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Dynamic Valuations in Incomplete Markets

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SUSANNE KLÖPPEL
M.Sc. in Mathematics NUIG
Dipl.-Math oec. University of Ulm
born April 18, 1978
citizen of Germany

accepted on the recommendation of
Prof. Dr. Martin Schweizer, examiner
Prof. Dr. Freddy Delbaen, co-examiner
Prof. Dr. Alexander Schied, co-examiner

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To my parents

and Stefan.

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Abstract

We study two dynamic approaches to the valuation of contingent claims in an incomplete market. The first is *indifference valuation* where at each time t, an agent determines for a random payoff X a value $p_t(X)$ by the requirement that she is indifferent between buying X for $p_t(X)$ or not doing so, provided she always trades optimally in the basic assets. We assume that the agent's time t preferences are given by a *monetary concave utility functional* (MCUF) U_t, i.e., that $-U_t$ is a (conditional) convex risk measure. The valuation functional $p_t(X)$ is then the *convolution* of U_t and a *market functional* constructed from the underlying financial market with the help of the optional decomposition under constraints. Our main goal is to show that the valuation functional $p_.(\cdot)$ is *time-consistent*, i.e., preserves (in a suitable sense) the ordering of payoffs over time. This is achieved by proving that the market functional is time-consistent and that the convolution of dynamic MCUFs preserves time-consistency. As an auxiliary result, we provide a representation for (conditional) MCUFs in terms of their concave conjugates and via equivalent probability measures. Moreover, we show how our results can be translated to dynamic MCUFs defined via backward stochastic differential equations.

Our second valuation approach is a bit less restrictive. We do not specify a unique value for X, but a whole interval of possible values which is still small enough to be useful in practice. This interval is obtained by taking for valuation those measures Q which yield neither arbitrage opportunities nor *good deals*. The latter are defined as investment opportunities with a (von Neumann-Morgenstern expected) utility which is "too high" in comparison with the maximal utility obtainable by trading in the basic assets. The main difficulty is the precise definition of the set $\mathcal{N}$ of *no-good-deal measures* which is very important for computational and dynamic properties of the *good deal bounds*, i.e., of the boundaries of the interval of possible values. In a Lévy setting, we define $\mathcal{N}$ via a restriction on an appropriate integrand, and we clarify the exact relation between this "local" and an economically more intuitive "global" restriction. The resulting valuation bounds are then time-consistent dynamic MCUFs.

In order to establish the relation between the local and global restrictions, we need to know that the Lévy structure of the underlying market is preserved under an optimal change of measure. This is proved in the last part of this thesis for optimal measures obtained from the *dual* problem of minimizing some f*-divergence* over a set of equivalent local martingale measures. These optimization problems naturally arise in utility maximization, and we establish the Lévy preservation result for the f-divergences corresponding to logarithmic, power and quadratic utility.

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Zusammenfassung

Diese Arbeit befasst sich mit zwei dynamischen Bewertungsmethoden für Derivate in unvollständigen Märkten. Die erste ist *Indifferenzbewertung*, bei der ein Agent zu jedem Zeitpunkt t den Wert $p_t(X)$ einer zufälligen Auszahlung X dadurch festlegt, dass es ihm gleichgültig ist, ob er X für den Preis $p_t(X)$ kauft oder nicht, sofern er stets optimal in den Basisanlagen handelt. Wir nehmen an, dass die Präferenzen des Agenten zum Zeitpunkt t durch eine *monetäre konkave Nutzenfunktion* (MCUF) U_t gegeben sind, d.h., dass $-U_t$ ein (bedingtes) konvexes Risikomaß ist. Das Bewertungsfunktional $p_t(\,\cdot\,)$ entspricht dann gerade der *Faltung* von U_t und einem *Marktfunktional*, welches mit Hilfe der optionalen Zerlegung unter Handelsbeschränkungen aus dem zugrunde liegenden Finanzmarkt konstruiert wird. Unser eigentliches Ziel ist es zu zeigen, dass $p_.(\,\cdot\,)$ *zeitkonsistent* ist, also (auf eine geeignete Art und Weise) die Ordnung zwischen verschiedenen Auszahlung über die Zeit erhält. Hierzu beweisen wir, dass das Marktfunktional zeitkonsistent ist, und dass bei der Faltung dynamischer MCUFs die Zeitkonsistenz erhalten bleibt. Als Hilfsresultat leiten wir eine Darstellung für (bedingte) MCUFs mit Hilfe der zugehörigen konkav konjugierten Funktion und äquivalenten Wahrscheinlichkeitsmaßen her. Zusätzlich zeigen wir, wie unsere Resultate auf solche dynamische MCUFs übertragen werden können, die durch stochastische Rückwärtsdifferenzialgleichungen definiert sind.

Die zweite hier betrachtete Methode ist etwas weniger restriktiv. Anstelle eines eindeutigen Wertes für X bestimmen wir ein ganzes Intervall möglicher Werte, welches klein genug ist, um praktischen Nutzen zu haben. Dieses Intervall erhalten wir durch die Verwendung all jener Bewertungsmaße Q, die weder zu Arbitragemöglichkeiten noch zu *good deals* führen. Letztere definieren wir als Auszahlungen mit einem (von Neumann-Morgenstern erwarteten) Nutzen, der im Vergleich mit dem maximal durch Handeln in den Basisanlagen erreichbaren Nutzen "zu hoch" ist. Die eigentliche Schwierigkeit besteht in der genauen Definition der Menge der *no-good-deal-Maße* $\mathcal{N}$. Sie ist maßgebend für die Berechenbarkeit und die dynamischen Eigenschaften der *good-deal bounds*, d.h. der Grenzen des Intervalls der möglichen Werte für X. In einem Lévy-Model definieren wir $\mathcal{N}$ über die Beschränkung eines geeigneten Integranden and klären den genauen Zusammenhang zwischen dieser "lokalen" und einer intuitiveren "globalen" Beschränkung. Die daraus resultierenden good-deal bounds sind zeitkonsistente dynamische MCUFs.

Um eine Beziehung zwischen den lokalen und globalen Beschränkungen herzustellen, benötigen wir, dass die Lévy-Struktur des zugrunde liegenden Marktes bei einem optimalen Maßwechsel erhalten bleibt. Dies wird im letzten Teil der vorliegenden Arbeit für optimale Maße gezeigt, welche das duale Problem der Minimierung einer f-Divergenz über eine Menge von äquivalenten lokalen Martingalmaßen lösen.

Solche Optimierungsprobleme treten bei der Nutzenmaximierung auf, und wir zeigen die Erhaltung der Lévy-Struktur für die f-Divergenzen, welche zu logarithmischen, quadratischen und Potenz-Nutzenfunktionen gehören.

Contents

Chapter 1

Introduction

This thesis is concerned with dynamic approaches to valuation in incomplete markets. To put our contributions in perspective in a larger context, it seems useful to start with a short overview.

1.1 Valuations in incomplete markets

Complete models of financial markets have nice conceptual properties. In particular, by no-arbitrage arguments alone, one can obtain a unique price for any random payoff X due at some time T in the future. This price is the expectation of X with respect to the unique martingale measure for the traded assets. However, in a complete model, every payoff is redundant, and this is in general not true for a real financial market. Therefore realistic models are incomplete, and if the (discounted) price processes of the traded assets are modelled by some semimartingale S, the set $\mathcal{M}^e(S)$ of equivalent local martingale measures for S contains infinitely many elements. Since any $E_Q[X]$ with $Q \in \mathcal{M}^e(S)$ is an arbitrage-free value for X, we obtain a whole interval of possible values; we refer to *values* for X rather than *prices* because they are not unique and their determination typically involves subjective preferences. Valuation in incomplete markets has been and still is being intensively studied, and we mention here only some of the main approaches with a few original references, and some overviews.

A frequently chosen approach is to specify unique values $E_Q[X]$ for all X by fixing one element Q of $\mathcal{M}^e(S)$ as pricing measure; see Bellini/Frittelli [BF02] for a discussion. Typical examples are the minimal entropy, the variance-optimal or the minimal martingale measure. However, such a choice can be rather arbitrary or restrictive, even if it is backed up by some specific subjective preferences. On the other hand, as discussed above, introducing no preferences at all only gives a rather big interval of arbitrage-free values for X. Its upper bound, $\sup_{Q\in\mathcal{M}^e(S)} E_Q[X]$, is the *superreplication value* (also known as *superhedging value*), which is the smallest amount of money from which it is possible to obtain, with a dynamic self-financing trading strategy in S, a payoff dominating X with probability one. This valuation principle was first studied by El Karoui/Quenez [EKQ95] and Kramkov [Kra96]. Like most other approaches, it is defined from the perspective of an agent who can trade in S and tries to find a suitable amount of money for which she is willing to sell (or buy) X. Superreplication is a very conservative approach since it allows the seller to implement a dominating trading strategy for X, thus eliminating all the risk involved in selling X. The drawback of this is that the capital requirement for the superreplication strategy is usually very big, so that the superreplication value is unrealistically high. However, if the agent is willing to accept some risk, the required capital can be reduced. This leads to the *quantile hedging* and *efficient hedging* approaches. The first looks for the smallest initial capital for which there exists a dynamic trading strategy in S whose probability of a shortfall, i.e., of a loss after selling X and implementing the trading strategy, stays below some bound; see Föllmer/Leukert [FL99]. In contrast, efficient hedging also takes into account the size of the shortfall and determines the smallest amount of money for which there is a trading strategy with shortfall risk below a fixed bound; see Föllmer/Leukert [FL00].

Perhaps the earliest approaches to valuation via hedging are those based on quadratic criteria. *Mean-variance hedging* is similar to efficient hedging, but instead of fixing a bound for the shortfall risk, one minimizes the $\mathbf{L}^2$-norm of the hedging error over all initial capitals and trading strategies; see Bouleau/Lamberton [BL89], Duffie/Richardson [DR91] or Schweizer [Sch94]. The value of X is then given by $E_{\tilde{Q}}[X]$, where $\tilde{Q}$ denotes the variance-optimal signed martingale measure for S; this is obtained via solving the *dual problem* of minimizing the variance (under the subjective measure P) of the density dQ/dP over all (signed) martingale measures Q. An even earlier quadratic approach is *(local) risk-minimization*; see Föllmer/Sondermann [FS86] and Schweizer [Sch91]. In contrast to the other approaches, one considers here trading strategies which perfectly replicate X. Due to the incompleteness of the market, these trading strategies cannot be self-financing in general so that intermediate costs occur. The value for X is then the initial capital of that trading strategy which minimizes the expected squared intermediate costs, and it turns out to be given by the expectation of X under the minimal signed martingale measure. An overview of these quadratic approaches is given by Schweizer [Sch01]. More generally, one can consider other than quadratic criteria, e.g., one can replace the $\mathbf{L}^2$-norm by the $\mathbf{L}^p$-norm. Viewing such a power function as a utility function then naturally leads us to utility based valuation approaches.

The first utility based approach is *utility indifference valuation* which (in the context of financial markets) was suggested by Hodges/Neuberger [HN89]; see Henderson/Hobson [HH04] for an overview. One assumes that the preferences of the agent can be captured by some utility functional $\mathbf{U}$. The value p of X is then determined by the condition that the agent is indifferent (according to $\mathbf{U}$) between buying X for p and not buying it, presuming she trades optimally in S in both cases. If one uses this approach with a von Neumann-Morgenstern expected utility, it is very difficult to obtain explicit results, and several variants have been suggested to overcome these problems. A very recent and promising method is *indifference valuation via convex risk measures*, where preferences are given by monetary utility functionals, i.e., minus convex risk measures; see Barrieu/El Karoui [BEK05], Xu [Xu06] and Chapter 2 of this thesis. The crucial difference is that monetary utility functionals measure, as their name suggests, utility in monetary units; they are therefore translation-invariant for the addition of cash, and this yields rather explicit results for the corresponding indifference value. Another, considerably earlier variant is the *fair price* (also known as *marginal utility based* or *shadow price*) which goes back to Davis [Dav98] and which is defined via a marginal substitutability condition on the expected utility. More precisely, it is that amount p of money which, given the possibility of buy-and-hold trading in X for the price p and dynamic trading in S, makes the agent's optimal demand for X (with respect to her utility functional $\mathbf{U}$) equal to zero.

All valuation concepts presented above aim to specify for the payoff X a unique value out of the usually very large interval of arbitrage-free values. As mentioned above, this can be quite restrictive, and one would perhaps like to have a middle way. Instead of a single value, one could therefore aim for an entire interval of values which at the same time is small enough to be useful in practice. This leads us to *good deal value bounds*, where in addition to arbitrage opportunities, one also rules out those investments which are too attractive (in an appropriate sense) in comparison with those traded in the market. This approach has originally been suggested by Cochrane/Saà-Requejo [CSR00] who measure attractiveness in terms of the Sharpe ratio; see also Černý/Hodges [CH02] and Jaschke/Küchler [JK01] for subsequent alternatives and generalizations.

1.2 Main results

A large part of the existing literature studies valuation in incomplete markets statically, i.e., at time $t = 0$; exceptions are notably works based on stochastic control theory. The main goal of this thesis is a *dynamic* study of value functionals and valuation bounds for some payoff X. More precisely, this is done for indifference valuation via convex risk measures, and for good deal price bounds. Both topics in the current general form have emerged only rather recently. In particular, indifference valuation via convex risk measures has been developped in parallel to this thesis by Barrieu/El Karoui [BEK05] and Xu [Xu06]. In our view, the most important dynamic property of both indifference valuation via risk measures and good deal bounds (in the sense they are presented here) is that they are *time-consistent*. This means that if at time t the value of X is higher than that of Y, the same holds true at any time $s \leq t$, i.e., when less information is available. For indifference valuation in particular, the study of time-consistency is one of the most important issues and contributions of this thesis.

1.2.1 Indifference valuation via convex risk measures

We first consider indifference valuation via convex risk measures in more detail and explain our main results. The mathematical formulation is as follows. We assume that our agent's time t preferences are given by a *monetary utility functional* U_t, i.e., minus a convex risk measure. The corresponding *indifference value* $p_t(X)$ for a payoff X is then defined by

$$\operatorname*{ess\,sup}_{G\in\mathcal{C}_t} U_t\big(G - p_t(X) + X\big) = \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} U_t(G), \qquad (1.2.1)$$

where the set $\mathcal{C}_t$ of payoffs superreplicable from time t with zero wealth encodes the trades available in the underlying financial market. Because U_t is *translation invariant* in the sense that $U_t(X+a_t) = U_t(X)+a_t$ if a_t is $\mathcal{F}_t$-measurable, we can solve (1.2.1) explicitly for $p_t(X)$ to get

$$p_t(X) = \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} U_t(G+X) - \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} U_t(G) =: U_t^{\text{opt}}(X) - U_t^{\text{opt}}(0). \qquad (1.2.2)$$

Here $U_t^{\text{opt}}(X)$ describes the agent's modified preferences when she takes into account her trading opportunities. Due to (1.2.2), it suffices to study U_t^{opt} instead of p_t itself. It turns out that U_t^{opt} is the *convolution* of U_t with another monetary utility functional, namely the so-called *market functional*. The latter is associated to the financial market via the set $\mathcal{C}_t$, and it is constructed like in Föllmer/Schied [FS02] with the help of the optional decomposition under constraints. More precisely, we show in Theorems 2.5.11 and 2.6.8 that the market functional for $\mathcal{C}_t$ exists and that it is time-consistent. Moreover, we study in Theorem 2.4.3 the convolution of two abstract dynamic convex risk measures and prove in particular that the convolution operation preserves time-consistency. Combining these results readily implies that the indifference valuation functional (p_t) itself is time-consistent, which achieves one of our major goals.

A second contribution in connection with the above approach concerns the structure of conditional convex risk measures. Because pricing in financial markets is usually done with the help of equivalent martingale measures, we want a *representation* for conditional convex risk measures in terms of their conjugate functionals via *equivalent* probability measures. In Theorem 2.3.16, we obtain such a result which is slightly sharper than those in the existing literature. Finally, we also look at examples. As shown in Rosazza Gianin [RG06], a large class of examples of time-consistent dynamic convex risk measures can be obtained via backward stochastic differential equations (BSDEs for short) and can be entirely described via some integrand. This allows rather explicit representations, at the cost of strong assumptions (Brownian filtration) on the information structure. We show in Theorems 2.7.15 and 2.7.17 for such a setting how U_t and the market functional can be expressed in terms of BSDEs, and that U_t^{opt} can be obtained by convoluting the respective integrands.

Although various aspects of our approach have appeared before, the combined and systematic treatment of all ideas at the general and conditional level seems to be new.

Most previous results are only given unconditionally for $t = 0$; this applies to the indifference valuation via risk measures mentioned in Barrieu/El Karoui [BEK05] and discussed in more detail in Xu [Xu06], to the construction of the market functional in Föllmer/Schied [FS02], or to the convolution in Barrieu/El Karoui [BEK05]. Some conditional results are available; Larsen/Pirvu/Shreve/Tütüncü [LPST05] treat indifference valuation for a special U_t, Detlefsen/Scandolo [DS05] and Cheridito/Delbaen/Kupper [CDK06] provide similar representations like here for conditional convex risk measures; see Section 2.3 for a more detailed comparison with these two papers. Jobert/Rogers [JR06] study several of the above issues in finite discrete time over a finite probability space. Barrieu/El Karoui [BEK04] discuss the convolution of DM-CUFs which are given by BSDEs. However, they work with a class of BSDEs which is not general enough to incorporate the market functional of an incomplete market, which is constructed as in Bender/Kohlmann [BK04].

1.2.2 Utility based good deal bounds

The second contribution of this thesis is in the area of good deal bounds. The main idea is as follows. We have already observed that the interval

$$\Big(\inf_{Q\in\mathcal{M}^e(S)} E_Q[X],\ \sup_{Q\in\mathcal{M}^e(S)} E_Q[X]\Big)$$

of all arbitrage-free values for a payoff X is usually too big to be useful in practice. One reason for this is that $\mathcal{M}^e(S)$ contains many pricing measures which are not very reasonable, because they yield investment opportunities which are too "good" compared with those traded in the market. By omitting these "unrealistic" measures, one obtains a smaller interval of values for X. This approach has the advantage that it is not as restrictive as singling out one particular pricing measure.

But how does one measure "good" or "unrealistic"? We quantify here attractiveness in terms of von Neumann-Morgenstern expected utility and relate this approach to the original definition of Cochrane/Saà-Requejo [CSR00] where the Sharpe ratio is used as performance measure. In order to obtain a mathematically tractable problem, [CSR00] use an inequality from Hansen/Jagannathan [HJ91] to bound the Sharpe ratio of any payoff by the variance of the density of the pricing measure being used. We prove that this upper bound is just the maximal attainable Sharpe ratio in an extended market, where the extension depends on the chosen pricing measure. In addition, we show that the set $\mathcal{N}$ of *no-good-deal measures* from [CSR00] can also be obtained by imposing an upper bound on the maximal quadratic utility attainable in the same extended market. This gives rise to a more general approach where we replace the quadratic by more general utility functions like power, exponential or logarithmic utility. A similar approach has been suggested by Černý [Cer03]; one major difference is that we provide here a more general and (we believe) more transparent treatment.

Our main goal is to study the no-good-deal values and value bounds as processes.

Because their computability and dynamic properties depend on the set $\mathcal{N}$ of no-good-deal measures, a key issue is to find an appropriate and yet workable definition for this set in a dynamic context. In a Lévy setting, we define $\mathcal{N}$ by a pointwise restriction on an appropriate integrand. This allows to apply dynamic programming techniques in the computation of the value bounds. Moreover, we show in Theorem 3.5.7 that this pointwise ("local") restriction implies a bound on the corresponding "global" criterion. The resulting dynamic value bounds are dynamic coherent risk measures and in particular time-consistent.

The main contributions of this thesis to the theory of good deal bounds are Theorem 3.5.7 and Proposition 3.5.10 which clarify the connection between the "global" and the pointwise restrictions. This intrinsically depends on the fact that we choose the pointwise bound for the integrand to be deterministic and time-independent. However, this bound cannot be chosen completely arbitrarily. In order to yield meaningful intervals for the values, it must depend on the maximal utility attainable from trading in the basic assets S only. It is well known that the latter can be calculated via a *dual problem*, namely minimizing some f-divergence, i.e., a functional $Q \mapsto E\left[f\left(\frac{dQ}{dP}\right)\right]$ where f is convex, over $\mathcal{M}^e(S)$. If the *f-minimal martingale measure* Q^f, i.e., the solution to this dual optimization problem, preserves the Lévy property of the process driving the financial market, then the pointwise bound can be chosen deterministic and time-independent. This is one motivation for us to study in general the question whether the Lévy property is preserved under an optimal change of measure; see below for details.

Pointwise restrictions in connection with the determination of good deal bounds have been suggested before in Cochrane/Saà-Requejo [CSR00], Černý [Cer03] and Björk/Slinko [BS06]. [CSR00] and [Cer03] work in a Brownian setting and obtain a sort of connection between the local and global restrictions by taking limits. [BS06] extend that model by adding a marked point process, but do not study the relation between the local and global restrictions. In contrast, Theorem 3.5.7 proves in a general setting that the local implies the global restriction, and Proposition 3.5.10 provides a precise description of a situation when the local and global restrictions coincide for the choice of no-good-deal pricing measures. Moreover, we also give a justification why a constant or deterministic local restriction is reasonable, and in particular show why it induces a non-empty set of no-good-deal measures.

1.2.3 Preservation of the Lévy property under an optimal change of measure

As explained above, we can obtain a nice relation between the global and local restrictions in the definition of the good deal bounds if we know that the *f-minimal martingale measure* Q^f preserves the Lévy property of the underlying financial market. Let us explain this in more detail. Suppose our filtration is generated by a d-dimensional semimartingale L which is a Lévy process under the subjective measure P, and consider the set $\mathcal{M}^e(\mathbf{M}L)$ of equivalent local martingale measures for $\mathbf{M}L$ with a fixed $d \times d$-matrix $\mathbf{M}$. We assume that $\mathbf{M}$ and the basic assets are chosen such that $\mathcal{M}^e(S) = \mathcal{M}^e(\mathbf{M}L)$; this includes for instance exponential Lévy models $S = \mathcal{E}(L)$ or suitable classes of stochastic volatility models, as explained in Esche/Schweizer [ES05]. The main result of [ES05] in this setting is then that L is still a Lévy process under the *minimal entropy* martingale measure $Q^e = \operatorname{argmin}_{Q \in \mathcal{M}^e(S)} E\left[(dQ/dP)\log(dQ/dP)\right]$. This Q^e is the f-minimal martingale measure for the function $f^e(z) = z\log z$ which comes up in the dual problem corresponding to exponential utility maximization.

In the final contribution of this thesis, we show that the approach and result in [ES05] can be generalized to the *f-minimal* martingale measures associated to the convex functions

$$\begin{aligned} f^\ell(z) &:= -\log z, \\ f^p(z) &:= z^{-\delta} \quad \text{for } \delta \in (0,\infty), \\ f^q(z) &:= z^2. \end{aligned} \tag{1.2.3}$$

These occur in the dual problems for logarithmic (ℓ), power (p) and quadratic (q) utility, respectively. More precisely, the allowed power utilities are the functions $\frac{\delta+1}{\delta}x^{\frac{\delta}{\delta+1}} = \frac{1}{\hat\delta}x^{\hat\delta}$ with $\hat\delta := \frac{\delta}{\delta+1} \in (0,1)$ and $x \in (0,\infty)$. The main idea for proving this generalization is the same as in [ES05], but the computations and technical details become a little bit more involved.

For completeness, let us explain how the argument works. Due to the underlying Lévy structure, any $Q \approx P$ can be described by two stochastic processes called the *Girsanov parameters* of Q. The f-divergence $f(Q|P) = E[f(dQ/dP)]$ of Q is then a convex functional of these Girsanov parameters and by Jensen's inequality can thus be reduced by averaging the Girsanov parameters. More precisely, the new parameters obtained by averaging define a measure $\overline{Q}$ with $f\left(\overline{Q}\middle|P\right) \le f(Q|P)$. Since we are interested in the measure $Q^f = \operatorname{argmin}_{Q \in \mathcal{M}^e(S)} f(Q|P)$, we should also like to have that $\overline{Q}$ is a local martingale measure if Q is. This is not true in general, but it does hold if we take Q from a suitable subset of $\mathcal{M}^e(\mathbf{M}L)$, specified via an additional integrability property for L. We then show that this subset is dense in $\mathcal{M}^e(\mathbf{M}L)$ in an appropriate sense, and this allows us to prove that for all f in (1.2.3), the f-minimal martingale measure Q^f has time-independent and deterministic Girsanov parameters. Since this is exactly the property which describes the measures preserving the Lévy property of L, one can conclude that L is indeed still a Lévy process under Q^f.

1.3 Links between indifference valuation and good deal bounds

At first sight, the above two valuation approaches may appear very different. However, there is a close relation between the indifference valuation considered in this thesis and good deal bounds, provided that the latter are defined slightly differently than here. Alternative definitions are discussed in detail in Section 3.3 below, and so we only outline the main idea.

In the literature, *good deals* are usually defined as payoffs with non-positive prices and which are contained in an (abstract) set of *desirable* claims; see, e.g., Černý/Hodges [CH02], Jaschke/Küchler [JK01] or Staum [Sta04]. The (lower) good deal value bound for a payoff X is then defined as the biggest amount a of money that can be subtracted from X so that the resulting payoff $X - a$ cannot be turned into a good deal by trading in the market with zero initial capital. Hence, buying X for a price below this bound allows to generate a good deal by trading. It remains to specify the set of desirable payoffs, and this is typically the *acceptance set* of a monetary utility functional U_t, i.e., the set of those payoffs X with $U_t(X) \geq 0$. This is different from our approach to good deal bounds because we work with a (non-monetary) von Neumann-Morgenstern expected utility.

Now suppose that U_t used for defining good deals as above also describes the preferences in our indifference valuation approach. If p_t is the corresponding indifference value, then the interval of no-good-deal values turns out to be $[p_t(X), -p_t(-X)]$; this is discussed in more detail at the end of Section 2.6 below. Hence we see that the good deal bounds in this general approach coincide with the seller and buyer values from the indifference valuation method.

1.4 Structure of the thesis

This thesis consists of an introduction, three chapters and an appendix. To keep each chapter self-contained, we have deliberately allowed redundancies. The structure of the thesis is as follows.

In Chapter 2 we study indifference valuation via convex risk measures, beginning with a detailed introduction and some notation. In Section 2.3 we then concentrate on monetary concave utility functionals (MCUFs) U_t. In particular, we provide a representation for U_t in terms of its concave conjugate and via equivalent probability measures. In addition, we give some results about time-consistency, inspired mainly by Delbaen [Del06]. Section 2.4 studies the convolution of general dynamic MCUFs (DMCUFs), showing in particular that this operation preserves time-consistency. The proof is an application of the representation theorem of Section 2.3. In Section 2.5, we adapt the results of Föllmer/Kramkov [FK97] about superhedging under constraints to our needs. We combine the above results in Section 2.6 to prove that U^{opt} from

(1.2.2) is the convolution of U and the market DMCUF given via the superhedging price. Then we show that the indifference valuation functional p is a DMCUF, give conditions for it to be time-consistent and consistent with the no-arbitrage principle, and relate it to good deal bounds. Section 2.7 presents three examples. The first deals with time-consistency and some properties of the convolution, the second with DMCUFs described by BSDEs, and the third illustrates that a static MCUF cannot always be extended to a dynamic MCUF.

In **Chapter 3** we turn to good deal bounds. After a general introduction, we recall in Section 3.2 the original definition of good deal bounds from Cochrane/Saà-Requejo [CSR00] and explain how it can be generalized. For clarity of presentation, this is done in a static setting. Section 3.3 explains the link between value bounds and monetary utility functionals, and discusses in more detail the connections between the different existing approaches on good deal bounds. We then turn to a dynamic setting to study good deal values and value bounds as processes. In order to have a nice parametrization for the set of all equivalent local martingale measures, we choose to work in a Lévy framework, and Section 3.4 collects some auxiliary results on this. Section 3.5 deals with the extension of no-good-deal valuation to a dynamic setting. The main difficulty is to find a reasonable definition for the set of no-good-deal measures which still leads to mathematically tractable problems. As explained above in Section 1.2.2, our definition is obtained from a pointwise restriction on an appropriate integrand. We explain how this "local" restriction is motivated by a "global" criterion, and how the two are connected. Section 3.6 discusses the properties of the resulting good deal prices and price bounds as processes. Finally, we present two explicit examples in Section 3.7.

Chapter 4 studies the preservation of the Lévy property for a P-Lévy process L under the f-minimal martingale measure for $\mathbf{M}L$, with f as in (1.2.3) and a fixed matrix $\mathbf{M}$. We first motivate our results and relate them to existing literature. In Section 4.3 we fix some notation and recall some important facts about Lévy processes and changes of measure. In particular, we explain how equivalent measures can be described by their Girsanov parameters and give conditions for the latter to describe a measure in $\mathcal{M}^e(\mathbf{M}L)$. Section 4.4 then contains the main results of this chapter. We explicitly define the averaging procedure for the Girsanov parameters and show how it reduces the f-divergence. Then we specify a dense subset of $\mathcal{M}^e(\mathbf{M}L)$ consisting of measures for which this averaging leads to measures again contained in $\mathcal{M}^e(\mathbf{M}L)$. This is subsequently exploited to prove our main result that L is still a Lévy process under the f-minimal martingale measure. Finally, Section 4.5 briefly discusses the quadratic case $f(z) = f^q(z) = z^2$. We show that if the f^q-minimal martingale measure and the *variance-optimal signed martingale measure* coincide, one can show directly that the f-minimal martingale measure preserves the Lévy property. This uses that in a Lévy setting the variance-optimal signed martingale measure agrees with the *minimal signed martingale measure*, for which an explicit formula is known.

Chapter 2

Dynamic indifference valuation via convex risk measures

2.1 Introduction

This chapter deals with the valuation of contingent claims in incomplete financial markets. We present a dynamic indifference valuation approach which stems from the basic economic concept of certainty equivalent, modified and extended to accommodate the market environment (an idea introduced by Hodges/Neuberger [HN89]). The agents' attitudes towards risk are incorporated to establish preferences over risk which cannot be eliminated by trading.

More precisely, our investor's preferences at each time t are given by some utility functional $U_t : \mathbf{L}^\infty \to \mathbf{L}^\infty(\mathcal{F}_t)$. The investor has at time t an $\mathcal{F}_t$-measurable initial endowment x_t and can trade in a financial market, possibly under constraints. We denote by $\mathcal{C}_t$ the set of payoffs she can superhedge by trading during $(t, T]$ with zero initial endowment. At each time $t \in [0, T]$, the *indifference value* $p_t(X)$ of a payoff $X \in \mathbf{L}^\infty$ due at time T is defined implicitly by

$$\operatorname*{ess\,sup}_{G \in \mathcal{C}_t} U_t(x_t + G) = \operatorname*{ess\,sup}_{G \in \mathcal{C}_t} U_t(x_t - p_t(X) + G + X), \qquad (2.1.1)$$

i.e., such that the agent is indifferent between buying X for the price $p_t(X)$ and not buying it, presuming she trades optimally in the market in both cases. U_t belongs to the class of *monetary concave utility functionals at time t* (MCUFs for short), which is defined axiomatically such that $-U_t$ is a ($\mathcal{F}_t$-conditional) convex risk measure. In particular, U_t is *$\mathcal{F}_t$-translation invariant* in the sense that

$$U_t(X + a_t) = U_t(X) + a_t \quad \text{for all } a_t \in \mathbf{L}^\infty(\mathcal{F}_t),$$

so that in (2.1.1) all $\mathcal{F}_t$-measurable quantities can be extracted. Hence

$$p_t(X) = U_t^{\mathrm{opt}}(X) - U_t^{\mathrm{opt}}(0), \tag{2.1.2}$$

where the operator

$$U_t^{\mathrm{opt}}(\,\cdot\,) := \operatorname*{ess\,sup}_{G \in \mathcal{C}_t} U_t(\,\cdot\, + G)$$

corresponds to the agent's market modified preferences when she takes into account her trading opportunities.

We show that similarly as in Barrieu/El Karoui [BEK05], U_t^{opt} is an MCUF and given by the *convolution* of U_t and the *market MCUF*; the latter is associated to $\mathcal{C}_t$ and constructed like in Föllmer/Schied [FS02] with the help of the optional decomposition under constraints. A key issue is to ensure (strong) time-consistency for the dynamic behaviour of $p = (p_t)$. Therefore we study the convolution of two abstract conditional risk measures and prove that this operation preserves (strong) time-consistency. In the same general setting, we give sufficient conditions to guarantee that $p_t(X)$ lies inside the interval of arbitrage-free prices so that it could be considered as a price for X, and we investigate the structure of p when there are no trading constraints. We briefly discuss the connection to good deal bounds. In the special case where U is given by a backward stochastic differential equation (BSDE), we also describe the market DMCUF, U^{opt} and p in this way, and we show that the driver for U^{opt} is the pointwise convolution of the drivers of U and of the market DMCUF. This extends results of Rosazza Gianin [RG06] and Barrieu/El Karoui [BEK04]. Finally, because pricing and valuation in financial markets is done with the help of equivalent martingale measures, we also want a *representation* for MCUFs in terms of their concave conjugate functionals via *equivalent* probability measures.

Although various aspects of our approach have appeared before, the combined treatment of all ideas at the general and conditional level seems to be new. Most previous results are only given unconditionally for $t = 0$; this applies to the indifference valuation via risk measures in Xu [Xu06] or (briefly) in Barrieu/El Karoui [BEK05], to the construction of the market functional in Föllmer/Schied [FS02], or to the convolution in Barrieu/El Karoui [BEK05]. Some conditional results are available; Larsen/Pirvu/Shreve/Tütüncü [LPST05] treat indifference valuation for a special U_t, Detlefsen/Scandolo [DS05] and Cheridito/Delbaen/Kupper [CDK06] provide similar representations for conditional convex risk measures; see Section 2.3 for a more detailed comparison with these two papers. Jobert/Rogers [JR06] study several of the above issues in finite discrete time over a finite probability space. Barrieu/El Karoui [BEK04] discuss the convolution of DMCUFs which are given by BSDEs. However, they work with a class of BSDEs which is not general enough to incorporate the market functional of an incomplete market, which is constructed as in Bender/Kohlmann [BK04]. Our general results that convolution preserves time-consistency and that the market functional in an incomplete market with trading constraints is time-consistent seem to be new.

The chapter is structured as follows. Notations and conventions are given in Section 2.2. Section 2.3 introduces (dynamic) MCUFs. We state a representation theorem for MCUFs similar to [DS05] but in terms of equivalent probability measures; a closely related result can be found in [CDK06]. Some results about (strong) time-consistency inspired mainly by [Del06] are also given. Section 2.4 introduces the convolution of general dynamic MCUFs and extends a result of [BEK05]. The proof is one application of the representation theorem of Section 2.3. In Section 2.5, we adapt the results of [FK97] about superhedging under constraints to our needs. We combine the above results in Section 2.6 to prove that U^{opt} is the convolution of U and the market DMCUF given via the superhedging price. Then we show that the indifference valuation functional p is a dynamic MCUF, give conditions when it is strongly time-consistent and consistent with the no-arbitrage principle, and relate it to good deal bounds. Section 2.7 presents three examples. The first deals with time-consistency and some properties of the convolution, the second with dynamic MCUFs described by backward stochastic differential equations, and the third illustrates that a static MCUF cannot always be extended to a dynamic MCUF.

