\bar{\beta}(z),\bar{\beta}(t)),
\end{aligned}$$

which is the desired functoriality. □

Example 5.1.17. The following are examples of balanced Jordan algebra homomorphisms (balanced morphisms of bounded symmetric domains):

- Jordan algebra homomorphisms $\alpha : V \to W$ between simple Jordan algebras (tight holomorphic morphisms between irreducible bounded symmetric domains) are balanced by Lemma 5.1.2.
- If $\mathrm{rk}(V) = \mathrm{rk}(W)$ then every injective Jordan algebra homomorphism $\alpha : V \to W$ is balanced. (Similarly for domains of equal rank.)
- In particular, maximal polydisc embeddings are balanced.
- Any Jordan algebra homomorphism $\alpha : \mathbb{R} \to W$ (any tight holomorphic disc) is balanced.
- Compositions of balanced Jordan algebra homomorphisms (or balanced tight holomorphic morphisms) are balanced.

5.1.3 Further properties of generalized cross ratios

We now collect a couple of further properties of generalized cross ratios. We will focus on those properties needed in the proof of Theorem 1.5.1 and those required in order to relate our cross ratios to strict cross ratios on the circle in the sense of Labourie [47]. Throughout this section we fix a Euclidean Jordan algebra V, associated bounded symmetric domain $\mathcal{D}$ and Shilov boundary $\check{S}$.

Lemma 5.1.18. *If X is a set and $k : X^2 \to \mathbb{C}^\times$ is an arbitrary function then*

$$b : \begin{cases} X^4 \to \mathbb{C}^\times \\ (x, y, z, t) \mapsto \frac{k(t,x)k(z,y)}{k(t,z)k(y,x)} \end{cases}$$

has the following properties:

$$\begin{aligned} b(x, y, z, t) &= b(z, t, x, y) && (5.11)\\ b(x, y, z, t) &= b(x, y, z, w)b(x, w, z, t) && (5.12)\\ b(x, y, z, t) &= b(x, y, w, t)b(w, y, z, t) && (5.13) \end{aligned}$$

Proof. Straightforward computation. □

Since the normalized kernel is only partially defined, this does not directly apply. Still we have:

Corollary 5.1.19. *Let $\mathcal{D}$ be a bounded symmetric domain of tube type with Shilov boundary $\check{S}$. Then the generalized cross ratio $B_{\check{S}} : \check{S}^{4+} \to \mathbb{C}^\times$ satisfies* (5.11)-(5.13) *above, whenever both sides of the equation are well-defined.*

Proof. Using Proposition 5.1.10 we can reduce to the irreducible case. In this case, Lemma 5.1.18 yields (5.11)-(5.13) for the weighted Bergman cross ratio $B_{\mathcal{D}}$, and by continuity these properties extend to $B_{\check{S}}$. □

Our next goal is to show that $B_{\check{S}}$ maps $\check{S}^{(4+)}$ to $\mathbb{R} \setminus \{0,1\}$. Thus let $(x,y,z,t) \in \check{S}^{(4+)}$; if (x,y,z) is maximal then we may apply Proposition 2.1.37 in order to find $g \in G$ and a Jordan frame $(c_1, \ldots, c_r)$ of V such that

$$g.(x,y,z,t) = (-e, -ie, e, \sum_{j=1}^{r} \lambda_j c_j).$$

Since the embedding of a maximal polydisc is balanced (Example 5.1.17), we can apply Proposition 5.1.12 to obtain

$$B_{\check{S}}(x,y,z,t) = B_{(S^1)^r}(-e, -ie, e, \lambda),$$

where $\lambda = (\lambda_j)$. Similary if (x,x,z) is minimal then we find $\lambda \in (S^1)^r$ with

$$B_{\check{S}}(x,y,z,t) = B_{(S^1)^r}(e, -ie, -e, \lambda).$$

In any case we may assume $V = \mathbb{R}^r$, $\mathcal{D} = \mathbb{D}^r$ and $\check{S} = (S^1)^r$ and either $(a,b,c) = (-e,-ie,e)$ or $(a,b,c) = (e,-ie,-e)$. We will only discuss the first case here, leaving the second (completely analogous) case to the reader. Since $(-e,-ie,e,\lambda)$ is assumed extremal, the possible values of λ are seriously restricted: Indeed, $(-1, \lambda_j, 1)$ is positive iff λ_j is contained in the lower half-circle and negative, iff λ_j is contained in the upper half-circle. Since $(-e, \lambda, e)$ is either maximal or minimal we see that either λ_j is contained in the lower half-circle for all $j = 1, \ldots, r$ or in the upper half-circle for all $j = 1, \ldots, r$. Correspondingly, let us call λ *positive* or *negative*. In the positive case, all the λ_j are contained in a fixed quarter circle. For special values of λ, the expression $B_{(S^1)^r}(-e,-ie,e,\lambda)$ is easy to compute:

Lemma 5.1.20. *If* $\lambda_1 = \cdots = \lambda_r$, *then*

$$B_{(S^1)^r}(-e, -ie, e, \lambda) = [-1 : -i : 1 : \lambda_1].$$

Proof. The Jordan algebra homomorphism $\mathbb{R} \to \mathbb{R}^r$ given by diagonal embedding is tight and balanced; its complexification maps

$(-1, -i, 1, \lambda_1)$ to $(-e, -ie, e, \lambda)$. Then the lemma follows from Proposition 5.1.12 and Example 5.1.8. $\square$

This is enough information to determine the sign of $B_{(S^1)^r}(-e, -ie, e, \lambda)$ in general:

Proposition 5.1.21. *The cross-ratio $B_{(S^1)^r}$ is real-valued on $((S^1)^r)^{(4+)}$. More precisely, $B_{(S^1)^r}(-e, -ie, e, \lambda)$ is positive/negative iff λ is positive/negative.*

Proof. Consider the function $f : (S^1 \setminus \{-1, -i, 1\})^r \to S^1$ given by

$$f(\lambda) := \frac{B_{(S^1)^r}(-e, -ie, e, \lambda)}{|B_{(S^1)^r}(-e, -ie, e, \lambda)|}.$$

We have $B_{(S^1)^r}(-e, -ie, e, \lambda)^r = \prod[-1, -i, 1, \lambda_j] \in \mathbb{R}$, hence $f(\lambda)^r \in \mathbb{R} \cap S^1 = \{\pm 1\}$. Therefore f takes values in the set R_{2r} of $2r$-th roots of unity. Since R_{2r} is discrete and f is continuous, f must be locally constant. In particular, if λ and μ are contained in the same connected component of $(S^1 \setminus \{-1, -i, 1\})^r$ and $B_{(S^1)^r}(-e, -ie, e, \mu)$ is a positive/negative real number, then $B_{(S^1)^r}(-e, -ie, e, \lambda)$ os also positive/negative. Combining this with Lemma 5.1.20 we obtain the proposition. $\square$

We can use the proposition to derive an explicit formula for the generalized cross ratio on the polydisc. Let us call an extremal quadruple (a, b, c, d) *positive/negative* if it is conjugate to $(-e, -ie, e, \lambda)$ for some positive/negative λ. Then Proposition 5.1.21 and Example 5.1.8 combine to the following formula:

Corollary 5.1.22. *Suppose (x, y, z) is maximal and $(x, y, z, t) \in ((S^1)^r)^{(4+)}$. Then*

$$B_{(S^1)^r}(x, y, z, t) = \epsilon(x, y, z, t) \cdot \sqrt[r]{\left|\prod_{j=1}^{r} [x_j : y_j : z_j : t_j]\right|},$$

where

$$\epsilon(x,y,z,t) = \begin{cases} +1 & (x,y,z,t) \textit{ positive} \\ -1 & (x,y,z,t) \textit{ negative} \end{cases}$$

In the case, where (x,y,z,t) is positive, there are two possibilities for t: Either, each t_j lies in between x_j and y_j or between y_j and t_j. This corresponds to the cases of (x,t,y) or (y,t,z) being maximal. These two cases can be distinguished by the cross ratio as follows:

Lemma 5.1.23. *If (x,y,z) and (x,t,y) are maximal, then $0 < B_{(S^1)^r}(x,y,z,t) < 1$. If (x,y,z) and $(y,t,z)$are maximal, then*

$$B_{(S^1)^r}(x,y,z,t) > 1.$$

Proof. The assumptions imply $0 < [x_j : y_j : z_j : t_j] < 1$, respectively $[x_j : y_j : z_j : t_j] > 1$ for each j, hence the lemma follows from the explicit formula in Corollary 5.1.22. □

We leave it to the reader to formulate the corresponding statements for the case where (x,y,z) in minimal. In any case we obtain:

Corollary 5.1.24. *We have $B_{\check{S}}(\check{S}^{(4+)}) = \mathbb{R} \setminus \{0,1\}$.*

Proof. Let $(x,y,z,t) \in \check{S}^{(4+)}$. If (x,y,z) is maximal, then depending on t we have either $B_{(S^1)^r}(x,y,z,t) < 0$ (if (x,y,z,t) is negative) or $B_{(S^1)^r}(x,y,z,t) < 1$ (if (x,t,y) is maximal) or $B_{(S^1)^r}(x,y,z,t) > 1$ (if (y,t,z) is maximal). If (x,y,z) is minimal one may argue similarly (or reduce to the former case by means of suitable cocycle properties). This shows the inclusion $\subset$. For the converse inclusion, it suffices to see that B_{S^1} is onto $\mathbb{R} \setminus \{0,1\}$ and $\mathbb{R}$ has a balanced embedding into every Euclidean Jordan algebra. □

As a consequence we obtain the following additional identity:

Corollary 5.1.25. *For all $(x,y,z,t) \in \check{S}^{(4+)}$ we have*

$$B_{\check{S}}(x,y,z,t) = B_{\check{S}}(y,x,t,z).$$

Proof. This follows immediately from the real-valuedness and the property $\overline{k_{\det}(z,w)} = k_{\det}(w,z)$. □

5.1.4 Proof of the functorial characterization

We claim that the family of normalized cross ratios $\{B_{\check{S}}\}$ satisfies Properties (i)-(iv) from Theorem 1.5.1 and is uniquely characterized by these properties. Indeed, Properties (i) and (iii) were proved in Proposition 5.1.10, Property (ii) was established in Proposition 5.1.12, and Property (iv) was checked in Example 5.1.8. It thus remains to establish uniqueness in order to prove Theorem 1.5.1. For this we argue as follows: Given a Shilov boundary $\check{S}$, any $(x, y, z, t) \in \check{S}^{(4+)}$ is contained in the boundary of a maximal polydisc by Proposition 2.1.37. Since the embedding of a maximal polydisc is balanced, the family $\{B_{\check{S}}\}$ is uniquely determined by the family $\{B_{(S^1)^r}\}$. Condition (iii) of Theorem 1.5.1 implies that

$$B_{(S^1)^r}(x, y, z, t)^r = \prod B_{S^1}(x_j, y_j, z_j, t_j).$$

Since $B_{\mathbb{R}}$ is determined by (iv), this determines $B^r_{\mathbb{R}^r}$ for every r. Since $B_{\mathbb{R}^r}$ is assumed real-valued, we have in fact determined $B_{\mathbb{R}^r}$ up to a locally constant function into $\{\pm 1\}$. To fix this sign, consider a diagonal disc embedding $\mathbb{R} \to \mathbb{R}^r$; the transversal quadruples of the Shilov boundary S^1 hit every connected component, and therefore determine the sign uniquely (see the proof of Proposition 5.1.21). This shows uniqueness and finishes the proof of Theorem 1.5.1.

