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MEAN-VARIANCE PORTFOLIO OPTIMISATION: TRADING CONSTRAINTS AND TIME CONSISTENCY

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To Corinne.

Abstract

This thesis studies *mean-variance portfolio selection (MVPS)* and *mean-variance hedging (MVH)* in a general semimartingale model under constraints and develops a time-consistent formulation for MVPS as a dynamic optimisation problem. The constraints are formulated via predictable correspondences; trading strategies are restricted to lie in a closed convex set $C(\omega, t)$ which may depend on the state ω and time t in a predictable way.

To obtain a solution for the constrained MVH problem, we establish the closedness in L^2 of the space G_T of all gains from trade (i.e. the terminal values of stochastic integrals with respect to the price process of the underlying assets). This is a first contribution which allows us to subsequently tackle the problem in a systematic and unified way, and to obtain more information on the structure of the solution by convex duality tools.

It turns out that the closedness of G_T in L^2 is related to the closedness, in the semimartingale topology $\mathcal{S}(P)$, for spaces of stochastic integrals with constrained (C-valued) integrands, and we provide necessary and sufficient conditions for the latter to hold. Applications to utility maximisation and superreplication under constraints often bring up spaces of stochastic integrals that are predictably convex. We show that such a space is closed in $\mathcal{S}(P)$ if and only if it is a space of stochastic integrals of C-valued integrands, where C is a predictable correspondence with closed and convex values.

If the constraints are given by closed cones, MVPS viewed as a *static* optimisation problem reduces to solving a particular MVH problem. Treating the latter as a stochastic optimal control problem allows us to characterise the value function by the maximal solutions of two coupled backward stochastic differential equations (BSDEs) and to describe the optimal strategy locally as the pointwise minimiser of the drift rates.

Viewed as a *dynamic* optimisation problem, MVPS is time inconsistent in the sense that it does not satisfy Bellman's optimality principle and the usual dynamic programming approach fails. We propose a time-consistent formulation of this problem, which is based on a local notion of optimality. To justify the continuous-time formulation, we prove that it is the continuous-time limit of that in discrete time. This exploits that we establish a global description of the locally optimal strategy in terms of the structure condition and the Föllmer–Schweizer decomposition of the mean-variance tradeoff.

Kurzfassung

Diese Arbeit befasst sich mit *Mean-Variance Portfolio Selection (MVPS)* and *Mean-Variance Hedging (MVH)* in einem allgemeinen Semimartingalmodell unter Handelseinschränkungen und entwickelt eine zeitkonsistente Formulierung von MVPS als ein dynamisches Optimierungsproblem. Die Einschränkungen werden durch previsible Korrespondenzen beschrieben; es werden nur Strategien zugelassen, die in einer abgeschlossenen konvexen Menge $C(\omega, t)$ liegen, die vom Zustand ω und Zeitpunkt t auf eine previsible Art abhängen kann.

Um eine Lösung für das eingeschränkte MVH-Problem zu erhalten, zeigen wir zunächst die L^2-Abgeschlossenheit des Raums G_T aller Handelserträge (d.h. aller Endwerte von stochastischen Integralen bezüglich des Preisprozesses der zugrundeliegenden Anlagen). Dies ist ein erste Neuerung, die im Folgenden erlaubt das Problem auf eine systematische und einheitliche Weise anzugehen und mehr Informationen über die Struktur der Lösung mittels konvexer Dualität zu erhalten.

Es zeigt sich, dass die L^2-Abgeschlossenheit von G_T mit der Abgeschlossenheit in der Semimartingaltopologie $\mathcal{S}(P)$ von Räumen von stochastischen Integralen mit eingeschränkten (C-wertigen) Integranden zusammenhängt, und wir geben notwendige und hinreichende Bedingungen für letztere Abgeschlossenheit. Anwendungen in der Nutzenmaximierung und Superreplikation unter Einschränkungen führen oft zu Räumen von stochastischen Integralen, die previsibel-konvex sind. Wir zeigen, dass ein solcher Raum genau dann in $\mathcal{S}(P)$ abgeschlossen ist, wenn er ein Raum stochastischer Integrale von C-wertigen Integranden ist, wobei C eine previsible Korrespondenz mit abgeschlossenen und konvexen Werten ist.

Falls die Einschränkungen durch abgeschlossene Kegel gegeben sind, reduziert sich MVPS, als ein *statisches* Optimierungsproblem verstanden, auf ein bestimmtes MVH-Problem. Wenn wir letzteres als ein stochastisches Kontrollproblem auffassen, können wir die zugehörige Wertfunktion durch die maximalen Lösungen von zwei gekoppelten stochastischen Rückwärtsdifferentialgleichungen charakterisieren und die optimale Strategie als die punktweisen Minimierer der Driftraten beschreiben.

Als *dynamisches* Optimierungsproblem ist MVPS zeitinkonsistent im Sinne, dass es Bellmans Optimalitätsprinzip nicht erfüllt und daher der

übliche dynamische Programmierungsansatz versagt. Wir schlagen deshalb eine zeitkonsistente Formulierung des Problems vor, die auf einem lokalen Optimalitätsbegriff beruht. Um die zeitstetige Formulierung zu rechtfertigen, zeigen wir, dass sie der zeitstetige Grenzwert derer in diskreter Zeit ist. Dies beruht auf einer globalen Beschreibung der lokal optimalen Strategie mittels der Structure Condition und der Föllmer–Schweizer Zerlegung des Mean-Variance Tradeoff.

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Contents

Chapter I

Introduction

This chapter describes the basic optimisation problems and gives an overview over the main results of this thesis.

I.1 Mean-variance portfolio selection and mean-variance hedging

Two central issues in mathematical finance are the optimisation of investments and the pricing and hedging of contingent claims. Analysing these by means of quadratic criteria immediately leads to the classical problems of mean-variance portfolio selection and mean-variance hedging. In simple terms, *mean-variance portfolio selection (MVPS)* consists of finding a self-financing portfolio whose terminal wealth has maximal mean and minimal variance. *Mean-variance hedging (MVH)* is the problem of approximating a given payoff by the terminal wealth of a self-financing trading strategy with minimal mean-squared hedging error. As both problems have met with great interest in both academia and practice, the literature is vast and we do not give a complete survey here but rather focus on introducing the basic problems and explaining the relation between them. The related literature is discussed in each chapter separately. For a broader overview and the history of both problems, we refer the reader to the survey articles [90], [87], [77] and [89].

Let $S = (S_t)_{0 \leq t \leq T}$ be an $\mathbb{R}^d$-valued semimartingale modelling the discounted prices of d risky assets and Θ a linear space of $\mathbb{R}^d$-valued, S-integrable, predictable processes $\vartheta = (\vartheta_t)_{0 \leq t \leq T}$ satisfying some technical conditions. As *trading strategies*, which are available for investment, we consider for the moment a set $\mathfrak{C} \subseteq \Theta$, where $\vartheta \in \mathfrak{C}$ describes the number of shares held dynamically over time. We call $\mathfrak{C}$ *unconstrained* if $\mathfrak{C} = \Theta$ is a linear subspace and *constrained* otherwise. The unconstrained case $\mathfrak{C} = \Theta$ corresponds to a frictionless financial market where the investor can use any linear combination of trading strategies. If $\mathfrak{C}$ is constrained, not all processes $\vartheta \in \Theta$

are available and the investor faces market frictions in the sense of trading constraints. By choosing a trading strategy $\vartheta \in \mathfrak{C}$ and trading up to time $t \in [0,T]$ in a self-financing way, an investor with initial capital $x \in \mathbb{R}$ can generate the *wealth*

$$V_t(x,\vartheta) := x + \int_0^t \vartheta_u dS_u =: x + \vartheta \bullet S_t. \tag{1.1}$$

Mean-variance portfolio selection can then be formulated as the problem to

$$\text{maximise } E[V_T(x,\vartheta)] - \frac{\gamma}{2}\operatorname{Var}[V_T(x,\vartheta)] \text{ over all } \vartheta \in \mathfrak{C}, \tag{1.2}$$

where $\gamma > 0$ denotes the *risk aversion* of the investor. An alternative formulation is to

$$\begin{aligned}&\text{minimise } \operatorname{Var}[V_T(x,\vartheta)] = E\big[|V_T(x,\vartheta)|^2\big] - m^2 \\ &\text{subject to } E[V_T(x,\vartheta)] = m > x \text{ and } \vartheta \in \mathfrak{C}.\end{aligned} \tag{1.3}$$

If $\mathfrak{C}$ is a *cone* which includes in particular the unconstrained case $\mathfrak{C} = \Theta$, it can be shown by elementary arguments that the solutions $\widetilde{\vartheta}$ and $\widetilde{\vartheta}^{(x,m)}$ to (1.2) and (1.3) are given by

$$\widetilde{\vartheta} = \frac{1}{\gamma}\frac{1}{E[1-\widetilde{\varphi}\bullet S_T]}\widetilde{\varphi} \quad \text{and} \quad \widetilde{\vartheta}^{(m,x)} = \frac{m-x}{E[1-\widetilde{\varphi}\bullet S_T]}\widetilde{\varphi} = (m-x)\gamma\widetilde{\vartheta}, \tag{1.4}$$

respectively, where $\widetilde{\varphi}$ is the solution to the auxiliary problem to

$$\text{minimise } E\big[|1-\vartheta\bullet S_T|^2\big] \text{ over all } \vartheta \in \mathfrak{C}. \tag{1.5}$$

The latter is a particular version of the *mean-variance hedging* problem to

$$\text{minimise } E\big[|H - V_T(x,\vartheta)|^2\big] \text{ over all } \vartheta \in \mathfrak{C}, \tag{1.6}$$

where H is a square-integrable random variable denoting the time-T payoff of some financial instrument. Mathematically, solving (1.6) corresponds to approximating $H - x$ in $L^2(P)$ by an element of the space of all terminal gains from trading given by

$$G_T(\mathfrak{C}) := \{\vartheta \bullet S_T \mid \vartheta \in \mathfrak{C}\}.$$

Therefore a solution to (1.6) exists by the best approximation theorem in Hilbert spaces if $G_T(\mathfrak{C})$ is convex and closed in $L^2(P)$. After establishing the existence of a solution the main challenge is to describe the optimal strategy more explicitly. It turns out that this is due to the combination of linear wealth dynamics (1.1) and quadratic objective function (1.6) very tractable in the *unconstrained case* $\mathfrak{C} = \Theta$. Here the description of the optimal strategy crucially relies on the fact that the best approximation with respect to a linear subspace is a linear projection. This can be exploited either by combining projection and martingale techniques or by using dynamic programming. For the latter approach we consider instead of the single static problem (1.6) the corresponding conditional problems to

$$\text{minimise } E\big[|H - V_T(x,\vartheta)|^2\big|\mathcal{F}_t\big] \text{ over all } \vartheta \in \Theta_t(\psi) \tag{1.7}$$

where $\Theta_t(\psi) := \{\vartheta \in \Theta \mid \vartheta \mathbb{1}_{[\![0,t]\!]} = \psi \mathbb{1}_{[\![0,t]\!]}\}$ denotes the set of all strategies $\vartheta \in \Theta$ that agree up to time t with a given $\psi \in \Theta$. Since (1.6) is a standard stochastic optimal control problem, the family of conditional problems (1.7) is time consistent in the sense that it satisfies *Bellman's optimality principle:* If $\widetilde{\vartheta}^H$ is the solution to (1.6), then it is also the solution to (1.7) with $\psi = \widetilde{\vartheta}^H$ for all $t \in [0,T]$. This dynamic characterisation of optimality then indeed allows to describe $\widetilde{\vartheta}^H$ more explicitly. It is worth pointing out that this works in semimartingale settings and does not need Markovian or Brownian frameworks, but only exploits the combination of linear wealth dynamics (1.1), quadratic objective function (1.6) and most important the linearity of the projection on a linear subspace; see [11] and [68] for example.

For applications, one would like to study MVPS and MVH also under trading constraints, by requiring the strategy to lie pointwise in some set $C(\omega,t)$ depending on the state and time which corresponds to choosing

$$\mathfrak{C} = \Theta(C) := \{\vartheta \in \Theta \mid \vartheta(\omega,t) \in C(\omega,t) \text{ for all } (\omega,t) \in \Omega \times [0,T]\}.$$

Examples of interest include no-shortselling or no-borrowing constraints; see e.g. [21] or [55]. While MVPS and MVH have been extensively studied in the unconstrained case in various settings, research on these problems under constraints in continuous time covers only Itô process models so far; see [66], [49], [63], [53] and [34]. In all these works, the characterisations of the solution as well as the proof of its existence rely on specific features of the underlying model. However, it is one advantage of MVPS and MVH that these approaches allow a good description of the structure of the optimal strategy even in general semimartingale models. So we ask if it is possible to obtain more general results under constraints in these models as well, and this is the question we deal with in the first part (Chapters II–IV) of this thesis.

In the formulation (1.2) with linear $\mathfrak{C} = \Theta$, mean-variance portfolio selection is understood like in the classical Markowitz problem as a *static* optimisation *problem* in the sense that one determines the optimal strategy $\widetilde{\vartheta}$ for the entire time interval $[0,T]$ with respect to the (static) mean-variance criterion at time 0. From the description of the solution to (1.6), one can then obtain a *dynamic description* of $\widetilde{\vartheta}$ via (1.5) and (1.4) as well. To study (1.2) as a *dynamic* optimisation *problem*, one would in analogy to (1.7) consider the conditional problems to

$$\text{maximise } U_t(\vartheta) := E[V_T(x,\vartheta)|\mathcal{F}_t] - \frac{\gamma}{2}\operatorname{Var}[V_T(x,\vartheta)|\mathcal{F}_t] \text{ over all } \vartheta \in \Theta_t(\psi). \tag{1.8}$$

However, plugging in for ψ the optimal strategy $\widetilde{\vartheta}$ to (1.2) yields that, in contrast to (1.7), this family of conditional problem is no longer time consistent and that Bellman's optimality principle fails: If we use the solution $\widetilde{\vartheta}$ to (1.2) on $[0, t]$ and then determine the corresponding conditionally optimal strategy by maximising in (1.8) over all $\vartheta \in \Theta_t(\widetilde{\vartheta})$, then this strategy is different from $\widetilde{\vartheta}$ on $(t, T]$. This time inconsistency leads us to the basic question how to obtain a *time-consistent* formulation of MVPS, i.e. a dynamic formulation that gives a solution which is in some reasonable sense optimal for the conditional criterion $U_t(\cdot)$ and time consistent in the sense that if a strategy is optimal at time 0 for the entire time interval it is at time t also conditionally optimal on the remaining time interval $(t, T]$. This question is studied in Chapter V of this thesis. We remark that it depends of course on the preferences of the investor whether he would like to have a (so-called pre-commitment) strategy which involves dynamic trading and is optimal for the static mean-variance criterion evaluated at time 0, or a strategy $\widehat{\vartheta}$ which is optimal for the conditional mean-variance criterion in a dynamic and time-consistent sense. One can find in the literature justifications for both approaches.

I.2 Overview of the thesis

The results obtained in this thesis are divided into four chapters which correspond to the articles [23], [25], [24] and [22]. To keep each chapter self-contained, we deliberately allowed for redundancies and discuss the related literature in each chapter seperately.

Convex duality in mean-variance hedging under constraints. As in the unconstrained case where $\mathfrak{C} = \Theta$ is linear, the Markowitz problem under constraints can be tackled by solving the particular mean-variance hedging problem of approximating constant payoffs. So we focus first on mean-variance hedging under constraints, and study this problem in a general semimartingale model. The constraints are formulated via predictable correspondences, meaning that the trading strategy is restricted to lie in a given closed convex set $C(\omega, t)$ which may depend on the state ω and time t in a predictable way. It is worth pointing out that this is a very general formulation for mean-variance hedging under constraints. To obtain the existence of a solution for mean-variance hedging, we first establish the closedness in L^2 of the space of all final gains from trade. This is a first main contribution which allows us to subsequently tackle the problem in a systematic and unified way. In addition, using the closedness allows us to obtain more information on the structure of the solution by convex duality tools. This explains and generalises the convex duality results that have been obtained previously by other authors via ad hoc methods in specific frameworks.

Closed spaces of stochastic integrals with constrained integrands. It turns out that the closedness in L^2 of the space of all final gains from trade is related to the closedness, in the semimartingale topology, for spaces of stochastic integrals (as processes) with constrained, i.e. C-valued integrands. On this issue, we are able to make both mathematical contributions to stochastic calculus, and financial contributions in the modelling and handling of trading constraints for optimisation problems from mathematical finance. We provide necessary and sufficient conditions for the closedness in the semimartingale topology, which are in some sense definitive results. In most cases of economic interest, it is easy to verify that these conditions are satisfied. Moreover, spaces of stochastic integrals that are predictably convex often appear in applications to utility maximisation and superreplication under constraints. We show that such a space is closed in the semimartingale topology if and only if it is a space of stochastic integrals of C-valued integrands, where C is a predictable correspondence with closed and convex values. This result indicates why predictable correspondences come up naturally in this context, and the necessary and sufficient condition makes it again essentially definitive.

On the Markowitz problem under cone constraints. In this chapter, we understand mean-variance portfolio selection as in the classical Markowitz problem, i.e. like a static optimisation problem. Although this formulation fails to produce a time-consistent solution in the sense that the initially optimal strategy is still conditionally optimal for the analogous conditional criterion at a later time, this is nevertheless the usual way used in the literature to avoid dealing with the time-inconsistency of the mean-variance criterion. As in the unconstrained case, the solution to the Markowitz problem under constraints can be obtained by solving the particular mean-variance hedging problem of approximating in L^2 constant payoffs by the terminal gains of a self-financing trading strategy. To approach the latter task we slightly generalise results on the closedness in L^2 of the space of constrained terminal gains by combining the space of admissible trading strategies of Černý and Kallsen [14] with the generalisation of martingales, the so-called $\mathcal{E}$-martingales, introduced by Choulli, Krawczyk and Stricker [16]. Actually, $\mathcal{E}$-martingales come up naturally in quadratic optimisation problems in mathematical finance due to the possibly negative "marginal utility" of the square function which makes this generalisation necessary. The closedness we obtain is sufficient to provide the existence of solutions to the approximation problems if the constraints are in addition convex.

By treating the approximation problems as stochastic optimal control problems in a semimartingale framework, we obtain a factorisation of the value function involving two auxiliary processes. This is similar to the results on the opportunity process by Černý and Kallsen [14] in the unconstrained case, but due to the constraints now requires *two* opportunity processes. Combining the martingale optimality principle with the factorisation of the value function allows us to describe the optimal strategy locally in feedback form via the pointwise minimisers of two predictable functions, which are given in terms of the joint differential semimartingale characteristics of the opportunity processes and the asset price process. Conversely, assuming the existence of solutions to the approximation problems enables us to characterise the opportunity processes as the maximal solutions of two coupled backward stochastic differential equations (BSDEs) for which we also provide verification theorems. This explains and generalises all existing results on the Markowitz problem under cone constraints in the literature so far; compare [66], [49], [63] and [53]. In particular, the generality of our framework allows us to capture a new behaviour of the optimal strategy: It jumps from the minimiser of one predictable function to that of a second one whenever the optimal wealth process of the approximation problem changes its sign. Because this phenomenon is related to jumps in the price process of the underlying assets, it could not be observed in earlier work since the Markowitz problem under constraints has only been studied in (continuous) Itô process models so far. Of course, the presence of jumps and the resulting coupling of the BSDEs make the situation more involved, also at a technical level.

Time-consistent mean-variance portfolio selection. As already explained, Mean-variance portfolio selection is a time-inconsistent optimal control problem in the sense that it does not satisfy Bellman's optimality principle. Therefore the usual dynamic programming approach fails to produce a time-consistent dynamic formulation of the optimisation problem. To overcome this, one has to use a weaker optimality criterion which consists of optimising the strategy only locally in some sense. We propose such a local notion of optimality, called local mean-variance efficiency, for the conditional mean-variance problem. This generalises recent results by Basak and Chabakauri [6] and the examples on MVPS in Björk and Murgoci [9] who developed such a formulation in Markovian settings. By exploiting that framework, they could characterise the local notion of optimality by system of partial differential equations. To develop a time-consistent formulation in a general semimartingale setting we start in discrete time where this is straightforward, and then find the natural extension to continuous time which is similar to the formulation of local risk minimisation in continuous time introduced by Schweizer in [85]. As we shall see, our formulation in discrete as well as in continuous time embeds time-consistent mean-variance portfolio selection in a natural way into the already existing quadratic optimisation problems in mathematical finance, i.e. the Markowitz problem, mean-variance hedging, and local risk minimisation; compare [87] and [89]. Moreover, we provide an alternative characterisation of the optimal strategy in terms of the structure condition and the Föllmer–Schweizer decomposition of the mean-variance tradeoff, which gives necessary and sufficient conditions for the existence of a solution. The link to the Föllmer–Schweizer decomposition allows us to exploit known results to give a recipe to obtain the solution in concrete models. Since the ingredients for this recipe can be obtained directly from the canonical decomposition of the asset price process, this can be seen as the analogue to the explicit solution in the one-period case. Additionally, we obtain an intuitive interpretation of the optimal strategy as follows. On the one hand, the investor maximises the conditional mean-variance criterion in a myopic way one step ahead. In the multiperiod setting, this generates a risk represented by the mean-variance tradeoff process which he then minimises on the other hand by local risk minimisation. Finally, using the alternative characterisation of the optimal strategy allows us to justify the continuous-time formulation by showing that it coincides with the continuous-time limit of that in discrete time.

Chapter II

Convex duality in mean-variance hedging under constraints

II.1 Introduction

Mean-variance hedging and mean-variance portfolio selection are two classical problems in finance. The latter is also called Markowitz problem and involves finding a trading strategy whose resulting final wealth has an optimal risk-reward profile, where reward and risk are measured via mean and variance. Understanding and solving this problem is vastly simplified by a good knowledge about the general mean-variance or quadratic hedging problem. We study this in a general semimartingale financial market with general convex constraints on strategies.

In more mathematical terms, let $S = (S_t)_{0 \leq t \leq T}$ be an $\mathbb{R}^d$-valued semimartingale modelling the discounted prices of d risky assets. A self-financing trading strategy is described by its initial wealth $x \in \mathbb{R}$ and an $\mathbb{R}^d$-valued predictable process $\vartheta = (\vartheta_t)_{0 \leq t \leq T}$ describing the numbers of shares held dynamically over time. Its resulting final wealth is

$$V_T(x, \vartheta) := x + \int_0^T \vartheta_s \, dS_s =: x + G_T(\vartheta),$$

and if the $\mathcal{F}_T$-measurable random variable H gives the time-T payoff of a financial product, *mean-variance hedging* for H is to solve the (linear-quadratic control) problem to

$$\text{minimise } E\big[|H - x - G_T(\vartheta)|^2\big]$$

either over $\vartheta \in \Theta(C)$ for fixed x or over $(x, \vartheta) \in \mathbb{R} \times \Theta(C)$. The space $\Theta(C)$ of allowed integrands of course imposes a square-integrability condition on the stochastic integral process $\int \vartheta \, dS$, and the argument in brackets indicates that we have *trading constraints* in the sense that $\vartheta_t(\omega)$ must lie in a convex closed subset $C(\omega, t)$ of $\mathbb{R}^d$. This can depend on ω and t in a predictable way, as made precise later, and it is worth pointing out that the $C(\omega, t)$ need not be cones in general. One strength of our contribution is that the above setup is essentially the most general formulation for mean-variance hedging under constraints. Under very weak local square-integrability and no-arbitrage-type assumptions on S, we give in Theorem 3.12 a necessary and sufficient condition (jointly on S and C) for the space $G_T\big(\Theta(C)\big)$ to be closed in $L^2(P)$. This allows us to prove easily in Theorem 4.1 the *existence* of a solution to the mean-variance hedging problem for any $H \in L^2(P)$. To obtain more information on the *structure* of this solution, we then use *convex duality* tools. We introduce a dual problem for which variables and objective function both involve the constraints C through their support function. We then prove in Theorems 5.7 and 5.13 the existence of a solution to the dual problem, show how it is related to the solution of the primal problem, and give properties of the corresponding (primal and dual) value functions. There are two results because we give two formulations — one in terms of static, the other in terms of dynamic variables.

Conceptually and result-wise, our duality approach is analogous to the classical convex duality techniques familiar from utility maximisation problems; see the work by Cvitanić and Karatzas [21], Kramkov and Schachermayer [62] and Karatzas and Žitković [56]. However, the mathematics are a bit different since our "quadratic random utility" $U(x, \omega) = -\frac{1}{2}|x - H(\omega)|^2$ is not increasing in x and the duality is taken in a different space. A fairly close precursor of our work is due to Labbé and Heunis [63] who studied the same problem when S is given by a complete Itô process model and the constraints do not depend on ω and t. Their duality is very similar to parts of our Theorem 5.13, but their formulations and arguments strongly depend on the availability and use of Itô's representation theorem. We do not need that at all, since S and the underlying filtration $\mathbb{F}$ are general in our setting.

This chapter is structured as follows. Section II.2 contains a precise problem formulation, including basic results on correspondences that we use for modelling constraints. Section II.3 contains the central closedness result for $G_T\big(\Theta(C)\big)$, and Section II.4 uses this to prove existence of a solution to the mean-variance hedging problem under constraints. Finally, Section II.5 presents the duality results. We first give a careful motivation for the way the dual problem is set up, both for static and dynamic variables. Then we prove the main duality theorems in those two settings, and we close the section with more detailed comments on and comparison to the literature.

II.2 Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions of completeness and right-continuity, where $T > 0$ is a fixed and finite time horizon. Hence we can and do choose an RCLL version, i.e. right-continuous with left limits (RCLL), of every local P-martingale. For all unexplained notions concerning stochastic integration, we refer to Protter [80].

We consider a *financial market* consisting of one riskless asset, whose (discounted) price is 1, and d risky assets described by an $\mathbb{R}^d$-valued semimartingale S. We denote by $\mathcal{H}^2(P)$ the Banach space of all square-integrable semimartingales, i.e. special semimartingales Y with canonical decomposition $Y = Y_0 + M^Y + A^Y$, where M^Y is a square-integrable martingale Y null at zero, $M^Y \in \mathcal{M}_0^2(P)$, and A^Y is a predictable finite variation RCLL process null at zero, such that

$$\|Y\|_{\mathcal{H}^2(P)} := \|Y_0\|_{L^2(P)} + \|([M^Y]_T)^{\frac{1}{2}}\|_{L^2(P)} + \big\| \int_0^T |dA_s^Y| \big\|_{L^2(P)} < \infty.$$

Note that $\mathcal{H}^2_{loc}(P)$ coincides with the semimartingale space $\mathcal{S}^2_{loc}(P)$. We suppose that S is a locally square-integrable semimartingale, for short $S \in \mathcal{H}^2_{loc}(P)$, with canonical decomposition $S = S_0 + M + A$. Then there exists a predictable increasing RCLL process B, e.g. $B = \sum_{i=1}^d (\langle M^i \rangle + \int |dA^i|)$, with $\langle M^i, M^j \rangle \ll B$ and $A^i \ll B$ for $i, j = 1, \dots, d$. We define an $\mathbb{R}^{d\times d}$-valued predictable process c^M and an $\mathbb{R}^d$-valued predictable process a by $(c^M)^{ij} = \frac{d\langle M^i, M^j\rangle}{dB}$ and $a^i = \frac{dA^i}{dB}$. We set $\overline{\Omega} := \Omega \times [0, T]$, $P_B := P \otimes B$ and view $\mathbb{R}^d$-valued predictable processes as $\mathcal{P}$-measurable random variables, i.e. elements of $\mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d)$. For *trading strategies*, we take

$$\Theta := \Theta_S := \{\vartheta \in \mathcal{L}(S) \mid \int \vartheta \, dS \in \mathcal{H}^2(P)\},$$

where $\mathcal{L}(Y)$ denotes the space of all $\mathbb{R}^d$-valued, Y-integrable, predictable processes for a semimartingale Y. Note that we work with processes without identifying ϑ and ϑ' when $\int \vartheta \, dS = \int \vartheta' \, dS$; hence we write $\mathcal{L}(S)$, not $L(S)$. By the uniqueness of the canonical decomposition, we have that $\Theta_S = \mathcal{L}^2(M) \cap \mathcal{L}^2(A)$ with

$$\mathcal{L}^2(M) := \Big\{ \vartheta \in \mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d) \,\Big|\, \|\vartheta\|_{\mathcal{L}^2(M)} := \big(E\big[\int_0^T \vartheta_s^\top c_s^M \vartheta_s \, dB_s \big] \big)^{\frac{1}{2}} < \infty \Big\},$$
$$\mathcal{L}^2(A) := \Big\{ \vartheta \in \mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d) \,\Big|\, \|\vartheta\|_{\mathcal{L}^2(A)} := \big(E\big[\big(\int_0^T |\vartheta_s^\top a_s| \, dB_s \big)^2 \big] \big)^{\frac{1}{2}} < \infty \Big\}.$$

The wealth process generated up to time $t \in [0, T]$ by a self-financing trading strategy ϑ with initial capital $x \in \mathbb{R}$ is

$$V_t(x, \vartheta) := x + \int_0^t \vartheta_s \, dS_s =: x + G_t(\vartheta),$$

where the process $G(\vartheta)$ denotes the cumulative gains from trading. The set of all outcomes of self-financing trading strategies with zero initial wealth is

$$G_T(\Theta) = \{G_T(\vartheta) \,|\, \vartheta \in \Theta\},$$

and the set of attainable payoffs is

$$\mathcal{A}(\Theta) = \mathbb{R} + G_T(\Theta).$$

In contrast to strategies, we identify here final wealths that are equal P-a.s. Due to the definition of Θ, the sets $G_T(\Theta)$ and $\mathcal{A}(\Theta)$ are thus linear subspaces of $L^2(P)$ by the following result.

Proposition 2.1. *Let Y be in $\mathcal{H}^2(P)$ and $Y_t^* := \sup_{0\le s\le t} |Y_s|$. Then*

$$E\Big[\big(Y_T^*\big)^2\Big] \le 8\|Y\|^2_{\mathcal{H}^2(P)}.$$

Proof. See Theorem IV.5 in [80]. □

A square-integrable $\mathcal{F}_T$-measurable random variable H is called a *contingent claim*. We assume that an investor wants to hedge a contingent claim by means of a self-financing trading strategy. However, since the market is usually incomplete, perfect replication of the contingent claim, in the sense that $H = V_T(x, \vartheta)$ P-a.s. for some x and ϑ, is in general impossible. So the investor wants to optimise the hedging performance of his trading strategy according to some criterion. One possible choice, especially when the investor simultaneously considers buying or selling H, is the minimisation of the mean squared hedging error, which leads to the approximation problem

$$E\left[\left|H - x - \int_0^T \vartheta_s \, dS_s\right|^2\right] = \min_{(x,\vartheta)\in\mathbb{R}\times\Theta}!$$

For a fixed initial capital $x \in \mathbb{R}$, one obtains the problem of *mean-variance hedging*, i.e.

$$E\left[\left|H - x - \int_0^T \vartheta_s \, dS_s\right|^2\right] = \min_{\vartheta\in\Theta}!$$

Mathematically, this amounts to projecting $H - x$ onto $G_T(\Theta)$ or H onto $\mathcal{A}(\Theta)$. Therefore a solution for every $H \in L^2(P)$ exists if and only if $G_T(\Theta)$ and $\mathcal{A}(\Theta)$ are closed in $L^2(P)$. Note that both problems are naturally studied in $L^2(P)$ rather than $\mathcal{L}^2(P)$.

Before we introduce the mean-variance hedging problem under *trading constraints*, we make the following simple observation. In the unconstrained case, where $G_T(\Theta)$ and $\mathcal{A}(\Theta)$ are closed linear subspaces, the problem admits a unique solution by elementary Hilbert space arguments. Under trading constraints, this is still true if the subsets in which we want to find the best approximation are closed and convex subsets of $L^2(P)$. Despite its simplicity, this observation is very useful. We shall see that mean-variance hedging problems can be embedded into this framework even under the additional constraint that the trading strategy only takes values in a closed convex set, which is allowed to depend on the state ω and time t in a predictable way. This allows us to treat these problems in a systematic and unified way.

To model *"predictable trading constraints"*, we formulate them via predictable correspondences. This idea is analogous to [54], where it is used to study the existence of the numéraire portfolio under predictable convex constraints.

Definition 2.2. A mapping $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ is called a *correspondence*. We say that a correspondence C is *predictable* if $C^{-1}(F) = \{(\omega, t) \,|\, C(\omega, t) \cap F \neq \emptyset\}$ is a predictable set (i.e. in $\mathcal{P}$) for all closed $F \subseteq \mathbb{R}^d$. The *domain* $\mathrm{dom}(C)$ of a correspondence is given by $\mathrm{dom}(C) = \{(\omega, t) \,|\, C(\omega, t) \neq \emptyset\}$. A *(predictable) selector* of a (predictable) correspondence C is a (predictable) process ψ with $\psi(\omega, t) \in C(\omega, t)$ for all $(\omega, t) \in \mathrm{dom}(C)$.

For convenience, we recall some results on predictable correspondences which ensure the existence of predictable selectors in all situations relevant for us.

Proposition 2.3 (Castaing)**.** *For a correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ with closed values, the following are equivalent:*

1) *C is predictable.*

2) $\mathrm{dom}(C)$ *is predictable and there exists a* Castaing representation *of C, i.e. a sequence (ψ^n) of predictable selectors of C such that*

$$C(\omega, t) = \overline{\{\psi^1(\omega, t), \psi^2(\omega, t), \dots\}} \qquad \textit{for each } (\omega, t) \in \mathrm{dom}(C).$$

Proof. See Corollary 18.14 in [2] or Theorem 1B in [83]. □

Proposition 2.4. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed values and $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^d$ and $g : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}^m$* Carathéodory functions*, which means that $f(\omega, t, y)$ and $g(\omega, t, x)$ are predictable with respect to (ω, t) and continuous in y and x. Then C' and C'' given by*

$$C'(\omega, t) = \{y \in \mathbb{R}^m \,|\, f(\omega, t, y) \in C(\omega, t)\}$$

and

$$C''(\omega, t) = \overline{\{g(\omega, t, x) \,|\, x \in C(\omega, t)\}}$$

are predictable correspondences (from $\overline{\Omega}$ to $2^{\mathbb{R}^m}$) with closed values.

Proof. See Corollaries 1P and 1Q in [83]. □

Proposition 2.5. *Let $C^n : \overline{\Omega} \to 2^{\mathbb{R}^d}$ for each $n \in \mathbb{N}$ be a predictable correspondence with closed values and define the correspondences C' and C'' by $C'(\omega,t) = \bigcap_{n\in\mathbb{N}} C^n(\omega,t)$ and $C''(\omega,t) = \bigcup_{n\in\mathbb{N}} C^n(\omega,t)$. Then C' and C'' are predictable and C' is closed-valued.*

Proof. See Theorem 1M in [83] and Lemma 18.4 in [2]. □

For a predictable correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$, we denote by

$$\mathcal{C} := \mathcal{C}^S := \{\psi \in \mathcal{L}(S) \mid \psi(\omega,t) \in C(\omega,t) \text{ for all } (\omega,t) \in \overline{\Omega}\}$$

the set of *C-valued* or *C-constrained integrands* for S and by

$$\Theta(C) = \Theta \cap \mathcal{C} = \{\vartheta \in \Theta \,|\, \vartheta(\omega,t) \in C(\omega,t) \text{ for all } (\omega,t) \in \overline{\Omega}\}$$

the set of all *C-constrained trading strategies*. With this formulation, the set of all outcomes of C-constrained self-financing trading strategies with zero initial wealth is

$$G_T(\Theta(C)) = \{G_T(\vartheta) \,|\, \vartheta \in \Theta(C)\},$$

and the set of payoffs that are attainable with C-constrained trading strategies is

$$\mathcal{A}(\Theta(C)) = \mathbb{R} + G_T(\Theta(C)).$$

Note that one can for example model prohibition of short-selling or rectangular constraints in this formulation; see Examples 4.1 in Section 5.4 of [55]. The *mean-variance hedging problem under trading constraints* is then formulated as

$$E\left[\left|H - x - \int_0^T \vartheta_s \, dS_s\right|^2\right] = \min_{\vartheta\in\Theta(C)}!$$

for a fixed initial capital $x \in \mathbb{R}$, and we also study

$$E\left[\left|H - x - \int_0^T \vartheta_s \, dS_s\right|^2\right] = \min_{(x,\vartheta)\in\mathbb{R}\times\Theta(C)}!$$

when including the initial capital into the approximation problem. As already explained above, these problems admit solutions if $G_T(\Theta(C))$ and $\mathcal{A}(\Theta(C))$ are closed and convex subsets of $L^2(P)$. By the convexity of C, the convexity of $G_T(\Theta(C))$ and $\mathcal{A}(\Theta(C))$ immediately follows. The closedness will be established in the next section.

II.3 The closedness of $G_T(\Theta(C))$ and $\mathcal{A}(\Theta(C))$

In this section, we show that the set of all outcomes of C-constrained self-financing trading strategies with zero initial wealth and the set of all payoffs that are attainable with C-constrained trading strategies are both closed in

$L^2(P)$. To that end, we use the concept of $\mathcal{E}$-martingales, which was introduced and developed by Choulli, Krawczyk and Stricker in [16] to deduce the closedness of the analogous subspaces in the unconstrained case. For easy reference, we start by briefly recalling some definitions and results.

For a semimartingale Y, we denote its stochastic exponential by $\mathcal{E}(Y)$. *Throughout this chapter, let N be a fixed local P-martingale starting at zero.* For any stopping time τ, we denote the process Y stopped at τ by Y^τ and the process Y started at τ by ${}^\tau Y = Y - Y^\tau$, but we set ${}^\tau\mathcal{E} = {}^\tau\mathcal{E}(N) = \mathcal{E}(N - N^\tau)$. So for the stochastic exponential, ${}^\tau\mathcal{E}(N)$ denotes a multiplicative rather than an additive restarting. In the sequel, we use the symbol $\mathcal{E}(N)$ (or even $\mathcal{E}$) for the family $\{{}^\tau\mathcal{E}(N) \,|\, \tau \text{ stopping time}\}$ of processes, rather than for the process ${}^0\mathcal{E}(N)$. Since N is RCLL, it has at most a finite number of jumps with $\Delta N = -1$, and so each ${}^\tau\mathcal{E}(N)$ has P-a.s. at most a finite number of times where it can jump to zero; this follows from the representation of the stochastic exponential in Theorem II.37 in [80]. Therefore we can define an increasing sequence of stopping times by $\widehat{T}_0 = 0$ and $\widehat{T}_{n+1} = \inf\{t > \widehat{T}_n \,|\, {}^{\widehat{T}_n}\mathcal{E}(N)_t = 0\} \wedge T$.

Definition 3.1. An adapted RCLL process Y is an *$\mathcal{E}$-local martingale* if the product of ${}^{\widehat{T}_n}Y$ and ${}^{\widehat{T}_n}\mathcal{E}$ is a local P-martingale for any $n \in \mathbb{N}$. It is an *$\mathcal{E}$-martingale* if for any $n \in \mathbb{N}$, we have $E\big[|Y_{\widehat{T}_n}\,{}^{\widehat{T}_n}\mathcal{E}_{\widehat{T}_{n+1}}|\big] < \infty$ and the above product is a (true) P-martingale.

The next two propositions, which are Corollaries 3.16 and 3.17 in [16], give some information about the structure of $\mathcal{E}$-martingales.

Proposition 3.2. *Let Y be a special semimartingale with canonical decomposition $Y = Y_0 + M^Y + A^Y$. Then Y is an $\mathcal{E}$-local martingale if and only if $[M^Y, N]$ is locally P-integrable and $A^Y = -\langle M^Y, N\rangle$.*

Proposition 3.3. *A semimartingale $Y = Y_0 + M^Y - \langle M^Y, N\rangle$ satisfying $E\big[Y_T^*({}^{\widehat{T}_n}\mathcal{E})_T^*\big] < \infty$ for any $n \in \mathbb{N}$ is an $\mathcal{E}$-martingale.*

We also need the following definitions.

Definition 3.4. We say that $\mathcal{E}$ is *regular* if ${}^{\widehat{T}_n}\mathcal{E}$ is a P-martingale for any n.

Definition 3.5. We say that $\mathcal{E}$ satisfies *the reverse Hölder inequality $R_2(P)$* if there exists a constant $c \geq 1$ such that $E\big[|{}^t\mathcal{E}_T|^2 \big| \mathcal{F}_t\big] \leq c$ for any t.

The next proposition is a partial statement of Proposition 3.9 in [16].

Proposition 3.6. *Assume that $\mathcal{E}$ satisfies $R_2(P)$. Then $\mathcal{E}$ is regular if and only if ${}^\tau\mathcal{E}$ is a P-martingale for any stopping time τ, and in that case, ${}^\tau\mathcal{E}$ is a P-square-integrable P-martingale.*

Finally, a combination of Theorem 4.9 in [16] and Proposition 2.1 gives the following equivalence of norms.

Proposition 3.7. *Assume that $\mathcal{E}$ is regular and satisfies $R_2(P)$. Then there exists a constant c such that*

$$\frac{1}{c}\|Y\|_{\mathcal{H}^2(P)} \leq \|Y_T\|_{L^2(P)} \leq c\|Y\|_{\mathcal{H}^2(P)}$$

for every $\mathcal{E}$-martingale Y. We write this for short as $\|Y\|_{\mathcal{H}^2(P)} \sim \|Y_T\|_{L^2(P)}$.

Note that when ${}^0\mathcal{E}(N)$ is a strictly positive P-martingale, the definition of an $\mathcal{E}$-local martingale coincides with the notion of a local martingale under the measure Q defined by $dQ = {}^0\mathcal{E}(N)_T\,dP$. This will be called the classical case.

As explained in the previous section, we consider a possibly incomplete financial market composed of one riskless asset, whose price is 1, and d risky assets described by an $\mathbb{R}^d$-valued semimartingale $S \in \mathcal{H}^2_{loc}(P)$ with canonical decomposition $S = S_0 + M + A$. We suppose that there exists $N \in \mathcal{M}_{0,loc}(P)$ such that S is an $\mathcal{E}$-local martingale. By Proposition 3.2, this implies that $\langle M, N\rangle$ exists and $A = -\langle M, N\rangle$. Moreover, we assume that $\mathcal{E}(N)$ satisfies $R_2(P)$, which gives that N is locally P-square integrable and in bmo_2, i.e. there exists a constant $c > 0$ such that $E[\langle N\rangle_T - \langle N\rangle_t|\mathcal{F}_t] \leq c$ for all $t \in [0, T]$; see Proposition 3.10 in [16]. An application of the Kunita–Watanabe decomposition yields $N = -\int \lambda\,dM + L$ with $\lambda \in L^2(M)$ and $L \in \mathcal{M}^2_0(P)$ strongly P-orthogonal to M, and hence S satisfies the *structure condition (SC)*, i.e.

$$S = S_0 + M + \int d\langle M\rangle\lambda.$$

Since N is in bmo_2, $\int \lambda\,dM$ is also in bmo_2, which implies by Theorem 3.3 in [29] the inequality $D_2(P)$, i.e. there exists a constant $c > 0$ such that $\|\vartheta\|_{\mathcal{L}^2(A)} \leq c\|\vartheta\|_{\mathcal{L}^2(M)}$ for all $\vartheta \in \mathcal{L}^2(M)$. As a consequence, we have $\Theta = \mathcal{L}^2(M)$. To motivate the closedness proof under trading constraints, we give below the argument for the unconstrained case, which is due to Choulli, Krawczyk and Stricker; see Theorem 5.2 in [16].

Proposition 3.8. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $S \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Then the following hold:*

1) *For each σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$ and each $Y_0 \in L^2(\mathcal{B}_0)$, the process $Y_0 + \int \vartheta\,dS$ in $\mathcal{H}^2(P)$ is an $\mathcal{E}$-martingale.*

2) *The spaces $G_T(\Theta)$, $\mathcal{A}(\Theta)$ and $L^2(\mathcal{B}_0) + G_T(\Theta)$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$.*

Proof. 1) The stochastic integral $\int \vartheta\,dS$ is for each $\vartheta \in \Theta$ in $\mathcal{H}^2(P)$ and hence special with canonical decomposition $\int \vartheta\,dS = \int \vartheta\,dM + \int \vartheta\,dA$. Since S is an $\mathcal{E}$-local martingale, we have $A = -\langle M, N\rangle$ by Proposition 3.2; so

$\int \vartheta\, dA = -\langle \int \vartheta\, dM, N\rangle$ and therefore $\int \vartheta\, dS$ is an $\mathcal{E}$-local martingale again by Proposition 3.2. Since $\mathcal{E}$ is regular and satisfies $R_2(P)$, Proposition 3.6 states that ${}^\tau\mathcal{E}$ is a square-integrable martingale for each stopping time τ. By Doob's inequality and Proposition 2.1, $\{{}^\tau\mathcal{E}\}^*_T$ and $\{G(\vartheta)\}^*_T$ are in $L^2(P)$ so that $\{G(\vartheta)\}^*_T\{{}^\tau\mathcal{E}\}^*_T$ is in $L^1(P)$ for every stopping time τ. Proposition 3.3 now implies that $G(\vartheta)$ is an $\mathcal{E}$-martingale. Replacing $G(\vartheta)$ by Y_0 shows in the same way that the constant process Y_0 is an $\mathcal{E}$-martingale for any $Y_0 \in L^2(\mathcal{B}_0)$, and hence so is $Y_0 + G(\vartheta)$.

2) Let $\big(Y_0^n + G_T(\vartheta^n)\big)$ be a sequence in $L^2(\mathcal{B}_0) + G_T(\Theta)$ converging to H in $L^2(P)$. By part 1), each $Y_0^n + G(\vartheta^n)$ is an $\mathcal{E}$-martingale and therefore the sequence $\big(Y_0^n + G(\vartheta^n)\big)$ is a Cauchy sequence in the Banach space $\mathcal{H}^2(P)$ by Proposition 3.7, hence convergent to some $Y \in \mathcal{H}^2(P)$ which satisfies $Y_T = H$. Since the space (of processes) $Y_0 + G(\Theta)$ is closed in $\mathcal{H}^2(P)$ (either by the construction of the stochastic integral as in Section IV.2 in [80] or by Theorem V.4 in [71]), there exists some $\vartheta \in \Theta$ with $Y = Y_0 + G(\vartheta)$, and therefore $L^2(\mathcal{B}_0) + G_T(\Theta)$ is closed in $L^2(P)$. Choosing above $\mathcal{B}_0 = \{\emptyset, \Omega\}$ and $Y_0^n = 0$ for all $n \in \mathbb{N}$ then implies the closedness of $\mathcal{A}(\Theta)$ and $G_T(\Theta)$ in $L^2(P)$, which completes the proof. □

Remark 3.9. Assuming that there exists $N \in \mathcal{M}_{0,loc}(P)$ such that $\mathcal{E}(N)$ is regular and P-square-integrable and such that S is an $\mathcal{E}$-local martingale implies the weak no-arbitrage condition that $G_T(\Theta)$ admits *no approximate profits in* L^2; this means that $1 \notin \overline{G_T(\Theta)}$, where $\overline{}$ denotes the closure in $L^2(P)$. See Section 4 in [87].

To obtain the closedness under constraints, we observe the following. If $\big(Y_0^n + G_T(\vartheta^n)\big)$ is a sequence in $L^2(\mathcal{B}_0) + G_T(\Theta(C))$ converging to some H in $L^2(P)$, then there exist under the assumptions of Proposition 3.8 some $Y_0 \in L^2(\mathcal{B}_0)$ and some $\vartheta \in \Theta$ such that $Y_0 + G_T(\vartheta) = H$ and $\big(Y_0^n + G(\vartheta^n)\big)$ converges to $Y_0 + G(\vartheta)$ even in $\mathcal{H}^2(P)$. The question is then whether ϑ can be chosen to be C-valued. In general (even if S is a martingale), the answer is negative, as a simple counterexample in Section 3 of [25] illustrates. The reason behind this is that the linear dependence of the different components of S can make some of the risky assets *redundant* in the sense that one of them can be replicated on some predictable set by trading in the other ones. As a consequence, there may exist different strategies with the same gains process, and so a trading strategy is not uniquely determined (even up to indistinguishability) by its gains process. Indeed, by the construction of the stochastic integral, two strategies ψ and φ in Θ have the same gains process up to indistinguishability if and only if $c^M(\psi - \varphi) = 0$ and $a^\top(\psi - \varphi) = 0$ P_B-a.e.; see Lemma 5.1 in Chapter III. Therefore the convergence of the gains processes $G(\vartheta^n)$ need not imply the pointwise convergence of the strategies ϑ^n, and so the pointwise closedness of $C(\omega, t)$ is not sufficient to deduce that ϑ can be chosen to be C-valued. However, the convergence of the gains processes is sufficient to prove that we do get a convergent sequence (ψ^n) of modified strategies which have the same gains processes as the strategies ϑ^n. This sequence is given by the projection of (ϑ^n) on the predictable range of S.

Proposition 3.10. *For each $\mathbb{R}^d$-valued semimartingale Y, there exists an $\mathbb{R}^{d\times d}$-valued predictable process Π^Y, called the* projection on the predictable range of Y*, which takes values in the orthogonal projections in $\mathbb{R}^d$ and has the following property: If $\vartheta \in \mathcal{L}(Y)$ and φ is predictable, then φ is in $\mathcal{L}(Y)$ with $\int \varphi dY = \int \vartheta dY$ (up to indistinguishability) if and only if $\Pi^Y \vartheta = \Pi^Y \varphi$ P_B-a.e. We choose and fix one version of Π^Y.*

Proof. See Lemma 5.3 in Chapter III. □

Remark 3.11. Suppose that S satisfies (SC). Then the construction of the stochastic integral gives that $\int \psi dS = 0$ if and only if $\int \psi\, dM = 0$ and therefore that Π^S and Π^M coincide (P_B-a.e.). Since $\int \psi\, dM = 0$ if and only if $c^M \psi = 0$ P_B-a.e., we see that $\psi = \vartheta - \Pi^M \vartheta$ is P_B-a.e. valued in $\mathrm{Ker}(c^M)$, the kernel of the matrix c^M. Moreover, since Π^M is an orthogonal projection, so is $\mathbb{1}_{d\times d} - \Pi^M$, and therefore $\Pi^M = \Pi^S$ can be chosen as the pointwise orthogonal projection on $\mathrm{range}(c^M(\omega,t))$, the range of the matrix $c^M(\omega,t)$, which is equal to $\mathrm{Ker}(c^M(\omega,t))^\perp$.

Using the projection on the predictable range, we can now prove the closedness in the constrained case.

Theorem 3.12. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that*
$S \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed values and such that $\Theta(C)$ is non-empty. Then the spaces $G_T\big(\Theta(C)\big)$, $\mathcal{A}(\Theta(C))$ and $L^2(\mathcal{B}_0) + G_T(\Theta(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$ if and only if the projection of C on the predictable range of S is closed, i.e. $\Pi^S(\omega,t)C(\omega,t)$ is closed P_B-a.e.

Proof. "$\Leftarrow$": Let $\big(Y_0^n + G_T(\vartheta^n)\big)$ be a sequence in $L^2(\mathcal{B}_0) + G_T(\Theta(C))$ converging to H in $L^2(P)$. Then there exist some $Y_0 \in L^2(\mathcal{B}_0)$ and some $\vartheta \in \Theta$ such that $Y_0 + G_T(\vartheta) = H$ P-a.s. and $\big(Y_0^n + G(\vartheta^n)\big)$ converges to $Y_0 + G(\vartheta)$ in $\mathcal{H}^2(P)$, by the proof of Proposition 3.8. The convergence in $\mathcal{H}^2(P)$ implies the convergence in the semimartingale topology by Theorem V.14 and the lemma preceding Theorem IV.12 in [80]. By Theorem 4.5 in Chapter III, the space of stochastic integrals of C-valued integrands is closed in the semimartingale topology if the projection of C on the predictable range of S is closed. Thus there exists $\widetilde{\vartheta} \in \Theta(C)$ such that $G(\widetilde{\vartheta}) = G(\vartheta)$, and therefore $L^2(\mathcal{B}_0) + G_T(\Theta(C))$ is closed in $L^2(P)$. As in the proof of Proposition 3.8, choosing $\mathcal{B}_0 = \{\emptyset, \Omega\}$ and $Y_0^n = 0$ for all $n \in \mathbb{N}$ gives the closedness of $\mathcal{A}(\Theta)$ and $G_T(\Theta)$.

"$\Rightarrow$": First note that for any stopping time τ, the projection Π^{S^τ} on the predictable range of S^τ is simply $\Pi^S \mathbb{1}_{[\![0,\tau]\!]}$. Recall that $S \in \mathcal{H}^2_{loc}(P)$. Arguing by contradiction, we choose a stopping time τ such that S^τ is in $\mathcal{H}^2(P)$ and Π^{S^τ} is not closed. Applying Lemma 4.4 in Chapter III with S^τ and using that $\int \varphi \, dS^\tau = \int \varphi \mathbb{1}_{[\![0,\tau]\!]} \, dS$ for any $\varphi \in \mathcal{L}(S)$ implies that there exist $\vartheta \in \mathcal{L}(S)$ and a sequence (ψ^n) of C-valued integrands such that $(\int \psi^n \mathbb{1}_{[\![0,\tau]\!]} \, dS)$ converges to $\int \vartheta \mathbb{1}_{[\![0,\tau]\!]} \, dS$ in the semimartingale topology, but there is no C-valued integrand ψ such that $\int \psi \mathbb{1}_{[\![0,\tau]\!]} \, dS = \int \vartheta \mathbb{1}_{[\![0,\tau]\!]} \, dS$. An inspection of the proof of Lemma 4.4 in Chapter III shows that we can choose ϑ and (ψ^n) such that $(\Pi^S \vartheta)\mathbb{1}_{[\![0,\tau]\!]}$ and $(\Pi^S \psi^n)\mathbb{1}_{[\![0,\tau]\!]}$ are uniformly bounded and $(\Pi^S \psi^n)\mathbb{1}_{[\![0,\tau]\!]} \to (\Pi^S \vartheta)\mathbb{1}_{[\![0,\tau]\!]}$ uniformly in (ω, t). Since $\int \psi^n \mathbb{1}_{[\![0,\tau]\!]} \, dS = \int (\Pi^S \psi^n)\mathbb{1}_{[\![0,\tau]\!]} \, dS$ and $\int \vartheta \mathbb{1}_{[\![0,\tau]\!]} \, dS = \int (\Pi^S \vartheta)\mathbb{1}_{[\![0,\tau]\!]} \, dS$, we have by dominated convergence that $\int (\psi^n \mathbb{1}_{[\![0,\tau]\!]} + \varphi \mathbb{1}_{]\!]\tau,T]\!]}) \, dS \to \int \widetilde{\vartheta} \, dS$ in $\mathcal{H}^2(P)$ for any $\varphi \in \Theta(C)$, with $\widetilde{\vartheta} = \vartheta \mathbb{1}_{[\![0,\tau]\!]} + \varphi \mathbb{1}_{]\!]\tau,T]\!]}$, and hence also that $G_T(\psi^n \mathbb{1}_{[\![0,\tau]\!]} + \varphi \mathbb{1}_{]\!]\tau,T]\!]}) \to G_T(\widetilde{\vartheta})$ in $L^2(P)$ by Proposition 2.1. But because there exists by construction of $\widetilde{\vartheta}$ no C-valued integrand ψ with $G(\psi) = G(\widetilde{\vartheta})$ and since $G(\widetilde{\vartheta})$ is uniquely determined in $\mathcal{H}^2(P)$ by its terminal value $G_T(\widetilde{\vartheta})$ by Proposition 3.7, there cannot be any $\psi \in \Theta(C)$ with $G_T(\psi) = G_T(\widetilde{\vartheta})$. This contradicts the closedness of $G_T\big(\Theta(C)\big)$ in $L^2(P)$ and therefore completes the proof. □

For a better understanding of our assumptions, we now spell them out in a multidimensional Itô process model. This is one standard example of a financial market, and it illustrates that our assumptions are weaker than those in [63], [49] and [53].

Example 3.13. Let W be an $\mathbb{R}^n$-valued Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a filtration satisfying the usual conditions. Note that for our results, $\mathbb{F}$ need not be the P-augmentation of the filtration generated by W; we do not use Itô's representation theorem. Let $\bar{S} = (\bar{S}^i)_{i=1,\dots,d}$ be the undiscounted price processes of the d risky assets and $\bar{S}^0$ the undiscounted price of the bank account. These processes are given as the solutions to the stochastic differential equations (SDEs)

$$d\bar{S}^i_t = \bar{S}^i_t \left(\mu^i_t \, dt + \sum_{j=1}^n \sigma^{ij}_t \, dW^j_t \right), \qquad \bar{S}^i_0 = s^i_0$$

for $i = 1, \dots, d$ with constants $s^i_0 > 0$ and

$$d\bar{S}^0_t = \bar{S}^0_t r_t \, dt, \qquad \bar{S}^0_0 = 1$$

with predictable $\mathbb{R}^d$-, $\mathbb{R}$- and $\mathbb{R}^{d \times n}$-valued processes μ, r and σ that are P-a.s. on $[0, T]$ Lebesgue-integrable and Lebesgue-square-integrable, respectively. In our abstract setup, we consider the discounted prices $S^i = \bar{S}^i / \bar{S}^0$.

The SDEs for the S^i then take the form

$$dS_t^i = S_t^i \left((\mu_t^i - r_t)\,dt + \sum_{j=1}^n \sigma_t^{ij}\,dW_t^j \right), \qquad S_0^i = s_0^i,$$

and we explicitly have

$$\begin{aligned} d\langle M\rangle_t &= \operatorname{diag}(S_t)\sigma_t\sigma_t^\top \operatorname{diag}(S_t)\,dt =: c_t^M\,dt, \\ dA_t &= \operatorname{diag}(S_t)(\mu_t - r_t\mathbf{1})\,dt =: a_t\,dt \end{aligned}$$

with $\mathbf{1} = (1\dots 1)^\top \in \mathbb{R}^d$. Up to integrability conditions on μ, r and σ, the process S satisfies (SC) if and only if $(\mu - r\mathbf{1}) \in \operatorname{range}(\sigma\sigma^\top)$ $dP \otimes dt$-a.e., since $S_t^i > 0$ for $i = 1, \dots, d$. For a sufficient condition for this, we assume that $\sigma\sigma^\top$ is $dP \otimes dt$-a.e. invertible, which means that $n \geq d$ and that σ has $dP \otimes dt$-a.e. full rank d. This condition also implies that $\Pi^S = \mathbb{1}_{d\times d}$ and therefore that the projection of *any* closed-valued predictable correspondence C on the predictable range of S is closed. A natural candidate to obtain a local martingale N such that S is an $\mathcal{E}(N)$-martingale is $N = -\int \lambda\,dM = -\int \varphi\,dW$, where $\varphi = \sigma^\top\lambda = \sigma^\top(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})$ is the *instantaneous market price of risk*. Here we make the frequently used assumption that the *mean-variance tradeoff (MVT) process*

$$K_t := \int_0^t \lambda_s^\top d\langle M\rangle_s \lambda_s = \left\langle \int \lambda\,dM \right\rangle_t = \left\langle \int \varphi\,dW \right\rangle_t = \int_0^t |\varphi_s|^2 ds,$$

which coincides in this setup with the integrated squared market price of risk, is uniformly bounded in t and ω. This is sufficient to guarantee that $\mathcal{E}(-\int \lambda\,dM)$ is a true martingale and satisfies $R_2(P)$ by Proposition 3.7 in [16]. As M is continuous, $\mathcal{E}(-\int \lambda\,dM)$ is strictly positive and $d\widehat{P} = \mathcal{E}(-\int \lambda\,dM)_T\,dP$ defines an equivalent local martingale measure (ELMM) for the process S, the so-called *minimal martingale measure*; see [87].

Thus we can conclude that if the MVT process K is uniformly bounded in t and ω and $\sigma\sigma^\top$ is $dP \otimes dt$-a.e. invertible, the assumptions of Theorem 3.12 are satisfied and $G_T(\Theta(C))$ and $\mathcal{A}(\Theta(C))$ are closed in $L^2(P)$ for *all* closed-valued, predictable correspondences C. If we suppose in addition that $\bar{S}_T^0$ and $1/\bar{S}_T^0$ are in $L^\infty(P)$, which holds for instance if r is uniformly bounded in t and ω, then also the corresponding sets $\bar{S}_T^0 G_T(\Theta(C))$, $\bar{S}_T^0\big(x + G_T(\Theta(C))\big)$ and $\bar{S}_T^0\mathcal{A}(\Theta(C))$ of undiscounted payoffs attainable with constrained trading strategies considered in [63], [49] and [53] are closed in $L^2(P)$.

Our assumptions are clearly far less restrictive than completeness of the (unconstrained) financial market. The latter is imposed in [63] and [49] by the conditions that μ, r and σ are uniformly bounded in t and ω, that σ^{-1} exists and is uniformly bounded in t and ω as well, and that $\mathbb{F}$ is the P-augmentation of the filtration generated by W. The last two assumptions allow to use Itô's representation theorem and then rewrite integrals of W as integrals of S.

In [63] and [49], the constraints are not formulated in terms of *number of shares* ϑ^i, but in terms of the *cash amounts* $\pi^i := \vartheta^i \bar{S}^i$ invested *in the different assets*. To see that this can also be handled in our formulation, let C^π be a closed-valued predictable correspondence which describes constraints on the cash amounts. Extending [63] and [49], this need not be deterministic. Since $\bar{S}^i > 0$, we can define the correspondence C^ϑ by $C^\vartheta(\omega,t) := \operatorname{diag}\big(\bar{S}^i(\omega,t)\big)^{-1} C^\pi(\omega,t)$, which is by Proposition 2.4 again a closed-valued predictable correspondence and describes by definition the same constraints as C^π, but in number of shares. Alternatively, we can consider the dynamics of the gains process parametrised in cash amounts, i.e.

$$\begin{aligned} dG_t(\vartheta) &= \vartheta_t^\top \operatorname{diag}(S_t)(\mu_t - r_t\mathbf{1})\,dt + \vartheta_t^\top \operatorname{diag}(S_t)\sigma_t\,dW_t \\ &= \pi_t^\top \frac{1}{\bar{S}_t^0}(\mu_t - r_t\mathbf{1})\,dt + \pi_t^\top \frac{1}{\bar{S}_t^0}\sigma_t\,dW_t = \pi_t^\top\,dX_t, \\ G_0(\vartheta) &= G_0(\pi) = 0 \end{aligned}$$

with the discounted returns process

$$dX_t = \frac{1}{\bar{S}_t^0}(\mu_t - r_t\mathbf{1})\,dt + \frac{1}{\bar{S}_t^0}\sigma_t\,dW_t, \qquad X_0 = \mathbf{1}$$

as integrator. Then we can apply our results to the stochastic integrals $\int \pi\,dX$ with

$$\pi \in \Theta_X(C^\pi) := \big\{\pi \in \mathcal{L}(X) \,\big|\, \textstyle\int \pi\,dX \in \mathcal{H}^2(P) \text{ and } \pi(\omega,t) \in C^\pi(\omega,t) \text{ for all } (\omega,t) \in \overline{\Omega}\big\}$$

rather than $\int \vartheta\,dS$ with $\vartheta \in \Theta(C^\vartheta)$ to obtain that the set $G_T\big(\Theta_X(C^\pi)\big) = G_T\big(\Theta_S(C^\vartheta)\big)$ is closed in $L^2(P)$. In this parametrisation, each (undiscounted) payoff in $\bar{S}_T^0\big(x + G_T\big(\Theta_X(C^\pi)\big)\big)$ is the final value $\bar{V}_T(x,\pi)$ of the wealth process of a self-financing trading strategy, where $\bar{V}(x,\pi)$ is given by the solution of the SDE

$$d\bar{V}_t(x,\pi) = \big(\bar{V}_t(x,\pi)r_t + \pi_t^\top(\mu_t - r_t\mathbf{1})\big)\,dt + \pi_t^\top\sigma_t\,dW_t, \qquad \bar{V}_0(x,\pi) = x.$$

For no short-selling constraints, i.e. $C^\pi = [0,+\infty)^d$, Jin and Zhou in [53] do not (need to) assume invertibility of $\sigma\sigma^\top$ to obtain a solution to the constrained Markowitz problem. The reason behind this is that as $[0,+\infty)^d$ is a polyhedral set, all its projections are closed. Of course, our results cover this case as well.

To obtain (the existence of) a solution to the mean-variance hedging problem under constraints, we assume in addition to the conditions of Theorem 3.12 that $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ also takes convex values. This gives the following relations to *predictable convexity* and *stability* which come up naturally in dynamic portfolio optimisation problems. The notion of predictable convexity was introduced in [44] to obtain an optional decomposition theorem under constraints, and stability of a set of strategies is usually assumed to establish a dynamic programming principle. The next result and its proof are inspired by Theorems 3 and 4 in [28].

Proposition 3.14. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $S \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $\mathfrak{C} \subseteq \Theta$ be non-empty and such that $G_T(\mathfrak{C})$ is closed in $L^2(P)$. Then the following are equivalent:*

1) *The set $\mathfrak{C}$ is* predictably convex, *i.e. for all ϑ and φ in $\mathfrak{C}$ and all $[0,1]$-valued predictable processes k, the strategy $k\vartheta + (1-k)\varphi$ is in $\mathfrak{C}$.*

2) *The set $\mathfrak{C}$ is convex and* stable, *i.e. for all ϑ and φ in $\mathfrak{C}$, all $t \in [0,T]$ and all $F \in \mathcal{F}_t$, the strategy $\vartheta \mathbb{1}_{F^c} + \big(\vartheta \mathbb{1}_{[\![0,t]\!]} + \varphi \mathbb{1}_{]\!]t,T]\!]}\big)\mathbb{1}_F$ is in $\mathfrak{C}$.*

3) *There exists a predictable correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ with non-empty, closed and convex values such that the projection of C on the predictable range of S is closed, i.e. $\Pi^S(\omega,t)C(\omega,t)$ is closed P_B-a.e., and*

$$G(\mathfrak{C}) = \{G(\varphi) \,|\, \varphi \in \mathfrak{C}\} = \{G(\vartheta) \,|\, \vartheta \in \Theta(C)\} = G_T(\Theta(C)).$$

Proof. The implication "1) $\Longrightarrow$ 2)" is obvious. For the remaining ones, we observe that by Proposition 3.7, the closedness of $G_T(\mathfrak{C})$ in $L^2(P)$ is equivalent to that of $G(\mathfrak{C})$ in $\mathcal{H}^2(P)$. The equivalence "3) $\Longleftrightarrow$ 1)" then follows from part 2) of Remark 4.12 in Chapter III, and "2) $\Longrightarrow$ 1)" from (the arguments in the proof of) Lemma 11 in [28]. □

II.4 Existence of a solution

Having established the closedness of $G_T(\Theta(C))$ and $\mathcal{A}(\Theta(C))$, we are able to prove the existence of a solution to the mean-variance hedging problem under trading constraints by using the best approximation theorem in Hilbert spaces; see Theorem 1.4.1 in [5]. Although this looks very easy, it is worth pointing out that our result is given for a very general framework. It covers for instance the existence of a solution in the Itô process setting of Labbé and Heunis [63], Hu and Zhou [49], and Jin and Zhou [53]. We also emphasise that our approach provides a unified treatment for the above papers, which use different and more situation-based arguments like convex duality for Itô processes, Itô's representation theorem, linear-quadratic optimal control and BSDE techniques.