2.2 Notations and conventions

Throughout this chapter, we work with a fixed probability space $(\Omega, \mathcal{F}, P)$ and a fixed filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$, where $T < \infty$ is a fixed finite time horizon. We assume that $\mathbb{F}$ satisfies the usual conditions of right-continuity and completeness. Hence we can and do choose for each semimartingale a right-continuous version with left limits (RCLL for short). For simplicity we let $\mathcal{F}_0$ be trivial and $\mathcal{F}_T = \mathcal{F}$. For $s < t$ an integral from s to t is defined on the half-open interval $(s, t]$. For $p \in [1, \infty]$, $\mathbf{L}^p(\Omega, \mathcal{G}, P)$ ($\mathbf{L}^p(\mathcal{G})$ or even $\mathbf{L}^p = \mathbf{L}^p(\mathcal{F})$ if no confusion is possible) denotes the space of all equivalence classes of real-valued, $\mathcal{G}$-measurable random variables with finite $\mathbf{L}^p(P)$-norm, where $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. By $\mathbf{L}^0(\mathcal{F}_t, Y)$ we denote the set of all equivalence classes of $\mathcal{F}_t$-measurable mappings $\Omega \to Y$. An $\mathcal{F}_t$-partition is a family of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}_t$ whose union is Ω. The transpose of a vector z is denoted by z^* and $\mathbf{1}_A$ denotes the indicator function for a set $A \in \mathcal{F}$. $\mathcal{P}$ denotes the set of all probability measures Q on $(\Omega, \mathcal{F})$, $\mathcal{P}^a$ the set of all $Q \in \mathcal{P}$ with $Q \ll P$ and $\mathcal{P}^e$ the set of all $Q \in \mathcal{P}^a$ with $Q \approx P$. Unless mentioned otherwise, all (in-)equalities which involve random variables hold almost surely with respect to P, (conditional) expectations and essential infima and suprema are taken with respect to P, a density Z_T of some measure $Q \in \mathcal{P}^a$ is its density with respect to P on $\mathcal{F} = \mathcal{F}_T$ and its density process $Z = (Z_t)_{0\leq t\leq T}$ consists of its densities Z_t with respect to P on $\mathcal{F}_t$. We frequently identify a probability measure $Q \in \mathcal{P}^a$ with its density $Z_T \in \mathbf{L}^1(\Omega, \mathcal{F}, P)$. When we say that a set $\mathcal{Q} \subseteq \mathcal{P}^a$ has a property in $\mathbf{L}^1$, we mean that the set of corresponding densities has this property. $\mathcal{Q}^e$ consists of all $Q \in \mathcal{Q}$ which are equivalent to P. We always work with equivalence classes of random variables and thus do not distinguish between different versions of, e.g., the essential infimum of a family of random variables. In particular, when defining some set depending on an equivalence class of random variables we take one (fixed) representative, in order to have that set well-defined. For the definition of processes having locally some property we refer to Definition VI.27 in [DM82]. As we consider processes on $[t, T]$ having some local properties, this definition has the advantage that a stopped process with starting point t need not have the required property on all of Ω but only on the sets $\{\tau_n > t\}$ where $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence. In particular, for any $t \geq 0$ we have $\{\tau_n > t\} \subseteq \{\tau_n > 0\}$ and this ensures that if S is a locally bounded semimartingale on $[0, T]$, so is S on $[t, T]$. Moreover, note that the assumption of $\mathcal{F}_0$ to be trivial implies boundedness of S_0. Since we are working with a finite time horizon T, a localizing sequence for some process $(S_t)_{0\leq t\leq T}$ is an increasing sequence of $[0, T]$-valued stopping times τ_n, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} P[\tau_n < T] = 0$ and such that for each $n \in \mathbb{N}$, the stopped process S^{τ_n} has the desired property.

2.3 Representations and time-consistency of dynamic MCUFs

In this section we introduce and study *(dynamic) monetary concave utility functionals* (MCUFs for short). This is the class of functionals we consider for indifference valuation in a later section. Their definition is very similar to that of convex risk measures, for which it is known that they can be equivalently described by their *acceptance set*, i.e., the set of payoffs to which they assign non-positive values. We state the analogous result for (dynamic) MCUFs and investigate the properties, in particular continuity, of (dynamic) MCUFs. The main result of this section gives an equivalence between continuity of an MCUF, its representability, and closedness of its acceptance set. This extends well-known results from the static case to a dynamic setting. Similar dynamic results can also be found in a recent work of Detlefsen/Scandolo [DS05], and in [CDK06] in a more general setting. Finally we investigate a property called *(strong) time-consistency* which ensures that the ordering on payoffs induced by a dynamic MCUF is consistent between different points in time.

Definition 2.3.1. Fix $t \in [0, T]$. We call a mapping

$$\Phi_t : \mathbf{L}^\infty(\Omega, \mathcal{F}, P) \to \mathbf{L}^\infty(\Omega, \mathcal{F}_t, P)$$

a *monetary concave utility functional at time* t (MCUF for short) if it satisfies

A) *Monotonicity:* $\Phi_t(X_1) \leq \Phi_t(X_2)$ for all $X_1, X_2 \in \mathbf{L}^\infty$ with $X_1 \leq X_2$.

B) *$\mathcal{F}_t$-translation invariance:* $\Phi_t(X + a_t) = \Phi_t(X) + a_t$ for all $X \in \mathbf{L}^\infty$, $a_t \in \mathbf{L}^\infty(\mathcal{F}_t)$.

C) *Concavity:* $\Phi_t(\beta X_1 + (1-\beta)X_2) \geq \beta\Phi_t(X_1) + (1-\beta)\Phi_t(X_2)$ for all X_1, $X_2 \in \mathbf{L}^\infty$ and $\beta \in [0, 1]$.

We say that an MCUF Φ_t is *normalized* if $\Phi_t(0) = 0$, and we call it a *monetary coherent utility functional at time* t (MCohUF for short) if it satisfies

D) *Positive homogeneity:* $\Phi_t(\lambda X) = \lambda\Phi_t(X)$ for all $X \in \mathbf{L}^\infty$ and $\lambda \geq 0$.

If Φ_t is an MCUF (respectively an MCohUF) at each time $t \in [0, T]$, we call the family $\Phi = (\Phi_t(\,.\,))_{0 \leq t \leq T}$ a *dynamic MCUF* (respectively a *dynamic MCohUF*) and use the abbreviation DMCUF (respectively DMCohUF).

An additional property one might require of Φ_t is

E) $\mathcal{F}_t$*-regularity:* $\Phi_t(\mathbf{1}_A X_1 + \mathbf{1}_{A^c} X_2) = \mathbf{1}_A\Phi_t(X_1) + \mathbf{1}_{A^c}\Phi_t(X_2)$ for all X_1, $X_2 \in \mathbf{L}^\infty$ and $A \in \mathcal{F}_t$.

But M. Kupper has pointed out to us that monotonicity and translation invariance already imply E) as follows; see also Proposition 3.3 of [CDK06]. First of all, we have $\mathbf{1}_A\Phi_t(X\mathbf{1}_A) = \mathbf{1}_A\Phi_t(X)$ for $X \in \mathbf{L}^\infty$ and $A \in \mathcal{F}_t$, because A) and B) yield

$$\mathbf{1}_A\Phi_t(X) \underset{\geq}{\overset{\leq}{}} \mathbf{1}_A\Phi_t\big(X\mathbf{1}_A \pm \|X\|_{\mathbf{L}^\infty}\mathbf{1}_{A^c}\big) = \mathbf{1}_A\Phi_t(X\mathbf{1}_A).$$

Applying this to $X = \mathbf{1}_A X_1 + \mathbf{1}_{A^c} X_2$ gives

$$\Phi_t(X) = \mathbf{1}_A\Phi_t(X\mathbf{1}_A) + \mathbf{1}_{A^c}\Phi_t(X\mathbf{1}_{A^c}) = \mathbf{1}_A\Phi_t(X_1) + \mathbf{1}_{A^c}\Phi_t(X_2).$$

Remark 2.3.2. i) An MCUF Φ_t at time t automatically satisfies not only C), but even the stronger property of $\mathcal{F}_t$-concavity, where $\beta \in \mathbf{L}^0(\mathcal{F}_t; [0, 1])$. This can be proved by the standard measure-theoretic induction, using the $\mathcal{F}_t$-regularity and Lipschitz-continuity of Φ_t. So $-\Phi_t$ is almost an $\mathcal{F}_t$-conditional convex risk measure in the sense of [DS05]; the only difference is that in [DS05] Φ_t is normalized. Also by standard measure-theoretic induction, one can show that an MCohUF automatically satisfies instead of D) the stronger property of $\mathcal{F}_t$*-positive homogeneity*, where $\lambda \in \mathbf{L}^\infty_+(\mathcal{F}_t)$.

ii) Since $\mathcal{F}_0$ is trivial, $-\Phi_0$ is simply a convex risk measure in the usual sense; see [FS04] for an comprehensive textbook account. We call $t = 0$ the *static* or *unconditional* case.

iii) In the literature, extensions from static to dynamic risk measures have been considered under two aspects. What we present here corresponds to the study of risk measures conditioned on some information. A second aspect is to define risk measures for payoff streams, i.e., on stochastic processes instead of random variables; see [Wan99], [Det03], [Sca03], [ADEHK04], [CDK04], [PR04], [Rie04], [CDK05], [CDK06], [Del06] or [Web06] for work on that topic. Despite the importance of the latter aspect, we restrict our considerations here to the first one.

iv) We note that MCohUFs are always normalized. Moreover, under the assumption of positive homogeneity, concavity is equivalent to

F) *Superadditivity:* $\Phi_t(X_1 + X_2) \geq \Phi_t(X_1) + \Phi_t(X_2)$ for all X_1, $X_2 \in \mathbf{L}^\infty$. ◇

An MCUF at time $t \leq T$ assigns to each discounted net payoff X due at time T another random variable $\Phi_t(X)$. We interpret $\Phi_t(X)$ as the (individual) utility, expressed in monetary units, that some agent assigns to X at time t. However, this does not imply that it is always possible to swap at time t the future payoff X for $\Phi_t(X)$ monetary units. In fact, this would require the existence of another agent who is willing to pay $\Phi_t(X)$ in exchange for the entitlement to X. Such an agent need not exist in general.

For an economic interpretation of the axioms, we assume that there is a non-risky investment opportunity where the agent can borrow or invest arbitrary amounts of money. Moreover, we assume that all payoffs are already discounted with respect to this non-risky asset. Then the interpretation of $\mathcal{F}_t$-translation invariance is of particular interest because it clarifies the idea behind the definition of $\Phi_t(X)$ and justifies the terminology of a *monetary* utility functional; see also [FS04]. In fact, it implies that $\Phi_t(X - \Phi_t(X)) = 0$. Hence $\Phi_t(X)$ is the maximal monetary amount that can be subtracted from X at time t such that the agent still assigns a non-negative utility to the resulting (discounted) payoff $X - \Phi_t(X)$ due at time T. (To be precise, the agent cannot take the money away from X; she must borrow it from the non-risky investment and pay this debt back at time T, thus changing the discounted payoff due at time T to $X - \Phi_t(X)$.)

We emphasize that translation invariance distinguishes the considered class of utility functionals from von Neumann-Morgenstern expected utility functionals, most of which do not have this property. In contrast, the economic interpretation of the other axioms is more familiar. The meaning of monotonicity is obvious, and concavity models the idea that diversification should not decrease the utility. The condition that $\Phi_t(X)$ is $\mathcal{F}_t$-measurable means that values only depend on information which is available at time t. $\mathcal{F}_t$-regularity implies that an event which can already be ruled out at time t does not influence the value of $\Phi_t(X)$. As utility may grow in a non-linear way with the size of the payoff, we usually do not insist on positive homogeneity.

The issue of normalization is a bit more subtle. It depends on the exact interpretation of the random variable X to which Φ_t is applied whether this assumption makes sense or not. If X expresses a change in wealth, assuming normalization seems reasonable. But if X is some payoff to which we want to apply some utility, normalization might be inappropriate. To see this, suppose the agent has the possibility to trade in some financial market. Then she might obtain with zero initial endowment a position she personally considers to be strictly preferable to the payoff 0. In this situation she might very well assign non-zero utility to 0. Note that this again uses the idea that $\Phi_t(X)$ should be viewed as a subjective *value* rather than a (market) price. Finally, we point out that normalization can always be achieved by subtracting $\Phi_t(0)$ from the original functional. This changes the initial level of utility, but has no influence on the ordering induced by Φ_t. Nevertheless, we will see that subtracting $\Phi_t(0)$ to obtain a normalized functional can yield some difficulties.

Example 2.3.3. a) A classical example of an MCUF at time 0 is the *exponential certainty equivalent* with risk aversion γ, i.e.,

$$\Phi_0(X) := -\frac{1}{\gamma}\log E\left[\exp\left(-\gamma X\right)\right] = \operatorname*{ess\,inf}_{Q\in\mathcal{P}^e}\left\{E_Q[X] + \gamma f^e(Q|P)\right\},$$

where for $Q \in \mathcal{P}^e$ with density Z_T the functional $f^e(Q|P) := E[Z_T \log Z_T]$ denotes the relative entropy of Q with respect to P; see for instance Example 4.105 in [FS04], Section 5 in [DS05], or Examples 3.2 and 3.4 in [BEK04]. This MCUF is not coherent.

b) It is well known and easy to verify that every non-empty set $\mathcal{Q} \subseteq \mathcal{P}^e$ defines an MCohUF by

$$\Phi_t(X) := \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} E_Q[X|\mathcal{F}_t]. \tag{2.3.1}$$

For $\mathcal{Q} = \{Q\}$, this is just the conditional expectation under some $Q \in \mathcal{P}^e$. If $\mathcal{Q}$ is not a singleton, Φ_t can be interpreted to express the preferences of a conservative agent who is uncertain about the underlying model and hence takes into account several possible models. For an extension to the convex case, see Remark 2.3.18 below.

Note that since it is only taken over measures *equivalent* to P, the P-essential infimum in (2.3.1) is well-defined. This need not be the case if $\mathcal{Q}$ were to contain probability measures which are only absolutely continuous with respect to P.

◇

An elementary consequence of the axioms is that every MCUF is Lipschitz-continuous for the $\mathbf{L}^\infty$-norm with Lipschitz coefficient 1. In fact, translation invariance and monotonicity are already sufficient to obtain this property.

Lemma 2.3.4. *For any MCUF Φ_t at time t and any $X, Y \in \mathbf{L}^\infty$ we have*

$$\|\Phi_t(X) - \Phi_t(Y)\|_{\mathbf{L}^\infty} \le \|X - Y\|_{\mathbf{L}^\infty}.$$

Proof. This can be shown exactly as in the static case; see Lemma 4.3 in [FS04]. □

It is well-known from the theory of static risk measures that an MCUF Φ_0 at time 0 can be equivalently described by its acceptance set; see Propositions 4.6 and 4.7 in [FS04]. This also holds true for the conditional case if we use (like in [CDK06]) a conditional form of the $\mathbf{L}^\infty$-norm as follows. For $X \in \mathbf{L}^\infty$ and $t \in [0, T]$, set

$$\|X\|_t := \text{ess inf}\left\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \,\middle|\, |X| \leq m_t\right\}$$

and call $\mathcal{B} \subseteq \mathbf{L}^\infty$ *closed with respect to* $\|\cdot\|_t$ if for any sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ such that $\lim_{n\to\infty} \|X_n - X\|_t = 0$ for some $X \in \mathbf{L}^\infty$, we also have $X \in \mathcal{B}$. This holds for instance if $\mathcal{B}$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$.

Definition 2.3.5. For a given MCUF Φ_t, the *acceptance set* is

$$\mathcal{A}_t := \left\{X \in \mathbf{L}^\infty \,\middle|\, \Phi_t(X) \geq 0\right\},$$

and elements of $\mathcal{A}_t$ are called *acceptable* (with respect to Φ_t, to be precise).

Lemma 2.3.6. *The acceptance set $\mathcal{A}_t$ of an MCUF Φ_t at time t has the following properties:*

a) *$\mathcal{A}_t$ is non-empty and convex;*

b) $\text{ess sup}\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid -m_t \in \mathcal{A}_t\} = \text{ess sup}\,(-\mathcal{A}_t \cap \mathbf{L}^\infty(\mathcal{F}_t)) \in \mathbf{L}^\infty$;

c) *$-\mathcal{A}_t$ is solid, i.e., $X \in \mathcal{A}_t$, $Y \in \mathbf{L}^\infty$ and $Y \geq X$ imply that $Y \in \mathcal{A}_t$;*

d) *$\mathcal{A}_t$ is $\mathcal{F}_t$-regular, i.e., $X, Y \in \mathcal{A}_t$ and $A \in \mathcal{F}_t$ implies that $\mathbf{1}_A X + \mathbf{1}_{A^c} Y \in \mathcal{A}_t$.*

Moreover, $\mathcal{A}_t$ is closed with respect to $\|\,.\,\|_t$. Finally, if Φ_t is an MCohUF, then $\mathcal{A}_t$ is a cone containing 0.

Proof. For the closedness property, see Proposition 3.6 in [CDK06] and Remark 2.3.7 below. The rest follows from the definition as in the static case; see Proposition 4.6 in [FS04]. □

Remark 2.3.7. Some of the results in the present section can be obtained as special cases from [CDK06]. This is not entirely obvious for two reasons. Like [DS05], [CDK06] impose in their axioms for MCUFs normalization and $\mathcal{F}_t$-concavity; see i) of Remark 2.3.2. This difference has no effect for those results we want to quote. More importantly, [CDK06] work more generally with DMCUFs defined on processes instead of random variables and therefore use more elaborate notations than we need here. To help readers in making the connection, we very briefly sketch here the main translations between [CDK06] and our setting.

When using [CDK06] results for random variables, replace $\mathcal{R}^\infty$ and $\mathcal{R}^\infty_{\tau,\theta}$ by $\mathbf{L}^\infty$; replace $\|\cdot\|_{\tau,\theta}$ by $\|\cdot\|_t$; replace $\mathbf{L}^\infty(\mathcal{F}_\tau)$ by $\mathbf{L}^\infty(\mathcal{F}_t)$; and omit all $\mathbf{1}_{[\tau,\infty)}$. Moreover, replace $\mathcal{D}_{\tau,\theta}$ by $\{Z \in \mathbf{L}^1_+(P) \mid E[Z|\mathcal{F}_t] = 1\}$ which corresponds to the set of densities of the elements in $\mathcal{P}^=_t$ defined in (2.3.21) below. Then $\mathcal{D}^{\text{rel}}_{\tau,\theta}$ corresponds to

$$\begin{aligned}
&\{Z \in \mathbf{L}^1_+(P) \mid E[Z|\mathcal{F}_t] = 1, Z > 0\ P\text{-a.s.}\} \\
&= \left\{ \frac{Z^Q_T}{Z^Q_t} \,\middle|\, Z^Q = (Z^Q_t)_{0\le t\le T} \text{ is the density process of some } Q \in \mathcal{P}^e \right\}.
\end{aligned}$$

Finally, $\langle X, a\rangle_{\tau,\theta}$ with $X \in \mathcal{R}^\infty$ and $a \in \mathcal{A}^1$ must be replaced by $E[Xa|\mathcal{F}_t]$ with $X \in \mathbf{L}^\infty$ and $a \in \{Z \in \mathbf{L}^1_+(P) \mid E[Z|\mathcal{F}_t] = 1\}$. ◇

Definition 2.3.8. A subset $\mathcal{B}$ of $\mathbf{L}^\infty$ satisfying the properties a) – d) in Lemma 2.3.6 is called a *pre-acceptance set at time t*.

Lemma 2.3.9. *Let $\mathcal{B} \subseteq \mathbf{L}^\infty$ be a pre-acceptance set at time t and define a mapping on $\mathbf{L}^\infty$ by*

$$\begin{aligned}
\Phi^{\mathcal{B}}_t(X) &:= \operatorname{ess\,sup}\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid X - m_t \in \mathcal{B}\} \\
&= \operatorname{ess\,sup}\big((X - \mathcal{B}) \cap \mathbf{L}^\infty(\mathcal{F}_t)\big). \qquad (2.3.2)
\end{aligned}$$

Then:

a) $\Phi^{\mathcal{B}}_t$ *is an MCUF at time t.*

b) If $\mathcal{B}$ *is in addition closed with respect to* $\|\,.\,\|_t$, *then* $\mathcal{B}$ *is the acceptance set of* $\Phi^{\mathcal{B}}_t$.

c) If $\mathcal{B}$ *is the acceptance set* $\mathcal{A}_t$ *of an MCUF* Φ_t *at time t, then* $\Phi_t = \Phi^{\mathcal{B}}_t$, *i.e., we can recover* Φ_t *from its acceptance set as* $\Phi_t = \Phi^{\mathcal{A}_t}_t$.

d) If $\mathcal{B}$ *is a cone containing 0, then* $\Phi^{\mathcal{B}}_t$ *is an MCohUF.*

Proof. This follows from Proposition 3.10 of [CDK06]; see Remark 2.3.7. □

Our next goal is now to provide a representation for an MCUF Φ_t via its concave conjugate functional, which is defined as follows.

Definition 2.3.10. The *concave conjugate functional* of an MCUF Φ_t at time t is the mapping $\alpha_t : \mathcal{P}^{\approx}_t \mapsto L^0(\mathcal{F}_t; [-\infty, +\infty))$,

$$Q \mapsto \alpha_t(Q) := \operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty}\{E_Q[X|\mathcal{F}_t] - \Phi_t(X)\} \qquad (2.3.3)$$

where $\mathcal{P}^{\approx}_t := \{Q \in \mathcal{P} \mid Q \approx P \text{ on } \mathcal{F}_t\}$.

Remark 2.3.11. $\mathcal{P}_t^{\approx}$ is the largest set on which the essential infimum in (2.3.3) is well-defined in the usual sense; see Section 5 in [Del06] for more general definitions. ◇

Lemma 2.3.12. *The concave conjugate α_t of an MCUF Φ_t at time t with acceptance set $\mathcal{A}_t$ can be written as*

$$\alpha_t(Q) = \operatorname*{ess\,inf}_{X \in \mathcal{A}_t} E_Q[X|\mathcal{F}_t] \quad \text{for } Q \in \mathcal{P}_t^{\approx}, \tag{2.3.4}$$

and it has the following σ-pasting property: If Q^n, $n \in \mathbb{N}$, are in $\mathcal{P}^e$ with density processes Z^n, if $(A_n)_{n \in \mathbb{N}}$ is an $\mathcal{F}_t$-partition of Ω and if $Q \in \mathcal{P}^e$ is defined by $\frac{dQ}{dP} := \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \frac{Z_T^n}{Z_t^n}$, then $\alpha_t(Q) = \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \alpha_t(Q^n)$.

Proof. We start by proving (2.3.4), i.e., by showing that

$$\operatorname*{ess\,inf}_{X' \in \mathbf{L}^\infty} \left\{ E_Q[X'|\mathcal{F}_t] - \Phi_t(X') \right\} = \operatorname*{ess\,inf}_{X' \in \mathcal{A}_t} E_Q[X'|\mathcal{F}_t].$$

As $\mathcal{A}_t \subseteq \mathbf{L}^\infty$ and Φ_t is non-negative on $\mathcal{A}_t$, we clearly have

$$\begin{aligned} \operatorname*{ess\,inf}_{X' \in \mathbf{L}^\infty} \left\{ E_Q[X'|\mathcal{F}_t] - \Phi_t(X') \right\} &\leq \operatorname*{ess\,inf}_{X' \in \mathcal{A}_t} \left\{ E_Q[X'|\mathcal{F}_t] - \Phi_t(X') \right\} \\ &\leq \operatorname*{ess\,inf}_{X' \in \mathcal{A}_t} E_Q[X'|\mathcal{F}_t]. \end{aligned}$$

Conversely, translation invariance implies for $X \in \mathbf{L}^\infty$ that $\hat{X} := X - \Phi_t(X) \in \mathcal{A}_t$ and hence that

$$\operatorname*{ess\,inf}_{X' \in \mathcal{A}_t} E_Q[X'|\mathcal{F}_t] \leq E_Q[\hat{X}|\mathcal{F}_t] = E_Q[X|\mathcal{F}_t] - \Phi_t(X).$$

Taking the essential infimum over all $X \in \mathbf{L}^\infty$ we obtain

$$\operatorname*{ess\,inf}_{X' \in \mathcal{A}_t} E_Q[X'|\mathcal{F}_t] \leq \operatorname*{ess\,inf}_{X' \in \mathbf{L}^\infty} \left\{ E_Q[X'|\mathcal{F}_t] - \Phi_t(X') \right\}.$$

For the second claim note that $E\left[\sum_{n=1}^{\infty} \mathbf{1}_{A_n} \frac{Z_T^n}{Z_t^n}\right] = E\left[\sum_{n=1}^{\infty} \mathbf{1}_{A_n} \frac{1}{Z_t^n} E[Z_T^n|\mathcal{F}_t]\right] = 1.$

Since $\mathcal{A}_t \subseteq \mathbf{L}^\infty$, (2.3.4) and the dominated convergence theorem imply that

$$\begin{aligned}
\alpha_t(Q) &= \operatorname*{ess\,inf}_{X' \in \mathcal{A}_t} E\left[\sum_{n=1}^{\infty} \mathbf{1}_{A_n} \frac{Z_T^n}{Z_t^n} X' \,\middle|\, \mathcal{F}_t\right] \\
&= \operatorname*{ess\,inf}_{X' \in \mathcal{A}_t} \sum_{n=1}^{\infty} \mathbf{1}_{A_n} E_{Q^n}[X' | \mathcal{F}_t] \\
&= \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \operatorname*{ess\,inf}_{X' \in \mathcal{A}_t} E_{Q^n}[X' | \mathcal{F}_t] \\
&= \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \alpha_t(Q^n)
\end{aligned}$$

This finishes the proof. □

In Lemma 2.3.9 b), we proved that if $\mathcal{B}$ is a pre-acceptance set at time t which is closed with respect to $\|\,.\,\|_t$ and thus in particular if it is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$, it is the acceptance set of $\Phi_t^{\mathcal{B}}$. We shall see that $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closedness of the acceptance set is equivalent to a continuity property of the corresponding MCUF. As in the static case, this continuity property will be required to obtain a structural characterization of MCUFs.

Definition 2.3.13. An MCUF Φ_t at time t is called *continuous from above (below)* if $\lim_{n\to\infty} \Phi_t(X_n) = \Phi_t(X)$ for any sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathbf{L}^\infty$ decreasing (increasing) to some $X \in \mathbf{L}^\infty$. (Note that monotonicity of Φ_t implies the almost sure existence of the limit.)

Like in the static case, continuity from below is stronger than continuity from above:

Lemma 2.3.14. *If Φ_t is an MCUF at time t and continuous from below, then it is also continuous from above.*

Proof. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{L}^\infty$ decreasing to some $X \in \mathbf{L}^\infty$ and for $n \in \mathbb{N}$ define $Z_n := X_n - X$. With $\overline{X} := X - \Phi_t(X)$, we obtain from B) and C) that

$$\begin{aligned}
0 &= \Phi_t(\overline{X}) \\
&= \Phi_t\Big(\frac{1}{2}(\overline{X} + Z_n) + \frac{1}{2}(\overline{X} - Z_n)\Big) \\
&\geq \frac{1}{2}\Phi_t(\overline{X} + Z_n) + \frac{1}{2}\Phi_t(\overline{X} - Z_n) \\
&= \frac{1}{2}\Big(\Phi_t(X + Z_n) - \Phi_t(X) + \Phi_t(X - Z_n) - \Phi_t(X)\Big)
\end{aligned}$$

so that

$$\Phi_t(X + Z_n) - \Phi_t(X) \leq \Phi_t(X) - \Phi_t(X - Z_n). \tag{2.3.5}$$

From this together with A), continuity from below and since $X - Z_n = 2X - X_n \nearrow X$, we obtain

$$0 \leq \Phi_t(X_n) - \Phi_t(X) \leq \Phi_t(X) - \Phi_t(X - Z_n) \searrow \Phi_t(X) - \Phi_t(X) = 0.$$

Hence $\Phi_t(X_n)$ decreases to $\Phi_t(X)$ as $n \to \infty$. □

Remark 2.3.15. The MCUFs in Example 2.3.3 a) and b) are always continuous from above. The exponential certainty equivalent is also continuous from below, but for the MCohUFs in part b) this depends on the choice of $\mathcal{Q}$; see Corollary 4.35 in [FS04]. ◇

The following Theorem 2.3.16 is the main result of this section. It shows that for an MCUF Φ_t, the existence of a representation via the concave conjugate functional, continuity from above, and $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closedness of its acceptance set $\mathcal{A}_t$ are all equivalent. A detailed discussion is given below.

Theorem 2.3.16. *For an MCUF Φ_t at time t with acceptance set $\mathcal{A}_t$, the following are equivalent:*

I) Φ_t *is continuous from above and* $\inf_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty$ *for some* $\tilde{Q} \in \mathcal{P}^e$.

II) Φ_t *can be represented as*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{P}^e} \left\{ E_Q[X|\mathcal{F}_t] - \alpha_t^0(Q) \right\} \tag{2.3.6}$$

for a mapping $\alpha_t^0 : \mathcal{P}^e \to \mathbf{L}^0(\mathcal{F}_t; [-\infty, +\infty))$ *which has the* σ*-pasting property.*

III) Φ_t *can be represented as*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{P}^e} \left\{ E_Q[X|\mathcal{F}_t] - \alpha_t(Q) \right\} \tag{2.3.7}$$

where α_t *is the concave conjugate of* Φ_t.

IV) $\mathcal{A}_t$ *is closed in* $\sigma\left(\mathbf{L}^\infty, \mathbf{L}^1\right)$ *and* $\inf_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty$ *for some* $\tilde{Q} \in \mathcal{P}^e$.

If Φ_t *satisfies one of the above properties and is in addition positively homogeneous, hence an MCohUF, it can be represented as*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{Q}^e} E_Q[X|\mathcal{F}_t] \tag{2.3.8}$$

for some set $\mathcal{Q} \subseteq \mathcal{P}^a$ *and with* $\mathcal{Q}^e = \mathcal{Q} \cap \mathcal{P}^e \neq \emptyset$. $\mathcal{Q}$ *can be chosen convex and closed in* $\mathbf{L}^1$.

Definition 2.3.17. If one of the equivalent properties I) – IV) is satisfied, we say that Φ_t is *well-representable*.

Remark 2.3.18. i) In analogy to Example 2.3.3 b), it is easy to see that any functional $\Phi_t : \mathbf{L}^\infty \to \mathbf{L}^\infty(\mathcal{F}_t)$ which can be represented as in (2.3.6) is an MCUF at time t. This does not require the σ-pasting property of α_t^0.

ii) In II) it suffices to have the σ-pasting property only for those $Q \in \mathcal{P}^e$ satisfying $\alpha_t^0(Q) \not\equiv -\infty$.

iii) Note that Theorem 2.3.16 also allows us to *define* an MCUF at time t from a suitable mapping α_t^0 by (2.3.6). This is particularly useful in the coherent case where Φ_t is specified via (2.3.8) entirely by the set $\mathcal{Q}$; see Example 2.3.3 b). A similar interpretation holds in the convex case, where $\alpha_t^0(Q)$ is a correction term which quantifies how the model Q is viewed. In Example 2.3.3 a), P can be seen as a reference model and the correction term is chosen proportional to the (entropic) deviation of Q from P; see also Section 4.3 in [FS04].

◇

Proof of Theorem 2.3.16. "III) $\Rightarrow$ II):" Obvious due to Lemma 2.3.12.

"II) $\Rightarrow$ I):" To see continuity from above, let $(X_n)_{n\in\mathbb{N}} \subseteq \mathbf{L}^\infty$ be a uniformly bounded sequence decreasing to some $X \in \mathbf{L}^\infty$. Then

$$\begin{aligned}
\searrow\text{-}\lim_{n\to\infty} \Phi_t(X_n) &= \inf_{n\in\mathbb{N}} \left\{ \operatorname*{ess\,inf}_{Q\in\mathcal{P}^e} \left\{ E_Q[X_n|\mathcal{F}_t] - \alpha_t^0(Q) \right\} \right\} \\
&= \operatorname*{ess\,inf}_{Q\in\mathcal{P}^e} \left\{ \inf_{n\in\mathbb{N}} \left\{ E_Q[X_n|\mathcal{F}_t] - \alpha_t^0(Q) \right\} \right\} \\
&= \operatorname*{ess\,inf}_{Q\in\mathcal{P}^e} \left\{ \searrow\text{-}\lim_{n\to\infty} E_Q[X_n|\mathcal{F}_t] - \alpha_t^0 Q) \right\} \\
&= \Phi_t(X),
\end{aligned}$$

where the last equality follows from the monotone convergence theorem and (2.3.6). It remains to prove the existence of $\tilde{Q}$ as desired. To this behalf choose a sequence (Q^n) in $\mathcal{P}^e$ and for $\varepsilon > 0$ an $\mathcal{F}_t$-partition (A_n) of Ω such that

$$-\Phi_t(0) = \operatorname*{ess\,sup}_{Q\in\mathcal{P}^e} \alpha_t^0(Q) = \sup_{n\in\mathbb{N}} \alpha_t^0(Q^n) \le \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \alpha_t^0(Q^n) + \varepsilon.$$

Define $\tilde{Q} \in \mathcal{P}^e$ by $\frac{d\tilde{Q}}{dP} := \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \frac{Z_T^n}{Z_t^n}$ and note that the σ-pasting property of α_t^0 gives $\alpha_t^0(\tilde{Q}) + \varepsilon \ge -\Phi_t(0) \in \mathbf{L}^\infty$. Using (2.3.4) and (2.3.6) yields

$$\begin{aligned}
\operatorname*{ess\,inf}_{X\in\mathcal{A}_t} E_{\tilde{Q}}[X|\mathcal{F}_t] &= \operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty} \left\{ E_{\tilde{Q}}[X|\mathcal{F}_t] - \operatorname*{ess\,inf}_{Q\in\mathcal{P}^e} \left\{ E_Q[X|\mathcal{F}_t] - \alpha_t^0(Q) \right\} \right\} \\
&\ge \alpha_t^0(\tilde{Q})
\end{aligned}$$

and therefore $\inf_{X\in\mathcal{A}_t} E_{\tilde{Q}}[X] \geq E_{\tilde{Q}}[\alpha_t^0(\tilde{Q})] \geq -E_{\tilde{Q}}[\Phi_t(0)] - \varepsilon > -\infty$.