5.2 Periods and translation lengths

5.2.1 Translation length for linear groups

We recall from the introduction that given an action of a group G on a metric space X the *translation length* $\tau_X(g)$ of $g \in G$ on X is defined by the formula

$$\tau_X(g) := \inf_{x \in X} d(x, g.x). \tag{5.14}$$

If $g \in$
$GL(V)$ is an element of the general linear group of some finite-dimensional Hilbert space V and $X = \mathcal{P}(V)$ is given by the space of positive definite symmetric endomorphisms of V (as described e.g. in [11, Ch. II.10]) this translation length can be estimated easily. Since $\tau_X(g) = \tau_X(g^{-1})$ we may assume $\det(g) \geq 1$. Then we have:

Lemma 5.2.1. *Let* $g \in \mathrm{GL}(V)$ *and assume* $\det(g) \geq 1$*. Then*

$$\tau_{\mathcal{P}(V)}(g) \geq \frac{1}{\sqrt{\dim V}} \cdot \log\det(g)^2.$$

If all eigenvalues of g *are of modulus* ≥ 1*, then*

$$\tau_{\mathcal{P}(V)}(g) \leq 2 \cdot \log\det(g)^2.$$

Proof. Let $p \in \mathcal{P}(V)$ and $c : [0, d(p, gp)] \to \mathcal{P}(V)$ a unit speed geodesic joining p with gp. We deduce from the description in [11, Ch. II.10] that there exists $h \in \mathrm{GL}(V)$ such that $p = hh^\top$ and a symmetric endomorphism X of V of norm 1 such that $c(t) = h\exp(tX)h^\top$. Moreover, $gp = ghh^\top g^\top$. Since $c(d(p, gp)) = gp$ we have

$$\begin{aligned}
& h\exp(d(p,gp)\cdot X)h^\top = ghh^\top g^\top \\
\Rightarrow \quad & \det(h\exp(d(p,gp)\cdot X)h^\top) = \det(ghh^\top g^\top) \\
\Rightarrow \quad & \exp(d(p,gp)\cdot \mathrm{tr}(X)) = \det(g)^2 \\
\Rightarrow \quad & \exp(d(p,gp)\cdot \mathrm{tr}(X)) = \exp(\log\det(g)^2)
\end{aligned}$$

Since both $d(p, gp) \cdot \mathrm{tr}(X)$ and $\log\det(g)^2$ are real this implies

$$d(p, gp) \cdot \mathrm{tr}(X) = \log\det(g)^2.$$

Since $\det(g) \geq 1$ this means

$$d(p, gp) \cdot |\operatorname{tr}(X)| = \log\det(g)^2. \tag{5.15}$$

Now observe that

$$|\operatorname{tr}(X)| = |(X|\mathbf{1})| \leq \|X\| \cdot \|\mathbf{1}\| = 1 \cdot \sqrt{\dim V} = \sqrt{\dim V}.$$

Inserting into (5.15) we obtain

$$d(p, gp) \geq \frac{1}{\sqrt{\dim V}} |\log\det(g)^2|.$$

Passing to the infimum over all $p \in \mathcal{P}(V)$ we obtain the first inequality.

For the converse inequality we use the following consequence of the existence of a real Jordan canonical form: Assume that the eigenvalues of g (with multiplicity) are given by $\lambda_1, \ldots, \lambda_m$. Then there exists a sequence $h_n \in \mathrm{GL}(V)$ such that $(h_n^{-1} g h_n)(h_n^{-1} g h_n)^\top$ converges to a diagonal matrix $\hat{g}$ with entries $|\lambda_1|^2, \ldots |\lambda_m|^2$. In particular we obtain

$$\tau(g) \quad = \quad \inf_{h \in \mathrm{GL}(V)} d(hh^\top, ghh^\top g^\top) \leq d(\mathrm{id}_V, \hat{g}).$$

Then [11, Cor. 10. 42] yields

$$\begin{aligned} \tau(g) \quad &\leq \quad \left(\sum_{j=1}^m (\log|\lambda_j|^2)^2 \right)^{\frac{1}{2}} \\ &\leq \quad 2 \cdot \sum_{j=1}^m |\log|\lambda_j||. \end{aligned}$$

Now, if $|\lambda_j| > 1$ for all $j = 1, \ldots, m$, then the right hand side is precisely given by $2 \cdot \log\det(g)^2$. $\square$

We can use the lemma to compute translation lengths for isometry groups of totally geodesic subspaces of $\mathcal{P}(V)$ by means of the following general result:

Lemma 5.2.2. *Let X be a complete $CAT(0)$-manifold. Let $Y \subset X$ a totally geodesic subspace and h be an isometry of X with $hY \subset Y$. Then*

$$\tau_X(h) = \tau_Y(h).$$

Proof. By assumption, Y is closed, convex and complete with respect to the induced metric. This implies [11, II.2.4] that there exists an orthogonal projection $\pi : X \to Y$. Given $x \in X \setminus Y$ we denote by σ_x the constant speed geodesic with $\sigma_x(0) = \pi(x)$, $\sigma_x(1) = x$. By construction, σ_x is the unique geodesic which contains x and intersects Y orthogonally. This description implies in particular that

$$h\sigma_x = \sigma_{hx} \quad (x \in X \setminus Y). \tag{5.16}$$

For any $y \in Y$ denote by τ_y the geodesic joining y and $h.y$. By assumption, τ_y is contained in Y for every $y \in Y$. In particular, given $x \in X \setminus Y$, the geodesic $\tau_{\pi(x)}$ is orthogonal to both σ_x and $h.\sigma_x$, whence the shortest connetion between these two geodesics. We deduce that

$$d(\sigma_x, h.\sigma_x) = d(\sigma_x \cap \tau_{\pi(x)}, h.\sigma_x \cap \tau_{\pi(x)}) = d(\sigma_x(0), h.\sigma_x(0)).$$

Combining this with (5.16) we obtain for all $x \in X \setminus Y$ the inequality

$$\begin{aligned} d(x, hx) = d(\sigma_x(1), \sigma_{hx}(1)) \geq \\ \geq d(\sigma_x, h.\sigma_x) = d(\sigma_x(0), h.\sigma_x(0)) = d(p(x), h.p(x)). \end{aligned}$$

Then the lemma follows by passing to the infimum. $\square$

We will apply the lemma in the following form:

Corollary 5.2.3. *Assume $G \subset \mathrm{GL}(V)$ is a reductive subgroup. Then $K := G \cap O(V)$ is a maximal compact subgroups and*

$$\iota : G/K \mapsto \mathcal{P}(V), \quad gK \mapsto g^\top g$$

is an embedding. If for all symmetric matrices $X \in \mathfrak{gl}(V)$ with $\exp(X) \in G$ already $\exp(tX) \in G$ for all $t \in \mathbb{R}$, then $\iota(G/K)$ is totally geodesic in $\mathcal{P}(V)$ and for $g \in G$,

$$\tau_{\iota(G/K)}(g) = \tau_{\mathcal{P}(V)}(g).$$

Proof. For the first statement see [11, Thm. II.10.58]. Then the second statement follows from Lemma 5.2.2. □

5.2.2 Special isometries and their periods

Throughout this section $\mathcal{D}$ is a bounded symmetric domain realized by means of a Euclidean Jordan algebra V and $G = G_V$. If $\mathcal{D} = \mathbb{D}$ is the Poincaré disc, then a period can be defined for all hyperbolic isometries g of $\mathbb{D}$; equivalently, g has two fixed points in S^1. In higher rank these two assumptions differ. Let us call an isometry $g \in G$ *special* if it admits a pair of transverse fixed points $g^\pm \in \check{S}$. This asumption is sufficient to define a period; no hyperbolicity assumptions are required. For the rest of this section let g be a special isometry and label the two fixed points in such a way that either g^- is non-attractive or g^+ is non-repellent. We then call $z \in \check{S}$ *admissible* if $(g^-, z, g^+, gz) \in \check{S}^{(4)}$. Given any admissible point z we define

$$\tau^\infty_{\mathcal{D}}(g, g^+, g^-)_z := \log B_{\check{S}}(g^-, z, g^+, gz).$$

Lemma 5.2.4. *The expression $\tau^\infty_{\mathcal{D}}(g, g^+, g^-)_z$ does not depend on the choice of admissible point z.*

Proof. Let $F(y) := B_{\check{S}}(g^-, y, g^+, gy)$. We claim that F is constant on the set X of admissible points. If $y, z \in X$ with $(g^-, y, g^+, z) \in$

$\check{S}^{(4)}$, then

$$\begin{aligned} F(z) = B_{\check{S}}(g^-, z, g^+, gz) &= B_{\check{S}}(g^-, z, g^+, y) \cdot B_{\check{S}}(g^-, y, g^+, gz) \\ &= B_{\check{S}}(gg^-, gz, gg^+, gy) \cdot B_{\check{S}}(g^-, y, g^+, gz) \\ &= B_{\check{S}}(g^-, y, g^+, gz) \cdot B_{\check{S}}(g^-, gz, g^+, gy) \\ &= B_{\check{S}}(g^-, y, g^+, gy) = F(y); \end{aligned}$$

otherwise we can find $w \in X$ with $(g^-, y, g^+, w), (g^-, z, g^+, w) \in \check{S}^{(4)}$ which then yields $F(y) = F(w) = F(z)$. □

In view of the lemma we may define

$$\tau_{\mathcal{D}}^{\infty}(g, g^+, g^-) := \tau_{\mathcal{D}}^{\infty}(g, g^+, g^-)_z$$

and refer to it as the *period of g with respect to the pair* (g^+, g^-). If g^+ and g^- are clear from the context, we will simply write $\tau_{\mathcal{D}}^{\infty}(g)$.

Remark 5.2.5. Strictly speaking, our axiomatically defined generalized cross ratio has domain $\check{S}^{(4+)}$, so that the period can only be defined if $(g^-, z, g^+, gz) \in \check{S}^{(4+)}$. However, we have constructed an explicit model of the cross ratio on all of $\check{S}^{(4)}$, which on the subset $\check{S}^{(4+)}$ agrees with the axiomatic one. The last lemma then implies that the period as defined above only depends on the axiomatically defined cross ratio, but in order to compute it we can use our concrete model as defined on all of $\check{S}^{(4)}$.

