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TWO PROBLEMS IN TRANSPORT THEORY: LOCALIZATION AND FRICTION

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Abstract. Transport theory is a broad field. In this thesis, we limit the scope of our investigations to two aspects of transport theory that are characterized by the very lack of any transport.

In a first part, we study the phenomenon that an electron traveling in a disordered solid can get trapped in exponentially sharply localized orbitals—it experiences so-called *Anderson localization.* Our investigation revolves around the series of alloys $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$ for which our experimental colleagues have found interesting transport properties. Specifically, they found a metal-insulator transition as the alloying parameter x is lowered below $x \simeq 0.3$ and observed colossal magnetoresistance effects for x in a range between 0.2 and 0.3. We show that these observations can be understood in terms of a localization-delocalization transition. We introduce a model for conduction electrons of $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$—simplified to the extent that we can treat some aspects of it related to Anderson localization in a mathematically rigorous way—reproducing the main features observed in the experiments.

Europium has a half-filled $4f$-shell and has therefore a large magnetic moment, whereas Calcium is non-magnetic. We place Europium and Calcium atoms on a cubic lattice according to a site percolation process with parameter x. Because of the experimental fact that the conduction band is very weakly populated we neglect electron-electron interactions and consider a one-particle Hamiltonian. In a Born-Oppenheimer approximation we freeze the dynamics of the magnetic moments of the Europium atoms, and because the moments are quite large we treat them as classical vectors. We propose to describe the exchange coupling of a conduction electron to the magnetic moments of the Europium atoms by a Zeeman term. The disorder comes about because the direction of the magnetic moments of Europium atoms that do not lie in a connected cluster will—in the absence of an external magnetic field—vary randomly.

We investigate various distributions of the magnetic moments: For temperatures above T_{Curie} and for pure EuB_6, neighboring magnetic moments are only weakly correlated, and we model this regime with a Gibbs distribution that slightly favors ferromagnetic alignment of neighboring moments. At low temperatures and for x below the percolation threshold, we expect ferromagnetic alignment of the Europium moments across connected clusters to prevail. Hence we assume that the directions of magnetic moments are fixed across connected Europium clusters but vary randomly over distinct clusters.

In these regimes, we prove Anderson localization, that is, almost-sure pure point spectrum of the Hamiltonian with exponentially decaying eigenfunctions, for energies in the band tails.

Finally, we show that the case of a large external magnetic field can be modeled with a Bernoulli-type random Schrödinger operator, where the random potential at a site takes values ± 1. Here we prove a weaker result, namely absence of diffusion for energies in the Lifshitz tails and outside a set of energies of very small measure.

In the second part of this thesis, we discuss another hindrance to transport, namely *friction.* We study the motion of a tracer particle that interacts with a dispersive medium, in our case a Bose-Einstein condensate. For the sake of mathematical rigor, we look at the case of a very heavy tracer particle in a very dense, non-interacting Bose gas. We argue heuristically that this mean-field limit corresponds to a classical limit and that the quantum dynamics reduces to a classical system of Hamiltonian equations of motion. We expect that the particle experiences friction by emission of Cerenkov radiation of gapless (Goldstone) modes into the Bose gas.

For these—as it turns out—semi-linear integro-differential equations describing the dynamics of the tracer particle and the medium, we prove that the particle velocity v_t decays like $|v_t| \lesssim t^{-1-\varepsilon}$ as $t \to \infty$, for some $\varepsilon > 0$, and that the gas forms a splash that follows the position of the particle. In particular, the decay of the particle speed is integrable and hence the particle comes to rest after having traveled a finite distance. We prove this result by expanding the propagator around its instantaneous value at a large but fixed time, and using asymptotic expansions of the resolvent of Schrödinger operators, standard dispersive estimates, and a contraction principle.

Kurzfassung. Die Transporttheorie ist ein weites Feld. Deshalb schränken wir uns in dieser Dissertation auf die Behandlung von zwei Aspekten des Transports ein, die gerade dadurch charakterisiert sind, dass gar kein Transport stattfindet.

In einem ersten Teil untersuchen wir die Quantendynamik von Elektronen, die sich in einem ungeordneten Festkörper bewegen und in exponentiell genau lokalisierten Orbitalen gefangen werden — man spricht dann von *Anderson-Lokalisierung*. Unsere Untersuchung dreht sich um die Reihe von Legierungen $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$, für die unsere Kollegen aus der Experimentalphysik interessante Transporteigenschaften gefunden haben. Konkret haben sie einen Metall-Isolator-Übergang gefunden, sobald der Legierungsparameter x unter $x \simeq 0.3$ fällt. Ausserdem haben sie für x in einem Bereich zwischen 0.2 und 0.3 kolossalen Magnetwiderstand beobachtet. Wir werden aufzeigen, dass diese Effekte im Rahmen eines Lokalisierungs-Delokalisierungsübergangs verstanden werden können. Dazu führen wir ein Modell für Leitungselektronen in $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$ ein, das so weit vereinfacht ist, dass wir einige seiner Eigenschaften mathematisch streng behandeln können, das aber trotzdem die wesentlichen Züge der Experimente wiedergibt.

Europium hat eine halbgefüllte $4f$-Schale, weshalb es ein grosses magnetisches Moment hat, wohingegen Kalzium nichtmagnetisch ist. Wir setzen Europium- und Kalziumatome auf ein kubisches Gitter gemäss einem Perkolationsprozess mit Parameter x. Weil, wie Experimente zeigen, das Leitungsband sehr schwach gefüllt ist, vernachlässigen wir Wechselwirkungen der Elektronen untereinander und betrachten einen Einteilchen-Hamilton-Operator. In der Born-Oppenheimer-Näherung frieren wir die Dynamik der magnetischen Momente der Europiumatome ein, und weil die Momente recht gross sind, behandeln wir sie als klassische Vektoren. Die Austauschkopplung zwischen einem Leitungselektron und dem magnetischen Moment eines Europiumatoms beschreiben wir mit einem Zeemanterm. Die Unordnung kommt daher, dass die Richtung der magnetischen Momente der Europiumatome, die nicht in einem zusammenhängenden Cluster liegen, zufällig variiert.

Wir werden veschiedene Verteilungen für die magnetischen Momente untersuchen: Bei Temperaturen über T_{Curie} des reinen EuB_6 sind benachbarte magnetische Momente nur schwach korreliert, weshalb wir diesen Parameterbereich mit einer Gibbsverteilung modellieren, die ferromagnetische Anordnung benachbarter Momente leicht bevorzugt. Bei tiefen Temperaturen und für x unterhalb der Perkolationsschwelle erwarten wir ferromagnetische Ordung der Momente in einem zusammenhängenden Europiumcluster. Wir nehmen also an, dass die Richtung der magnetischen Momente in einem Europiumcluster fixiert ist, aber zwischen verschiedenen Clustern zufällig ändert.

In diesen Bereichen beweisen wir Anderson-Lokalisierung, das heisst fast sicher reines Punktspektrum des Hamilton-Operators mit exponentiell lokalisierten Eigenfunktionen für Energien nahe der Ränder des Leitungsbandes.

Schliesslich zeigen wir, dass sich der Fall eines grossen äusseren magnetischen Feldes auf einen Bernoulli-Schrödinger-Operator zurückführen lässt, bei dem das Zufallspotential an jedem Gitterplatz die Werte ± 1 annehmen kann. In diesem Falle beweisen wir etwas weniger, nämlich Abwesenheit von Diffusion für Energien in den Lifshitz-Rändern und ausserhalb einer Menge von Energien von sehr kleinem Mass.

Im zweiten Teil dieser Dissertation diskutieren wir ein weiteres Hindernis für Transport, nämlich *Reibung*. Wir analysieren ein Teilchen das mit einem dispergierenden Medium wechselwirkt, in unserem Fall mit einem Bose-Einstein-Kondensat. Aus Gründen der mathematischen Strenge schauen wir uns den Fall eines sehr schweren Teilchens in einem sehr dichten, nicht wechselwirkenden Bose-Gas an. Wir argumentieren heuristisch, dass dieser Molekularfeld-Limes einem klassischen Limes entspricht und dass die Quantendynamik so zu einem System von klassischen Hamiltongleichungen führt. Wir erwarten, dass das Teilchen durch Aussenden von Cerenkovstrahlung von Goldstone-Moden in das Bose-Gas Reibung erfährt.

Für die — wie es sich herausstellt — semilinearen Integro-Differentialgleichungen, die die Dynamik des Teilchens und des Mediums beschreiben, beweisen wir, dass die Geschwindigkeit des Teilchens wie $v_t \lesssim t^{-1-\varepsilon}$, $\varepsilon > 0$ abnimmt, wenn $t \to \infty$, und dass das Gas eine Wolke bildet, die der Position des Teilchens folgt. Insbesondere ist die zeitliche Abnahme der Teilchengeschwindigkeit integrierbar, weshalb das Teilchen schliesslich zur Ruhe kommt, nachdem es eine endliche Strecke zurückgelegt hat. Wir beweisen dieses Resultat, indem wir den Propagator um seinen instantanen Wert zu einer grossen aber festen Zeit entwickeln, und indem wir asymptotische Entwicklungen der Resolvente eines Schrödinger-Operators, gewisse Streuabschätzungen und ein Kontraktionsprinzip benützen.

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I was very lucky to have had the help of three distinguished mathematicians, Jean Bourgain, Tom Spencer and Gang Zhou, without whose mathematical help this thesis would never have been written. I thank Gang Zhou in particular for being very patient while explaining to me his ingenious tricks in handling PDEs, and I am looking forward to collaborating with him in the future.

Many people at the ETH made my PhD time a very enjoyable one, and I want to extend my heartfelt thanks to them. The beginning of my PhD was dominated by Latin Swiss culture, and I am very glad to have had around me my good old friends Sven Bachmann, impulsive but kind, Antti Knowles, pedantic but kind, and Gregorio Ortelli, a kind man of few but weighty words, to discuss any sort of problem or to go for a beer when the problems seemed unsurmountable. In particular, I thank Antti for lifting my confusion about mathematical physics on numerous occasions.

My later years were the Germanic era, as I was joined by my new friends Wojciech de Roeck, Kevin Schnelli, André Lisibach, Samuel Bieri, Gang Zhou, Cyril Stark, Sebastian Gerigk and recently Michael Walter, Marcello Porta and Wolfgang Unger. I thank Kevin for sharing the ups and downs of academic life as my one true little PhD brother and for the many fruitful and not so fruitful discussions in our office, Wojciech for his expertise on cluster expansions, and André for his upbeat nature and his big-eyed admiration for his senior PhD.

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Finally, my dearest thanks go to my family and friends without whose belief in me I would never even have started my PhD.

Contents

Introduction and organization of the thesis

The Latin root of the word transport, *trans-portare*, meaning “to carry over”, neatly summarizes what happens during transport; and the sheer amount of goods and people that are carried, ferried and shipped around the globe each day is a testimony to its importance for the functioning of our world. But also when we zoom in on the atomic scale of matter, the realm of condensed matter physics, transport theory is an essential field of study: transport of matter (diffusion of a solute in a solvent), energy (heat conduction) or charge (electrical conductivity) are physical phenomena of prime interest and importance.

In this thesis, we focus on two phenomena where transport is actually hampered—*localization* and *friction*. In the former, charge transport (for instance) is reduced when electrons traveling through a disordered crystal lattice of a solid get trapped and do not contribute to electrical conductivity. In the latter, matter transport is reduced as particles traversing a medium are slowed down because of interaction with their environment. The aim of this thesis is to further the understanding of the mathematics that may lie at the bottom of these phenomena. That is, we will present and study mathematical models of localization and friction that are simplified to an extent where we can analyze them rigorously but where they still retain the essential phenomenological features.

Localization. If we were to describe a conduction electron in a metal as one traveling in the periodic potential of the atomic cores that are arrayed in a perfect crystal lattice we would run immediately into troubles: The electron is described by a Bloch wave and as such has a non-vanishing mean velocity that persists forever—the conductivity is infinite. In a more realistic model, scattering of the electron on lattice vibrations (phonons) and defects in the crystal lattice reduce conductivity to a finite amount seen in experiments. Let us concentrate on the second mechanism, the influence on conductivity of lattice imperfections; this is reasonable since we will only consider very low temperatures so that phonons can be neglected. It is probably too difficult to describe mathematically the influence on an electron of a combination of interstitials, impurities, dislocation etc. at prescribed locations in the lattice. P.W. Anderson’s fruitful idea was to treat the imperfections in a summary fashion by replacing the perfect periodic potential by a random potential, where an increasing amount of lattice imperfections is modeled by increasing randomness. In his famous 1958 paper [**6**] he argued that in the presence of strong disorder caused by impurities and/or defects and neglecting electron-electron interactions, electrons populating a weakly filled conduction band of a metal get trapped in exponentially sharply localized one-particle orbitals. This is the phenomenon known as *Anderson localization.* A consequence of localization is that the conductivity of such a material very nearly vanishes at low temperatures. If disorder is described by an on-site random potential with bounded probability density and short-range correlations in a one-electron tight-binding Hamiltonian then Anderson’s arguments can be made precise, mathematically, for one-dimensional systems with arbitrarily weak disorder [**30, 42**], and for higher-dimensional systems provided that disorder is strong enough, or the energy of the one-electron orbital lies in the band tails [**27, 25**]. It is generally expected—but not rigorously proven—that in two-dimensional systems of this kind, too, all states are localized, no matter how weak the disorder. In contrast, in three or more dimensions, localized states with energies in the band tails are expected to coexist with extended states (generalized eigenstates of the model Hamiltonian) corresponding to energies in the continuous spectrum near the center of the band, provided the disorder is sufficiently weak. It is expected that wave packets made from superpositions of such extended states exhibit diffusive propagation corresponding to a non-zero conductivity [**60, 1**]. One is led to predict that, at very low temperatures, a three-dimensional disordered semiconductor exhibits a transition from an insulating state (all electrons in the conduction band occupy localized states) to a conducting state (some fraction of the electrons populate extended states), as the density of electrons in the conduction band is increased or the strength of disorder is lowered. This transition from an insulator to a metal is called a *Mott transition.*

In the first part of this thesis, we investigate a different kind of disorder, namely spin disorder. More precisely, we will analyse a mathematical model of the alloy $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$ where experimenters have done interesting measurements the results of which we propose to explain in the framework of Anderson localization. In the same way the electron feels the potential of its surrounding lattice of atomic cores because it is charged, its spin feels the magnetic moments of the atomic cores. So if the magnetic moments are disordered, we may expect similar localization effects to take place. It is the main achievement of the first part of this thesis to show that, indeed, completely analogous results to the case of a random potential can be proven.

Friction. Friction is arguably too all-pervading as a physical phenomenon to allow for a unified mathematical description. We will therefore restrict our attention to one instance of it, namely the friction a particle experiences while traveling through a dispersive medium. Dispersive here means that the medium is "forgetful" in some sense: It forgets quickly where the particle has been by propagating the disturbance caused by the particle to infinity in such a way that if the particle revisits a place, (almost) all trace of its former visit is gone. Again, to be able to prove a mathematical theorem we will have to study a simple model for the medium; we will take it to be non-interacting, for instance. However, our methods should apply also to the interacting case. In the rigorous study of friction effects in a model of a particle interacting with a dispersive medium, different models for the medium have been investigated, mostly wave fields (that is, the field satisfies a wave equation) such as the electro-magnetic field or a scalar wave field, but also the mean-field limit of an ideal gas of small particles (obeying the Vlasov equation). In the second part of this thesis, we will derive, through physical considerations, a model describing the motion of a heavy particle in a very dense Bose-Einstein condensate. We will prove that the particle loses kinetic energy (and hence speed) by engendering so-called Cerenkov radiation in the condensate. Its velocity as a function of time decays as a power law, indicative of memory effects. If the medium were more forgetful we would expect exponential decay.

The thesis is organized as follows. First part: In Chapter 1, we give a summary of experimental results concerning a particlar alloy, $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$, which we aim to explain in the framework of Anderson localization. Moreover, we introduce the mathematical model, state the main results and give some definitions used throughout the first part. In Chapter 2, we discuss, as a preliminary example, the "classical" mathematical theory of Anderson localization with random potentials.. In Chapter 3, we prove the main results outlined in Chapter 1. Second part: In Chapter 4, we give an introduction to the mathematical treatment of friction and introduce our model in greater generality. In Chapter 5, we present and prove the main theorem for the free Bose gas, and in Chapter 6 we describe some technical proofs. The appendix contains results about the cluster expansion used to handle correlations of the magnetic moments, a matrix-valued Cartan-type theorem that lies at the core of the proof of localization, and some musings about the thermodynamic limit.

Part 1

Localization

CHAPTER 1

Introduction

In the first part of the thesis, we study the phenomenon of Anderson localization, both from a physical and a mathematical perspective. Regarding the former, we try to elucidate later in this Chapter intriguing experimental findings concerning the electric properties of Europium-based hexaborides ($\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$), [**62**]. A Mott transition is found experimentally as the concentration, x, of the magnetic Europium atoms is varied, and considerable magneto-resistance effects are observed. To account for these properties we introduce a tight-binding model for a Mott transition where the disorder is caused by indirect exchange interactions between the electrons in a conduction band and a dilute array of localized atoms with large magnetic moments. The model is only physically motivated, appealing to heuristic arguments. The mathematical perspective enters with the analysis of the model, which occupies Chapter 3. Since the mathematics of Anderson localization is quite involved, we provide an introduction to it in Chapter 2.

Fairly simple arguments presented below lead us to introduce a model given in terms of a one-electron tight-binding Hamiltonian with a *random Zeeman interaction term* acting on electron spin. This term describes indirect exchange interactions between an electron in the conduction band and electrons in the half-filled $4f$-shell of a Eu-atom located nearby. It takes the form of a ferromagnetic coupling of the spin of the electron in the conduction band to the static total spin of electrons in the $4f$-shell of a Eu-atom. Because the latter is quite large, $S = 7/2$, it can be described, in good approximation, by a classical unit vector, $\vec{m}$ [**44**]. However, if a unit cell of the simple cubic lattice of a $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$ alloy contains a Ca-atom then $\vec{m} = 0$, because a Ca-atom has spin 0. At low temperatures, the direction of $\vec{m}$ is approximately constant throughout a connected Eu-cluster, because of indirect ferromagnetic exchange interactions between the spins of different Eu-atoms in the cluster. The direction of $\vec{m}$ varies randomly, however, from one Eu-cluster to another, as long as the external magnetic field vanishes. Thus, an electron in the conduction band of a $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$ alloy in zero magnetic field propagates in a *disordered* quasi-static background of essentially classical spins located in those unit cells that contain a Eu-atom. These spins are ferromagnetically coupled to the spin operator of the electron.

One of our main results is that, as long as there is no ferromagnetic long-range order (unit cells containing a Eu-atom do not percolate), but the concentration of Eu-atoms is not too small, in zero magnetic field, this type of magnetic disorder causes *Anderson localization* in the tails of the conduction band.

If the concentration, x, of Eu-atoms is brought above the percolation threshold then there is an infinite connected cluster of positive density of unit cells containing a Eu-atom, and the alloy is observed to order ferromagnetically at low enough temperatures [**63**]. Most Eu-spins are then aligned in a fixed direction. The same happens if a sufficiently strong external magnetic field is applied. Finally, if x is very small most unit cells exhibit a vanishing spin, that is, the vector $\vec{m}$ vanishes in most unit cells. In all these three situations, the disorder felt by electrons in the conduction band is weak, so that the localization threshold (or "mobility edge") moves towards the band edges. We thus expect to observe a delocalization– or Mott transition to a conducting state, as x increases across the percolation threshold, x_{c}, or if the external magnetic field is increased.

It is not understood, at present, how to *prove* the existence of such a transition and analyze its characteristics, although *heuristically* it is fairly well understood. But the existence of Anderson localization in the band tails, for x below the percolation threshold but non-zero, and for a sufficiently weak external magnetic field, can be proven rigorously.

The main mathematical results presented in this first part of the thesis may not be particulary suprising, but they concern examples of Anderson localization that have *not* previously been studied mathematically.

Our preliminary discussion is summarized in Figures 1 and 2.

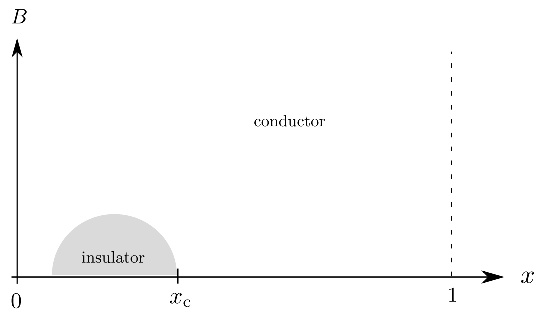

FIGURE 1. x denotes the concentration of Eu-atoms, B the value of a homogeneous external magnetic field. The concentration of conduction electrons is assumed to be approximately constant. The shaded area corresponds to an insulating state; a Mott transition to a semi-metal is expected to be observed at its boundary. Rigorous results are proven for a subset of the parameter values inside the shaded area.

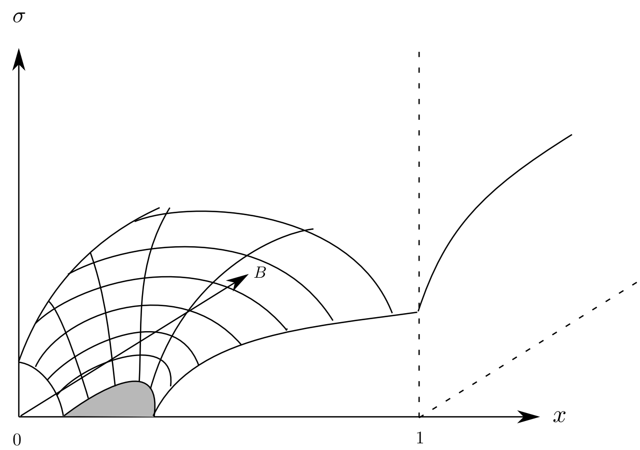

FIGURE 2. σ denotes the conductivity of the alloy. The figure provides a *qualitative* plot of σ.

1. Summary of experimental results concerning $Eu_xCa_{1-x}B_6$, and physical mechanisms

We begin by recalling some essential properties of EuB_6. This binary compound crystallizes in a simple cubic lattice. At the center of each unit cell of the crystal there is a divalent Eu atom, at every corner of a unit cell there is an octahedron of B-ions; see Figure 3 below.

The $4f$-shell of a Eu-atom is half filled, which, according to Hund's rule, implies that the total spin is $S = 7/2$. Electron transport is dominated by defect-state conduction with a low concentration, n_c, of around 10^{-3} charge carriers per unit cell [**64**]. At low temperatures, EuB_6 orders ferromagnetically

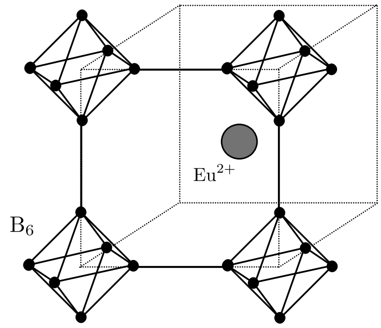

FIGURE 3. Schematic unit cell of EuB_6; each cube corner is the centre of a Boron octahedron which make up a rigid cage of covalent bonds

at a Curie temperature $T_C \simeq 12$ K, accompanied by a significant reduction of the resistivity, ρ, in the ordered phase [**22**]. The isostructural compound CaB_6 is obtained by replacing Eu by isoelectronic but non-magnetic Ca, which leads to a further reduction of n_c by an order of magnitude [**62**]. In the series $Eu_xCa_{1-x}B_6$, T_C decreases monotonically with decreasing x, down to $x \simeq 0.3$. At lower values of x, no onset of long-range magnetic order is observed. Instead spin-glass type features dominate the magnetic response at low temperatures [**63**]. For the simple cubic lattice, $x_c = 0.31$ is the site percolation limit [**63**]. In the concentration range $0.2 < x < 0.3$, significant localization and colossal magnetoresistance effects, such as shown in Figure 4, have been observed. For $x = 0.27$, the enhancement of the low-temperature resistivity by six orders of magnitude below 10 K may be quenched by rather moderate magnetic fields of the order of 1 T. Detailed investigations using selected-area electron diffraction

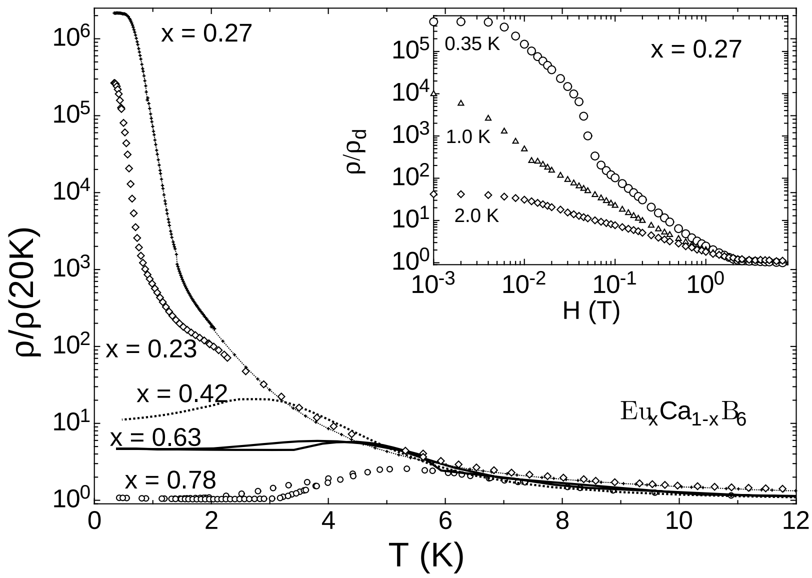

FIGURE 4. Main panel: $\rho(T)$ of $Eu_xCa_{1-x}B_6$ for various values of x. The thin solid line for $x = 0.27$ is to guide the eye. Inset: Magnetoresistance of $Eu_{0.27}Ca_{0.73}B_6$ at low temperatures. Reprinted from [**62**]. Reprinted with permission.

patterns and high-resolution transmission electron-microscopy (HRTEM) have shown that also for large concentrations of Ca for Eu, the structural quality, that is, the perfect atomic arrangement in a simple cubic lattice is preserved and the disorder is simply in the spins on the sites of the Eu clusters.

Energy-filtered TEM reveals a phase separation into microscopically small Ca- and Eu-rich regions, respectively. This implies that the material is magnetically and electronically inhomogeneous [**62**].

Next, we sketch some ideas on a possible mechanism that may explain the long-range ferromagnetic order observed in EuB_6 at temperatures below T_C. See also [**48**] for a similar discussion. The large size of the unit cells of EuB_6 (as compared to the size of a Eu atom) and numerical simulations [**41**] suggest that ferromagnetic order is established through *indirect exchange* mediated by electrons in a somewhat less than half-filled valence band, with strong on-site Coulomb repulsion preventing double occupancy; see Figure 5. For a non-vanishing density of holes in the valence band [**64**], the spins of the electrons in the valence band are expected to order ferromagnetically at very low temperatures. For the groundstate, this is a prediction of the Thouless-Nagaoka theorem [**55, 46, 5**]; (see also [**28**] for an analysis of ferromagnetism in the Hubbard model). Because of overlap of the orbitals of electrons in the valence band with those in the $4f$-shells of Eu-atoms, the spin of a valence electron in a unit cell has a tendency of being "anti-parallel" to the total spin of the Eu-atom in the same unit cell, provided the temperature is low. Appealing to Hund's rule, this is seen to be a consequence of Pauli's exclusion principle and of the half-filling of the $4f$-shell. Hopping processes of valence electrons into either an empty orbital of the $4f$-shell of a Eu-atom or to an empty orbital of the valence band thus give rise to ferromagnetic order among the spins of the Eu-atoms and those of the valence electrons, the latter being "anti-parallel" to the spins of the Eu-atoms. Because the orbitals of conduction electrons overlap with

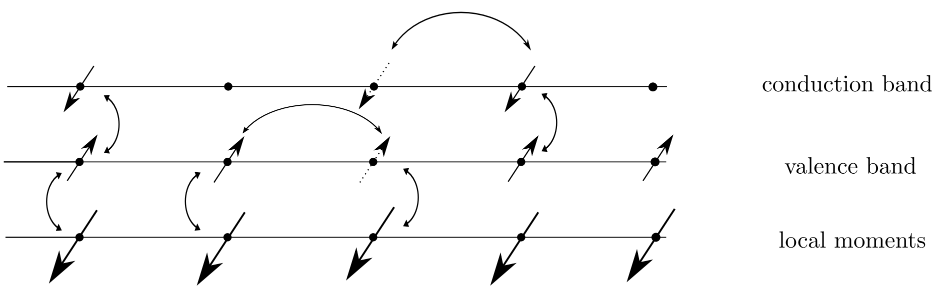

FIGURE 5. Spin ordering in EuB_6.

those of valence electrons, there are exchange interactions between conduction– and valence electrons that, again because of the Pauli principle, favor anti-ferromagnetic order between conduction– and valence electrons. Thus, the spins of conduction electrons have a tendency of being aligned with the spins of the Eu-atoms. We will describe this tendency by a Heisenberg term that couples the spin of a conduction electron in a unit cell ferromagnetically to the spin of the Eu-atom in the same unit cell. Since the spin of the Eu-atom is rather large ($S = 7/2$), we propose to describe it as a static classical spin, $\vec{m}$. It would be of considerable interest to improve the theoretical understanding of ferromagnetism in a one-band Hubbard model coupled to a lattice of large localized spins.

In our somewhat idealized theoretical description of $Eu_xCa_{1-x}B_6$, we place Eu- and Ca-atoms at the centers of the unit cells of the simple cubic lattice $\mathbb{Z}^3$ according to a site percolation process with probability x to find a Eu-atom at a given site. The mechanism for ferromagnetic order through indirect exchange described above suggests that, within connected clusters of unit cells filled with Eu-atoms, the spins of the Eu-atoms are ferromagnetically ordered. Since different Eu-clusters are separated by regions filled with non-magnetic Ca-atoms, one expects that the directions in which the spins of Eu-atoms are aligned vary randomly from one Eu-cluster to the next, as long as the external magnetic field vanishes (or is very small). If there is no infinite cluster of Eu-atoms this introduces disorder, and, because the conduction electrons are scattered at the spins of the Eu-atoms, it enhances a tendency towards Anderson localization of conduction electrons.

The threshold for the emergence of an infinite connected cluster in a site percolation process on $\mathbb{Z}^3$ is $x_c \simeq 0.31$. For x above x_c, one expects that there exists an infinite connected cluster of Eu-atoms. At low temperature, the spins of the Eu-atoms in the infinite cluster are all aligned, so that spin-disorder is weak. And if x is very small there is an infinite cluster of non-magnetic Ca-atoms, while Eu-clusters are tiny, on average, and sparse. Hence, spin-disorder is again weak. However, for x in some range below x_c, and in zero external magnetic field, there is considerable disorder in the way spins in different Eu-clusters are aligned. This enhances scattering of conduction electrons at different Eu-clusters, and one expects that the mobility edge, E_*, separating low-lying *localized* orbitals from *extended states* near the center of the conduction band is shifted away from the band edge towards the center of the band. If the Fermi energy in the conduction band is approximately constant as x varies one is led to predict that Mott transitions may be observed at some $x^* \simeq x_c$ and some $x_* \ll x_c$; see Figure 6.

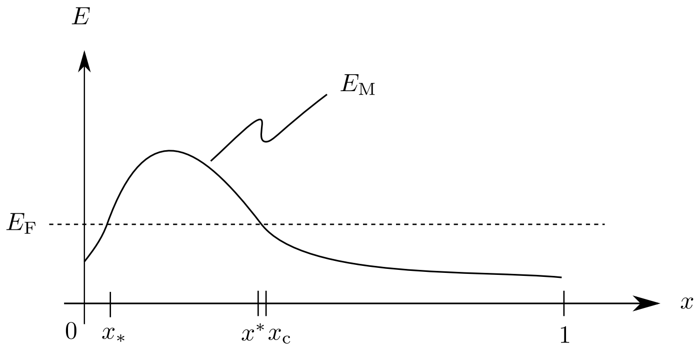

FIGURE 6. Mobility edge, $E_{\mathrm{M}}(x)$, as a function of x. There is no solid mathematical understanding of E_{M}, and the figure is purely qualitative.

2. Mathematical model and results

In this section, we propose a model expected to exhibit some of the phenomena described in the last section, namely the Mott transition and the colossal magnetoresistance observed in $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$ alloys. Our model is idealized to an extent that some of its properties, in particular Anderson localization, can be established rigorously.

Because, experimentally, the conduction band of $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$ is only weakly populated, $n_c \lesssim O(10^{-3})$, we neglect interactions among conduction electrons and describe the propagation of a conduction electron with the help of a one-particle model. It is convenient to make use of a tight-binding approximation. The Hilbert space of pure state vectors of a conduction electron is then given by

$$\mathcal{H} = \ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^2 \,. \tag{1.1}$$

Although valence electrons mediate an indirect exchange interaction between conduction electrons and the electrons in the half-filled $4f$-shells of Eu atoms, they do not appear explicitly in our model. Instead, the interactions of conduction electrons with the local Eu spins are described by a Heisenberg term coupling the spin of a conduction electron ferromagnetically to the spin of a Eu atom localized in the same unit cell. Since the latter is quite large ($S = 7/2$), we describe it by a *classical unit vector*, m. The Heisenberg term then takes the form of a Zeeman term, $-Jm \cdot \sigma$, where σ is the vector of Pauli matrices associated with a conduction electron, and $J > 0$ is a constant. If a unit cell $j \in \mathbb{Z}^3$ is filled with a Eu atom then $|m_j| = 1$; if it is filled with a Ca-atom then $m_j = 0$. Eu- and Ca-atoms are assumed to be distributed over the unit cells of $\mathbb{Z}^3$ by a *site percolation process*, with probability x to place a Eu atom at any given site. The configuration, $(m_j)_{j\in\mathbb{Z}^3}$, of classical spins is treated as quenched (in particular time-independent). Because of the observed tendency of Eu-spins in a connected Eu-cluster to order ferromagnetically, the distribution of the configurations $(m_j)_{j\in\mathbb{Z}^3}$ of Eu-spins in every connected Eu-cluster, $\mathcal{C}$, is chosen to be given by a Gibbs measure

$$\mathrm{d}\mathbb{P}_{\mathcal{C}}(m) := Z_{\mathcal{C}}^{-1} \exp\{\kappa \sum_{\substack{i,j\in\mathcal{C} \\ |i-j|=1}} m_i \cdot m_j + \beta B \sum_j m_j^{(z)}\} \prod_{j\in\mathcal{C}} \delta(|m_j|^2 - 1)\mathrm{d}^3 m_j \,. \tag{1.2}$$

Here, $\kappa = \kappa(T)$ is a temperature-dependent, positive constant ($\kappa(T)$ is decreasing with T), β is proportional to the inverse temperature, B is the strength of a uniform external magnetic field in the z-direction, $m_j^{(z)}$ is the z-component of m_j, and $Z_{\mathcal{C}}$ (the cluster partition function) is chosen such that $\mathrm{d}\mathbb{P}_{\mathcal{C}}$ is a probability measure. *Distinct clusters are taken to be independent.* This accounts for the fact that the indirect exchange mechanism that is responsible for the ferromagnetic order across connected Eu-clusters is not effective between two clusters that are separated by non-magnetic Calcium. It is appropriate to draw attention to the paper [**29**] (and references therein) where a Schrödinger operator with random vector potentials is studied. However, in their case the direction of the vectors is fixed and only their length is varied randomly, which is arguably an easier problem and not suited to our physical situation. See also [**19**] for a more elaborate continuum version.

One might envisage to combine the distribution of the Eu-clusters $\mathcal{C}$ and of the configurations $(m_j)_{j\in\mathbb{Z}^3}$ (with $m_j = 0$ if j is occupied by a Ca atom) into a single probability distribution that would then describe a tendency towards Eu-Ca phase segregation. We will not consider this possibility in this thesis.

The one-particle tight-binding Hamiltonian is chosen to be given by

$$H(\omega) := T + v_j(\omega) - J m_j(\omega) \cdot \sigma \,, \tag{1.3}$$

where $T \overset{e.g.}{=} -\Delta$ is a short-range hopping term (Δ is the discrete Laplacian), ω denotes the randomness of the interaction terms, $(m_j(\omega))$ is distributed according to (1.2), and $v(\omega)$ is a Bernoulli random potential with distribution

$$v_j(\omega) = \begin{cases} v & \text{if } m_j \neq 0 \text{ (that is, } j \text{ occupied by Eu)} \\ -v & \text{if } m_j = 0 \text{ (that is, } j \text{ occupied by Ca)} \end{cases} . \tag{1.4}$$

The potential v is incorporated in (1.3) because the potential energy of a conduction electron at a site j may depend on whether j is occupied by a Eu atom or a Ca atom.

The physical quantity of main interest is the electrical conductivity, σ, given in linear response theory by

$$\sigma = \frac{e^2}{h} D \,, \tag{1.5}$$

where D is the diffusion constant of conduction electrons. At temperature $T = 0$ and for a given Fermi energy E_{F}, D is given by

$$D = \int_{-\infty}^{E_{\mathrm{F}}} \mathrm{d}E \rho(E) D(E) \,, \tag{1.6}$$

where $\rho(E)$ is the density of states and $D(E)$ is given by the Kubo formula

$$\rho(E)D(E) = \lim_{\varepsilon\to 0} \frac{2\varepsilon^2}{3\pi} \sum_{j\in\mathbb{Z}^3} |j|^2 \, \mathbb{E}|\langle 0|(H(\omega) - E - \mathrm{i}\varepsilon)^{-1}|j\rangle|^2 \,, \tag{1.7}$$

where $\mathbb{E}$ denotes an expectation with respect to the distributions given in (1.2) and (1.4).

In Chapter 3, we consider various limiting regimes of the model introduced in (1.2)–(1.4) of varying mathematical difficulty:

(A) κ small, B small.

This regime is appropriate to describe electronic properties of $Eu_xCa_{1-x}B_6$ in the absence of magnetic order (e.g. well above the Curie temperature of the magnetic transition). Mathematically, this is the easiest regime. In Chapter 3, Section 3 through 6 we will focus on the case $x = 1$ of pure EuB_6 above the Curie temperature. Using methods developed in [**8**] and a cluster expansion to treat the weak correlations of the magnetic moments, it is not very difficult to establish Anderson localization, provided the energy lies sufficiently close to the band edges (depending on B and β in (1.2)). Our results are in agreement with experiments (see Figure 7), which find that for moderate temperatures above T_C (in our model, "moderate" means not so high as to invalidate our not including phonons) resistivity drops with increasing B and decreasing temperature, suggesting spin disorder as the localizing agent. (In contrast, below T_C resistivity in EuB_6 even increases with increasing B.) Coexistence

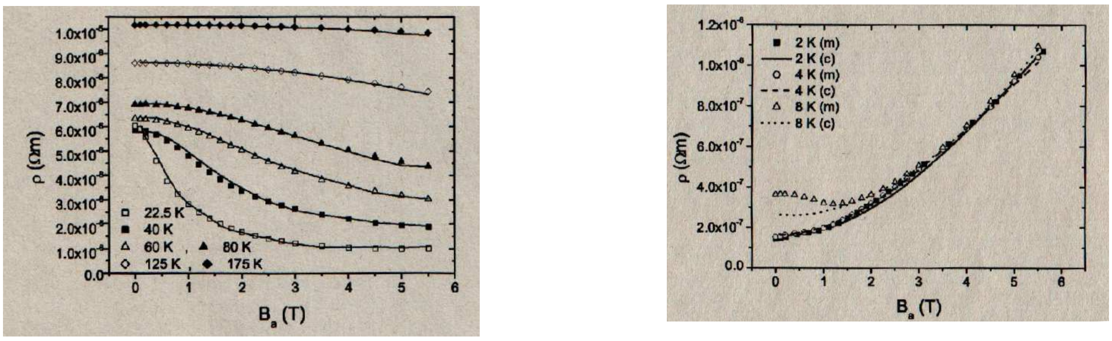

FIGURE 7. Resistivity ρ of EuB_6 as a function of applied external magnetic field, at various temperatures both above (left) and below (right) T_C. Reprinted from [**64**]. Reprinted with permission.

of localized states corresponding to energies in the band tails and extended states corresponding to energies near the center of the conduction band is expected for J small enough. However, the nature of the spectrum of the random Schrödinger Hamiltonian $H(\omega)$ defined in (1.3) near the center of the energy band is very poorly understood, at present.

(B) $\kappa \to \infty$, B small.

In this regime, the spins of Eu atoms in every connected Eu-cluster are completely aligned, but their direction can vary arbitrarily from one such cluster to another one. In Chapter 3, Section 7 we prove Anderson localization fo $x < x_c$, so Europium does not percolate, and for energies sufficiently close to the band edges. For x sufficiently close to 1 an infinite, ferromagnetically ordered Eu-cluster of density fairly close to x exists, and we expect to find two mobility edges close to the band edges. (A mobility edge separates energies corresponding to localized states from energies corresponding to extended states.) Our results suggest that the putative mobility edge moves towards the band edges when B is increased, in accordance with the experimental fact that colossal negative magnetoresistance is observed in $Eu_xCa_{1-x}B_6$, see Figure 4, inset. Mathematically, the existence of mobility edges remains, however, an open issue.

(C) $B \to \infty$.

In this limit, all the spins m_j are aligned in the positive z-direction. The conduction band then splits into two independent subbands for electrons with spin in the negative z-direction and those with spin in the positive z-direction, respectively. Within each subband, the Hamiltonian $H(\omega)$ is then equivalent to a "Bernoulli Hamiltonian"

$$H(\omega) = T + v(\omega)\,, \tag{1.8}$$

where

$$v_j(\omega) \equiv v_j^{\pm}(\omega) := \begin{cases} w_j \pm J\,, & m_j \neq 0 \\ -w_j\,, & m_j = 0 \end{cases}\,. \tag{1.9}$$

Adapting methods developed in [**54**], we show that, at all energies in small intervals adjacent to the band edges, except possibly in subsets of those intervals of very small Lebesgue measure, the quantity $\rho(E)D(E)$ introduced in (1.7) vanishes, as long as $x \neq 0, 1$. However, it is not known whether corresponding eigenstates are exponentially localized. It should be pointed out that the localization effect in this regime seems to be very weak, as can be seen in the inset of Figure 4.

Assuming that mobility edges, E_*, exist—separating energies E with $\rho(E)D(E) = 0$ from energies E' closer to the center of the band, where $\rho(E')D(E') > 0$—we expect (on the basis of our mathematical analysis of regimes (A) and (B)) that, as a function of B, $E_* = E_*(B)$ moves ever closer to a band edge, as B increases (that is, as magnetic disorder decreases). Thus, for $0 < x \lesssim 0.3$ and for a small, but positive density of conduction electrons, it can be expected that, at zero temperature, our model describes a Mott transition from an insulating state at small values of the magnetic field B to a conducting state at large values of B. If correct this conjecture would explain the colossal (negative) magnetoresistance observed at $x = 0.27$ and very low temperatures; (recall Figure 4).

3. Definitions and notation

We find it convenient to gather the most frequently used definitions in this section, for ease of reference.

Since we are dealing exclusively with the lattice Anderson model it is convenient to use the maximum norm as a distance measure in $\mathbb{Z}^d$, that is,

$$|x| := \max_{1 \le i \le d} |x_i|\,.$$

We denote the matrix elements of an operator H on $\ell^2(\mathbb{Z}^d;\mathbb{C}^r)$—which are $r \times r$-matrices—by

$$H(x,y)_{ij} := \langle \delta_{x,i}\,, H\delta_{y,j}\rangle\,,$$

where $\delta_{x,i}(z)_k = \delta_{xz}\delta_{ik}$. For an $r \times r$-matrix A we denote its operator norm also by $|\cdot|$, that is,

$$|A| := \sup_{|v|=1} |Av|\,.$$

This should not give rise to confusion, and we have of course $|\mathrm{tr}\, A| \le r|A|$. The reason for the nonstandard notation is that we want it to be distinguished from the operator norm of the operators on $\ell^2(\mathbb{Z}^d;\mathbb{C}^r)$, which we denote

$$\|B\| := \sup_{\|f\|=1} |Bf|\,, \quad f \in \ell^2(\mathbb{Z}^d;\mathbb{C}^r)\,.$$

Sometimes, for brevity, we write $a-$ to denote a not further specified real number *strictly lower* than, but arbitrarily close to a.

The discrete analog of the Laplacian, acting on $\ell^2(\mathbb{Z}^d)$, is also denoted by Δ and is defined by

$$(\Delta f)(j) = \sum_{|j-j'|=1} f(j') - f(j)\,.$$

By Fourier transform, the spectrum of $-\Delta$ is easily found to be $[0, 4d]$:

$$\begin{aligned}(\widehat{\Delta f})(k) &= \sum_{j\in\mathbb{Z}^d}(\Delta f)(j)\mathrm{e}^{\mathrm{i}j\cdot k} = \sum_{j\in\mathbb{Z}^d}\sum_{\alpha=1}^{d}(f(j-e_\alpha)-f(j)+f(j+e_\alpha)-f(j))\mathrm{e}^{\mathrm{i}j\cdot k} \\ &= \sum_{j\in\mathbb{Z}^d}\sum_{\alpha=1}^{d} f(j)(\mathrm{e}^{\mathrm{i}(j+e_\alpha)\cdot k}-\mathrm{e}^{\mathrm{i}j\cdot k}+\mathrm{e}^{\mathrm{i}(j-e_\alpha)\cdot k}-\mathrm{e}^{\mathrm{i}j\cdot k}) = \sum_{j\in\mathbb{Z}^d}\sum_{\alpha=1}^{d} f(j)\mathrm{e}^{\mathrm{i}j\cdot k}(\mathrm{e}^{\mathrm{i}k_\alpha}-1+\mathrm{e}^{-\mathrm{i}k_\alpha}-1) \\ &= \widehat{f}(k)\sum_{\alpha=1}^{d}(2\cos k_\alpha - 2)\,.\end{aligned}$$

If not stated otherwise, we will consider the Laplacian with diagonal elements removed, that is,

$$(\Delta f)(j) = \sum_{|j-j'|=1} f(j')\,,$$

and so its spectrum is $[-2d, 2d]$.

We will also mainly consider the Laplacian restricted to finite subsets $\Lambda \subset \mathbb{Z}^d$ with zero Dirichlet boundary conditions,

$$\Delta_\Lambda(j, j') = 0$$

if $j \notin \Lambda$ or $j' \notin \Lambda$. For a Schrödinger operator $H = -\Delta + v_j$ we define

$$H_\Lambda := -\Delta_\Lambda + \mathbb{1}_\Lambda(j)v_j\,,$$

where $\mathbb{1}_\Lambda$ denotes the characteristic function of the set Λ, and we denote the corresponding Green function by $G_\Lambda(E)$,

$$G_\Lambda(E) := (H_\Lambda - E)^{-1}\,.$$

The tool for handling these restricted Green functions is the second resolvent identity

$$R_A - R_B = R_B(B-A)R_A = R_A(B-A)R_B\,,$$

for operators A, B and their resolvents R_A, R_B.

Next, we introduce some geometric notions that will be used throughout this first part of the thesis.

DEFINITION.
- An *l-cube* with center $x \in \mathbb{Z}^d$ is the set $\Lambda_l(x) := \{y \in \mathbb{Z}^d : |x-y| \le l\}$.
- An *elementary l-region* is a difference of two l-cubes, that is, a set $R \subset \mathbb{Z}^d$ such that there exist two l-cubes C_1, C_2 satisfying $R = C_1 \backslash C_2$. The set of all elementary l-regions is denoted by $\mathcal{E}_l$.
- An elementary l-region R_l is called *regular at energy E* (sometimes, to be very clear, *c-regular*) if there exists a constant $c > 0$ such that

$$\|G_{R_l}(E)\| < \mathrm{e}^{l^{1/2}} \tag{1.10}$$

$$\text{and}\quad |G_{R_l}(E; x, y)| < \mathrm{e}^{-c|x-y|} \qquad \text{for}\ \ |x-y| \ge \frac{l}{10}\,. \tag{1.11}$$

  If an elementary region is not regular, it is called *singular*.
- For a set $X \subset \mathbb{Z}^d$ we call its *boundary* ∂X the set $\{(x, y) : x \in X, y \notin X, |x-y| = 1\}$, and its *inner boundary* ∂X^- the set $\{x \in X : \exists y \notin X, |x-y| = 1\}$

Finally, we explain how to use the second resolvent identity for the basic perturbation of the resolvent in the hopping terms: Consider a set $X \subset \mathbb{Z}^d$ and a subset $Y \subset X$. Clearly,

$$H_X = H_Y \oplus H_{X\backslash Y} - \Gamma\,,$$

where

$$\Gamma(x,y) = \begin{cases} 1 & x \in X\backslash Y, y \in Y, |x-y| = 1 \\ 1 & y \in X\backslash Y, x \in Y, |x-y| = 1 \\ 0 & \text{otherwise} \end{cases},$$

that is, Γ is the operator coupling Y along the boundary to its complement in X. The resolvent equation reads $G_X = G_Y \oplus G_{X\backslash Y} + G_Y \oplus G_{X\backslash Y}\Gamma G_X$. Note that $(G_A \oplus G_B)(x,y) = 0$ unless both x and y belong to A (or B), and that in this case $(G_A \oplus G_B)(x,y) = G_A(x,y)$ (or $G_B(x,y)$), for instance

- $x, y \in Y$
$$G_X(x,y) = G_Y(x,y) + \sum_{(z,z')\in\partial Y} G_Y(x,z) G_X(z',y)$$
- $x, y \in X\backslash Y$
$$G_X(x,y) = G_{X\backslash Y}(x,y) + \sum_{(z',z)\in\partial Y} G_{X\backslash Y}(x,z) G_X(z',y)$$
- $x \in Y, y \in X\backslash Y$
$$G_X(x,y) = \sum_{(z,z')\in\partial Y} G_Y(x,z) G_X(z',y)$$

CHAPTER 2

Preliminary example: random potentials with bounded density

The first mathematically rigorous results on Anderson localization in arbitrary dimensions were established by Fröhlich and Spencer [**27**] in 1983—a quarter of a century after the inception of the field by Anderson. The multi-scale analysis devised in their work has the reputation of being arcane and difficult to understand. We will show in this chapter that—at least for random potentials with bounded density—this is not the case. Because we will use many of the techniques, later on in Chapter 3, when we treat the random Zeeman interaction model, this chapter also serves to set the stage for subsequent arguments. The original method [**27**] has been simplified by Spencer and Dreifus (see [**58, 53**]) and applied to a plethora of problems by many people, see [**38**] for a recent review.

We study the Anderson Hamiltonian

$$H(\omega) = -\Delta + v_j(\omega) \tag{2.1}$$

acting on $\ell^2(\mathbb{Z}^d)$, where $-\Delta$ is the finite-difference Laplacian and $(v_j)_{j\in\mathbb{Z}^d}$ is a collection of independent identically distributed random variables, the density $g = \frac{d\lambda(v)}{\mathrm{d}v}$ of which satisfies

$$\|g\|_\infty < \infty\,.$$

The random potential is supposed to model the effects of defects and impurities in a real solid, as opposed to a periodic potential representing the perfect crystal structure of ideal solids covered by Bloch/Floquet-theory. The physically relevant result in the mathematical theory of localization is to prove absence of conduction, represented by the vanishing of the conductivity, σ, given by

$$\sigma(E) = \frac{e^2}{h}\rho(E)D(E)\,,$$

where e is the electric charge, h is Planck's constant, $\rho(E)$ is the density of states, and the diffusion constant, D, is given by the Kubo formula (see e.g. [**45**])

$$\rho(E)D(E) := \lim_{\varepsilon\to 0}\frac{2\varepsilon^2}{\pi d}\sum_{x\in\mathbb{Z}^d}|x|^2\,\mathbb{E}|G(E+\mathrm{i}\varepsilon,\omega;0,x)|^2\,, \tag{2.2}$$

$$D = \int^{E_\mathrm{F}} \mathrm{d}E\rho(E)D(E)\,,$$

where

$$G(z,\omega;x,y) = \langle\delta_x\,,(H(\omega)-z)^{-1}\delta_y\rangle\,.$$

As suggested by (2.2), analysis of the behaviour of the Green function $G(E+\mathrm{i}\varepsilon)$ as $\varepsilon\searrow 0$ is essential in the investigation of the transport properties of a Hamiltonian. For instance, it is easy to see (wait for the next chapter for precise definitions) that the density of states $\rho(E)$ has the following representation,

$$\rho(E) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\mathrm{Im}\,\mathbb{E}\,G(E+\mathrm{i}\varepsilon;0,0)\,.$$

By the second resolvent formula, we compute

$$\pi\rho_\varepsilon(E) = \mathbb{E}\,\frac{1}{2\mathrm{i}}[G(E+\mathrm{i}\varepsilon)-G(E-\mathrm{i}\varepsilon)](0,0) = \mathbb{E}\,\varepsilon[G(E+\mathrm{i}\varepsilon)G(E-\mathrm{i}\varepsilon)](0,0)\,,$$

so that

$$\sum_x \mathbb{E}\,|G(E+\mathrm{i}\varepsilon;0,x)|^2 = \frac{\pi\rho_\varepsilon(E)}{\varepsilon}\,.$$

If we define

$$f_\varepsilon(x) := \mathbb{E}\,|G(E+\mathrm{i}\varepsilon;0,x)|^2\,,$$

then we can think of the Fourier transform

$$\hat{f}_\varepsilon(p) = \sum_x \mathrm{e}^{\mathrm{i}p\cdot x} f_\varepsilon(x)\,,$$

so that the above formula reads

$$\frac{\pi\rho_\varepsilon(E)}{\varepsilon} = \hat{f}_\varepsilon(0)\,.$$

Formal computations (see [**45**]) lead us to the following expectation in what concerns the telltale signature of localization and delocalization, respectively:

$$\hat{f}_\varepsilon(p) = \frac{\pi\rho_\varepsilon(E)}{D_\varepsilon(p)+\varepsilon}\,,$$

where for small p

$$D_\varepsilon(p) \propto \begin{cases} D(E)p^2 & \text{diffusive regime} \\ \varepsilon p^2 & \text{localized regime} \end{cases}.$$

The form of $D_\varepsilon(p)$ in the diffusive regime is easily explained: From the Kubo formula above we obtain

$$\rho_\varepsilon(E)D(E) = \frac{2\varepsilon^2}{\pi d}\sum_x |x|^2 f(x) = \frac{2\varepsilon^2}{\pi d}\widehat{|x|^2 f(x)}(0) = -\frac{2\varepsilon^2}{\pi d}\Delta\hat{f}_\varepsilon(0)\,.$$

On the other hand, using $D_\varepsilon(0)=0$,

$$\Delta\hat{f}_\varepsilon(0) = -\pi\rho_\varepsilon(E)\frac{1}{\varepsilon^2}\Delta D_\varepsilon(0)\,,$$

so that

$$\Delta D_\varepsilon(0) = \frac{d}{2}D(E)\,.$$

Summarizing, precise knowledge of the function $f_\varepsilon(x,E)$ is all we need in the study of the transport properties of a Hamiltonian. Unfortunately, the analysis of the average $\mathbb{E}\,|G(E+\mathrm{i}\varepsilon,\omega;0,x)|$ is fraught with small denominator problems coming from configurations $\omega=(v_j)_{j\in\mathbb{Z}^d}$ which are near-resonant, as exemplified by the rank-one perturbation formula

$$G(z;x,x) = \frac{1}{\widetilde{G}(z;x,x)^{-1}+v_x}\,, \tag{2.3}$$

where in $\widetilde{G}$ the value of v_x is set to 0. Since $\widetilde{G}$ is independent of v_x, we see that for some "resonant" values of v_x there is a non-integrable $\frac{1}{v}$-divergence. One way to overcome this so-called small denominator problem was introduced in [**3**] (see also [**4**] for a simplified version), and goes under the name of "fractional moments method". The main idea boils down to considering $\mathbb{E}\,|G(E+\mathrm{i}0,\omega;0,x)|^s$ for some positive $s<1$, thus rendering integrable the $\frac{1}{v}$-divergences alluded to above, and the basic inequality

$$\left(\sum_j |a_j|\right)^{\frac{s}{2}} \le \sum_j |a_j|^{\frac{s}{2}}\,,\quad \forall s\le 2\,.$$

We will not discuss this method here because the other method, the multi-scale analysis, is better suited to the random Zeeman interaction terms considered in Chapter 3.