"I) $\Rightarrow$ III):" First we show that "$\leq$" holds in (2.3.7), i.e., that

$$\Phi_t(X) \leq \operatorname*{ess\,inf}_{Q\in\mathcal{P}^e} \{E_Q[X|\mathcal{F}_t] - \alpha_t(Q)\} \tag{2.3.9}$$

for all $X \in \mathbf{L}^\infty$. Indeed, for any $Q \in \mathcal{P}^e$ and any $X \in \mathbf{L}^\infty$, (2.3.3) gives

$$\begin{aligned} E_Q[X|\mathcal{F}_t] - \alpha_t(Q) &= E_Q[X|\mathcal{F}_t] - \operatorname*{ess\,inf}_{X'\in\mathbf{L}^\infty} \{E_Q[X'|\mathcal{F}_t] - \Phi_t(X')\} \\ &\geq E_Q[X|\mathcal{F}_t] - \big(E_Q[X|\mathcal{F}_t] - \Phi_t(X)\big) \\ &= \Phi_t(X). \end{aligned}$$

(2.3.9) follows if we take the essential infimum over all $Q \in \mathcal{P}^e$. Inequality (2.3.9) implies (2.3.7) if we show that for any $X \in \mathbf{L}^\infty$

$$E_{\tilde{Q}}[\Phi_t(X)] = E_{\tilde{Q}}\big[\operatorname*{ess\,inf}_{Q\in\mathcal{P}^e} \{E_Q[X|\mathcal{F}_t] - \alpha_t(Q)\}\big]. \tag{2.3.10}$$

Similarly to, e.g., [Det03], this will be done by exploiting the well-known representation results for the static case. To derive from Φ_t an MCUF at time 0, we define the mapping $\tilde{\Phi}_0 : \mathbf{L}^\infty \to I\!R$ by $\tilde{\Phi}_0(X) := E_{\tilde{Q}}[\Phi_t(X)]$. This is an MCUF at time 0, and continuous from above because Φ_t is. Hence Theorem 4.31 and Remark 4.16 of [FS04] imply that it can be represented as

$$\tilde{\Phi}_0(X) = \inf_{Q\in\mathcal{P}^a} \{E_Q[X] - \tilde{\alpha}_0(Q)\}, \tag{2.3.11}$$

where

$$\tilde{\alpha}_0(Q) = \inf_{Y\in\mathbf{L}^\infty} \Big\{E_Q[Y] - \tilde{\Phi}_0(Y)\Big\}. \tag{2.3.12}$$

We argue below that $\tilde{\alpha}_0(\tilde{Q}) > -\infty$, and because $\tilde{Q} \in \mathcal{P}^e$, this implies that we have

$$\tilde{\Phi}_0(X) = \inf_{Q\in\mathcal{P}^e} \{E_Q[X] - \tilde{\alpha}_0(Q)\}. \tag{2.3.13}$$

Similarly to [DS05], we show next that (2.3.13) remains true if we take the infimum only over all Q in

$$\tilde{\mathcal{Q}}_t := \Big\{Q \in \mathcal{P}^e \;\Big|\; Q[A] = \tilde{Q}[A] \text{ for all } A \in \mathcal{F}_t\Big\},$$

i.e., we claim that

$$\inf_{Q\in\mathcal{P}^e} \{E_Q[X] - \tilde{\alpha}_0(Q)\} = \inf_{Q\in\tilde{\mathcal{Q}}_t} \{E_Q[X] - \tilde{\alpha}_0(Q)\}. \tag{2.3.14}$$

It is clear that "$\leq$" holds, and "$\geq$" will follow once we show that

$$\tilde{\alpha}_0(Q) = -\infty \quad \text{for any } Q \in \big(\mathcal{P}^e \setminus \tilde{\mathcal{Q}}_t\big). \tag{2.3.15}$$

But if $Q \notin \tilde{\mathcal{Q}}_t$, there exists $A \in \mathcal{F}_t$ such that $Q[A] \neq \tilde{Q}[A]$. As $\mathcal{F}_t$-translation invariance of Φ_t implies that

$$\tilde{\Phi}_0(\lambda \mathbf{1}_A) = E_{\tilde{Q}}[\Phi_t(\lambda \mathbf{1}_A + 0)] = E_{\tilde{Q}}[\lambda \mathbf{1}_A] + E_{\tilde{Q}}[\Phi_t(0)],$$

we obtain from (2.3.12)

$$\begin{aligned}\tilde{\alpha}_0(Q) &\leq \inf_{\lambda \in \mathbb{R}} \left\{ E_Q[\lambda \mathbf{1}_A] - \tilde{\Phi}_0(\lambda \mathbf{1}_A) \right\} \\ &= \inf_{\lambda \in \mathbb{R}} \left\{ \lambda Q[A] - \lambda \tilde{Q}[A] - E_{\tilde{Q}}[\Phi_t(0)] \right\} = -\infty.\end{aligned}$$

Hence (2.3.14) follows. Now we show that

$$E_{\tilde{Q}}[\alpha_t(Q)] = \tilde{\alpha}_0(Q) \quad \text{for all } Q \in \tilde{\mathcal{Q}}_t. \tag{2.3.16}$$

In fact, $\mathcal{F}_t$-regularity of Φ_t implies that the set $\left\{ E_Q[X|\mathcal{F}_t] - \Phi_t(X) \;\middle|\; X \in \mathbf{L}^\infty \right\}$ is a lattice. Hence ([Nev75]) there exists a sequence $(X_n)_{n \in \mathbb{N}} \subseteq \mathbf{L}^\infty$ such that

$$\operatorname*{ess\,inf}_{X \in \mathbf{L}^\infty} \left\{ E_Q[X|\mathcal{F}_t] - \Phi_t(X) \right\} = \searrow\text{-}\lim_{n \to \infty} \left(E_Q[X_n|\mathcal{F}_t] - \Phi_t(X_n) \right) \tag{2.3.17}$$

so that by the monotone convergence theorem

$$\begin{aligned} & E_{\tilde{Q}}\left[\operatorname*{ess\,inf}_{X \in \mathbf{L}^\infty} \left\{ E_Q[X|\mathcal{F}_t] - \Phi_t(X) \right\} \right] \\ &\qquad = \searrow\text{-}\lim_{n \to \infty} E_{\tilde{Q}}\left[E_Q[X_n|\mathcal{F}_t] - \Phi_t(X_n) \right] \\ &\qquad \geq \inf_{X \in \mathbf{L}^\infty} E_{\tilde{Q}}\left[E_Q[X|\mathcal{F}_t] - \Phi_t(X) \right]; \end{aligned}$$

clearly we then even have "=" in the last line. This together with (2.3.3), $Q \in \tilde{\mathcal{Q}}_t$ and (2.3.12) yields

$$\begin{aligned} E_{\tilde{Q}}[\alpha_t(Q)] &= \inf_{X \in \mathbf{L}^\infty} \left\{ E_Q[X] - E_{\tilde{Q}}[\Phi_t(X)] \right\} \\ &= \inf_{X \in \mathbf{L}^\infty} \left\{ E_Q[X] - \tilde{\Phi}_0(X) \right\} \\ &= \tilde{\alpha}_0(Q) \end{aligned}$$

and hence (2.3.16). Combining this with (2.3.9), (2.3.16), (2.3.13) and (2.3.14) we can finish the proof of (2.3.10) as follows:

$$\begin{aligned}
E_{\tilde{Q}}\left[\Phi_t(X)\right] &\leq E_{\tilde{Q}}\left[\operatorname*{ess\,inf}_{Q\in\mathcal{P}^e}\left\{E_Q[X|\mathcal{F}_t]-\alpha_t(Q)\right\}\right]\\
&\leq E_{\tilde{Q}}\left[\operatorname*{ess\,inf}_{Q\in\tilde{\mathcal{Q}}_t}\left\{E_Q[X|\mathcal{F}_t]-\alpha_t(Q)\right\}\right]\\
&\leq \inf_{Q\in\tilde{\mathcal{Q}}_t} E_{\tilde{Q}}\left[\left\{E_Q[X|\mathcal{F}_t]-\alpha_t(Q)\right\}\right]\\
&= \inf_{Q\in\tilde{\mathcal{Q}}_t}\left\{E_Q[X]-E_{\tilde{Q}}\left[\alpha_t(Q)\right]\right\}\\
&= \inf_{Q\in\tilde{\mathcal{Q}}_t}\left\{E_Q[X]-\tilde{\alpha}_0(Q)\right\}\\
&= \tilde{\Phi}_0(X)\\
&= E_{\tilde{Q}}\left[\Phi_t(X)\right].
\end{aligned}$$

Finally, to see that $\tilde{\alpha}_0(\tilde{Q}) > -\infty$, note that $Y - \Phi_t(Y) \in \mathcal{A}_t$ for any $Y \in \mathbf{L}^\infty$. Hence (2.3.12) gives

$$\tilde{\alpha}_0(\tilde{Q}) = \inf_{Y\in\mathbf{L}^\infty} E_{\tilde{Q}}[Y - \Phi_t(Y)] \geq \inf_{X\in\mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty.$$

"I) $\Rightarrow$ IV):" Closedness of the acceptance set can be shown as in the static case, see [FS04], Theorem 4.31, c) $\Rightarrow$ e) $\Rightarrow$ f) together with Lemma 4.20.

"IV) $\Rightarrow$ I):" To see continuity from above, let $(X_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $\mathbf{L}^\infty$ decreasing to some $X \in \mathbf{L}^\infty$ so that

$$\searrow\text{-}\lim_{n\to\infty} \Phi_t(X_n) = Z \tag{2.3.18}$$

for some $Z \in \mathbf{L}^\infty(\mathcal{F}_t)$. Then $Y_n := X_n - \Phi_t(X_n)$ converges to $X - Z$ P-a.s. and is uniformly bounded as well. By dominated convergence, $(Y_n)_{n\in\mathbb{N}}$ thus also converges to $X - Z$ in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$. But by translation invariance, $Y_n \in \mathcal{A}_t$ for all n and $\mathcal{A}_t$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ so that $X - Z$ is in $\mathcal{A}_t$ as well. From this together with translation invariance and since $Z \in \mathbf{L}^\infty(\mathcal{F}_t)$, we obtain that $\Phi_t(X) \geq Z$. Hence monotonicity implies by (2.3.18)

$$\lim_{n\to\infty} \Phi_t(X_n) = Z \leq \Phi_t(X) = \Phi_t\big(\lim_{n\to\infty} X_n\big) \leq \lim_{n\to\infty} \Phi_t(X_n).$$

To finish the proof of Theorem 2.3.16, it remains to show that if Φ_t is positively homogeneous, there exists a set $\mathcal{Q} \subseteq \mathcal{P}^e$ such that

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} E_Q[X|\mathcal{F}_t].$$

By positive homogeneity, the acceptance set $\mathcal{A}_t$ is closed under multiplication with non-negative scalars and in particular $0 \in \mathcal{A}_t$. Therefore $\alpha_t(Q)$ from (2.3.4) is $\{0, -\infty\}$-valued for each $Q \in \mathcal{P}^e$. Next we show that there exists $\hat{Q} \in \mathcal{P}^e$ such that $\alpha_t(\hat{Q}) = 0$. In fact, as any MCohUF is normalized, we obtain from III) that

$$0 = \Phi_t(0) = \operatorname*{ess\,inf}_{Q \in \mathcal{P}^e}\{-\alpha_t(Q)\} = \inf_{n \in \mathbb{N}}\{-\alpha_t(Q_n)\}$$

for some sequence $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}^e$ (see [Nev75]). Hence there exists an $\mathcal{F}_t$-partition $(A_n)_{n\in\mathbb{N}}$ and a sequence $(Q^n)_{n\in\mathbb{N}}$, $Q^n \in \mathcal{P}^e$ with density processes Z^n, such that

$$\sum_{n=1}^{\infty} \mathbf{1}_{A_n}\, \alpha_t(Q^n) = 0;$$

this uses that each $\alpha_t(Q^n)$ only takes the values 0 and $-\infty$. We define the measure $\hat{Q} \in \mathcal{P}^e$ via its density

$$\hat{Z}_T := \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \frac{Z^n_T}{Z^n_t}.$$

Lemma 2.3.12 then implies that

$$\alpha_t(\hat{Q}) = \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \alpha_t(Q^n) = 0. \tag{2.3.19}$$

Now fix $Q' \in \mathcal{P}^e$ and let

$$A := \{\alpha_t\left(Q'\right) = 0\} \in \mathcal{F}_t$$

(where, as usual, we consider a fixed version of $\alpha_t\left(Q'\right)$). If $\hat{Z}$ and Z' denote the density processes of $\hat{Q}$ and Q', we define a new measure $\tilde{Q} \in \mathcal{P}^e$ via its density $\tilde{Z}_T$ as

$$\tilde{Z}_T := \mathbf{1}_A \frac{Z'_T}{Z'_t} + \mathbf{1}_{A^c} \frac{\hat{Z}_T}{\hat{Z}_t}.$$

Then Lemma 2.3.12 implies that

$$\alpha_t(\tilde{Q}) = \mathbf{1}_A\, \alpha_t\left(Q'\right) + \mathbf{1}_{A^c}\, \alpha_t(\hat{Q}) = 0.$$

Because $\mathbf{1}_A E_{Q'}[\,.\,|\mathcal{F}_t] = \mathbf{1}_A E_{\tilde{Q}}[\,.\,|\mathcal{F}_t]$ and $\alpha_t(Q') = -\infty$ on A^c, we obtain

$$E_{Q'}[X|\mathcal{F}_t] - \alpha_t(Q') \geq E_{\tilde{Q}}[X|\mathcal{F}_t] - \alpha_t(\tilde{Q})$$

by looking separately at A and A^c. In other words, when taking the essential infimum in (2.3.7) it is enough to restrict attention to measures like $\tilde{Q}$ that have $\alpha_t(\tilde{Q}) = 0$. So if we define

$$\mathcal{Q} := \left\{ Q \in \mathcal{P}^e \;\middle|\; \alpha_t(Q) \equiv 0 \right\},$$

we obtain

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} E_Q[X|\mathcal{F}_t] = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^e} E_Q[X|\mathcal{F}_t].$$

In order to have $\mathcal{Q}$ convex and closed in $\mathbf{L}^1$, we can replace $\mathcal{Q}$ by its $\mathbf{L}^1$-closed convex hull and recall that for convex sets the norm closure and the weak closure are the same. □

The papers [DS05] and [CDK06] contain closely related representation results; the relations and differences will be discussed below after we have introduced some additional concepts. Another representation for conditional convex risk measures can be found in Rosazza Gianin [RG06] in the context of BSDEs. In the coherent case, things become simpler; see for instance [Rie04], [ADEHK04] or [RSE05]. The recent work of Weber [Web06] is less relevant for our goals, because law-invariance does not fit well with the notion of hedging.

For comparison purposes, let us first give a slight variation of Theorem 2.3.16; without IV′), this is simply Theorem 1 of [DS05] in our notation.

Theorem 2.3.19. *For an MCUF Φ_t at time t with acceptance set $\mathcal{A}_t$, the following are equivalent:*

I′) Φ_t is continuous from above.

II′) Φ_t can be represented as

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{P}_t^=} \left\{ E_Q[X|\mathcal{F}_t] - \alpha_t^0(Q) \right\} \tag{2.3.20}$$

for a mapping $\alpha_t^0 : \mathcal{P}_t^= \to \mathbf{L}^0\big(\mathcal{F}_t; [-\infty, +\infty)\big)$ and where

$$\mathcal{P}_t^= := \{Q \ll P \mid Q = P \text{ on } \mathcal{F}_t\}. \tag{2.3.21}$$

III′) Φ_t can be represented as

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{P}_t^=} \left\{ E_Q[X|\mathcal{F}_t] - \alpha_t(Q) \right\} \tag{2.3.22}$$

where α_t is the concave conjugate of Φ_t.

IV′) $\mathcal{A}_t$ is closed in $\sigma\big(\mathbf{L}^\infty, \mathbf{L}^1\big)$.

Definition 2.3.20. If one of the equivalent properties I′) – IV′) is satisfied, we say that Φ_t is *representable.*

One difference to other related representation results is our use of the condition

$$\inf_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty \quad \text{for some } \tilde{Q} \in \mathcal{P}^e. \tag{2.3.23}$$

Before discussing this difference in more detail, let us first show how (2.3.23) can be ensured from a relevance condition on Φ_t; see Definition 4.32 and Corollaries 4.34 and 9.30 in [FS04] for this economically very natural concept.

Definition 2.3.21. If an MCUF Φ_t at time t satisfies $P\big[\Phi_t(-\mathbf{1}_B) < \Phi_t(0)\big] > 0$ for any $B \in \mathcal{F}$ with $P[B] > 0$, it is called *relevant* or *sensitive*.

Proposition 2.3.22. *Let Φ_t be an MCUF at time t.*

a) If Φ_t is continuous from above and relevant, then (2.3.23) holds. In particular, Φ_t is well-representable.

b) If Φ_t is well-representable and an MCohUF at time t, then Φ_t is relevant.

Proof. b) (2.3.8) gives $\Phi_t(-\mathbf{1}_B) \le -E_Q[\mathbf{1}_B|\mathcal{F}_t]$ for some $Q \in \mathcal{P}^e$, and $\Phi_t(0) = 0$. Hence Φ_t is relevant.

a) Almost like in the proof of Theorem 2.3.16, " I) $\Longrightarrow$ III)", we define and represent an MCUF $\overline{\Phi}_0$ at time 0 by

$$\overline{\Phi}_0(X) := E[\Phi_t(X)] = \inf_{Q \in \mathcal{P}^a} \big\{E_Q[X] - \overline{\alpha}_0(Q)\big\} = \inf_{\substack{Q \in \mathcal{P}^a, \\ \overline{\alpha}_0(Q) > -\infty}} \big\{E_Q[X] - \overline{\alpha}_0(Q)\big\} \tag{2.3.24}$$

with

$$\overline{\alpha}_0(Q) = \inf_{Y \in \mathbf{L}^\infty} \big\{E_Q[Y] - \overline{\Phi}_0(Y)\big\};$$

the last equality in (2.3.24) holds since $\overline{\Phi}_0$ is finite-valued. Because Φ_t is relevant, so is $\overline{\Phi}_0$. To construct $\tilde{Q} \in \mathcal{P}^e$ with

$$\overline{\alpha}_0(\tilde{Q}) > -\infty, \tag{2.3.25}$$

we define $B \in \mathcal{F}$ up to nullsets by

$$\mathbf{1}_B := \operatorname{ess\,sup} \Big\{\mathbf{1}_{\{Z_T^Q > 0\}} \,\Big|\, Q \in \mathcal{P}^a \text{ and } \overline{\alpha}_0(Q) > -\infty\Big\}.$$

By the definition of B, for $Q \in \mathcal{P}^a$ with $\overline{\alpha}_0(Q) > -\infty$, we must have $E_Q[\mathbf{1}_{B^c}] = 0$, so that by (2.3.24) we have $\overline{\Phi}_0(-\mathbf{1}_{B^c}) = \overline{\Phi}_0(0)$. Hence $P[B] = 1$ by relevance of $\overline{\Phi}_0$. Now choose $Q^n \in \mathcal{P}^a$ with density processes Z^n and $\overline{\alpha}_0(Q^n) > -\infty$ such that $\sup_{n \in \mathbb{N}} \mathbf{1}_{\{Z_T^n > 0\}} = \mathbf{1}_B = 1$ P-a.s., and $\beta_n > 0$ with $\sum_{n=1}^\infty \beta_n = 1$ and $\sum_{n=1}^\infty \beta_n \overline{\alpha}_0(Q^n) > -\infty$. Then $\frac{d\tilde{Q}}{dP} := \sum_{n=1}^\infty \beta_n Z_T^n$ defines a measure $\tilde{Q} \in \mathcal{P}^e$ which satisfies (2.3.25). With the same arguments as for (2.3.15) and (2.3.16) one can first prove that $\overline{\alpha}_0(Q) = -\infty$ for any $Q \in \left(\mathcal{P}^e \setminus \mathcal{P}_t^=\right)$ which implies that $\tilde{Q} \in \mathcal{P}_t^=$ and then conclude that $\overline{\alpha}_0(\tilde{Q}) = E_P[\alpha_t(\tilde{Q})]$ so that (2.3.4) yields

$$\inf_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X] \geq E_{\tilde{Q}}\Big[\operatorname*{ess\,inf}_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X|\mathcal{F}_t]\Big] = E_P[\alpha_t(\tilde{Q})] = \overline{\alpha}_0(\tilde{Q}) > -\infty.$$

Hence $\tilde{Q}$ does the job. □

Now we can discuss the differences between Theorem 2.3.16, Theorem 2.3.19 (which corresponds to Theorem 1 in [DS05]) and Theorems 3.16, 3.18 and 3.23 in [CDK06]. Obvious differences are changes of signs in [DS05] and that [CDK06] work with MCUFs on processes instead of only random variables like here. In Remark 2.3.7, we have briefly sketched how their notation can be translated to our setting. But the main difference is that [DS05] and [CDK06] assume in I) only that Φ_t is continuous from above. They then obtain representations like in (2.3.22), where the set of measures is $\mathcal{P}_t^=$ which explicitly depends on $\mathcal{F}_t$. By imposing the additional condition (2.3.23) on Φ_t, we have in contrast a representation with one set $\mathcal{P}^e$ for all t and, more importantly, a representation in terms of measures which are equivalent to P. The term "well-representable" is meant to highlight this difference.

To be accurate, things are even more subtle. In their Theorem 3.23, [CDK06] also provide a representation like (2.3.7) in terms of $\mathcal{P}^e$. However, they assume for this that Φ_t is relevant, which by Lemma 2.3.22 is sufficient (but not necessary) for (2.3.23). In contrast, we show that the weaker condition (2.3.23) is already sufficient for the representation in (2.3.7), and that (together with continuity from above) it is actually also necessary.

None of the properties imposed on DMCUFs so far requires any relation between the MCUFs at different points in time. To actually study the *dynamic* behavior of DMCUFs, we now introduce a notion of time-consistency.

Definition 2.3.23. A DMCUF $\Phi := (\Phi_t)_{0 \leq t \leq T}$ is called *time-consistent* if for any $X, Y \in \mathbf{L}^\infty$ and $s \leq t$,

$$\Phi_t(X) = \Phi_t(Y) \qquad \text{implies that} \qquad \Phi_s(X) = \Phi_s(Y). \tag{2.3.26}$$

Φ is called *strongly time-consistent* if in addition its acceptance sets $(\mathcal{A}_t)_{0 \leq t \leq T}$ satisfy

$$\mathcal{A}_t \subseteq \mathcal{A}_s \quad \text{for } t \geq s.$$

In the literature, one can find several differing definitions of time-consistency; see for instance [Pen04], [Web06], or [ADEHK04] for an overview. For our purposes, (2.3.26) means that indifference at time t between two payoffs X and Y is carried over to any time $s < t$, i.e., when less information is available. Because the "=" signs could obviously be replaced by "≥" signs in (2.3.26), time-consistency preserves the ordering between payoffs over time, but does not fix the level at which this occurs. Unless all Φ_t are normalized, (2.3.26) therefore does not guarantee that an X acceptable in t is also acceptable at time $s < t$; this requires strong time-consistency. We do not impose normalization here since we later consider operations on DMCUFs which preserve (strong) time-consistency, but may change the initial utility level; see the remark after Theorem 2.4.3 and Example 2.7.1.

Remark 2.3.24. i) In Section 2.7.2 we investigate DMCUFs which are defined via solutions of backward stochastic differential equations. As they are always time-consistent, these provide us with a big class of examples for time-consistent DMCUFs.

ii) Epstein and Schneider's Example 4.1 in [ES03] illustrates that under ambiguity aversion, a rational agent might well exhibit a time-inconsistent behavior. Like for all axioms concerning decision making, it is thus important to be aware of situations where seemingly natural rules are violated.

◇

For a DMCUF $(\Phi_t)_{0\leq t\leq T}$ with acceptance sets $(\mathcal{A}_t)_{0\leq t\leq T}$ and for $s \leq t$, we use the notation $\mathcal{A}_s(\mathcal{F}_t) := \mathcal{A}_s \cap \mathbf{L}^\infty(\mathcal{F}_t)$. We note that $\Phi_{s\circ t} := \Phi_s \circ \Phi_t$ is an MCUF at time s and denote by $\mathcal{A}_{s\circ t}$ its acceptance set. Similarly as in Theorems 12 and 16 in [Del06], time-consistency can then be characterized as follows; see also Proposition 8 of [DS05].

Lemma 2.3.25. *For a DMCUF* $\Phi = (\Phi_t)_{0\leq t\leq T}$, *the properties*

a) $\Phi_s = \Phi_{s\circ t}$ *for all* $s \leq t$,

b) $\mathcal{A}_s = \mathcal{A}_{s\circ t}$ *for all* $s \leq t$,

c) $\mathcal{A}_s = \mathcal{A}_s(\mathcal{F}_t) + \mathcal{A}_t$ *for all* $s \leq t$,

are all equivalent and imply

d) Φ *is time-consistent.*

If Φ *is normalized, i.e.,* $\Phi_t(0) \equiv 0$ *for all* $t \in [0, T]$, *then d) is equivalent to a) – c).*

Proof. a) implies d) and by c) of Lemma 2.3.9 is equivalent to b). If $\Phi_t(0) \equiv 0$, take $X \in \mathbf{L}^\infty$ and define $Y := \Phi_t(X)$ to get by translation invariance $\Phi_t(Y) = \Phi_t\big(0 + \Phi_t(X)\big) = \Phi_t(X)$. Time-consistency then yields $\Phi_s(X) = \Phi_s(Y) = \Phi_{s\circ t}(X)$ so that d) implies a).

"b) $\Rightarrow$ c)": To show the inclusion "$\supseteq$" in c), let $X = X_1 + X_2$ with $X_1 \in \mathcal{A}_s(\mathcal{F}_t)$, $X_2 \in \mathcal{A}_t$ and use translation invariance and that $X_2 \in \mathcal{A}_t$ to get $\Phi_t(X) = X_1 + \Phi_t(X_2) \geq X_1$. Monotonicity and $X_1 \in \mathcal{A}_s(\mathcal{F}_t)$ thus yield $\Phi_{s\circ t}(X) \geq \Phi_s(X_1) \geq 0$

so that $X \in \mathcal{A}_{s \circ t} = \mathcal{A}_s$ by b). For the converse inclusion, write $X \in \mathcal{A}_s$ as $X = \Phi_t(X) + \big(X - \Phi_t(X)\big)$. The second summand is in $\mathcal{A}_t$, and the first is in $\mathcal{A}_s(\mathcal{F}_t)$ since $\Phi_s\big(\Phi_t(X)\big) = \Phi_{s \circ t}(X) \geq 0$ because $X \in \mathcal{A}_s = \mathcal{A}_{s \circ t}$ by b).

"c) $\Rightarrow$ b)": To show "$\subseteq$", write $X \in \mathcal{A}_s$ by c) as $X = X_1 + X_2$ with $X_1 \in \mathcal{A}_s(\mathcal{F}_t)$ and $X_2 \in \mathcal{A}_t$. As above, this yields $\Phi_t(X) \geq X_1$ and hence by monotonicity of Φ_s that $\Phi_s\big(\Phi_t(X)\big) \geq \Phi_s(X_1) \geq 0$ since $X_1 \in \mathcal{A}_s$. Thus $\Phi_t(X) \in \mathcal{A}_s$ which is equivalent to $X \in \mathcal{A}_{s \circ t}$. To obtain "$\supseteq$", note that $X \in \mathcal{A}_{s \circ t}$ gives $\Phi_t(X) \in \mathcal{A}_s(\mathcal{F}_t)$ so that $X = \big(X - \Phi_t(X)\big) + \Phi_t(X) \in \mathcal{A}_t + \mathcal{A}_s(\mathcal{F}_t) = \mathcal{A}_s$ by c). □

For a normalized DMCUF, time-consistency and strong time-consistency are the same. In fact, $\Phi_s(0) = 0$ implies $0 \in \mathcal{A}_s(\mathcal{F}_t)$ and therefore $\mathcal{A}_t \subseteq \mathcal{A}_s$ by c) of Lemma 2.3.25. Moreover, each of the equivalent properties a) – c) in Lemma 2.3.25 implies that Φ is normalized. To see this for a), simply write

$$\Phi_t(0) = \Phi_{t \circ t}(0) = \Phi_t\big(0 + \Phi_t(0)\big) = \Phi_t(0) + \Phi_t(0) = 2\Phi_t(0).$$

Moreover, an arbitrary DMCUF $\Phi := (\Phi_t)_{0 \leq t \leq T}$ is time-consistent if and only if the normalized DMCUF $\Phi' := (\Phi'_t)_{0 \leq t \leq T}$ defined by $\Phi'_t(\cdot) := \Phi_t(\cdot) - \Phi_t(0)$ is (strongly) time-consistent. The acceptance set of Φ'_t is $\mathcal{A}'_t := \mathcal{A}_t + \Phi_t(0)$, where $\mathcal{A}_t$ denotes the acceptance set of Φ_t. However, we illustrate in Example 2.7.1 below that a DMCUF can be strongly time-consistent without being normalized.

Suppose that a DMCUF Φ satisfies $\mathcal{A}_t \subseteq \mathcal{A}_s$ for $t \geq s$. Then $t \mapsto \inf_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X]$ is increasing and thus $\inf_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty$ holds for all t as soon as we have this for $t = 0$, i.e., if $\alpha_0(\tilde{Q}) = \inf_{X \in \mathcal{A}_0} E_{\tilde{Q}}[X] > -\infty$. Hence condition I) in Theorem 2.3.16 simplifies in this case. Similarly, a time-consistent DMCUF Φ with Φ_0 relevant has Φ_t relevant for all t.

For the economic interpretation of property a) in Lemma 2.3.25, note that for any normalized DMCUF Φ we have $\Phi_t(X) = \Phi_t(\Phi_t(X))$. This means that the agent assigns at time t the same monetary utility to X and to $\Phi_t(X)$. If she acts in a time-consistent way, she should stick to this indifference at time s, which yields exactly property a). Clearly property b) is just a reformulation of a). For property c), we note that $\mathcal{A}_s \subseteq \mathcal{A}_s(\mathcal{F}_t) + \mathcal{A}_t$ means that we can split any payoff acceptable at time s into the sum of a payoff X_1 which is acceptable at time s when the observation period ends at time t, and a payoff X_2 which is acceptable if the observation period starts at time t. Conversely, let a payoff X be the sum of such X_1 and X_2. Normalization and translation invariance imply that $\Phi_t(X_1 + X_2) = \Phi_t(X_2) + X_1 \geq X_1 = \Phi_t(X_1)$, i.e., at time t the agent prefers the payoff $X = X_1 + X_2$ to X_1. If she acts in a time-consistent way, she should have the same ordering at time s; see the comment after Definition 2.3.23. As X_1 is acceptable at time s, this shows that the converse inclusion should hold as well. Note that the above interpretations all use that Φ is normalized.

Remark 2.3.26. Until now, we have considered DMCUFs for the time horizon T. To emphasize the dependence on T we write

$$\Phi_{s,T}(\,\cdot\,) \text{ instead of } \Phi_s(\,\cdot\,).$$

In view of a possible study of indifference valuation functionals for intermediate time horizons $t < T$ one could also look at DMCUFs $(\Phi_{s,t}(\,\cdot\,))_{0\le s\le t}$ for all $t < T$, the idea being that $\Phi_{s,t}(X)$ is the value at time s for the payoff $X \in \mathbf{L}^\infty(\mathcal{F}_t)$ due at time t (instead of T). Where such a $\Phi_{\cdot,t}$ comes from will be discussed later. In general, a property one might want to have for such families of functionals (in addition to (strong) time-consistency of $\Phi_{\cdot,T}$) is

($\mathcal{R}$) *Recursiveness:* $\quad \Phi_{s,t}(\Phi_{t,T}(X)) = \Phi_{s,T}(X) \quad$ for any $X \in \mathbf{L}^\infty(\mathcal{F}_T)$.

(This could also be called Bellman's principle.) Note the difference between recursiveness and property a) in Lemma 2.3.25, where we have $\Phi_{s,T}$ instead of $\Phi_{s,t}$. In economic terms, recursiveness means that if we want to value the time T payoff X at time s, we can either do this directly or first value it at time $t \ge s$ and then value that result at time s. This can also be desirable for non-normalized functionals. The concept of recursiveness seems to go back to Peng who studied it in the context of non-linear expectations; see [Pen04] for a comprehensive overview. The following considerations are motivated by a discussion with S. Peng.

The aim of the present work is to obtain a valuation functional from indifference considerations. Among other things, we assume that there exists a bank account with zero interest rate, so that money can be freely transferred over time. Hence an investor should be indifferent between receiving a payoff $X \in \mathbf{L}^\infty(\mathcal{F}_t)$ at time $t < T$ or at time T. If the indifference valuation functionals over time are given by a family p, we should therefore have

$$p_{s,t}(X) = p_{s,T}(X) \quad \text{for all } s \le t \le T \text{ and } X \in \mathbf{L}^\infty(\mathcal{F}_t). \tag{2.3.27}$$

In addition, indifference valuation functionals should be normalized, i.e., $p_{s,t}(0) = 0$ for all $s \le t \le T$. With this and (2.3.27), time-consistency and recursiveness are equivalent; see Lemma 2.3.25. Moreover, ($\mathcal{R}$) for p then also holds for any time horizon $u \ge t$ instead of T.

The indifference valuation DMCUF $p_{\cdot,T}$ will be obtained from a (strongly) time-consistent DMCUF $\Phi_{\cdot,T}$ via normalization, i.e., $p_{s,T}(\,\cdot\,) = \Phi_{s,T}(\,\cdot\,) - \Phi_{s,T}(0)$. Here difficulties can arise if we want to valuate also for intermediate time horizons $t < T$ but do not start with normalized families Φ. In fact, for all $s \le t \le T$ we want to have MCUFs (with time horizon t) $\Phi_{s,t} : \mathbf{L}^\infty(\mathcal{F}_t) \to \mathbf{L}^\infty(\mathcal{F}_s)$ and then to set

$$p_{s,t}(X) := \Phi_{s,t}(X) - \Phi_{s,t}(0) \quad \text{for all } X \in \mathbf{L}^\infty(\mathcal{F}_t). \tag{2.3.28}$$

With this construction, we can assume (2.3.27) if and only if

$$\Phi_{s,t}(X) - \Phi_{s,t}(0) = \Phi_{s,T}(X) - \Phi_{s,T}(0) \quad \text{for all } s \le t \le T,\ X \in \mathbf{L}^\infty(\mathcal{F}_t). \tag{2.3.29}$$

This holds, e.g., if the indifference valuation DMCUF is constructed from the conditional exponential certainty equivalent; see Example 2.7.19. If Φ satisfies (2.3.29) and $\Phi_{\cdot,T}$ is time-consistent, then $\Phi_{\cdot,t}$ is also time-consistent for each t. $p_{\cdot,t}$ from (2.3.28) is then normalized and strongly time-consistent for each t, and hence the family p also satisfies $(\mathscr{R})$.

Since the family Φ is the basic building block in the above construction, we now have to ask where $\Phi_{\cdot,t}$ comes from. $\Phi_{\cdot,T}$ is always given, and the simplest way to obtain some $\Phi_{\cdot,t}$ satisfying (2.3.29) is the brute force definition

$$\Phi_{s,t}(X) := \Phi_{s,T}(X) \quad \text{for } s \leq t \leq T \text{ and } X \in \mathbf{L}^\infty(\mathcal{F}_t).$$

This will always work, but is not always reasonable. Suppose for instance that $\Phi_{s,t}$ should represent some maximal subjective utility achievable between s and t. Then another reasonable definition could be

$$\Phi_{s,t}(X) := \Phi_{s,T}(X) - \Phi_{t,T}(0) \quad \text{for all } s \leq t < T \text{ and } X \in \mathbf{L}^\infty(\mathcal{F}_t). \tag{2.3.30}$$

The loose argument for subtracting the second term is that since X is known at time t, it is by translation invariance irrelevant for the maximal utility achievable during the period from t to T. (But of course such an "argument" via splitting $(s, T]$ into $(s, t]$ and $(t, T]$ is based on the intuition from recursiveness and thus has a taste of circularity.) It is straightforward to check that (2.3.30) implies (2.3.29). However, $\Phi_{s,t}(X)$ is not $\mathcal{F}_s$-measurable unless $\Phi_{t,T}(0)$ is, and if this should hold for all s, we must require that $(\Phi_{s,T}(0))_{0\leq s\leq T}$ is a deterministic process. In that case, (2.3.30) gives a good definition.