Theorem 4.1. *Assume $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $S \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed convex values such that $\Theta(C)$ is non-empty. Then the following hold for every $H \in L^2(P)$:*

1) *There exists a solution* $\widehat{\vartheta}(x) \in \Theta(C)$ *to the problem*

$$E\left[|x + G_T(\vartheta) - H|^2\right] = \min_{\vartheta\in\Theta(C)} !$$

2) *There exists a solution* $\big(\widehat{x}, \widehat{\vartheta}(\widehat{x})\big) \in \mathbb{R} \times \Theta(C)$ *to the problem*

$$E\left[|x + G_T(\vartheta) - H|^2\right] = \min_{(x,\vartheta)\in\mathbb{R}\times\Theta(C)} !$$

Proof. By Theorem 3.12, $G_T(\Theta(C))$ and $\mathcal{A}(\Theta(C))$ are closed and convex subsets of $L^2(P)$. Therefore Theorem 1.4.1 in [5] implies the existence of a unique best approximation of $H - x$ by an element in $G_T(\Theta(C))$. This can be identified uniquely with an element $G\big(\widehat{\vartheta}(x)\big)$ in $G\big(\Theta(C)\big)$ which gives some $\widehat{\vartheta}(x) \in \Theta(C)$ and proves 1). In the same way, we get a unique element $\widehat{v}$ in $\mathcal{A}(\Theta(C))$ which is the best approximation to H in $L^2(P)$, and $\widehat{v}$ can again be identified with an element $\big(\widehat{x}, \widehat{\vartheta}(\widehat{x})\big)$ in $\mathbb{R} \times \Theta(C)$. □

Remark 4.2. 1) As explained in Example 3.13, the assumptions of Theorem 4.1 are satisfied in the Itô process framework of [49] and [53]. By the argument in the proof of Theorem 11 in [53], obtaining a solution to the *constrained Markowitz problem*, i.e.

$$\begin{aligned} &\text{minimise } \operatorname{Var}[\bar{V}_T(x,\pi)] = E\big[|\bar{V}_T(x,\pi)|^2\big] - z^2 \\ &\text{subject to } \pi \in \Theta_X(K^\pi) \text{ and } E[\bar{V}_T(x,\pi)] = z \end{aligned} \tag{4.1}$$

for $z \geq xE[\bar{S}^0_T]$ and a predictable correspondence K^π with closed and convex *cones* as values, is equivalent to finding a solution to

$$E\Big[\Big(\bar{V}_T(x,\pi) - \big(m_1 - m_2\mathcal{E}(-\textstyle\int \lambda\, dM - \int r\, dt)_T\big)\Big)^2\Big] = \min_{\pi\in\Theta_X(K^\pi)} !$$

for a suitable pair $(m_1, m_2) \in \mathbb{R}^2$ of Lagrange multipliers. Therefore the existence of a solution to (4.1) follows from Theorem 4.1 above. The dynamic structure of this solution in a general semimartingale framework is established in Chapter IV, which generalises the results obtained for a complete Itô process model in Theorem 6.3 in [49].

2) The problem studied in [63] is

$$E\big[\tfrac{1}{2}\left(\bar{a}|\bar{V}_T(x,\pi)|^2 + \bar{c}V_T(x,\pi)\right)\big] + q = \min_{\pi\in\Theta_X(C^\pi)} !$$

where $\bar{a} > 0$ and $1/\bar{a}$ are in $L^\infty(P)$, $\bar{c} \in L^2(P)$, $q \in \mathbb{R}$ and $C^\pi \equiv K \subseteq \mathbb{R}^d$ is a fixed closed and convex set; see Problem (5.2) in [63]. To obtain a solution to this problem, we observe that $\bar{a}\bar{S}^0_T\big(x + G_T\big(\Theta_X(C^\pi)\big)\big)$ is convex and closed in $L^2(P)$, since $\bar{a}$ and $1/\bar{a}$ are in $L^\infty(P)$, and that we can write

$$E\big[\tfrac{1}{2}\left(\bar{a}|\bar{V}_T(x,\pi)|^2 + \bar{c}V_T(x,\pi)\right)\big] + q = \tfrac{1}{2}E\big[\,\big|\bar{a}\bar{V}_T(x,\pi) + \tfrac{\bar{c}}{2\bar{a}}\big|^2\,\big] - E\big[\,\big|\tfrac{\bar{c}}{2\bar{a}}\big|^2\,\big] + q.$$

Then the existence of a solution follows as in the proof of Theorem 4.1 by the best approximation theorem.

In order to handle also constraints on the trajectory of the wealth process, we use a simple martingale argument which already appears in [7]. For that, we define the set of equivalent local martingale measures (ELMMs) for S with P-square-integrable density by

$$\mathbb{P}^2_e(S) = \left\{ Q \sim P \,\middle|\, S \text{ is a local } Q\text{-martingale and } \tfrac{dQ}{dP} \in L^2(P) \right\}.$$

Proposition 4.3. *Suppose that $S \in \mathcal{H}^2_{loc}(P)$ and that $\mathbb{P}^2_e(S) \neq \emptyset$. Let J be a closed interval in $\mathbb{R}$. Then the following hold for any $\vartheta \in \Theta$ and any $x \in \mathbb{R}$:*

1) *$G(\vartheta)$ takes values in J P-a.s. if and only if its final value $G_T(\vartheta)$ does.*

2) *The wealth process $V(x,\vartheta)$ takes values in J P-a.s. if and only if its final value $V_T(x,\vartheta)$ does.*

Proof. Since $V(x,\vartheta) = x + G(\vartheta)$, the proofs for 1) and 2) are completely analogous, and the "only if" part is obvious. For the "if" part, choose $Q \in \mathbb{P}^2_e(S)$ and write $J = [b_1, b_2]$ with $b_1, b_2 \in \mathbb{R}$. Because Q and P are equivalent, we can write a.s. without specifying which measure is meant. Moreover, the density process Z^Q is strictly positive and can be represented as a stochastic exponential $Z^Q = \mathcal{E}(L^Q)$ with $L^Q = \int \frac{1}{Z^Q_-}\, dZ^Q$. In the proof of part 1) of Proposition 3.8, the only point which uses the assumption that $R_2(P)$ is satisfied is to ensure that ${}^{\tau}\mathcal{E}(N)$ is a P-square-integrable P-martingale for all stopping times τ by Proposition 3.6. However, as this is already known for $Z^Q = \mathcal{E}(L^Q)$, we can apply the same arguments here to obtain that $G(\vartheta)$ is a Q-martingale for all $\vartheta \in \Theta$. Hence $b_1 \leq G_T(\vartheta) \leq b_2$ a.s. implies that $b_1 \leq G_t(\vartheta) \leq b_2$ for all $t \in [0,T]$ a.s. For an infinite interval, the argument is analogous. □

The previous result allows us to solve the mean-variance hedging problem also under constraints on the trajectory of the wealth process, again via the best approximation theorem.

Proposition 4.4. *Suppose that $S \in \mathcal{H}^2_{loc}(P)$ and that there exists $Q \in \mathbb{P}^2_e(S)$ such that its density process Z^Q satisfies $R_2(P)$. Then the following hold for every $H \in L^2(P)$ and every closed interval J in $\mathbb{R}$:*

1) *With*

$$\mathcal{G}_c(\Theta) := \{ G_T(\vartheta) \in G_T(\Theta) \,|\, G_t(\vartheta) \in J \text{ for all } t \in [0,T] \ P\text{-a.s.} \},$$

there exists a unique solution $\widehat{g}(x) \in \mathcal{G}_c(\Theta)$ to the problem

$$E\left[|x + g - H|^2 \right] = \min_{g \in \mathcal{G}_c(\Theta)} !$$

2) *With*

$$\mathcal{A}_c(\Theta) := \{V_T(x,\vartheta) \in \mathcal{A}(\Theta) \,|\, V_t(x,\vartheta) \in J \text{ for all } t \in [0,T] \ P\text{-a.s.}\},$$

there exists a unique solution $\widehat{v} \in \mathcal{A}_c(\Theta)$ *to the problem*

$$E\left[|v - H|^2\right] = \min_{v \in \mathcal{A}_c(\Theta)} !$$

Proof. Thanks to Proposition 3.8, $G_T(\Theta)$ and $\mathcal{A}(\Theta)$ are closed in $L^2(P)$. By Proposition 4.3, we have that $\mathcal{G}_c(\Theta) = \{g \in G_T(\Theta) \,|\, g \in J \ P\text{-a.s.}\}$ and $\mathcal{A}_c(\Theta) = \{a \in \mathcal{A}(\Theta) \,|\, a \in J \ P\text{-a.s.}\}$. Moreover, we have

$$\{g \in G_T(\Theta) \,|\, g \in J \ P\text{-a.s.}\} = G_T(\Theta) \cap \{f \in L^2(P) \,|\, f \in J \ P\text{-a.s.}\}$$
$$\{v \in \mathcal{A}(\Theta) \,|\, v \in J \ P\text{-a.s.}\} = \mathcal{A}(\Theta) \cap \{f \in L^2(P) \,|\, f \in J \ P\text{-a.s.}\}.$$

Since $J \subseteq \mathbb{R}$ is closed, the set $\{f \in L^2(P) \,|\, f \in J \ P\text{-a.s.}\}$ is closed in $L^2(P)$ and so are $\{g \in G_T(\Theta) \,|\, g \in J \ P\text{-a.s.}\}$ and $\{a \in \mathcal{A}(\Theta) \,|\, a \in J \ P\text{-a.s.}\}$. An application of the best approximation theorem completes the proof. □

II.5 Convex duality

While the existence in Theorem 4.1 is valid in a general framework, its easy proof has the drawback that it only gives the existence of a solution without any further properties. This is one motivation to study mean-variance hedging problems under trading constraints by means of convex duality. Typically, this provides additional insights into the structure of the solution, e.g. that the value functions of the primal and dual problems are continuously differentiable, strictly concave or convex, respectively, and conjugate to each other. Moreover, the solution of the primal problem is linked via the inverse of the "marginal utility" to the solution of the dual problem.

The general outline of these arguments follows the classical approach of Kramkov and Schachermayer [62] to maximising the expected utility from terminal wealth. The main idea we adopt from there is to treat the problem first as a static optimisation problem. This can be handled easily since we can apply duality theory in a Hilbert space. The obtained duality and existence results are then transferred back to the level of stochastic processes. Like for the existence of a solution, the required condition to establish this dual formulation is the closedness of the set $G_T(\Theta(C))$. Of course, there is a lot of related work in the literature; we discuss this in more detail in Section II.5.3 below.

To emphasise the analogy with utility maximisation, we rewrite the mean-variance hedging problem as a maximisation problem. This is then our primal problem and consists of finding the optimal trading strategy over time.

Primal problem (stochastic processes):

$$E\left[-\tfrac{1}{2}|x+G_T(\vartheta)-H|^2\right] = \max_{\vartheta\in\Theta(C)}! \tag{5.1}$$

The objective function in (5.1) is $U(x,\omega) = -\frac{1}{2}|x-H(\omega)|^2$, which depends on the state ω and is strictly convex and continuously differentiable in x. Its derivative and the inverse of that are $U'(x,\omega) = -x+H(\omega)$ and $I(y,\omega) = (U')^{-1}(y,\omega) = -y+H(\omega)$. Since U fails to be monotonic in x, it is not a utility function in the proper sense. But as it satisfies the other properties of a utility function and represents our preferences, we call it a *"quadratic utility function"*.

Nw observe that (5.1) only involves the terminal wealth $x+G_T(\vartheta)$. Hence we do not change the optimal value if we regard (5.1) as an optimisation problem over the set of square-integrable random variables defined by

$$\mathcal{C}(x) = \{f\in L^2(P)\,|\, f = x+G_T(\vartheta) \text{ for some } \vartheta\in\Theta(C)\} = x+G_T(\Theta(C)).$$

This leads to the corresponding static optimisation problem, which runs only over a set of random variables.

Primal problem (random variables):

$$E[U(f)] = E\left[-\tfrac{1}{2}|f-H|^2\right] = \max_{f\in\mathcal{C}(x)}! \tag{5.2}$$

By construction, both problems have the same value function

$$u(x) = \sup_{\vartheta\in\Theta(C)} E\big[U\big(x+G_T(\vartheta)\big)\big] = \sup_{f\in\mathcal{C}(x)} E[U(f)].$$

II.5.1 Duality for static variables

If $C:\overline{\Omega}\to 2^{\mathbb{R}^d}$ is a predictable correspondence with closed convex values and the assumptions of Theorem 3.12 are satisfied, we obtain from there that $G_T(\Theta(C))$ is a closed convex subset of $L^2(P)$. Thus the set of primal variables has the general structure

$$\mathcal{C}(x) = x+\mathcal{G}_c,$$

where

$$\mathcal{G}_c \text{ is a non-empty, closed and convex subset of } L^2(P). \tag{5.3}$$

Moreover, the assumptions of Theorem 3.12 imply that with $\mathcal{G} = \operatorname{span}\{\mathcal{G}_c\}$, the set

$$\mathbb{P}^2_s(\mathcal{G}) = \Big\{Q\ll P \,\Big|\, Q \text{ is a signed measure with, } Q[\Omega]=1,\ \frac{dQ}{dP}\in L^2(P) \text{ and } \frac{dQ}{dP}\in\mathcal{G}^\perp\Big\}$$

of signed $\mathcal{G}$-martingale measures is non-empty, where $\mathcal{G}^\perp$ denotes the orthogonal complement of $\mathcal{G}$ in $L^2(P)$. Hence we also suppose in our abstract static setting that

$$\mathbb{P}^2_s(\mathcal{G}) \neq \emptyset. \tag{5.4}$$

We emphasise that these simple structural properties will be enough to establish the desired duality results in the static setting. To obtain a dual characterisation of the primal variables, we use the following characterisation of closed convex sets $\mathfrak{K}$ in a Hilbert space $\mathfrak{H}$; see Theorem 2.5.1 in [5]. For any $k \in \mathfrak{H}$,

$$k \in \mathfrak{K} \quad \Longleftrightarrow \quad (h,k)_{\mathfrak{H}} \leq \sup_{k' \in \mathfrak{K}} (h,k')_{\mathfrak{H}} =: \delta(h|\mathfrak{K}), \quad \forall h \in \mathfrak{H},$$

where $(\cdot,\cdot)_{\mathfrak{H}}$ denotes the scalar product in $\mathfrak{H}$ and $\delta(h|\mathfrak{K}) := \sup_{k' \in \mathfrak{K}}(h,k')_{\mathfrak{H}}$ is the *support function* of $\mathfrak{K}$. It is easy to see that the support function of a general non-empty set is positively homogeneous, convex, lower semicontinuous and bounded from below by $-\min_{k \in \mathfrak{K}} \|k\|_{\mathfrak{H}} \|h\|_{\mathfrak{H}}$, which is finite if $\mathfrak{K}$ is non-empty; see Proposition 2.5.1 in [5]. Applying this characterisation to $\mathcal{G}_c$ and $L^2(P)$, we obtain for any $g \in L^2(P)$ that

$$g \in \mathcal{G}_c \quad \Longleftrightarrow \quad E[hg] \leq \sup_{g' \in \mathcal{G}_c} E[hg'] =: \delta(h|\mathcal{G}_c), \quad \forall h \in L^2(P), \tag{5.5}$$

where $\delta(\cdot|\mathcal{G}_c)$ is the support function of the set $\mathcal{G}_c$.

To deduce dual variables from the characterisation (5.5), we observe from [62], [72] and [78] that for the general outline of the arguments for the dual approach to hold, the dual variables should have the following properties. First, they should be defined in such a way that the dual problem, which is an optimisation problem over the set of dual variables, attains its solution. Since the primal problem is a maximisation for a concave function, the dual problem is a minimisation for a convex function. Thus it should be enough for the existence of a solution that the set of dual variables is convex and closed. Second, one should be able to establish a duality relation between the set of primal and dual variables that allows one to show that the natural candidate for the dual solution lies in the set of primal variables. This candidate is given by the inverse I of the quadratic “marginal utility" applied to the dual solution, as follows typically from the first order condition for optimality in the dual problem. Third, to obtain that the value functions of the primal and dual problems are conjugate, the product of the parameters x and y of the primal and dual problem should appear in the upper bound for the expectation of a primal and a dual variable for the corresponding parameters.

Let us start with the last point. For a primal variable $f \in \mathcal{C}(x)$ and an element h of $L^2(P)$, the general structure of the primal variables gives

$$E[fh] = E[(x+g)h] \leq xE[h] + \delta(h|\mathcal{G}_c).$$

This motivates us to define the static dual variables by

$$\mathcal{D}(y) = \{h \in L^2(P) \,|\, E[h] = y, \delta(h|\mathcal{G}_c) < \infty\} \qquad \text{for } y \in \mathbb{R},$$

because this gives the third of the above properties, i.e.

$$f \in \mathcal{C}(x), h \in \mathcal{D}(y) \quad \Longrightarrow \quad E[fh] \leq xy + \delta(h|\mathcal{G}_c). \tag{5.6}$$

By continuity and linearity of the expectation and lower semicontinuity and convexity of the support function, the set $\mathcal{D}(y)$ is closed and convex in $L^2(P)$ and thus also likely to satisfy the first property listed above. Note that $\mathcal{D}(y)$ contains $y\frac{dQ}{dP}$ for any $Q \in \mathbb{P}^2_s(\mathcal{G})$ so that it is non-empty due to (5.4). The second property will follow via the dual characterisation of convex closed sets in $L^2(P)$; see the proof of Theorem 5.7 later.

Remark 5.1. 1) For a linear subspace $\mathcal{G}_c = \mathcal{G}$, the characterisation (5.5) simplifies to orthogonality and the dual domain becomes $\mathcal{D}(y) = \{h \in L^2(P) \,|\, E[h] = y, h \in \mathcal{G}_c^{\perp}\}$. Moreover, we have $\mathcal{D}(y) = y\mathbb{P}^2_s(\mathcal{G})$ for any $y \neq 0$. This is exploited in [48] for the dual formulation in the unconstrained case.

2) If $\mathcal{G}_c$ is a *cone*, the support function $\delta(\cdot\,|\mathcal{G}_c)$ only takes the values 0 and ∞ and (5.5) therefore reduces to the *bipolar relation*

$$g \in \mathcal{G}_c \quad \Longleftrightarrow \quad E[hg] \leq 0, \quad \forall h \in \mathcal{G}_c^{\circ},$$

where $\mathcal{G}_c^{\circ} = \{h \in L^2(P) \mid E[hg] \leq 0,\ \forall g \in \mathcal{G}_c\}$ is the polar of $\mathcal{G}_c$. Since $\mathcal{G}_c^{\circ}$ is again a cone, we have $\mathcal{D}(y) = y\mathcal{D}(1)$ for $y > 0$ and $\mathcal{D}(y) = |y|\mathcal{D}(-1)$ for $y < 0$. The sets $\mathcal{D}(1)$ and $\mathcal{D}(-1)$ can then be interpreted as the sets of all Radon–Nikodým derivatives of signed $\mathcal{G}_c$-super- and $\mathcal{G}_c$-submartingale measures, respectively. The above simplification explains why the majority of papers concentrates on constraints given by closed convex *cones.*

Returning to the general case, we work as usual with the Legendre transform V in x of $-U(-\,\cdot, \omega)$ to derive the formulation of the dual problem. The function V is given by

$$V(y, \omega) = \sup_{x \in \mathbb{R}} \{U(x, \omega) - xy\} = U\big(I(y, \omega)\big) - I(y, \omega)y = \tfrac{1}{2}y^2 - yH(\omega);$$

it depends on the state ω and is continuously differentiable and strictly convex in y. The motivation for using the Legendre transform comes from looking for the sharpest inequality such that

$$U(x, \omega) \leq V(y, \omega) + xy, \qquad \forall x, y \in \mathbb{R}, \forall \omega \in \Omega.$$

Plugging in $f \in \mathcal{C}(x)$ and $h \in \mathcal{D}(y)$ for x and y in the above inequality, taking expectations and optimising on both sides gives via (5.6)

$$\begin{aligned} u(x) = \sup_{f \in \mathcal{C}(x)} E[U(f)] &\leq \inf_{y \in \mathbb{R}} \Big\{ \inf_{h \in \mathcal{D}(y)} E[V(h) + gh] \Big\} \\ &\leq \inf_{y \in \mathbb{R}} \Big\{ \inf_{h \in \mathcal{D}(y)} \{E[V(h)] + \delta(h|\mathcal{G}_c)\} + xy \Big\} = \inf_{y \in \mathbb{R}} \{v(y) + xy\}, \end{aligned} \tag{5.7}$$

where the value function of the dual problem on the level of random variables is

$$v(y) = \inf_{h \in \mathcal{D}(y)} \{E[V(h)] + \delta(h|\mathcal{G}_c)\}.$$

Note that the objective function of the dual problem explicitly involves the constraints via the support function δ.

Dual problem (random variables):

$$\Psi(h) := E[V(h)] + \delta(h|\mathcal{G}_c) = \min_{h \in \mathcal{D}(y)} ! \tag{5.8}$$

From the inequalities in (5.7), we see that if we can find for a given x a pair $\big(\widehat{f}(x), \widehat{h}(y)\big)$ of primal and dual variables such that equality holds in (5.7), we have also found a solution to the primal problem (5.2), as $\widehat{f}(x)$ attains the supremum. Of course, y will then also depend on x. So an abstract recipe for solving the primal problem looks as follows:

1) Find the solution $\widehat{h}(y)$ to the dual problem (5.8) for any $y \in \mathbb{R}$.

2) Find the minimiser $\widehat{y}(x)$ for the indirect dual problem $v(y) + xy = \min_{y \in \mathbb{R}}!$, for any $x \in \mathbb{R}$.

3) Define $\widehat{h} := \widehat{h}\big(\widehat{y}(x)\big)$, $\widehat{f}(x) := I(\widehat{h})$ and show that $E[I(\widehat{h})\widehat{h}] = x\widehat{y}(x) + \delta(\widehat{h}|\mathcal{G}_c)$.

4) If we can show that $\widehat{f}(x) \in \mathcal{C}(x)$, then $\widehat{f}(x)$ solves (5.2), since we have by combining (5.7) with steps 1)–3) that

$$\begin{aligned} u(x) &\leq \inf_{y \in \mathbb{R}} \{v(y) + xy\} = v\big(\widehat{y}(x)\big) + x\widehat{y}(x) \\ &= E[V(\widehat{h})] + \delta(\widehat{h}|\mathcal{G}_c) + x\widehat{y}(x) \\ &= E\big[U\big(I(\widehat{h})\big) - I(\widehat{h})\widehat{h}\big] + \delta(\widehat{h}|\mathcal{G}_c) + x\widehat{y}(x) = E\big[U\big(\widehat{f}(x)\big)\big] \leq u(x). \end{aligned}$$

To solve the primal problem (5.2), it now remains to implement the above recipe. We start by solving the dual problem, making use of the following result from convex analysis; see Proposition 1.2 in [39].

Proposition 5.2. *Let B be a reflexive Banach space, K a non-empty closed convex subset of B and F a strictly convex, coercive and lower semicontinuous function from B into $\mathbb{R} \cup \{+\infty\}$ that is proper on K. Then there exists a unique solution $\widehat{b} \in K$ to*

$$F(b) = \min_{b \in K}!$$

Taking $B = L^2(P)$ and $K = \mathcal{D}(y)$, we only need to check that the dual objective function Ψ satisfies the properties of F to apply the proposition.

Lemma 5.3. *For every $H \in L^2(P)$, the mapping*

$$h \mapsto \Psi(h) = E[V(h)] + \delta(h|\mathcal{G}_c) = E\left[\tfrac{1}{2}h^2 - hH\right] + \delta(h|\mathcal{G}_c)$$

from $L^2(P)$ into $\mathbb{R}\cup\{+\infty\}$ is strictly convex, lower semicontinuous, coercive and uniformly bounded from below by $-\frac{1}{2}\big(\|H\|_{L^2(P)} + \min_{g\in\mathcal{G}_c}\|g\|_{L^2(P)}\big)^2$.

Proof. We begin by proving that the mapping $h \mapsto E[V(h)]$ from $L^2(P)$ into $\mathbb{R}$ is strictly convex and continuous. The first property follows immediately from the strict convexity of the function $V(\cdot,\omega)$ for all $\omega \in \Omega$. If (h_n) converges to h in $L^2(P)$, then (h_n) is bounded in $L^2(P)$ and the Cauchy–Schwarz inequality gives

$$\begin{aligned}&\left|E\left[\tfrac{1}{2}h_n^2 - h_nH\right] - E\left[\tfrac{1}{2}h^2 - hH\right]\right| = \left|E\left[(h_n - h)\left(\tfrac{1}{2}(h_n + h) - H\right)\right]\right|\\ &\le \|h_n - h\|_{L^2(P)}\Big(\tfrac{1}{2}\big(\sup_{n\in\mathbb{N}}\|h_n\|_{L^2(P)} + \|h\|_{L^2(P)}\big) + \|H\|_{L^2(P)}\Big) \overset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

which proves the claimed continuity. Since $\delta(\cdot\,|\mathcal{G}_c)$ is convex and lower semicontinuous, the sum $\Psi(\cdot) = E[V(\cdot)] + \delta(\cdot\,|\mathcal{G}_c)$ is strictly convex and lower semicontinuous. Moreover, Cauchy–Schwarz implies that

$$\begin{aligned}\Psi(h) &= E\left[\tfrac{1}{2}h^2 - hH\right] + \delta(h|\mathcal{G}_c) \\ &\ge \tfrac{1}{2}\|h\|^2_{L^2(P)} - \|h\|_{L^2(P)}\big(\|H\|_{L^2(P)} + \min_{g\in\mathcal{G}_c}\|g\|_{L^2(P)}\big)\end{aligned} \tag{5.9}$$

which gives coercivity, since the right-hand side tends to ∞ as $\|h\|_{L^2(P)} \to \infty$ because $\min_{g\in\mathcal{G}_c}\|g\|_{L^2(P)}$ is finite. Minimising the right-hand side over $\|h\|_{L^2(P)}$ also gives the asserted lower bound, which completes the proof. □

From the definition of $\mathcal{D}(y)$, we have that $y\frac{dQ}{dP}$ is in $\mathcal{D}(y)$ for every $y \in \mathbb{R}$ and $Q \in \mathbb{P}^2_s(\mathcal{G})$. Since $\delta\big(y\frac{dQ}{dP}\big|\mathcal{G}_c\big) = 0$, assumption (5.4) implies that Ψ is proper on $\mathcal{D}(y)$ for each $y \in \mathbb{R}$. Therefore all the conditions of Proposition 5.2 are satisfied in the setting of the dual problem, and the existence of a solution to the dual problem follows by combining Lemma 5.3 with Proposition 5.2. This gives

Proposition 5.4. *Under the assumptions* (5.3) *and* (5.4)*, there exists a unique solution $\widehat{h}(y) \in \mathcal{D}(y)$ to the dual problem* (5.8) *for every $H \in L^2(P)$ and each $y \in \mathbb{R}$, i.e.*

$$\Psi\big(\widehat{h}(y)\big) = E\big[V\big(\widehat{h}(y)\big)\big] + \delta\big(\widehat{h}(y)\big|\mathcal{G}_c\big) = \inf_{h\in\mathcal{D}(y)} E[V(h)] + \delta(h|\mathcal{G}_c) = v(y)$$

By Proposition 5.4, the function v inherits all nice properties of Ψ, which enables us to solve the indirect dual problem by again using Proposition 5.2. More precisely:

Lemma 5.5. *Under the assumptions* (5.3) *and* (5.4)*, the function v is strictly convex, continuous and coercive.*

Proof. If y_1, $y_2 \in \mathbb{R}$ and $\mu \in (0,1)$, then $\mu\widehat{h}(y_1) + (1-\mu)\widehat{h}(y_2)$ is in $\mathcal{D}\big(\mu y_1 + (1-\mu)y_2\big)$; so

$$\begin{aligned}\mu v(y_1) + (1-\mu)v(y_2) &= \mu\Psi\big(\widehat{h}(y_1)\big) + (1-\mu)\Psi\big(\widehat{h}(y_2)\big)\\ &> \Psi\big(\mu\widehat{h}(y_1) + (1-\mu)\widehat{h}(y_2)\big) \geq v\big(\mu y_1 + (1-\mu)y_2\big)\end{aligned}$$

by Proposition 5.4 and the strict convexity of Ψ. Hence v is strictly convex, and continuous like any convex function on $\mathbb{R}$ with finite values; see Corollary II.10.1.1 in [82]. By Jensen's inequality, $\|\widehat{h}(y)\|_{L^2(P)} \geq E[\widehat{h}(y)] = y$ tends to ∞ as $y \to \infty$. Thus coercivity of Ψ implies coercivity of v, again by Proposition 5.4. Note that in view of 5.9, $v(y)$ even grows quadratically as $|y| \to \infty$. □

Since a continuous function is obviously proper, applying Proposition 5.2 to the strictly convex, continuous and coercive mapping $y \mapsto v(y) + xy$ on $\mathbb{R}$ immediately gives

Corollary 5.6. *Assume* (5.3) *and* (5.4). *For every $x \in \mathbb{R}$, there exists a unique $\widehat{y}(x) \in \mathbb{R}$ that solves*

$$v(y) + xy = \min_{y\in\mathbb{R}}!$$

Now we have everything in place to formulate and prove the abstract static version of the main result of this section.

Theorem 5.7. *Suppose as in* (5.3) *that $\mathcal{G}_c$ is a non-empty, convex and closed subset of $L^2(P)$, and impose the assumption* (5.4) *that $\mathbb{P}^2_s(\mathcal{G}) \neq \emptyset$. Then:*

1) *For every $x \in \mathbb{R}$, there exists a unique solution $\widehat{f}(x) \in \mathcal{C}(x)$ to*

$$E\left[-\tfrac{1}{2}|f - H|^2\right] = \max_{f\in\mathcal{C}(x)}!$$

It is given by

$$\widehat{f}(x) = I\Big(\widehat{h}\big(\widehat{y}(x)\big)\Big) = -\widehat{h}\big(\widehat{y}(x)\big) + H,$$

where $\widehat{h}\big(\widehat{y}(x)\big) \in \mathcal{D}\big(\widehat{y}(x)\big)$ and $\widehat{y}(x) \in \mathbb{R}$ are the unique solutions, respectively, to

$$\Psi(h) = E\left[\tfrac{1}{2}h^2 - hH\right] + \delta(h|\mathcal{G}_c) = \min_{h\in\mathcal{D}(\widehat{y}(x))}!$$

and

$$v(y) + xy = \min_{y\in\mathbb{R}}! \tag{5.10}$$

2) *The value functions u and v are conjugate, i.e.*

$$u(x) = \inf_{y\in\mathbb{R}}\{v(y) + xy\},$$
$$v(y) = \sup_{x\in\mathbb{R}}\{u(x) - xy\},$$

and continuously differentiable. u is strictly concave and v is strictly convex.

3) *Furthermore, we have the equivalent relations*

$$\begin{aligned} E\big[\widehat{f}(x)\widehat{h}\big(\widehat{y}(x)\big)\big] &= x\widehat{y}(x) + \delta\big(\widehat{h}\big(\widehat{y}(x)\big)\big|\mathcal{G}_c\big), \\ E\big[\widehat{f}(x)U'\big(\widehat{f}(x)\big)\big] &= xu'(x) + \delta\big(U'\big(\widehat{f}(x)\big)\big|\mathcal{G}_c\big), \\ E\big[\widehat{h}\big(\widehat{y}(x)\big)V'\big(\widehat{h}\big(\widehat{y}(x)\big)\big)\big] &= \widehat{y}(x)v'\big(\widehat{y}(x)\big) - \delta\big(\widehat{h}\big(\widehat{y}(x)\big)\big|\mathcal{G}_c\big). \end{aligned} \tag{5.11}$$

Proof. 1) Since $\widehat{y}(x)$ and $\widehat{h} := \widehat{h}\big(\widehat{y}(x)\big)$ solve the problems (5.10) and (5.8), the definition of $\mathcal{D}(y)$ implies that $\widehat{h}\big(\widehat{y}(x)\big)$ is also the solution to

$$\Psi(h) + xE[h] = \min_{h\in L^2(P)}! = \min_{h\in\mathcal{D}(\widehat{y}(x))}! \tag{5.12}$$

For $\varepsilon \in (0,1)$ and $h \in L^2(P)$, set $h_\varepsilon = \widehat{h} + \varepsilon h$. Then optimality of $\widehat{h}$ for (5.12) gives

$$\begin{aligned} 0 &\le \liminf_{\varepsilon\searrow 0} \frac{\Psi(h_\varepsilon) + xE[h_\varepsilon] - \big(\Psi(\widehat{h}) + xE[\widehat{h}]\big)}{\varepsilon} \\ &= \liminf_{\varepsilon\searrow 0}\Big\{E[(\widehat{h} - H + x)h] + \frac{1}{2}\varepsilon E[h^2] + \frac{\delta(\widehat{h}+\varepsilon h|\mathcal{G}_c) - \delta(\widehat{h}|\mathcal{G}_c)}{\varepsilon}\Big\}, \end{aligned} \tag{5.13}$$

where the last expression is well defined as $\delta(\widehat{h}|\mathcal{G}_c)$ is finite. Hence we obtain by using $I(\widehat{h}) = -\widehat{h} + H$ and the sublinearity of $\delta(\cdot\,|\mathcal{G}_c)$ that

$$E\big[\big(I(\widehat{h}) - x\big)h\big] \le \delta(h|\mathcal{G}_c), \qquad \forall h \in L^2(P) \tag{5.14}$$

and thus $I(\widehat{h}) - x \in \mathcal{G}_c$, i.e. $I(\widehat{h}) \in \mathcal{C}(x)$, by the characterisation of closed convex sets in (5.5). Plugging $h = -\widehat{h}$ into (5.13) and using the positive homogeneity of $\delta(\cdot\,|\mathcal{G}_c)$ gives

$$E\big[\big(I(\widehat{h}) - x\big)(-\widehat{h})\big] \le -\delta(\widehat{h}|\mathcal{G}_c).$$

Combining this with (5.14) for $h = \widehat{h}$ gives

$$\delta(\widehat{h}|\mathcal{G}_c) = E\big[\big(I(\widehat{h}) - x\big)\widehat{h}\big] = E[I(\widehat{h})\widehat{h}] - x\widehat{y}(x). \tag{5.15}$$

Hence we obtain from (5.15) as in step 4) of the recipe that

$$\begin{aligned} u(x) &\geq E\big[U\big(I(\widehat{h})\big)\big] = E[V(\widehat{h}) + I(\widehat{h})\widehat{h}] \\ &= E[V(\widehat{h})] + \delta(\widehat{h}|\mathcal{G}_c) + x\widehat{y}(x) = v\big(\widehat{y}(x)\big) + x\widehat{y}(x) \geq u(x), \end{aligned} \tag{5.16}$$

which shows that $\widehat{f}(x) := I\left(\widehat{h}\big(\widehat{y}(x)\big)\right)$ indeed maximises $E[U(f)]$ over $\mathcal{C}(x)$.

2) Since we have equality in (5.16) and $\widehat{y}(x)$ attains $\inf_{y\in\mathbb{R}}\{v(y) + xy\}$, we also have that $u(x) = \inf_{y\in\mathbb{R}}\{v(y) + xy\}$ for all $x \in \mathbb{R}$ and then $v(y) = \sup_{x\in\mathbb{R}}\{u(x) - xy\}$ by the biconjugate property of the Legendre transform; see Theorem III.12.2 in [82]. To show the strict concavity of u, we fix $x_1, x_2 \in \mathbb{R}$ and $\mu \in (0,1)$. Then $\mu\widehat{f}(x_1) + (1-\mu)\widehat{f}(x_2)$ is in $\mathcal{C}(\mu x_1 + (1-\mu)x_2)$ and so part 1) yields by the strict concavity of $U(\,\cdot\,,\omega)$ that

$$\begin{aligned} \mu u(x_1) + (1-\mu)u(x_2) &= E\big[\mu U\big(\widehat{f}(x_1)\big) + (1-\mu)U\big(\widehat{f}(x_2)\big)\big] \\ &< E\big[U\big(\mu\widehat{f}(x_1) + (1-\mu)\widehat{f}(x_2)\big)\big] \leq u(\mu x_1 + (1-\mu)x_2). \end{aligned}$$

Continuous differentiability of u and v follows since the Legendre transform of a strictly convex function is differentiable; see Theorems V.24.1 and V.26.3 in [82]. Since v is continuously differentiable, we obtain for the minimiser $\widehat{y}(x)$ of $v(y) + xy$ over $y \in \mathbb{R}$ the relation $v'\big(\widehat{y}(x)\big) = -x$. Again by general results on the Legendre transform, we have $V'(\,\cdot\,,\omega) = -(U')^{-1}(\,\cdot\,,\omega) = -I(\,\cdot\,,\omega)$ and $v' = -(u')^{-1}$; see Theorem V.23.5 in [82]. Combining this with $v'\big(\widehat{y}(x)\big) = -x$, $\widehat{f}(x) = I(\widehat{h})$ and (5.15) gives the relations (5.11). This completes the proof. □

II.5.2 Duality for dynamic variables

Under the assumptions of Theorem 3.12, Theorem 5.7 already implies the existence of a unique solution to the primal problem (5.1) by choosing $\mathcal{G}_c = G_T(\Theta(C))$, i.e. there exists an optimal trading strategy $\widehat{\vartheta}(x) \in \Theta(C)$ such that $\widehat{f}(x) = x + G_T\big(\widehat{\vartheta}(x)\big)$. In particular, we recover part 2) of Theorem 4.1.

To establish an analogous duality result on the level of stochastic processes, we need a dynamic version for the dual variables. If we assume for simplicity that $\mathcal{F} = \mathcal{F}_T$, we can identify every $h \in L^2(P)$ with a square-integrable RCLL martingale $Z = Z(h)$ given by $Z_t = E[h|\mathcal{F}_t]$ for $t \in [0,T]$. The Kunita–Watanabe decomposition then yields

$$Z_t = E[h|\mathcal{F}_0] + \int_0^t \eta_s \, dM_s + R_t, \qquad t \in [0,T],$$

with $\eta \in L^2(M)$ and $R \in \mathcal{M}_0^2(P)$ strongly P-orthogonal to M. We choose this parametrisation because it makes it easy to calculate the dynamics of the product of a gains process and a dual variable. Moreover, it is similar to [21], where dual variables are supermartingale measures for the gains processes of constrained trading strategies. The parametrisation in [21] can be obtained by applying the Kunita–Watanabe decomposition to the stochastic logarithm of the density process, and in the Brownian filtration of [21], this decomposition can of course be replaced by Itô's representation theorem; see [21] and Example 3.2 in [72].

Lemma 5.8. *Suppose that S is in $\mathcal{H}^2_{loc}(P)$ and satisfies the structure condition (SC). For every $\vartheta \in \Theta$ and every $Z \in \mathcal{M}^2(P)$, the process*

$$\big(G_t(\vartheta)Z_t - \int_0^t (\eta_s + Z_{s-}\lambda_s)^\top c^M_s \vartheta_s \, dB_s\big)_{0\le t\le T} \tag{5.17}$$

is a P-martingale with P-integrable supremum, i.e. a martingale in $\mathcal{H}^1(P)$.

Proof. Applying the product rule and using that S satisfies (SC) gives that

$$\begin{aligned} d\big(G(\vartheta)Z\big) &= Z_-\vartheta \, dM + Z_-\vartheta^\top d\langle M\rangle\lambda + G_-(\vartheta)\, dZ \\ &\quad + d[Z, \textstyle\int \vartheta \, dA] + d[Z, \int \vartheta \, dM]. \end{aligned}$$

Clearly, $\int Z_-\vartheta \, dM$ and $\int G_-(\vartheta)\, dZ$ are local P-martingales. Moreover, $\langle Z, \int \vartheta \, dM\rangle$ exists because $Z \in \mathcal{M}^2(P)$ and $\vartheta \in \mathcal{L}^2(M)$, and $[Z, \int \vartheta \, dA]$ is a local martingale by Yoeurp's lemma. Writing $\overset{\text{mart}}{=}$ for equality up to a local P-martingale and using that R is strongly P-orthogonal to M, we thus obtain $d\big(G(\vartheta)Z\big) \overset{\text{mart}}{=} Z_-\vartheta^\top d\langle M\rangle\lambda + \vartheta^\top d\langle Z, M\rangle \overset{\text{mart}}{=} (\eta + Z_-\lambda)^\top d\langle M\rangle\vartheta$. This shows that the process in (5.17) is a local P-martingale. To check integrability, we first observe that by Doob's inequality and Proposition 2.1, $\big(ZG(\vartheta)\big)^*_T$ is in $L^1(P)$ since $Z \in \mathcal{M}^2(P)$ and $G(\vartheta) \in \mathcal{H}^2(P)$, and that the Kunita–Watanabe inequality gives

$$E\left[(\textstyle\int \eta^\top d\langle M\rangle\vartheta)^*_T\right] \le E\big[\int_0^T |\eta_s^\top c^M_s \vartheta_s| \, dB_s\big] \le \|\eta\|_{\mathcal{L}^2(M)} \|\vartheta\|_{\mathcal{L}^2(M)} < \infty.$$

Moreover, using $\int |\vartheta \, dA| = \int |\vartheta^\top c^M \lambda| \, dB$ and the Cauchy–Schwarz and Doob inequalities allows us to estimate the remaining term by

$$E\left[(\textstyle\int Z_-\vartheta^\top d\langle M\rangle\lambda)^*_T\right] \le E\big[Z^*_T \int_0^T |\vartheta_s^\top c^M_s \lambda_s| \, dB_s\big] \le 2\|Z_T\|_{L^2(P)} \|\vartheta\|_{\mathcal{L}^2(A)}.$$

Replacing $c^M_t \, dB_t$ by $d\langle M\rangle_t$ and using the estimate

$$\begin{aligned} &\big(G(\vartheta)Z - \textstyle\int (\eta + Z_-\lambda)^\top c^M \vartheta \, dB\big)^*_T \\ &\qquad \le \big(ZG(\vartheta)\big)^*_T + (\textstyle\int \eta^\top d\langle M\rangle\vartheta)^*_T + (\int Z_-\vartheta^\top d\langle M\rangle\lambda)^*_T \end{aligned}$$

then shows that the local P-martingale in (5.17) has a P-integrable supremum. □

Using Lemma 5.8 and optimising over ϑ immediately gives for every $Z \in \mathcal{M}^2(P)$

$$\begin{aligned} \sup_{\vartheta\in\Theta(C)} E[G_T(\vartheta)Z_T] &= \sup_{\vartheta\in\Theta(C)} E\big[\textstyle\int_0^T (\eta_s + Z_{s-}\lambda_s)^\top c^M_s \vartheta_s \, dB_s\big] \\ &\le E\big[\textstyle\int_0^T \delta\big(c^M_s(\eta_s + Z_s\lambda_s)\big|C\big)\, dB_s\big] \end{aligned} \tag{5.18}$$

by definition of the support function $\delta(\cdot\,|C)$, because each ϑ_s has values in C. The next result shows that we even have equality in (5.18). Note that we use the same symbol δ for support functions in two different Hilbert spaces — $L^2(P)$ on the left-hand and $\mathbb{R}^d$ on the right-hand side of (5.19).

Lemma 5.9. *Suppose that S is in $\mathcal{H}^2_{loc}(P)$ and satisfies the structure condition (SC). For every $Z \in \mathcal{M}^2(P)$,*

$$\delta\big(Z_T\big|G_T(\Theta(C))\big) = \sup_{\vartheta\in\Theta(C)} E[G_T(\vartheta)Z_T] = E\big[\int_0^T \delta\big(c^M_s(\eta_s+Z_{s-}\lambda_s)\big|C\big)\,dB_s\big]. \tag{5.19}$$

Proof. Without loss of generality, we can assume that $0 \in C(\omega,t)$. Indeed, let φ be in $\Theta(C)$ and set $C' = C-\varphi$, which is by Proposition 2.3 a predictable correspondence with $0 \in C'(\omega,t)$. Then $\Theta(C) = \Theta(C') + \varphi$ and therefore $\delta\big(Z_T\big|G_T\big(\Theta(C)\big)\big) = \delta\big(Z_T\big|G_T\big(\Theta(C')\big)\big) + E[Z_T G_T(\varphi)]$ and

$$\begin{aligned}&E\big[\int_0^T \delta\big(c^M_s(\eta_s + Z_{s-}\lambda_s)\big|C\big)\,dB_s\big]\\ &= E\big[\int_0^T \delta\big(c^M_s(\eta_s + Z_{s-}\lambda_s)\big|C'\big)\,dB_s + \int_0^T \varphi_s^\top c^M_s(\eta_s + Z_{s-}\lambda_s)\,dB_s\big].\end{aligned}$$

Since $E[Z_T G_T(\varphi)] = E[\int_0^T \varphi_s^\top c^M_s(\eta_s+Z_{s-}\lambda_s)\,dB_s]$ by Lemma 5.8, we obtain that (5.19) holds for C if and only if it holds for C'.

In view of (5.18), it remains to show that

$$\sup_{\vartheta\in\Theta(C)} E[G_T(\vartheta)Z_T] \geq E\big[\int_0^T \delta\big(c^M_s(\eta_s + Z_{s-}\lambda_s)\big|C\big)\,dB_s\big].$$

To that end, we construct a sequence (ϑ^n) of C-constrained trading strategies such that $\lim_{n\to\infty} E[G_T(\vartheta^n)Z_T] = E\big[\int_0^T \delta\big(c^M_s(\eta_s + Z_{s-}\lambda_s)\big|C\big)\,dB_s\big]$. Define a function $f : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ by

$$f\big((\omega,t),z\big) := \big(\eta(\omega,t) + Z(\omega,t-)\lambda(\omega,t)\big)^\top c^M(\omega,t)z$$

and for each $n \in \mathbb{N}$ a predictable correspondence C^n by

$$C^n(\omega,t) := C(\omega,t) \cap \overline{B_n(0)} \subseteq \mathbb{R}^d$$

for $(\omega,t) \in \overline{\Omega}$, where $\overline{B_n(0)}$ denotes the closure of the ball of radius n in $\mathbb{R}^d$. Note that the C^n have convex and compact values and that $0 \in C^n(\omega,t)$ for $(\omega,t) \in \overline{\Omega}$ and each $n \in \mathbb{N}$. Moreover, $f(\,\cdot\,,z)$ is predictable for $z \in \mathbb{R}^d$ and $f\big((\omega,t),\,\cdot\,\big)$ is continuous for $(\omega,t) \in \overline{\Omega}$, i.e. f is a Carathéodory function. Let $\{x^{n,m}\,|\,m \in \mathbb{N}\}$ be a Castaing representation for C^n as in Proposition (2.3) and define

$$\begin{aligned}g^n(\omega,t) &:= \delta\big(c^M(\omega,t)\big(\eta(\omega,t) + Z(\omega,t-)\lambda(\omega,t)\big)\big|C^n(\omega,t)\big)\\ &= \sup_{x\in C^n(\omega,t)} f\big((\omega,t),x\big).\end{aligned}$$

As $g^n(\omega,t) = \sup_{m\in\mathbb{N}} f\big((\omega,t), x^{n,m}(\omega,t)\big)$ by Proposition 2.3, g^n is predictable and finite-valued by the compactness of $C^n(\omega,t)$. Combining Propositions 2.4 and 2.5 gives that

$$D^n(\omega,t) = \big\{z \in C^n(\omega,t) \,\big|\, f\big((\omega,t),z\big) = g^n(\omega,t)\big\}$$

is a predictable correspondence with non-empty, convex and compact values. Let y^n be a predictable selector of D^n. Then $g^n(\omega,t) = f\big((\omega,t),y^n(\omega,t)\big)$ and

$$\delta\big(c^M(\omega,t)\big(\eta(\omega,t)+Z(\omega,t-)\lambda(\omega,t)\big)\big|C(\omega,t)\big) = \lim_{n\to\infty} f\big((\omega,t),y^n(\omega,t)\big),$$

where the limit is increasing since $C(\omega,t) = \bigcup_{n\in\mathbb{N}} C^n(\omega,t)$. Let $(\tau_m)_{m\in\mathbb{N}}$ be a localising sequence such that $S^{\tau_m} \in \mathcal{H}^2(P)$. Since $|y^n(\omega,t)| \le n$ and each $C(\omega,t)$ contains zero, the process $\vartheta^n := y^n I_{[\![0,\tau_n]\!]}$ is in $\Theta(C)$ for each $n \in \mathbb{N}$. Hence Lemma 5.8 and monotone integration yield

$$\begin{aligned}\lim_{n\to\infty} E[G_T(\vartheta^n)Z_T] &= \lim_{n\to\infty} E\big[\textstyle\int_0^{\tau_n} (y^n_s)^\top c^M_s(\eta_s + Z_{s-}\lambda_s)\,dB_s\big]\\ &= E\big[\textstyle\int_0^T \delta\big(c^M_s(\eta_s+Z_{s-}\lambda_s)\big|C\big)\,dB_s\big],\end{aligned}$$

which completes the proof. □

As Lemma 5.9 relates the support function $\delta\big(\cdot\,\big|G_T(\Theta(C))\big)$ to the expectation of the terminal value of a stochastic process, we are led to reformulate the dual problem (5.8) on the level of stochastic processes in the following way.

Dual problem (stochastic processes):

$$\Psi(Z_T) = E\big[\tfrac{1}{2}Z_T^2 - Z_T H + \textstyle\int_0^T \delta\big(c^M_s(\eta_s+Z_{s-}\lambda_s)\big|C\big)\,dB_s\big] = \min_{Z\in\mathcal{Y}(y)} !, \quad (5.20)$$

where

$$\begin{aligned}\mathcal{Y}(y) = \big\{&Z \in \mathcal{M}^2(P) \,\big|\, Z = Z_0 + \textstyle\int \eta\,dM + R \text{ with } Z_0 \in L^2(P,\mathcal{F}_0),\\ &\eta \in L^2(M),\ R \in \mathcal{M}^2_0(P) \text{ strongly } P\text{-orth. to } M \text{ and } E[Z_T] = y\big\}.\end{aligned}$$

Remark 5.10. In the Itô process framework of Example 3.13 and if $\mathbb{F}$ is the P-augmentation of the filtration generated by W, the above dual problem (5.20) for stochastic processes specialises to the dual problem (5.37) considered in [63]; this only needs some adjustments for notation, along the lines of part 2) of Remark 4.2).

Now set $\mathcal{G}_c := G_T(\Theta(C))$ so that Lemma 5.9 gives an explicit representation of the support function $\delta(\cdot\,|\mathcal{G}_c)$. Then the functions Ψ in (5.8) and (5.20) coincide, and identifying each $h \in L^2(P)$ with the corresponding square-integrable martingale also yields that (5.20) and (5.8) have the same optimal value and therefore the same value function

$$\begin{aligned} v(y) &= \inf_{Z \in \mathcal{Y}(y)} E\left[\tfrac{1}{2} Z_T^2 - Z_T H + \textstyle\int_0^T \delta\big(c_s^M(\eta_s + Z_{s-}\lambda_s)\big|C\big)\, dB_s\right] \\ &= \inf_{h \in \mathcal{D}(y)} \{E[V(h)] + \delta(h|\mathcal{G}_c)\}\,. \end{aligned} \tag{5.21}$$

Moreover, we have the following relation between the primal and dual variables on the level of stochastic processes.

Lemma 5.11. *Suppose that S is in $\mathcal{H}^2_{loc}(P)$ and satisfies the structure condition (SC). For every $x \in \mathbb{R}$, $\vartheta \in \Theta(C)$ and $Z \in \mathcal{M}^2(P)$ with $\delta(Z_T|\mathcal{G}_c) < \infty$, the process*

$$\Big(\big(x + G_t(\vartheta)\big) Z_t - \textstyle\int_0^t \delta\big(c_s^M(\eta_s + Z_{s-}\lambda_s)\big|C\big)\, dB_s\Big)_{0 \le t \le T}$$

is a P-supermartingale.

Proof. The process $\big(x + G(\vartheta)\big)Z_- \int_0^t (\eta + Z_-\lambda)^\top c^M \vartheta\, dB$ is a P-martingale by Lemma 5.8 and because Z is a P-martingale. Moreover,

$$\int \delta\big(c^M(\eta + Z_-\lambda)\big|C\big)\, dB - \int (\eta + Z_-\lambda)^\top c^M \vartheta\, dB$$

is adapted and increasing by the definition of the support function $\delta(\cdot\,|\mathcal{G}_c)$, and integrable due to Lemma 5.9 since $\delta(Z_T|\mathcal{G}_c) < \infty$. Taking the difference gives the result. □

Remark 5.12. In our formulation, the process $\big(\int \delta\big(c^M(\eta + Z_-\lambda)|C\big)\, dB\big)$ plays a similar role as the upper variation process $A(Q)$ in the optional decomposition of Föllmer and Kramkov in [44]; see also Example 3.2 in [72].

Combining Lemma 5.9 with Lemma 5.11 gives a result for stochastic processes which is analogous to Theorem 5.7.

Theorem 5.13. *Assume $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $S \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed convex values such that $\Theta(C)$ is non-empty and the projection of C on the predictable range of S is closed, i.e. $\Pi^S(\omega,t)C(\omega,t)$ is closed P_B-a.e. Then:*

1) *For every $x \in \mathbb{R}$ and $H \in L^2(P)$, there exists a solution $\widehat{\vartheta}(x) \in \Theta(C)$ to*

$$E\left[-\tfrac{1}{2}|x + G_T(\vartheta) - H|^2\right] = \max_{\vartheta \in \Theta(C)}!$$

All solutions $\widehat{\vartheta}(x)$ have the same gains process $G\big(\widehat{\vartheta}(x)\big)$ and satisfy

$$x + G_T\big(\widehat{\vartheta}(x)\big) = I(\widehat{Z}_T) = -\widehat{Z}_T + H,$$

where $\widehat{Z} \in \mathcal{Y}\big(\widehat{y}(x)\big)$ *and* $\widehat{y}(x) \in \mathbb{R}$ *are the unique solutions, respectively, to*

$$\Psi(Z_T) = E\big[\tfrac{1}{2} Z_T^2 - Z_T H + \textstyle\int_0^T \delta\big(c_s^M(\eta_s + Z_{s-}\lambda_s)\big|C\big)\, dB_s\big] = \min_{Z \in \mathcal{Y}(\widehat{y}(x))}!$$

and

$$v(y) + xy = \min_{y \in \mathbb{R}}!$$

2) *The value functions* u *and* v *are conjugate, i.e.*

$$u(x) = \inf_{y \in \mathbb{R}} \{v(y) + xy\},$$
$$v(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\},$$

and continuously differentiable. u *is strictly concave and* v *is strictly convex.*

3) *The process*

$$\big(x + G(\widehat{\vartheta}(x))\big)\widehat{Z} - \int \delta\big(c^M(\widehat{\eta} + \widehat{Z}_- \lambda)\big|C\big)\, dB$$

is a P*-martingale for all solutions* $\widehat{\vartheta}(x)$*, and*

$$(\widehat{\eta}_s + \widehat{Z}_{s-}\lambda_s)^\top c_s^M \widehat{\vartheta}_s(x) = \delta\big(c_s^M(\widehat{\eta}_s + \widehat{Z}_{s-}\lambda_s)\big|C\big) \qquad P_B\text{-a.e.}$$

Proof. 1) By part 3) of Theorem 3.12, we obtain that $\mathcal{G}_c = G_T(\Theta(C))$ is a non-empty, closed, convex subset of $L^2(P)$. Hence we can apply Theorem 5.7 to obtain unique solutions $\widehat{f}(x) \in \mathcal{C}(x)$ to (5.2) and $\widehat{h}\big(\widehat{y}(x)\big) \in \mathcal{D}\big(\widehat{y}(x)\big)$ to (5.8). Since $\widehat{f}(x) - x$ is in $\mathcal{G}_c$, there exists some $\widehat{\vartheta}(x) \in \Theta(C)$ with $x + G_T(\widehat{\vartheta}(x)) = \widehat{f}(x)$ which is a solution to (5.1), and since $\widehat{f}(x)$ is unique, this equality must hold for all solutions. As $G\big(\widehat{\vartheta}(x)\big)$ is an $\mathcal{E}$-martingale, it is uniquely determined by its terminal value and so all solutions $\widehat{\vartheta}(x)$ have this as gains process. Identifying $\widehat{h}\big(\widehat{y}(x)\big)$ with $\widehat{Z}$ shows that $\widehat{Z}$ solves (5.20); this uses the observation before (5.21) that the functions Ψ in (5.20) and (5.8) coincide due to Lemma 5.9.

2) Since the value functions of (5.1) and (5.2) and (5.20) and (5.8), respectively, coincide, the assertion follows from part 2) of Theorem 5.7.

3) By Lemma 5.11, the process $\big(x + G\big(\widehat{\vartheta}(x)\big)\big)\widehat{Z} - \int \delta\big(c^M(\widehat{\eta} + \widehat{Z}_-\lambda)\big|C\big)\, dB$ is a P-supermartingale with initial value $x\widehat{y}(x)$ and final value

$$\begin{aligned}&\big(x + G_T\big(\widehat{\vartheta}(x)\big)\big)\widehat{Z}_T - \textstyle\int_0^T \delta\big(c_s^M(\widehat{\eta}_s + \widehat{Z}_{s-}\lambda_s)\big|C\big)\, dB_s \\ &= \widehat{f}(x)\widehat{h}\big(\widehat{y}(x)\big) - \textstyle\int_0^T \delta\big(c_s^M(\widehat{\eta}_s + \widehat{Z}_{s-}\lambda_s)\big|C\big)\, dB_s.\end{aligned}$$

Moreover, Lemma 5.9 shows that

$$E\big[\int_0^T \delta\big(c_s^M(\widehat{\eta}_s+\widehat{Z}_{s-}\lambda_s)\big|C\big)\,dB_s\big]=\delta\big(\widehat{Z}_T\big|G_T(\Theta(C))\big)=\delta\big(\widehat{h}\big(\widehat{y}(x)\big)\big|\mathcal{G}_c\big).$$

Hence the first relation in (5.11) implies that the above process has constant expectation and is therefore a P-martingale. Combining this with Lemma 5.8 yields that the increasing process $\int\delta\big(c^M(\widehat{\eta}+\widehat{Z}_-\lambda)\big|C\big)\,dB - \int(\widehat{\eta}+\widehat{Z}_-\lambda)^\top c^M\widehat{\vartheta}(x)\,dB$ is a martingale null at zero and hence indistinguishable from the zero process. Since the definition of the support function yields $(\widehat{\eta}_s+\widehat{Z}_{s-}\lambda_s)^\top c_s^M\widehat{\vartheta}_s(x)\le\delta\big(c_s^M(\widehat{\eta}_s+\widehat{Z}_{s-}\lambda_s)\big|C\big)$, we must have equality P_B-a.e., and this completes the proof. □

II.5.3 Related work

Our approach combines duality techniques and constraints with quadratic optimisation problems and so has connections to several areas, in particular utility maximisation under constraints. Very informally, our results can be viewed as the special case of a state-dependent quadratic utility $U(x,\omega)=-\frac{1}{2}|x-H(\omega)|^2$. But they cannot be deduced directly because this "utility function" is not increasing in x and since the duality must be taken in a different setting (L^2 instead of L^0_+). Let us explain the relations in more detail.

The oldest neoclassical work on utility maximisation under constraints is probably by Cvitanić and Karatzas [21]. In an Itô process setting, they introduced the basic ideas of using convex duality and working with the support function of the constraint set to describe the dual variables and also the dual criterion. The seminal work of Kramkov and Schachermayer [62] extended the duality idea to general semimartingale models without trading constraints. One key idea there was to separate the duality arguments into a static level of random variables and a dynamic level of stochastic processes, like in Sections II.5.1 and II.5.2. For the static level, this also needed a bipolar theorem in L^0_+. In Karatzas and Žitković [56], general semimartingale models were combined with *cone* constraints on trading strategies, and the optional decomposition theorem under constraints from [44] was used to obtain the basic duality characterisation of superreplicable consumption-investment pairs. In contrast to [21], the support function δ of the constraint set did not show up explicitly since the latter was a cone; see Remark 5.1. However, [56] obtained a full duality result in the sense that like [62], they could prove the existence of an optimiser for the dual problem and then use that to construct an optimiser for the primal problem. The paper by Mnif and Pham [72] is more general in that it allows convex (not necessarily conic) constraints and does not impose non-negativity for (intermediate values of) the wealth process. The last fact makes it impossible to parametrise strategies by fractions of wealth, and this in turn forces one to use the additive form of the optional decomposition under constraints. Together with the general convex constraints, this leads to an additional term in the objective function for the dual problem. Due to these complications, [72] only obtain a partial (verification) duality result; they show how to construct a primal from a dual optimiser, but do not prove existence of a dual optimiser.

The utility paper closest to our results is probably the one of Pham [76]. It works in finite discrete time with cone constraints (so that, as explained in Remark 5.1, the dual objective function has no explicit extra term), and the key (superreplication) duality rests on the monotonicity of the utility function. But like our approach, it does not impose non-negativity constraints on wealth, and the underlying duality is formulated in an (L^p, L^q)-setting.

The second area of related work is mean-variance hedging and mean-variance portfolio selection. Like utility maximisation, this is huge, and we only focus on a small sample of papers. (An attempt at a broader overview can be found in [89]). Duality for mean-variance hedging without constraints is discussed in Hou and Karatzas [48]. An abstract and static formulation of Markowitz-type problems under cone constraints is given in Sun and Wang [92]; this is similar to Section II.5.1, but gives no duality and is considerably simpler since constraints are conic. Labbé and Heunis [63] study quadratic utility maximisation problems in an Itô process model whose completeness is destroyed by having convex constraints on trading strategies. They introduce (in a fairly complicated way, to our mind) a dual problem for certain processes, show that this has a solution and construct from that a solution to the original problem. Via Itô' s representation theorem, the last step crucially exploits the completeness of the unconstrained market. The existence proof for the dual optimiser is analogous to our Proposition 5.4, and as in Lemma 5.3, the objective function involves an extra term from the support function of the constraint set. It is a matter of taste whether our results are simpler or more natural than those in [63]; but they are definitely much more general.

Markowitz problems in complete and incomplete Itô process models are also studied in Hu and Zhou [49] and Jin and Zhou [53]. The former has cone constraints on strategies, the latter imposes no short sale constraints (which are also described by cones), and both use (quadratic or linear) BSDEs to obtain a solution. This setup has a lot of extra structure, and the continuity of asset prices simplifies matters considerably. For an extension to general semimartingale models with cone constraints and a more detailed discussion, we refer to Chapter IV.

Chapter III

Closed spaces of stochastic integrals with constrained integrands

This chapter corresponds to the article [25] which has been published in the Séminaire de Probabilités XLIII. I would like to thank an anonymous referee for careful reading and helpful suggestions.

III.1 Introduction

In mathematical finance, proving the existence of a solution to optimisation problems like superreplication, utility maximisation or quadratic hedging usually boils down to the same abstract problem: One must show that a subsequence of (predictably) convex combinations of an optimising sequence of wealth processes, i.e. stochastic integrals with respect to the underlying price process S, converges and that the limit is again a wealth process, i.e. can be represented as a stochastic integral with respect to S. As the space of *all* stochastic integrals is closed in the semimartingale topology, this is the suitable topology to work with.

For applications, it is natural to include trading constraints by requiring the strategy (integrand) to lie pointwise in some set C; this set is usually convex to keep the above procedure applicable, and one would like it to depend on the state and time as well. Examples of interest include no shortselling, no borrowing or nonnegative wealth constraints; see e.g. [21, 54]. As pointed out by Delbaen [28] and Karatzas and Kardaras [54], a natural and convenient formulation of constraints is in terms of *correspondences*, i.e. set-valued functions. This is the approach we also advocate and use here.

For motivation, consider a sequence of (predictably convex combinations of) strategies and suppose (as usually happens by the convexification trick) that this converges pointwise. Each strategy is predictable, so constraints should also be "predictable" in some sense. To have the limit still satisfy the same restrictions as the sequence, the constraints should moreover be of the form "closure of a sequence $(\psi^n(\omega,t))$ of random points", since this is where the limit will lie. But if each $\psi^n(\omega,t)$ is a predictable process, the above closure is then a predictable correspondence by the Castaing representation (see Proposition 2.3). This explains why correspondences come up naturally.

In our constrained optimisation problem, assuming that we have predictable, convex, closed constraints, the same procedure as in the unconstrained case yields a sequence of wealth processes (integrals) converging to some limit which is a candidate for the solution of our problem. (We have cheated a little in the motivation — the integrals usually converge, not the integrands.) This limit process is again a stochastic integral, but it still remains to check that the corresponding trading strategy also satisfies the constraints. In abstract terms, one asks whether the limit of a sequence of stochastic integrals of constrained integrands can again be represented as a stochastic integral of some constrained integrand or, equivalently, if the space of stochastic integrals of constrained integrands is closed in the semimartingale topology. We illustrate by a *counterexample* that this is not true in general, since it might happen that some assets become redundant, i.e. can be replicated on some predictable set by trading in the remaining ones. This phenomenon occurs when there is linear dependence between the components of S.

As in [21, 20, 63, 72], one could resolve this issue by simply assuming that there are no redundant assets; then the closedness result is true for all constraints formulated via closed (and convex) sets. Especially in Itô process models with a Brownian filtration, such a non-redundancy condition is useful (e.g. when working with artificial market completions), but it can be restrictive. Alternatively, as in [53, 93, 26], one can study only constraints given by polyhedral or continuous convex sets. While most constraints of practical interest are indeed polyhedral, this is conceptually unsatisfactory as one does not recover all results from the case when there are no redundant assets. A good formulation should thus account for the interplay between the constraints C and redundancies in the assets S.

To realise this idea, we use the *projection on the predictable range* of S. This is a predictable process taking values in the orthogonal projections in $\mathbb{R}^d$; it has been introduced in [84, 32, 28], and allows us to uniquely decompose each integrand into one part containing all relevant information for its stochastic integral and another part having stochastic integral zero. This reduces our problem to the question whether or not the projection of the constraints on the predictable range is closed. Convexity is not relevant for that aspect. Since that approach turns out to give a necessary and sufficient condition, we recover all previous results in [21, 63, 72, 53, 26] as special cases; and in addition, we obtain for constant constraints $C(\omega,t) \equiv C$ that closedness of the space of C-constrained integrands holds for *all* semimartingales if and only if all projections of C in $\mathbb{R}^d$ are closed. The well-known characterisation of polyhedral cones thus implies in particular that the closedness result for *constant* convex cone constraints is true for arbitrary semimartingales if and only if the constraints are polyhedral.

For a general constraint set $C(\omega, t)$ which is closed and convex, the set of stochastic integrals of C-constrained integrands is the prime example of a predictably convex space of stochastic integrals. By adapting arguments from [28], we show that this is in fact the *only* class of predictably convex spaces of stochastic integrals which are closed in the semimartingale topology. So in this chapter we make both mathematical contributions to stochastic calculus and financial contributions in the modelling and handling of trading constraints for optimisation problems from mathematical finance.

The remainder of the chapter is organised as follows. In Section III.2, we formulate the problem in the terminology of stochastic processes and provide some results on measurable correspondences and measurable selectors. These are needed to introduce and handle the constraints. Section III.3 contains a counterexample which illustrates where the difficulties arise and motivates in a simple setting the definition of the projection on the predictable range. The main results discussed above are established in Section III.4. Section III.5 gives the construction of the projection on the predictable range as well as two proofs omitted in Section III.4. Finally, Section III.6 briefly discusses some related work.

III.2 Problem formulation and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t < \infty}$ satisfying the usual conditions of completeness and right-continuity. For all notation concerning stochastic integration, we refer to the book of Jacod and Shiryaev [52].

Set $\overline{\Omega} := \Omega \times [0, \infty)$. The space of all $\mathbb{R}^d$-valued semimartingales is denoted by $\mathcal{S}^{0,d}(P) := \mathcal{S}^0(P; \mathbb{R}^d)$, or simply $\mathcal{S}(P)$ if the dimension is clear. The *Émery distance* (see [41]) of two semimartingales X and Y is $d(X, Y) = \sup_{|\vartheta| \leq 1} \big(\sum_{n \in \mathbb{N}} 2^{-n} E\big[1 \wedge |(\vartheta \cdot (X - Y))_n|\big]\big)$, where $(\vartheta \cdot X)_t := \int_0^t \vartheta_s dX_s$ stands for the *vector stochastic integral*, which is by construction a real-valued semimartingale, and the supremum is taken over all $\mathbb{R}^d$-valued predictable processes ϑ bounded by 1. With this metric, $\mathcal{S}(P)$ is a complete topological vector space, and the corresponding topology is called the *semimartingale topology*. For brevity, we say "in $\mathcal{S}(P)$" for "in the semimartingale topology". For a given $\mathbb{R}^d$-valued semimartingale S, we write $\mathcal{L}(S)$ for the space of $\mathbb{R}^d$-valued, S-integrable, predictable processes ϑ and $L(S)$ for the space of equivalence classes $[\vartheta] = [\vartheta]^S = \{\varphi \in \mathcal{L}(S) \mid \varphi \cdot S = \vartheta \cdot S\}$ of processes in $\mathcal{L}(S)$ which yield the same stochastic integral with respect to S, identifying processes equal up to P-indistinguishability. By Theorem V.4 in [71], the space of stochastic integrals $\{\vartheta \cdot S \mid \vartheta \in \mathcal{L}(S)\}$ is closed in $\mathcal{S}(P)$. Equivalently, $L(S)$ is a complete topological vector space with respect to $d_S\big([\vartheta],[\varphi]\big) = d(\vartheta \cdot S, \varphi \cdot S)$, where ϑ and φ are representatives of the equivalence classes $[\vartheta]$ and $[\varphi]$.