1. Multi-scale analysis

The multi-scale analysis is an inductive scheme to handle the small denominators due to resonant configurations. The Hamiltonian H is regularized by restricting it to finite-size cubes Λ; induction is then on the size of these cubes, the "scales". The main idea is to prove exponential decay of the off-diagonal elements of the regularized Green function, with probability approaching 1 for $\Lambda \nearrow \mathbb{Z}^d$. The induction proceeds roughly as follows: The strongly resonant configurations (they turn out to be very rare) are avoided at scale $n+1$, and the decay of the Green function at scale n can then be used to offset the small denominators coming from mildy resonant configurations to prove the decay at scale $n+1$.

A typical result that we can prove is a theorem such as

THEOREM 2.1. *Consider the random Schrödinger operator (2.1) on the lattice $\mathbb{Z}^d$. Let $E \in \mathbb{R}$, $k > 2d$ and $0 < \eta < 1$. There is an L such that if for $l_0 \geq L$ we have*

(i)

$$\mathbb{P}\left[\sum_{y \in \partial\Lambda^-_{l_0}(0)} |G_{\Lambda_{l_0}}(E + \mathrm{i}\varepsilon; 0, y)| \leq \eta\right] \geq 1 - l_0^{-k} \qquad \forall \varepsilon \neq 0$$

(ii)

$$\mathbb{P}\left[\operatorname{dist}\left(E, \sigma(H_{\Lambda_l(0)})\right) < \kappa\right] \leq C|\Lambda_l|\kappa\,,$$

for all $\kappa > 0$ small enough and all $l \geq l_0$,

then for $m_0 = -\log \eta / 2l_0$ we have that

$$\mathbb{P}[|G(E + \mathrm{i}\varepsilon; 0, x)| \leq \mathrm{e}^{m_0(N - |x|)}\,, \varepsilon \neq 0] \geq 1 - C_k N^{-k}$$

holds for all $x \in \mathbb{Z}^d$.

REMARKS.

- The so-called Wegner estimate (ii) in the form stated is too strong a condition. All that is needed is
$$\mathbb{P}\left[\operatorname{dist}\left(E, \sigma(H_{\Lambda_l(0)})\right) < \mathrm{e}^{-l^b}\right] < l^{-k}\,,$$
for a $0 < b < 1$. Using the selfadjointess of $H_{\Lambda_l(0)}$ we write this as
$$\mathbb{P}\left[\|G_{\Lambda_l(0)}(E)\| < \mathrm{e}^{l^b}\right] > 1 - l^{-k}\,. \tag{2.4}$$
- The Wegner estimate is actually not needed for all $l \geq l_0$, but only for an infinite sequence of scales $l_i = l_0^{\alpha^i}$, $i \in \mathbb{N}$, where $1 < \alpha < 2$.

In [**27**] it was shown how Theorem 2.1 implies absence of diffusion. We reproduce their argument here for the sake of completeness. Decompose $\mathbb{Z}^d$ into annuli, $A_i, i = 0, 1, 2, \ldots$ where

$$A_0 := \{x : |x| < R\}$$
$$A_i := \{x : R2^{i-1} \leq |x| < R2^i\} \quad i = 1, 2, \ldots$$

with an R to be chosen later on. Define the events

$$V_N := \{\omega : |G(E + \mathrm{i}\varepsilon, \omega; 0, x)| \leq \mathrm{e}^{m_0(N - |x|)}, \varepsilon \neq 0\}\,,$$

and estimate by (2.2)

$$\rho(E)D(E) \leq \lim_{\varepsilon \to 0} \frac{2}{\pi d} \sum_{i=0}^{\infty} \left(\varepsilon^2 \mathrm{e}^{m_0 N_i} \mathbb{P}[V_{N_i}] \sum_{x \in A_i} |x|^2 \mathrm{e}^{-m_0|x|} + C_d (R2^i)^{d+2} \mathbb{P}[V^{\mathrm{c}}_{N_i}]\right)\,,$$

where C_d is a constant, and in the second term on the right hand side we used the trivial upper bound

$$\varepsilon^2 |G(E+\mathrm{i}\varepsilon,\omega;0,x)|^2 \leq 1\,;$$

the sequence $(N_i)_{i=0}^{\infty}$ can be chosen at convenience. Clearly

$$\sum_{x\in\Lambda_i} |x|^2 \mathrm{e}^{-m_0|x|} \leq \text{const } \mathrm{e}^{-\frac{m_0}{4}R2^i}\,, \quad j\geq 1\,,$$

and by Theorem 2.1

$$\mathbb{P}[V_{N_i}^{\mathrm{c}}] = 1-\mathbb{P}[V_{N_i}] \leq C_k N_i^{-k}\,,$$

for any $k>2d$. We now choose $N_i = \frac{1}{8}R2^i$. It follows then that

$$\rho(E)D(E) \leq \lim_{\varepsilon\to 0} \text{const}\left[\varepsilon^2\left(\mathrm{e}^{\frac{Rm_0}{8}}R^{d+2} + \sum_{i=1}^{\infty}\mathrm{e}^{-\frac{m_0}{8}R2^i}\right) + \sum_{i=0}^{\infty} C_d (R2^i)^{d+2-k} C_k 8^{k-d}\right].$$

Choosing now $k>d+2$, letting first ε tend to 0 and then R to ∞ we conclude that $\rho(E)D(E)=0$, that is, $\sigma(E)=0$. ☐

We can hope to prove Theorem 2.1 for all energies E if the disorder is large, that is $\|g\|_\infty \ll 1$, or in general for energies in the band tails. To make the proof more transparent we prove first the following "finite-volume version",

Theorem 2.2. *Consider the random Schrödinger operator (2.1), and fix $1<\alpha<2$, $\mathbb{N}\ni k>\frac{2d\alpha}{2-\alpha}$. There exists an $L>0$ such that if for $l_0>L$, and $c>0$ we have that*

$$\mathbb{P}[|G_{\Lambda_{l_0}}(E;x,y)| \leq \mathrm{e}^{-c|x-y|}, \quad \forall |x-y|\geq \frac{l_0}{10}] \geq 1-l_0^{-k}\,,$$

then, for all $l_n = l_0^{\alpha^n}$,

$$\mathbb{P}[|G_{\Lambda_{l_n}}(E;x,y)| \leq \mathrm{e}^{-\frac{c}{2}|x-y|}, \quad \forall |x-y|\geq \frac{l_n}{10}] \geq 1-l_n^{-k}\,. \tag{2.5}$$

Definition. We call a cube of size l that satisfies (2.5) "l-good" (or simply "good"), and otherwise "bad".

The proof of this theorem uses an expansion of the resolvent (see Section 3 for notation), as in

$$G_\Lambda = G_B + G_B \Gamma_B G_\Lambda\,,$$

where B is a suitable subset of Λ. It is clear from the above formula that we need an a priori estimate on the dangling factor of G_Λ. This estimate is provided by the fact that $\|G_\Lambda(E)\| = 1/\mathrm{dist}(E,\sigma(H_\Lambda))$, and the following elementary result,

Lemma 2.3 (Wegner estimate). *If the random potential has a bounded density g, then*

$$\mathbb{P}[\mathrm{dist}(E,\sigma(H_\Lambda))\leq\kappa] \leq 2\|g\|_\infty \kappa|\Lambda|\,. \tag{2.6}$$

Proof. The core of the proof is the following observation,

$$N_\Lambda(E;\{v_j\}) = N_\Lambda(0,\{v_j-E\})\,,$$

where $N_\Lambda(E)$ denotes the number of eigenvalues of H_Λ less than E. Thus, we can write

$$\begin{aligned}
\mathbb{P}[\mathrm{dist}(\sigma(E,H_\Lambda))\leq\kappa] &\leq \mathbb{E}\left[N_\Lambda(E+\kappa)-N_\Lambda(E-\kappa)\right] = \mathbb{E}\int_{|E-E'|\leq\kappa}\frac{\mathrm{d}N_\Lambda(E')}{\mathrm{d}E'}\mathrm{d}E' \\
&= -\sum_{j\in\Lambda}\mathbb{E}\int_{|E-E'|\leq\kappa}\frac{\partial N_\Lambda(E')}{\partial v_j}\mathrm{d}E' = -\sum_{j\in\Lambda}\int_{|E-E'|\leq\kappa}\mathrm{d}E'\int\prod_{j'\in\Lambda}\mathrm{d}v_{j'}\,g(v_{j'})\frac{\partial N_\Lambda(E')}{\partial v_j} \\
&\leq \|g\|_\infty\sum_{j\in\Lambda}\int_{|E-E'|\leq\kappa}\mathrm{d}E'\int\prod_{j'\neq j}\mathrm{d}v_{j'}\,g(v_{j'})[N_\Lambda(E',v_j=-\infty)-N_\Lambda(E',v_j=\infty)]\,,
\end{aligned}$$

because $\frac{-\partial N_\Lambda(E')}{\partial v_j}$ is clearly positive. Now, by the minmax principle

$$0 \le N_\Lambda(E', v_j = -\infty) - N_\Lambda(E', v_j = \infty) \le 1\,,$$

so

$$\begin{aligned}\mathbb{P}[\mathrm{dist}(\sigma(E, H_\Lambda)) \le \kappa] &\le \|g\|_\infty \sum_{j\in\Lambda} \int_{|E-E'|\le\kappa} \mathrm{d}E' \int \prod_{j'\ne j} \mathrm{d}v_{j'}\, g(v_{j'}) \\ &= 2\|g\|_\infty \kappa |\Lambda|\,.\end{aligned}$$

□

Remark. Much effort has been devoted to the study of how much the conditions on the probability density can be relaxed such that a Wegner estimate can still be proven. The most natural condition is probably Hölder continuity of the probability measure: $\mathbb{P}$ is called Hölder continuous of order $\alpha > 0$ if

$$C_\alpha^{-1} := \inf_{\tau>0} \sup_{|b-a|\le\tau} \frac{\mathbb{P}([a,b])}{|b-a|^\alpha} < \infty\,.$$

In such a case one can prove [**13**] in a very similar fashion the following Wegner estimate,

$$\mathbb{P}[\mathrm{dist}(\sigma(E, H_\Lambda)) \le \kappa] \le \frac{2^\alpha}{C} |\Lambda|^{1+\alpha} \kappa^\alpha\,,$$

for all $0 < C < C_\alpha$ and κ small enough. This is good enough to prove the exact same things we are now going to prove for a bounded probability density.

Since the volume factor $|\Lambda_l|$ grows only polynomially in l, we see from the Wegner estimate that the probability that E is exponentially close (in l) to the spectrum of H_{Λ_l} is exponentially small.

Proof. (Of Theorem 2.2) The proof is by induction, and to keep notation light we will show the step from l_0 to $l_1 = l_0^\alpha$. As a first step, denote by Ω_{W} the set of configurations of the random potentials where $\|G_{\Lambda_{l_1}}(E)\| \ge \mathrm{e}^{l_1^{1/2}}$. Because of the Wegner estimate we know

$$\mathbb{P}[\Omega_{\mathrm{W}}] \le C\mathrm{e}^{-l_1^{1/2-}}\,.$$

Denote further by Ω_{2+} the set of configurations where there are *two or more* disjoint bad cubes of size l_0 in Λ_{l_1}. Because of the induction hypothesis we know

$$\mathbb{P}[\Omega_{2+}] \le C l_1^{2d} l_0^{-2k} = C l_1^{2d-\frac{2k}{\alpha}} \le \frac{1}{2} l_1^{-k}\,,$$

for l_0 large enough. So the probability of these problematic configurations is small,

$$\mathbb{P}[\Omega_{\mathrm{W}} \cup \Omega_{2+}] \le l_1^{-k}\,,$$

which means that the proof is finished if we can show the exponential decay for the unproblematic configurations. Hence, for the rest of the proof, consider only configurations in $\Omega\backslash(\Omega_{\mathrm{W}} \cup \Omega_{2+})$, that is, configurations for which $\|G_{\Lambda_{l_1}}\| \le \mathrm{e}^{l_1^{1/2}}$, and for which there is at most one bad l_0-cube in Λ_{l_1}. (Actually, there can be more, but they all have to intersect, so that they are contained in a slightly larger cube of size $3l_0$.)

First, assume that there is *no* bad l_0-cube in Λ_{l_1}. We expand the resolvent along a sequence of nested l_0-cubes C_i from x to y.

$$G_{\Lambda_{l_1}}(E; x, y) = \sum_{(z_1, z_1')\in\partial C_1} G_{C_1}(E; x, z_1) G_{\Lambda_{l_1}}(E; z_1', y)$$

for $x \in C_1$ but $y \notin C_1$. Find z_1' that maximizes $|G_{\Lambda_{l_1}}(E; z_1', y)|$ in the expression above, and denote it by $\widetilde{z}_1'$. Continue expanding around the cube C_2:

$$|G_{\Lambda_{l_1}}(E;x,y)| \leq \sum_{\substack{(z_1,z_1')\in\partial C_1\\(z_2,z_2')\in\partial C_2}} |G_{C_1}(E;x,z_1)||G_{C_2}(E;\widetilde{z}_1',z_2)||G_{\Lambda_{l_1}}(E;z_2',y)|\,,$$

for $y \notin C_2$. We can continue this procedure until $y \in C_{k+1}$, and obtain

$$|G_{\Lambda_{l_1}}(E;x,y)| \leq \sum_{(z_i,z_i')\in\partial C_i} |G_{C_1}(E;x,z_1)|\ldots|G_{C_k}(E;\widetilde{z}_{k-1}',z_k)|\|G_{\Lambda_{l_1}}(E)\|\,.$$

For the configurations under consideration, we know that $\|G_{\Lambda_{l_1}}(E)\| \leq \mathrm{e}^{l_1^{1/2}}$. So,

$$|G_{\Lambda_{l_1}}(E;x,y)| \leq (Cl_0^{d-1})^k \mathrm{e}^{-cl_0k}\mathrm{e}^{l_1^{1/2}}\,.$$

Since $k \geq |x-y|/l_0$ we obtain

$$\begin{aligned}|G_{\Lambda_{l_1}}(E;x,y)| &\leq \mathrm{e}^{\frac{\log(Cl_0^{d-1})}{l_0}|x-y|}\mathrm{e}^{-c|x-y|}\mathrm{e}^{l_1^{1/2}}\\ &\leq \mathrm{e}^{-(c-\frac{\log(Cl_0^{d-1})}{l_0}-\frac{l_0^{\alpha/2}}{|x-y|})|x-y|}\\ &\leq \mathrm{e}^{-(c-\frac{\log(Cl_0^{d-1})}{l_0}-l_0^{\alpha/2-1})|x-y|} =: \mathrm{e}^{-c_1|x-y|}\,,\end{aligned}$$

for $|x-y| \geq l_0$.

Next, we will prove exponential decay of the resolvent if there is *one* bad l_0-cube in Λ_{l_1}, denote it by Q. Consider first $\Lambda_{l_1}\backslash Q$, and note that by construction there is no bad l_0-cube in $\Lambda_{l_1}\backslash Q$, so that we can prove as above that

$$|G_{\Lambda_{l_1}\backslash Q}(E;x,y)| \leq \mathrm{e}^{-c_1|x-y|}\,, \qquad |x-y| \geq l_0\,.$$

Restoring the couplings to the bad cube Q is only a small pertubation and does not destroy the exponential decay:

$$G_{\Lambda_{l_1}}(x,y) = [G_{\Lambda_{l_1}\backslash Q} + G_{\Lambda_{l_1}\backslash Q}\Gamma G_{\Lambda_{l_1}}\Gamma G_{\Lambda_{l_1}\backslash Q}](x,y)\,,$$

where Γ here denotes the operator corresponding to ∂Q. Using the estimates established up to now we obtain

$$|G_{\Lambda_{l_1}}(x,y)| \leq \mathrm{e}^{-c_1|x-y|} + \mathrm{e}^{l_1^{1/2}} \sum_{\substack{(z_1,z_1')\in\partial Q\\(z_2,z_2')\in\partial Q}} \mathrm{e}^{-c_1(|x-z_1|+|z_2-y|)}\,.$$

(A little nicety was swept under the rug here, namely what if the distance of either x or y to ∂Q is less than l_0 (it is not possible that both are since $|x-y| > l_1/10$). However, by the Wegner estimate we can again establish a very crude bound $|G_{\Lambda_{l_1}\backslash Q}(E;x,y)| \leq \|G_{\Lambda_{l_1}\backslash Q}(E)\| \leq \mathrm{e}^{l_1^{1/2}}$ for any Q and $|x-y|$, having to discard configurations of measure of order $\mathrm{e}^{-l_1^{1/2-}}$.)

Because

$$\begin{aligned}|x-y| &\leq |x-z_1| + |z_1-z_2| + |z_2-y| \qquad \text{and}\\ |z_1-z_2| &\leq l_0\end{aligned}$$

we get

$$\begin{aligned}|G_{\Lambda_{l_1}}(x,y)| &\leq \mathrm{e}^{-c_1|x-y|}(1+|\Gamma|^2\mathrm{e}^{l_1^{1/2}+l_0})\\ &\leq \mathrm{e}^{-(c_1-\frac{l_1^{1/2}+l_0+\log C|\Gamma|^2}{|x-y|})|x-y|}\\ &\leq \mathrm{e}^{-(c_1-10\frac{l_1^{1/2}+l_1^{1/\alpha}+\log C|\Gamma|^2}{l_1})|x-y|} =: \mathrm{e}^{-c_1'|x-y|}\,,\end{aligned}$$

for $|x-y| \geq l_1/10$ and l_0 large enough.

The induction step is now proven, and the claim follows since it is clear that $c_n' > \frac{c}{2}$ for all n. $\square$

From this result of exponential decay for finite-size cubes, it is straightforward to go to the limit $\Lambda \nearrow \mathbb{Z}^d$. We shall now recall (see [**58**]) how to prove Theorem 2.1 from the finite-volume version Theorem 2.2.

Proof of Theorem 2.1. We have for an $E \in \mathbb{R}$, all scales l and all cubes Λ_l with probability at least $1 - l^{-k}$

$$|G_{\Lambda_l}(E + \mathrm{i}0; x, y)| < \mathrm{e}^{-\frac{c}{2}|x-y|} \qquad \text{for } \ |x-y| \geq \frac{l}{10}\,.$$

Fix an $x \in \mathbb{Z}^d$ and consider a sequence of cubes Λ_n centered at the origin with sides of length $l_n = 2^n(2|x|+L)$. We expand $G(E+\mathrm{i}\varepsilon; 0, x)$ around this sequence of nested cubes using the second resolvent identity and obtain

$$G(E+\mathrm{i}\varepsilon; 0, x) = [G_{\Lambda_0} + G_{\Lambda_1}\Gamma_0 G_{\Lambda_0} + G_{\Lambda_2}\Gamma_1 G_{\Lambda_1}\Gamma_0 G_{\Lambda_0} + \ldots](E+\mathrm{i}\varepsilon; 0, x)\,.$$

Since the distance between $\partial\Lambda_i$ and $\partial\Lambda_{i+1}$ is greater than l_i we get that

$$|G_{\Lambda_k}\Gamma_{k-1}\ldots G_{\Lambda_2}\Gamma_1 G_{\Lambda_1}\Gamma_0(E+\mathrm{i}\varepsilon; 0, x)| \leq \prod_{i=0}^{k-1} |\Lambda_i| \mathrm{e}^{-\frac{c}{2} l_i}$$

holds with probability at least

$$1 - \sum_{i=1}^{k} l_i^{-k}\,.$$

For the innermost cube, if $|x| \geq L/2$ we get

$$|G_{\Lambda_0}(E+\mathrm{i}\varepsilon; 0, x)| \leq \mathrm{e}^{-\frac{c}{2}|x|}$$

with probability at least

$$1 - l_0^{-k} = 1 - \left(\frac{1}{2|x|+L}\right)^k \geq 1 - (2L)^{-k}\,.$$

For $|x| < L/2$, the Wegner estimate gives

$$|G_{\Lambda_0}(E+\mathrm{i}\varepsilon; 0, x)| \leq \mathrm{e}^{L^{1/2}} < \mathrm{e}^{\frac{c}{4}L}$$

with probability at least

$$1 - L^{-k}\,.$$

Combined, the above estimates say that

$$|G_{\Lambda_0}(E+\mathrm{i}\varepsilon; 0, x)| \leq \mathrm{e}^{\frac{c}{2}(L-|x|)}$$

with probability at least

$$1 - (2L)^{-k} - L^{-k} \geq 1 - \widetilde{C}_k L^{-k}\,.$$

In a completely analogous way we get an estimate for $G_{\Lambda_0}(E+\mathrm{i}\varepsilon; x, y_0)$ for a $y_0 \in \partial\Lambda_0$:

$$|G_{\Lambda_0}(E+\mathrm{i}\varepsilon; x, y_0)| \leq \mathrm{e}^{\frac{c}{2}(L-|x|)}$$

with probability at least

$$1 - \widetilde{C}_k L^{-k}\,.$$

So Theorem 2.1 follows observing that first

$$\sum_{k=0}^{\infty}\prod_{i=0}^{k-1}|\Lambda_i|\mathrm{e}^{-\frac{c}{2}l_i}\leq \mathrm{const}\,,$$

and second that

$$\sum_{i=1}^{\infty} l_i^{-k}+\widetilde{C}_k L^{-k}\leq C_k L^{-k}$$

for some $C_k>0$. □

2. Pure point spectrum

Although we have established absence of diffusion, one might argue that a mathematically more satisfactory result would be a more precise knowledge of the spectrum of the random Schrödinger operator (2.1). Note that the group of translations acts ergodically on the probability space of configurations $(v_j)_{j\in\mathbb{Z}^d}$, allowing us to conclude [**42**] that there exist fixed sets σ_{pp}, σ_{ac}, $\sigma_{\mathrm{sc}}\subset\mathbb{R}$ such that

$$\sigma_{\mathrm{pp}}(H(\omega))=\sigma_{\mathrm{pp}}\quad \sigma_{\mathrm{ac}}(H(\omega))=\sigma_{\mathrm{ac}}\quad \sigma_{\mathrm{sc}}(H(\omega))=\sigma_{\mathrm{sc}}\quad \text{almost surely.}$$

In the same work it is shown that we can determine the spectrum of (2.1) with probability one,

$$\sigma(H(\omega))=[-2d,2d]+\operatorname{supp} g\qquad \text{almost surely,}\tag{2.7}$$

where $A+B=\{a+b: a\in A, b\in B\}$ for sets $A,B\subset\mathbb{R}$.

With Theorem 2.1 at hand, there are several ways to show pure point spectrum with exponentially decaying eigenfunctions, the easiest of which is due to Simon and Wolff [**52**],

THEOREM 2.4. *If for almost every $E\in(a,b)$ and almost every ω*

$$\lim_{\varepsilon\to 0}\sum_{x\in\mathbb{Z}^d}|G(E+\mathrm{i}\varepsilon;0,x)|^2<\infty$$

then, with probability one, H has pure point spectrum in (a,b) with exponentially decaying eigenfunctions.

Theorem 2.1 and the lemma of Borel-Cantelli imply that for almost every ω there is an L_∞ not depending on ε such that

$$|G(E+\mathrm{i}\varepsilon;0,x)|\leq \mathrm{e}^{m(L_\infty-|x|)}\,.$$

Thus Theorem 2.4 implies that H has almost surely pure point spectrum with exponentially decaying eigenfunctions in the range of energies E where Theorem 2.1 holds, that is where we can establish the initial estimate of Theorem 2.2. We show in the following how this can be done in various energy regimes.

Large disorder. For large disorder, that is, $\|g\|_\infty\ll 1$, the condition on the smallness of off-diagonal elements of the resolvent in Theorem 2.2 is established simply with a norm bound: Consider G_Λ for some finite subset $\Lambda\subset\mathbb{Z}^d$. If we have for some E

$$|v_x(\omega)-E|\geq M+2d\qquad \forall x\in\Lambda\,,$$

then

$$\|G_\Lambda(E,\omega)\|=(\operatorname{dist}(E,\sigma(H_\Lambda(\omega))))^{-1}\leq\frac{1}{M}\,.$$

Now,

$$\begin{aligned}\mathbb{P}[|v_x(\omega) - E| \geq M + 2d \quad \forall x \in \Lambda] &= (\mathbb{P}[|v_x(\omega) - E| \geq M + 2d])^{|\Lambda|} \\ =&(1 - \mathbb{P}[|v_x(\omega) - E| < M + 2d])^{|\Lambda|} \geq (1 - 2(M + 2d)\|g\|_\infty)^{|\Lambda|} \underset{\|g\|_\infty \to 0}{\longrightarrow} 1\,.\end{aligned}$$

As the above holds for any M, the condition of Theorem 2.2 is verified for all $E \in \mathbb{R}$ provided $\|g\|_\infty$ is small enough. (Note that $\|g\|_\infty \to 0$ implies in particular that the support of g gets infinite.)

Low energy. In the same vein for low energy: Suppose that for some N we have $|v_x(\omega)| \leq N$ for all $x \in \Lambda$. Then

$$\sigma(H_\Lambda(\omega)) \subset [-N - 2d, N + 2d]\,,$$

and for $|E| \geq N + 2d + M$ we have

$$\|G_\Lambda(E, \omega)\| = (\operatorname{dist}(E, \sigma(H_\Lambda(\omega))))^{-1} \leq \frac{1}{M}\,.$$

Because Λ is finite it is clear from the Markov inequality that

$$\begin{aligned}&\mathbb{P}[|v_x| \leq N \quad \forall x \in \Lambda] = \mathbb{P}[|v_x| \leq N]^{|\Lambda|} \\ &\geq \left(1 - \frac{\mathbb{E}\,|v_x|}{N}\right)^{|\Lambda|} \underset{N \to \infty}{\longrightarrow} 1\,,\end{aligned}$$

for $v_x \in L^1(\Omega_x)$. So the initial condition of Theorem 2.2 is fulfilled for $|E|$ large enough. Obviously, this is only of interest when the support of the probability density is infinite.

Band tails. We will treat this case extensively in Chapter 3, so the reader is asked for patience.

CHAPTER 3

Mathematics of random Zeeman interaction

1. Introduction

Now that we have explained the key ideas and main techniques used in the mathematical treatment of Anderson localization, we are ready to discuss the random Schrödinger operator that describes the evolution of a conduction electron in $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$. Recall from Chapter 1 that it is given by

$$H(\omega) = -\Delta + v_j(\omega) - Jm_j(\omega)\cdot\sigma\,. \tag{3.1}$$

As before, we have that there exist fixed sets σ_{pp}, σ_{ac}, $\sigma_{\mathrm{sc}} \subset \mathbb{R}$ such that

$$\sigma_{\mathrm{pp}}(H(\omega)) = \sigma_{\mathrm{pp}} \quad \sigma_{\mathrm{ac}}(H(\omega)) = \sigma_{\mathrm{ac}} \quad \sigma_{\mathrm{sc}}(H(\omega)) = \sigma_{\mathrm{sc}} \quad \text{almost surely,}$$

and

$$\sigma(H(\omega)) = [-2d, 2d] + \operatorname{supp}_{\mathrm{diag}}(H) \qquad \text{almost surely,}$$

where the "support of the diagonal part" of H is defined by

$$\operatorname{supp}_{\mathrm{diag}}(H) := \{\alpha \in \mathbb{R} : \forall\varepsilon > 0 \quad \mathbb{P}[\operatorname{dist}(\sigma(H(\omega)(0,0)), \alpha) < \varepsilon] > 0\}\,.$$

It is easy to see that $\operatorname{supp}_{\mathrm{diag}}(H) = \{\pm v \pm J\}$ so that

$$\sigma(H(\omega)) = [-2d - v - J, 2d + v + J] \qquad \text{almost surely,}$$

for v, J small, whereas the band splits into several bands if v and/or J are large. Our results hold only for the band edges of the extreme bands, so we will not distinguish the cases.

The proof of localization could run along the very same lines of the case of a random potential with bounded density, if it were not for the fact that it is not known how to prove an a priori (Wegner-type) estimate concerning the distribution of eigenvalues of the Hamiltonian restricted to finite-size cubes, such as

$$\mathbb{P}[\operatorname{dist}(E, \sigma(H_\Lambda)) \le \kappa] \le C\kappa|\Lambda|\,. \tag{3.2}$$

Recall that in the case of a bounded probability density the constant is given by $C = 2\|g\|_\infty$, where g is the density, explaining why the estimate is useless for the Bernoulli potential. The random Zeeman interaction term $m \cdot \sigma$ is similar to the Bernoulli potential in the sense that is has the same spectrum $\{\pm 1\}$, so that we could naively expect to encounter the same difficulties. However, as it will turn out, the continuous distribution of the random vector m on the unit sphere makes life a lot easier. It is still not known, though, how to prove a Wegner estimate (3.2) for a Schrödinger operator with random Zeeman interaction term (it may even be wrong in the form stated), but a new scheme introduced in [**8**] circumvents this problem by establishing a Wegner-type estimate inductively, starting from an initial scale, that will therefore hold only for energies E where we can prove the initial estimate. (The classic Wegner estimate holds for all energies E, even if these energies lie in the region that has supposedly absolutely continuous spectrum in the thermodynamic limit.)

Recall the main theorem from Chapter 2,

Theorem (2.1). *Consider the random Schrödinger operator (2.1) on the lattice $\mathbb{Z}^d$. Let $E \in \mathbb{R}$, $k > 2d$ and $0 < \eta < 1$. There is an L such that if for $l_0 \ge L$ we have*

(i)

$$\mathbb{P}\left[\sum_{y\in\partial\Lambda_{l_0}(0)} |G_{\Lambda_{l_0}}(E+\mathrm{i}\varepsilon;0,y)| \le \eta\right] \ge 1 - l_0^{-k} \qquad \forall\varepsilon\neq 0$$

(ii)

$$\mathbb{P}\left[\mathrm{dist}\left(E,\sigma(H_{\Lambda_l(0)})\right) < \kappa\right] \le C|\Lambda_l|\kappa\,,$$

for all $\kappa > 0$ small enough and all $l \ge l_0$,

then for $m_0 = -\log\eta/2l_0$ we have that

$$\mathbb{P}[|G(E+\mathrm{i}\varepsilon;0,x)| \le \mathrm{e}^{m_0(N-|x|)}\,, \varepsilon\neq 0] \ge 1 - C_k N^{-k}$$

holds for all $x \in \mathbb{Z}^d$.

There is a certain dichotomy regarding the conditions of this Theorem. The first condition, about the off-diagonal decay of the Green function, is to be verified only at an initial scale, whereas the Wegner estimate (ii) has to be established on all scales. However, recent efforts of Jean Bourgain [**8**] have shown that under some analyticity assumptions the Wegner estimate, too, has to be verified only at the initial scale. Using his techniques we can prove the following reslts; consult Section 3 for notation.

Recall from Chapter 1 the different physical regimes (A), (B), (C) for which we want to prove Anderson localization.

Remark. For both the regimes (A) and (B) we will discard the additional Bernoulli potential $v_j(\omega)$. It would only burden the notation, without really changing any proof. This is clear for the case of $v_j = v_j^{\mathrm{bdd}} + v_j^{\mathrm{Bernoulli}}$ (so the random potential is a sum of independent random variables, one of which has a bounded probability density g, and the other one is Bernoulli). The joint probability density g_v is given by the convolution, $g_v(x) = g * (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)(x) = \frac{1}{2}g(x+1) + \frac{1}{2}g(x-1)$, so it is bounded as well, yielding the Wegner estimate and thus the pure point spectrum.

In this chapter, we prove the following results for the various regimes described in the introduction:

(A) κ small, B small, $x = 1$

Theorem 3.1. *Consider the random Schrödinger operator (3.1), with the distribution of ω given by*

$$\mathrm{d}\mathbb{P}(m) = Z^{-1}\exp\{\kappa\sum_{|i-j|=1} m_i\cdot m_j + \beta B\sum_j m_j^{(z)}\}\prod_j \delta(m_j^2-1)\mathrm{d}^3 m_j\,. \tag{3.3}$$

There is a $\kappa_0 > 0$ such that for $\kappa < \kappa_0$ the following holds true. There exists a $\delta = \delta(\kappa, B)$ such that $H(\omega)$ has, with probability one, pure point spectrum for $E \in [\overline{E}-\delta, \overline{E}]$ with exponentially decaying eigenfunctions, where $\overline{E}$ denotes the upper spectral edge (and symmetrically for the lower spectral edge).

(B) $\kappa \to \infty$, B small.

Theorem 3.2. *For $x < x_{\mathrm{c}}$ there exists a $\delta = \delta(B, x)$ such that the random Schrödinger operator (3.1) with magnetic moments distributed "of percolation type" according to Chapter 1 has, with probability one, pure point spectrum for $E \in [\overline{E}-\delta, \overline{E}]$ with exponentially decaying eigenfunctions, where $\overline{E}$ denotes the upper spectral edge (and symmetrically for the lower spectral edge).*

(C) $B \to \infty$.

THEOREM 3.3. *Consider the Bernoulli random Schrödinger operator*

$$H(\omega) = -\Delta + \lambda v_j(\omega)\,,$$

where

$$v_j(\omega) = \begin{cases} 1\,, & \text{with probability } x \\ -1\,, & \text{with probability } 1-x \end{cases}. \tag{3.4}$$

The v_j are assumed to be independent. Then, in the band tail from $[-\lambda, -c\lambda^2 + O(\lambda^{4-})]$, $\rho(E)D(E)$ vanishes for all energies outside a set of the order of $\exp(-\frac{1}{2}\exp(\frac{1}{2}\lambda^{-1/2-}))$.

This is a perturbative result, and is meaningful only for small values of λ.

2. Wegner estimate

Since the main new difficulty when considering a random Zeeman interaction term instead of a random potential with bounded density comes from the Wegner estimate, we devote this section to the latter.

The Wegner estimate Lemma 2.3 says that the eigenvalues of H_Λ do not "clump together" too seriously. This is closely related to the density of states, to which we will now give a precise mathematical meaning. Denote by $N_\Lambda(E,\omega)$ the number of eigenvalues of $H_\Lambda(\omega)$ less than, or equal to E. Denote further by $\mathrm{P}_\Lambda(E,\omega)$ the spectral projection of H_Λ onto the interval $(-\infty, E)$. Then Stone's formula reads

$$\mathrm{P}_\Lambda(E,\omega) = \mathrm{s}\text{-}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{E} \mathrm{Im}\,(H_\Lambda(\omega) - E' - \mathrm{i}\varepsilon)^{-1}\mathrm{d}E'\,.$$

We can express the number of eigenvalues using the projection,

$$\begin{aligned} N_\Lambda(E,\omega) &= \mathrm{tr}\ \mathrm{P}_\Lambda(E,\omega) = \sum_{x\in\Lambda} \mathrm{tr}\ \langle \delta_x\,, \mathrm{P}_\Lambda(E,\omega)\delta_x\rangle \\ &= \lim_{\varepsilon\to 0}\frac{1}{\pi}\sum_{x\in\Lambda}\int_{-\infty}^{E} \mathrm{tr}\ \mathrm{Im}\, G_\Lambda(E' + \mathrm{i}\varepsilon; x, x)\mathrm{d}E'\,. \end{aligned}$$

By the subadditive ergodic theorem, see for instance [**14**],

$$n(E) := \lim_{\Lambda\to\mathbb{Z}^3}\frac{N_\Lambda(E,\omega)}{|\Lambda|}\,,$$

the *integrated density of states*, exists and is independent of ω with probability one. Since $n(E)$ is monotone, its derivative exists almost everywhere and is denoted by $\rho(E)$, the density of states. From the formulae above we get

$$\rho(E) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\mathbb{E}\,\mathrm{tr}\ \mathrm{Im}\, G(E + \mathrm{i}\varepsilon; 0, 0)\,.$$

Here we used translation invariance, details are discussed in Appendix C. At finite scales, we can define

$$\rho_\Lambda(E) = \frac{1}{2|\Lambda|}\sum_{x\in\Lambda}\lim_{\varepsilon\to 0}\frac{1}{\pi}\mathbb{E}\,\mathrm{tr}\ \mathrm{Im}\, G_\Lambda(E + \mathrm{i}\varepsilon; x, x)\,.$$

The connection to the Wegner estimate is obvious from the relation $\rho_\Lambda(E) = \frac{\mathrm{d}}{\mathrm{d}E}N_\Lambda(E)/|\Lambda|$, where $N_\Lambda(E) = \mathbb{E}\,N_\Lambda(E,\omega)$. If, for instance, we know that $\rho_\Lambda(E)$ is bounded by C uniformly in Λ then this provides us immediately with the Wegner estimate:

$$\begin{aligned} &\mathbb{P}\,[\mathrm{dist}\,(E, \sigma(H_\Lambda)) < \kappa] \le \mathbb{E}N_\Lambda(E+\kappa) - \mathbb{E}N_\Lambda(E-\kappa) = \\ &= \int_{|E-E'|\le\kappa}\frac{\mathrm{d}}{\mathrm{d}E'}\mathbb{E}N_\Lambda(E')\mathrm{d}E' = 2|\Lambda|\int_{|E-E'|\le\kappa}\rho_\Lambda(E')\mathrm{d}E' \le 4C|\Lambda|\kappa\,. \end{aligned}$$

The regularity of the (averaged) finite-scale density of states $\rho_\Lambda(E)$ is thus crucial for a Wegner estimate to hold. This is an explanation for why the Bernoulli random potential is so difficult: The unaveraged quantity $\rho_\Lambda(E,\omega)$ is simply a sum of δ-functions, but taking the expectation value has a smoothing effect for a random potential with a density. In the Bernoulli case, however, taking the expectation amounts to taking a finite average as long as Λ is finite,

$$\mathbb{E}\,\rho_\Lambda(E,\omega) = \frac{1}{2^{|\Lambda|}} \sum_{\text{configurations } \omega} \rho_\Lambda(E,\omega)$$

so that $\rho_\Lambda(E)$ still is a sum of δ-functions.

In our case of a random Zeeman interaction term, $m(\omega)\cdot\sigma$, things look much brighter—even though its spectrum is as singular as Bernoulli's—because taking the average entails integrating m over the unit sphere, thus making $\rho_\Lambda(E)$ much more regular than in the Bernoulli case.

However, it is still not straightforward to come up with a Wegner estimate, and it is only an inductive scheme described in the next section that bears fruit. To see why the matrix-nature of randomness complicates matters, we look back at the case of a random potential with bounded density: In this case, the Wegner estimate is obtained by first-order eigenvalue perturbation (as long as the probability is bounded): For $\Lambda \subset \mathbb{Z}^d$ finite consider an interval, centered at an eigenvalue of $H_\Lambda = -\Delta + v$, of length $2\kappa \ll \min|E_\alpha - E_{\alpha'}|$, for E_α, E'_α different eigenvalues. Compute for $H(v)\xi(v) = E(v)\xi(v)$

$$\frac{\partial E(v)}{\partial v_j} = |\xi_j(v)|^2\,,$$

and note that as $\|\xi\| = 1$ there is at least one site j_0 with $|\xi_{j_0}(v)|^2 \geq 1/|\Lambda|$. Thus by varying v_{j_0} of order κ we get

$$\delta E(v) = \mathcal{O}\left(\frac{\kappa}{|\Lambda|}\right)\,,$$

indicating that $\mathbb{P}[\mathrm{dist}(\sigma(H_\Lambda(v),E)) \leq \kappa/|\Lambda|]$ is of order κ (for a bounded probability density). We have given a rigorous proof in Lemma 2.3. An essential ingredient in the proof is the trivial identity $N(E,v) = N(0, v-E)$. Such an identity does *not* hold in general for matrix-valued potentials, explaining why it is not possible to generalize the above proof in a straight-forward way. The difficulty that arises when treating matrix-valued potentials can also be exemplified by doing an eigenvalue variation. Consider the Hamiltonian $H(\omega) = -\Delta + \omega^{-1}A\omega$, where $A = \mathrm{diag}(\lambda_1,\ldots,\lambda_r)$ and $\omega \in \mathrm{SU}(r)$ random, that is, $\omega^{-1}A\omega$ a random hermitian matrix with fixed spectrum. Then

$$\frac{\partial E(\omega)}{\partial \omega_j}\bigg|_{\omega_j=\mathbb{1}} = \frac{\partial}{\partial \omega_j}\bigg|_{\omega_j=\mathbb{1}} \langle H(\omega)\xi(\omega)\,,\xi(\omega)\rangle = 0\,.$$

Thus for an eigenfunction that is concentrated on sites j with $\omega_j \simeq \mathbb{1}$, it is not possible to move the corresponding eigenvalue by small variations of ω_j.

However, if we allow for some analyticity in the distribution of magnetic moments, it is very easy—using a path expansion—to prove the following lemma (see Appendix 6, where the proof is done for correlated magnetic moments using a cluster expansion.)

LEMMA 3.4. *Consider the random Schrödinger operator (3.1) with magnetic moments distributed according to*

$$\mathrm{d}\mathbb{P}(m) = Z^{-1}\exp\{\kappa \sum_{|i-j|=1} m_i\cdot m_j + \beta B \sum_j m_j^{(z)}\} \prod_j A(\gamma)\mathrm{e}^{-\gamma(m_j^2-\mu^2)^2}\mathrm{d}^3 m_j\,.$$

Then, for $|E|+\gamma$ large and κ small, the finite-scale density of states, defined by

$$\rho_\Lambda(E) = \frac{1}{2|\Lambda|}\sum_{x\in\Lambda}\lim_{\varepsilon\to 0}\frac{1}{\pi}\mathrm{tr}\ \mathrm{Im}\,\mathbb{E}G_\Lambda(E+\mathrm{i}\varepsilon;x,x)\,,$$

is bounded uniformely in Λ.

With the Wegner estimate at hand, we can proceed as in Chapter 2 to prove pure point spectrum with exponentially decaying eigenfunctions.

The Wegner estimate in the fractional-moments approach. Although it does not figure as prominently as in the multi-scale analysis, a Wegner-type estimate *does* enter the estimates of the fractional-moments method. A uniform estimate on the average of a fractional power of the diagonal elements of the resolvent is needed, see [**4**],

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \mathbb{E}\,|G(z;x,x)|^s \leq C\,.$$

For random potentials with e.g. bounded density this is a simple consequence of (2.3). For the random Zeeman interaction we have an analogous formula,

$$G(z;x,x) = (\widetilde{G}(z;x,x)^{-1} + \sigma\cdot m_x)^{-1}\,,$$

the quantities here being 2×2-matrices, and in $\widetilde{G}$ the value of m_x has been set to zero. Unfortunately, it is not easy to get a uniform estimate from this formula; and again the problem is the fixed spectrum of $\sigma\cdot m_x$. If we try to compute the expectation by conditioning on the values of $(m_y)_{y\neq x}$,

$$\mathbb{E}\,|G(z;x,x)|^s = \int \mathrm{d}\mu^{\perp}(m_x)\int \mathrm{d}\mu(m_x)|(\widetilde{G}(z;x,x)^{-1}+\sigma\cdot m_x)^{-1}|^s\,,$$

then we run into problems because

$$\int \mathrm{d}\mu(m_x)|(A+\sigma\cdot m_x)^{-1}|^s \to \infty, \quad A\to \pm\mathbb{1}\,.$$

We see that the fractional-moments approach is plagued with the same difficulties, and we will not pursue it any further.

3. Multi-scale analysis with inductive Wegner estimate

The method devised by Fröhlich and Spencer [**27**] to prove Anderson localization for a random Schrödinger operator H consists of proving exponential decay of the Green function of the operator restricted to finite subsets Λ, with probability approaching one on ever larger scales. As discussed in Chapter 2, two ingredients are needed for their inductive scheme to work. First, a certain control over the off-diagonal elements of the Green function at an initial scale, and second, an a priori control over the clumping together of eigenvalues of H_Λ on all scales—the Wegner estimate. The main technical innovation in [**8**] is to prove also the Wegner-type estimate inductively, so that it, too, has to be verified only at an initial scale.

We turn now our attention to the proof of Theorems 3.1 and 3.2. Its most technical part is contained in the following lemma (see page 13 for definitions),

Lemma 3.5. *Let $E\in\mathbb{R}$. Consider the random Schrödinger operator (3.1). There are N_0 and k such that if for all $R_l\in\mathcal{E}_l$, $\frac{1}{2}N_0^{1/4d}\leq l\leq N_0$,*

$$\mathbb{P}[R_l \text{ is } c\text{-regular at energy } E] \geq 1-l^{-k}\,,$$

then

$$\mathbb{P}[R_N \text{ is } \tfrac{c}{2}\text{-regular at energy } E] \geq 1-N^{-k}\,,$$

for all $R_N\in\mathcal{E}_N$, for all $N\geq N_0$.

3.1. Sketch of proof. Recall the main strategy of the multi-scale analysis: We establish existence of "good" cubes with high probability at an initial scale, and use this information to show existence of good cubes with even higher probability, inductively, at larger scales. The probability that a cube Λ_l is good should behave like $1-p_l$, where $p_l \propto l^{-k}$, and k is some integer. The key point is that, at finite scales, we have to prove existence of good cubes only with high probability and not with probability one: Denote by Ω the whole probability space, that is, the set of all configurations ω of the random magnetic moments $(m_j)_{j\in\mathbb{Z}^3}$. *From the outset, we can discard configurations ω that are difficult to handle, as long as they have small probability.*

The induction step from scale l_n to $l_{n+1} \lesssim l_n^2$ proceeds as follows: Consider a cube $\Lambda_{l_{n+1}}$ of size l_{n+1}. Since "bad" cubes of size l_n are, by the induction hypothesis, improbable, the probability that there are *many*, say N, bad l_n-cubes in a l_{n+1}-cube is even smaller, of the order of $p_{l_n}^N$. The integer N is chosen such that $p_{l_n}^N < p_{l_{n+1}}$. *Therefore, only configurations ω where the l_{n+1}-cube contains not too many bad l_n-cubes need to be considered.*

To prove that $\Lambda_{l_{n+1}}$ is good we proceed in the following way. In a first step, we excise the bad l_n-cubes (of which there are not too many) from $\Lambda_{l_{n+1}}$. Iterating the resolvent identity along a sequence of l_n-cubes and using the induction hypothesis at scale l_n we prove that $\Lambda_{l_{n+1}}\backslash\{\text{bad cubes}\}$ is good. The difficulty is now that, to couple a bad cube Λ_{l_n} back to $\Lambda_{l_{n+1}}$, we need a slightly larger cube covering Λ_{l_n} that satisfies a Wegner-type estimate. But by the induction hypothesis, the Wegner estimate holds for cubes of this size only with probability $1-p_{l_n}$, whereas we should like to establish it with the much larger probability $1-p_{l_{n+1}}$.

As discussed in detail in Chapter 2, for random potentials with a *bounded* probability density it has been known for a long time [**61**] how to establish a Wegner-type estimate *simultaneously* on all scales l. Recent mathematical work [**8**], triggered by our study of the hexaboride alloys, shows how to establish a Wegner estimate inductively.

We have already argued that we can restrict our attention to configurations ω where there are not too many bad cubes of size l_n in $\Lambda_{l_{n+1}}$—call these bad cubes B_k. The key idea is to modify each configuration ω by changing in $\omega = (m_j)_{j\in\Lambda_{l_{n+1}}}$ only the values of m_j for j *inside* the B_k such that, for this new configuration ω', each bad cube has a neighbourhood satisfying a Wegner estimate. In a second step, one shows with the help of complex analysis that these configurations ω' have actually very large probability.

Technically, this is done as follows (see also Figure 1): Pick one of the bad cubes, call it B. Now, cover the bad cube B with a slightly larger cube $B_{(1)}$. Then the probability that the configuration in the ring $B_{(1)}\backslash B$ makes the cube $B_{(1)}$ bad, no matter what the configuration inside B, is smaller than p_{l_n}. Thus, the probability that there are *many*, say N, equicentered cubes $B_{(i)}$ of increasing diameter such that, for all i, the configuration in the ring $B_{(i)}\backslash B_{(i-1)}$ makes the cube $B_{(i)}$ bad, no matter what the configuration in the interior, $B_{(i-1)}$, is very small, namely of the order of $p_{l_n}^N$. We can therefore, from the outset, restrict considerations to configurations ω where each bad cube B can be replaced by a larger cube $B_{(i_0-1)}$ with the property that the configuration ω can be modified *inside the cube* $B_{(i_0-1)}$ *alone* such that the cube $B_{(i_0-1)}$ has a good neighborhood $B_{(i_0)}$. The modified configuration shall be denoted by ω'.

We can now use again the resolvent identity to establish the desired bound on $\|G_{\Lambda_{l_{n+1}}}(E,\omega')\|$. We have thus found that $\|G_{\Lambda_{l_{n+1}}}(E)\|$ is bounded at scale l_{n+1} for *one fixed configuration* ω'. One may think that this is far too little, since a single point ω' has zero probability in Ω. However, in a last step, one shows, using a matrix-valued Cartan-type lemma, that the probability of configurations for which $\|G_{\Lambda_{l_{n+1}}}(E)\|$ satisfies the desired estimate is at least $1-p_{n+1}$.

To understand the last step we need to recall a result from complex analysis, known as Cartan's lemma. The precise mathematical statement can be found in the appendix. For the purpose of this sketch, a rough understanding of the lemma will suffice: The lemma says that an analytic function

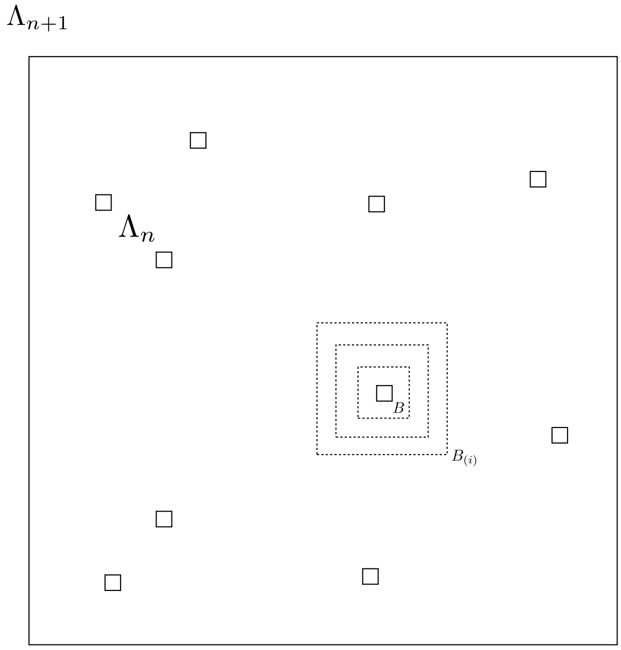

FIGURE 1. Generic cube of size l_{n+1}. The bad cubes of size l_n, of which we suppose there are not too many, are depicted as smaller cubes. The construction of a good neighborhood of a bad cube is hinted at.

that is bounded away from zero at *one* point of its domain of definition is at *most* points not too close to zero. We need a higher-dimensional matrix-valued analog of this lemma due to Bourgain [**8**], which we relegate to the appendix. This generalized lemma says that if $\|G_{\Lambda_{n+1}}(E;\omega)\|$ is not too large for *one* configuration ω', it is not too large for most configurations; actually only exponentially few (in l_n) configurations have to be excluded. *It is important to point out that in order to apply this lemma the distribution of magnetic moments on the unit sphere needs to have a bounded density with respect to the uniform measure on the sphere.*

We can apply this lemma, for we have explicitly constructed a configuration ω' with the desired properties.