In Section 2.7.2, we shall examine functionals $\Phi_{\cdot,T}$ defined via backward stochastic differential equations (BSDEs). In that case, the BSDE also produces a natural definition for $\Phi_{\cdot,t}$ for each $t < T$, and one can show that if $\Phi_{\cdot,T}(0)$ is deterministic, those $\Phi_{\cdot,t}$ must be of the form (2.3.30). In that sense, this definition is also natural. $\diamond$

Although time-consistency is desirable in most situations, it is also quite restrictive as we shall illustrate by an example in Section 2.7.3. In preparation and to complete the results here, we provide another equivalent description of time-consistency for the case where the DMCUF is coherent. Since DMCohUFs are always normalized, this description is also equivalent to strong time-consistency.

Definition 2.3.27. A set $\mathcal{Q} \subseteq \mathcal{P}^a$ is called *weakly multiplicatively stable* (*weakly m-stable* for short) if $\mathcal{Q} \cap \mathcal{P}^e \neq \emptyset$ and $\mathcal{Q}$ has the following property: If we take any $Q^0, Q^1, Q^2 \in \mathcal{Q}$ with associated density processes Z^0, Z^1, Z^2, fix $t \in [0, T]$ and $A \in \mathcal{F}_t$, impose that $Q^1, Q^2 \in \mathcal{P}^e$ and define

$$Z_T := \mathbf{1}_A \frac{Z_t^0}{Z_t^1} Z_T^1 + \mathbf{1}_{A^c} \frac{Z_t^0}{Z_t^2} Z_T^2,$$

then Z_T is the density of some element in $\mathcal{Q}$.

Remark 2.3.28. i) Intuitively, weak m-stability means that $\mathcal{Q}$ is closed under the following operation: We pick any time t and construct from $Q^0, Q^1, Q^2 \in \mathcal{Q}$ a new probability measure $Q \in \mathcal{Q}$ which agrees with Q^0 on $\mathcal{F}_t$ and has after t on A the same $\mathcal{F}_t$-conditional behavior as Q^1 and on A^c the same as Q^2.

ii) Definition 2.3.27 is similar to the definition of m-stable sets given in [Del06]. However, we only paste together probability measures at deterministic times, whereas Delbaen also considers stopping times. Therefore we have to introduce the set A to ensure that the set $\{E_Q[X|\mathcal{F}_t] \mid Q \in \mathcal{Q} \cap \mathcal{P}^e\}$ is a lattice for each $X \in \mathbf{L}^\infty$, whereas this holds automatically when stopping-times are allowed; see Proposition 1 in [Del06]. Moreover, in [Del06] it is assumed that $P \in \mathcal{Q}$; but since P is required only to specify the nullsets and to ensure that $\mathcal{Q} \cap \mathcal{P}^e \neq \emptyset$, the latter condition is already sufficient. Moreover, our assumption that $\mathcal{F}_0$ is trivial simplifies the definition slightly.

◇

The following Lemma 2.3.29 is a slight improvement of Theorem 12 of [Del06] as it does not only give (in part a)) a structural description of time-consistent DMCohUFs of a particular form, but also shows (in part b)) that every normalized time-consistent DMCUF which is well-representable and positively homogeneous at time 0 is of this form, and gives an explicit representation. This will prove helpful in the abovementioned example of Section 2.7.3.

Lemma 2.3.29. *a) Define a family of mappings $\Phi = (\Phi_t)_{0\le t\le T}$ on $\mathbf{L}^\infty$ by*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^e} E_Q[X \mid \mathcal{F}_t] \tag{2.3.31}$$

for some $\mathbf{L}^1$-closed and convex set $\mathcal{Q} \subseteq \mathcal{P}^a$ with $\mathcal{Q}^e = \mathcal{Q} \cap \mathcal{P}^e \neq \emptyset$. Then Φ is a well-representable strongly time-consistent DMCohUF if and only if $\mathcal{Q}$ is weakly m-stable.

b) Conversely, let $\Phi = (\Phi_t)_{0\le t\le T}$ be a normalized time-consistent DMCUF such that Φ_0 is positively homogeneous and well-representable. Then Φ can be represented as in (2.3.31) and is in particular a DMCohUF, i.e., positively homogeneous for all $t \in [0, T]$. Moreover, $\mathcal{Q}$ is unique, weakly m-stable and consists of all $Q \in \mathcal{P}^a$ whose densities are elements of the polar cone of $-\mathcal{A}_0$ where $\mathcal{A}_0$ is the acceptance set of Φ_0, i.e.,

$$\mathcal{Q} = \left\{ Q \in \mathcal{P}^a \;\middle|\; dQ = Z_T\, dP,\ Z_T \in (-\mathcal{A}_0)^\circ \cap \mathcal{B}(\mathbf{L}^1) \right\}. \tag{2.3.32}$$

Here $\mathcal{B}(\mathbf{L}^1)$ is the unit ball in $\mathbf{L}^1$, and the polar cone of $-\mathcal{A}_0$ is given by

$$(-\mathcal{A}_0)^\circ = \left\{ Z \in \mathbf{L}^1 \;\middle|\; E[ZX] \le 0 \text{ for all } X \in (-\mathcal{A}_0) \right\}.$$

Remark 2.3.30. Lemma 2.3.29 shows that a DMCUF which is positively homogeneous at time 0 can only be time-consistent if the set of representing measures at time 0 or the acceptance set $\mathcal{A}_0$ at time 0 (more precisely, the polar cone of $-\mathcal{A}_0$) has an appropriate structure. Moreover, it shows that there exists at most one normalized time-consistent DMCUF which extends a given static MCohUF at time 0. In Section 2.7.3 we consider an example of a static MCohUF at time 0 which cannot be extended to a time-consistent DMCUF. ◇

Proof of Lemma 2.3.29. a) This follows similarly as in the proof of Theorem 12 in [Del06]. The assumption that $\mathcal{Q}^e \neq \emptyset$ is obviously necessary from the definition of a (weakly) m-stable set.

b) By the proof of Theorem 3.2 in [Del02], with $\mathcal{Q}$ from (2.3.32),

$$\Phi_0(\cdot) = \inf_{Q\in\mathcal{Q}} E_Q[\cdot] \tag{2.3.33}$$

on $\mathbf{L}^\infty$, and $\mathcal{Q}$ is $\mathbf{L}^1$-closed and convex. To show that Φ_t can be represented by (2.3.31), we define a DMCUF $\hat{\Phi} = (\hat{\Phi}_t)_{0\leq t\leq T}$ as the RHS of (2.3.31), i.e.,

$$\hat{\Phi}_t(X) := \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^e} E_Q[X|\mathcal{F}_t] \quad \text{for all } X \in \mathbf{L}^\infty,$$

and we show that $\Phi = \hat{\Phi}$; weak m-stability of $\mathcal{Q}$ then follows from a) and the time-consistency of Φ. Since Φ_0 is well-representable, there exists $\overline{Q} \in \mathcal{P}^e$ such that $\Phi_0(\cdot) \leq E_{\overline{Q}}[\cdot]$. Hence the density $\overline{Z}_T$ of $\overline{Q}$ is in $(-\mathcal{A}_0)^\circ$ so that $\overline{Q} \in \mathcal{Q}^e$ and we can replace $\mathcal{Q}$ by $\mathcal{Q}^e$ in (2.3.33) which then implies that

$$\Phi_0 = \hat{\Phi}_0 \tag{2.3.34}$$

on $\mathbf{L}^\infty$. In fact, fix $Q' \in \mathcal{Q}$ with density Z'_T and define for each $\varepsilon > 0$ a measure $Q^\varepsilon \in \mathcal{Q}^e$ by its density $Z^\varepsilon_T := \varepsilon\overline{Z}_T + (1-\varepsilon)Z'_T$. Clearly, as ε tends to zero, Z^ε_T converges to Z'_T in $\mathbf{L}^1$ and hence also weakly in $\mathbf{L}^1$. This shows (2.3.34).

Lemma 2.3.9 implies that two DMCohUFs Φ^1 and Φ^2 are equal if and only if they have the same acceptance set at each time t. Therefore we are left to show that for all $t \in (0, T]$

$$\mathcal{A}_t := \left\{X \in \mathbf{L}^\infty \,\middle|\, \Phi_t(X) \geq 0\right\} = \left\{X \in \mathbf{L}^\infty(\mathcal{F}) \,\middle|\, \hat{\Phi}_t(X) \geq 0\right\} =: \hat{\mathcal{A}}_t.$$

Fix $t \in (0, T]$ and let $X \in \hat{\mathcal{A}}_t$, i.e., $\hat{\Phi}_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^e} E_Q[X \mid \mathcal{F}_t] \geq 0$. Then (2.3.34) yields

$$0 \leq \inf_{Q\in\mathcal{Q}^e} E_Q[\mathbf{1}_A X] = \Phi_0(\mathbf{1}_A X) \quad \text{for all } A \in \mathcal{F}_t. \tag{2.3.35}$$

As Φ is time-consistent and normalized, Lemma 2.3.25 and $\mathcal{F}_t$-regularity imply that

$$\Phi_0\left(\mathbf{1}_A X\right) = \Phi_0(\Phi_t(\mathbf{1}_A X)) = \Phi_0\big(\mathbf{1}_A \Phi_t(X)\big) \quad \text{for all } A \in \mathcal{F}_t.$$

From this, (2.3.35), (2.3.34) and since $\overline{Q} \in \mathcal{Q}^e$, we obtain that

$$0 \leq \Phi_0\left(\mathbf{1}_A \Phi_t(X)\right) = \inf_{Q \in \mathcal{Q}^e} E_Q\left[\mathbf{1}_A \Phi_t(X)\right] \leq E_{\overline{Q}}\left[\mathbf{1}_A \Phi_t(X)\right] \quad \text{for all } A \in \mathcal{F}_t.$$

But since $\Phi_t(X)$ is $\mathcal{F}_t$-measurable, this implies that $\Phi_t(X) \geq 0$. Hence $X \in \mathcal{A}_t$ and $\hat{\mathcal{A}}_t \subseteq \mathcal{A}_t$. To show the converse inclusion, suppose that $\Phi_t(X) \geq 0$. Then $\mathcal{F}_t$-regularity and normalization yield

$$\Phi_t(\mathbf{1}_A X) = \mathbf{1}_A \Phi_t(X) \geq 0 \quad \text{for all } A \in \mathcal{F}_t.$$

Hence time-consistency, monotonicity and normalization imply that

$$\Phi_0(\mathbf{1}_A X) = \Phi_0\big(\Phi_t(\mathbf{1}_A X)\big) \geq 0 \quad \text{for all } A \in \mathcal{F}_t.$$

Consequently, we have by (2.3.34) that

$$\inf_{Q \in \mathcal{Q}^e} E_Q[\mathbf{1}_A X] = \Phi_0(\mathbf{1}_A X) \geq 0 \quad \text{for all } A \in \mathcal{F}_t$$

and obtain

$$\hat{\Phi}_t(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{Q}^e} E_Q[X|\mathcal{F}_t] \geq 0.$$

This shows that $\mathcal{A}_t = \hat{\mathcal{A}}_t$.

We are left to show uniqueness of $\mathcal{Q}$, and it suffices to prove that there is a unique representing set at time 0. Suppose there exists another $\mathbf{L}^1$-closed and convex set $\tilde{\mathcal{Q}} \neq \mathcal{Q} \subseteq \mathcal{P}^a$ such that

$$\Phi_0(\,\cdot\,) = \inf_{Q \in \mathcal{Q}^e} E_Q[\,\cdot\,] = \inf_{Q \in \tilde{\mathcal{Q}}^e} E_Q[\,\cdot\,] \tag{2.3.36}$$

on $\mathbf{L}^\infty$. Then we apply the same arguments as in the proof of (2.3.34) to conclude that also

$$\inf_{Q \in \mathcal{Q}} E_Q[\,\cdot\,] = \inf_{Q \in \tilde{\mathcal{Q}}} E_Q[\,\cdot\,] \tag{2.3.37}$$

on $\mathbf{L}^\infty$. Without loss of generality there exists $\tilde{Q} \in \tilde{\mathcal{Q}} \backslash \mathcal{Q}$, and so the Hahn-Banach theorem yields some $X \in \mathbf{L}^\infty$ such that

$$\Phi_0(X) = \inf_{Q \in \mathcal{Q}} E_Q[X] > E_{\tilde{Q}}[X] \geq \inf_{Q \in \tilde{\mathcal{Q}}} E_Q[X] = \Phi_0(X).$$

This being a contradiction, $\mathcal{Q}$ must be unique.

□

2.4 Convolution

In this section we study an operation on MCUFs called *convolution*. We know from the preceding section that an MCUF models the preferences of an agent. If this agent gets the possibility to trade in some financial market, this will affect her preference ordering. We shall see that this can be captured by convoluting appropriate MCUFs. From a purely mathematical point of view, the convolution is an operation on two MCUFs at time t which defines a new MCUF. If Φ^1 and Φ^2 are two (strongly) time-consistent DMCUFs, then we can obtain a new DMCUF by convoluting Φ_t^1 and Φ_t^2 at each time t. An important property of the convolution is that this DMCUF is again (strongly) time-consistent.

Definition 2.4.1. Let Φ_t^1 and Φ_t^2 be two MCUFs at time t. The *convolution of* Φ_t^1 *and* Φ_t^2 is defined as

$$\Phi_t^1 \Box \Phi_t^2(X) := \operatorname*{ess\,sup}_{Y \in \mathbf{L}^\infty} \left\{ \Phi_t^1(X+Y) + \Phi_t^2(-Y) \right\} \quad \text{for all } X \in \mathbf{L}^\infty. \tag{2.4.1}$$

If $\mathcal{B} \subseteq \mathbf{L}^\infty$ is non-empty, convex and $\mathcal{F}_t$-regular, the *convolution* of Φ_t^1 and $\mathcal{B}$ is defined as

$$\Phi_t^1 \Box \mathcal{B}(X) := \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y) \quad \text{for } X \in \mathbf{L}^\infty. \tag{2.4.2}$$

Remark 2.4.2. i) The convolution is obviously symmetric, i.e.,

$$\Phi_t^1 \Box \Phi_t^2(X) = \Phi_t^2 \Box \Phi_t^1(X) \quad \text{for all } X \in \mathbf{L}^\infty.$$

ii) Since $\mathbf{L}^\infty$ is a linear space, we could equivalently define the convolution by

$$\Phi_t^1 \Box \Phi_t^2(X) := \operatorname*{ess\,sup}_{Y \in \mathbf{L}^\infty} \left\{ \Phi_t^1(X-Y) + \Phi_t^2(Y) \right\}.$$

This looks more natural because of the analogy to classical convolution operations. We deliberately choose the formulation (2.4.1) because it will turn out to be more convenient for subsequent interpretations. ◇

For a brief overview of the development of this type of convolution, we should probably start with Rockafellar. In his book [Roc70], he studied the *infimal convolution* of two convex functions f and g, defined as

$$f \Box g(x) := \inf_{y \in \mathbb{R}} \{ f(x-y) + g(y) \}. \tag{2.4.3}$$

The terminology arises from the obvious analogy to the formula for classical integral convolutions. The convolution (2.4.3) is dual to the operation of addition for convex functions in the sense that the convex conjugate of $f+g$ is equal to the convolution of the conjugates of f and of g. The convolution in (2.4.1) was introduced and studied by [Del00] for static and coherent risk measures; see also [Del06a]. One motivation for studying $\Phi_t^1 \Box \Phi_t^2$ comes from a problem of risk transfer between two agents with preferences given by Φ_t^1 and Φ_t^2; see Barrieu/El Karoui ([BEK04], [BEK05]). We will show below that convoluting Φ_0^1 and Φ_0^2 also corresponds to finding a Pareto-efficient exchange between two individuals with preferences Φ_0^1 and Φ_0^2. This has been pointed out to us by N. Touzi; see also [JST05].

The main result of this section is an extension of Theorem 3.6 in [BEK05] in several directions. We show that the convolution operation produces a new MCUF and also preserves the dynamic property of (strong) time-consistency. All this is done in a conditional and abstract setting. This is in contrast to [BEK05] who only treat the static abstract case, and also to [BEK04] who study in the dynamic case a class of DMCUFs defined via BSDEs; we will come back to this in Section 2.7.2. Moreover, the question of time-consistency for convolutions of DMCUFs seems not to have been addressed so far in a general setting. In technical terms, the main difficulty here is related to closedness properties of acceptance sets; this comes up when we need to identify the acceptance set of the convolution $\Phi_t^1 \Box \Phi_t^2$. Before we state the main result of this section, we recall from Lemma 2.3.14 that any MCUF which is continuous from below is also continuous from above, and hence representable due to Theorem 2.3.19.

Theorem 2.4.3. *For $i = 1, 2$, let Φ_t^i be MCUFs at time t with acceptance sets $\mathcal{A}_t^i$ and concave conjugates α_t^i. Assume that $\Phi_t^1 \Box \Phi_t^2(0) \in \mathbf{L}^\infty$. Then:*

a) *$\Phi_t^1 \Box \Phi_t^2$ is an MCUF at time t, and for all $X \in \mathbf{L}^\infty$*

$$\Phi_t^1 \Box \Phi_t^2(X) = \Phi_t^1 \Box \mathcal{A}_t^2(X) = \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \left\{ \Phi_t^1(X+Y) + \Phi_t^2(-Y) \right\}, \qquad (2.4.4)$$

where $\mathcal{B}$ is an arbitrary subset of $\mathbf{L}^\infty$ containing $\mathcal{A}_t^2$.

b) *If Φ_t^1 and Φ_t^2 are both coherent, so is $\Phi_t^1 \Box \Phi_t^2$.*

c) *If Φ_t^1 or Φ_t^2 is continuous from below, then $\Phi_t^1 \Box \Phi_t^2$ is continuous from below and in particular representable. Its concave conjugate $\alpha_t^{1\Box 2}$ is then given by*

$$\alpha_t^{1\Box 2}(Q) = \alpha_t^1(Q) + \alpha_t^2(Q) \quad \text{for } Q \in \mathcal{P}_t^{\approx}, \qquad (2.4.5)$$

and its acceptance set $\mathcal{A}_t^{1\Box 2}$ is given by

$$\mathcal{A}_t^{1\Box 2} = \overline{\mathcal{A}_t^1 + \mathcal{A}_t^2}, \qquad (2.4.6)$$

where the closure is taken in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$. If in addition we have

$$\inf_{X \in \mathcal{A}_t^1 + \mathcal{A}_t^2} E_{\tilde{Q}}[X] > -\infty \quad \text{for some } \tilde{Q} \in \mathcal{P}^e, \qquad (2.4.7)$$

then $\Phi_t^1 \square \Phi_t^2$ *is also well-representable.*

d) *Suppose that* $\Phi^i = (\Phi_t^i)_{0 \le t \le T}$ *for* $i = 1, 2$ *are (strongly) time-consistent DMCUFs such that for each* $t \in [0, T]$, Φ_t^1 *is continuous from below and* $\Phi_t^1 \square \Phi_t^2(0) \in \mathbf{L}^\infty$. *Then* $\Phi^1 \square \Phi^2 = (\Phi_t^1 \square \Phi_t^2)_{0 \le t \le T}$ *is also a (strongly) time-consistent DMCUF.*

Remark 2.4.4. i) Like in Section 2.3, condition (2.4.7) simplifies if $\Phi^1 \square \Phi^2$ is strongly time-consistent; it is then enough if

$$\inf_{X \in \mathcal{A}_0^1 + \mathcal{A}_0^2} E_{\tilde{Q}}[X] = \alpha_0^1(\tilde{Q}) + \alpha_0^2(\tilde{Q}) > -\infty \quad \text{for some } \tilde{Q} \in \mathcal{P}^e.$$

ii) We illustrate in Example 2.7.1 below that $\Phi_t^1 \square \Phi_t^2$ need not be normalized even if Φ_t^1 and Φ_t^2 both are. This is our main reason for abandoning the requirement of normalization.

◇

As mentioned above, convoluting the MCUFs Φ_0^1 and Φ_0^2 corresponds to finding a Pareto-efficient exchange between two individuals with preferences corresponding to Φ_0^1 respectively Φ_0^2. To see this, denote by

$$K_0 := \left\{ (Y^1, Y^2) \in \mathbf{L}^\infty \times \mathbf{L}^\infty \,\middle|\, Y^1 + Y^2 = X \right\}$$

the set of all *feasible exchanges*. Then (2.4.1) for $t = 0$ can equivalently be written as

$$\sup_{(Y^1, Y^2) \in K_0} \left\{ \Phi_0^1(Y^1) + \Phi_0(Y^2) \right\}. \tag{2.4.8}$$

A feasible exchange $(\hat{Y}^1, \hat{Y}^2) \in K_0$ is called *Pareto-efficient* if no $(Y^1, Y^2) \in K_0$ satisfies

$$\Phi_0^i(Y^i) \ge \Phi_0^i(\hat{Y}^i), \;\; \Phi_0^j(Y^j) > \Phi_0^j(\hat{Y}^j) \text{ for } (i, j) = (1, 2) \text{ or } (i, j) = (2, 1). \tag{2.4.9}$$

$(\hat{Y}^1, \hat{Y}^2)$ is called *weakly Pareto-efficient* if "$\ge$" is replaced by "$>$" in (2.4.9). It is well known (see, e.g., Proposition 2.8 in [IBK02]) that $(\hat{Y}^1, \hat{Y}^2) \in K_0$ is weakly Pareto-efficient if and only if there exists $(\lambda^1, \lambda^2) \in \mathbb{R}_+^2 \backslash \{(0, 0)\}$ such that

$$(\hat{Y}^1, \hat{Y}^2) \quad \text{maximizes} \quad (Y^1, Y^2) \mapsto \lambda^1 \Phi_0^1(Y^1) + \lambda^2 \Phi_0^2(Y^2) \quad \text{over } K_0. \tag{2.4.10}$$

If $\lambda^1 > 0$ and $\lambda^2 > 0$ then $(\hat{Y}^1, \hat{Y}^2)$ is even Pareto-efficient.

Note that for any $c \in \mathbb{R}$ and $(Y^1, Y^2) \in K_0$ also $(Y^1 + c, Y^2 - c) \in K_0$ and that by translation invariance of Φ_0^i, we have

$$\lambda^1 \, \Phi_0^1(Y^1 + c) + \lambda^2 \, \Phi_0^2(Y^2 - c) = \lambda^1 \, \Phi_0^1(Y^1) + \lambda^2 \, \Phi_0^2(Y^2) + c(\lambda^1 - \lambda^2).$$

But if $\lambda^1 \neq \lambda^2$, then this tends to $+\infty$ if $c \to +\infty$ or if $c \to -\infty$. Thus $(\hat{Y}^1, \hat{Y}^2) \in K_0$ is a Pareto-efficient exchange if and only if it satisfies (2.4.10) for $\lambda^1 = \lambda^2 > 0$, i.e., if it maximizes (2.4.8). This explains the connection between the convolution and Pareto-efficient exchanges.

There is another economic interpretation for the convolution which comes from the second expression in (2.4.4) and was suggested in [BEK05]. Consider two individuals I_1 and I_2 with preferences corresponding to Φ_t^1 and Φ_t^2 who want to maximize their monetary utilities. Suppose that I_1 owns at time $t < T$ some asset with payoff X at time T. She might try to increase her utility by exchanging at time t with I_2 some payoff Y due at time T. But of course, I_2 will only agree to hand over Y to I_1 if he deems the for him resulting payoff $-Y$ acceptable. This gives a constraint for the maximization problem of I_1 exactly as in (2.4.4). In particular, if the preferences of agent I_2 correspond to a normalized MCUF so that $\Phi_t^2(0) = 0$, he will agree to handing over Y if and only if this does not decrease his utility.

In the proof of Theorem 2.4.3, we use the following auxiliary result.

Lemma 2.4.5. *Take an MCUF Φ_t^1 at time t and a non-empty, convex and $\mathcal{F}_t$-regular set $\mathcal{B} \subseteq \mathbf{L}^\infty$. If $\Phi_t^1 \Box \mathcal{B}(0) \in \mathbf{L}^\infty$, then:*

a) $\Phi_t^1 \Box \mathcal{B}$ is an MCUF at time t.

b) If Φ_t^1 is coherent and $\mathcal{B}$ a convex cone containing 0, then $\Phi_t^1 \Box \mathcal{B}$ is an MCohUF at time t.

c) If Φ_t^1 is continuous from below, so is $\Phi_t^1 \Box \mathcal{B}$.

Proof. To shorten notation we write $\Phi_t := \Phi_t^1 \Box \mathcal{B}$.

a) Properties A) and B) of Definition 2.3.1 are obvious. To see C), let $\beta \in [0, 1]$ and $X_1, X_2 \in \mathbf{L}^\infty$. Since $\mathcal{B}$ is convex and Φ_t^1 is concave, we get

$$\begin{aligned}
&\Phi_t\big(\beta X_1 + (1-\beta)X_2\big) \\
&= \operatorname*{ess\,sup}_{Y_1, Y_2 \in -\mathcal{B}} \Phi_t^1\big(\beta(X_1+Y_1) + (1-\beta)(X_2+Y_2)\big) \\
&\geq \beta \operatorname*{ess\,sup}_{Y_1 \in -\mathcal{B}} \Phi_t^1(X_1+Y_1) + (1-\beta) \operatorname*{ess\,sup}_{Y_2 \in -\mathcal{B}} \Phi_t^1(X_2+Y_2) \\
&= \beta\Phi_t(X_1) + (1-\beta)\Phi_t(X_2).
\end{aligned}$$

Finally, A) and B) imply

$$\|\Phi_t(X)\|_{\mathbf{L}^\infty} \leq \|\Phi_t(0)\|_{\mathbf{L}^\infty} + \|X\|_{\mathbf{L}^\infty} < \infty$$

so that $\Phi_t(X) \in \mathbf{L}^\infty$ for each $X \in \mathbf{L}^\infty$.

b) To see that Φ_t is coherent, we first show that

$$\Phi_t(0) = 0. \tag{2.4.11}$$

Suppose this is not true. Since Φ_t^1 as MCohUF is normalized so that $0 \in \mathcal{B}$ implies $\Phi_t(0) \geq \Phi_t^1(0+0) = 0$, we then must have $\Phi_t(0) > 0$ with positive probability. Because the essential supremum in the definition of Φ_t can be written as the pointwise supremum over a countable number of elements of $-\mathcal{B}$ ([Nev75]), there exist $\overline{Y} \in -\mathcal{B}$ and $A \in \mathcal{F}_t$ with $P[A] > 0$ such that

$$\Phi_t(0) \geq \Phi_t^1(\overline{Y}) > 0 \quad \text{on } A.$$

Now replace $\overline{Y}$ by $n\overline{Y}$ and use positive homogeneity of Φ_t^1 and that $\mathcal{B}$ is a convex cone to obtain for $n \to \infty$ that $\Phi_t(0) = +\infty$ on A, contradicting $\Phi_t(0) \in \mathbf{L}^\infty$. This establishes (2.4.11). Now let $\lambda > 0$. For any $X \in \mathbf{L}^\infty$ we obtain by using positive homogeneity of Φ_t^1 and the fact that $\mathcal{B}$ is a convex cone that

$$\begin{aligned} \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(\lambda X + Y) &= \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \left\{ \lambda \Phi_t^1\left(X + \frac{Y}{\lambda}\right)\right\} \\ &= \lambda \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y). \end{aligned}$$

Hence Φ_t is positively homogeneous.

c) To see that continuity from below of Φ_t^1 carries over to Φ_t, let $(X_n)_{n \in \mathbb{N}}$ be a uniformly bounded sequence increasing to some $X \in \mathbf{L}^\infty$. Then monotonicity of Φ_t yields

$$\begin{aligned} \lim_{n\to\infty} \Phi_t(X_n) &= \sup_{n \in \mathbb{N}} \Phi_t(X_n) \\ &= \sup_{n \in \mathbb{N}} \left\{ \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X_n + Y) \right\} \\ &= \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \left\{ \sup_{n \in \mathbb{N}} \Phi_t^1(X_n + Y) \right\} \\ &= \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y) \\ &= \Phi_t(X), \end{aligned}$$

which shows that Φ_t is continuous from below. □

Proof of Theorem 2.4.3. To shorten notation we write $\Phi_t := \Phi_t^1 \Box \Phi_t^2$ for $t \in [0, T]$.

a) Once we have shown (2.4.4), the rest follows from Lemma 2.4.5 a). We begin by proving the first equality in (2.4.4), i.e., that

$$\operatorname*{ess\,sup}_{Y\in\mathbf{L}^\infty}\left\{\Phi_t^1(X+Y)+\Phi_t^2(-Y)\right\}=\operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_t^2}\Phi_t^1(X+Y').$$

For arguing "$\leq$", we fix $Y\in\mathbf{L}^\infty$ and show that there exists $Y'\in-\mathcal{A}_t^2$ such that

$$\Phi_t^1(X+Y)+\Phi_t^2(-Y)=\Phi_t^1(X+Y').$$

In fact, translation invariance implies that $Y':=Y+\Phi_t^2(-Y)$ is in $-\mathcal{A}_t^2$ and also yields $\Phi_t^1(X+Y')=\Phi_t^1(X+Y)+\Phi_t^2(-Y)$. To see "$\geq$", note that $Y'\in-\mathcal{A}_t^2$ yields $\Phi_t^2(-Y')\geq 0$ and therefore

$$\Phi_t^1(X+Y')\leq\Phi_t^1(X+Y')+\Phi_t^2(-Y').$$

This shows the first equality in (2.4.4) which then immediately implies the second by

$$\begin{aligned}
\Phi_t^1\square\Phi_t^2(X) &\geq \operatorname*{ess\,sup}_{Y'\in-\mathcal{B}}\left\{\Phi_t^1(X+Y')+\Phi_t^2(-Y')\right\}\\
&\geq \operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_t^2}\left\{\Phi_t^1(X+Y')+\Phi_t^2(-Y')\right\}\\
&\geq \operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_t^2}\Phi_t^1(X+Y')\\
&= \Phi_t^1\square\Phi_t^2(X),
\end{aligned}$$

where we used again that $\Phi_t^2(-Y')\geq 0$ for all $Y'\in-\mathcal{A}_t^2$.

b) This follows immediately from (2.4.4) and Lemma 2.4.5 b), since $\mathcal{A}_t^2$ is by Lemma 2.3.6 a convex cone containing 0.

c) Continuity from below follows immediately from (2.4.4) and Lemma 2.4.5. From this together with a) and Lemma 2.3.14, we can apply Theorem 2.3.19 which implies that Φ_t is representable. If in addition (2.4.7) holds, Φ_t is even well-representable by Theorem 2.3.16. Moreover, (2.4.5) holds since by Definition 2.3.10 for any $Q \in \mathcal{P}_t^{\approx}$

$$\begin{aligned}
\alpha_t^{1\Box 2}(Q) &= \operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty}\left\{E_Q[X|\mathcal{F}_t] - \operatorname*{ess\,sup}_{Y\in\mathbf{L}^\infty}\left\{\Phi_t^1(X+Y)+\Phi_t^2(-Y)\right\}\right\}\\
&= \operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty}\left\{\operatorname*{ess\,inf}_{Y\in\mathbf{L}^\infty}\right.\\
&\quad \left.\left\{E_Q[X+Y|\mathcal{F}_t]+E_Q[-Y|\mathcal{F}_t]-\Phi_t^1(X+Y)-\Phi_t^2(-Y)\right\}\right\}\\
&= \operatorname*{ess\,inf}_{Y\in\mathbf{L}^\infty}\left\{E_Q[-Y|\mathcal{F}_t]-\Phi_t^2(-Y)\right.\\
&\qquad\qquad \left.+\operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty}\left\{E_Q[X+Y|\mathcal{F}_t]-\Phi_t^1(X+Y)\right\}\right\}\\
&= \operatorname*{ess\,inf}_{Y\in\mathbf{L}^\infty}\left\{E_Q[-Y|\mathcal{F}_t]-\Phi_t^2(-Y)+\alpha_t^1(Q)\right\}\\
&= \alpha_t^2(Q)+\alpha_t^1(Q).
\end{aligned}$$

The proof of the assertion that $\mathcal{A}_t^{1\Box 2} = \overline{\mathcal{A}_t^1 + \mathcal{A}_t^2}$ is a bit more involved. If $X_i \in \mathcal{A}_t^i$ for $i = 1, 2$, then $\Phi_t(X_1 + X_2) \geq \Phi_t^1(X_1) + \Phi_t^2(X_2) \geq 0$ shows that $X_1 + X_2 \in \mathcal{A}_t^{1\Box 2}$, and because $\mathcal{A}_t^{1\Box 2}$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ by Theorem 2.3.19, we obtain

$$\overline{\mathcal{A}_t^1 + \mathcal{A}_t^2} \subseteq \mathcal{A}_t^{1\Box 2}.$$

For the converse inclusion, we claim that for all $Z \in \mathbf{L}_+^1$

$$\inf_{X\in\mathcal{A}_t^{1\Box 2}} E[ZX] = \inf_{X\in\mathcal{A}_t^1+\mathcal{A}_t^2} E[ZX] = \inf_{X\in\overline{\mathcal{A}_t^1+\mathcal{A}_t^2}} E[ZX]; \qquad (2.4.12)$$

note that the second equality follows from the first since we already know that $\mathcal{A}_t^1 + \mathcal{A}_t^2 \subseteq \overline{\mathcal{A}_t^1 + \mathcal{A}_t^2} \subseteq \mathcal{A}_t^{1\Box 2}$. Then if the inclusion "$\subseteq$" in (2.4.6) is not true, there exists some $X' \in \mathcal{A}_t^{1\Box 2} \setminus \overline{\mathcal{A}_t^1 + \mathcal{A}_t^2}$, and the Hahn-Banach theorem yields some $Z' \in \mathbf{L}^1$ with

$$\inf_{X\in\overline{\mathcal{A}_t^1+\mathcal{A}_t^2}} E[XZ'] > E[X'Z'] > -\infty. \qquad (2.4.13)$$

But since $-\big(\overline{\mathcal{A}_t^1 + \mathcal{A}_t^2}\big)$ is solid, we must have $Z' \geq 0$, and so (2.4.13) contradicts (2.4.12).