We now choose some auxiliary data which will allow us to express $\tau_{\mathcal{D}}^{\infty}(g)$ in more explicit terms. Since $g^{\pm}$ are transverse we can choose some $h \in G$ with $hg^{\pm} = \pm e$. We then define

$$g_1 := hgh^{-1} \in L(Q^+). \tag{5.17}$$

By Proposition 2.1.41 we then have

$$g_2 := \hat{c}(g_1) = c \circ g_1 \circ c^{-1} \in G(\Omega) \subset \mathrm{GL}(V). \tag{5.18}$$

Note that our enumeration of fixed points implies $\det(g_2) \geq 1$. Now we claim:

Proposition 5.2.6. *Assume that $\mathcal{D}$ is irreducible. Then the period of g with respect to (g^+, g^-) is given in terms of the above auxiliary data by*

$$\tau_{\mathcal{D}}^{\infty}(g) \quad = \quad \frac{1}{2 \cdot \dim V} \cdot \log \det(g_2)^2.$$

Proof. It follows from the G-invariance of the generalized cross ratio that

$$\tau_{\mathcal{D}}^{\infty}(g) = \log B_{\check{S}}(-e, z, e, g_1 z)$$

for all z in some dense subset of $\check{S}$. Since $\mathcal{D}$ is irreducible, we have $B_{\mathcal{D}} = B_{\mathcal{D}}^{(-\frac{1}{2\dim V})}$. We then use Proposition 5.1.11 to deduce that

$$B_{\mathcal{D}}(x, y, z, t) = B_{T_\Omega}^{(-\frac{1}{2\dim V})}(c(x), c(y), c(z), c(t)) \quad ((x, y, z, t) \in \mathcal{D}^4),$$

where T_Ω is the tube over Ω and $B_{T_\Omega}^{(-\frac{1}{2\dim V})}$ is the associated weighted Bergman kernel. By continuity we deduce that if x_n is any sequence in $\mathcal{D}$ converging to e then for all w in a dense open subset of V we have

$$\tau_{\mathcal{D}}^{\infty}(g) \quad = \quad \frac{1}{2 \cdot \dim V} \cdot \log \left(\frac{k_{T_\Omega}(w, 0)}{k_{T_\Omega}(g_2 w, 0)} \cdot \lim_{n \to \infty} \frac{k_{T_\Omega}(g_2 w, c(x_n))}{k_{T_\Omega}(w, c(x_n))} \right) \tag{5.19}$$

Now we claim that

$$\lim_{n \to \infty} \frac{k_{T_\Omega}(g_2 w, c(x_n))}{k_{T_\Omega}(w, c(x_n))} = 1. \tag{5.20}$$

Indeed, let $\lambda \in [0, 1)$. Then

$$c(\lambda \cdot e) = i \frac{1 + \lambda}{1 - \lambda} e.$$

Using [29, X.1.3] we obtain

$$\lim_{n \to \infty} \frac{k_{T_\Omega}(g_2 w, c(x_n))}{k_{T_\Omega}(w, c(x_n))} \quad = \quad \lim_{\lambda \to 1} \left(\frac{\det(g_2 w - i \frac{1+\lambda}{1-\lambda} e)}{\det(w - i \frac{1+\lambda}{1-\lambda} e)} \right)^{-\frac{2n}{r}}$$

$$= \lim_{\lambda\to 1}\left(\frac{\det(\frac{1-\lambda}{1+\lambda}g_2 w - ie)}{\det(\frac{1-\lambda}{1+\lambda}w - ie)}\right)^{-\frac{2n}{r}} = 1.$$

In view of (5.19) and (5.20) it remains to show that

$$\frac{k_{T_\Omega}(w,0)}{k_{T_\Omega}(g_2 w,0)} = \det(g_2)^2. \tag{5.21}$$

Since $g_2 : T_\Omega \to T_\Omega$ is biholomorphic, we see from [29, Prop. IX.2.4] that

$$k_{T_\Omega}(w,0) = k_{T_\Omega}(g_2 w, g_2 0)\det{}_{\mathbb{C}} J_{g_2}(w)\overline{\det{}_{\mathbb{C}} J_{g_2}(0)},$$

where J_{g_2} denotes the complex Jacobi matrix of g_2. Note that g_2 is a real matrix, because it is in $G(\Omega)^0 \subset \mathrm{GL}(V)$. Since it is linear, we have $J_{g_2} \equiv g_2$ and $g_2 0 = 0$, whence

$$\begin{aligned} k_{T_\Omega}(w,0) =& k_{T_\Omega}(g_2 w, g_2 0)\det{}_{\mathbb{C}} J_{g_2}(w)\overline{\det{}_{\mathbb{C}} J_{g_2}(0)} \\ =& k_{T_\Omega}(g_2 w, 0)\det(g_2)^2. \end{aligned}$$

This establishes (5.21) and finishes the proof. □

5.2.3 Translation lengths of special isometries

We now want to estimate the translation lengths $\tau_{\mathcal{D}}(g)$ of a special isometry g of a bounded symmetric domain $\mathcal{D}$ of tube type with respect to the (unnormalized) Bergman metric on $\mathcal{D}$. All quantitative statements in this section depend on this choice of metric. We keep the notation introduced in the last section. In particular, g^+, g^-, g_1, g_2 are defined as before. Since $\det(g_2) \geq 1$, we deduce that g_2 has at least one eigenvalue of modulus ≥ 1, and we will get the strongest estimates if actually all eigenvalues of g_2 are ≥ 1. In our applications we will always be in a situation, where all eigenvalues are in fact strictly greater than 1; we will then call (g^+, g^-) an *attractor-repellor pair* for g. Nevertheless, we find it worthwhile to point out that some of our estimates work without this hyperbolicity assumption. In the irreducible case we are going to prove the following:

Theorem 5.2.7. *Let $\mathcal{D}$ be an irreducible bounded symmetric domain of tube type and $g \in G = G(\mathcal{D})$ with two transverse fixed points $g^{\pm}$ labelled as above. Then*

$$\tau_{\mathcal{D}}(g) \geq \sqrt{\dim_{\mathbb{C}} \mathcal{D}} \cdot \tau_{\mathcal{D}}^{\infty}(g, g^+, g^-),$$

and if all eigenvalues of the auxiliary matrix g_2 have modulus ≥ 1, then

$$\tau_{\mathcal{D}}(g) \leq 2 \dim_{\mathbb{C}} \mathcal{D} \cdot \tau_{\mathcal{D}}^{\infty}(g, g^+, g^-).$$

The restriction to irreducible bounded symmetric domains is easy to remove:

Corollary 5.2.8. *Let $\mathcal{D}$ be a bounded symmetric domain of tube type which decomposes as $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_m$ into irreducible bounded symmetric domains. Assume that $g \in G(\mathcal{D})^0$ admits two transverse fixed points $g^{\pm} \in \check{S}$ labeled as above. Then*

$$\tau_{\mathcal{D}}(g) \geq \sqrt{\min_j \dim_{\mathbb{C}} \mathcal{D}_j} \cdot \tau_{\mathcal{D}}^{\infty}(g, g^+, g^-),$$

and if all eigenvalues of the auxiliary matrix g_2 have modulus ≥ 1, then

$$\tau_{\mathcal{D}}(g) \leq 2 \cdot \operatorname{rk} \mathcal{D} \cdot \max_j \frac{\dim_{\mathbb{C}} \mathcal{D}_j}{\operatorname{rk} \mathcal{D}_j} \cdot \tau^{\infty}(g).$$

This is notably the case if (g^+, g^-) is an attractor-repellor pair for g.

Proof. Identifying $G(\mathcal{D})^0$ with the product of the group $G(\mathcal{D}_j)^0$ we can then write $g = (g_1, \ldots, g_m)$ for some $g_j \in G(\mathcal{D}_j)^0$. Let us abbreviate $r_j := \operatorname{rk} \mathcal{D}_j$, $n_j := \dim_{\mathbb{C}} \mathcal{D}_j$, $r := \operatorname{rk} \mathcal{D}$, $\tau_j := \tau_{\mathcal{D}_j}(g_j)$ and

$\tau_j^\infty := \tau_{\mathcal{D}_j}^\infty(g_j)$ so that

$$\tau_{\mathcal{D}}(\varrho(\gamma)) = \sqrt{\sum_{j=1}^m \tau_j^2}, \quad \tau_{\mathcal{D}}^\infty(\varrho(\gamma)) = \sum_{j=1}^m \frac{r_j}{r}\tau_j^\infty.$$

By Theorem 5.2.7 we thus obtain

$$\begin{aligned}
\tau_{\mathcal{D}}^\infty(g) &= \sum_{j=1}^m \frac{r_j}{r}\tau_j^\infty \leq \left(\sum_{j=1}^m \frac{r_j}{r}\right) \cdot \max_j \tau_j^\infty \\
&\leq \sqrt{\sum_{j=1}^m (\tau_j^\infty)^2} \leq \max_j \frac{1}{\sqrt{n_j}} \cdot \sqrt{\sum_{j=1}^m n_j(\tau_j^\infty)^2} \\
&\leq \frac{1}{\sqrt{\min_j n_j}}\sqrt{\sum_{j=1}^m \tau_j^2} = \frac{1}{\sqrt{\min_j n_j}}\tau_{\mathcal{D}}(g).
\end{aligned}$$

For the other inequality Theorem 5.2.7 yields

$$\begin{aligned}
\tau_{\mathcal{D}}(g) &= \sqrt{\sum_{j=1}^m \tau_j^2} \leq \sum_{j=1}^m \tau_j \leq 2 \cdot \sum_{j=1}^m \dim_{\mathbb{C}} \mathcal{D}_j \cdot \tau_j^\infty \\
&= 2 \cdot \sum_{j=1}^m \dim_{\mathbb{C}} \mathcal{D}_j \cdot r \cdot \frac{n_j}{r_j} \cdot \frac{r_j}{r} \cdot \tau_j^\infty \\
&\leq 2 \cdot r \cdot \max_j \frac{n_j}{r_j} \cdot \tau^\infty.
\end{aligned}$$

□

Theorem 5.2.7. Since translation length is invariant under both conjugation and isometries we have

$$\tau_{\mathcal{D}}(g) = \tau_{\mathcal{D}}(g_1) = \tau_{T_\Omega}(g_2), \tag{5.22}$$

where both $\mathcal{D}$ and T_Ω are equipped with the respective Bergman metrics. Since the inclusion $i\Omega \subset T_\Omega$ is totally geodesic (see [54, p. 361]), Lemma 5.2.2 yields

$$\tau_{T_\Omega}(g_2) = \tau_{i\Omega}(g_2) \tag{5.23}$$

for the restriction of the Bergman metric to $i\Omega$. Now $G(\Omega)$ acts transitively on $i\Omega$ with stabilizer of ie_V given by $K := G(\Omega) \cap O(V)$ [29, Prop. I.4.3]. We deduce that the inclusion $G(\Omega) \subset \mathrm{GL}(V)$ induce an embedding $\iota : i\Omega \cong G(\Omega)/K \to \mathcal{P}(V)$. We want to apply Corollary 5.2.3 to this inclusion, thus let us check the assumptions of that corollary. Denote by $\mathfrak{p}(\Omega)$ the symmetric matrices in the Lie algebra $\mathfrak{g}(\Omega) \subset \mathfrak{gl}(V)$ of $G(\Omega)$. Then $G(\Omega)$ admits a polar decomposition $G(\Omega) = K(\Omega)\exp(\mathfrak{p}(\Omega))$ [29, Prop. I.1.9, I.4.3 and Thm. III.5.1]. In particular, if X is a symmetric matrix with $\exp(X) \in G$, then there exist $k \in K(\Omega)$ and $Y \in \mathfrak{p}(\Omega)$ such that

$$e^X = ke^Y \Rightarrow e^{2X} = (e^X)^\top e^X = (ke^Y)^\top ke^Y = e^{2Y}.$$

Then the uniqueness of the Polar decomposition in $\mathrm{GL}(V)$ yields $2X = 2Y$, whence $X \in \mathfrak{p}(\Omega)$ and $\exp(tX) \in G$ for all $t \in \mathbb{R}$. Thus Corollary 5.2.3 applies and yields

$$\tau_{\iota(i\Omega)}(g_2) = \tau_{\mathcal{P}(V)}(g_2). \tag{5.24}$$

Now on $i\Omega$ we have two metrics, one induced from the restriction of the Bergman metric H of T_Ω, and one given by pullback of the metric on $\mathcal{P}(V)$ via ι. We claim that the latter metric is twice the former. Denote by $\mathbf{1} \in \mathcal{P}(V)$ the identity matrix. Under the canonical identifications $T_e\Omega \cong V$ and $T_{\mathbf{1}}\mathcal{P}(V) = Sym_{\dim V}(\mathbb{R})$ the differential of the embedding ι at e is given by [29, Thm. III.3.1]

$$d\iota_e : V \to Sym_{\dim V}(\mathbb{R}), \quad x \mapsto L(x).$$

The Bergman metric in x is given by the formula [29, X.1.3 and Ch. III.4]

$$H_x(u,v) := \frac{r}{2n}\,\mathrm{tr}_V((P(x)^{-1}u)v),$$

where $r := \mathrm{rk}\, V$, $n := \dim V$ and tr_V is again the Jordan algebra trace. On the other hand, the metric of $\mathcal{P}(V)$ is given by [11, Ch. II.10]

$$g_x(X, Y) = \mathrm{tr}(x^{-1}Xx^{-1}Y),$$

where tr is the usual matrix trace. Since $\mathrm{tr}_V(x) = \frac{r}{n} \cdot \mathrm{tr}(L(x))$ [29, III.4.2] we have

$$H_e(u, v) = \frac{1}{2} g_I(L(u), L(v)).$$

Both the Bergman metric and the restriction of the metric on $\mathcal{P}(V)$ to the image of Ω are invariant under $G(\Omega)$; we thus deduce that the Riemannian metrics on Ω and $\iota(\Omega)$ coincide up to a global factor of $\frac{1}{2}$. This proves the claim and shows that

$$\tau_{i\Omega}(g_2) = \frac{1}{2} \cdot \tau_{\iota(i\Omega)}(g_2). \tag{5.25}$$