The off-diagonal decay is standard now, using again iteration of the resolvent equation along a sequence of nested cubes.

3.2. Proof of Lemma 3.5. We are now ready to discuss the proof with all technical details.

PROOF. To simplify the presentation we will first do the proof for regime (A), more precisely for $\kappa = 0, x = 1$. We are thus looking at independent unit vectors m_j, $j \in \mathbb{Z}^d$, that are uniformly distributed on the unit sphere,

$$\mathrm{d}\mathbb{P}(m) = \prod_j \delta(m_j^2 - 1)\mathrm{d}^3 m_j \,.$$

We will show later on how the proof has to be modified to cover correlations and percolation effects.

The proof is by induction. For sake of readability we perform the step from N_0 to N_1. Take $n < N_0 < N_1$ with $N_1 < N_0^2$ and $n = N_1^{1/8d} < N_0^{1/4d}$.

Induction hypothesis: Elementary l-regions are c-regular with probability at least $1 - l^{-k}$ for scales $\frac{n}{2} \leq l \leq N_0$.

First step: Let B be an elementary N_1-region, and fix an integer $M > 32d$. The probability that there are many singular n-regions is small: Denote by $\Omega_1 \subset \Omega$ the set of all configurations ω such that there are M or more disjoint *singular* n-regions $R \subset B$. Thanks to the induction assumption, its probability is bounded by

$$\mathbb{P}[\Omega_1] \leq C N_1^{2dM} n^{-kM} < n^{-\frac{1}{2}kM}\,, \tag{3.5}$$

if $k > 32d^2$. The first factor is an upper bound on the number of possibilities to place M n-regions in B, since $|B| \leq C N_1^d$ and elementary regions are differences of two cubes; the second factor is due to independence. We restrict to configurations $\omega \in \Omega_1^{\mathrm{c}}$, that is, to configurations where there are at most M singular n-regions $R \subset B$ all of which are disjoint, or where there are (possibly more than M) *intersecting* singular n-regions no M of which are disjoint. In the latter case, these n-regions are contained in a cube of diameter $(2M-1)2n$. In the former case, group regions R that have a mutual distance less than $10Mn$ and cover them with minimal cubes Q. In addition, if a cube Q happens to lie at a distance $\leq n$ from ∂B then replace it by a minimal covering cube (again denoted by Q) of which each side either intersects ∂B or is at a distance at least n from it. These cubes Q satisfy by construction

$$4n \leq \operatorname{diam} Q \leq 20M^2 n \tag{3.6}$$

$$\operatorname{dist}(Q, Q') \geq 10Mn \qquad \text{for distinct } Q, Q'\,. \tag{3.7}$$

Denote the union of these new (and at most M) cubes by Λ. The first result is that a Wegner estimate holds on $B\backslash\Lambda$,

LEMMA 3.6. *There is an n such that*

$$\|G_{B\backslash\Lambda}(E)\| \leq 2n^{d+1}\mathrm{e}^{n^{1/2}}\,.$$

PROOF. By the resolvent identity, we have

$$G_{B\backslash\Lambda}(E;x,y) = G_{W(x)}(E;x,y) + \sum_{(z,z')\in\partial W(x)} G_{W(x)}(E;x,z)G_{B\backslash\Lambda}(E;z',y)\,,$$

where $W(x) = C(x) \cap (B\backslash\Lambda)$ with $C(x)$ an n-cube with center x. Summing over $y \in B\backslash\Lambda$ and taking the supremum we obtain

$$\begin{aligned}\sup_{x\in B\backslash\Lambda}\sum_{y\in B\backslash\Lambda}|G_{B\backslash\Lambda}(E;x,y)| &\leq \sup_{x\in B\backslash\Lambda}\sum_{y\in W(x)}\|G_{W(x)}(E)\| \\ &+ \sup_{x\in B\backslash\Lambda}\sum_{(z,z')\in\partial W(x)}|G_{W(x)}(E;x,z)|\sup_{w\in B\backslash\Lambda}\sum_{y\in B\backslash\Lambda}|G_{B\backslash\Lambda}(E;w,y)|\,.\end{aligned}$$

Since the $W(x)$ are regular n-regions for all $x \in B\backslash\Lambda$ by construction of Λ (if x is in an outer corner of B, $W(x)$ is actually a $n/2$-region, which explains the lower bound in the induction hypothesis) we can use (1.10) and (1.11) at scale n to get

$$\|G_{B\backslash\Lambda}(E)\| \leq \sup_{x\in B\backslash\Lambda}\sum_{y\in B\backslash\Lambda}|G_{B\backslash\Lambda}(E;x,y)| \leq 2n^{d+1}\mathrm{e}^{n^{1/2}}\,,$$

for $\kappa_d n^{d-1}\mathrm{e}^{-cn} < 1/2$, where the constant κ_d is such that the magnitude of the boundary of an n-cube is bounded by $\kappa_d n^{d-1}$. ☐

The second result ist the off-diagonal decay,

LEMMA 3.7. *There is an n and a $c_1 < c$ such that*

$$|G_{B\backslash\Lambda}(E;x,y)| \leq \mathrm{e}^{-c_1|x-y|}\,,$$

for $|x-y| \geq n$.

Proof. We expand the resolvent along a sequence of regular n-cubes C_i as follows:

$$G_{B\backslash\Lambda}(E;x,y) = \sum_{(z_1,z_1')\in\partial C_1} G_{C_1}(E;x,z_1)G_{B\backslash\Lambda}(E;z_1',y)$$

for $x \in C_1$ but $y \notin C_1$. Find z_1' that maximizes $|G_{B\backslash\Lambda}(E;z_1',y)|$ in the expression above, and denote it by $\widetilde{z}_1'$. Continue expanding around the cube C_2:

$$|G_{B\backslash\Lambda}(E;x,y)| \leq \sum_{\substack{(z_1,z_1')\in\partial C_1\\(z_2,z_2')\in\partial C_2}} |G_{C_1}(E;x,z_1)||G_{C_2}(E;\widetilde{z}_1',z_2)||G_{B\backslash\Lambda}(E;z_2',y)|\,,$$

for $y \notin C_2$. We can continue this procedure until $y \in C_{k+1}$, and obtain

$$\begin{aligned}|G_{B\backslash\Lambda}(E;x,y)| &\leq \sum_{(z_i,z_i')\in\partial C_i} |G_{C_1}(E;x,z_1)|\dots|G_{C_k}(E;\widetilde{z}_{k-1}',z_k)|\|G_{B\backslash\Lambda}(E)\|\\ &\leq (\kappa_d n^{d-1})^k \mathrm{e}^{-cnk} 2n^{d+1}\mathrm{e}^{n^{1/2}}\,,\end{aligned}$$

Since $k \geq |x-y|/n$ we are left with

$$\begin{aligned}|G_{B\backslash\Lambda}(E;x,y)| &\leq C\mathrm{e}^{-c|x-y|+\frac{\log(\kappa_d n^{d-1})}{n}|x-y|+n^{1/2}}\\ &\leq C\mathrm{e}^{-(c-\frac{\log(\kappa_d n^{d-1})}{n}-n^{-1/3})|x-y|}\mathrm{e}^{-n^{2/3}+n^{1/2}}\\ &\leq \mathrm{e}^{-(c-\frac{\log(\kappa_d n^{d-1})}{n}-n^{-1/3})|x-y|} =: \mathrm{e}^{-c_1|x-y|}\,,\end{aligned}$$

for $|x-y| \geq n$ and n large enough. □

We have thus completed the first step, namely we constructed with probability at least $1-n^{-\frac{1}{2}kM}$ a decomposition of $B = \Lambda \cup (B\backslash\Lambda)$ such that

$$\|G_{B\backslash\Lambda}(E)\| \leq 2n^{d+1}\mathrm{e}^{n^{1/2}} \tag{3.8}$$

$$|G_{B\backslash\Lambda}(E;x,y)| \leq \mathrm{e}^{-c_1|x-y|} \qquad \text{for } |x-y| \geq n\,. \tag{3.9}$$

Remark. By repeating the above argument, (3.8), (3.9) hold for any N-region $\subset B\backslash\Lambda$, $N \geq n$.

Second step: After showing that the Green function is well-behaved outside the cubes constituting Λ, all we are wanting for a norm estimate on the whole of B is a norm estimate on neighbourhoods of these cubes. To achieve this, we make the following construction. Let Q be one of the components of Λ, and let

$$Q = Q_{(0)} \subset Q_{(1)} \subset Q_{(2)} \subset \dots \subset Q_{(M)}$$

be a sequence of equicentered cubes with $\mathrm{dist}(\partial Q_{(i-1)}, \partial Q_{(i)}) = n$. Next, we introduce an auxiliary notion: Call a ring $Q_{(i)}\backslash Q_{(i-1)}$ *ring-singular* if the configuration in the ring $Q_{(i)}\backslash Q_{(i-1)}$ is such that $Q_{(i)}$ is singular for all configurations in $Q_{(i-1)}$. Since $\mathbb{P}[Q_{(i)} \text{ is singular}] \leq n^{-k}$ (it is here that we need the induction hypothesis to hold over a *range* of initial scales), the probability that $Q_{(i)}\backslash Q_{(i-1)}$ is ring-singular is at most n^{-k}. Denote with $\Omega_2 \subset \Omega$ the set of configurations ω where for some cube $Q \subset B$, $Q \subset Q_{(1)} \subset Q_{(2)} \subset \dots \subset Q_{(M)}$ as above, we have that each ring $Q_{(i)}\backslash Q_{(i-1)}$ is ring-singular. Its probability is bounded by

$$\mathbb{P}[\Omega_2] \leq N_1^{d+1} n^{-kM} < n^{-\frac{1}{2}kM}\,.$$

Hence let us restrict further to configurations $\omega \in (\Omega_1 \cup \Omega_2)^{\mathrm{c}}$, that is, to configurations where for each component Q of Λ there is some $1 \leq i \leq M$ such that the ring $Q_{(i)}\backslash Q_{(i-1)}$ is *not* ring-singular, and denote this $Q_{(i-1)}$ by $\overline{Q}$. The new cubes are at least $8Mn$-separated by definition, see (3.7). The union of these larger cubes is called $\overline{\Lambda}$, and it is clear that (3.8),(3.9) still hold. By construction, $\overline{\Lambda}$ has the property that each of its components $\overline{Q}$ has an n-neighbourhood Q' such that the ring $Q'\backslash\overline{Q}$ is not ring-singular. Hence we can modify any given configuration $\omega \in (\Omega_1 \cup \Omega_2)^{\mathrm{c}}$ on each $\overline{Q}$ such that the resulting ω' satisfies

$$G_{B\setminus\overline{\Lambda}}(E,\omega') = G_{B\setminus\overline{\Lambda}}(E,\omega) \tag{3.10}$$

$$\|G_{Q'}(E,\omega')\| < \mathrm{e}^{10Mn^{1/2}} \quad \text{for all } Q' , \tag{3.11}$$

due to (3.6) and $(20M^2n)^{1/2} < 10Mn^{1/2}$. Estimates (3.10),(3.11) and the so-called Fröhlich-Spencer coupling lemma (since it "couples" singular subsets to their complement) allow us to establish a bound on $\|G_B(E,\omega')\|$,

Lemma 3.8. *There is an n such that*

$$\|G_B(E,\omega')\| \le \mathrm{e}^{11Mn^{1/2}} .$$

Proof. Fix $x, y \in B$. We have to distinguish three cases:

(a) $x, y \in \overline{\Lambda}$
(b) $x, y \in B\setminus\overline{\Lambda}$
(c) $x \in \overline{\Lambda},\ y \in B\setminus\overline{\Lambda}$

(a): Expanding G_B alternatingly along $\partial\overline{\Lambda}$ and $\partial\Lambda'$ (where $\Lambda' := \bigcup Q'$), and denoting the corresponding operators $\overline{\Gamma}$ and Γ', respectively, we obtain

$$G_B(x,y) = [G_{\Lambda'} + G_{\Lambda'}\Gamma' G_{B\setminus\overline{\Lambda}}\overline{\Gamma} G_{\Lambda'} + G_{\Lambda'}\Gamma' G_{B\setminus\overline{\Lambda}}\overline{\Gamma} G_{\Lambda'}\Gamma' G_{B\setminus\overline{\Lambda}}\overline{\Gamma} G_{\Lambda'} + \ldots](x,y) ,$$

since here $G_{B\setminus\overline{\Lambda}}(\cdot,y) = 0$. Now, clearly $\|G_{\Lambda'}\| \le \max_{Q'}\|G_{Q'}\|$ as $H_{\Lambda'}$ is block diagonal. Thus, we estimate each factor of $G_{\Lambda'}(u,v)$ by (3.11), and because

$$\operatorname{dist}(\partial\Lambda', \partial\overline{\Lambda}) \ge n$$

we estimate with (3.10), that is, (3.9),

$$|[\Gamma' G_{B\setminus\overline{\Lambda}}\overline{\Gamma}](u,v)| \le \mathrm{e}^{-c_1|u-v|} \le \mathrm{e}^{-c_1 n}$$

to get

$$|G_B(x,y)| \le \mathrm{e}^{10Mn^{1/2}}(1 - |\partial\overline{\Lambda}||\partial\Lambda'|\mathrm{e}^{-c_1 n + 10Mn^{1/2}})^{-1} \le 2\mathrm{e}^{10Mn^{1/2}} ,$$

for n large enough.

(b): Since now $G_{\Lambda'}(\cdot,y) = 0$ a similar expansion looks like

$$G_B(x,y) = [G_{B\setminus\Lambda'} + G_{B\setminus\Lambda'}\Gamma' G_{B\setminus\overline{\Lambda}} + G_{B\setminus\Lambda'}\Gamma' G_{B\setminus\overline{\Lambda}}\overline{\Gamma} G_{\Lambda'}\Gamma' G_{B\setminus\overline{\Lambda}} + \ldots](x,y) .$$

By the remark after (3.9) we have $\|G_{B\setminus\Lambda'}\| \le 2n^{d+1}\mathrm{e}^{n^{1/2}}$, and thus

$$\begin{aligned} |G_B(x,y)| &\le 2n^{d+1}\mathrm{e}^{n^{1/2}} + |\Gamma'|(2n^{d+1}\mathrm{e}^{n^{1/2}})^2(1 - |\partial\overline{\Lambda}||\partial\Lambda'|\mathrm{e}^{-c_1 n + 10Mn^{1/2}})^{-1} \\ &\le \mathrm{e}^{3n^{1/2}} , \end{aligned}$$

for n large enough.

(c): A very similar expansion again is in this case

$$G_B(x,y) = [G_{\Lambda'} + G_{\Lambda'}\Gamma' G_{B\setminus\overline{\Lambda}} + G_{\Lambda'}\Gamma' G_{B\setminus\overline{\Lambda}}\overline{\Gamma} G_{\Lambda'}\Gamma' G_{B\setminus\overline{\Lambda}} + \ldots](x,y) ,$$

and yields

$$\begin{aligned} |G_B(x,y)| &\le \mathrm{e}^{10Mn^{1/2}} + |\Gamma'|2n^{d+1}\mathrm{e}^{n^{1/2}}(1 - |\partial\overline{\Lambda}||\partial\Lambda'|\mathrm{e}^{-c_1 n + 10Mn^{1/2}})^{-1} \\ &\le 2e^{10Mn^{1/2}} , \end{aligned}$$

for n large enough.

As seen previously, we can bound the norm via matrix elements,

$$\|G_B(E,\omega')\| \leq \sup_{x\in B} \sum_{y\in B} |G_B(E,\omega';x,y)| \,,$$

and thus obtain the desired result

$$\|G_B(E,\omega')\| \leq N_1^{d+1} 2\mathrm{e}^{10Mn^{1/2}} \leq \mathrm{e}^{11Mn^{1/2}} \,. \tag{3.12}$$

□

Third step: Now that we have a bound on $\|G_B(E)\|$ for at least one point in probability space, we have all the requirements to invoke the Cartan-type lemma proven in Appendix B to prove the norm estimate with high probability on the whole of B,

Lemma 3.9. *There is an n such that*

$$\mathbb{P}[\|G_B(E)\| > \mathrm{e}^{N_1^{1/2}}] \leq N_1^{-2k} \,.$$

Proof. We apply Lemma B.4 (with $n = m = |\overline{\Lambda}|, N = |B|$) to the real analytic matrix function on $(\mathbb{R}^3)^{\overline{\Lambda}}$

$$A(x_j, j \in \overline{\Lambda}) = H_B(\omega_j \ (j \in B\backslash\overline{\Lambda}), x_j \ (j \in \overline{\Lambda})) - E$$

to get

$$\begin{aligned}&\mathbb{P}[x \in (\mathbb{R}^3)^{\overline{\Lambda}} : \|G_B(E;\omega_j \ (j \in B\backslash\overline{\Lambda}), x_j \ (j \in \overline{\Lambda}))\| > \mathrm{e}^t] \\ &< C|\overline{\Lambda}|\mathrm{e}^{C|\overline{\Lambda}|}\mathrm{e}^{-\frac{c' t}{|\overline{\Lambda}|Mn^{1/2}}} < M(21M^2 n)^d \mathrm{e}^{-C(M^2n)^{-d-1}t} \,.\end{aligned}$$

Denote by $\Omega_3 \subset \Omega$ the set

$$\{\omega \in (\Omega_1 \cup \Omega_2)^{\mathrm{c}} : \omega\big|_{\overline{\Lambda}} \text{ is such that } \|G_B(E;\omega)\| > \mathrm{e}^{N_1^{1/2}}\} \,.$$

Combining all of the above we obtain

$$\begin{aligned}\mathbb{P}[\|G_B(E)\| > \mathrm{e}^{N_1^{1/2}}] &\leq \mathbb{P}[\Omega_1 \cup \Omega_2 \cup \Omega_3] \\ &\leq n^{-\frac{1}{2}kM} + n^{-\frac{1}{2}kM} + (2N_1)^{dM} M(21M^2n)^d \mathrm{e}^{-C(M^2n)^{-d-1}N_1^{1/2}} \\ &\leq 2N_1^{-\frac{1}{16d}kM} + CN_1^{dM+\frac{1}{8}}\mathrm{e}^{-CN_1^{\frac{3}{8}-\frac{1}{8d}}} \leq N_1^{-2k} \,,\end{aligned}$$

for $M > 32d$, and n (and thus N_1) large enough. □

All that remains is to verify the off-diagonal decay at scale N_1,

Lemma 3.10. *There is an n such that*

$$|G_B(E;x,y)| \leq \mathrm{e}^{-c_1|x-y|} \quad \textit{for } |x-y| \geq \frac{N_1}{10} \,,$$

Proof. Recall Λ, the set introduced in the beginning of the proof, which is the union of at most M cubes of size $4n \leq \operatorname{diam}(Q) \leq 20M^2 n$, and let us restrict to configurations where $\|G_B(E)\| \leq \mathrm{e}^{N_1^{1/2}}$. We use again the F-S coupling lemma, but this time in an inductive manner, for to get information on the decay of off-diagonal elements it is required to couple *one* singular cube at a time. For, consider the situation where we would like to couple a singular set S that consists of two or more disjoint cubes of some fixed size. Choosing two points that are far apart compared to the size of a cube does then *not* imply that at least one of the points is far away from ∂S, which is needed for the off-diagonal decay. Therefore, define $A := (B\backslash\Lambda) \cup Q$ for some cube $Q \subset \Lambda$.

Let $x, y \in A : |x-y| > N_1/(3^{M-1}10)$. (We need the factor of 3^{M-1} in the denominator of the separation of x and y for the coupling of the next $M-1$ cubes.)

(a): $x, y \notin Q$

$$G_A(x,y) = [G_{A\backslash Q} + G_{A\backslash Q}\Gamma G_A \Gamma G_{A\backslash Q}](x,y)\,,$$

where Γ here denotes the operator corresponding to ∂Q. Using the estimates established up to now (if the distance of either x or y to ∂Q is less than n - it is not possible that both are since $|x-y|$ is too large - estimate the corresponding term with the norm bound, the rest goes through analogously) we obtain

$$|G_A(x,y)| \leq \mathrm{e}^{-c_1|x-y|} + \mathrm{e}^{N_1^{1/2}} \sum_{\substack{(z_1,z_1')\in\partial Q\\(z_2,z_2')\in\partial Q}} \mathrm{e}^{-c_1(|x-z_1|+|z_2-y|)}\,.$$

It is clear that the norm estimate on G_A is established in exactly the same way as for G_B (possibly having to exclude events of measure N_1^{-2k}, but $MN_1^{-2k} < N_1^{-k}$, for N_1 large enough). Because

$$\begin{aligned} |x-y| &\leq |x-z_1| + |z_1-z_2| + |z_2-y| + 2 \qquad \text{and} \\ |z_1-z_2| &\leq 20M^2 n \end{aligned}$$

we get

$$\begin{aligned} |G_A(x,y)| &\leq \mathrm{e}^{-c_1|x-y|}(1+|\Gamma|^2 \mathrm{e}^{N_1^{1/2}+20M^2n+2}) \\ &\leq \mathrm{e}^{-(c_1-N_1^{-1/3})|x-y|} C|\Gamma|^2 \mathrm{e}^{N_1^{1/2}+20M^2n+2-N_1^{2/3}/(3^M 10)} \\ &\leq \mathrm{e}^{-(c_1-N_1^{-1/3})|x-y|} =: \mathrm{e}^{-c_1'|x-y|}\,, \end{aligned}$$

for $N_1 = n^{8d}$ large enough.

(b): $x \in Q,\ y \notin Q$

$$G_A(x,y) = [G_A \Gamma G_{A\backslash Q}](x,y)$$

implies

$$\begin{aligned} |G_A(x,y)| &\leq \mathrm{e}^{N_1^{1/2}} \sum_{(z,z')\in\partial Q} |G_{A\backslash Q}(z',y)| \\ &\leq \mathrm{e}^{N_1^{1/2}} |\Gamma| \mathrm{e}^{-c_1|z'-y|} \leq \mathrm{e}^{-c_1'|x-y|}\,, \end{aligned}$$

by using $|x-y| \leq |x-z| + |z'-y| + 1$ and the same arguments as in (a).

Repeat now the above steps for $A \to A \cup Q$ for some other cube $Q \subset \Lambda$, and $|x-y| > N_1/(3^{M-2}10)$, and so on until all singular cubes have been "coupled", that is, $A = B$. In this way we obtain the sought after decay

$$|G_B(E;x,y)| \leq \mathrm{e}^{-c_1|x-y|} \quad \text{ for } |x-y| \geq \frac{N_1}{10}\,,$$

with probability at least $1 - N_1^{-k}$. We denoted the resulting "mass" again by c_1. □

Thus, the induction step is performed, and Lemma 3.5 proven as it is apparent that $\lim_{i\to\infty} c_i > c/2$, for N_0 large enough. □

4. Pure Point Spectrum

The vanishing of the diffusion constant so far proven is a pointwise result, that is we have showed $D(E) = 0$ for energies E at which $G(E)$ satisfies certain hypotheses. As argued in Chapter 2 it may be mathematically more satisfactory to determine the nature of the spectrum, or at least part of it. There are several ways of deducing pure point spectrum from the exponential decay of the Green function. We have used the simplest, due to Simon and Wolff [**52**], in Chapter 2, but their method is unfortunately not applicable for matrix-valued potentials or potentials with too singular a probability density. We will pursue another method using generalized eigenfunctions, closely following [**9**].

The problem with the direct application of Lemma 3.5 is that for proving almost sure point spectrum in an interval I the Green function $G(E,\omega)$ needs to be small for all energies $E \in I$ *simultaneously* (essentially because uncountable intersections of sets of measure 1 can have measure < 1). The following remark is essential and follows easily by considering (1.10):

Remark. Regularity at scale l is stable under perturbation of the energy of order $\mathrm{e}^{-l^{\frac{1}{2}}+}$.

The key observation for proving the absence of continuous spectrum is a representation of the spectrum of a Schrödinger operator [**51**] using generalized eigenfunctions.

Definition 4.1. A function ψ on $\mathbb{Z}^d$ is called a *generalized eigenfunction with generalized eigenvalue E of a self-adjoint operator H, if ψ is a polynomially bounded solution to the equation $H\psi = E\psi$.*

In the same work it is proven that

$$\sigma(H) = \overline{\{E : E \text{ is a generalized eigenvalue of } H\}}\,.$$

All we need to show is therefore that any generalized eigenfunction of $H(\omega)$ is actually a true eigenfunction. Consider thus a polynomially bounded solution $\psi(\omega)$ of the equation

$$H(\omega)\psi(\omega) = \mathcal{E}(\omega)\psi(\omega)\,,$$

where $\mathcal{E} \in I$, the set where we can prove the initial estimate of Lemma 3.5. To reduce notation, we will in the following pretend that $I = \sigma(H)$.

The problem is that $\mathcal{E}$ and ψ depend on the configuration ω. We want to get rid of that dependence by finding a finite set of "deterministic" energies that are close to the generalized eigenvalues $\mathcal{E}(\omega)$. Because regularity is stable under slight perturbation of the energy, this will enable us to prove regularity for all energies simultaneously.

Fix therefore a scale l and evenly spaced points $(E_i)_{i=1}^N \in \sigma(H)$ with spacing $\mathrm{e}^{-l^{\frac{2}{3}}}$ such that $E_1 = \inf \sigma(H)$ and $|E_N - \sup \sigma(H)| \le \mathrm{e}^{-l^{\frac{2}{3}}}$ (N is of order $\mathrm{e}^{l^{\frac{2}{3}}}$, since the spectrum is compactly supported), and consider cubes centered at 0 of size $L := \ell^2$ and $2L$, denoted by Λ_L and Λ_{2L}, respectively. Consider a tiling of Λ_{2L} with l-cubes, denote it by $\mathcal{F}$. Define

$$\mathcal{F}_{\mathrm{bad}} := \{\Lambda \in \mathcal{F} : G_\Lambda(E,\omega) \text{ is not regular}\}\,,$$

and denote by $\mathcal{C}$ $(= \mathcal{C}(E))$ the connected component of

$$\Lambda_L \cup \bigcup_{\Lambda \in \mathcal{F}_{\mathrm{bad}}} \Lambda\,. \tag{3.13}$$

Lemma 3.11. *With probability larger than $1 - l^{-\frac{kl}{4}}$ we have that for all energies E*

$$\mathcal{C} \subset \Lambda_{\frac{3}{2}L}$$

with obvious notation.

Proof. Assume that (3.13) is violated. Then there is a chain

$$\Lambda^{(1)}, \Lambda^{(2)}, \cdots, \Lambda^{(r)}$$

of distinct l-cubes $\in \mathcal{F}_{\text{bad}}$ such that $\Lambda^{(1)} \cap \Lambda_L \neq \varnothing$, $\Lambda^{(i)} \cap \Lambda^{(i+1)} \neq \varnothing$, and $\Lambda^{(r)} \cap \Lambda^{\mathrm{c}}_{\frac{3}{2}L} \neq \varnothing$. In particular,

$$r \geq \ell^2 \frac{1}{2\ell} = \frac{\ell}{2}.$$

The number of such r-chains is bounded by

$$(2\ell^2)^{d-1} C_d^r ,$$

since there are at most C_d possibilities for $\Lambda^{(i+1)}$ given $\Lambda^{(i)}$. The probability of a given chain of non-regular cubes is at most

$$\ell^{-kr} ,$$

so that the probability of (3.13) going awry is at most

$$(2\ell^2)^{d-1} C_d^r \ell^{-kr} \leq \ell^{2d} (C\ell^{-k})^{l/2} < \ell^{-\frac{k\ell}{3}} ,$$

for l large enough. Adding these exceptional sets over all $\mathrm{e}^{l^{\frac{2}{3}}}$ energies E_i (because of the aforementioned stability of the regularity of cubes under perturbation of the energy) we get the final bound on the probability

$$\mathrm{e}^{l^{\frac{2}{3}}} l^{-\frac{kl}{3}} = \mathrm{e}^{l^{\frac{2}{3}} - \log l \frac{kl}{3}} < l^{-\frac{kl}{4}} ,$$

for l large enough. □

Denote by $\Omega_{\mathcal{C}}$ the set of configurations such that for all energies E we have that $\mathcal{C}(E) \subset \Lambda_{\frac{3}{2}L}$. For ω in $\Omega_{\mathcal{C}}$ consider a polynomially bounded solution of $H(\omega)\psi(\omega) = \mathcal{E}(\omega)\psi(\omega)$. For $x \in \mathbb{Z}^d$ the center of a $\mathcal{E}$-regular cube Λ_l, and $|x| \leq l^N$ for some N, $|\psi(x)|$ is small for l large enough:

$$\psi(x) = \sum_{(z,z') \in \partial \Lambda_l} G_{\Lambda_l}(\mathcal{E}; x, z) \psi(z') ,$$

and hence

$$|\psi(x)| \leq |\partial \Lambda_l| \mathrm{e}^{-c|x-z|} (1 + |z'|)^m \leq C l^{d-1} \mathrm{e}^{-cl} l^{Nm} \leq \mathrm{e}^{-c^{-} l} . \tag{3.14}$$

For $\omega \in \Omega_{\partial \Lambda_L}$, setting to zero the wave function $\psi(\omega)$ outside $\mathcal{C}$ yields an approximate eigenfunction of $H_{\Lambda_{2L}}(\omega)$ to the generalized eigenvalue $\mathcal{E}(\omega)$ of the full Hamiltonian $H(\omega)$:

$$\begin{aligned} &(H_{\Lambda_{2L}} - \mathcal{E}) \mathbb{1}_{\mathcal{C}} \psi = (H - \mathcal{E}) \mathbb{1}_{\mathcal{C}} \psi \\ =&[H - \mathcal{E}, \mathbb{1}_{\mathcal{C}}] \psi = [H, \mathbb{1}_{\mathcal{C}}] \psi = [-\Delta, \mathbb{1}_{\mathcal{C}}] \psi . \end{aligned}$$

The last commutator is small because it is supported on $\partial \mathcal{C}$ and because cubes intersecting $\partial \mathcal{C}$ are by construction regular:

$$\begin{aligned} &|[-\Delta, \mathbb{1}_{\mathcal{C}}] \psi(j)| \\ =& |\sum_{|j-j'|=1} \mathbb{1}_{\mathcal{C}}(j') \psi(j') - \mathbb{1}_{\mathcal{C}}(j) \sum_{|j-j'|=1} \psi(j')| \leq \begin{cases} 0 & j \notin \partial \mathcal{C} \\ dM & j \in \partial \mathcal{C} \end{cases} , \end{aligned}$$

where $M := \max\{|\psi(j')| : j' \in \partial \mathcal{C}, |j - j'| = 1\}$. From (3.14) we get

$$\|(H_{\Lambda_{2L}} - \mathcal{E}) \mathbb{1}_{\mathcal{C}} \psi\| = \|[-\Delta, \mathbb{1}_{\mathcal{C}}] \psi\| \leq d |\partial \mathcal{C}|^{1/2} \mathrm{e}^{-c^{-} l} \leq d |\Lambda_{2L}|^{1/2} \mathrm{e}^{-c^{-} l} \leq \mathrm{e}^{-c^{-} l} . \tag{3.15}$$

This implies that $H_{\Lambda_{2L}}$ must have an eigenvalue close to $\mathcal{E}$: Let ζ_α be the normalized eigenfunctions of $H_{\Lambda_{2L}}$ and E_α the corresponding eigenvalues. We write

$$\mathbb{1}_{\Lambda_L}\psi = \sum c_\alpha \zeta_\alpha$$

and obtain with (3.15)

$$\begin{aligned} \mathrm{e}^{-c^- l} &\geq \|\sum c_\alpha (E_\alpha - \mathcal{E})\zeta_\alpha\|^2 = \sum_\alpha c_\alpha^2 (E_\alpha - \mathcal{E})^2 \\ &\geq \sum_{\alpha : |E_\alpha - \mathcal{E}| > \mathrm{e}^{-\frac{c^-}{4} l}} c_\alpha^2 (E_\alpha - \mathcal{E})^2 \geq \sum_{\alpha : |E_\alpha - \mathcal{E}| > \mathrm{e}^{-\frac{c^-}{4} l}} c_\alpha^2 \mathrm{e}^{-\frac{c^-}{2} l}\,, \end{aligned}$$

hence

$$\sum_{\alpha : |E_\alpha - \mathcal{E}| > \mathrm{e}^{-\kappa l}} c_\alpha^2 \leq \mathrm{e}^{-\frac{c^-}{2} l}\,,$$

We can assume $\psi(0) = 1$, so in particular $\sum_\alpha c_\alpha^2 \geq 1$, and therefore

$$\operatorname{dist}(\mathcal{E}, \sigma(H_{\Lambda_L})) \leq \mathrm{e}^{-\frac{c^-}{4} l}\,. \tag{3.16}$$

Obviously, $\sigma(H_{\Lambda_{2L}})$ depends only on the random variables inside Λ_{2L}, that is on $\{\omega_j : j \in \Lambda_{2L}\}$. This is the (almost) "deterministic" set that lies close to $\mathcal{E}$, alluded to above. Fix an energy $E \in \sigma(H_{\Lambda_{2L}})$ and consider an overlapping covering of the ring $\Lambda_{10L}\backslash\Lambda_{2L}$ with cubes of size l (such that the allowed perturbation of the energy is greater than $\mathrm{e}^{-\frac{c^-}{4} l}$). The number of cubes in this covering is bounded by $(10L)^d/l^d = Cl^d$, and so the probability that all cubes in the covering are E-regular is at least

$$1 - Cl^d l^{-k} = 1 - Cl^{d-k}\,.$$

Adding these exceptional sets over all energies in $\sigma(H_{\Lambda_{2L}})$ (so that the cubes covering the ring are in particular $\mathcal{E}(\omega)$-regular *for all* $\mathcal{E}(\omega)$) we get an exceptional set of size

$$Cl^{d-k}L^d = Cl^{3d-k}\,.$$

In the proof of Lemma 3.5 we have shown how to prove regularity of the ring $\Lambda_{10L}\backslash\Lambda_{2L}$ for the case where we have an overlapping covering of regular cubes. We have thus reached our goal: For an $\omega \in \Omega_{\mathcal{C}}$ we have that for $(\omega_j)_{j\in\Lambda_{10L}\backslash\Lambda_{2L}}$ of probability (restricted to the ring) at least $1 - Cl^{3d-k}$ the ring $\Lambda_{10L}\backslash\Lambda_{2L}$ is $\mathcal{E}(\omega)$-regular for all $\mathcal{E}(\omega)$. Because of independence we finally obtain

$$\begin{aligned} &\mathbb{P}[\omega : \Lambda_{10L}\backslash\Lambda_{2L} \textit{ is } \mathcal{E}(\omega)\textit{-regular for all } \mathcal{E}(\omega)] \\ &\geq (1 - l^{-\frac{kl}{4}})(1 - Cl^{3d-k}) \geq 1 - Cl^{4d-k}\,. \end{aligned} \tag{3.17}$$

Now, with the exponential decay of the Green function established for rings on all scales with probability approaching 1, the proof that H has almost surely pure point spectrum is straight-forward (and does not need independence).

DEFINITION 4.2. For an orthonormal basis $\{\psi_i\}_{i=1}^\infty$ of Hilbert space and a sequence (a_i) with $a_i > 0, \sum_i a_i < \infty$, define the Borel measure $\rho(B) := \sum_i a_i \langle \psi_i\,, \mathbb{1}_B(H)\psi_i\rangle = \sum_i a_i \mu_{\psi_i}(B)$, whith μ_ψ the spectral measure of H associated to ψ. Any Borel measure equivalent to ρ is called a *spectral measure*. This equivalence class is independent of $\{\psi_i\}$ and $\{a_i\}$ as $\rho(B) = 0$ if and only if $\mathbb{1}_B(H) = 0$.

PROOF OF POINT SPECTRUM. Define the sequence of events

$$A_n := \{\omega : \Lambda_{5L_n}\backslash\Lambda_{L_n} \textit{ is } \mathcal{E}(\omega)\textit{-regular for all } \mathcal{E}(\omega)\}\,,$$

where $L_n = 2^n L$ for some L large enough, and observe that

$$\sum_n \mathbb{P}[A_n^{\mathrm{c}}] \leq \sum_n C L_n^{\frac{1}{2}(4d-k)} < \infty\,,$$

for k large enough. Hence the Borel-Cantelli lemma says that for $\Omega_\infty := \cap_{n\geq 1} \cup_{k\geq n} A_n^{\mathrm{c}}$, that is, "$A_n^{\mathrm{c}}$ infinitely often", we have $\mathbb{P}[\Omega_\infty] = 0$. So, for every $\omega \in \Omega_\infty^{\mathrm{c}}$ there is an $N(\omega)$ such that $R_n := \Lambda_{5L_n} \backslash \Lambda_{L_n}$ is regular for all $n \geq N(\omega)$ and all $\mathcal{E}(\omega)$.

Let ψ be a generalized eigenfuction of $H(\omega)$ with generalized eigenvalue $\mathcal{E}(\omega)$. Using the resolvent identity we can write

$$\psi(x) = \sum_{(z,z')\in\partial R_n} G_{R_n}(E(\omega); x, z)\psi(z') \qquad \forall x \in R_n \,.$$

Choose an arbitrary x with $|x| \geq L_{N(\omega)+1}$. It is easy to see that there exists an n such that $\mathrm{dist}(x, \partial R_n^-) \geq |x|/3 \geq L_{n-1}$. We get

$$|\psi(x)| \leq |\partial R_n| \mathrm{e}^{-c|x|/3}(1 + |z'|)^m \leq C L_n^{(d-1)} \mathrm{e}^{-c|x|/3} L_n^m \,,$$

from the polynomial boundedness of ψ. Apparently, $C L_n^{(d-1+m)} \leq \mathrm{e}^{cL_{n-1}/2} \leq \mathrm{e}^{c|x|/6}$, provided L is large enough. We get

$$|\psi(x)| \leq \mathrm{e}^{-c|x|/6}$$

for all x outside a cube of size $L_{N(\omega)+1}$. We have therefore proven that for almost every configuration ω any generalized eigenfunction corresponding to a generalized eigenvalue decays exponentially and is thus a true eigenfuction.

To conclude the proof we need the following well-known fact:

LEMMA 3.12. *For almost every $\omega \in \Omega$ there exists a spectral measure ρ_ω such that almost every energy E, with respect to ρ_ω, is a generalized eigenvalue.*

Denote the set of ω such that there exists a spectral measure ρ_ω with ρ_ω-a.e. energy a generalized eigenvalue by Ω_{gen}. We have

$$\mathbb{P}[\Omega_\infty^{\mathrm{c}} \cap \Omega_{\mathrm{gen}}] = 1 \,.$$

Fix now an $\omega \in \Omega_\infty^{\mathrm{c}} \cap \Omega_{\mathrm{gen}}$. If $\sigma_{\mathrm{cont}} \cap I := \sigma_{\mathrm{ac}} \cup \sigma_{\mathrm{sc}} \cap I \neq \varnothing$ there exists a $\psi \in \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}}$ such that $\mu_\psi(\sigma_{\mathrm{cont}} \cap I) > 0$, in particular $\rho_\omega^{\mathrm{cont}}(\sigma_{\mathrm{cont}} \cap I) > 0$. But since ρ_ω-a.e. energy is a generalized eigenvalue, and thus because of what we have proven above a true eigenvalue, we have $\rho_\omega^{\mathrm{cont}}(\{E \in \sigma_{\mathrm{cont}} \cap I \text{ is an eigenvalue }\}) > 0$, and because $\rho_\omega^{\mathrm{cont}}$ is a continuous measure, we have that the number of eigenvalues is uncountable, which is a contradiction as $\mathcal{H}$ is separable. Therefore $\sigma_{\mathrm{cont}} \cap I = \varnothing$ almost surely. $\square$

4.1. Dynamical Localization. So far we have used only the polynomial boundedness of the generalized eigenfunctions, and not yet the full eigenfunction expansion. The latter can be used to prove *dynamical* localization, as opposed to the previously established spectral localization. We discuss the case of a random potential in order not to burden the notation with spin indices. For an interval I we define

$$r^2(t,\omega) := \langle \mathrm{e}^{-\mathrm{i}tH_\omega} \mathbb{1}_I(H_\omega)\delta_0 \,, x^2 \mathrm{e}^{-\mathrm{i}tH_\omega} \mathbb{1}_I(H_\omega)\delta_0 \rangle \,,$$

the mean square of the distance from the origin of a particle, initially localized (for simplicity $\psi(0) = \delta_0$) with spectral support in I. We say that H_ω exhibits dynamical localization in I if

$$\sup_{t\in\mathbb{R}} r^2(t,\omega) < \infty \qquad \text{almost surely.}$$

See [**16**] for a discussion of the subtle difference between pure point spectrum with exponentially decaying eigenfunctions and actual dynamical localization.

Use the generalized eigenfunction expansion to write

$$\mathrm{e}^{-\mathrm{i}tH_\omega} \mathbb{1}_I(H_\omega)\delta_0 \; (x) = \int_I \mathrm{d}\rho_\omega(E) \mathrm{e}^{-\mathrm{i}tE} F(x, 0; E, \omega) \,,$$

where $F(x,0;E,\omega)$ is defined $\rho(E)$-almost everywhere by

$$(1+x^2)^{\delta/2}\sum_{j=1}^{N(E)} f_j(x,E)\overline{f_j}(0,E)\,,$$

with $\delta > d/2$ and $\{f_j\}_{j=1}^{N(E)}$ orthogonal functions in $\ell^2(\mathbb{Z}^d)$ such that $\varphi_j(x,E) := (1+x^2)^{\delta/2}f_j(x,E)$ are solutions of

$$(H_\omega - E)\varphi = 0\,,$$

and $N(E)$ counts multiplicity. Normalization is such that

$$\sum_{j=1}^{N(E)} \|f_j(E)\|^2 = 1\,.$$

We can thus estimate

$$\begin{aligned} r^2(t,\omega) &= \|x\,\mathrm{e}^{-\mathrm{i}tH_\omega}\mathbb{1}_I(H_\omega)\delta_0\,(x)\|^2 \\ &\le \rho_\omega(I)\int_I \mathrm{d}\rho_\omega(E)\|xF(0,x;E,\omega)\|^2\,. \end{aligned}$$

The almost sure exponential decay of the $\varphi_j(x)$ established in the last section allows one to conclude $\sup_t r^2(t,\omega) < \infty$ with probability one. For details see [**35**].

5. Proof for Correlated Variables

In the proofs of the foregoing sections independence of the random variables is used only when estimating the probability of occurence of several singular regions. All we need is that this probability is low, which is particularly easy in the independent case, but independence is not essential. In regime (A), for $\kappa \gtrsim 0$, the random variables are weakly correlated, and our proof goes through essentially unchanged. Using a cluster expansion we will prove in Appendix A, for the case of the particular Gibbs measure (3.3) under consideration, the following estimate,

$$|\mathbb{P}[\bigcap_{i=1}^k \Omega_{S_i}] - \prod_{i=1}^k \mathbb{P}[\Omega_{S_i}]| \le (k-1)\mathrm{e}^{-mL}\sum_{i=1}^k |S_i|\,, \tag{3.18}$$

for all events Ω_{S_i} supported on subsets S_i which satisfy $\mathrm{dist}(S_i,S_j) > L$ (see Appendix A for results and definitions). If the random variables are weakly correlated in this sense, then the proofs still hold. See [**59**] for the first result on localization for correlated random potentials. Their results are not strong enough for the inductive Wegner scheme, though.

Proof of Lemma 3.5 for correlated variables. The proof proceeds along the very same lines, albeit with the following modifications:

- In the first step, the singular n-regions are required to be at a mutual distance of at least n, rather than being merely disjoint. So $\widetilde{\Omega}_1 \subset \Omega$ is defined as the set of configurations where there are M or more singular n-regions $R \subset B$ that are at least n-separated. Then we have with (3.18)
$$\mathbb{P}[\widetilde{\Omega}_1] \le N_1^{(d+1)M}\left(n^{-kM} + (M-1)\mathrm{e}^{-mn}M(2n+1)^d\right) \le n^{-\frac{1}{2}kM}\,,$$
for $k > 16d(d+1)$ and n large enough. We restrict then to configurations $\omega \in \widetilde{\Omega}_1^{\mathrm{c}}$ where there are at most M singular n-regions that are n-separated, or where there are (possibly more than M) singular n-regions no M of which have mutual distance greater than n. In the latter case, these cubes are contained in a cube of size $(4M-3)n$. The estimates of the first step are unchanged.

- In the second step, modify the construction by considering n-separated rings $\widetilde{Q}_{(i)}, i = 1, \ldots, M$ of thickness n around each cube $Q \subset \Lambda$, rather than touching rings. That is, consider a sequence of equicentered cubes
$$Q = Q_{(0)} \subset \mathring{Q}_{(1)} \subset Q_{(1)} \subset \cdots \subset \mathring{Q}_{(M)} \subset Q_{(M)} ,$$
with $\mathring{Q}_{(i)}$ an n-neighbourhood of $Q_{(i-1)}$, and $Q_{(i)}$ an n-neighbourhood of $\mathring{Q}_{(i)}$. Define the ring $\widetilde{Q}_{(i)}$ to be $Q_{(i)} \backslash \mathring{Q}_{(i)}$.

Denote by $\Omega_{Q_{(i)}}$ the set of configurations ω such that $Q_{(i)}$ is singular. By the induction hypothesis we have $\mathbb{P}[\Omega_{Q_{(i)}}] \leq n^{-k}$. Introduce now the set
$$\widetilde{\Omega}_{Q_{(i)}} := \{\omega : Q_{(i)}(\omega|_{\widetilde{Q}_{(i)}}, \omega'|_{\mathring{Q}_{(i)}}) \text{ is singular } \forall \omega'|_{\mathring{Q}}\} .$$
Evidently, $\widetilde{\Omega}_{Q_{(i)}} \subset \Omega_{Q_{(i)}}$, that is,
$$\mathbb{P}[\widetilde{\Omega}_{Q_{(i)}}] \leq \mathbb{P}[\Omega_{Q_{(i)}}] \leq n^{-k} ,$$
and $\widetilde{\Omega}_{Q_{(i)}}$ has support in $\widetilde{Q}_{(i)}$. Thus, define by $\widetilde{\Omega}_2 \subset \Omega$ the set of configurations where for some cube $Q \subset \Lambda$ every ring $\widetilde{Q}_{(i)}$ is singular, in the sense that the cube $Q_{(i)}$ is singular no matter the configuration in $\mathring{Q}_{(i)}$. Its probability is bounded by
$$\mathbb{P}[\widetilde{\Omega}_2] \leq N_1^{d+1}\left(n^{-kM} + (M-1)\mathrm{e}^{-mn} M(2n+1)^d\right) \leq n^{-\frac{1}{2}kM} .$$
So as in the independent case, we restrict to configurations in $\widetilde{\Omega}_2^{\mathrm{c}}$ where for each Q there is an i such that the ring $\widetilde{Q}_{(i)}$ is not singular. Replace each Q by its $\mathring{Q}_{(i)}$. Hence the new cubes are still $6Mn$-separated, and the rest of step two remains unchanged.
- The third step and the off-diagonal decay do not require independence and remain unchanged.

□

Proof of the analogon of Theorem 2.1 from Lemma 3.5. This proof does not need independence and remains thus unchanged. □

Proof of absence of diffusion. Here, too, independence is not needed, and we may infer absence of diffusion as we did for independent random variables. □

Proof of point spectrum. Independence is used when estimating the probability of occurence of a chain of non-regular cubes. Simply modify the notion of connectedness in the definition of $\mathcal{C}$: Two l-cubes Λ, Λ' are called "connected" if $|\partial\Lambda - \partial\Lambda'| \leq l$. The chain is modified to consist of
$$\Lambda^{(1)}, \Lambda^{(2)}, \cdots, \Lambda^{(r)} ,$$
such that $|\Lambda^{(1)} - \Lambda_L| \leq l$, $|\Lambda^{(i)} - \Lambda^{(i+1)}| \leq l$, and $|\Lambda^{(r)} - \Lambda^{\mathrm{c}}_{\frac{3}{2}L}| \leq l$. In particular,
$$(2\ell)^{2d} \geq r \geq \ell^2 \frac{1}{4\ell} = \frac{\ell}{4} .$$
The number of such r-chains is bounded by
$$(2\ell^2)^{d-1} C_d^{2r} .$$
The probability of a given chain of non-regular cubes is at most
$$l^{-kr} + (r-1)\mathrm{e}^{-ml} r l^d \leq \mathrm{e}^{-\frac{m}{2}l} ,$$
for l large enough. This estimate is good enough to proceed with the proof.

The second instance where independence is used is (3.17), and since $\Omega_{\mathcal{C}}$ and the set of configurations where the ring $\Lambda_{10L}\backslash\Lambda_{2L}$ is covered by only regular l-cubes are supported on sets a distance $\simeq L/2$ apart we have

$$\begin{aligned}&\mathbb{P}[\omega : \Lambda_{10L}\backslash\Lambda_{2L} \textit{ is } \mathcal{E}(\omega)\textit{-regular for all } \mathcal{E}(\omega) \in I_{E_1}]\\ &\geq (1 - l^{-\frac{kl}{4}})(1 - Cl^{3d-k}) - \mathrm{e}^{-mL/2}Cl^{2d} \geq 1 - Cl^{4d-k}\,.\end{aligned}$$

for l large enough. The rest of the proof carries over verbatim. □

6. Verification of the initial conditions

All that is left to do is to establish the initial estimate of Lemma 3.5. Bourgain [**8**] showed how the estimates can be proved for energies in the band tail. For notational convenience, set $J = 1$ and $v = 0$. The upper edge of the spectrum is then $\overline{E} = 2d + 1$. The Neumann series expansion of the Green function

$$G_N(E;\omega) = \frac{1}{E}\frac{1}{1 - E^{-1}H_N(\omega)}$$

shows that we have to prove that the probability of $\|H_N(\omega)\| > \overline{E} - \kappa$, for some small $\kappa > 0$, is low. The basic idea is that for a $\xi \in \ell^2(Q_N;\mathbb{C}^2)$ with $\langle H_N(\omega)\xi,\xi\rangle > \overline{E} - \kappa$, neighbouring $\xi_j, j \in Q_N$ have to be close, and each ξ_j has to be close to the eigenvector with eigenvalue $+1$ of the random matrix $\sigma \cdot h_j$. This implies that the $+1$-eigenvectors of the random matrices on different lattice sites have to be close which happens with low probability as long as they are only weakly correlated.