To complete the proof, it remains to establish (2.4.12). To that end, we first use Lemma 2.3.12, (2.4.5) and again Lemma 2.3.12 to obtain

$$\begin{aligned}
\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{1\Box 2}} E_Q[X|\mathcal{F}_t] &= \alpha_t^{1\Box 2}(Q)\\
&= \operatorname*{ess\,inf}_{X_1\in\mathcal{A}_t^1} E_Q[X_1|\mathcal{F}_t] + \operatorname*{ess\,inf}_{X_2\in\mathcal{A}_t^2} E_Q[X_2|\mathcal{F}_t]\\
&= \operatorname*{ess\,inf}_{X\in\mathcal{A}_t^1+\mathcal{A}_t^2} E_Q[X|\mathcal{F}_t] \quad \text{for all } Q \in \mathcal{P}_t^{\approx}. \qquad (2.4.14)
\end{aligned}$$

Now up to normalization, $\mathcal{P}_t^{\approx} \cap \mathcal{P}^a$ can be identified with

$$\mathcal{Z}_t := \left\{ Z \in \mathbf{L}^1_+ \,\middle|\, \text{ for all } A \in \mathcal{F}_t,\, P[A] > 0 \text{ implies } E[Z\mathbf{1}_A] > 0 \right\}.$$

Hence (2.4.14) implies that

$$\operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{1\Box 2}} E[ZX|\mathcal{F}_t] = \operatorname*{ess\,inf}_{X \in \mathcal{A}_t^1 + \mathcal{A}_t^2} E[ZX|\mathcal{F}_t] \quad \text{for all } Z \in \mathcal{Z}_t. \tag{2.4.15}$$

To extend this to all $Z \in \mathbf{L}^1_+$, fix $Z \in \mathbf{L}^1_+$ and define $B \in \mathcal{F}_t$ up to nullsets by $\mathbf{1}_B := \operatorname{ess\,sup}\{\mathbf{1}_A \mid A \in \mathcal{F}_t \text{ and } Z\mathbf{1}_A = 0\}$ so that $Z\mathbf{1}_{B^c} = Z$. Because Φ_t is representable, we have by Lemma 2.3.12 that

$$\mathbf{L}^\infty \ni -\Phi_t(0) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t^=} \alpha_t^{1\Box 2}(Q) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t^=} \Big(\operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{1\Box 2}} E_Q[X|\mathcal{F}_t] \Big)$$

and so there exists $Q' \in \mathcal{P}_t^=$ such that $\operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{1\Box 2}} E_{Q'}[X|\mathcal{F}_t] \in \mathbf{L}^\infty$. If Z'_T denotes the density of Q', then $\hat{Z} := Z'_T\mathbf{1}_B + Z\mathbf{1}_{B^c}$ is in $\mathcal{Z}_t$ and

$$\mathbf{1}_{B^c} E[ZX|\mathcal{F}_t] = \mathbf{1}_{B^c} E[\hat{Z}X|\mathcal{F}_t] \quad \text{for all } X \in \mathbf{L}^\infty. \tag{2.4.16}$$

Using $Z = Z\mathbf{1}_{B^c}$, (2.4.16), (2.4.15) for $\hat{Z}$ and then reversing the steps again yields

$$\operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{1\Box 2}} E[ZX|\mathcal{F}_t] = \operatorname*{ess\,inf}_{X \in \mathcal{A}_t^1 + \mathcal{A}_t^2} E[ZX|\mathcal{F}_t]$$

as desired. Because $\{E[ZX|\mathcal{F}_t] \,|\, X \in \mathcal{B}\}$ is a lattice for $\mathcal{B} \in \{\mathcal{A}_t^{1\Box 2}, \mathcal{A}_t^1 + \mathcal{A}_t^2\}$ by $\mathcal{F}_t$-regularity, we can interchange infimum and expectation to obtain

$$\inf_{X \in \mathcal{A}_t^{1\Box 2}} E[ZX] = \inf_{X \in \mathcal{A}_t^1 + \mathcal{A}_t^2} E[ZX]$$

for every $Z \in \mathbf{L}^1_+$. This establishes (2.4.12).

d) Suppose first that Φ^1 and Φ^2 are time-consistent. We may also assume that they are normalized, because the MCUFs $\hat{\Phi}_u^i(X) := \Phi_u^i(X) - \Phi_u^i(0)$ for $i = 1, 2$ are, we have $\Phi_u(X) = \hat{\Phi}_u^1 \Box \hat{\Phi}_u^2(X) + (\Phi_u^1(0) + \Phi_u^2(0))$, and time-consistency is not affected by translation. So let $s \le t$ and X_1, X_2 be such that

$$\Phi_t(X_1) = \Phi_t(X_2) = \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^2} \Phi_t^1(X_2 + Y). \tag{2.4.17}$$

By (2.4.4) it suffices to show that we then have

$$\Phi_s^1 \Box \mathcal{A}_s^2(X_1) = \operatorname*{ess\,sup}_{Y' \in -\mathcal{A}_s^2} \Phi_s^1(X_1 + Y') = \operatorname*{ess\,sup}_{Y' \in -\mathcal{A}_s^2} \Phi_s^1(X_2 + Y').$$

Now Lemma 2.3.25 implies that

$$\Phi_s^1(X) = \Phi_s^1\big(\Phi_t^1(X)\big) \quad \text{for } X \in \mathbf{L}^\infty, \tag{2.4.18}$$

$$\mathcal{A}_s^2 = \mathcal{A}_s^2(\mathcal{F}_t) + \mathcal{A}_t^2, \tag{2.4.19}$$

and Lemma 2.3.6 applied to $\mathcal{A}_t^2$ together with the $\mathcal{F}_t$-regularity of Φ_t^1 yields that $\{\Phi_t^1(X+Y) \mid Y \in -\mathcal{A}_t^2\}$ is a lattice for any $X \in \mathbf{L}^\infty$. Hence there is a sequence (Y_n) in $-\mathcal{A}_t^2$ such that $\operatorname*{ess\,sup}_{Y\in-\mathcal{A}_t^2} \Phi_t^1(X+Y) = \nearrow\text{-}\lim_{n\to\infty} \Phi_t^1(X+Y_n)$. Moreover, $\big(\Phi_t^1(X+Y_n)\big)_{n\in\mathbb{N}}$ is uniformly bounded due to (2.4.4) because

$$-\|X+Y_1\|_{\mathbf{L}^\infty} \le \Phi_t^1(X+Y_1) \le \Phi_t^1(X+Y_n) \le \operatorname*{ess\,sup}_{Y\in-\mathcal{A}_t^2} \Phi_t^1(X+Y) = \Phi_t(X)$$

and since $\Phi_t(X) \in \mathbf{L}^\infty$. Hence translation invariance and continuity from below of Φ_s^1 imply for any element $\hat{Y}$ of $\mathcal{A}_s^2(\mathcal{F}_t)$ that

$$\begin{aligned}
\Phi_s^1\Big(\operatorname*{ess\,sup}_{Y\in-\mathcal{A}_t^2} \Phi_t^1(X+Y+\hat{Y})\Big) &= \nearrow\text{-}\lim_{n\to\infty} \Phi_s^1\big(\Phi_t^1(X+Y_n)+\hat{Y}\big) \\
&\le \operatorname*{ess\,sup}_{Y\in-\mathcal{A}_t^2} \Phi_s^1\big(\Phi_t^1(X+Y)+\hat{Y}\big),
\end{aligned}$$

and by monotonicity of Φ_s^1, we even must have equality. Combining this with (2.4.18), (2.4.19) and using (2.4.17) to exchange X_1 for X_2, we get

$$\begin{aligned}
\operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_s^2} \Phi_s^1(X_1+Y') &= \operatorname*{ess\,sup}_{\hat{Y}\in-\mathcal{A}_s^2(\mathcal{F}_t)} \operatorname*{ess\,sup}_{Y\in-\mathcal{A}_t^2} \Phi_s^1\big(\Phi_t^1(X_1+Y+\hat{Y})\big) \\
&= \operatorname*{ess\,sup}_{\hat{Y}\in-\mathcal{A}_s^2(\mathcal{F}_t)} \Phi_s^1\Big(\operatorname*{ess\,sup}_{Y\in-\mathcal{A}_t^2} \Phi_t^1(X_2+Y)+\hat{Y}\Big) \\
&= \operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_s^2} \Phi_s^1(X_2+Y'),
\end{aligned}$$

where the last equality is obtained by doing the same steps in reverse order with X_1 replaced by X_2. This shows that Φ is time-consistent. If Φ^1, Φ^2 are strongly time-consistent, we have in addition $\mathcal{A}_t^i \subseteq \mathcal{A}_s^i$ for $t \ge s$ and $i = 1, 2$, and thus also $\overline{\mathcal{A}_t^1 + \mathcal{A}_t^2} \subseteq \overline{\mathcal{A}_s^1 + \mathcal{A}_s^2}$. Hence (2.4.6) implies that Φ is strongly time-consistent as well, and so d) is proved.

□

Remark 2.4.6. Parts of the proof of Theorem 2.4.3 are a straightforward generalization of the arguments for the (static) Theorem 3.6 in [BEK05]; this extends smoothly because thanks to the preparations in Section 2.3, we can appeal to the dynamic representations in Theorem 2.3.16 and 2.3.19 instead of their static counterpart. Exceptions are the parts where we show (2.4.6) and the assertions b) and d). ◇

If Φ_t^1 is an MCUF and $\mathcal{B}$ a pre-acceptance set at time t, Lemma 2.4.5 implies that $\Phi_t := \Phi_t^1 \square \mathcal{B}$ is again an MCUF, provided that $\Phi_t(0) \in \mathbf{L}^\infty$. In the sequel, we want to have a maximum of good properties for that Φ_t with a minimum of assumptions on $\mathcal{B}$. To make this more precise, recall from Lemma 2.3.9 the MCUF $\Phi_t^{\mathcal{B}}$ associated to $\mathcal{B}$. By (2.4.4), it seems natural to expect that $\Phi_t^1 \square \mathcal{B} = \Phi_t^1 \square \Phi_t^{\mathcal{B}}$ and that the acceptance set of Φ_t should be $\overline{\mathcal{A}_t^1 + \mathcal{B}}$ in view of (2.4.6). However, this can be deduced from the preceding results only if Φ_t^1 is continuous from below and $\mathcal{B}$ is the acceptance set of $\Phi_t^{\mathcal{B}}$, e.g., if $\mathcal{B}$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$. Because the latter is often hard to check, we do not want to make that assumption. So we first work with the $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closure $\overline{\mathcal{B}}$ of $\mathcal{B}$ since we have precise results for $\Phi_t^1 \square \Phi_t^{\overline{\mathcal{B}}}$, and then show that the latter coincides with $\Phi_t^1 \square \mathcal{B}$.

The program sketched above is carried out in the next result. This in turn is used below in Section 2.6 when we study indifference valuation.

Proposition 2.4.7. *Let $\mathcal{B}$ be a pre-acceptance set and Φ_t^1 an MCUF at time t with acceptance set $\mathcal{A}_t^1$ and concave conjugate α_t^1. Denote by $\overline{\mathcal{B}}$ the closure of $\mathcal{B}$ in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$. If $\Phi_t^1 \square \mathcal{B}(0) = \operatorname{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(Y) \in \mathbf{L}^\infty$, then*

$$\Phi_t^1 \square \mathcal{B} = \Phi_t^1 \square \Phi_t^{\mathcal{B}}. \tag{2.4.20}$$

If in addition Φ_t^1 is continuous from below and

$$\operatorname{ess\,sup}\big(- \overline{\mathcal{B}} \cap \mathbf{L}^\infty(\mathcal{F}_t)\big) \in \mathbf{L}^\infty, \tag{2.4.21}$$

then

$$\Phi_t^1 \square \mathcal{B} = \Phi_t^1 \square \Phi_t^{\overline{\mathcal{B}}}. \tag{2.4.22}$$

In particular, $\Phi_t := \Phi_t^1 \square \mathcal{B}$ is then continuous from below with concave conjugate

$$\alpha_t(Q) = \alpha_t^1(Q) + \alpha_t^{\overline{\mathcal{B}}}(Q) := \alpha_t^1(Q) + \operatorname*{ess\,inf}_{Y \in \overline{\mathcal{B}}} E_Q[Y \mid \mathcal{F}_t] \tag{2.4.23}$$

and acceptance set

$$\mathcal{A}_t = \overline{\mathcal{A}_t^1 + \overline{\mathcal{B}}} = \overline{\mathcal{A}_t^1 + \mathcal{B}}.$$

Proof. If $\mathcal{A}_t^{\mathcal{B}}$ denotes the acceptance set of $\Phi_t^{\mathcal{B}}$, then $\mathcal{B} \subseteq \mathcal{A}_t^{\mathcal{B}}$ so that (2.4.4) implies

$$\Phi_t^1 \square \Phi_t^{\mathcal{B}}(X) = \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^{\mathcal{B}}} \Phi_t^1(X+Y) \geq \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y) = \Phi_t^1 \square \mathcal{B}(X). \tag{2.4.24}$$

Since $\Phi_t^{\mathcal{B}}$ is non-negative on $\mathcal{A}_t^{\mathcal{B}}$, (2.4.4) also yields

$$\begin{aligned}\Phi_t^1 \square \Phi_t^{\mathcal{B}}(X) &\leq \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^{\mathcal{B}}} \left(\Phi_t^1(X+Y) + \Phi_t^{\mathcal{B}}(-Y)\right) \\ &\leq \operatorname*{ess\,sup}_{Y \in \mathbf{L}^\infty} \left(\Phi_t^1(X+Y) + \Phi_t^{\mathcal{B}}(-Y)\right) \\ &= \Phi_t^1 \square \Phi_t^{\mathcal{B}}(X)\end{aligned}$$

so that $\Phi_t^1 \square \Phi_t^{\mathcal{B}}(X) = \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^{\mathcal{B}}} \left(\Phi_t^1(X+Y) + \Phi_t^{\mathcal{B}}(-Y)\right)$. In view of (2.4.24), it thus suffices to show that for each $Y' \in -\mathcal{A}_t^{\mathcal{B}}$,

$$\Phi_t^1(X+Y') + \Phi_t^{\mathcal{B}}(-Y') \leq \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y). \tag{2.4.25}$$

Pick a sequence (m_t^n) in $\mathbf{L}^\infty(\mathcal{F}_t)$ and an $\mathcal{F}_t$-partition (A_n) with $-Y' - m_t^n \in \mathcal{B}$ and

$$\Phi_t^{\mathcal{B}}(-Y') \leq \sum_{n=1}^{\infty} \mathbf{1}_{A_n} m_t^n + \varepsilon,$$

for a fixed $\varepsilon > 0$. Then translation invariance of Φ_t^1 implies that

$$\begin{aligned}\operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y) &= \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y) \\ &\geq \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \Phi_t^1(X+Y'+m_t^n) \\ &= \Phi_t^1(X+Y') + \sum_{n=1}^{\infty} \mathbf{1}_{A_n} m_t^n \\ &\geq \Phi_t^1(X+Y') + \Phi_t^{\mathcal{B}}(-Y') - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves (2.4.25) and hence (2.4.20).

If we now assume (2.4.21), $\overline{\mathcal{B}}$ is like $\mathcal{B}$ acceptable at time t and thus by Lemma 2.3.9 the acceptance set of the MCUF $\Phi_t^{\overline{\mathcal{B}}}$. So it is enough to prove (2.4.22) because all claimed properties then follow from Theorem 2.4.3 and Lemma 2.3.12, and as $\Phi_t := \Phi_t^1 \square \mathcal{B}$ and $\Phi_t^1 \square \Phi_t^{\overline{\mathcal{B}}}$ both are MCUFs at time t, they coincide if their acceptance sets $\mathcal{A}_t$ and $\overline{\mathcal{A}_t^1 + \mathcal{B}} = \mathcal{A}_t^1 + \overline{\mathcal{B}}$ agree. By the assumptions and Lemma 2.4.5, Φ_t is continuous from below, so that $\mathcal{A}_t$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ by Lemma 2.3.14 and Theorem 2.3.19. Because the definition of Φ_t gives $\mathcal{A}_t^1 + \mathcal{B} \subseteq \mathcal{A}_t$, we obtain $\overline{\mathcal{A}_t^1 + \mathcal{B}} \subseteq \mathcal{A}_t$, and the converse inclusion is trivial since (2.4.2) and (2.4.4) with $\mathcal{A}_t^2 = \overline{\mathcal{B}}$ give

$$\Phi_t(X) \leq \operatorname*{ess\,sup}_{Y \in -\overline{\mathcal{B}}} \Phi_t^1(X+Y) = \Phi_t^1 \square \Phi_t^{\overline{\mathcal{B}}}(X) \quad \text{for } X \in \mathbf{L}^\infty.$$

This completes the proof. □

2.5 Superhedging under constraints

This section deals with superhedging under constraints. The results presented here are slight modifications of those Föllmer and Kramkov proved in [FK97]. We obtain the existence of a minimal hedging portfolio for a given payoff if trading is constrained, and we provide a representation of the value process corresponding to this portfolio. These results will be very helpful in Section 2.6. There we consider a DMCUF Φ representing the preferences of some agent and assume that she gets the possibility to trade in a financial market, possibly under some constraints. Then we use the value process of the minimal hedging portfolio to construct a strongly time-consistent DMCUF which allows us to capture the effects on the agent's preference order of the trading opportunities.

In this section, all processes (except for integrands of stochastic integrals) are assumed to be RCLL and adapted with respect to the given filtration $\mathbb{F}$. For two such processes U and V, the relation $U \preceq V$ means that $V - U$ is an increasing process. We model the discounted price process of some traded assets by a locally bounded $\mathbb{R}^d$-valued P-semimartingale $S = (S_t)_{0 \le t \le T}$. Before we can state the main theorem of this section, we need to specify the set of strategies allowed for trading and provide some technical results which are required for its proof.

Definition 2.5.1. We denote by $L(S)$ the set of all $\mathbb{R}^d$-valued predictable processes $H = (H_t)_{0 \le t \le T}$ which are S-integrable, and call $H \in L(S)$ an *admissible strategy* if the process $(\int_0^t H_u \, dS_u)_{0 \le t \le T}$ is locally bounded from below. The set of all admissible strategies is denoted by $L^a_{\mathrm{loc}}(S)$. We call a triple (x, H, K) an *admissible portfolio* if $x \in \mathbb{R}$, $H \in L^a_{\mathrm{loc}}(S)$ and $K = (K_t)_{0 \le t \le T}$ is an adapted RCLL increasing process with $K_0 = 0$. The corresponding *value process* is defined by

$$V_t = x + \int_0^t H_s \, dS_s - K_t, \quad t \in [0, T].$$

The economic interpretation of an admissible portfolio (x, H, K) is very simple: x gives the initial capital of the portfolio, H specifies the number of units of each asset held in the portfolio, and K models cumulative consumption.

If trading is not constrained, every admissible strategy can be used for trading. However, we want to allow for trading constraints. For technical reasons we need to impose some closedness properties on the set of allowed hedging strategies. To that end, we recall the *Émery distance* between two real-valued semimartingales N^1 and N^2, defined as

$$D(N^1, N^2) = \sup_{|J| \le 1} E\left[1 \wedge \int_0^T J_s \, d(N^1 - N^2)_s\right],$$

where the supremum is taken over all predictable processes J which are uniformly bounded by 1. By Theorem 5.4 of [Mem80], the space $L(S)$ is complete with respect to the metric

$$d_S(H^1, H^2) = D\left(\int H^1\,dS, \int H^2\,dS\right).$$

Definition 2.5.2. We call a subset $\mathcal{H}$ of $L^a_{\mathrm{loc}}(S)$ an *admissible hedging set* if it contains $H \equiv 0$, is closed in $L^a_{\mathrm{loc}}(S)$ with respect to the metric d_S and is *predictably convex*, i.e., for any $H^1, H^2 \in \mathcal{H}$ and any $[0,1]$-valued predictable process $h = (h_t)_{0\le t\le T}$, the process $hH^1 + (1-h)H^2$ belongs to $\mathcal{H}$. An admissible portfolio (x, H, K) is called *$\mathcal{H}$-constrained* if $H \in \mathcal{H}$.

Remark 2.5.3. i) Note that $\mathcal{H}$ need not be closed under addition or multiplication by scalars in general.

ii) Since $\mathcal{H}$ is predictably convex and contains 0, we have for every $H \in \mathcal{H}$ and every stopping time τ that also $H' := H\mathbf{1}_{]\!]\tau,T]\!]} \in \mathcal{H}$. In addition, for any $H \in L^a_{\mathrm{loc}}(S)$, such H' is also in $L^a_{\mathrm{loc}}(S)$. More generally, if N is any process which is locally bounded from below and τ is any stopping time, then $N' := N - N^\tau$ is again locally bounded from below. To see this, assume for simplicity that $N_0 = 0$, $N \ge 0$ and for some $n \in \mathbb{N}$ define $\sigma := \inf\{t \ge 0 \mid N_t \ge n\}$. Then on $\{t \ge \tau\}$ we have $N'_{t\wedge\sigma} = N_{t\wedge\sigma} - N_{\tau\wedge\sigma} \ge 0 - n$, since on $\{\tau < \sigma\}$ we have $N_{\tau\wedge\sigma} \le n$ and on $\{t \ge \tau \ge \sigma\}$ we have $N_{t\wedge\sigma} - N_{\tau\wedge\sigma} = N_\sigma - N_\sigma = 0$. ◇

Definition 2.5.4. For a payoff $X \in \mathbf{L}^\infty$, we call an $\mathcal{H}$-constrained portfolio (x, H, K) an *$\mathcal{H}$-constrained hedging portfolio* for X if its value process V is uniformly bounded from below and satisfies $V_T \ge X$. An $\mathcal{H}$-constrained portfolio $(\hat{x}, \hat{H}, \hat{K})$ for X with value process $\hat{V}$ is called *minimal $\mathcal{H}$-constrained hedging portfolio* for X if

$$\hat{V}_t \le V_t \quad \text{for all } t \in [0,T]$$

for any $\mathcal{H}$-constrained hedging portfolio for X with value process V.

One central auxiliary result is a characterization of value processes corresponding to $\mathcal{H}$-constrained portfolios. For its formulation, we need to introduce some additional notation. Moreover, we make the following assumption to ensure that the market does not provide any arbitrage opportunities.

Assumption (NFLVR): There exists $\hat{Q} \in \mathcal{P}^e$ such that S is a local $\hat{Q}$-martingale.

Let us fix an admissible hedging set $\mathcal{H}$ and introduce the family of semimartingales

$$\mathcal{S} = \left\{ \int H\,dS \,\middle|\, H \in \mathcal{H} \right\}.$$

Definition 2.5.5. Let $\mathcal{R}(\mathcal{S})$ denote the class of all $Q \in \mathcal{P}^e$ for which there exists an increasing predictable process A (depending on Q and $\mathcal{S}$) such that $N - A$ is a local Q-supermartingale for any $N \in \mathcal{S}$, i.e.,

$$A^N(Q) \preceq A \text{ for all } N \in \mathcal{S}, \tag{2.5.1}$$

where $A^N(Q)$ is the predictable process of finite variation in the canonical decomposition of N under Q. Then we call an increasing predictable process $A^{\mathcal{S}}(Q)$ the *upper variation process* of $\mathcal{S}$ under Q if it satisfies (2.5.1) and is minimal with respect to this property in the sense that $A^{\mathcal{S}}(Q) \preceq A$ for any increasing predictable process A satisfying (2.5.1).

Remark 2.5.6. a) The set $\mathcal{R}(\mathcal{S})$ is denoted by $\mathcal{P}(\mathcal{S})$ in [FK97]. However, we changed notation to avoid confusion with the set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$.

b) Note that **(NFLVR)** ensures that $\mathcal{R}(\mathcal{S}) \neq \emptyset$. In fact, since $\mathcal{H} \subseteq L^a_{\text{loc}}(S)$, if $\hat{Q} \in \mathcal{P}^e$ is a local martingale measure for S, then each $N \in \mathcal{S}$ is even a local $\hat{Q}$-martingale by Corollary 3.5 in [AS94]. Hence $A^N(\hat{Q}) \equiv 0$ so that $A^{\mathcal{S}}(\hat{Q}) \equiv 0$. ◇

Example 2.5.7. If $\mathcal{H} = L^a_{\text{loc}}(S)$, i.e., in the case of unconstrained trading, it is well known that $\mathcal{R}(\mathcal{S})$ is just the set $\mathcal{M}^e(S)$ of all equivalent local martingale measures for S. Indeed, this can also be seen from Remark 2.5.6 and Lemma 2.6.15 below. As shown in Remark 2.5.6, we then have $A^{\mathcal{S}}(Q) \equiv 0$ for all $Q \in \mathcal{R}(\mathcal{S}) = \mathcal{M}^e(S)$. Further examples can be found in [FK97]. ◇

Lemma 2.1 of [FK97], which characterizes $\mathcal{R}(\mathcal{S})$ and the upper variation processes $A^{\mathcal{S}}(Q)$, reads as follows:

Lemma 2.5.8. *A probability measure $Q \in \mathcal{P}^e$ belongs to $\mathcal{R}(\mathcal{S})$ if and only if all $N \in \mathcal{S}$ are special semimartingales under Q and* $\operatorname{ess\,sup}_{N \in \mathcal{S}} A^N(Q)_t < \infty$ *P-a.s. for all $t \in [0, T]$. In this case the upper variation process exists and is uniquely determined by the equations*

$$A^{\mathcal{S}}(Q)_\tau = \operatorname*{ess\,sup}_{N \in \mathcal{S}} A^N(Q)_\tau, \tag{2.5.2}$$

$$E\left[A^{\mathcal{S}}(Q)_\tau\right] = \sup_{N \in \mathcal{S}} E\left[A^N(Q)_\tau\right] \tag{2.5.3}$$

for all stopping times $\tau \leq T$. Moreover, there exists a sequence $(N^n)_{n \in \mathbb{N}} \subseteq \mathcal{S}$ such that the compensators $A^n := A^{N^n}(Q)$ satisfy $A^n \preceq A^{n-1}$ and

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} \left(A^{\mathcal{S}}(Q)_t - A^n_t\right) = 0 \quad P\text{-a.s.}$$

Remark 2.5.9. Equation (2.5.3) is not really required for the characterization of $A^{\mathcal{S}}(Q)$ as it is a consequence of (2.5.2). In fact, Föllmer and Kramkov show in the proof of their Lemma 2.1 that for fixed $Q \in \mathcal{R}(\mathcal{S})$, the space of compensators $\{A^N(Q) \mid N \in \mathcal{S}\}$ is directed upwards. This implies that $\{A^N(Q)_\tau \mid N \in \mathcal{S}\}$ is directed upwards for any stopping time $\tau \leq T$ so that (2.5.2) implies (2.5.3). ◇

In order to manipulate the upper variation process we require the following result:

Lemma 2.5.10. *Fix a stopping time $\tau \leq T$, a set $B \in \mathcal{F}_\tau$ and probability measures Q^1, Q^2, $\tilde{Q} \in \mathcal{R}(\mathcal{S})$, and denote by $\tilde{Z}^1$, $\tilde{Z}^2$ the density processes of Q^1, Q^2 with respect to $\tilde{Q}$. Then*

$$\frac{d\overline{Q}}{d\tilde{Q}} := \tilde{Z}^1_T \mathbf{1}_B + \tilde{Z}^1_\tau \frac{\tilde{Z}^2_T}{\tilde{Z}^2_\tau} \mathbf{1}_{B^c}$$

defines a probability measure $\overline{Q} \in \mathcal{R}(\mathcal{S})$ such that $\overline{Q} = Q^1$ on $\mathcal{F}_\tau$ and

$$E_{\overline{Q}}[\,.\,|\mathcal{F}_t] \;=\; E_{Q^1}[\,.\,|\mathcal{F}_t]\mathbf{1}_B + E_{Q^2}[\,.\,|\mathcal{F}_t]\mathbf{1}_{B^c} \quad \text{on } \{t > \tau\}. \tag{2.5.4}$$

The upper variation process of $\mathcal{S}$ under $\overline{Q}$ can be written as

$$\begin{aligned} A^{\mathcal{S}}(\overline{Q})_u \;=\; & \Big(\Big(A^{\mathcal{S}}(Q^1)_u - A^{\mathcal{S}}(Q^1)_\tau\Big)\mathbf{1}_B + \Big(A^{\mathcal{S}}(Q^2)_u - A^{\mathcal{S}}(Q^2)_\tau\Big)\mathbf{1}_{B^c}\Big)\mathbf{1}_{\{u>\tau\}} \\ & + A^{\mathcal{S}}(Q^1)_{u\wedge\tau}. \end{aligned} \tag{2.5.5}$$

Proof. That $\overline{Q} = Q^1$ on $\mathcal{F}_\tau$ is obvious. To see (2.5.4), denote by $\overline{Z}$ the density process of $\overline{Q}$ with respect to $\tilde{Q}$ and note that $\tilde{Z}^i_\tau \mathbf{1}_{\{t>\tau\}}$ is $\mathcal{F}_t$-measurable for $i = 1, 2$, so that

$$\overline{Z}_t \mathbf{1}_{\{t>\tau\}} = \left(\tilde{Z}^1_t \mathbf{1}_B + \tilde{Z}^1_\tau \frac{\tilde{Z}^2_t}{\tilde{Z}^2_\tau} \mathbf{1}_{B^c}\right)\mathbf{1}_{\{t>\tau\}}.$$

Then

$$\frac{\overline{Z}_T}{\overline{Z}_t}\mathbf{1}_{\{t>\tau\}} = \left(\frac{\tilde{Z}^1_T}{\tilde{Z}^1_t}\mathbf{1}_B + \frac{\tilde{Z}_T}{\tilde{Z}^2_t}\mathbf{1}_{B^c}\right)\mathbf{1}_{\{t>\tau\}}$$

yields (2.5.4). From this it is easy to check that for any $N \in \mathcal{S}$ the finite variation process in the canonical decomposition of N under $\overline{Q}$ is given by

$$\begin{aligned} A^N(\overline{Q})_u \;=\; & \Big(\Big(A^N(Q^1)_u - A^N(Q^1)_\tau\Big)\mathbf{1}_B + \Big(A^N(Q^2)_u - A^N(Q^2)_\tau\Big)\mathbf{1}_{B^c}\Big)\mathbf{1}_{\{u>\tau\}} \\ & + A^N(Q^1)_{u\wedge\tau}. \end{aligned} \tag{2.5.6}$$

By Lemma 2.5.8, if $A^{\mathcal{S}}(\overline{Q})$ exists, then it is given by (2.5.2). Hence (2.5.6) implies that $A^{\mathcal{S}}(\overline{Q}) = A^{\mathcal{S}}(Q^1)$ on the stochastic interval $[\![0, \tau]\!]$ and we are left to consider the increments after τ, i.e., to show that

$$\begin{aligned}&\left(A^{\mathcal{S}}(\overline{Q})_u - A^{\mathcal{S}}(\overline{Q})_\tau\right)\mathbf{1}_{\{u>\tau\}} \\ &\quad= \left(\left(A^{\mathcal{S}}(Q^1)_u - A^{\mathcal{S}}(Q^1)_\tau\right)\mathbf{1}_B + \left(A^{\mathcal{S}}(Q^2)_u - A^{\mathcal{S}}(Q^2)_\tau\right)\mathbf{1}_{B^c}\right)\mathbf{1}_{\{u>\tau\}}.\end{aligned} \tag{2.5.7}$$

For this, we note that for every $N^1, N^2 \in \mathcal{S}$, we also have $N^1\mathbf{1}_{[\![0,\tau]\!]} + N^2\mathbf{1}_{]\!]\tau,T]\!]} \in \mathcal{S}$ since $\mathcal{H}$ is predictably convex. This yields for any $Q \in \mathcal{R}(\mathcal{S})$ that

$$\begin{aligned}\operatorname*{ess\,sup}_{N\in\mathcal{S}} A^N(Q)_u &= \operatorname*{ess\,sup}_{N\in\mathcal{S}}\left\{A^N(Q)_u - A^N(Q)_\tau + A^N(Q)_\tau\right\} \\ &= \operatorname*{ess\,sup}_{N\in\mathcal{S}}\left\{A^N(Q)_u - A^N(Q)_\tau\right\} + \operatorname*{ess\,sup}_{N\in\mathcal{S}} A^N(Q)_\tau\end{aligned}$$

on $\{u > \tau\}$ so that (2.5.2) and (2.5.6) imply that on $\{u > \tau\} \cap B^c$, we have

$$\begin{aligned}A^{\mathcal{S}}(\overline{Q})_u &= \operatorname*{ess\,sup}_{N\in\mathcal{S}}\left\{A^N(Q^2)_u - A^N(Q^2)_\tau + A^N(Q^1)_\tau\right\} \\ &= \operatorname*{ess\,sup}_{N\in\mathcal{S}}\left\{A^N(Q^2)_u - A^N(Q^2)_\tau\right\} + \operatorname*{ess\,sup}_{N\in\mathcal{S}} A^N(Q^1)_\tau \\ &= \operatorname*{ess\,sup}_{N\in\mathcal{S}}\left\{A^N(Q^2)_u - A^N(Q^2)_\tau\right\} + A^{\mathcal{S}}(Q^1)_\tau \\ &= A^{\mathcal{S}}(Q^2)_u - A^{\mathcal{S}}(Q^2)_\tau + A^{\mathcal{S}}(Q^1)_\tau.\end{aligned}$$

As an analogous equality holds on $\{u > \tau\} \cap B$, this proves (2.5.7) and hence (2.5.5). In addition, existence of the upper variation process of $\mathcal{S}$ under $\overline{Q}$ implies by Lemma 2.5.8 that $\overline{Q} \in \mathcal{R}(\mathcal{S})$. □

One of our goals in the next section is the construction of a certain DMCUF from the minimal $\mathcal{H}$-constrained hedging portfolio. The key tool for this is the main result of this section, which is a slight modification of Proposition 4.1 in [FK97]:

Theorem 2.5.11. *For any $X \in \mathbf{L}^\infty$ there exists a minimal $\mathcal{H}$-constrained hedging portfolio $(\hat{x}, \hat{H}, \hat{K})$. Its value process equals*

$$\begin{aligned}\hat{V}_t &= \hat{x} + \int_0^t \hat{H}_s\, dS_s - \hat{K}_t \\ &= \operatorname*{ess\,sup}_{Q\in\mathcal{R}(\mathcal{S})}\left\{E_Q[X|\mathcal{F}_t] - E_Q\left[A^{\mathcal{S}}(Q)_T - A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t\right]\right\}\end{aligned} \tag{2.5.8}$$

and is in particular uniformly bounded.

Remark 2.5.12. i) We can immediately see from Example 2.5.7 that in the unconstrained case, (2.5.8) becomes the well-known representation of the superhedging price process as

$$\hat{V}_t = \operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(S)} E_Q[X|\mathcal{F}_t].$$

ii) There are some differences between our work and [FK97]. First of all, Föllmer and Kramkov consider non-negative random variables as payoffs whereas we impose that payoffs are in $\mathbf{L}^\infty$. A more significant difference is that we allow the value process of $\mathcal{H}$-constrained hedging portfolios for a payoff X to be bounded from below by an arbitrary constant (depending on X), whereas Föllmer and Kramkov fix the lower bound at 0. This causes some changes in the results, and some arguments in the proof become a bit more involved. For related results see also Theorem 5.5 in [DS98] and Theorem 5.1 in [DS99].

◇

The proof of Theorem 2.5.11 strongly relies on Theorem 4.1 of [FK97] which we state next:

Theorem 2.5.13. *Consider a process V which is locally bounded from below. Then the following statements are equivalent:*

a) V is the value process of some $\mathcal{H}$-constrained portfolio (V_0, H, K), i.e.,

$$V = V_0 + \int H\,dS - K.$$

b) For all $Q \in \mathcal{R}(\mathcal{S})$, the process $V - A^{\mathcal{S}}(Q)$ is a local Q-supermartingale.

As a second auxiliary result for the proof of Theorem 2.5.11, we require the following Lemma 2.5.14, which is similar to Lemma A.1 from [FK97].