In this chapter, we generalise the above closedness result from [71] to integrands restricted to lie in a given closed set, in the following sense. Let $C(\omega, t)$ be a non-empty, closed subset of $\mathbb{R}^d$ which may depend on ω and t in a predictably measurable way. Definition 2.2 below makes this precise: C should be a *predictable correspondence with closed values*. Denote by

$$\mathcal{C} := \mathcal{C}^S := \big\{\psi \in \mathcal{L}(S) \bigm| \psi(\omega,t) \in C(\omega,t) \text{ for all } (\omega,t)\big\} \tag{2.1}$$

the set of *C-valued* or *C-constrained integrands* for S. If (ψ^n) is a sequence in $\mathcal{C}^S$ such that $(\psi^n \cdot S)$ converges to some X in the semimartingale topology, does there exist a ψ in $\mathcal{C}^S$ such that $X = \psi \cdot S$? In general, the answer is negative, as a simple counterexample in the next section illustrates, and so we ask under which conditions the above is true. By the closedness in $\mathcal{S}(P)$ of the space of all stochastic integrals, the limit X can always be represented as some stochastic integral $\vartheta \cdot S$. Thus it is enough to decide whether or not there exists for the limit class $[\vartheta]$ a representative ψ which is C-valued. Equivalently, one can ask whether $\mathcal{C}^S \cdot S$ is closed in $\mathcal{S}(P)$ or if the corresponding set

$$[\mathcal{C}] := [\mathcal{C}]^S := \big\{[\vartheta] \in L(S) \bigm| [\vartheta] \cap \mathcal{C} \neq \emptyset\big\}$$

of equivalence classes of elements of $\mathcal{C}^S$ is closed in $\big(L(S), d_S\big)$.

As already explained, this question arises naturally in mathematical finance for various optimisation problems under trading constraints; see [44], [72], [78], [63], [53] and [24]. But not all papers make it equally clear whether the procedure outlined in the introduction can be or is being used. For [63] and [53], this is clarified in Chapter II. Under additional assumptions, the closedness of $\mathcal{C}^S \cdot S$ in the semimartingale topology is sufficient to apply the results of Föllmer and Kramkov [44] on the optional decomposition under constraints, which give a dual characterisation of payoffs that can be super-replicated by constrained trading strategies. This is used in [72], [78] and [56] to prove the existence of solutions to constrained utility maximisation problems. The results in [44] are formulated more generally for sets of (special) semimartingales which are predictably convex.

Definition 2.1. A set $\mathfrak{S}$ of semimartingales is *predictably convex* if $h \cdot X + (1-h) \cdot Y \in \mathfrak{S}$ for all X and Y in $\mathfrak{S}$ and all $[0,1]$-valued predictable processes h. Analogously, a set $\mathfrak{C} \subseteq \mathcal{L}(S)$ of integrands is *predictably convex* if $h\vartheta + (1-h)\varphi \in \mathfrak{C}$ for all ϑ and φ in $\mathfrak{C}$ and all $[0,1]$-valued predictable processes h.

The prime example of predictably convex sets of integrands is given by C-constrained integrands when C is convex-valued. Theorem 4.11 below shows that *all* predictably convex spaces $\mathfrak{C}$ of integrands must be of this form if $\mathfrak{C} \cdot S$ is in addition closed in $\mathcal{S}(P)$.

To formulate precisely the assumptions on the (random and time-dependent) set C, we adapt the language of measurable correspondences to our framework of predictable measurability and recall for later use some of the results in this context. Note that the general results we exploit do not depend on special properties of the predictable σ-field on $\overline{\Omega}$. However, we do use that the range space $\mathbb{R}^d$ is metric and σ-compact; this ensures by Proposition 1A in [83] or the proof of Lemma 18.2 in [2] that weak measurability and measurability for a closed-valued correspondence coincide in our setting.

Definition 2.2. A mapping $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ is called an *($\mathbb{R}^d$-valued) correspondence.* Its *domain* is $\operatorname{dom}(C) := \{(\omega,t) \mid C(\omega,t) \neq \emptyset\}$. We call a correspondence C *predictable* if $C^{-1}(F) := \{(\omega,t) \mid C(\omega,t) \cap F \neq \emptyset\}$ is a predictable set for each closed $F \subseteq \mathbb{R}^d$. A correspondence has *predictable graph* if its graph $\operatorname{gr}(C) := \{(\omega,t,x) \in \overline{\Omega} \times \mathbb{R}^d \mid x \in C(\omega,t)\}$ is in $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$. A *predictable selector* of a predictable correspondence C is a predictable process ψ which satisfies $\psi(\omega,t) \in C(\omega,t)$ for all $(\omega,t) \in \operatorname{dom}(C)$.

The following results ensure the existence of predictable selectors in all situations relevant for us.

Proposition 2.3 (Castaing)**.** *For a correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ with closed values, the following are equivalent:*

1) *C is predictable.*

2) $\operatorname{dom}(C)$ *is predictable and there exists a* Castaing representation *of C, i.e. a sequence (ψ^n) of predictable selectors of C such that*

$$C(\omega,t) = \overline{\{\psi^1(\omega,t), \psi^2(\omega,t), \dots\}} \qquad \text{for each } (\omega,t) \in \operatorname{dom}(C).$$

Proof. See Corollary 18.14 in [2] or Theorem 1B in [83]. □

Proposition 2.4 (Aumann)**.** *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a correspondence with non-empty values and predictable graph and μ a finite measure on $(\overline{\Omega}, \mathcal{P})$. Then there exists a predictable process ψ with $\psi(\omega,t) \in C(\omega,t)$ μ-a.e.*

Proof. See Corollary 18.27 in [2]. □

The proof of Proposition 2.4 is based on the following result on projections to which we refer later.

Proposition 2.5. *Let $(R, \mathcal{R}, \mu)$ be a σ-finite measure space, $\mathcal{R}_\mu$ the σ-field of μ-measurable sets and A in $\mathcal{R}_\mu \otimes \mathcal{B}(\mathbb{R}^d)$. Then the projection $\pi_R(A)$ of A on R belongs to $\mathcal{R}_\mu$.*

Proof. See Theorem 18.25 in [2]. □

Measurability and graph measurability of a correspondence are linked as follows.

Proposition 2.6. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ be a correspondence. If C is predictable, its closure correspondence $\overline{C}$ given by $\overline{C}(\omega,t) := \overline{C(\omega,t)}$ has a predictable graph.*

Proof. See Theorem 18.6 in [2]. □

Since we require in (2.1) for our integrands ψ that $\psi(\omega,t) \in C(\omega,t)$ for all (ω,t), we shall assume, as motivated in the introduction, that C *is predictable and has closed values.* Then Proposition 2.3 guarantees the existence of predictable selectors. Moreover, we shall use that predictable measurability of a correspondence is preserved under transformations by Carathéodory functions and is stable under countable unions and intersections. Recall that a function $f : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^m$ is called *Carathéodory* if $f(\omega,t,x)$ is predictable with respect to (ω,t) and continuous in x.

Proposition 2.7. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed values and $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^d$ and $g : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}^m$ Carathéodory functions. Then C' and C'' given by $C'(\omega,t) = \{y \in \mathbb{R}^m \mid f(\omega,t,y) \in C(\omega,t)\}$ and $C''(\omega,t) = \overline{\{g(\omega,t,x) \mid x \in C(\omega,t)\}}$ are predictable correspondences with closed values.*

Proof. See Corollaries 1P and 1Q in [83]. □

Proposition 2.8. *Let $C^n : \overline{\Omega} \to 2^{\mathbb{R}^d}$ for each $n \in \mathbb{N}$ be a predictable correspondence with closed values and define the correspondences C' and C'' by $C'(\omega,t) = \bigcap_{n\in\mathbb{N}} C^n(\omega,t)$ and $C''(\omega,t) = \bigcup_{n\in\mathbb{N}} C^n(\omega,t)$. Then C' and C'' are predictable and C' is closed-valued.*

Proof. See Theorem 1M in [83] and Lemma 18.4 in [2]. □

To establish a relation between predictably convex spaces of integrands and C-valued integrands, we later use the following result, which is a reformulation of the contents of Theorem 5 in [28]. We view an $\mathbb{R}^d$-valued predictable process on Ω as a $\mathcal{P}$-measurable $\mathbb{R}^d$-valued mapping on $\overline{\Omega}$, take some probability μ on $(\overline{\Omega}, \mathcal{P})$ and denote by $\overline{B(0,r)}^{L^\infty}$ and $\overline{B(0,r)}$ the closures of a ball of radius r in $L^\infty(\overline{\Omega}, \mathcal{P}, \mu; \mathbb{R}^d)$ and in $\mathbb{R}^d$, respectively. Predictable convexity is understood as in the second part of Definition 2.1.

Proposition 2.9. *Let $\mathfrak{K}$ be a predictably convex and μ-weak*-compact subset of $\overline{B(0,r)}^{L^\infty}$ with $0 \in \mathfrak{K}$. Then there exists a predictable correspondence*

$K : \overline{\Omega} \to 2^{\overline{B(0,r)}} \setminus \{\emptyset\}$, *whose values are convex and compact and contain zero, such that*

$$\mathfrak{K} = \Big\{ \vartheta \in L^\infty(\overline{\Omega}, \mathcal{P}, \mu; \mathbb{R}^d) \;\Big|\; \vartheta(\omega,t) \in K(\omega,t) \ \mu\text{-a.e.} \Big\}.$$

Proof. In the proof of Theorem 5 in [28], the set $\mathcal{C}^\lambda$ defined there for $\lambda > 0$ contains zero and is by Lemmas 10 and 11 in [28] a predictably convex and weak*-compact subset of $\overline{B(0,\lambda)}^{L^\infty}$. No other properties of $\mathcal{C}^\lambda$ are used. So we can modify the proof of Theorem 5 in [28] by replacing the use of the Radon–Nikodým theorem of Debreu and Schmeidler (Theorem 2 in [27]) with that of Artstein (Theorem 9.1 in [4]). This yields that $K := \Phi^r$ constructed in that proof is predictably measurable and has not only (as argued in [28]) predictable graph. Replacing the correspondence K coming from this construction by $K \cap \overline{B(0,r)}$ then gives that K is valued in $2^{\overline{B(0,r)}}$. □

III.3 A motivating example

In this section, we give a simple example of a semimartingale Y and a predictable correspondence C with non-empty, closed, convex cones as values such that $\mathcal{C}^Y \cdot Y$ is not closed in $\mathcal{S}(P)$. This illustrates where the problems with our basic question arise and suggests a way to overcome them. The example is the same as Example 2.2 in [26], but we use it here for a different purpose and with different emphasis.

Let $W = (W^1, W^2, W^3)^\top$ be a 3-dimensional Brownian motion and $Y = \sigma \cdot W$, where

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The matrix σ and hence $\hat{c} = \sigma\sigma^\top$ have a non-trivial kernel spanned by $w = \frac{1}{\sqrt{2}}(0,1,1)^\top$, i.e. $\mathrm{Ker}(\hat{c}) = \mathrm{Ker}(\sigma) = \mathbb{R}w = \mathrm{span}\{w\}$. By construction, the stochastic integral of each $\mathbb{R}^3$-valued predictable process valued in $\mathrm{Ker}(\hat{c})$ $dP \otimes dt$-a.e. is zero, and vice versa. Thus the equivalence class $[\vartheta]^Y$ of any given $\vartheta \in \mathcal{L}(Y)$ is given by

$$[\vartheta]^Y = \{\vartheta + hw \mid h \text{ is a real-valued predictable process}\}$$

up to $dP \otimes dt$-a.e. equality, since adding a representative of 0 to some element of $\mathcal{L}(Y)$ does not change its equivalence class. Let K be the closed and convex cone

$$K = \big\{(x,y,z)^\top \in \mathbb{R}^3 \;\big|\; x^2 + y^2 \le z^2,\ z \ge 0\big\}$$

and C the (constant) predictable correspondence with non-empty and closed values given by $C(\omega,t) = K$ for all $(\omega,t) \in \overline{\Omega}$. Define the sequence of (constant) processes (ψ^n) by $\psi^n = (1, \sqrt{n^2-1}, n)^\top$ for each $n \in \mathbb{N}$. In geometric terms, K is a circular cone around the z-axis, and (ψ^n) is a sequence of points on its surface going to infinity. (Instead of n, any sequence $z_n \to \infty$ in $[1,\infty)$ would do as well.) Each ψ^n is C-valued, and we compute $\psi^n \cdot Y = (\sigma\psi^n) \cdot W = W^1 + \big(\sqrt{n^2-1} - n\big)(W^2 - W^3)$. Using this explicit expression yields by a simple calculation that $\psi^n \cdot Y \to W^1$ locally in $\mathcal{M}^2(P)$ and therefore in $\mathcal{S}(P)$; see [26] for details. However, the (constant) process $e_1 := (1,0,0)^\top$ leading to the limiting stochastic integral $e_1 \cdot Y = W^1$ does not have values in C, and since its equivalence class is $\big\{e_1 + hw \bigm| h \text{ is a real-valued predictable process}\big\}$, also no other integrand equivalent to e_1 does. Thus $\mathcal{C}^Y \cdot Y$ is not closed in $\mathcal{S}(P)$.

To see why this causes problems, define $\tau := \inf\big\{t > 0 \bigm| |W_t| = 1\big\}$ and set $S := Y^\tau$. The arguments above then imply that the sequence $(\psi^n \cdot Y^\tau)$ is bounded from below (uniformly in n, t, ω) and converges in $\mathcal{S}(P)$ to $(W^1)^\tau$, which cannot be represented as $\psi \cdot S$ for any C-valued integrand ψ. Thus the set $\mathcal{C}^S \cdot S$ does not satisfy Assumption 3.1 of the optional decomposition theorem under constraints in [44]. But for instance the proof of Proposition 2.13 in [56] (see p. 1835) explicitly uses that result of [44] in a setting where constrained integrands could be given by C-valued integrands as above. So technically, the argument in [56] is not valid without further assumptions (and Theorem 4.5 and Corollary 4.9 below show ways to fix this).

What can we learn from the counterexample? The key point is that *the convergence of stochastic integrals $\psi^n \cdot Y$ need not imply the pointwise convergence of their integrands.* Without constraints, this causes no problems; by Mémin's theorem, the limit is still *some* stochastic integral of Y, here $e_1 \cdot Y$. But if we insist on having C-valued integrands, the example shows that we ask for too much. Since K is closed, we can deduce above that $(|\psi^n|)$ must diverge (otherwise we should get along a subsequence a limit, which would be C-valued by closedness), and in fact $|\psi^n| = \sqrt{2}\,n \to \infty$. But at the same time, $(\sigma\psi^n)$ converges to $e_1 = (1,0,0)^\top$ — and this observation brings up the key idea of not looking at ψ^n, but at suitable projections of ψ^n linked (via σ) to the integrator Y.

To make this precise, denote the orthogonal projection on $\mathrm{Im}(\sigma\sigma^\top)$ by

$$\Pi^Y = \mathbb{1}_{d\times d} - ww^\top = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then $\Pi^Y\psi^n = \big(1, \frac{1}{2}(\sqrt{n^2-1} - n), -\frac{1}{2}(\sqrt{n^2-1} - n)\big)^\top$ converges to the limit integrand $(1,0,0)^\top = e_1$. We might worry about the obvious fact that $\Pi^Y\psi^n$ does not take values in C; but for the stochastic integrals, this does not matter because $(\Pi^Y\psi^n) \cdot Y = \psi^n \cdot Y$. Indeed, any $\vartheta \in \mathcal{L}(Y)$ can be written as a sum $\vartheta = \Pi^Y\vartheta + (ww^\top)\vartheta$ of one part with values in $\mathrm{Im}(\sigma\sigma^\top)$ and another part orthogonal to the first one; and since $\sigma^\top w = 0$ implies that $\big((ww^\top)\vartheta\big) \cdot Y = (\vartheta^\top ww^\top\sigma)^\top \cdot W = 0$, the claim follows. Going a little further, we even have for any $\vartheta \in \mathcal{L}(Y)$ and any $\mathbb{R}^d$-valued predictable process φ that

$$\varphi \in \mathcal{L}(Y) \text{ with } \varphi \cdot Y = \vartheta \cdot Y \iff \Pi^Y\varphi = \Pi^Y\vartheta \ dP \otimes dt\text{-a.e.,} \tag{3.1}$$

by using that $\mathrm{Ker}(\sigma\sigma^\top) \cap \mathrm{Im}(\sigma\sigma^\top) = \{0\}$ and that $\sigma^\top(\Pi^Y v) = \sigma^\top v$ for all $v \in \mathbb{R}^d$ to check the Y-integrability of φ. The significance of (3.1) is that the stochastic integral $\vartheta \cdot Y$ is uniquely determined by $\Pi^Y\vartheta$, and so $\Pi^Y\vartheta$ gives a "minimal" choice of a representative of the equivalence class $[\vartheta]^Y$. Moreover, Π^Y gives via (3.1) a simple way to decide whether or not a given $\mathbb{R}^d$-valued predictable process φ belongs to the equivalence class $[\vartheta]^Y$.

Coming back to the set K, we observe that

$$\Pi^Y K = \left\{ \left(x, \frac{1}{2}(y-z), -\frac{1}{2}(y-z)\right)^\top \,\middle|\, x^2 + y^2 \le z^2,\ z \ge 0 \right\}$$

is the projection of the cone K on the plane through the origin and with the normal vector $(0,1,1)^\top$. In geometric terms, the projection of each horizontal slice of the cone transforms the circle above the x-y-plane into an ellipse in the projection plane having the origin as a point of its boundary. As we move up along the z-axis, the circles become larger, and so do the ellipses which in addition flatten out towards the line through the origin and the point $e_1 = (1,0,0)^\top$. But since they never reach that line although they come arbitrarily close, $\Pi^Y K$ is not closed in $\mathbb{R}^d$ — and this is the source of all problems in our counterexample. It explains why the limit $e_1 = \lim_{n\to\infty} \Pi^Y \psi^n$ is not in $\Pi^Y K$, which implies by (3.1) that there cannot exist any C-valued integrand ψ such that $\Pi^Y\psi = e_1$. But the insight about $\Pi^Y K$ also suggests that if we assume for a predictable correspondence C that

$$\Pi^Y C(\omega, t) \text{ is closed } dP \otimes dt\text{-a.e.,} \tag{3.2}$$

we ought to get that $\mathcal{C}^Y \cdot Y$ is closed in $\mathcal{S}(P)$. This indeed works (see Theorem 4.5), and it turns out that condition (3.2) is not only sufficient, but also necessary.

The above explicit computations rely on the specific structure of Y, but they nevertheless motivate the approach for a general semimartingale S. We are going to define a predictable process Π^S taking values in the orthogonal projections in $\mathbb{R}^d$ and satisfying (3.1) with $dP \otimes dt$ replaced by a suitable measure on $\big(\overline{\Omega}, \mathcal{P}\big)$ to control the stochastic integrals with respect to S. The process Π^S will be called the *projection on the predictable range* and will allow us to formulate and prove our main results in the next section.

III.4 Main results

This section contains the main results (Theorems 4.5 and 4.11) as well as some consequences and auxiliary results. Before we can formulate and prove them, we need some facts and results about the projection on the predictable range of S. For the reader's convenience, the actual construction of Π^S is postponed to Section III.5.

As in [52], Theorem II.2.34, each semimartingale S has the *canonical representation*

$$S = S_0 + S^c + \widetilde{A} + [x \mathbb{1}_{\{|x| \leq 1\}}] * (\mu - \nu) + [x \mathbb{1}_{\{|x|>1\}}] * \mu$$

with the jump measure μ of S and its predictable compensator ν. Then the triplet (b, c, F) of predictable characteristics of S consists of a predictable $\mathbb{R}^d$-valued process b, a predictable nonnegative-definite matrix-valued process c and a predictable process F with values in the set of Lévy measures such that

$$\widetilde{A} = b \cdot B, \qquad [S^c, S^c] = c \cdot B \qquad \text{and} \qquad \nu = F \cdot B, \tag{4.1}$$

where $B := \sum_{i=1}^d \big([S^c, S^c]^{i,i} + \mathrm{Var}(\widetilde{A}^i)\big) + (|x|^2 \wedge 1) * \nu$.

Note that B is locally bounded since it is predictable and increasing. Therefore $P \otimes B$ is σ-finite on $\big(\overline{\Omega}, \mathcal{P}\big)$ and there exists a probability measure P_B equivalent to $P \otimes B$. By the construction of the stochastic integral, S-integrable, predictable processes which are P_B-a.e. equal yield the same stochastic integral with respect to S (up to P-indistinguishability). Put differently, $\varphi = \vartheta$ P_B-a.e. implies for the equivalence classes in $L(S)$ that $[\varphi] = [\vartheta]$. But the converse is not true; a sufficient and necessary condition involves the projection Π^S on the predictable range of S, as we shall see below. Because S is now (in contrast to Section III.3) a general semimartingale, the actual construction of Π^S and the proof of its properties become more technical and are postponed to the next section. We give here merely the definition and two auxiliary results.

Definition 4.1. The *projection on the predictable range of* S is a predictable process $\Pi^S : \overline{\Omega} \to \mathbb{R}^{d \times d}$ which takes values in the orthogonal projections in $\mathbb{R}^d$ and has the following property: If $\vartheta \in \mathcal{L}(S)$ and φ is predictable, then φ is in $\mathcal{L}(S)$ with $\varphi \cdot S = \vartheta \cdot S$ if and only if $\Pi^S \vartheta = \Pi^S \varphi$ P_B-a.e. We choose and fix one version of Π^S.

Remark 4.2. There are many possible choices for a process B satisfying (4.1). However, the definition of Π^S is independent of the choice of B in the sense that (with obvious notation) $\Pi^{S,B} \vartheta = \Pi^{S,B} \varphi$ P_B-a.e. if and only if $\Pi^{S,B'} \vartheta = \Pi^{S,B'} \varphi$ $P_{B'}$-a.e. This is because stochastic integrals of S do not depend on the choice of B.

As illustrated by the example in Section III.3, the convergence in $\mathcal{S}(P)$ of stochastic integrals does not imply in general that the integrands converge

P_B-a.e. But like in the example, a subsequence of the projections of the integrands on the predictable range does.

Lemma 4.3. *Let* (ϑ^n) *be a sequence in* $\mathcal{L}(S)$ *such that* $\vartheta^n \cdot S \to \vartheta \cdot S$ *in* $\mathcal{S}(P)$. *Then there exists a subsequence* (n_k) *such that* $\Pi^S \vartheta^{n_k} \to \Pi^S \vartheta$ P_B*-a.e.*

Lemma 4.4. *Let* $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ *be a predictable correspondence with closed values and such that the projection on the predictable range of* S *is* not closed, *i.e.*

$$\widetilde{F} = \big\{(\omega, t) \in \overline{\Omega} \;\big|\; \Pi^S(\omega,t)C(\omega,t) \text{ is not closed}\big\}$$

has outer P_B*-measure* > 0. *Then there exist* $\vartheta \in \mathcal{L}(S)$ *and a sequence* (ψ^n) *of* C*-valued integrands such that* $\psi^n \cdot S \to \vartheta \cdot S$ *in* $\mathcal{S}(P)$, *but there is no* C*-valued integrand* ψ *such that* $\psi \cdot S = \vartheta \cdot S$. *Equivalently, there exists a sequence* $\big([\psi^n]\big)$ *in* $[\mathcal{C}]^S$ *such that* $[\psi^n] \overset{L(S)}{\to} [\vartheta]$ *but* $[\vartheta] \notin [\mathcal{C}]^S$, *i.e.* $[\mathcal{C}]^S$ *is not closed in* $L(S)$.

Lemmas 4.3 and 4.4 as well as the existence of Π^S will be shown in Section III.5. Admitting that, we can now prove our first main result; related work in [54] is discussed in Section III.6. Recall the definition of $\mathcal{C} := \mathcal{C}^S$ from (2.1).

Theorem 4.5. *Let* $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ *be a predictable correspondence with closed values. Then* $\mathcal{C}^S \cdot S$ *is closed in* $\mathcal{S}(P)$ *if and only if the projection of* C *on the predictable range of* S *is closed, i.e.* $\Pi^S(\omega,t)C(\omega,t)$ *is closed* P_B*-a.e. Equivalently: There exists a* C*-valued integrand* ψ *with* $X = \psi \cdot S$ *for any sequence* (ψ^n) *of* C*-valued integrands with* $\psi^n \cdot S \to X$ *in* $\mathcal{S}(P)$ *if and only if the projection of* C *on the predictable range of* S *is closed.*

Proof. "$\Rightarrow$": This implication follows immediately from Lemma 4.4.
"$\Leftarrow$": Let (ψ^n) be a sequence in $\mathcal{C}$ with $\psi^n \cdot S \to X$ in $\mathcal{S}(P)$. Then there exist by Mémin's theorem $\vartheta \in \mathcal{L}(S)$ with $X = \vartheta \cdot S$ and by Lemma 4.3 a subsequence, again indexed by n, with $\Pi^S \psi^n \to \Pi^S \vartheta$ P_B-a.e. So it remains to show that we can find a C-valued representative ψ of the limit class $[\vartheta] = [\Pi^S \vartheta]$. To that end, we observe that the P_B-a.e. closedness of $\Pi^S(\omega,t)C(\omega,t)$ implies that $\Pi^S \vartheta = \lim_{n\to\infty} \Pi^S \psi^n \in \Pi^S C$ P_B-a.e. By Proposition 2.7, the correspondences given by $\{\Pi^S(\omega,t)\vartheta(\omega,t)\}$, $C'(\omega,t) = \{\Pi^S(\omega,t)\vartheta(\omega,t)\} \cap \Pi^S(\omega,t)C(\omega,t)$ and $C''(\omega,t) = \big\{z \in \mathbb{R}^d \;\big|\; \Pi^S(\omega,t)z \in C'(\omega,t)\big\} \cap C(\omega,t)$ are predictable and closed-valued. Indeed, $\Pi^S \vartheta$ is a predictable process, and $\big\{z \in \mathbb{R}^d \;\big|\; \Pi^S(\omega,t)z \in C'(\omega,t)\big\}$ and $\Pi^S C = \overline{\Pi^S C}$ are the pre-image and (the closure of) the image of a closed-valued correspondence under a Carathéodory function, respectively. Thus C' and C'' are the intersections of two predictable and closed-valued correspondences and therefore predictable by Proposition 2.8. So there exists by Proposition 2.3 a predictable selector ψ of C'' on $\mathrm{dom}(C'') = \big\{(\omega,t) \;\big|\; \Pi^S(\omega,t)\vartheta(\omega,t) \in \Pi^S(\omega,t)C(\omega,t)\big\}$. This ψ

can be extended to a C-valued integrand by using any predictable selector on the P_B-nullset $\big(\mathrm{dom}(C'')\big)^c$. By construction, ψ is then in $\mathcal{C}$ and satisfies $\Pi^S\psi = \Pi^S\vartheta$ P_B-a.e., so that $\psi \in [\vartheta]$ by the definition of Π^S. This completes the proof. □

Theorem 4.5 gives as necessary and sufficient condition for the closedness of the space of C-constrained integrals of S that the projection of the constraint set C on the predictable range of S is closed. This uses information from both the semimartingale S and the constraints C, as well as their interplay. We shall see below how this allows to recapture several earlier results as special cases.

Corollary 4.6. *Suppose that $S = S_0 + M + A$ is in $\mathcal{S}^2_{loc}(P)$ and define the process a via $A = a \cdot B$. If*

$$[0]^M = \big\{ha \bigm| h \text{ is real-valued and predictable}\big\} \tag{4.2}$$

up to P_B-a.e. equality, then $\mathcal{C}^S \cdot S$ is closed in $\mathcal{S}(P)$ for all predictable correspondences $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ with closed values.

Proof. Lemma 5.1 below shows that (4.2) implies $[0]^S = [0]^M \cap [0]^A = \{0\}$ and therefore $\Pi^S = \mathbb{1}_{d\times d}$ by (5.2) below. So the projection of any closed-valued correspondence C on the predictable range of S is closed, which gives the assertion by Theorem 4.5. □

In applications from mathematical finance, S often satisfies the so-called *structure condition (SC)*, i.e. $S = S_0 + M + A$ is in $\mathcal{S}^2_{loc}(P)$ and there exists an $\mathbb{R}^d$-valued predictable process $\lambda \in \mathcal{L}^2_{loc}(M)$ such that $A = \lambda \cdot \langle M, M\rangle$ or, equivalently, $a = \hat{c}\lambda$ P_B-a.e.; this is a weak no-arbitrage type condition. In this situation, Lemma 5.1 below gives $[0]^M \subseteq [0]^A$, and thus condition (4.2) holds if and only if $[0]^M = \{0\}$ (up to P_B-a.e. equality), which means that $\hat{c}$ is P_B-a.e. invertible. This is the case covered in Lemma 3.1 in [72], where one has conditions only on S but not on C. Basically this ensures that there are no redundant assets, i.e. every stochastic integral is realised by exactly one integrand (up to P_B-a.e. equality).

The opposite extreme is to place conditions only on C that ensure closedness of $\mathcal{C}^S \cdot S$ for arbitrary semimartingales S, as in Theorem 3.5 of [26]. We recover this as a special case in the following corollary; note that in a slight extension over [26], the constraints need not be convex. Recall that a closed convex set $K \subseteq \mathbb{R}^d$ is called *continuous* if its support function $\delta(v|K) = \sup_{w\in K} w^\top v$ is continuous for all vectors $v \in \mathbb{R}^d$ with $|v| = 1$; see [47].

Corollary 4.7. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ be a predictable correspondence with closed values. Then $\mathcal{C}^Y \cdot Y$ is closed in $\mathcal{S}(P)$ for all semimartingales Y if with probability 1, for all $t \geq 0$ all projections $\Pi C(\omega, t)$ of $C(\omega, t)$ are closed*

in $\mathbb{R}^d$.
In particular, if with probability 1*, every* $C(\omega, t)$*,* $t \geq 0$*, is compact, or polyhedral, or a continuous and convex set, then* $\mathcal{C}^Y \cdot Y$ *is closed in* $\mathcal{S}(P)$ *for all semimartingales* Y*.*

Proof. If a set is compact or polyhedral, all its projections have the same property (see Corollary 2.15 in [58]) and are thus closed. For a continuous convex set, every projection is closed by Theorem 1.3 in [47]. Now if with probability 1, for all $t \geq 0$ all projections $\Pi C(\omega, t)$ of $C(\omega, t)$ are closed, the projection $\Pi^Y C$ of C on the predictable range of every semimartingale Y is closed $P \otimes B^Y$-a.e. So $\mathcal{C}^Y \cdot Y$ is closed in $\mathcal{S}(P)$ by Theorem 4.5. □

Combining Theorem 4.5 with the example in Section III.3, we obtain the following corollary. It is formulated for fixed sets K, but can probably be generalised to predictable correspondences C by using measurable selections.

Corollary 4.8. *Suppose* $(\Omega, \mathcal{F}, P)$ *is sufficiently rich. Fix* $K \subseteq \mathbb{R}^d$ *and define as in* (2.1) $\mathcal{K}^Y = \{\psi \in \mathcal{L}(Y) \mid \psi(\omega, t) \in K \text{ for all } (\omega, t)\}$*. Then* $\mathcal{K}^Y \cdot Y$ *is closed in* $\mathcal{S}(P)$ *for all* $\mathbb{R}^d$*-valued semimartingales* Y *if and only if all projections* ΠK *of* K *in* $\mathbb{R}^d$ *are closed.*

Proof. The "if" part follows immediately from Theorem 4.5. For the converse, assume by way of contradiction that there is a projection Π in $\mathbb{R}^d$ such that ΠK is not closed. Let W be a d-dimensional Brownian motion and set $Y = \Pi^\top \cdot W$. Then Π is the projection on the predictable range of Y, and therefore $\mathcal{K}^Y \cdot Y$ is not closed by Theorem 4.5. □

If the constraints are not only convex, but also cones, a characterisation of convex polyhedra due to Klee [58] gives an even sharper result.

Corollary 4.9. *Let* $K \subseteq \mathbb{R}^d$ *be a closed convex cone. Then* $\mathcal{K}^Y \cdot Y$ *is closed in* $\mathcal{S}(P)$ *for all* $\mathbb{R}^d$*-valued semimartingales* Y *if and only if* K *is polyhedral.*

Proof. By Corollary 4.8, $\mathcal{K}^Y \cdot Y$ is closed in $\mathcal{S}(P)$ if and only if all projections ΠK are closed in $\mathbb{R}^d$. But Theorem 4.11 in [58] says that all projections of a convex cone are closed in $\mathbb{R}^d$ if and only if that cone is polyhedral. □

Remark 4.10. Armed with the last result, we can briefly come back to the proof of Proposition 2.13 in [56]. We have already pointed out in Section III.3 that the argument in [56] uses the optional decomposition under constraints from [44], without verifying its Assumption 3.1. In view of Corollary 4.9, we can now be more precise: The argument in [56] as it stands (i.e. without assumptions on S) only works for *polyhedral* cone constraints; for others, one could by Corollary 4.9 construct a semimartingale S giving a contradiction.

We now turn to our second main result. Recall again the definition of $\mathcal{C}$ from (2.1) and note that for a correspondence C with convex values, $\mathcal{C}$ is the prime example of a predictably convex space of integrands. The next theorem shows that this is actually the only class of predictably convex integrands if we assume in addition that the resulting space $\mathcal{C} \cdot S$ of stochastic integrals is closed in $\mathcal{S}(P)$. The result and its proof are inspired from Theorems 3 and 4 in [28], but require quite a number of modifications.

Theorem 4.11. *Let $\mathfrak{C} \subseteq \mathcal{L}(S)$ be non-empty. Then $\mathfrak{C} \cdot S$ is predictably convex and closed in the semimartingale topology if and only if there exists a predictable correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ with closed convex values such that the projection of C on the predictable range of S is closed, i.e. $\Pi^S(\omega,t)C(\omega,t)$ is closed P_B-a.e., and with $\mathfrak{C} \cdot S = \mathcal{C}^S \cdot S$, i.e.*

$$\begin{aligned}\mathfrak{C} \cdot S &= \{\psi \cdot S \mid \psi \in \mathfrak{C}\} \\ &= \{\psi \cdot S \mid \psi \in \mathcal{L}(S) \text{ and } \psi(\omega,t) \in C(\omega,t) \text{ for all } (\omega,t)\}.\end{aligned}$$

Proof. "$\Leftarrow$": The pointwise convexity of C immediately implies that $\mathcal{C}^S \cdot S$ is predictably convex, and closedness follows from Theorem 4.5.
"$\Rightarrow$": Like at the end of Section III.2, we view predictable processes on Ω as $\mathcal{P}$-measurable random variables on $\overline{\Omega} = \Omega \times [0,\infty)$. Since we are only interested in a non-empty space of stochastic integrals with respect to S, we lose no generality if we replace $\mathfrak{C}$ by $\mathfrak{C} - \varphi := \{\vartheta - \varphi \in \mathcal{L}(S) \mid \vartheta \in [\mathfrak{C}]\}$ for some $\varphi \in \mathfrak{C}$ and identify this with a subspace of $L^0(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d)$ which contains zero. Indeed, if the assertion is true for $\mathfrak{C} - \varphi$ with a correspondence $\widetilde{C}$, it is also true for $\mathfrak{C}$ with $C = \widetilde{C} + \varphi$, which is a predictable correspondence by Proposition 2.7. In order to apply Proposition 2.9, we truncate $\mathfrak{C}$ to get

$$\mathfrak{C}^q = \big\{\psi \in \mathfrak{C} \bigm| \|\psi\|_{L^\infty} \le q\big\} = \mathfrak{C} \cap \overline{B(0,q)}^{L^\infty} \qquad \text{for } q \in \mathbb{Q}_+.$$

Then $\mathfrak{C}^q$ inherits predictable convexity from $\mathfrak{C}$ and is thus a convex subset of $\overline{B(0,q)}^{L^\infty}$. Moreover, $\mathfrak{C}^q$ is closed with respect to convergence in P_B-measure since its elements are uniformly bounded by q and $\mathfrak{C} \cdot S$ is closed in $\mathcal{S}(P)$; this uses the fact, easily proved via dominated convergence separately for the M- and A-integrals, that for any uniformly bounded sequence of integrands (ψ^n) converging pointwise, the stochastic integrals converge in $\mathcal{S}(P)$. By a well-known application of the Krein–Šmulian and Banach–Alaoglu theorems (see Theorems A.62 and A.63 and Lemma A.64 in [45]), $\mathfrak{C}^q$ is thus weak*-compact, and Proposition 2.9 gives a predictable correspondence $C^q : \overline{\Omega} \to 2^{\overline{B(0,q)}} \setminus \{\emptyset\}$ with convex compact values containing zero such that

$$\mathfrak{C}^q = \big\{\psi \in L^0(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d) \bigm| \psi(\omega,t) \in C^q(\omega,t)\ P_B\text{-a.e.}\big\}.$$

By the definition of $\mathfrak{C}^q$ we obtain, after possibly modifying the sets on a P_B-nullset, that

$$C^{q_2}(\omega,t) \cap \overline{B(0,q_1)} = C^{q_1}(\omega,t) \qquad \text{for all } (\omega,t) \in \overline{\Omega} \tag{4.3}$$

for $0 < q_1 \leq q_2 < \infty$ by Lemma 12 in [28], since the graph of each C^q is predictable by Proposition 2.6. Using the characterisation of closed sets in metric spaces as limit points of converging sequences implies with (4.3) that the correspondence C given by

$$C(\omega, t) := \bigcup_{q \in \mathbb{Q}_+} C^q(\omega, t)$$

has closed values. Moreover, each $C(\omega, t)$ is convex as the union of an increasing sequence of convex sets, and it only remains to show that $\mathfrak{C} \cdot S = \mathcal{C} \cdot S$.

Suppose first that ψ is in $\mathfrak{C}$. By predictable convexity and since $0 \in \mathfrak{C}$, $\psi^n := \mathbb{1}_{\{|\psi| \leq n\}} \psi$ is in $\mathfrak{C}^n$ and therefore C^n- and hence C-valued. Since (ψ^n) converges pointwise to ψ, the closedness of C implies that ψ is C-valued, so that $\psi \in \mathcal{C}$ and $\mathfrak{C} \cdot S \subseteq \mathcal{C} \cdot S$. Conversely, if ψ is in $\mathcal{C}$, then $\psi^n := \mathbb{1}_{\{|\psi| \leq n\}} \psi$ is C^n-valued and hence in $\mathfrak{C}^n \subseteq \mathfrak{C}$. But $(\psi^n \cdot S)$ converges to $\psi \cdot S$ in $\mathcal{S}(P)$ and $\mathfrak{C} \cdot S$ is closed in $\mathcal{S}(P)$. So the limit $\psi \cdot S$ is in $\mathfrak{C} \cdot S$ and hence $\psi \in \mathfrak{C}$ and $\mathcal{C} \cdot S \subseteq \mathfrak{C} \cdot S$. Finally, $\mathcal{C} \cdot S = \mathfrak{C} \cdot S$ is closed in $\mathcal{S}(P)$, and therefore $\Pi^S C$ is closed P_B-a.e. by Theorem 4.5. This completes the proof. □

Remark 4.12. 1) Theorem 4.11 can be used as follows. Start with any convex-valued correspondence C, form the space $\mathcal{C} \cdot S$ of corresponding stochastic integrals and take its closure in $\mathcal{S}(P)$. Then Theorem 4.11 tells us that we can realise this closure as a space of stochastic integrals from $\widetilde{C}$-constrained integrands, for some predictable correspondence $\widetilde{C}$ with convex and closed values. In other words, $\overline{\mathcal{C} \cdot S}^{\mathcal{S}(P)} = \widetilde{\mathcal{C}} \cdot S$; and one possible choice of $\widetilde{C}$ is $\widetilde{C} = \left(\Pi^S\right)^{-1}(\overline{C})$. Another possible choice would be $\widetilde{C} = \overline{\overline{C} + \mathfrak{N}}$, where $\mathfrak{N}$ denotes the correspondence of null investments for S; see Section III.6.

2) If we assume in Theorem 4.11 that $\mathfrak{C} \subseteq \mathcal{L}^p_{loc}(S)$ for $p \in [1, \infty)$, then $\mathfrak{C} \cdot S \subseteq \mathcal{S}^p_{loc}(P)$, and $\mathfrak{C} \cdot S$ is closed in $\mathcal{S}^p(P)$ if and only if there exists C as in the theorem. This can be useful for applications (e.g., mean-variance hedging under constraints, with $p = 2$).

III.5 Projection on the predictable range

In this section, we construct the projection Π^S on the predictable range of a general semimartingale S in continuous time. The idea to introduce such a projection comes from [84] and [32], where it was used to prove the fundamental theorem of asset pricing in discrete time. It was also used for a continuous local martingale in [28] to investigate the structure of m-stable sets and in particular the set of risk-neutral measures.

As already explained before Definition 4.1, a *sufficient* condition for $\varphi \cdot S = \vartheta \cdot S$ (up to P-indistinguishability) or, equivalently, $\varphi = \vartheta$ in $L(S)$ or $[\varphi] = [\vartheta]$, is that $\varphi = \vartheta$ P_B-a.e. If we again view predictable processes on Ω as $\mathcal{P}$-measurable random variables on $\overline{\Omega} = \Omega \times [0, \infty)$, i.e. elements of $\mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d)$, then $\varphi = \vartheta$ P_B-a.e. is the same as saying that $\varphi = \vartheta$ in $L^0(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d)$. But to get a *necessary and sufficient* condition for $[\vartheta] = [\varphi]$, we need to understand not only what $0 \in \mathcal{L}(S)$ looks like, but rather the precise structure of (the equivalence class) $[0]$. This is achieved by Π^S.

The construction of Π^S basically proceeds by generalising that of Π^Y in the example in Section III.3 and adapting the steps in [32] to continuous time. The idea is as follows. We start by characterising the equivalence class $[0]$ as a linear subspace of $L^0(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d)$. Since this subspace satisfies a certain stability property, we can construct predictable processes $e^1, \dots, e^d$ which form an "orthonormal basis" of $[0]$ in the sense that $[0]$ equals up to P_B-a.e. equality their linear combinations with predictable coefficients, i.e.

$$[0] = \left\{ \sum_{j=1}^{d} h^j e^j \;\middle|\; h^1, \dots, h^d \text{ are real-valued predictable} \right\} \tag{5.1}$$

up to P_B-a.e. equality. But these linear combinations contribute 0 to the integral with respect to S; so we filter them out to obtain the part of the integrand which determines the stochastic integral, by defining

$$\Pi^S := \mathbb{1}_{d \times d} - \sum_{j=1}^{d} e^j (e^j)^\top. \tag{5.2}$$

This construction then yields the projection on the predictable range as in Definition 4.1.

To describe $[0] = [0]^S$ as a linear subspace of $L^0(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d)$, we exploit that although we work with a general semimartingale S, we can by Lemma I.3 in [71] switch to an equivalent probability Q under which S is locally square-integrable. Since the stochastic integral and hence $[0]^S$ are invariant under a change to an equivalent measure, any representation we obtain $Q \otimes B$-a.e. also holds P_B-a.e., as $P_B \sim P \otimes B \sim Q \otimes B$. Let $S = S_0 + M^Q + A^Q$ be the canonical decomposition of S under Q into an $\mathbb{R}^d$-valued square-integrable Q-martingale $M^Q \in \mathcal{M}_0^{2,d}(Q)$ null at 0 and an $\mathbb{R}^d$-valued predictable process $A^Q \in \mathcal{A}^{1,d}(Q)$ of Q-integrable variation $\mathrm{Var}(A^Q)$ also null at 0. By Propositions II.2.9 and II.2.29 in [52], there exist an increasing, locally Q-integrable, predictable process B^Q, an $\mathbb{R}^d$-valued process a^Q and a predictable $\mathbb{R}^{d \times d}$-valued process $\hat{c}^Q$ whose values are positive semidefinite symmetric matrices such that

$$(A^Q)^i = (a^Q)^i \cdot B^Q \quad \text{and} \quad \langle (M^Q)^i, (M^Q)^j \rangle^Q = (\hat{c}^Q)^{ij} \cdot B^Q \tag{5.3}$$

for $i, j = 1, \dots, d$. By expressing the semimartingale characteristics of S under Q by those under P via Girsanov's theorem, writing A^Q and $\langle M^Q, M^Q \rangle^Q$

in terms of semimartingale characteristics and then passing to differential characteristics with B as predictable increasing process, we obtain that we can and do choose $B^Q = B$ in (5.3); see Theorem III.3.24 and Propositions II.2.29 and II.2.9 in [52]. Using the canonical decomposition of S under Q as auxiliary tool then allows us to give the following characterisation of $[0]^S$.

Lemma 5.1. *Let $Q \sim P$ such that $S = S_0 + M^Q + A^Q \in \mathcal{S}^2_{loc}(Q)$. Then*

1) $[0]^{M^Q} = \{\varphi \in \mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d) \mid \hat{c}^Q \varphi = 0 \ P_B\text{-a.e.}\}$.

2) $[0]^{A^Q} = \{\varphi \in \mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d) \mid (a^Q)^\top \varphi = 0 \ P_B\text{-a.e.}\}$.

3) $[0]^S = [0]^{M^Q} \cap [0]^{A^Q}$.

Moreover, $[0]^{M^Q}$, $[0]^{A^Q}$ and $[0]^S$ all do not depend on Q.

Proof. The last assertion is clear since the stochastic integral of a semimartingale (like M^Q, A^Q, S) is invariant under a change to an equivalent measure. Because also $P_B \sim Q \otimes B$, we can argue for the rest of the proof under the measure Q. Then the inclusions "$\supseteq$" follow immediately from the definition of the stochastic integral with respect to a square-integrable martingale and a finite variation process, since the conditions on the right-hand side ensure that φ is in $\mathcal{L}^2(M^Q)$ and $\mathcal{L}^1(A^Q)$. For the converse, we start with $\varphi \in [0]^S$ and set $\varphi^n := \mathbb{1}_{\{|\varphi| \le n\}}\varphi$. Then $\varphi^n \cdot S = 0$ implies that $\varphi^n \cdot M^Q = 0$ and $\varphi^n \cdot A^Q = 0$ by the uniqueness of the Q-canonical decomposition of $\varphi^n \cdot S$; this uses that φ^n is bounded. Therefore we can reduce the proof of "$\subseteq$" for 3) to that for 1) and 2). So assume now that φ is in either $[0]^{M^Q}$ or $[0]^{A^Q}$ so that $\varphi^n \cdot M^Q = 0$ or $\varphi^n \cdot A^Q = 0$. But φ^n is bounded, hence in $\mathcal{L}^2(M^Q)$ or $\mathcal{L}^1(A^Q)$, for each n, and by the construction of the stochastic integral, we obtain that $\hat{c}^Q \varphi^n = 0$ or $(a^Q)^\top \varphi^n = 0$ $Q \otimes B$-a.e. and hence P_B-a.e. Since (φ^n) converges pointwise to φ, the inclusions "$\subseteq$" for 1) and 2) follow by passing to the limit. □

The following technical lemma, which is a modification of Lemma 6.2.1 in [32], gives the announced "orthonormal basis" of $[0]^S$ in the sense of (5.1).

Lemma 5.2. *Let $U \subseteq L^0(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d)$ be a linear subspace which is closed with respect to convergence in P_B-measure and satisfies the following stability property:*

$$\varphi^1 \mathbb{1}_F + \varphi^2 \mathbb{1}_{F^c} \in U \qquad \text{for all } \varphi^1 \text{ and } \varphi^2 \text{ in } U \text{ and } F \in \mathcal{P}.$$

Then there exist $e^j \in L^0(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d)$ for $j = 1, \dots, d$ such that

1) $\{e^{j+1} \neq 0\} \subseteq \{e^j \neq 0\}$ *for $j = 1, \dots, d-1$;*

2) $|e^j(\omega, t)| = 1$ *or* $|e^j(\omega, t)| = 0$*;*

3) $(e^j)^\top e^k = 0$ *for* $j \neq k$*;*

4) $\varphi \in U$ *if and only if there are* $h^1, \dots, h^d$ *in* $L^0\big(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}\big)$ *with* $\varphi = \sum_{j=1}^d h^j e^j$*, i.e.*

$$U = \left\{ \sum_{j=1}^d h^j e^j \;\middle|\; h^1, \dots, h^d \text{ are real-valued predictable} \right\}.$$

Proof. The predictable processes $e^1, \dots, e^d$ with the properties 1)–4) are the column vectors of the measurable projection-valued mapping constructed in Lemma 6.2.1 in [32]. Therefore their existence follows immediately from the construction given there. □

By Lemma I.3 in [71], there always exists a probability measure Q as in Lemma 5.1, and therefore the space $[0]^S$ satisfies the assumptions of Lemma 5.2. So we take a "basis" $e^1, \dots, e^d$ as in the latter result and define Π^S as in (5.2) by

$$\Pi^S := \mathbb{1}_{d\times d} - \sum_{j=1}^d e^j (e^j)^\top.$$

Then $\Pi^S(\omega, t)$ is the projection on the orthogonal complement of the linear space spanned in $\mathbb{R}^d$ by $e^1(\omega,t), \dots, e^d(\omega,t)$ so that $\Pi^S(\omega,t)\gamma$ is orthogonal to all $e^i(\omega,t)$ for each $\gamma \in \mathbb{R}^d$; and Lemma 5.2 says that each element of $[0]^S$ is a (random and time-dependent) linear combination of $e^1, \dots, e^d$, and vice versa. In particular, $\vartheta - \Pi^S\vartheta$ is in $[0]^S$ for every predictable $\mathbb{R}^d$-valued ϑ. The next result shows that Π^S satisfies the properties required in Definition 4.1. Note that Π^S is only defined up to P_B-nullsets since the e^j are; so we have to choose one version for Π^S to be specific.

Lemma 5.3 (Projection on the predictable range of S)**.** *For a semimartingale S, the projection Π^S on the predictable range of S exists, i.e. there exists a predictable process $\Pi^S : \overline{\Omega} \to \mathbb{R}^{d\times d}$ which takes values in the orthogonal projections in $\mathbb{R}^d$ and has the following property: If $\vartheta \in \mathcal{L}(S)$ and ψ is an $\mathbb{R}^d$-valued predictable process, then*

$$\psi \in \mathcal{L}(S) \text{ with } \psi \cdot S = \vartheta \cdot S \quad \Longleftrightarrow \quad \Pi^S\psi = \Pi^S\vartheta \;\; P_B\text{-a.e.} \tag{5.4}$$

Proof. If we define Π^S as above, Lemma 5.2 implies that Π^S is predictable and valued in the orthogonal projections in $\mathbb{R}^d$, and it only remains to check (5.4). So take $\vartheta \in \mathcal{L}(S)$ and assume first that $\Pi^S\vartheta = \Pi^S\psi$ P_B-a.e. The definition of Π^S and Lemma 5.1 then yield that $\vartheta - \Pi^S\vartheta$ and $\Pi^S\vartheta - \Pi^S\psi$ are in $[0]^S$, which implies that $\Pi^S\vartheta = \vartheta - \big(\vartheta - \Pi^S\vartheta\big)$ and $\Pi^S\psi$ are in $\mathcal{L}(S)$ and also that $\vartheta \cdot S = (\Pi^S\vartheta) \cdot S = (\Pi^S\psi) \cdot S$. Because also $\psi - \Pi^S\psi$ is in $[0]^S \subseteq \mathcal{L}(S)$, we conclude that $\psi \in \mathcal{L}(S)$ with $\vartheta \cdot S = \psi \cdot S$. Conversely, if

$\psi \cdot S = \vartheta \cdot S$, then $\psi - \vartheta \in [0]^S$, and we always have $(\psi - \vartheta) - \Pi^S(\psi - \vartheta) \in [0]^S$. Therefore $\Pi^S(\psi - \vartheta) \in [0]^S$ which says by Lemma 5.2 that for P_B-a.e. (ω, t), $\Pi^S(\psi - \vartheta)(\omega, t)$ is a linear combination of the $e^i(\omega, t)$. But the column vectors of Π^S are orthogonal to $e^1, \dots, e^d$ for each fixed (ω, t), and so we obtain $\Pi^S(\psi - \vartheta) = 0$ P_B-a.e., which completes the proof. □

With the existence of the projection on the predictable range established, it remains to prove Lemmas 4.3 and 4.4, which we recall for convenience.

Lemma 4.3. *Let (ϑ^n) be a sequence in $\mathcal{L}(S)$ such that $\vartheta^n \cdot S \to \vartheta \cdot S$ in $\mathcal{S}(P)$. Then there exists a subsequence (n_k) such that $\Pi^S \vartheta^{n_k} \to \Pi^S \vartheta$ P_B-a.e.*

Proof. As in the proof of Theorem V.4 in [71], we can switch to a probability measure $Q \sim P$ such that $\frac{dQ}{dP}$ is bounded, $S - S_0 = M^Q + A^Q$ is in $\mathcal{M}^{2,d}(Q) \oplus \mathcal{A}^{1,d}(Q)$ and $\vartheta^n \cdot S \to \vartheta \cdot S$ in $\mathcal{M}^{2,d}(Q) \oplus \mathcal{A}^{1,d}(Q)$ along a subsequence, again indexed by n. Since $\vartheta^n \cdot S \to \vartheta \cdot S$ in $\mathcal{M}^{2,1}(Q) \oplus \mathcal{A}^{1,1}(Q)$, we obtain by using (4.1) with $B^Q = B$ that

$$E_Q\left[\int_0^\infty (\vartheta^n_s - \vartheta_s)^\top \hat{c}^Q_s (\vartheta^n_s - \vartheta_s) dB_s + \int_0^\infty \left|(\vartheta^n_s - \vartheta_s)^\top a^Q_s\right| dB_s\right] \longrightarrow 0$$

as $n \to \infty$, which implies that there exists a subsequence, again indexed by n, such that

$$(\vartheta^n - \vartheta)^\top \hat{c}^Q (\vartheta^n - \vartheta) \to 0 \quad \text{and} \quad |(\vartheta^n - \vartheta)^\top a^Q| \to 0 \quad Q \otimes B\text{-a.e.} \tag{5.5}$$

Since $P_B \sim Q \otimes B$, Lemma 5.1 gives

$$[0]^S = \left\{\varphi \in \mathcal{L}^0\big(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d\big) \;\middle|\; \hat{c}^Q \varphi = 0 \text{ and } (a^Q)^\top \varphi = 0 \quad Q \otimes B\text{-a.e.}\right\}.$$

Let $e^1, \dots, e^d$ be predictable processes from Lemma 5.2 which satisfy properties 1)–4) for $[0]^S$ and set

$$U = \left\{\psi \in \mathcal{L}^0\big(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d\big) \;\middle|\; \psi^\top \varphi = 0 \quad Q \otimes B\text{-a.e. for all } \varphi \in [0]^S\right\},$$
$$V = \left\{\psi \in \mathcal{L}^0\big(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d\big) \;\middle|\; \psi^\top \varphi = 0 \quad Q \otimes B\text{-a.e. for all } \varphi \in [0]^{M^Q}\right\}$$

so that loosely speaking, $U^\perp = [0]^S$ and $V^\perp = [0]^{M^Q}$. Then $[0]^{M^Q} \cap U$ and $[0]^{A^Q} \cap V$ satisfy the assumptions of Lemma 5.2 and thus there exist predictable processes $u^1, \dots, u^d$ and $v^1, \dots, v^d$ with the properties 1)–4) for $[0]^{M^Q} \cap U$ and $[0]^{A^Q} \cap V$, respectively. By the definition of U and V we also obtain, using $[0]^S = [0]^{M^Q} \cap [0]^{A^Q}$, that

$$(e^j)^\top u^k = (e^j)^\top v^k = (u^j)^\top v^k = 0 \quad Q \otimes B\text{-a.e. for } j, k = 1, \dots, d$$

and

$$
\begin{aligned}
[0]^{MQ} &= \left\{ \sum_{j=1}^{d} h^j e^j + \sum_{k=1}^{d} h^{d+k} u^k \;\middle|\; h^1, \dots, h^{2d} \text{ real-valued predictable} \right\}, \\
[0]^{AQ} &= \left\{ \sum_{j=1}^{d} h^j e^j + \sum_{k=1}^{d} h^{d+k} v^k \;\middle|\; h^1, \dots, h^{2d} \text{ real-valued predictable} \right\}
\end{aligned}
$$

up to $Q \otimes B$-a.e. equality. Therefore Π^{MQ} and Π^{AQ} can be written as

$$
\begin{aligned}
\Pi^{MQ} &= \mathbb{1}_{d\times d} - \sum_{j=1}^{d} e^j (e^j)^\top - \sum_{k=1}^{d} u^k (u^k)^\top, \\
\Pi^{AQ} &= \mathbb{1}_{d\times d} - \sum_{j=1}^{d} e^j (e^j)^\top - \sum_{k=1}^{d} v^k (v^k)^\top,
\end{aligned}
$$

and we have

$$
\left(\sum_{k=1}^{d} v^k (v^k)^\top \right) \Pi^{AQ} \vartheta^n = \left(\sum_{k=1}^{d} v^k (v^k)^\top \right) \vartheta^n, \tag{5.6}
$$

all up to $Q \otimes B$-a.e. equality. Since $\Pi^{MQ}(\vartheta^n - \vartheta)$ and $\Pi^{AQ}(\vartheta^n - \vartheta)$ are by Lemma 5.1 $Q \otimes B$-a.e. valued in $\mathrm{Im}(\hat{c}^Q)$ and $\mathrm{Im}\big((a^Q)^\top\big)$, respectively, (5.5) yields $\Pi^{MQ}\vartheta^n \to \Pi^{MQ}\vartheta$ and $\Pi^{AQ}\vartheta^n \to \Pi^{AQ}\vartheta$ $\;Q \otimes B$-a.e. From the latter convergence and (5.6), it follows that

$$
\left(\sum_{k=1}^{d} v^k (v^k)^\top \right) \vartheta^n \to \left(\sum_{k=1}^{d} v^k (v^k)^\top \right) \vartheta \quad Q \otimes B\text{-a.e.},
$$

and since $Q \otimes B \sim P_B$ and

$$
\Pi^S = \Pi^{MQ} + \sum_{k=1}^{d} v^k (v^k)^\top \qquad Q \otimes B\text{-a.e.},
$$

we obtain that $\Pi^S \vartheta^n \to \Pi^S \vartheta$ P_B-a.e. by combining everything. □

The only result whose proof is now still open is Lemma 4.4. This provides the general (and fairly abstract) version of the counterexample in Section III.3, as well as the necessity part for the equivalence in Theorem 4.5.

Lemma 4.4. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ be a predictable correspondence with closed values and such that the projection on the predictable range of S is* not closed, *i.e.*

$$
\widetilde{F} = \big\{ (\omega, t) \in \overline{\Omega} \;\big|\; \Pi^S(\omega, t) C(\omega, t) \textit{ is not closed} \big\}
$$

has outer P_B-measure > 0. Then there exist $\vartheta \in \mathcal{L}(S)$ and a sequence (ψ^n) of C-valued integrands such that $\psi^n \cdot S \to \vartheta \cdot S$ in $\mathcal{S}(P)$, but there is no C-valued integrand ψ such that $\psi \cdot S = \vartheta \cdot S$. Equivalently, there exists a sequence $\big([\psi^n]\big)$ in $[\mathcal{C}]^S$ such that $[\psi^n] \overset{L(S)}{\to} [\vartheta]$ but $[\vartheta] \notin [\mathcal{C}]^S$, i.e. $[\mathcal{C}]^S$ is not closed in $L(S)$.

Proof. The basic idea is to construct a $\vartheta \in \mathcal{L}(S)$ which is valued in $\overline{\Pi^S C} \setminus \Pi^S C$ on some $F \in \mathcal{P}$ with $F \subseteq \widetilde{F}$ and $P_B(F) > 0$, and in C on F^c. Then there exists no C-valued integrand $\psi \in [\vartheta]$ by the definition of Π^S since $\Pi^S \vartheta \notin \Pi^S C$ on F; but one can construct a sequence (ψ^n) of C-valued integrands with $\Pi^S \psi^n \to \Pi^S \psi$ pointwise since $\Pi^S \vartheta \in \overline{\Pi^S C}$. However, this is technically a bit more involved for several reasons: While C, $\Pi^S C$ and $\overline{\Pi^S C}$ are all predictable, $(\Pi^S C)^c$ need not be; so $\widetilde{F}$ need not be predictable, and one cannot use Proposition 2.3 to obtain a predictable selector. In addition, $\overline{\Pi^S C} \setminus \Pi^S C$ need not be closed-valued.

We first argue that $\widetilde{F}$ is $\mathcal{P}_{P_B}$-measurable. Let $\overline{B(0,n)}$ be a closed ball of radius n in $\mathbb{R}^d$. Then $\Pi^S\big(C \cap \overline{B(0,n)}\big)$ is compact-valued as C is closed-valued. Since C is predictable and $\Pi^S(\omega,t)x$ with $x \in \mathbb{R}^d$ is a Carathéodory function, $\overline{\Pi^S C}$ is predictable by Proposition 2.7. By the same argument, $\Pi^S\big(C \cap \overline{B(0,n)}\big) = \overline{\Pi^S\big(C \cap \overline{B(0,n)}\big)}$ is predictable since $C \cap \overline{B(0,n)}$ is, and then so is $\Pi^S C = \bigcup_{n=1}^{\infty} \Pi^S\big(C \cap \overline{B(0,n)}\big)$ as a countable union of predictable correspondences; see Proposition 2.8. Then Proposition 2.6 implies that $\overline{\Pi^S C}$ and $\Pi^S\big(C \cap \overline{B(0,n)}\big)$ have predictable graph; hence so does $\Pi^S C$. Therefore $\mathrm{gr}(\overline{\Pi^S C}) \cap \big(\mathrm{gr}(\Pi^S C)\big)^c$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and so by Proposition 2.5,

$$
\begin{aligned}
\widetilde{F} &= \big\{(\omega,t) \in \overline{\Omega} \bigm| \Pi^S(\omega,t)C(\omega,t) \text{ is not closed}\big\} \\
&= \big\{(\omega,t) \in \overline{\Omega} \bigm| \overline{\Pi^S(\omega,t)C(\omega,t)} \setminus \Pi^S(\omega,t)C(\omega,t) \neq \emptyset\big\} \\
&= \pi_{\overline{\Omega}}\Big(\mathrm{gr}(\overline{\Pi^S C}) \cap \big(\mathrm{gr}(\Pi^S C)\big)^c\Big)
\end{aligned}
$$

is indeed $\mathcal{P}_{P_B}$-measurable. Thus there exists a predictable set $F \subseteq \widetilde{F}$ with $P_B(F) > 0$.

Now fix some C-valued integrand $\widetilde{\psi} \in \mathcal{L}(S)$ and define the correspondence C' by

$$
C'(\omega,t) = \begin{cases} \overline{\Pi^S(\omega,t)C(\omega,t)} \setminus \Pi^S(\omega,t)C(\omega,t) & \text{for } (\omega,t) \in F, \\ \widetilde{\psi}(\omega,t) & \text{else.} \end{cases}
$$

Then C' has non-empty values and predictable graph and therefore admits a P_B-a.e. predictable selector ϑ by Proposition 2.4. By possibly subtracting a predictable P_B-nullset from F, we can without loss of generality assume that ϑ takes values in C'. Moreover, the predictable sets $F_n := F \cap \{|\vartheta| \leq n\}$ increase to F and so we can, by shrinking F to some F_n if necessary, assume that ϑ is uniformly bounded in (ω, t) on F. Let $\{\varphi^m \mid m \in \mathbb{N}\}$ be a Castaing representation of C as in Proposition 2.3. Then $\overline{\Pi^S C} = \overline{\{\Pi^S \varphi^m \mid m \in \mathbb{N}\}}$, and because $\vartheta \in \overline{\Pi^S C}$, we can find for each $n \in \mathbb{N}$ a predictable process ψ^n such that $\Pi^S(\omega, t)\psi^n(\omega, t) \in \vartheta(\omega, t) + \overline{B(0, \frac{1}{n})}$ on F and $\psi^n = \widetilde{\psi}$ on F^c. Note that on F, we have $\vartheta \in \overline{\Pi^S C} \subseteq \Pi^S \mathbb{R}^d$ and therefore $\Pi^S \vartheta = \vartheta$; so $\Pi^S \vartheta = \mathbb{1}_F \vartheta + \mathbb{1}_{F^c} \Pi^S \widetilde{\psi}$ and this shows that $\Pi^S \psi^n \to \Pi^S \vartheta$ uniformly in (ω, t) by construction. Since $\Pi^S \vartheta \in \mathcal{L}(S)$ because ϑ is bounded on F, we thus first get $\Pi^S \psi^n \in \mathcal{L}(S)$, hence $\psi^n \in \mathcal{L}(S)$, and then also that $\psi^n \cdot S \to \vartheta \cdot S$ in $\mathcal{S}(P)$ by dominated convergence. But now $\{\Pi^S \vartheta\} \cap \Pi^S C = \emptyset$ on F shows by Lemma 5.3 that there exists no C-valued integrand $\psi \in [\vartheta]$ and therefore $[\vartheta] \notin [\mathcal{C}]$. This ends the proof. □

III.6 Related work

We have already explained how our results generalise most of the existing literature on optimisation problems under constraints. In this section, we discuss the relation to the work of Karatzas and Kardaras [54].

We start by introducing the terminology of [54]. For a given S with triplet (b, c, F), the linear subspace of *null investments* $\mathfrak{N}$ is given by the predictable correspondence

$$\mathfrak{N}(\omega, t) := \big\{ z \in \mathbb{R}^d \bigm| z^\top c(\omega, t) = 0,\ z^\top b(\omega, t) = 0 \\ \text{and } F(\omega, t)(\{x \mid z^\top x \neq 0\}) = 0 \big\}$$

(see Definition 3.6 in [54]). Note that we use F instead of ν and that our B is slightly different than in [54]. But this does not affect the definition of $\mathfrak{N}$. As in Definition 3.7 in [54], a correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ is said to *impose predictable closed convex constraints* if

0) $\mathfrak{N}(\omega, t) \subseteq C(\omega, t)$ for all $(\omega, t) \in \overline{\Omega}$,

1) $C(\omega, t)$ is a closed and convex set for all $(\omega, t) \in \overline{\Omega}$, and

2) C is predictable.

To avoid confusion, we call constraints with 0)–2) *KK-constraints* in the sequel.

In the comment following their Theorem 4.4 on p. 467 in [54], Karatzas and Kardaras (KK) remark that $\mathcal{C} \cdot S$ is closed in $\mathcal{S}(P)$ if C describes KK-constraints. For comparison, our Theorem 4.5 starts with C which is predictable and has closed values, and shows that $\mathcal{C} \cdot S$ is then closed in $\mathcal{S}(P)$ if and only if $\Pi^S C$ is closed P_B-a.e. So we do not need convexity of C, and our condition on C and S is not only sufficient, but also necessary.

Before explaining the connections in more detail, we make the simple but important observation that

$$\text{0) plus 1) imply that } C + \mathfrak{N} = C \text{ (for all } (\omega, t) \in \overline{\Omega}). \tag{6.1}$$

Indeed, each $\mathfrak{N}(\omega, t)$ is a linear subspace, hence contains 0, and so $C \subseteq C + \mathfrak{N}$. Conversely, $\frac{1}{\varepsilon} z \in \mathfrak{N} \subseteq C$ for every $z \in \mathfrak{N}$ and $\varepsilon > 0$ due to 0); so for every $c \in C$, $(1-\varepsilon)c + z \in C$ by convexity and hence $c + z = \lim_{\varepsilon \searrow 0} (1-\varepsilon)c + z$ is in C by closedness, giving $C + \mathfrak{N} \subseteq C$.

As a matter of fact, KK say, but do not explicitly prove, that $\mathcal{C} \cdot S$ is closed in $\mathcal{S}(P)$. However, the clear hint they give suggests the following reasoning. Let (ϑ^n) be a sequence in $\mathcal{C}$ such that $(\vartheta^n \cdot S) \to X$ in $\mathcal{S}(P)$. By the proof of Theorem V.4 in [71], there exist $\widetilde{\vartheta}^n \in [\vartheta^n]$ and $\vartheta \in \mathcal{L}(S)$ such that $\vartheta \cdot S = X$ and $\widetilde{\vartheta}^n \to \vartheta$ P_B-a.e. From the description of $\mathfrak{N}$ in Section 3.3 in [54], $\widetilde{\vartheta}^n \in [\vartheta^n]$ translates into $\widetilde{\vartheta}^n - \vartheta^n \in \mathfrak{N}$ P_B-a.e. or $\widetilde{\vartheta}^n \in \vartheta^n + \mathfrak{N}$ P_B-a.e. Because each ϑ^n has values in C, (6.1) thus shows that each $\widetilde{\vartheta}^n$ can be chosen to be C-valued, and by the closedness of C, the same is then true for the limit ϑ of $(\widetilde{\vartheta}^n)$. Hence we are done.