Combining (5.22), (5.23), (5.24) and (5.25) we obtain

$$\tau_{\mathcal{D}}(g) = \frac{1}{2} \cdot \tau_{\mathcal{P}(V)}(g_2). \tag{5.26}$$

In view of Proposition 5.2.1 and Proposition 5.2.6 we now deduce

$$\tau_{\mathcal{D}}(g) \geq \frac{1}{2 \cdot \sqrt{\dim V}} \log \det(g_2)^2 = \sqrt{\dim_{\mathbb{C}} \mathcal{D}} \cdot \tau_{\mathcal{D}}^{\infty}(g).$$

and, if all eigenvalues of g_2 are ≥ 1,

$$\tau_{\mathcal{D}}(g) \leq \log \det(g_2)^2 = 2 \cdot \dim_{\mathbb{C}} \mathcal{D} \cdot \tau_{\mathcal{D}}^{\infty}(g).$$

□

Remark 5.2.9. Analyzing the proof we see that the key estimate is provided by applying Lemma 5.2.1 to a faithful linear representation for the Levi factor of a Shilov parabolic. We have used the representation constructed in Corollary 2.1.42, which has the advantage that it can be defined in a uniform way for all classical and exceptional bounded symmetric domains of tube type. It is easy to see that for e.g. the symplectic case there exist faithful linear representations of much smaller dimension. One can use these representations to obtain better constants in the above estimate; in particular, the above constants are not sharp. This possible improvement requires, however, a careful case by case analyis, and since the above estimates are sufficient for our purposes, we will not carry out the necessary case by case considerations here.

5.3 Maximal representations, strict cross ratios and well-displacing

5.3.1 Maximal representations, limit curves and strict cross ratios

The aim of this section is the construction of a strict cross ratio for maximal representation of Γ_g

Now let $\Gamma := \Gamma_g$ with $g \geq 2$, $\varrho : \Gamma \to G$ be a maximal representation and $\varphi : S^1 \to \check{S}$ the associated monotone continuous limit curve. As a consequence of monotonicity two distinct points $x \neq y \in S^1$ are mapped to transverse points under φ; thus if

$$(S^1)^{4*} := \{(x, y, z, t) \in (S^1)^4 \,|\, x \neq t, y \neq z\}$$

denotes the domain of the classical cross ratio, then we obtain a map

$$\varphi^{(4)} : (S^1)^{4*} \to \check{S}^{(2)} \times \check{S}^{(2)}, \quad (x, y, z, t) \mapsto (\varphi(x), \varphi(y), \varphi(z), \varphi(t)).$$

Moreover, the dense subset $(S^1)^{(4)} \subset (S^1)^{4*}$ satisfies

$$\varphi^{(4)}((S^1)^{(4)}) \subset \check{S}^{(4+)}, \tag{5.27}$$

where the right hand side is precisely the domain of definition of $B_{\check{S}}$. We may thus define a function

$$b_\varrho := (\varphi^{(4)})^* B_{\check{S}} : \begin{cases} (S^1)^{(4)} \to \mathbb{R} \setminus \{0,1\} \\ (x,y,z,t) \mapsto B_{\check{S}}(\varphi(x), \varphi(y), \varphi(z), \varphi(t)). \end{cases}$$

Since $B_{\check{S}}$ extends continuously to $\check{S}^{(2)} \times S^{(2)}$, we may also extend b_ϱ to a continuous function

$$b_\varrho : (S^1)^{4*} \to \mathbb{R}.$$

Definition 5.3.1. Let $\varrho : \Gamma \to G$ be a maximal representation and $\varphi : S^1 \to \check{S}$ an associated limit curve. Then the function $b_\varrho : (S^1)^{4*} \to \mathbb{R}$ defined above is called the *cross ratio of the maximal representation* ϱ.

Because of our functorial construction of generalized cross ratios the following functoriality of the b_ϱ comes for free:

Proposition 5.3.2. *Let G, H be Hermitian Lie groups of tube type, $\varrho : \Gamma \to H$ a maximal representation and $t : H \to G$ a homomorphism inducing a tight holomorphic morphism of the underlying bounded symmetric domains. Then $t \circ \varrho$ is maximal and $b_\varrho = b_{t\circ\varrho}$.*

Proof. The homomorphism t induces a map $t_* : \check{S}_H \to \check{S}_G$ of the corresponding Shilov boundaries [16]. Now if φ is a limit curve for ϱ, then $t_* \circ \varphi$ is a limit curve for $t \circ \varrho$. Thus the proposition follows from Property (ii) of Theorem 1.5.1. □

The main properties of cross ratios of maximal representations are collected in the following theorem:

Theorem 5.3.3. *The cross ratio $b_\varrho : (S^1)^{4*} \to \mathbb{R}$ is a continuous Γ-invariant function satisfying the following properties:*

$$b_\varrho(x,y,z,t) \quad = \quad b_\varrho(z,t,x,y) \tag{5.28}$$

$$\begin{aligned}
b_\varrho(x,y,z,t) &= b_\varrho(x,y,z,w)b_\varrho(x,w,z,t) && (5.29)\\
b_\varrho(x,y,z,t) &= b_\varrho(x,y,w,t)b_\varrho(w,y,z,t) && (5.30)\\
x=z \ or \ y=t &\Leftrightarrow b_\varrho(x,y,z,t)=1 && (5.31)\\
t=x \ or \ z=y &\Leftrightarrow b_\varrho(x,y,z,t)=0 && (5.32)
\end{aligned}$$

Proof. Γ-invariance on $(S^1)^{(4)}$ follows from Γ-equivariance of φ and G-invariance of $B_{\check{S}}$ on $S^{(4+)}$ (Proposition 5.1.10). By continuity we obtain Γ-invariance on all of $(S^1)^{4*}$. By a similar extension argument, Properties (5.28)-(5.30) follow from Corollary 5.1.19. For $(x,y,z,t) \in (S^1)^{(4)}$ the inclusion (5.27) together with Proposition 5.1.24 implies $b_\varrho(x,y,z,t) \notin \{0,1\}$. It thus remains to consider the cases $x=z$, $y=t$, $t=x$ and $z=y$. In the last two cases the vanishing of $\varphi^* k_V$ along the diagonal implies $b_\varrho(x,y,z,t)=0$. In the first two cases we get $b_\varrho(x,y,z,t)=1$ as a consequence of the similar property for $B_{\check{S}}$. This establishes (5.31)-(5.32) and finishes the proof. □

In fact, it follows from Corollary 5.1.25 that b_ϱ also satisfies

$$b_\varrho(x,y,z,t) = b_\varrho(y,x,t,z).$$

However, we are not going to use this property in the sequel. In the language of [46, Def. 3.2] the theorem says precisely that b_ϱ is a *strict weak cross ratio.* Concerning such cross ratios we have the following equivalence theorem of Labourie:

Theorem 5.3.4 (Labourie). *Let b_1 and b_2 be strict weak cross ratios. Then there exist $C, D > 0$ such that for all quadruples (x,y,z,t) with (x,y,z) and (x,t,z) positively oriented*

$$D\log b_1(x,y,z,t) - D \le \log b_2(x,y,z,t) \le C\log b_1(x,y,z,t) + C$$

The proof is based on the following lemma, which is contained in [47, Prop. 3.3.7][1].

[1]The proposition is originally stated for cross ratios in the sense of [47], i.e.

Lemma 5.3.5. *For every strict weak cross ratio b there exists a Γ-equivariant continuous map (with respect to the trivial action on $\mathbb{R}$)*

$$\psi : \mathbb{R} \times (S^1)^{3+} \to S^1$$

such that $\log b(x, y, z, \psi_s(x, y, z)) = s$. *This function satisfies*

$$\psi_{s+t}(x, y, z) = \psi_t(x, \psi_s(x, y, z), z). \tag{5.33}$$

Proof of Theorem 5.3.4. Let ψ_s^1 and ψ_s^2 be maps associated to b_1 and b_2 by means of Lemma 5.3.5. Define a function $T : (S^1)^{3+} \to \mathbb{R}$ by

$$T(x, y, z) := \log b_1(x, y, z, \psi_1^2(x, y, z))$$

This map is positive and continuous. Since ψ_s^2 is Γ-equivariant, T is Γ-invariant. Furthermore it satisfies

$$\psi_1^2(x, y, z) = \psi^1_{T(x,y,z)}(x, y, z). \tag{5.34}$$

Since $(S^1)^{3+}/\Gamma$ is compact, $|T|$ has a global maximum A. Now consider the function $f : \mathbb{R} \to \mathbb{R}$ (depending on x, y, z) given by

$$f(s) := \log b_1(x, y, z, \psi_s^2(x, y, z))$$

For $n \in \mathbb{Z}$ we have

$$\begin{aligned} |f(n)| &= \left|\log b_1(x, y, z, \psi_n^2(x, y, z))\right| \\ &= \left|\sum_{i=0}^{n-1} \log b_1(x, \psi_i(x, y, z), z, \psi_{i+1}(x, y, z))\right| \\ &= \left|\sum_{i=0}^{n-1} \log b_1(x, \psi_i(x, y, z), z, \psi_1(x, \psi_i(x, y, z), z))\right| \\ &\leq A \cdot |n| \end{aligned}$$

strict weak cross ratios which are additionally Hölder continuous. However, Hölder continuity is actually never used in the proof.

Because of monotonicity of f we get for $0 \le s \in [n, n+1)$:

$$f(s) \le f(n+1) \le An + A \le As + A$$

and for $0 \ge s \in [n, n+1)$:

$$|f(s)| \le |f(n)| \le A|n| \le A(|s|+1) = A|s| + A.$$

We can summarize these inequalities by saying that for all $s \in \mathbb{R}$

$$|\log b_1(x,y,z,\psi_s^2(x,y,z))| = |f(s)| \le A \cdot |s| + A.$$

Now given any $t \in S^1$ we can choose $s \in \mathbb{R}$ such that $\psi_s^2(x,y,z) = t$. By Lemma 5.3.5 we have $s = \log b_2(x,y,z,t)$, hence

$$\begin{aligned} |\log b_1(x,y,z,t)| &\le A \cdot |s| + A \\ &= A \cdot |\log b_2(x,y,z,t)| + A. \end{aligned}$$

If (x,y,z) and (x,t,z) are positively oriented then $\log b_j(x,y,z,t) > 0$, $j = 1,2$. This proves the upper bound, and the lower bound is obtained by reversing the roles of b_1 and b_2. $\square$

Remark 5.3.6. Note that the compactness of Σ is crucial for the proof of Theorem 5.3.4.