Lemma 2.5.14. *For each $X \in \mathbf{L}^\infty$, there exists a uniformly bounded (RCLL adapted) process $V = (V_t)_{0 \le t \le T}$ such that for all stopping times $\tau \le T$*

$$V_\tau = \operatorname*{ess\,sup}_{Q \in \mathcal{R}(\mathcal{S})} \left\{ E_Q[X|\mathcal{F}_\tau] - E_Q\left[A^{\mathcal{S}}(Q)_T - A^{\mathcal{S}}(Q)_\tau \middle| \mathcal{F}_\tau \right] \right\} \quad P\text{-a.s.} \qquad (2.5.9)$$

Moreover, the process $V - A^{\mathcal{S}}(\tilde{Q})$ is a local $\tilde{Q}$-supermartingale for each $\tilde{Q} \in \mathcal{R}(\mathcal{S})$.

Proof. Define via the RHS of (2.5.9) a family of random variables U_τ, indexed by the set of all stopping times $\tau \le T$. Note that the family U_τ is uniformly bounded. Indeed, by (**NFLVR**) there exists an equivalent local martingale measure $\hat{Q}$ for S so that in particular $\hat{Q} \in \mathcal{R}(\mathcal{S})$ and $A^{\mathcal{S}}(\hat{Q}) \equiv 0$. Then boundedness follows immediately from the definition of U_τ since

$$-\|X\|_{\mathbf{L}^\infty} \leq E_{\hat{Q}}[X|\mathcal{F}_\tau] \leq U_\tau \leq \|X\|_{\mathbf{L}^\infty}. \tag{2.5.10}$$

1) Fix $\tilde{Q} \in \mathcal{R}(\mathcal{S})$, $t \in [0, T]$ and stopping times $\sigma \leq t$, $\tau \leq T$ such that the stopped upper variation process $A^{\mathcal{S}}(\tilde{Q})^\tau$ is bounded. We first show that

$$\begin{aligned} E_{\tilde{Q}}[U_{t\wedge\tau}|\mathcal{F}_\sigma] &= \operatorname*{ess\,sup}_{Q\in\mathcal{R}(\mathcal{S})_{t\wedge\tau}} E_Q\left[X - A^{\mathcal{S}}(Q)_T \,\middle|\, \mathcal{F}_{\sigma\wedge\tau}\right] \\ &\quad + E_{\tilde{Q}}\left[A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau} \,\middle|\, \mathcal{F}_{\sigma\wedge\tau}\right], \end{aligned} \tag{2.5.11}$$

where

$$\mathcal{R}(\mathcal{S})_{t\wedge\tau} := \left\{ Q \in \mathcal{R}(\mathcal{S}) \,\middle|\, Q = \tilde{Q} \text{ on } \mathcal{F}_{t\wedge\tau} \right\}.$$

For abbreviation we introduce on $\mathcal{R}(\mathcal{S})$ the operator

$$F(Q) := E_Q[X|\mathcal{F}_{t\wedge\tau}] - E_Q\left[A^{\mathcal{S}}(Q)_T - A^{\mathcal{S}}(Q)_{t\wedge\tau} \,\middle|\, \mathcal{F}_{t\wedge\tau}\right]. \tag{2.5.12}$$

Moreover, in part 1) of this proof we express all densities and density processes with respect to $\tilde{Q}$. To see (2.5.11), we first note that

$$U_{t\wedge\tau} = \operatorname*{ess\,sup}_{Q\in\mathcal{R}(\mathcal{S})} F(Q) = \operatorname*{ess\,sup}_{Q\in\mathcal{R}(\mathcal{S})_{t\wedge\tau}} F(Q). \tag{2.5.13}$$

In fact, take $Q \in \mathcal{R}(\mathcal{S})$ with density process $(Z_u)_{0\leq u\leq T}$ and define a new measure $\overline{Q}$ by the density

$$\overline{Z}_T := \frac{Z_T}{Z_{t\wedge\tau}} \tag{2.5.14}$$

with respect to $\tilde{Q}$. Then Lemma 2.5.10 implies that $\overline{Q} \in \mathcal{R}(\mathcal{S})_{t\wedge\tau}$ and

$$A^{\mathcal{S}}(\overline{Q})_u = \left(A^{\mathcal{S}}(Q)_u - A^{\mathcal{S}}(Q)_{t\wedge\tau}\right)\mathbf{1}_{\{u>(t\wedge\tau)\}} + A^{\mathcal{S}}(\tilde{Q})_{\{u\wedge(t\wedge\tau)\}},$$

and we have $E_{\overline{Q}}[\,.\,|\mathcal{F}_{t\wedge\tau}] = E_Q[\,.\,|\mathcal{F}_{t\wedge\tau}]$ so that (2.5.13) holds. Next we note that the set $\{F(Q) \mid Q \in \mathcal{R}(\mathcal{S})_{t\wedge\tau}\}$ is a lattice, since for any $Q^1, Q^2 \in \mathcal{R}(\mathcal{S})_{t\wedge\tau}$ with densities Z_T^1, Z_T^2 and any $B \in \mathcal{F}_{t\wedge\tau}$ we can define a probability measure Q' by $Z'_T := Z_T^1\mathbf{1}_B + Z_T^2\mathbf{1}_{B^c}$ to obtain from Lemma 2.5.10 that $Q' \in \mathcal{R}(\mathcal{S})_{t\wedge\tau}$ and that $F(Q') = F(Q^1)\mathbf{1}_B + F(Q^2)\mathbf{1}_{B^c}$. This guarantees ([Nev75]) by (2.5.13) the existence of some sequence $(Q^m)_{m\in\mathbb{N}} \subseteq \mathcal{R}(\mathcal{S})_{t\wedge\tau}$ such that

$$U_{t\wedge\tau} = \operatorname*{ess\,sup}_{Q\in\mathcal{R}(\mathcal{S})_{t\wedge\tau}} F(Q) = \nearrow\text{-}\lim_{m\to\infty} F(Q^m). \tag{2.5.15}$$

To finish the proof of (2.5.11), we recall that by **(NFLVR)** there exists an equivalent local martingale measure $\hat{Q}$ for S. By Remark 2.5.6, we have $\hat{Q} \in \mathcal{R}(\mathcal{S})$ and $A^{\mathcal{S}}(\hat{Q}) \equiv 0$. If we denote the density process of $\hat{Q}$ by $\hat{Z}$, we can define as in (2.5.14) a probability measure $\overline{Q} \in \mathcal{R}(\mathcal{S})_{t\wedge\tau}$ by the density

$$\overline{Z}_T := \frac{\hat{Z}_T}{\hat{Z}_{t\wedge\tau}}.$$

Since $\{ F(Q) \mid Q \in \mathcal{R}(\mathcal{S})_{t\wedge\tau}\}$ is a lattice, we can assume without loss of generality that in the above sequence $Q^1 = \overline{Q}$. By Lemma 2.5.10,

$$A^{\mathcal{S}}(\overline{Q})_T - A^{\mathcal{S}}(\overline{Q})_{t\wedge\tau} = A^{\mathcal{S}}(\hat{Q})_T - A^{\mathcal{S}}(\hat{Q})_{t\wedge\tau} = 0$$

so that we can apply the monotone convergence theorem to obtain from (2.5.15), (2.5.12) and since $\tilde{Q} = Q^m$ on $\mathcal{F}_{t\wedge\tau}$ that

$$\begin{aligned} &E_{\tilde{Q}}[U_{t\wedge\tau}|\mathcal{F}_\sigma] \\ &\quad= E_{\tilde{Q}}\left[\nearrow\text{-}\lim_{m\to\infty} F(Q^m)\middle|\,\mathcal{F}_\sigma\right] \\ &\quad= \nearrow\text{-}\lim_{m\to\infty} E_{\tilde{Q}}\left[F(Q^m)\middle|\,\mathcal{F}_\sigma\right] \\ &\quad= \nearrow\text{-}\lim_{m\to\infty} E_{Q^m}\left[F(Q^m)\middle|\,\mathcal{F}_\sigma\right] \\ &\quad\leq \operatorname*{ess\,sup}_{Q\in\mathcal{R}(\mathcal{S})_{t\wedge\tau}} E_Q\left[X - A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_{t\wedge\tau}\middle|\,\mathcal{F}_{\sigma\wedge\tau}\right]. \end{aligned} \tag{2.5.16}$$

As the converse inequality is trivial due to (2.5.13), we even get equality in (2.5.16). This implies (2.5.11) since for any $Q \in \mathcal{R}_{t\wedge\tau}$ with density process Z we have $Q = \tilde{Q}$ on $\mathcal{F}_{t\wedge\tau}$ and $Z_T = \frac{Z_T}{Z_{t\wedge\tau}}$ so that by Lemma 2.5.10

$$A^{\mathcal{S}}(Q)_{t\wedge\tau} = A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau}. \tag{2.5.17}$$

2) As in 1), we fix $\tilde{Q} \in \mathcal{R}(\mathcal{S})$, $t \in [0, T]$ and stopping times $\sigma \leq t$, $\tau \leq T$ such that the stopped process $A^{\mathcal{S}}(\tilde{Q})^\tau$ is uniformly bounded. We show the following supermartingale property for the family $\left(U_{t\wedge\tau} - A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau}\right)_{0\leq t\leq T}$:

$$E_{\tilde{Q}}\left[U_{t\wedge\tau} - A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau}\middle|\,\mathcal{F}_\sigma\right] \leq U_{\sigma\wedge\tau} - A^{\mathcal{S}}(\tilde{Q})_{\sigma\wedge\tau}. \tag{2.5.18}$$

Indeed, since $\sigma \leq t$ implies that $\mathcal{R}(\mathcal{S})_{t\wedge\tau} \subseteq \mathcal{R}(\mathcal{S})_{\sigma\wedge\tau}$, we get from (2.5.11) that

$$\begin{aligned} E_{\tilde{Q}}[U_{t\wedge\tau}|\mathcal{F}_\sigma] \leq\ & \operatorname*{ess\,sup}_{Q\in\mathcal{R}(\mathcal{S})_{\sigma\wedge\tau}} E_Q\left[X - A^{\mathcal{S}}(Q)_T\middle|\,\mathcal{F}_{\sigma\wedge\tau}\right] \\ & + E_{\tilde{Q}}\left[A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau}\middle|\,\mathcal{F}_{\sigma\wedge\tau}\right]. \end{aligned} \tag{2.5.19}$$

Because $A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau}$ is $\mathcal{F}_\tau$-measurable, we have

$$\begin{aligned} E_{\tilde{Q}}\left[A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau}\middle|\mathcal{F}_{\sigma\wedge\tau}\right] &= E_{\tilde{Q}}\left[E_{\tilde{Q}}\left[A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau}\middle|\mathcal{F}_\tau\right]\middle|\mathcal{F}_\sigma\right] \\ &= E_{\tilde{Q}}\left[A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau}\middle|\mathcal{F}_\sigma\right]. \end{aligned}$$

From this together with (2.5.19), (2.5.13) and (2.5.17), we get

$$\begin{aligned} E_{\tilde{Q}}\left[U_{t\wedge\tau} - A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau}\middle|\mathcal{F}_\sigma\right] &\le \operatorname*{ess\,sup}_{Q\in\mathcal{R}(\mathcal{S})_{\sigma\wedge\tau}} E_Q\left[X - A^{\mathcal{S}}(Q)_T\middle|\mathcal{F}_{\sigma\wedge\tau}\right] \\ &= U_{\sigma\wedge\tau} - A^{\mathcal{S}}(\tilde{Q})_{\sigma\wedge\tau} \qquad (2.5.20) \end{aligned}$$

and hence (2.5.18).

3) Our next goal is to show that for a sequence of stopping times $\sigma_n \le T$ decreasing to another stopping time $\sigma \le T$, we have

$$E_{\hat{Q}}[U_\sigma] = \lim_{n\to\infty} E_{\hat{Q}}[U_{\sigma_n}] \qquad (2.5.21)$$

for any equivalent local martingale measure $\hat{Q}$ for S. Indeed, (2.5.18) yields for $\tilde{Q} = \hat{Q}$, $t = T$ and $\tau = \sigma_n$ that

$$U_\sigma \ge E_{\hat{Q}}[U_{\sigma_n}|\mathcal{F}_\sigma] \qquad (2.5.22)$$

and hence also

$$E_{\hat{Q}}[U_\sigma] \ge \limsup_{n\to\infty} E_{\hat{Q}}[U_{\sigma_n}]. \qquad (2.5.23)$$

To prove the converse inequality, we fix $\varepsilon > 0$. From (2.5.11), we get for $\tilde{Q} = \hat{Q}$, $\tau = \sigma$, $t = T$ and $\sigma = 0$ there that

$$E_{\hat{Q}}[U_\sigma] = \sup_{Q\in\hat{\mathcal{R}}(\mathcal{S})_\sigma} E_Q\left[X - A^{\mathcal{S}}(Q)_T\right],$$

where $\hat{\mathcal{R}}(\mathcal{S})_\sigma = \{Q \in \mathcal{R}(\mathcal{S}) \mid Q = \hat{Q} \text{ on } \mathcal{F}_\sigma\}$. Hence there exists $Q' \in \hat{\mathcal{R}}(\mathcal{S})_\sigma$ such that

$$E_{\hat{Q}}[U_\sigma] \le E_{Q'}[X - A^{\mathcal{S}}(Q')_T] + \varepsilon. \qquad (2.5.24)$$

Note that in particular $0 \le A^{\mathcal{S}}(Q')_T \in \mathbf{L}^1(Q')$. For the rest of this proof, all densities and density processes are expressed with respect to $\hat{Q}$. Denote the density process of Q' by $Z' = (Z'_s)_{0\le s\le T}$ and set $\nu := \inf\{u > 0 \mid Z'_u \le 0.1\} \wedge T$. As Z' is right-continuous and $Z' = 1$ on $[\![0,\sigma]\!]$, we have $\nu > \sigma$. For each $n \in \mathbb{N}$ we define a measure Q^n by

$$Z^n_T := \frac{Z'_T}{Z'_{\sigma_n}} \mathbf{1}_{\{\sigma_n < \nu\}} + \mathbf{1}_{\{\sigma_n \ge \nu\}}.$$

By Lemma 2.5.10, $Q^n \in \hat{\mathcal{R}}(\mathcal{S})_{\sigma_n}$ and

$$A^{\mathcal{S}}(Q^n)_T = \left(A^{\mathcal{S}}(Q')_T - A^{\mathcal{S}}(Q')_{\sigma_n}\right)\mathbf{1}_{\{\sigma_n<\nu\}}.$$

Hence we can apply (2.5.11) for each $n \in \mathbb{N}$ with $\tilde{Q} = \hat{Q}$, $t = T$, $\sigma = 0$ and $\tau = \sigma_n$ to obtain

$$\begin{aligned}\liminf_{n\to\infty} E_{\hat{Q}}[U_{\sigma_n}] &\geq \liminf_{n\to\infty} E_{Q^n}\left[X - A^{\mathcal{S}}(Q^n)_T\right] \\ &\geq \liminf_{n\to\infty} E_{\hat{Q}}\left[\frac{Z'_T}{Z'_{\sigma_n}}\left(X - A^{\mathcal{S}}(Q')_T + A^{\mathcal{S}}(Q')_{\sigma_n}\right)\mathbf{1}_{\{\sigma_n<\nu\}}\right] \\ &\quad + \liminf_{n\to\infty} E_{\hat{Q}}[X\mathbf{1}_{\{\sigma_n\geq\nu\}}]. \end{aligned} \tag{2.5.25}$$

Because $\sigma_n \searrow \sigma < \nu$, the second summand is zero by dominated convergence. For the first summand, we note that $Z'_{\sigma_n} > 0.1$ on $\{\sigma_n < \nu\}$, so that a lower bound for the sequence $\left(\frac{1}{Z'_{\sigma_n}}\left(X - A^{\mathcal{S}}(Q')_T + A^{\mathcal{S}}(Q')_{\sigma_n}\right)\mathbf{1}_{\{\sigma_n<\nu\}}\right)_{n\in\mathbb{N}}$ is given by $10\left(-\|X\|_{\mathbf{L}^\infty} - A^{\mathcal{S}}(Q')_T\right) \in \mathbf{L}^1(Q')$. This allows us to apply Fatou's lemma to get from (2.5.25), (2.5.24) and since $\hat{Q} = Q'$ on $\mathcal{F}_\sigma$ that

$$\begin{aligned}\liminf_{n\to\infty} E_{\hat{Q}}[U_{\sigma_n}] &\geq E_{Q'}\left[X - A^{\mathcal{S}}(Q')_T + A^{\mathcal{S}}(Q')_\sigma\right] && (2.5.26)\\ &= E_{Q'}\left[X - A^{\mathcal{S}}(Q')_T\right] \\ &\geq E_{\hat{Q}}[U_\sigma] - \varepsilon; \end{aligned}$$

we used like in (2.5.17) that $Q' \in \mathcal{R}(\mathcal{S})_\sigma$ so that $A^{\mathcal{S}}(Q')_\sigma = A^{\mathcal{S}}(\hat{Q})_\sigma = 0$. Since $\varepsilon > 0$ was arbitrary, this together with (2.5.23) implies (2.5.21).

4) Next we deduce that $U := (U_t)_{0\leq t\leq T}$ admits an RCLL modification V. Denote by $\hat{Q}$ an equivalent local martingale measure for S and note that with $s \leq t \leq T$, (2.5.18) yields for $\tilde{Q} = \hat{Q}$, $\sigma = s$ and $\tau = T$ that

$$E_{\hat{Q}}[U_t|\mathcal{F}_s] \leq U_s.$$

Hence the family $U = (U_t)_{0\leq t\leq T}$ satisfies under $\hat{Q}$ the supermartingale property and is by (2.5.21) right-continuous in expectation. This implies by Theorem VI.3 of [DM82] the existence of an RCLL modification $V = (V_t)_{0\leq t\leq T}$ of U.

5) By the definition of a right-continuous modification, (2.5.9) holds for any deterministic time t. However, we still have to show that it remains true for any stopping time $\sigma \leq T$. To see this, take a sequence of stopping times $\sigma_n \leq T$ decreasing to σ and taking only rational values. If we can show that

$$\lim_{n\to\infty} E_{\hat{Q}}[\,|U_\sigma - U_{\sigma_n}|\,] = 0, \tag{2.5.27}$$

then we are done since right-continuity and boundedness of V then imply that

$$U_\sigma = \lim_{n\to\infty} U_{\sigma_n} = \lim_{n\to\infty} V_{\sigma_n} = V_\sigma,$$

where the limits are taken in $\mathbf{L}^1(\hat{Q})$. However, (2.5.18) yields for $\tilde{Q} = \hat{Q}$, $t = T$, $\sigma = \sigma_{n+1}$ there and $\tau = \sigma_n$ that

$$E_{\hat{Q}}[U_{\sigma_n} | \mathcal{F}_{\sigma_{n+1}}] \le U_{\sigma_{n+1}}.$$

Since $(\sigma_n)_{n\in\mathbb{N}}$ is decreasing and U is uniformly bounded, this means that $(U_{\sigma_n})_{n\in\mathbb{N}}$ is a backward supermartingale under $\hat{Q}$. By Theorem V.30 of [DM82], $(U_{\sigma_n})_{n\in\mathbb{N}}$ therefore converges in $\mathbf{L}^1(\hat{Q})$ to some $\overline{U}$. Clearly $\overline{U}$ is measurable with respect to $\mathcal{F}_\sigma = \bigcap_{n\in\mathbb{N}} \mathcal{F}_{\sigma_n}$ so that the sequence $(E_{\hat{Q}}[U_{\sigma_n}|\mathcal{F}_\sigma])_{n\in\mathbb{N}}$ also converges to $\overline{U}$ in $\mathbf{L}^1(\hat{Q})$, and so it remains to show that $(E_{\hat{Q}}[U_{\sigma_n}|\mathcal{F}_\sigma])_{n\in\mathbb{N}}$ converges to U_σ in $\mathbf{L}^1(\hat{Q})$. But this follows immediately from (2.5.21) and (2.5.22).

6) Finally we want to conclude that $V - A^{\mathcal{S}}(\tilde{Q})$ is a local $\tilde{Q}$-supermartingale for each $\tilde{Q} \in \mathcal{R}(\mathcal{S})$. To that end let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence such that the upper variation process $A^{\mathcal{S}}(\tilde{Q})^{\tau_n}$ is bounded for each n. Then 5) implies that

$$V_t^{\tau_n} - A^{\mathcal{S}}(\tilde{Q})_t^{\tau_n} = U_{t\wedge\tau_n} - A^{\mathcal{S}}(\tilde{Q})_{t\wedge\tau_n},$$

which together with 2) and boundedness of V implies that $(V - A^{\mathcal{S}}(\tilde{Q}))^{\tau_n}$ is a bounded $\tilde{Q}$-supermartingale.

□

Proof of Theorem 2.5.11. Use Lemma 2.5.14 to define $\hat{V}$ as a uniformly bounded (RCLL) process satisfying for each $t \in [0, T]$

$$\hat{V}_t = \operatorname*{ess\,sup}_{Q\in\mathcal{R}(\mathcal{S})} E_Q\left[X - A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t \right]. \tag{2.5.28}$$

Then Lemma 2.5.14 and Theorem 2.5.13 imply that $\hat{V}$ is the value process of some $\mathcal{H}$-constrained hedging portfolio $(\hat{x}, \hat{H}, \hat{K})$ for X. To prove that $\hat{V}$ is minimal, we first show that in (2.5.28) we can replace $\mathcal{R}(\mathcal{S})$ by

$$\mathcal{R}(\mathcal{S})^b := \left\{ Q \in \mathcal{R}(\mathcal{S}) \,\middle|\, E_Q\left[-A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t \right] \ge -2\|X\|_{\mathbf{L}^\infty} - 1 \right\}.$$

By **(NFLVR)** there exists an equivalent local martingale measure $\hat{Q}$ for S. As $A^{\mathcal{S}}(\hat{Q}) \equiv 0$, we have $\hat{Q} \in \mathcal{R}(\mathcal{S})^b$. Since $\hat{V}_t \ge E_{\hat{Q}}[X|\mathcal{F}_t] \ge -\|X\|_{\mathbf{L}^\infty}$ and $X \le \|X\|_{\mathbf{L}^\infty}$, we claim that a measure $Q \in \mathcal{R}(\mathcal{S})$ cannot contribute to the essential supremum in (2.5.28) on the set

$$B := \left\{ E_Q\left[-A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t \right] < -2\|X\|_{\mathbf{L}^\infty} - 1 \right\} \in \mathcal{F}_t.$$

In fact, if $Z = (Z_t)_{0 \leq t \leq T}$ denotes the density process of Q with respect to $\hat{Q}$, we can construct a measure $\overline{Q}$ via its density

$$\overline{Z}_T := \mathbf{1}_B + \frac{Z_T}{Z_t}\mathbf{1}_{B^c}$$

with respect to $\hat{Q}$ to obtain from Lemma 2.5.10 that $\overline{Q} \in \mathcal{R}(\mathcal{S})$ and that

$$\begin{aligned}
&E_{\overline{Q}}\left[-A^{\mathcal{S}}(\overline{Q})_T + A^{\mathcal{S}}(\overline{Q})_t \,\middle|\, \mathcal{F}_t \right] \\
&= E_{\hat{Q}}\left[-A^{\mathcal{S}}(\hat{Q})_T + A^{\mathcal{S}}(\hat{Q})_t \,\middle|\, \mathcal{F}_t \right]\mathbf{1}_B + E_Q\left[-A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t \right]\mathbf{1}_{B^c} \\
&\geq -2\|X\|_{\mathbf{L}^\infty} - 1,
\end{aligned}$$

where the inequality holds by the definition of B and because $A^{\mathcal{S}}(\hat{Q}) \equiv 0$. This shows that $\overline{Q}$ is in $\mathcal{R}(\mathcal{S})^b$ and also that in (2.5.28) we can indeed replace $\mathcal{R}(\mathcal{S})$ by $\mathcal{R}(\mathcal{S})^b$ since

$$\begin{aligned}
&E_{\hat{Q}}\left[-A^{\mathcal{S}}(\hat{Q})_T + A^{\mathcal{S}}(\hat{Q})_t \,\middle|\, \mathcal{F}_t \right]\mathbf{1}_B + E_Q\left[-A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t \right]\mathbf{1}_{B^c} \\
&\geq E_Q\left[-A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t \right],
\end{aligned}$$

where we used again that $A^{\mathcal{S}}(\hat{Q}) \equiv 0$.

Now we can prove that $\hat{V}$ is a lower bound for the value process V of any $\mathcal{H}$-constrained hedging portfolio (x, H, K) for X. To that end fix $Q \in \mathcal{R}(\mathcal{S})^b$ and let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence of stopping times such that $A^{\mathcal{S}}(Q)$ is bounded on $[\![0, \tau_n]\!]$. By the definition of $A^{\mathcal{S}}(Q)$, the process $V - A^{\mathcal{S}}(Q)$ is a local Q-supermartingale. On $[\![0, \tau_n]\!]$, it is bounded from below and hence a Q-supermartingale, and therefore

$$V_{t\wedge\tau_n} \geq E_Q\left[V_{\tau_n} - A^{\mathcal{S}}(Q)_{\tau_n} + A^{\mathcal{S}}(Q)_{t\wedge\tau_n} \,\middle|\, \mathcal{F}_t \right]$$

for each $n \in \mathbb{N}$. Moreover, $Q \in \mathcal{R}(\mathcal{S})^b$ implies that $-A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t$ is Q-integrable and hence an integrable lower bound for $\left(-A^{\mathcal{S}}(Q)_{\tau_n} + A^{\mathcal{S}}(Q)_{t\wedge\tau_n}\right)_{n\in\mathbb{N}}$. This allows us to apply Fatou's lemma to obtain

$$\begin{aligned}
V_t &\geq E_Q\left[\liminf_{n\to\infty} \left(V_{\tau_n} - A^{\mathcal{S}}(Q)_{\tau_n} + A^{\mathcal{S}}(Q)_{t\wedge\tau_n} \right) \,\middle|\, \mathcal{F}_t \right] \\
&= E_Q\left[V_T - A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t \right] \\
&\geq E_Q\left[X - A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t \right].
\end{aligned}$$

Because $Q \in \mathcal{R}(\mathcal{S})^b$ was arbitrary, this implies

$$V_t \geq \underset{Q \in \mathcal{R}(\mathcal{S})^b}{\text{ess sup}}\ E_Q\left[X - A^{\mathcal{S}}(Q)_T + A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t\right] = \hat{V}_t.$$

□

2.6 Dynamic indifference valuation

Asset valuation in incomplete markets is still an important problem in mathematical finance. One approach is the dynamic indifference valuation method which we consider in this section. After defining the indifference value for each time $t \in [0, T]$, we investigate its properties as a functional on $\mathbf{L}^\infty$, in particular with respect to continuity and time-consistency. For this we observe that the indifference valuation functional is obtained by normalization of the convolution of the DMCUF corresponding to the agent's preferences and the *market DMCUF* whose acceptance sets consist (up to sign) of exactly those payoffs that can be superhedged at zero cost. We extend an idea of Föllmer/Schied [FS02] by using the optional decomposition under constraints dynamically over time to construct the market DMCUF, and notably show that this DMCUF is strongly time-consistent. Moreover, we discuss the connections between this indifference valuation approach and arbitrage opportunities, explain the link to good deal bounds, and examine the special case when trading in the market is possible without constraints.

Valuation by indifference with respect to an expected utility is an old theme and has been much studied again in the last years. An early reference is Hodges/Neuberger [HN89]; Frittelli [Fri00] and Rouge/El Karoui [REK00] are at the start of the recent resurgence of activity, and Becherer [Bec03] and Henderson/Hobson [HH04] contain overviews and many more references. However, explicit results are hard to obtain because except for the exponential case, the utility-based certainty equivalent is not translation invariant.

The idea of replacing expected utility by a monetary (hence translation invariant) utility functional and the naturally ensuing link to the convolution with the market functional have only emerged rather recently. Perhaps the earliest reference where a similar idea can be found in a general abstract (but static) form is Jaschke/Küchler [JK01], even though the formulation there is for coherent risk measures and cast in terms of good-deal bounds. Indifference valuation proper is mentioned in [BEK05] and discussed in more detail in Xu [Xu06] which also contains a number of worked examples. However, both deal only with the static case, and [Xu06] has no constraints in the market. Larsen/Pirvu/Shreve/Tütüncü [LPST05] contains a dynamic treatment for a particular class of examples where the monetary utility functional is given via a finite set of scenario and stress measures, generalizing an idea from Carr/Geman/Madan [CGM01]. None of these works study the issue of time-consistency.

The underlying idea is the following. For each $t \in [0, T]$, let U_t be a functional which maps $\mathbf{L}^\infty$ into $\mathbf{L}^\infty(\mathcal{F}_t)$. We assume that $U_t(X)$ models the utility that some (fixed) agent assigns at time $t \le T$ to the payoff X which is due at time T. We suppose that she can trade in a financial market and denote by $\mathcal{C}_t$ the set of payoffs due at time T that she can superhedge by trading during $(t, T]$ with zero initial capital. If the agent has at time t an initial endowment $x_t \in \mathbf{L}^\infty(\mathcal{F}_t)$, she can implicitly determine a time t value $p_t(X)$ for the payoff $X \in \mathbf{L}^\infty$ by the indifference requirement

$$\operatorname*{ess\,sup}_{G\in\mathcal{C}_t} U_t(x_t + G) = \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} U_t(x_t - p_t(X) + G + X) \tag{2.6.1}$$

(presuming that $p_t(X)$ is well-defined). We call $p(X) = (p_t(X))_{0\le t\le T}$ the *indifference value process* for X since it makes the agent at each time t indifferent (according to U_t) between buying the asset X or not, provided that she always optimally exploits her trading opportunities.

Remark 2.6.1. i) The set $\mathcal{C}_t$ consists of all payoffs that the agent can superhedge by trading during $(t, T]$ from zero initial capital. Hence $\mathcal{C}_t$ is solid. This will be required later when we assume that $-\mathcal{C}_t$ is a pre-acceptance set. Note that we assumed implicitly in the definition of p that the initial endowment x_t can be transferred from t to T, i.e., the existence of a bank account with zero interest rate. However, besides from this, we did not impose any conditions on the structure of $\mathcal{C}_t$ so far. In fact, $\mathcal{C}_t$ can be used to incorporate transaction costs or bid and ask prices for the traded assets. However, when we specify $\mathcal{C}_t$ later in this section, we do not make use of this.

ii) In analogy to the value $p_t(X)$ for buying the asset X, we can define a value $p_t^s(X)$ for selling X by

$$\operatorname*{ess\,sup}_{G\in\mathcal{C}_t} U_t(x_t + G) = \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} U_t(x_t + p_t^s(X) + G - X). \tag{2.6.2}$$

All results will be stated for $p_t(X)$ only, since $p_t^s(X) = -p_t(-X)$ so that the value of selling the asset X can easily be deduced from $p_t(X)$. ◇

Let us first consider the indifference value p_t for a fixed time t. Throughout this section we assume that the functional U_t is *$\mathcal{F}_t$-translation invariant* in the sense of Definition 2.3.1, i.e., we make the standing assumption

Assumption (TI): The functional $U_t : \mathbf{L}^\infty \to \mathbf{L}^\infty(\mathcal{F}_t)$ satisfies

$$U_t(X + a_t) = U_t(X) + a_t \quad \text{for all } X \in \mathbf{L}^\infty \text{ and } a_t \in \mathbf{L}^\infty(\mathcal{F}_t).$$

This assumption implies (like the notation suggests) that $p_t(X)$ does not depend on the initial endowment $x_t \in \mathbf{L}^\infty(\mathcal{F}_t)$, since this can be pulled out on both sides of equation (2.6.1). If in addition

$$U_t^{\text{opt}}(X) := \operatorname*{ess\,sup}_{G \in \mathcal{C}_t} U_t(X+G) \in \mathbf{L}^\infty \quad \text{for all } X \in \mathbf{L}^\infty, \tag{2.6.3}$$

then translation invariance ensures that $p_t(X)$ is well-defined in $\mathbf{L}^\infty(\mathcal{F}_t)$ and given by

$$p_t(X) = U_t^{\text{opt}}(X) - U_t^{\text{opt}}(0). \tag{2.6.4}$$

$U_t^{\text{opt}}(X)$ is the maximal utility the agent can achieve from the payoff X by trading optimally in the market. It is clear from (2.6.4) that this operator is a key tool in the investigation of the indifference value.

When defining the value $p_t(X)$ as in (2.6.1), we implicitly assume that the agent does not yet hold any other assets due at time T. In fact, such assets might cause diversification effects which she should take into account for the valuation. Suppose the agent already holds in her portfolio an asset with payoff $Y \in \mathbf{L}^\infty$ due at time T. Then she should define $p_t^Y(X)$, the indifference value at time t for buying the asset X when holding Y, implicitly by

$$\operatorname*{ess\,sup}_{G \in C_t} U_t\big(x_t + G + Y\big) = \operatorname*{ess\,sup}_{G \in C_t} U_t\big(x_t - p_t^Y(X) + G + Y + X\big). \tag{2.6.5}$$

In other words, she should compare the maximal utility she can achieve by trading optimally when she has only Y with the maximal utility she can obtain when her portfolio consists of X and Y (and when she has to pay $p_t^Y(X)$ at time t). Analogously to (2.6.4), provided that U_t^{opt} maps $\mathbf{L}^\infty$ into $\mathbf{L}^\infty(\mathcal{F}_t)$, we can resolve (2.6.5) for $p_t^Y(X)$ to obtain

$$p_t^Y(X) = U_t^{\text{opt}}(X+Y) - U_t^{\text{opt}}(Y). \tag{2.6.6}$$

The following result shows that our approach has the pleasant property that this leads to a consistent valuation principle, in the sense that the value for $X+Y$ coincides with the sum of the value for Y and the value for X when holding Y. Put differently, it does not matter whether the agent buys the assets one after another or in bulk, always provided that she properly takes into account what has already been bought.