Thus let us assume $\|H_N(\omega)\| > \overline{E} - \kappa$. Then there is a $\xi \in \ell^2(Q_N;\mathbb{C}^2)$, $|\xi| = 1$ such that

$$\langle H_N(\omega)\xi\,,\xi\rangle = \sum_{j\in Q_N} \langle(\sigma \cdot h_j)\xi_j\,,\xi_j\rangle + \sum_{|j-j'|=1} \langle\xi_j\,,\xi_{j'}\rangle > 2d + 1 - \kappa\,.$$

It follows that

$$\sum_{j\in Q_N} \langle(\sigma \cdot h_j)\xi_j\,,\xi_j\rangle > 1 - \kappa \tag{3.19}$$

$$\sum_{|j-j'|=1} \langle\xi_j\,,\xi_{j'}\rangle > 2d - \kappa\,.$$

From the latter follows the closeness of neighbouring ξ_j: For

$$\sum_j \langle\xi_j\,,\xi_{j+e_\alpha}\rangle > 1 - \kappa\,,$$

with e_α the unit vectors of $\mathbb{Z}^d$, implies

$$\sum_j |\xi_j - \xi_{j+e_\alpha}|^2 = 2 - \sum_j 2\mathrm{Re}\langle\xi_j\,,\xi_{j+e_\alpha}\rangle < 2\kappa\,. \tag{3.20}$$

Furthermore, for $K \in \mathbb{Z}^d$, $|K_\alpha| \leq l$ we have

$$\sum_j |\xi_j - \xi_{j+K}|^2 \leq \sum_j (|\xi_j - \xi_{\omega_1}| + |\xi_{\omega_1} - \xi_{\omega_2}| + \cdots + |\xi_{\omega_n} - \xi_{j+K}|)^2\,,$$

where the $\omega_i = \omega_i(j)$ describe a shortest path from j to $j + K$, and $n = |K_1| + \cdots + |K_d| - 1$. We obtain

$$\begin{aligned}\sum_j |\xi_j - \xi_{j+K}|^2 &\leq \sum_j (n + 1)|\xi_j - \xi_{\omega_i}|^2 + \cdots + (n + 1)|\xi_{\omega_n} - \xi_{j+K}|^2\\ &\leq (n + 1)^2 2\kappa \leq 2d^2\ell^2\kappa\,. \end{aligned} \tag{3.21}$$

If we denote the eigenvectors corresponding to the eigenvalues $+1$ and -1 of $\sigma \cdot h_j$ with η_j^+ and η_j^-, respectively, we have the decomposition

$$\xi_j = \alpha_j \eta_j^+ + \beta_j \eta_j^- \Rightarrow \langle (\sigma \cdot h_j)\xi_j\,, \xi_j\rangle = (|\alpha_j|^2 - |\beta_j^2|)\,,$$

and from (3.19) we get

$$\sum_j |\alpha_j|^2 - |\beta_j^2| > 1 - \kappa\,. \tag{3.22}$$

Hence we obtain that up to scaling and a phase factor the ξ_j are close to the eigenvectors η_j^+:

$$\begin{aligned}
&\sum_j |\xi_j - \mathrm{e}^{\mathrm{i}\varphi_j}|\xi_j|\eta_j^+|^2 = \sum |(\alpha_j - \mathrm{e}^{\mathrm{i}\varphi_j}|\xi_j|)\eta_j^+ + \beta_j\eta_j^-)|^2 \\
=&\sum_j |\alpha_j - \mathrm{e}^{\mathrm{i}\varphi_j}|\xi_j||^2 + |\beta_j|^2 = 1 + \sum_j |\alpha_j - \mathrm{e}^{\mathrm{i}\varphi_j}|\xi_j||^2 - |\alpha_j|^2 \\
=&1 - \sum_j |\alpha_j|^2 - (\sqrt{|\alpha_j|^2 + |\beta_j|^2} - \underbrace{\mathrm{e}^{-\mathrm{i}\varphi_j}\alpha_j}_{|\alpha_j|})^2 \\
&\leq 1 - \sum_j |\alpha_j|^2 - (|\alpha_j| + |\beta|_j - |\alpha_j|)^2 \leq \kappa\,,
\end{aligned}$$

where in the last step we used (3.22). Combining this last result with (3.20) we find

$$\begin{aligned}
&\sum_j |\xi_{j+e_\alpha} - |\xi_j|\mathrm{e}^{\mathrm{i}\varphi_j}\eta_j^+|^2 \\
\leq& 2\sum_j |\xi_{j+e_\alpha} - \xi_j|^2 + |\xi_j - |\xi_j|\mathrm{e}^{\mathrm{i}\varphi_j}\eta_j^+|^2 \leq 2(2\kappa + \kappa) = C\kappa\,,
\end{aligned}$$

and using this

$$\begin{aligned}
&\sum_j ||\xi_{j+e_\alpha}|\mathrm{e}^{\mathrm{i}\varphi_{j+e_\alpha}}\eta_{j+e_\alpha}^+ - |\xi_j|\mathrm{e}^{\mathrm{i}\varphi_j}\eta_j^+|^2 \leq \\
&2\sum_j ||\xi_{j+e_\alpha}|\mathrm{e}^{\mathrm{i}\varphi_{j+e_\alpha}}\eta_{j+e_\alpha}^+ - \xi_{j+e_\alpha}|^2 + |\xi_{j+e_\alpha} - |\xi_j|\mathrm{e}^{\mathrm{i}\varphi_j}\eta_j^+|^2 \leq 2(\kappa + C\kappa) = C\kappa\,.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&\sum_j |\mathrm{e}^{\mathrm{i}\varphi_{j+e_\alpha}}|\xi_{j+e_\alpha}|\eta_{j+e_\alpha}^+ - \mathrm{e}^{\mathrm{i}\varphi_{j+e_\alpha}}|\xi_j|\eta_{j+e_\alpha}^+|^2 \\
=&\sum_j (|\xi_{j+e_\alpha}| - |\xi_j|)^2|\eta_{j+e_\alpha}^+|^2 \leq \sum_j |\xi_{j+e_\alpha} - \xi_j|^2 \leq 2\kappa\,,
\end{aligned}$$

and hence finally

$$\begin{aligned}
&\sum_j ||\xi_j|\mathrm{e}^{\mathrm{i}\varphi_{j+e_\alpha}}\eta_{j+e_\alpha}^+ - |\xi_j|\mathrm{e}^{\mathrm{i}\varphi_j}\eta_j^+|^2 \\
\leq& 2\sum_j ||\xi_j|\mathrm{e}^{\mathrm{i}\varphi_{j+e_\alpha}}\eta_{j+e_\alpha}^+ - |\xi_{j+e_\alpha}|\mathrm{e}^{\mathrm{i}\varphi_{j+e_\alpha}}\eta_{j+e_\alpha}^+|^2 + ||\xi_{j+e_\alpha}|\mathrm{e}^{\mathrm{i}\varphi_{j+e_\alpha}}\eta_{j+e_\alpha}^+ - |\xi_j|\mathrm{e}^{\mathrm{i}\varphi_j}\eta_j^+|^2 \\
&\leq 2(2\kappa + C\kappa) = C\kappa\,.
\end{aligned}$$

Since $\sum_j |\xi_j|^2 = 1$ we rewrite the last result as

$$\sum_j |\xi_j|^2 |\langle \eta_j^+\,, \eta_{j+e_\alpha}^+\rangle| > 1 - C\kappa\,.$$

From the bound (3.21) on the distance of ξ_j and ξ_{j+K} for K as above we get

$$\sum_j ||\xi_j|^2 - |\xi_{j-K}|^2||\langle\eta_j^+, \eta_{j+e_\alpha}^+\rangle| \le \sum_j (|\xi_j| + |\xi_{j-K}|)||\xi_j| - |\xi_{j-K}||$$
$$\le 2\sqrt{\sum_j |\xi_j|^2}\sqrt{\sum_j |\xi_j - \xi_{j-K}|^2} \le 2\sqrt{2}dl\sqrt{\kappa}\,.$$

Hence we find

$$\sum_j |\xi_j|^2|\langle\eta_{j+K}^+, \eta_{j+e_\alpha+K}^+\rangle| = \sum_j |\xi_{j-K}|^2|\langle\eta_j^+, \eta_{j+e_\alpha}^+\rangle| > 1 - Cl\sqrt{\kappa}\,. \tag{3.23}$$

If we take $\kappa \propto l^{-3}$ and average (3.23) over $K \in \{0, \ldots, l-1\}^d$ there is a $j \in Q_N$ such that

$$\sum_{0\le K_\alpha<l} |\langle\eta_{j+K}^+, \eta_{j+K+e_\alpha}^+\rangle| > ql^d\,,$$

for any $q < 1$, for l and hence N large enough. Now, since $\mathbb{E}|\langle\eta_1^+, \eta_2^+\rangle| < 1$, a large deviation estimate (see, for instance, [**18**]) gives

$$\mathbb{P}\left[\frac{1}{l^d}\sum_{0\le K_\alpha<l} |\langle\eta_{j+K}^+, \eta_{j+K+e_\alpha}^+\rangle| > q\right] \le \mathrm{e}^{-cl^d}\,.$$

Thus it follows that for $\kappa \propto (\frac{k}{c}\log N)^{-3/d}$

$$\mathbb{P}\left[\|H_N(\omega)\| > \overline{E} - \kappa\right] \le N^{-k}\,.$$

With this information we can easily verify the conditions of Lemma 3.5 for energies in the band tail. Let I be the interval $[\overline{E} - (\log N_0)^{-4}, \overline{E}]$ and $\delta = (\log N_0)^{-3/d} - (\log N_0)^{-4}$. Since $\mathrm{dist}(\sigma(H_{N_0}), E)) > \delta$ with probability at least $1 - N_0^{-k}$ we obtain for $E \in I$

$$\|G_{N_0}(E)\| \le \frac{1}{\delta} \le \frac{(\log N_0)^4}{(4-3/d)\log\log N_0} < \mathrm{e}^{N_0^{1/2}} \text{ and}$$
$$|G_{N_0}(E + \mathrm{i}0; x, y)| \le \mathrm{e}^{-c|x-y|}\,,$$

where $c < \frac{1}{2}((4-3/d)\log\log N_0)^{1/2}/(\log N_0)^2$, the latter from the Combes-Thomas argument. Thus it is apparent that we have established the conditions for Lemma 3.5 in the required range of inital scales between n and N_0.

The proof of Theorem 3.1 is finished. □

7. Percolation

Let us now turn to regime (B)—that is to the proof of Theorem 3.2. Magnetic moments are placed according to a site percolation process with parameter x. The moments are distributed uniformely on a sphere of fixed radius, but are required to be equal in a cluster of neighbouring moments. Although it might seem that this perfect correlation of the magnetic moments across connected clusters throws us well out of the regime of weak correlations where we could prove Anderson localization, there is a result on the cluster size distribution which is equivalent to the exponential decorrelation used in section 5.

Indeed, as long as we are below the percolation threshold, $x < x_c$, we have the following result about the cluster size distribution [**31, 33**],

$$\mathbb{P}[\mathrm{diam}\ C(0) \ge m] \le \mathrm{e}^{-\alpha(x)m}\,,$$

where $\alpha(x) > 0$, $C(0)$ denotes the cluster 0 is in. Because of translation invariance

$$\mathbb{P}[\exists C \subset \Lambda : \operatorname{diam} C \geq m] \leq \mathbb{P}[\bigcup_{j\in\Lambda} \{\operatorname{diam} C(j) \geq m\}] \leq |\Lambda| \mathrm{e}^{-\alpha(x)m} . \tag{3.24}$$

There is a similar bound on the probability of large clusters for $1 > x > x_{\mathrm{c}}$, namely

$$\mathbb{P}[|C(0)| = m] \leq \exp(-\beta(x) m^{\frac{d-1}{d}}) ,$$

with $\beta(x) > 0$. However,

$$\mathbb{P}[|C(0)| \geq m] = \sum_{k\geq m} \mathbb{P}[|C(0)| = k] + \mathbb{P}[0 \text{ lies in infinite cluster}] ,$$

and the last term is a constant > 0 since $x > x_{\mathrm{c}}$, rendering the estimate useless. For the rest of this section, we restrict ourselves therefore to $x < x_{\mathrm{c}}$.

With (3.24) at hand, we discard in the proof of Lemma 3.5 from the beginning a small set Ω_{perc} of configurations where there are clusters in the N_1-region B of size larger than $n/10$, where $n = N_1^{1/8d} < N_0^{1/4d}$, with N_0 the initial scale in Lemma 3.5. We have

$$\mathbb{P}[\Omega_{\mathrm{perc}}] \leq n^{8d^2} \mathrm{e}^{-\alpha(p)n/10} < n^{-\frac{1}{2}kM} ,$$

for n large enough. For configurations in $\Omega_{\mathrm{perc}}^{\mathrm{c}}$ we can repeat the proof of Lemma 3.5 that we did for the correlated case, as n-regions that are a distance n apart are independent because there are no clusters of size bigger than $n/10$, and distinct clusters are by construction independent.

To see that the Cartan-type Lemma B.4 is still applicable we divide probability space into $2^{|B|}$ parts, one for each percolation configuration in B. Fix one of the $2^{|B|}$ "percolation pictures". As the moments in one cluster are all identical we take as domain $(\mathrm{S}^2)^{\#\{\text{clusters}\}}$ instead of $(\mathrm{S}^2)^{|\Lambda|}$, and the smallness of the exceptional set follows as in the appendix. Since the bound on the exceptional set is certainly bounded by the one where each site is occupied Lemma B.4 carries over verbatim.

The deduction of pure point spectrum in the band tails from Lemma 3.5 is done as in the case of correlated magnetic moments.

It remains to check whether the initial-scale estimates can still be established. Take Q_N an N-region. For a normalized function ξ to have $\langle H_{Q_N} \xi , \xi \rangle$ κ-close to the upper spectral edge, $2d + 1$, neighboring ξ_j have to be close to maximize the hopping term, and each ξ_j has to be close to the eigenvector with eigenvalue $+1$ of the random matrix $\sigma \cdot m_j$. In addition, sites j with $m_j = 0$ should be avoided: As in the previous section, we have

$$1 - \kappa < \sum_j \langle (\sigma \cdot m_j) \xi_j , \xi_j \rangle = \sum_{j:m_j\neq 0} \langle (\sigma \cdot m_j) \xi_j , \xi_j \rangle .$$

On the other hand, we know that $\sum_j \langle \xi_j , \xi_j \rangle = 1$, so that subtraction yields

$$\sum_{j:m_j\neq 0} \langle (\mathbb{1} - \sigma \cdot m_j) \xi_j , \xi_j \rangle + \sum_{j:m_j=0} \langle \xi_j , \xi_j \rangle < \kappa .$$

Since $\mathbb{1} - \sigma \cdot m_j$ is a positive matrix, we get the desired result

$$\sum_{j:m_j=0} \langle \xi_j , \xi_j \rangle < \kappa . \tag{3.25}$$

In the previous section, we established for any $K \in \mathbb{Z}^d$ with $|K_\alpha| \leq \ell$

$$\sum_j |\xi_j - \xi_{j+K}|^2 \leq 2d^2 \ell^2 \kappa , \tag{3.26}$$

Since $\sum_j |\xi_j|^2 = 1$ we know that there exists a j such that

$$|\xi_j|^2 > \frac{1}{|Q_N|} = \frac{1}{N^d} =: \frac{1}{\ell^{2d}} .$$

Set $\kappa = \frac{1}{8d^2}\ell^{-2d-2}$. Because $\sum_j |\xi_j - \xi_{j+K}|^2 \le 2d^2\ell^2\kappa$ we have

$$|\xi_j - \xi_{j+K}|^2 \le 2d^2\ell^2\kappa = \frac{1}{4}\ell^{-2d}\,,$$

and therefore

$$|\xi_{j+K}| = |\xi_j - (\xi_j - \xi_{j+K})| \ge |\xi_j| - |\xi_j - \xi_{j+K}| > \frac{1}{2}\ell^{-d}\,.$$

By (3.25) this implies that $m_{j+K} \ne 0$. As this is true for all $K \in \mathbb{Z}^d$ with $|K_\alpha| \le \ell$, we get that there exists a cluster of size ℓ, but by the percolation estimate we know that with probability larger than $1 - \ell^{2d}\exp(-\alpha(x)\ell)$ there are no clusters of size larger than ℓ in Q_N. We have therefore proven that

$$\mathbb{P}[\|H_{Q_N}(\omega)\| > \overline{E} - \kappa] \le N\exp(-\alpha(x)N^{1/2}) \le N^{-k}\,,$$

for N large enough. The proof of Theorem 3.2 is finished as in the previous section. □

8. Bernoulli in the Lifshitz tails

In our analysis of $\mathrm{Eu}_x\mathrm{Ca}_{1-x}\mathrm{B}_6$ we have investigated so far both the regimes x above and below the percolation threshold, but in either case we restricted attention to a weak external magnetic field. In this section, we will analyze regime (C) ($B \to \infty$) introduced at the end of Chapter 1. As argued there, the spin up and spin down bands split, and each is described by a Bernoulli Hamiltonian

$$H(\omega) = -\Delta + v(\omega)\,,$$

where

$$v_j(\omega) \equiv v_j^{\pm}(\omega) := \begin{cases} v \pm J\,, & m_j \ne 0 \\ -v\,, & m_j = 0 \end{cases}\,.$$

We will concentrate on one subband, and for the sake of simplicity we choose $x = \frac{1}{2}$ and shift the energy such that

$$H(\omega) = -\Delta + \lambda v_j(\omega)\,,$$

where

$$v_j(\omega) := \begin{cases} 1\,, & \text{with probability } \frac{1}{2} \\ -1\,, & \text{with probability } \frac{1}{2} \end{cases}\,.$$

The singular nature of the Bernoulli probability "density" makes it hard to prove a Wegner estimate (see the discussion in Section 2), and not even the inductive scheme that we used to our advantage in Section 3 is applicable (recall that a bounded probability density is required). We will therefore pursue an old method [**54**] that gives at least the partial results which constitute Theorem 3.3. We should also mention that the Bernoulli problem has been solved in the continuum [**9**], but with methods that do not extend to the lattice.

We have explained in Section 2 why it is difficult to get a good estimate on the density of states for the Bernoulli Hamiltonian. However, it stands to reason that in the extreme band tails—the so-called Lifshitz tails—things might be easier. Lifshitz argued that near $E_{\mathrm{inf}} := \inf\sigma(H)$ the integrated density of states $n(E)$ should behave like $\exp(-c(E - E_{\mathrm{inf}})^{-d/2})$. His heuristic argument (see, for instance, [**14**]) is very similar to the ideas we have encountered in the previous section: In order for the Hamiltonian $H(\omega)$ to have an eigenvalue close to E_{inf} the corresponding eigenfunction ξ has to minimize both $\langle -\Delta\xi\,,\xi\rangle$ and $\langle v\xi\,,\xi\rangle$. To minimize the former, the eigenfunction ξ has to be spread out over a region of size at least $|E - E_{\mathrm{inf}}|^{-1/2}$, see (3.26), comprising $|E - E_{\mathrm{inf}}|^{d/2}$ sites. To minimize the potential term, v_j has to equal -1 for all the sites in the support of ξ, which happens with probability $(1/2)^{|E-E_{\mathrm{inf}}|^{d/2}}$ because of independence.

In this section, we will prove *rigorously* that for a Bernoulli potential the integrated density of states is indeed exponentially small:

LEMMA 3.13. *For $d = 3$ and $E \in [-\lambda, -c\lambda^2 + \mathcal{O}(\lambda^{4-})]$, the integrated density of states satisfies the estimate*

$$n(E) \leq C \exp(-\frac{1}{2}(2\lambda)^{-\frac{1}{2}}),$$

where $c, C > 0$ are constants. We defer the proof for the moment and will instead give a proof of Theorem 3.3 using the smallness of the integrated density of states in the Lifshitz tails. See Section 2 for notions and definitions.

Proof of Theorem 3.3. As noted earlier on in Chapter 2, we can determine the spectrum of a random Schrödinger operator with probability one,

$$\sigma(H(\omega)) = [-\lambda, 4d + \lambda] \qquad \text{almost surely}.$$

(From the point of view of notation, it is convenient to work in this section without the subtraction of $2d$ from the discrete Laplacian). We focus on the lower edge of the spectrum, that is, $[-\lambda, 0]$. The task at hand is to establish conditions (i), (ii) of Theorem 2.1, as Lemma 3.5 is not applicable for too singular a probability density.

The following considerations show that the task indeed boils down to finding a good bound on the integrated density of states $n(E)$. Recall that we denoted by $N_\Lambda(E)$ the number of eigenvalues of H_Λ less than E, for some $\Lambda \subset \mathbb{Z}^d$. The probability that there is an eigenvalue below E is bounded in terms of $n(E)$:

$$\mathbb{P}[N_\Lambda(E) \geq 1] = \mathbb{E}\mathbb{1}_{\{N_\Lambda(E)\geq 1\}} \leq \mathbb{E}N_\Lambda(E)\mathbb{1}_{\{N_\Lambda(E)\geq 1\}} \leq \mathbb{E}\, N_\Lambda(E). \tag{3.27}$$

Since Dirichlet boundary conditions raise eigenvalues [**14**] we have

$$\mathbb{E}\, N_\Lambda(E) \leq |\Lambda| n(E). \tag{3.28}$$

We combine this result with the following lemma to obtain exponential off-diagonal decay with probability at least $1 - |\Lambda| n(E)$.

LEMMA (Combes-Thomas). *Whenever H_Λ has no spectrum below E_1 then for $E < E_1$,*

$$|G_\Lambda(E; x, y)| \leq \frac{2}{\delta} \exp(-\frac{1}{2}\sqrt{\delta}|x - y|), \tag{3.29}$$

where $\delta := |E - E_1|$.

PROOF. Let $U(a)$ be the operator of multiplication by $\mathrm{e}^{a\cdot x}$ (actually $\mathrm{e}^{a\cdot x} \otimes \mathbb{1}$, but we drop the identity matrix for sake of clarity), and compute

$$\begin{aligned}
&[U(-a)(H_\Lambda - E)U(a)]f(x) = \\
&[H_\Lambda - E]f(x) + \sum_{j=1}^{d} \big((\mathrm{e}^{a_j} - 1)f(x + e_j) + (\mathrm{e}^{-a_j} - 1)f(x - e_j)\big) \\
&=: [H_\Lambda - E]f(x) + [Q_\Lambda(a)]f(x),
\end{aligned}$$

where e_j denote the unit vectors in $\mathbb{Z}^d$. Thus, $Q_\Lambda(a)$ is a bounded operator with

$$\|Q_\Lambda(a)\| \leq \sum_{j=1}^{d} 2(\cosh a_j - 1) \leq C \sum_{j=1}^{d} a_j^2 = C|a|^2,$$

with $C < 2$ for a small. Hence, if we choose $|a| = \sqrt{\delta/2C}$, we obtain

$$\mathrm{dist}(\sigma(H_\Lambda + Q_\Lambda(a)), E) \geq \frac{\delta}{2},$$

and hence

$$\|[U(-a)(H_\Lambda - E)U(a)]^{-1}\| = \|[H_\Lambda - E + Q_\Lambda(a)]^{-1}\| \le \frac{2}{\delta}\,.$$

Therefore we can bound the matrix elements

$$\frac{2}{\delta} \ge |[U(-a)G_\Lambda(E)U(a)](x,y)| = |\mathrm{e}^{a\cdot(x-y)}G_\Lambda(E;x,y)|\,,$$

from which the claim follows choosing $a \parallel (x-y)$. ☐

Since the spectrum of $H(\omega)$ is compactly supported we can easily get a bound for $n(E)$: Using

$$\frac{2}{\pi}\int_{-\infty}^{E} \mathrm{d}E' \frac{\varepsilon}{(x-E')^2+\varepsilon^2} = \frac{2}{\pi}\arctan(\frac{E-x}{\varepsilon}) + 1 \ge \Theta(E-x) \qquad \forall \varepsilon > 0\,,$$

where Θ is the Heaviside function, the spectral theorem provides the bound

$$\frac{2}{\pi}\int_{-\infty}^{E} \mathrm{d}E' \,\mathrm{Im}\,\mathbb{E}\, G(E'+\mathrm{i}\varepsilon;0,0) \ge \mathbb{E}\,\langle \delta_0\,, \mathrm{P}(E,\omega)\delta_0\rangle = n(E) \qquad \forall\varepsilon>0\,. \tag{3.30}$$

The last equality is due to translation invariance and is discussed in Appendix C.

Remark. Observe that the above estimate on $n(E)$ holds for any $\varepsilon > 0$. It will turn out to be essential to choose ε small but not too small, depending on λ.

This estimate implies that in order to get a bound on the integrated density of states, we have to bound the imaginary part of the averaged Green function. The strategy is simple. If λ is small the first thought is to expand the resolvent $(-\Delta + \lambda v(\omega) - E - \mathrm{i}\varepsilon)^{-1}$ in powers of λ around $(-\Delta - E)^{-1}$. But since we consider the *average* $\mathbb{E}\,(-\Delta - \lambda v(\omega) - E)^{-1}$, there may be an optimal energy E_0 around which to expand. This energy is found as follows. Formally, we have that

$$\begin{aligned} G(E+\mathrm{i}\varepsilon) &= \frac{1}{-\Delta + \lambda v - E - \mathrm{i}\varepsilon} = \frac{1}{-\Delta + E_0 - \mathrm{i}\varepsilon + \lambda v + (-E_0 - E)} \\ &= G_0 \sum_{n\ge 0} ((-\lambda v + E + E_0)G_0)^n\,, \end{aligned}$$

where we have introduced the unperturbed Green function

$$G_0(-E_0 + \mathrm{i}\varepsilon) = (-\Delta + E_0 - \mathrm{i}\varepsilon)^{-1}\,.$$

The first terms are

$$G = G_0 - G_0[\lambda v - (E+E_0)]G_0 + G_0[\lambda v - (E+E_0)]G_0[\lambda v - (E+E_0)]G_0 + \ldots$$

Using that $\mathbb{E}\, v = 0$ and $\mathbb{E}\, v^2 = 1$, we obtain

$$\mathbb{E}\, G = G_0 + G_0^2[E+E_0] + \lambda^2 G_0(0,0)G_0^2 + G_0^3[E+E_0]^2 + \ldots\,.$$

Thus, we see that, in order for the λ^2-term to vanish, we must choose

$$E + E_0 = -\lambda^2 G_0(-E_0 + \mathrm{i}\varepsilon;0,0) = \mathcal{O}(\lambda^2) \tag{3.31}$$

to arrive at $\mathbb{E}\, G = G_0 + \mathcal{O}(\lambda^4)$.

To make the above considerations mathematically respectable, we iterate the second resolvent identity

$$G(E+\mathrm{i}\varepsilon;x_0,x) = G_0(-E_0+\mathrm{i}\varepsilon;x_0,x) + \sum_y G_0(-E_0+\mathrm{i}\varepsilon;x_0,y)[-\lambda v + (E+E_0)]G(E+\mathrm{i}\varepsilon;y,x)\,,$$

with $0 < E_0 = -E - \lambda^2\Sigma(E)$, where $\lambda^2\Sigma(E)$ is motivated by (3.31) and is defined in a self-consistent way by

$$\lambda^2\Sigma(E) = \lambda^2(-\Delta - E - \lambda^2\Sigma(E) - \mathrm{i}\varepsilon)^{-1}(0,0)\,.$$

Since E_0 has to be positive in order for $G_0(E_0)$ to exist as a bounded operator we have as an upper limit for E

$$E^* = \lambda^2 \Delta^{-1}(0,0) < 0\,. \tag{3.32}$$

Indeed, we have that $E_0 \geq C\lambda^{4-\delta}$ if $E \leq E^* - \lambda^{4-\delta}$ for a positive constant C, any small $\delta > 0$ and λ small enough, see [**17**]. (It will become clear later on why we need a lower bound on E_0.) In Figure 2 we depict the expected nature of the spectrum. We will prove absence of diffusion for (most) energies $E \leq E^* - \lambda^{4-\delta}$. It is expected that H_ω has a mobility edge, separating pure point from absolutely continuous spectrum, slightly above E_*, but this is well outside the scope of this thesis. Next, we

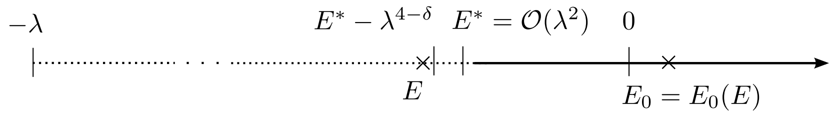

FIGURE 2. Expected nature of the spectrum of the Bernoulli Hamiltonian $-\Delta + \lambda v_j$ for small values of λ.

iterate the resolvent identity M times with the intention of optimizing the truncation parameter M, later on, to minimize the remainder term. Setting $W := -\lambda v - \lambda^2 \Sigma$ and $G^\varepsilon := G(E + \mathrm{i}\varepsilon)$ we get

$$G^\varepsilon = \sum_{m=0}^{M} G_0^\varepsilon \,[W G_0^\varepsilon]^m + G_0^\varepsilon \,[W G_0^\varepsilon]^M \, W G^\varepsilon\,. \tag{3.33}$$

The dangling factor G^ε in the remainder is estimated trivially by $1/\varepsilon$. The key observation is that the imaginary part of the first $M+1$ terms on the right-hand side can be shown to be proportional to ε, whereas the remainder term multiplying G^ε is of order $\mathrm{e}^{-(2\lambda)^{-1/2}}$. Thus we choose $\varepsilon^2 = \mathrm{e}^{-(2\lambda)^{-1/2}}$ and get the estimate

$$\operatorname{Im} \mathbb{E}\, G(E' + \mathrm{i}\varepsilon; 0, 0) \leq \text{const}\ \mathrm{e}^{-\frac{1}{2}(2\lambda)^{-1/2}}\,,$$

which implies Lemma 3.13 by (3.30), as the spectrum is compactly supported.

Summarizing, with probability larger than

$$1 - |\Lambda| C \mathrm{e}^{-\frac{1}{2}(2\lambda)^{-1/2}} \tag{3.34}$$

there is off-diagonal exponential decay of the resolvent $G_\Lambda(E)$, for $E \in [-\lambda, -c\lambda^2 + \mathcal{O}(\lambda^{-4-})]$, see (3.32). Therefore, for any initial scale l_0 there is a λ such that (1.11) holds with high probability. Having established condition (1.11) of the multiscale analysis, we are looking for a Wegner estimate. Because of the very singular nature of the Bernoulli potential, the inductive scheme devised in [**8**] does not work. In [**54**], the following trick was introduced: For each scale l_n, we define $\mu(J) = \mu_{l_n}(J)$ to be the expected number of eigenvalues of $H_{\Lambda_{l_n}}$ in an interval J. The following easy estimate shows how we have to proceed:

$$\mathbb{P}[\operatorname{dist}(E, \sigma(H_{\Lambda_l})) \leq \kappa] \leq \mathbb{E} N_{\Lambda_l}(E+\kappa) - \mathbb{E} N_{\Lambda_l}(E-\kappa) = \mu_l(E-\kappa, E+\kappa)\,.$$

We see that (1.10) is fulfilled if we exclude a set of "singular" energies, and the following lemma shows that this set of energies has very small measure.

LEMMA. *Let μ be a measure on an interval I. Let S be the set of energies E for which the measure is singular at scale ε, that is at which*

$$\mu(E - \varepsilon, E + \varepsilon) \geq \varepsilon^{1/2}\,.$$

If $|S|$ denotes the Lebesgue measure of S then

$$|S| \leq 2\mu(I)\varepsilon^{1/2}\,.$$

PROOF. An easy application of Fubini's theorem shows that

$$\begin{aligned}\varepsilon^{1/2}|S| &\le \int_S \mathrm{d}E \int_I \mathrm{d}\mu(x) 1_{[E-\varepsilon,E+\varepsilon]}(x) \\ &\le \int_I \mathrm{d}\mu \int_I \mathrm{d}E 1_{[E-\varepsilon,E+\varepsilon]}(x) = \mu(I)2\varepsilon\,.\end{aligned}$$

□

Appealing to multiscale analysis, we see that, at each scale $l_n \simeq l_0^{2^n}$, we have to exclude energies of measure $C\exp(-\frac{1}{2}l_0^{\frac{1}{2}2^n})$. Thus the total measure of energies we might have to excise is

$$|E^{\mathrm{exc}}| = |\bigcup_n E_n^{\mathrm{exc}}| \le \sum_{n=0}^{\infty} C\exp(-\frac{1}{2}l_0^{2^{n-1}}) \le 2C\exp(-\frac{1}{2}l_0^{\frac{1}{2}})\,.$$

Because the integrated density of states is so small, we can choose l_0 to be exponentially large in $\lambda^{-1/2-}$ (but not larger because of the factor $\Lambda_l \propto l^3$ in (3.34)), and hence the set of energies we have to excise is of order $\exp(-\frac{1}{2}\exp(\frac{1}{2}\lambda^{-1/2-}))$. Summarizing, for all energies in $[-\lambda, -c\lambda^2 + \mathcal{O}(\lambda^{4-})]\backslash E^{\mathrm{exc}}$ we have proven—provided of course that Lemma 3.13 holds—that the initial conditions for Theorem 2.1 hold and we can prove absence of diffusion as in Chapter 2. □

Proof of Lemma 3.13. Recall that we have to estimate

$$\operatorname{Im}\mathbb{E}\,G(E+\mathrm{i}\varepsilon;0,0)\,.$$

We use the expansion (3.33) of the Green function,

$$G^\varepsilon = \sum_{m=0}^{N} G_0^\varepsilon\,[WG_0^\varepsilon]^m + G_0^\varepsilon\,[WG_0^\varepsilon]^N\,WG^\varepsilon\,,$$

which we write in abbreviated form

$$G^\varepsilon(0,0) =: A_N^\varepsilon(0,0) + \sum_y B_N^\varepsilon(0,y)G^\varepsilon(y,0)\,.$$

Thus we can estimate

$$\begin{aligned}\operatorname{Im} G^\varepsilon(0,0) &\le \operatorname{Im} A_N^\varepsilon(0,0) + |\operatorname{Im}\sum_y B_N^\varepsilon(0,y)G^\varepsilon(y,0)| \\ &\le \operatorname{Im} A_N^\varepsilon(0,0) + \left(\sum_y |B_N^\varepsilon(0,y)|^2\right)^{1/2}\left(\sum_y |G^\varepsilon(y,0)|^2\right)^{1/2} \\ &\le \operatorname{Im} A_N^\varepsilon(0,0) + \left(\sum_y |B_N^\varepsilon(0,y)|^2\right)^{1/2}\frac{1}{\varepsilon}\,, \qquad (3.35)\end{aligned}$$

where we used that $\operatorname{Im} G^\varepsilon(0,0) \ge 0$ (which follows easily from the spectral theorem), the Schwartz inequality and the self-adjointness of H. Next, we average over the random vectors and obtain with Jensen's inequality

$$\operatorname{Im}\mathbb{E}\,G^\varepsilon(0,0) \le \operatorname{Im}\mathbb{E}A_N^\varepsilon(0,0) + \left(\mathbb{E}\sum_y |B_N^\varepsilon(0,y)|^2\right)^{1/2}\frac{1}{\varepsilon}\,. \qquad (3.36)$$

Similarly, we have

$$\mathbb{E}\,|G^\varepsilon(x,z)| \le \mathbb{E}|A_N^\varepsilon(x,z)| + \left(\mathbb{E}\sum_y |B_N^\varepsilon(x,y)|^2\right)^{1/2}\frac{1}{\varepsilon}\,. \qquad (3.37)$$

From now on we drop the superscript ε for sake of readability.

Self-energy correction. We elucidate the simplification due to the energy shift $\lambda^2\Sigma$ by noting the effect of averaging in the resolvent expansion. A first and second application of the resolvent equation yields

$$\begin{aligned} G &= G_0 - \lambda G_0 v G - \lambda^2 G_0 \Sigma G \\ &= G_0 - \lambda G_0 v G_0 + \lambda^2 G_0 v G_0 v G - \lambda^2 G_0 \Sigma G + \lambda^3 G_0 v G_0 \Sigma G \,. \end{aligned}$$

By definition of Σ it is clear that the "descendants" of the third term will cancel the ones of the fourth upon averaging. This cancelation holds of course at every renewed application of the resolvent equation such that the average of B_N will in the end contain but one (of order λ^{N+1} and thus subleading) term containing Σ. The order N of the perturbation expansion will be chosen to minimize the remainder term.

In computing the terms of the form

$$\begin{aligned} &\mathbb{E}[G_0(\lambda v G_0)^m](x,y) \\ =&\mathbb{E}[\sum_{x_1,\dots,x_m} G_0(x,x_1)\lambda v_{x_1} G_0(x_1,x_2)\dots\lambda v_{x_m} G_0(x_m,y)] \,, \end{aligned}$$

it is easiest to use a graphical representation: $G_0(x,y)$ corresponds to a line joining x and y, while the interaction v corresponds to a vertex. Averaging over the randomness yields terms represented by graphs obtained by fusing an even number (since expressions involving an odd number of v vanish upon averaging because of $\mathbb{E}\, v = 0$) of vertices at a time until none remains unpaired (in particular, m has to be even). Because

$$1 = \mathbb{E} v^{2n} \ll \gamma(n) \left(\mathbb{E} v^2\right)^n = \gamma(n)$$
$$\gamma(n) = \tfrac{(2n)!}{2^n n!} = \text{number of full pair contractions}$$

we can use Wick's theorem to get an upper bound by considering only fusions of *pairs* of vertices. The self-energy correction alluded to above means on the graphic level that we do not have to consider graphs containing tadpoles, as they get canceled by the $\lambda^2\Sigma$-shift in energy.

We begin with the term B_N. In the graphical representation, $|B_N(0,y)|^2 = B_N(0,y)B_N(0,y)^*$ is given by Figure 3. The leading diagram that results upon averaging is the ladder diagram, see Figure 4. The resulting graph is depicted in Figure 5 and estimated in the following.

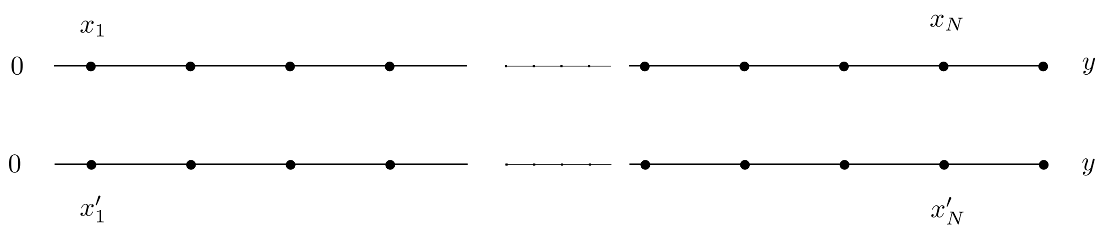

FIGURE 3. Graphical representation of the remainder term $|B_N|^2$.

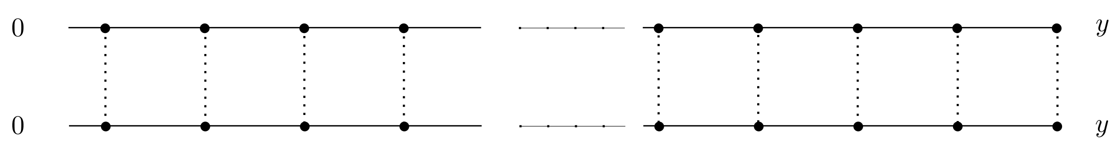

FIGURE 4. Leading diagram in $|B_N(0,y)|^2$. Contraction is along dotted lines.

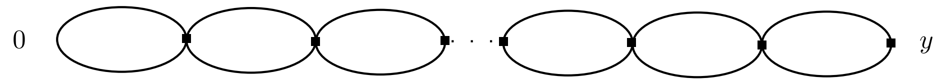

FIGURE 5. The resulting graph.

Bounds on graphs. The asymptotic behaviour of the free propagator G_0 for $E_0 > 0$ is [**36**]:

$$|G_0(E_0; 0, x)| = \frac{C}{|x|} \mathrm{e}^{-\sqrt{E_0}|x|}(1 + o(|x|)) , \quad |x| \to \infty$$

Since ultraviolet regularity is trivial because of the lattice we can bound the propagator by

$$|G_0(E_0; 0, x)| \leq \frac{C}{|x| + 1} \mathrm{e}^{-\sqrt{E_0}|x|} . \tag{3.38}$$

Here and in the following, C denotes different, largely irrelevant constants. The ladder diagram is now easily estimated:

$$\begin{aligned}\lambda^2 \sum_y |G_0(x_N, y)|^2 &= \lambda^2 \sum_y |G_0(0, y)|^2 \leq C\lambda^2 \int \mathrm{d}^3 y \frac{\mathrm{e}^{-2\sqrt{E_0}|y|}}{(1 + |y|)^2} \\ &= \lambda^2 \frac{C}{\sqrt{E_0}} \int \mathrm{d}^3 y \frac{\mathrm{e}^{-2|y|}}{(\sqrt{E_0} + |y|)^2} \leq C \left(\frac{\lambda^2}{\sqrt{E_0}} \right) =: A .\end{aligned}$$

Summing next over $x_N, x_{N-1}, \ldots$ we get as a bound on the ladder diagram contributing to $\mathbb{E}\,|B_N(0, y)|^2$ the value A^{N+1}. We see that in order to make the remainder term $\mathbb{E}\,|B_N|^2$ small we need to put a lower bound on E_0, namely $E_0 > \lambda^{4-\delta}$, for some $\delta > 0$. In particular, we have from the discussion after (3.32) as an upper bound for E

$$E + \lambda^{4-\delta} \leq E^* . \tag{3.39}$$

By construction, any graph contributing to $\mathbb{E}\,|B_N|^2$ is one-particle irreducible. Tadpole (self-energy) corrections to a propagator line are canceled by the energy shift $\lambda^2\Sigma$, as explained above. The next-order subgraph is shown Figure 6 and is estimated as follows:

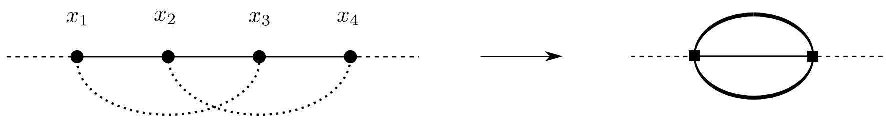

FIGURE 6. Contraction along the dotted lines yields the lip diagram.

$$\begin{aligned}\lambda^4 \sum_{x_4} |G_0(x_1, x_4)|^3 &\leq C\lambda^4 \int \mathrm{d}x^3 |G_0(0, x)|^3 \leq C\lambda^4 \int \mathrm{d}^3 x \frac{\mathrm{e}^{-3\sqrt{E_0}|x|}}{(|x| + 1)^3} \\ &= C\lambda^4 \int \mathrm{d}^3 x \frac{\mathrm{e}^{-3|x|}}{(|x| + \sqrt{E_0})^3} \leq C\lambda^4 \log(E_0^{-1}) .\end{aligned}$$

The next-order subgraph that might occur is shown in Figure 7 and is bounded by $C\lambda^6 E_0^{-1/2}$,

$$\begin{aligned}&\lambda^6 \sum_{x,y} G_0(0, y)^2 G_0(y, x)^2 G_0(0, x) \\ &\leq \lambda^6 E_0^{-1/2} \int \mathrm{d}x^3 \mathrm{d}y^3 \frac{\mathrm{e}^{-2|y|}}{(|y| + \sqrt{E_0})^2} \frac{\mathrm{e}^{-2|y-x|}}{(|y - x| + \sqrt{E_0})^2} \frac{\mathrm{e}^{-|x|}}{|x| + \sqrt{E_0}} \\ &\leq C\lambda^6 E_0^{-1/2} \int^{\mathrm{cutoff}} \mathrm{d}x^3 \mathrm{d}y^3 \frac{1}{(|y| + \sqrt{E_0})^2} \frac{1}{(|x| + \sqrt{E_0})^2} \frac{1}{|x - y| + \sqrt{E_0}} .\end{aligned}$$

Now use

$$
\begin{aligned}
&|x-y| = \sqrt{x^2+y^2-2|x||y|\cos\theta} = \sqrt{2|x||y|}\sqrt{\frac{x^2+y^2}{2|x||y|}-\cos\theta} \\
&\geq \sqrt{2|x||y|}\sqrt{1-\cos\theta} \\
&\Longrightarrow \frac{\sin\theta}{|x-y|} \leq \frac{1}{\sqrt{2|x||y|}}\frac{\sin\theta}{\sqrt{1-\cos\theta}} = \frac{\sqrt{1+\cos\theta}}{\sqrt{2|x||y|}} \leq \frac{1}{\sqrt{|x||y|}}
\end{aligned}
$$

to see that the integral remains bounded for $E_0 \to 0$:

$$
\begin{aligned}
&C\lambda^6 E_0^{-1/2}\int^{\mathrm{cutoff}} \mathrm{d}x^3\mathrm{d}y^3 \frac{1}{(|y|+\sqrt{E_0})^2}\frac{1}{(|x|+\sqrt{E_0})^2}\frac{1}{|x-y|+\sqrt{E_0}} \\
&\leq C\lambda^6 E_0^{-1/2}\int^{\mathrm{cutoff}} \mathrm{d}|x|\mathrm{d}|y| \frac{|y|^2}{|y|^2}\frac{|x|^2}{|x|^2}\frac{\sin\theta\mathrm{d}\theta}{|x-y|} \\
&\leq C\lambda^6 E_0^{-1/2}\int^{\mathrm{cutoff}} \mathrm{d}|x|\mathrm{d}|y| \frac{1}{|x|^{1/2}|y|^{1/2}} \leq C\lambda^6 E_0^{-1/2}.
\end{aligned}
$$

We can easily convince ourselves (see [**20, 17**] for a more thorough discussion of the various graphs

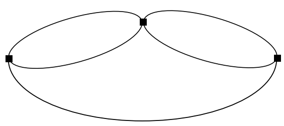

FIGURE 7. The moustache graph.

that occur) that the worst kind of graph is depicted in Figure 8, where the beads stand for one-particle irreducible subgraphs. If we can bound the subgraphs between u_{2i-1} and u_{2i} (the value of which we

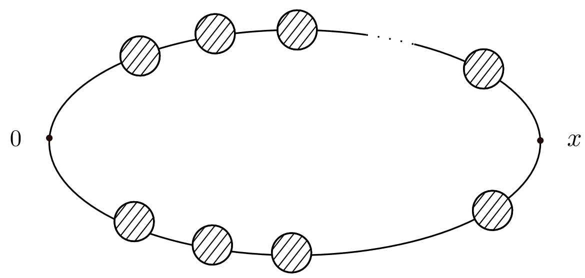

FIGURE 8. Worst type of diagram in $|B_N|^2$.

denote by $f_i(u_{2i-1}-u_{2i})$) we go to Fourier space to localize it:

$$
\begin{aligned}
&|\int \mathrm{d}^3x \prod_i \mathrm{d}^3u_i\, G_0(0-u_1)f_1(u_1-u_2)\dots f_n(u_{2n-1}-u_{2n})G_0(u_{2n}-y)G_0(y-v_i)\dots G_0(v_j-0)| \\
&= |(G_0 * f_1 * G_0 * f_2 * \dots * G_0)(0)| \\
&= C|\int \mathrm{d}^3k\, \widehat{G}_0(k)\hat{f}_1(k)\widehat{G}_0(k)\dots \hat{f}_n(k)\widehat{G}_0(k)| \\
&\le C\|f_1\|_1\dots\|f_n\|_1 \int \mathrm{d}^3k|\widehat{G}_0(k)|^{n+2} \le C\|f_1\|_1\dots\|f_n\|_1 \int \mathrm{d}^3k \left(\frac{1}{k^2+E_0}\right)^{n+2} \\
&\le C\|f_1\|_1\dots\|f_n\|_1 \frac{1}{E_0^{n+1/2}}\,.
\end{aligned}
$$

As we have seen before, for $E_0 > \lambda^{4-\delta}$ the worst subgraph goes like $\lambda^4\log(E_0^{-1})$, so the worst graph is bounded by

$$
C\left(\frac{\lambda^4\log(E_0^{-1})}{E_0}\right)^{N/2}\frac{1}{\sqrt{E_0}} = C\left(\frac{\lambda^2|\log E_0|^{1/2}}{\sqrt{E_0}}\right)^N\frac{1}{\sqrt{E_0}}\,.
$$

The number of graphs contributing to $\mathbb{E}\,|B_N(0,x)|^2$ is less than $\frac{(2N)!}{2^N N!} \simeq 2^N(N/e)^N\sqrt{2} < 2^N N!$, so

$$
\left(\sum_y \mathbb{E}|B_N(0,y)|^2\right) \le 2^N N!\frac{C}{\sqrt{E_0}}\left(\frac{\lambda^2|\log E_0|^{1/2}}{\sqrt{E_0}}\right)^N \le N!((C\lambda)^{\delta^+})^N\frac{1}{\lambda^{2-\delta}}\,.
$$

Choosing now $N \cong (C\lambda)^{-\delta}$ renders this remainder term exponentially small in $\lambda^{-\delta}$:

$$
\left(\sum_y \mathbb{E}|B_N(0,y)|^2\right) \le C'\frac{1}{\lambda^{2-\delta}}N^Ne^{-N}\sqrt{N}N^{-N} \le C'e^{-\widetilde{C}\lambda^{-\delta}}\,. \tag{3.40}
$$

Equation (3.36) leaves us with bounding $\operatorname{Im}\mathbb{E}A_N^\varepsilon(0,0)$. From the discussion of the term B_N it is clear that the worst graph is of the form depicted in Figure 9, and we estimate

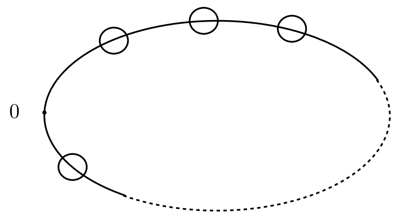

FIGURE 9. Worst type of graph in A_N.

$$
\begin{aligned}
T(m) :=&\operatorname{Im}\sum_{x_i} G_0(0-x_1)(G_0^2\overline{G}_0)(x_1-x_2)\dots(G_0^2\overline{G}_0)(x_{m/2-1}-x_{m/2})G_0(x_{m/2}-0) \\
=&\operatorname{Im} G_0 * (G_0^2\overline{G}_0) * \dots * (G_0^2\overline{G}_0) * G_0\ (0) = \operatorname{Im}\int_{[-\pi,\pi]^3}\mathrm{d}^3k\,\widehat{G_0}(k)\widehat{G_0^3}(k)\dots\widehat{G_0}(k)\,.
\end{aligned}
$$

Considering

$$
\operatorname{Im}\widehat{G_0}(k) = \operatorname{Im}\frac{1}{\epsilon(k)+E_0-\mathrm{i}\varepsilon} = \frac{\varepsilon}{(\epsilon(k)+E_0)^2+\varepsilon^2} < \frac{\varepsilon}{(\epsilon(k)+E_0)^2}\,,
$$

and

$$
\operatorname{Im}\widehat{G_0^3}(k) = \operatorname{Im}\left(\frac{1}{\epsilon(k)+E_0-\mathrm{i}\varepsilon}\right)^3 = \operatorname{Im}\left(\frac{\epsilon(k)+E_0+\mathrm{i}\varepsilon}{(\epsilon(k)+E_0)^2+\varepsilon^2}\right)^3 < C\frac{\varepsilon}{(\epsilon(k)+E_0)^6}\,,
$$

heeding the following calculation for the imaginary part of the product of complex numbers (x and y denote the maximum of the $|x_k|$ and $|y_k|$, respectively),

$$\begin{aligned}|\mathrm{Im} \prod_{k=1}^{m}(x_k+\mathrm{i}y_k)| &\leq \sum_{j\,\mathrm{odd}} \binom{m}{j} x^{m-j}y^j \leq \max\{1,x^m\}\sum_j \frac{(my)^j}{j!} \\ &< C\sum_{j=1}^{\infty}\frac{(my)^j}{j!} = C(e^{my}-1) \leq 2Cmy\,,\end{aligned}$$

for y small enough, and using the bounds on the diagram established above, we find

$$T(m) \leq C\varepsilon m\left(\frac{\lambda^2}{\sqrt{E_0}}\right)^m \leq Cm\varepsilon\lambda^{\delta m}\,,$$

where we discard logarithmic factors. We have thus, recalling that there are less than $2^{m/2}(m/2)!$ graphs with $m/2$ vertices,

$$\mathrm{Im}\,\mathbb{E}\,A_N^{\varepsilon}(0,0) = \mathrm{Im}\,\mathbb{E}\,\sum_{m=0}^{N} G_0^{\varepsilon}(\lambda v G_0^{\varepsilon})^m(0,0) \leq \sum_{m=0}^{N} 2^{m/2}(m/2)!\lambda^{\delta m/2}C\frac{m}{2}\varepsilon\,.$$

Recalling $N \cong (C\lambda)^{-\delta}$ we get

$$\mathrm{Im}\,\mathbb{E}\,A_N^{\varepsilon}(0,0) \leq C\varepsilon\sum_{m=0}^{N} 2^{m/2}(m/2)!\,m\frac{1}{N^{m/2}} \leq C\varepsilon\sum_{m=0}^{N} m^{3/2}\left(\frac{m}{N\mathrm{e}}\right)^{m/2} \leq C\varepsilon\,. \tag{3.41}$$

Recalling (3.35) and combining (3.40) with (3.41) we obtain as a bound on the imaginary part of the averaged Green function

$$\mathrm{Im}\,\mathbb{E}\,G^{\varepsilon}(0,0) \leq C\varepsilon + C'e^{-c\lambda^{-\delta}}\frac{1}{\varepsilon}\,.$$

If $\varepsilon \gtrsim e^{-\frac{c}{2}\lambda^{-\delta}}$ then

$$\mathrm{Im}\,\mathbb{E}\;G^{\varepsilon}(0,0) \leq C\varepsilon + C'e^{-\frac{c}{2}\lambda^{-\delta}} \leq C\varepsilon\,.$$

From equation (3.30) we get, for all E satisfying condition (3.39),

$$n(E) \leq \frac{2}{\pi}\int^{E}\mathrm{d}E'\,\mathrm{Im}\,\mathbb{E}\,G^{\varepsilon}(E';0,0) \leq \mathrm{const}\,\varepsilon = \mathrm{const}\,e^{-c\lambda^{-\delta}}\,, \tag{3.42}$$

which is our desired bound on the integrated density of states. By a similar computation we get for $\mathbb{E}\,|A_N(x,z)|$

$$\mathbb{E}\,|A_N(x,z)| \leq C\mathrm{e}^{-\frac{1}{2}\sqrt{E_0}|x-z|}$$

and hence

$$\mathbb{E}\,|G(x,z)| \leq C\mathrm{e}^{-\frac{1}{2}\sqrt{E_0}|x-z|} + 2Ce^{-\frac{\tilde{C}}{2}\lambda^{-\delta}}\,,$$

for $\varepsilon \gtrsim e^{-\frac{c}{2}\lambda^{-\delta}}$. □

Part 2

Friction

CHAPTER 4

Introduction

Although friction is ubiquitous as a phenomenon in the physical world it does not often make an appearance in the mathematical literature. Indeed, we know of only a few instances of a rigorous mathematical analysis of friction. The physics approach usually consists of adding explicitely a term that mimicks the effect of friction to the equations governing the evolution of the system under consideration. The physical system that we have in mind in the context of friction is that of a tracer particle traveling in a dispersive medium. In a very crude approximation, we could study Newton's equation of motion for the particle where we lump together all the interaction of the tracer particle with the medium in an effective friction term that is usually chosen proportional to the velocity,

$$m\ddot{x}(t) = -\nabla V(x(t)) - \eta\dot{x}(t)\,, \quad \eta > 0\,. \tag{4.1}$$

For a confining potential, for instance $V(x) = |x|^2$, the particle approaches the minimum of the potential well, $x = 0$, exponentially fast with rate $\frac{\eta}{2m}$, as one computes easily. For a constant external force, $V(x) = -F \cdot x$, the particle reaches a limiting speed $\frac{F}{\eta}$ proportional to the external force, with exponential rate $\frac{\eta}{m}$. In particular, the particle comes to a full stop exponentially fast if the external force vanishes.