In order to relate the KK result to our work, we now observe that

$$\text{0) plus 1) imply that } \Pi^S C \text{ is closed } P_B\text{-a.e.}$$

To see this, we start with the fact that the null investments $\mathfrak{N}$ and $[0]^S$ are linked by

$$[0]^S = \{\varphi \mid \varphi \text{ is } \mathbb{R}^d\text{-valued predictable with } \varphi \in \mathfrak{N} \ P_B\text{-a.e.}\}; \tag{6.2}$$

see Section 3.3 in [54]. Recalling that Π^S is the projection on the orthogonal complement of $[0]^S$, we see from (6.2) that the column vectors of Π^S are P_B-a.e. a generating system of $\mathfrak{N}^\perp$ so that the projection of $\vartheta \in \mathcal{L}(S)$ on the predictable range of S can be alternatively defined P_B-a.e. as a predictable selector of the closed-valued predictable correspondence $\{\vartheta + \mathfrak{N}\} \cap \mathfrak{N}^\perp$ or P_B-a.e. as the pointwise projection $\Pi^{\mathfrak{N}(\omega,t)} \vartheta(\omega, t)$ in $\mathbb{R}^d$ of $\vartheta(\omega, t)$ on $\mathfrak{N}(\omega, t)$, which is always a predictable process. This yields $\Pi^S C = \{C + \mathfrak{N}\} \cap \mathfrak{N}^\perp$ P_B-a.e.; but by (6.1), $C + \mathfrak{N} = C$ due to 0) and 1), and so $\Pi^S C$ is P_B-a.e. closed like C and $\mathfrak{N}^\perp$.

In the KK notation, we could reformulate our Theorem 4.5 as saying that for a predictable and closed-valued C, the space $\mathcal{C} \cdot S$ is closed in $\mathcal{S}(P)$ if and only if $C + \mathfrak{N}$ is closed P_B-a.e. This is easily seen from the argument above showing that $\Pi^S C = \{C + \mathfrak{N}\} \cap \mathfrak{N}^\perp$ P_B-a.e. If C is also convex-valued, 0) is a simple and intuitive sufficient condition; it seems however more difficult to find an elegant formulation without convexity.

The difference between our constraints and the KK formulation in [54] is as follows. We fix a set C of constraints and demand that the strategies should lie in C pointwise, so that $\vartheta(\omega, t) \in C(\omega, t)$ for all (ω, t). KK in contrast only stipulate that $\vartheta(\omega, t) \in C(\omega, t) + \mathfrak{N}(\omega, t)$ or, equivalently, that $[\vartheta] \in [\mathcal{C}]$. At the level of wealth (which is as usual in mathematical finance modelled by the stochastic integral $\vartheta \cdot S$), this makes no difference since all $\mathfrak{N}$-valued processes have integral zero. But for practical checking and risk management, it is much simpler if one can just look at the strategy ϑ and tick off pointwise whether or not it lies in C. If S has complicated redundancy properties, it may be quite difficult to see whether one can bring ϑ into C by adding something from $\mathfrak{N}$. Of course, when discussing the closedness of the space of integrals $\vartheta \cdot S$, we face the same level of difficulty when we have to check whether $\Pi^S C$ is closed P_B-a.e. But for actually working with given strategies, we believe that our formulation of constraints is more natural and simpler to handle.

Chapter IV

On the Markowitz problem under cone constraints

IV.1 Introduction

Mean-variance portfolio selection is a classical problem in finance. It consists of finding in a financial market a self-financing trading strategy whose final wealth has maximal mean and minimal variance. It is often called the *Markowitz problem* after its inventor Harry Markowitz who proposed it in a one-period setting as a formulation for portfolio optimisation; see [69] and [70]. We study this problem here in continuous time in a general semimartingale model and under *cone constraints*, meaning that each allowed trading strategy is restricted to always lie in a closed cone which might depend on the state and time in a predictable way. For applications in the management of pension funds and insurance companies, the inclusion of such constraints into the setup is very useful as they allow to model regulatory restrictions, like for example no shortselling.

As in the unconstrained case, the solution to the Markowitz problem can be obtained by solving the particular mean-variance hedging problem of approximating in L^2 a constant payoff by the terminal gains of a self-financing trading strategy. To get existence of a solution to the latter problem, we show first that the space $G_T(\mathfrak{C})$ of constrained terminal gains is closed in L^2; this is sufficient if the constraints, and hence $G_T(\mathfrak{C})$, are in addition convex. Our approach here combines the space of (L^2-)admissible trading strategies of Černý and Kallsen [14] with $\mathcal{E}$-martingales, a generalisation of martingales introduced by Choulli, Krawczyk and Stricker [16]. The latter notion comes up naturally in quadratic optimisation problems in mathematical finance due to the negative "marginal utility" of the square function. The *closedness* result and hence the *existence* of optimal strategies for the constrained Markowitz problem constitute a first major contribution, especially in view of the generality of our setting.

Our main focus and achievement, however, is the subsequent *structural description* of the optimal strategy by its local properties. This is made possible by treating the approximation in L^2 as a problem in stochastic optimal control and systematically using ideas and results from there. By exploiting the quadratic and conic structure of our task, we first obtain a decomposition of its value process $J(x, \vartheta)$ into a sum involving two auxiliary coefficient processes. This is similar to the results by Černý and Kallsen [14] in the unconstrained case, but now requires two *opportunity processes* $L^{\pm}$, due to the constraints. An analogous opportunity process also plays a central role in the analysis by Nutz [75] of power utility maximisation, and some of the ideas and techniques are similar. Using the martingale optimality principle for $J(x, \vartheta)$ next allows us to describe first the drift of $L^{\pm}$ and from there the optimal strategy locally in feedback form via the pointwise minimisers of two predictable functions $\mathfrak{g}^{\pm}$; these are given in terms of the joint differential semimartingale characteristics of the opportunity processes $L^{\pm}$ and the price process S. The drift equations can also be rewritten as a system of coupled backward stochastic differential equations (BSDEs) for $L^{\pm}$, and we show that the opportunity processes are the maximal solutions of this system. This is motivated by a similar result in [75]. Conversely, we also prove verification results saying that if we have minimisers of $\mathfrak{g}^{\pm}$ (or a solution to the BSDE system), then we can construct from there an optimal strategy. This explains and generalises all results so far in the literature on the Markowitz problem under cone constraints; see [66], [49], [63] and [53].

The generality of our framework allows us to capture a new behaviour of the optimal strategy: It jumps from the minimiser of one predictable function to that of a second one, whenever the optimal wealth process of the approximation problem changes sign. Because this phenomenon is due to jumps in the price process S of the underlying assets, it could not be observed in earlier work since the Markowitz problem under constraints has so far only been studied in (continuous) Itô process models. Not surprisingly, the presence of jumps and the resulting nontrivial coupling of the BSDEs make the situation more involved; we explain in Section IV.6 how things quickly simplify if S is continuous. The usefulness of our general results can also be illustrated by applying them to Lévy processes. Here the two random equations for the joint differential characteristics of $L^{\pm}$ and S reduce to two coupled ordinary differential equations. These allow us to describe the solution explicitly, and it turns out that its behaviour is quite different than in the unconstrained case; the details and examples illustrating the various effects have been worked out and will be presented elsewhere.

The chapter is organised as follows. Section IV.2 gives a precise formulation of the problem, recalls basic results on predictable correspondences and proves the closedness in L^2 of the space of constrained terminal gains. In Section IV.3, we use dynamic programming arguments to establish the general structure of the value process $J(x, \vartheta)$ in terms of the opportunity processes

$L^{\pm}$. Section IV.4 exploits this via the martingale optimality principle to derive the local description of the optimal strategy and the characterisation of the opportunity processes via coupled BSDEs. Section IV.5 contains the more computational parts of the proofs from Section IV.4, and Section IV.6 concludes with a comparison to related work.

IV.2 Formulation of the problem and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions of completeness and right-continuity, where $T > 0$ is a fixed and finite time horizon. We can and do choose for every local P-martingale a right-continuous version with left limits (RCLL for short). All unexplained notation concerning stochastic integration can be found in the books of Jacod and Shiryaev [52] and Protter [80]. For local martingales, we use the definition in [80].

We consider a *financial market* consisting of one riskless asset, whose (discounted) price is 1, and d risky assets described by an $\mathbb{R}^d$-valued RCLL semimartingale $S = (S_t)_{0 \le t \le T}$. We suppose that S is *locally square-integrable*, $S \in \mathcal{H}^2_{\mathrm{loc}}(P)$, in the sense that S is special with canonical decomposition $S = S_0 + M + A$, where M is an $\mathbb{R}^d$-valued locally square-integrable local martingale null at zero, $M \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$, and A is an $\mathbb{R}^d$-valued predictable RCLL process of finite variation and null at zero. Using semimartingale characteristics, we write $\langle M \rangle = \widetilde{c}^M \bullet B$ and $A = b^S \bullet B$, where all processes are predictable, B is RCLL and strictly increasing and null at 0, and $\widetilde{c}^M$ is $d \times d$-matrix-valued. For details, see Section II.2 in [52] or Section IV.4 below. On the product space $\overline{\Omega} := \Omega \times [0, T]$ with the predictable σ-field $\mathcal{P}$, define $P_B := P \otimes B$. As *trading strategies* available for investment, we consider a set $\mathfrak{C}$ of S-integrable, $\mathbb{R}^d$-valued, predictable processes; this will be specified more precisely later. We call $\mathfrak{C}$ *unconstrained* if $\mathfrak{C}$ is a linear subspace and *constrained* otherwise. By trading with a strategy $\vartheta \in \mathfrak{C}$ up to time $t \in [0, T]$ in a self-financing way, an investor with initial capital $x \in \mathbb{R}$ can generate the *wealth*

$$V_t(x, \vartheta) := x + \textstyle\int_0^t \vartheta_u \, dS_u =: x + \vartheta \bullet S_t.$$

In this chapter, we understand *mean-variance portfolio selection* as in the usual *Markowitz problem*, i.e. as the *static* optimisation problem of finding a (dynamic) self-financing trading strategy whose final wealth has maximal mean and minimal variance. This is static in the sense that we only consider the optimisation at the initial time 0 without looking at intermediate conditional versions. Mathematically, this can be formulated as

$$\text{maximise } E[V_T(x, \vartheta)] - \frac{\gamma}{2} \operatorname{Var}[V_T(x, \vartheta)] \text{ over all } \vartheta \in \mathfrak{C}, \tag{2.1}$$

where the parameter $\gamma > 0$ describes the *risk aversion* of the investor. The most common alternative formulation is to

$$\begin{aligned}&\text{minimise } \operatorname{Var}[V_T(x,\vartheta)] = E\big[|V_T(x,\vartheta)|^2\big] - m^2\\&\text{subject to } E[V_T(x,\vartheta)] = m > x \text{ and } \vartheta \in \mathfrak{C}.\end{aligned} \tag{2.2}$$

If $\mathfrak{C} = \mathfrak{K}$ is a *cone*, we obtain from the purely geometric structure of the optimisation problems the following global description of the solutions to (2.1) and (2.2).

Lemma 2.1. *If $\mathfrak{C} = \mathfrak{K}$ is a* cone*, the solutions to* (2.1) *and* (2.2) *are given by*

$$\widetilde{\vartheta} = \frac{1}{\gamma}\frac{1}{E[1-\widetilde{\varphi}\bullet S_T]}\widetilde{\varphi} \qquad \text{and} \qquad \widetilde{\vartheta}^{(m,x)} = \frac{m-x}{E[1-\widetilde{\varphi}\bullet S_T]}\widetilde{\varphi}, \tag{2.3}$$

respectively, where $\widetilde{\varphi}$ is the solution to

$$\text{minimise } E\big[|V_T(-1,\vartheta)|^2\big] = E\big[|1-\vartheta\bullet S_T|^2\big] \text{ over all } \vartheta \in \mathfrak{C}. \tag{2.4}$$

Proof. This follows from the arguments in the proof of Proposition 3.1 and Theorem 4.2 in [92] which are derived in an abstract L^2-setting by Hilbert space arguments. Note that the convexity assumed in [92] is not necessary for the equations (2.3) to hold; it is used in [92] only for the existence of a solution to (2.4), which we do not assert here. □

If $\mathfrak{C}$ is a *convex set*, but not necessarily a cone, one can under suitable feasibility conditions still establish the *existence* of a solution to (2.1) and (2.2) by using Lagrange multipliers; see [63] and [34]. However, these solutions admit less *structure* so that their dynamic behaviour over time cannot be described very explicitly. We therefore concentrate from Section IV.3 onwards on constraints which are given by *cones*. Before that, however, we want to prove existence of an optimal strategy in a continuous-time setting.

We first observe that despite its simplicity, Lemma 2.1 is very useful as it relates the solution to the Markowitz problems (2.1) and (2.2) to the solution of a *constrained mean-variance hedging problem*, namely minimising the mean-squared hedging error between a given payoff $H \in L^2(P)$ and a constrained self-financing trading strategy, i.e. to

$$\text{minimise } E\left[|V_T(x,\vartheta) - H|^2\right] = E\left[|x+\vartheta\bullet S_T - H|^2\right] \text{ over all } \vartheta \in \mathfrak{C}. \tag{2.5}$$

Indeed, (2.4) corresponds to the very particular version of this problem with $H \equiv 0$ and $x = -1$, or $H \equiv 1$ and $x = 0$. Since (2.5) is an approximation problem in the Hilbert space $L^2(P)$, it admits a solution for arbitrary $H \in L^2(P)$ if the space

$$G_T(\mathfrak{C}) = \{\vartheta\bullet S_T \mid \vartheta \in \mathfrak{C}\}$$

of terminal constrained gains is convex and *closed* in $L^2(P)$. Such constrained mean-variance hedging problems in a general semimartingale framework have been studied in Chapter II. As explained there, one can formulate constraints on trading strategies and then adapt closedness results from the unconstrained case to obtain closedness under constraints as well. This needs a suitable choice of strategies and constraints which we now introduce.

Conceptually, our choice of space of strategies can be traced back to Černý and Kallsen [14]. They start with simple integrands of the form $\vartheta = \sum_{i=1}^{m-1} \xi_i I_{]\!]\sigma_i, \sigma_{i+1}]\!]}$ with stopping times $0 \leq \sigma_1 \leq \cdots \leq \sigma_m \leq \tau_n \leq T$ for some $n \in \mathbb{N}$ and bounded $\mathbb{R}^d$-valued $\mathcal{F}_{\sigma_i}$-measurable random variables ξ_i for $i = 1, \ldots, m-1$, where (τ_n) is a localising sequence of stopping times with $S^{\tau_n} \in \mathcal{H}^2(P)$. Their ($L^2$-)admissible strategies are then those integrands $\vartheta \in \mathcal{L}(S)$ for which there exists a sequence $(\vartheta^n)_{n \in \mathbb{N}}$ of simple integrands such that

1) $\vartheta^n \bullet S_T \xrightarrow{L^2(P)} \vartheta \bullet S_T$.

2) $\vartheta^n \bullet S_t \xrightarrow{P} \vartheta \bullet S_t$ for all $t \in [0, T]$.

A discussion why such a class of strategies is economically reasonable and mathematically useful can be found in [14]. For our purposes, we need to modify that definition a little.

Instead of simple strategies, another natural space of strategies coming from the construction of the stochastic integral is $\Theta := \Theta_S := \mathcal{L}^2(M) \cap \mathcal{L}^2(A)$ with

$$\mathcal{L}^2(M) := \big\{\vartheta \in \mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d) \,\big|\, \|\vartheta\|_{L^2(M)} := \big(E\big[\textstyle\int_0^T \vartheta_s^\top d\langle M\rangle_s \vartheta_s\big]\big)^{\frac{1}{2}} < \infty\big\},$$
$$\mathcal{L}^2(A) := \big\{\vartheta \in \mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d) \,\big|\, \|\vartheta\|_{L^2(A)} := \big(E\big[\big(\textstyle\int_0^T |\vartheta_s^\top dA_s|\big)^2\big]\big)^{\frac{1}{2}} < \infty\big\}.$$

Next, the trading constraints we consider are formulated via predictable correspondences.

Definition 2.2. A *correspondence* is a mapping $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$. We call a correspondence C *predictable* if $C^{-1}(F) := \{(\omega, t) \,|\, C(\omega, t) \cap F \neq \emptyset\}$ is a predictable set for all closed sets $F \subseteq \mathbb{R}^d$. The *domain* of a correspondence C is $\mathrm{dom}(C) := \{(\omega, t) \,|\, C(\omega, t) \neq \emptyset\}$. A *(predictable) selector* of a (predictable) correspondence C is a (predictable) process ψ with $\psi(\omega, t) \in C(\omega, t)$ for all $(\omega, t) \in \mathrm{dom}(C)$.

For a correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$, the sets of *$C$-valued* or *$C$-constrained integrands* and of *square-integrable C-constrained trading strategies* are given by

$$\mathcal{C} := \mathcal{C}^S := \{\vartheta \in \mathcal{L}(S) \,|\, \vartheta(\omega, t) \in C(\omega, t) \text{ for all } (\omega, t) \in \overline{\Omega}\},$$
$$\Theta(C) := \Theta \cap \mathcal{C} = \{\vartheta \in \Theta \,|\, \vartheta(\omega, t) \in C(\omega, t) \text{ for all } (\omega, t) \in \overline{\Omega}\}.$$

Definition 2.3. A trading strategy $\vartheta \in \mathcal{C}$ is called *C-admissible (in $L^2(P)$)* if there exists a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ in $\Theta(C)$, called *approximating sequence for* ϑ, such that

1) $\vartheta^n \bullet S_T \xrightarrow{L^2(P)} \vartheta \bullet S_T$.

2) $\vartheta^n \bullet S_\tau \xrightarrow{P} \vartheta \bullet S_\tau$ for all stopping times τ.

The set of all C-admissible trading strategies is called $\overline{\Theta(C)}$, and we set $\overline{\Theta} := \overline{\Theta(\mathbb{R}^d)}$.

In comparison to Černý and Kallsen [14], there are two differences. Instead of using simple strategies for the approximation, we use strategies from $\Theta(C)$; the reason is that it can easily happen with time-dependent constraints that no simple strategy satisfies them. (The constraints can also be so bad that no strategy in Θ satisfies them either; but such situations are almost pathological.) The second difference is that we stipulate 2) for all stopping times τ and not only for deterministic times t; this is needed for dynamic programming arguments, as explained at the end of this section.

Before addressing the issue of closedness of $G_T(\overline{\Theta(C)})$ in $L^2(P)$, we recall some results on predictable correspondences, used later to ensure the existence of predictable selectors.

Proposition 2.4 (Castaing)**.** *For a correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ with closed values, the following are equivalent:*

1) *C is predictable.*

2) dom(C) *is predictable and there exists a* Castaing representation *of C, i.e. a sequence (ψ^n) of predictable selectors of C such that*

$$C(\omega,t) = \overline{\{\psi^1(\omega,t), \psi^2(\omega,t), \dots\}} \quad \text{for each } (\omega,t) \in \mathrm{dom}(C).$$

In particular, every predictable C admits a predictable selector ψ.

Proof. See Corollary 18.14 in [2] or Theorem 1B in [83]. □

Proposition 2.5. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed values and $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^d$ and $g : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}^m$* Carathéodory functions*, which means that $f(\omega,t,y)$ and $g(\omega,t,x)$ are predictable with respect to (ω,t) and continuous in y and x. Then the mappings C' and C'' given by*

$$C'(\omega,t) = \{y \in \mathbb{R}^m \,|\, f(\omega,t,y) \in C(\omega,t)\}$$

and

$$C''(\omega,t) = \overline{\{g(\omega,t,x) \,|\, x \in C(\omega,t)\}}$$

are predictable correspondences (from $\overline{\Omega}$ to $2^{\mathbb{R}^m}$) with closed values.

Proof. See Corollaries 1P and 1Q in [83]. □

Proposition 2.6. *Let $C^n : \overline{\Omega} \to 2^{\mathbb{R}^d}$ for each $n \in \mathbb{N}$ be a predictable correspondence with closed values and define the correspondences C' and C'' by $C'(\omega, t) = \bigcap_{n\in\mathbb{N}} C^n(\omega, t)$ and $C''(\omega, t) = \bigcup_{n\in\mathbb{N}} C^n(\omega, t)$. Then C' and C'' are predictable and C' is closed-valued.*

Proof. See Theorem 1M in [83] and Lemma 18.4 in [2]. □

Now we aim to prove closedness in $L^2(P)$ of the space $G_T(\overline{\Theta(C)})$ of constrained terminal gains. To that end, we combine the (modified) space of (L^2-)admissible trading strategies of Černý and Kallsen, studied in [14] under the assumption that there exists an equivalent local martingale measure (ELMM) Q for S with $\frac{dQ}{dP} \in L^2(P)$, with the more general absence-of-arbitrage condition that S is an $\mathcal{E}$-local martingale, introduced by Choulli, Krawczyk and Stricker in [16]. Since we are interested in solving (2.4), such a generalisation is useful as a solution to (2.4) can exist even if there is no ELMM for S. The simplest example for this occurs when S is a Poisson process. Then one can compute straightforwardly that the solution to (2.4) is given by $\widetilde{\varphi} = \mathbb{1}_{[\![0,\tau]\!]}$, where $\tau = \inf\{t > 0 \,|\, \Delta S_t = 1\} \wedge T$. To see that there exists no E(L)MM, we simply observe that each integrand $\vartheta \equiv c > 0$ is an arbitrage opportunity.

Let us first recall the notion of an $\mathcal{E}$-martingale. For a semimartingale Y, we denote its stochastic exponential by $\mathcal{E}(Y)$. **Throughout this chapter, we let N stand for a local P-martingale null at zero and Z^N for a strictly positive adapted RCLL process.** We shall see below how N and Z^N are related. For any stopping time τ, we denote the process Y stopped at τ by Y^τ and the process Y started at τ by ${}^\tau Y := Y - Y^\tau$; but we set ${}^\tau\mathcal{E}(N) := \mathcal{E}(N - N^\tau)$. So for stochastic exponentials, ${}^\tau\mathcal{E}(N)$ denotes a multiplicative rather than an additive restarting. Since N is RCLL, it has at most a finite number of jumps with $\Delta N = -1$, and so each ${}^\tau\mathcal{E}(N)$ has P-a.s. at most a finite number of times where it can jump to zero; this follows from the representation of the stochastic exponential in Theorem II.37 in [80]. Therefore we can define an increasing sequence of stopping times by $T_0 = 0$ and $T_{m+1} = \inf\{t > T_m \,|\, {}^{T_m}\mathcal{E}(N)_t = 0\} \wedge T$.

Definition 2.7. An adapted RCLL process Y is an *$\mathcal{E}$-local martingale* if the product of ${}^{T_m}Y$ and ${}^{T_m}\mathcal{E}(N)$ is a local P-martingale for any $m \in \mathbb{N}$. It is an *$(\mathcal{E}, Z^N)$-martingale* if for any $m \in \mathbb{N}$ we have $E\big[|Y_{T_m} Z^N_{T_m}\, {}^{T_m}\mathcal{E}(N)_{T_{m+1}}|\big] < \infty$ and the product of ${}^{T_m}Y$ and $Z^N_{T_m}\, {}^{T_m}\mathcal{E}(N)$ is a (true) P-martingale.

In comparison to Definition 3.11 in [16], we have generalised the definition of $\mathcal{E}$-martingales to $(\mathcal{E}, Z^N)$-martingales by introducing the process Z^N. This is needed for a clean formulation of our results, but it also makes intuitive sense. Suppose Q is an equivalent martingale measure for Y and write its density process with respect to P as $Z^Q = Z_0^Q \mathcal{E}(N^Q)$. By the Bayes rule, the product YZ^Q is then a P-martingale and so is ${}^0YZ^Q = (Y - Y_0)Z^Q$. One consequence is that the product of ${}^0Y$ and $\mathcal{E}(N^Q)$ is a local P-martingale so that Y is an $\mathcal{E}(N^Q)$-local martingale. (Of course, $Z^Q > 0$ implies that $T_m \equiv T$ for $m \geq 1$.) We also have that ${}^0YZ_0^Q\mathcal{E}(N^Q)$ is a true P-martingale so that Y is an $(\mathcal{E}(N^Q), Z^Q)$-martingale. But unless we know more about Z_0^Q, we cannot assert that the product ${}^0Y\mathcal{E}(N^Q)$ is a true P-martingale (since it need not be P-integrable); so Y is not an $\mathcal{E}(N^Q)$-martingale in the sense of [16]. Hence we see that in the abstract definition, $Z_{T_m}^N$ plays a similar role at time T_m as the density Z_0^Q of Q at time 0, and its main role is to ensure integrability properties. (This is not needed in [16] because the authors there work with $Y = \vartheta \bullet S \in \mathcal{H}^2(P)$ and assume that $\mathcal{E}(N)$ satisfies the reverse Hölder inequality $R_2(P)$. In our notation, this allows to take $Z^N \equiv 1$.)

Remark 2.8. If N is as above a local martingale, then $J^m := \mathbb{1}_{]\!]T_m,T]\!]} \bullet \mathcal{E}(\mathbb{1}_{]\!]T_m,T]\!]} \bullet N)$ is for each m also a local martingale; if N is in addition locally square-integrable, then so is J^m; and both statements still hold if we multiply J^m by a strictly positive $\mathcal{F}_{T_m}$-measurable random variable. There is no problem with adaptedness since $J^m = 0$ on $]\!]T_m, T]\!]$.

Conversely – and this will be used later – suppose N is a semimartingale. If J^m is for each m a local martingale, then writing $J^m = (\mathcal{E}(\mathbb{1}_{]\!]T_m,T]\!]} \bullet N)_- \mathbb{1}_{]\!]T_m,T]\!]}) \bullet N$ and observing that $\mathcal{E}(\mathbb{1}_{]\!]T_m,T]\!]} \bullet N)_- \neq 0$ on $]\!]T_m, T_{m+1}]\!]$ by the definition of T_m shows that $\mathbb{1}_{]\!]T_m,T_{m+1}]\!]} \bullet N$ is a local martingale for each m, and then so is N. Again this still holds if we replace J^m by $\beta_m J^m$ for an $\mathcal{F}_{T_m}$-measurable $\beta_m > 0$, and again local square-integrability transfers, from J^m (or $\beta_m J^m$) to N.

The next two propositions give some information about the structure of $\mathcal{E}$-local martingales and $(\mathcal{E}, Z^N)$-martingales. The results are almost literally taken from Corollaries 3.16 and 3.17 in [16]; the proofs there still work for our generalisation.

Proposition 2.9. *Let Y be a special semimartingale and $Y = Y_0 + M^Y + A^Y$ its canonical decomposition. Then Y is an $\mathcal{E}$-local martingale if and only if $[M^Y, N]$ is locally P-integrable and $A^Y = -\langle M^Y, N\rangle$.*

Proposition 2.10. *A semimartingale $Y = Y_0 + M^Y - \langle M^Y, N\rangle$ satisfying $E\big[Y_T^* \big(Z_{T_m}^N {}^{T_m}\mathcal{E}(N)\big)_T^*\big] < \infty$ for any $m \in \mathbb{N}$ is an $(\mathcal{E}, Z^N)$-martingale.*

We also need the following definitions.

Definition 2.11. We say that $(\mathcal{E}, Z^N)$ with $\mathcal{E} = \mathcal{E}(N)$ is *regular and square-integrable* if $\mathbb{1}_{]\!]T_m,T]\!]} \bullet (Z_{T_m}^N {}^{T_m}\mathcal{E}(N))$ is a square-integrable (true) P-martingale and $Z_{T_m}^N$ is square-integrable for any m.

Lemma 2.12. *Suppose $(\mathcal{E}, Z^N)$ with $\mathcal{E} = \mathcal{E}(N)$ is regular and square-integrable. Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of $(\mathcal{E}, Z^N)$-martingales with $X^n_T \in L^2(P)$ and $X^n_T \to H$ in $L^2(P)$ as $n \to \infty$. Then there exist a subsequence $(X^{n_\ell})_{\ell\in\mathbb{N}}$ and an $\mathcal{E}$-local martingale X given by $X_T = H$ and*

$$X_t := \frac{E[H\,{}^{T_m}\mathcal{E}(N)_T|\mathcal{F}_t]}{{}^{T_m}\mathcal{E}(N)_t} \quad \text{on } [\![T_m, T_{m+1}[\![\tag{2.6}$$

such that $X^{n_\ell} \to X$ in the semimartingale topology (in $\mathcal{S}(P)$, for short) as $\ell \to \infty$. If $\mathcal{E}(N)$ satisfies the reverse Hölder inequality $R_1(P)$, then X is an $(\mathcal{E}, Z^N)$-martingale.

Proof. 1) To show that X above is an $\mathcal{E}$-local martingale with $X_T = H$, we argue similarly as in the proof of Proposition 3.12.iii) in [16]. More precisely, we exploit that we need not assume $\mathcal{E}(N)$ to satisfy $R_q(P)$ with $q = 2$ as used there; it is sufficient to exploit that $\mathcal{E}(N)$ always satisfies $R_1(P)$ in a local sense. We define for each $m \in \mathbb{N}_0$ a sequence of stopping times $\tau^m_k = T_m \mathbb{1}_{F^c_k} + T\mathbb{1}_{F_k}$ with $F_k := \big\{E\big[|{}^{T_m}\mathcal{E}(N)_T|\big|\mathcal{F}_{T_m}\big] \le k\big\}$ for $k \in \mathbb{N}$. Then we rewrite (2.6) after multiplication with $Z^N_{T_m}$ as

$$L_t := X_t Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_t = E[X_T Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_T|\mathcal{F}_t] \quad \text{on } [\![T_m, T_{m+1}[\![\tag{2.7}$$

and note that the right-hand side is in $L^1(P)$ since $X_T = H$ and $Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_T$ are both in $L^2(P)$. Hence $L_t\mathbb{1}_{\{T_m\le t<T_{m+1}\}}$ is in $L^1(P)$ and so is then $X_{T_m} Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{T_m}$. To argue that X is an $\mathcal{E}$-local martingale, we want to prove that $({}^{T_m}X Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N))^{\tau^m_k}$ is a P-martingale, and (2.7) already gives the martingale property for the unstopped process ${}^{T_m}L$. So due to ${}^{T_m}X = X - X^{T_m}$, the P-integrability of L_t and $\tau^m_k \ge T_m$, it only remains to show that

$$X_{T_m} Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{t\wedge\tau^m_k}\mathbb{1}_{\{T_m\le t<T_{m+1}\}} \in L^1(P). \tag{2.8}$$

But $Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)$ is a P-martingale and remains so after stopping by τ^m_k, and the final value of that stopped process is

$$Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{\tau^m_k} = Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{T_m}\mathbb{1}_{F^c_k} + Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_T\mathbb{1}_{F_k}.$$

Multiplying by X_{T_m}, conditioning on $\mathcal{F}_{T_m}$ and using the definition of F_k hence gives (2.8); indeed, we have

$$\begin{aligned} &E\big[|X_{T_m} Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_T\mathbb{1}_{F_k}|\big] \\ &\quad\le E\big[|X_{T_m} Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{T_m}|E[|\mathcal{E}(N)_T|\,|\,\mathcal{F}_{T_m}]\mathbb{1}_{F_k}\big] < \infty. \end{aligned}$$

This shows that X is an $\mathcal{E}$-local martingale; and if $\mathcal{E}(N)$ satisfies $R_1(P)$, we have $F_k = \Omega$, hence $\tau^m_k = T$, for k large enough so that X is even an $(\mathcal{E}, Z^N)$-martingale.

2) Now fix $m \in \mathbb{N}$ and take any subsequence of (X^n), again denoted by (X^n) in this step for ease of notation. Set $Y^{n,m} := {}^{T_m}X^n = X^n - (X^n)^{T_m}$ so that by the definition of $(\mathcal{E}, Z^N)$-martingales, the product of $Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)$ and $Y^{n,m}$ is a martingale. Note that $(Y^{n,m})^{\tau^m_k} = (X^n - (X^n)^{T_m})\mathbb{1}_{F_k}$ and $(Y^m)^{\tau^m_k} = (X - X^{T_m})\mathbb{1}_{F_k}$ for each $k \in \mathbb{N}$. Since $X^n_T \to X_T = H$ in $L^2(P)$ and

$$X^n_t\,{}^{T_m}\mathcal{E}(N)_t = E[X^n_T\,{}^{T_m}\mathcal{E}(N)_T|\mathcal{F}_t] \quad \text{on } [\![T_m, T_{m+1}[\![\tag{2.9}$$

for the $(\mathcal{E}, Z^N)$-martingales X^n by (2.7), we obtain for $n \to \infty$ that

$$\begin{aligned} &E\big[|(X^n_{T_{m+1}\wedge\tau^m_k} - X_{T_{m+1}\wedge\tau^m_k})Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{T_{m+1}\wedge\tau^m_k}|\big] \\ &\quad\le E\big[|(X^n_T - H)Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_T|\big] \\ &\quad\le \|X^n_T - H\|_{L^2(P)}\|Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_T\|_{L^2(P)} \end{aligned}$$

tends to 0, and from the definition of τ^m_k that for $n \to \infty$,

$$\begin{aligned} &E\big[|(X^n_{T_{m+1}\wedge\tau^m_k} - X_{T_{m+1}\wedge\tau^m_k})Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{T_{m+1}\wedge\tau^m_k}|\big] \\ &\quad= E\big[|E[(X^n_T - H)\,{}^{T_m}\mathcal{E}(N)_T|\mathcal{F}_{T_m}]Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{T_{m+1}\wedge\tau^m_k}|\big] \\ &\quad\le E\big[|(X^n_T - H)Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_T|\big]k \longrightarrow 0. \end{aligned}$$

This gives $Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{T\wedge\tau^m_k}\,Y^{n,m}_{T\wedge\tau^m_k} \to Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_{T\wedge\tau^m_k}\,Y^m_{T\wedge\tau^m_k}$ in $L^1(P)$ as $n \to \infty$ because ${}^{T_m}\mathcal{E}(N)_T = 0$ on $\{T_{m+1} < T\}$. Theorem 4.21 in [50] then yields a subsequence $(Y^{n_j,m})_{j\in\mathbb{N}}$ such that

$$\big(Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)\,Y^{n_j,m}\big)^{\tau^m_k} \longrightarrow \big(Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)\,Y^m\big)^{\tau^m_k} \quad \text{locally in } \underline{\underline{H}}^1_{\text{loc}}(P)$$

as $j \to \infty$ and therefore $Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)\,Y^{n_j,m} \to Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)\,Y^m$ in $\mathcal{S}(P)$ as $j \to \infty$ by Theorem V.14 in [80]. Because $\frac{1}{Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)}\mathbb{1}_{[\![T_m,T_{m+1}[\![}$ is a semimartingale and the multiplication of semimartingales is continuous in $\mathcal{S}(P)$, we get $Y^{n_j,m}\mathbb{1}_{[\![T_m,T_{m+1}[\![} \to Y^m\mathbb{1}_{[\![T_m,T_{m+1}[\![}$ in $\mathcal{S}(P)$ as $j \to \infty$. Note that the subsequence $(n_j)_{j\in\mathbb{N}}$ depends on m.

3) Now we construct the desired subsequence $(n_\ell)_{\ell\in\mathbb{N}}$ by a diagonal argument, as follows. Start with $m = 0$ and the original sequence (X^n) to obtain from step 2) a subsequence $(n_j(0))_{j\in\mathbb{N}}$, and take $n_1 := n_1(0)$. Then take $m = 1$, apply step 2) for the subsequence $(X^{n_j(0)})_{j\in\mathbb{N}}$ to get a new subsequence $(n_j(1))_{j\in\mathbb{N}}$, and take $n_2 := n_1(1)$. Iterating this procedure yields our subsequence $(n_\ell)_{\ell\in\mathbb{N}}$, and we claim that $X^{n_\ell} \to X$ in $\mathcal{S}(P)$ as $\ell \to \infty$. To see this, use the definition of $Y^{n,m}$ to write

$$X^{n_\ell} = \sum_{m=0}^{\infty} Y^{n_\ell}\mathbb{1}_{[\![T_m,T_{m+1}[\![} + \sum_{m=0}^{\infty} X^{n_\ell}_{T_m}\mathbb{1}_{[\![T_m,T_{m+1}[\![} + X^{n_\ell}_T\mathbb{1}_{[\![T]\!]}. \tag{2.10}$$

Since $Y^{n_j(m),m}\mathbb{1}_{[\![T_m,T_{m+1}[\![} \to Y^m\mathbb{1}_{[\![T_m,T_{m+1}[\![}$ as $j \to \infty$, the first sum converges in $\mathcal{S}(P)$ to

$$\sum_{m=0}^{\infty} Y^m\mathbb{1}_{[\![T_m,T_{m+1}[\![} = X - \sum_{m=0}^{\infty} X_{T_m}\mathbb{1}_{[\![T_m,T_{m+1}[\![} - X_T\mathbb{1}_{[\![T]\!]},$$

where the equality now uses the definition of $Y^m = X - X^{T_m}$. To obtain the convergence of the second sum in (2.10), we observe that

$$E\big[|X^n_{T_m} - X_{T_m}|Z^N_{T_m}\big] = E\big[|E[(X^n_T - H)\,{}^{T_m}\mathcal{E}(N)_T|\mathcal{F}_{T_m}]|Z^N_{T_m}\big] \\ \leq \|X^n_T - H\|_{L^2(P)} \|Z^N_{T_m}\,{}^{T_m}\mathcal{E}(N)_T\|_{L^2(P)}$$

by (2.9) for all $m \in \mathbb{N}_0$ and for $m = \infty$ with $T_\infty := T$ and therefore as $\ell \to \infty$,

$$\sum_{m=0}^{\infty} Z^N_{T_m} X^{n_\ell}_{T_m} \mathbb{1}_{[\![T_m,T_{m+1}[\![} + X^{n_\ell}_T \mathbb{1}_{[\![T]\!]} \longrightarrow \sum_{m=0}^{\infty} Z^N_{T_m} X_{T_m} \mathbb{1}_{[\![T_m,T_{m+1}[\![} + X_T \mathbb{1}_{[\![T]\!]} \tag{2.11}$$

locally in $\underline{\underline{H}}^1(P)$ with the localising sequence (T_m). As local convergence in $\underline{\underline{H}}^1(P)$ implies convergence in $\mathcal{S}(P)$ again by Theorem V.14 in [80], (2.11) also holds in $\mathcal{S}(P)$. Because $\sum_{m=0}^{\infty} \frac{1}{Z^N_{T_m}} \mathbb{1}_{[\![T_m,T_{m+1}[\![}$ is a semimartingale and the multiplication of semimartingales is continuous in $\mathcal{S}(P)$, this completes the proof. □

Corollary 2.13. *Suppose that $(\mathcal{E}, Z^N)$ with $\mathcal{E} = \mathcal{E}(N)$ is regular and square-integrable, $S = S_0 + M - \langle M, N\rangle$ is in $\mathcal{H}^2_{\mathrm{loc}}(P)$ and $(\vartheta^n)_{n\in\mathbb{N}}$ is a sequence in Θ such that $\vartheta^n \bullet S_T \to H$ in $L^2(P)$. Then $\vartheta^n \bullet S$ is an $(\mathcal{E}, Z^N)$-martingale for each $n \in \mathbb{N}$, and there exist $\vartheta \in \overline{\Theta}$ with $\vartheta \bullet S_T = H$ and*

$$\vartheta \bullet S_t = \frac{E[(\vartheta \bullet S_T)\,{}^{T_m}\mathcal{E}(N)_T|\mathcal{F}_t]}{{}^{T_m}\mathcal{E}(N)_t} = \frac{E[H\,{}^{T_m}\mathcal{E}(N)_T|\mathcal{F}_t]}{{}^{T_m}\mathcal{E}(N)_t} \quad \text{on } [\![T_m, T_{m+1}[\![$$

and a subsequence $(\vartheta^{n_k})_{k\in\mathbb{N}}$ in Θ such that $\vartheta^{n_k} \bullet S \to \vartheta \bullet S$ in $\mathcal{S}(P)$ as $k \to \infty$.

Proof. By Proposition 2.10, S is an $\mathcal{E}$-local martingale and $\vartheta^n \bullet S$ is an $(\mathcal{E}, Z^N)$-martingale for each n. Then Lemma 2.12 gives the existence of an $\mathcal{E}$-local martingale X and a subsequence (ϑ^{n_k}) in Θ such that $X_T = H$ and $X_t = \frac{E[H\,{}^{T_m}\mathcal{E}(N)_T|\mathcal{F}_t]}{{}^{T_m}\mathcal{E}(N)_t}$ on $[\![T_m, T_{m+1}[\![$ and $\vartheta^{n_k} \bullet S \to X$ in $\mathcal{S}(P)$. As the space of stochastic integrals is closed under convergence in $\mathcal{S}(P)$ by Theorem V.4 in [71], there exists some $\vartheta \in \mathcal{L}(S)$ with $\vartheta \bullet S = X$. Since convergence in $\mathcal{S}(P)$ implies ucp-convergence and therefore that $\vartheta^{n_k} \bullet S_\tau \to \vartheta \bullet S_\tau$ in probability for all stopping times τ, we obtain that $\vartheta \in \overline{\Theta}$ which completes the proof. □

To deal with the fact that different integrands may lead to the same stochastic integral (or, in financial terms, that we may have redundant assets), we introduce the *projection on the predictable range*. For a detailed explanation of the related issues of selecting particular representatives of equivalence classes of integrands as well as for sufficient conditions for the closedness of the projection on the predictable range for certain correspondences, we refer the reader to Chapter III.

Proposition 2.14. *For each* $\mathbb{R}^d$*-valued semimartingale* Y*, there exists an* $\mathbb{R}^{d\times d}$*-valued predictable process* Π^Y*, called the* projection on the predictable range of Y*, which takes values in the orthogonal projections in* $\mathbb{R}^d$ *and has the following property: If* ϑ *is in* $\mathcal{L}(Y)$ *and* φ *is predictable, then* φ *is in* $\mathcal{L}(Y)$ *with* $\varphi \bullet Y = \vartheta \bullet Y$ *(up to indistinguishability) if and only if* $\Pi^Y\vartheta = \Pi^Y\varphi$ P_B*-a.e. We choose and fix one version of* Π^Y*.*

Proof. See Lemma 5.3 in Chapter III. □

Example 2.15. For the frequently used Itô process models of the form

$$\frac{dY_t^i}{Y_t^i} = (\mu_t^i - r_t)\,dt + \sum_{k=1}^m \sigma_t^{ik}\,dW_t^k,$$

Π^Y is the projection on the orthogonal complement of the kernel of $\sigma\sigma^\top$. If each $\sigma_t\sigma_t^\top$ is invertible (as is usually assumed), Π^Y is just the identity. This holds in particular when $m = d$ and each σ_t is invertible, i.e. when the model is complete without the constraints.

After these preparations, we obtain the closedness of $G_T(\overline{\Theta(C)})$ by the following theorem. We recall that this implies the *existence of a solution* to the constrained mean-variance hedging problem (2.5), for any payoff $H \in L^2(P)$, if C has also convex values.

Theorem 2.16. *Suppose that* $(\mathcal{E}, Z^N)$ *with* $\mathcal{E} = \mathcal{E}(N)$ *is regular and square-integrable and* $S = S_0 + M - \langle M, N\rangle$ *is in* $\mathcal{H}^2_{\mathrm{loc}}(P)$ *so that* S *is an* $\mathcal{E}$*-local martingale by Proposition 2.9. Let* $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ *be a predictable correspondence with closed values such that the projection of* C *on the predictable range of* S *is closed, i.e.* $\Pi^S(\omega, t)C(\omega, t)$ *is* P_B*-a.e. closed. Then* $G_T(\overline{\Theta(C)})$ *is closed in* $L^2(P)$*.*

Proof. Let (ϑ^n) be a sequence in $\overline{\Theta(C)}$ with $\vartheta^n \bullet S_T \to H$ in $L^2(P)$. Using the definition of $\overline{\Theta(C)}$ and a diagonal argument yields a sequence (φ^n) in $\Theta(C)$ with $\varphi^n \bullet S_T \to H$ in $L^2(P)$. Then Corollary 2.13 implies that there exist $\vartheta \in \overline{\Theta}$ with $\vartheta \bullet S_T = H$ and a subsequence, again indexed by n, with $\varphi^n \bullet S \to \vartheta \bullet S$ in $\mathcal{S}(P)$. Since $\mathcal{C} \bullet S = \{\psi \bullet S \,|\, \psi \in \mathcal{C}\}$ is closed in $\mathcal{S}(P)$ by Theorem 4.5 in Chapter III, the integrand ϑ can be chosen C-valued; this uses the assumption on $\Pi^S C$. As convergence in $\mathcal{S}(P)$ implies ucp-convergence, we obtain $\varphi^n \bullet S_\tau \to \vartheta \bullet S_\tau$ in probability for all stopping times τ, and therefore ϑ is in $\overline{\Theta(C)}$. This completes the proof. □

Remark 2.17. Theorem 2.16 should be compared to the main result of Theorem 3.12 in Chapter II. It has a considerably weaker assumption than the latter and is therefore a stronger result in that respect. On the other hand, $\overline{\Theta(C)}$ is bigger than the space $\Theta(C)$ considered in Chapter II so that one feels it could be easier for $G_T(\overline{\Theta(C)})$ to be closed in $L^2(P)$.

Having clarified the existence of a solution to (2.5) or (2.4), our goal in the sequel is to describe its *structure* in more detail. This is done via stochastic control techniques and in particular dynamic programming, and for that, we need certain properties for the space $\overline{\Theta(C)}$ of strategies we work with. This is the reason why we slightly changed the definition in comparison to [14]: We want to show, without assuming that there exists an ELMM Q for S with $\frac{dQ}{dP} \in L^2(P)$, that $\overline{\Theta(C)}$ is stable under bifurcation and almost stable.

Lemma 2.18. *For any predictable correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$, the space $\overline{\Theta(C)}$ has the following properties:*

1) $\overline{\Theta(C)}$ *is* stable under bifurcation*: If ϑ, φ are in $\overline{\Theta(C)}$, σ is a stopping time, $F \in \mathcal{F}_\sigma$ and $\vartheta \mathbb{1}_{[\![0,\sigma]\!]} = \varphi \mathbb{1}_{[\![0,\sigma]\!]}$, then $\psi = \vartheta \mathbb{1}_F + \varphi \mathbb{1}_{F^c}$ is also in $\overline{\Theta(C)}$.*

2) $\overline{\Theta(C)}$ *is* almost stable*: For all ϑ, φ in $\overline{\Theta(C)}$, stopping times σ and $F \in \mathcal{F}_\sigma$ with $P[F] > 0$, there is for each $\varepsilon \in (0, P[F])$ a set $F_\varepsilon \subseteq F$ in $\mathcal{F}_\sigma$ with $P[F \setminus F_\varepsilon] \leq \varepsilon$ such that*

$$\psi := \vartheta \mathbb{1}_{F_\varepsilon^c} + (\vartheta \mathbb{1}_{[\![0,\sigma]\!]} + \varphi \mathbb{1}_{]\!]\sigma,T]\!]}) \mathbb{1}_{F_\varepsilon} \text{ is in } \overline{\Theta(C)}$$

and $\vartheta \bullet S_\sigma$ is uniformly bounded on F_ε.

Proof. By the definition of $\overline{\Theta(C)}$, we must in both cases find a sequence (ψ^n) in $\Theta(C)$ such that $\psi^n \bullet S_T \xrightarrow{L^2(P)} \psi \bullet S_T$ and $\psi^n \bullet S_\tau \xrightarrow{P} \psi \bullet S_\tau$ for all stopping times τ. We start with approximating sequences (ϑ^n) and (φ^n) in $\Theta(C)$ for $\vartheta, \varphi \in \overline{\Theta(C)}$.

1) For $\psi^n := \vartheta^n \mathbb{1}_F + \varphi^n \mathbb{1}_{F^c} \in \Theta(C)$, the local character of stochastic integrals yields

$$\begin{aligned} &\|\psi^n \bullet S_T - \psi \bullet S_T\|_{L^2(P)} \\ &\quad = \|(\vartheta^n \bullet S_T - \vartheta \bullet S_T)\mathbb{1}_F + (\varphi^n \bullet S_T - \varphi \bullet S_T)\mathbb{1}_{F^c}\|_{L^2(P)} \overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

and, for all stopping times τ,

$$\psi^n \bullet S_\tau = (\vartheta^n \bullet S_\tau)\mathbb{1}_F + (\varphi^n \bullet S_\tau)\mathbb{1}_{F^c} \xrightarrow{P} (\vartheta \bullet S_\tau)\mathbb{1}_F + (\varphi \bullet S_\tau)\mathbb{1}_{F^c} = \psi \bullet S_\tau.$$

2) By Egorov's theorem, we can find for each $\varepsilon \in (0, P[F])$ a set $F_\varepsilon \in \mathcal{F}_\sigma$ with $P[F \setminus F_\varepsilon] \leq \varepsilon$ such that $\vartheta^n \bullet S_\sigma \to \vartheta \bullet S_\sigma$ and $\varphi^n \bullet S_\sigma \to \varphi \bullet S_\sigma$ uniformly on F_ε. For the sequence $\psi^n := \vartheta^n \mathbb{1}_{F_\varepsilon^c} + (\vartheta^n \mathbb{1}_{[\![0,\sigma]\!]} + \varphi^n \mathbb{1}_{]\!]\sigma,T]\!]}) \mathbb{1}_{F_\varepsilon}$ in $\Theta(C)$, we obtain again from the local character of stochastic integrals that

$$\begin{aligned} &\left\|\psi^n \bullet S_T - \left(\vartheta \mathbb{1}_{F_\varepsilon^c} + (\vartheta \mathbb{1}_{[\![0,\sigma]\!]} + \varphi \mathbb{1}_{]\!]\sigma,T]\!]}) \mathbb{1}_{F_\varepsilon}\right) \bullet S_T\right\|_{L^2(P)} \\ &\leq \|(\vartheta^n \bullet S_T - \vartheta \bullet S_T)\mathbb{1}_{F_\varepsilon^c}\|_{L^2(P)} + \|(\vartheta^n \bullet S_\sigma - \vartheta \bullet S_\sigma)\mathbb{1}_{F_\varepsilon}\|_{L^2(P)} \\ &\quad + \|(\varphi^n \bullet S_\sigma - \varphi \bullet S_\sigma)\mathbb{1}_{F_\varepsilon}\|_{L^2(P)} + \|(\varphi^n \bullet S_T - \varphi \bullet S_T)\mathbb{1}_{F_\varepsilon}\|_{L^2(P)} \overset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

where the first and the last term on the right-hand side converge to zero by the choice of (ϑ^n) and (φ^n) and the two middle terms by the uniform convergence on F_ε. Since

$$\psi^n \bullet S_\tau = (\vartheta^n \bullet S_\tau)\mathbb{1}_{F_\varepsilon^c} + (\vartheta^n \bullet S_{\sigma\wedge\tau})\mathbb{1}_{F_\varepsilon} + (\varphi^n \bullet S_\tau - \varphi^n \bullet S_{\sigma\wedge\tau})\mathbb{1}_{F_\varepsilon},$$
$$\psi \bullet S_\tau = (\vartheta \bullet S_\tau)\mathbb{1}_{F_\varepsilon^c} + (\vartheta \bullet S_{\sigma\wedge\tau})\mathbb{1}_{F_\varepsilon} + (\varphi \bullet S_\tau - \varphi \bullet S_{\sigma\wedge\tau})\mathbb{1}_{F_\varepsilon}$$

for all stopping times τ again by the local character of stochastic integrals, we obtain that $\psi^n \bullet S_\tau \xrightarrow{P} \psi \bullet S_\tau$ for all stopping times τ.

Finally, to get $\vartheta \bullet S_\sigma$ uniformly bounded on F_ε as well, one starts instead of F with some $F'_N := F \cap \{|\vartheta \bullet S_\sigma| \le N\} \in \mathcal{F}_\sigma$. Then $F'_N \nearrow F$, so $P[F'_N]$ increases to $P[F]$ as $N \to \infty$, and taking $N(\varepsilon)$ large enough will give the result. This completes the proof. □

IV.3 Dynamic programming

In this section, we establish a dynamic description of the optimal strategy for (2.4) by dynamic programming. To that end, we consider the problem to

$$\text{minimise } E\big[|V_T(x,\vartheta)|^2\big] = E\big[|x + \vartheta \bullet S_T|^2\big] \text{ over all } \vartheta \in \overline{\Theta(K)} \tag{3.1}$$

for a fixed $x \in \mathbb{R}$ and a predictable correspondence $K : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ with *closed cones* as values. We view (3.1) as a stochastic optimal control problem and want to study the corresponding value process.

We first need some notation. For any stopping time τ with values in $[0,T]$, we denote by $\mathcal{S}_{\tau,T}$ the family of all stopping times σ with $\tau \le \sigma \le T$ (so that $\tau \in \mathcal{S}_{0,T}$). In order to describe the optimisation starting at time τ with wealth x, we define for $\tau \in \mathcal{S}_{0,T}$, $\sigma \in \mathcal{S}_{\tau,T}$ and $\vartheta \in \overline{\Theta(K)}$ with $\vartheta = 0$ on $[\![0,\tau]\!]$ the space

$$\begin{aligned}\mathfrak{K}(\vartheta,\sigma;\tau) &:= \big\{\varphi \in \overline{\Theta(K)} \,\big|\, \varphi = 0 \text{ on } [\![0,\tau]\!] \text{ and } \varphi\mathbb{1}_{]\!]\tau,\sigma]\!]} = \vartheta\mathbb{1}_{]\!]\tau,\sigma]\!]}\big\} \\ &= \big\{\varphi \in \overline{\Theta(K)} \,\big|\, \varphi\mathbb{1}_{[\![0,\sigma]\!]} = \vartheta\mathbb{1}_{[\![0,\sigma]\!]}\big\}.\end{aligned}$$

Note that $\mathfrak{K}(\vartheta,\sigma;\sigma) = \mathfrak{K}(0,\sigma;\sigma)$. We then define for $\varphi \in \mathfrak{K}(\vartheta,\sigma;\tau)$ the random variables

$$\Gamma(\varphi,\sigma;x,\tau,\vartheta) := E\big[|V_T(x,\varphi)|^2\big|\mathcal{F}_\sigma\big] = E\big[|x + \textstyle\int_\tau^\sigma \vartheta_u\,dS_u + \int_\sigma^T \varphi_u\,dS_u|^2\big|\mathcal{F}_\sigma\big],$$

and for $\sigma \in \mathcal{S}_{\tau,T}$ and $\vartheta \in \overline{\Theta(K)}$ with $\vartheta = 0$ on $[\![0,\tau]\!]$

$$\bar{J}(\sigma;x,\tau,\vartheta) := \operatorname*{ess\,inf}_{\varphi \in \mathfrak{K}(\vartheta,\sigma;\tau)} \Gamma(\varphi,\sigma;x,\tau,\vartheta).$$

Because the family $\{\Gamma(\varphi,\sigma;x,\tau,\vartheta) \,|\, \varphi \in \mathfrak{K}(\vartheta,\sigma;\tau)\}$ is stable under taking minima by part 1) of Lemma 2.18, the family $\{\bar{J}(\sigma;x,\tau,\vartheta) \,|\, \sigma \in \mathcal{S}_{\tau,T}\}$ for any fixed $\tau \in \mathcal{S}_{0,T}$ is a submartingale system for any $\vartheta \in \overline{\Theta(K)}$ with $\vartheta = 0$ on $[\![0,\tau]\!]$. It is a martingale system for $\widetilde{\vartheta} \in \overline{\Theta(K)}$ with $\widetilde{\vartheta} = 0$ on $[\![0,\tau]\!]$ if and only if $\widetilde{\vartheta} = \widetilde{\varphi}^{(x,\tau)}$ is optimal for the problem to

$$\text{minimise } E\big[|x + \textstyle\int_\tau^T \varphi_u \, dS_u|^2\big] = E\big[|x + \varphi \bullet S_T|^2\big] \text{ over all } \varphi \in \mathfrak{K}(0,\tau;\tau). \tag{3.2}$$

These facts follow by standard arguments as e.g. in Chapter 1 of [40] or the proof of Theorem 4.1 in [64]. We now exploit the quadratic and conic structure of our problem to obtain a decomposition of $\bar{J}$.

Proposition 3.1. *For any stopping time $\tau \in \mathcal{S}_{0,T}$, there exist families of random variables $\{\bar{L}^\pm(\sigma) \,|\, \sigma \in \mathcal{S}_{\tau,T}\}$ such that*

$$\begin{aligned} \bar{J}(\sigma; x, \tau, \vartheta) &= \operatorname*{ess\,inf}_{\varphi \in \mathfrak{K}(\vartheta,\sigma;\tau)} E\big[|x + \textstyle\int_\tau^\sigma \vartheta_u \, dS_u + \int_\sigma^T \varphi_u \, dS_u|^2 \big| \mathcal{F}_\sigma\big] \\ &= \big((x + \textstyle\int_\tau^\sigma \vartheta_u \, dS_u)^+\big)^2 \bar{L}^+(\sigma) + \big((x + \int_\tau^\sigma \vartheta_u \, dS_u)^-\big)^2 \bar{L}^-(\sigma) \end{aligned} \tag{3.3}$$

for any $\sigma \in \mathcal{S}_{\tau,T}$ and any $\vartheta \in \overline{\Theta(K)}$ with $\vartheta = 0$ on $[\![0,\tau]\!]$. The random variables $\bar{L}^\pm(\sigma)$ do not depend on x, τ or ϑ and are explicitly given by

$$\bar{L}^\pm(\sigma) := \operatorname*{ess\,inf}_{\varphi \in \mathfrak{K}(0,\sigma;\sigma)} E\big[|1 \pm \textstyle\int_\sigma^T \varphi_u \, dS_u|^2 \big| \mathcal{F}_\sigma\big] = \bar{J}(\sigma; \pm 1, \sigma, 0). \tag{3.4}$$

In particular, all the $\bar{L}^\pm(\sigma)$ are $[0,1]$-valued, and $\bar{L}^\pm(T) = 1$.

Proof. Fix x, τ, ϑ and σ and define $\bar{L}^\pm(\sigma)$ by (3.4). The last assertion is then obvious, and the intuition for (3.3) is that the quadratic structure of our problem and the fact that the constraints are given by cones allow us to pull out an $\mathcal{F}_\sigma$-measurable factor. Note that we can also write $\vartheta \bullet S_\sigma$ instead of $\int_\tau^\sigma \vartheta_u \, dS_u$ because $\vartheta = 0$ on $[\![0,\tau]\!]$. For the detailed proof of (3.3), we argue by contradiction. Suppose first that

$$\bar{J}(\sigma; x, \tau, \vartheta) < \big((x + \vartheta \bullet S_\sigma)^+\big)^2 \bar{L}^+(\sigma) + \big((x + \vartheta \bullet S_\sigma)^-\big)^2 \bar{L}^-(\sigma) \quad \text{on } F'$$

for some set $F' \in \mathcal{F}_\sigma$ with $P[F'] > 0$. Then there exist $\varphi \in \mathfrak{K}(\vartheta, \sigma; \tau)$ and $F \in \mathcal{F}_\sigma$ with $F \subseteq F'$ and $P[F] > 0$ such that

$$E\big[|x + \varphi \bullet S_T|^2 \big| \mathcal{F}_\sigma\big] < \big((x + \vartheta \bullet S_\sigma)^+\big)^2 \bar{L}^+(\sigma) + \big((x + \vartheta \bullet S_\sigma)^-\big)^2 \bar{L}^-(\sigma) \tag{3.5}$$

on F. Since $\bar{J}(\sigma; x, \tau, \vartheta) \geq 0$, we have $F \subseteq \{0 < |x + \vartheta \bullet S_\sigma|\}$ and can write

$$\begin{aligned} E\big[|x + \varphi \bullet S_T|^2 \big| \mathcal{F}_\sigma\big] &= \big((x + \vartheta \bullet S_\sigma)^+\big)^2 E\bigg[\bigg(1 + \frac{\mathbb{1}_{]\!]\sigma,T]\!]}\varphi}{(x + \vartheta \bullet S_\sigma)^+} \bullet S_T\bigg)^2 \bigg| \mathcal{F}_\sigma\bigg] \\ &\quad + \big((x + \vartheta \bullet S_\sigma)^-\big)^2 E\bigg[\bigg(1 - \frac{\mathbb{1}_{]\!]\sigma,T]\!]}\varphi}{(x + \vartheta \bullet S_\sigma)^-} \bullet S_T\bigg)^2 \bigg| \mathcal{F}_\sigma\bigg] \end{aligned} \tag{3.6}$$

on F. Plugging the last expression into (3.5), we obtain

$$E\bigg[\bigg(1 \pm \frac{\mathbb{1}_{]\!]\sigma,T]\!]}\varphi}{(x+\vartheta \bullet S_\sigma)^\pm} \bullet S_T\bigg)^2 \bigg|\mathcal{F}_\sigma\bigg] < \bar{L}^\pm(\sigma)$$

on $F^\pm := F \cap \{x + \vartheta \bullet S_\sigma \gtrless 0\}$. To derive a contradiction to the definition of $\bar{L}^\pm(\sigma)$, it remains to show that

$$\psi^\pm := \frac{\mathbb{1}_{]\!]\sigma,T]\!]}\varphi}{(x+\vartheta \bullet S_\sigma)^\pm}\mathbb{1}_{G^\pm} \in \mathfrak{K}(0,\sigma;\sigma)$$

for some sets $G^\pm \in \mathcal{F}_\sigma$ with $G^\pm \subseteq F^\pm$ and $P[G^\pm] > 0$. To that end, let $(\varphi^n)_{n\in\mathbb{N}}$ be an approximating sequence in $\Theta(K)$ for φ. By passing to a subsequence again indexed by n, we can assume that $\varphi^n \bullet S_\sigma \to \varphi \bullet S_\sigma$ P-a.s. Then we can find $G^+ \in \mathcal{F}_\sigma$ with $G^+ \subseteq F^+$ and $P[G^+] > 0$ such that $m \geq |x + \vartheta \bullet S_\sigma| \geq \frac{1}{m}$ on G^+ for some $m \in \mathbb{N}$, by continuity of P from below, and $\varphi^n \bullet S_\sigma \to \varphi \bullet S_\sigma$ uniformly on G^+, by Egorov's theorem. Moreover, we obtain that $\psi^n := \frac{\mathbb{1}_{]\!]\sigma,T]\!]}\varphi^n}{(x+\vartheta\bullet S_\sigma)^+}\mathbb{1}_{G^+} \in \Theta(K)$ because K is cone-valued, and

$$|\psi^n \bullet S_\varrho - \psi^+ \bullet S_\varrho| \leq (|\varphi^n \bullet S_\varrho - \varphi \bullet S_\varrho| + |\varphi^n \bullet S_\sigma - \varphi \bullet S_\sigma|)\frac{1}{m}\mathbb{1}_{G^+}$$

for all stopping times ϱ. By the choice of (φ^n) and the local character of stochastic integrals, the right-hand side converges to zero in probability for all stopping times ϱ, and in $L^2(P)$ for $\varrho = T$. Since $\psi^n \bullet S = 0 = \psi^+ \bullet S$ on $[\![0,\tau]\!]$, we have that $\psi^+ \in \mathfrak{K}(0,\sigma;\sigma)$. By analogous arguments, we can also establish that $\psi^- \in \mathfrak{K}(0,\sigma;\sigma)$.

To complete the proof of (3.3), we now assume that

$$\bar{J}(\sigma;x,\tau,\vartheta) > \big((x+\vartheta \bullet S_\sigma)^+\big)^2\bar{L}^+(\sigma) + \big((x+\vartheta \bullet S_\sigma)^-\big)^2\bar{L}^-(\sigma) \quad \text{on } F$$

for some set $F \in \mathcal{F}_\sigma$ with $P[F] > 0$. Then there exist φ^+ and φ^- in $\mathfrak{K}(0,\sigma;\sigma)$, some $\varepsilon > 0$ and $F_\varepsilon \in \mathcal{F}_\sigma$ with $F_\varepsilon \subseteq F$ and $P[F_\varepsilon] > 2\varepsilon$ such that

$$\begin{aligned}\bar{J}(\sigma;x,\tau,\vartheta) \geq{}& \big((x+\vartheta \bullet S_\sigma)^+\big)^2 E\big[|1+\varphi^+ \bullet S_T|^2\big|\mathcal{F}_\sigma\big] \\ &+ \big((x+\vartheta \bullet S_\sigma)^-\big)^2 E\big[|1-\varphi^- \bullet S_T|^2\big|\mathcal{F}_\sigma\big] + 2\varepsilon \quad \text{on } F_\varepsilon. \qquad (3.7)\end{aligned}$$

By the definition of the essential infimum, there exists $\varphi^\varepsilon \in \mathfrak{K}(\vartheta,\sigma;\tau)$ with

$$E\big[|x+\varphi^\varepsilon \bullet S_T|^2\big] < E\big[\bar{J}(\sigma;x,\tau,\vartheta)\big] + \varepsilon^2. \qquad (3.8)$$

Since $\{|x + \vartheta \bullet S_\sigma| \leq m\} \nearrow \Omega$ for $m \to \infty$, there exists $G_\varepsilon \in \mathcal{F}_\sigma$ with $G_\varepsilon \subseteq F_\varepsilon$ and $P[G_\varepsilon] > \varepsilon$ and such that $|x + \vartheta \bullet S_\sigma| \leq m$ on G_ε, and therefore

$$\chi := \big((x+\vartheta \bullet S_\sigma)^+\varphi^+ + (x+\vartheta \bullet S_\sigma)^-\varphi^-\big)\mathbb{1}_{G_\varepsilon} \in \mathfrak{K}(0,\sigma;\sigma)$$

by the local character of stochastic integrals. Moreover, we can by part 2) of Lemma 2.18 without loss of generality choose G_ε such that

$$\psi := \varphi^\varepsilon\mathbb{1}_{G_\varepsilon^c} + (\vartheta\mathbb{1}_{[\![0,\sigma]\!]} + \chi\mathbb{1}_{]\!]\sigma,T]\!]})\mathbb{1}_{G_\varepsilon} \in \mathfrak{K}(\vartheta,\sigma;\tau).$$

Then we use that $\varphi^\varepsilon \in \mathfrak{K}(\vartheta, \sigma; \tau)$, the definitions of ψ and χ, and (3.7) to write

$$\begin{aligned} E\big[|x + \varphi^\varepsilon \bullet S_T|^2 \big| \mathcal{F}_\sigma\big] &\geq \mathbb{1}_{G_\varepsilon^c} E\big[|x + \psi \bullet S_T|^2 \big| \mathcal{F}_\sigma\big] + \mathbb{1}_{G_\varepsilon} \bar{J}(\sigma; x, \tau, \vartheta) \\ &\geq \mathbb{1}_{G_\varepsilon^c} E\big[|x + \psi \bullet S_T|^2 \big| \mathcal{F}_\sigma\big] + \mathbb{1}_{G_\varepsilon}\big(E\big[|x + \vartheta \bullet S_\sigma + \chi \bullet S_T|^2 \big| \mathcal{F}_\sigma\big] + 2\varepsilon\big). \end{aligned}$$

In view of (3.8), the definition of ψ and since $P[G_\varepsilon] > \varepsilon$ and $\psi \in \mathfrak{K}(\vartheta, \sigma; \tau)$, we obtain after taking expectations that

$$\begin{aligned} E\big[\bar{J}(\sigma; x, \tau, \vartheta)\big] &> E\big[|x + \varphi^\varepsilon \bullet S_T|^2\big] - \varepsilon^2 \\ &\geq E\big[|x + \psi \bullet S_T|^2\big] + 2\varepsilon^2 - \varepsilon^2 \geq E\big[\bar{J}(\sigma; x, \tau, \vartheta)\big] + \varepsilon^2 \end{aligned}$$

which is a contradiction. So (3.3) must hold. □

Our next result shows that the random variables $\bar{L}^\pm(\sigma)$ and $\bar{J}(\sigma; x, \tau, \vartheta)$ can be aggregated into nice RCLL processes.