Proposition 2.6.2. *If $U_t^{\text{opt}}(X) \in \mathbf{L}^\infty$ for all $X \in \mathbf{L}^\infty$ then*

$$p_t(X+Y) = p_t(Y) + p_t^Y(X).$$

Proof. This follows immediately from (2.6.4) and (2.6.6). □

From now on we do not only assume that U_t is translation invariant, but that $U_t = \Phi_t$ is an MCUF at time t. The analogue of U_t^{opt} from (2.6.3) is then

$$\Phi_t^{\text{opt}}(X) := \operatorname*{ess\,sup}_{G \in \mathcal{C}_t} \Phi_t(X+G), \tag{2.6.7}$$

and the corresponding indifference value functional p_t from (2.6.4) is

$$p_t(X) = \Phi_t^{\text{opt}}(X) - \Phi_t^{\text{opt}}(0). \tag{2.6.8}$$

Monotonicity and translation invariance of an MCUF imply that Φ_t^{opt} maps $\mathbf{L}^\infty$ into $\mathbf{L}^\infty(\mathcal{F}_t)$ if and only if $\Phi_t^{\text{opt}}(0)$ is bounded, i.e., if

$$\Phi_t^{\text{opt}}(0) = \operatorname*{ess\,sup}_{G \in \mathcal{C}_t} \Phi_t(G) = \Phi_t \square (-\mathcal{C}_t)(0) \in \mathbf{L}^\infty. \tag{2.6.9}$$

Recall that we have studied the operator Φ_t^{opt} in detail in Lemma 2.4.5 and Proposition 2.4.7. In particular, we have given conditions for when it is an MCUF and also for when it corresponds to the convolution of Φ_t and

$$\Phi_t^{-\mathcal{C}_t}(X) := \operatorname{ess\,sup}\left\{ m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \,\middle|\, X - m_t \in -\mathcal{C}_t \right\}, \tag{2.6.10}$$

the *market MCUF* induced by $\mathcal{C}_t$. This name is justified by the observation that in view of the interpretation of $\mathcal{C}_t$ as superhedgeable payoffs, $-\Phi_t^{-\mathcal{C}_t}(-X)$ is the minimal amount required at time t that allows to superhedge X. Lemma 2.4.5 and Proposition 2.4.7 together with (2.6.8) immediately yield the following result:

Proposition 2.6.3. *Let Φ_t be an MCUF at time t and $\mathcal{C}_t \subseteq \mathbf{L}^\infty$ a non-empty convex and $\mathcal{F}_t$-regular set such that (2.6.9) holds. Then:*

a) $p_t(\,.\,)$ is a normalized MCUF at time t, which is continuous from below if $\Phi_t(\,.\,)$ is.

b) If $-\mathcal{C}_t$ is a pre-acceptance set at time t then

$$\Phi_t^{\text{opt}}(X) = \Phi_t \square \Phi_t^{-\mathcal{C}_t}(X) \quad \textit{for all } X \in \mathbf{L}^\infty \tag{2.6.11}$$

so that

$$p_t(X) = \Phi_t \square \Phi_t^{-\mathcal{C}_t}(X) - \Phi_t \square \Phi_t^{-\mathcal{C}_t}(0) \quad \textit{for all } X \in \mathbf{L}^\infty. \tag{2.6.12}$$

Remark 2.6.4. A sufficient condition for (2.6.9) is that

$$\mathcal{C}_t \cap \left\{ X \in \mathbf{L}^\infty \mid P[\Phi_t(X) > 0] > 0 \right\} = \emptyset, \tag{2.6.13}$$

since this implies that ess $\sup_{G \in \mathcal{C}_t} \Phi_t(G) \leq 0$. If Φ_t is coherent and $\mathcal{C}_t$ is a non-empty $\mathcal{F}_t$-regular convex cone containing 0, then (2.6.13) is even necessary for (2.6.9). In fact, if (2.6.13) does not hold, then there exist $X \in \mathcal{C}_t$ and $\varepsilon > 0$ such that for the set $A := \{\Phi_t(X) \geq \varepsilon\} \in \mathcal{F}_t$ we have $P[A] > 0$. But since for all $n \in \mathbb{N}$ also $nX\mathbf{1}_A \in \mathcal{C}_t$, positive homogeneity and $\mathcal{F}_t$-regularity of Φ_t imply that

$$\operatorname*{ess\,sup}_{G \in \mathcal{C}_t} \Phi_t(G) \geq \Phi_t(nX\mathbf{1}_A) \geq n\varepsilon\mathbf{1}_A.$$

Taking the limit for $n \to \infty$, this shows that (2.6.9) cannot hold true. $\diamond$

It seems natural to ask if we can consider $p_t(X)$ not only as a value for X, but also as a price for (buying) X. A minimal requirement for this is clearly that $p_t(X)$ should not lead to arbitrage opportunities. Before we make this more precise we should first ensure that the market itself does not contain arbitrage opportunities. Therefore we impose that

$$\Phi_t^{-\mathcal{C}_t}(0) = \text{ess sup}\left(\mathcal{C}_t \cap \mathbf{L}^\infty(\mathcal{F}_t)\right) \leq 0, \tag{2.6.14}$$

i.e., that one cannot superhedge from t on at zero cost something known at time t and positive. In particular (2.6.14) ensures that the interval $[\Phi_t^{-\mathcal{C}_t}(X), -\Phi_t^{-\mathcal{C}_t}(-X)]$ from the subhedging to the superhedging price is non-empty. Then for $p_t(X)$ respectively $p_t^s(X)$ not to yield arbitrage opportunities they should lie inside the interval $(\Phi_t^{-\mathcal{C}_t}(X), -\Phi_t^{-\mathcal{C}_t}(-X))$; for an early work on this see [Fri00]. By Proposition 2.6.3, $p_t(X)$ is a normalized MCUF at time t, and since $p_t^s(X) = -p_t(-X)$, this implies that $p_t(X) \leq p_t^s(X)$ so that the value (or price) for buying X does not exceed the value for selling X. In fact, normalization and concavity imply that

$$0 = p_t\left(\frac{1}{2}X - \frac{1}{2}X\right) \geq \frac{1}{2}p_t(X) + \frac{1}{2}p_t(-X),$$

so that

$$-p_t(X) \geq p_t(-X)$$

on $\mathbf{L}^\infty$. Consequently, we seek for conditions which ensure that $p_t(X)$ and $p_t^s(X)$ yield arbitrage-free bid and ask prices for X in the sense that

$$[p_t(X), p_t^s(X)] \subseteq [\Phi_t^{-\mathcal{C}_t}(X), -\Phi_t^{-\mathcal{C}_t}(-X)]. \tag{2.6.15}$$

But a violation of condition (2.6.15) does not necessarily lead to an arbitrage opportunity. Indeed, to exclude arbitrage, it would already suffice to have the two interlocking inequalities

$$p_t(X) \leq -\Phi_t^{-\mathcal{C}_t}(-X) \quad \text{and} \quad p_t^s(X) \geq \Phi_t^{-\mathcal{C}_t}(X). \tag{2.6.16}$$

However, if for instance $p_t^s(X)$, the value for selling X, exceeds the superhedging price $-\Phi_t^{-\mathcal{C}_t}(-X)$ for buying X, nobody would agree to pay this as a price. Therefore we consider the stronger condition (2.6.15) to be desirable. The next result gives sufficient conditions for (2.6.15).

Proposition 2.6.5. *Let Φ_t be an MCUF at time t and $-\mathcal{C}_t \subseteq \mathbf{L}^\infty$ a pre-acceptance set at time t such that (2.6.9) and (2.6.14) hold. Then we have absence of arbitrage in the sense of (2.6.15) if one of the following conditions holds:*

a) $-\mathcal{C}_t$ *is a convex cone containing 0.*

b) 0 *is in the acceptance set of* Φ_t *and the MCUF* Φ_t^{opt} *is normalized, i.e.,* $\Phi_t(0) \geq 0$ *and* $\text{ess sup}_{G \in \mathcal{C}_t} \Phi_t(G) = 0$.

In particular, if a) or b) holds and if X satisfies $\Phi_t^{-\mathcal{C}_t}(X) = -\Phi_t^{-\mathcal{C}_t}(-X)$, then

$$\Phi_t^{-\mathcal{C}_t}(X) = p_t(X) = p_t^s(X) = -\Phi_t^{-\mathcal{C}_t}(-X).$$

Thus for an asset which is traded in the market, value and market price must coincide.

Proof. Since $p_t^s(\cdot) = -p_t(-\cdot)$, it suffices to show that

$$\Phi_t^{-\mathcal{C}_t}(X) \leq p_t(X). \tag{2.6.17}$$

a) If $-\mathcal{C}_t$ is a convex cone containing 0, then $\Phi_t^{-\mathcal{C}_t}$ is by Lemma 2.3.9 positively homogeneous and therefore by Remark 2.3.2 iv) superadditive, i.e., it satisfies $\Phi_t^{-\mathcal{C}_t}(X+Y) \geq \Phi_t^{-\mathcal{C}_t}(X) + \Phi_t^{-\mathcal{C}_t}(Y)$. Hence Proposition 2.6.3 and the symmetry of the convolution imply that

$$\begin{aligned} p_t(X) &= \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) - \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(0) \qquad (2.6.18)\\ &= \operatorname*{ess\,sup}_{Y \in \mathbf{L}^\infty} \left(\Phi_t^{-\mathcal{C}_t}(X+Y) + \Phi_t(-Y) \right) - \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(0) \\ &\geq \Phi_t^{-\mathcal{C}_t}(X) + \operatorname*{ess\,sup}_{Y \in \mathbf{L}^\infty} \left(\Phi_t^{-\mathcal{C}_t}(Y) + \Phi_t(-Y) \right) - \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(0) \\ &= \Phi_t^{-\mathcal{C}_t}(X). \end{aligned}$$

b) If $\Phi_t \Box \Phi_t^{-\mathcal{C}_t}(0) = \Phi_t^{\mathrm{opt}}(0) = 0$, then (2.6.18) simplifies to

$$\begin{aligned} p_t(X) &= \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) \\ &= \operatorname*{ess\,sup}_{Y \in \mathbf{L}^\infty} \left(\Phi_t^{-\mathcal{C}_t}(X+Y) + \Phi_t(-Y) \right) \\ &\geq \Phi_t^{-\mathcal{C}_t}(X) + \Phi_t(0) \\ &\geq \Phi_t^{-\mathcal{C}_t}(X), \end{aligned}$$

where the last inequality holds since $\Phi_t(0) \geq 0$.

□

When $\mathcal{C}_t$ is only convex but not a convex cone containing 0, even the weaker no-arbitrage condition (2.6.16) can be violated. This can be explained as follows. Our definition (2.6.1) of the indifference value uses the same set $\mathcal{C}_t$ of gains from strategies irrespective of whether the agent owns X or not, and so we implicitly assume that buying X does not change the set of possible strategies. Note that X is here viewed as a new financial *instrument*; like in a market with transaction costs, this must be distinguished from a *portfolio* generating the same payoff as X, but formed from the primary assets in the market. The following example explicitly illustrates how buying or owning such a portfolio can change the set $\mathcal{C}_0$ of allowed gains into a new set $\mathcal{C}_0^X$, and how this makes it reasonable for the agent to pay more for X than the $\mathcal{C}_0$-superhedging price. Indeed, although $p_0(X)$ is bigger than $-\Phi_0^{-\mathcal{C}_0}(-X)$, the agent cannot increase her maximal attainable utility by superhedging X via the portfolio instead of buying it directly for $p_0(X)$, because she may only work with $\mathcal{C}_0^X$ after the superhedging.

The above discussion shows that one must be very careful when introducing a new instrument X in the market, because (especially with constraints) this may affect the set of allowed trades. However, we do not pursue this delicate issue any further.

Example 2.6.6. For simplicity we consider a one-step discrete time model with only two possible states. There exists a bank account with zero interest rate and one risky asset S with net payoff $S_1 - S_0 = \left(-1, \frac{1}{4}\right)$. Trading is restricted in that the agent is not allowed to hold strictly less than -1 units of the risky asset. Hence the set of payoffs which can be superhedged by trading from zero initial capital is

$$\mathcal{C}_0 = \left\{ \beta \left(-1, \frac{1}{4}\right) \middle| \beta \geq -1 \right\} - \mathbb{R}^2_+.$$

We consider the payoff $X := \left(\frac{1}{2}, -\frac{1}{4}\right)$. Its superhedging price is

$$\begin{aligned} -\Phi_0^{-\mathcal{C}_0}(-X) &= \inf\left\{ c \in \mathbb{R} \,\middle|\, \left(\frac{1}{2}, -\frac{1}{4}\right) \leq c + \beta\left(-1, \frac{1}{4}\right) \text{ for some } \beta \geq -1 \right\} \\ &= \inf_{\beta \geq -1} \left\{ \max\left\{ \frac{1}{2} + \beta, -\frac{1}{4} - \frac{1}{4}\beta \right\} \right\} \\ &= -\frac{1}{10}, \end{aligned}$$

since it is easy to check that the infimum is attained for $\beta = -\frac{3}{5}$. Note that the corresponding superhedging strategy is even a hedging strategy as it perfectly replicates X. The preferences of the agent correspond to the exponential certainty equivalent from Example 2.3.3 with risk aversion $\frac{1}{4}$ so that

$$\Phi_0(X) = -4 \log E\left[e^{-\frac{1}{4}X}\right],$$

where the probability measure P assigns to both possible states the same probability.

Hence the maximal attainable monetary utility without owning X is

$$\begin{aligned}\sup_{G\in\mathcal{C}_0}\Phi_0(G) &= \sup_{\beta\geq -1}\Phi_0\left(\beta\left(-1,\frac{1}{4}\right)\right)\\ &= \sup_{\beta\geq -1}\left\{-4\log\left\{\frac{1}{2}e^{\frac{1}{4}\beta}+\frac{1}{2}e^{-\frac{1}{16}\beta}\right\}\right\}\\ &= -4\log\left(\frac{1}{2}\left(e^{-\frac{1}{4}}+e^{\frac{1}{16}}\right)\right)\\ &\approx 0.3264,\end{aligned}$$

where the supremum is attained for $\beta = -1$. Along the same lines, the maximal attainable monetary utility when holding X is

$$\begin{aligned}\sup_{G\in\mathcal{C}_0}\Phi_0(X+G) &= \sup_{\beta\geq -1}\left\{-4\log\left(\frac{1}{2}e^{-\frac{1}{4}\left(\frac{1}{2}-\beta\right)}+\frac{1}{2}e^{-\frac{1}{4}\left(-\frac{1}{4}+\frac{1}{4}\beta\right)}\right)\right\}\\ &= -4\log\left(\frac{1}{2}\left(e^{-\frac{3}{8}}+e^{\frac{1}{8}}\right)\right)\\ &\approx 0.3763,\end{aligned}$$

where again the supremum is attained for $\beta = -1$. By (2.6.8),

$$p_0(X) = \sup_{G\in\mathcal{C}_0}\Phi_0(X+G) - \sup_{G\in\mathcal{C}_0}\Phi_0(G) \approx 0.050 > -\frac{1}{10} = -\Phi_0^{-\mathcal{C}_0}(-X) \quad (2.6.19)$$

so that even the weak no-arbitrage condition (2.6.16) is violated. Moreover, we can immediately see why this happens. In fact, the argument why prices should be consistent with the no-arbitrage principle is that instead of buying X for a price exceeding its superhedging price, it would be cheaper to buy the assets required to superhedge X. However, the situation is slightly different here. For superhedging X, the agent needs to sell short $\frac{3}{5}$ units of the risky asset, and then she can go short only $\frac{2}{5}$ further units in the risky asset. Therefore her maximal attainable monetary utility after implementing the (super-)hedging strategy for $X = -\frac{1}{10} - \frac{3}{5}\left(-1, \frac{1}{4}\right)$ is

$$\begin{aligned}&\sup_{\beta\geq -\frac{2}{5}}\Phi_0\left(X-\left(-\Phi_0^{-\mathcal{C}_0}(-X)\right)+\beta\left(-1,\frac{1}{4}\right)\right)\\ &\qquad = \sup_{\beta\geq -\frac{2}{5}}\Phi_0\left(-\frac{3}{5}\left(-1,\frac{1}{4}\right)+\beta\left(-1,\frac{1}{4}\right)\right)\\ &\qquad = \Phi_0^{\mathrm{opt}}(0)\\ &\qquad \approx 0.3264.\end{aligned}$$

Note how the initial trade to superhedge X has explicitly changed the set of strategies from $\mathcal{C}_0 \widehat{=} \{\beta \geq -1\}$ to $\mathcal{C}_0^X \widehat{=} \{\beta \geq -\frac{2}{5}\}$. On the other hand, if directly buying X for $p_0(X)$ does not change the set of possible trading strategies, then the maximal monetary utility after that purchase is

$$\begin{aligned}\sup_{G \in \mathcal{C}_0} \Phi_0(X - p_0(X) + G) &= \Phi_0^{\mathrm{opt}}(X) - p_0(X) \\ &= \Phi_0^{\mathrm{opt}}(0) \\ &\approx 0.3264.\end{aligned}$$

Hence acting upon the apparent arbitrage opportunity does not yield a higher attainable utility than buying X for $p_0(X)$, since the former trade changes the set of admissible strategies. ◇

To specify the representation and the acceptance set of the convolution $\Phi_t \square \Phi_t^{-\mathcal{C}_t}$ and hence of p_t more precisely, we require some additional properties. The following result follows immediately from Proposition 2.4.7 (with $\mathcal{B} = -\mathcal{C}_t$) and (2.6.12).

Proposition 2.6.7. *Let $-\mathcal{C}_t$ be a pre-acceptance set at time t and Φ_t an MCUF at time t which is continuous from below with acceptance set $\mathcal{A}_t$ and concave conjugate α_t. If (2.6.9) holds and if*

$$\text{ess sup}\left\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid m_t \in \overline{\mathcal{C}_t}\right\} \in \mathbf{L}^\infty,$$

where the closure is taken in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$, then the MCUF $\Phi_t^{\mathrm{opt}} = \Phi_t \square \Phi_t^{-\mathcal{C}_t}$ is continuous from below and its concave conjugate is

$$\alpha_t(Q) + \alpha_t^{\overline{-\mathcal{C}_t}}(Q),$$

where $\alpha_t^{\overline{-\mathcal{C}_t}}(Q) := \underset{Y \in \overline{-\mathcal{C}_t}}{\text{ess inf}}\, E_Q[Y \mid \mathcal{F}_t]$. Its acceptance set is

$$\overline{\mathcal{A}_t + \overline{-\mathcal{C}_t}} = \overline{\mathcal{A}_t - \mathcal{C}_t}.$$

In particular, the indifference value functional

$$p_t(\,.\,) = \Phi_t^{\mathrm{opt}}(\,.\,) - \Phi_t^{\mathrm{opt}}(0)$$

is an MCUF which is continuous from below with acceptance set

$$\overline{\mathcal{A}_t - \mathcal{C}_t} + \Phi_t^{\mathrm{opt}}(0).$$

Having discussed the properties of p_t for fixed t, we now investigate the dynamic aspects of the indifference valuation DMCUF $p = (p_t)_{0 \leq t \leq T}$. In particular, we turn our attention to time-consistency. Under the assumptions of Proposition 2.6.3 b), p_t is obtained at each time t from the convolution $\Phi_t \Box \Phi_t^{-\mathcal{C}_t}$ by normalization, and we know that normalization turns a time-consistent DMCUF into a strongly time-consistent one. We also know from Theorem 2.4.3 that the convolution of (strongly) time-consistent DMCUFs is again a (strongly) time-consistent DMCUF. Hence the obvious idea to ensure that the indifference valuation DMCUF p is strongly time-consistent is to choose Φ and the sets $(\mathcal{C}_t)_{0\le t\le T}$ such that both Φ and the market DMCUF $(\Phi_t^{-\mathcal{C}_t})_{0\le t\le T}$ are time-consistent. To achieve the latter by defining $\mathcal{C}_t$ in an appropriate way, we have to specify the structure of the financial market in more detail.

As in Section 2.5 we model the discounted price process of the basic traded assets by a locally bounded RCLL P-semimartingale $S = (S_t)_{0\le t\le T}$. We assume that **(NFLVR)** holds and fix an admissible hedging set $\mathcal{H} \subseteq L^a_{\mathrm{loc}}(S)$. For each time t we define the set of payoffs superhedgeable from zero initial endowment via trading during $(t, T]$ by

$$\mathcal{C}_t := \left(\left\{ \int_t^T H_s\, dS_s \,\middle|\, H \in \mathcal{H}_t \right\} - \mathbf{L}^0_+ \right) \cap \mathbf{L}^\infty \tag{2.6.20}$$

with

$$\mathcal{H}_t := \left\{ H \in \mathcal{H} \,\middle|\, \int_t^{\cdot} H_s\, dS_s \text{ is uniformly bounded from below} \right\}. \tag{2.6.21}$$

Each $H \in \mathcal{H}_t$ describes a self-financing trading strategy on $(t, T]$ with a wealth process which is uniformly bounded from below. The subtraction of $\mathbf{L}^0_+$ economically means that we are always allowed to "throw away" money. In the following result we apply Theorem 2.5.11 to prove that the above sets $\mathcal{C}_t$ yield a strongly time-consistent market DMCUF $(\Phi_t^{-\mathcal{C}_t})_{0\le t\le T}$.

Theorem 2.6.8. *For $X \in \mathbf{L}^\infty$ and each $t \in [0, T]$ define*

$$\hat{\Phi}_t(X) := -\hat{V}_t, \tag{2.6.22}$$

where $(\hat{V}_t)_{0\le t\le T}$ is the value process of the minimal $\mathcal{H}$-constrained hedging portfolio for $-X$ from Theorem 2.5.11. Then $(\hat{\Phi}_t)_{0\le t\le T}$ is a well-representable strongly time-consistent DMCUF. Its acceptance set at any time t is $-\mathcal{C}_t$ so that $\hat{\Phi}_t = \Phi_t^{-\mathcal{C}_t}$ on $\mathbf{L}^\infty$. In particular, each $\mathcal{C}_t$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$.

Proof. Clearly, $\hat{\Phi}_t(X) \in \mathbf{L}^\infty$ by uniform boundedness of $\hat{V}$. By (2.5.8) we can write

$$\hat{\Phi}_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{R}(\mathcal{S})} \left(E_Q[X|\mathcal{F}_t] + E_Q\left[A^{\mathcal{S}}(Q)_T - A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t \right] \right);$$

note that we construct $\hat{V}$ from $-X$. Hence Remark 2.3.18 i) yields that $\hat{\Phi}_t$ is indeed an MCUF at time t since in (2.3.6) we can set $\alpha_t^0(Q) := -E_Q\left[A^{\mathcal{S}}(Q)_T - A^{\mathcal{S}}(Q)_t \,\middle|\, \mathcal{F}_t\right]$ if $Q \in \mathcal{R}(\mathcal{S})$ and $\alpha_t^0(Q) := -\infty$ otherwise. Now Theorem 2.3.16 together with Remark 2.3.18 ii) imply that $\hat{\Phi}_t$ is well-representable and in particular that its acceptance set $\hat{\mathcal{A}}_t$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$. Next we show that $\hat{\mathcal{A}}_t = -\mathcal{C}_t$. To see that $-\mathcal{C}_t \subseteq \hat{\mathcal{A}}_t$, note that for any $H \in \mathcal{H}_t$ and $Y \in \mathbf{L}^0_+$ such that $G := \int_t^T H_s\,dS_s - Y \in \mathcal{C}_t$, we can construct an $\mathcal{H}$-constrained hedging portfolio $(0, H', K')$ for G by choosing $H' := H\mathbf{1}_{]\!]t,T]\!]}$ and $K' := Y\mathbf{1}_{[\![T]\!]}$, where $H' \in \mathcal{H}$ by Remark 2.5.3 ii). The value process V' corresponding to $(0, H', K')$ is zero at time t and, as required for an $\mathcal{H}$-constrained hedging portfolio, uniformly bounded from below since $H \in \mathcal{H}_t$ and $G \in \mathbf{L}^\infty$. This implies that the value process $\tilde{V}$ of the minimal $\mathcal{H}$-constrained hedging portfolio for G satisfies $\tilde{V}_t \leq V'_t = 0$ so that $\hat{\Phi}_t(-G) = -\tilde{V}_t \geq 0$, i.e., $-G \in \hat{\mathcal{A}}_t$. To see that also $\hat{\mathcal{A}}_t \subseteq -\mathcal{C}_t$, fix $X \in \hat{\mathcal{A}}_t$ and denote by $(\hat{x}, \hat{H}, \hat{K})$ the minimal $\mathcal{H}$-constrained hedging portfolio for $-X$ and the corresponding (uniformly bounded) value process by $\hat{V}$. Since $(\hat{K}_u - \hat{K}_t)_{t \leq u \leq T}$ is an increasing process, we obtain from

$$\hat{V}_u = \hat{V}_t + \int_t^u \hat{H}_s\,dS_s - (\hat{K}_u - \hat{K}_t), \quad t \leq u \leq T \tag{2.6.23}$$

that $\hat{H} \in \mathcal{H}_t$. Moreover, if we take $u = T$ in (2.6.23) and recall that $\hat{V}_t \leq 0$ (since $X \in \hat{\mathcal{A}}_t$) and $\hat{V}_T \geq -X$, this also shows that $-X \in \mathcal{C}_t$. Hence we have proved that $-\mathcal{C}_t$ is the acceptance set of $\hat{\Phi}_t$.

Since clearly $-\mathcal{C}_t \subseteq -\mathcal{C}_s$ for $t \geq s$, it only remains to show time-consistency. So let $s < t$ and suppose that $\hat{\Phi}_t(X) = \hat{\Phi}_t(Y)$, but $P[\hat{\Phi}_s(X) > \hat{\Phi}_s(Y)] > 0$ for some $X, Y \in \mathbf{L}^\infty$. Denote by (x^X, H^X, K^X), (x^Y, H^Y, K^Y) the minimal $\mathcal{H}$-constrained hedging portfolios for $-X$ and $-Y$ with value processes $V^X = -\hat{\Phi}(X)$ and $V^Y = -\hat{\Phi}(Y)$. Then we can define another $\mathcal{H}$-constrained hedging portfolio (x', H', K') for $-Y$ (by essentially switching from (x^X, H^X, K^X) to (x^Y, H^Y, K^Y) at time t) via

$$\begin{aligned} x' &:= x^X, \\ H'_u &:= H_u^X \mathbf{1}_{\{u \leq t\}} + H_u^Y \mathbf{1}_{\{u > t\}}, \\ K'_u &:= K_u^X \mathbf{1}_{\{u \leq t\}} + (K_u^Y - K_t^Y + K_t^X)\mathbf{1}_{\{u > t\}}. \end{aligned}$$

Note that $H' \in \mathcal{H}$ by predictable convexity and that the value process corresponding to (x', H', K') is given by $V' := V^X \mathbf{1}_{[\![0,t]\!]} + V^Y \mathbf{1}_{[\![t,T]\!]}$ (since $V_t^X = V_t^Y$). Hence, V' is in particular uniformly bounded (from below) so that (x', H', K') is an $\mathcal{H}$-constrained hedging portfolio for $-Y$. Since $P[V'_s < V_s^Y] > 0$ we get a contradiction to the minimality of (x^Y, H^Y, K^Y). Therefore $\hat{\Phi}_s(X) = \hat{\Phi}_s(Y)$ and $\hat{\Phi}$ is time-consistent. □

Combining Theorems 2.6.8 and 2.4.3 immediately shows that we can extend Proposition 2.6.7 to obtain strong time-consistency as well:

Proposition 2.6.9. *Let Φ be a time-consistent DMCUF which is continuous from below, and such that (2.6.9) is satisfied for each $t \in [0, T]$ with $\mathcal{C}_t$ from (2.6.20). Then the indifference valuation DMCUF $p(\,.\,)$ is also continuous from below and strongly time-consistent.*

Remark 2.6.10. i) The idea for using the optional decomposition under constraints to construct an MCUF describing a financial market is due to Föllmer/Schied [FS02] in the static case; see also Section 4.8 in [FS04]. But, time-consistency aspects have apparently not been studied or proved so far.

ii) Note that (**NFLVR**) implies that $\mathcal{C}_t$ from (2.6.20) always satisfies the no-arbitrage condition (2.6.14); see also Lemma 2.6.15 below.

◇

As mentioned in Section 2.3, one might be interested in finding an indifference value $p_{s,t}(X)$ for all intermediate time horizons $t < T$ and $s \leq t$, $X \in \mathbf{L}^\infty(\mathcal{F}_t)$. This requires a definition for $\Phi^{\text{opt}}_{s,t} : \mathbf{L}^\infty(\mathcal{F}_t) \to \mathbf{L}^\infty(\mathcal{F}_s)$ so that we can set

$$p_{s,t}(X) := \Phi^{\text{opt}}_{s,t}(X) - \Phi^{\text{opt}}_{s,t}(0).$$

We have argued in Section 2.3 that the existence of a bank account with zero interest rate implies that we should have

$$p_{s,t}(X) = p_{s,T}(X) \quad \text{for all } X \in \mathbf{L}^\infty(\mathcal{F}_t) \tag{2.6.24}$$

since money can be freely transferred between t and T. (2.6.24) holds if and only if

$$\Phi^{\text{opt}}_{s,t}(X) - \Phi^{\text{opt}}_{s,t}(0) = \Phi^{\text{opt}}_{s,T}(X) - \Phi^{\text{opt}}_{s,T}(0) \text{ for all } s \leq t \leq T,\ X \in \mathbf{L}^\infty(\mathcal{F}_t), \tag{2.6.25}$$

and in this case, time-consistency of the family p is equivalent to its recursiveness. The natural choice $\Phi^{\text{opt}}_{s,t}(X) := \Phi^{\text{opt}}_{s,T}(X) - \Phi^{\text{opt}}_{t,T}(0)$ satisfies (2.6.25) and makes sense if $(\Phi^{\text{opt}}_{s,T}(0))_{0 \leq s \leq T}$ is a *deterministic* process, hence in particular if the process is constantly zero. This occurs for instance if all sets $\mathcal{C}_t$ from (2.6.20) are convex cones containing 0, so that the market functional is a time-consistent DMCohUF, and if in addition Φ is a time-consistent DMCohUF. Their convolution Φ^{opt} is then by Theorem 2.4.3 a strongly time-consistent DMCohUF and in particular normalized. Hence, in this coherent setting, the valuation family p is recursive as in $(\mathcal{R})$.

Remark 2.6.11. Let us indicate why we used the results of Section 2.5 about superhedging under constraints to prove that $(\Phi_t^{-\mathcal{C}_t})_{0 \leq t \leq T}$ is time-consistent. In Theorem 2.6.8, we have seen that those results imply that $-\mathcal{C}_t$ is a $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closed pre-acceptance set at time t, so that it is the acceptance set of $\Phi_t^{-\mathcal{C}_t}$ and the essential supremum in the definition of $\Phi_t^{-\mathcal{C}_t}(X)$ is attained by some $m_t \in \mathbf{L}^\infty(\mathcal{F}_t)$. Moreover, Theorem 2.6.8 tells us that there exists an $\mathcal{H}$-constrained hedging portfolio for $-X$ such that $\Phi_t^{-\mathcal{C}_t}(X)$ corresponds at each time t to minus the value $\hat{V}_t$ of this portfolio. In particular, this value process is uniformly bounded.

Now suppose we try to find a set $\mathcal{H}$ of integrands such that $(\Phi_t^{-\mathcal{C}_t})_{0\le t\le T}$ becomes time-consistent, without using the results from Section 2.5. From Lemma 2.3.25, we basically have two possibilities to prove time-consistency. Since the set of payoffs which can be superhedged from zero initial capital by trading during $(t, T]$ corresponds to a set of stochastic integrals with respect to the price process of the traded assets S, it seems natural to try and prove that the acceptance sets $(\mathcal{A}_t)$ of $(\Phi_t^{-\mathcal{C}_t})_{0\le t\le T}$ have the decomposition property

$$\mathcal{A}_s = \mathcal{A}_s(\mathcal{F}_t) + \mathcal{A}_t \quad \text{for all } s \le t. \tag{2.6.26}$$

But then the following problems occur:

- It is difficult to find conditions on the set $\mathcal{H}$ of integrands allowed for trading so that $-\mathcal{C}_t$ is an acceptance set of some MCUF at each time t, e.g., conditions which ensure that $-\mathcal{C}_t$ from (2.6.20) is a $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closed pre-acceptance set at each time t. But if this fails, the acceptance set of $\Phi_t^{-\mathcal{C}_t}$ differs from $-\mathcal{C}_t$ and we cannot expect that it has the nice structure as a set of integrals which we would like to exploit to prove (2.6.26). Replacing $\mathcal{C}_t$ by its closure $\overline{\mathcal{C}_t}$ in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ at each time t we lose the above integral structure. So the difficulty here is that a closure operation in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ does not fit well with stochastic integrals.

- Even if $-\mathcal{C}_t$ is for each time t the acceptance set of $\Phi_t^{-\mathcal{C}_t}$ and has a nice integral structure as above, we have not finished. Indeed, if $\int_s^T H\,dS$ is an element of $\mathcal{C}_s$, we can clearly split it for any $s \le t \le T$ into the sum of $\int_s^t H\,dS$ and $\int_t^T H\,dS$. But unfortunately, uniform boundedness from below of $(\int_s^u H\,dS)_{s\le u\le T}$ need not carry over to $(\int_t^u H\,dS)_{t\le u\le T}$, which is required if we want the latter to correspond to an element of $-\mathcal{C}_t$. So here the difficulty is to handle lower bounds on varying time intervals.

Alternatively, we might try to prove time-consistency directly from its definition, i.e., to show that

$$\Phi_t^{-\mathcal{C}_t}(X) = \Phi_t^{-\mathcal{C}_t}(Y) \quad \text{implies} \quad \Phi_s^{-\mathcal{C}_s}(X) = \Phi_s^{-\mathcal{C}_s}(Y) \tag{2.6.27}$$

for all $s \le t$, where $\Phi_u^{-\mathcal{C}_u}(X) = \operatorname{ess\,sup}\{m_u \in \mathbf{L}^\infty(\mathcal{F}_u) \mid X - m_u \in -\mathcal{C}_u\}$. It looks natural to try this by a contradiction argument, and that involves the construction of a hedging strategy starting at time s by pasting together at time t the strategies which are associated with $\Phi_s^{-\mathcal{C}_s}(X)$ and $\Phi_t^{-\mathcal{C}_t}(Y)$. But then similar problems as above occur:

- If $-\mathcal{C}_t$ is not a $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closed pre-acceptance set at time t, the supremum in the definition of $\Phi_t^{-\mathcal{C}_t}$ need not be attained. Hence we cannot relate to it one single hedging strategy, but need an entire sequence. However, pasting together countably many strategies is not feasible in general since we lose control over the required uniform lower bound for the corresponding value process.
- Even if $-\mathcal{C}_t$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$, so that the essential supremum is attained, it is not clear how $\Phi_s^{-\mathcal{C}_s}(X)$ and $\Phi_t^{-\mathcal{C}_t}(X)$ are related. The problem is that the value at time t of the hedging strategy corresponding to $\Phi_s^{-\mathcal{C}_s}$ need not be in $\mathbf{L}^\infty(\mathcal{F}_t)$, since it is not necessarily bounded from above.

This discussion explains why we decided to provide and work with the results about superhedging under constraints. ◇

We now turn to a discussion of the special case of unconstrained trading. In particular, we examine the effect of unconstrained trading on the MCUF Φ_t which expresses the preferences of an investor. For a static MCUF Φ_0, it is known (see e.g., Chapter 4.8 in [FS04] or [BEK05]) that this is captured by taking the infimum in the representation of Φ_0 only over all $Q \in \mathcal{M}^a(S)$, the set of all $Q \in \mathcal{P}^a$ which are local martingale measures for S, instead of taking it over the whole set $\mathcal{P}^a$. In other words, if α_0 is the concave conjugate of Φ_0, then the new MCUF $\Phi_0^{\mathrm{opt}}(\,.\,) = \sup_{G \in \mathcal{C}_0} \Phi_0(\,.\,+G)$ can be represented as

$$\Phi_0^{\mathrm{opt}}(X) = \inf_{Q \in \mathcal{M}^a(S)} \left\{ E_Q[X] - \alpha_0(Q) \right\}. \tag{2.6.28}$$

We shall obtain an analogous result in the dynamic case. One might expect that at time t, we have to take the essential supremum over the set of all local martingale measures for the process $(S_u)_{t \le u \le T}$, but we shall see that it is even possible to take the set of all equivalent local martingale measures for S (considered on all of $[\![0, T]\!]$).