A more elaborate approach is the Langevin equation that models the effects of the medium by a memory and a random term, see [**57**] or for the quantum case [**23**]. The Langevin equation in one dimension is given by

$$m\ddot{x}(t) = -\nabla V(x(t)) - \eta^2 \int_0^t \mu(t-\tau)\dot{x}(\tau)\mathrm{d}\tau + \eta\zeta(t)\,,$$

where Newton's equation has been modified by two terms, a mean force characterized by a memory function $\mu(t)$, and a random force ζ_t given by a stationary Gaussian process with mean zero and covariance

$$\begin{aligned}\langle\zeta_t\zeta_s\rangle &= \frac{1}{2\pi}\int |\hat{\rho}(k)|^2\cos(k(t-s))\mathrm{d}k\\ \mu(t) = -\partial_t\langle\zeta_t\zeta_0\rangle &= \frac{1}{2\pi}\int k|\hat{\rho}(k)|^2\sin(kt)\mathrm{d}k\,.\end{aligned}$$

Here, η is a friction constant, and $\rho(x)$ is a function describing the coupling between particle and medium. The non-linear effective equation we will end up studying is structurally similar to the Langevin equation insofar as there will also be a linear "memory term", whereas non-linear (in the velocity) terms will take the place of the random term ζ_t.

Rigorous results. In [**10**] the authors investigate how realistic the aforementioned "linear" model of friction (4.1) is. They study rigorously a Hamiltonian model of a classical particle interacting with a homogeneous dissipative medium. The medium consists of an independent scalar wave field *at each point in space*. They prove that for wave medium dimension $d = 3$, and for c sufficiently large (where c denotes the "speed of light" in the medium), the particle indeed exhibits the above behaviour of linear friction. More precisely, they prove that the particle approaches a minimum of a confining potential with exponential rate $\frac{\eta}{2}$, and a limiting velocity $v(F)$ with exponential rate η in the case of a constant external force F. Here, η is some effective friction parameter depending on the form factor of the interaction between the particle and the wave fields.

The assumption of independent scalar wave fields at each point in space seems too strong a simplification, and it might be precisely this complete lack of memory effects that leads to the exponential relaxation rates. In an earlier work, [**40**], a Hamiltonian model of a classical particle (with relativistic kinetic energy) in a confining potential interacting with a scalar wave field is investigated. It is proven that solutions of finite energy converge in some local sense to the set of stationary solutions as $t \to \infty$. The relaxation rate depends on the spatial decay of the initial conditions and is exponential only for rapid spatial decay of the latter.

The earliest attempt at understanding rigorously the friction effect felt by a particle in a dispersive medium seems to be [**15, 7**]. There, the Pauli-Fierz model for a charged extended particle in a central potential, interacting with the electro-magnetic field is studied. The authors find that, under some conditions on the radius and the frequency of revolution of the particle, orbits that are initially close to those allowing the purely mechanical circular motion stay close to these orbits for exponentially long times.

More recently, Pulvirenti et al. [**12**] analyzed the time-evolution of a disk subjected to a constant force interacting with a gas of free particles in the mean-field (Vlasov) approximation. They find that for sufficiently small positive values of $v_\infty - v_0$, the difference between the equilibrium velocity and the initial velocity of the disk, the disk reaches v_∞ with a power law $t^{-(d+2)}$, where d is the dimension of the physical space in which the disk moves. They find that the power law is due to recollisions, and that any Markovian approximation (which amounts to neglecting the recollisions) yields an exponential rate.

The phenomenon of runaway particles is investigated in [**47**]. The authors look at a Hamiltonian model of a charged particle under the influence of a strong constant electric field and interacting with a medium that is described as a Vlasov fluid that is not perturbed by the particle. They find that if the singularity of the particle-medium interaction is integrable and the electric field is large enough then the particle escapes to infinity with a quasi-uniformely accelerated motion.

New Hamiltonian model of friction. Recently, some Hamiltonian models of friction have been introduced in [**26**] and analyzed in [**24**]. This second part of the thesis is a contribution to that body of work. In this introduction, we follow closely [**24**].

The goal is not to study an *effective* model for friction but to understand how friction comes about in a Hamiltonian system. We shoot a tracer particle into a dispersive medium (here taken to be a Bose-Einstein condensate) and investigate its trajectory. Because the tracer particle interacts with the medium there should be some energy transfer from the particle to the medium where the energy is dissipated, hence the particle should be slowed down—it experiences friction. As discussed above, the rate of approach to some equilibrium position seems to depend sensitively on how "forgetful" the medium is. We will not make any assumptions weakening the memory of the bath—such as independent baths at each point in space, or a wave bath with high velocity of light such that energy is rapidly dissipated away from the particle—so that it is not too surprising that we will find only a power-law approach to equilibrium.

Mathematically, we consider a tracer particle of Mass M coupled to a bath of n identical bosons of mass m confined to some cubic region $\Lambda \subset \mathbb{R}^3$,

$$H = -\frac{1}{2M}\Delta_X^{(\Lambda)} + V(X) - \sum_{i=1}^{n} \frac{1}{2m}\Delta_{x_i}^{(\Lambda)} + g\sum_{i=1}^{n} W(x_i - X) + \lambda \sum_{i<j} \phi(x_i - x_j)\,, \tag{4.2}$$

acting on the Hilbert space $\mathcal{H}^{(n)} \otimes \mathcal{H}_{\mathrm{P}} := (P_+ L^2(\Lambda)^{\otimes n}) \otimes L^2(\Lambda)$. P_+ denotes the projection onto symmetric wavefunctions and is used because we are considering Bosons. The parameters $g, \lambda \geq 0$ denote coupling constants, W is a fast decaying interaction potential between tracer particle and medium, ϕ is a potential, of positive type and also of fast decay at infinity, describing two-body interactions between medium particles, and V is an external potential affecting only the tracer particle. The Laplacians are required to have, for instance, periodic boundary conditions on $\partial\Lambda$.

We are interested in the thermodynamic limit $\Lambda \nearrow \mathbb{R}^3$ of this system, with the particle density $\rho = \frac{n}{|\Lambda|}$ kept fixed. It is therefore convenient to pass to a second quantized description on Fock space $\mathscr{F}_+ \otimes L^2(\Lambda) := \left(\bigoplus_{n\geq 0} \mathcal{H}^{(n)}\right) \otimes L^2(\Lambda)$ in order to accomodate an arbitrary number of bosons. To understand the thermodynamic limit $\Lambda \nearrow \mathbb{R}^3$ of this system is a mathematical problem that is not at all understood rigorously. For our purposes it is enough, though, to carry out the following computations on a formal level. Only the resulting model will be analyzed in a mathematically rigorous way. We therefore pretend that we have given some sense to the limit $\Lambda \nearrow \mathbb{R}^3$ and drop the reference to the finite cube Λ in the following.

In second quantized notation, the Hamiltonian (4.2) takes the following form,

$$\begin{aligned} H = &-\frac{1}{2M}\Delta_X + V(X) + \int \frac{1}{2m}\nabla a^\dagger(x)\nabla a(x)\mathrm{d}^3x \\ &+ g\int W(x-X)\left(a^\dagger(x)a(x)-\rho\right)\mathrm{d}^3x \\ &+ \frac{\lambda}{2}\int \left(a^\dagger(x)a(x)-\rho\right)\phi(x-y)\left(a^\dagger(y)a(y)-\rho\right)\mathrm{d}^3x\mathrm{d}^3y\,, \end{aligned}$$

where we modified the Hamiltonian so as to give states of small total energy the desired value $\approx \rho$. Note that we just added a (infinite) constant and a chemical potential, so that the new Hamiltonian describes the same physics. Here, $a^\dagger, a$ are creation and annihilation operators,

$$[a(x), a^\dagger(y)] = \delta(x-y)\mathbb{1}\,, \qquad [a(x)^\sharp, a(y)^\sharp] = 0\,.$$

We introduce a mean-field parameter $N > 0$

$$\lambda = \frac{\lambda_0 g^2}{N}\,, \qquad \rho = \frac{N\rho_0}{g^2}\,,$$

as we are interested in a high-density and weak-interaction limit, $N \to \infty$. For our purposes, it is more natural to work with rescaled and shifted creation and annihilation operators,

$$b_N^\sharp := \frac{1}{\sqrt{N}}a^\sharp - \sqrt{\frac{\rho_0}{g^2}}\,.$$

In the following, we consider a very heavy tracer particle with mass $M = NM_0$ moving in an external potential $V(X) = Nv(X)$. In the new variables, the Hamiltonian is given by $H = NH^{(N)}$, where

$$\begin{aligned} H^{(N)} = &-\frac{1}{N^2}\frac{1}{2M_0}\Delta_X + v(X) + \int \frac{1}{2m}\nabla b_N^\dagger(x)\nabla b_N(x)\mathrm{d}^3x \\ &+ g\int W(x-X)[b_N^\dagger(x)b_N(x) + \sqrt{\frac{\rho_0}{g^2}}(b_N^\dagger(x)+b_N(x))]\mathrm{d}^3x \\ &+ \frac{\lambda_0 g^2}{2}\int \left(b_N^\dagger(x)b_N(x) + \sqrt{\frac{\rho_0}{g^2}}(b_N^\dagger(x)+b_N(x))\right)\times \\ &\times \phi(x-y)\left(b_N^\dagger(y)b_N(y) + \sqrt{\frac{\rho_0}{g^2}}(b_N^\dagger(y)+b_N(y))\right)\mathrm{d}^3x\mathrm{d}^3y\,. \end{aligned}$$

Recall the Schrödinger equation,

$$\mathrm{i}\hbar\partial_t\psi_t = H\psi_t\,,$$

which in our case reads

$$\mathrm{i}\partial_t\psi_t = H\psi_t \Longleftrightarrow \mathrm{i}\frac{1}{N}\partial_t\psi_t = H^{(N)}\psi_t\,,$$

so that $\frac{1}{N}$ is seen to play the role of $\hbar$. Note that the kinetic energy is usually proportional to $\hbar^2$. This is why we chose to scale the mass as $M = M_0 N$.

The "mean-field limit" $N \to \infty$ corresponds therefore to a "classical limit" $\hbar \to 0$, and we obtain the following classical Hamiltonian system. For a system in a finite periodic box, this could probably be made rigorous, see [**32, 39**]. But again, for our purposes a heuristic derivation is sufficient. The phase space of the system is given by

$$\Gamma = \mathbb{R}^6 \times H^1(\mathbb{R}^3)\,,$$

where $H^1(\mathbb{R}^3)$ is the complex Sobolev space over $\mathbb{R}^3$. The Poisson brackets are

$$\begin{aligned}&\{X, X\} = \{P, P\} = 0\,, \{X_i, P_j\} = \delta_{ij}\\ &\{\beta^\sharp, \beta^\sharp\} = 0\,, \{\beta(x), \bar{\beta}(y)\} = \mathrm{i}\delta(x-y)\,.\end{aligned}$$

Finally, the Hamilton functional is given by

$$\begin{aligned}\mathscr{H} =&\frac{P^2}{2M_0} + v(X) + \int \frac{|\nabla\beta(x)|^2}{2m}\mathrm{d}^3x + g\int W(x-X)\left(|\beta(x)|^2 + 2\sqrt{\frac{\rho_0}{g^2}}\mathrm{Re}\,\beta(x)\right)\mathrm{d}^3x\\ &+ \frac{\lambda_0 g^2}{2}\int\left(|\beta(x)|^2 + 2\sqrt{\frac{\rho_0}{g^2}}\mathrm{Re}\,\beta(x)\right)\phi(x-y)\left(|\beta(y)|^2 + 2\sqrt{\frac{\rho_0}{g^2}}\mathrm{Re}\,\beta(y)\right)\mathrm{d}^3x\mathrm{d}^3y\,.\end{aligned} \tag{4.3}$$

Setting $\alpha(x) = \beta(x) + \sqrt{\frac{\rho_0}{g^2}}$ we see that the Hamiltonian takes the form

$$\begin{aligned}\mathscr{H} =&\frac{P^2}{2M_0} + v(X) + \int \frac{|\nabla\alpha(x)|^2}{2m}\mathrm{d}^3x + g\int W(x-X)\left(|\alpha(x)|^2 - \frac{\rho_0}{g^2}\right)\mathrm{d}^3x\\ &+ \frac{\lambda_0 g^2}{2}\int\left(|\alpha(x)|^2 - \frac{\rho_0}{g^2}\right)\phi(x-y)\left(|\alpha(y)|^2 - \frac{\rho_0}{g^2}\right)\mathrm{d}^3x\mathrm{d}^3y\,,\end{aligned}$$

and for $W = 0$ we can read off the explicit ground state,

$$\begin{aligned}&P_t \equiv 0\\ &X_t \equiv X_0 : \text{a minimum of } v(x)\\ &\alpha_t \equiv \alpha_0 = \mathrm{e}^{\mathrm{i}\theta}\sqrt{\frac{\rho_0}{g^2}}\,,\end{aligned}$$

where $\mathrm{e}^{\mathrm{i}\theta}$ is an arbitrary phase. Note that, for $\rho_0 > 0$, the continuous gauge symmetry

$$\alpha(x) \to \mathrm{e}^{\mathrm{i}\theta}\alpha(x), \quad \overline{\alpha}(x) \to \mathrm{e}^{-\mathrm{i}\theta}\overline{\alpha}(x)$$

of the Hamilton functional is spontaneously broken in the ground states, which corresponds to Bose-Einstein condensation. It is expected that the Bose gas exhibits gapless (Goldstone) modes.

In the β-variables, the equations of motion are found to be

$$\begin{aligned}\dot{X}_t =&\frac{P_t}{M},\\ \dot{P}_t =& - \nabla_X V(X_t) - g\int \nabla_X W(x-X_t)(|\beta_t(x)|^2 + 2\sqrt{\frac{\rho_0}{g^2}}\mathrm{Re}\,\beta_t(x))dx,\\ i\dot{\beta}_t(x) =&(-\frac{1}{2m}\Delta + gW(x-X_t))\beta_t(x) + \sqrt{\rho_0}W(x-X_t)\\ &+\lambda_0 g^2[\phi * (|\beta_t|^2 + 2\sqrt{\frac{\rho_0}{g^2}}\mathrm{Re}\,\beta_t)](x)(\beta_t(x) + \sqrt{\frac{\rho_0}{g^2}}).\end{aligned}$$

For a pure Bose gas (that is, $W = 0$) and in the weak coupling limit $\lambda_0 \to 0$ while $\lambda_0\rho_0 =$const, the frequency spectrum of the fluctuations β can be found: The equation for β takes the form

$$\mathrm{i}\dot{\beta}_t = -\frac{1}{2m}\Delta\beta_t + 2\lambda_0\rho_0\phi * \mathrm{Re}\,\beta_t\,,$$

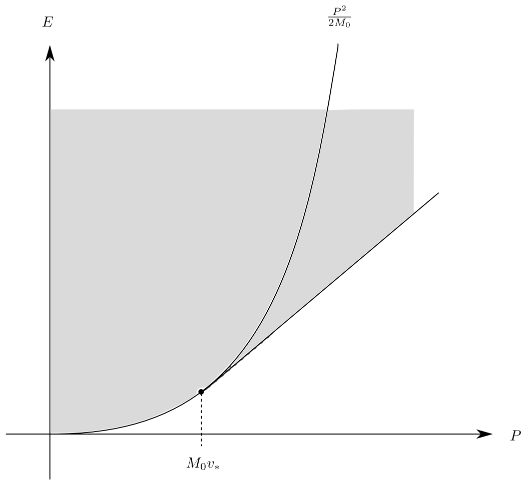

FIGURE 1. Interacting Bose gas: linear dispersion relation for small wave vectors. Momentum states with $|P| > M_0 v_*$ decay by emitting Cerenkov radiation, that is, the particle experiences friction. Momentum states with $|P| \leq M_0 v_*$ describe stable, ballistic propagation.

and writing $\beta = \xi + \mathrm{i}\eta$ we find

$$\dot{\xi}_t = -\frac{1}{2m}\Delta\eta_t$$
$$\dot{\eta}_t = \frac{1}{2m}\Delta\xi_t - 2\lambda_0\rho_0\phi * \xi_t\,.$$

Differentiating the lower line with respect to time and plugging in the upper yields

$$\ddot{\eta}_t = -\frac{1}{4m^2}\Delta^2\eta_t + \frac{1}{m}\lambda_0\rho_0\phi * \Delta\eta_t\,.$$

By Fourier transform we obtain

$$-\Omega^2\widehat{\eta}(k) = -\frac{1}{4m^2}k^4\widehat{\eta}(k) - \frac{k^2}{m}\lambda_0\rho_0(2\pi)^{\frac{3}{2}}\widehat{\phi}\,\widehat{\eta}(k)\,.$$

The dispersion relation found is *linear* for small $|k|$:

$$\Omega(k) = |k|\sqrt{\frac{k^2}{4m^2} + \frac{(2\pi)^{\frac{3}{2}}\lambda_0\rho_0}{m}\widehat{\phi}(k)} \simeq |k|\sqrt{\frac{(2\pi)^{\frac{3}{2}}\lambda_0\rho_0\widehat{\phi}(0)}{m}} =: |k|v_*\,,$$

where v_* is the speed of sound in the Bose gas. In Figure 1 we plot the joint energy-momentum spectrum of the tracer particle and the medium (for $W = v = 0$). We see that for momenta $|P| > M_0 v_*$ the energy, $\frac{P^2}{2M_0}$, of the particle is embedded in the continuous energy spectrum. Resonance theory suggests that as soon as the interaction is switched on (that is, $W \neq 0$) the state of the particle decays into states of smaller velocity by exciting gapless modes in the Bose gas (Cerenkov radiation). The speed of the particle diminishes until it has dropped below v_*. For the non-interacting Bose gas, the speed of sound equals zero, so that the tracer particle comes to a full stop eventually.

In summary, the physical interpretation is that a particle shot into a condensed Bose gas experiences *friction by emission of Cerenkov radiation* and is decelerated until its speed comes to lie below the speed of sound in the Bose gas.

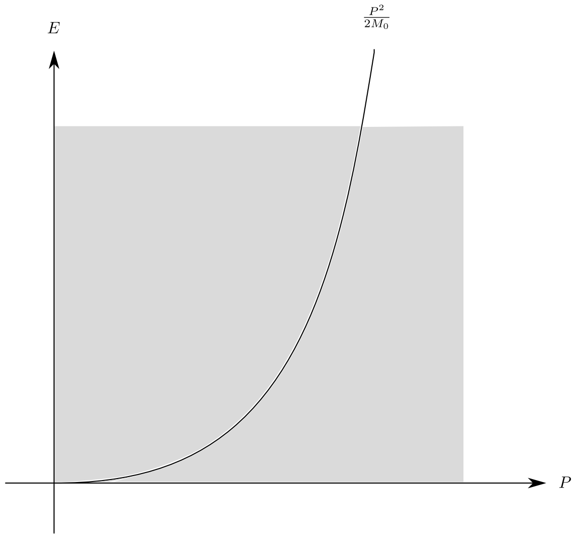

FIGURE 2. Non-interacting Bose gas: quadratic dispersion relation, $v_* = 0$, so that all momentum states decay.

CHAPTER 5

Free Bose gas

1. Outline of strategy

We do not intend to study the full model but certain limiting regimes. We consider the free Bose gas, that is the limit $\lambda_0 = 0$. Furthermore, we consider the model without forcing, so $v = 0$. In this case, the equations of motion are

$$
\begin{aligned}
\dot{X}_t &= \frac{P_t}{M_0} \\
\dot{P}_t &= -g \int_{\mathbb{R}^3} \nabla W^{X_t}(x) \left(|\beta_t(x)|^2 + 2\sqrt{\frac{\rho_0}{g^2}} \operatorname{Re} \beta_t(x) \right) \mathrm{d}^3 x \\
\mathrm{i}\dot{\beta}_t(x) &= h^{X_t} \beta_t(x) + \sqrt{\rho_0} W^{X_t}(x) \,,
\end{aligned}
\tag{5.1}
$$

where $W^{X_t}(x) := W(x - X_t)$, and $h^{X_t} := \left(-\frac{1}{2m}\Delta + gW^{X_t}\right)$.

For simplicity, we choose $m = \frac{1}{2}$ and require the potential W

(A1) to be smooth,
(A2) to decay exponentially at infinity,
(A3) to be spherically symmetric,
(A4) to satisfy $\widehat{W}(0) \neq 0$.

Remark. Under these assumptions there are no bound states, nor zero-energy resonances for g small.

In the main part of this chapter we *prove* that the tracer particle experiences friction and is decelerated to a full stop in accordance with the heuristic findings of the previous chapter. We prove a lower bound for the strength of this friction mechanism, namely $|P_t| \leq ct^{-1-\varepsilon}$, $t \to \infty$, for some explicit $\varepsilon > 0$ depending on the initial conditions. At large times, the medium is shown to exhibit the expected behavior: It forms a "splash" that follows the motion of the tracer particle. Remarkably, even though initial conditions β_0 can be chosen to be very small (in L^2-sense), the splash that the medium forms is not square integrable. This is a consequence of the fact that we chose the medium to be non-interacting. This fact is also responsible for making it difficult to "guess" the right asymptotic behaviour of $|P_t|$ on a heuristic level. See [**26, 24**] for a more thorough discussion.

In [**24**] the problem is analysed in the weak-coupling limit $g \to 0$. They find completely analogous results. Nevertheless, our findings are interesting in their own right as we treat a particle coupled fully to the medium (as opposed to a weak coupling limit), which is usually a much harder problem. The main technical difference is that the generator of time evolution of the reservoir, $h^{X_t} = -\Delta + gW^{X_t}$, depends on time, for $g \neq 0$, through the position X_t of the particle. In addition, the generator of translations, ∂_x, no longer commutes with h^{X_t}.

In order to be able to state a precise theorem, introduce the continuous, monotonically increasing function $\Omega : (-\infty, 1) \to \mathbb{R}^+$,

$$
\Omega(\delta) := \frac{1}{\pi} \int_0^1 \frac{1}{1 + (1-r)^{\frac{1}{2}}} (1-r)^{-\frac{1}{2}} \left(\frac{1}{1-2\delta} (r^{-\frac{1}{2}} - r^{-\delta}) + r^{\frac{1}{2}-\delta} \right) \mathrm{d}r \,,
$$

and denote by δ^* the value at which $\Omega = 1$,

$$\Omega(\delta^*) = 1\,.$$

The computer says

$$\delta^* \simeq 0.66\,.$$

In the subsequent sections, we will prove the following main theorem.

THEOREM 5.1. *For any $\delta \in I := (\frac{1}{2}, \delta^*)$ there exists a $g_0 > 0$ and an $\varepsilon_0 > 0$ such that if $0 \le g \le g_0$ and $\|\langle x\rangle^3 \beta_0\|_2, |P_0| \le \varepsilon_0$ and $\|\langle x\rangle^3 \partial_x \beta_0\|_2 < \infty$ then*

$$|P_t| \le C_\delta t^{-\frac{1}{2}-\delta} \text{as } t \to \infty,$$

and

$$\lim_{t\to\infty} \|\beta_t + \sqrt{\rho_0}(h^{X_t})^{-1} W^{X_t}\|_\infty = 0\,.$$

In particular, the particle comes to rest after a finite distance: There is a $X_\infty \in \mathbb{R}^3$ such that $X_t \to X_\infty$, and

$$\beta_t \to -\sqrt{\rho_0}(h^{X_\infty})^{-1} W^{X_\infty} \notin L^2(\mathbb{R}^3)\,.$$

REMARKS.

- Assumption (A4), that is, $\widehat{W}(0) \neq 0$, is essential for the theorem to hold. However, it is also this assumption which makes the fluctuation field β_t evolve to a not square-integrable function, as

$$\widehat{(-\Delta)^{-1}W} = O(k^{-2}), \quad |k| \to 0\,.$$

- δ^* is critical in the following sense,

$$\delta \to \delta^* \quad \text{as } g_0, \varepsilon_0 \to 0\,.$$

  Unfortunately, we do not know whether the decay exponent is universal (that is, independent of initial conditions).
- It is *not* easy to guess the right power law from some linearization of the equations of motion. This is essentially because the eventual configuration of the fluctuation field β_t is, as mentioned, not square-integrable, has therefore infinite energy and is unphysical as an initial condition.

The main strategy for proving Theorem 5.1 is easily explained. We split the equation for $\dot{P}_\cdot$ into a part linear in $P_\cdot$ and a non-linear part,

$$\dot{P}_t = L_1(P)(t) + L_2(P)(t) + \mathcal{N}(P, X, \beta)(t)\,.$$

Then we establish a decay estimate for the solution of one of the linear parts of the equation, call it $K(t)$:

$$\dot{K}(t) = L_1(K)(t)\,.$$

We can then rewrite P in terms of this decaying solution and the remaining terms

$$P_t = K(t)P_0 + \int_0^t K(t-s)(L_2(P)(s) + \mathcal{N}(P, X, \beta)(s))\mathrm{d}s\,.$$

Using the decay of $K(t)$, the local existence of small solutions to the full equation and a bootstrap argument (in the form of a contraction lemma) we prove the theorem.

A little more detailed: By some elementary manipulations that we will do in the next section we obtain the following equation for $\dot{P}_t$,

$$\begin{aligned}\dot{P}_t &= 2\rho_0 \mathrm{Re}\,\langle(1+gh^{-1}W)\partial_x W\,,\mathrm{e}^{-iht}(X_0-X_t)\cdot\partial_x(h)^{-1}W\rangle\\ &- 2\frac{\rho_0}{M_0}\mathrm{Re}\,\langle(1+gh^{-1}W)\partial_x W\,,\int_0^t \mathrm{e}^{-ih(t-s)}P_s\cdot\partial_x(h)^{-1}W\mathrm{d}s\rangle\\ &+ \mathcal{N}(P,X,\beta)(t)\,,\end{aligned}$$

where $\mathcal{N}(P,X,\beta)(t)$ is a non-linear remainder. To reduce the amount of notation define

$$f(t) = 2\frac{\rho_0}{M_0}\mathrm{Re}\,\langle(1+gh^{-1}W)\partial_{x_1}W\,,\mathrm{e}^{-iht}\partial_{x_1}(h)^{-1}W\rangle\,.$$

It is easy to see that $f(t)$ is of order $t^{-\frac{3}{2}}$ as $t\to\infty$. We have thus

$$\dot{P}_t = f(t)\int_0^t P_s\mathrm{d}s - \int_0^t f(t-s)P_s\mathrm{d}s + \mathcal{N}(P,X,\beta)(t)\,.$$

We turn now to the linear equation

$$\dot{K}(t) = -\int_0^t f(t-s)K(s)\mathrm{d}s\,.$$

Essentially by Fourier transformation we prove

LEMMA 5.2. *The function $K:\mathbb{R}^+\to\mathbb{R}$ satisfies*

$$K(t) = Ct^{-\frac{1}{2}} + O(t^{-1}) \quad \textit{as } t\to\infty.$$

Note that in the proof of the lemma use is made of the fact that $\widehat{W}(0)=\int W\mathrm{d}^3x\neq 0$.

As advertised, we can now express P_t as

$$P_t = K(t)P_0 + \int_0^t K(t-s)f(s)\int_0^s P_{s_1}\mathrm{d}s_1\mathrm{d}s + \int_0^t K(t-s)\mathcal{N}(P,X,\beta)(s)\mathrm{d}s\,. \tag{5.2}$$

REMARK. Superficially, all three terms are only of order $t^{-\frac{1}{2}}$, that is, if P_t is to decay faster there must be some cancelation.

We effect this cancelation in the following way. Integrate the equation for $\dot{P}$ from 0 to t to obtain the equivalent equation

$$\begin{aligned}P_t = P_0 &- \int_0^t\int_0^s f(s-s_1)P_{s_1}\mathrm{d}s_1\mathrm{d}s + \int_0^t f(s)\int_0^s P_{s_1}\mathrm{d}s_1\mathrm{d}s\\ &+ \int_0^t \mathcal{N}(P,X,\beta)(s)\mathrm{d}s\,.\end{aligned}$$

Multiply it by $K(t)$ and subtract it from (5.2) to obtain

$$\begin{aligned}P_t(1-K(t)) = &\int_0^t K(t)\int_0^s f(s-s_1)P_{s_1}\mathrm{d}s_1\mathrm{d}s\\ &+ \int_0^t (K(t-s)-K(t))f(s)\int_0^s P_{s_1}\mathrm{d}s_1\mathrm{d}s\\ &+ \int_0^t (K(t-s)-K(t))\mathcal{N}(P,X,\beta)(s)\mathrm{d}s\,.\end{aligned}$$

The first term still does not have the required decay. We cancel the leading term of it by regrouping the terms and using the explicit form of $f(t)$ to obtain

$$\begin{aligned}P_t(1-K(t)) &= K(t)\mathrm{Re}\,\langle \partial_{x_1}\widetilde{W}\,,(-\mathrm{i}h)^{-1}\int_0^t [\mathrm{e}^{-\mathrm{i}h(t-s)}-\mathrm{e}^{-\mathrm{i}ht}]\partial_{x_1}(h)^{-1}WP_s\mathrm{d}s\rangle\\ &+\int_0^t (K(t-s)-K(t))f(s)\int_s^t P_{s_1}\mathrm{d}s_1\mathrm{d}s\\ &-\int_0^t K(t-s)f(s)\mathrm{d}s\int_0^t P_{s_1}\mathrm{d}s_1+\int_0^t (K(t-s)-K(t))\mathcal{N}(P,X,\beta)(s)\mathrm{d}s\,.\end{aligned}$$

Remark. The third term of the right hand side has good decay, as seen by

$$\int_0^t K(t-s)f(s)\mathrm{d}s=\int_0^t K(s)f(t-s)\mathrm{d}s=\dot{K}(t)$$

We know that $K(t)=O(t^{-\frac{1}{2}})$ as $t\to\infty$, so we may hope that $\dot{K}(t)=O(t^{-\frac{3}{2}})$ as $t\to\infty$, and this will indeed turn out to be the case.

We make the self-consistent assumption that $P_t=O(t^{-\frac{1}{2}-\delta})$ for some $\delta>\frac{1}{2}$ and find that all terms on the right hand side have the appropriate decay. To invoke a contraction principle we would like to divide by $1-K(t)$ to get an integral equation for P_t. Since $K(t)\to 1$ as $t\to 0$ we need to be careful for small values of t. However, since we know that $K(t)$ is eventually small we can adopt the strategy of "waiting for long enough". We divide the time axis $[0,\infty)$ into two parts, a finite one $[0,T]$, and an infinite one $[T,\infty)$, such that $K(t)\ll 1$ for $t\geq T$. In the finite part, standard local existence proofs provide a solution P_t, $t\in[0,T]$, and in the infinite one we have—after division by $1-K(t)$—an integral equation for $(P_t)_{T\leq t\leq\infty}$ with an inhomogeneous part depending on $(P_t)_{0\leq t\leq T}$,

$$P_t=Y(\chi_{[T,\infty)}P_\cdot)(t)+G(\chi_{[0,T)}P_\cdot)(t)\,,$$

which is amenable to a contraction principle. For this purpose, introduce the family of Banach spaces

$$B_{\delta,T}:=\{f:t^{\frac{1}{2}+\delta}f\in L^\infty[T,\infty)\}$$

with norm

$$\|f\|_{\delta,T}:=\|t^{\frac{1}{2}+\delta}f\|_\infty\,.$$

The main technical task is now to show that, for T large enough, first the inhomogeneous term $G(t)$ is small in $B_{\delta,T}$, and second that $Y(\cdot):B_{\delta,T}\to B_{\delta,T}$ is a contraction. The contraction lemma then guarantees the existence of a solution in $B_{\delta,T}$, which is the claim.

To prove these estimates we use the following asymptotic expansion of the propagator,

$$\mathrm{e}^{-\mathrm{i}ht}=Ct^{-\frac{3}{2}}B_1+O(t^{-\frac{5}{2}})\,,$$

valid in the topology of $\mathcal{B}(L^{2,3},L^{2,-3})$, where $L^{2,s}$ denotes a weighted L^2 space,

$$L^{2,s}=\{f:\|\langle x\rangle^s f\|_2<\infty\}\,.$$

This expansion is correct if $h=-\Delta+gW$ has no eigenvalues and no zero resonance, which is the case for our choice of W and g small enough. The operator B_1 is given by

$$B_1(\cdot)=\langle\cdot\,,(1+(-\Delta)^{-1}gW)^{-1}1\rangle(1+(-\Delta)^{-1}gW)^{-1}1\,,$$

so that it is easy to see that in a term of the form

$$\langle\partial_x W\,,\mathrm{e}^{-\mathrm{i}ht}\ldots\rangle\,,$$

the leading term is in effect of order $t^{-\frac{5}{2}}$ for spherically symmetric W.

To facilitate later discussions we rescale the equation such that

$$2m=1,\ |\hat{W}(0)|=1.$$

2. Main theorem

Recall from Section 1 the continuous, monotonically increasing function $\Omega : (-\infty, 1) \to \mathbb{R}^+$,

$$\Omega(\delta) := \frac{1}{\pi}\int_0^1 \frac{1}{1+(1-r)^{\frac{1}{2}}}(1-r)^{-\frac{1}{2}}\left(\frac{1}{1-2\delta}(r^{-\frac{1}{2}}-r^{-\delta})+r^{\frac{1}{2}-\delta}\right)\mathrm{d}r\,,$$

and recall that we have denoted by δ^* the value at which $\Omega = 1$,

$$\Omega(\delta^*) = 1\,.$$

We have

$$\delta^* \simeq 0.66\,.$$

For the system of equations (5.1) we prove the following main theorem,

THEOREM 5.3. *Suppose that in* (5.1) *the external potential vanishes,* $V = 0$, *and the potential* W *is smooth, spherically symmetric, decays rapidly at* $|x| = \infty$, *and satisfies*

$$|\widehat{W}(0)| \neq 0\,.$$

Then, for any $\delta \in I := (\frac{1}{2}, \delta^*)$ *there exists a* $g_0 > 0$ *and an* $\varepsilon_0 > 0$ *such that if* $0 \le g \le g_0$ *and* $\|\langle x\rangle^3\beta_0\|_2, |P_0| \le \varepsilon_0$ *and* $\|\langle x\rangle^3\partial_x\beta_0\|_2 < \infty$ *then*

$$|P_t| \le ct^{-\frac{1}{2}-\delta} \text{as } t \to \infty, \tag{5.3}$$

and

$$\lim_{t\to\infty}\|\beta_t + \sqrt{\rho_0}(h^{X_t})^{-1}W^{X_t}\|_\infty = 0\,. \tag{5.4}$$

In particular, the particle comes to rest after a finite distance: There is a $X_\infty \in \mathbb{R}^3$ *such that* $X_t \to X_\infty$, *and*

$$\beta_t \to -\sqrt{\rho_0}(h^{X_\infty})^{-1}W^{X_\infty} \notin L^2(\mathbb{R}^3)\,.$$

The theorem will be proved in section 5.

Now we present the main difficulties in the proof and the strategies of overcoming them. Similar to what was proved in [**24**], we start with decomposing the equation for $\dot{P}_t$ into a linear and a non-linear part. The linear equations can be solved explicitly, and we use the solution to rewrite the equation for P_t in terms of this solution and the non-linear part. Since we expect that the momentum P_t decays for large times t it is reasonable to assume that eventually the dynamics is dominated by the linear part. The detailed knowledge of the decay properties of the solution to the linear part and standard dispersive estimates enable us to use a contraction principle to establish the claim.

There is one major technical difference to the model studied in [**24**], namely that the generator of time evolution, $h^{X_t} = -\Delta + gW^{X_t}$, depends on time through the position X_t of the particle. Mathematically, this makes it more involved to cancel various terms by symmetry considerations, and, as additional complication, the generator of translations, ∂_x, no longer commutes with h^{X_t}. We treat this as follows. Since we expect that the particle will come to rest at some $X_\infty \in \mathbb{R}^3$, we expand the propagator $U(t,s)$ gererated by $h^{X_t} = -\Delta + gW^{X_t}$ around the "instantaneous" propagator $\mathrm{e}^{-\mathrm{i}h^{X_t}t}$, at some large time t where

$$\mathrm{e}^{-\mathrm{i}h^{X_t}t} = \mathrm{e}^{-\mathrm{i}h^{X_T}t}\big|_{t=T}$$

is to be understood. By Duhamel's principle we obtain

$$U(t,0) = \mathrm{e}^{-\mathrm{i}h^{X_t}t} - \mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}h^{X_t}(t-s)}(X_s - X_t)\cdot\partial_x W^{X_t}\mathrm{e}^{-\mathrm{i}h^{X_t}s}\mathrm{d}s + \dots\,.$$

To facilitate later discussions we rescale the equation such that

$$2m = 1,\ |\widehat{W}(0)| = 1.$$

3. Reformulation of the problem and the local wellposedness

Recall the general plan outlined above of decomposing the equation for $\dot{P}_t$ into linear and non-linear parts. In the following, we carry out this plan albeit in a slightly more elaborate form: We first decompose the fluctuation field β_t into a main part and a remainder, the $\|\cdot\|_\infty$-norm of which will go to 0 as $t \to \infty$.

Since the generator h^{X_t} depends on the position X_t of the particle, we expand it around its value at a position X_T for some large time T. Define $\bar{\beta}^X := -\sqrt{\rho_0}(h^X)^{-1}W^X$ and introduce a new function δ_t by

$$\beta_t =: \bar{\beta}^{X_T} + \sqrt{\rho_0} \sum_{|\alpha|=1}^{N_0} \frac{1}{\alpha!}(X_t - X_T)^\alpha \partial_x^\alpha (h^{X_T})^{-1} W^{X_T} + \delta_t \,. \tag{5.5}$$

Then δ_t satisfies the equation

$$\begin{aligned} \mathrm{i}\dot{\delta}_t &= h^{X_T}\delta_t + g(W^{X_t} - W^{X_T})\delta_t - \mathrm{i}\frac{\sqrt{\rho_0}}{M} P_t \cdot \sum_{|\alpha|=1}^{N_0} \frac{1}{\alpha!}\alpha(X_t - X_T)^{\alpha-1}\partial_x^\alpha (h^{X_T})^{-1} W^{X_T} - G_1 \\ \delta_0 &= \beta_0 - \bar{\beta}^{X_T} - \sqrt{\rho_0}\sum_{|\alpha|=1}^{N_0} \frac{1}{\alpha!}(X_0 - X_T)^\alpha \partial_x^\alpha (h^{X_T})^{-1} W^{X_T} \,, \end{aligned} \tag{5.6}$$

where $\alpha X^{\alpha-1}$ means the vector $X = (\alpha_1 X^{(\alpha_1-1,\alpha_2,\alpha_3)}, \alpha_2 X^{(\alpha_1,\alpha_2-1,\alpha_3)}, \alpha_3 X^{(\alpha_1,\alpha_2,\alpha_3-1)})$, and the term G_1 is defined as

$$G_1 := h^{X_t} r_{N_0} \,,$$

with r_N defined by

$$\bar{\beta}^{X_t} =: \bar{\beta}^{X_T} + \sqrt{\rho_0}\sum_{|\alpha|\le N} \frac{1}{\alpha!}(X_t - X_T)^\alpha \partial_x^\alpha (h^{X_T})^{-1} W^{X_T} + r_N \,,$$

and estimated in the following lemma,

LEMMA 5.4. *For any* $N \in \mathbb{N}$

$$\bar{\beta}^{X_t} = \bar{\beta}^{X_T} + \sqrt{\rho_0}\sum_{|\alpha|\le N} \frac{1}{\alpha!}(X_t - X_T)^\alpha \partial_x^\alpha (h^{X_T})^{-1} W^{X_T} + r_N \,, \tag{5.7}$$

where $\|\langle x\rangle^3 h^{X_t} r_N\|_2 \le C_N |X_t - X_T|^{N+1}$.

PROOF. The claim follows immediately by Taylor-expanding the function $\bar{\beta}_t$ around $\bar{\beta}_T$ in the vector-variable $X_t - X_T$. To control the remainder we used the fact that $(h^X)^{-1}$ is a bounded operator from $L^{2,3}$ to $L^{2,-3}$, and the exponential decay of W. □

We take the first N_0 terms in the expansion of $\bar{\beta}^{X_t}$, where $N_0 := \min\{n \in \mathbb{N} : (n+1)(\delta - \frac{1}{2}) \ge \frac{3}{2}\}$, because for $T \ge t$, $|X_t - X_T| = O(t^{\frac{1}{2}-\delta})$, hence $|X_t - X_T|^{N_0+1} = O(t^{-\frac{3}{2}})$.

Using Duhamel's principle we can rewrite δ_t in the form

$$\begin{aligned} \delta_t \;=\; & \mathrm{e}^{-\mathrm{i}h^{X_T}t}\delta_0 - \mathrm{i}g\int_0^t \mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}[W^{X_s} - W^{X_T}]\delta_s \mathrm{d}s \\ & - \frac{\sqrt{\rho_0}}{M}\sum_{|\alpha|=1}^{N_0}\frac{1}{\alpha!}\int_0^t \mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}\partial_x^\alpha (h^{X_T})^{-1}W^{X_T} P_s \alpha(X_s - X_T)^{\alpha-1}\mathrm{d}s + \mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)} G_1(s)\mathrm{d}s \end{aligned} \tag{5.8}$$

The function δ_t admits the following estimate: Define an estimating function $\mu:\ \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\mu(t) := \max_{0\le s\le t}(1+s)^{\frac{1}{2}+\delta}|P_s| \,. \tag{5.9}$$

PROPOSITION 5.5. *If $\mu(T) \le 1$ then for any $\tau \le T$ we have*

$$\|\langle x\rangle^{-3}\delta_\tau\|_2 \lesssim (1+\tau)^{-\frac{1}{2}}\,. \tag{5.10}$$

The proposition will be proved in Section 6.

In what follows we derive an equation for $\dot{P}_t$. To this end, we rewrite equation (5.8) for δ_t as

$$\begin{aligned}
\delta_t \quad = \quad & \mathrm{e}^{-\mathrm{i}h^{X_T}t}\sqrt{\rho_0}(X_0-X_T)\cdot\partial_x(h^{X_T})^{-1}W^{X_T} - \frac{\sqrt{\rho_0}}{M}\int_0^t \mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}P_s\cdot\partial_x(h^{X_T})^{-1}W^{X_T}\mathrm{d}s \\
& + \mathrm{e}^{-\mathrm{i}h^{X_T}t}(\beta_0-\bar\beta^{X_T}+\sqrt{\rho_0}\sum_{|\alpha|=2}^{N_0}\frac{1}{\alpha!}(X_0-X_T)^\alpha\partial_x^\alpha(h^{X_T})^{-1}W^{X_T}) \\
& - \mathrm{i}g\int_0^t \mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}[W^{X_s}-W^{X_T}]\delta_s\mathrm{d}s \\
& - \frac{\sqrt{\rho_0}}{M}\sum_{|\alpha|=2}^{N_0}\frac{1}{\alpha!}\int_0^t \mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}\partial_x^\alpha(h^{X_T})^{-1}W^{X_T}P_s\alpha(X_s-X_T)^{\alpha-1}\mathrm{d}s + \mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}G_1(s)\mathrm{d}s
\end{aligned}\tag{5.11}$$

$$=: \quad \sum_{n=1}^{6} D_n(t)\,,$$

where D_1 and D_2 will be the main terms (being linear in P_t) in the equation for $\dot{P}_t$, whereas D_3 through D_6 will constitute remainder terms.

Recalling (5.1) and using $\beta_T = \bar\beta^{X_T}+\delta_T$ we thus arrive at the following equation for $\dot{P}_t$, where we evaluate at $t=T$ to effect the cancelations due to spherical symmetry, which is only perfect when all centers agree:

$$\begin{aligned}
\dot{P}_t\big|_{t=T} \quad = \quad & -2\rho_0\mathrm{Re}\,\langle\partial_x W^{X_T}, \mathrm{e}^{-\mathrm{i}h^{X_T}T}(X_0-X_T)\cdot\partial_x(h^{X_T})^{-1}W^{X_T}\rangle \\
- \quad & 2g\sqrt{\rho_0}\mathrm{Re}\,\langle\bar\beta^{X_T}\partial_x W^{X_T}, \mathrm{e}^{-\mathrm{i}h^{X_T}T}(X_0-X_T)\cdot\partial_x(h^{X_T})^{-1}W^{X_T}\rangle \\
+ \quad & 2\frac{\rho_0}{M}\mathrm{Re}\,\langle\partial_x W^{X_T}, \int_0^T \mathrm{e}^{-\mathrm{i}h^{X_T}(T-s)}P_s\cdot\partial_x(h^{X_T})^{-1}W^{X_T}\mathrm{d}s\rangle \\
+ \quad & 2g\frac{\sqrt{\rho_0}}{M}\mathrm{Re}\,\langle\bar\beta^{X_T}\partial_x W^{X_T}, \int_0^T \mathrm{e}^{-\mathrm{i}h^{X_T}(T-s)}P_s\cdot\partial_x(h^{X_T})^{-1}W^{X_T}\mathrm{d}s\rangle \\
+ \quad & R(P,T)\,,
\end{aligned}$$

with $R(P,T)$ defined as

$$R(P,T) = -2\sqrt{\rho_0}\langle(1+\frac{g}{\sqrt{\rho_0}}\bar\beta^{X_T})\partial_x W^{X_T}, \sum_{n=3}^{6}D_n\rangle - g\langle\partial_x W^{X_T}, |\delta_T|^2\rangle\,. \tag{5.12}$$

By shifting the center of integration and using the spherical symmetry of W the above equation is equivalent to $(k=1,2,3)$

$$\begin{aligned}
\dot{P}_T^{(k)} \quad = \quad & -2\rho_0\mathrm{Re}\,\langle(1+\frac{g}{\sqrt{\rho_0}}\bar\beta)\partial_{x_k}W, \mathrm{e}^{-\mathrm{i}hT}(X_0-X_T)_k\partial_{x_k}(h)^{-1}W\rangle \\
+ \quad & 2\frac{\rho_0}{M}\mathrm{Re}\,\langle(1+\frac{g}{\sqrt{\rho_0}}\bar\beta)\partial_{x_k}W, \int_0^T \mathrm{e}^{-\mathrm{i}h(T-s)}P_s^{(k)}\partial_{x_k}(h)^{-1}W\mathrm{d}s\rangle \\
+ \quad & R(P,T)_k\,,
\end{aligned}$$

or

$$\dot{P}_T = L(P)(T) + R(P,T)\,, \tag{5.13}$$

where

$$L(P) := \begin{pmatrix} L(P^{(1)}) \\ L(P^{(2)}) \\ L(P^{(3)}) \end{pmatrix}.$$

Remark: From now on, we will write t for T for esthetic reasons.

The local well-posedness of this equation is standard, as summarized in

THEOREM 5.6. *The equation (5.13) is locally well-posed: If $P_0 \in \mathbb{R}^3$ and $\langle x\rangle^3\beta_0 \in L^2(\mathbb{R}^3)$ then there exists a time $T_{\rm loc} = T_{\rm loc}(|P_0|, \|\langle x\rangle^3\beta_0\|_2)$ such that $|P_t| < \infty$ for any time $t \leq T_{\rm loc}$. Moreover, for any $T_{\rm loc} > 0$ there exists an $\varepsilon_0(T_{\rm loc})$ such that if $|P_0|, \|\langle x\rangle^3\beta_0\|_2 \leq \varepsilon_0(T_{\rm loc})$ then P satisfies the estimate*

$$|P_t| \leq T_{\rm loc}^{-2} \quad t \in [0, T_{\rm loc}]\,. \tag{5.14}$$

PROOF. The local well-posedness of (5.13) can be proved by standard techniques, hence we omit the details here. The second assertion follows considering that if $P_0 = 0$ and $\beta_0 = 0$ then $P_t = 0$ is a global solution. □

4. The existence of the solution in the infinite time interval

It is hard to derive a decay estimate for P_t directly from (5.13). In what follows we will rearrange terms until a fixed point theorem becomes applicable.

We will express the solution of the full equation (5.13) in terms of the solution $K(t)$ of one part of the linear equation,

$$\begin{aligned} \dot K(t) &= Z\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}K(s)\partial_{x_1}(h)^{-1}W\mathrm{d}s\rangle\,, \\ K(0) &= 1. \end{aligned} \tag{5.15}$$

Here the constant $Z \in \mathbb{R}^+$ is defined as

$$Z := 2\frac{\rho_0}{M}.$$

In Section 7 we prove the following lemma,

LEMMA 5.7. *Let $K(t)$ be a solution of equation (5.15) with $K(0) = 1$. Then there exists a real constant C such that*

$$ZK(t) = \frac{3}{\sqrt{2}}\pi^{-\frac{5}{2}}(1 + Cg)t^{-\frac{1}{2}} + O(t^{-1})\,. \tag{5.16}$$

With $K(t)$ at hand, we can write the Duhamel-like formula

$$P_t = K(t)P_0 + Z\int_0^t K(t-s)\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\int_0^s P_{s_1}\mathrm{d}s + \int_0^t K(t-s)R(P,s)\mathrm{d}s\,.$$

We now repeat the same steps as in [**24**] to arrive at

$$P_t = \frac{1}{1-K(t)}A(P)(t) + \frac{1}{1-K(t)}\int_0^t [K(t-s) - K(t)]R(P,s)\mathrm{d}s\,, \tag{5.17}$$

where the linear operator A is defined by

$$\begin{aligned} A(P)(t) = &-Z\int_0^t [K(t-s)-K(t)]\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\int_s^t P_{s_1}\mathrm{d}s_1\mathrm{d}s \\ &+ Z\int_0^t K(t-s)\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\mathrm{d}s\int_0^t P_{s_1}\mathrm{d}s_1 \\ &+ ZK(t)\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,(-\mathrm{i}h)^{-1}\int_0^t[\mathrm{e}^{-\mathrm{i}h(t-s)} - \mathrm{e}^{-\mathrm{i}ht}]P_s\mathrm{d}s\partial_{x_1}(h)^{-1}W\rangle\,. \end{aligned} \tag{5.18}$$

Since we plan to use a fixed point theorem, we introduce a family of suitable Banach spaces,

$$B_{\delta,T_{\rm loc}} := \{f : t^{\frac{1}{2}+\delta} f \in L^\infty[T_{\rm loc},\infty)\}$$

with norm

$$\|f\|_{\delta,T_{\rm loc}} := \|t^{\frac{1}{2}+\delta} f\|_\infty \,.$$

We divide the time interval $[0,\infty)$ into two parts $[0,T_{\rm loc})$ and $[T_{\rm loc},\infty)$. Introduce the Heaviside function $\chi_{T_{\rm loc}} := \mathbb{1}_{[0,T_{\rm loc})}$ and rewrite (5.17) as

$$P_t = \Upsilon((1-\chi_{T_{\rm loc}})P)(t) + G_t, \tag{5.19}$$

where

$$\Upsilon((1-\chi_{T_{\rm loc}})P)(t) := \frac{1}{1-K(t)} A((1-\chi_{T_{\rm loc}})P)(t) + \frac{1}{1-K(t)}\int_0^t [K(t-s)-K(t)][R(P,s)-R(\chi_{T_{\rm loc}}P,s)]{\rm d}s$$

$$G_t := \frac{1}{1-K(t)} A(\chi_{T_{\rm loc}}P)(t) + \frac{1}{1-K(t)}\int_0^t [K(t-s)-K(t)]R(\chi_{T_{\rm loc}}P,s){\rm d}s\,.$$

The following two propositions, proven in Sections 8 and 9, show that for $T_{\rm loc}$ large enough, $\Upsilon((1-\chi_{T_{\rm loc}})P)(t) : B_{\delta,T_{\rm loc}} \to B_{\delta,T_{\rm loc}}$ is a contraction, and G_t is small in $B_{\delta,T_{\rm loc}}$ if the initial conditions for P and β are small enough, which will allow us to prove the main theorem.

PROPOSITION 5.8. *There is an $M>0$ such that for $T_{\rm loc}\ge M$ the mapping $\Upsilon((1-\chi_{T_{\rm loc}})P)(t) : B_{\delta,T_{\rm loc}} \to B_{\delta,T_{\rm loc}}$ is a contraction, or more precisely:*

(1) *For any function $q\in B_{\delta,T_{\rm loc}}$*

$$t^{\frac{1}{2}+\delta}\Big|\frac{1}{1-K(t)}A((1-\chi_{T_{\rm loc}})q_t)\Big| \le [\frac{1}{\pi}\Omega(\delta)+\varepsilon(T_{\rm loc})]\|q_t\|_{\delta,T_{\rm loc}}\,,$$

where $\varepsilon(T_{\rm loc})\to 0$ as $T_{\rm loc}\to\infty$.

(2) *Recall that the solution P exists in the time interval $[0,T_{\rm loc}]$ according to Theorem 5.6. Suppose that $Q_1,Q_2:[0,\infty)\to\mathbb{R}^3$ are two functions satisfying*

$$Q_1(t)=Q_2(t)=P_t \quad \textit{for } t\in[0,T_{\rm loc}]\,,$$

and in the interval $[T_{\rm loc},\infty)$

$$\|Q_1\|_{\delta,T_{\rm loc}}, \|Q_2\|_{\delta,T_{\rm loc}} \ll 1\,.$$

Then,

$$t^{\frac{1}{2}+\delta}\Big|\frac{1}{1-K(t)}\int_0^t [K(s-t)-K(t)][R(Q_1,s)-R(Q_2,s)]{\rm d}s\Big| \lesssim (\|Q_1\|_{\delta,T_{\rm loc}}+\|Q_2\|_{\delta,T_{\rm loc}})\,\|Q_1-Q_2\|_{\delta,T_{\rm loc}}\,.$$

PROPOSITION 5.9. *Suppose that $T_{\rm loc}\ge M$ (see Proposition 5.8) and $|P_0|, \|\langle x\rangle^3\beta_0\|_2 \le \varepsilon_0(T_{\rm loc})$ (see Theorem 5.6). Then G_t is in the Banach space $B_{\delta,T_{\rm loc}}$, and its norm is small. Specifically, for any $t\ge T_{\rm loc}$*

$$\begin{array}{rcl} t^{\frac{1}{2}+\delta}\big|\frac{1}{1-K(t)}A(\chi_{T_{\rm loc}}P)(t)\big| & \le & \varepsilon(T_{\rm loc}) \\ t^{\frac{1}{2}+\delta}\big|\frac{1}{1-K(t)}\int_0^t [K(t-s)-K(t)]R(\chi_{T_{\rm loc}}P,s){\rm d}s\big| & \le & \varepsilon(T_{\rm loc})\,, \end{array}$$

with $\varepsilon(T_{\rm loc})\to 0$ as $T_{\rm loc}\to\infty$.