Proposition 3.2. 1) *There exist RCLL submartingales $(L_t^\pm)_{0\leq t\leq T}$, called* opportunity processes*, such that*

$$L_\sigma^\pm = \bar{L}^\pm(\sigma) \quad P\text{-a.s. for each } \sigma \in \mathcal{S}_{0,T}. \tag{3.9}$$

2) *Fix $x \in \mathbb{R}$ and $\tau \in \mathcal{S}_{0,T}$. Define the RCLL process $(J_t(\vartheta; x, \tau))_{0\leq t\leq T}$ for every $\vartheta \in \overline{\Theta(K)}$ with $\vartheta = 0$ on $[\![0, \tau]\!]$ by*

$$J_t(\vartheta; x, \tau) = \big((x + \textstyle\int_\tau^t \vartheta_u \, dS_u)^+\big)^2 L_t^+ + \big((x + \int_\tau^t \vartheta_u \, dS_u)^-\big)^2 L_t^-. \tag{3.10}$$

Then we have for each $\vartheta \in \overline{\Theta(K)}$ with $\vartheta = 0$ on $[\![0, \tau]\!]$ that

$$J_\sigma(\vartheta; x, \tau) = \bar{J}(\sigma; x, \tau, \vartheta) \quad P\text{-a.s. for each } \sigma \in \mathcal{S}_{\tau,T}. \tag{3.11}$$

Moreover, $J(\vartheta; x, \tau)$ is a submartingale for every $\vartheta \in \overline{\Theta(K)}$ with $\vartheta = 0$ on $[\![0, \tau]\!]$, and $J(\widetilde{\vartheta}; x, \tau)$ is a martingale for $\widetilde{\vartheta} \in \overline{\Theta(K)}$ with $\widetilde{\vartheta} = 0$ on $[\![0, \tau]\!]$ if and only if $\widetilde{\vartheta} = \widetilde{\varphi}^{(x,\tau)}$ is optimal for (3.2).

Proof. 1) For $\tau \equiv 0$, $(\bar{L}^\pm(t))_{0\leq t\leq T}$ are submartingales by Proposition 3.1 and they have by Theorem VI.4 in [33] RCLL versions if the mappings $t \mapsto E[\bar{L}^\pm(t)]$ are right-continuous. We only prove this for $\bar{L}^-$ as the argument for $\bar{L}^+$ is completely analogous, but argue a bit more generally than directly needed. Fix a stopping time $\sigma \in \mathcal{S}_{\tau,T}$. By (3.4) and the definition of the essential infimum, there exists for each $\varepsilon > 0$ some $\vartheta^\varepsilon \in \mathfrak{K}(0, \sigma; \sigma)$ with

$$E\big[\bar{L}^-(\sigma)\big] > E\big[|1 - \vartheta^\varepsilon \bullet S_T|^2\big] - \varepsilon,$$

and ϑ^ε can be chosen in Θ as the $L^2(P)$-closure of $G_T(\Theta(K))$ contains $G_T(\overline{\Theta(K)})$. Let (σ_n) be a sequence in $\mathcal{S}_{\sigma,T}$ with $\sigma_n \searrow \sigma$. Then

$$(\mathbb{1}_{]\!]\sigma_n,T]\!]}\vartheta^\varepsilon) \bullet S \overset{\mathcal{H}^2(P)}{\longrightarrow} (\mathbb{1}_{]\!]\sigma,T]\!]}\vartheta^\varepsilon) \bullet S$$

and thus $E[|1-(\mathbb{1}_{]\!]\sigma_n,T]\!]}\vartheta^\varepsilon)\bullet S_T|^2]\to E[|1-(\mathbb{1}_{]\!]\sigma,T]\!]}\vartheta^\varepsilon)\bullet S_T|^2]$ by Theorem IV.5 in [80]. Therefore

$$E\big[\bar{L}^-(\sigma)\big] > \lim_{n\to\infty} E\big[|1-(\mathbb{1}_{]\!]\sigma_n,T]\!]}\vartheta^\varepsilon)\bullet S_T|^2\big]-\varepsilon \geq \lim_{n\to\infty} E\big[\bar{L}^-(\sigma_n)\big]-\varepsilon,$$

which yields $E[\bar{L}^-(\sigma)]\geq \lim_{n\to\infty} E[\bar{L}^-(\sigma_n)]$ as $\varepsilon>0$ was arbitrary. Conversely, the submartingale property of $\bar{L}^-$ gives $E[\bar{L}^-(\sigma)]\leq \lim_{n\to\infty} E[\bar{L}^-(\sigma_n)]$, where the limit exists by monotonicity. So we get $E[\bar{L}^-(\sigma)] = \lim_{n\to\infty} E[\bar{L}^-(\sigma_n)]$, completing the proof of right-continuity.

2) Thanks to step 1), we can take as $L^\pm$ an RCLL version of $(\bar{L}^\pm(t))_{0\leq t\leq T}$. To prove (3.9), take $\sigma,\sigma_n\in\mathcal{S}_{\tau,T}$ such that $\sigma_n\searrow\sigma$ and each σ_n takes only finitely many values. Then (3.9) holds for each σ_n and so $\lim_{n\to\infty}\bar{L}^\pm(\sigma_n) = \lim_{n\to\infty} L^\pm_{\sigma_n} = L^\pm_\sigma$ because $L^\pm$ are RCLL. Since all processes take values in $[0,1]$, dominated convergence yields $E[L^\pm_\sigma] = \lim_{n\to\infty} E[\bar{L}^\pm(\sigma_n)] = E[\bar{L}^\pm(\sigma)]$ by the argument in step 1), and since the submartingale property, (3.9) for σ_n and again dominated convergence give

$$\bar{L}^\pm(\sigma)\leq \lim_{n\to\infty} E\big[\bar{L}^\pm(\sigma_n)\big|\mathcal{F}_\sigma\big] = \lim_{n\to\infty} E\big[L^\pm_{\sigma_n}\big|\mathcal{F}_\sigma\big] = L^\pm_\sigma,$$

we obtain (3.9) for σ as well. This proves part 2).

3) The equality in (3.11) follows directly from the definition (3.10), (3.9) and the decomposition (3.3) in Proposition 3.1. The properties of the $\bar{J}$-family then immediately give the remaining assertion in part 2). □

The next result gives an alternative description of the processes $L^\pm$ and some further useful properties.

Lemma 3.3. *Suppose that there exists a solution* $\widetilde{\varphi}^{(x,\tau)}$ *to* (3.2)*. Then:*

1) *We have the decomposition*

$$\widetilde{\varphi}^{(x,\tau)} = x^+\widetilde{\varphi}(1,\tau) + x^-\widetilde{\varphi}^{(-1,\tau)}. \tag{3.12}$$

2) *For any* $\sigma\in\mathcal{S}_{\tau,T}$*, we have on* $\{V_\sigma(x,\widetilde{\varphi}^{(x,\tau)})\gtrless 0\}$ *that*

$$L^\pm_\sigma = E\left[\left(1\pm\frac{\mathbb{1}_{]\!]\sigma,T]\!]}\widetilde{\varphi}^{(x,\tau)}}{V^\pm_\sigma(x,\widetilde{\varphi}^{(x,\tau)})}\bullet S_T\right)^2\middle|\mathcal{F}_\sigma\right] = E\left[1\pm\frac{\mathbb{1}_{]\!]\sigma,T]\!]}\widetilde{\varphi}^{(x,\tau)}}{V^\pm_\sigma(x,\widetilde{\varphi}^{(x,\tau)})}\bullet S_T\middle|\mathcal{F}_\sigma\right].$$

3) *The process* ${}^\tau\widetilde{M}^{(x,\tau)} = \mathbb{1}_{]\!]\tau,T]\!]}\bullet\widetilde{M}^{(x,\tau)}$ *with*

$$\widetilde{M}^{(x,\tau)} := (x+\widetilde{\varphi}^{(x,\tau)}\bullet S)^+L^+ - (x+\widetilde{\varphi}^{(x,\tau)}\bullet S)^-L^-$$

is a square-integrable martingale.

4) *If* $K:\overline{\Omega}\to 2^{\mathbb{R}^d}\setminus\{\emptyset\}$ *is convex-valued, then* $(\vartheta\bullet S)\widetilde{M}^{(x,\tau)}$ *is a submartingale for all* $\vartheta\in\Theta(K)$ *with* $\vartheta=0$ *on* $[\![0,\tau]\!]$.

Proof. 1) The decomposition (3.12) of the optimal strategy is obtained like (3.6) directly from the fact that our optimisation problem is quadratic and the constraints are conic.

2) If there exists a solution $\widetilde{\varphi}^{(x,\tau)}$ to (3.2), we obtain by part 2) of Proposition 3.2 that $J_\sigma(\widetilde{\varphi}^{(x,\tau)}; x, \tau) = E[|x + \widetilde{\varphi}^{(x,\tau)} \bullet S_T|^2 | \mathcal{F}_\sigma]$ and therefore

$$L^+_\sigma = E\left[\left(1 + \frac{\mathbb{1}_{]\!]\sigma,T]\!]}\widetilde{\varphi}^{(x,\tau)}}{V^+_\sigma(x, \widetilde{\varphi}^{(x,\tau)})} \bullet S_T\right)^2 \middle| \mathcal{F}_\sigma\right] \quad \text{on } F := \{V_\sigma(x, \widetilde{\varphi}^{(x,\tau)}) > 0\} \in \mathcal{F}_\sigma$$

by dividing in (3.3). For the proof of the second equality, we can assume that the process $\vartheta := \frac{\mathbb{1}_{]\!]\sigma,T]\!]}\widetilde{\varphi}^{(x,\tau)}}{V^+_\sigma(x,\widetilde{\varphi}^{(x,\tau)})}\mathbb{1}_F$ is in $\overline{\Theta(K)}$ by part 2) of Lemma 2.18 and by possibly shrinking F. Then the first equality implies for all $\varepsilon > -1$ that

$$\begin{aligned} 0 &\le \frac{E\big[|1 + ((1+\varepsilon)\vartheta) \bullet S_T|^2 \big| \mathcal{F}_\sigma\big] - E\big[|1 + \vartheta \bullet S_T|^2 \big| \mathcal{F}_\sigma\big]}{|\varepsilon|} \\ &= -\operatorname{sign}(\varepsilon) E[(\vartheta \bullet S_T)(1 + \vartheta \bullet S_T) | \mathcal{F}_\sigma] + |\varepsilon| E\big[|\vartheta \bullet S_T|^2 \big| \mathcal{F}_\sigma\big]. \end{aligned} \tag{3.13}$$

Taking $\underline{\lim}_{\varepsilon \nearrow 0}$ and $\underline{\lim}_{\varepsilon \searrow 0}$ in (3.13) yields $E[(\vartheta \bullet S_T)(1 + \vartheta \bullet S_T) | \mathcal{F}_\sigma] = 0$, which implies that $E[|1 + \vartheta \bullet S_T|^2 | \mathcal{F}_\sigma] = E[1 + \vartheta \bullet S_T | \mathcal{F}_\sigma]$ and therefore the second asserted equality. The argument for L^-_σ is completely analogous and therefore omitted.

3) Using the second equalities in part 2), we can write for $\sigma \in \mathcal{S}_{\tau,T}$ that

$$E[x + \widetilde{\varphi}^{(x,\tau)} \bullet S_T | \mathcal{F}_\sigma] = (x + \widetilde{\varphi}^{(x,\tau)} \bullet S_\sigma)^+ L^+_\sigma - (x + \widetilde{\varphi}^{(x,\tau)} \bullet S_\sigma)^- L^-_\sigma,$$

which immediately gives that ${}^\tau\widetilde{M}^{(x,\tau)} = \mathbb{1}_{]\!]\tau,T]\!]} \bullet \widetilde{M}^{(x,\tau)}$ is a square-integrable martingale.

4) Since $\vartheta \in \Theta(K)$ implies that $\mathbb{1}_{F\times(s,t]\cap]\!]\tau,T]\!]}\vartheta$ is in $\mathfrak{K}(0,\tau)$ for all $s \le t$ and $A \in \mathcal{F}_s$, it follows from the first order condition of optimality for (3.2) that

$$\begin{aligned} &E\big[\mathbb{1}_F\big((\mathbb{1}_{]\!]\tau,T]\!]}\vartheta) \bullet S_t - (\mathbb{1}_{]\!]\tau,T]\!]}\vartheta) \bullet S_s\big)(x + \widetilde{\varphi}^{(x,\tau)} \bullet S_T)\big] \\ &\quad = E\big[\big((\mathbb{1}_{F\times(s,t]\cap]\!]\tau,T]\!]}\vartheta) \bullet S_T\big)(x + \widetilde{\varphi}^{(x,\tau)} \bullet S_T)\big] \ge 0 \end{aligned}$$

and therefore that $((\mathbb{1}_{]\!]\tau,T]\!]}\vartheta) \bullet S_t) E[(x + \widetilde{\varphi}^{(x,\tau)} \bullet S_T) | \mathcal{F}_t]$, $0 \le t \le T$, is a submartingale. □

The martingale optimality principle in Proposition 3.2 gives a dynamic description of the solution $\widetilde{\varphi} = \widetilde{\varphi}^{(x,0)}$ only for $J(\widetilde{\varphi}; x, 0) \neq 0$. This can cause problems. But (3.10) shows that if $J(\widetilde{\varphi}; x, 0)$ becomes 0, then either $V(x, \widetilde{\varphi}) = 0$ or $L^+ = 0$ or $L^- = 0$. In the latter two cases, the payoffs $\mathbb{1}_{\{L^+_\tau = 0\}}$ or $-\mathbb{1}_{\{L^-_\tau = 0\}}$ with $\tau = \inf\{t > 0 \,|\, J_t(\widetilde{\varphi}; x, 0) = 0\} \wedge T$ are in $G_T(\overline{\Theta(K\mathbb{1}_{]\!]\tau,T]\!]})})$, and in the terminology of Section 4 in [87], these random variables provide approximate profits in L^2 which is a weak form of arbitrage. So intuitively, we have difficulties with describing $\widetilde{\varphi}$ only if the basic model allows some kind of arbitrage. The next result, which generalises Lemma 3.10 in [14], gives a sufficient condition to prevent such problems.

Lemma 3.4. *Suppose that there exist $N \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ and Z^N such that $(\mathcal{E}, Z^N)$ with $\mathcal{E} = \mathcal{E}(N)$ is regular and square-integrable and S is an $\mathcal{E}$-local martingale. Then $L^\pm$ and their left limits $L^\pm_-$ are $(0,1]$-valued.*

Proof. We prove the assertion for L^+ and L^+_- by way of contradiction; the completely analogous proof for L^- and L^-_- is omitted. Define the stopping time $\tau := \inf\{t > 0 \,|\, L^+_t = 0\} \wedge T$ and suppose that $P[L^+_\tau = 0] > 0$. By (3.4), (3.9) and the definition of τ, $\operatorname*{ess\,inf}_{\varphi \in \mathfrak{K}(0,\tau;\tau)} E\big[|1 + \varphi \bullet S_T|^2 \big| \mathcal{F}_\tau\big] \mathbb{1}_{\{L^+_\tau = 0\}} = L^+_\tau \mathbb{1}_{\{L^+_\tau = 0\}} = 0$ and so there exists a sequence (ϑ^n) in $\mathfrak{K}(0,\tau;\tau)$ such that $((\vartheta^n \bullet S_T)\mathbb{1}_{\{L^+_\tau = 0\}})$ converges to $-\mathbb{1}_{\{L^+_\tau = 0\}}$ in $L^2(P)$. Since $L^+_T = 1$, we have that

$$\{L^+_\tau = 0\} = \{L^+_\tau = 0, \tau < T\} = \bigcup_{m=0}^{\infty} \{L^+_\tau = 0, T_m \le \tau < T_{m+1}\}$$

and hence $P[L^+_\tau = 0, T_m \le \tau < T_{m+1}] > 0$ for some $m \in \mathbb{N}_0$. But each $\vartheta^n \bullet S$ is an $(\mathcal{E}, Z^N)$-martingale by Corollary 2.13, and since $Z^N_{T_m}\, {}^{T_m}\mathcal{E}(N)$ is square-integrable, we get for every $F \in \mathcal{F}_\tau$ that

$$\begin{aligned} 0 &= \lim_{n\to\infty} E\big[Z^N_{T_m}\, {}^{T_m}\mathcal{E}(N)_T (\vartheta^n \bullet S_T) \mathbb{1}_{\{L^+_\tau = 0, T_m \le \tau < T_{m+1}\} \cap F}\big] \\ &= -E\big[Z^N_{T_m}\, {}^{T_m}\mathcal{E}(N)_\tau \mathbb{1}_{\{L^+_\tau = 0, T_m \le \tau < T_{m+1}\} \cap F}\big]. \end{aligned}$$

Since ${}^{T_m}\mathcal{E}(N) \neq 0$ on $[\![T_m, T_{m+1}[\![$, choosing $F := \{{}^{T_m}\mathcal{E}(N)_\tau > 0\}$ or $F := \{{}^{T_m}\mathcal{E}(N)_\tau < 0\}$ gives a contradiction to the assumption that $P[L^+_\tau = 0] > 0$. So we get $L^+ > 0$.

To prove $L^+_- > 0$, define the stopping time $\sigma := \inf\{t > 0 \,|\, L^+_{t-} = 0\} \wedge T$ and assume that $F_\infty := \{L^+_{\sigma-} = 0\}$ has $P[F_\infty] > 0$. Because ${}^{T_m}\mathcal{E}(N) \neq 0$ on $[\![T_m, T_{m+1}[\![$ and

$$\{L^+_{\sigma-} = 0\} = \{L^+_{\sigma-} = 0, \sigma > 0\} = \bigcup_{m=0}^{\infty} \{L^+_{\sigma-} = 0, T_m < \sigma \le T_{m+1}\},$$

there exists some $m \in \mathbb{N}_0$ with $P[F^{m,+}_\infty] > 0$ or $P[F^{m,-}_\infty] > 0$, where

$$F^{m,\pm}_\infty := F_\infty \cap \{T_m < \sigma \le T_{m+1}\} \cap \{{}^{T_m}\mathcal{E}(N)_{\sigma-} \gtrless 0\}.$$

We fix m and treat without loss of generality only the "+" case here so that $P[F^{m,+}_\infty] > 0$. Setting $\sigma_n := \inf\{t > 0 \,|\, L^+_t \le \frac{1}{n}\} \wedge T$ gives $\sigma_n < \sigma$ and $\sigma_n \nearrow \sigma$ P-a.s. on F_∞, and defining

$$F^{m,+}_n := \{0 < L^+_{\sigma_n} \le \frac{1}{n}\} \cap \{T_m \le \sigma_n < T_{m+1}\} \cap \{{}^{T_m}\mathcal{E}(N)_{\sigma_n} > 0\} \in \mathcal{F}_{\sigma_n}$$

yields by the definition of σ_n that

$$E\Big[\operatorname*{ess\,inf}_{\varphi\in\mathfrak{K}(0,\sigma_n;\sigma_n)} E\big[|1+\varphi\bullet S_T|^2\big|\mathcal{F}_{\sigma_n}\big]\mathbb{1}_{F_n^{m,+}}\Big] = E[L^+_{\sigma_n}\mathbb{1}_{F_n^{m,+}}] \leq \frac{1}{n}P[F_n^{m,+}].$$

Thus there exist $\varphi^n \in \mathfrak{K}(0,\sigma_n;\sigma_n)$ such that $\lim_{n\to\infty} E\big[|1+\varphi^n\bullet S_T|^2\mathbb{1}_{F_n^{m,+}}\big] = 0$. This implies as above via Corollary 2.13 and the square-integrability of $Z^{N\ T_m}_{T_m}\mathcal{E}(N)$ that

$$\begin{aligned} 0 &= \lim_{n\to\infty} E\left[Z^{N\ T_m}_{T_m}\mathcal{E}(N)_T(\varphi^n\bullet S_T)\mathbb{1}_{F_n^{m,+}}\right] \\ &= -\lim_{n\to\infty} E\left[Z^{N\ T_m}_{T_m}\mathcal{E}(N)_{\sigma_n}\mathbb{1}_{F_n^{m,+}}\right] = -E\big[Z^{N\ T_m}_{T_m}\mathcal{E}(N)_{\sigma-}\mathbb{1}_{F_\infty^{m,+}}\big]. \end{aligned}$$

This contradicts the fact that $P[F_\infty^{m,+}] > 0$ so that we must have $P[F_\infty] = 0$. $\square$

The lemma below allows us to parametrise the optimal strategy in terms of units of wealth. The proof uses the technique in [31], which also appears in [16] and [14].

Lemma 3.5. *Suppose that $L^\pm$ and their left limits $L^\pm_-$ are $(0,1]$-valued and that there exists a solution $\widetilde{\varphi}^{(x,\tau)}$ to (3.2). Then there exists $\widetilde{\psi}^{(x,\tau)} \in \mathcal{L}(S)$ such that*

$$V(x,\widetilde{\varphi}^{(x,\tau)}) = x + \widetilde{\varphi}^{(x,\tau)}\bullet S = x\,\mathcal{E}(\widetilde{\psi}^{(x,\tau)}\bullet S) \tag{3.14}$$

and

$$L^\pm_t = E\big[|\mathcal{E}(\widetilde{\psi}^{(x,\tau)}\mathbb{1}_{]\!]t,T]\!]}\bullet S)_T|^2\big|\mathcal{F}_t\big] \quad \text{on } \{x+\widetilde{\varphi}^{(x,\tau)}\bullet S_t \gtrless 0\}. \tag{3.15}$$

Proof. Define the stopping times $\sigma_n = \inf\{t>0\,|\,|V_t(x,\widetilde{\varphi}^{(x,\tau)})| \leq \frac{|x|}{n+1}\}\wedge T$ for $n\in\mathbb{N}$, set $\sigma = \lim_{n\to\infty}\sigma_n$ and $F = \bigcap_{n\in\mathbb{N}}\{\sigma_n<\sigma\} \in \bigvee_{n=1}^\infty \mathcal{F}_{\sigma_n} = \mathcal{F}_{\sigma-}$ and consider the square-integrable martingale $M^{(x,\tau)}_t = E[V_T(x,\widetilde{\varphi}^{(x,\tau)})|\mathcal{F}_t]$ for $t\in[0,T]$. Lemma 3.3 yields

$$\begin{aligned} M^{(x,\tau)}_t &= (x+\widetilde{\varphi}^{(x,\tau)}\bullet S_t)^+L^+_t - (x+\widetilde{\varphi}^{(x,\tau)}\bullet S_t)^-L^-_t \text{ for } t\geq\tau, \\ E\big[(M^{(x,\tau)}_T)^2\big|\mathcal{F}_t\big] &= \big((x+\widetilde{\varphi}^{(x,\tau)}\bullet S_t)^+\big)^2L^+_t \\ &\quad + \big((x+\widetilde{\varphi}^{(x,\tau)}\bullet S_t)^-\big)^2L^-_t \text{ for } t\geq\tau, \end{aligned} \tag{3.16}$$

and since $L^\pm$ are $(0,1]$-valued and $\sigma_n\geq\tau$, we get $|M^{(x,\tau)}_{\sigma_n}| \leq \frac{|x|}{n+1}$, $|M^{(x,\tau)}_{\sigma_n}| > 0$ on $\{\sigma_n<\sigma\}$, $F = \{M^{(x,\tau)}_{\sigma-} = 0\}$ and $\mathbb{1}_F E[M^{(x,\tau)}_T|\mathcal{F}_{\sigma-}] = 0$. Then the martingale property of $M^{(x,\tau)}$, conditioning on $\mathcal{F}_{\sigma-}$, and using Cauchy–Schwarz and (3.16) yields

$$\begin{aligned}\mathbb{1}_{\{\sigma_n<\sigma\}} &= E\left[\frac{M_T^{(x,\tau)}}{M_{\sigma_n}^{(x,\tau)}}\mathbb{1}_{\{\sigma_n<\sigma\}}\middle|\mathcal{F}_{\sigma_n}\right] = E\left[\frac{M_T^{(x,\tau)}}{M_{\sigma_n}^{(x,\tau)}}\mathbb{1}_{\{\sigma_n<\sigma\}}\mathbb{1}_{F^c}\middle|\mathcal{F}_{\sigma_n}\right]\\ &\leq E\left[\left(\frac{M_T^{(x,\tau)}}{M_{\sigma_n}^{(x,\tau)}}\right)^2\mathbb{1}_{\{\sigma_n<\sigma\}}\middle|\mathcal{F}_{\sigma_n}\right]^{\frac{1}{2}} P[F^c|\mathcal{F}_{\sigma_n}]^{\frac{1}{2}}\\ &\leq \left(\frac{1}{L_{\sigma_n}^+}+\frac{1}{L_{\sigma_n}^-}\right)^{\frac{1}{2}}\mathbb{1}_{\{\sigma_n<\sigma\}}P[F^c|\mathcal{F}_{\sigma_n}]^{\frac{1}{2}}.\end{aligned}$$

Since

$$\begin{aligned}\mathbb{1}_F &= \lim_{n\to\infty}\mathbb{1}_{\{\sigma_n<\sigma\}}\mathbb{1}_F \leq \lim_{n\to\infty}\left(\frac{1}{L_{\sigma_n}^+}+\frac{1}{L_{\sigma_n}^-}\right)^{\frac{1}{2}}\mathbb{1}_F\mathbb{1}_{\{\sigma_n<\sigma\}}P[F^c|\mathcal{F}_{\sigma_n}]^{\frac{1}{2}}\\ &= \left(\frac{1}{L_{\sigma-}^+}+\frac{1}{L_{\sigma-}^-}\right)^{\frac{1}{2}}\mathbb{1}_F\mathbb{1}_{F^c} = 0,\end{aligned}$$

this gives $P[F] = 0$ and therefore that $V_-(x,\widetilde{\varphi}^{(x,\tau)}) \neq 0$ on $[\![0,\sigma]\!]$ and $V(x,\widetilde{\varphi}^{(x,\tau)}) = 0$ on $[\![\sigma,T]\!]$. Therefore $\widetilde{\psi}^{(x,\tau)} := \frac{\widetilde{\varphi}^{(x,\tau)}}{V_-(x,\widetilde{\varphi}^{(x,\tau)})}\mathbb{1}_{[\![0,\sigma]\!]}$ is well defined and satisfies (3.14). Plugging (3.14) into the equations of part 2) of Lemma 3.3 yields (3.15) and completes the proof. □

IV.4 Local description and structure

In this section, we use the dynamic characterisation of the solution of (3.1) to derive a local description for the structure of the optimal strategy. To that end, we first give a local description of the underlying processes by their differential semimartingale characteristics.

As in [52], Theorem II.2.34, each $\mathbb{R}^d$-valued semimartingale X has, with respect to some truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$, the *canonical representation*

$$X = X_0 + X^c + A^{X,h} + h(x) * (\mu^X - \nu^X) + [x - h(x)] * \mu^X$$

with the jump measure μ^X of X and its predictable compensator ν^X. The quadruple (b^X, c^X, F^X, B) of *differential characteristics* of X then consists of a predictable $\mathbb{R}^d$-valued process b^X, a predictable nonnegative-definite symmetric matrix-valued process c^X, a predictable process F^X with values in the set of Lévy measures on $\mathbb{R}^d$, and a predictable increasing RCLL process B null at zero such that

$$A^{X,h} = b^X \bullet B, \qquad \langle X^c\rangle = c^X \bullet B, \qquad \nu^X = F^X \bullet B.$$

We use the same predictable process B for all the finitely many semimartingales appearing in this chapter, and since they are all special, we can and do always work with the (otherwise forbidden) truncation function $h(x) = x$, which simplifies computations considerably. We then write A^X instead of $A^{X,h}$. For two (special) semimartingales X and Y, we denote their joint differential characteristics by

$$(b^{X,Y}, c^{X,Y}, F^{X,Y}, B) = \left(\begin{pmatrix} b^X \\ b^Y \end{pmatrix}, \begin{pmatrix} c^X & c^{XY} \\ c^{YX} & c^Y \end{pmatrix}, F^{X,Y}, B \right).$$

By adding t to B, we can assume that B is strictly increasing. Recall that $P_B = P \otimes B$. For the locally square-integrable semimartingale S, there exists by Proposition II.2.29 in [52] a predictable nonnegative-definite symmetric matrix-valued process $\widetilde{c}^M$ such that $\langle M \rangle = \widetilde{c}^M \bullet B$, and it is given by $\widetilde{c}^M = c^S + \int x x^\top F^S(dx) - b^S (b^S)^\top \Delta B$.

To prepare for the local description of the optimal strategy, we need some notation. For two $[0,1]$-valued (hence special) semimartingales ℓ^+ and ℓ^-, we look at their joint differential characteristics with S and define the predictable functions

$$\mathfrak{g}^{1,\pm}(\psi) := \mathfrak{g}^{1,\pm}(\psi; S, \ell^+, \ell^-) := \ell^\pm_- \psi^\top c^S \psi \pm 2 \ell^\pm_- \psi^\top b^S \pm 2 \psi^\top c^{S \ell^\pm}, \tag{4.1}$$

$$\begin{aligned} \mathfrak{g}^{2,\pm}(\psi) := \mathfrak{g}^{2,\pm}(\psi; S, \ell^+, \ell^-) := \ell^\pm_- \int & \left(\left\{ (1 \pm \psi^\top u)^+ \right\}^2 - 1 \mp 2 \psi^\top u \right) F^S(du) \\ + \int & \left(\left\{ (1 \pm \psi^\top u)^+ \right\}^2 - 1 \right) y F^{S,\ell^\pm}(du, dy) \\ + \int & \left\{ (1 \pm \psi^\top u)^- \right\}^2 (\ell^\mp_- + z) F^{S,\ell^\mp}(du, dz), \end{aligned} \tag{4.2}$$

$$\mathfrak{g}^\pm(\psi) := \mathfrak{g}^\pm(\psi; S, \ell^+, \ell^-) := \mathfrak{g}^{1,\pm}(\psi; S, \ell^+, \ell^-) + \mathfrak{g}^{2,\pm}(\psi; S, \ell^+, \ell^-). \tag{4.3}$$

All these functions have $\psi \in \mathbb{R}^d$ as arguments and depend on ω, t via $\ell^\pm_{t-}(\omega)$ and the joint characteristics of S and $\ell^\pm$. For ease of notation, we shall drop in the proofs all superscripts $^\top$, writing xy instead of $x^\top y$ for the scalar product of two vectors x, y.

Our first main result is now a local description of the optimal strategy $\widetilde{\varphi}$ for (3.1). It is obtained by examining the drift rate of $J(\vartheta)$, as follows. Recall that the constraints are given by a predictable correspondence K with closed cones as values.

Theorem 4.1. *For each $\vartheta \in \overline{\Theta(K)}$, define a K-valued predictable process ψ via*

$$\psi := \mathbb{1}_{\{V_-(x,\vartheta) \neq 0\}} \frac{\vartheta}{|V_-(x,\vartheta)|} + \mathbb{1}_{\{V_-(x,\vartheta) = 0\}} \vartheta \tag{4.4}$$

or equivalently

$$\vartheta =: V_-^+(x,\vartheta)\psi + V_-^-(x,\vartheta)\psi + \mathbb{1}_{\{V_-(x,\vartheta)=0\}}\psi.$$

Then:

1) *The finite variation part of $J(\vartheta)$ is given by $A(\vartheta) = b^{J(\vartheta)} \bullet B$ with*

$$\begin{aligned} b^{J(\vartheta)} &= \big(V_-^+(x,\vartheta)\big)^2\big\{\mathfrak{g}^+(\psi;S,\ell^+,\ell^-) + b^{\ell^+}\big\} \\ &\quad + \big(V_-^-(x,\vartheta)\big)^2\big\{\mathfrak{g}^-(\psi;S,\ell^+,\ell^-) + b^{\ell^-}\big\} \\ &\quad + \mathbb{1}_{\{V_-(x,\vartheta)=0\}}\Big(\int \big((\psi^\top u)^+\big)^2(\ell_-^+ + y)F^{S,\ell^+}(du,dy) \\ &\quad + \ell_-^-\psi^\top c^S\psi + \int \big((\psi^\top u)^-\big)^2(\ell_-^- + z)F^{S,\ell^-}(du,dz)\Big) \\ &\geq 0. \end{aligned}$$

2) *If there exists a solution $\widetilde{\varphi} = \widetilde{\varphi}^{(x,0)} \in \overline{\Theta(K)}$ to problem* (3.1) *with the property that*

$$V(x,\widetilde{\varphi}) = x + \widetilde{\varphi} \bullet S = x\,\mathcal{E}(\widetilde{\psi} \bullet S),$$

then the joint differential characteristics of (S, L^+, L^-) satisfy the two coupled equations

$$b^{L^\pm} = -\min_{\psi\in K}\mathfrak{g}^\pm(\psi;S,L^+,L^-) = -\mathfrak{g}^\pm(\pm\widetilde{\psi};S,L^+,L^-) \text{ on } \{V_-(x,\widetilde{\varphi}) \gtrless 0\}. \tag{4.5}$$

Proof. 1) Since $J(\vartheta)$ is given by (3.10), finding its drift rate $b^{J(\vartheta)}$ is a straightforward, but lengthy computation; this is done in Lemma 5.2 below. Then $b^{J(\vartheta)}$ is nonnegative because $J(\vartheta)$ is a submartingale by the martingale optimality principle in Proposition 3.2.

2) The basic idea to prove the first equality is (as usual) to assume that the set

$$D := \big\{(\omega,t)\,\big|\,b^{L^+} > -\min_{\psi\in K}\mathfrak{g}^+(\psi;S,L^+,L^-)\big\} \cap \{x\,\mathcal{E}(\widetilde{\psi} \bullet S)_- > 0\}$$

has $P_B(D) > 0$ and then to construct from D via measurable selection a strategy ϑ in $\overline{\Theta(K)}$ which violates the submartingale property of $J(\vartheta)$. This simple idea is technically a bit involved because one must ensure that ϑ is K-admissible and that there exists a set $D' \in \mathcal{P}$ with $D' \subseteq D$, $P_B(D') > 0$ and $V_-(x,\vartheta) > 0$ on D'. The details are as follows.

Since $V(x,\widetilde{\varphi}) = x\,\mathcal{E}(\widetilde{\psi} \bullet S)$ is a stochastic exponential, it changes sign only at jumps with $\widetilde{\psi}\Delta S < -1$, which P-a.s. can only happen a finite number of times. So there exist stopping times $\tau_1 \leq \tau_2$ such that $P_B(D \cap]\!]\tau_1,\tau_2]\!]) > 0$ and $x\,\mathcal{E}(\widetilde{\psi} \bullet S)_- > 0$ on $]\!]\tau_1,\tau_2]\!]$. By part 2) of Lemma 2.18, we can choose $F_\varepsilon \in \mathcal{F}_{\tau_1}$ such that $\widetilde{\varphi}\mathbb{1}_{[\![0,\sigma_1]\!]} \in \overline{\Theta(K)}$ and $(x + \widetilde{\varphi} \bullet S_{\sigma_1})\mathbb{1}_{F_\varepsilon} \geq 0$ is uniformly bounded and $D_\varepsilon := D \cap]\!]\sigma_1, \sigma_2]\!]$ has $P_B(D_\varepsilon) > 0$, where $\sigma_i := \tau_i \mathbb{1}_{F_\varepsilon} + T\mathbb{1}_{F_\varepsilon^c}$ for $i = 1, 2$ are stopping times. Because $\mathfrak{g}^+$ is a Carathéodory function by Lemma 5.1 below and K is a predictable correspondence, we can construct by Propositions 2.5 and 2.4 a K-valued predictable process φ with $\mathfrak{g}^+(\varphi) < -b^{L^+}$ on D_ε and $\mathfrak{g}^+(\varphi) = 0$ else. After possibly shrinking D_ε, we can also assume without loss of generality that φ is bounded, which implies that φ is in $\mathcal{L}(S)$ so that $\varphi \bullet S$ is well defined and has P-a.s. only a finite number of jumps with $\varphi\Delta S < -1$. Thus there exists stopping times $\varrho_1 \leq \varrho_2$ such that $D' := D_\varepsilon \cap]\!]\varrho_1, \varrho_2]\!]$ has $P_B(D') > 0$ and $\mathcal{E}(\psi \bullet S)_- > 0$ on $]\!]\varrho_1, \varrho_2]\!]$, where $\psi := \varphi \mathbb{1}_{]\!]\varrho_1, \varrho_2]\!]}$. By stopping $\mathcal{E}(\psi \bullet S)_-$ and S, we can even choose ϱ_2 such that $\mathcal{E}(\psi \bullet S)_-$ is bounded and $\mathcal{E}(\psi \bullet S)_-\psi \in \Theta(K)$; this uses that K is cone-valued. Moreover, since $(x + \widetilde{\varphi} \bullet S_{\sigma_1})\mathbb{1}_{F_\varepsilon}$ is bounded, also $(x + \widetilde{\varphi} \bullet S_{\sigma_1})\mathbb{1}_{F_\varepsilon}\mathcal{E}(\psi \bullet S)_-\psi$ is in $\overline{\Theta(K)}$. Therefore the sum $\vartheta := \widetilde{\varphi}\mathbb{1}_{[\![0,\sigma_1]\!]} + (x + \widetilde{\varphi} \bullet S_{\sigma_1})\mathbb{1}_{F_\varepsilon}\mathcal{E}(\psi \bullet S)_-\psi$ is in $\overline{\Theta(K)}$ and has $(x + \vartheta \bullet S)_- > 0$ and $\mathfrak{g}^+(\frac{\vartheta}{(x+\vartheta \bullet S)_-}) = \mathfrak{g}^+(\psi) < -b^{L^+}$ on D'. In view of part 1), $\mathbb{1}_{D'} \bullet A(\vartheta) = (\mathbb{1}_{D'} b^{J(\vartheta)}) \bullet B = (\mathbb{1}_{D'}(x + \vartheta \bullet S)_- \{\mathfrak{g}^+(\psi) + b^{L^+}\}) \bullet B$ is strictly decreasing on a non-negligible set, and so $J(\vartheta)$ cannot be a submartingale. This contradicts the martingale optimality principle and thus establishes the equality for b^{L^+}. The argument for b^{L^-} is completely analogous and therefore omitted. □

To explain the significance as well as the limitations of Theorem 4.1, let us suppose that we have an optimal strategy $\widetilde{\varphi}$ for problem (3.1). Then part 2) of Theorem 4.1 gives a kind of BSDE description for the pair (L^+, L^-) since it expresses their drift rates in terms of their joint semimartingale characteristics with S. However, this description is not yet fully informative on its own. A closer look at (4.5) shows that we only have a description of the drift of L^+ (or L^-) when $V_-(x, \widetilde{\varphi})$ is positive (or negative). Once $V(x, \widetilde{\varphi})$ hits 0, it stays there, being a stochastic exponential, and we can no longer tell how $L^\pm$ behave. Even worse, $V(x, \widetilde{\varphi})$ might jump across 0 so that we immediately lose track of the drift of L^+ or L^-, depending on whether the jump goes downwards or upwards. To overcome this difficulty and obtain a full characterisation of $L^\pm$, we must be able to "restart $V(x, \widetilde{\varphi})$ whenever it jumps across or to 0". This can be achieved by assuming that not only (3.1), but each problem (3.2) for x and τ has a solution. This key insight can be traced back to Černý and Kallsen [14].

The second condition we need to get a description of $L^\pm$ is that these processes as well as their left limits are strictly positive. As already explained before Lemma 3.4, this can be interpreted as a kind of absence-of-arbitrage condition. In fact, if – as in [14] – there exists an equivalent local martingale measure for S with density in $L^2(P)$, that condition is automatically satisfied; a slightly more general result is given in Lemma 3.4 above. For the case without constraints, we provide a sharper result in Theorem 6.2 below.

Corollary 4.2. *Suppose that $L^\pm$ and their left limits $L^\pm_-$ are all $(0, 1]$-valued* *and that there exists a solution $\widetilde{\varphi}^{(x,\tau)}$ to (3.2) for any $x \in \mathbb{R}$ and any stopping time τ. Then the joint differential characteristics of (S, L^+, L^-) satisfy*

$$b^{L^+} = -\min_{\psi \in K} \mathfrak{g}^+(\psi; S, L^+, L^-) \quad \textit{and} \quad b^{L^-} = -\min_{\psi \in K} \mathfrak{g}^-(\psi; S, L^+, L^-). \tag{4.6}$$

Moreover, for all $x \in \mathbb{R}$ and all stopping times τ, there exists a solution to the SDE

$$dV_t^{(x,\tau)} = \big((V_{t-}^{(x,\tau)})^+ \widetilde{\psi}_t^+ + (V_{t-}^{(x,\tau)})^- \widetilde{\psi}_t^-\big) \mathbb{1}_{\rrbracket \tau, T \rrbracket}\, dS_t, \quad V_0^{(x,\tau)} = V_\tau^{(x,\tau)} = x \tag{4.7}$$

such that $\widetilde{\psi}^\pm$ are in $\operatorname{argmin}_{\psi \in K} \mathfrak{g}^\pm(\psi; S, L^+, L^-)$ on $\{V_-^{(x,\tau)} \gtrless 0\} \cap \rrbracket \tau, T \rrbracket$ and $\widetilde{\psi}^\pm \mathbb{1}_{\{V_-^{(x,\tau)} \gtrless 0\} \cap \rrbracket \tau, T \rrbracket} \in \mathcal{L}(S)$, and we have

$$\widetilde{\varphi}^{(x,\tau)} = \big((V_-^{(x,\tau)})^+ \widetilde{\psi}^+ + (V_-^{(x,\tau)})^- \widetilde{\psi}^-\big) \mathbb{1}_{\rrbracket \tau, T \rrbracket}. \tag{4.8}$$

Note that $\widetilde{\psi}^\pm$ are not the positive and negative parts of the process $\widetilde{\psi}$ from Theorem 4.1.

Proof. By Lemma 3.5, we have $V(x, \widetilde{\varphi}^{(x,\tau)}) = x\,\mathcal{E}(\widetilde{\psi}^{(x,\tau)} \bullet S)$ for some $\widetilde{\psi}^{(x,\tau)} \in \mathcal{L}(S)$ with $\widetilde{\psi}^{(x,\tau)} = \widetilde{\psi}^{(x,\tau)} \mathbb{1}_{\rrbracket \tau, T \rrbracket}$ so that $\widetilde{\psi}^\pm := \widetilde{\psi}^{(x,\tau)} \mathbb{1}_{\{V_-(x, \widetilde{\varphi}^{(x,\tau)}) \gtrless 0\}}$ are in $\mathcal{L}(S)$ and yield (4.7) with $V^{(x,\tau)} := V(x, \widetilde{\varphi}^{(x,\tau)})$. Moreover, (4.5) in Theorem 4.1 shows that $\widetilde{\psi}^\pm$ are minimisers for $\mathfrak{g}^\pm$ on $\{V_-(x, \widetilde{\varphi}^{(x,\tau)}) \gtrless 0\} \cap \rrbracket \tau, T \rrbracket$, and finally (4.8) holds by construction because we have $V^{(x,\tau)} = V(x, \widetilde{\varphi}^{(x,\tau)}) = x + \widetilde{\varphi}^{(x,\tau)} \bullet S$. □

Remark 4.3. For the purpose of *constructing* an optimal strategy, the result in Corollary 4.2 is not yet optimal. Ideally, one would like to take any minimisers $\widetilde{\psi}^\pm$ for $\mathfrak{g}^\pm$, solve the SDE (4.7) and obtain that $\widetilde{\varphi}^{(x,\tau)}$ *defined* by (4.8) is optimal. However, it is not obvious whether these $\widetilde{\psi}^\pm$ are automatically in $\mathcal{L}(S)$. (That would of course imply solvability of (4.7), and even optimality of $\widetilde{\varphi}^{(x,\tau)}$ if that strategy is K-admissible.)

Before we proceed with our BSDE descriptions, let us briefly return to the classical (but constrained) Markowitz problem in (2.2). For given initial wealth x and target mean m, we know from Lemma 2.1 that the optimal strategy is given by $\widetilde{\vartheta}^{(m,x)} = \frac{m-x}{E[1-\widetilde{\varphi} \bullet S_T]} \widetilde{\varphi}$, where $\widetilde{\varphi} = \widetilde{\varphi}^{(-1,0)}$ solves (3.2) for $x = -1, \tau = 0$. To express $\widetilde{\vartheta}^{(m,x)}$ in feedback form, write

$$V(x, \widetilde{\vartheta}^{(x,m)}) = x + \tfrac{m-x}{E[1-\widetilde{\varphi} \bullet S_T]} \big(V(-1, \widetilde{\varphi}) + 1\big) \quad = \widetilde{m} + \tfrac{m-x}{E[1-\widetilde{\varphi} \bullet S_T]} V(-1, \widetilde{\varphi}) \tag{4.9}$$

with

$$\widetilde{m} := x + \frac{m-x}{E[1 - \widetilde{\varphi} \bullet S_T]} = \frac{m - xE[\widetilde{\varphi} \bullet S_T]}{E[1 - \widetilde{\varphi} \bullet S_T]}.$$

By Corollary 4.2, we have $\widetilde{\varphi}^{(-1,0)} = (V_-^{(-1,0)})^+\widetilde{\psi}^+ + (V_-^{(-1,0)})^-\widetilde{\psi}^-$ and therefore

$$\widetilde{\vartheta}^{(m,x)} = \big(V_-(x, \widetilde{\vartheta}^{(m,x)}) - \widetilde{m}\big)^+\widetilde{\psi}^+ + \big(V_-(x, \widetilde{\vartheta}^{(m,x)}) - \widetilde{m}\big)^-\widetilde{\psi}^-$$

by plugging in for $V^{(-1,0)} = V(-1, \widetilde{\varphi})$ from (4.9). This shows that $\widetilde{\vartheta}^{(m,x)}$ is indeed a *state feedback control*, and it also makes it clear that the critical level for switching between the "positive and negative case strategies" $\widetilde{\psi}^+$ and $\widetilde{\psi}^-$ is not zero (as one might think from the appearance of positive and negative parts), but rather $\widetilde{m}$.

Having found in Theorem 4.1 and Corollary 4.2 necessary conditions for optimality, we now turn to sufficient ones.

Theorem 4.4 (Verification theorem I). *Let $\ell^\pm$ be semimartingales such that*

1) *$\ell^\pm$ and their left limits $\ell^\pm_-$ are all $(0,1]$-valued and $\ell^\pm_T = 1$.*

2) *The joint differential characteristics of (S, ℓ^+, ℓ^-) satisfy*

$$b^{\ell^+} = -\min_{\psi\in K} \mathfrak{g}^+(\psi; S, \ell^+, \ell^-) \quad \text{and} \quad b^{\ell^-} = -\min_{\psi\in K} \mathfrak{g}^-(\psi; S, \ell^+, \ell^-). \tag{4.10}$$

3) *The solution to the SDE*

$$dV_t = (V_{t-}^+\widetilde{\psi}_t^+ + V_{t-}^-\widetilde{\psi}_t^-)\, dS_t, \quad V_0 = x \tag{4.11}$$

with $\widetilde{\psi}^\pm \in \operatorname{argmin}_{\psi\in K} \mathfrak{g}^\pm(\psi)$ on $\{V_- \gtrless 0\}$ exists and satisfies that

$$\bar{\varphi} := V_-^+\widetilde{\psi}^+ + V_-^-\widetilde{\psi}^- \in \overline{\Theta(K)}. \tag{4.12}$$

Then $\widetilde{\varphi} := \bar{\varphi}$ is the solution to (3.1). In particular, $(V^+)^2\ell^+ + (V^-)^2\ell^-$ is of class (D).

To better explain the significance of our results, let us rewrite the drift descriptions (4.6) and (4.10) into a BSDE as follows. Consider the pair of coupled backward equations

$$\begin{aligned} \ell^\pm &= -\inf_{\psi\in K} \mathfrak{g}^\pm(\psi; S, \ell^+, \ell^-) \bullet B + H^{\ell^\pm} \bullet S^c + W^{\ell^\pm} * (\mu^S - \nu^S) + N^{\ell^\pm}, \\ \ell^\pm_T &= 1, \end{aligned} \tag{4.13}$$

where a solution is a tuple $(\ell^\pm, H^{\ell^\pm}, W^{\ell^\pm}, N^{\ell^\pm})$ satisfying suitable properties; see below for a more precise formulation. Then Corollary 4.2 says that the opportunity processes $L^\pm$ from (3.9) satisfy the BSDE system (4.13), and Theorem 4.4 conversely allows us to construct from a solution to (4.13) a solution to the basic problem (3.1), if the natural candidate strategy $\bar{\varphi}$ from (4.12) has sufficiently good properties.

Remark 4.5. More generally, we could use Theorem 4.4 to construct solutions to (3.2) for any $x \in \mathbb{R}$ and stopping time τ. Indeed, if we replace the SDE (4.11) with (4.7), the definition of $\bar{\varphi}$ in (4.12) by (4.8) and assume that $\bar{\varphi}^{(x,\tau)}$ is in $\overline{\Theta(K)}$, then $\bar{\varphi}^{(x,\tau)}$ is the solution to (3.2). The argument is exactly the same as below for problem (3.1).

Proof of Theorem 4.4. For $\vartheta \in \overline{\Theta(K)}$, define

$$j(\vartheta) = (V^+(x,\vartheta))^2\ell^+ + (V^-(x,\vartheta))^2\ell^-$$

and a K-valued predictable process ψ by (4.4) so that

$$\vartheta = V_-^+(x,\vartheta)\psi + V_-^-(x,\vartheta)\psi + \mathbb{1}_{\{V_-(x,\vartheta)=0\}}\psi.$$

If $\vartheta \in \Theta(K)$, then $\sup_{0\le t\le T}|V_t(x,\vartheta)| \in L^2(P)$. Since $\ell^\pm$ are $(0,1]$-valued, we then have $\sup_{0\le t\le T}|j_t(\vartheta)| \in L^1(P)$ and so $j(\vartheta)$ is a special semimartingale with canonical decomposition $j(\vartheta) = j_0(\vartheta) + M^{j(\vartheta)} + A^{j(\vartheta)}$. Lemma 5.2 below gives $A^{j(\vartheta)} = b^{j(\vartheta)} \bullet B$ with

$$\begin{aligned} b^{j(\vartheta)} = \bar{b}^\vartheta = &\big(V_-^+(x,\vartheta)\big)^2\big\{\mathfrak{g}^+(\psi;S,\ell^+,\ell^-) + b^{\ell^+}\big\} \\ &+ \big(V_-^-(x,\vartheta)\big)^2\big\{\mathfrak{g}^-(\psi;S,\ell^+,\ell^-) + b^{\ell^-}\big\} \\ &+ \mathbb{1}_{\{V_-(x,\vartheta)=0\}}\Big(\int \big((\psi u)^+\big)^2(\ell_-^+ + y)F^{S,\ell^+}(du,dy) \\ &+ \ell_-^-\psi c^S\psi + \int \big((\psi u)^-\big)^2(\ell_-^- + z)F^{S,\ell^-}(du,dz)\Big). \end{aligned}$$

Since $\bar{b}^\vartheta \ge 0$ by the BSDE (4.10) in 2) and because $\ell^\pm$ are nonnegative, $j(\vartheta)$ is therefore a submartingale, and using $|V_T(x,\vartheta)|^2 = j_T(\vartheta)$ due to $\ell_T^\pm = 1$ gives

$$E\big[|V_T(x,\vartheta)|^2\big] \ge E\big[(x^+)^2\ell_0^+ + (x^-)^2\ell_0^-\big]. \tag{4.14}$$

Because $\vartheta \in \Theta(K)$ was arbitrary and the closure in L^2 of $G_T(\Theta(K))$ contains $G_T(\overline{\Theta(K)})$, by definition, (4.14) extends to all $\vartheta \in \overline{\Theta(K)}$.

To show that $\bar{\varphi}$ is optimal, we want to argue that $j(\bar{\varphi})$ is a supermartingale, since we then get the reverse inequality in (4.14) which is enough to conclude. Because $\bar{\varphi}$ is only in $\overline{\Theta(K)}$, however, we do not know a priori if $j(\bar{\varphi})$ is special and thus must localise as in Lemma 5.2. So we define for each $n \in \mathbb{N}$ the set $D_n := \{|\bar{\varphi}| \le n\} \in \mathcal{P}$ and $X^n := \mathbb{1}_{D_n} \bullet j(\bar{\varphi}) = j^n(\bar{\varphi})$. We first note that (4.12) and (4.11) imply that $V = V(x,\bar{\varphi})$. The SDE (4.11) then implies that V remains at 0 after V_- hits zero, and so $\bar{\varphi}\mathbb{1}_{\{V_-=0\}} = 0$ by (4.12). For $\bar{\psi}$ defined from $\bar{\varphi}$ via (4.4) or (5.1) in Lemma 5.2 below, we then get $\bar{\psi} = \bar{\varphi} = 0$ on $\{V_- = 0\} = \{V_-(x,\bar{\varphi}) = 0\}$ and therefore from (5.2) below that

$$\begin{aligned} \bar{b}^{\bar{\varphi}} = &\big(V_-^+(x,\bar{\varphi})\big)^2\big\{\mathfrak{g}^+(\bar{\psi};S,\ell^+,\ell^-) + b^{\ell^+}\big\} \\ &+ \big(V_-^-(x,\bar{\varphi})\big)^2\big\{\mathfrak{g}^-(\bar{\psi};S,\ell^+,\ell^-) + b^{\ell^-}\big\}. \end{aligned}$$

But (4.4) also gives that $\bar{\varphi} = V_-^+(x,\bar{\varphi})\bar{\psi} + V_-^-(x,\bar{\varphi})\bar{\psi} = V_-^+\bar{\psi} + V_-^-\bar{\psi}$, and comparing this to (4.12) shows that $\bar{\psi} = \widetilde{\psi}^+$ on $\{V_- > 0\}$ and $\bar{\psi} = \widetilde{\psi}^-$ on $\{V_- < 0\}$. Because $\widetilde{\psi}^\pm$ are minimisers for $\mathfrak{g}^\pm$, we obtain that $\bar{b}^{\bar{\varphi}} \equiv 0$.

Now each X^n is by Lemma 5.2 below and the above argument a special semimartingale with finite variation part $A^{X^n} = A^{j^n(\bar{\varphi})} = b^{j^n(\bar{\varphi})} \bullet B = (\mathbb{1}_{D_n}\bar{b}^{\bar{\varphi}}) \bullet B \equiv 0$. So each X^n is a local martingale, which means that $j(\bar{\varphi})$ is a σ-martingale. Since $j(\bar{\varphi}) \geq 0$, it is therefore a supermartingale and so $\bar{\varphi}$ solves (3.1). By part 2) of Proposition 3.2, $j(\bar{\varphi})$ is then even a martingale on $[0,T]$ and hence in particular of class (D). □

If we assume in addition that the constraints are *convex*, we can prove our verification result in a second, different way. This approach can be viewed as a version of the maximum principle, in the sense that global optimality is deduced from the fact that local optimality as in (4.15) below implies that certain derivatives vanish. The same comment as in Remark 4.5 applies here as well.

Theorem 4.6 (Verification theorem II)**.** *Let $K : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ be a predictable correspondence with closed and convex cones as values and $\ell^\pm$ semimartingales such that*

1) *$\ell^\pm$ and their left limits $\ell_-^\pm$ are all $(0,1]$-valued and $\ell_T^\pm = 1$.*

2) *The joint differential characteristics of (S,ℓ^+,ℓ^-) satisfy*

$$b^{\ell^+} = -\min_{\psi\in K}\mathfrak{g}^+(\psi; S,\ell^+,\ell^-) \quad \text{and} \quad b^{\ell^-} = -\min_{\psi\in K}\mathfrak{g}^-(\psi; S,\ell^+,\ell^-). \tag{4.15}$$

3) *The solution to the SDE*

$$dV_t = (V_{t-}^+\widetilde{\psi}_t^+ + V_{t-}^-\widetilde{\psi}_t^-)\,dS_t, \quad V_0 = x$$

with $\widetilde{\psi}^\pm \in \operatorname{argmin}_{\psi\in K}\mathfrak{g}^\pm(\psi)$ on $\{V_- \gtrless 0\}$ exists and satisfies that

$$\bar{\varphi} := V_-^+\widetilde{\psi}^+ + V_-^-\widetilde{\psi}^- \in \overline{\Theta(K)}$$

and $V^+\ell^+ - V^-\ell^-$ is of class (D).

Then $\widetilde{\varphi} := \bar{\varphi}$ is the solution to (3.1)*.*

Proof. In abstract terms, problem (3.1) consists of minimising $\|h-g\|_{L^2}$, for $h \equiv -x$, over the convex set of all $g \in G_T(\overline{\Theta(K)})$. The necessary and sufficient first order condition for the optimal $\bar{g}$ is then that $(h-\bar{g}, \bar{g}-g)_{L^2} \geq 0$ for all g, which can be rewritten as

$$E[V_T(x,\bar{\varphi})(\bar{\varphi} \bullet S_T)] \leq E[V_T(x,\bar{\varphi})(\vartheta \bullet S_T)] \tag{4.16}$$

for all $\vartheta \in \overline{\Theta(K)}$. Since $G_T(\overline{\Theta(K)})$ is contained in the $L^2(P)$-closure of $G_T(\Theta(K))$, it is even sufficient to have (4.16) only for all $\vartheta \in \Theta(K)$. Because $\ell^{\pm}_T = 1$, we can write $Z^{(x,0)}_T := V_T(x,\bar{\varphi}) = V^+_T(x,\bar{\varphi})\ell^+_T - V^-_T(x,\bar{\varphi})\ell^-_T$. So if we define the process $Z^{(x,0)} := V^+(x,\bar{\varphi})\ell^+ - V^-(x,\bar{\varphi})\ell^-$, then (4.16) follows if we show that for each $\vartheta \in \Theta(K)$, the product $Z^{(x,0)}(\vartheta \bullet S)$ is a submartingale, and $Z^{(x,0)}(\bar{\varphi} \bullet S)$ is a supermartingale. This is done in Lemma 5.4 below. □

Remark 4.7. 1) Because $\ell^{\pm}$ take values in $[0,1]$, Theorem 4.4 plus the estimate $|V^+\ell^+ - V^-\ell^-| \leq \big(1+(V^+)^2\big)\ell^+ + \big(1+(V^-)^2\big)\ell^-$ show that we do not need the assumption that $V^+\ell^+ - V^-\ell^-$ is of class (D). But of course, we do not want to base our second proof for the verification result on the verification result itself, via Theorem 4.4, so that making the assumption in Theorem 4.6 is reasonable.

2) The above choice of $Z^{(x,0)}$ may look ad hoc. One can show by considering dynamic versions of (3.1) that on the contrary, it is actually very natural (and this can in turn be used for more results). But we refrain from doing this for reasons of space.

We now return to the formulation of the equations (4.6) or (4.10) as a coupled system of BSDEs. We first recall that by Proposition II.2.29 and Lemma III.4.24 in [52], any special semimartingale ℓ can be decomposed as

$$\ell = A^\ell + H^\ell \bullet S^c + W^\ell * (\mu^S - \nu^S) + N^\ell \tag{4.17}$$

with $H^\ell \in L^2_{\mathrm{loc}}(S^c)$, $W^\ell \in G_{\mathrm{loc}}(\mu)$ and $N^\ell \in \mathcal{M}_{0,\mathrm{loc}}(P)$ where $\langle S^c, (N^\ell)^c\rangle = 0$ and $M^P_\mu(\Delta N^\ell|\widetilde{\mathcal{P}}) = 0$. Then

$$\Delta\ell = \Delta A^\ell + (W^\ell - \widehat{W^\ell})\mathbb{1}_{\{\Delta S \neq 0\}} + \Delta N^\ell$$

and therefore

$$\mathrm{p}(\Delta\ell\Delta S) = \int \big(\Delta A^\ell + (W^\ell(u) - \widehat{W^\ell})\big)uF^S(du). \tag{4.18}$$

This allows us to rewrite the functions $\mathfrak{g}^{\pm}$ from (4.1)–(4.3) as

$$\begin{aligned}
&\mathfrak{g}^{\pm}(\psi; S, \ell^+, \ell^-)\\
&\quad= \ell^{\pm}_-\psi^\top c^S\psi \pm 2\ell^{\pm}_-\psi^\top b^S \pm 2\psi^\top c^S H^{\ell^\pm}\\
&\qquad+ \ell^{\pm}_- \int \big(\{(1 \pm \psi^\top u)^+\}^2 - 1 \mp 2\psi^\top u\big)F^S(du)\\
&\qquad+ \int \big(\{(1 \pm \psi^\top u)^+\}^2 - 1\big)\big(\Delta A^{\ell^\pm} + W^{\ell^\pm}(u) - \widehat{W^{\ell^\pm}}\big)F^S(du)\\
&\qquad+ \int \{(1 + \psi^\top u)^-\}^2\big(\ell^{\mp}_- + \Delta A^{\ell^\mp} + W^{\ell^\mp}(u) - \widehat{W^{\ell^\mp}}\big)F^S(du)\\
&\quad=: \mathfrak{h}^{\pm}(\psi; S, \ell^+, \ell^-).
\end{aligned} \tag{4.19}$$

We now consider the coupled system of backward equations

$$\begin{aligned}\ell^{\pm} &= -\inf_{\psi\in K} \mathfrak{h}^{\pm}(\psi; S, \ell^{+}, \ell^{-}) \bullet B + H^{\ell^{\pm}} \bullet S^{c} + W^{\ell^{\pm}} * (\mu^{S} - \nu^{S}) + N^{\ell^{\pm}},\\ \ell^{\pm}_{T} &= 1.\end{aligned} \tag{4.20}$$

A *solution* of (4.20) consists of tuples $(\ell^{\pm}, H^{\ell^{\pm}}, W^{\ell^{\pm}}, N^{\ell^{\pm}})$ such that $H^{\ell^{\pm}}$ are in $L^2_{\mathrm{loc}}(S^c)$, $W^{\ell^{\pm}}$ are in $G_{\mathrm{loc}}(\mu)$, $N^{\ell^{\pm}}$ are in $\mathcal{M}_{0,\mathrm{loc}}(P)$ with $\langle S^c, (N^{\ell^{\pm}})^c\rangle = 0$ and $M^P_\mu(\Delta N^{\ell^{\pm}}|\widetilde{\mathcal{P}}) = 0$, and $\ell^{\pm}$ are (special) semimartingales with values in $[0,1]$. Moreover, being a solution also includes the condition that $\inf_{\psi\in K} \mathfrak{h}^{\pm}(\psi; S, \ell^{+}, \ell^{-})$ are finite-valued processes. For brevity, we sometimes call only (ℓ^{+}, ℓ^{-}) a solution. Then Corollary 4.2 can be restated as

Corollary 4.8. *Suppose that $L^{\pm}$ and their left limits $L^{\pm}_{-}$ are all $(0,1]$-valued and that there exists a solution to (3.2) for any $x \in \mathbb{R}$ and any stopping time τ. Then the opportunity processes satisfy the coupled BSDE system*

$$\begin{aligned}L^{\pm} &= -\inf_{\psi\in K} \mathfrak{h}^{\pm}(\psi; S, L^{+}, L^{-}) \bullet B + H^{L^{\pm}} \bullet S^{c} + W^{L^{\pm}} * (\mu^{S} - \nu^{S}) + N^{L^{\pm}},\\ L^{\pm}_{T} &= 1.\end{aligned} \tag{4.21}$$

Moreover, there exist K-valued processes $\widetilde{\psi}^{\pm}$ such that

$$\mathfrak{h}^{\pm}(\widetilde{\psi}^{\pm}; S, L^{+}, L^{-}) = \inf_{\psi\in K} \mathfrak{h}^{\pm}(\psi; S, L^{+}, L^{-}).$$

From Example 3.26 in [14] and the counterexample in [15], one can deduce that the opportunity processes $L^{\pm}$ are not the only solution to the BSDE system (4.20), not even in the unconstrained case and if S is continuous and under uniform integrability assumptions. However, it turns out that $L^{\pm}$ are the *maximal* processes which satisfy (4.20). This result is motivated by similar ones in [75].

Lemma 4.9. *The opportunity processes $L^{\pm}$ satisfy $L^{\pm} \geq \ell^{\pm}$ for any solution (ℓ^{+}, ℓ^{-}) of the BSDE (4.20). In particular, under the assumptions of Corollary 4.2, (L^{+}, L^{-}) is the maximal solution of (4.20).*

Proof. This argument only uses the definitions of $L^{\pm}$ in (3.9) and (3.4) as essential infima. Let (ℓ^{+}, ℓ^{-}) be any solution to (4.20) and define the stopping time $\tau := \inf\{t > 0 \,|\, \ell^{+}_t > L^{+}_t\} \wedge T$. By (3.9), there exists a sequence (ϑ^n) in $\Theta(K\mathbb{1}_{]\!]\tau,T]\!]})$ such that $\lim_{n\to\infty} E[|V_T(1,\vartheta^n)|^2|\mathcal{F}_\tau] = L^{+}_\tau$ P-a.s. The same argument as in the proof of Lemma 5.2 then shows that the process $j(\vartheta^n) = (V^{+}(1,\vartheta^n))^2\ell^{+} + (V^{-}(1,\vartheta^n))^2\ell^{-}$ is a submartingale, and so we obtain from $\ell^{+}_T = 1$ and $V_\tau(1,\vartheta^n) = 1$ that $\ell^{+}_\tau \leq \lim_{n\to\infty} E\big[|V_T(1,\vartheta^n)|^2\big|\mathcal{F}_\tau\big] = L^{+}_\tau$. By the definition of τ, this implies that $P[\tau < T] = 0$ and therefore that $L^{+} \geq \ell^{+}$ P-a.s. The proof of $L^{-} \geq \ell^{-}$ P-a.s. is analogous and therefore omitted. □

IV.5 Proofs

This section contains the more technical proofs. Several results and computations do not use the precise definition (3.9) of the processes $L^{\pm}$, but only some of their properties. To emphasise this, we formulate the corresponding results here for generic processes $\ell^{\pm}$. Recall that we drop the superscript $^{\top}$ in all proofs.

We first show that the predictable functions in (4.1)–(4.3) are well defined and have nice properties.

Lemma 5.1. *Let $\ell^{\pm}$ be two $[0,1]$-valued semimartingales. Then the predictable functions $\mathfrak{g}^{1,\pm}$, $\mathfrak{g}^{2,\pm}$ and $\mathfrak{g}^{\pm}$ defined in (4.1)–(4.3) are Carathéodory functions, which are convex and continuously differentiable in ψ with*

$$\begin{aligned}\nabla\mathfrak{g}^{1,\pm}(\psi) &= 2\ell^{\pm}_{-}c^{S}\psi \pm 2\ell^{\pm}_{-}b^{S} \pm 2c^{S\ell^{\pm}},\\ \nabla\mathfrak{g}^{2,\pm}(\psi) &= 2\ell^{\pm}_{-}\int\big((1\pm\psi^{\top}u)^{+}u-u\big)F^{S}(du)\\ &\quad\pm 2\int(1\pm\psi^{\top}u)^{+}uyF^{S,\ell^{\pm}}(du,dy)\\ &\quad\mp 2\int(1\pm\psi^{\top}u)^{-}u(\ell^{\mp}_{-}+z)F^{S,\ell^{\mp}}(du,dz).\end{aligned}$$

Proof. We only prove the assertion for $\mathfrak{g}^{2,-}$ as the arguments for the other functions are completely analogous or obvious. So we write $\mathfrak{g}^{2,-}$ as

$$\begin{aligned}\mathfrak{g}^{2,-}(\psi;S,\ell^{+},\ell^{-}) = \ell^{-}_{-}\int f_1(\psi,u)F^{S}(du) + \int f_2(\psi,u,y)F^{S,\ell^{-}}(du,dy)\\ +\int\big(f_3(\psi,u)\ell^{+}_{-}+f_4(\psi,u,z)\big)F^{S,\ell^{+}}(du,dz)\end{aligned}$$

with

$$\begin{aligned}f_1(\psi,u) &= \big\{(1-\psi u)^{+}\big\}^2 - 1 + 2\psi u,\\ f_2(\psi,u,y) &= \big(\big\{(1-\psi u)^{+}\big\}^2-1\big)y,\\ f_3(\psi,u) &= \big\{(1-\psi u)^{-}\big\}^2,\\ f_4(\psi,u,z) &= \big\{(1-\psi u)^{-}\big\}^2 z.\end{aligned}$$

Since S is in $\mathcal{H}^2_{\mathrm{loc}}(P)$ and the jumps of $\ell^{\pm}$ are bounded by 1, we obtain that $\int|u|^2F^{S}(du)$, $\int|u|^2|y|F^{S,\ell^{-}}(du,dy)$, $\int|u|^2|y|^2F^{S,\ell^{-}}(du,dy)$ and $\int|u|^2|z|F^{S,\ell^{+}}(du,dz)$ are finite. Combining this with the estimates

$$\begin{aligned}|f_1(\psi,u)| &= |\psi u|^2\mathbb{1}_{\{\psi u\le 1\}} + |2\psi u-1|\mathbb{1}_{\{\psi u>1\}} \le 2|\psi|^2|u|^2,\\ |f_2(\psi,u,y)| &= \big|\big((\psi u)^2-2\psi u\big)y\mathbb{1}_{\{\psi u\le 1\}} - y\mathbb{1}_{\{\psi u>1\}}\big| \le |\psi|^2|u|^2(|y|+|y|^2),\\ |f_3(\psi,u)| &= |\psi u-1|^2\mathbb{1}_{\{\psi u\le 1\}} \le |\psi|^2|u|^2,\\ |f_4(\psi,u,z)| &= |\psi u-1|^2|z|\mathbb{1}_{\{\psi u\le 1\}} \le |\psi|^2|u|^2|z|\end{aligned}$$

gives that $\mathfrak{g}^{2,-}$ is finite-valued for all $\psi \in \mathbb{R}^d$. The convexity of $\mathfrak{g}^{2,-}$ then follows immediately from the convexity of $f_1, \dots, f_4$ in ψ. To verify the continuous differentiability of $\mathfrak{g}^{2,-}$, we want to differentiate under the integrals via an appeal to dominated convergence. To that end, we fix $\psi \in \mathbb{R}^d$, take an open ball $B_\varepsilon(\psi)$ of radius $\varepsilon > 0$ around ψ and estimate for $\xi \in B_\varepsilon(\psi)$ the partial derivatives

$$\begin{aligned}
|\nabla_\psi f_1(\xi, u)| &= |-2(1-\xi u)^+ u + 2u| \le 2|\xi u u| \mathbb{1}_{\{\xi u \le 1\}} + 2|u| \mathbb{1}_{\{\xi u > 1\}} \\
&\le 2(|\psi| + \varepsilon)|u|^2 + 2|u| \mathbb{1}_{\left\{|u| > \frac{1}{|\psi|+\varepsilon}\right\}} \le 4(|\psi| + \varepsilon)|u|^2 =: h_1(u), \\
|\nabla_\psi f_2(\xi, u, y)| &= |-2(1-\xi u)^+ u y| = 2|\xi u||u||y| \mathbb{1}_{\{\xi u \le 1\}} \\
&\le 2(|\psi| + \varepsilon)|u|^2 |y| =: h_2(u, y), \\
|\nabla_\psi f_3(\xi, u)| &= |2(1-\xi u)^- u| = 2|1 - \xi u||u| \mathbb{1}_{\{\xi u \le 1\}} \\
&\le 2(|\psi| + \varepsilon)|u|^2 =: h_3(u), \\
|\nabla_\psi f_4(\xi, u, z)| &= |2(1-\xi u)^- u z| = 2|1 - \xi u| \mathbb{1}_{\{\xi u \le 1\}} |u| z| =: h_4(u, z).
\end{aligned}$$

Since $h_1, \dots, h_4$ are all integrable, we may indeed interchange differentiation and integration, and so $\mathfrak{g}^{2,-}$ is continuously differentiable in ψ. In particular, $\mathfrak{g}^{2,-}$ is continuous in ψ and a Carathéodory function. □

We next want to compute the drift of $J(\vartheta)$ for Theorem 4.1. Note below that the superscripts $\pm$ for ℓ only serve as indices; they do not denote positive and negative parts, unlike $V^\pm(x, \vartheta)$. While this notation may be slightly ambiguous, we found $\ell^{(\pm)}$ too heavy.