Corollary 5.3.7. *Let $\varrho : \Gamma \to G$ be a maximal representation with associated cross ratio b_ϱ. Then there exists $D > 0$ such that for all $(x,y,z,t) \in (S^1)^{4*}$ with (x,y,z) and (x,t,z) positively oriented,*

$$\log b_\varrho(x,y,z,t) \ge D \cdot \log[x:y:z:t] - D.$$

5.3.2 Explicit formulas in the symplectic case

For maximal representations into symplectic groups there is a more classical construction of an associated weak cross ratio described in [47, Ch. 3.2.5]. This construction is based on the identification of the Shilov boundary associated with $\mathrm{Sp}(2n)$ with the Lagrangian

Grassmannian $\mathcal{L}(V)$ of $V = (\mathbb{R}^n \times \mathbb{R}^n, \omega)$, where $\omega(x,y) = x^\top J y$ with

$$J = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}.$$

The classical cross ratio $B_{class} : \mathcal{L}(V)^{(4*)} \to \mathbb{C}$ is given as follows: Let $L^{(j)} = \langle l_1^{(j)}, \ldots, l_n^{(j)} \rangle$, $j = 1, \ldots, 4$, be pairwise transverse Lagrangians and define

$$A^{ij}_{ab} := \omega(l_a^{(i)}, l_b^{(j)}).$$

Then

$$B_{class}(L^{(1)}, L^{(2)}, L^{(3)}, L^{(4)}) := \frac{\det(A^{12})\det(A^{34})}{\det(A^{14})\det(A^{32})}.$$

Given a maximal representation $\varrho : \Gamma \to \mathrm{Sp}(2n)$ with associated limit curve $\varphi : S^1 \to \mathcal{L}(V)$ one then defines a function

$$b_{\varrho,class}(x,y,z,t) : \begin{cases} (S^1)^{4*} \to \mathbb{R} \\ (x,y,z,t) \mapsto B_{class}(\varphi(x), \varphi(y), \varphi(z), \varphi(t)) \end{cases}$$

which turns out to be a weak cross ratio. We claim that our cross ratio b_ϱ is related to the classical cross ratio by the formula

$$b_{\varrho,class} = b_\varrho^n. \tag{5.35}$$

It n is even, then this formula implies in particular that, contrary to the claims in [47] the weak cross ratio $b_{\varrho,class}$ is not strict. To establish (5.35) it suffices to show that $B^n_{\check{S}}$ and B_{class} coincide on $\mathcal{L}(V)^{(4+)}$; by conjugation-invariance and Proposition 2.1.37 it then suffices to show that they coincide on the Shilov boundary $\check{S}_P$ of a given maximal polydisc P in $\mathcal{L}(V)$. Let $G := \mathrm{Sp}(2n)$ and define an embedding $\iota : H := \mathrm{SL}(2,\mathbb{R})^n \to G$ by

$$g = (g_1, \dots, g_n) \mapsto \begin{pmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_n & & & d_n \end{pmatrix};$$

then every H-orbits in $\mathcal{L}(V)$ is the Shilov boundary of a maximal polydisc. We choose the basepoint $L_0 = \langle (e_1, e_1), (e_2, e_2), \dots, (e_n, e_n)\rangle \in \mathcal{L}(V)$, where e_k denotes the kth standard basis vector of $\mathbb{R}^n$, and set $\check{S}_P := \iota(H).L_0$. We now provide an H-equivariant identification $\nu : (S^1)^n \to \check{S}_P$: The action of $g \in H$ on the former is given by $g.\lambda = (S^{-1}gS) \cdot \lambda$, where

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

and the $\cdot$-action is given by Möbius transformations. In particular,

$$\begin{aligned} g.(1, \dots, 1) =& ((S^{-1}g_1S) \cdot 1, \dots, (S^{-1}g_nS) \cdot 1) \\ =& (S^{-1} \cdot (g_1 \cdot 1), \dots, S^{-1} \cdot (g_n \cdot 1)). \end{aligned}$$

On the other hand, the action on $\check{S}_P$ is given by

$$\begin{aligned} g \cdot L_0 =& \langle \big((a_1 + b_1)e_1, (c_1 + d_1)e_1\big), \dots, \big((a_n + b_n)e_n, \big((c_n + d_n)e_n\big)\rangle \\ =& \langle \big((a_1 + b_1)(c_1 + d_1)^{-1}e_1, e_1\big), \dots, \big((a_n + b_n)(c_n + d_n)^{-1}e_n, \big(e_n\big)\rangle \\ =& \langle \big((g_1 \cdot 1)e_1, e_1\big), \dots, \big((g_n \cdot 1)e_n, e_n\big)\rangle, \end{aligned}$$

The points $(1, \dots, 1)$ and L_0 have the same stabilizer in H. Thus the desired identification is given by

$$\nu(\lambda_1, \dots, \lambda_n) = (((S \cdot \lambda_1)e_1, e_1), \dots, ((S \cdot \lambda_n)e_n, e_n)).$$

By functoriality the pullback $\nu^* B_{\check{S}}$ to $(S^1)^n$ satisfies

$$\begin{aligned}&\left(\nu^* B_{\check{S}}((\lambda_1^{(1)}, \ldots, \lambda_n^{(1)}), \ldots, (\lambda_1^{(4)}, \ldots, \lambda_n^{(4)}))\right)^n \\ =&\prod_{j=1}^{n} [\lambda_j^{(1)} : \lambda_j^{(2)} : \lambda_j^{(3)} : \lambda_j^{(4)}].\end{aligned}$$

We now compare this to the classical cross ratio: If $L^{(j)} \in \mathcal{L}(V)$ is of the form $L^{(j)} = \langle(\alpha_1^{(j)} e_1, e_1), \ldots, (\alpha_n^{(j)} e_n, e_n)\rangle$, then a direct calculation shows that

$$A^{ij}_{ab} := (\alpha_a^{(j)} - \alpha_b^{(i)}) \cdot \delta_{ab} \;\Rightarrow\; \det(A^{ij}) = \prod_{k=1}^{n} (\alpha_k^{(j)} - \alpha_k^{(i)}),$$

and thus

$$\begin{aligned} B_{class}(L^{(1)}, L^{(2)}, L^{(3)}, L^{(4)}) &= \frac{\prod_{k=1}^n (\alpha_k^{(2)} - \alpha_k^{(1)}) \prod_{k=1}^n (\alpha_k^{(4)} - \alpha_k^{(3)})}{\prod_{k=1}^n (\alpha_k^{(4)} - \alpha_k^{(1)}) \prod_{k=1}^n (\alpha_k^{(2)} - \alpha_k^{(3)})} \\ &= \prod_{k=1}^{n} [\alpha_k^{(1)} : \alpha_k^{(2)} : \alpha_k^{(3)} : \alpha_k^{(4)}]. \end{aligned}$$

In particular we finally obtain

$$\begin{aligned} & \nu^* B_{class}((\lambda_1^{(1)}, \ldots, \lambda_n^{(1)}), \ldots, (\lambda_1^{(4)}, \ldots, \lambda_n^{(4)})) \\ = & \prod_{j=1}^{n} [S \cdot \lambda_j^{(1)} : S \cdot \lambda_j^{(2)} : S \cdot \lambda_j^{(3)} : S \cdot \lambda_j^{(4)}] \\ = & \prod_{j=1}^{n} [\lambda_j^{(1)} : \lambda_j^{(2)} : \lambda_j^{(3)} : \lambda_j^{(4)}], \end{aligned}$$

which establishes $(\nu^* B_{\check{S}})^n = \nu^* B_{class}$ and thus $B_{\check{S}}^n = B_{class}$ on $\mathcal{L}(V)^{(4+)}$. In particular, on this domain we obtain

$$|B_{\check{S}}(L^{(1)}, L^{(2)}, L^{(3)}, L^{(4)})| = \left| \frac{\det(A^{12}) \det(A^{34})}{\det(A^{14}) \det(A^{32})} \right|^{\frac{1}{n}}.$$

Since the sign of $B_{\check{S}}$ can always be determined combinatorially, this provides an explicit formula for the functorial cross ratio in the symplectic case.

5.3.3 Translation lengths under maximal representations

We now apply cross ratios of maximal representations for estimates of the corresponding translation lengths. For this we fix a maximal representation $\varrho : \Gamma \to G$ and denote by φ the associated continuous monotone limit curve. Since every $\gamma \in \Gamma \setminus \{e\}$ is hyperbolic when considered as an element of $PU(1,1)$, it has a unique attractive fixed point γ^+ and a unique repellent fixed point γ^-. We may thus define

$$g^{\pm} := \varphi(\gamma^{\pm}). \tag{5.36}$$

Then we have:

Proposition 5.3.8. *The pair (g^+, g^-) is an attractor-repellor pair for $\varrho(\gamma)$.*

Proof. By Lemma A.3.2 the element $\varrho(\gamma)$ contracts a dense open subset of $\check{S}$ to g^+. This implies that the corresponding element g_1 contracts a dense open subset of $\check{S}$ to e, and thus for every $v \in V$ we have $g_2^n.v \to \infty$. This implies that every eigenvalue of g_2 has modulus > 1. □

In particular, we can define the associated period

$$\tau_{\mathcal{D}}^{\infty}(\varrho(\gamma)) := \tau_{\mathcal{D}}^{\infty}(\varrho(\gamma), g^+, g^-);$$

we then have for any $\xi \in S^1 \setminus \{\gamma^{\pm}\}$,

$$\tau_{\mathcal{D}}^{\infty}(\varrho(\gamma)) = b_{\varrho}(\gamma^-, \xi, \gamma^+, \gamma.\xi). \tag{5.37}$$

Now we have the following special case of Corollary 5.2.8:

Theorem 5.3.9. *Let Σ be a closed oriented surface of negative Euler characteristic and $\Gamma = \pi_1(\Sigma)$ its fundamental group. Let G be a semisimple Hermitian Lie group with finite center and associated bounded symmetric domain $\mathcal{D}$ and $\varrho : \Gamma \to G$ a maximal representation. Then there exist positive constants $C_1(\mathcal{D}), C_2(\mathcal{D})$ depending only on $\mathcal{D}$ such that for all $\gamma \in \Gamma$,*

$$C_1(\mathcal{D}) \cdot \tau_{\mathcal{D}}^{\infty}(\varrho(\gamma)) \leq \tau_{\mathcal{D}}(\varrho(\gamma)) \leq C_2(\mathcal{D}) \cdot \tau_{\mathcal{D}}^{\infty}(\varrho(\gamma)),$$

where the period is given by (5.37) *and the translation length is taken with respect to the (unnormalized) Bergman metric on $\mathcal{D}$.*

Indeed, if $\mathcal{D}_1, \dots, \mathcal{D}_l$ are the irreducible factors of $\mathcal{D}$ then we can choose

$$C_1(\mathcal{D}) := \sqrt{\min_j \dim_{\mathbb{C}} \mathcal{D}_j}$$

and

$$C_2(\mathcal{D}) := 2 \cdot \mathrm{rk}\ \mathcal{D} \cdot \max_j \frac{\dim_{\mathbb{C}} \mathcal{D}_j}{\mathrm{rk}\ \mathcal{D}_j}.$$

5.3.4 Well-displacing

The following theorem is the main result of this section:

Theorem 5.3.10 (Well-displacing)**.** *Let Γ be the fundamental group of a closed oriented surface Σ of genus ≥ 2, $\mathcal{D}$ a bounded symmetric domain and S a finite generating set for Γ. Then for every maximal representation $\varrho : \Gamma \to G(\mathcal{D})^0$ there exist $A, B > 0$ such that for all $\gamma \in \Gamma$,*

$$\tau_{\mathcal{D}}(\varrho(\gamma)) \geq A \cdot l_S(\gamma) - B.$$

The proof is based on Theorem 5.3.9 and Labourie's equivalence theorem for cross ratios (in the form of Corollary 5.3.7). The third ingredient is a version of the Milnor-Švarc lemma for translation length, which we describe briefly. Let Γ be the fundamental group of a closed oriented surface of genus ≥ 2 as in the theorem and S a finite generating set for Γ. We then denote by $\|\cdot\|_S$ the word length with respect to S and by

$$d_S(\gamma_1, \gamma_2) := \|\gamma_2^{-1}\gamma_1\|_S$$

the associated word metric. We also recall that the translation length l_S of the pair (Γ, S) is defined by (1.8) (see p. 24).