5. Proof of main theorem

As discussed before, we divide the time interval $[0,\infty)$ into two parts, $[0,T_{\mathrm{loc}}]$ and $[T_{\mathrm{loc}},\infty)$. The existence of the solution in the finite domain was proven in Theorem 5.6. For the infinite domain, Propositions 5.8 and 5.9 enable us to apply the contraction lemma on (5.19) to see that $P\in B_{\delta,T_{\mathrm{loc}}}$. By the definition of $B_{\delta,T_{\mathrm{loc}}}$ we have proven (5.3).

To prove (5.4), we estimate (5.20) in $\|\cdot\|_\infty$-norm and use standard $L^1\to L^\infty$-bounds [**65**],

$$\begin{aligned}\|\delta_t\|_\infty \quad &\leq \quad \|\mathrm{e}^{-\mathrm{i}h^{X_T}t}\delta_0\|_\infty + g\int_0^t \|\mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}[W^{X_s}-W^{X_T}]\delta_s\|_\infty \mathrm{d}s \\ &\quad + \frac{\sqrt{\rho_0}}{M}\sum_{|\alpha|=1}^{N_0}\frac{1}{\alpha!}\int_0^t \|\mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}\partial_x^\alpha (h^{X_T})^{-1}W^{X_T}P_s\alpha(X_s-X_T)^{\alpha-1}\|_\infty \mathrm{d}s \\ &\quad + \int_0^t \|\mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}G_1(s)\|_\infty \mathrm{d}s\,.\end{aligned}$$

To control the various terms forming the first term on the right hand side we use

$$\|\mathrm{e}^{\mathrm{i}ht}\partial_x^\alpha (h)^{-1}W\|_\infty = O(t^{-\frac{1}{2}})\,,$$

which is implied by the observation that

$$\partial_{x_i}(h)^{-1}W = (h)^{-1}\partial_{x_i}W - (h)^{-1}g\partial_{x_i}W(h)^{-1}W\,,$$

so that each term of $\mathrm{e}^{\mathrm{i}ht}\partial_x^\alpha (h)^{-1}W$ is of the form

$$\mathrm{e}^{\mathrm{i}ht}h^{-1}f\,,\quad f\in L^1\,,$$

and the estimate

$$\|e^{iht}h^{-1}\|_{L^1\to L^\infty} = O(t^{-\frac{1}{2}}).$$

For the second term on the right hand side use

$$\|\mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}[W^{X_s}-W^{X_T}]\|_\infty \leq (t-s)^{-\frac{3}{2}}\|W^{X_s}-W^{X_T}\|_1 \lesssim (t-s)^{-\frac{3}{2}}\,.$$

For the term containing G_1 use the Hölder inequality:

$$\|G_1(s)\|_1 = \|\langle x\rangle^3 G_1(s)\langle x\rangle^{-3}\|_1 \leq \|\langle x\rangle^3 G_1(s)\|_2\|\langle x\rangle^{-3}\|_2 = \frac{\pi}{2}\|\langle x\rangle^3 G_1(s)\|_2 \lesssim (1+s)^{-\frac{3}{2}}\,,$$

where we used the definition of $G_1 = h^X r_{N_0}$ and its estimation in Lemma 5.4.

To deal with the appearance of δ_s on the right hand side, introduce the function

$$\widetilde{Q}(t) := \max_{0\leq s\leq t} s^{\frac{1}{2}}\|\delta_s\|_\infty\,,$$

and use the above preparations to obtain

$$\begin{aligned}\|\delta_t\|_\infty \quad &\leq \quad \|\mathrm{e}^{-\mathrm{i}h^{X_T}t}\delta_0\|_\infty + g\int_0^t (t-s)^{-\frac{3}{2}}\|\delta_s\|_\infty \mathrm{d}s \\ &\quad + \frac{\sqrt{\rho_0}}{M}\sum_{|\alpha|=1}^{N_0}\frac{1}{\alpha!}\int_0^t \|\mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}\partial_x^\alpha (h^{X_T})^{-1}W^{X_T}\|_\infty (1+s)^{-\frac{1}{2}-\delta}\mathrm{d}s + \int_0^t \|\mathrm{e}^{-\mathrm{i}h^{X_T}(t-s)}G_1(s)\|_\infty \mathrm{d}s \\ &\lesssim \|\mathrm{e}^{-\mathrm{i}h^{X_T}t}\beta_0\|_\infty + \|\mathrm{e}^{-\mathrm{i}h^{X_T}t}\bar{\beta}^{X_T}\|_\infty + \sum_{|\alpha|=1}^{N_0}\|\mathrm{e}^{-\mathrm{i}h^{X_T}t}\frac{1}{\alpha!}(X_0-X_T)^\alpha\partial_x^\alpha (h^{X_T})^{-1}W^{X_T}\|_\infty \\ &\quad + g\int_0^t (t-s)^{-\frac{3}{2}}\|[W^{X_s}-W^{X_T}]\|_1\|\delta_s\|_\infty \mathrm{d}s + \int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}-\delta}\mathrm{d}s + \int_0^t (t-s)^{-\frac{3}{2}}\|G_1(s)\|_1 \mathrm{d}s \\ &\lesssim t^{-\frac{3}{2}}\|\beta_0\|_1 + t^{-\frac{1}{2}} + t^{-\frac{1}{2}} + g\widetilde{Q}(t)\int_0^t (t-s)^{-\frac{3}{2}}(1+s)^{-\frac{1}{2}}\mathrm{d}s + t^{-\frac{1}{2}} + \int_0^t (t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}}\mathrm{d}s \\ &\lesssim t^{-\frac{3}{2}}\varepsilon_0 + g\widetilde{Q}(t)t^{-\frac{1}{2}} + 3t^{-\frac{1}{2}} + t^{-\frac{3}{2}}\,.\end{aligned}$$

Combining the above estimates we obtain, for $t = T$,

$$T^{\frac{1}{2}} \|\delta_T\|_\infty \lesssim g\widetilde{Q}(T) + 1\,,$$

or

$$\widetilde{Q}(T) \lesssim g\widetilde{Q}(T) + 1\,,$$

and so since g is small

$$\widetilde{Q}(T) \lesssim 1\,.$$

Since $\beta_T = -\sqrt{\rho_0}(h^{X_T})^{-1}W^{X_T} + \delta_T$, this proves (5.4).

The proof of the main theorem is finished. □

6. Proof of Proposition 5.5

For any time $\tau \leq T$ we apply Duhamel's principle to rewrite (5.6) as

$$\begin{aligned}
\delta_\tau \quad &= \quad \mathrm{e}^{-\mathrm{i}h^{X_T}\tau}\delta_0 - \mathrm{i}g\int_0^\tau \mathrm{e}^{-\mathrm{i}h^{X_T}(\tau-s)}[W^{X_s} - W^{X_T}]\delta_s \mathrm{d}s \\
&\quad - \frac{\sqrt{\rho_0}}{M}\sum_{|\alpha|=1}^{N_0}\frac{1}{\alpha!}\int_0^\tau \mathrm{e}^{-\mathrm{i}h^{X_T}(\tau-s)}\partial_x^\alpha (h^{X_T})^{-1}W^{X_T}P_s\alpha(X_s - X_T)^{\alpha-1}\mathrm{d}s + \mathrm{i}\int_0^\tau \mathrm{e}^{-\mathrm{i}h^{X_T}(\tau-s)}G_1(s)\mathrm{d}s
\end{aligned} \tag{5.20}$$

$$=: \quad \sum_{n=1}^{4} B_n\,.$$

Now we estimate each term on the right hand side of (5.20). Recall the definition of $\mu(T)$ in (5.9) and the assumption $\mu(T) \leq 1$. By the definition of δ_0 and the propagator estimates of Proposition 5.15 we have

$$\begin{aligned}
\|\langle x\rangle^{-3}B_1\|_2 \quad &= \quad \|\langle x\rangle^{-3}\mathrm{e}^{-\mathrm{i}h^{X_T}\tau}[\beta_0 - \bar{\beta}^{X_T} - \sqrt{\rho_0}\sum_{|\alpha|=1}^{N_0}\frac{1}{\alpha!}(X_0 - X_T)^\alpha \partial_x^\alpha (h^{X_T})^{-1}W^{X_T}]\|_2 \\
&\leq \quad \|\langle x\rangle^{-3}\mathrm{e}^{-\mathrm{i}h^{X_T}\tau}\beta_0\|_2 + \|\langle x\rangle^{-3}\mathrm{e}^{-\mathrm{i}h^{X_T}\tau}\bar{\beta}^{X_T}\|_2 \\
&\quad + \sqrt{\rho_0}\sum_{|\alpha|=1}^{N_0}\frac{1}{\alpha!}|X_0 - X_T|^\alpha \|\langle x\rangle^{-3}\mathrm{e}^{-\mathrm{i}h^{X_T}\tau}\partial_x^\alpha (h^{X_T})^{-1}W^{X_T}\|_2 \\
&\lesssim \quad (1+\tau)^{-\frac{3}{2}}\|\langle x\rangle^3\beta_0\|_2 + (1+\tau)^{-\frac{1}{2}} + (1+\tau)^{-\frac{3}{2}}\mu(T) \\
&\leq \quad (1+\tau)^{-\frac{1}{2}}[1 + \varepsilon_0 + \mu(T)]\,,
\end{aligned}$$

where in the third line we used the fact

$$|X_0 - X_T| \leq \int_0^T |P_s|\mathrm{d}s \lesssim \mu(T)\,.$$

For the last line we recall the overarching hypothesis of Theorem 5.3 $\|\langle x\rangle^3\beta_0\|_2 \leq \varepsilon_0$.

For B_3 we have

$$\begin{aligned}
\|\langle x\rangle^{-3}B_3\|_2 \quad &\lesssim \quad \mu(T)\int_0^\tau (1+\tau-s)^{-\frac{3}{2}}(1+s)^{-\frac{1}{2}-\delta}\mathrm{d}s \\
&\lesssim \quad \mu(T)(1+\tau)^{-\frac{1}{2}-\delta}\,;
\end{aligned}$$

recall that we only consider $\delta \in (\frac{1}{2}, \delta^*)$ and $\delta^* < 1$. Similarly for B_4

$$\begin{aligned}\|\langle x\rangle^{-3} B_4\|_2 &\lesssim \mu^{N_0+1}(T)\int_0^\tau (1+\tau-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}}\mathrm{d}s \\ &\lesssim \mu^{N_0+1}(T)(1+\tau)^{-\frac{3}{2}}\,.\end{aligned}$$

Since B_2 depends on δ_τ, we have to proceed in a different way. Define the function Q by

$$Q(\tau) := \max_{0\le s\le\tau\le T}(1+s)^{\frac{1}{2}}\|\langle x\rangle^{-3}\delta_s\|_2\,.$$

Then B_2 admits the estimate

$$\begin{aligned}\|\langle x\rangle^{-3} B_2\|_2 &\lesssim g\int_0^\tau (1+\tau-s)^{-\frac{3}{2}}\|\langle x\rangle^{3}(W^{X_\tau}-W^{X_s})\delta_s\|_2\mathrm{d}s \\ &\lesssim gQ(\tau)\int_0^\tau (1+\tau-s)^{-\frac{3}{2}}(1+s)^{-\frac{1}{2}}\mathrm{d}s \\ &\lesssim gQ(\tau)(1+\tau)^{-\frac{1}{2}}\,.\end{aligned}$$

In the first line, we use the fact

$$|\langle x\rangle^{3} W^{X.}| \lesssim \langle x\rangle^{-3}\,,$$

which holds since $|X.|$ is bounded.

Collecting the estimates above we find

$$(1+\tau)^{\frac{1}{2}}\|\langle x\rangle^{-3}\delta_\tau\| \lesssim gQ(\tau)+1+\varepsilon_0+\mu(T)\,,$$

which by definition of $Q(\tau)$ yields for any $0\le\tau\le T$

$$Q(\tau) \lesssim gQ(\tau)+1+\varepsilon_0+\mu(T)\,.$$

As g is small we obtain

$$Q(\tau) \lesssim 1+\varepsilon_0+\mu(T) \lesssim 1\,,$$

which is the claim.

7. Proof of Lemma 5.7

We follow the strategy of [**24**]. Define $Z := \frac{2\rho_0}{M}$ and a function $G:\mathbb{R}\to\mathbb{C}$ by

$$\begin{aligned}G(k+\mathrm{i}0) :=& \frac{\mathrm{i}}{2}\langle (h+k+\mathrm{i}0)^{-1}\partial_{x_1}(h)^{-1}W\,,[1-g(h)^{-1}W]\partial_{x_1}W\rangle \\ &-\frac{\mathrm{i}}{2}\langle [1-g(h)^{-1}W]\partial_{x_1}W\,,(h-k-\mathrm{i}0)^{-1}\partial_{x_1}(h)^{-1}W\rangle\end{aligned} \tag{5.21}$$

Now, we relate G to the solution K:

Lemma 5.10. *The solution K of (5.15) takes the form*

$$K(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{Re}\,\frac{1}{\mathrm{i}k+ZG(k+\mathrm{i}0)}\cos kt\mathrm{d}k.$$

In particular,

$$K(t)=0 \qquad \textit{for } t<0.$$

The proof of Lemma 5.10 is done as in [**24**] and is not repeated here. With this explicit expression for K, we can prove Lemma 5.7 with the help of the following lemma,

Lemma 5.11. *The function $G(k+\mathrm{i}0)$ satisfies*

$$G(k+\mathrm{i}0)=\begin{cases}(\mathrm{i}-1)\frac{\pi^2}{3}(1+O(g))k^{\frac{1}{2}}+C_1k+O(k^{\frac{3}{2}}) & \text{if } k>0\\ (-\mathrm{i}-1)\frac{\pi^2}{3}(1+O(g))|k|^{\frac{1}{2}}+C_2k+O(|k|^{\frac{3}{2}}) & \text{if } k<0\end{cases},$$

with C_1, C_2 being some constants.

Lemma 5.11 is proven at the end of this section.

Proof of Lemma 5.7. Decompose $K(t)$ into two parts,

$$K(t)=K_+(t)+K_-(t)\,,$$

with

$$K_+(t):=-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\,\frac{1}{\mathrm{i}k+ZG(k+\mathrm{i}0)}\cos kt\mathrm{d}k$$

and

$$K_-(t):=-\frac{1}{\pi}\int_{-\infty}^0 \mathrm{Re}\,\frac{1}{\mathrm{i}k+ZG(k+\mathrm{i}0)}\cos kt\mathrm{d}k$$

Define a new function $g:\mathbb{R}^+\to\mathbb{R}$ by

$$\begin{aligned}|k|^{-\frac{1}{2}}g(|k|^{\frac{1}{2}}):&=-\frac{1}{\pi}\mathrm{Re}\,\frac{1}{\mathrm{i}k+ZG(k+\mathrm{i}0)}\\ &=-\frac{1}{Z\pi}\frac{\mathrm{Re}\,G}{(\frac{k}{Z}+\mathrm{Im}\,G)^2+(\mathrm{Re}\,G)^2}\\ &=\frac{3(1+O(g))}{2\pi^3Z}|k|^{-\frac{1}{2}}(1+O(k^{\frac{1}{2}}))\,,\end{aligned}$$

where we used the explicit form of $G(k+\mathrm{i}0)$ of Lemma 5.11. By construction, the function g is smooth on $[0,\infty)$ and satisfies (because $G(k)$ is bounded as $k\to\infty$, see [**49**])

$$|g(\rho)|\le C(1+\rho)^{-3}\,.$$

We can now directly compute as in [**24**]

$$\begin{aligned}K_+(t)&=\int_0^\infty k^{-\frac{1}{2}}g(k^{\frac{1}{2}})\cos kt\mathrm{d}k\\ &=2\int_0^\infty g(\rho)\cos(\rho^2t)\mathrm{d}\rho\\ &=2g(0)\int_0^\infty\cos(\rho^2t)+D\end{aligned}$$

with D defined as

$$D:=2\int_0^\infty[g(\rho)-g(0)]\cos(\rho^2t)\mathrm{d}\rho\,.$$

The first term on the right hand side is the dominant one:

$$2g(0)\int_0^\infty\cos(\rho^2t)\mathrm{d}\rho=2g(0)t^{-\frac{1}{2}}\int_0^\infty\cos x^2\mathrm{d}x=\frac{3(1+O(g))}{2\sqrt{2}Z}\pi^{-\frac{5}{2}}t^{-\frac{1}{2}}\,,$$

where we used the Fresnel integral $\int_0^\infty\cos x^2\mathrm{d}x=(\pi/8)^{\frac{1}{2}}$.

We prove now that D is a correction of order $t^{-\frac{3}{2}}$. This implies

$$K_+=\frac{3(1+O(g))}{2\sqrt{2}Z}\pi^{-\frac{5}{2}}t^{-\frac{1}{2}}+O(t^{-1})\,.$$

Since we find by completely analogous computation

$$K_- = \frac{3(1+O(g))}{2\sqrt{2}Z}\pi^{-\frac{5}{2}}t^{-\frac{1}{2}} + O(t^{-1})$$

the claim follows.

To estimate D first integrate by parts:

$$\begin{aligned}|D| &= t^{-1}|\int_0^\infty \rho^{-1}[g(\rho)-g(0)]\partial_\rho \sin(\rho^2 t)\mathrm{d}\rho| \\ &= t^{-1}|\int_0^\infty H(\rho)\sin(\rho^2 t)\mathrm{d}\rho|\end{aligned}$$

with $H(\rho) := \partial_\rho \rho^{-1}[g(\rho)-g(0)]$ a smooth function satisfying $|H(\rho)| \lesssim (1+\rho)^{-2}$. Write $H(\rho) = H(0) + \rho[\rho^{-1}(H(\rho)-H(0))]$ and perform again integration by parts to obtain

$$|D| = t^{-1}|H(0)||\int_0^\infty \sin(\rho^2 t)\mathrm{d}\rho| + \frac{1}{2}t^{-2}\lim_{\rho\to 0}\frac{|H(\rho)-H(0)|}{\rho} + \frac{1}{2}t^{-2}|\int_0^\infty \partial_\rho[\rho^{-1}(H(\rho)-H(0))]|\mathrm{d}\rho\,.$$

The first term on the right hand side can be computed explicitely,

$$t^{-1}|H(0)||\int_0^\infty \sin(\rho^2 t)\mathrm{d}\rho| = t^{-\frac{3}{2}}|H(0)|\sqrt{\frac{\pi}{8}}\,,$$

and the second term is obviously of order t^{-2}. The last term is controlled by

$$t^{-2}\int_0^\infty (1+\rho)^{-2}\mathrm{d}\rho \lesssim t^{-2}$$

by the fact that $|\partial_\rho[\rho^{-1}(H(\rho)-H(0))]| \lesssim (1+\rho)^{-2}$. $\square$

Proof of Lemma 5.11. We start by studying the function $G(k),\ k\in\mathbb{C}\backslash\mathbb{R}^+$ whose limit on the real axis is $G(k+i0),\ k\in\mathbb{R}$. Define a variable ζ by $\zeta := k^{\frac{1}{2}}$, where k is in the domain $\mathbb{C}\backslash\mathbb{R}^+$, and $k^{\frac{1}{2}} = +k^{\frac{1}{2}} > 0$ for $k>0$. By standard theory we know that $G(k),\ k\in\mathbb{C}\backslash\mathbb{R}^+$ is analytic in ζ.

The claim follows by expanding $(h+k)^{-1}$ around h^{-1}. By classical results, see for example [**34**], if the constant $|g|$ in $h=-\Delta+gW$ is sufficiently small and W decays sufficiently fast at ∞, then h has no zero-resonance or eigenvectors. This together with the discussions above and results in [**34**] imply that

$$(h+k)^{-1} = B_0 + \zeta B_1 + \zeta^2 B_2 + O(\zeta^3)\,, \tag{5.22}$$

in the topology of $\mathcal{B}(L^{2,3},L^{2,-3})$, B_i being operators in $\mathcal{B}(L^{2,3},L^{2,-3})$, namely

$$\begin{aligned}B_0 &= (1+(-\Delta)^{-1}gW)^{-1}(-\Delta)^{-1} \\ B_1 &= \frac{1}{4\pi}\langle\cdot,(1+(-\Delta)^{-1}gW)1\rangle(1+(-\Delta)^{-1}gW)1\,.\end{aligned}$$

We cannot apply this expansion directly because $\partial_{x_1}h^{-1}W \notin L^{2,3}$. So first, observe that $1-gh^{-1}W$ is bounded and define for brevity

$$\partial_{x_1}W_1 := [1-g(h)^{-1}W]\partial_{x_1}W\,,$$

so that W_1 is rapidly decaying. Then use the second resolvent identity to rewrite the terms of $G(k)$,

$$\frac{\mathrm{i}}{2}\langle[(-\Delta \pm k+\mathrm{i}0)^{-1} - (h \pm k + \mathrm{i}0)^{-1} gW(-\Delta \pm k + \mathrm{i}0)^{-1}]\partial_{x_1}[(-\Delta)^{-1} - (-\Delta)^{-1} gWh^{-1}]W\,,\partial_{x_1}W_1\rangle$$

$$= \frac{\mathrm{i}}{2}\langle(-\Delta \pm k+\mathrm{i}0)^{-1}\partial_{x_1}(-\Delta)^{-1}W\,,\partial_{x_1}W_1\rangle \tag{5.23}$$

$$- \frac{\mathrm{i}}{2}\langle(-\Delta \pm k+\mathrm{i}0)^{-1}\partial_{x_1}(-\Delta)^{-1}gWh^{-1}W\,,\partial_{x_1}W_1\rangle \tag{5.24}$$

$$+ \frac{\mathrm{i}}{2}\langle(h \pm k+\mathrm{i}0)^{-1}gW(-\Delta \pm k+\mathrm{i}0)^{-1}\partial_{x_1}(-\Delta)^{-1}W\,,\partial_{x_1}W_1\rangle \tag{5.25}$$

$$- \frac{\mathrm{i}}{2}\langle(h \pm k+\mathrm{i}0)^{-1}gW(-\Delta \pm k+\mathrm{i}0)^{-1}\partial_{x_1}(-\Delta)^{-1}gWh^{-1}W\,,\partial_{x_1}W_1\rangle\,. \tag{5.26}$$

The term (5.23) is rewritten as

$$\frac{\mathrm{i}}{2}\langle(-\Delta \pm k+\mathrm{i}0)^{-1}\partial_{x_1}(-\Delta)^{-1}W\,,\partial_{x_1}W_1\rangle = \frac{\mathrm{i}}{6}\langle(-\Delta \pm k+\mathrm{i}0)^{-1}W\,,W_1\rangle\,,$$

for which we can now use (5.22). The constant term in the expansion vanishes in the difference (5.21). For the $k^{1/2}$-term we get (consider first $k>0$)

$$\begin{aligned}&\frac{1}{24\pi}k^{1/2}(\mathrm{i}-1)\,\langle W\,,(1+(-\Delta)^{-1}gW\rangle\langle(1+(-\Delta)^{-1}gW\,,W_1\rangle\\ =&\frac{1}{24\pi}k^{1/2}(\mathrm{i}-1)\big[\langle W\,,1\rangle\langle 1\,,W\rangle + O(g)\big]\,,\end{aligned}$$

where we used $W_1 = W + O(g)$ in $\|\cdot\|_\infty$. Using $\langle W\,,1\rangle = \langle 1\,,W\rangle = (2\pi)^{\frac{3}{2}}\widehat{W}(0)$ the last line equals

$$\frac{\pi^2}{3}k^{1/2}(\mathrm{i}-1)(1+O(g))\,.$$

The term (5.24) is treated analogously and gives a contribution of order $k^{\frac{1}{2}}O(g)$.

The term (5.25) is rewritten as

$$\begin{aligned}&\frac{\mathrm{i}}{2}\langle(h \pm k+\mathrm{i}0)^{-1}gW(-\Delta \pm k+\mathrm{i}0)^{-1}\partial_{x_1}(-\Delta)^{-1}W\,,\partial_{x_1}W_1\rangle\\ =&g\frac{\mathrm{i}}{2}\langle(-\Delta \pm k+\mathrm{i}0)^{-1}\partial_{x_1}(-\Delta)^{-1}W\,,W(h \pm k-\mathrm{i}0)^{-1}\partial_{x_1}W_1\rangle\,.\end{aligned} \tag{5.27}$$

On the right hand side of the scalar product, we can use the expansion for $(h+k-\mathrm{i}0)$, as $\partial_{x_1}W_1$ is in the appropriate weighted L^2-space:

$$W(h+k-\mathrm{i}0)^{-1}\partial_{x_1}W_1 = WB_0(\partial_{x_1}W_1) + k^{\frac{1}{2}}WB_1(\partial_{x_1}W_1) + \dots \tag{5.28}$$

Note that each factor in the expansion is a rapidly decaying function of x. We consider thus

$$\begin{aligned}&\langle(-\Delta \pm k+\mathrm{i}0)^{-1}\partial_{x_1}(-\Delta)^{-1}W\,,V\rangle = \langle(\rho^2 \pm k+\mathrm{i}0)^{-1}\rho_1\rho^{-2}\widehat{W}\,,\widehat{V}\rangle\\ =&2\pi\int_0^\pi\int_0^\infty(\rho^2 \pm k+\mathrm{i}0)^{-1}\rho\cos\vartheta\widehat{W}(\rho)\widehat{V}(\rho,\vartheta)\mathrm{d}\rho\sin\vartheta\mathrm{d}\vartheta = O(1)\end{aligned}$$

by scaling $\rho \to \rho k^{\frac{1}{2}}$. On the other hand, we know from general theory that

$$\langle(-\Delta + k+\mathrm{i}0)^{-1}\partial_{x_1}(-\Delta)^{-1}W\,,V\rangle \tag{5.29}$$

is analytic in $\zeta = k^{\frac{1}{2}} \in \{\mathrm{Im}\,\zeta > 0\}$, so that the next order is $k^{\frac{1}{2}}$. Plugging this and (5.28) into (5.27), and noting that the constant term vanishes in the difference (5.21), we are left with a contribution of order $k^{\frac{1}{2}}O(g)$ for the term (5.25).

The term (5.26) can be treated in the exact same way and yields a contribution of order $k^{\frac{1}{2}}O(g^2)$. The case $k<0$ is treated analogously. $\square$

8. Proof of Point (1) of Proposition 5.8

Similar to [**24**] we decomposition the linear operator A, defined in (5.18), to be

$$A((1-\chi_{T_{\rm loc}})q) := \sum_{k=1}^{3}\Gamma_k((1-\chi_{T_{\rm loc}})q)\,,$$

where the terms Γ_k are defined as

$$\Gamma_1 := -Z\int_0^t [K(t-s)-K(t)]\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\int_s^t q_{s_1}(1-\chi_{T_{\rm loc}}(s_1))\mathrm{d}s_1\mathrm{d}s$$

$$\Gamma_2 := Z\int_0^t K(t-s)\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\mathrm{d}s\int_0^t q_{s_1}(1-\chi_{T_{\rm loc}}(s_1))\mathrm{d}s_1$$

$$\Gamma_3 := ZK(t)\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,(-\mathrm{i}h)^{-1}\int_0^t[\mathrm{e}^{-\mathrm{i}h(t-s)}-\mathrm{e}^{-\mathrm{i}ht}]q_s(1-\chi_{T_{\rm loc}}(s))\mathrm{d}s\partial_{x_1}(h)^{-1}W\rangle\,.$$

Define two continuous functions $\Omega_1,\ \Omega_2:\ (-\infty,1)\to\mathbb{R}^+$ by

$$\Omega_1(\delta) := \frac{1}{(1-2\delta)\pi}\int_0^1\frac{1}{1+(1-r)^{\frac{1}{2}}}(1-r)^{-\frac{1}{2}}[r^{-\frac{1}{2}}-r^{-\delta}]\mathrm{d}r \tag{5.30}$$

$$\Omega_2(\delta) := \frac{1}{\pi}\int_0^1\frac{1}{1+(1-r)^{\frac{1}{2}}}(1-r)^{-\frac{1}{2}}r^{\frac{1}{2}-\delta}\mathrm{d}r\,. \tag{5.31}$$

Recall the function $\Omega(\delta)$ introduced before Theorem 5.3. It is given by the sum $\Omega(\delta)=\Omega_1(\delta)+\Omega_2(\delta)$, and we compute

$$\Omega(\delta)=\frac{1}{\pi d(2d-1)}+\frac{1}{2\sqrt{\pi}}\left(\frac{2\Gamma(\frac{1}{2}-\delta)}{\Gamma(1-\delta)}-\frac{\Gamma(-\delta)}{\Gamma(\frac{3}{2}-\delta)}\right)\,.$$

Note that $\Omega(\delta)$ has only apparent singularities at $\delta=0,\frac{1}{2}$. It is a continuous, monotonically increasing function

$$\Omega:(-\infty,1)\to\mathbb{R}^+$$

satisfying

$$\begin{aligned}\lim_{\delta\to-\infty}\Omega(\delta)&=0\\ \Omega(0)&=1-\frac{\log 4}{\pi}\simeq 0.56\\ \Omega(\frac{1}{2})&=1+\frac{\log 4-2}{\pi}\simeq 0.8\,.\\ \lim_{\delta\to 1}\Omega(\delta)&=\infty\end{aligned}$$

Numerical analysis suggests that $\Omega(\delta)<1$ for all $\delta<0.66$.

Point (1) of Proposition 5.8 is covered by the following lemma,

LEMMA 5.12. *If $q_t\in B_{\delta,T_{\rm loc}}$ then there is a small constant $\varepsilon(T_{\rm loc})$ satisfying $\varepsilon(\infty)=0$ such that*

$$\begin{aligned}|\Gamma_1|&\le t^{-\frac{1}{2}-\delta}[\Omega_1(\delta)+\varepsilon(T_{\rm loc})](1+O(g))\|q\|_{\delta,T_{\rm loc}}\\ |\Gamma_3|&\le t^{-\frac{1}{2}-\delta}[\Omega_2(\delta)+\varepsilon(T_{\rm loc})](1+O(g))\|q\|_{\delta,T_{\rm loc}}\\ |\Gamma_2|&\le t^{-\frac{1}{2}-\delta}\varepsilon(T_{\rm loc})\|q\|_{\delta,T_{\rm loc}}\,,\end{aligned}$$

PROOF. We start with Γ_2. The second term in the product is easy to estimate,

$$|\int_0^t q_{s_1}(1-\chi_{T_{\mathrm{loc}}}(s_1))\mathrm{d}s_1| \leq (1+t)^{\frac{1}{2}-\delta}\|q\|_{\delta,T_{\mathrm{loc}}}\,. \tag{5.32}$$

The first term is estimated as follows. Apply the Fourier transform to the convolution function, then inverse Fourier transform to find

$$Z\int_0^t K(t-s)\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\mathrm{d}s = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(k)}{\mathrm{i}k+ZG(k+\mathrm{i}0)}\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}k\,,$$

where $F(k)$ is defined as

$$\begin{aligned}F(k) :&= Z\int_0^\infty \mathrm{e}^{\mathrm{i}ks}\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\mathrm{d}s\\ &= \frac{Z}{2}\int_0^\infty \mathrm{e}^{\mathrm{i}ks}[\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle+\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle]\mathrm{d}s\\ &= \frac{Z}{2}[\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,(-ih+ik-0)^{-1}\partial_{x_1}(h)^{-1}W\rangle+\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,(ih+ik-0)^{-1}\partial_{x_1}(h)^{-1}W\rangle]\\ &= -ZG(k+\mathrm{i}0)\,.\end{aligned}$$

Around $k=0$, the term $\frac{F(k)}{\mathrm{i}k+ZG(k+\mathrm{i}0)}$ has the expansion

$$\frac{F(k)}{\mathrm{i}k+ZG(k+\mathrm{i}0)} = -1+Ck^{\frac{1}{2}}+O(k)\,.$$

The constant term does not contribute, as is seen by integration by parts,

$$\int_{-\infty}^{\infty}\frac{F(k)}{\mathrm{i}k+ZG(k+\mathrm{i}0)}\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}k = \int_{-\infty}^{\infty}\frac{1}{\mathrm{i}t}\partial_k\left(\frac{F(k)}{\mathrm{i}k+ZG(k+\mathrm{i}0)}\right)\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}k\,,$$

and the Fourier transform of $k^{\frac{1}{2}}$ is of order $t^{-\frac{3}{2}}$. The detailed computations are identical to [**24**] and thus omitted. We obtain

$$|Z\int_0^t K(t-s)\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\mathrm{d}s| \lesssim (1+t)^{-\frac{3}{2}}\,. \tag{5.33}$$

Combining (5.32) and (5.33), we have

$$|\Gamma_2| \lesssim (1+t)^{-\frac{3}{2}}(1+t)^{\frac{1}{2}-\delta}\|q\|_{\delta,T_{\mathrm{loc}}} = (1+t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}-\delta}\|q\|_{\delta,T_{\mathrm{loc}}} \leq T_{\mathrm{loc}}^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}-\delta}\|q\|_{\delta,T_{\mathrm{loc}}}\,,$$

which is the desired estimate.

Now, we turn to Γ_1. Recall the asymptotic expression for K in Lemma (5.16),

$$ZK(t) = \frac{3}{\sqrt{2}}\pi^{-\frac{5}{2}}(1+Cg)t^{-\frac{1}{2}}+O(t^{-1})\,,$$

and observe that by Duhamel's principle and the second resolvent identity there exists a $\widetilde{C}\in\mathbb{R}$ such that

$$\begin{aligned}\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}ht}\partial_{x_1}(h)^{-1}W\rangle &= \frac{1}{3}\mathrm{Re}\,\langle W\,,\mathrm{e}^{\mathrm{i}\Delta t}W\rangle+O(gt^{-\frac{3}{2}})\\ &= -\frac{1}{3\sqrt{2}}\pi^{\frac{3}{2}}t^{-\frac{3}{2}}(1+\widetilde{C}g)+o(t^{-\frac{3}{2}})\,.\end{aligned} \tag{5.34}$$

Now to the proof of (5.34):

$$\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}ht}\partial_{x_1}(h)^{-1}W\rangle = \langle\partial_{x_1}W\,,\mathrm{e}^{\mathrm{i}\Delta t}\partial_{x_1}(-\Delta)^{-1}W\rangle - g\langle(h^{-1}W)\partial_{x_1}W\,,\mathrm{e}^{\mathrm{i}\Delta t}\partial_{x_1}(-\Delta)^{-1}W\rangle$$
$$-g\langle V\,,\mathrm{e}^{\mathrm{i}\Delta t}\partial_{x_1}(-\Delta)^{-1}Wh^{-1}W\rangle + g\langle V\,,\int_0^t \mathrm{e}^{\mathrm{i}\Delta(t-s)}W\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}h^{-1}W\rangle\,,$$

where we used the abbreviation

$$V = [1 - g(h)^{-1}W]\partial_{x_1}W\,.$$

Concerning the various terms on the right hand side we use the propagator estimates in 5.15 to get:

$$\begin{aligned}
&\langle \partial_{x_1}W\,, \mathrm{e}^{\mathrm{i}\Delta t}\partial_{x_1}(-\Delta)^{-1}W\rangle = \frac{1}{3}\langle W\,, \mathrm{e}^{\mathrm{i}\Delta t}W\rangle \\
&g|\langle (h^{-1}W)\partial_{x_1}W\,, \mathrm{e}^{\mathrm{i}\Delta t}\partial_{x_1}(-\Delta)^{-1}W\rangle| \le gt^{-\frac{3}{2}}\|\langle x\rangle^3(h^{-1}W)\partial_{x_1}W\|_2\|\langle x\rangle^3 W\|_2 \\
&g|\langle V\,, \mathrm{e}^{\mathrm{i}\Delta t}\partial_{x_1}(-\Delta)^{-1}Wh^{-1}W\rangle| \le gt^{-\frac{3}{2}}\|\langle x\rangle^3 V\|_2\|\langle x\rangle^3 Wh^{-1}W\|_2 \\
&g|\langle V\,, \int_0^t \mathrm{e}^{\mathrm{i}\Delta(t-s)}W\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}h^{-1}W\rangle| \le g\|\langle x\rangle^3 V\|_2\int_0^t (1+t-s)^{-\frac{3}{2}}\|\langle x\rangle^3 W\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}h^{-1}W\|_2\mathrm{d}s \\
\lesssim{}& g\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}}\|\langle x\rangle^4 W\|_2\mathrm{d}s \lesssim gt^{-\frac{3}{2}}\,.
\end{aligned}$$

This concludes the proof of (5.34).

We take the leading order terms $\widetilde{K}, \widetilde{M}, \widetilde{\Gamma}_1$ to approximate these functions,

$$\begin{aligned}
Z\widetilde{K}(t) &:= \frac{3}{\sqrt{2}}\pi^{-\frac{5}{2}}t^{-\frac{1}{2}}(1+Cg) \\
\widetilde{M} &:= -\frac{1}{3\sqrt{2}}\pi^{\frac{3}{2}}t^{-\frac{3}{2}}(1+\widetilde{C}g) \\
\widetilde{\Gamma}_1 &:= -Z\int_0^t [\widetilde{K}(t-s) - \widetilde{K}(s)]\widetilde{M}(s)\int_s^t q_{s_1}[1-\chi_T(s_1)]\mathrm{d}s_1\mathrm{d}s\,.
\end{aligned}$$

Now compute

$$\begin{aligned}
|\widetilde{\Gamma}_1| &\le \frac{1+O(g)}{2\pi}\int_0^t [(t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}}]s^{-\frac{3}{2}}\int_s^t |q_{s_1}|\mathrm{d}s_1\mathrm{d}s \\
&\le \frac{1+O(g)}{(1-2\delta)\pi}\int_0^t [(t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}}]s^{-\frac{3}{2}}(t^{\frac{1}{2}-\delta} - s^{\frac{1}{2}-\delta})\mathrm{d}s\|q_t\|_{\delta,T} \\
&= \frac{1+O(g)}{(1-2\delta)\pi}\int_0^t (t-s)^{-\frac{1}{2}}t^{-\frac{1}{2}}\frac{1}{(t-s)^{\frac{1}{2}}+t^{\frac{1}{2}}}s^{-\frac{1}{2}}(t^{\frac{1}{2}-\delta} - s^{\frac{1}{2}-\delta})\mathrm{d}s\|q_t\|_{\delta,T}\,,
\end{aligned}$$

Change variables $s = rt$ to obtain

$$|\widetilde{\Gamma}_1| \le t^{-\frac{1}{2}-\delta}(1+O(g))\Omega_1(\delta)\|q_t\|_{\delta,T}\,,$$

where the constant Ω_1 is defined in (5.30). In what follows we estimate $|\Gamma_1 - \widetilde{\Gamma}_1|$. Divide the integration region $[0,t]$ into three parts, $[0, T_{\mathrm{loc}}^{\frac{1}{3}}]$,$[T_{\mathrm{loc}}^{\frac{1}{3}}, t - T_{\mathrm{loc}}^{\frac{1}{3}}]$, and $[t - T_{\mathrm{loc}}^{\frac{1}{3}}, t]$.

$$\begin{aligned}
I_1 :={}& Z\int_0^{T_{\mathrm{loc}}^{\frac{1}{3}}}[K(t-s) - K(s)]\mathrm{Re}\,\langle [1-g(h)^{-1}W]\partial_{x_1}W\,, \mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\int_s^t q_{s_1}[1-\chi_{T_{\mathrm{loc}}}(s_1)]\mathrm{d}s_1\mathrm{d}s \\
&- Z\int_0^{T_{\mathrm{loc}}^{\frac{1}{3}}}[\widetilde{K}(t-s) - \widetilde{K}(s)]\widetilde{M}(s)\int_s^t q_{s_1}[1-\chi_{T_{\mathrm{loc}}}(s_1)]\mathrm{d}s_1\mathrm{d}s\,.
\end{aligned}$$

We have

$$\begin{aligned}
&|K(t-s) - K(t)|, |\widetilde{K}(t-s) - \widetilde{K}(t)| \\
\lesssim{}& t^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}\frac{s}{t^{\frac{1}{2}} + (t-s^{\frac{1}{2}})} + (t-s)^{-\frac{3}{2}} - t^{-\frac{3}{2}} \\
\lesssim{}& t^{-\frac{3}{2}}(1+s)
\end{aligned} \tag{5.35}$$

because $s \le T_{\mathrm{loc}}^{\frac{1}{3}}$, and $t \ge T_{\mathrm{loc}}$. And consequently

$$|K(t-s)-K(t)|\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}ht}\partial_{x_1}(h)^{-1}W\rangle+|\widetilde{K}(t-s)-\widetilde{K}(t)||\widetilde{M}(s)|\lesssim t^{-\frac{3}{2}}s^{-\frac{1}{2}}(1+O(g))\,.$$

Plug this into I_1 to obtain

$$|I_1|\lesssim t^{-1-\delta}(1+O(g))\int_0^{T_{\mathrm{loc}}^{\frac{1}{3}}}s^{-\frac{1}{2}}\mathrm{d}s\|q_t\|_{\delta,T_{\mathrm{loc}}}=t^{-1-\delta}(1+O(g))2T_{\mathrm{loc}}^{\frac{1}{6}}\|q_t\|_{\delta,T_{\mathrm{loc}}}\lesssim t^{-\frac{1}{2}-\delta}T_{\mathrm{loc}}^{-\frac{1}{3}}(1+O(g))\|q_t\|_{\delta,T_{\mathrm{loc}}}\,.$$

Now we turn to the second interval, $[T_{\mathrm{loc}}^{\frac{1}{3}}, t-T_{\mathrm{loc}}^{\frac{1}{3}}]$.

$$\begin{aligned}
I_2:=&Z\int_{T_{\mathrm{loc}}^{\frac{1}{3}}}^{t-T_{\mathrm{loc}}^{\frac{1}{3}}}[K(t-s)-K(t)]\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\int_s^t(1-\chi_{T_{\mathrm{loc}}}(s_1))q_{s_1}\mathrm{d}s_1\mathrm{d}s\\
&-Z\int_{T_{\mathrm{loc}}^{\frac{1}{3}}}^{t-T_{\mathrm{loc}}^{\frac{1}{3}}}[\widetilde{K}(t-s)-\widetilde{K}(t)]\widetilde{M}(s)\int_s^t(1-\chi_{T_{\mathrm{loc}}}(s_1))q_{s_1}\mathrm{d}s_1\mathrm{d}s\\
=&Z\int_{T_{\mathrm{loc}}^{\frac{1}{3}}}^{t-T_{\mathrm{loc}}^{\frac{1}{3}}}[(K(t-s)-K(t))-(\widetilde{K}(t-s)-\widetilde{K}(t))]\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\int_s^t(1-\chi_{T_{\mathrm{loc}}}(s_1))q_{s_1}\mathrm{d}\\
&+Z\int_{T_{\mathrm{loc}}^{\frac{1}{3}}}^{t-T_{\mathrm{loc}}^{\frac{1}{3}}}[\widetilde{K}(t-s)-\widetilde{K}(t)][\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle-\widetilde{M}(s)]\int_s^t(1-\chi_{T_{\mathrm{loc}}}(s_1))q_{s_1}\mathrm{d}s_1\mathrm{d}s\,.
\end{aligned}$$

Using

$$\begin{array}{rcc}
|K(t-s)-\widetilde{K}(t-s)| & \lesssim & (1+t-s)^{-1}\\
|K(t)-\widetilde{K}(t)| & \lesssim & (1+t)^{-1}\\
|\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle-\widetilde{M}(s)| & \lesssim & s^{-\frac{3}{2}}\,,
\end{array}$$

and

$$|\widetilde{K}(t-s)-\widetilde{K}(t)|\lesssim t^{-1}(t-s)^{-\frac{1}{2}}s\,,$$

we obtain

$$|I_2|\lesssim T_{\mathrm{loc}}^{-\frac{1}{3}}t^{-\frac{1}{2}-\delta}\|q_t\|_{\delta,T_{\mathrm{loc}}}\,.$$

In the third interval, $s\in[t-T_{\mathrm{loc}}^{\frac{1}{3}},t]$, we have

$$\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\lesssim t^{-\frac{3}{2}}\,,$$

and hence

$$\begin{aligned}
|I_3|=&|Z\int_{t-T_{\mathrm{loc}}^{\frac{1}{3}}}^{t}[K(t-s)-K(t)]\mathrm{Re}\,\langle[1-g(h)^{-1}W]\partial_{x_1}W\,,\mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\int_s^t(1-\chi_{T_{\mathrm{loc}}}(s_1))q_{s_1}\mathrm{d}s_1\mathrm{d}s\\
&-Z\int_{t-T_{\mathrm{loc}}^{\frac{1}{3}}}^{t}[\widetilde{K}(t-s)-\widetilde{K}(t)]\widetilde{M}(s)\int_s^t(1-\chi_{T_{\mathrm{loc}}}(s_1))q_{s_1}\mathrm{d}s_1\mathrm{d}s|\\
\lesssim&\int_{t-T_{\mathrm{loc}}^{\frac{1}{3}}}^{t}(|t-s|^{-\frac{1}{2}}+t^{-\frac{1}{2}})\mathrm{d}s\;t^{-1-\delta}\|q_t\|_{\delta,T_{\mathrm{loc}}}\\
\lesssim&T_{\mathrm{loc}}^{1/6}t^{-1-\delta}\|q_t\|_{\delta,T_{\mathrm{loc}}}\\
\le&T_{\mathrm{loc}}^{-1/3}t^{-\frac{1}{2}-\delta}\|q_t\|_{\delta,T_{\mathrm{loc}}}\,.
\end{aligned}$$

Putting together, we have shown that

$$|\Gamma_1|\lesssim t^{-\frac{1}{2}-\delta}[\Omega_1(\delta)+\varepsilon(T_{\mathrm{loc}})](1+O(g))\|q_t\|_{\delta,T_{\mathrm{loc}}}\,,$$

where $\varepsilon(T_{\mathrm{loc}})\to 0$, as $T_{\mathrm{loc}}\to\infty$.

Finally, we turn to Γ_3. Similar to the strategy in estimating Γ_1, we start with retrieving the main part. Define a new function $\widetilde{V}$ to approximate the function $V(t) := \mathrm{Re}\,\langle [1-g(h)^{-1}W]\partial_{x_1}W\,,(-\mathrm{i}h)^{-1}\mathrm{e}^{-\mathrm{i}ht}\partial_{x_1}(h)^{-1}W\rangle$ when t is large (simply by integrating (5.34)),

$$\widetilde{V} := \frac{\sqrt{2}}{3}\pi^{\frac{3}{2}}(1+\widetilde{C}g)t^{-\frac{1}{2}}\,.$$

Now, define an approximation $\widetilde{\Gamma}_3$ of Γ_3,

$$\widetilde{\Gamma}_3 := Z\widetilde{K}(t)\int_0^t [\widetilde{V}(t-s)-\widetilde{V}(t)](1-\chi_{T_{\mathrm{loc}}}(s))q_s\mathrm{d}s$$

Compute

$$\begin{aligned}|\widetilde{\Gamma}_3| \leq & t^{-\frac{1}{2}}\frac{1}{\pi}(1+O(g))\int_0^t [(t-s)^{-\frac{1}{2}}-t^{-\frac{1}{2}}]s^{-\frac{1}{2}-\delta}\mathrm{d}s\|q_t\|_{\delta,T_{\mathrm{loc}}}\\ =& t^{-1}\frac{1}{\pi}(1+O(g))\int_0^t \frac{s}{(t-s)^{\frac{1}{2}}+t^{\frac{1}{2}}}(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}-\delta}\mathrm{d}s\|q_t\|_{\delta,T_{\mathrm{loc}}}\,.\end{aligned}$$

The change variables $s=tr$ yields

$$|\widetilde{\Gamma}_3| \leq t^{-\frac{1}{2}-\delta}\frac{1}{\pi}(1+O(g))\int_0^1 (1-r)^{-\frac{1}{2}}\frac{1}{1+(1-r)^{\frac{1}{2}}}r^{\frac{1}{2}-\delta}\mathrm{d}r\|q_t\|_{\delta,T_{\mathrm{loc}}} = t^{-\frac{1}{2}-\delta}\Omega_2(\delta)(1+O(g))\|q_t\|_{\delta,T_{\mathrm{loc}}}\,.$$

Next, we write the difference $\Gamma_3-\widetilde{\Gamma}_3$ as

$$\begin{aligned}\Gamma_3-\widetilde{\Gamma}_3 =& Z[K(t)-\widetilde{K}(t)]\int_0^t [V(t-s)-V(t)](1-\chi_{T_{\mathrm{loc}}}(s))q_s\mathrm{d}s\\ &+Z\widetilde{K}(t)\int_0^t [V(t-s)-\widetilde{V}(t-s)](1-\chi_{T_{\mathrm{loc}}}(s))q_s\mathrm{d}s\\ &+Z\widetilde{K}(t)[\widetilde{V}(t)-V(t)]\int_0^t (1-\chi_{T_{\mathrm{loc}}}(s))q_s\mathrm{d}s\,.\end{aligned}\tag{5.36}$$

We know

$$|K(t)-\widetilde{K}(t)| \lesssim t^{-1}\,,$$

and

$$|V(t-s)-\widetilde{V}(t-s)| \lesssim (1+t-s)^{-\frac{3}{2}}\,.$$

Plug these into (5.36) to find the desired estimate

$$\begin{aligned}|\Gamma_3-\widetilde{\Gamma}_3| \lesssim & t^{-1}\int_0^t (1+s)^{-\frac{1}{2}-\delta}\mathrm{d}s\|q_t\|_{\delta,T_{\mathrm{loc}}} + t^{-\frac{1}{2}}(1+O(g))\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{1}{2}-\delta}\mathrm{d}s\|q_t\|_{\delta,T_{\mathrm{loc}}}\\ &+t^{-2}\int_0^t (1+s)^{-\frac{1}{2}-\delta}\mathrm{d}s\|q_t\|_{\delta,T_{\mathrm{loc}}}\\ \lesssim & t^{-1-\delta}(1+O(g)))\|q_t\|_{\delta,T_{\mathrm{loc}}} \leq (1+O(g))T_{\mathrm{loc}}^{-\frac{1}{2}}t^{-\frac{1}{2}-\delta}\|q_t\|_{\delta,T_{\mathrm{loc}}}\,.\end{aligned}$$

Putting it together, we have shown

$$|\Gamma_3| \lesssim t^{-\frac{1}{2}-\delta}[\Omega_2(\delta)+\varepsilon(T_{\mathrm{loc}})](1+O(g))\|q_t\|_{\delta,T_{\mathrm{loc}}}\,,$$

which finishes the proof. □

9. Proof of Point (2) of Proposition 5.8

Point (2) is proven by the next result,

LEMMA 5.13. *Let Q_1 and Q_2 be as in Proposition (5.8), and recall the definition of $R(P,t)$ in (5.12). If $|P_0| \leq T_{\rm loc}^{-3}$ then, for $k = 3,4,5,6$,*

$$\int_0^t |K(t-s) - K(t)||\widetilde{D}_k(Q_1) - \widetilde{D}_k(Q_2)|(s)\mathrm{d}s \leq t^{-\frac{1}{2}-\delta}\varepsilon(T_{\rm loc})\|Q_1 - Q_2\|_{\delta,T_{\rm loc}}(\|Q_1\|_{\delta,T_{\rm loc}} + \|Q_2\|_{\delta,T_{\rm loc}})\,.$$

PROOF. Because the proof is rather long we divide it into sections.

9.1. The term $\widetilde{D}_3$. Because of the spherical symmetry of W certain terms in the definition vanish. For notational convenience, we define

$$(1 + \frac{g}{\sqrt{\rho_0}}\bar{\beta}^{X_t})\partial_x W^{X_t} := V^{X_t}\,.$$

Note that V is rapidly decaying. This makes $\widetilde{D}_3$ take the form

$$\begin{aligned}\widetilde{D}_3(t) &= -2\sqrt{\rho_0}\mathrm{Re}\,\langle V^{X_t}\,, \mathrm{e}^{-\mathrm{i}h^{X_t}t}\beta_0\rangle - 2\rho_0\mathrm{Re}\,\langle V^{X_t}\,, \mathrm{e}^{-\mathrm{i}h^{X_t}t}\sum_{|\alpha|=2}^{N_0}\frac{1}{\alpha!}(X_0 - X_t)^\alpha\partial_x^\alpha(h^{X_t})^{-1}W^{X_t}\rangle\\ &=: \widetilde{D}_{31} + \widetilde{D}_{32}\,.\end{aligned}$$

For $\widetilde{D}_{31}$ we write

$$\langle V^{X_t}\,, \mathrm{e}^{-\mathrm{i}h^{X_t}t}\beta_0\rangle = \langle V\,, \mathrm{e}^{-\mathrm{i}ht}\beta_0^{-X_t}\rangle\,,$$

and compute

$$\begin{aligned}&|\langle V\,, \mathrm{e}^{-\mathrm{i}ht}(\beta_0^{-X_t} - \beta_0^{-\widetilde{X}_t})\rangle| \leq \|\langle x\rangle^3 V\|_2 t^{-\frac{3}{2}}\|\langle x\rangle^3(\beta_0^{-X_t} - \beta_0^{-\widetilde{X}_t})\|_2 \lesssim t^{-\frac{3}{2}}\|\langle x\rangle^3(\beta_0 - \beta_0^{\int_{T_{\rm loc}}^t (Q_1-Q_2)(s)\mathrm{d}s})\|_2\\ =&t^{-\frac{3}{2}}\|\langle x\rangle^3\int_{T_{\rm loc}}^t \partial_s\beta_0^{\int_{T_{\rm loc}}^s (Q_1-Q_2)(s_1)\mathrm{d}s_1}\mathrm{d}s\|_2 = t^{-\frac{3}{2}}\|\langle x\rangle^3\partial_x\beta_0\cdot\int_{T_{\rm loc}}^t (Q_1 - Q_2)(s)\mathrm{d}s\|_2\\ \leq&t^{-\frac{3}{2}}\|\langle x\rangle^3\partial_x\beta_0\|_2\int_{T_{\rm loc}}^t |Q_1 - Q_2|(s)\mathrm{d}s \lesssim t^{-1-\delta}\|Q_1 - Q_2\|_{\delta,T_{\rm loc}}\,,\end{aligned}$$

where we used $X_t = X_{T_{\rm loc}} + \int_{T_{\rm loc}}^t Q_1(s)\mathrm{d}s$ and $\widetilde{X}_t = X_{T_{\rm loc}} + \int_{T_{\rm loc}}^t Q_2(s)\mathrm{d}s$.