Lemma 5.2. *Let $\ell^\pm$ be $[0,1]$-valued semimartingales and set*

$$j(\vartheta) := \big(V^+(x, \vartheta)\big)^2 \ell^+ + \big(V^-(x, \vartheta)\big)^2 \ell^-.$$

For each $\vartheta \in \overline{\Theta(K)}$, we define the K-valued predictable process ψ as in (4.4) via

$$\vartheta =: V_-^+(x, \vartheta)\psi + V_-^-(x, \vartheta)\psi + \mathbb{1}_{\{V_-(x,\vartheta)=0\}}\psi. \tag{5.1}$$

Then the process $j^n(\vartheta) := \mathbb{1}_{D_n} \bullet j(\vartheta)$ is a special semimartingale for each $D_n := \{|\vartheta| \le n\} \in \mathcal{P}$ and $n \in \mathbb{N}$. In the canonical decomposition $j^n(\vartheta) = j_0^n(\vartheta) + M^{j^n(\vartheta)} + A^{j^n(\vartheta)}$, we have $A^{j^n(\vartheta)} = (\mathbb{1}_{D_n} \bar{b}^\vartheta) \bullet B$ with

$$\begin{aligned}
\bar{b}^\vartheta = {} & \big(V_-^+(x, \vartheta)\big)^2 \big\{\mathfrak{g}^+(\psi; S, \ell^+, \ell^-) + b^{\ell^+}\big\} \\
& + \big(V_-^-(x, \vartheta)\big)^2 \big\{\mathfrak{g}^-(\psi; S, \ell^+, \ell^-) + b^{\ell^-}\big\} \\
& + \mathbb{1}_{\{V_-(x,\vartheta)=0\}} \Big(\int \big((\psi^\top u)^+\big)^2 (\ell_-^+ + y) F^{S,\ell^+}(du, dy) \\
& + \ell_-^- \psi^\top c^S \psi + \int \big((\psi^\top u)^-\big)^2 (\ell_-^- + z) F^{S,\ell^-}(du, dz) \Big).
\end{aligned} \tag{5.2}$$

If $j(\vartheta)$ is special, then $b^{j(\vartheta)} = \bar{b}^\vartheta$.

Proof. The Meyer–Itô formula (Theorem IV.71 in [80]) and integration by parts give

$$\begin{aligned} d\big(V^+(x,\vartheta)\big)^2 &= 2V_-^+(x,\vartheta)\vartheta\,dS + \mathbb{1}_{\{V_-(x,\vartheta)>0\}}\vartheta\,d[S^c]\vartheta \\ &\quad + \Delta\big(V^+(x,\vartheta)\big)^2 - 2V_-^+(x,\vartheta)\vartheta\Delta S, \\ d\big(V^-(x,\vartheta)\big)^2 &= -2V_-^-(x,\vartheta)\vartheta\,dS + \mathbb{1}_{\{V_-(x,\vartheta)\le 0\}}\vartheta\,d[S^c]\vartheta \\ &\quad + \Delta\big(V^-(x,\vartheta)\big)^2 + 2V_-^-(x,\vartheta)\vartheta\Delta S \end{aligned}$$

and

$$\begin{aligned} &\mathbb{1}_{D_n} d\big\{\ell^+\big(V^+(x,\vartheta)\big)^2\big\} \\ &\quad = \mathbb{1}_{D_n}\big(V_-^+(x,\vartheta)\big)^2 d\ell^+ + \mathbb{1}_{D_n}\ell_-^+\Big(2V_-^+(x,\vartheta)\vartheta\,dS \\ &\qquad + \mathbb{1}_{\{V_-(x,\vartheta)>0\}}\vartheta\,d[S^c]\vartheta + \big\{\Delta\big(V^+(x,\vartheta)\big)^2 - 2V_-^+(x,\vartheta)\vartheta\Delta S\big\}\Big) \\ &\qquad + 2\mathbb{1}_{D_n}V_-^+(x,\vartheta)\vartheta\,d[S^c,(\ell^+)^c] + \mathbb{1}_{D_n}\Delta\big(V^+(x,\vartheta)\big)^2\Delta\ell^+, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\mathbb{1}_{D_n} d\big\{\ell^-\big(V^-(x,\vartheta)\big)^2\big\} \\ &\quad = \mathbb{1}_{D_n}\big(V_-^-(x,\vartheta)\big)^2 d\ell^- + \mathbb{1}_{D_n}\ell_-^-\Big(-2V_-^-(x,\vartheta)\vartheta\,dS \\ &\qquad + \mathbb{1}_{\{V_-(x,\vartheta)\le 0\}}\vartheta\,d[S^c]\vartheta + \big\{\Delta\big(V^-(x,\vartheta)\big)^2 + 2V_-^-(x,\vartheta)\vartheta\Delta S\big\}\Big) \\ &\qquad - 2\mathbb{1}_{D_n}V_-^-(x,\vartheta)\vartheta\,d[S^c,(\ell^-)^c] + \mathbb{1}_{D_n}\Delta\big(V^-(x,\vartheta)\big)^2\Delta\ell^-. \end{aligned} \tag{5.4}$$

Since $\Delta V(x,\vartheta) = \vartheta\Delta S$, $S \in \mathcal{H}^2_{\mathrm{loc}}(P)$, $|\Delta\ell^\pm| \le 1$ and ϑ is bounded on D_n, the supremum of the jumps of each term in (5.3) and (5.4) is locally integrable. So Theorem III.36 in [80] implies that these terms are all special and we can calculate their compensators as

$$\begin{aligned} &\mathbb{1}_{D_n} \bullet \big\{\ell^+\big(V^+(x,\vartheta)\big)^2\big\} \\ &\overset{\mathrm{mart}}{=} \mathbb{1}_{D_n}\big(V_-^+(x,\vartheta)\big)^2 \bullet A^{\ell^+} \\ &\quad + (\mathbb{1}_{D_n}\ell_-^+) \bullet \big((2V_-^+(x,\vartheta)\vartheta) \bullet A^S + \mathbb{1}_{\{V_-(x,\vartheta)>0\}} \bullet [\vartheta \bullet S^c]\big) \\ &\quad + \mathbb{1}_{D_n}\ell_-^+\big\{\big((V_-(x,\vartheta)+\vartheta u)^+\big)^2 - \big(V_-^+(x,\vartheta)\big)^2 - 2V_-^+(x,\vartheta)\vartheta u\big\} * \nu^S \\ &\quad + \mathbb{1}_{D_n}\big\{\big((V_-(x,\vartheta)+\vartheta u)^+\big)^2 - \big(V_-^+(x,\vartheta)\big)^2\big\} y * \nu^{S,\ell^+} \\ &\quad + 2\mathbb{1}_{D_n}V_-^+(x,\vartheta) \bullet [\vartheta \bullet S^c, (\ell^+)^c], \\ &\mathbb{1}_{D_n} \bullet \big\{\ell^-\big(V^-(x,\vartheta)\big)^2\big\} \\ &\overset{\mathrm{mart}}{=} \mathbb{1}_{D_n}\big(V_-^-(x,\vartheta)\big)^2 \bullet A^{\ell^-} \\ &\quad + (\mathbb{1}_{D_n}\ell_-^-) \bullet \big(-(2V_-^-(x,\vartheta)\vartheta) \bullet A^S + \mathbb{1}_{\{V_-(x,\vartheta)\le 0\}} \bullet [\vartheta \bullet S^c]\big) \\ &\quad + \mathbb{1}_{D_n}\ell_-^-\big\{\big((V_-(x,\vartheta)+\vartheta u)^-\big)^2 - \big(V_-^-(x,\vartheta)\big)^2 + 2V_-^-(x,\vartheta)\vartheta u\big\} * \nu^S \\ &\quad + \mathbb{1}_{D_n}\big\{\big((V_-(x,\vartheta)+\vartheta u)^-\big)^2 - \big(V_-^-(x,\vartheta)\big)^2\big\} z * \nu^{S,\ell^-} \\ &\quad - 2\mathbb{1}_{D_n}V_-^-(x,\vartheta) \bullet [\vartheta \bullet S^c, (\ell^-)^c], \end{aligned}$$

where we denote by $\overset{\text{mart}}{=}$ equality up to a local martingale. Adding both equations and passing to differential characteristics gives

$$\begin{aligned}
&A^{j^n(\vartheta)} \\
&= \mathbb{1}_{D_n}\Big(\mathbb{1}_{\{V_-(x,\vartheta)>0\}}\ell_-^+\vartheta c^S\vartheta + 2V_-^+(x,\vartheta)\vartheta\big(\ell_-^+ b^S + c^{S,\ell^+}\big) + \big(V_-^+(x,\vartheta)\big)^2 b^{\ell^+} \\
&\quad + \ell_-^+\int\big\{\big((V_-(x,\vartheta)+\vartheta u)^+\big)^2 - \big(V_-^+(x,\vartheta)\big)^2 - 2V_-^+(x,\vartheta)\vartheta u\big\}F^S(du) \\
&\quad + \int\big\{\big((V_-(x,\vartheta)+\vartheta u)^+\big)^2 - \big(V_-^+(x,\vartheta)\big)^2\big\}yF^{S,\ell^+}(du,dy) \\
&\quad + \mathbb{1}_{\{V_-(x,\vartheta)\le 0\}}\ell_-^-\vartheta c^S\vartheta - 2V_-^-(x,\vartheta)\vartheta(\ell_-^- b^S + c^{S,\ell^-}) + \big(V_-^-(x,\vartheta)\big)^2 b^{\ell^-} \\
&\quad + \ell_-^-\int\big\{\big((V_-(x,\vartheta)+\vartheta u)^-\big)^2 - \big(V_-^-(x,\vartheta)\big)^2 + 2V_-^-(x,\vartheta)\vartheta u\big\}F^S(du) \\
&\quad + \int\big\{\big((V_-(x,\vartheta)+\vartheta u)^-\big)^2 - \big(V_-^-(x,\vartheta)\big)^2\big\}zF^{S,\ell^-}(du,dz)\Big)\bullet B.
\end{aligned}$$

By plugging in (5.1), we obtain first

$$\begin{aligned}
\big((V_-(x,\vartheta)+\vartheta u)^\pm\big)^2 - \big(V_-^\pm(x,\vartheta)\big)^2 &= \big(V_-^\pm(x,\vartheta)\big)^2\big\{\big((1\pm\psi u)^+\big)^2 - 1\big\} \\
&\quad + \big(V_-^\mp(x,\vartheta)\big)^2\big((1\mp\psi u)^-\big)^2 + \mathbb{1}_{\{V_-(x,\vartheta)=0\}}\big((\psi u)^\pm\big)^2
\end{aligned}$$

and therefore also $A^{j^n(\vartheta)} = (\mathbb{1}_{D_n}\bar{b}^\vartheta)\bullet B$ with

$$\begin{aligned}
\bar{b}^\vartheta = \big(V_-^+(x,\vartheta)\big)^2\Big\{&\ell_-^+\psi c^S\psi + 2\psi(\ell_-^+ b^S + c^{S\ell^+}) + b^{\ell^+} \\
&+ \ell_-^+\int\big\{\big((1+\psi u)^+\big)^2 - 1 - 2\psi u\big\}F^S(du) \\
&+ \int\big\{\big((1+\psi u)^+\big)^2 - 1\big\}yF^{S,\ell^+}(du,dy) \\
&+ \int\big((1+\psi u)^-\big)^2(\ell_-^- + z)F^{S,\ell^-}(du,dz)\Big\} \\
+ \big(V_-^-(x,\vartheta)\big)^2\Big\{&\ell_-^-\psi c^S\psi - 2\psi(\ell_-^- b^S + c^{S\ell^-}) + b^{\ell^-} \\
&+ \ell_-^-\int\big\{\big((1-\psi u)^+\big)^2 - 1 + 2\psi u\big\}F^S(du) \\
&+ \int\big\{\big((1-\psi u)^+\big)^2 - 1\big\}zF^{S,\ell^-}(du,dz) \\
&+ \int\big((1-\psi u)^-\big)^2(\ell_-^+ + y)F^{S,\ell^+}(du,dy)\Big\} \\
+ \mathbb{1}_{\{V_-(x,\vartheta)=0\}}\Big\{&\int\big((\psi u)^+\big)^2(\ell_-^+ + y)F^{S,\ell^+}(du,dy) + \ell_-^-\psi c^S\psi \\
&+ \int\big((\psi u)^-\big)^2(\ell_-^- + z)F^{S,\ell^-}(du,dz)\Big\}
\end{aligned}$$

after collecting terms. The assertion then follows by inserting the definitions of $\mathfrak{g}^\pm$. □

The following result should be folklore, but we have not found a direct reference.

Lemma 5.3. *Any stochastic exponential has local time* 0 *at the origin.*

Proof. If $L^0(X)$ denotes the local time at 0 of the semimartingale X, the Meyer–Tanaka formula (Theorem IV.70 in [80]) gives

$$\begin{aligned}\tfrac{1}{2}dL^0(X) &= dX^+ - \mathbb{1}_{\{X_->0\}}\,dX - \Delta(X^+) + \mathbb{1}_{\{X_->0\}}\Delta X\\ &= dX^+ - \mathbb{1}_{\{X_->0\}}\,dX - \mathbb{1}_{\{X_->0\}}X^- - \mathbb{1}_{\{X_-\le 0\}}X^+. \qquad (5.5)\end{aligned}$$

If X is of finite variation, then $L^0(X)\equiv 0$ by construction (e.g. from Theorem IV.66 in [80]). Now if $X=\mathcal{E}(N)$, then (the proof of) Theorem II.37 in [80] allows us to write $X=UV$ with $U>0$, V of finite variation and $[U,V]\equiv 0$. So using the product rule and (5.5) for V with $L^0(V)\equiv 0$ yields

$$\begin{aligned}dX^+ &= d(UV^+)\\ &= U_-\mathbb{1}_{\{V_->0\}}\,dV + \mathbb{1}_{\{V_->0\}}U_-V^- + \mathbb{1}_{\{V_-\le 0\}}U_-V^+\\ &\quad + V_-^+\,dU + \Delta U\Delta(V^+)\\ &= \mathbb{1}_{\{V_->0\}}(U_-\,dV + V_-\,dU) + \mathbb{1}_{\{V_->0\}}\big(U_-V^- + \Delta U(V^+-V_-)\big)\\ &\quad + \mathbb{1}_{\{V_-\le 0\}}\big(U_-V^+ + (\Delta U)V^+\big).\end{aligned}$$

But $U_-V^- + \Delta U(V^+-V_-) = (U_-+\Delta U)V^- + \Delta U\Delta V = UV^-$ because $[U,V]\equiv 0$, and $\operatorname{sign} V = \operatorname{sign} X$ since $U>0$. So we get

$$\begin{aligned}dX^+ &= \mathbb{1}_{\{X_->0\}}\,d(UV) + \mathbb{1}_{\{X_->0\}}UV^- + \mathbb{1}_{\{X_-\le 0\}}UV^+\\ &= \mathbb{1}_{\{X_->0\}}\,dX + \mathbb{1}_{\{X_->0\}}X^- + \mathbb{1}_{\{X_-\le 0\}}X^+,\end{aligned}$$

and (5.5) thus yields that $L^0(X)\equiv 0$. □

The next result proves a bit more than Theorem 4.6.

Lemma 5.4. *Let* $K:\overline{\Omega}\to 2^{\mathbb{R}^d}\setminus\{\emptyset\}$ *be a predictable correspondence with closed and convex cones as values and* $\ell^\pm$ *semimartingales such that*

1) $\ell^\pm$ *and their left limits* $\ell^\pm_-$ *are all* $(0,1]$*-valued and* $\ell^\pm_T=1$.

2) *The joint differential characteristics of* (S,ℓ^+,ℓ^-) *satisfy*

$$b^{\ell^+} = -\min_{\psi\in K}\mathfrak{g}^+(\psi;S,\ell^+,\ell^-) \quad \text{and} \quad b^{\ell^-} = -\min_{\psi\in K}\mathfrak{g}^-(\psi;S,\ell^+,\ell^-).$$

3) *For each* $x\in\mathbb{R}$ *and each stopping time* τ, *there exists a solution to the SDE*

$$\begin{aligned}dV_t^{(x,\tau)} &= \big((V_{t-}^{(x,\tau)})^+\widetilde{\psi}_t^+ + (V_{t-}^{(x,\tau)})^-\widetilde{\psi}_t^-\big)\mathbb{1}_{]\!]\tau,T]\!]}\,dS_t =: \bar{\varphi}_t^{(x,\tau)}\,dS_t,\\ V_0 &= V_\tau = x \qquad (5.6)\end{aligned}$$

with $\widetilde{\psi}^{\pm} \in \operatorname*{argmin}_{\psi \in K} \mathfrak{g}^{\pm}(\psi)$ *on* $\{V_-^{(x,\tau)} \gtrless 0\} \cap]\!]\tau, T]\!]$ *such that the processes* $\widetilde{\psi}^{\pm} \mathbb{1}_{\{V_-^{(x,\tau)} \gtrless 0\} \cap]\!]\tau,T]\!]}$ *are in* $\mathcal{L}(S)$, $V_T^{(x,\tau)}$ *is in* $L^2(P)$ *and*

$$Z^{(x,\tau)} := Z^{(x,\tau)}(\ell^+, \ell^-) := (V^{(x,\tau)})^+ \ell^+ - (V^{(x,\tau)})^- \ell^- \tag{5.7}$$

is of class (D).

Then $\mathbb{1}_{]\!]\tau,T]\!]} \bullet Z^{(x,\tau)}$ *is a square-integrable martingale,* $(\vartheta \bullet S)Z^{(x,\tau)}$ *is a submartingale for every* $\vartheta \in \Theta(K)$ *with* $\vartheta = 0$ *on* $[\![0,\tau]\!]$ *and* $(\bar{\varphi}^{(x,\tau)} \bullet S)Z^{(x,\tau)}$ *is a local martingale, which is uniformly bounded from below by a square-integrable random variable and hence a supermartingale. Moreover, there exists* $N^{(x,\tau)} \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ *such that* $Z^{(x,\tau)} = (Z^{(x,\tau)})^\tau \mathcal{E}(N^{(x,\tau)})$.

Proof. Throughout this proof, we fix x and τ and drop all superscripts (x,τ) to alleviate the notation. We also point out that the superscripts $^+$ and $^-$ have different meanings for different processes; they are just indices for ℓ and $\widetilde{\psi}$, but denote positive and negative parts for V and for all quantities involving jumps.

1) First note that by its definition, V equals $x + \bar{\varphi} \bullet S = V(x,\bar{\varphi})$. Moreover, the definition of Z gives that $VZ = (V^+)^2\ell^+ + (V^-)^2\ell^-$ is nonnegative like $\ell^{\pm}$. To argue that $\mathbb{1}_{]\!]\tau,T]\!]} \bullet Z$ is a σ-martingale, we first compute (as in Lemma 5.2) its semimartingale characteristics and then argue (as in Theorem 4.4) that the assumptions imply that its drift term vanishes, at least σ-locally. This is very computational and so we only give the key steps below.

2) We first need the dynamics of $V^{\pm}$. By its definition, V is a stochastic exponential so that its local time at 0 vanishes, by Lemma 5.3. The Itô–Meyer–Tanaka formula (see Theorem IV.68 in [80]) therefore simplifies as in (5.5) with $L^0(X) \equiv 0$ to

$$\begin{aligned} dV^+ &= V_-^+ \widetilde{\psi}^+ \mathbb{1}_{]\!]\tau,T]\!]} dS + \mathbb{1}_{\{V_- > 0\}} V^- + \mathbb{1}_{\{V_- \le 0\}} V^+, \\ dV^- &= -V_-^- \widetilde{\psi}^- \mathbb{1}_{]\!]\tau,T]\!]} dS + \mathbb{1}_{\{V_- > 0\}} V^- + \mathbb{1}_{\{V_- \le 0\}} V^+. \end{aligned}$$

Using the SDE (5.7) to compute ΔV and plugging that into $V^{\pm} = (V_- + \Delta V)^{\pm}$ gives after some calculations that

$$\begin{aligned} \mathbb{1}_{\{V_- > 0\}} V^- &= V_-^+ (1 + \widetilde{\psi}^+ \mathbb{1}_{]\!]\tau,T]\!]} \Delta S)^-, \\ \mathbb{1}_{\{V_- \le 0\}} V^+ &= V_-^- (1 - \widetilde{\psi}^- \mathbb{1}_{]\!]\tau,T]\!]} \Delta S)^-, \end{aligned}$$

and therefore by plugging in that

$$dV^{\pm} = \pm V_-^{\pm} \big(\psi^{\pm} \mathbb{1}_{]\!]\tau,T]\!]} dS \pm (1 \pm \psi^{\pm} \mathbb{1}_{]\!]\tau,T]\!]} \Delta S)^- \big) + V_-^{\mp} (1 \mp \psi^{\mp} \mathbb{1}_{]\!]\tau,T]\!]} \Delta S)^-. \tag{5.8}$$

In particular, computing the jumps of $V^{\pm}$ and using $u+(1+u)^{-} = (1+u)^{+}-1$ and $u-(1-u)^{-} = -\big((1-u)^{+}-1\big)$ gives

$$\Delta V^{\pm} = V_{-}^{\pm}\big((1 \pm \widetilde{\psi}^{\pm}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{+} - 1\big) + V_{-}^{\mp}(1 \mp \widetilde{\psi}^{\mp}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{-}. \tag{5.9}$$

3) Because $Z = V^{+}\ell^{+} - V^{-}\ell^{-}$, we next need to compute the products $V^{\pm}\ell^{\pm}$. Using the product rule, (5.8) and (5.9) and collecting terms gives (after a while)

$$\begin{aligned} d(V^{\pm}\ell^{\pm}) = V_{-}^{\pm}\big\{&d\ell^{\pm} \pm \ell_{-}^{\pm}\big(\widetilde{\psi}^{\pm}\mathbb{1}_{\rrbracket\tau,T\rrbracket}dS \pm (1 \pm \widetilde{\psi}^{\pm}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{-}\big) \\ &\pm \widetilde{\psi}^{\pm}\mathbb{1}_{\rrbracket\tau,T\rrbracket}d[S^{c},(\ell^{\pm})^{c}] + \Delta\ell^{\pm}\big((1 \pm \widetilde{\psi}^{\pm}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{+} - 1\big)\big\} \\ &+ V_{-}^{\mp}(\ell_{-}^{\pm} + \Delta\ell^{\pm})(1 \mp \widetilde{\psi}^{\mp}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{-} \end{aligned}$$

and therefore

$$\begin{aligned} dZ = V_{-}^{+}\big\{&d\ell^{+} + \ell_{-}^{+}\big(\widetilde{\psi}^{+}\mathbb{1}_{\rrbracket\tau,T\rrbracket}dS + (1 + \widetilde{\psi}^{+}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{-}\big) \\ &+ \widetilde{\psi}^{+}\mathbb{1}_{\rrbracket\tau,T\rrbracket}d[S^{c},(\ell^{+})^{c}] + \Delta\ell^{+}\big((1 + \widetilde{\psi}^{+}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{+} - 1\big) \\ &- (\ell_{-}^{-} + \Delta\ell^{-})(1 + \widetilde{\psi}^{+}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{-}\big\} \\ - V_{-}^{-}\big\{&d\ell^{-} - \ell_{-}^{-}\big(\widetilde{\psi}^{-}\mathbb{1}_{\rrbracket\tau,T\rrbracket}dS - (1 - \widetilde{\psi}^{-}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{-}\big) \\ &- \widetilde{\psi}^{-}\mathbb{1}_{\rrbracket\tau,T\rrbracket}d[S^{c},(\ell^{-})^{c}] + \Delta\ell^{-}\big((1 - \widetilde{\psi}^{-}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{+} - 1\big) \\ &- (\ell_{-}^{+} + \Delta\ell^{+})(1 - \widetilde{\psi}^{-}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{-}\big\}. \end{aligned} \tag{5.10}$$

For later use, we already note that this yields

$$dZ^{c} = V_{-}^{+}\big(d(\ell^{+})^{c} + \ell_{-}^{+}\widetilde{\psi}^{+}\mathbb{1}_{\rrbracket\tau,T\rrbracket}dS^{c}\big) - V_{-}^{-}\big(d(\ell^{-})^{c} - \ell_{-}^{-}\widetilde{\psi}^{-}\mathbb{1}_{\rrbracket\tau,T\rrbracket}dS^{c}\big) \tag{5.11}$$

and by using (5.9) that

$$\begin{aligned} \Delta Z = V_{-}^{+}\big\{&\ell_{-}^{+}\big((1 + \widetilde{\psi}^{+}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{+} - 1\big) + \Delta\ell^{+}(1 + \widetilde{\psi}^{+}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{+} \\ &- (\ell_{-}^{-} + \Delta\ell^{-})(1 + \widetilde{\psi}^{+}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{-}\big\} \\ - V_{-}^{-}\big\{&\ell_{-}^{-}\big((1 - \widetilde{\psi}^{-}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{+} - 1\big) + \Delta\ell^{-}(1 - \widetilde{\psi}^{-}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{+} \\ &- (\ell_{-}^{+} + \Delta\ell^{+})(1 - \widetilde{\psi}^{-}\mathbb{1}_{\rrbracket\tau,T\rrbracket}\Delta S)^{-}\big\}. \end{aligned} \tag{5.12}$$

4) For each n, we define the set $D_n := \{|V_{-}^{+}\widetilde{\psi}^{+} + V_{-}^{-}\widetilde{\psi}^{-}| \le n\} \cap \rrbracket\tau,T\rrbracket \in \mathcal{P}$ and the process $Z^{n} := \mathbb{1}_{D_n} \bullet Z$. Then $\sup_{0\le s\le T}|\Delta Z_s^{n}|$ is like S locally square-integrable so that Z^{n} is special, and we even have this integrability for each term from above in $\Delta Z^{n} = \mathbb{1}_{D_n}\Delta Z$. To compute the finite variation part $A^{Z^{n}}$ from the canonical decomposition of Z^{n}, we can therefore compensate each summand separately, and so we find from (5.10) that

$A^{Z^n} = (\mathbb{1}_{D_n}\beta) \bullet B$ with

$$\begin{aligned}\beta = V_-^+ \Big\{ & b^{\ell^+} + \ell_-^+ \Big(\widetilde{\psi}^+ b^S + \int (1+\widetilde{\psi}^+ u)^- F^S(du) \Big) + \widetilde{\psi}^+ c^{S\ell^+} \\ & + \int y\big((1+\widetilde{\psi}^+ u)^+ - 1\big) F^{S,\ell^+}(du,dy) \\ & - \int (\ell_-^- + z)(1+\widetilde{\psi}^+ u)^- F^{S,\ell^-}(du,dz) \Big\} \\ - V_-^- \Big\{ & b^{\ell^-} - \ell_-^- \Big(\widetilde{\psi}^- b^S - \int (1-\widetilde{\psi}^- u)^- F^S(du) \Big) - \widetilde{\psi}^- c^{S\ell^-} \\ & + \int z\big((1-\widetilde{\psi}^- u)^+ - 1\big) F^{S,\ell^-}(du,dz) \\ & - \int (\ell_-^+ + y)(1-\widetilde{\psi}^- u)^- F^{S,\ell^+}(du,dy) \Big\}. \end{aligned} \tag{5.13}$$

But by the assumption in 2), we have $b^{\ell^\pm} = -\mathfrak{g}^\pm(\widetilde{\psi}^\pm; S, \ell^+, \ell^-)$. Plugging that into (5.13), using the definition of $\mathfrak{g}^\pm$ in (4.3), collecting terms and using Lemma 5.1 leads after lengthy but straightforward computations to $\beta = -V_-^+ \frac{1}{2}\widetilde{\psi}^+ \nabla\mathfrak{g}^+(\widetilde{\psi}^+) + V_-^- \frac{1}{2}\widetilde{\psi}^- \nabla\mathfrak{g}^-(\widetilde{\psi}^-)$. Because $\widetilde{\psi}^\pm$ is by assumption 2) the minimiser for $\mathfrak{g}^\pm$ on $\{V_- \gtrless 0\} \cap \rrbracket\tau, T\rrbracket$, both summands are 0 on $\rrbracket\tau, T\rrbracket$ by the first order conditions for optimality. Hence each Z^n is a local martingale so that $\mathbb{1}_{\rrbracket\tau,T\rrbracket} \bullet Z$ is a σ-martingale. But Z is of class (D) by assumption 3) and therefore $\mathbb{1}_{\rrbracket\tau,T\rrbracket} \bullet Z$ is a true martingale, and it is then even square-integrable because $\mathbb{1}_{\rrbracket\tau,T\rrbracket} \bullet Z_T = V_T - (x^+\ell_\tau^+ - x^-\ell_\tau^-)$ is in $L^2(P)$ due to $\ell_T^\pm = 1$ and $\ell_\tau^\pm \in (0,1]$.

5) Now we look at the product of $\vartheta \bullet S$ and Z for either $\vartheta \in \Theta(K)$ or $\vartheta = \bar{\varphi}$. By the product rule,

$$X := (\vartheta \bullet S)Z = (Z_-\vartheta) \bullet S + (\vartheta \bullet S)_- \bullet Z + [\vartheta \bullet S, Z],$$

where $Z = (x^+\ell^+ - x^-\ell^-)^\tau + \mathbb{1}_{\rrbracket\tau,T\rrbracket} \bullet Z \in \mathcal{H}^2(P)$. For $\vartheta \in \Theta(K)$, we have $\vartheta \bullet S \in \mathcal{H}^2(P)$ so that X is special and all the jumps appearing when we compute the above expressions have integrable suprema. For $\vartheta = \bar{\varphi}$, we set $D_n := \{|\bar{\varphi}| \le n\} \in \mathcal{P}$ and $X^n := \mathbb{1}_{D_n} \bullet X$. Then X^n is special and all its jump terms have locally integrable suprema, as can be seen from the explicit expression (5.12) for ΔZ. So setting $D_n := \overline{\Omega}$ for $\vartheta \in \Theta(K)$, we can in both cases compute the finite variation term A^{X^n} from the canonical decomposition of X^n as

$$\begin{aligned} A^{X^n} &= \mathbb{1}_{D_n} \bullet \Big\{ (Z_-\vartheta) \bullet A^S + \langle (\vartheta \bullet S)^c, Z^c \rangle + \Big(\sum \Delta(\vartheta \bullet S)\Delta Z \Big)^{\mathbf{P}} \Big\} \\ &= (\mathbb{1}_{D_n}\vartheta) \bullet \Big\{ Z_- \bullet A^S + \langle S^c, Z^c \rangle + \sum{}^{\mathbf{P}}(\Delta S \Delta Z) \Big\} = (\mathbb{1}_{D_n}\vartheta\gamma) \bullet B. \end{aligned}$$

By using (5.11) and (5.12) as well as the definition (5.7) of Z, we explicitly obtain after collecting terms that

$$\begin{aligned}\gamma = V_-^+\Big\{&\ell_-^+ b^{\ell+} + c^{S\ell+} + \ell_-^+ c^S \widetilde{\psi}^+ + \ell_-^+ \int \big((1+\widetilde{\psi}^+ u)^+ - 1\big) F^S(du) \\ &+ \int y(1+\widetilde{\psi}^+ u)^+ u F^{S,\ell^+}(du,dy) - \int (\ell_-^- + z)(1+\widetilde{\psi}^+ u)^- u F^{S,\ell^-}(du,dz)\Big\} \\ - V_-^-\Big\{&\ell_-^- b^{\ell^-} + c^{S\ell^-} - \ell_-^- c^S \widetilde{\psi}^- + \ell_-^- \int \big((1-\widetilde{\psi}^- u)^+ - 1\big) F^S(du) \\ &+ \int z(1-\widetilde{\psi}^- u)^- u F^{S,\ell^-}(du,dz) - \int (\ell_-^+ + y)(1-\widetilde{\psi}^- u)^- F^{S,\ell^+}(du,dy)\Big\}\end{aligned}$$

on $]\!]\tau, T]\!]$. By comparing this to the expression for $\nabla\mathfrak{g}^\pm$ in Lemma 5.1, we see that

$$\vartheta\gamma = \tfrac{1}{2}\big(V_-^+ \vartheta\nabla\mathfrak{g}^+(\widetilde{\psi}^+) + V_-^- \vartheta\nabla\mathfrak{g}^-(\widetilde{\psi}^-)\big),$$

and both summands are nonnegative by the first order conditions for optimality since $\widetilde{\psi}^\pm$ are the minimisers of $\mathfrak{g}^\pm$ on $\{V_- \gtrless 0\}\cap]\!]\tau, T]\!]$ by assumption 2) and $\vartheta = 0$ on $[\![0,\tau]\!]$. For $\vartheta = \bar\varphi = (V_-^+\widetilde{\psi}^+ + V_-^-\widetilde{\psi}^-)\mathbb{1}_{]\!]\tau,T]\!]}$, we get as in step 4) that $\bar\varphi\gamma = 0$. So X^n is a local submartingale for $\vartheta \in \Theta(K)$, and because $D_n = \overline{\Omega}$ here, $X^n = X$ is actually even a true submartingale since its supremum is integrable. For $\vartheta = \bar\varphi$, X^n is a local martingale; so $X = (V^+)^2\ell^+ + (V^-)^2\ell^- - xZ \geq -|x|\sup_{0\leq s\leq T}|Z_s|$ is a σ-martingale which is bounded from below by an square-integrable random variable by step 1) and 5) and Doob's maximal inequality so that it is a local martingale by Proposition 3.3 in [3] and hence a supermartingale by Fatou's Lemma.

6) Finally, it remains to argue that $Z = Z^\tau\,\mathcal{E}(N)$ for some local martingale N, which is then in $\mathcal{M}^2_{0,\mathrm{loc}}(P)$ by Remark 2.8 because $\mathbb{1}_{]\!]\tau,T]\!]} \bullet Z$ is in $\mathcal{M}^2(P)$. To that end, we define $\sigma := \inf\{t > 0\,|\,V_t = 0\}$ and note from the SDE (5.6) that $\sigma \geq \tau$ for $x \neq 0$, $V_- \neq 0$ on $[\![0,\sigma]\!]$ and $V = 0$ on $[\![\sigma, T]\!]$. Hence the definition (5.7) of Z gives that $Z_- \neq 0$ on $[\![0,\sigma]\!]$ and $Z = 0$ on $[\![\sigma, T]\!]$ so that $N := (\mathbb{1}_{]\!]\tau,\sigma]\!]}\frac{1}{Z_-}) \bullet Z$ is well defined and gives $Z = Z^\tau\,\mathcal{E}(N)$. This completes the proof. □

IV.6 Related work

To round off the chapter and put our contribution into perspective, we finally discuss the connections of our work to the existing literature. This naturally splits in two parts.

IV.6.1 The unconstrained case

For (semimartingale) models without constraints, one key motivation to study the Markowitz problem has been the mean-variance hedging problem (2.5). The solution of (2.5), for an arbitrary payoff H, can be described more explicitly if one knows the variance-optimal martingale measure or the opportunity-neutral measure; see for example Theorem 4.6 in [87] and Theorem 4.10 in [14]. Finding those measures is intimately linked to the approximation in $L^2(P)$ of the constant 1 by stochastic integrals of S, i.e. to (2.4). While there is a vast literature on mean-variance hedging, the most general results for these problems without constraints have been obtained by Černý and Kallsen [14], and their work has also provided a lot of inspiration for our approach. We now quickly explain how the main results of [14] can be obtained directly as special cases of our setting.

Suppose that there are no constraints so that $C \equiv K \equiv \mathbb{R}^d$. The first key simplification is then that the opportunity processes $L^\pm$ agree so that we can write $L := L^+ = L^-$. One way to see this is to look at the proof of Proposition 3.1 and note there that the distinction according to the sign of $x + \vartheta \bullet S_\sigma$ becomes superfluous since K is symmetric. Alternatively, one can look at the definitions of $\bar{L}^\pm(\sigma)$ in (3.4) and observe that they agree for $+$ and $-$ because $\mathfrak{K}(0,\sigma;\sigma)$ contains with φ also $-\varphi$. Again this only needs that K is a cone and symmetric around 0, but we shall exploit $K \equiv \mathbb{R}^d$ later. Recall that $\overline{\Theta} = \overline{\Theta(\mathbb{R}^d)}$.

To get good properties for the (single) opportunity process L, we next suppose as in [14] that there exists an equivalent σ-martingale measure (EσMM) Q for S with $\frac{dQ}{dP} \in L^2(P)$. (Because $S \in \mathcal{H}^2_{\mathrm{loc}}(P)$, we then have that $\sup_{0 \le t \le \tau_n} |S_t| \in L^1(Q)$ so that Q is actually an equivalent local martingale measure (ELMM) for S.) Lemma 3.4 then tells us that both L and L_- are strictly positive; this recovers Lemma 3.10 from [14]. A substantial sharpening is given in Theorem 6.2 below.

Moving on to the local description in Section IV.4, we see from $L^+ = L^- = L$ that we only need to consider a setting with $\ell^+ = \ell^- =: \ell$. Then (4.2) reduces to

$$\begin{aligned}
\mathfrak{g}^{2,+}(\psi) &= \ell_- \int \big((1+\psi^\top u)^2 - 1 - 2\psi^\top u\big) F^S(du) \\
&\quad + \int \big((1+\psi^\top u)^2 - 1\big) y F^{S,\ell}(du,dy) \\
&= \int (\psi^\top u)^2 (\ell_- + y) F^{S,\ell}(du,dy) + \int 2\psi^\top u y F^{S,\ell}(du,dy) \\
&= \mathfrak{g}^{2,-}(-\psi),
\end{aligned}$$

and therefore (4.3) yields

$$\mathfrak{g}^+(\psi) = \ell_- \psi^\top c^S \psi + 2\ell_- \psi^\top b^S + 2\psi^\top c^{S\ell} + \mathfrak{g}^{2,+}(\psi) = \mathfrak{g}^-(-\psi).$$

If in addition ℓ_- is strictly positive, we can rewrite this as

$$\mathfrak{g}^+(\psi) = \ell_-(\psi^\top \bar{c} \psi + 2\psi^\top \bar{b}) = \mathfrak{g}^-(-\psi)$$

with

$$\bar{c} := \bar{c}(S, \ell) := c^S + \int uu^\top \Big(1 + \frac{y}{\ell_-}\Big) F^{S,\ell}(du, dy), \tag{6.1}$$

$$\bar{b} := \bar{b}(S, \ell) := b^S + \frac{c^{S\ell}}{\ell_-} + \int u \frac{y}{\ell_-} F^{S,\ell}(du, dy), \tag{6.2}$$

as in (3.25) and (3.23) in [14]. So $\mathfrak{g}^\pm$ are quadratic functions and we can easily, by completing squares, find their minimisers and minimal values in explicit form. The result is

$$\min_{\psi \in \mathbb{R}^d} \mathfrak{g}^+(\psi) = \mathfrak{g}^+(\widetilde{\psi}^+) = -\ell_- \bar{b}^\top (\bar{c})^{-1} \bar{b} = \min_{\psi \in \mathbb{R}^d} \mathfrak{g}^-(\psi) = \mathfrak{g}^-(\widetilde{\psi}^-) \tag{6.3}$$

with

$$\widetilde{\psi}^+ = -\widetilde{\psi}^- =: \widetilde{\psi} = -(\bar{c})^{-1}\bar{b} =: -\bar{a}, \tag{6.4}$$

where $(\bar{c})^{-1}$ denotes the Moore–Penrose pseudoinverse of $\bar{c}$. We remark that this is well defined whenever a minimiser exists, hence in particular if there is an optimal strategy.

Under the assumption (made in [14]) that there is an EσMM Q for S with $\frac{dQ}{dP} \in L^2(P)$, Theorem 2.16 for $C \equiv \mathbb{R}^d$ tells us that $G_T(\overline{\Theta})$ is closed in $L^2(P)$. The same is true for

$$G_T(\overline{\Theta}\mathbb{1}_{\rrbracket \tau, T \rrbracket}) = G_T\big(\overline{\Theta(\mathbb{R}^d \mathbb{1}_{\rrbracket \tau, T \rrbracket})}\big)$$

for any stopping time τ, and so (3.2) has a solution $\widetilde{\varphi}^{(x,\tau)}$ for every pair (x, τ). Corollary 4.2 thus allows us to identify $\widetilde{\varphi}^{(x,\tau)}$; indeed, $\widetilde{\psi}^+ = -\widetilde{\psi}^- = \widetilde{\psi}$ reduces the SDE (4.7) to

$$dV_t^{(x,\tau)} = V_{t-}^{(x,\tau)} \widetilde{\psi}_t \mathbb{1}_{\rrbracket \tau, T \rrbracket}\, dS_t, \quad V_0^{(x,\tau)} = V_\tau^{(x,\tau)} = x$$

whose solution is of course

$$V^{(x,\tau)} = x\, \mathcal{E}\big((\widetilde{\psi}\mathbb{1}_{\rrbracket \tau, T \rrbracket}) \bullet S\big) = x\, \mathcal{E}\big((-\bar{a}\mathbb{1}_{\rrbracket \tau, T \rrbracket}) \bullet S\big),$$

and so (4.8) yields

$$\widetilde{\varphi}^{(x,\tau)} = V_-^{(x,\tau)} \widetilde{\psi} \mathbb{1}_{\rrbracket \tau, T \rrbracket} = -x\, \mathcal{E}\big((-\bar{a}\mathbb{1}_{\rrbracket \tau, T \rrbracket}) \bullet S\big)_- \bar{a} \mathbb{1}_{\rrbracket \tau, T \rrbracket}. \tag{6.5}$$

This recovers Lemma 3.7 from [14].

One major simplification in the unconstrained case is that we no longer need to distinguish between the cases $V_-(x, \widetilde{\varphi}) > 0$ and $V_-(x, \widetilde{\varphi}) < 0$ because there is only one opportunity process L. In terms of the discussion before Corollary 4.2, we no longer need to worry about jumps of $V(x, \widetilde{\varphi})$ across 0 since these do not affect the description of L. All we need is to be able to "restart $V(x, \widetilde{\varphi})$ when it jumps to 0", which is the important insight obtained by Černý and Kallsen [14]. The adjustment process $\widetilde{a}$ from [14] is moreover seen to be given by $\widetilde{a} = \bar{a} = -(\bar{c})^{-1}\bar{b} = -\widetilde{\psi}$, by comparing (6.5) to (3.12) in [14].

The above result highlights an important difference between our approach and that in [14]. We obtain our results by systematically using stochastic control ideas and in particular the martingale optimality principle (MOP). To illustrate this with an example, we see from the above that $\widetilde{a} = -\widetilde{\psi}$ is obtained as the minimiser of the function $\mathfrak{g}$, which means that we exploit the MOP by using that the drift of $J(\vartheta)$ must vanish for the optimal strategy. In contrast, Černý and Kallsen [14] obtain $\widetilde{a}$ by closely examining the structure of the optimal strategies $\widetilde{\varphi}^{(x,\tau)}$ for variable τ, and they prove its properties using the optimality of $\widetilde{\varphi}^{(x,\tau)}$ via martingale orthogonality conditions. They do not explicitly use dynamic programming and never mention the MOP.

The next proposition summarises the most important results for the unconstrained case $C \equiv \mathbb{R}^d$. We give no proof; this all follows directly by specialising our earlier results.

Proposition 6.1. *Suppose that S is in $\mathcal{H}^2_{\mathrm{loc}}(P)$. Then:*

1) *There exists an RCLL submartingale $L = (L_t)_{0 \le t \le T}$, called* opportunity process*, such that for each $x \in \mathbb{R}$ and $\tau \in \mathcal{S}_{0,T}$, the process*

$$J_t(\vartheta; x, \tau) = \big(x + \textstyle\int_\tau^t \vartheta_u dS_u\big)^2 L_t, \quad 0 \le t \le T$$

is a submartingale for every $\vartheta \in \overline{\Theta}$ with $\vartheta = 0$ on $[\![0,\tau]\!]$. Moreover, $J(\widetilde{\vartheta}; x, \tau)$ is a martingale for $\widetilde{\vartheta} \in \overline{\Theta}$ with $\widetilde{\vartheta} = 0$ on $[\![0,\tau]\!]$ if and only if $\widetilde{\vartheta} = \widetilde{\varphi}^{(x,\tau)}$ is optimal for (3.2). The process L is given explicitly as an RCLL version of

$$\bar{L}(t) := \operatorname{ess\,inf}\big\{E\big[|1 - \textstyle\int_t^T \varphi_u \, dS_u|^2 \big| \mathcal{F}_t\big] \;\big|\; \varphi \in \overline{\Theta} \text{ with } \varphi = 0 \text{ on } [\![0,t]\!]\big\},$$

$0 \le t \le T$.

2) *Suppose that L and L_- are both > 0 and that there exists a solution $\widetilde{\varphi}^{(1,\tau)}$ to (3.2) with $x = 1$ for any stopping time τ. Then the joint differential characteristics of (S, L) satisfy*

$$b^L = L_- \bar{b}^\top (\bar{c})^{-1} \bar{b} \tag{6.6}$$

and we have $V(1, \widetilde{\varphi}^{(1,\tau)}) = \mathcal{E}\big((-\bar{a}\mathbb{1}_{]\!]\tau,T]\!]}) \bullet S\big)$ with $\bar{a} = (\bar{c})^{-1}\bar{b}$. A sufficient condition for the assumptions in 2) is that there exists an EσMM Q for S with $\frac{dQ}{dP} \in L^2(P)$.

3) *Conversely, let ℓ be a semimartingale such that*

a) *ℓ and its left limit ℓ_- are $(0,1]$-valued and $\ell_T = 1$.*

b) *The joint differential characteristics of* (S, ℓ) *satisfy*

$$b^\ell = \ell_- \bar{b}^\top (\bar{c})^{-1} \bar{b}.$$

c) *For* $\bar{a} := (\bar{c})^{-1}\bar{b}$*, we have that*

$$\bar{\lambda}^{(\tau)} := \mathcal{E}\big((-\bar{a}\mathbb{1}_{\rrbracket\tau,T\rrbracket}) \bullet S\big)_- \bar{a}\mathbb{1}_{\rrbracket\tau,T\rrbracket}) \in \overline{\Theta}.$$

Then $\widetilde{\varphi}^{(1,\tau)} := -\bar{\lambda}^{(\tau)}$ *is the solution to* (3.1) *with* $x = 1$ *for each* $\tau \in \mathcal{S}_{0,T}$*, and* $L := \ell$ *is the opportunity process.*

Note that the equation (6.6) for the joint differential characteristics of (S, ℓ) is the same as (3.32) in [14]. Moreover, parts 2) and 3) of Proposition 6.1 essentially recover Theorem 3.25 of [14]; our result is actually even stronger since we do not need the assumption from [14] that the process $\mathcal{E}((-\bar{a}\mathbb{1}_{\rrbracket\tau,T\rrbracket}) \bullet S)\ell$ is of class (D) for each stopping time $\tau \in \mathcal{S}_{0,T}$.

The results of Černý and Kallsen [14] show (as repeated in part 2) of Proposition 6.1) that a sufficient condition for the existence of all optimal strategies $\widetilde{\varphi}^{(1,\tau)}$ for $\tau \in \mathcal{S}_{0,T}$ as well as for strict positivity of L and L_- is the existence of an EσMM Q for S with $\frac{dQ}{dP} \in L^2(P)$. Our next theorem sharpens this into a precise characterisation by giving *necessary and sufficient* conditions. This result is also one reason why we have introduced the notion of $(\mathcal{E}, Z^N)$-martingales in the precise form of Section IV.2.

Theorem 6.2. *For* $S \in \mathcal{H}^2_{\mathrm{loc}}(P)$*, the following are equivalent:*

1) The opportunity process L *and its left limit* L_- *are* $(0,1]$*-valued and there exists a solution* $\widetilde{\varphi}^{(1,\tau)}$ *to* (3.2) *with* $x = 1$ *for any stopping time* $\tau \in \mathcal{S}_{0,T}$.

2) There exist $N \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ *and* Z^N *such that* $(\mathcal{E}, Z^N)$ *with* $\mathcal{E} = \mathcal{E}(N)$ *is regular and square-integrable and* $S = S_0 + M - \langle M, N\rangle$ *is an* $\mathcal{E}$*-local martingale.*

Proof. The implication "2) $\Longrightarrow$ 1)" is easy. Indeed, the closedness in $L^2(P)$ of $G_T(\overline{\Theta(C)})$ obtained from Theorem 2.16 implies the existence of all the $\widetilde{\varphi}^{(1,\tau)}$ by taking $C = \mathbb{R}^d \mathbb{1}_{\rrbracket\tau,T\rrbracket}$, and strict positivity of L and L_- is from Lemma 3.4. We prove the converse implication "1) $\Longrightarrow$ 2)" in several steps.

1) Fix τ and use Lemma 3.5 to write $V(1, \widetilde{\varphi}^{(1,\tau)}) = \mathcal{E}(\widetilde{\psi}^{(1,\tau)} \bullet S) = \mathcal{E}((\widetilde{\psi}^{(1,\tau)}\mathbb{1}_{\rrbracket\tau,T\rrbracket}) \bullet S)$. As in Lemma 3.3, using that $L^+ = L^- = L$, consider the process $\widetilde{M}^{(1,\tau)} = V(1, \widetilde{\varphi}^{(1,\tau)})L$ and the square-integrable martingale $\mathbb{1}_{\rrbracket\tau,T\rrbracket} \bullet \widetilde{M}^{(1,\tau)} = \mathbb{1}_{\rrbracket\tau,T\rrbracket} \bullet (V(1, \widetilde{\varphi}^{(1,\tau)})L)$. Because $L_- > 0$, we can write $L = L_0\,\mathcal{E}(K')$. Moreover, Corollary 4.2 and its proof give that $\widetilde{\psi}^{(1,\tau)}$ coincides on the set $\rrbracket\tau,T\rrbracket \cap \{V_-(1, \widetilde{\varphi}^{(1,\tau)}) \neq 0\}$ with the minimiser $\widetilde{\psi}$ of

the function $\mathfrak{g}$, which is $\widetilde{\psi} = -\bar{a} = -(\bar{c})^{-1}\bar{b}$ by (6.4), so that $V(1, \widetilde{\varphi}^{(1,\tau)}) = \mathcal{E}((-\bar{a}\mathbb{1}_{]\!]\tau,T]\!]}) \bullet S)$. This implies

$$\begin{aligned}\widetilde{M}^{(1,\tau)} &= L^\tau + \mathbb{1}_{]\!]\tau,T]\!]} \bullet \big(V(1, \widetilde{\varphi}^{(1,\tau)}) L_0\, \mathcal{E}(K')\big) \\ &= L^\tau + \mathbb{1}_{]\!]\tau,T]\!]} \bullet \Big(\mathcal{E}\big((-\bar{a}\mathbb{1}_{]\!]\tau,T]\!]}) \bullet S\big) L_\tau\, \mathcal{E}\big(\mathbb{1}_{]\!]\tau,T]\!]} \bullet K'\big)\Big) \\ &= L^\tau \mathcal{E}\big(\mathbb{1}_{]\!]\tau,T]\!]} \bullet N\big) \end{aligned} \tag{6.7}$$

by Yor's formula, with $N := -\bar{a} \bullet S + K' - [\bar{a} \bullet S, K']$. Moreover, by Lemma 3.3 for $\vartheta := \pm\mathbb{1}_{]\!]\tau_n,\tau_{n+k}]\!]}$ for a localising sequence with $S^{\tau_m} \in \mathcal{H}^2(P)$ for all m, we obtain that the product of ${}^{\tau_n}S$ and $\widetilde{M}^{(1,\tau)}$ is for each n a local martingale (with $(\tau_{n+k})_{k\in\mathbb{N}}$ as localising sequence).

2) At the end of step 1), we have glossed over a point that we must settle now. While (6.7) is correct as it stands, the subsequent definition of N on all of $[\![0,T]\!]$ requires us to show that $\bar{a}$ is in $\mathcal{L}(S)$. To do that, we recall that $K' = \frac{1}{L_-} \bullet L$ (this is called the extended mean-variance tradeoff process in Definition 3.11 in [14]) and introduce the opportunity-neutral measure $P^* \approx P$ by $\frac{dP^*}{dP} := \frac{L_T}{E[L_0]\,\mathcal{E}(A^{K'})_T}$. Then Girsanov's theorem (see Lemma A.9 in [14]) gives as in the proof of Lemma 3.17 in [14] that $b^{S,P^*} = \frac{\bar{b}}{1+\Delta A^{K'}}$ and $[S]^{\mathbf{p},P^*} = \widetilde{c}^{S,P^*} \bullet B = \frac{\bar{c}}{1+\Delta A^{K'}} \bullet B$. Note that $A^{K'}$ is increasing because L is a submartingale, and Corollary 4.2 with (6.3) gives

$$A^{K'} = \tfrac{1}{L_-} \bullet A^L = \tfrac{b^L}{L_-} \bullet B = \big(-\tfrac{1}{L_-} \min_{\psi\in\mathbb{R}^d} \mathfrak{g}(\psi; L)\big) \bullet B = (\bar{b}^\top(\bar{c})^{-1}\bar{b}) \bullet B.$$

So we obtain from $\bar{a} = -(\bar{c})^{-1}\bar{b}$ and since $[S]^{\mathbf{p},P^*} - \langle M^{S,P^*}\rangle$ is nonnegative definite that

$$\begin{aligned}\textstyle\int |\bar{a}\, dA^{S,P^*}| + \int \bar{a}^\top d\langle M^{S,P^*}\rangle \bar{a} &= (|\bar{a}^\top b^{S,P^*}| + \bar{a}^\top \widetilde{c}^{M,P^*}\bar{a}) \bullet B \\ &\le 2\frac{\bar{b}^\top(\bar{c})^{-1}\bar{b}}{1+\Delta A^{K'}} \bullet B \le 2A^{K'},\end{aligned}$$

which shows that $\bar{a}$ is in both $\mathcal{L}(A^{S,P^*})$ and $\mathcal{L}^2_{\mathrm{loc}}(M^{S,P^*})$ and therefore in $\mathcal{L}(S)$. Hence N is well defined and a semimartingale. As in Section IV.2, define the stopping times $T_0 := 0$ and $T_{m+1} = \inf\{t > T_m \mid {}^{T_m}\mathcal{E}(N)_t = 0\} \wedge T$, and note that (T_m) increases to T stationarily.

3) Step 1) with $\tau = T_m$ implies that

$$\mathbb{1}_{]\!]T_m,T]\!]} \bullet \widetilde{M}^{(1,T_m)} = L_{T_m}\mathbb{1}_{]\!]T_m,T]\!]} \bullet \mathcal{E}(\mathbb{1}_{]\!]T_m,T]\!]} \bullet N)$$

is for each m a square-integrable martingale. By Remark 2.8, this implies that N is in $\mathcal{M}^2_{0,\mathrm{loc}}(P)$ because $L > 0$. Then step 1) also shows that $(\mathcal{E}, Z^N)$ with $\mathcal{E} = \mathcal{E}(N)$ and $Z^N = L$ is regular and square-integrable, since the product of L^{T_m} and ${}^{T_m}\mathcal{E}(N) = \mathcal{E}(\mathbb{1}_{]\!]T_m,T]\!]} \bullet N)$ is $\widetilde{M}^{(1,T_m)}$. Finally, step 1) with τ_n replaced by $\tau_n \wedge T_m$ yields for $n \to \infty$ that S is an $\mathcal{E}$-local martingale. This ends the proof. □

An alternative description of L and hence of the optimal strategies is via the BSDE (4.21) in Corollary 4.8. Combining (6.3) with the fact that $\mathfrak{h}^{\pm} = \mathfrak{g}^{\pm}$ in Section IV.4, we obtain that the BSDE system (4.21) (for $L^{\pm}$) collapses to the single BSDE (for L)

$$L = (L_- \bar{b}^\top (\bar{c})^{-1} \bar{b}) \bullet B + H^L \bullet S^c + W^L * (\mu^S - \nu^S) + N^L, \quad L_T = 1.$$

By also using (6.1), (6.2) and (4.17)–(4.19), we can rewrite the drift term (with respect to B) into a more explicit form and obtain

$$\begin{aligned} L &= H^L \bullet S^c + W^L * (\mu^S - \nu^S) + N^L \\ &\quad + \left\{ \left(b^S + c^S \frac{H^L}{L_-} + \int \frac{\Delta A^L + W^L(u) - \widehat{W^L}}{L_-} u F^S(du) \right)^\top \right. \\ &\quad \times \left(c^S + \int u u^\top \left(1 + \frac{\Delta A^L + W^L(u) - \widehat{W^L}}{L_-} \right) F^S(du) \right)^{-1} \\ &\quad \times \left. \left(b^S + c^S \frac{H^L}{L_-} + \int \frac{\Delta A^L + W^L(u) - \widehat{W^L}}{L_-} u F^S(du) \right) L_- \right\} \bullet B, \\ L_T &= 1. \end{aligned} \tag{6.8}$$

This is much simpler than the constrained case because we no longer have a coupled system of BSDEs (for $L^{\pm}$). Note that (6.8) has one more term than the otherwise identical equation (3.37) in [14]; it seems that Černý and Kallsen [14] have somewhere lost ΔA^L, as has also been noted by other authors.

IV.6.2 The continuous case

To the best of our knowledge, all results on the Markowitz problem under constraints in continuous-time models have been obtained when S is *continuous*. Before discussing individual papers, we therefore explain how our results simplify for continuous S.

First of all, Lemma 3.5 yields that $V(x, \widetilde{\varphi}^{(x,\tau)}) = x\,\mathcal{E}(\widetilde{\psi}^{(x,\tau)} \bullet S)$. So if (3.1) (when we start from $\tau = 0$) has a solution, the process $V(x, \widetilde{\varphi}^{(x,0)})$ has a *unique sign* on all of $[\![0, T]\!]$ because the stochastic exponential of a continuous process never hits 0. One can then show with some extra work that

$$\bar{\varphi}^{(x,\tau)} := x\,\mathcal{E}\big((\widetilde{\psi}^{(x,0)} \mathbb{1}_{]\!]\tau,T]\!]}) \bullet S\big) \widetilde{\psi}^{(x,0)} \mathbb{1}_{]\!]\tau,T]\!]} = \frac{x}{V_\tau(x, \widetilde{\varphi}^{(x,0)})} \widetilde{\varphi}^{(x,0)} \mathbb{1}_{]\!]\tau,T]\!]}$$

is optimal for (3.2) (when we start from τ); more precisely, this can be done if we have the existence of an optimal strategy $\widetilde{\varphi}^{(x,\tau)}$ for all (x, τ) or if the constraints correspondence C has convex closed cones as values. So if S is continuous, we basically do not need to study all the conditional problems; it is enough to understand and describe $\widetilde{\varphi}^{(x,0)}$.

In the local description in Section IV.4, we next see in (4.2) that $\mathfrak{g}^{2,\pm} \equiv 0$ when S has no jumps; so (4.3) gives $\mathfrak{g}^{\pm} = \mathfrak{g}^{1,\pm}$ and (4.1) shows that $\mathfrak{g}^{+}$ and $\mathfrak{g}^{-}$ only depend on ℓ^{+} and ℓ^{-}, respectively. This implies in turn that the two coupled equations in (4.5) in Theorem 4.1 *decouple*; and since we have already seen above that $V(x, \widetilde{\varphi})$ has a unique sign on $[\![0, T]\!]$, we need in fact only one of those two equations (depending on the sign of x).

To describe the optimal strategy $\widetilde{\varphi}^{(x,0)}$, we must find the minimiser $\widetilde{\psi}^{(x,0)}$ of $\mathfrak{g}^{+}$ or $\mathfrak{g}^{-}$ (depending on the sign of x). Because $\mathfrak{g}^{\pm}$ are simple quadratic functions of ψ, as the terms $\mathfrak{g}^{2,\pm}$ are absent, finding their minimisers over the constraint set K is straightforward in principle. But explicit (closed form) expressions can be expected only in special cases.

Conversely, Theorem 4.4 allows us to construct a solution $\widetilde{\varphi}^{(x,0)}$ to (3.1) from a solution to the BSDEs in (4.20). Those equations take the more explicit form

$$\ell^{\pm} = -\inf_{\psi \in K} \mathfrak{h}^{\pm}(\psi; S, \ell^{\pm}) \bullet B + H^{\ell^{\pm}} \bullet M + N^{\ell^{\pm}}, \quad \ell^{\pm}_T = 1 \tag{6.9}$$

with

$$\mathfrak{h}^{\pm}(\psi; S, \ell^{\pm}) = \ell^{\pm}_{-} \psi^\top c^S \psi \pm 2\ell^{\pm}_{-} \psi^\top b^S \pm 2\psi^\top c^S H^{\ell^{\pm}}.$$

In the unconstrained case $C \equiv K \equiv \mathbb{R}^d$, we can find the minimal value of $\mathfrak{h}^{\pm}$ explicitly by completing the square. Since we then also need not distinguish between ℓ^{+} and ℓ^{-}, as seen in Section IV.6.1, the BSDE (6.9) becomes (after doing the computations)

$$\begin{aligned} L &= H^L \bullet M + N^L + \left\{ \left(b^S + c^S \frac{H^L}{L_-} \right) (c^S)^{-1} \left(b^S + c^S \frac{H^L}{L_-} \right) L_- \right\} \bullet B, \\ L_T &= 1. \end{aligned} \tag{6.10}$$

This equation can also be found in Kohlmann and Tang [60], Mania and Tevzadze [68] or Bobrovnytska and Schweizer [11], among others. Of course, (6.10) can also be obtained as a special case of (6.8) by simply dropping there all the jump terms. Note that even if S is continuous, L need not be, due to the presence of the orthogonal martingale term N^L.

After these general remarks, let us now discuss and compare the most important results in the literature so far.

We start with Hu and Zhou [49], Labbé and Heunis [63] and Li, Zhou and Lim [66]. They all use for S a multidimensional Itô process model as in Example 2.15 of the form

$$dS_t = \operatorname{diag}(S_t)\big((\mu_t - r_t \mathbf{1})\, dt + \sigma_t\, dW_t\big) \tag{6.11}$$

with a vector drift process μ and a matrix volatility process σ. An important assumption is that $\dim S = \dim W$ and that σ is invertible (even uniformly elliptic); this means that the model without constraints is complete and implies that the projection Π^S on the predictable range of S is simply the identity. Finally, the constraints are given by closed convex cones K which are constant (i.e. do not depend on t or ω).

In [49], the approach is to first study a more general constrained stochastic linear-quadratic (LQ) control problem and then derive results for the Markowitz problem as a special case. One inherent disadvantage is that this usually provides less intuition and insight than a direct approach as in this chapter. At the more abstract level, [49] prove verification theorems; they show how solutions to certain BSDEs induce solutions to certain LQ control problems and also prove existence of solutions to their BSDEs under suitable conditions. In the context of the model (6.11), one key assumption is that the instantaneous *Sharpe ratio* process $\bar{\lambda} := \sigma^{-1}(\mu - r\mathbf{1}) = \sigma^\top(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})$ is uniformly *bounded*; this is exploited to prove solvability of the BSDEs by using results of Kobylanski [59]. Moreover, the arguments exploit (via the use of BSDE comparison theorems) that the opportunity processes $L^\pm$ are continuous since the filtration generated by the driving Brownian motion has no discontinuous martingales. Boundedness of $\bar{\lambda}$ also implies the existence of an EσMM Q for S with $\frac{dQ}{dP} \in L^2(P)$; in fact, one can take for Q the minimal martingale measure given by $dQ = \mathcal{E}(-\bar{\lambda} \bullet W)_T \, dP$. Theorem 2.16 then implies the closedness in $L^2(P)$ of $G_T(\overline{\Theta(K)})$ and hence the solvability of (3.1). For applications, one drawback of assuming $\bar{\lambda}$ bounded is that this restrictive condition is often hard to check or even not satisfied in specific (e.g. Markovian) models for S. Moreover, we could not find in [49] any explanation where the BSDEs come from so that the presentation seems to us not fully transparent. One simple illustration is that the authors of [49] also observe that one needs only one of the two BSDEs; but their explanation seems to miss that this is directly due to the continuity of S, as explained above before (6.9).

In [63], the final setting is even more special since the coefficients μ, r, σ in (6.11) are all deterministic functions. Labbé and Heunis [63] use convex duality to obtain existence and the structure of the solution to the Markowitz problem, by first solving a dual problem and then constructing from that the desired primal solution. More precisely, existence is proved for random coefficients and even (fixed) convex closed, but not necessarily conic, constraints if $\bar{\lambda} = \sigma^{-1}(\mu - r\mathbf{1})$ is bounded (as in [49]). However, the results on the *structure* of the optimal portfolio are obtained by first studying and solving the HJB equation for the dual problem, and this hinges crucially on the assumption of deterministic coefficients. It also needs closed convex cones for the constraints. From our perspective, the use of duality is in general not really necessary to obtain the structure of the solution to the primal problem. Duality is very often useful for proving the existence of a (primal) solution; but if that is achieved differently (or assumed), structural results about the solution can usually be derived directly in the primal setting, as we have done here.

Finally, one of the earliest papers on the Markowitz problem under constraints in a continuous-time setting is due to Li, Zhou and Lim [66]. The coefficients μ, r, σ there are deterministic functions, $\bar{\lambda} = \sigma^{-1}(\mu - r\mathbf{1})$ is again bounded, and constraints are given by $C \equiv K \equiv \mathbb{R}^d_+$ (no shortselling). The treatment in [66] combines LQ control with Markovian and PDE techniques; instead of working with BSDEs as in [49], the authors of [66] study the (primal) HJB equation associated to the Markowitz problem, construct for that a viscosity solution, and use a verification result to then derive the optimal strategy. A major step in their proof is to deal with a potential irregularity in the HJB equation (the set Γ_3 in [66], where $v(t, x) = 0$). From our general perspective, there are two comments. One is that a (well-hidden) assumption in [66] is that the vector $\mu - r\mathbf{1}$ is in $\mathbb{R}^d_+$ (since the coefficient B in the abstract problem (3.1) in [66] must lie in the positive orthant). By looking at our functions $\mathfrak{g}^{\pm} = \mathfrak{g}^{1,\pm}$ in (4.1) and using that $K \equiv \mathbb{R}^d_+$, we then directly obtain the minimisers $\widetilde{\psi}^+ = 0$, $\widetilde{\psi}^- = (\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1}) = (\sigma^\top)^{-1}\bar{\lambda}$, so that the optimal strategy is directly given. Secondly, the fact that $V(x, \widetilde{\varphi})$ has a unique sign implies that the potential irregularity in the HJB equation is actually not relevant since the optimiser will not go there; this explains why there is no genuine smoothness problem in [66].