Lemma 5.3.11. *Let S be an arbitrary finite generating set for Γ. Then there exist constants $A, B > 0$ such that for every $\gamma \in \Gamma$*

$$\tau_{\mathbb{D}}(\gamma) \geq A \cdot l_S(\gamma) - B.$$

Proof. We fix a compact fundamental domain F for the Γ-action on $\mathbb{D}$. We know that every $\gamma \in \Gamma$ is hyperbolic, i.e. there exists a geodesic σ on which γ acts by translation and we have $\gamma \cdot \sigma(t) = \sigma(t + \tau_{\mathbb{D}}(\gamma))$ for all t. There exists $\eta \in \Gamma$ such that $\eta\sigma$ intersects F, say $y := \eta\sigma(t_0) \in F$. Then we have for any $x \in F$:

$$\begin{aligned} d(x, \eta\gamma\eta^{-1}x) &\leq d(x, y) + d(y, \eta\gamma\eta^{-1}y) + d(\eta\gamma\eta^{-1}y, \eta\gamma\eta^{-1}x) \\ &\leq 2\mathrm{diam}(F) + \tau_{\mathbb{D}}(\eta\gamma\eta^{-1}). \end{aligned}$$

Now we fix a basepoint $x \in F$ and apply the Milnor-Švarc lemma [11, Prop. I.8.19] with respect to this base point. We then obtain positive constants A, B' satisfying

$$d(x, \gamma x) = d(ex, \gamma x) \geq A \cdot d_S(e, \gamma) - B' = A \cdot l_S(\gamma) - B'$$

for all $\gamma \in \Gamma$. We deduce that

$$\begin{aligned} \tau_{\mathbb{D}}(\gamma) &= \tau_{\mathbb{D}}(\eta\gamma\eta^{-1}) \geq d(x, \eta\gamma\eta^{-1}x) - 2\mathrm{diam}(F) \\ &\geq A \cdot l_S(\eta\gamma\eta^{-1}) - B' - 2\mathrm{diam}(F) = A \cdot l_S(\gamma) - (B' + 2\mathrm{diam}(F)). \end{aligned}$$

$\square$

Now the proof of Theorem 5.3.10 follows easily.

Theorem 5.3.10. Given $\gamma \in \Gamma$, choose $\xi \in S^1$ such that $(\gamma^-, \xi, \gamma^+)$ is positively oriented. Then also $(\gamma^-, \gamma.\xi, \gamma^+)$ is positively oriented, and using Corollary 5.3.7, Theorem 5.3.9, Equation (1.4) and Lemma 5.3.11 we find positive constants $C_1, \ldots, C_4$ such that

$$\begin{aligned} \tau_{\mathcal{D}}(\varrho(\gamma)) &\geq C_1 \cdot \tau_{\mathcal{D}}^{\infty}(\varrho(\gamma)) \\ &= C_1 \cdot \log b_{\varrho}(\gamma^-, \xi, \gamma^+, \gamma\xi) \\ &\geq C_2 \cdot \log[\gamma^- : \xi : \gamma^+ : \gamma\xi] - C_2 \\ &= C_2 \cdot \tau_{\mathbb{D}}^{\infty}(\gamma) - C_2 \\ &= C_2 \cdot \tau_{\mathbb{D}}(\gamma) - C_2 \\ &\geq C_3 \cdot \ell_S(\gamma) - C_4. \end{aligned}$$

□

Note that compactness of Σ was indispensable for the proof of Theorem 5.3.10.

5.3.5 Proofs of Corollaries 1.5.3-1.5.6

All three corollaries are well-known consequences of the well-displacing property established in Theorem 5.3.10. For the convenience of the reader we provide some explicit references:

Corollary 1.5.3 follows from [27, Prop. 4.2.1] and [27, Lemma 4.0.4], since higher genus surface groups are hyperbolic.

Corollary 1.5.4 follows from [58, Lemma 2.7] (or Corollary 1.5.3 and the Milnor-Švarc lemma) and the proof of Theorem 5.3.10.

Corollary 1.5.5 follows from Corollary 1.5.4 and [58, Prop. 2.4].

Finally, Corollary 1.5.6 follows from [47, Thm. 5.2.2].

5.4 Determining Maximal Representations via Cross Ratios

This section is also joint work with Tobias Hartnick, even if it does not appear in [38]. It uses ideas communicated to us by Marc Burger. In [38] we associated a cross ratio with any maximal representation into a Lie group of Hermitian type of tube type. In this section we show that this cross ratio characterizes the representation in some cases.

Proposition 5.4.1. *Let G_1 and G_2 two simple groups and ϱ_1 and ϱ_2 two representations of Γ into G_1 resp. G_2 which are both maximal and Zariski dense. Assume $b_{\varrho_1} = b_{\varrho_2}$. Then G_1 and G_2 and ϱ_1 and ϱ_2 are isomorphic under some algebraic morphism.*

Proof. Define:

$$M := \overline{\{(\varrho_1(\gamma), \varrho_2(\gamma))|\gamma \in \Gamma\}}^{Z} \subset G_1 \times G_2$$

and denote by p_1 and p_2 the projection on the respective factors. The algebraic group M has only finitely many connected components and the same is true for the image of p_i. Since $p_i(M)$ contains a $\varrho_i(\Gamma)$ (hence infinitely many points), it cannot be discrete. By Lemma 5.4.2 the $p_i(M) = G_i$. Now let $K_j := p_j(\ker p_3 - j)$. As a kernel K_j is a normal subgroup of G_j and since G_j is simple K_j is either trivial or equal to G_j.

Assume that $K_1 = G_1$. Then $G_1 \times \{e\} \subset \ker p_2 \subset M$. Since p_2 is surjective, we can find for any given $g_2 \in G_2$ a $g_1 \in G_1$ such that $(g_1, g_2) \in M$ and if $G_1 \times \{e\} \subset M$ we get $M = G_1 \times G_2$. But in this case the Zariski closure of the diagonal limit curve $(\varphi_1(\gamma), \varphi_2(\gamma))$ is equal to $\check{S} \times \check{S}$. Hence the two cross ratios b_{ϱ_1} and b_{ϱ_2} can not coincide.

Therefore K_1 and K_2 are trivial and M is the graph of an algebraic transformation between G_1 and G_2. $\square$

Lemma 5.4.2. *Let H be a Zariski dense subgroup of a simple algebraic group G. Then H is discrete or equal to G.*

Proof. Use the Lie algebra of the Zariski closure of H. By simplicity, it is either trivial or equal to the Lie algebra of G. □

Appendix A

Appendix

A.1 Matrix calculations

Lemma A.1.1. *Let* $A \in \mathrm{GL}(n, \mathbb{R})$*. Then it is conjugate in* $\mathrm{GL}(n, \mathbb{R})$ *to a block matrix*

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}$$

such that the characteristic polynomial of A_i *is a maximal power of an irreducible factor of the characteristic polynomial of* A*. In particular the eigenvalues of* A_i *have the same absolute value.*

This lemma follows immediately from Cayley-Hamilton and the following lemma:

Lemma A.1.2. *Let* $A \in Mat_{n,n}(k)$ *and* p *a polynomial over* k*. Let* $p = p_1 \cdot p_2$ *with* p_1 *and* p_2 *relatively prime. Then*

$$\ker p(A) = \ker p_1(A) \oplus \ker p_2(A).$$

Proof. It is clear that $\ker p(A) \supset \ker p_1(A) + \ker p_2(A)$ and it remains to show that the sum is direct and that equality holds. Since p_1 and p_2 are relatively prime, we find polynomials r_1 and r_2 such that $1 = r_1 p_1 + r_2 p_2$, i.e. $\mathrm{id} = r_1(A) r_1(A) + r_2(A) p_2(A)$.

If $v \in \ker p_1(A) \cap \ker p_2(A)$, then

$$v = \mathrm{id}(v) = r_1(A) r_1(A) v + r_2(A) p_2(A) v = 0,$$

hence the sum is direct.

Let $v \in \ker p(A)$ then $r_i(A) p_i(A) v \in \ker p_{3-i}(A)$, hence $\ker p(A) \subset \ker p_1(A) + \ker p_2(A)$. This finishes the proof. $\square$

Remark A.1.3. Lemma A.1.1 implies that A is conjugate to the matrix

$$\begin{pmatrix} A_> & & \\ & A_< & \\ & & A_= \end{pmatrix}$$

where the absolute values of the eigenvalues of $A_>$ are strictly bigger than 1, the absolute values of the eigenvalues of $A_<$ are strictly smaller than 1 and the absolute values of $A_=$ are equal to 1.

Proposition A.1.4. *The map* $(N_1, N_2, N_3) \mapsto (N_1 N_2^\top, N_2 N_3^\top, N_3 N_1^\top)$ *induces a bijection between* $\mathrm{O}(n) \backslash \mathrm{GL}(n, \mathbb{R})^3 / \mathrm{O}(n)$ *and*

$R = \{(X_1, X_2, X_3) | X_3 (X_2^\top)^{-1} X_1$ *sym. pos. definite*, $i = 1, 2, 3\} / \mathrm{O}(n)$,

where $\mathrm{O}(n) \times \mathrm{O}(n)$ *acts on* $\mathrm{GL}(n, \mathbb{R})^3$ *by right and left multiplication and* $\mathrm{O}(n)$ *acts on the last set by conjugation.*

Proof. First note that $X_3 (X_2^\top)^{-1} X_1$ is positive definite and symmetric if and only if $X_{i+2} (X_{i+1}^\top)^{-1} X_i$ is positive definite and symmetric for all $i \pmod 3$.