The term $\widetilde{D}_{32}$ is treated similarly,

$$\begin{aligned}|\widetilde{D}_{32}(Q_1) - \widetilde{D}_{32}(Q_2)|(t) &\lesssim |\langle V\,, \mathrm{e}^{-\mathrm{i}ht}\sum_{|\alpha|=2}^{N_0}(\widetilde{X}_t - X_t)^\alpha\partial_x^\alpha(h)^{-1}W\rangle|\\ &\leq \|V\|_1\sum_{|\alpha|=2}^{N_0}\|\mathrm{e}^{-\mathrm{i}ht}\partial_x^\alpha(h)^{-1}W\|_\infty|\int_{t_{\rm loc}}^t [Q_1(s) - Q_2(s)]^\alpha|\\ &\lesssim t^{-\frac{3}{2}}\sum_{|\alpha|=2}^{N_0}\int_{t_{\rm loc}}^t |[Q_1(s) - Q_2(s)]^\alpha| \lesssim t^{-1-\delta}\sum_{|\alpha|=2}^{N_0}\|Q_1^\alpha - Q_2^\alpha\|_{\delta,T_{\rm loc}}\\ &\lesssim t^{-1-\delta}(\|Q_1^2\|_{\delta,T_{\rm loc}} + \|Q_2^2\|_{\delta,T_{\rm loc}})\|Q_1 - Q_2\|_{\delta,T_{\rm loc}}\,.\end{aligned}$$

The estimate for K is as in [**24**],

$$|K(t-s) - K(t)| \lesssim (1 + t - s)^{-\frac{1}{2}}(1+t)^{-1}s\,,$$

so the claim follows for $\widetilde{D}_3$.

9.2. The term $\widetilde{D}_4$. Consider

$$\begin{aligned}\widetilde{D}_4(t) &= -\,\mathrm{i}g\langle V^{X_t}\,, \int_0^t \mathrm{e}^{-\mathrm{i}h^{X_t}(t-s)}[W^{X_s} - W^{X_t}]\delta_s \mathrm{d}s\rangle \\ &= -\,\mathrm{i}g\langle V^{X_t}\,, \int_0^t \mathrm{e}^{-\mathrm{i}h^{X_t}(t-s)}\partial_x W^{X_t}\cdot (X_s - X_t)\delta_s \mathrm{d}s\rangle \\ &\quad -\,\mathrm{i}g\langle V^{X_t}\,, \int_0^t \mathrm{e}^{-\mathrm{i}h^{X_t}(t-s)}O(|X_s - X_t|^2)\sum_{|\alpha|=2}\partial_x^\alpha W^S \delta_s \mathrm{d}s\rangle \\ &=: \widetilde{D}_{41} + \widetilde{D}_{42}\,.\end{aligned}$$

The second term is easier and thus omitted. Two terms depend on Q, namely $(X_s - X_t)$ and $\delta_s^{-X_t}$. We use the expansion [**34**] $\mathrm{e}^{-\mathrm{i}h\tau} = \tau^{-\frac{3}{2}}B_1 + \tau^{-\frac{5}{2}}B_2 + \dots$ in $\mathcal{B}(L^{2,-6}, L^{2,6})$ and the fact that V is odd in x so that the $t^{-\frac{3}{2}}$-term vanishes to compute

$$\begin{aligned}|\widetilde{D}_{41}(Q_1) - \widetilde{D}_{41}(Q_2)|(t) &\lesssim g|\langle V\,, \int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}\partial_x W\cdot(\widetilde{X}_t - \widetilde{X}_s + X_s - X_t)\delta_s^{-X_t}\mathrm{d}s\rangle| \\ &\lesssim g\|V\langle x\rangle^6\|_2 \int_0^t (1+t-s)^{-\frac{5}{2}}\|\langle x\rangle^{-6}(\widetilde{X}_t - \widetilde{X}_s + X_s - X_t)\cdot B_2(\partial_x W\delta_s^{-X_t})\|_2 \mathrm{d}s \\ &\lesssim g\int_0^t |\widetilde{X}_t - \widetilde{X}_s + X_s - X_t|(1+t-s)^{-\frac{5}{2}}\|\langle x\rangle^6\partial_x W\delta_s^{-X_t}\|_2\mathrm{d}s \\ &\lesssim g\int_0^t |\widetilde{X}_t - \widetilde{X}_s + X_s - X_t|(1+t-s)^{-\frac{5}{2}}(1+s)^{-\frac{1}{2}}\mathrm{d}s\,.\end{aligned}$$

Both Q_1 and Q_2 are in $B_{\delta,T_{\mathrm{loc}}}$, and thus

$$\begin{aligned}|\widetilde{X}_t - \widetilde{X}_s + X_s - X_t| = |\int_s^t [Q_1 - Q_2](s_1)\mathrm{d}s_1| &\le \|Q_1 - Q_2\|_{\delta,T_{\mathrm{loc}}}\int_s^t s_1^{-\frac{1}{2}-\delta}\mathrm{d}s_1 \\ &\lesssim \|Q_1 - Q_2\|_{\delta,T_{\mathrm{loc}}}(t^{\frac{1}{2}-\delta} - s^{\frac{1}{2}-\delta}).\end{aligned}$$

Now write

$$(t^{\frac{1}{2}-\delta} - s^{\frac{1}{2}-\delta}) = -t^{\frac{1}{2}-\delta}s^{\frac{1}{2}-\delta}(t^{\delta-\frac{1}{2}} - s^{\delta-\frac{1}{2}})\,,$$

and use that for $s \le t$ and any $\varepsilon > 0$

$$t^\varepsilon - s^\varepsilon \le \frac{t-s}{t^{1-\varepsilon}}$$

to estimate

$$\begin{aligned}|\widetilde{D}_{41}(Q_1) - \widetilde{D}_{41}(Q_2)|(t) &\lesssim g\|Q_1 - Q_2\|_{\delta,T_{\mathrm{loc}}}\int_0^t (1+t-s)^{-\frac{3}{2}}t^{-1}s^{\frac{1}{2}-\delta}(1+s)^{-\frac{1}{2}}\mathrm{d}s \\ &\lesssim g t^{-1-\delta}\|Q_1 - Q_2\|_{\delta,T_{\mathrm{loc}}}\,.\end{aligned}$$

Since also δ_s depends on Q we have to estimate also

$$\begin{aligned}&g|\langle V\,, \int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}\partial_x W\cdot(\widetilde{X}_t - \widetilde{X}_s)(\delta_s^{-X_t} - \widetilde{\delta}_s^{-\widetilde{X}_t})\mathrm{d}s\rangle| \\ &\lesssim g\int_0^t (1+t-s)^{-\frac{5}{2}}|\widetilde{X}_t - \widetilde{X}_s|\|\langle x\rangle^6\partial_x W(\delta_s^{-X_t} - \widetilde{\delta}_s^{-\widetilde{X}_t})\|_2\mathrm{d}s \qquad (5.37)\end{aligned}$$

This is more involved. Introduce a new function η,

$$\eta_s := \delta_s^{-X_t} - \widetilde{\delta}_s^{-\widetilde{X}_t}\,.$$

By (5.6), η satisfies the following equation,

$$\begin{aligned}
\mathrm{i}\dot{\eta}_s &= h\eta_s + g(W^{X_s - X_t} - W^{\widetilde{X}_s - \widetilde{X}_t})\delta_s^{-X_t} + gW^{\widetilde{X}_s - \widetilde{X}_t}\eta_s - gW\eta_s \\
&\quad + \mathrm{i}\frac{\sqrt{\rho_0}}{M}(Q_1 - Q_2)(s)\sum_{|\alpha|=1}^{N_0}\frac{1}{\alpha!}\alpha(X_s - X_t)^{\alpha-1}\partial_x^\alpha(h)^{-1}W \\
&\quad + \mathrm{i}\frac{\sqrt{\rho_0}}{M}Q_2(s)\sum_{|\alpha|=1}^{N_0}\frac{1}{\alpha!}\alpha[X_s - X_t - \widetilde{X}_s + \widetilde{X}_t]^{\alpha-1}\partial_x^\alpha(h)^{-1}W \\
&\quad - (h^{X_s - X_t} - h^{\widetilde{X}_s - \widetilde{X}_t})r_{N_0} - h^{\widetilde{X}_s - \widetilde{X}_t}(r_{N_0} - \widetilde{r}_{N_0}),
\end{aligned}$$

which implies by Duhamel's principle

$$\begin{aligned}
\eta_s = \mathrm{e}^{-\mathrm{i}hs}\eta_0 - ig\int_0^s \mathrm{e}^{-\mathrm{i}h(s-s_1)}(W^{X_{s_1} - X_t} - W^{\widetilde{X}_s - \widetilde{X}_t})\delta_s^{-X_t}\mathrm{d}s_1 + g\int_0^s \mathrm{e}^{-\mathrm{i}h(s-s_1)}W^{\widetilde{X}_{s_1} - \widetilde{X}_t}\eta_{s_1}\mathrm{d}s_1 \\
+ g\int_0^s \mathrm{e}^{-\mathrm{i}h(s-s_1)}W\eta_{s_1}\mathrm{d}s_1 + \dots .
\end{aligned}$$

We will only treat the displayed terms, the others are similar but easier and hence omitted. From (5.37) and the discussion of the previous term it is clear that we are done if we can prove

$$\|\langle x\rangle^{-3}\eta_t\|_2 \lesssim t^{-\frac{1}{2}}\|Q_1 - Q_2\|_{\delta,T_{\mathrm{loc}}}.$$

With the usual estimates we obtain

$$\|\langle x\rangle^{-3}\mathrm{e}^{-\mathrm{i}ht}\eta_0\|_2 \lesssim t^{-\frac{1}{2}}\|Q_1 - Q_2\|_{\delta,t_{\mathrm{loc}}}$$

and

$$\begin{aligned}
&\|\langle x\rangle^{-3}g\int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}(W^{X_s - X_t} - W^{\widetilde{X}_s - \widetilde{X}_t})\delta_s^{-X_t}\mathrm{d}s\|_2 \\
\lesssim& g\int_0^t (1+t-s)^{-\frac{3}{2}}\|\langle x\rangle^3(W^{X_s - X_t} - W^{\widetilde{X}_s - \widetilde{X}_t})\|_2(1+s)^{-\frac{1}{2}}\mathrm{d}s \\
\lesssim& g\int_0^t (1+t-s)^{-\frac{3}{2}}\|\langle x\rangle^3\partial_x W\|_2\int_s^t |Q_1 - Q_2|(s_1)\mathrm{d}s_1(1+s)^{-\frac{1}{2}}\mathrm{d}s \\
\lesssim& g\|Q_1 - Q_2\|_{\delta,T_{\mathrm{loc}}}\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{\frac{1}{2}-\delta}(1+s)^{-\frac{1}{2}}\mathrm{d}s \\
\lesssim& gt^{-\delta}\|Q_1 - Q_2\|_{\delta,T_{\mathrm{loc}}}.
\end{aligned}$$

The next term contains η implicitely and is therefore dealt with in the by now familiar way. Introduce a new function L,

$$L(t) := \max_{s\le t} s^{\frac{1}{2}}\|\langle x\rangle^{-3}\eta_s\|_2.$$

Then

$$\begin{aligned}
\|\langle x\rangle^{-3}g\int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}W^{\widetilde{X}_s - \widetilde{X}_t}\eta_s\mathrm{d}s\|_2 &\lesssim g\int_0^t (1+t-s)^{-\frac{3}{2}}\|\langle x\rangle^3 W^{\widetilde{X}_s - \widetilde{X}_t}\eta_s\|_2\mathrm{d}s \\
&\lesssim gL(t)\int_0^t (1+t-s)^{-\frac{3}{2}}s^{-\frac{1}{2}}\mathrm{d}s \\
&\lesssim gL(t)t^{-\frac{1}{2}}.
\end{aligned}$$

The last term is treated in the same way

$$\|\langle x\rangle^{-3}g\int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}W\eta_s\mathrm{d}s\|_2 \lesssim gL(t)t^{-\frac{1}{2}},$$

so that we obtain in the end

$$t^{\frac{1}{2}}\|\langle x\rangle^{-3}\eta_t\|_2 \lesssim \|Q_1-Q_2\|_{\delta,T_{\rm loc}} + gt^{\frac{1}{2}-\delta}\|Q_1-Q_2\|_{\delta,T_{\rm loc}} + 2gL(t)\,.$$

Maximizing both sides over t yields

$$L(t) \lesssim 2\|Q_1-Q_2\|_{\delta,T_{\rm loc}} + 2gL(t)\,,$$

and because g is small

$$L(t) \lesssim \|Q_1-Q_2\|_{\delta,T_{\rm loc}}\,,$$

which finishes the proof for $\widetilde{D}_4$.

Consider now the term $\widetilde{D}_5$. The treatment of it is very similar to $\widetilde{D}_4$ but much easier, and hence omitted.

9.3. The term $\widetilde{D}_6$. Consider finally

$$\mathrm{i}\langle V^{X_t}\,, \int_0^t \mathrm{e}^{-\mathrm{i}h^{X_t}(t-s)}G_1(s)\mathrm{d}s\rangle\,,$$

so that

$$|\widetilde{D}_6(Q_1)-\widetilde{D}_6(Q_2)|(t) = |\langle\partial_x W\,, \int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}(h^{X_s-X_t}r_{N_0}^{-X_t} - h^{\widetilde{X}_s-\widetilde{X}_t}\widetilde{r}_{N_0}^{-\widetilde{X}_t})\mathrm{d}s\rangle|\,.$$

With the usual procedure there are two terms to be estimated,

$$|\langle\partial_x W\,, \int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}(h^{X_s-X_t} - h^{\widetilde{X}_s-\widetilde{X}_t})r_{N_0}^{-X_t}(s)\mathrm{d}s\rangle|\,,$$

and

$$|\langle\partial_x W\,, \int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}h^{\widetilde{X}_s-\widetilde{X}_t}(r_{N_0}^{-X_t} - \widetilde{r}_{N_0}^{-\widetilde{X}_t})(s)\mathrm{d}s\rangle|\,.$$

For the first, use

$$h^{X_s-X_t} - h^{\widetilde{X}_s-\widetilde{X}_t} = g(W^{X_s-X_t} - W^{\widetilde{X}_s-\widetilde{X}_t})$$

to obtain

$$\begin{aligned}
&|\langle\partial_x W\,, \int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}(h^{X_s-X_t} - h^{\widetilde{X}_s-\widetilde{X}_t})r_{N_0}^{-X_t}(s)\mathrm{d}s\rangle|\\
\lesssim& g\int_0^t (1+t-s)^{-\frac{3}{2}}\|\langle x\rangle^3(W^{X_s-X_t}-W^{\widetilde{X}_s-\widetilde{X}_t})(h^{X_s-X_t})^{-1}\langle x\rangle^{-3}\|_2\|\langle x\rangle^3 h^{X_s-X_t}r_{N_0}^{-X_t}\|_2\mathrm{d}s\\
\lesssim& g\int_0^t (1+t-s)^{-\frac{3}{2}}\|\langle x\rangle^3(W^{X_s-X_t}-W^{\widetilde{X}_s-\widetilde{X}_t})(h^{X_s-X_t})^{-1}\langle x\rangle^{-3}\|_2(1+s)^{-\frac{3}{2}}\mathrm{d}s\\
\lesssim& g\|Q_1-Q_2\|_{T_{\rm loc},\delta}\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{\frac{1}{2}-\delta}\|\langle x\rangle^3\partial_x W(h^{X_s-X_t})^{-1}\langle x\rangle^{-3}\|_2(1+s)^{-\frac{3}{2}}\mathrm{d}s\\
\lesssim& g\|Q_1-Q_2\|_{T_{\rm loc},\delta}\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{\frac{1}{2}-\delta}(1+s)^{-\frac{3}{2}}\mathrm{d}s\\
\lesssim& g\|Q_1-Q_2\|_{T_{\rm loc},\delta}t^{-1-\delta}\,,
\end{aligned}$$

where in the second step we used

$$\|\langle x\rangle^3 h^{X_s}r_{N_0}(s)\|_2 \lesssim (1+s)^{-\frac{3}{2}}\,.$$

We are left with a last term,

$$|\langle\partial_x W\,, \int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}h^{\widetilde{X}_s-\widetilde{X}_t}(r_{N_0}^{-X_t} - \widetilde{r}_{N_0}^{-\widetilde{X}_t})(s)\mathrm{d}s\rangle|\,. \tag{5.38}$$

Recall that $r_{N_0}(s)$ is the remainder term in the Taylor expansion of $(h^{X_s})^{-1}W^{X_s}$, so we can write

$$r_{N_0}(s) = (-1)^{N_0+1}\int_t^s\int_t^{s_1}\dots\int_t^{s_{N_0}} \partial_x^{j_1}\dots\partial_x^{j_{N_0+1}}(h^{X_{s_{N_0+1}}})^{-1}W^{X_{s_{N_0+1}}}\dot X_{s_{N_0+1}}^{j_{N_0+1}}\dots\dot X_{s_1}^{j_1}\mathrm{d}\underline{s}\,.$$

In (5.38) we use $h^Y = h + g(W^Y - W)$, the W-term of which is easy to estimtate, and for the h-term we observe

$$i\mathrm{e}^{-\mathrm{i}ht}hf(t) = \mathrm{i}\partial_t(\mathrm{e}^{-\mathrm{i}ht}f(t)) - \mathrm{i}\mathrm{e}^{\mathrm{i}hs}\partial_t f(t)\,,$$

of which the first term leads even to improved decay, and the second is by now standard. Each term $\dot X(s) = Q_1(s)$, and $\dot{\tilde X}(s) = Q_2(s)$ gives a decay factor of order $s^{-\frac12-\delta}$, and the many derivatives render $(h)^{-1}$ harmless. So with the usual estimates we obtain

$$\begin{aligned}&|\langle\partial_x W\,, \int_0^t \mathrm{e}^{-\mathrm{i}h(t-s)}h^{\widetilde X_s-\widetilde X_t}(r_{N_0}^{-X_t} - \widetilde r_{N_0}^{-\widetilde X_t})(s)\mathrm{d}s\rangle|\\ \lesssim&(\|Q_1\|_{\delta,T_{\mathrm{loc}}} + \|Q_2\|_{\delta,T_{\mathrm{loc}}})\|Q_1 - Q_2\|_{\delta,T_{\mathrm{loc}}}t^{-1-\delta}\,.\end{aligned}$$

The last step is to incorporate the term $g\langle\partial_x W^{X_t}\,, |\delta_t|^2\rangle$. But this is straight forward, because

$$|\delta_t(Q_1)|^2 - |\delta_t(Q_2)|^2 = (\delta_t(Q_1) - \delta_t(Q_2))\delta_t^*(Q_1) + \delta_t(Q_2)(\delta_t^*(Q_1) - \delta_t^*(Q_2))\,,$$

and so, using the estimates from above,

$$\begin{aligned}|\langle\partial_x W^{X_t}\,, |\delta_t(Q_1)|^2 - |\delta_t(Q_2)|^2\rangle| &\lesssim \|\langle x\rangle^3\partial_x W^{X_t}\delta_t(Q_1) - \delta_t(Q_2)\|_1\|\langle x\rangle^{-3}\delta_t^*(Q_1)\|_\infty\\ &\lesssim (\|Q_1\|_{T_{\mathrm{loc}},\delta} + \|Q_2\|_{T_{\mathrm{loc}},\delta})\|Q_1 - Q_2\|_{T_{\mathrm{loc}},\delta}t^{-1-\delta}t^{-\frac12}\,,\end{aligned}$$

where we also used

$$\|\langle x\rangle^{-3}\delta_t^*(Q_1)\|_\infty \lesssim t^{-\frac12}\,,$$

which is proven in complete analogy to (5.10).

□

10. Proof of Proposition 5.9

As in [**24**], we prove first

LEMMA 5.14.

$$|A(\chi_{T_{\mathrm{loc}}}(P))| \le \varepsilon(T_{\mathrm{loc}})t^{-\frac12-\delta}\,.$$

PROOF. Recall the definition of A in (5.18) and the local existence estimate $|P_t| \le T_{\mathrm{loc}}^{-2}$ for $t \in [0, T_{\mathrm{loc}}]$. Then compute

$$\begin{aligned}|A(\chi_{T_{\mathrm{loc}}}P)| \le& Z\varepsilon(T_{\mathrm{loc}})[\int_0^{T_{\mathrm{loc}}} |K(t-s) - K(t)||\mathrm{Re}\,\langle[1 - g(h)^{-1}W]\partial_{x_1}W\,, \mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle|\mathrm{d}s\\ &+|\int_0^t K(t-s)\mathrm{Re}\,\langle[1 - g(h)^{-1}W]\partial_{x_1}W\,, \mathrm{e}^{-\mathrm{i}hs}\partial_{x_1}(h)^{-1}W\rangle\mathrm{d}s|\\ &+|K(t)|\int_0^{T_{\mathrm{loc}}} |\mathrm{Re}\,\langle[1 - g(h)^{-1}W]\partial_{x_1}W\,, (-\mathrm{i}h)^{-1}[\mathrm{e}^{-\mathrm{i}h(t-s)} - \mathrm{e}^{-\mathrm{i}ht}]\partial_{x_1}(h)^{-1}W\rangle\mathrm{d}s]\,.\end{aligned}$$

As proved in (5.33) the second term on the right hand side is of order $t^{-\frac32}$. For the third term, we have by a computation similar to (5.35)

$$|\mathrm{Re}\,\langle[1 - g(h)^{-1}W]\partial_{x_1}W\,, (-\mathrm{i}h)^{-1}[\mathrm{e}^{-\mathrm{i}h(t-s)} - \mathrm{e}^{-\mathrm{i}ht}]\partial_{x_1}(h)^{-1}W\rangle| \lesssim (1+t-s)^{-\frac12}(1+t)^{-1}s + (1+t-s)^{-\frac32}\,.$$

So we obtain

$$
\begin{aligned}
|A(\chi_{T_{\rm loc}}P)| \lesssim & \varepsilon(T_{\rm loc})[(1+t)^{-1}\int_0^{T_{\rm loc}}(1+t-s)^{-\frac{1}{2}}\frac{s}{(1+s)^{\frac{3}{2}}}\mathrm{d}s+\int_0^{T_{\rm loc}}(1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}}\mathrm{d}s \\
&+(1+t)^{-\frac{3}{2}}+(1+t)^{-\frac{3}{2}}\int_0^{T_{\rm loc}}(1+t-s)^{-\frac{1}{2}}s\mathrm{d}s+(1+t)^{-\frac{1}{2}}\int_0^{T_{\rm loc}}(1+t-s)^{-\frac{3}{2}}\mathrm{d}s] \\
&\leq \varepsilon(T_{\rm loc})(1+t)^{-\frac{1}{2}-\delta}\,,
\end{aligned}
$$

where $\varepsilon(T_{\rm loc}) \to 0$ as $T_{\rm loc} \to \infty$. □

We are left with proving

$$
\int_0^t [K(t-s)-K(t)]\widetilde{D}_k(\chi_{T_{\rm loc}}P,s)\mathrm{d}s\big| \leq t^{-\frac{1}{2}-\delta}\varepsilon(T_{\rm loc})\,,
$$

for $k=3,4,5,6$. As in in the proof of Proposition 5.8, all we have to show is

$$
|\widetilde{D}_k(\chi_{T_{\rm loc}}P,s)| \leq s^{-1-\delta}\varepsilon(T_{\rm loc})\,.
$$

The estimates are very similar to the ones in the proof of Proposition 5.8, so we will do only the first two, $\widetilde{D}_3$ and $\widetilde{D}_4$.

$$
\begin{aligned}
|\widetilde{D}_3(\chi_{T_{\rm loc}}P,t)| \lesssim & |\langle\partial_x W^{X_t}, \mathrm{e}^{-\mathrm{i}h^{X_t}t}\beta_0\rangle| + |\langle \partial_x W^{X_t}, \mathrm{e}^{-\mathrm{i}h^{X_t}t}\sum_{|\alpha|=2}^{N_0}\frac{1}{\alpha!}(X_0-X_t)^\alpha\partial_x^\alpha(h^{X_t})^{-1}W^{X_t}\rangle| \\
\lesssim & \|\partial_x W^{X_t}\langle x\rangle^3\|_2 t^{-\frac{3}{2}}\|\langle x\rangle^3\beta_0\| + \|\partial_x W\langle x\rangle^3\|_2 t^{-\frac{3}{2}}\,.
\end{aligned}
$$

This, together with (5.16) and the fact that we consider $t \geq T_{\rm loc}$, imply the claim.
For $\widetilde{D}_4$, observe

$$
\begin{aligned}
|\widetilde{D}_4(\chi_{T_{\rm loc}}P,t)| &\lesssim g|\langle\partial_x W^{X_t}, \int_0^{T_{\rm loc}}\mathrm{e}^{-\mathrm{i}h^{X_t}(t-s)}\partial_x W^{X_t}\cdot(X_s-X_{T_{\rm loc}})\delta_s\mathrm{d}s\rangle| \\
&\lesssim g\int_0^{T_{\rm loc}}(1+t-s)^{-\frac{5}{2}}\int_s^{T_{\rm loc}}|P_{s_1}|\mathrm{d}s_1\|\langle x\rangle^{-3}\delta_s^{-X_t}\|_2\mathrm{d}s\,.
\end{aligned}
$$

Because of the local existence estimate (5.14), we have

$$
\int_s^{T_{\rm loc}}|P_{s_1}|\mathrm{d}s_1 \leq T_{\rm loc}^{-2}\,.
$$

So we can estimate

$$
|\widetilde{D}_4(\chi_{T_{\rm loc}}P,t)| \lesssim gT_{\rm loc}^{-2}\int_0^{T_{\rm loc}}(1+t-s)^{-\frac{5}{2}}(1+s)^{-\frac{1}{2}}\mathrm{d}s \lesssim gT_{\rm loc}^{-\frac{3}{2}}(1+t)^{-\frac{5}{2}}\,.
$$

The term $g\langle\partial_x W^{X_t}, |\delta_t|^2(\chi_{T_{\rm loc}}P)\rangle$ is treated as follows,

$$
\begin{aligned}
|g\langle\partial_x W^{X_t}, \delta_t(\chi_{T_{\rm loc}}P)\delta_t^*(\chi_{T_{\rm loc}}P)\rangle| &\lesssim \|\langle x\rangle^3\partial_x W\delta_t(\chi_{T_{\rm loc}}P)\|_1\|\langle x\rangle^{-3}\delta_t^*(\chi_{T_{\rm loc}}P)\|_\infty \\
&\lesssim \int_0^{T_{\rm loc}}(1+t-s)^{-\frac{3}{2}}T_{\rm loc}^{-2}\mathrm{d}st^{-\frac{1}{2}} \\
&\lesssim T_{\rm loc}^{-1}t^{-2}\,.
\end{aligned}
$$

□

11. Propagator Estimates

In this section we prove the propagator estimates used throughout the second part of the thesis. Define $h := -\Delta + gW$, with $g \in \mathbb{R}$ small and $W(x) = W(|x|)$.

PROPOSITION 5.15. *If $W : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and decays exponentially fast at ∞ we have*

$$\|\langle x\rangle^{-3}\mathrm{e}^{\mathrm{i}th}(h)^{-1+\varepsilon}\phi\|_2 \le C(1+t)^{-\frac{1}{2}(1+2\varepsilon)}\|\langle x\rangle^3\phi\|_2\,, \quad \varepsilon\in[0,1] \tag{5.39}$$

$$\|\langle x\rangle^{-5}\mathrm{e}^{\mathrm{i}th}\partial_x(h)^{-1}W\|_2 \le C(1+t)^{-\frac{3}{2}} \tag{5.40}$$

$$\|\langle x\rangle^{-5}\mathrm{e}^{\mathrm{i}th}\partial_x^\alpha(h)^{-1}W\|_2 \le C(1+t)^{-\frac{3}{2}}\,, \quad |\alpha|\ge 2\,. \tag{5.41}$$

Estimate (5.39) is a classic result, see e.g. [**34**]. In the proof of the remaining assertions we will use the following

LEMMA 5.16. *For any smooth, spherically symmetric and fast decaying function φ we have*

$$\|\langle x\rangle^{-5}\mathrm{e}^{-\mathrm{i}t\Delta}\partial_x(-\Delta)^{-1}\varphi\|_2 \le C(1+t)^{-\frac{3}{2}}\|\langle x\rangle^4\varphi\|_2\,.$$

PROOF. By Fourier transform we obtain

$$\mathrm{e}^{-\mathrm{i}t\Delta}\partial_x(-\Delta)^{-1}\varphi = C\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}k\cdot x}\mathrm{e}^{\mathrm{i}t|k|^2}\frac{k}{|k|^2}\hat{\varphi}(k)\mathrm{d}k\,,$$

for some constant $C\in\mathbb{R}$. Since φ is spherically symmetric, so is $\hat{\varphi}$. Using polar coordinates ($\mathbb{R}^3 \ni k = \rho g(\alpha,\vartheta)$) we find

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}t\Delta}\partial_x(-\Delta)^{-1}\varphi &= C\int_{-1}^1\int_0^{2\pi}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho|x|\cos\vartheta}\mathrm{e}^{\mathrm{i}t\rho^2}\rho g(\alpha,\vartheta)\hat{\varphi}(\rho)\mathrm{d}\rho\mathrm{d}\alpha\mathrm{d}\cos\vartheta\\
&= C\int_{-1}^1\int_0^{2\pi}\int_0^\infty \mathrm{e}^{\mathrm{i}t\rho^2}\rho g(\alpha,\vartheta)\hat{\varphi}(\rho)\mathrm{d}\rho\mathrm{d}\alpha\mathrm{d}\cos\vartheta\\
&+ C\int_{-1}^1\int_0^{2\pi}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\rho|x|\cos\vartheta}-1}{\rho}\rho^2\mathrm{e}^{\mathrm{i}t\rho^2}g(\alpha,\vartheta)\hat{\varphi}(\rho)\mathrm{d}\rho\mathrm{d}\alpha\mathrm{d}\cos\vartheta\\
&= C\int_{-1}^1\int_0^{2\pi}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\rho|x|\cos\vartheta}-1}{\rho}\rho^2\mathrm{e}^{\mathrm{i}t\rho^2}g(\alpha,\vartheta)\hat{\varphi}(\rho)\mathrm{d}\rho\mathrm{d}\alpha\mathrm{d}\cos\vartheta\,,
\end{aligned}$$

as the unit vector $g(\alpha,\vartheta)$ averages to zero over the unit sphere. Denote by $f_x(\rho)$ the smooth function $\frac{\mathrm{e}^{\mathrm{i}\rho|x|\cos\vartheta}-1}{\rho}$ and evaluate the ρ-integral by scaling $\rho\to t^{\frac{1}{2}}\rho$ as follows:

$$\int_0^\infty f_x(\rho)\rho^2\mathrm{e}^{\mathrm{i}t\rho^2}\hat{\varphi}(\rho)\mathrm{d}\rho = t^{-\frac{3}{2}}\int_0^\infty f_x(\rho t^{-\frac{1}{2}})\rho^2\mathrm{e}^{\mathrm{i}\rho^2}\hat{\varphi}(\rho t^{-\frac{1}{2}})\mathrm{d}\rho\,.$$

Since $\rho^2\mathrm{e}^{\mathrm{i}\rho^2}$ is not integrable we integrate by parts which yields

$$\begin{aligned}
t^{-\frac{3}{2}}\int_0^\infty f_x(\rho t^{-\frac{1}{2}})\rho^2\mathrm{e}^{\mathrm{i}\rho^2}\hat{\varphi}(\rho t^{-\frac{1}{2}})\mathrm{d}\rho &= -t^{-\frac{3}{2}}\frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho^2}\partial_\rho(\rho f_x(\rho t^{-\frac{1}{2}})\hat{\varphi}(\rho t^{-\frac{1}{2}}))\mathrm{d}\rho\\
= -t^{-\frac{3}{2}}\frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho^2}f_x(\rho t^{-\frac{1}{2}})\hat{\varphi}(\rho t^{-\frac{1}{2}})\mathrm{d}\rho &- t^{-\frac{3}{2}}\frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho^2}\rho\partial_\rho(f_x(\rho t^{-\frac{1}{2}})\hat{\varphi}(\rho t^{-\frac{1}{2}}))\mathrm{d}\rho\,.
\end{aligned}$$

The first term on the second line is easily seen to be given by

$$t^{-\frac{3}{2}}\frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho^2}f_x(\rho t^{-\frac{1}{2}})\hat{\varphi}(\rho t^{-\frac{1}{2}})\mathrm{d}\rho = Ct^{-\frac{3}{2}}(f_x(0)\hat{\varphi}(0)+o(1)) = Ct^{-\frac{3}{2}}(\cos\vartheta|x|\hat{\varphi}(0)+o(1))\,,$$

where $o(1)$ is short for $o(1)\,,\ t\to\infty$. In the second term we integrate by parts again to get

$$\begin{aligned}&t^{-\frac{3}{2}}\frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho^2}\rho\partial_\rho(f_x(\rho t^{-\frac{1}{2}})\hat\varphi(\rho t^{-\frac{1}{2}}))\mathrm{d}\rho\\ &=-Ct^{-\frac{3}{2}}(f_x'(0)\hat\varphi(0)+f_x(0)\hat\varphi'(0))-t^{-\frac{3}{2}}\frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho^2}\partial_\rho^2(f_x(\rho t^{-\frac{1}{2}})\hat\varphi(\rho t^{-\frac{1}{2}}))\mathrm{d}\rho\,.\end{aligned}$$

The last term is given by

$$t^{-\frac{3}{2}}\frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho^2}\partial_\rho^2(f_x(\rho t^{-\frac{1}{2}})\hat\varphi(\rho t^{-\frac{1}{2}}))\mathrm{d}\rho=Ct^{-\frac{3}{2}}(\partial_r^2\Big|_{r=0}f_x(r)\hat\varphi(r)+o(1))\,.$$

Summarizing, we have shown

$$|\mathrm{e}^{-\mathrm{i}t\Delta}\partial_x(-\Delta)^{-1}\varphi(x)|\le Ct^{-\frac{3}{2}}(|x|^3\|y^2\varphi\|_1+o(1))\,,$$

because $f_x''(0)=-\mathrm{i}\cos^3\vartheta|x|^3$, and $\hat\varphi''(0)=\int y^2\varphi\mathrm{d}^3y$. Using Hölder's inequality we arrive at

$$\|\langle x\rangle^{-5}\mathrm{e}^{-\mathrm{i}t\Delta}\partial_x(-\Delta)^{-1}\varphi\|_2\le Ct^{-\frac{3}{2}}\|\langle x\rangle^4\varphi\|_2\,,$$

which is the claim. □

Proof of Proposition 5.15. We only prove (5.40), the proof of (5.41) is easier and hence omitted. Define the function

$$\xi:=(1+gW(-\Delta)^{-1})^{-1}W\,.$$

The function ξ is spherically symmetric, and from the equation $(-\Delta)^{-1}\xi=h^{-1}W$ we get

$$\xi=(-\Delta)h^{-1}W=W-gWh^{-1}W\,,$$

from wich it is easy to see that ξ decays exponentially fast at ∞, since h^{-1} is a bounded operator $\mathscr{H}^{2,s}\to\mathscr{H}^{2,-s}$ for $s>\frac{1}{2}$.
By Duhamel's principle, we rewrite $\mathrm{e}^{\mathrm{i}th}\partial_xh^{-1}W$ as

$$\begin{aligned}\mathrm{e}^{\mathrm{i}th}\partial_xh^{-1}W&=\mathrm{e}^{\mathrm{i}th}\partial_x(-\Delta)^{-1}\xi\\ &=\mathrm{e}^{-\mathrm{i}t\Delta}\partial_x(-\Delta)^{-1}\xi+\int_0^t\mathrm{e}^{\mathrm{i}h(t-s)}gW\mathrm{e}^{-\mathrm{i}s\Delta}\partial_x(-\Delta)^{-1}\xi\mathrm{d}s\,.\end{aligned}$$

The desired estimate follows from (5.39) and Lemma 5.16. □

12. Weighted L^2 spaces

Weighted L^2 spaces are a useful tool when dealing with Schrödinger operators $-\Delta+V(x)$, since the resolvent $(-\Delta+V-z)^{-1}$ may not remain bounded as an operator on L^2 as z approaches the real axis. It remains, however, bounded as an operator between certain weighted L^2 spaces.

Definition 12.1. For any $s\in\mathbb{R}$, we denote by $L^{2,s}$ the set of functions

$$\langle x\rangle^{-s}L^2\,,$$

which is the set of all measurable functions f that satisfy

$$\|\langle x\rangle^sf\|_2<\infty\,.$$

Clearly, $L^{2,s}\subset L^{2,s'}$ for $s'<s$, so that we define

$$L^{2,s-0}:=\bigcap_{\gamma<s}L^{2,\gamma}\,.$$

It is a classic result (see e.g. [**2**]), sometimes called the limiting absorption principle, that for sufficiently fast decaying potential V, the limits

$$(-\Delta+V-E\pm\mathrm{i}0)^{-1}\qquad E\neq\text{eigenvalue}$$

exist as bounded operators from $L^{2,s}$ to $L^{2,-s}$ for $s > \frac{1}{2}$ ($s > 1$ for $E = 0$).

13. Absence of Eigenvalues and Zero Resonance

In this section, we show that the operator $H = -\Delta + gW$ has no eigenvalues or zero resonance for g small enough. There are many results in the literature that cover the case of a smooth, exponentially decaying potential (see for instance [**50**]), but since part of the proofs are elementary we give them here for completeness.

(i) Absence of eigenvalues ≤ 0 and zero resonance:
This is elementary to show. Recall that a zero resonance is defined as a solution of $H\psi = 0$ that is not in L^2 but in $L^{2,-\frac{1}{2}-0}$. Let thus $E \leq 0$ and assume that $\psi \in L^{2,-\frac{1}{2}-0}$ satisfies the equation

$$(-\Delta + gW - E)\psi = 0 \,.$$

Then we have

$$\begin{aligned} (\mathbb{1} + (-\Delta - E)^{-1} gW)\psi &= 0 \\ (\mathrm{e}^{\varepsilon|x|} + g(-\Delta - E)^{-1} W \mathrm{e}^{\varepsilon|x|})\mathrm{e}^{-\varepsilon|x|}\psi &= 0 \\ (\mathbb{1} + g\mathrm{e}^{-\varepsilon|x|}(-\Delta - E)^{-1} W \mathrm{e}^{\varepsilon|x|})\mathrm{e}^{-\varepsilon|x|}\psi &= 0 \,, \end{aligned} \tag{5.42}$$

for some small $\varepsilon > 0$. For brevity, set $A_E := \mathrm{e}^{-\varepsilon|x|}(-\Delta - E)^{-1} W \mathrm{e}^{\varepsilon|x|}$. By looking at the integral kernel of A_E,

$$A_E(x,y) = \mathrm{e}^{-\varepsilon|x|} \frac{\mathrm{e}^{-|E|^{\frac{1}{2}}|x-y|}}{|x-y|} W(y) \mathrm{e}^{\varepsilon|y|} \quad \in L^2(\mathbb{R}^6) \,,$$

we see that A_E is Hilbert-Schmidt, so in particular compact (uniformely in E). By the analytic Fredholm theorem, and the fact that $\mathbb{1} + gA_E$ is invertible for $g = 0$, we conclude that there exists an $r > 0$ such that for all $|g| < r$ and all $E \leq 0$, the resolvent $(\mathbb{1} + gA_E)^{-1}$ exists. Equation (5.42) thus implies $\psi = 0$, so that $E \leq 0$ is *not* an eigenvalue, nor $E = 0$ a zero resonance.

(ii) Absence of Positive Eigenvalues:
This is considerably harder than the above. See [**50**] for the classic results.

APPENDIX A

Cluster expansion

In regime (A), that is, for pure EuB_6 above the Curie temperature, we should allow for weak correlations between the random magnetic moments at different sites, if we want our model of a disordered solid to be (more) realistic. The cluster expansion is a well established, robust method suiting this purpose. In the following, we give an introduction to this topic and state the most important results.

1. Connected parts

Let Λ be a finite set. For a symmetric function $\varphi : \{a \subset \Lambda\} \to \mathbb{C}$ define the "connected parts" ("truncated function","Ursell function") φ^{T} recursively by

$$\varphi(a) =: \sum_{\Pi \in \text{partitions of } a} \prod_{b \in \Pi} \varphi^{\mathrm{T}}(b)\,. \tag{A.1}$$

We will be dealing with φ of the form $\varphi = \mathrm{e}^{-V}$ where V is some other symmetric function modeling the interaction between elements of Λ. Note that any symmetric function $V : \{a \subset \Lambda\} \to \mathbb{C}$ can be written as a sum of "atoms" v as follows

$$\begin{aligned} V(a) &= \sum_{b \subset a} v_b \\ v_a &:= V(a) - \sum_{b \subsetneq a} v_b \qquad \forall \varnothing \neq a \subset \Lambda\,. \end{aligned}$$

To understand the denomination "connected parts" for φ^{T} compute

$$\varphi(\Lambda) = \exp(-V(\Lambda)) = \exp(-\sum_{a \in \mathcal{P}_*(\Lambda)} v_a) = \prod_{a \in \mathcal{P}_*(\Lambda)} \exp(-v_a)\,,$$

where $\mathcal{P}_*(\Lambda)$ denotes the power set of Λ without the empty set. We rewrite this with the usual trick

$$\prod_{a \in \mathcal{P}_*(\Lambda)} \exp(-v_a) = \prod_{a \in \mathcal{P}_*(\Lambda)} (\exp(-v_a) - 1 + 1) = \sum_{A \subset \mathcal{P}_*(\Lambda)} \prod_{a \in A} \zeta(a)\,, \tag{A.2}$$

where we introduced the standard abbreviation $\zeta(a) := \exp(-v_a) - 1$.

Any $A \subset \mathcal{P}_*(\Lambda)$ is decomposed uniquely into sets $C = \{a_1, \ldots, a_n\}$, $a_i \subset \Lambda$ that are $\bigcup_i a_i$-*connected* in the following sense: For any two points $p, q \in \bigcup_i a_i \subset \Lambda$ there exists a sequence of $a_i \in C$ such that $p \in a_1, a_1 \cap a_2 \neq \varnothing, \ldots, a_{m-1} \cap a_m \neq \varnothing, q \in a_m$. The empty set is, by definition, a-connected if and only if $|a| = 1$. We therefore have

$$\sum_{A \subset \mathcal{P}_*(\Lambda)} \prod_{a \in A} \zeta(a) = \sum_{\Pi \in \text{partitions of } \Lambda} \prod_{y \in \Pi} \sum_{\substack{C \subset \mathcal{P}_*(y) \\ C\ y\text{-connected}}} \prod_{a \in C} \zeta(a)\,, \tag{A.3}$$

where the sets $\bigcup_i a_i$ from the decomposition of A constitute the sets y of the partition. An empty product is defined to be 1. Comparing (A.3) to (A.1) we find an explicit expression for the connected parts:

$$\varphi^{\mathrm{T}}(y) = \exp(-V)^{\mathrm{T}}(y) = \sum_{\substack{C \subset \mathcal{P}_*(y) \\ C\ y\text{-connected}}} \prod_{a \in C} \zeta(a)\,. \tag{A.4}$$

The expansion in connected parts is particularly transparent if V is a sum of two-body potentials v:

$$V(\Lambda) = V(p_1, p_2, \ldots, p_n) := \sum_{1 \le i < j \le n} v(p_i, p_j)\,,$$

so the "atoms" v_a are nonzero only for two-element subsets $a \subset \Lambda$:

$$v_a = \begin{cases} v(p,q) & a = \{p,q\} \\ 0 & \text{otherwise}\,. \end{cases}$$

The sets $A \subset \mathcal{P}_*(\Lambda)$ in (A.2) are sets of two-element subsets of Λ, that is, *graphs*:

$$\mathrm{e}^{-V} = \prod_{1 \le i < j \le n} \mathrm{e}^{-v(p_i, p_j)} = \sum_{G \in \mathcal{G}_n} \prod_{(i,j) \in G} \zeta(p_i, p_j)\,,$$

where $\mathcal{G}_n$ denotes the set of all (unoriented) graphs with n vertices. Our notion of connectedness for a set $C \in \mathcal{P}(\mathcal{P}_*(\Lambda))$ clearly coincides for two-element subsets with the natural one for graphs. Thus we get the name-giving formula

$$\varphi^{\mathrm{T}}(y) = \exp(-V)^{\mathrm{T}}(y) = \sum_{G \in \mathcal{C}_y} \prod_{(i,j) \in G} \zeta(p_i, p_j)\,,$$

where $\mathcal{C}_y$ denotes the set of *connected* graphs with the elements of y as vertices.

Thus we understand the general situation: One is given a set of objects Λ furnished with a notion of connectedness by the interaction V that gives rise to a natural defition of connected parts of a function defined on subsets of Λ. For example, in the case of $\Lambda \subset \mathbb{Z}^d$ and v a nearest-neighbor interaction it is clear that the induced notion of connectedness is the same as the natural one of the lattice $\mathbb{Z}^d$, thus implying

$$\varphi^{\mathrm{T}}(y) = \exp(-V)^{\mathrm{T}}(y) = 0\,, \quad \text{if } y \text{ is not connected.}$$

2. Logarithm of partition function

Let (Ω, μ) be a measure space and μ a finite measure. Ω is a set of objects, such as spins on a lattice or coordinates of particles in a box, which interact with a symmetric Gibbs factor $\mathrm{e}^{-\beta V(x_1, x_2, \ldots, x_n)}$ defined on finite sequences of Ω, where β is a small parameter such as the inverse temperature.

The cluster expansion is a means to write the logarithm of the grand canonical partition function as a sum over clusters. The partition function Z is defined as

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \ldots \mathrm{d}\mu(x_n) \mathrm{e}^{-\beta V(x_1, x_2, \ldots, x_n)}$$

The goal of the cluster expansion is to use combinatorics and analysis to write Z as the exponential of an absolutely convergent sum over "simpler terms".

Upon substituting the definition (A.1) of connected parts into the partition function (with $\varphi = \mathrm{e}^{-\beta V}$), we obtain

$$\begin{aligned} Z &= \sum_n \frac{1}{n!} \sum_{\Pi \in \text{partitions of } \{1,2,\ldots,n\}} \prod_{Y \in \Pi} \int \mathrm{d}^Y \mu \, \varphi^{\mathrm{T}}(Y) \\ &= \sum_n \frac{1}{n!} \sum_{m \geq 1} \sum_{\Pi = \{X_1,\ldots,X_m\}} \prod_{Y \in \Pi} \int \mathrm{d}^Y \mu \, \varphi^{\mathrm{T}}(Y) \\ &= \sum_n \frac{1}{n!} \sum_{m \geq 1} \frac{1}{m!} \sum_{\Pi = (X_1,\ldots,X_m)} \prod_{Y \in \Pi} \int \mathrm{d}^Y \mu \, \varphi^{\mathrm{T}}(Y) \,. \end{aligned}$$

As φ, and hence φ^{T}, is symmetric the integral $\int \mathrm{d}^Y \mu \, \varphi^{\mathrm{T}}(Y)$ depends only on $|Y|$. The number of ordered partitions $\Pi = (X_1, \ldots, X_m)$, with $|X_i| = n_i$ fixed, $\sum_i n_i = n \,, n_i \geq 1$, m fixed, is $\frac{n!}{n_1! \cdots n_i!}$, so

$$Z = \sum_n \frac{1}{n!} \sum_{m \geq 1} \frac{1}{m!} \sum_{\sum_i n_i = n} n! \prod_{i=1}^m \frac{1}{n_i!} \int \mathrm{d}^{n_i} \mu \, \varphi^{\mathrm{T}}(x_1, \ldots, x_{n_i}) \,.$$

Formally, we can carry out the sum over n and get

$$Z = \sum_{m \geq 1} \frac{1}{m!} \sum_{n_1,\ldots,n_m} \prod_{i=1}^m \frac{1}{n_i!} \int \mathrm{d}^{n_i} \mu \varphi^{\mathrm{T}}(x_1, \ldots, x_{n_i}) \,.$$

Note that

$$\sum_{n_1,\ldots,n_m \geq 1} \prod_{i=1}^m a_{n_i} = \left(\sum_{n \geq 1} a_n \right)^m ,$$

to get

$$Z = \exp \left(\sum_{n \geq 1} \frac{1}{n!} \int \mathrm{d}^n \mu \, \varphi^{\mathrm{T}}(x_1, \ldots, x_n) \right) . \tag{A.5}$$

Written for the case of two-body interactions, the partition function and its logarithm take the form

$$\begin{aligned} Z &= \sum_{n \geq 0} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \ldots \mathrm{d}\mu(x_n) \sum_{G \in \mathcal{G}_n} \prod_{(i,j) \in G} \zeta(x_i, x_j) \,, \\ \log Z &= \sum_{n \geq 1} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \ldots \mathrm{d}\mu(x_n) \sum_{G \in \mathcal{C}_n} \prod_{(i,j) \in G} \zeta(x_i, x_j) \,. \end{aligned}$$

Remark. Compare also to the probabilistic version. Let $X_1, \ldots, X_n$ be random variables on a probability space Ω and consider the moment generating function

$$\mathbb{E}\, \mathrm{e}^{X \cdot t} = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \mu_\alpha t^\alpha \,,$$

where α is a multiindex and $\mu_\alpha := \mathbb{E}\, X^\alpha$. Its logarithm is the cumulant generating function

$$\log \mathbb{E}\, \mathrm{e}^{X \cdot t} = \sum_{|\alpha| \geq 1} \frac{1}{\alpha!} \kappa_\alpha t^\alpha \,,$$

where κ_α are the cumulants, related to the moments via

$$\mu_\alpha = \sum_{\Pi \in \text{partitions of } \alpha} \prod_{Y \in \Pi} \kappa_Y \,.$$

3. Polymer models

Let us now turn to the random Zeeman interaction model of regime (A), that is, we look at an electron on the lattice $\mathbb{Z}^d$ interacting with a static background of random magnetic moments, represented by unit vectors $\{m\}_{j\in\mathbb{Z}^d}$ distributed as

$$\begin{aligned}\mathrm{d}\mathbb{P}(m) &= Z^{-1}\exp\{\kappa\sum_{|x-y|=1} m_x\cdot m_y + \beta B\sum_x m_x^{(3)}\}\prod_x \delta(m_x^2-1)\mathrm{d}^3 m_x\\ &=: Z^{-1}\exp(-V(\mathbf{m}))\prod_x \mathrm{d}\mu(m_x)\,.\end{aligned}$$

Since the number of lattice sites is constant for a fixed finite $\Lambda\subset\mathbb{Z}^d$ considered in our application, we first have to find a suitable grand canonical formulation of the problem. Expanding the Gibbs factor as in (A.2) yields

$$\begin{aligned}Z_\Lambda &= \int\prod_{x\in\Lambda}\mathrm{d}\mu(m_x)\exp(-V(\mathbf{m})) = \int\prod_{x\in\Lambda}\mathrm{d}\mu(m_x)\sum_{A\subset\mathcal{P}_*(\Lambda)}\prod_{a\in A}\zeta(a)\,.\\ &= \int\prod_{x\in\Lambda}\mathrm{d}\mu(m_x)\sum_n\frac{1}{n!}\sum_{\substack{C_1,\dots,C_n\subset\mathcal{P}_*(\Lambda)\\ C_i\text{ connected}}}\prod_{i<j}(1+\widetilde{\zeta}(C_i,C_j))\prod_{i=1}^n\prod_{a\in C_i}\zeta(a)\,,\end{aligned}$$

where $\widetilde{\zeta}$ takes care of the non-intersection property of the C_i:

$$\widetilde{\zeta}(C_1,C_2) := \begin{cases}-1 & (\bigcup_{c\in C_1} c)\cap(\bigcup_{c\in C_2} c)\neq\varnothing\\ 0 & \text{otherwise}\,.\end{cases}$$

The notion of connectedness of the C_i is the one introduced in section 1.3.1. Note that such a $\widetilde{\zeta}$ corresponds to a two-body hard-core interaction,

$$\begin{aligned}\widetilde{V}(C_1,C_2,\dots,C_n) &= \sum_{1\le i<j\le n}\widetilde{v}(C_i,C_j)\\ \widetilde{v}(C_1,C_2) &= \begin{cases}\infty & (\bigcup_{c\in C_1} c)\cap(\bigcup_{c\in C_2} c)\neq\varnothing\\ 0 & \text{otherwise}\end{cases}\\ \widetilde{\zeta}(C_1,C_2) &= \exp(-\widetilde{v}(C_1,C_2))-1\,.\end{aligned}$$

Since we chose the C_i in such a way that they do not share elements the integral factorizes (in addition, we use here that $\mathrm{d}\mu$ is a probability measure)

$$Z_\Lambda = \sum_n\frac{1}{n!}\sum_{\substack{C_1,\dots,C_n\subset\mathcal{P}_*(\Lambda)\\ C_i\text{ connected}}}\prod_{i<j}(1+\widetilde{\zeta}(C_i,C_j))\prod_{i=1}^n\underbrace{\int\prod_{x\in\underline{C_i}}\mathrm{d}\mu(m_x)\prod_{a\in C_i}\zeta(a)}_{=:z(C_i)}\,,$$

and we have arrived at a so-called polymer system:

$$Z_\Lambda = \sum_n\frac{1}{n!}\sum_{\substack{C_1,\dots,C_n\subset\mathcal{P}_*(\Lambda)\\ C_i\text{ connected}}}\prod_{i<j}(1+\widetilde{\zeta}(C_i,C_j))\prod_{i=1}^n z(C_i)$$

$$\log Z_\Lambda = \sum_{n\ge1}\frac{1}{n!}\sum_{\substack{C_1,\dots,C_n\subset\mathcal{P}_*(\Lambda)\\ C_i\text{ connected}}}\widetilde{\varphi}^{\mathrm{T}}(C_1,\dots,C_n)\prod_{i=1}^n z(C_i)\,, \tag{A.6}$$

where $\widetilde{\varphi} = \mathrm{e}^{-\widetilde{V}}$. The above expression for the logarithm of Z_Λ is just formula (A.5) with weighted counting measure and hard-core interaction $\widetilde{V}$. To be very clear we write $\underline{C}$ for the "support" ($\subset \Lambda$) of the polymer $C \subset \mathcal{P}_*(\Lambda)$.