While all the above papers consider models which are complete without constraints, there has also been some recent work going beyond such restrictive setups; we mention here Jin and Zhou [53] and Donnelly [34]. Both use duality techniques to prove the existence of a solution; [34] has an Itô process model with regime-switching coefficients and (deterministic and constant) convex constraints, while [53] studies no-shortselling constraints ($C \equiv K \equiv \mathbb{R}^d_+$) in an incomplete Itô process model. The latter paper also obtains the optimal strategy more explicitly for the special case of deterministic parameters μ, r, σ; this is possible because (like in [63]) the dual problem becomes much simpler under that condition. All in all, it seems fair to say that even for continuous S, our results on the *structure* of the optimal strategy in the Markowitz problem under constraints contain and substantially extend all the available literature so far.

The last statement needs an important clarification. We focus here on constraints on *strategies* and there in particular on the *structure of the optimiser* for the Markowitz problem. There have been quite a few papers on the Markowitz problem (usually in the form (2.2) of minimising the variance subject to a given mean for the final wealth) with the additional constraint of having a *nonnegative wealth* process. One of the earliest papers on this topic is due to Korn and Trautmann [61], and more recent contributions include Bielecki, Jin, Pliska and Zhou [7] and Xia [94]. In most cases, the discussion and solution goes as follows. If one has a good equivalent martingale measure Q, say, then nonnegative wealth $V(x, \vartheta) \geq 0$ as a process is equivalent to having nonnegative final wealth, $V_T(x, \vartheta) \geq 0$. If one also has a complete model, every final payoff is replicable and so it is enough to solve the static Markowitz problem over (nonnegative) final wealth only. This is done in [61] via duality and utility-based techniques and in [7] via Lagrange multipliers. The paper by Xia [94] is a little different; it actually reduces the problem of minimising $E[|y - V_T(x, \vartheta)|^2]$ for continuous S and $y > x$ by observing (and proving) that it is optimal to first minimise the expected squared shortfall $E[|(y - V_T(x, \vartheta))^+|^2]$ and then stop the corresponding wealth process as soon as it hits y. But in all these cases, a nonnegative wealth constraint is substantially easier to deal with than constraints imposed on strategies.

Chapter V

Time-consistent mean-variance portfolio selection

V.1 Introduction

In his seminal paper "Portfolio selection" [69], Harry Markowitz gave to the common wisdom that investors try to maximise return and minimise risk a quantitative description by saying that the return should be measured by the expectation and the risk by the variance. In a one period financial market, *mean-variance portfolio selection* then simply consists of finding the self-financing portfolio whose one-period terminal wealth has maximal mean and minimal variance. Since the mean-variance criterion is quadratic with respect to the strategy, we can calculate the solution, the so-called *mean-variance efficient strategy*, directly and explicitly. Apart from the appealing and immediate interpretation of the optimisation criterion this probably explains its popularity.

Although one can obtain explicit formulas in one period, a multiperiod or continuous-time treatment is considerably more delicate; this has already been observed by Mossin in [74]. The reason is the well-known fact that the mean-variance criterion does not satisfy Bellman's optimality principle.

One way to deal with this issue is to treat mean-variance portfolio selection as in the *Markowitz problem* considered by Richardson [81], Schweizer [86] and Li and Ng in [65]. It consists of simply plugging in the multiperiod or continuous-time terminal wealth into the one period criterion and to maximise that with respect to the strategy over the entire time interval. Although this formulation fails to produce a time-consistent solution in the sense that it is optimal for the conditional criterion at a later time, this is nevertheless a common way to avoid dealing with the time inconsistency of the mean-variance criterion used in the literature. There it is sometimes referred to as mean-variance portfolio selection under precommitment, as the investor commits to follow the strategy which is optimal at time zero even though it is not (conditionally) optimal later on.

In this chapter, we approach the time inconsistency of the mean-variance criterion in a different way. We try to find a solution which is in some reasonable way optimal for the conditional mean-variance criterion and time-consistent in the sense that if it is optimal at time zero, it is also optimal on any remaining time interval. In a Markovian framework, such a time-consistent formulation has been introduced by Basak and Chabakauri in [6]. However, to find a time-consistent formulation in general is an open problem as pointed out by Schweizer at the end of the survey article [89]. As the failure of Bellman's optimality principle indicates, we have to use a different notion of optimality for the dynamic criterion than the classical one used in dynamic programming. As in [6], we follow Robert Strotz who suggested in [91] (for a different time-inconsistent deterministic optimisation problem) to maximise not over all possible future strategies, but only those one is actually going to follow. In discrete time, this leads to determining the optimal strategy by a backward recursion starting from the terminal date. For a continuous-time formulation one has to combine this *recursive approach to time inconsistency* with a limit argument. In a Markovian framework, for optimal consumption problems with non-exponential discounting this has recently been studied by Ekeland and Lazrak in [36] and [35] and Ekeland and Pirvu in [37] and [38] and for mean-variance portfolio selection problems by Basak and Chabakauri [6] and Björk, Murgoci and Zhou [10]. These authors give the definition of the time-consistent solution via a backward recursion the interpretation of a Nash subgame perfect equilibrium strategy for an interpersonal game. Building on these specific cases, Björk and Murgoci developed in [9] a "general theory of Markovian time inconsistent stochastic control problems" for various forms of time inconsistency in a Markovian setting. In all these problems one exploits that the underlying Markovian structure turns all quantities of interest into deterministic functions. Then recursive optimality can be characterised by a system of partial differential equations (PDEs), so-called extended Hamilton–Jacobi–Bellman equations, and one can provide verification theorems which allow to deduce that if one has a smooth solution to the PDE, this gives the solution to the optimal control problem.

Although it is known how to formulate and handle time-inconsistent optimal control problems in a Markovian framework, it is still an open question how to do this in a more general setting and how to apply martingale techniques to these kind of problems (see for example page 35 in [8]). For the problem of mean-variance portfolio selection, we answer these open questions in this chapter. In discrete time, obtaining the time-consistent solution by recursive optimisation is straightforward. To find the natural extension of this formulation to continuous time, we introduce a local notion of optimality called *local mean-variance efficiency*; this is a first main result. In continuous time, the definition of local mean-variance efficiency is of course inspired by the concept of continuous-time local risk minimisation introduced by Schweizer in [85]. As we shall see, our formulation in discrete as well as in continuous time embeds time-consistent mean-variance portfolio selection in a natural way into the already existing quadratic optimisation problems in mathematical finance, i.e. the Markowitz problem, mean-variance hedging, and local risk minimisation; see [87] and [89]. Moreover, we provide an alternative characterisation of the optimal strategy in terms of the structure condition and the Föllmer–Schweizer decomposition of the mean-variance tradeoff process. This is a second main result and gives necessary and sufficient conditions for the existence of a solution. The link to the Föllmer–Schweizer decomposition allows us to exploit known results and to give a recipe to obtain the solution in concrete models. Since the ingredients for this recipe can be obtained directly and explicitly from the canonical decomposition of the price process, this can be seen as the analog to the explicit solution in the one-period case. Besides this, we obtain an intuitive interpretation of the optimal strategy. On the one hand the investor maximises the conditional mean-variance criterion in a myopic way one step ahead. This generates a risk represented by the mean-variance tradeoff process which he then minimises by local risk minimisation on the other hand. Using the alternative characterisation of the optimal strategy allows us to justify the continuous-time formulation by showing that it coincides with the continuous-time limit of the discrete-time formulation. This underlines that our reasoning in discrete time, where the solution is determined by a backward recursion, is consistent with the way of defining optimality in continuous time and is our third main result. On the technical side, the link to the Föllmer–Schweizer decomposition exploit and extend known results.

Recently Cui et al. proposed in [19] an alternative way to deal with the time inconsistency of the mean-variance criterion. Relaxing the self-financing condition by allowing the withdrawal of money out of the market, they obtain a strategy which dominates the solution for the Markowitz problem in the sense that while both strategies achieve the same mean-variance pair for the terminal wealth their optimal strategy enables the investor to receive a free cash flow stream during the investment process. Compared to our study their reasoning and techniques are different. In particular, their solution is not time-consistent in our sense.

The remainder of the chapter is organised as follows. In the next section we explain the basic problem and the issue of time inconsistency of the mean-variance criterion and introduce the required notation for this. To establish the time-consistent formulation, we start in Section V.3 in discrete time and then find the natural extension of that to continuous time in Section V.4. The convergence of the solutions obtained in discretisations of a continuous-time model to the solution in continuous time is shown in the last section.

V.2 Formulation of the problem and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of completeness and right-continuity, where $T \in (0, \infty)$ is a fixed and finite time horizon. For all unexplained notation concerning stochastic integration we refer to the book of Dellacherie and Meyer [33]. Our presentation of the basic problem here builds upon that in Basak and Chabakauri, [6], Björk, Murgoci and Zhou [10] and Schweizer [89].

We consider a *financial market* consisting of one riskless asset whose price is 1 and d risky assets described by an $\mathbb{R}^d$-valued semimartingale S. As set of *trading strategies* we choose $\Theta := \Theta_S := \{\vartheta \in L(S) \mid \int \vartheta dS \in \mathcal{H}^2(P)\}$ where $L(S)$ is the space of all $\mathbb{R}^d$-valued, S-integrable, predictable processes and $\mathcal{H}^2(P)$ the space of all square-integrable semimartingales, i.e. special semimartingales X with canonical decomposition $X = X_0 + M^X + A^X$ such that

$$\|X\|_{\mathcal{H}^2(P)} := \|X_0\|_{L^2(P)} + \big\| \big([M^X, M^X]_T\big)^{\frac{1}{2}} \big\|_{L^2(P)} + \big\| \int_0^T |dA_s^X| \big\|_{L^2(P)} < +\infty.$$

The *wealth* generated by using the self-financing trading strategy $\vartheta \in \Theta$ up to time $t \in [0, T]$ and starting from initial capital $x \in \mathbb{R}$ is given by

$$V_t(x, \vartheta) := x + \int_0^t \vartheta_u dS_u =: x + \vartheta \bullet S_t.$$

Note that we use the notation above also for the stochastic integral in discrete time. Since we work with Θ_S, we can always find *representative square-integrable portfolios* for the financial market (S, Θ_S) as explained in the appendix. These are portfolios $\varphi^i \in \Theta$ for $i = 1, \ldots, d$ such that the financial market $(\widetilde{S}, \Theta_{\widetilde{S}})$ with $\widetilde{S}^i := \varphi^i \bullet S$ for $i = 1, \ldots, d$ satisfies $\widetilde{S} \in \mathcal{H}^2(P)$ and which are representative in the sense that $(\widetilde{S}, \Theta_{\widetilde{S}})$ generates the same wealth processes as (S, Θ_S), i.e. $\Theta_S \bullet S = \Theta_{\widetilde{S}} \bullet \widetilde{S}$. We can and do therefore assume without loss of generality that S is in $\mathcal{H}^2(P)$ and hence special with canonical decomposition $S = S_0 + M + A$, where M is an $\mathbb{R}^d$-valued square-integrable martingale null at zero, i.e. $M \in \mathcal{M}_0^2(P)$, and A is an $\mathbb{R}^d$-valued predictable RCLL process null at zero with square-integrable variation. Besides simplifying the presentation this allows to refer directly to the standard literature on quadratic optimisation in mathematical finance which usually assumes (local) square-integrability of S. Conversely, this change of parameterisation of the financial market can be used to generalise local risk minimisation and quadratic hedging to the case where S is a general semimartingale and not necessarily locally square-integrable; this will be explained in more detail in future work.

In the one-period case, where $T = 1$, $\vartheta \bullet S_1 = \vartheta_1^\top (S_1 - S_0) =: \vartheta_1^\top \Delta S_1$ and ϑ_1 is an $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random vector, *mean-variance portfolio selection (MVPS)* with risk aversion $\gamma > 0$ can be formulated as the problem to

$$\text{maximise } E[x + \vartheta_1^\top \Delta S_1] - \frac{\gamma}{2}\operatorname{Var}[x + \vartheta_1^\top \Delta S_1] \text{ over all } \mathcal{F}_0\text{-measurable } \vartheta_1. \tag{2.1}$$

The solution, the so-called *mean-variance efficient strategy*, is then

$$\widetilde{\vartheta}_1 := \frac{1}{\gamma}\operatorname{Cov}[\Delta S_1|\mathcal{F}_0]^{-1}E[\Delta S_1|\mathcal{F}_0] =: \widehat{\vartheta}_1 \tag{2.2}$$

which, as already explained in the introduction, is given by an explicit formula in terms of the risk aversion and the conditional mean and variance of the stock price changes. Note that $\operatorname{Cov}[\Delta S_1|\mathcal{F}_0]^{-1}$ denotes the Moore-Penrose pseudoinverse (see [1]) and therefore the solution exists if and only if $E[\Delta S_1|\mathcal{F}_0]$ is in the range of $\operatorname{Cov}[\Delta S_1|\mathcal{F}_0]$.

Having obtained the formulation and the explicit form of the solution in one period, we ask ourselves how the two extend to multiperiod or continuous time. An immediate extension of the formulation is simply to plug in the multiperiod or continuous-time terminal wealth into the one-period criterion. This corresponds to considering MVPS as in the classical *Markowitz problem* which is to

$$\text{maximise } E[V_T(x,\vartheta)] - \frac{\gamma}{2}\operatorname{Var}[V_T(x,\vartheta)] \text{ over all } \vartheta \in \Theta. \tag{2.3}$$

In this setup, MVPS is a *static* optimisation problem as one determines the optimal strategy $\widetilde{\vartheta}$ with respect to the criterion evaluated at time 0. This typically leads to a characterisation of the optimal strategy $\widetilde{\vartheta}$ via its terminal gains $\widetilde{\vartheta} \bullet S_T = \int_0^T \widetilde{\vartheta}_u dS_u$. As this means that we determine a predictable process $\widetilde{\vartheta}$ on $[0,T]$ implicitly by the terminal value of its stochastic integral $\int_0^T \widetilde{\vartheta}_u dS_u$, the question is then how to obtain a more explicit dynamic description of $\widetilde{\vartheta}$ on $[0,T]$. A natural idea to do this is to use instead of the single static the family of corresponding *dynamic* formulations of (2.3). For this, one would consider for any $t \in [0,T]$ to

$$\text{maximise } U_t(\vartheta) := E[V_T(x,\vartheta)|\mathcal{F}_t] - \frac{\gamma}{2}\operatorname{Var}[V_T(x,\vartheta)|\mathcal{F}_t] \text{ over all } \vartheta \in \Theta_t(\psi), \tag{2.4}$$

where $\Theta_t(\psi) = \{\vartheta \in \Theta \mid \vartheta \mathbb{1}_{[\![0,t]\!]} = \psi \mathbb{1}_{[\![0,t]\!]}\}$ denotes all strategies $\vartheta \in \Theta$ that agree up to time t with a given $\psi \in \Theta$. Then one uses the optimal strategy $\widetilde{\vartheta}$ for (2.3) on $[0,t]$ and determines the optimal strategy on $(t,T]$ by maximising (2.4) over $\Theta_t(\widetilde{\vartheta})$. However, it is well known that this produces a strategy which is different from $\widetilde{\vartheta}$ on $(t,T]$ which basically means that the formulation (2.4) is time inconsistent in the sense that it does not satisfy *Bellman's optimality principle*. This time inconsistency leads us to the basic question we study in this chapter, namely how to obtain a *time-consistent* formulation of MVPS, i.e. a dynamic formulation that gives a solution which is in some reasonable sense optimal for the dynamic criterion and time consistent.

The reason for the time inconsistency of the formulation (2.4) is the conditional variance term. Due to the total variance formula

$$\begin{aligned}\mathrm{Var}[V_T(x,\vartheta)|\mathcal{F}_t] = E\big[\,\mathrm{Var}[V_T(x,\vartheta)|\mathcal{F}_{t+h}]\big|\mathcal{F}_t\big] \\ + \mathrm{Var}\left[E\big[\textstyle\int_{t+h}^T \vartheta dS\big|\mathcal{F}_{t+h}\big] + V_{t+h}(x,\vartheta)\Big|\mathcal{F}_t\right],\end{aligned}$$

we see that the objective function at time t is given by the conditional expectation of the objective function at time $t+h$ and some adjustment term, i.e.

$$U_t(\vartheta) = E\big[U_{t+h}(\vartheta)\big|\mathcal{F}_t\big] - \frac{\gamma}{2}\,\mathrm{Var}\left[E\big[\textstyle\int_{t+h}^T \vartheta dS\big|\mathcal{F}_{t+h}\big] + V_{t+h}(x,\vartheta)\Big|\mathcal{F}_t\right] \quad (2.5)$$

for all $\vartheta \in \Theta$. As this adjustment term does not only depend on the strategy via its behaviour on $(t, t+h]$ but also on $(t+h, T]$, it cannot be interpreted as a running cost term, and therefore the objective function is not of the "standard form" which is crucial for the dynamic programming approach to work; see for instance [9], or [43] for a textbook account. The economic explanation for the time-inconsistent behaviour of the investor is as follows. At time t, the investor uses the strategy on $(t+h, T]$ not only to maximise the time $(t+h)$ objective function $U_{t+h}(\vartheta)$, but also to minimise the second term. This means that he tries to hedge some of the risk coming from the strategy used on $(t, t+h]$. At time $t+h$, the outcome of the trading on $(t, t+h]$ is already known and there is no need to hedge against it. Therefore the investor at time $t+h$ chooses the trading strategy on $(t+h, T]$ only to maximise $U_{t+h}(\vartheta)$, and so his objective and hence his choice will be in general different from that at time t.

An alternative explanation for the failure of the time consistency of the dynamic formulation (2.4) is of course that already the underlying mean-variance preferences are time inconsistent due to their non-monotonicity; see for example [67].

To handle the inconsistency of the criterion, we follow, as already explained in the introduction, the recursive approach to time inconsistency proposed by Strotz in [91] for the deterministic optimal consumption problem with non-exponential discounting. This suggests to choose the best strategy not among all available strategies, but among those one is actually going to follow. For the discrete-time case, this is formulated straightforwardly by recursively optimising backward starting from T, as we illustrate in the next section.

V.3 Discrete time

In this section, we develop a time-consistent formulation for the mean-variance portfolio selection problem in discrete time and derive the general structure of the solution. As this mainly serves for the motivation of the continuous-time case, we restrict our presentation here for simplicity to the one dimensional case $d = 1$.

Let $T \in \mathbb{N}$ and assume that trading only takes place at fixed times $k = 0, 1, \ldots, T$, where we choose at time k the number of shares ϑ_{k+1} to be held over the time period $(k, k+1]$. In this setting, we obtain an optimal strategy by recursively optimising starting from T, which is equivalent to optimality with respect to local perturbations. This is then a time-consistent solution to MVPS in the recursively optimal sense introduced by Strotz [91]. Due to the local nature of optimisation we call this notion of optimality local mean-variance efficiency, which is formulated as follows.

Definition 3.1. Let $\psi \in \Theta$ be a strategy and $k \in \{1, \ldots, T\}$. A *local perturbation of* ψ *at time* k is any strategy $\vartheta \in \Theta$ with $\vartheta_j = \psi_j$ for all $j \neq k$. We call a trading strategy $\widehat{\vartheta} \in \Theta$ *locally mean-variance efficient (LMVE)* if

$$U_{k-1}(\widehat{\vartheta}) \geq U_{k-1}(\vartheta) \qquad \text{P-a.s.} \tag{3.1}$$

for all $k = 1, \ldots, T$ and any local perturbation $\vartheta \in \Theta$ of $\widehat{\vartheta}$ at time k or, equivalently,

$$U_{k-1}(\widehat{\vartheta}) \geq U_{k-1}(\widehat{\vartheta} + \delta \mathbb{1}_{\{k\}}) \qquad \text{P-a.s.} \tag{3.2}$$

for all $k = 1, \ldots, T$ and any $\delta \in \Theta$.

Note that since $U_t(\vartheta) = V_t(x, \vartheta) + U_t(\mathbb{1}_{]\!]t,T]\!]}\vartheta) =: V_t(x, \vartheta) + \overline{U}_t(\vartheta)$, the structure of mean-variance preferences implies that conditions (3.1) and (3.2) do not depend for fixed k on the strategy used on $\{0, \ldots, k-1\}$. This allows us to derive the following recursive formula for the LMVE strategy $\widehat{\vartheta}$, which underlines the time-consistency of the solution. This formula already appeared in a Markovian framework in Proposition 5 in [6] and in a slightly different semimartingale setting in an unpublished Master thesis by Sigrid Källblad.

Lemma 3.2. *A strategy* $\widehat{\vartheta} \in \Theta$ *is LMVE if and only if it satisfies*

$$\widehat{\vartheta}_k = \frac{1}{\gamma} \frac{E[\Delta S_k | \mathcal{F}_{k-1}]}{\operatorname{Var}[\Delta S_k | \mathcal{F}_{k-1}]} - \frac{\operatorname{Cov}\left[\Delta S_k, \sum_{i=k+1}^{T} \widehat{\vartheta}_i \Delta S_i | \mathcal{F}_{k-1}\right]}{\operatorname{Var}[\Delta S_k | \mathcal{F}_{k-1}]} \tag{3.3}$$

for $k = 1, \ldots, T$.

Proof. Plugging $\widehat{\vartheta}$ and $\widehat{\vartheta} + \delta \mathbb{1}_{\{k\}}$ into (2.5), we obtain that (3.2) is equivalent to

$$-\delta_k \left(E[\Delta S_k | \mathcal{F}_{k-1}] - \gamma \operatorname{Cov}\left[\Delta S_k, \sum_{i=k}^{T} \widehat{\vartheta}_i \Delta S_i \middle| \mathcal{F}_{k-1}\right]\right) + \frac{\gamma}{2} \operatorname{Var}[\delta_k \Delta S_k | \mathcal{F}_{k-1}] \geq 0 \tag{3.4}$$

for all $k = 1, \dots, T$ and any $\delta \in \Theta$. Since $\mathrm{Var}\,[\delta_k \Delta S_k | \mathcal{F}_{k-1}] \geq 0$ for all $k = 1, \dots, T$ and any $\delta \in \Theta$, it follows immediately that $\widehat{\vartheta}$ satisfies (3.2) if (3.3) holds. For the converse, we argue by backward induction; so assume that (3.3) holds for $j = k+1, \dots, T$. Because the conditional covariance term in (3.4) vanishes on $D := \{\mathrm{Var}[\Delta S_k | \mathcal{F}_{k-1}] = 0\}$, we set $\varepsilon = E[\Delta S_k | \mathcal{F}_{k-1}] \mathbb{1}_D$ and

$$\varphi = \left(\frac{1}{\gamma} \frac{E[\Delta S_k | \mathcal{F}_{k-1}]}{\mathrm{Var}\left[\Delta S_k \middle| \mathcal{F}_{k-1}\right]} - \frac{\mathrm{Cov}\left[\Delta S_k, \sum_{i=k+1}^T \widehat{\vartheta}_i \Delta S_i \middle| \mathcal{F}_{k-1}\right]}{\mathrm{Var}\left[\Delta S_k \middle| \mathcal{F}_{k-1}\right]} - \widehat{\vartheta}_k \right) \mathbb{1}_{D^c}.$$

Then choosing $\delta = \varepsilon \mathbb{1}_{D_n} \mathbb{1}_{\{k\}} \in \Theta$ with $D_n = \{E[(\varepsilon \Delta S_k)^2 | \mathcal{F}_{k-1}] \leq n\}$ and $\delta = \varphi \mathbb{1}_{D_n} \mathbb{1}_{\{k\}} \in \Theta$ with $D_n \{E[(\varphi \Delta S_k)^2 | \mathcal{F}_{k-1}] \leq n\}$ for each $n \in \mathbb{N}$ implies that $\varepsilon = 0$ and $\varphi = 0$, as we could otherwise derive a contradiction to (3.4). By the Cauchy–Schwarz inequality and since $\varepsilon = 0$, the right-hand side of (3.3) is always well defined by setting $\frac{0}{0} = 0$, and equal to $\widehat{\vartheta}$ since $\varphi = 0$. This completes the proof. □

To simplify (3.3), we use the canonical decomposition of $S = S_0 + M + A$ into a martingale M and a predictable process A, which is in discrete time given by the *Doob decomposition*, i.e. $M_0 := 0 =: A_0$, $\Delta A_k = E[\Delta S_k | \mathcal{F}_{k-1}]$ and $\Delta M_k = \Delta S_k - E[\Delta S_k | \mathcal{F}_{k-1}]$ for $k = 1, \dots, T$. Then (3.3) can be written as

$$\widehat{\vartheta}_k = \frac{1}{\gamma} \frac{\Delta A_k}{E\left[(\Delta M_k)^2 | \mathcal{F}_{k-1}\right]} - \frac{\mathrm{Cov}\left[\Delta M_k, \sum_{i=k+1}^T \widehat{\vartheta}_i \Delta A_i | \mathcal{F}_{k-1}\right]}{E\left[(\Delta M_k)^2 | \mathcal{F}_{k-1}\right]} \tag{3.5}$$

for $k = 1, \dots, T$. From this it follows by the Cauchy–Schwarz inequality that the existence of a LMVE strategy $\widehat{\vartheta}$ implies that S satisfies the *structure condition (SC)*, i.e. there exists a predictable process λ given by

$$\lambda_k := \frac{\Delta A_k}{E\left[(\Delta M_k)^2 | \mathcal{F}_{k-1}\right]} = \frac{E[\Delta S_k | \mathcal{F}_{k-1}]}{\mathrm{Var}\left[\Delta S_k | \mathcal{F}_{k-1}\right]} \qquad \text{for } k = 1, \dots, T$$

such that the *mean-variance tradeoff (MVT) process*

$$K_k := \sum_{i=1}^k \frac{\left(E[\Delta S_i | \mathcal{F}_{i-1}]\right)^2}{\mathrm{Var}\left[\Delta S_i | \mathcal{F}_{i-1}\right]} = \sum_{i=1}^k \lambda_i^2 E\left[(\Delta M_i)^2 | \mathcal{F}_{i-1}\right] = \sum_{i=1}^k \lambda_i \Delta A_i$$

for $k = 0, \dots, T$ is finite-valued. This is not surprising, as these quantities also appear naturally in other quadratic optimisation problems in mathematical finance; see [87]. For each $\vartheta \in \Theta$, we define the process of *expected future gains* $Z(\vartheta)$ and the square integrable martingale $Y(\vartheta)$ of its canonical decomposition by

$$\begin{aligned} Z_k(\vartheta) :&= E\left[\sum_{i=k+1}^{T} \vartheta_i \Delta S_i \middle| \mathcal{F}_k\right] = E\left[\sum_{i=k+1}^{T} \vartheta_i \Delta A_i \middle| \mathcal{F}_k\right] \\ &= E\left[\sum_{i=1}^{T} \vartheta_i \Delta A_i \middle| \mathcal{F}_k\right] - \sum_{i=1}^{k} \vartheta_i \Delta A_i \\ &=: Y_k(\vartheta) - \sum_{i=1}^{k} \vartheta_i \Delta A_i \end{aligned}$$

for $k = 0, 1, \dots, T$. Note that for the LMVE strategy $\widehat{\vartheta}$, the process $Z(\widehat{\vartheta})$ has already been introduced in a discrete-time semimartingale setting in Sigrid Källblad's Master thesis and in the Markovian framework in [6] in discrete and continuous time, where it is a function $Z_t(\widehat{\vartheta}) = f(W_t, S_t, X_t, t)$ of time t and the underlying state variables, i.e. current wealth W_t, stock S_t and hidden Markov factor X_t. Using the *Galtchouk–Kunita–Watanabe (GKW) decomposition*

$$\sum_{i=1}^{T} \vartheta_i \Delta A_i = Y_0(\vartheta) + \sum_{i=1}^{T} \xi_i(\vartheta) \Delta M_i + L_T(\vartheta)$$

of $Y(\vartheta)$ with a square-integrable martingale $L(\vartheta)$ strongly orthogonal to M, we can rewrite $Z(\vartheta)$ as

$$Z_k(\vartheta) = Y_k(\vartheta) - \sum_{i=1}^{k} \vartheta_i \Delta A_i = Y_0(\vartheta) + \sum_{i=1}^{k} \xi_i(\vartheta) \Delta M_i + L_k(\vartheta) - \sum_{i=1}^{k} \vartheta_i \Delta A_i \tag{3.6}$$

for $k = 0, 1, \dots, T$. Inserting the last expression into (3.5), we can reformulate Lemma 3.2 by combining the above as follows.

Lemma 3.3. *The LMVE strategy $\widehat{\vartheta}$ exists if and only if we have both*

1) *S satisfies (SC) with $\lambda \in L^2(M)$, i.e. $K_T \in L^1(P)$.*

2) *There exists $\widehat{\psi} \in \Theta$ such that*

$$\widehat{\psi} = \frac{1}{\gamma}\lambda - \xi(\widehat{\psi}), \tag{3.7}$$

where $\xi(\widehat{\psi})$ is the integrand in the GKW decomposition of $\sum_{i=1}^{T} \widehat{\psi}_i \Delta A_i$.

In that case, $\widehat{\vartheta} = \widehat{\psi}$.

Proof. By Lemma 3.2 the existence of a LMVE strategy $\widehat{\vartheta}$ and a strategy satisfying (3.5) are equivalent. As already explained, (3.5) implies by the Cauchy–Schwarz inequality that S satisfies (SC). Since we obtain

$$\begin{aligned}\mathrm{Cov}\left[\Delta M_k, \textstyle\sum_{i=k+1}^T \widehat{\vartheta}_i \Delta A_i \middle| \mathcal{F}_{k-1}\right] &= \mathrm{Cov}\left[\Delta M_k, Z_k(\widehat{\vartheta}) \middle| \mathcal{F}_{k-1}\right] \\ &= \xi_k(\widehat{\vartheta}) E\left[(\Delta M_k)^2 | \mathcal{F}_{k-1}\right] \quad (3.8)\end{aligned}$$

by simply plugging into (3.5) the definition of $Z(\widehat{\vartheta})$ and (3.6), it follows that $\widehat{\vartheta}$ satisfies (3.7) and, conversely, that each strategy $\widehat{\psi} \in \Theta_S$ satisfying (3.7) is LMVE. Moreover, since $\widehat{\vartheta} \in \Theta = L^2(M) \cap L^2(A)$, we have that $Y_T(\widehat{\vartheta}) = \sum_{i=1}^T \widehat{\vartheta}_i \Delta A_i \in L^2(P)$ and therefore that $\xi(\widehat{\vartheta}) \in L^2(M)$ by construction. Rewriting (3.7), this implies that $\lambda = \gamma\widehat{\vartheta} + \xi(\widehat{\vartheta})$ is in $L^2(M)$ and $K_T \in L^1(P)$, which completes the proof. □

Integrating both sides of (3.7) with $\widehat{\psi} = \widehat{\vartheta}$ with respect to M and plugging in the GKW decomposition then gives

$$\begin{aligned}\sum_{i=1}^T \widehat{\vartheta}_i \Delta M_i &= \frac{1}{\gamma} \sum_{i=1}^T \lambda_i \Delta M_i - \sum_{i=1}^T \xi_i(\widehat{\vartheta}) \Delta M_i \\ &= \frac{1}{\gamma} \sum_{i=1}^T \lambda_i \Delta M_i + Y_0(\widehat{\vartheta}) + L_T(\widehat{\vartheta}) - \sum_{i=1}^T \widehat{\vartheta}_i \Delta A_i.\end{aligned}$$

After rearranging terms and adding $\frac{1}{\gamma} K_T = \frac{1}{\gamma} \sum_{i=1}^T \lambda_i \Delta A_i$ on both sides we arrive at

$$\frac{1}{\gamma} K_T = Y_0(\widehat{\vartheta}) + \sum_{i=1}^T \left(\frac{1}{\gamma}\lambda_i - \widehat{\vartheta}_i\right) \Delta M_i + \sum_{i=1}^T \left(\frac{1}{\gamma}\lambda_i - \widehat{\vartheta}_i\right) \Delta A_i + L_T(\widehat{\vartheta}), \quad (3.9)$$

which means that the terminal value of the MVT process K_T admits a decomposition

$$K_T = \widehat{K}_0 + \sum_{i=1}^T \widehat{\xi}_i \Delta S_i + \widehat{L}_T \quad (3.10)$$

into a square-integrable $\mathcal{F}_0$-measurable random variable $\widehat{K}_0$, the terminal value $\sum_{i=1}^T \widehat{\xi}_i \Delta S_i$ of a stochastic integral with respect to the price process, and the terminal value of a square-integrable martingale $\widehat{L}$ strongly orthogonal to M. If the integrand $\widehat{\xi}$ is in Θ and one replaces the left-hand side by any $H \in L^2(\Omega, \mathcal{F}, P)$, a decomposition of the form

$$H = \widehat{H}_0 + \sum_{i=1}^T \widehat{\xi}_i^H \Delta S_i + \widehat{L}_T^H$$

is called the *Föllmer–Schweizer (FS) decomposition* of H, and the integrand $\widehat{\xi}^H$ yields the so-called *locally risk minimising strategy* for the contingent claim H; see e.g. [87] and [88]. However, it turns out that $\widehat{\xi} = \lambda - \gamma\widehat{\vartheta}$ is in general not in Θ and therefore (3.10) does not necessarily coincide with the FS decomposition of K_T. But nevertheless, (3.10) gives a nice explanation what the investor is doing in order to invest optimally. On the one hand, he is optimising the conditional mean-variance criterion for the wealth of the next period in a myopic way one step ahead by choosing $\frac{1}{\gamma}\lambda_k = \frac{1}{\gamma}\frac{E[\Delta S_k|\mathcal{F}_{k-1}]}{\mathrm{Var}[\Delta S_k|\mathcal{F}_{k-1}]}$ for $k = 1, \dots, T$; see (2.1) and (2.2). In the multiperiod setting, this generates a risk given by $\frac{1}{\gamma}K_T$. This risk is then minimised in the sense of local risk minimisation by the investor on the other hand which leads to the intertemporal hedging demand $\frac{1}{\gamma}\widehat{\xi} = \xi(\widehat{\vartheta})$ in the solution. Besides this interpretation we also obtain an alternative, in some sense global, characterisation of the LMVE strategy in terms of the structure condition and the MVT process, which is summarised in the next lemma.

Lemma 3.4. *There exists a LMVE strategy $\widehat{\vartheta}$ if and only if S satisfies (SC) and (the terminal value of) the MVT process K_T is in $L^1(P)$ and can be written as*

$$K_T = \widehat{K}_0 + \sum_{i=1}^{T} \widehat{\xi}_i \Delta S_i + \widehat{L}_T \tag{3.11}$$

with $\widehat{K}_0 \in L^2(\Omega, \mathcal{F}_0, P)$, $\widehat{\xi} \in L^2(M)$ such that $\widehat{\xi} - \lambda \in L^2(A)$, and $\widehat{L} \in \mathcal{M}_0^2(P)$ strongly orthogonal to M. In that case, $\widehat{\vartheta}$ is given by $\widehat{\vartheta} = \frac{1}{\gamma}\big(\lambda - \widehat{\xi}\big)$.
If K_T is in $L^2(P)$ and admits a decomposition (3.11)*, the integrand $\widehat{\xi}$ is in Θ and* (3.11) *coincides with the Föllmer–Schweizer decomposition of K_T.*

Proof. By plugging (3.7) into (3.9) and comparing the resulting equation with (3.10), we obtain that $\widehat{\xi} = \lambda - \gamma\widehat{\vartheta} = \gamma\xi(\widehat{\vartheta})$ and therefore the first assertion. If K_T is in $L^2(P)$, this gives that $\lambda \in \Theta_S$, which implies that $\widehat{\xi} \in \Theta$ and completes the proof. □

V.4 Continuous time

In continuous time, we should like to obtain the time-consistent solution in analogy to discrete time by optimising the mean-variance criterion with respect to local perturbations. For a precise formulation of this we need a local description of the underlying quantities and a limit argument. To that end, let us fix some terminology first.

Recall from Section V.2 that we can and do assume that S is square-integrable with canonical decomposition $S = S_0 + M + A$, where M is an $\mathbb{R}^d$-valued square-integrable martingale null at zero, i.e. $M \in \mathcal{M}_0^2(P)$, and A is an $\mathbb{R}^d$-valued predictable finite variation RCLL process null at zero. By Propositions II.2.9 and II.2.29 in [52], there exist an increasing, integrable, predictable process B, an $\mathbb{R}^d$-valued predictable process a and a predictable $\mathbb{R}^{d\times d}$-valued process c^M whose values are positive semidefinite symmetric matrices such that

$$\vartheta \bullet A = (\vartheta^\top a) \bullet B \quad \text{and} \quad \langle \vartheta \bullet M\rangle = (\vartheta^\top c^M \vartheta) \bullet B \quad \text{for all } \vartheta \in \Theta. \tag{4.1}$$

By adding t to B, we can assume that B is strictly increasing. Set $P_B := P \otimes B$. There exist many processes B, a and c^M satisfying (4.1), but our results do not depend on the specific choice we make. Using the Moore–Penrose pseudoinverse $(c^M)^{-1}$ of c^M (see [1]) or the arguments preceding Theorem 2.3 in [30], we define a predictable process $\lambda := (c^M)^{-1}a$ which gives a decomposition

$$a = c^M\lambda + \eta \tag{4.2}$$

such that η is valued in $\mathrm{Ker}(c^M)$. Then S satisfies the *structure condition (SC)* if and only if $\eta = 0$ and $\lambda \in L^2_{loc}(M)$, i.e. the *mean-variance tradeoff (MVT) process* K given by $K_t = \int_0^t \lambda_u^\top d\langle M\rangle_u \lambda_u = \langle \lambda \bullet M\rangle_t$ for $t \in [0,T]$ is P-a.s. finite. In continuous time, the process of *expected future gains* $Z(\vartheta)$ and the square-integrable martingale $Y(\vartheta)$ of its canonical decomposition are given by

$$\begin{aligned} Z_t(\vartheta) &:= E\left[\int_t^T \vartheta_u dS_u \middle| \mathcal{F}_t\right] = E\left[\int_0^T \vartheta_u dA_u \middle| \mathcal{F}_t\right] - \int_0^t \vartheta_u dA_u \\ &=: Y_t(\vartheta) - \int_0^t \vartheta_u dA_u \end{aligned}$$

for $t \in [0,T]$ and each strategy $\vartheta \in \Theta$. Using the (continuous-time) GKW decomposition

$$\int_0^T \vartheta_u dA_u = Y_0(\vartheta) + \int_0^T \xi_u(\vartheta) dM_u + L_T(\vartheta)$$

of $Y(\vartheta)$, we can rewrite $Z(\vartheta)$ as

$$Z_t(\vartheta) = Y_t(\vartheta) - \int_0^t \vartheta_u dA_u = Y_0(\vartheta) + \int_0^t \xi_u(\vartheta) dM_u + L_t(\vartheta) - \int_0^t \vartheta_u dA_u \tag{4.3}$$

for $t \in [0,T]$, exactly as in discrete time.

A *partition* of $[0,T]$ is a finite set $\tau = \{t_0, t_1, \dots, t_m\}$ with $0 = t_0 < t_1 < \dots < t_m = T$, and its *mesh size* is $|\tau| := \max_{t_i,t_{i+1}}(t_{i+1} - t_i)$. A sequence of partitions $(\tau_n)_{n\in\mathbb{N}}$ is *increasing* if $\tau_n \subseteq \tau_{n+1}$ for all n; it *tends to the identity* if $\lim_{n\to\infty} |\tau_n| = 0$. For later use, we associate to each partition τ the σ-field

$$\mathcal{P}^\tau := \sigma\big(\{F_0 \times \{0\}, F_i \times (t_i, t_{i+1}] \big| t_i, t_{i+1} \in \tau, F_0 \in \mathcal{F}_0, F_{t_i} \in \mathcal{F}_{t_i}\}\big)$$

on $\Omega \times [0,T]$. Note for any sequence of partitions $(\tau_n)_{n\in\mathbb{N}}$ tending to the identity that $\sigma\Big(\bigcup_{n\in\mathbb{N}} \mathcal{P}^{\tau_n}\Big)$ is equal to the predictable σ-field $\mathcal{P}$ and that $\mathcal{P}^{\tau_n}$

increases to $\mathcal{P}$ if $(\tau_n)_{n\in\mathbb{N}}$ is in addition increasing. The optimality with respect to local perturbations can then be formulated in continuous time as follows. Recall the notations $U_t(\vartheta)$ from (2.4) and $\overline{U}_t(\vartheta) = U_t\big(\mathbb{1}_{]\!]t,T]\!]}\vartheta\big)$.

Definition 4.1. For $\vartheta, \delta \in \Theta$ and a partition τ of $[0,T]$, we set

$$u^\tau[\vartheta,\delta] := \sum_{t_i,t_{i+1}\in\tau} \frac{U_{t_i}(\vartheta) - U_{t_i}(\vartheta + \delta\mathbb{1}_{(t_i,t_{i+1}]})}{E[B_{t_{i+1}} - B_{t_i}|\mathcal{F}_{t_i}]}\mathbb{1}_{(t_i,t_{i+1}]} \tag{4.4}$$
$$= \sum_{t_i,t_{i+1}\in\tau} \frac{\overline{U}_{t_i}(\vartheta) - \overline{U}_{t_i}(\vartheta + \delta\mathbb{1}_{(t_i,t_{i+1}]})}{E[B_{t_{i+1}} - B_{t_i}|\mathcal{F}_{t_i}]}\mathbb{1}_{(t_i,t_{i+1}]}.$$

A strategy $\widehat{\vartheta} \in \Theta$ is called *locally mean-variance efficient (in continuous time)* if

$$\liminf_{n\to\infty} u^{\tau_n}[\widehat{\vartheta},\delta] \geq 0 \quad P_B\text{-a.e.} \tag{4.5}$$

for any increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of partitions tending to the identity and any $\delta \in \Theta$.

Intuitively, $u^\tau[\vartheta,\delta]$ measures the change in the tradeoff between mean and variance when we perturb ϑ locally by δ along τ. Condition (4.5) then says that perturbing the optimal stratetgy $\widehat{\vartheta}$ locally should always decrease this tradeoff, at least asymptotically. The appropriate "time scale" for this asymptotic is given by the process B which is sometimes also referred to in the literature as operational time. In analogy to discrete time, finding the time-consistent solution by recursive optimisation is captured by comparing at time t_i strategies which differ only on $(t_i, t_{i+1}]$ but are equal on $(t_{i+1}, T]$. Passing to the limit then takes this recursive optimisation to continuous time. By the usual embedding of the discrete-time case into the continuous-time setting (as for example explained in Section I.1f in [52]) it is straightforward to see that the continuous-time formulation (4.5) coincides with that in discrete time (3.2), since we can choose $B_t = \sum_{k=1}^T \mathbb{1}_{\{k\leq t\}}$ in this situation (see Section II.3 in [52]).

The definition of local mean-variance efficiency above as well as the subsequent treatment are inspired by the concept of local risk minimisation in continuous time introduced by Schweizer in [85]; see also [87] and [88]. To obtain a characterisation of the LMVE strategy $\widehat{\vartheta}$ we need to derive the asymptotics of (4.5). As in [88], the first ingredient for this is a decomposition of u^τ into three terms A_1^τ, A_2^τ and A_3^τ for which we can control the asymptotics of each one separately. This follows by using the same arguments as in [88] which we give here for completeness.

Proposition 4.2. *For all strategies $\vartheta, \delta \in \Theta$ and every partition τ of $[0,T]$, we have*

$$u^\tau[\vartheta,\delta] = A_1^\tau + A_2^\tau + A_3^\tau,$$

where

$$A_1^\tau = E_B\left[\left(\gamma\big(\xi(\vartheta)+\vartheta\big)-\lambda-\tfrac{\gamma}{2}\delta\right)^\top c^M\delta+\delta^\top\eta\,\Big|\,\mathcal{P}^\tau\right]$$
$$A_2^\tau = \frac{\gamma}{2}\sum_{t_i,t_{i+1}\in\tau}\frac{\operatorname{Var}\left[\int_{t_i}^{t_{i+1}}\delta dA\,\Big|\,\mathcal{F}_{t_i}\right]}{E[B_{t_{i+1}}-B_{t_i}|\mathcal{F}_{t_i}]}\mathbb{1}_{(t_i,t_{i+1}]}$$
$$A_3^\tau = \gamma\sum_{t_i,t_{i+1}\in\tau}\frac{\operatorname{Cov}\left[L_{t_{i+1}}(\vartheta)-L_{t_i}(\vartheta)+\int_{t_i}^{t_{i+1}}(\xi(\vartheta)+\vartheta+\delta)dM,\int_{t_i}^{t_{i+1}}\delta dA\,\Big|\,\mathcal{F}_{t_i}\right]}{E[B_{t_{i+1}}-B_{t_i}|\mathcal{F}_{t_i}]}\mathbb{1}_{(t_i,t_{i+1}]}.$$

Proof. Plugging ϑ and $\vartheta+\delta\mathbb{1}_{(t_i,t_{i+1}]}$ into the definition of $U(\cdot)$ gives that

$$\begin{aligned}U_{t_i}(\vartheta)-U_{t_i}(\vartheta+\delta\mathbb{1}_{(t_i,t_{i+1}]}) &= -E\left[\int_{t_i}^{t_{i+1}}\delta_u dS_u\,\Big|\,\mathcal{F}_{t_i}\right]\\ &\quad+\gamma\operatorname{Cov}\left[\int_0^T\vartheta_u dS_u+\frac{1}{2}\int_{t_i}^{t_{i+1}}\delta_u dS_u,\int_{t_i}^{t_{i+1}}\delta_u dS_u\,\Big|\,\mathcal{F}_{t_i}\right]. \quad (4.6)\end{aligned}$$

Using $S=S_0+M+A$ and the definition of $Y(\vartheta)$ we can write

$$\int_0^T\vartheta_u dS_u - E\left[\int_0^T\vartheta_u dS_u\,\Big|\,\mathcal{F}_{t_i}\right] = Y_T(\vartheta)-Y_{t_i}(\vartheta)+\int_{t_i}^T\vartheta_u dM_u,$$

which gives

$$\begin{aligned}&\operatorname{Cov}\left[\int_0^T\vartheta_u dS_u+\frac{1}{2}\int_{t_i}^{t_{i+1}}\delta_u dS_u\,,\int_{t_i}^{t_{i+1}}\delta_u dS_u\,\Big|\,\mathcal{F}_{t_i}\right]\\ &=\operatorname{Cov}\left[Y_T(\vartheta)-Y_{t_i}(\vartheta)+\int_{t_i}^{t_{i+1}}\left(\vartheta_u+\frac{1}{2}\delta_u\right)dM_u,\int_{t_i}^{t_{i+1}}\delta_u dM_u\,\Big|\,\mathcal{F}_{t_i}\right]\\ &\quad+\operatorname{Cov}\left[Y_T(\vartheta)-Y_{t_i}(\vartheta)+\int_{t_i}^{t_{i+1}}(\vartheta_u+\delta_u)\,dM_u,\int_{t_i}^{t_{i+1}}\delta_u dA_u\,\Big|\,\mathcal{F}_{t_i}\right]\\ &\quad+\frac{1}{2}\operatorname{Var}\left[\int_{t_i}^{t_{i+1}}\delta_u dA_u\,\Big|\,\mathcal{F}_{t_i}\right]. \quad (4.7)\end{aligned}$$

Since $Y(\vartheta)$ and $\int\vartheta dM$ are martingales, the second term on the right-hand side above equals

$$\operatorname{Cov}\left[Y_{t_{i+1}}(\vartheta)-Y_{t_i}(\vartheta)+\int_{t_i}^{t_{i+1}}(\vartheta_u+\delta_u)\,dM_u,\int_{t_i}^{t_{i+1}}\delta_u dA_u\,\Big|\,\mathcal{F}_{t_i}\right]. \quad (4.8)$$

With an analogous argument and inserting the Galtchouk–Kunita–Watanabe decomposition $Y(\vartheta) = Y_0(\vartheta) + \int \xi(\vartheta)dM + L(\vartheta)$, we obtain

$$\begin{aligned}
&\operatorname{Cov}\left[Y_T(\vartheta) - Y_{t_i}(\vartheta) + \int_{t_i}^{T} \vartheta dM + \frac{1}{2}\int_{t_i}^{t_{i+1}} \delta dM, \int_{t_i}^{t_{i+1}} \delta dM \middle| \mathcal{F}_{t_i}\right] \\
&= \operatorname{Cov}\left[\, \textstyle\int_{t_i}^{t_{i+1}} (\xi(\vartheta) + \vartheta) dM + \int_{t_i}^{t_{i+1}} dL(\vartheta) + \frac{1}{2}\int_{t_i}^{t_{i+1}} \delta dM, \int_{t_i}^{t_{i+1}} \delta dM \middle| \mathcal{F}_{t_i}\right] \\
&= E\left[\int_{t_i}^{t_{i+1}} d\langle \textstyle\int \left(\xi(\vartheta) + \vartheta + \frac{1}{2}\delta\right) dM, \int \delta dM \rangle | \mathcal{F}_{t_i}\right] \\
&= E\left[\int_{t_i}^{t_{i+1}} \left(\xi(\vartheta) + \vartheta + \tfrac{1}{2}\delta\right)^\top c^M \delta dB \middle| \mathcal{F}_{t_i}\right].
\end{aligned} \tag{4.9}$$

By the martingale property of $\int \delta dM$ and using $a = c^M \lambda + \eta$ we have

$$E\left[\int_{t_i}^{t_{i+1}} \delta_u dS_u \middle| \mathcal{F}_{t_i}\right] = E\left[\int_{t_i}^{t_{i+1}} (\delta_u^\top c_u^M \lambda_u + \delta_u^\top \eta_u) dB_u \middle| \mathcal{F}_{t_i}\right]. \tag{4.10}$$

Combining (4.6)–(4.10) we conclude that

$$\begin{aligned}
&U_{t_i}(\vartheta) - U_{t_i}(\vartheta + \delta|_{(t_i,t_{i+1}]}) \\
&= E\left[\int_{t_i}^{t_{i+1}} \left(\left(\gamma\big(\xi(\vartheta)_u + \vartheta_u\big) - \lambda_u + \tfrac{\gamma}{2}\delta_u\right)^\top c_u^M \delta_u - \delta_u^\top \eta_u\right) dB_u \middle| \mathcal{F}_{t_i}\right] \\
&\quad + \gamma \operatorname{Cov}\left[Y_{t_{i+1}}(\vartheta) - Y_{t_i}(\vartheta) + \int_{t_i}^{t_{i+1}} (\vartheta_u + \delta_u)\, dM_u, \int_{t_i}^{t_{i+1}} \delta_u dA_u \middle| \mathcal{F}_{t_i}\right] \\
&\quad + \frac{\gamma}{2} \operatorname{Var}\left[\int_{t_i}^{t_{i+1}} \delta_u dA_u \middle| \mathcal{F}_{t_i}\right].
\end{aligned}$$

After dividing by $E[B_{t_{i+1}} - B_{t_i} | \mathcal{F}_{t_i}]$, multiplying by $\mathbb{1}_{(t_i,t_{i+1}]}$ and summing over $t_i, t_{i+1} \in \tau$, we obtain $u^\tau[\vartheta, \delta]$ on the left-hand side and A_1^τ, A_3^τ and A_2^τ on the right-hand side, as

$$\begin{aligned}
&\sum_{t_i,t_{i+1}\in\tau} \frac{E\left[\int_{t_i}^{t_{i+1}} \left(\left(\gamma\big(\xi(\vartheta)_u + \vartheta_u\big) - \lambda_u + \frac{\gamma}{2}\delta_u\right)^\top c_u^M \delta_u - \delta_u^\top \eta_u\right) dB_u \middle| \mathcal{F}_{t_i}\right]}{E[B_{t_{i+1}} - B_{t_i} | \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i,t_{i+1}]} \\
&= E_B\left[\left(\xi(\vartheta) + \vartheta - \lambda + \tfrac{1}{2}\delta\right)^\top c^M \delta + \delta^\top \eta \middle| \mathcal{P}^\tau\right] = A_1^\tau,
\end{aligned}$$

which completes the proof. □

Since A_1^τ is of the same form as the corresponding term in Proposition 2.2 in [88], we obtain its asymptotic behaviour by the same argument as in Lemma 3.1 in [88]. The additional term $\delta^\top \eta$ is not relevant for this.

Lemma 4.3. *Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions tending to the identity. Then*

$$\lim_{n\to\infty} A_1^{\tau_n} = \left(\gamma\big(\xi(\vartheta) + \vartheta\big) - \lambda + \frac{\gamma}{2}\delta\right)^\top c^M \delta - \delta^\top \eta \qquad P_B\text{-a.e.} \tag{4.11}$$

Proof. We observe that $\left(\gamma\left(\xi(\vartheta)+\vartheta\right)-\lambda+\frac{1}{2}\delta\right)^\top c^M\delta-\delta^\top\eta\in L^1(P_B)$, since ϑ and δ are in Θ, and recall that $(\mathcal{P}^{\tau_n})_{n\in\mathbb{N}}$ increases to the predictable σ-field $\mathcal{P}$, since $(\tau_n)_{n\in\mathbb{N}}$ is increasing and tending to the identity. As $A_1^{\tau_n}=E_B\left[\left(\gamma\left(\xi(\vartheta)+\vartheta\right)-\lambda+\frac{1}{2}\delta\right)^\top c^M\delta-\delta^\top\eta\middle|\mathcal{P}^{\tau_n}\right]$ by definition, $(A_1^{\tau_n})_{n\in\mathbb{N}}$ is a uniformly integrable P_B-martingale and (4.11) follows from the martingale convergence theorem, since $\left(\gamma\left(\xi(\vartheta)+\vartheta\right)-\lambda+\frac{1}{2}\delta\right)^\top c^M\delta-\delta^\top\eta$ is predictable. □

To show that the term $A_2^{\tau_n}$ is asymptotically negligible, we establish the following general convergence result. For this we argue with the predictable measurability of X and need not assume continuity of X as in Proposition 3.5 in [88]. Applying our techniques to *local risk minimisation* enables us to generalise this concept and some related results to a general semimartingale setting. In particular, we are able to drop the continuity of A and (SC) in Theorem 1.6 and Proposition 5.2 in [88]; this will be explained in more detail in future work.

Lemma 4.4. *Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $[0,T]$ tending to the identity and $X\in\mathcal{H}^2(P)$ a predictable finite variation process such that $X=\int\alpha dB$ for $\alpha\in L^0(B)$. Then*

$$\lim_{n\to\infty}\sum_{t_{i-1},t_i\in\tau_n}\frac{\operatorname{Var}\left[X_{t_i}-X_{t_{i-1}}|\mathcal{F}_{t_{i-1}}\right]}{E[B_{t_i}-B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]}\mathbb{1}_{(t_{i-1},t_i]}=0\qquad P_B\text{-a.e.}\tag{4.12}$$

Proof. We first decompose

$$\begin{aligned}&\sum_{t_{i-1},t_i\in\tau_n}\frac{\operatorname{Var}\left[X_{t_i}-X_{t_{i-1}}|\mathcal{F}_{t_{i-1}}\right]}{E[B_{t_i}-B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]}\mathbb{1}_{(t_{i-1},t_i]}\\&=\sum_{t_{i-1},t_i\in\tau_n}\frac{E\left[(X_{t_i}-X_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}\right]}{E[B_{t_i}-B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]}\mathbb{1}_{(t_{i-1},t_i]}\\&\quad-\sum_{t_{i-1},t_i\in\tau_n}\frac{\left(E[X_{t_i}-X_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]\right)^2}{E[B_{t_i}-B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]}\mathbb{1}_{(t_{i-1},t_i]}.\end{aligned}$$

For the proof of (4.12) we then only need to show that both sums on the right-hand side converge to the same limit $\alpha\Delta X$. To that end, set $t^{\tau_n}=\inf\{s\in\tau_n\mid s\geq t\}$ and $t^{\tau_n-}=\sup\{s\in\tau_n\mid s<t\}$ for each $t\in[0,T]$, and $X^n(\omega,t)=(X_{t^{\tau_n}}-X_{t^{\tau_n-}})(\omega)$ and $\widetilde{X}^n(\omega,t)=E[X^n_t|\mathcal{F}_{t^{\tau_n-}}](\omega)$ for all $(\omega,t)\in\Omega\times[0,T]$. Using $X=\int\alpha dB$ we can write

$$\begin{aligned}&\sum_{t_{i-1},t_i\in\tau_n}\frac{E[(X_{t_i}-X_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}]}{E[B_{t_i}-B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]}\mathbb{1}_{(t_{i-1},t_i]}\\&=\sum_{t_{i-1},t_i\in\tau_n}\frac{E[(X_{t_i}-X_{t_{i-1}})\int_{t_{i-1}}^{t_i}\alpha_u dB_u|\mathcal{F}_{t_{i-1}}]}{E[B_{t_i}-B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]}\mathbb{1}_{(t_{i-1},t_i]}=E_B[X^n\alpha|\mathcal{P}^{\tau_n}]\end{aligned}$$

and

$$\sum_{t_{i-1},t_i\in\tau_n} \frac{\big(E[X_{t_i}-X_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]\big)^2}{E[B_{t_i}-B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]}\mathbb{1}_{(t_{i-1},t_i]}$$

$$= \sum_{t_{i-1},t_i\in\tau_n} E[X_{t_i}-X_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]\frac{E\left[\int_{t_{i-1}}^{t_i}\alpha_u dB_u\Big|\mathcal{F}_{t_{i-1}}\right]}{E[B_{t_i}-B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]}\mathbb{1}_{(t_{i-1},t_i]}$$

$$= \widetilde{X}^n E_B[\alpha|\mathcal{P}^{\tau_n}].$$

By estimating $\sup_{n\in\mathbb{N}}|X^n\alpha| \le 2|\alpha|\sup_{0\le s\le T}|X_s| \le 2|\alpha|\int_0^T|dX_u|$ we obtain that $\sup_{n\in\mathbb{N}}|X^n\alpha|\in L^1(P_B)$ as $\int_0^T(\int_0^T|dX_s|)|\alpha_u|dB_u = \big(\int_0^T|dX_s|\big)^2 \in L^1(P)$. Since X is RCLL and $t^{\tau_n}\searrow t$ and $t^{\tau_n-}\nearrow t$ as $n\to\infty$, it follows that X^n converges pointwise to ΔX. Combining this with the integrability of $\sup_{n\in\mathbb{N}}|X^n\alpha|$ gives that $E_B[X^n\alpha|\mathcal{P}^{\tau_n}]$ tends to $\alpha\Delta X$ P_B-a.e. by Hunt's lemma (see [33], V.45), since $\mathcal{P}^{\tau_n}$ increases to $\mathcal{P}$ and $\alpha\Delta X$ is predictable. As the P_B-a.e. convergence of $E_B[\alpha|\mathcal{P}^{\tau_n}]$ to α already follows by the martingale convergence theorem, it remains to show that $\widetilde{X}^n$ converges to ΔX P_B-a.e. for the convergence of the second sum. Since $\sup_{n\in\mathbb{N}}|X_{t^{\tau_n}}-X_{t^{\tau_n}-}|\le 2\int_0^T|dX_s|\in L^2(P)$ for all $t\in[0,T]$ and X^n converges pointwise to ΔX, it follows by Hunt's lemma that

$$\widetilde{X}^n_t \longrightarrow E[\Delta X_t|\mathcal{F}_{t-}] \quad P\text{-a.s. for each } t\in[0,T]. \tag{4.13}$$

By Theorem III.5 in [80] the limit coincides with ΔX_t, as ΔX is predictable. Since $\{\lim_{n\to\infty}\widetilde{X}^n\ne\Delta X\}\in\mathcal{F}\otimes\mathcal{B}([0,T])$, we obtain that $\widetilde{X}^n$ converges to ΔX P_B-a.e. from (4.13) by Fubini's theorem. This completes the proof. □

With this we have now everything in place to derive the asymptotics of $u^\tau[\vartheta,\delta]$.

Lemma 4.5. *Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $[0,T]$ tending to the identity. Then*

$$\lim_{n\to\infty} u^{\tau_n}[\vartheta,\delta] = \Big(\gamma\big(\xi(\vartheta)+\vartheta\big)-\lambda+\frac{\gamma}{2}\delta\Big)^\top c^M\delta-\delta^\top\eta \qquad P_B\text{-a.e.}$$

for all $\vartheta,\delta\in\Theta$.

Proof. The proof follows immediately by combining Proposition 4.2 and Lemma 4.3 after we have shown that $A_2^{\tau_n}$ and $A_3^{\tau_n}$ converge to 0 P_B-a.e. To that end, we estimate

$$\left|\operatorname{Cov}\left[Y_{t_{i+1}}(\vartheta)-Y_{t_i}(\vartheta)+\int_{t_i}^{t_{i+1}}(\vartheta+\delta)\,dM,\int_{t_i}^{t_{i+1}}\delta dA\Big|\mathcal{F}_{t_i}\right]\right|^2$$

$$\le \operatorname{Var}\left[Y_{t_{i+1}}(\vartheta)-Y_{t_i}(\vartheta)+\int_{t_i}^{t_{i+1}}(\vartheta+\delta)\,dM\Big|\mathcal{F}_{t_i}\right]\operatorname{Var}\left[\int_{t_i}^{t_{i+1}}\delta dA\Big|\mathcal{F}_{t_i}\right]$$

$$= E\big[X_{t_{i+1}}-X_{t_{i+1}}\big|\mathcal{F}_{t_i}\big]\operatorname{Var}\left[\int_{t_i}^{t_{i+1}}\delta dA\Big|\mathcal{F}_{t_i}\right]$$

by using the Cauchy-Schwarz inequality and $X := \langle Y + \int(\vartheta+\delta)dM\rangle$. Again by the Cauchy-Schwarz inequality we obtain from the above that

$$
\begin{aligned}
|A_3^{\tau_n}| &\leq \gamma \left(\sum_{t_i,t_{i+1}\in\tau_n} \frac{E[X_{t_{i+1}} - X_{t_{i+1}}|\mathcal{F}_{t_i}]}{E[B_{t_{i+1}} - B_{t_i}|\mathcal{F}_{t_i}]} \mathbb{1}_{(t_i,t_{i+1}]} \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{t_i,t_{i+1}\in\tau_n} \frac{\operatorname{Var}\left[\int_{t_i}^{t_{i+1}} \delta_u dA_u \middle| \mathcal{F}_{t_i}\right]}{E[B_{t_{i+1}} - B_{t_i}|\mathcal{F}_{t_i}]} \mathbb{1}_{(t_i,t_{i+1}]} \right)^{\frac{1}{2}} \\
&= \sqrt{2\gamma} \left(\frac{dP_X}{dP_B}\Big|_{\mathcal{P}^{\tau_n}} \right)^{\frac{1}{2}} (A_2^{\tau_n})^{\frac{1}{2}},
\end{aligned}
\tag{4.14}
$$

where $P_X := P \otimes X$ and $\frac{dP_X}{dP_B}\big|_{\mathcal{P}^{\tau_n}} = \sum_{t_i,t_{i+1}\in\tau_n} \frac{E[X_{t_{i+1}}-X_{t_i}|\mathcal{F}_{t_i}]}{E[B_{t_{i+1}}-B_{t_i}|\mathcal{F}_{t_i}]} \mathbb{1}_{(t_i,t_{i+1}]}$. It is straightforward to verify that $\left(\frac{dP_X}{dP_B}\big|_{\mathcal{P}^{\tau_n}}\right)_{n\in\mathbb{N}}$ is a P_B-martingale by simply checking the definition; see Lemma 3.4 in [88]. Since $\frac{dP_X}{dP_B}\big|_{\mathcal{P}^{\tau_n}}$ is non-negative, it follows directly by the martingale convergence theorem that $\left(\frac{dP_X}{dP_B}\big|_{\mathcal{P}^{\tau_n}}\right)_{n\in\mathbb{N}}$ is P_B-a.e. convergent and hence P_B-a.e. bounded in n. (Moreover, the limit coincides with the Radon–Nikodým derivative of the absolutely continuous part of P_X with respect to P_B.) Since $\int \delta dA = \int \delta^\top a\, dB$, applying Lemma 4.4 with $\alpha = \delta^\top a$ yields that $\lim_{n\to\infty} A_2^{\tau_n} = 0$ P_B-a.e. and therefore also that $\lim_{n\to\infty} A_3^{\tau_n} = 0$ P_B-a.e. by (4.14). This completes the proof. □

Having the representation of our criterion above, we can now describe the solution.

Theorem 4.6. *The LMVE strategy $\widehat{\vartheta}$ exists if and only if we have both*

1) *S satisfies (SC) with $\lambda \in L^2(M)$, i.e. $K_T \in L^1(P)$.*

2) *There exists $\widehat{\psi} \in \Theta$ such that*

$$
\widehat{\psi} = \frac{1}{\gamma}\lambda - \xi(\widehat{\psi}),
\tag{4.15}
$$

where $\xi(\widehat{\psi})$ is the integrand in the GKW decomposition of $\int_0^T \widehat{\psi}_u dA_u$.

In that case, $\widehat{\vartheta} = \widehat{\psi}$.

Proof. Using Lemma 4.5 it follows by definition that $\widehat{\vartheta}$ is LMVE if and only if

$$
\left(\gamma\big(\xi(\widehat{\vartheta}) + \widehat{\vartheta}\big) - \lambda + \frac{\gamma}{2}\delta\right)^\top c^M \delta - \delta^\top \eta \geq 0 \qquad P_B\text{-a.e.}
\tag{4.16}
$$

for all $\delta \in \Theta$. If 1) and 2) hold, (4.16) reduces to $\frac{\gamma}{2}\delta^\top c^M \delta \geq 0$ for $\widehat{\vartheta} := \widehat{\psi} = \frac{1}{\gamma}\lambda - \xi(\widehat{\psi})$ and all $\delta \in \Theta$, which immediately gives that this strategy $\widehat{\vartheta}$

is LMVE. For the converse, we first observe that since $c^M\eta = 0$, choosing $\delta = \eta 1\!\!1_{\{|\eta^\top a|\le n\}}$ for each $n \in \mathbb{N}$ gives that $\delta \in \Theta$ and $-\delta^\top\delta \ge 0$ in (4.16). This implies that $\eta = 0$ P_B-a.e. and therefore that S satisfies (SC). Set $\varphi = \frac{1}{\gamma}\lambda - \big(\xi(\widehat{\vartheta}) + \widehat{\vartheta}\big)$. Then plugging $\delta = \varphi 1\!\!1_{\{\varphi^\top c^M\varphi + |\varphi^\top a|\le n\}} \in \Theta$ into (4.16) for each $n \in \mathbb{N}$ yields that $-\frac{\gamma}{2}\varphi^\top c^M \varphi \ge 0$ P_B-a.e. so that $\varphi = 0$ in $L^2(M)$, which gives that $\lambda = \gamma\big(\xi(\widehat{\vartheta}) + \vartheta\big) \in L^2(M)$. This completes the proof. □

As in discrete time, we say that a random variable $H \in L^2(\Omega, \mathcal{F}_T, P)$ admits a *Föllmer–Schweizer decomposition* if it can be written as

$$H = \widehat{H}_0 + \int_0^T \widehat{\xi}^H_u \, dS_u + \widehat{L}^H_T,$$

where $\widehat{H}_0 \in L^2(\Omega, \mathcal{F}_0, P)$, $\widehat{\xi}^H \in \Theta$ and $\widehat{L}^H \in \mathcal{M}^2_0(P)$ is strongly P-orthogonal to M. Using this notion we can then give the following alternative characterisation of the LMVE. Note that in contrast to the definition of optimality, this alternative description is global.