Since $X_{i+2} (X_{i+1}^\top)^{-1} X_i$ is positive definite and symmetric there exists canonical $\tilde{N}_i$ such that $X_{i+2} (X_{i+1}^\top)^{-1} X_i = \tilde{N}_i \tilde{N}_i^\top$. Since $N_i N_i^\top = \tilde{N}_i \tilde{N}_i^\top$, there exist $k_i \in \mathrm{O}(n)$ such that $N_i = \tilde{N}_i k_i$. Therefore

$N_i^{-1}X_i(N^\top)^{-1} \in \mathrm{O}(n)$ and a direct calculation shows that $l_i = \tilde{N}_{i+1}^{-1}X_i(\tilde{N}_i^\top)^{-1} \in \mathrm{O}(n)$ and $l_i = k_i k_{i+1}^{-1}$, hence $l_1 l_2 l_3 = \mathrm{id}$. This gives $N_1 = \tilde{N}_1 k_1$, $N_2 = \tilde{N}_2 l_3 l_2 k_1$ and $N_3 = \tilde{N}_3 l_3 l_2 k_1$. This provides an inverse to the map given above. □

Lemma A.1.5. *Let k be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Every matrix $X \in \mathrm{GL}(n,k)$. Then X is conjugate in $\mathrm{GL}(n,k)$ to its transpose $X^\top$.*

Proof. First let $X \in \mathrm{GL}(n,k)$ be a single Jordan block, i.e.

$$X = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

for some $\lambda \in \mathbb{C}$. Then one can easily check that $X^\top = hXh^{-1}$ with

$$h = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

Therefore the statement is true for Jordan blocks and hence for matrices which consists of Jordan blocks.

Now let $X \in \mathrm{GL}(n,\mathbb{C})$ and J its Jordan canonical form. Then there exists $l \in \mathrm{GL}(n,\mathbb{C})$ such that $X = lJl^{-1}$ and we have $X^\top = (l^\top)^{-1}J^\top l^\top$. Furthermore there exists h such that $J^\top = hJh^{-1}$, hence $X^\top = (l^\top)^{-1}hl^{-1}Xlh^{-1}l^\top$.

Let $X \in \mathrm{GL}(n,\mathbb{R})$. Then X and $X^\top$ are conjugate over $\mathbb{C}$. But two real matrices conjugate over $\mathbb{C}$ are also conjugate over $\mathbb{R}$. This finishes the proof. □

A.2 Complements to the theory of Euclidean Jordan algebras

Throughout this article we have made essential use of results from the theory of Euclidean Jordan algebras. Most of these results are standard and can be found in the literature, see in particular [29, 10]. However, we need two results on Euclidean Jordan algebras, which are not covered by these references and appear to be partly new. These are discussed in this appendix. Our first result concerns morphisms of Euclidean Jordan algebras and is needed to complete the proof of Proposition 2.1.39.

Proposition A.2.1. *Let $\mathcal{D}_1, \mathcal{D}_2$ be bounded symmetric domains of tube type with respective Shilov boundaries $\check{S}_1$ and $\check{S}_2$, and $\beta : \mathcal{D}_1 \to \mathcal{D}_2$ be a boundary morphism. Then there exist Euclidean Jordan algebras V_1, V_2, a Jordan algebra homomorphism $\alpha : V_1 \to V_2$ and isomorphisms $\mathcal{D}_j \cong \mathcal{D}_{V_j}$ intertwining β and $\alpha^{\mathbb{C}}$.*

The proof uses the theory of positive Hermitian Jordan triple systems (pHJts'). We refer the reader to [22] for background. We recall that the unit balls of such triples systems (always with respect to the spectral norm) are circled (i.e. invariant under the diagonal multiplication with elements of S^1) and symmetric (see [50, Thm. 4.1]), and that every bounded symmetric domain arises as the unit ball of a pHJts (see [50, Thm. 1.6 and Thm. 4.1] and [22]). Every morphism of pHJts' induces a morphism of the corresponding unit balls. Conversely we have:.

Lemma A.2.2. *Let W_1, W_2 be positive Hermitian Jordan triple systems and $\mathcal{D}_1, \mathcal{D}_2$ their unit balls with respect to the respective spectral norms. Then every morphism $\beta : \mathcal{D}_1 \to \mathcal{D}_2$ with $\beta(0) = 0$ extends to a morphism $W_1 \to W_2$ of pHJts.*

Proof. We adapt an argument of Loos [50] going back to Cartan [19, p. 30] (see also [29, L. X.5.2]): Consider the maps $\beta_t^{(1)}(z) := \beta(e^{it}z)$

and $\beta_t^{(2)}(z) := e^{it}\beta(z)$ for $t \in \mathbb{R}$. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are circled, these map $\mathcal{D}_1$ into $\mathcal{D}_2$; moreover, both maps are affine, since β is, and share the same z-derivative at the origin. Since also $\beta_t^{(1)}(0) = \beta_t^{(2)}(0) = 0$ we deduce [53, Prop. 3.2] that $\beta_t^{(1)} = \beta_t^{(2)}$; comparing Taylor expansions, we see that β is linear and thus extends to $\beta : W_1 \to W_2$. Since the derivative of a morphism of bounded symmetric domains is a morphism of Jordan triple systems [2, Thm. III.2.8] and the exponential map intertwines the Jordan triple structures on W_j and T_0W_j, the lemma follows. ☐

Now we can deduce Proposition A.2.1:

Proposition A.2.1. By applying suitable isomorphisms we may assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are the unit balls of pHJts' W_1, W_2 with respect to the corresponding spectral norms and that $\beta(0) = 0$. Then Lemma A.2.2 applies and provides a linear extension $\beta : W_1 \to W_2$, which is a morphism of Euclidean Jordan triple systems. Note that by uniqueness, $\beta|_{\check{S}_1}$ is the boundary extension of β. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are of tube type, the elements of $\check{S}_j$ are precisely the maximal tripotents of the Jordan triple system W_j [22, Thm. 4.2]. Now pick $e_1 \in \check{S}_1$ arbitrarily and define $e_2 := \beta(e_1)$. Out of the respective triple products $\{\cdot, \cdot, \cdot\}$ we then obtain complex Jordan algebra structures on W_1 and W_2 by

$$x \cdot y := \{x, e_j, y\};$$

by construction, β is a morphism $(W_1, \cdot) \to (W_2, \cdot)$ and maps the Euclidean real forms given by

$$V_j := \{z \in W_j \mid \{e_j, z, e_j\} = z\}$$

to each other. Then the restriction $\alpha : V_1 \to V_2$ is the desired morphism of Euclidean Jordan algebras with $\alpha^{\mathbb{C}}|_{\mathcal{D}_1} = \beta$. ☐

Our second result concerns a refinement of results from [25] concerning transversality on Shilov boundaries. We denote by K :

$V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathrm{End}(V^{\mathbb{C}})$ the canonical automorphy kernel of V and by $\det_V$ the *Jordan algebra determinant* (see [29, Ch. II.2]). We also denote by $Str(V^{\mathbb{C}})$ the *structure group* of $V^{\mathbb{C}}$ ([29, p.147]). Then we have the following characterization of transversality:

Proposition A.2.3. *Let V be a Euclidean Jordan algebra, $\mathcal{D}$ the associated bounded symmetric domain and $\check{S}$ its Shilov boundary. Then $z, w \in \check{S}$ are transverse iff one of the following equivalent conditions holds true:*

(i) $\det_V(z - w) \neq 0$.

(ii) $K(z, w)$ *is invertible.*

(iii) $K(z, w) \in Str(V^{\mathbb{C}})$.

(iv) $\det K(z, w) \neq 0$.

(v) (z, w) *is the unique open G-orbit in $\check{S} \times \check{S}$.*

The lion's share of the proof is provided in [25]. In order to complete the arguments given there, we need to understand the transformation behavior of the automorphy kernel. For this we remark that by [55, Ch. II, Sec. 5] there exists a function $J : G \times \overline{\mathcal{D}} \to Str(V^{\mathbb{C}})$, called the *canonical automorphy factor*, satisfying

$$K(gz, gw) = J(g, z)K(z, w)J(g, w)^* \tag{A.1}$$

for $g \in G$, $z, w \in \overline{\mathcal{D}}$.

Proposition A.2.3. Let us first prove equivalence of the statements (i)-(iv): The implication (i) $\Rightarrow$ (ii) is provided in [25, Lemma 5.1]. The implication (ii) $\Rightarrow$ (iii) follows from the fact that $K(z_0, w_0) \in Str(V^{\mathbb{C}})$ for $z_0, w_0 \in \mathcal{D}$ together with the continuity of K and the fact that $Str(V^{\mathbb{C}})$ is closed in $GL(V^{\mathbb{C}})$. Finally, the implication (iii) $\Rightarrow$ (iv) is obvious.

Thus it remains to show (iv) $\Rightarrow$ (i). Let $w, z \in \check{S}$ be arbitrary and assume $\det K(z,w) \neq 0$. Recall that

$$\det K(z,w) = k_{\det}(z,w) = k_V^{2\dim V}(z,w)$$

We know that there exists $g \in G$ and a Jordan frame (c_1) such that

$$gz = \sum z_i c_i, \quad gw = \sum w_i c_i.$$

Using Lemma 5.1.15 we get

$$\begin{aligned}\det K(gz,gw) =& k_V^{2\dim V}(gz,gw) = \prod (1 - z_i \bar{w}_i)^{2\frac{\dim V}{\operatorname{rk} V}} \\ =& \prod \left((w_i - z_i)\bar{w}_i\right)^{2\frac{\dim V}{\operatorname{rk} V}},\end{aligned}$$

since $|w_i| = 1$ Furthermore from [29, Thm III.1.2] we get

$$\det{}_{V^{\mathbb{C}}}(gz - gw) = \prod (z_i - w_i),$$

which shows that $\det K(gz,gw) \neq 0$ if and only if $\det_{V^{\mathbb{C}}}(gz - gw) \neq 0$. But by (A.1) this is the case if and only if $\det K(z,w) \neq 0$ and by [25, Ch. 4] $\det_{V^{\mathbb{C}}}(gz-gw) \neq 0$ if and only if $\det_{V^{\mathbb{C}}}(z-w) \neq 0$. This finishes the proof of the equivalence of (i)-(iv).

We deduce in particular that

$$\check{S}^{[2]} := \{(z,w) \in \check{S} \mid \det K(z,w) \neq 0\} = \{(z,w) \in \check{S} \mid \det{}_V(z-w) \neq 0\}$$

The first description together with (A.1) and the continuity of $\det K(\cdot,\cdot)$ imply already that $\check{S}^{[2]}$ is G-invariant and open; the second description together with [25, Prop. 3.4] shows that $\check{S}^{(2)}$ is even a G-orbit. Since $\check{S}^{[2]}$ is the unique open G-orbit in $\check{S}^2$ we obtain $\check{S}^{(2)} = \check{S}^{[2]}$, which finishes the proof. $\square$

As a corollary of the characterization (i) of transversality in Proposition A.2.3 we get:

Corollary A.2.4. *The image $p(V)$ of V under the inverse Cayley transform is precisely the subset of points in $\check{S}$, which are transverse to e.*

Proof. First note that the classical Cayley transform $c_{\mathbb{R}}$ maps $S^1 \setminus \{1\}$ homeomorphically onto $\mathbb{R}$. We denote its inverse by $p_{\mathbb{R}} : \mathbb{R} \to S^1 \setminus \{1\}$. Now let $v \in V$. By the spectral theorem (in the version of [29, Thm. III.1.]) there exists a Jordan frame $(c_1, \ldots, c_n)$ such that $v = \sum \lambda_i c_i$ with λ_i real. Then

$$p(v) = \sum p_{\mathbb{R}}(\lambda_i) c_i \in \check{S},$$

is transversal to e, because $p_{\mathbb{R}}(\lambda_i) \neq 1$ for all i. Conversely, every $w \in \check{S}$ can be writen as $\sum \mu_i c_i$ for some Jordan frame $(c_1, \ldots, c_n)$ and complex numbers $\mu_i \in S^1$. If w is transverse to e then $\mu_i \neq 1$ and

$$c(w) = \sum c_{\mathbb{R}}(\mu_i) c_i.$$

Since $c_{\mathbb{R}}^{-1}(\mu_i) \in \mathbb{R}$ we have $c(w) \in V$. □

A.3 Uniqueness of limit curves of Anosov representations

The purpose of this appendix is to sketch a proof of Proposition 2.4.6, which claims that the continuous monotone limit curve associated with a maximal representation is unique. As mentioned in the body of the text this is a consequence of the Anosov property of maximal representations as established in [15] (see also Theorem A.3.3).