Sometimes in the literature (see e.g. [**11**]), this procedure of creating a polymer model is considered a first instance of "taking connected parts" with respect to connectedness induced by V. That both are the same has been made clear in the section on connected parts:

$$Z_\Lambda = \int \prod_{x\in\Lambda} \mathrm{d}\mu(m_x) \exp(-V(\mathbf{m})) = \sum_{\Pi\in\text{partitions of }\Lambda} \prod_{Y\in\Pi} \int \mathrm{d}^Y \mu \exp(-V)^{\mathrm{T}}(Y)\,.$$

$$= \sum_n \frac{1}{n!} \sum_{\substack{Y_1,\dots,Y_n\subset\Lambda \\ \bigcup_i Y_i=\Lambda}} \prod_{i<j} (1+\widetilde{\zeta}(Y_i,Y_j)) \prod_{i=1}^n \int \mathrm{d}^{Y_i}\mu \exp(-V)^{\mathrm{T}}(Y_i)\,,$$

where

$$\widetilde{\zeta}(Y,Z) := \begin{cases} -1 & Y\cap Z \neq \varnothing \\ 0 & \text{otherwise}\,. \end{cases}$$

Since (A.4) yields $\exp(-V)^{\mathrm{T}}(Y) = 1 + (\exp(-v(Y)) - 1)$ for $|Y| = 1$, and $\int \mathrm{d}\mu(h) = 1$, it is clear that we can drop the condition $\cup_i Y_i = \Lambda$ if we modify the weights of Y's with $|Y| = 1$:

$$Z_\Lambda = \sum_n \frac{1}{n!} \sum_{Y_1,\dots,Y_n} \prod_{i<j} (1+\widetilde{\zeta}(Y_i,Y_j)) \prod_{i=1}^n w(Y_i)\,,$$

where

$$w(Y) := \begin{cases} \int \mathrm{d}^Y \mu \exp(-V)^{\mathrm{T}}(Y) & \text{if } |Y| > 1 \\ \int \mathrm{d}^Y \mu \exp(-V)^{\mathrm{T}}(Y) - 1 & \text{if } |Y| = 1\,. \end{cases}$$

In the general terms of the previous section, $\Omega = \{\text{subsets of } \Lambda\}$, and μ is the counting measure multiplied by the weight factor w. Thus we write

$$Z_\Lambda = \sum_n \frac{1}{n!} \int \mathrm{d}\mu(Y_1)\dots\mathrm{d}\mu(Y_n) \prod_{i<j} (1+\widetilde{\zeta}(Y_i,Y_j))$$

and consequently, (A.5) yields

$$\log Z_\Lambda = \sum_{n\geq 1} \frac{1}{n!} \int \mathrm{d}\mu(Y_1)\dots\mathrm{d}\mu(Y_n) \sum_{G\in\mathcal{C}_n} \prod_{(i,j)\in G} \widetilde{\zeta}(Y_i,Y_j)$$

$$= \sum_{n\geq 1} \frac{1}{n!} \sum_{Y_1,\dots,Y_n} \prod_{i=1}^n w(Y_i) \sum_{G\in\mathcal{C}_n} \prod_{(i,j)\in G} \widetilde{\zeta}(Y_i,Y_j)\,,$$

where we used graph-notation since we are dealing with a polymer system and hence with a two-point interaction (hard-core). Because of the simple form of the function $\widetilde{\zeta}$, we can further simplify things by singling out a subset $\widetilde{\mathcal{G}}_n \subset \mathcal{G}_n$ of graphs of polymers where two vertices Y_i, Y_j are connected with an edge iff $Y_i \cap Y_j \neq \varnothing$, the polymers are then called "incompatible", $Y_i \not\sim Y_j$. The above expressions become

$$Z_\Lambda = \sum_n \frac{1}{n!} \sum_{Y_1,\dots,Y_n} \prod_{i=1}^n w(Y_i) \sum_{G\in\widetilde{\mathcal{G}}_n} (-1)^{|E(G)|}\,,$$

$$\log Z_\Lambda = \sum_{n\geq 1} \frac{1}{n!} \sum_{Y_1,\dots,Y_n} \prod_{i=1}^n w(Y_i) \sum_{G\in\widetilde{\mathcal{G}}_n\cap\mathcal{C}_n} (-1)^{|E(G)|} \tag{A.7}$$

$$= \sum_{n\geq 1} \frac{1}{n!} \sum_{Y_1,\dots,Y_n} \prod_{i=1}^n w(Y_i)\, \widetilde{\varphi}^{\mathrm{T}}(Y_1,\dots,Y_n)\,. \tag{A.8}$$

Note that we have now two types of Ursell functions, one, $\varphi^{\mathrm{T}}(Y) = \exp(-V)^{\mathrm{T}}(Y)$, hidden in the weights $w(Y)$, and another $\widetilde{\varphi}^{\mathrm{T}}(\{Y_i\}) = \exp(-\widetilde{V})^{\mathrm{T}}(\{Y_i\})$. This is at the first glance slightly confusing. The former stems from the connectedness induced by V, and the latter from the one induced by $\widetilde{V}$, that is, non-intersection of the Y_i.

4. Convergence

There are two recent papers that deal comprehensively with the convergence issues [**56, 21**], the former introducing a generalized form of the Kotecky-Preiss convergence criterion for discrete *and* continuous systems, and the latter providing refined convergence criteria, though only for polymer systems. Since in our application we are dealing with a polymer system we recall the useful notion of (in)compatibility between polymers:

$$
\begin{aligned}
C_i &\sim C_j \qquad \text{if } \bigcup_{c\in C_i} \cap \bigcup_{c\in C_j} = \varnothing \qquad \text{“}C_i \text{ and } C_j \text{ are compatible”} \\
C_i &\not\sim C_j \qquad \text{if } \bigcup_{c\in C_i} \cap \bigcup_{c\in C_j} \neq \varnothing \qquad \text{“}C_i \text{ and } C_j \text{ are incompatible”}\,.
\end{aligned}
$$

Of the various criteria entailing convergence of (A.6) the one of Kotecky and Preiss suits our needs best: If there exists a function $a : \mathcal{P}^0_*(\mathcal{P}^0(\mathbb{Z}^d)) \to [0,\infty)$ such that

$$
\sum_{C\not\sim C_0} |z(C)| e^{a(C)} \le a(C_0) \qquad \forall C_0\,, \tag{A.9}
$$

then (A.6) converges absolutely. (With $\mathcal{P}^0(X)$ we denote the set of finite subsets of X.) Furthermore, we have the following uniform bound in Λ,

$$
1 + \sum_{n\ge 1} \frac{1}{n!} \sum_{C_1,\dots,C_n} |\varphi^{\mathrm{T}}(C_0, C_1, \dots, C_n)| \prod_{i=1}^{n} |z(C_i)| \le \mathrm{e}^{a(C_0)} \qquad \forall C_0\,,
$$

implying the existence of the thermodynamic limit $\Lambda \to \mathbb{Z}^d$ of derivatives of $\log Z_\Lambda$, that is, correlation functions. In accordance with literature, we write φ^{T} to mean the Ursell function associated to the hard-core repulsion between polymers, formerly denoted by $\widetilde{\varphi}^{\mathrm{T}}$. From the above inequality follows, with (A.9), the useful inequality

$$
\sum_{n\ge 1} \frac{1}{n!} \sum_{\substack{C_1,\dots,C_n \\ \not\sim C}} |\varphi^{\mathrm{T}}(C_1, \dots, C_n)| \prod_{i=1}^{n} |z(C_i)| \le a(C) \qquad \forall C\,, \tag{A.10}
$$

where by $C_1, \dots C_n \not\sim C$ we mean $\exists i : C_i \not\sim C$.

Since the weight of a polymer decreases rapidly with size it is possible to obtain refined estimates if the cluster-size is bounded from below: If we can show that

$$
\sum_{C\not\sim C_0} \underbrace{\mathrm{e}^{m|C|}|z(C)|}_{|\overline{z}(C)|} e^{a(C)} \le a(C_0) \qquad \forall C_0\,,
$$

holds for some $m > 0$, then we can extract exponential decay and obtain, for $\sum_i |C_i| \ge L$,

$$
\sum_{n\ge 1} \frac{1}{n!} \sum_{\substack{C_1,\dots,C_n \\ \not\sim C}} |\varphi^{\mathrm{T}}(C_1, \dots, C_n)| \prod_{i=1}^{n} |z(C_i)| \le \mathrm{e}^{-mL} a(C) \qquad \forall C\,. \tag{A.11}
$$

5. Decay of correlations

In this section we use the results about the cluster expansion proved so far to show that for β small enough, events relating to subsets of the lattice that are distant are only weakly correlated under the Gibbs measure (3.3). So in the following, $(\Omega, \mathbb{P})$ denotes the probability space of configurations of magnetic moments $\{m_j\}_{j\in\mathbb{Z}^d}$ with distribution

$$\mathrm{d}\mathbb{P}(m) = Z^{-1}\exp\{\kappa\sum_{|i-j|=1} m_i\cdot m_j + \beta B\sum_j m_j^{(3)}\}\prod_j \delta(m_j^2-1)\mathrm{d}^3 m_j\,.$$

DEFINITION 5.1. An event $A\subset\Omega$ is said *to have support* in $S\subset\mathbb{Z}^d$ if

$$\omega\in A \Longrightarrow (\omega\big|_S, \omega'\big|_{S^c})\in A\quad \forall\omega'\big|_{S^c}\,.$$

LEMMA A.1. *Consider events Ω_{S_i} with support in $S_i\subset\mathbb{Z}^d$, satisfying dist$(S_i,S_j)>L$, for $i\neq j$. Then we have that for any $\eta>0$ there is a $\kappa_0>0$ such that for all $0\le\kappa<\kappa_0$*

$$|\mathbb{P}_\kappa[\Omega_{S_1}\cap\Omega_{S_2}] - \mathbb{P}_\kappa[\Omega_{S_1}]\mathbb{P}_\kappa[\Omega_{S_2}]| \le \mathrm{e}^{-\eta L}(|S_1|+|S_2|)\,,$$

and inductively

$$|\mathbb{P}_\kappa[\bigcap_{i=1}^n\Omega_{S_i}] - \prod_{i=1}^n\mathbb{P}_\kappa[\Omega_{S_i}]| \le (n-1)\mathrm{e}^{-\eta L}\sum_{i=1}^n|S_i|\,. \qquad (3.18)$$

REMARK. We also prove that $\kappa_0=\kappa_0(\beta,B)$ is a decreasing function of both the inverse temperature β and the strength B of the external magnetic field, so that at fixed correlation parameter κ the maximal decay parameter η is a decreasing function of both β and B. Going through the proof of pure point spectrum in the band tails, we see that the region where we can prove point spectrum *shrinks* with increasing β and B, suggesting that resistivity drops with lowering temperatures and increasing the external magnetic field—in agreement with the experimental results shown in Figure 7, left.

PROOF. We drop the subscript κ in the following.

$$\mathbb{P}\left[\Omega_{S_1}\cap\Omega_{S_2}\right] = \mathbb{E}\left[1_{\Omega_{S_1}}1_{\Omega_{S_2}}\right] = \frac{1}{Z}\int\mathrm{d}\mu\,\mathrm{e}^{\sum_{\langle x,y\rangle}\kappa m_x\cdot m_y+\beta B\sum_x m_x^{(3)}}1_{\Omega_{S_1}}1_{\Omega_{S_2}}\,,$$

with the one-site measure

$$\mathrm{d}\mu(m_x) = \delta(m_x^2-1)\mathrm{d}^3 m_x\,.$$

The covariance is obtained by derivation of the logarithm of the following partition function.

$$Z_\lambda = \int\mathrm{d}\mu\,\mathrm{e}^{\sum_{\langle x,y\rangle}\kappa m_x\cdot m_y+\beta B\sum_x m_x^{(3)}+\lambda_1 1_{\Omega_{S_1}}(m)+\lambda_2 1_{\Omega_{S_2}}(m)}\,.$$

We have thus $Z_0=Z$, and

$$\frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\lambda_2}\bigg|_0 \log Z_\lambda = \mathbb{E}\left[1_{\Omega_{S_1}}1_{\Omega_{S_2}}\right] - \mathbb{E}\left[1_{\Omega_{S_1}}\right]\mathbb{E}\left[1_{\Omega_{S_2}}\right]\,.$$

As we have seen, the cluster expansion is a means to write the logarithm of Z_λ as a restricted sum, where one sums only over polymers that are connected in some sense, that is, over polymers that constitute a cluster. The evaluation of the derivatives at $\lambda_i=0$ implies that the sum extends only over clusters that contain exactly one factor of $1_{\Omega_{S_1}}$ and one of $1_{\Omega_{S_2}}$,

$$\text{(A.12)}\qquad \frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\lambda_2}\bigg|_0 \log Z_\lambda = \sum_{n\ge1}\frac{1}{n!}\sum_{\substack{C_1,\dots,C_n\\ \not\sim S_1,\not\sim S_2}}\varphi^{\mathrm{T}}(C_1,\dots,C_n)\prod_i z(C_i)\,.$$

The weight $z(C)$ of a polymer C is given by

$$z(C) = \int \prod_{x\in\underline{C}} \mathrm{d}^3 m_x\, \delta(m_x^2 - 1) \prod_{\langle x,y\rangle \in C} (\mathrm{e}^{\kappa m_x\cdot m_y} - 1) \prod_{x\in C} (\mathrm{e}^{\beta B m_x^{(3)}} - 1)\,.$$

We use

$$\begin{aligned} |\mathrm{e}^{\beta m_x\cdot m_y} - 1| &\le \mathrm{e}^{\beta |m_x||m_y|} - 1 \le \mathrm{e}^{\beta(m_x^2+m_y^2)} - 1 \\ &= (\mathrm{e}^{\beta m_x^2} - 1)(\mathrm{e}^{\beta m_y^2} - 1) + (\mathrm{e}^{\beta m_x^2} - 1) + (\mathrm{e}^{\beta m_y^2} - 1) \end{aligned}$$

to disentangle the integration variables. Thus we have

$$\begin{aligned} |\prod_{\langle x,y\rangle\in C} (\mathrm{e}^{\beta m_x\cdot m_y} - 1)| &\le \prod_{\langle x,y\rangle\in C} \Big((\mathrm{e}^{\beta m_x^2} - 1)(\mathrm{e}^{\beta m_y^2} - 1) + (\mathrm{e}^{\beta m_x^2} - 1) + (\mathrm{e}^{\beta m_y^2} - 1) \Big) \\ &= \sum_i \prod_{x\in\underline{C}} \Big(\mathrm{e}^{\beta m_x^2} - 1 \Big)^{n_x(i)}\,, \end{aligned}$$

where the summation ranges from 1 to $3^{|C|}$ (here, $|C|$ shall denote the number of bonds in C), and $0 \le n_i(x) \le 2d$. Clearly, $\sum_x n_x(i) \ge |C|$ for all i. For the factor $\prod_{x\in C}(\mathrm{e}^{\beta B m_x^{(3)}} - 1)$ we consider the regime of $\mathrm{e}^{\beta B} \ge 2$ and estimate its absolute value by $(\mathrm{e}^{\beta B m_x^{(3)}} - 1)^{|\underline{C}|}$. For small values of the product βB, we get additional decay factors and we could estimate the product by 1.

We obtain

$$|z(C)| \le 3^{|C|} \max_i \prod_{x\in\underline{C}} \int \mathrm{d}^3 m_x \delta(m_x^2 - 1)(\mathrm{e}^{\kappa m_x^2} - 1)^{n_i(x)} (\mathrm{e}^{\beta B |m_x^{(3)}|} - 1) \tag{A.13}$$

$$= (3(\mathrm{e}^{\beta B} - 1))^{|C|} \max_i \prod_{x\in\underline{C}} (\mathrm{e}^{\kappa} - 1)^{n_i(x)} \le [3(\mathrm{e}^{\beta B} - 1)(\mathrm{e}^{\kappa} - 1)^{\frac{1}{d}}]^{|C|}\,, \tag{A.14}$$

since $|C| < d|\underline{C}|$.

We see that condition (A.9) is fulfilled, for it is easy to see that it is enough to show

$$\sup_{x\in\underline{C_0}} \sum_{C \not\sim \{x\}} |z(C)| \mathrm{e}^{a|C|} \le a\,,$$

where we have already included the customary ansatz $a(C) = a|C|$. For any $a, \eta > 0$ there exists a $\kappa = \kappa(\beta, B) > 0$ small enough, such that

$$\sup_{x\in\underline{C_0}} \sum_{C\not\sim\{x\}} \mathrm{e}^{\eta|C|} |z(C)| \mathrm{e}^{a|C|} \le \sum_{n\ge 1} 2^{n/d} (2d)^{2n} [3(\mathrm{e}^{\beta B} - 1)(\mathrm{e}^{\kappa} - 1)^{\frac{1}{d}}]^n \mathrm{e}^{(\eta + a)n} \le a\,,$$

since the number of polymers of size n is less than $(2d)^{2n} 2^{n/d}$.

Equation (A.12) tells us that our sum extends only over clusters that meet both S_1 *and* S_2, that is, the diameter of the clusters is at least L, since the distance of S_1 and S_2 is at least L. Therefore (see (A.11)) we get

$$|\frac{\partial}{\partial\lambda_1} \frac{\partial}{\partial\lambda_2}\Big|_0 \log Z_\lambda| \le \mathrm{e}^{-\eta L} a |S_1 \cup S_2|\,.$$

Taking $a = 1$ concludes the proof. □

6. Proof of Lemma 3.4

Using a path expansion we write formally

$$\begin{aligned}\rho_\Lambda(E) &= \lim_{\varepsilon\to 0}\frac{1}{|\Lambda|}\sum_{\substack{x\in\Lambda\\ s=\pm}}\frac{1}{\pi}\mathrm{Im}\sum_{\substack{\omega:x\to x\\ \omega\subset\Lambda}}\int \mathrm{d}\mu(\mathbf{m})\left(\overrightarrow{\prod_{j\in\omega}}\frac{1}{\sigma\cdot m_j - E - \mathrm{i}\varepsilon}\right)_{ss}\\ &= \lim_{\varepsilon\to 0}\frac{1}{|\Lambda|}\sum_{\substack{x\in\Lambda\\ s=\pm}}\frac{1}{\pi}\mathrm{Im}\sum_{\substack{\omega:x\to x\\ \omega\subset\Lambda}}\sum_{\substack{\{s_i=\pm\}\\ s_1=s,s_{|\omega|+1}=s}}\int \mathrm{d}\mu(\mathbf{m})\prod_{j=1}^{|\omega|}\left(\frac{1}{\sigma\cdot m_{\omega(j)} - E - \mathrm{i}\varepsilon}\right)_{s_j s_{j+1}},\end{aligned}$$

where the arrow indicates a path-ordered product since the Pauli matrices do not commute. Recall our choice of probability measure

$$\begin{aligned}\mathrm{d}\mu_\Lambda(\mathbf{m}) &= \frac{1}{Z_\Lambda}\prod_{x\in\Lambda}\mathrm{d}\mu(m_x)\mathrm{e}^{-V(\mathbf{m})}\\ V(\mathbf{m}) &= \sum_{\langle x,y\rangle\in\Lambda\times\Lambda}\kappa m_x\cdot m_y, \qquad \kappa>0\,,\end{aligned}\tag{A.15}$$

where $\langle x,y\rangle$ denote nearest neighbors (we forget about the external magnetic field, but it easily be incorporated as in the previous section). We want to use the cluster expansion to estimate the contribution F_ω of a path ω with fixed spin configuration to $\rho_\Lambda(E)$,

$$F_\omega := \frac{1}{Z_\Lambda}\int\prod_{x\in\Lambda}\mathrm{d}\mu(m_x)\,\mathrm{e}^{-V(\mathbf{m})}\underbrace{\prod_{j=1}^{|\omega|}\left(\frac{1}{\sigma\cdot m_{\omega(j)}-E-\mathrm{i}\varepsilon}\right)_{s_j s_{j+1}}}_{=:f_\omega}.$$

We proceed as in the previous section, starting with

$$F_\omega = \frac{\partial}{\partial\lambda}\bigg|_{\lambda=0}\log\int\prod_{x\in\Lambda}\mathrm{d}\mu(m_x)\,\mathrm{e}^{-V(\mathbf{m})+\lambda f_\omega}.$$

As in the previous section, evaluation of the derivative at $\lambda=0$ means that only clusters with exactly one factor of f_ω survive. So

$$F_\omega = \sum_{n\geq 1}\sum_{\substack{C_1,\dots,C_n\\ \not\sim\omega}}\varphi^{\mathrm{T}}(C_1,\dots,C_n)\prod_i z(C_i)\,,$$

where

$$z(C) = \int\prod_{x\in\underline{C}}\mathrm{d}\mu(m_x)\prod_{\langle x,y\rangle\in C}(\mathrm{e}^{\kappa m_x\cdot m_y}-1)\,,$$

and

$$z(C) = \int\prod_{x\in\underline{C}}\mathrm{d}\mu(m_x)\prod_{\langle x,y\rangle\in C}(\mathrm{e}^{\kappa m_x\cdot m_y}-1)\prod_{x\in\underline{\omega}}\left(\frac{1}{\sigma\cdot m_x - E - \mathrm{i}0}\right)_{s_j s_{j+1}}$$

for the one polymer containing ω. So if we can show that the weight of the polymer containing ω goes like $\alpha^{|\omega|}$ for some small $\alpha>0$, and the Kotecky-Preiss criterion holds for these weights $z(C)$ we have with (A.10)

$$|F_\omega|\leq\alpha^{|\omega|}a|\omega|\,.$$

Thus, for α small enough we are able to resum the path expansion

$$\pi\rho_\Lambda(E) \le \frac{1}{|\Lambda|}\sum_{\substack{x\in\Lambda\\ s_i=\pm}}\sum_{\substack{\omega:x\to x\\ \omega\subset\Lambda}}\sum_{\substack{\{s=\pm\}\\ s_1=s,s_{|\omega|+1}=s}}\alpha^{|\omega|}\mathrm{e}^{a|\omega|}|\omega| \tag{A.16}$$

$$= \sum_{\substack{\omega:x\to x\\ \omega\subset\Lambda}}\sum_{\substack{\{s_i=\pm\}\\ s_1=s,s_{|\omega|+1}=s}}\alpha^{|\omega|}\mathrm{e}^{a|\omega|}|\omega| \le \sum_{|\omega|=1}^{\infty}(2d)^{|\omega|}2^{|\omega|}\alpha^{|\omega|}\mathrm{e}^{a|\omega|}|\omega| \le C\,. \tag{A.17}$$

The constant C goes to 0 for $|E|, \gamma \to \infty$.

Remark. Exactly the same computation for $x \neq y$ yields $|\mathbb{E}G(E+\mathrm{i}0;x,y)| \le C\mathrm{e}^{-m|x-y|}$ (because the sum over $|\omega|$ starts only at $|\omega| = |x-y|$).

As we have seen in the section on the decay of correlations, the following condition suffices:

$$\sup_{x\in\underline{C_0}}\sum_{C\not\sim\{x\}}|z'(C)|\mathrm{e}^{a|C|} \le a\,, \tag{A.18}$$

Next, we estimate the weights of the polymer C containing the path ω:

$$z(C) = \int\prod_{x\in\underline{C}}\mathrm{d}^3m_x\, g(m_x^2)\prod_{\langle x,y\rangle\in C}(\mathrm{e}^{\kappa m_x\cdot m_y}-1)\prod_{x\in\underline{C}\cap\underline{\omega}}\left(\frac{1}{\sigma\cdot m_x - E - \mathrm{i}0}\right)_{s_js_{j+1}}\prod_{x\in\underline{\omega}\setminus\underline{C}}\alpha_x\,,$$

where

$$\alpha_x := \int\mathrm{d}\mu(m_0)\left(\frac{1}{m_0\cdot\sigma - E - \mathrm{i}0}\right)_{s_js_{j+1}}\qquad (x=\omega(j))\,.$$

In finding bounds for α, we take advantage of the analyticity of g, the one-site distribution of the random vectors,

$$|\alpha_x| = |\int\mathrm{d}^3m_x g(m_x^2)\left(\frac{1}{m_x\cdot\sigma - E - \mathrm{i}0}\right)_{s_js_{j+1}}| \tag{A.19}$$

We write $m_x = m_x n$, where n varies over the, say, north hemisphere of S^2, and m_x runs from $-\infty$ to ∞. So we have

$$\alpha := \max_x|\alpha_x| = \max_{ss'}|\int\mathrm{d}^2n\int_{-\infty}^{\infty}\mathrm{d}m_0\, g(m_0^2)\left(\frac{1}{m_0n\cdot\sigma - E - \mathrm{i}0}\right)_{ss'}|\,.$$

We think of m_0 as a complex variable and, since the integrand is meromorphic with single poles at $\pm(E+\mathrm{i}0)$, we deform the contour Γ so as to avoid the poles and get, with $D := \min_{+,-}\,\mathrm{dist}\,(\Gamma, \pm(E+\mathrm{i}0))$,

$$\alpha \le \int\mathrm{d}^2n\int_\Gamma\mathrm{d}m_0A(\gamma)|\mathrm{e}^{-\gamma(m_0^2-m^2)^2}|\frac{1}{D}\,,$$

since $||(A-z)^{-1}|| = (\mathrm{dist}\,(\mathrm{spec}\,A, z))^{-1}$ for a normal matrix A. It is now apparent that for either $|E| >> m$ (that is, energy in the band tail) or $\gamma << 1$ (that is, large disorder), we can deform the contour so as to get

$$\alpha \le \frac{2}{D}\,,$$

as small as we desire.

REMARK. For sake of simplicity we consider a self-avoiding path ω. The calculations carry over to the general case since

$$|a_{i_1j_1}\dots a_{i_pj_p}| \le (\max|a_{ij}|)^p \le \sum_{ij}|a_{ij}|^p \le n||A||^p \tag{A.20}$$

for any $n\times n$-matrix $A=(a_{ij})$. For a path ω that intersects itself, say, p times at x, α_x in $(A.19)$ changes into

$$\int \mathrm{d}^3 m_x g(m_x^2)\left(\frac{1}{m_x\cdot\sigma - E - \mathrm{i}0}\right)_{s_{j_1}s_{j_1+1}}\cdots\left(\frac{1}{m_x\cdot\sigma - E - \mathrm{i}0}\right)_{s_{j_p}s_{j_p+1}},$$

which is estimated by contour deformation in precisely the same way as before, using (A.20) in addition.

For the term $z(C)$ we do the same trick, deform the contour to avoid the poles and get

$$\begin{aligned} z(C) = &\prod_{x\in\underline{\omega}\setminus\underline{C}}\alpha_x\int\prod_{x\in\underline{C}}\mathrm{d}^2 n_x\iint\dots\\ &\int_\Gamma \mathrm{d}h_{x_2}g(h_{x_2}^2)t(x_2)\int_\Gamma \mathrm{d}h_{x_1}g(h_{x_1}^2)t(x_1)\prod_{\langle x,y\rangle\in C}(\mathrm{e}^{\kappa m_x m_y n_x\cdot n_y}-1)\,,\end{aligned}$$

where $\{x_1,\dots,x_{|\underline{C}|}\}=\underline{C}$, and

$$t(x)=\begin{cases}\left(\frac{1}{\sigma\cdot m_x - E - \mathrm{i}0}\right)_{ss'} & x\in\omega\\ 1 & \text{otherwise}\end{cases}$$

Hence,

$$|z(C)|\le\left(\frac{2}{D}\right)^{|\omega|}\int\prod_{x\in\underline{C}}\mathrm{d}^2 n_x\int_\Gamma\prod_{x\in\underline{C}}g(m_x^2)\prod_{\langle x,y\rangle\in C}(\mathrm{e}^{\kappa|m_x||m_y|}-1)\,.$$

The remaining integral is estimated as in the previous section:

$$|z(C)|\le\left(\frac{2}{D}\right)^{|\omega|}3^{|C|}\max_i\prod_{x\in\underline{C}}\int\mathrm{d}^3 m_x A(\gamma)\mathrm{e}^{-\gamma(m_x^2-\mu^2)^2}(\mathrm{e}^{\kappa m_x^2}-1)^{n_i(x)}\,. \tag{A.21}$$

Divide the radial integration region into two parts, the interval $[0,R]$ and its complement. The tail is estimated as follows:

$$\begin{aligned}&\int\mathrm{d}\Omega_x\int_R^\infty\mathrm{d}|m_x|m_x^2A(\gamma)\mathrm{e}^{-\gamma(m_x^2-m^2)^2}(\mathrm{e}^{\kappa m_x^2}-1)^{n(x)}\le\\ &4\pi\int_R^\infty\mathrm{d}|m_x|m_x^2A(\gamma)\mathrm{e}^{-\gamma(m_x^2-m^2)^2+\kappa 2dm_x^2}\,.\end{aligned}$$

The exponential above is a Gaussian in m_x^2 with mean $m^2+\kappa d/\gamma$, and thus the integral can be made as small as we like by choosing R large enough: For any $\varepsilon>0$ there is an R independent of C such that

$$\int\mathrm{d}\Omega_x\int_R^\infty\mathrm{d}m_x^3A(\gamma)\mathrm{e}^{-\gamma(m_x^2-m^2)^2}(\mathrm{e}^{\kappa m_x^2}-1)^{n(x)}\le\varepsilon\,.$$

Thus we are left with the integral over $[0,R]$:

$$\begin{aligned}&\int\mathrm{d}\Omega_x\int_0^R\mathrm{d}|m_x||m_x|^2A(\gamma)\mathrm{e}^{-\gamma(m_x^2-m^2)^2}(\mathrm{e}^{\kappa m_x^2}-1)^{n(x)}\le\\ &4\pi\int_0^R\mathrm{d}|m_x||m_x|^2A(\gamma)\mathrm{e}^{-\gamma(m_x^2-m^2)^2}(\mathrm{e}^{\kappa R^2}-1)^{2d}\,.\end{aligned}$$

Since the Gaussian is normalized to 1 if integrated over the whole $\mathbb{R}^3$ we obtain

$$\int \mathrm{d}\Omega_x \int_0^R \mathrm{d}|m_x||m_x|^2 A(\gamma)\mathrm{e}^{-\gamma(m_x^2-m^2)^2}(\mathrm{e}^{\kappa m_x^2}-1)^{n(x)} \leq (\mathrm{e}^{\kappa R^2}-1)^{2d} \leq (2\kappa R^2)^{2d}\,,$$

for κ small enough. Recalling equation (A.13), we have that for any $\varepsilon > 0$ there exists a $\kappa > 0$ such that (by choosing first R large and then κ small enough)

$$|z(C)| \leq \left(\frac{2}{D}\right)^{|\omega|} 3^{|C|}(2\varepsilon)^{|\underline{C}|} \leq \left(\frac{2}{D}\right)^{|\omega|}(3^d 2\varepsilon)^{|\underline{C}|}\,.$$

The polymers not containing the path ω are estimated analogously.

We see that condition (A.18) is fulfilled: For any $a > 0$ there exists an $\varepsilon = \varepsilon(\kappa, \gamma \text{ or } E)$ small enough, such that

$$\sup_{x\in\underline{C_0}} \sum_{C \not\sim \{x\}} |z(C)|\mathrm{e}^{a|C|} \leq \sum_{n\geq 1}(2d)^{2n}\varepsilon^n \mathrm{e}^{an} \leq a\,.$$

$\square$

APPENDIX B

A matrix-valued Cartan-type theorem

In this appendix we recall a theorem proven by Bourgain in [**8**] that lies at the core of the inductive Wegner estimate used in Chapter 3, and show how to modify it so that we can apply it to the random Zeeman interaction problem.

Some generality does not hurt, so consider the random Schrödinger operator

$$H(\omega) = -\Delta + A(\omega)\,, \tag{B.1}$$

acting on the Hilbert space $l^2(\mathbb{Z}^d;\mathbb{C}^r)$, with

$$A(\omega) = \sum_{x\in\mathbb{Z}^d} a(\omega_x) e_x \otimes e_x\,,$$

where a is a real-analytic function ranging in the hermitian $r \times r$-matrices, defined on a convex, bounded subset $U \subset \mathbb{R}^\nu$, that extends to an analytic function on $U + \mathrm{e}\,\mathrm{diam}(U)\mathrm{D}^\nu \subset \mathbb{C}^\nu$, where $\mathrm{D} = \{z \in \mathbb{C} : |z| \le 1\}$ is the unit disk in the complex plane. The random variables ω_x are independent and identically distributed according to

$$\mathrm{d}\mathbb{P}(\omega_x) = g(\omega_x)\mathrm{d}^\nu\omega_x \tag{B.2}$$

with $g : U \to \mathbb{R}^+$ a *bounded* density with respect to Lebesgue measure.

Theorem B.1. *There exists an N_0 such that the random Schrödinger operator (B.1) with random variables distributed according to (B.2) with g a bounded density has, with probability one, pure point spectrum for $E \in [\overline{E} - (\log N_0)^{-4}, \overline{E}]$ with exponentially decaying eigenfunctions, where $\overline{E}$ denotes the upper spectral edge.*

To start we state without proof (see, for instance, [**43**]) the classical result usually called a Cartan estimate.

Lemma B.2. *Let $f(z)$ be a function analytic in the disc $\{z : |z| \le \mathrm{e}R\}$, $|f(0)| = 1$, and let η be an arbitrary small positive number. Then the estimate*

$$\log|f(z)| \ge -\log\frac{15e^3}{\eta}\log M_f(\mathrm{e}R)\,,$$

where $M_f(r) = \max_{|z|=r}|f(z)|$, is valid everywhere in the disc $\{z : |z| \le R\}$ outside a set of discs with sum of radii

$$\sum r_j \le \eta R\,.$$

In particular, we have

$$|\{x \in [-R,R] : |f(x)| < \delta\}| \le 30\mathrm{e}^3 R\delta^{\frac{1}{\log M_f}}\,,$$

where $|\cdot|$ denotes Lebesgue measure.

The next lemma is a higher-dimensional generalization [**8**], Lemma B.3. *Let F be a real-analytic function on $\Omega = [a,b]^n$ which extends to an analytic function on D^n, $\mathrm{D} = \{z \in \mathbb{C} : |z - \frac{a+b}{2}| < \frac{\mathrm{e}}{2}|a-b|\}$, with the bound*

$$M_F := \max_{z \in \mathrm{D}^n} |F(z_1, \ldots, z_n)| < \infty .$$

Assume furthermore that there is an $a \in \Omega$ such that

$$|F(a)| > \varepsilon ,$$

where $0 < \varepsilon < 1/2$. Denoting for $\delta > 0$

$$E_\delta = \{x \in \Omega : |F(x)| < \delta\}$$

we have

$$\mu(\{x \in \Omega : |F(x)| < \delta\}) < C\|g\|_\infty n|a-b|^n \delta^{\frac{c}{\log M_F/\varepsilon}} ,$$

where μ denotes a measure with bounded density g with respect to Lebesgue measure, and $C, c > 0$ are constants.

Crucial is the dependence on the dimension n. For the normalized Lebesgue measure (uniform probability distribution) one can even drop the volume factor $|a-b|^n$ as it is cancelled by $\|g\|_\infty$.

Proof. The proof is derived from the classical statement for $n = 1$. Using polar coordinates with the origin at a we write

$$\mu(E_\delta) = \int \mathrm{d}^n x \, g(x) 1_{E_\delta}(x) = \int_{S_{n-1}} \mathrm{d}\zeta \int_0^{r(\zeta)} \mathrm{d}r \, r^{n-1} g(a + r\zeta) 1_{E_\delta}(a + r\zeta) , \tag{B.3}$$

where $r(\zeta)$ is defined by $\Omega = \{a + r\zeta : \zeta \in S_{n-1}, 0 \le r \le r(\zeta)\}$. Now, for a fixed $\zeta \in S_{n-1}$, $f(r) := F(a + r\zeta)$ is a real analytic function of $r \in I = [0, r(\zeta)]$. It extends to an analytic function $f(z)$ on the neighbourhood $\mathcal{D} = I + \frac{\mathrm{e}}{2}\max(1, r(\zeta)) \cdot \mathrm{D} \subset \mathbb{C}$ of I, where $\max_{z \in \mathcal{D}} |f(z)| \le M_F$ and $|f(0)| = |F(a)| > \varepsilon$. From Cartan's lemma, it follows that

$$\int_I \mathrm{d}r \, 1_{E_\delta}(a + r\zeta) = |\{r \in I : |f(r)| < \delta\}| < C\delta^{\frac{c}{\log M_F/\varepsilon}} r(\zeta) ,$$

with $C, c > 0$ constants. Substituting into (B.3) we obtain the desired bound

$$\mu(E_\delta) < C\|g\|_\infty \delta^{\frac{c}{\log M_F/\varepsilon}} \underbrace{\int_{S_{n-1}} \mathrm{d}\zeta \, r(\zeta)^n}_{n|\Omega|} = n\|g\|_\infty |a-b|^n C\delta^{\frac{c}{\log M_F/\varepsilon}} .$$

□

The next lemma generalizes the result to matrix-valued functions [**8**],

Lemma B.4. *Let $A(x)$ be a real analytic self-adjoint $N \times N$-matrix function of $x \in \Omega = [a,b]^n$, satisfying the following conditions (with $m \ll N$, $B_1, B_2, B_3 > 1$)*

(1) *$A(x)$ has an analytic extension $A(z)$ to $z \in \mathrm{D}^n$ (D as in the previous lemma) with*

$$\|A(z)\| < B_1 \qquad z \in \mathrm{D} \tag{B.4}$$

(2) *There is a subset Λ of $\{1, 2, \ldots, N\}$ such that $|\Lambda| \le m$ and for all $z \in \mathrm{D}^n$.*

$$\|\big(R_{\{1,\ldots,N\}\setminus\Lambda} A(z) R_{\{1,\ldots,N\}\setminus\Lambda}\big)^{-1}\| < B_2 , \tag{B.5}$$

where R_S denotes coordinate restriction to S.

(3) *For some $a \in \Omega$ we have*

$$\|A(a)^{-1}\| < B_3 . \tag{B.6}$$

Then

$$\mu(E_K) := \mu(\{x \in \Omega : \|A(x)^{-1}\| > K\}) < C\|g\|_\infty n|a-b|^n K^{-\frac{c}{m \log B_1 B_2 B_3}}\,. \tag{B.7}$$

Proof. The main idea is to reduce the inversion of $A(x)$ to that of a smaller matrix, its Schur complement. Consider the following analytic matrix-valued function on D^n with index set Λ,

$$B(z) = R_\Lambda A(z) R_\Lambda - (R_\Lambda A(z) R_{\Lambda^c})(R_{\Lambda^c} A(z) R_{\Lambda^c})^{-1}(R_{\Lambda^c} A(z) R_\Lambda)\,, \tag{B.8}$$

satisfying by (B.4) and (B.5)

$$\|B(z)\| < 2B_1^2 B_2 \qquad \text{for } z \in \mathrm{D}^n\,. \tag{B.9}$$

The invertibility of $A(x)$ is equivalent to that of $B(x)$, and more precisely

$$\begin{aligned} \|B(x)^{-1}\| \lesssim \|A(x)^{-1}\| &\lesssim \left(1 + \|(R_{\Lambda^c} A(z) R_{\Lambda^c})^{-1}\|\right)^2 (1 + \|B(x)^{-1}\|) \\ &\lesssim B_2^2(1 + \|B(x)^{-1}\|)\,. \end{aligned} \tag{B.10}$$

Because $B(x)$ is self-adjoint and (B.6,B.10)

$$|\det B(a)| = \prod_{\lambda \in \sigma(B(a))} |\lambda| \geq \|B(a)^{-1}\|^{-m} > (CB_3)^{-m}\,. \tag{B.11}$$

Also,

$$\|B(x)^{-1}\| \leq \frac{\|B(x)\|^{m-1}}{|\det B(x)|} < \frac{(2B_1^2 B_2)^{m-1}}{|\det B(x)|}\,. \tag{B.12}$$

Now, consider the analytic function on D^n

$$F(z) = (2B_1^2 B_2)^{-m} \det B(z)\,. \tag{B.13}$$

We have $|F(z)| \leq 1$ by (B.9) and $|F(a)| > (CB_1^2 B_2 B_3)^{-m}$ by (B.11). Application of Lemma B.3 with $\varepsilon = (CB_1^2 B_2 B_3)^{-m}$ to F yields

$$\mu(\{x \in \Omega : |\det B(x)| < K\}) < C\|g\|_\infty n|a-b|^n \left(\frac{K}{(2B_1^2 B_2)^m}\right)^{\frac{c}{m \log cB_1^2 B_2 B_3}} \tag{B.14}$$

$$< C\|g\|_\infty n|a-b|^n K^{\frac{c}{m \log B_1 B_2 B_3}}\,. \tag{B.15}$$

If $|\det B(x)| > K$ then $\|A(x)^{-1}\| < (2B_1^2)^m B_2^{m+2} K^{-1}$ by (B.10,B.12), so that the claim follows from (B.14). $\square$

It is not essential that the function A be defined on $[a,b]^n$. Consider the random Zeeman interaction case, for instance. Here, $\Omega = (\mathrm{S}^2)^n$ with the normalised uniform measure. For each $i \in \Lambda \subset \mathbb{Z}^d$ we have spherical coordinates $\psi_i : (0, 2\pi) \times (0, \pi) \to \mathrm{S}^2$ which parametrize the sphere outside half a great arc in a real-analytic way. We then apply Lemma B.4 to the function

$$\begin{aligned} G := A \circ \psi : ((0,2\pi) \times (0,\pi))^n &\to \{\text{hermitian } 2n \times 2n\text{-matrices}\} \\ (\varphi_i, \vartheta_i) &\mapsto -\Delta + \psi_i(\varphi_i, \vartheta_i) \cdot \sigma\,. \end{aligned}$$

The analyticity assumptions of the lemma a certainly fulfilled, as A and ψ and hence G are entire functions. On $((0,2\pi) \times (0,\pi))^n$ the pull-back measure is $\mathrm{d}\mu^* = \prod_i \sin\vartheta_i \mathrm{d}\vartheta_i \mathrm{d}\varphi_i$, so $\|g\|_\infty = 1$, and we have the estimate

$$\mu(E_K) = \mu^*(\psi^{-1}(E_K)) < Cn(2\pi^2)^n K^{-\frac{c}{m \log B_1 B_2 B_3}}\,,$$

for the case the function $G = A \circ \psi$ fulfills assumptions (1)-(3). Here we have used that the half great arc that is not parametrized by the spherical coordinates has zero measure.

Remark. An interesting class of functions $A(\omega)$ is given by

$$A(\omega) = U(\omega)^{-1}\mathrm{diag}(\lambda_1, \ldots, \lambda_r)U(\omega)\,,$$

where the U are distributed according to the normalized Haar measure on $\mathrm{SU}(r)$. The required chart is the exponential map $\exp : U \subset \mathfrak{su}(r) \to \mathrm{SU}(r)$. For example, the exponential map maps the open ball of radius π centered at 0 diffeomorphically onto $\mathrm{SU}(2)\backslash\{-\mathbb{1}\}$.

APPENDIX C

Thermodynamic limit

For a cube Λ_L, define

$$n_\Lambda(E,\omega) := \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \operatorname{tr}\,\langle \delta_x\,, \mathrm{P}_\Lambda(E,\omega)\delta_x\rangle\,.$$

We want to prove

Lemma C.1.

$$\lim_{L\to\infty} \mathbb{E}\, n_{\Lambda_L}(E,\omega) = \operatorname{tr}\,\mathbb{E}\,\langle \delta_{x_0}\,, \mathrm{P}(E,\omega)\delta_{x_0}\rangle\,.$$

Proof. The claim is intuitively clear from translation invariance in the limit of the whole lattice $\mathbb{Z}^d$. Divide the cube Λ_L into an interior region Λ_0 and a boundary of thickness $L^{1/2}$. Then

$$\mathbb{E}\, n_{\Lambda_L}(E,\omega) = \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_0} \operatorname{tr}\,\mathbb{E}\,\langle \delta_x\,, \mathrm{P}_{\Lambda_L}(E,\omega)\delta_x\rangle + \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L\setminus\Lambda_0} \operatorname{tr}\,\mathbb{E}\,\langle \delta_x\,, \mathrm{P}_{\Lambda_L}(E,\omega)\delta_x\rangle\,,$$

and as $\|P_{\Lambda_L}\| = 1\ \forall L$ the second term goes to 0 as $L\to\infty$, and we drop it from now on. Denote the lower and upper edge of the (almost sure) spectrum of H by E_0 and E_1, respectively. By assumption, $|E_0|, |E_1| < \infty$. We introduce an approximation of $\mathbb{1}_{[E_0,E]}$ by continuous functions χ_n with

$$\chi_n \geq \mathbb{1}_{[E_0,E]}\,,\quad \operatorname{supp}\,\chi_n \subset [E_0,E_1]$$
$$\lim_{n\to\infty}\|\chi_n - \mathbb{1}_{[E_0,E]}\|_\infty = 0\,.$$

Since $\chi_n(x)$ is continuous and defined on a compact interval we can by Weierstrass approximate it uniformely with polynomials p_{nm} in $(x-z)^{-1}$, $\operatorname{Im} z \neq 0$. Therefore

$$\begin{aligned}\mathbb{E}\, n_{\Lambda_L}(E,\omega) &= \lim_{n\to\infty}\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_0} \operatorname{tr}\,\mathbb{E}\,\langle \delta_x\,, \chi_n(H_{\Lambda_L}(\omega))\delta_x\rangle \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_0} \operatorname{tr}\,\mathbb{E}\,\langle \delta_x\,, p_{nm}(G_{\Lambda_L}(z,\omega))\delta_x\rangle\,.\end{aligned}$$

For a summand of $p_{nm}(G_{\Lambda_L}(z,\omega))$ we use a path expansion: For $|z|$ large enough we have

$$G_{\Lambda_L}(z,\omega)^k(x,x) = \sum_{y_1,\ldots,y_{k-1}}\prod_{i=0}^{k-1}\sum_{\gamma:y_i\to y_{i+1}}\overrightarrow{\prod_{y\in\gamma}} G^0_{\Lambda_L}(y)\ ,,$$

where γ is a path in Λ_L, $y_0 = y_k = x$, and $G^0_{\Lambda_L}(y) = (A_y - z)^{-1}$. It is thus clear that $\mathbb{E}\,G_{\Lambda_L}(z,\omega)^k(x,x)$ depends only negligibly on the base point x: For any other base point $x_0\in\Lambda_0$ we have the translated paths $\tau_{x-x_0}\gamma$ that give a differing contribution only if they intersect Λ_L^{c}. But in this case they are of combined length $|\Gamma|$ at least $2L^{1/2}$, and thus their contribution is estimated from above by

$$L^{d(k-1)}(2d)^{|\Gamma|}\mathrm{e}^{-C(z)|\Gamma|} \leq CL^{d(k-1)}\mathrm{e}^{-C'(z)L^{1/2}} \leq \mathrm{e}^{-CL^{1/2}}\,,$$

as we can choose $|z|$ large enough for each fixed L. Therefore we have

$$\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_0} \operatorname{tr}\,\mathbb{E}\,\langle \delta_x\,, p_{nm}(G_{\Lambda_L}(z,\omega))\delta_x\rangle = \frac{|\Lambda_0|}{|\Lambda_L|}\operatorname{tr}\,\mathbb{E}\,\langle \delta_{x_0}\,, p_{nm}(G_{\Lambda_L}(z,\omega))\delta_{x_0}\rangle + \mathrm{O}(L^d\mathrm{e}^{-CL^{1/2}})\,,$$

where the error term does not depend on n or m. Taking now the limits $m, n \to \infty$ yields

$$\mathbb{E}\, n_{\Lambda_L}(E,\omega) = \frac{|\Lambda_0|}{|\Lambda_L|}\mathbb{E}\,\langle \delta_{x_0}\,, \mathrm{P}_{\Lambda_L}(E,\omega)\delta_{x_0}\rangle + \mathrm{O}(L^d \mathrm{e}^{-CL^{1/2}})\,,$$

and hence

$$\begin{aligned}&\lim_{L\to\infty} \mathbb{E}\, n_{\Lambda_L}(E,\omega) = \\ &\lim_{L\to\infty} (1-2L^{-1/2})^d\, \mathbb{E}\,\langle \delta_{x_0}\,, \mathrm{P}_{\Lambda_L}(E,\omega)\delta_{x_0}\rangle + \mathrm{O}(L^d \mathrm{e}^{-CL^{1/2}}) = \mathbb{E}\,\langle \delta_{x_0}\,, \mathrm{P}(E,\omega)\delta_{x_0}\rangle\,.\end{aligned}$$

□

A second proof [**37**] appeals to Birkhoff's ergodic theorem. Define

$$\widetilde{n}_\Lambda(E,\omega) := \frac{1}{|\Lambda|}\mathrm{tr}\,(\mathrm{P}(E,\omega)\mathbb{1}_\Lambda) = \frac{1}{|\Lambda|}\sum_{j\in\Lambda}\mathrm{tr}\,\langle \delta_j\,, \mathrm{P}(E,\omega)\delta_j\rangle\,,$$

and note that $(X_j)_{j\in\mathbb{Z}^d}$ for $X_j := \mathrm{tr}\,\langle \delta_j\,, \mathrm{P}(E,\omega)\delta_j\rangle$ is an ergodic stochastic process, as obviously

$$X_j(T_i\omega) = X_{j-i}(\omega)$$

for $(T_j\omega)_i = \omega_{i-j}$. Thus apply the ergodic theorem to $(X_j)_{j\in\mathbb{Z}^d}$ to get

$$\widetilde{n}_\Lambda(E,\omega) = \frac{1}{|\Lambda|}\sum_{j\in\Lambda} X_j \;\longrightarrow\; \mathbb{E}\, X_0 = \mathbb{E}\,\mathrm{tr}\,\langle \delta_0\,, \mathrm{P}(E,\omega)\delta_0\rangle\,.$$

It remains to show that $\mathrm{d}\widetilde{n}_\Lambda(E)$ converges vaguely to the density of states measure $\mathrm{d}n(E) = \rho(E)\mathrm{d}E$, that is,

$$\int \mathrm{d}\widetilde{n}_\Lambda(E)\varphi(E) \;\longrightarrow\; \int \mathrm{d}n(E)\varphi(E)\,, \quad \forall\varphi\in C_0(\mathbb{R})\,,$$

where $C_0(\mathbb{R})$ denotes the continuous complex-valued functions vanishing at infinity. For the same reasons as above, it is enough to take $\varphi(E)$ of the form $1/(E-z)$ for some $z\in\mathbb{C}\backslash\mathbb{R}$. We obtain

$$\int \mathrm{d}\widetilde{n}_\Lambda(E)\frac{1}{E-z} = \frac{1}{|\Lambda|}\mathrm{tr}\,\left(\frac{1}{H-z}\mathbb{1}_\Lambda\right) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{tr}\,G(x,x)\,,$$

and

$$\int \mathrm{d}n_\Lambda(E)\frac{1}{E-z} = \frac{1}{|\Lambda|}\mathrm{tr}\,\left(\frac{1}{H_\Lambda-z}\right) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{tr}\,G_\Lambda(x,x)\,.$$

The resolvent equation yields

$$\begin{aligned}|\sum_{x\in\Lambda} G_\Lambda(x,x) - G(x,x)| &= |\sum_{x\in\Lambda}\sum_{\partial\Lambda} G_\Lambda(x,z)G(z,x)| \\ &\le \sum_{\partial\Lambda}\|G_\Lambda(z,\cdot)\|\|G(z,\cdot)\| \le \sum_{\partial\Lambda}\|G_\Lambda\|\|G\| \le C|\Lambda|^{\frac{d-1}{d}}\frac{1}{(\mathrm{Im}\,z)^2}\,,\end{aligned}$$

implying

$$|\int \mathrm{d}\widetilde{n}_\Lambda(E)\frac{1}{E-z} - \int \mathrm{d}n_\Lambda(E)\frac{1}{E-z}| \le C|\Lambda|^{-\frac{1}{d}}\frac{1}{(\mathrm{Im}\,z)^2} \overset{\Lambda\nearrow\mathbb{Z}^d}{\longrightarrow} 0\,.$$

As $\mathrm{d}n_\Lambda$ converges vaguely to $\mathrm{d}n$ the claim is (again) proven.

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