Theorem 4.7. *There exists a LMVE strategy $\widehat{\vartheta}$ if and only if S satisfies (SC) and (the terminal value of) the MVT process K_T is in $L^1(P)$ and can be written as*

$$K_T = \widehat{K}_0 + \int_0^T \widehat{\xi} dS + \widehat{L}_T \tag{4.17}$$

with $\widehat{K}_0 \in L^2(\Omega, \mathcal{F}_0, P)$, $\widehat{\xi} \in L^2(M)$ such that $\widehat{\xi} - \lambda \in L^2(A)$, and $\widehat{L} \in \mathcal{M}^2_0(P)$ strongly P-orthogonal to M. In that case, $\widehat{\vartheta}$ is given by $\widehat{\vartheta} = \frac{1}{\gamma}\big(\lambda - \widehat{\xi}\big)$, $\xi(\widehat{\vartheta}) = \frac{1}{\gamma}\widehat{\xi}$,

$$Z_t(\widehat{\vartheta}) = \frac{1}{\gamma}\left(\widehat{K}_0 + \int_0^t \widehat{\xi} dS + \widehat{L}_t - K_t\right) \tag{4.18}$$

and

$$\begin{aligned} U_t(\widehat{\vartheta}) &= x + \int_0^t \left(\widehat{\vartheta} + \frac{1}{\gamma}\widehat{\xi}\right) dS \\ &\quad + \frac{1}{\gamma}\left(\widehat{K}_0 + \widehat{L}_t - \frac{1}{2}E\left[K_T - K_t + \langle \widehat{L}\rangle_T - \langle \widehat{L}\rangle_t \middle| \mathcal{F}_t\right]\right) \end{aligned} \tag{4.19}$$

with canonical decomposition

$$\begin{aligned} U_t(\widehat{\vartheta}) &= x + \frac{1}{\gamma}\left(\widehat{K}_0 + \int_0^t \lambda dM + \widehat{L}_t - \frac{1}{2}E\left[K_T + \langle \widehat{L}\rangle_T \middle| \mathcal{F}_t\right]\right) \\ &\quad + \frac{1}{2\gamma}\left(K_t + \langle \widehat{L}\rangle_t\right). \end{aligned} \tag{4.20}$$

If K_T is in $L^2(P)$ and admits a decomposition (4.17)*, the integrand $\widehat{\xi}$ is in Θ and* (4.17) *coincides with the Föllmer–Schweizer decomposition of K_T.*

Proof. The equivalence between the existence of the LMVE strategy $\widehat{\vartheta}$ and the decomposition (4.17) follows from Theorem 4.6 by the same arguments as in discrete time given in the proof of Lemma 3.4 and before. Indeed by comparing (3.9) and (3.10), the integrability properties can be ticked off from the corresponding parts in the decomposition, since $K_T = \int_0^T \lambda_u^\top d\langle M\rangle_u \lambda_u$ is in $L^1(P)$ or $L^2(P)$, respectively. This also yields (4.18) by simply plugging $\widehat{\vartheta} = \frac{1}{\gamma}(\lambda - \widehat{\xi})$ and the parts of (4.17) into (4.3). For the proof of (4.20), we observe that the square-integrable martingale $R(\widehat{\vartheta})$ given by $R_t(\widehat{\vartheta}) = E\big[\int_0^T \widehat{\vartheta}_u dS_u \big| \mathcal{F}_t\big]$ for $t \in [0,T]$ is equal to $\frac{1}{\gamma}(\widehat{K}_0 + \lambda \bullet M + \widehat{L})$. Inserting this into the definition of $U_t(\widehat{\vartheta})$ gives

$$\begin{aligned} U_t(\widehat{\vartheta}) &= x + R_t(\widehat{\vartheta}) - \frac{\gamma}{2} E\Big[\big(R_T(\widehat{\vartheta}) - R_t(\widehat{\vartheta})\big)^2 \Big| \mathcal{F}_t\Big] \\ &= x + R_t(\widehat{\vartheta}) - \frac{\gamma}{2} E\Big[\big\langle R(\widehat{\vartheta})\big\rangle_T - \big\langle R(\widehat{\vartheta})\big\rangle_t \Big| \mathcal{F}_t\Big] \\ &= x + \frac{1}{\gamma}(\widehat{K}_0 + \lambda \bullet M_t + \widehat{L}_t) \\ &\quad - \frac{1}{2\gamma} E\Big[\langle \lambda \bullet M\rangle_T - \langle \lambda \bullet M\rangle_t + \langle \widehat{L}\rangle_T - \langle \widehat{L}\rangle_t \Big| \mathcal{F}_t\Big] \end{aligned}$$

and therefore (4.20). Since $R_t(\widehat{\vartheta}) = \int_0^t \widehat{\vartheta}_u dS_u + \frac{1}{\gamma}\Big(\widehat{K}_0 + \int_0^t \widehat{\xi}_u dS_u + \widehat{L}_t - K_t\Big)$ by (4.18), we then obtain (4.19) from (4.20), which completes the proof. □

In specific Markovian frameworks, relations like in Theorem 4.7 have been obtained in [6] and [9] by arguments using the Feynman-Kac formula, which are available there. The link between the LMVE strategy $\widehat{\vartheta}$ and the FS decomposition now allows us to exploit known results on the FS decomposition to give a sufficient condition for the existence of $\widehat{\vartheta}$. To formulate this, we first need to introduce some of the terminology used in [16]. Since the existence of $\widehat{\vartheta}$ implies that S satisfies (SC) with $\lambda \in L^2(M)$, we have that $-\lambda \bullet M$ is a square-integrable martingale. For any stopping time σ we denote ${}^\sigma\mathcal{E}(-\lambda \bullet M) = \mathcal{E}\big(-(\lambda \mathbb{1}_{]\!]\sigma,T]\!]}) \bullet M\big)$. Since $-\lambda \bullet M$ is RCLL, it has P-a.s. at most a countable number of jumps with $\Delta(-\lambda \bullet M) = -1$, and so we can define an increasing sequence of stopping times $\hat{T}_n$ by $\hat{T}_0 = 0$ and $\hat{T}_{n+1} = \inf\{t > \hat{T}_n \mid {}^{\hat{T}_n}\mathcal{E}(-\lambda \bullet M)_t = 0\} \wedge T$.

Definition 4.8. We call $\mathcal{E}(-\lambda \bullet M)$ *regular* if for any n, ${}^{\hat{T}_n}\mathcal{E}(-\lambda \bullet M)$ is a martingale.

Definition 4.9. We say that $\mathcal{E}(-\lambda \bullet M)$ satisfies *the reverse Hölder inequality* $R_2(P)$, if there exists a constant $c \geq 1$ such that for any t,

$$E\left[\left|{}^t\mathcal{E}(-\lambda \bullet M)_T\right|^2 \middle| \mathcal{F}_t\right] \leq c.$$

Definition 4.10. We say that an RCLL process X is an $\mathcal{E}(-\lambda \bullet M)$-*martingale*, if for any $n \in \mathbb{N}$, $E\big[|X_{\hat{T}_n}{}^{\hat{T}_n}\mathcal{E}(-\lambda \bullet M)_{\hat{T}_{n+1}}|\big] < +\infty$ and $(\mathbb{1}_{]\!]\hat{T}_n,T]\!]} \bullet X)^{\hat{T}_n}\mathcal{E}(-\lambda \bullet M)$ is a martingale.

Definition 4.11. A local martingale $N \in \mathcal{M}^2_{loc}(P)$ is in bmo_2, if there exists a constant c such that

$$E\left[\langle N\rangle_T - \langle N\rangle_t | \mathcal{F}_t\right] \leq c^2$$

for all $t \in [0,T]$. The smallest such constant c is denoted by $\|N\|_{bmo_2}$.

With the definitions above we can give the following sufficient condition for the existence of the LMVE strategy.

Corollary 4.12. *Suppose that S satisfies (SC) and that $\mathcal{E}(-\lambda \bullet M)$ is regular and satisfies $R_2(P)$. Then the LMVE strategy $\widehat{\vartheta}$ exists and is given by $\widehat{\vartheta} = \frac{1}{\gamma}\big(\lambda - \widehat{\xi}\big)$, where $\widehat{\xi} \in \Theta$ is the integrand in the FS decomposition of $K_T \in L^2(P)$, and*

$$Z_t(\widehat{\vartheta}) = \frac{1}{\gamma}E\big[\mathcal{E}\big(-(\lambda \mathbb{1}_{]\!]t,T]\!]}) \bullet M\big)_T (K_T - K_t)\big|\mathcal{F}_t\big] \tag{4.21}$$

for $t \in [0,T]$.

Proof. By Proposition 3.10 in [16], we have that $-\lambda \bullet M$ is in bmo_2 and therefore that $K_T = \langle \lambda \bullet M\rangle_T$ is in $L^2(P)$ because $\mathcal{E}(-\lambda \bullet M)$ is regular and satisfies $R_2(P)$. Moreover, by Theorem 5.5 in [16], S admits an FS decomposition (in the stronger sense of Definition 5.4 in [16]), which implies in particular that every $H \in L^2(P)$ has an FS decomposition, if and only if $\mathcal{E}(-\lambda \bullet M)$ is regular and satisfies $R_2(P)$. Combining this with Theorem 4.7 we obtain that the LMVE strategy $\widehat{\vartheta}$ exists and can be represented as above in terms of the FS decomposition of K_T. Since a random variable admits an FS decomposition if and only if it is the terminal value of an $\mathcal{E}$-martingale in $\mathcal{H}^2(P,\mathbb{F})$ (see the discussion preceding Theorem 5.5 in [16]), we obtain that

$$E\big[\mathcal{E}\big(-(\lambda \mathbb{1}_{]\!]t,T]\!]}) \bullet M\big)_T K_T\big|\mathcal{F}_t\big] = \widehat{K}_0 + \int_0^t \widehat{\xi}_u dS_u + \widehat{L}_t$$

by Proposition 3.12.i) in [16] and therefore (4.21) via (4.18), which completes the proof. □

Remark 4.13. 1) If $\mathcal{E}(-\lambda \bullet M)$ is strictly positive in addition to the assumptions above, then it is the density process of an equivalent martingale measure for S, the so-called *minimal martingale measure (MMM)* $\widehat{P}$; see [46]. In this case, (4.21) can be written as $Z_t(\widehat{\vartheta}) = \frac{1}{\gamma}\widehat{E}[K_T - K_t|\mathcal{F}_t]$. This relation has been obtained in [6] and [9] in the specific Markovian frameworks used there by arguments using the Feynman-Kac formula.

2) If the MMM exists and its density process satisfies $R_2(P)$ and S is continuous, then the FS decomposition coincides with the GKW decomposition under $\widehat{P}$; see [17]. In the case, where S is discontinuous, the relation between the two decompositions is more complicated and has recently been established in [18].

3) Applying the previous results allows us to obtain the LMVE strategy in concrete models in the following way. First, we check if S satisfies (SC) by using its canonical decomposition. If this is true, we obtain λ and therefore K and $\mathcal{E}(-\lambda \bullet M)$ directly from the canonical decomposition of S. If $\mathcal{E}(-\lambda \bullet M)$ is regular and satisfies $R_2(P)$, we can try to obtain the FS decomposition of K_T via Theorem 4.3 in [18], which gives the LMVE strategy by Theorem 4.7. Moreover, if $\mathcal{E}(-\lambda \bullet M)$, the candidate for the density process of the MMM, is strictly positive in addition to the previous assumptions, the MMM exists and we can derive the FS decomposition as explained in the previous remark from the GKW decomposition of K_T under $\widehat{P}$.

4) Since one can obtain the ingredients λ, K and $\mathcal{E}(-\lambda \bullet M)$ directly and explicitly from the canonical decomposition of S, obtaining (a candidate for) the LMVE strategy as explained in 3) is more explicit than solving the static but multiperiod or continuous-time Markowitz problem via finding the *variance-optimal martingale measure*; see [88] and compare Section 3 of [6].

The optimality condition (4.15) basically tells us that the locally mean-variance efficient strategy $\widehat{\vartheta}$ is a fixed point of the mapping $\widehat{J} : \Theta \to \Theta$ given by

$$\widehat{J}(\vartheta) = \frac{1}{\gamma}\lambda - \xi(\vartheta). \tag{4.22}$$

Exploiting again the relation to the FS decomposition, we can show that this fixed point can be obtained by an iteration. Since the iteration algorithm reduces to a backward recursion in discrete time, this can be seen as the continuous-time analogue of the recursive derivation of the LMVE strategy in Lemma 3.2 in discrete time.

Lemma 4.14. *If the mean-variance tradeoff process K is bounded and continuous, the mapping $\widehat{J}(\vartheta) = \frac{1}{\gamma}\lambda - \xi(\vartheta)$ is a contraction on $(\Theta, \|.\|_{\beta,\infty})$ with modulus of contraction $c \in (0,1)$ where*

$$\|\vartheta\|_{\beta,\infty} := \left\| \left(\int_0^T \frac{1}{\mathcal{E}(-\beta K)_u} \vartheta_u^\top d\langle M\rangle_u \vartheta_u \right)^{\frac{1}{2}} \right\|_{L^2(P)}.$$

In particular, the locally mean-variance efficient strategy $\widehat{\vartheta}$ is given as the limit

$$\widehat{\vartheta} = \lim_{n\to\infty} \vartheta^n$$

in $(\Theta, \|.\|_{\beta,\infty})$, where $\vartheta^{n+1} = \widehat{J}(\vartheta^n)$ for $n \geq 1$, for any starting value $\vartheta^0 = \vartheta \in \Theta$.

Proof. Integrating both sides of (4.22) with respect to M and using the definition of $\xi(\vartheta)$ we obtain

$$\int_0^T \widehat{J}_u(\vartheta)dM_u = \int_0^T \frac{1}{\gamma}\lambda_u dM_u + Y_0(\vartheta) + L_T(\vartheta) - \int_0^T \vartheta_u dA_u$$

and from this

$$\frac{1}{\gamma}K_T - \int_0^T \left(\frac{1}{\gamma}\lambda_u - \vartheta_u\right) dA_u = Y_0(\vartheta) + \int_0^T \left(\frac{1}{\gamma}\lambda_u - \widehat{J}_u(\vartheta)\right) dM_u + L_T(\vartheta)$$

after rearranging terms and inserting the zero term $\frac{1}{\gamma}K_T - \frac{1}{\gamma}\int_0^T \lambda_u dA_u$. Comparing the last equation with the definition of the mapping J in the proof of Corollary 5 in [79] gives that $\widehat{J}(\vartheta) = \frac{1}{\gamma}\lambda - J\left(\frac{1}{\gamma}\lambda - \vartheta\right)$, as $L(\vartheta)$ is strongly orthogonal to M and therefore the right-hand side is the GKW decomposition of the left-hand side. If K is bounded and continuous, it follows from the arguments in the proof of Corollary 5 in [79] that $J : (\Theta, \|.\|_{\beta,\infty}) \to (\Theta, \|.\|_{\beta,\infty})$, and hence also $\widehat{J}$, is a contraction with modulus of contraction $c \in (0,1)$, which immediately implies that the sequence (ϑ^n) converges to $\widehat{\vartheta}$ for any starting value $\vartheta^0 = \vartheta \in \Theta$ by Banach's fixed point theorem. □

Remarks 4.15. 1) Note that this proves that in our setting, the locally mean-variance efficient strategy $\widehat{\vartheta}$ can indeed be obtained by the iteration procedure suggested in [9].

2) If the jumps of K are uniformly bounded by some constant $b \in (0,1)$, it follows from the remark following Corollary 5 in [79] that J and therefore $\widehat{J}$ are still contractions on $(\Theta, \|.\|_{\beta,\infty})$ with modulus of contraction $c \in (0,1)$; see also Lemma 5.6 later.

3) Using the "salami technique" in [73], one can show that the iterations still converge if K is only bounded, even though the modulus of contraction c is then not necessarily in $(0,1)$.

V.5 Convergence of solutions

To establish a link between the intuitive situation in discrete time, where the time-consistent optimal strategy is found by a backward recursion, and the continuous-time formulation given by a limiting argument, we show that the solutions obtained in discretisations of a continuous-time model converge to the solution in continuous time. This underlines that our formulation in continuous time is indeed the natural extension of that in discrete time. For this result, however, we need to discretise in an appropriate sense.

Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $[0,T]$ such that $|\tau_n| \to 0$ and assume for simplicity that S is one dimensional, i.e. $d = 1$. Then we choose $B = \langle M \rangle$ and set $P_B = P_{\langle M \rangle}$ which we deliberately denote by P_M in this section. Moreover, we denote by S^n the RCLL discretisation of S with respect to the partition τ_n, which is given by $S^n_{t_i} = S_{t_i}$ for all $t_i \in \tau_n$ and constant on $[t_i, t_{i+1})$, and by $\mathbb{F}^n = (\mathcal{F}^n_t)_{0\leq t\leq T}$ the filtration given by $\mathcal{F}^n_t = \mathcal{F}_{t_i}$ for $t \in [t_i, t_{i+1})$. This discretisation corresponds to the situation that we only trade at a finite number of given trading dates $t_i \in \tau_n$. Under the assumption that $S = S_0 + M + A$ is square-integrable, all S^n are square-integrable semimartingales on $(\Omega, \mathcal{F}, \mathbb{F}^n, P)$ with Doob decompositions $S^n = S_0 + \bar{M}^n + \bar{A}^n$ in $\mathbb{F}^n$ as constructed in Section V.3. Since the processes $\bar{M}^n$ and $\bar{A}^n$ there are a priori only defined on τ_n, we extend them to piecewise constant right-continuous processes on $[0,T]$ by taking $\bar{M}^n_t = \bar{M}^n_{t_i}$ and $\bar{A}^n_t = \bar{A}^n_{t_i}$ for $t \in [t_i, t_{i+1})$ and $t_i \in \tau_n$, which is consistent with the Doob–Meyer decomposition of the semimartingale S^n with respect to the filtration $\mathbb{F}^n$. This will be the usual embedding we use to include the discrete-time case into the continuous-time framework (as for example explained in Sections I.1f and I.4g in [52]). Note that $\bar{M}^n$ and $\bar{A}^n$ are not obtained by discretising the continuous-time processes M and A in the same way as we obtain S^n from S; this explains the choice of notation, and it is the source of the difficulties in proving our result. For later references we denote by $\mathcal{M}^2_0(P, \mathbb{F}^n)$ the space of all square-integrable $\mathbb{F}^n$-martingales null at zero and by $\mathcal{H}^2(P, \mathbb{F}^n)$ the space of all special $\mathbb{F}^n$-semimartingales with finite $\mathcal{H}^2(\mathbb{F}^n)$-norm.

To ensure the existence of a solution in the continuous-time setting, we assume the conditions of Corollary 4.12. These also yield the existence of solutions in all discretised settings, in which we have

$$\lambda^n = \sum_{t_i, t_{i+1} \in \tau_n} \frac{\Delta \bar{A}^n_{t_{i+1}}}{E[(\Delta \bar{M}^n_{t_{i+1}})^2 | \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]}$$

and

$$K^n_T = \sum_{t_i, t_{i+1} \in \tau_n} \frac{\Delta \bar{A}^n_{t_{i+1}}}{E[(\Delta \bar{M}^n_{t_{i+1}})^2 | \mathcal{F}_{t_i}]} \Delta \bar{A}^n_{t_{i+1}}.$$

Since we are changing our optimisation criterion each time we increase the partition, we cannot use the elegant approximation techniques for standard utility maximisation problems as in [57] to obtain the convergence of the solutions. Instead, we have to work directly with the structure of the solution. We exploit that we have $\widehat{\vartheta}^n = \frac{1}{\gamma}(\lambda^n - \widehat{\xi}^n)$ and $\widehat{\vartheta} = \frac{1}{\gamma}(\lambda - \widehat{\xi})$ as global descriptions in discrete as well as in continuous time, where $\widehat{\xi}^n$ is the integrand in the discrete-time Föllmer–Schweizer decomposition of K^n_T with respect to S^n and $(\Omega, \mathcal{F}, \mathbb{F}^n, P)$, i.e.

$$K^n_T = \widehat{K}^n_0 + \int_0^T \widehat{\xi}^n_u dS^n_u + \widehat{L}^n_T = \widehat{K}^n_0 + \sum_{t_i \in \tau_n} \widehat{\xi}^n_{t_i} \Delta S^n_{t_i} + \widehat{L}^n_T$$

for $n \in \mathbb{N}$, and $\widehat{\xi}$ is the integrand in the continuous-time Föllmer–Schweizer decomposition of K_T with respect to S, i.e.

$$K_T = \widehat{K}_0 + \int_0^T \widehat{\xi}_u dS_u + \widehat{L}_T.$$

For the proof of the convergence $\widehat{\vartheta}^n = \frac{1}{\gamma}(\lambda^n - \widehat{\xi}^n) \overset{L^2(M)}{\longrightarrow} \widehat{\vartheta} = \frac{1}{\gamma}(\lambda - \widehat{\xi})$ we then show that

$$\lambda^n \overset{L^2(M)}{\longrightarrow} \lambda^\infty := \lambda \tag{5.1}$$

and

$$\widehat{\xi}^n \overset{L^2(M)}{\longrightarrow} \widehat{\xi}^\infty := \widehat{\xi} \tag{5.2}$$

separately. For the latter we also need to establish that

$$K_T^n \overset{L^2(P)}{\longrightarrow} K_T^\infty := K_T. \tag{5.3}$$

The main difficulty is that the canonical decomposition is not stable under discretisation in the following sense. As already pointed out, $\bar{M}^n$ and $\bar{A}^n$ are not simply obtained by discretising M and A to $M_t^n := M_{t_i}$ and $A_t^n := A_{t_i}$ for $t \in [t_i, t_{i+1})$. From the discrete-time Doob decomposition, they are rather given by the processes $\bar{M}_t^n := M_t^n + M_t^{A,n}$, where $M_t^{A,n} := \sum_{k=1}^i (\Delta A_{t_k}^n - E[\Delta A_{t_k}^n | \mathcal{F}_{t_{k-1}}])$, and $\bar{A}_t^n := \sum_{k=1}^i E[\Delta A_{t_k}^n | \mathcal{F}_{t_{k-1}}]$ for $t \in [t_i, t_{i+1})$. Note that we set $\langle M^n \rangle := \langle M^n \rangle^{\mathbb{F}^n}$, $\langle \bar{M}^n \rangle := \langle \bar{M}^n \rangle^{\mathbb{F}^n}$ and $\langle M^{A,n} \rangle := \langle M^{A,n} \rangle^{\mathbb{F}^n}$ to simplify notation. For the $\mathbb{F}^n$-martingale $M^{A,n}$, which represents the "discretisation error" in the canonical decomposition, we already know from Lemma 4.4 that

$$\lim_{n\to\infty} \frac{d\langle M^{A,n}\rangle}{d\langle M^n \rangle} = \lim_{n\to\infty} \sum_{t_{i-1},t_i \in \tau_n} \frac{\operatorname{Var}\left[A_{t_i} - A_{t_{i-1}} | \mathcal{F}_{t_{i-1}}\right]}{E[\langle M\rangle_{t_i} - \langle M\rangle_{t_{i-1}} | \mathcal{F}_{t_{i-1}}]} \mathbb{1}_{(t_{i-1},t_i]} = 0$$

P_M-a.e. Moreover, if $\lambda \bullet M \in bmo_2$, we have

$$\begin{aligned}
\operatorname{Var}\left[A_{t_i} - A_{t_{i-1}} | \mathcal{F}_{t_{i-1}}\right] &\le E\left[(A_{t_i} - A_{t_{i-1}})^2 \middle| \mathcal{F}_{t_{i-1}}\right] \\
&= E\left[\left(\int_{t_i}^{t_{i-1}} \lambda_u d\langle M\rangle_u\right)^2 \middle| \mathcal{F}_{t_{i-1}}\right] \\
&= E\left[\left(\int_{t_i}^{t_{i-1}} \lambda_u^2 d\langle M\rangle_u\right)\left(\int_{t_i}^{t_{i-1}} d\langle M\rangle_u\right) \middle| \mathcal{F}_{t_{i-1}}\right] \\
&\le \left\|(\mathbb{1}_{(t_{i-1},t_i]}\lambda) \bullet M\right\|_{bmo_2}^2 E\left[\int_{t_{i-1}}^{t_i} d\langle M\rangle_u \middle| \mathcal{F}_{t_{i-1}}\right]
\end{aligned} \tag{5.4}$$

by applying Jensen's inequality and the definition of the bmo_2-norm, which gives

$$\left\|\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}\right\|_{L^\infty(P_M)} \le \sup_{t_{i-1},t_i \in \tau_n} \left\|(\mathbb{1}_{(t_{i-1},t_i]}\lambda) \bullet M\right\|_{bmo_2}^2 \le \left\|\lambda \bullet M\right\|_{bmo_2}^2. \tag{5.5}$$

However, to obtain the convergences (5.1)–(5.3) above, we shall finally need to use that $\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle} \longrightarrow 0$ in $L^\infty(P_M)$, and we also need a tight control in $L^\infty(P_M)$ on the K^n_T and their jumps, for an arbitrary increasing sequence of partitions tending to the identity. A sufficient condition for this is given in the following lemma.

Lemma 5.1. *Assume that* $K = \int \mu^K dt$ *and that* μ^K *is uniformly bounded in* ω *and* t *by some constant* $c_\mu > 0$. *Then:*

1) $\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle} \overset{L^\infty(P_M)}{\longrightarrow} 0$, *which implies* $\frac{d\langle \bar{M}^n\rangle}{d\langle M^n\rangle} \overset{L^\infty(P_M)}{\longrightarrow} 1$ *and* $\frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle} \overset{L^\infty(P_M)}{\longrightarrow} 1$.
2) *There exist* $n_0 \in \mathbb{N}$ *and* $b \in (0,1)$ *such that* $\sup_{n\geq n_0} \|K^n_T\|_{L^\infty(P)}$ *is finite and* $\sup_{n\geq n_0} \|(\Delta K^n)^*_T\|_{L^\infty(P)} \leq b$, *and moreover* $(\Delta K^n)^*_T \to 0$ *in* $L^\infty(P)$.

Proof. 1) This immediately follows from (5.5) above and observing that

$$\|(\lambda \mathbb{1}_{(s,t]}) \bullet M\|^2_{bmo_2} \leq \sup_{s\leq u\leq t} \|E[K_t - K_u|\mathcal{F}_u]\|_{L^\infty(P)} \leq c_\mu(t-s).$$

From $\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle} \overset{L^\infty(P_M)}{\longrightarrow} 0$ we then obtain that $\frac{d\langle \bar{M}^n\rangle}{d\langle M^n\rangle} \overset{L^\infty(P_M)}{\longrightarrow} 1$ by using $\bar{M}^n = M^n + M^{A,n}$ and the Cauchy–Schwarz inequality. The latter convergence also implies that $\frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle} \overset{L^\infty(P_M)}{\longrightarrow} 1$.
2) Since $\frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle} \longrightarrow 1$ in $L^\infty(P_M)$, we can choose $n_0 \in \mathbb{N}$ such that we have $\sup_{n\geq n_0} \left\|\frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle}\right\|_{L^\infty(P_M)} \leq c$ for some $c > 0$. By the Cauchy–Schwarz inequality we can estimate

$$\begin{aligned}(\Delta \bar{A}^n_{t_{i+1}})^2 &= \left(E\left[\int_{t_i}^{t_{i+1}} \lambda_u d\langle M\rangle_u \middle| \mathcal{F}_{t_i}\right]\right)^2 \\ &\leq E[K_{t_{i+1}} - K_{t_i}|\mathcal{F}_{t_i}] E\left[\int_{t_i}^{t_{i+1}} d\langle M\rangle_u \middle| \mathcal{F}_{t_i}\right],\end{aligned}$$

which gives for $n \geq n_0$ that

$$\begin{aligned}\|(\Delta K^n)^*_T\|_{L^\infty(P)} &= \left\| \sup_{t_i,t_{i+1}\in\tau_n} \frac{(\Delta \bar{A}^n_{t_{i+1}})^2}{E[(\Delta \bar{M}^n_{t_{i+1}})^2|\mathcal{F}_{t_i}]} \right\|_{L^\infty(P)} \\ &\leq \left\| \frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle} \right\|_{L^\infty(P_M)} \left\| \sup_{t_i,t_{i+1}\in\tau_n} E[K_{t_{i+1}} - K_{t_i}|\mathcal{F}_{t_i}] \right\|_{L^\infty(P)} \\ &\leq c_\mu c|\tau_n| \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

By the same arguments we obtain $\|\Delta K^n_{t_{i+1}}\|_{L^\infty(P)} \leq c_\mu c(t_{i+1}-t_i)$ for $n \geq n_0$ and therefore $\sup_{n\geq n_0} \|K^n_T\|_{L^\infty(P)} \leq c_\mu cT$ after summing up. This completes the proof. □

Because $\frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle} \overset{L^\infty(P_M)}{\longrightarrow} 1$ implies the existence of some $n_0 \in \mathbb{N}$ and $c > 0$ such that $\sup_{n\geq n_0} \|\frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle}\|_{L^\infty(P_M)} \leq c$, we can already prove (5.1) via the next lemma.

Lemma 5.2. *Let $\lambda \in L^2(M)$ and assume that $\left\|\frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle}\right\|_{L^\infty(P_M)} \leq c$ for some $c > 0$. Then $\lambda^n \overset{L^2(M)}{\longrightarrow} \lambda$.*

Proof. Using (SC), we can write

$$\begin{aligned}\lambda^n &= \sum_{t_i,t_{i+1}\in\tau_n} \frac{E[\int_{t_i}^{t_{i+1}} \lambda_u d\langle M\rangle_u|\mathcal{F}_{t_i}]}{E[\int_{t_i}^{t_{i+1}} d\langle M\rangle_u|\mathcal{F}_{t_i}]} \frac{E[(\Delta M^n_{t_{i+1}})^2|\mathcal{F}_{t_i}]}{E[(\Delta \bar{M}^n_{t_{i+1}})^2|\mathcal{F}_{t_i}]} \mathbb{1}_{(t_i,t_{i+1}]} \\ &= E_M\left[\,\lambda\,\middle|\,\mathcal{P}^{\tau_n}\,\right] \frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle}.\end{aligned}$$

Since the σ-fields $\mathcal{P}^{\tau_n}$ increase to the predictable σ-field $\mathcal{P}$ and λ is in $L^2(P_M)$ and predictable, $\big(E_M[\,\lambda\,|\,\mathcal{P}^{\tau_n}\,]\big)_{n\in\mathbb{N}}$ is a square-integrable martingale on $\big(\Omega\times[0,T],\mathcal{P},(\mathcal{P}^{\tau_n})_{n\in\mathbb{N}},P_M\big)$ which converges to λ P_M-a.e. and in $L^2(P_M)$ by the martingale convergence theorem. To conclude the assertion, we use the following simple fact with $X^n = \lambda^n$, $Y^n = \frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle}$ and $P = P_M$. Let (X^n) and (Y^n) be two sequences of random variables such that $X^n \to X$ P-a.s. and in $L^2(P)$, $Y^n \to Y$ P-a.s. and $\|Y^n\|_{L^\infty(P)} \leq c$ and $\|Y\|_{L^\infty(P)} \leq c$ for some $c > 0$. Then $X^nY^n \to XY$ P-a.s. and in $L^2(P)$. Due to the estimate

$$\begin{aligned}\|X^nY^n - XY\|_{L^2(P)} &\leq \|(X^n - X)Y^n\|_{L^2(P)} + \|X(Y^n - Y)\|_{L^2(P)} \\ &\leq c\|X^n - X\|_{L^2(P)} + 2c\|X\|_{L^2(P)}\end{aligned}$$

this can be seen by using that $X^nY^n \to XY$ P-a.s. and Lebesgue's dominated convergence with majorant $2c|X| \in L^2(P)$, which completes the proof. □

For the proof of (5.3) we establish the following result which is slightly more general than we actually need.

Lemma 5.3. *Let $\lambda \bullet M \in bmo_2$ and assume that $\xi^n \overset{L^2(M)}{\longrightarrow} \xi$ and that ξ^n is $\mathcal{P}^{\tau_n}$-measurable for each $n \in \mathbb{N}$. Then $\xi^n \bullet \bar{A}^n_T \to \xi \bullet A_T$ in $L^2(P)$.*

Proof. As each ξ^n is piecewise constant along τ_n, we obtain

$$
\begin{aligned}
E\left[\left(\xi^n \bullet \bar{A}^n_T - \xi \bullet A_T\right)^2\right] &= E\left[\left(\sum_{t_i\in\tau_n} \xi^n_{t_i}(\Delta\bar{A}^n_{t_i} - \Delta A^n_{t_i}) + (\xi^n - \xi)\bullet A_T\right)^2\right] \\
&= E\left[\left(\sum_{t_i\in\tau_n} -\xi^n_{t_i}\Delta M^{A,n}_{t_i} - (\xi^n - \xi)\bullet A_T\right)^2\right] \\
&\leq 2E\left[\left(\xi^n \bullet M^{A,n}_T\right)^2\right] + 2E\left[\left((\xi^n - \xi)\bullet A_T\right)^2\right]
\end{aligned}
$$

and therefore that

$$
E\left[\left(\xi^n \bullet \bar{A}^n_T - \xi \bullet A_T\right)^2\right] \leq 2E\big[(\xi^n)^2 \bullet \langle M^{A,n}\rangle_T\big] + 2\|\xi^n - \xi\|^2_{L^2(A)} \tag{5.6}
$$

by Itô's isometry, since $\xi^n \bullet M^{A,n} \in \mathcal{M}^2_0(P,\mathbb{F}^n)$. Replacing $\langle M^{A,n}\rangle$ by $\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle} \bullet \langle M^n\rangle$ and using that $\xi^n \in L^2(M)$ and $\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}$ are piecewise constant along τ_n, we can write

$$
\begin{aligned}
E\big[(\xi^n)^2 \bullet \langle M^{A,n}\rangle_T\big] &= E\left[\left(\xi^n\sqrt{\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}}\right)^2 \bullet \langle M\rangle_T\right] \\
&= E_M\left[\left(\xi^n\sqrt{\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}}\right)^2\right].
\end{aligned}
$$

Moreover, $\left(\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}\right)_{n\in\mathbb{N}}$ is bounded in $L^\infty(P_M)$ due to (5.5). Applying again the simple fact from the proof of the previous lemma, this time with $X^n = \xi^n$, $Y^n = \sqrt{\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}}$ and $P = P_M$, we obtain that $\left(\xi^n\sqrt{\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}}\right)$ converges to 0 in $L^2(P_M)$. To complete the proof we observe that the second term on the right-hand side of (5.6) also vanishes, since we have $\|\xi^n - \xi\|^2_{L^2(A)} \leq 8\|\lambda\cdot M\|_{bmo_2}\big\|\xi^n - \xi\big\|^2_{L^2(M)}$ by Theorem 3.3 in [29]. By combining Jensen's inequality with the definition of the bmo_2-norm as in the last line of (5.4), we can replace the constant 8 actually by 1. □

Now (5.3) follows immediately by combining the two previous lemmas.

Corollary 5.4. *Let $\lambda\cdot M \in bmo_2$ and assume that $\left\|\frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle}\right\|_{L^\infty(P_M)} \leq c$ for some $c > 0$. Then $K^n_T \overset{L^2(P)}{\longrightarrow} K_T$.*

To conclude the convergence of the LMVE strategies, it then remains to show (5.2). For this we establish the convergence of the discrete Föllmer–Schweizer decompositions obtained in a sequence of discretisations of a financial market as the partitions tend to the identity. More precisely, we want to prove the following result.

Theorem 5.5. *Suppose that K is bounded, $\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle} \overset{L^\infty(P_M)}{\longrightarrow} 0$ and that there exist $n_0 \in \mathbb{N}$ and $b \in (0,1)$ such that $\sup_{n\geq n_0} \|K_T^n\|_{L^\infty(P)} < \infty$ and $\sup_{n\geq n_0} \|(\Delta K^n)_T^*\|_{L^\infty(P)} \leq b$. Let $H^n, H \in L^2(P)$ be contingent claims and $(\tau_n)_{n\in\mathbb{N}}$ an increasing sequence of partitions of $[0,T]$. Write the Föllmer–Schweizer decompositions of H^n and H with respect to S^n on $(\Omega, \mathcal{F}, \mathbb{F}^n, P)$ and S on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ as*

$$H^n = \widehat{H}_0^n + \int_0^T \widehat{\xi}_u^n dS_u^n + \widehat{L}_T^n = \widehat{H}_0^n + \sum_{t_i\in\tau_n} \widehat{\xi}_{t_i}^n \Delta S_{t_i}^n + \widehat{L}_T^n \tag{5.7}$$

and

$$H = \widehat{H}_0 + \int_0^T \widehat{\xi}_u dS_u + \widehat{L}_T. \tag{5.8}$$

Then $\widehat{\xi}^n$ converges to $\widehat{\xi}$ in $L^2(P_M)$, if $H^n \to H$ in $L^2(P)$ and $|\tau_n| \to 0$.

For the rest of the section, we always work under the assumptions of Theorem 5.5. To simplify notation we set $H^\infty := H$, $S^\infty := S$, $\widehat{\xi}^\infty := \widehat{\xi}$, $\bar{M}^\infty := M^\infty = M$, $\bar{A}^\infty := A$, $K^\infty := K$ etc. Note that $M^{A,\infty} = 0$. As we deal with GKW decompositions with respect to different martingales, we denote the GKW decomposition of a random variable $H \in L^2(P)$ with respect to $X \in \mathcal{M}_0^2(P, \mathbb{F}^n)$ for some $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$ by

$$H = E[H|\mathcal{F}_0] + \int_0^T \xi_u(X,H) dX_u + L_T(X,H),$$

if we need to clarify the dependence on H and X. If $n \in \mathbb{N}$, i.e. in discrete time, we have

$$\xi_t(X,H) = \frac{E\big[H\Delta X_{t_i}\big|\mathcal{F}_{t_{i-1}}\big]}{E\big[(\Delta X_{t_i})^2\big|\mathcal{F}_{t_{i-1}}\big]}$$

for $t \in [t_i, t_{i+1})$. The first step in the proof of Theorem 5.5 is then to observe that the Föllmer–Schweizer decomposition can be obtained under our assumptions by a fixed point iteration, as is shown in Lemma 5.6 below. This is basically the proof of Corollary 5 in [79] and the remark following that. However, as we are interested in the convergence of different Föllmer–Schweizer decompositions, we need to establish that several constants are independent of n. This allows us to adapt the method of proof of [12] and [13] to our situation. That method is used there to show the convergence of solutions to discretisations of a continuous-time BSDE to the solution in continuous time. Denoting by $\xi^{\infty,p}$ the p-th step of the fixed point iteration leading to $\widehat{\xi}^n$, for $n \in \overline{\mathbb{N}}$, (where $\widehat{\xi}^\infty = \widehat{\xi}$) gives the decomposition

$$\widehat{\xi}^n - \widehat{\xi} = (\widehat{\xi}^n - \xi^{n,p}) + (\xi^{n,p} - \xi^{\infty,p}) + (\xi^{\infty,p} - \widehat{\xi}).$$

To establish the convergence of the FS decompositions, it then remains to show that $\xi^{n,p}$ converges to $\widehat{\xi}^n$ in $L^2(M)$ for sufficiently large n uniformly in

n as $p \to \infty$, and that $\xi^{n,p}$ converges to $\xi^{\infty,p}$ in $L^2(M)$ for each $p \in \mathbb{N}_0$ as $n \to \infty$, which will be done in Propositions 5.7 and 5.8.

Lemma 5.6. *Under the assumptions of Theorem 5.5 there exist $n_0 \in \mathbb{N}$ and $b \in (0,1)$ such that the following hold for all $n \in \overline{\mathbb{N}}_{\geq n_0} := \{n \in \overline{\mathbb{N}} \mid n \geq n_0\}$:*

1) $\Theta_{S^n} = L^2(\bar{M}^n)$, *and*

$$\|\vartheta\|_{\beta,n} := \left\| \left(\int_0^T \frac{1}{\mathcal{E}(-\beta K^n)_u} \vartheta_u d\langle \bar{M}^n \rangle_u \vartheta_u \right)^{\frac{1}{2}} \right\|_{L^2(P)}$$

defines a norm on Θ_{S^n} which is equivalent to $\|.\|_{L^2(\bar{M}^n)}$ for any $\beta \in (0, \frac{1}{b})$, where the equivalence constant k can be chosen independent of n, e.g.

$$k = \max\left(\exp\left(\frac{\beta}{1-\beta b} \sup_{n \geq n_0} \|K_T^n\|_{L^\infty(P)} \right), \left\| \frac{1}{\mathcal{E}(-\beta K^\infty)_T} \right\|_{L^\infty(P)} \right).$$

2) *The mapping $J^n : \Theta_{S^n} \to \Theta_{S^n}$ which maps $\vartheta \in \Theta_{S^n}$ into the integrand*

$$\xi\left(\bar{M}^n, H^n - \int_0^T \vartheta_u d\bar{A}_u^n \right)$$

of $\bar{M}^n$ in the Galtchouk-Kunita-Watanabe decomposition of $H^n(\vartheta) := H^n - \int_0^T \vartheta_u d\bar{A}_u^n$, i.e.,

$$H^n(\vartheta) = E\left[H^n(\vartheta)|\mathcal{F}_0\right] + \int_0^T \xi_u(\bar{M}^n, H^n(\vartheta)) d\bar{M}_u^n + L_T^n(\bar{M}^n, H^n(\vartheta)),$$

is a contraction on $(\Theta_{S^n}, \|.\|_{\beta,n})$ with a modulus of contraction $c \in (0,1)$ that can be chosen independent of n, for any $\beta \in (1, \frac{1}{b})$.

3) *The integrand $\widehat{\xi}^n$ in the Föllmer–Schweizer decomposition is given as the limit*

$$\widehat{\xi}^n = \xi^{n,\infty} = \lim_{p \to \infty} \xi^{n,p}$$

in $(\Theta_{S^n}, \|.\|_{\beta,n})$, where $\xi^{n,0} = 0$ and $\xi^{n,p} = J^n(\xi^{n,p-1})$ for all $p \in \mathbb{N}$.

Proof. 1) Under the assumptions of Theorem 5.5, there exists $n_0 \in \mathbb{N}$ with

$$\sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \|K_T^n\|_{L^\infty(P)} < \infty$$

and therefore

$$\|\vartheta\|_{L^2(\bar{M}^n)} \leq \|\vartheta\|_{\Theta_{S^n}} \leq \left(1 + \sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \|K_T^n\|_{L^\infty(P)}^{\frac{1}{2}}\right) \|\vartheta\|_{L^2(\bar{M}^n)},$$

which implies that $\Theta_{S^n} = L^2(\bar{M}^n)$ for all $n \in \overline{\mathbb{N}}_{\geq n_0}$. Moreover, since there exists $b \in (0,1)$ such that $\sup_{n \geq n_0} \|(\Delta K^n)^*_T\|_{L^\infty(P)} \leq b$, the process $\frac{1}{\mathcal{E}(-\beta K^n)} = \frac{1}{\prod_{0<s\leq \cdot}(1-\beta \Delta K^n_s)}$ is increasing such that $\frac{1}{\mathcal{E}(-\beta K^n)} \geq 1$ and

$$\left\| \left(\frac{1}{\mathcal{E}(-\beta K^n)} \right)^*_T \right\|_{L^\infty(P)} \leq \left\| \exp\left(\sum_{0<s\leq T} -\beta \log(1 - \beta \Delta K^n_s) \right) \right\|_{L^\infty(P)}$$
$$\leq \exp\left(\frac{\beta}{1-\beta b} \sup_{n \geq n_0} \|K^n_T\|_{L^\infty(P)} \right) < \infty$$

for all $n \geq n_0$ and any $\beta \in (0, \frac{1}{b})$. Since K^∞ is of finite variation, both parts of the decomposition $K^\infty = \sum \Delta K^\infty + (K^\infty - \sum \Delta K^\infty)$ exist. By estimating $1 \leq \frac{1}{\mathcal{E}(-\beta K^\infty)} = \frac{1}{\mathcal{E}(-\beta \sum \Delta K^\infty - \beta(K^\infty - \sum \Delta K^\infty))} \leq e^{\left(\frac{\beta}{1-\beta b}+\beta\right)\|K^\infty_T\|_{L^\infty(P)}}$ we therefore obtain that the increasing process $\frac{1}{\mathcal{E}(-\beta K^\infty)}$ is uniformly bounded and

$$\frac{1}{k}\|\vartheta\|_{L^2(\bar{M}^n)} \leq \|\vartheta\|_{\beta,n} \leq k\|\vartheta\|_{L^2(\bar{M}^n)}$$

holds with $k = \max\left(\exp\left(\frac{\beta}{1-\beta b} \sup_{n\geq n_0} \|K^n_T\|_{L^\infty(P)} \right), \left\| \frac{1}{\mathcal{E}(-\beta K^\infty)_T} \right\|_{L^\infty(P)} \right)$ for all $\vartheta \in \Theta_{S^n}$, for all $n \in \overline{\mathbb{N}}_{\geq n_0}$.

2) Following the remark after the proof of Corollary 5 in [79], we apply Proposition 1 in [79] with $\beta > \mu^2 > 1$, $\vartheta = \vartheta^1 - \vartheta^2$, $\psi = J^n(\vartheta^1) - J^n(\vartheta^2)$, $V_0 = H^n_0(\vartheta^1) - H^n_0(\vartheta^2)$, $L = L^n(\bar{M}^n, H(\vartheta^1)) - L^n(\bar{M}^n, H(\vartheta^2))$ and $C = \frac{1}{\mathcal{E}(-\beta K^n)}$ which gives that

$$\|J^n(\vartheta^1) - J^n(\vartheta^2)\|^2_{\beta,n} = E\left[\int_0^T \frac{1}{\mathcal{E}(-\beta K^n)_s} \psi_s d\langle \bar{M}^n \rangle_s \psi_s \right]$$
$$\leq \frac{1}{\mu^2} E\left[\int_0^T \frac{1}{\mathcal{E}(-\beta K^n)_s} \vartheta_s d\langle \bar{M}^n \rangle_s \vartheta_s \right]$$
$$= \frac{1}{\mu^2} \|\vartheta^1 - \vartheta^2\|^2_{\beta,n},$$

and therefore that J^n is a contraction on $(\Theta_{S^n}, \|.\|_{\beta,n})$ with $c := \frac{1}{\mu^2}$ as modulus of contraction for all $n \in \overline{\mathbb{N}}_{\geq n_0}$.

3) This is an immediate consequence of 2) and Banach's fixed point theorem. □

By part 3) of Lemma 5.6 each Föllmer–Schweizer decomposition can be obtained for sufficiently large n by a fixed point iteration in p. Then the next proposition says that these fixed point iterations converge for $p \to \infty$ even uniformly in n.

Proposition 5.7. *Under the assumptions of Theorem 5.5, there exists* $n_0 \in \mathbb{N}$ *such that*

$$\sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \|\xi^{n,p} - \widehat{\xi}^n\|_{L^2(M)} \overset{p\to\infty}{\longrightarrow} 0.$$

Proof. Using that there exist $n_0 \in \mathbb{N}$ and $b \in (0,1)$ by Lemma 5.6 such that the J^n are contractions on $(\Theta_{S^n}, \|.\|_{\beta,n})$ with a common modulus of contraction $c \in (0,1)$ independent of n, for any $\beta \in (1, \frac{1}{b})$, and that $\xi^{n,0} = 0$ for each $n \in \overline{\mathbb{N}}_{\geq n_0}$, we obtain that

$$\begin{aligned}\sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \left\|\xi^{n,p} - \widehat{\xi}^n\right\|_{L^2(\bar{M}^n)} &\leq k \sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \left\|\xi^{n,p} - \widehat{\xi}^n\right\|_{\beta,n} \\ &\leq kc^p \sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \left\|\widehat{\xi}^n\right\|_{\beta,n} \leq k^2 c^p \sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \left\|\widehat{\xi}^n\right\|_{L^2(\bar{M}^n)}.\end{aligned} \tag{5.9}$$

To get an estimate for the right-hand side of (5.9), we are going to use the continuity of the Föllmer–Schweizer decomposition and results on the equivalence of norms for $\mathcal{E}$-local martingales. To that end, we view each S^n on $(\Omega, \mathcal{F}, \mathbb{F}^n, P)$. There we have that $S^n = S_0 + \bar{M}^n + \lambda^n \bullet \langle \bar{M}^n \rangle$ is an $\mathcal{E}(-\lambda^n \bullet \bar{M}^n)$-martingale (recall Definition 4.10) by Corollary 3.17 in [16], and $\mathcal{E}(-\lambda^n \bullet \bar{M}^n)$ is regular and satisfies $R_2(P)$ with the same constant $\exp\left(\sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \|K^n_T\|_{L^\infty(P)}\right)$ for each $n \in \overline{\mathbb{N}}_{\geq n_0}$ by Proposition 3.7 in [16]. Therefore S^n admits a Föllmer–Schweizer decomposition by and in the sense of Theorem 5.5 in [16], which implies that $\|\widehat{\xi}^n \bullet S^n_T\|_{L^2(P)} \leq \|H^n\|_{L^2(P)}$ for all $n \in \overline{\mathbb{N}}_{\geq n_0}$. As the constant in $R_2(P)$ is the same for all $n \in \overline{\mathbb{N}}_{\geq n_0}$, an inspection of the proof of Theorem 4.9 in [16] yields that

$$\left\|\widehat{\xi}^n \bullet S^n\right\|_{\mathcal{H}^2(\mathbb{F}^n)} \leq \bar{c} \left\|\widehat{\xi}^n \bullet S^n_T\right\|_{L^2(P)}$$

also holds with the same constant $\bar{c} > 0$ for all $n \in \overline{\mathbb{N}}_{\geq n_0}$, which implies

$$\sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \left\|\widehat{\xi}^n\right\|_{L^2(\bar{M}^n)} \leq \sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \left\|\widehat{\xi}^n \bullet S^n\right\|_{\mathcal{H}^2(\mathbb{F}^n)} \leq \bar{c} \sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \|H^n\|_{L^2(P)}. \tag{5.10}$$

Moreover, as $\frac{d\langle \bar{M}^n \rangle}{d\langle M^n \rangle} \to 1$ in $L^\infty(P_M)$ by our assumptions and part 1) of Lemma 5.1, there exists a constant $\tilde{c} > 0$ such that

$$\frac{1}{\tilde{c}} \|\vartheta\|_{L^2(\bar{M}^n)} \leq \|\vartheta\|_{L^2(M)} \leq \tilde{c} \|\vartheta\|_{L^2(\bar{M}^n)}$$

for all $\vartheta \in \Theta_{S^n} = L^2(\bar{M}^n)$ and all $n \in \overline{\mathbb{N}}_{\geq n_0}$ by possibly enlarging n_0. Combining this with (5.9) and (5.10) gives that

$$\sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \left\|\xi^{n,p} - \widehat{\xi}^n\right\|_{L^2(M)} \leq k^2 c^p \bar{c} \tilde{c} \sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \left\|H^n\right\|_{L^2(P)} \overset{p\to\infty}{\longrightarrow} 0,$$

since $\sup_{n \in \overline{\mathbb{N}}_{\geq n_0}} \left\|H^n\right\|_{L^2(P)}$ is bounded because $H^n \to H$ in $L^2(P)$. This completes the proof. □

Before we can conclude the proof of Theorem 5.5, we need to establish not only the convergence of the fixed point iterations as the number of iterations p tends to infinity, but also at each step as the mesh of the partitions goes to 0.

Proposition 5.8. *Under the assumptions of Theorem 5.5,*

$$\|\xi^{n,p} - \xi^{\infty,p}\|_{L^2(M)} \overset{n\to\infty}{\longrightarrow} 0 \tag{5.11}$$

for each $p \in \mathbb{N}_0$.

Proof. We prove this by induction on $p \in \mathbb{N}_0$. To that end, we observe that (5.11) is clearly true for $p = 0$, as we have $\xi^{n,0} = \xi^{\infty,0} = 0$, and so we assume as induction hypothesis that (5.11) holds for $p \in \mathbb{N}_0$. By Lemma 5.2 this implies that

$$H^{n,p} := H^n - \int_0^T \xi_u^{n,p} d\bar{A}_u^n \longrightarrow H^{\infty,p} := H - \int_0^T \xi_u^{\infty,p} dA_u$$

in $L^2(P)$ as $n \to \infty$. For each $n \geq n_0$ we can write

$$\begin{aligned}
\xi_t^{n,p+1} &= \xi_t(\bar{M}^n, H^{n,p}) = \frac{E\left[H^{n,p}\Delta\bar{M}_{t_i}^n \middle| \mathcal{F}_{t_{i-1}}\right]}{E\left[(\Delta\bar{M}_{t_i}^n)^2 \middle| \mathcal{F}_{t_{i-1}}\right]} \\
&= \Bigg(\frac{E\big[H^{n,p}\Delta M_{t_i}^n \big| \mathcal{F}_{t_{i-1}}\big]}{E\big[(\Delta M_{t_i}^n)^2 \big| \mathcal{F}_{t_{i-1}}\big]} + \frac{E\big[H^{n,p}\Delta M_{t_i}^{A,n} \big| \mathcal{F}_{t_{i-1}}\big]}{E\big[(\Delta M_{t_i}^{A,n})^2 \big| \mathcal{F}_{t_{i-1}}\big]} \\
&\qquad \times \frac{E\big[(\Delta M_{t_i}^{A,n})^2 \big| \mathcal{F}_{t_{i-1}}\big]}{E\big[(\Delta M_{t_i}^n)^2 \big| \mathcal{F}_{t_{i-1}}\big]}\Bigg) \frac{\Delta\langle M^n\rangle_{t_i}}{\Delta\langle\bar{M}^n\rangle_{t_i}} \\
&= \left(\xi_t(M^n, H^{n,p}) + \xi_t(M^{A,n}, H^{n,p}) \left(\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}\right)_t\right) \left(\frac{d\langle M^n\rangle}{d\langle\bar{M}^n\rangle}\right)_t
\end{aligned} \tag{5.12}$$

for $t \in [t_i, t_{i+1})$ by plugging in $\bar{M}^n = M^n + M^{A,n}$ and the definition of the discrete-time GKW decomposition. Since

$$\|\xi(M^n, H^{n,p}) - \xi(M^n, H^{\infty,p})\|_{L^2(M)} \leq \|H^{n,p} - H^{\infty,p}\|_{L^2(P)} \to 0$$

as $n \to \infty$ by the orthogonality of the terms in the GKW decomposition and

$$\xi(M^n, H^{\infty,p}) \to \xi(M, H^{\infty,p}) = \xi^{\infty,p+1} \qquad \text{as } n \to \infty$$

in $L^2(M)$ by Theorem 3.1 in [51], we obtain that

$$\xi(M^n, H^{n,p}) \to \xi^{\infty,p+1} \qquad \text{as } n \to \infty \tag{5.13}$$

in $L^2(M)$. Moreover,

$$\begin{aligned}
&\left\|\xi(M^{A,n},H^{n,p})\tfrac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}\right\|_{L^2(M)}\\
&\leq \left\|\xi(M^{A,n},H^{n,p})\right\|_{L^2(M^{A,n})}\left\|\sqrt{\tfrac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}}\right\|_{L^\infty(P_M)}\\
&\leq \left\|H^{n,p}\right\|_{L^2(P)}\left\|\sqrt{\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}}\right\|_{L^\infty(P_M)} \longrightarrow 0 \qquad \text{as } n\to\infty
\end{aligned} \tag{5.14}$$

by our assumptions. Since these also give via part 1) of Lemma 5.1 that $\frac{d\langle M^n\rangle}{d\langle \bar{M}^n\rangle}\to 1$ in $L^\infty(P_M)$, combining (5.12)–(5.14) implies that

$$\xi^{n,p+1} \overset{L^2(M)}{\longrightarrow} \xi^{\infty,p+1} \qquad \text{as } n\to\infty,$$

which completes the proof. □

Now we have everything in place to finish the proof of Theorem 5.5.

Proof of Theorem 5.5. The only remaining point is to show that we can control each of the terms in the decomposition

$$\widehat{\xi}^n-\widehat{\xi}=(\widehat{\xi}^n-\xi^{n,p})+(\xi^{n,p}-\xi^{\infty,p})+(\xi^{\infty,p}-\widehat{\xi})$$

in a sufficient way. To that end, fix an arbitrary $\varepsilon>0$. Then we choose n_0 and p in $\mathbb{N}$ such that $\sup_{n\geq n_0}\|\xi^{n,p}-\widehat{\xi}^n\|_{L^2(M)}\leq\varepsilon$ and $\|\xi^{\infty,p}-\widehat{\xi}\|_{L^2(M)}\leq\varepsilon$ by Lemma 5.6 and Proposition 5.7. By possibly enlarging n_0, Proposition 5.8 allows us to obtain that $\|\xi^{n,p}-\xi^{\infty,p}\|_{L^2(M)}\leq\varepsilon$ for all $n\geq n_0$ and therefore that

$$\begin{aligned}
\|\widehat{\xi}^n-\widehat{\xi}\|_{L^2(M)}\leq \sup_{n\geq n_0}&\|\xi^{n,p}-\widehat{\xi}^n\|_{L^2(M)}\\
&+\|\xi^{n,p}-\xi^{\infty,p}\|_{L^2(M)}+\|\xi^{\infty,p}-\widehat{\xi}\|_{L^2(M)}\leq 3\varepsilon,
\end{aligned}$$

which completes the proof. □

Combining the previous results then gives the convergence of the LMVE strategies.

Theorem 5.9. *Suppose that K is bounded, $\frac{d\langle M^{A,n}\rangle}{d\langle M^n\rangle}\overset{L^\infty(P_M)}{\longrightarrow}0$ and that there exist $n_0\in\mathbb{N}$ and $b\in(0,1)$ such that $\sup_{n\geq n_0}\|K^n_T\|_{L^\infty(P)}<\infty$ and $\sup_{n\geq n_0}\|(\Delta K^n)^*_T\|_{L^\infty(P)}\leq b$. Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $[0,T]$ and $\widehat{\vartheta}^n$ be the LMVE strategy with respect to S^n on $(\Omega,\mathcal{F},\mathbb{F}^n,P)$ and $\widehat{\vartheta}$ the LMVE strategy with respect to S on $(\Omega,\mathcal{F},\mathbb{F},P)$. Then $\widehat{\vartheta}^n$ converges to $\widehat{\vartheta}^n$ in $L^2(M)$ as $|\tau_n|\to 0$.*

Proof. Since $K^n = \langle \lambda^n \bullet M^n \rangle^{\mathbb{F}^n}$ and $K = \langle \lambda \bullet M \rangle$ are bounded, $\mathcal{E}(\lambda^n \bullet M^n)$ and $\mathcal{E}(\lambda \bullet M)$ satisfy $R_2(P)$ and are regular with respect to $\mathbb{F}^n$ and $\mathbb{F}$, respectively, by Proposition 3.7 in [16]. By Corollary 4.12 this implies that $\widehat{\vartheta}^n$ and $\widehat{\vartheta}$ exist and are given by $\widehat{\vartheta}^n = \frac{1}{\gamma}(\lambda^n - \widehat{\xi}^n)$ and $\widehat{\vartheta} = \frac{1}{\gamma}(\lambda - \widehat{\xi})$, where $\widehat{\xi}^n$ and $\widehat{\xi}$ denote the integrand of the Föllmer–Schweizer decomposition of K_T^n and K_T. Since $K = \langle \lambda \bullet M \rangle$ is bounded and hence $\lambda \bullet M$ is in bmo_2, the convergence of $\widehat{\vartheta}^n$ to $\widehat{\vartheta}$ in $L^2(M)$ follows by combining Lemma 5.2, Corollary 5.4 and Theorem 5.5, which completes the proof. □

V.6 Appendix: Representative square-integrable portfolios

In this appendix we show the existence of representative square-integrable portfolios as announced in Section V.2. As stated in Lemma 6.1 below, these are strategies $\varphi^i \in \Theta_S$ for $i = 1, \ldots, d$, which are representative in the sense that the financial market $(\widetilde{S}, \Theta_{\widetilde{S}})$ with $\widetilde{S}^i := \varphi^i \bullet S$ for $i = 1, \ldots, d$ generates the same wealth processes as the financial market (S, Θ_S), i.e. $\Theta_S \bullet S = \Theta_{\widetilde{S}} \bullet \widetilde{S}$. To obtain these we use the notion of σ-square-integrability: A semimartingale X is *σ-square-integrable*, which we denote by $X \in \mathcal{H}^2_\sigma(P)$, if there exists an increasing sequence (D_n) of predictable sets such that $D_n \uparrow \overline{\Omega}$ and $\mathbb{1}_{D_n} \bullet X \in \mathcal{H}^2(P)$ for each n. The basic idea for the proof is then the following. Even though square-integrability is a global property of the strategy ϑ it implies that ϑ is σ-square-integrable, i.e. $\vartheta \bullet S \in \mathcal{H}^2_\sigma(P)$, which can be characterised (ω, t)-pointwise. Since there exists a one-to-one correspondence between σ-square-integrable and square-integrable integrands by Proposition 2 in [42] (see below), the (ω, t)-pointwise characterisation of σ-square-integrability is sufficient to find the representative square-integrable portfolios. To derive this characterisation we need to work with the notion of predictable characteristics which we introduce next.

As in [52], Theorem II.2.34, each semimartingale S has the *canonical representation*

$$S = S_0 + S^c + \widetilde{A} + [x\mathbb{1}_{\{|x|\leq 1\}}] * (\mu - \nu) + [x\mathbb{1}_{\{|x|>1\}}] * \mu$$

with the jump measure μ of S and its predictable compensator ν. Then the quadruple (b, c, F, B) of *predictable characteristics* of S consists of a predictable $\mathbb{R}^d$-valued process b, a predictable nonnegative-definite symmetric matrix-valued process c, a predictable process F with values in the set of Lévy measures and a predictable non-decreasing process B null at zero such that

$$\widetilde{A} = b \bullet B, \qquad [S^c, S^c] = c \bullet B \qquad \text{and} \qquad \nu = F \bullet B. \tag{6.1}$$

Using this local description of the semimartingale S we can then prove the existence of representative square-integrable portfolios.

Lemma 6.1. *There exist strategies $\varphi^i \in \Theta_S$ for $i = 1, \dots, d$ such that the financial markets (S, Θ_S) and $(\widetilde{S}, \Theta_{\widetilde{S}})$ with $\widetilde{S}^i = \varphi^i \bullet S$ for $i = 1, \dots, d$ admit the same wealth processes, i.e. $\Theta_S \bullet S = \Theta_{\widetilde{S}} \bullet \widetilde{S}$.*

Proof. By Proposition 2 in [42] (and the paragraph preceding that), σ-square-integrablity of a semimartingale X is equivalent to the existence of a strictly positive, bounded predictable process ψ such that $\psi \bullet X \in \mathcal{H}^2(P)$. As ψ is bounded and strictly positive, we can therefore always switch back and forth between σ-square-integrable X and square-integrable semimartingales Y by using the associativity of the stochastic integral, i.e. $Y = \psi \bullet X$ and $X = \frac{1}{\psi} \bullet (\psi \bullet X) = \frac{1}{\psi} \bullet Y$. Moreover, this also allows to reduce our problem to σ-square-integrability, which we consider first. Like any semimartingale, a stochastic integral $\vartheta \bullet S$ of an S-integrable process ϑ is σ-square-integrable if and only if the sum of its squared jumps, $\sum_{0<s\leq\cdot}(\vartheta_s^\top \Delta S_s)^2$, is σ-integrable. By Theorem II.1.8 in [52], the latter condition is equivalent to $\int_0^\cdot \int_{\mathbb{R}^d} (\vartheta_s^\top x)^2 F_s(dx) dB_s$ being σ-integrable, which holds if and only if $\int_{\mathbb{R}^d} (\vartheta_s^\top x)^2 F_s(dx) < +\infty$ P_B-a.e. If S is one dimensional, i.e. $d = 1$, we can write $\vartheta_s^2 \int_{\mathbb{R}^d} x^2 F_s(dx) = \int_{\mathbb{R}^d} (\vartheta_s^\top x)^2 F_s(dx) < +\infty$ P_B-a.e., which basically tells us that we must have $\vartheta = 0$ P_B-a.e. on the set $D^c := \{\int_{\mathbb{R}^d} x^2 F(dx) = +\infty\} \in \mathcal{P}$. Therefore setting $\varphi^1 := \psi \mathbb{1}_D$, where ψ is the integrand from Proposition 2 in [42], gives the desired strategy.

In the multidimensional case, the situation is more involved due to the linear dependence between the different components of S. To deal with this issue, we use similar techniques as in Chapter III, where we also refer the reader to for more explanations on problems arising from this. For the rest of the proof, we consider integrands $\vartheta \in L(S)$ as elements of $L^0(\Omega \times [0,T], \mathcal{P}, P_B; \mathbb{R}^d)$ and define the linear subspace V by

$$V = \left\{ \vartheta \in L^0(\Omega \times [0,T], \mathcal{P}, P_B; \mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d} (\vartheta^\top x)^2 F(dx) < +\infty \quad P_B\text{-a.e.} \right\}.$$

By definition, V satisfies the stability property that $\vartheta^1 \mathbb{1}_D + \vartheta^2 \mathbb{1}_{D^c} \in V$ for all $\vartheta^1, \vartheta^2 \in V$ and $D \in \mathcal{P}$, and it is closed with respect to convergence in P_B-measure by Fatou's lemma. So there exist by Lemma 6.2.1 in [32] (see also Lemma 5.2 in Chapter III) $v^i \in V$ for $i = 1, \dots, d$ such that

1) $\{v^{i+1} \neq 0\} \subseteq \{v^i \neq 0\}$ for $i = 1, \dots, d-1$,

2) $|v^i(\omega, t)| = 1$ or $|v^i(\omega, t)| = 0$,

3) $(v^i)^\top v^k = 0$ for $i \neq k$,

4) $\vartheta \in V$ if and only if $\vartheta = \sum_{i=1}^d (\vartheta^\top v^i) v^i$ P_B-a.e.

Since v^i is in V and bounded by 2), $v^i \in L(S)$ and $v^i \bullet S$ is σ-square-integrable for $i = 1, \dots, d$. By Proposition 2 in [42], there exist strictly positive, bounded predictable processes ψ^i such that $(\psi^i v^i) \bullet S \in \mathcal{H}^2(P)$ for $i = 1, \dots, d$, and we set $\varphi^i = \psi^i v^i$ and $\widetilde{S}^i = \varphi^i \bullet S$. Since we can write each $\vartheta \in \Theta_S \subseteq V$ by 4) as $\vartheta = \sum_{i=1}^d (\vartheta^\top v^i) v^i = \sum_{i=1}^d \frac{(\vartheta^\top v^i)}{\psi^i} \varphi^i$ P_B-a.e., we obtain that $\widetilde{\vartheta} = (\frac{(\vartheta^\top v^1)}{\psi^1}, \dots, \frac{(\vartheta^\top v^d)}{\psi^d}) =: \Psi\vartheta$ is in $\Theta_{\widetilde{S}}$, where $\Psi := \left(\frac{v^1}{\psi^1}, \dots, \frac{v^d}{\psi^d}\right)^\top$ is an $\mathbb{R}^{d\times d}$-valued predictable process, and that $\vartheta \bullet S = \widetilde{\vartheta} \bullet \widetilde{S}$ by the associativity of the stochastic integral. Conversely, we have for each $\widetilde{\vartheta} \in \Theta_{\widetilde{S}}$ that $\vartheta = \sum_{i=1}^d \widetilde{\vartheta}^i \varphi^i = \Phi\widetilde{\vartheta} \in \Theta_S$ with $\vartheta \bullet S = \widetilde{\vartheta} \bullet \widetilde{S}$, where $\Phi := \left(\varphi^1, \dots, \varphi^d\right)$ is an $\mathbb{R}^{d\times d}$-valued predictable process, which allows us to conclude that $\Theta_S \bullet S = \Theta_{\widetilde{S}} \bullet \widetilde{S}$ and completes the proof. □

Remark 6.2. As an alternative to the proof above one can introduce a predictable correspondence C by

$$C(\omega, t) := \left\{ y \in \mathbb{R}^d \;\middle|\; \int_{\mathbb{R}^d} (y^\top x)^2 F(dx) < +\infty \right\}$$

for all $(\omega, t) \in \Omega \times [0, T]$. Then the condition $\vartheta \in V$ can be formulated as the pointwise constraint that $\vartheta(\omega, t) \in C(\omega, t)$ P_B-a.e. As the values of C are linear subspaces, one can deduce the existence of representative σ-square-integrable portfolios by using (the arguments in the proof of) Theorem B.3 in Nutz [75]. The correspondence of the transformed constraints $\widetilde{C}$ is then of course equal to $\mathbb{R}^d$ for all $(\omega, t) \in \Omega \times [0, T]$ and the representative σ-square-integrable portfolios are the representative portfolios.

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Curriculum Vitae

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Education

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– Diploma with distinction	
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High School Friedrich–Hecker–Gymnasium, Radolfzell	09/1992–06/2001
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Academic Employment

Teaching assistant in Mathematics, ETH Zurich	07/2006–03/2011
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