Before we start the proof, we shortly state the definition of Anosov representations following [36]. Let (N, g) be a closed negatively curved Riemannian manifold and $M := T^1N$ its unit tangent bundle equipped with the geodesic flow φ_t for the metric g. We denote by

$\widehat{M} := T^1N$ the $\pi_1(N)$-cover of M and by abuse of notation by φ_t the geodesic flow on $\widehat{M}$.

Let G be a semisimple Lie group and (P^+, P^-) be a pair of opposite parabolic subgroups of G and set $\mathcal{F}^\pm := G/P^\pm$. The subgroup $L := P^+ \cap P^-$ is the Levi factor of both P^+ and P^-. The homogeneous space $\mathcal{X} := G/L$ is the unique open G-orbit in the product $\mathcal{F}^+ \times \mathcal{F}^-$. From this product structure $\mathcal{X}$ inherits two G-invariant distributions E^+ and E^-: $(E^\pm)_{(x_+, x_-)} = T_{x_\pm}\mathcal{F}^\pm$. As a consequence any $\mathcal{X}$-bundle is equipped with two distributions which are denoted also by E^+ and E^-.

Let $varrho : \pi_1(N) \to G$ be a representation. We set

$$\mathcal{X}_\varrho := \pi_1(N)\backslash(\widehat{M} \times \mathcal{X}),$$

where $\pi_1(N)$ acts diagonally on $\widehat{M} \times \mathcal{X}$. The space $\mathcal{X}_\varrho$ is a $\mathcal{X}$-bundle over M.

Definition A.3.1. A representation $\varrho : \pi_1(N) \to G$ is said to be (P^+, P^-)-Anosov if

(i) the flat bundle $\mathcal{X}_\varrho$ admits a section $\sigma : M \to \mathcal{X}_\varrho$ which is flat along flow lines (i.e. the restriction of σ to any geodesic leaf is flat).

(ii) The (lifted) action of φ_t on $\sigma^* E^+$ (resp. $\sigma^* E^-$) is dilating (resp. contracting).

The section σ will be called *Anosov section.*

In writing this appendix we profited from a manuscript on Anosov representations by O. Guichard and A. Wienhard, which by now has appeared as part of [36]. Since the latter article discusses in detail various generalizations of Proposition 2.4.6, we will only provide a brief outline of the argument. We recommend the reader to consult [36] for more details. Throughout this appendix we fix a maximal representation $\varrho : \Gamma \to G$.

Lemma A.3.2. *Let $\gamma \in \Gamma - \{id\}$ and $\gamma^+ \in S^1$ its attractive fixed point. Then for any limit curve φ the sequence $\varrho(\gamma)^n$ contracts an open and dense set $U = U(\varphi, \gamma)$ of the Shilov boundary to $\varphi(\gamma^+)$.*

Let us first ensure that this indeed yields the desired conclusion:

Proposition 2.4.6. Assume φ_1 and φ_2 are two limit curves for the maximal representation ϱ and let $x \in U = U(\varphi_1, \gamma) \cap U(\varphi_2, \gamma)$, which is non-empty by the lemma. Then $\varrho(\gamma)^n x$ converges to both $\varphi_1(\gamma^+)$ and $\varphi_2(\gamma^+)$, whence $\varphi_1(\gamma^+) = \varphi_2(\gamma^+)$. Since $\{\gamma^+ | \gamma \in \Gamma\}$ is dense in S^1, we have $\varphi_1 = \varphi_2$. □

We will now sketch the proof of Lemma A.3.2. Throughout we fix $\gamma \in \Gamma - \{id\}$ with repellent fixpoint γ^- and attractive fixed point γ^+ in S^1; to simplify notation we will assume $\varphi(\gamma_+) = e_V$; since the statement of the lemma is conjugation-invariant, this is no loss of generality. To formulate the Anosov property, on which the proof relies, we have to introduce two bundles and an associated flow. Thus let $M := T^1\Sigma$ the unit tangent bundle of Σ and $\bar{M} := T^1\tilde{\Sigma}$ the unit tangent bundle of its universal covering; denote by $p : \bar{M} \to M$ the canonical projection. We identify points of $\bar{M}$ with positive triples in $(S^1)^3$ in such a way that (v_-, v_0, v_+) corresponds to the projection of v_0 onto the geodesic v_-v_+. As before we denote by $\check{S}^{(2)}$ the space of transverse pairs in the Shilov boundary of G. Then we define a flat $\check{S}^{(2)}$-bundle over M by

$$E_\varrho := \Gamma \backslash (\bar{M} \times \check{S}^{(2)}).$$

Since $\overline{E_\varrho} := p^* E_\varrho$ is a trivial bundle we have a splitting

$$T\overline{E_\varrho} \cong T\overline{M} \oplus T\check{S}^{(2)} \cong T\overline{M} \oplus (T\check{S} \oplus T\check{S})|_{\check{S}^{(2)}}.$$

To distinguish the second and the third summand in the last decomposition we denote them by $\bar{E}^+_\varrho$ and $\bar{E}^-_\varrho$ respectively. By definition the fiber of $\bar{E}^\pm_\varrho$ over $(v_-, v_0, v_+) \in \overline{M}$ is $T_{v_\pm}\check{S}$.

Every continuous limit curve $\varphi : S^1 \to \check{S}$ defines a continuous section σ_φ of E_ϱ, whose lift $\bar{\sigma}_\varphi : \bar{M} \to \overline{E_\varrho}$ is given by the formula

$$(v_-, v_0, v_+) \mapsto ((v_-, v_0, v_+), (\varphi(v_-), \varphi(v_+))).$$

We introduce the notations

$$\pi_+ : \sigma_\varphi^* E_\varrho^+ \to M, \quad \pi_- : \sigma_\varphi^* E_\varrho^- \to M$$

for the canonical projections of the bundles $\sigma_\varphi^* E_\varrho^\pm$.

There are natural flows on these bundles given as follows: On M and $\bar{M}$ there are the geodesic flow φ_t resp. $\bar{\varphi}_t$. The flow $\bar{\varphi}_t$ lifts to a flow on $\bar{E}_\varrho$ via

$$\overline{\hat{\varphi}_t}(v, s) := (\bar{\varphi}_t(v), s).$$

By construction this flow is compatible with the Γ-action on $\bar{E}_\varrho$, hence it descends to a flow $\hat{\varphi}_t$ on E_ϱ. Furthermore the maps $\bar{\sigma}_\varphi$ and σ_φ are invariant under the respective flows. From the explicit description

$$\sigma_\varphi^* E_\varrho^\pm = \{(m, e) \in M \times E_\varrho^\pm \mid \sigma_\varphi(m) = p_\varrho^\pm(e)\}$$

we see that the bundles $\sigma_\varphi^* E^\pm$ are invariant under the flow $\psi_t := \varphi_t \times \hat{\varphi}_t|_{M \times E_\varrho^\pm}$. We denote by $\bar{\psi}_t$ the corresponding flow on $\bar{\sigma}_\varphi^* \bar{E}_\varrho^\pm$. Now the main technical result of [15] reads as follows:

Theorem A.3.3 (Burger-Iozzi-Wienhard)**.** *The section $\sigma_\varphi : M \to E_\varrho$ is an Anosov section, i.e. for any continuous family of norms $(\|\cdot\|_m)_{m \in M}$ on $\sigma_\varphi^* E_\varrho^\pm$ there exist constants $A, a > 0$ such that for every $m \in M$, $v^\pm \in (\sigma_\varphi^* E_\varrho^\pm)_m$ and $t > 0$,*

$$\|\psi_{\pm t}(v^\pm)\|_{\pi_\pm(\psi_{\pm t}(v^\pm))} \leq A \exp(-at) \|v^\pm\|_m.$$

Returning to our isometry γ, denote by ω a geodesic in $\tilde{\Sigma}$ joining γ^- and γ^+. Note that γ acts on ω by translation. In particular if $\tau = \tau_{\tilde{\Sigma}}(\gamma)$, then

$$\gamma \cdot \omega(t) = \omega(t + \tau), \quad (t \in \mathbb{R}),$$

and

$$d\gamma \cdot \dot{\omega}(t) = \dot{\omega}(t+\tau) = \varphi_\tau(\dot{\omega}(t)). \tag{A.2}$$

One can now deduce Lemma A.3.2 along the following lines.

(i) We see from the definition of $\bar{\sigma}_\varphi$ that the fibers of $\bar{\sigma}_\varphi^* \bar{E}_\varrho^\pm$ along $\dot{\omega}(t)$ are canonically isomorphic with $T_{\varphi(\gamma^+)}\check{S}$; moreover the canonical isomorphisms $\iota_t : \bar{\sigma}_\varphi^* \bar{E}_\varrho^\pm \to T_{\varphi(\gamma^+)}\check{S}$ intertwine with the flow ψ_t, i.e. if $v \in (\bar{\sigma}_\varphi^*(\overline{E_\varrho^+}))_{\dot{\omega}(t_0)}$, then

$$\iota_{t_0}(v) = \iota_{t_0+t}(\psi_t v).$$

(ii) The flow ψ_t and the action of γ on $\dot{\omega}$ intertwine, i.e.

$$\varrho(\gamma)v = \psi_\tau(v)$$

for all v in the fiber over $\dot{\omega}(\mathbb{R}) \subset \bar{M}$.

(iii) The contraction property carries over to $T_{\varphi(\gamma^+)}\check{S}$. Let $v \in T_{\varphi(\gamma^+)}\check{S}$ and $x = \iota_0^{-1}v$. Then:

$$\begin{aligned}
\|d\varrho(\gamma).v\| &= \|d\varrho(\gamma).\iota_0(x))\| = \|\iota_\tau(\varrho(\gamma).x))\| \\
&= \|\iota_0^{-1}\iota_\tau(\varrho(\gamma).x))\|_0 = \|\varrho(\gamma).x\|_\tau \\
&= \|\bar{\psi}_\tau(x)\|_\tau = \|\psi_\tau(\Gamma x)\|_{p(\dot{\omega}(\tau))} \\
&\leq A\exp(-a\tau)\|\Gamma x\|_{p(\dot{\omega}(0))} = A\exp(-a\tau)\|x\|_0 \\
&= A\exp(-a\tau)\|v\|.
\end{aligned}$$

In particular, $\lim_{n\to\infty} \|d\varrho(\gamma^n).v\| = 0$ for all $v \in T_{\varphi(\gamma^+)}\check{S}$.

(iv) We now claim that γ acts contractingly on $U := \exp(T_{e_V}\check{S})$, which is open and dense in $\check{S}$. (Recall our assumption that $e_V = \varphi(\gamma^+)$.) Indeed, γ acts contractingly on $T_{\varphi(\gamma^+)}\check{S}$, and the exponential map intertwines this action with the action on U, whence $\varrho(\gamma)$ contracts U to $\varphi(\gamma^+)$ as claimed.

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