Before we can state our result, we have to introduce some notation. *Unconstrained trading* means that we allow all admissible strategies for trading, i.e., we use the admissible hedging set $\mathcal{H} = L^a_{\mathrm{loc}}(S)$. We denote by $L^a_t(S) := \mathcal{H}_t$ the set of all processes H in $L^a_{\mathrm{loc}}(S)$ which are *(uniformly) admissible from time t* in the sense that the process $(\int_t^s H_u\, dS_u)_{t \le s \le T}$ is uniformly bounded from below, and by

$$\mathcal{D}_t := \left\{ \int_t^T H_u\, dS_u \,\middle|\, H \in L^a_t(S) \right\}$$

we denote the corresponding set of terminal values. Furthermore we distinguish between several sets of martingale measures:

Definition 2.6.12. For any $t \in [0, T]$ and $A \in \mathcal{F}_t$ we denote by $\mathcal{M}_t^{e,A}(S)$ the set of all $Q \in \mathcal{P}^e$ such that $(S_s \mathbf{1}_A)_{t \le s \le T}$ is a local martingale under Q, i.e., there exists an increasing sequence of $[t, T]$-valued stopping times τ_n with $\lim_{n\to\infty} P[\tau_n < T] = 0$ such that $(S_s^{\tau_n} \mathbf{1}_A \mathbf{1}_{\{\tau_n > t\}})_{t\le s\le T}$ is a uniformly integrable Q-martingale for each $n \in \mathbb{N}$. For $A = \Omega$ we write $\mathcal{M}_t^e(S) := \mathcal{M}_t^{e,\Omega}(S)$. In particular $\mathcal{M}^e(S) := \mathcal{M}_0^e(S)$ denotes the set of all equivalent local martingale measures for $S = (S_s)_{0\le s\le T}$.

Theorem 2.6.13. *Let Φ_t be an MCUF at time t with acceptance set $\mathcal{A}_t$ and concave conjugate α_t. Assume that Φ_t is continuous from below, $\inf_{X\in\mathcal{A}_t} E_Q[X] > -\infty$ for some $Q \in \mathcal{M}^e(S)$ and that (2.6.9) holds with*

$$\mathcal{C}_t = \left(\mathcal{D}_t - \mathbf{L}_+^0\right) \cap \mathbf{L}^\infty. \tag{2.6.29}$$

Then we have the representation

$$\Phi_t^{\text{opt}}(X) = \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{M}^e(S)} \left\{E_Q[X|\mathcal{F}_t] - \alpha_t(Q)\right\}. \tag{2.6.30}$$

Remark 2.6.14. Both (2.6.9) and the assumption that $\inf_{X\in\mathcal{A}_t} E_Q[X] > -\infty$ for some $Q \in \mathcal{M}^e(S)$ formalize the intuitive requirement that the a priori preferences Φ_t should fit together with the financial market. Like in the comment after Lemma 2.3.25, the second condition (involving Q) need only hold for $t = 0$ if Φ is strongly time-consistent. ◇

In order to prove Theorem 2.6.13, we need to characterize the set $\mathcal{M}^e(S)$ of equivalent local martingale measures in terms of $\mathcal{D}_t$.

Lemma 2.6.15. *Let $t \in [0, T]$, $A \in \mathcal{F}_t$, $Q \in \mathcal{P}^e$. Then*

$$\begin{aligned} Q \in \mathcal{M}_t^{e,A}(S) \quad &\Longleftrightarrow \quad E_Q[G\mathbf{1}_A|\mathcal{F}_t] \le 0 \quad Q\text{ - a.s. for all } G \in \mathcal{D}_t \cap \mathbf{L}^\infty \\ &\Longleftrightarrow \quad E_Q[G\mathbf{1}_A] \le 0 \quad \text{for all } G \in \mathcal{D}_t \cap \mathbf{L}^\infty. \end{aligned}$$

Proof. The second equivalence is trivial since $\mathcal{D}_t$ is closed under multiplication with $\mathbf{1}_B$, $B \in \mathcal{F}_t$. Hence we only have to prove the first equivalence.

"$\Rightarrow$": Let $Q \in \mathcal{M}_t^{e,A}(S)$. Then $(S_s\mathbf{1}_A)_{t\le s\le T}$ is a local Q-martingale. Each element G of $\mathcal{D}_t$ satisfies $G = \int_t^T H_s\,dS_s$ for some $H \in L_t^a(S)$. By Corollary 3.5 of [AS94], the uniform boundedness from below of $(\mathbf{1}_A \int_t^s H_s\,dS_s)_{t\le s\le T}$ implies that $(\mathbf{1}_A \int_t^s H_s\,dS_s)_{t\le s\le T}$ is also a local Q-martingale and hence, again by uniform boundedness from below, a Q-supermartingale. Thus $E_Q[G\mathbf{1}_A|\mathcal{F}_t] \le 0$ Q - a.s.

"$\Leftarrow$": Since $(S_s \mathbf{1}_A)_{t \leq s \leq T}$ is locally bounded, it is a local Q-martingale if and only if $(S_s^\tau \mathbf{1}_A \mathbf{1}_{\{\tau > t\}})_{t \leq s \leq T}$ is a Q-martingale for each stopping time $t \leq \tau \leq T$ such that $(S_s^\tau \mathbf{1}_A \mathbf{1}_{\{\tau > t\}})_{t \leq s \leq T}$ is uniformly bounded. For $t \leq s_1 \leq s_2 \leq T$ and $B \in \mathcal{F}_{s_1}$, define $H := \mathbf{1}_{\rrbracket \tau \wedge s_1, \tau \wedge s_2 \rrbracket} \mathbf{1}_B$ which is in $L_t^a(S)$. By assumption,

$$
\begin{aligned}
0 \geq E_Q\left[\mathbf{1}_A \int_t^T H_s \, dS_s\right] &= E_Q[\mathbf{1}_B(\mathbf{1}_A S_{s_2}^\tau - \mathbf{1}_A S_{s_1}^\tau)] \\
&= E_Q[\mathbf{1}_B(\mathbf{1}_A S_{s_2}^\tau \mathbf{1}_{\{\tau > t\}} - \mathbf{1}_A S_{s_1}^\tau \mathbf{1}_{\{\tau > t\}})],
\end{aligned}
$$

and since $B \in \mathcal{F}_{s_1}$ is arbitrary, we get that $E_Q[\mathbf{1}_A S_{s_2}^\tau \mathbf{1}_{\{\tau > t\}} | \mathcal{F}_{s_1}] \leq \mathbf{1}_A S_{s_1}^\tau \mathbf{1}_{\{\tau > t\}}$ Q- a.s. Because we also have $-H \in L_t^a(S)$, we even get equality, and so $(S_s^\tau \mathbf{1}_A \mathbf{1}_{\{\tau > t\}})_{t \leq s \leq T}$ is a Q-martingale.

□

Proof of Theorem 2.6.13. 1) By Theorem 2.6.8, $\Phi_t^{-\mathcal{C}_t}$ is a well-representable MCUF at time t with acceptance set $-\mathcal{C}_t$. Because $\mathcal{C}_t$ is a convex cone containing 0, the concave conjugate $\alpha_t^{-\mathcal{C}_t}$ of $\Phi_t^{-\mathcal{C}_t}$ only takes the values 0 and $-\infty$. We claim that we have for each $Q \in \mathcal{P}^e$ the explicit expression (with $\infty \cdot 0 := 0$)

$$
\alpha_t^{-\mathcal{C}_t}(Q) = -\infty \, \mathbf{1}_{(A^Q)^c}, \tag{2.6.31}
$$

where $A^Q \in \mathcal{F}_t$ is defined up to nullsets by

$$
\mathbf{1}_{A^Q} = \text{ess sup} \left\{ \mathbf{1}_A \,\middle|\, A \in \mathcal{F}_t \text{ and } Q \in \mathcal{M}_t^{e,A}(S) \right\}.
$$

Intuitively, A^Q is the largest $\mathcal{F}_t$-measurable set on which $(S_s)_{t \leq s \leq T}$ is a local Q-martingale. To see (2.6.31), note first that $Q \in \mathcal{M}_t^{e,A^Q}(S)$. Since $0 \in \mathcal{C}_t$ and $\mathcal{D}_t \cap \mathbf{L}^\infty \subseteq \mathcal{C}_t$, Lemma 2.6.15 implies that

$$
\mathbf{1}_{A^Q} \operatorname*{ess\,inf}_{G \in -\mathcal{C}_t} E_Q[G | \mathcal{F}_t] \equiv 0 \quad P\text{ - a.s.},
$$

which means by Lemma 2.3.12 that $\alpha_t^{-\mathcal{C}_t}(Q) = 0$ on A^Q. To prove (2.6.31), it thus only remains to show that

$$
\operatorname*{ess\,inf}_{G \in -\mathcal{C}_t} E_Q[G | \mathcal{F}_t] = -\infty \quad P\text{ - a.s. on } (A^Q)^c.
$$

For this, we may assume that $P[(A^Q)^c] > 0$ so that $(S_s)_{t \leq s \leq T}$ with positive probability fails to be a local Q-martingale. By Lemma 2.6.15, we can thus find a $B \in \mathcal{F}_t$ with $P[B] > 0$ and $B \subseteq (A^Q)^c$ and some $G_0 \in \mathcal{D}_t \cap \mathbf{L}^\infty \subseteq \mathcal{C}_t$ such that $E_Q[-G_0 | \mathcal{F}_t] \leq -\varepsilon$ on B for some $\varepsilon > 0$. Closedness of $\mathcal{C}_t$ under multiplication with non-negative scalars then implies that $\operatorname*{ess\,inf}_{G \in -\mathcal{C}_t} E_Q[G|\mathcal{F}_t] = -\infty$ on B. But this must even hold on the whole set $(A^Q)^c$. In fact, if it does not, we obtain some set $\tilde{B} \in \mathcal{F}_t$ with $P[\tilde{B}] > 0$ and $\tilde{B} \subseteq (A^Q)^c$ such that

$$0 \geq \operatorname*{ess\,inf}_{G \in -\mathcal{C}_t} E_Q[G|\mathcal{F}_t] \geq -m > -\infty \text{ on } \tilde{B} \quad \text{for some } m > 0.$$

Closedness of $\mathcal{C}_t$ under multiplication with non-negative scalars now implies that $\operatorname*{ess\,inf}_{G \in -\mathcal{C}_t} E_Q[G|\mathcal{F}_t] = 0$ on $\tilde{B}$ and therefore by Lemma 2.6.15 that $Q \in \mathcal{M}_t^{e, A^Q \cup \tilde{B}}(S)$. But this contradicts the definition of A^Q, and hence we have proved (2.6.31).

2) From (2.6.31), our assumptions and Theorem 2.4.3, the MCUF $\Phi_t \Box \Phi_t^{-\mathcal{C}_t}$ is well-representable and since convoluting two MCUFs means adding their concave conjugates we obtain

$$\Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{P}^e} \left\{ E_Q[X|\mathcal{F}_t] - \alpha_t(Q) + \infty\, \mathbf{1}_{(A^Q)^c} \right\}. \qquad (2.6.32)$$

This suggests that it should be enough to take the above essential infimum only over those $Q \in \mathcal{P}^e$ that have $P[A^Q] = 1$, which means that Q should be in $\mathcal{M}_t^e(S)$. We now prove that this is true by showing that

$$\Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) = \operatorname*{ess\,inf}_{Q' \in \mathcal{M}_t^e(S)} \left\{ E_{Q'}[X|\mathcal{F}_t] - \alpha_t(Q') \right\}. \qquad (2.6.33)$$

By **(NFLVR)**, there exists $\hat{Q} \in \mathcal{M}^e(S) \subseteq \mathcal{M}_t^e(S)$ with density process $\hat{Z}$. For any $Q \in \mathcal{P}^e$ with density process Z^Q, define a new measure $Q' \in \mathcal{P}^e$ with density process Z' by

$$\frac{dQ'}{dP} := \mathbf{1}_{A^Q} \frac{Z_T^Q}{Z_t^Q} + \mathbf{1}_{(A^Q)^c} \frac{\hat{Z}_T}{\hat{Z}_t}$$

so that $Q' \in \mathcal{M}_t^e(S)$ by the definition of A^Q. Since

$$E_{Q'}[\,.\,|\mathcal{F}_t] = \mathbf{1}_{A^Q}\, E_Q[\,.\,|\mathcal{F}_t] + \mathbf{1}_{(A^Q)^c}\, E_{\hat{Q}}[\,.\,|\mathcal{F}_t]$$

we obtain from Lemma 2.3.12 and (2.6.31) that

$$\begin{aligned}
& E_{Q'}[X|\mathcal{F}_t] - \alpha_t(Q') - \alpha_t^{-\mathcal{C}_t}(Q') \\
&= \mathbf{1}_{A^Q} \left(E_Q[X|\mathcal{F}_t] - \alpha_t(Q) - \alpha_t^{-\mathcal{C}_t}(Q) \right) \\
&\qquad + \mathbf{1}_{(A^Q)^c} \left(E_{\hat{Q}}[X|\mathcal{F}_t] - \alpha_t(\hat{Q}) - \alpha_t^{-\mathcal{C}_t}(\hat{Q}) \right) \\
&= \mathbf{1}_{A^Q} \left(E_Q[X|\mathcal{F}_t] - \alpha_t(Q) + \infty\, \mathbf{1}_{(A^Q)^c} \right) \\
&\qquad + \mathbf{1}_{(A^Q)^c} \left(E_{\hat{Q}}[X|\mathcal{F}_t] - \alpha_t(\hat{Q}) + \infty\, \mathbf{1}_{(A^{\hat{Q}})^c} \right).
\end{aligned}$$

But $(A^{\hat{Q}})^c$ is a P-nullset since $\hat{Q} \in \mathcal{M}_t^e(S)$ and so $\alpha_t^{-C_t}(\hat{Q}) = 0 = -\infty \mathbf{1}_{(A^{\hat{Q}})^c}$ by (2.6.31). The same is true for Q'. Hence, using $A^Q \cap (A^Q)^c = \emptyset$ and (2.6.31) for Q', we get

$$\begin{aligned} & E_{Q'}[X|\mathcal{F}_t] - \alpha_t(Q') \\ &= E_{Q'}[X|\mathcal{F}_t] - \alpha_t(Q') + \infty \mathbf{1}_{(A^{Q'})^c} \\ &= \mathbf{1}_{A^Q}\Big(E_Q[X|\mathcal{F}_t] - \alpha_t(Q)\Big) + \mathbf{1}_{(A^Q)^c}\Big(E_{\hat{Q}}[X|\mathcal{F}_t] - \alpha_t(\hat{Q})\Big) \\ &\leq E_Q[X|\mathcal{F}_t] - \alpha_t(Q) + \infty \mathbf{1}_{(A^Q)^c} \end{aligned}$$

by looking separately at A^Q and $(A^Q)^c$. This shows that we can replace any $Q \in \mathcal{P}^e$ by a corresponding $Q' \in \mathcal{M}_t^e(S)$ when taking the essential infimum in (2.6.32) and thus establishes (2.6.33).

3) In view of (2.6.33), it only remains to show that

$$\operatorname*{ess\,inf}_{Q \in \mathcal{M}_t^e(S)} \big\{E_Q[X|\mathcal{F}_t] - \alpha_t(Q)\big\} = \operatorname*{ess\,inf}_{Q \in \mathcal{M}^e(S)} \big\{E_Q[X|\mathcal{F}_t] - \alpha_t(Q)\big\}.$$

The inequality "$\leq$" is clear since $\mathcal{M}_t^e(S) \supseteq \mathcal{M}^e(S)$. To prove the converse, we show that for any $Q \in \mathcal{M}_t^e(S)$ with density process Z, there exists $Q' \in \mathcal{M}^e(S)$ with density process Z' such that

$$Z_T = h_t Z'_T$$

for some $\mathcal{F}_t$-measurable $h_t > 0$. Because then we have from $\frac{Z_T}{Z_t} = \frac{Z'_T}{Z'_t}$ and by using (2.3.3) that

$$E_Q[X|\mathcal{F}_t] - \alpha_t(Q) = E_{Q'}[X|\mathcal{F}_t] - \alpha_t(Q'),$$

and obtain "$\geq$". To construct Q', take some $\hat{Q} \in \mathcal{M}^e(S)$ with density process $\hat{Z}$ and define

$$Z'_T := \hat{Z}_t \frac{Z_T}{Z_t} = \frac{1}{h_t} Z_T$$

with $h_t = \frac{Z_t}{\hat{Z}_t}$. Then $Q' \in \mathcal{M}^e(S)$ because $Z'S$ is a local P-martingale on all of $[\![0,T]\!]$: on $[\![0,t]\!]$ because $Z' = \hat{Z}$ on $[\![0,t]\!]$ and $\hat{Q} \in \mathcal{M}^e(S)$, and on $[\![t,T]\!]$ because

$$Z' = \frac{1}{h_t} Z \quad \text{on } [\![t,T]\!]$$

and ZS is a local P-martingale on $[\![t,T]\!]$ since $Q \in \mathcal{M}_t^e(S)$. This completes the proof.

□

As an immediate consequence we get the following no-arbitrage result for the indifference value in the case of unconstrained trading:

Corollary 2.6.16. *Under the assumptions of Theorem 2.6.13 and with $\mathcal{C}_t$ as in (2.6.29), the valuations p_t and p_t^s are consistent with the no-arbitrage principle in the following two senses:*

a) If $X \in \mathbf{L}^\infty$ is attainable from time t in the sense that $X = x_t + \int_t^T H_s\,dS_s$ with $x_t \in \mathbf{L}^\infty(\mathcal{F}_t)$ and $H \in L_t^a(S)$ such that $\left(\int_t^u H_s\,dS_s\right)_{t\le u\le T}$ is uniformly bounded, then

$$p_t(X) = p_t\left(x_t + \int_t^T H_s\,dS_s\right) = p_t^s\left(x_t + \int_t^T H_s\,dS_s\right) = p_t^s(X) = x_t.$$

b) Both, p_t and p_t^s, take values in the interval of possible arbitrage-free valuations, i.e.,

$$\operatorname*{ess\,inf}_{Q\in\mathcal{M}^e(S)} E_Q[X|\mathcal{F}_t] \le p_t(X) \le p_t^s(X) \le \operatorname*{ess\,sup}_{Q\in\mathcal{M}^e(S)} E_Q[X|\mathcal{F}_t] \quad \text{for all } X \in \mathbf{L}^\infty.$$

Proof. a) Since we have $E_Q[X|\mathcal{F}_t] = x_t$ for any $Q \in \mathcal{M}^e(S)$, this follows immediately from (2.6.12) and the representation (2.6.30).

b) Since $-\mathcal{C}_t$ is a convex cone containing 0, this follows from Proposition 2.6.5 and Remark 2.5.12.

□

In all of Section 2.6, we have assumed that it is the MCUF Φ representing the agent's preferences which is continuous from below, and not the market MCUF $\Phi_t^{-\mathcal{C}_t}$. (Note that for Theorem 2.4.3 it is enough if one of the two is continuous from below.) The reason is the following. It is known that in the unconstrained case we can represent $\Phi_0^{-C_0}$ analogously to (2.6.28) as

$$\Phi_0^{-C_0}(X) = \inf_{Q\in\mathcal{M}^a(S)} E_Q[X],$$

where $\mathcal{M}^a(S)$ denotes the set of all local martingale measures $Q \in \mathcal{P}^a$ for S. It follows from Corollary 4.35 of [FS04] that continuity from below of $\Phi_0^{-C_0}$ implies that $\mathcal{M}^a(S)$ is weakly compact (since it is weakly closed). But if the price process S is continuous and the filtration is quasi left-continuous, Corollary 7.2 of [Del92] then implies that $\mathcal{M}^a(S)$ is a singleton so that the market must be complete. This shows that it may be rather restrictive to insist on a market DMCUF which is continuous from below.

We finish this section with a comment about the connection between the indifference values $p_t(X)$, $p_t^s(X)$ and *good deal bounds.*

The no-arbitrage price bounds $\Phi_t^{-\mathcal{C}_t}(\,.\,)$ and $-\Phi_t^{-\mathcal{C}_t}(-\,.\,)$ induced by superhedging are usually not sharp enough to be useful for pricing in practice. Therefore several approaches have been suggested to define tighter price bounds which are less restrictive than the choice of one pricing measure; see, e.g., [BL00], [CSR00] or [CGM01]. In particular, Cochrane/Saà-Requejo [CSR00] introduced the concept of *good deal bounds.* These price bounds are obtained by ruling out not only arbitrage opportunities but also *good deals*, which are in [CSR00] defined as investment opportunities with a high Sharpe ratio. This procedure is justified by arguing that Sharpe ratios observed in the market tend to be rather low. Subsequently, the good deal pricing approach has been generalized by many authors; see, e.g., [JK01], [CH02], [Cer03] or [Sta04]. In particular, they defined good deals more generally as investment opportunities which are in some sense desirable and do not necessarily have a high Sharpe ratio. To justify the exclusion of good deals, it is argued like for arbitrage opportunities that they would vanish immediately from the market by trading.

For these good deal price bounds, it is well known that they correspond to risk measures (and hence to MCUFs). However, the literature often creates the impression that they are somehow generic and independent of individual preferences. This is not the case: One has to specify the set of good deals, and we shall see presently that this basically corresponds to the choice of an MCUF and hence of a specification of utility.

The following definition of good deals (in a static and coherent framework) is taken from Jaschke/Küchler [JK01]. They fix $\mathcal{C}_0$, a convex cone containing zero of payoffs which can be superhedged with zero initial capital, and in addition a coherent acceptance set $\mathcal{A}_0 \subseteq \mathbf{L}^\infty$, i.e., $\mathcal{A}_0$ is the acceptance set of some MCohUF at time 0. This specifies the set of desirable payoffs, and the most conservative choice is $\mathcal{A}_0 = \mathbf{L}^\infty_+$. In this latter case, the good deal price bounds correspond to those obtained by excluding arbitrage opportunities only.

Definition 2.6.17. An element $X \in \mathcal{C}_0$ is called *good deal of the first kind* if X is contained in $\mathcal{A}_0$ and $X \neq 0$, and *good deal (of the second kind)* if there exists $\varepsilon > 0$ such that $X - \varepsilon \mathbf{1}_\Omega \in \mathcal{A}_0$.

Whereas good deals of the first kind represent opportunities to get something good for free, where the good part may or may not come, those of the second kind are "cash-and-carry good deals" and yield a sure profit. Jaschke and Küchler consider the second concept to be much more important. They argue that any arbitrage transaction in practice involves some risks or costs that cannot be captured in a model. Therefore arbitrageurs will only act if the anticipated gain is substantial enough. As a consequence, they only consider good deals of the second kind, and we do the same here. The lower bound for prices for X obtained by excluding these good deals is given by

$$\pi_0^{\mathcal{A}_0}(X) := \sup\{m_0 \in I\!R \mid X - m_0 \mathbf{1}_\Omega + G \in \mathcal{A}_0 \quad \text{for some } G \in \mathcal{C}_0\}.$$

In fact, if the agent could buy the future payoff X for a price $\pi_0(X) < \pi_0^{\mathcal{A}_0}(X)$, then there exist $G \in \mathcal{C}_0$ and $\varepsilon > 0$ with $\pi_0(X) + \varepsilon \leq \pi_0^{\mathcal{A}_0}(X) - \varepsilon$ and such that $X - (\pi_0(X) + \varepsilon)\mathbf{1}_\Omega + G$ is contained in the set of desirable payoffs $\mathcal{A}_0$. Hence the agent could buy X for $\pi_0(X)$, use the superhedging strategy corresponding to G and obtain a resulting payoff $X - \pi_0(X) + G$ which is a good deal. As before, selling X corresponds to buying $-X$, and so the good deal price bounds are given by

$$\left[\pi_0^{\mathcal{A}_0}(X), -\pi_0^{\mathcal{A}_0}(-X)\right].$$

The above concept of good deal price bounds can immediately be generalized to a dynamic and convex framework. For a convex (but still static) setting this can also be found in Staum [Sta04]. However, he works with a slightly different definition, and the one given in [JK01] fits better into our framework. We model the set $\mathcal{C}_t$ of payoffs which are superhedgeable via trading during $(t, T]$ by a non-empty, convex and $\mathcal{F}_t$-regular subset of $\mathbf{L}^\infty$; compare Lemma 2.4.5. The set of desirable payoffs is given by some pre-acceptance set $\mathcal{B}_t$ at time t. In analogy to the static case, we then define a good deal as follows:

Definition 2.6.18. Fix $Y \in \mathcal{B}_t$. Then $X \in \mathcal{C}_t$ is called a *good deal at time t* if there exists a constant $\varepsilon > 0$ and a set $A \in \mathcal{F}_t$, $P[A] > 0$ such that $(X - \varepsilon\mathbf{1}_\Omega)\,\mathbf{1}_A + Y\mathbf{1}_{A^c}$ is contained in $\mathcal{B}_t$.

Note that $\mathcal{B}_t$ is $\mathcal{F}_t$-regular so that the definition does not depend on the choice of the element $Y \in \mathcal{B}_t$; this is introduced since whether X is a good deal at time t or not should not depend on events which can already be ruled out at this time. Note that also $\mathcal{B}_t$ need not contain 0 which is otherwise a natural choice for Y. The lower price bound obtained from excluding good deals is then given by

$$\pi_t^{\mathcal{B}_t}(X) := \text{ess sup}\left\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t)\,\middle|\, X - m_t + G \in \mathcal{B}_t \text{ for some } G \in \mathcal{C}_t\right\}. \quad (2.6.34)$$

The reasoning is similar to the static case. Indeed, if the agent could buy the future payoff X for a price $\pi_t(X)$ which is not greater or equal to $\pi_t^{\mathcal{B}_t}(X)$, then there exist $\varepsilon > 0$ and a set $A \in \mathcal{F}_t$ with $P[A] > 0$ such that $\pi_t(X) + \varepsilon\mathbf{1}_\Omega \leq \pi_t^{\mathcal{B}_t}(X) - \varepsilon\mathbf{1}_\Omega$ on A. By (2.6.34) we can find a subset $B \in \mathcal{F}_t$ of A with $P[B] > 0$, $m_t \in \mathbf{L}^\infty(\mathcal{F}_t)$ and $G \in \mathcal{C}_t$ such that $Y' := X - m_t + G \in \mathcal{B}_t$ and $m_t \geq \pi_t^{\mathcal{B}_t}(X) - \varepsilon\mathbf{1}_\Omega$ on B. The $\mathcal{F}_t$-regularity of $\mathcal{B}_t$ implies that also $Y'\mathbf{1}_B + Y\mathbf{1}_{B^c} \in \mathcal{B}_t$. But since $\pi_t(X) + \varepsilon\mathbf{1}_\Omega \leq m_t$ on B and $-\mathcal{B}_t$ is solid, we now obtain that $((X - \pi_t(X) + G) - \varepsilon\mathbf{1}_\Omega)\,\mathbf{1}_B + Y\mathbf{1}_{B^c} \in \mathcal{B}_t$, i.e., that $X - \pi_t(X) + G$ is a good deal.

Next we show how the above price bound is connected to an indifference valuation functional $p_t(X)$. To this end, we recall from (2.3.2) in Lemma 2.3.9 that $\mathcal{B}_t$ induces an MCUF Φ_t at time t by

$$\Phi_t(X) := \Phi_t^{\mathcal{B}_t}(X) = \text{ess sup}\left\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t)\,\middle|\, X - m_t \in \mathcal{B}_t\right\}.$$

This representation implies that

$$\begin{aligned}
\Phi_t^{\mathrm{opt}}(X) &= \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} \Phi_t(X+G) \\
&= \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} \operatorname{ess\,sup}\left\{ m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \,\middle|\, X+G-m_t \in \mathcal{B}_t \right\} \\
&= \operatorname{ess\,sup}\left\{ m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \,\middle|\, X-m_t+G \in \mathcal{B}_t \quad \text{for some } G \in \mathcal{C}_t \right\} \\
&= \pi_t^{\mathcal{B}_t}(X).
\end{aligned}$$

Hence if $\Phi_t^{\mathrm{opt}}(0) = 0$ so that $p_t(X) = \Phi_t^{\mathrm{opt}}(X)$, the lower good deal bound is the indifference value $p_t(X)$ and

$$\left[\pi^{\mathcal{B}_t}(X), -\pi^{\mathcal{B}_t}(-X)\right] = \left[p_t(X), p_t^s(X)\right]$$

is the interval of possible prices for X which do not yield a good deal. We recall from Proposition 2.6.5 that we might need additional assumptions to have price bounds which are actually tighter than those obtained by excluding arbitrage opportunities.

Using that $p_t(\,.\,)$ is defined as the indifference value, we can also give another interpretation for why $p_t(\,.\,)$ can be viewed as a lower price bound obtained by excluding (slightly differently defined) good deals. We fix a set $\mathcal{C}_t$ of superhedgeable payoffs and an MCUF Φ_t. Then we might call $X \in \mathcal{C}_t$ *useful deal* if it increases the maximal attainable utility, i.e., if

$$\Phi_t^{\mathrm{opt}}(X) = \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} \Phi_t(X+G) \geq \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} \Phi_t(G) = \Phi_t^{\mathrm{opt}}(0)$$

and the inequality is strict with strictly positive probability. This implies that

$$\left[p_t(X),\ p_t^s(X)\right]$$

is the interval of all prices for X which do not yield a useful deal.

Staum [Sta04] proves fundamental theorems of asset pricing for good deal bounds. In particular, he gives in his Theorem 6.1 an equivalent condition for the weak no-arbitrage condition (2.6.16). This theorem and its proof can easily be adapted to our framework; we simply state the result without giving a proof.

Theorem 2.6.19. *Let $-\mathcal{C}_t \subseteq \mathbf{L}^\infty$ be a pre-acceptance set at time t containing* 0 *such that $\Phi_t^{-\mathcal{C}_t}(0) = 0$. Let Φ_t be an MCUF at time t with acceptance set $\mathcal{A}_t$ such that $\Phi_t(0) \geq 0$. Then*

$$p_t(X) \leq -\Phi_t^{-\mathcal{C}_t}(-X) \text{ for all } X \in \mathbf{L}^\infty \quad \text{and} \quad \Phi_t^{\mathrm{opt}}(0) = \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} \Phi_t(G) = 0$$

if and only if

$$(\mathcal{C}_t - \mathcal{A}_t) \cap \left\{ X \in \mathbf{L}^\infty \,\middle|\, P\left[\Phi_t^{-\mathcal{C}_t}(X) > 0\right] > 0 \right\} = \emptyset.$$

2.7 Examples

2.7.1 Time-consistency and normalization

This example illustrates several points we have discussed in this chapter. For the exponential utility function $\mathbf{U}(x) = -e^{-x}$ we define by

$$\Phi_t(X) := \mathbf{U}^{-1}\big(E[\mathbf{U}(X)|\mathcal{F}_t]\big) = -\log E\big[e^{-X}\big|\mathcal{F}_t\big] \quad \text{for } X \in \mathbf{L}^\infty \tag{2.7.1}$$

the corresponding $\mathcal{F}_t$-conditional exponential certainty equivalent; see Example 2.3.3. Then $\Phi = (\Phi_t)_{0\le t\le T}$ is a DMCUF, each Φ_t is clearly continuous from below, and the concave conjugate functional of Φ_t is

$$\alpha_t(Q) = -E_Q\left[\log \frac{Z_T^Q}{Z_t^Q}\,\middle|\,\mathcal{F}_t\right] =: -f_t^e(Q|P), \tag{2.7.2}$$

i.e., minus the $\mathcal{F}_t$-conditional relative entropy of Q with respect to P. This is shown in Section 4 of [DS05]; see also Example 4.33 in [FS04]. The DMCUF $\Phi = (\Phi_t)_{0\le t\le T}$ is clearly normalized and time-consistent due to the explicit expression (2.7.1); hence Φ is strongly time-consistent. Moreover, each Φ_t is well-representable since Lemma 2.3.14 and (2.7.1) imply I) of Theorem 2.3.16. In fact, from Jensen's inequality we obtain $E[\mathbf{U}(X)|\mathcal{F}_t] \le \mathbf{U}(E[X|\mathcal{F}_t])$, hence $E[X|\mathcal{F}_t] \ge \Phi_t(X) \ge 0$ for all $X \in \mathcal{A}_t$ and therefore $\inf_{X\in\mathcal{A}_t} E[X] \ge 0 > -\infty$. From Theorem 2.3.16 and (2.7.2), we thus have

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{P}^e} \big\{E_Q[X|\mathcal{F}_t] + f_t^e(Q|P)\big\}.$$

Consider next a financial market as in Section 2.6. Choose $\mathcal{H} = L^a_{\text{loc}}(S)$ so that we have no constraints, define $\mathcal{C}_t$ by (2.6.29) and $\hat{\Phi} = (\hat{\Phi}_t)_{0\le t\le T}$ by (2.6.22) so that $\hat{\Phi}_t$ is by Theorem 2.6.8 the market MCUF induced by $\mathcal{C}_t$. Moreover, $\hat{\Phi}$ is also normalized, well-representable and strongly time-consistent by Theorem 2.6.8. Define

$$\Phi_t^{\text{opt}}(X) := \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} \Phi_t(X+G) \quad \text{for } t \in [0,T] \text{ and } X \in \mathbf{L}^\infty$$

and assume that

$$\Phi_t^{\text{opt}}(0) = \operatorname*{ess\,sup}_{G\in\mathcal{C}_t} \Phi_t(G) \in \mathbf{L}^\infty. \tag{2.7.3}$$

We give below a sufficient condition on S to ensure (2.7.3). Due to (2.7.2) and Theorem 2.6.13, we have

$$\begin{aligned}\Phi_t^{\text{opt}}(X) &= \Phi_t\Box(-\mathcal{C}_t)(X) = \Phi_t\Box\hat{\Phi}_t(X)\\ &= \operatorname*{ess\,inf}_{Q\in\mathcal{M}^e(S)} \big\{E_Q[X|\mathcal{F}_t] + f_t^e(Q|P)\big\}\end{aligned} \tag{2.7.4}$$

and by Theorem 2.4.3, Φ^{opt} is then again a strongly time-consistent DMCUF.

Now impose on the financial market the assumptions that $P \notin \mathcal{M}^e(S)$ (so S is not a local P-martingale) and that $\inf_{Q\in\mathcal{M}^e(S)} f_0^e(Q|P) < \infty$, so that there exists an equivalent local martingale measure for S with finite relative entropy with respect to P. Then it is well known that the minimal entropy martingale measure

$$Q^e := \operatorname{argmin}\{f_0^e(Q|P) \mid Q \in \mathcal{M}^e(S)\}$$

exists in $\mathcal{M}^e(S)$ and is unique, and we have

$$f_0^e(Q^e|P) > 0 \tag{2.7.5}$$

because P is not in $\mathcal{M}^e(S)$. But (2.7.5) implies by (2.7.4) that

$$\Phi_0^{\mathrm{opt}}(0) = \inf_{Q\in\mathcal{M}^e(S)} f_0^e(Q|P) = f_0^e(Q^e|P) > 0,$$

and therefore Φ^{opt} is not normalized. Hence this example illustrates that

- a DMCUF may be strongly time-consistent without being normalized.
- the convolution of two normalized DMCUFs may fail to be normalized.

To finish the example, let us briefly discuss how to guarantee the condition (2.7.3). By the explicit expression (2.7.1) for Φ_t, (2.7.3) is equivalent to

$$\operatorname*{ess\,sup}_{G\in\mathcal{C}_t} E[\mathbf{U}(G)|\mathcal{F}_t] \in \mathbf{L}^\infty, \tag{2.7.6}$$

and since $G \equiv 0$ is in $\mathcal{C}_t$, it is enough to have an upper bound for $E[\mathbf{U}(G)|\mathcal{F}_t]$ uniformly over $G \in \mathcal{C}_t$. Applying Fenchel's inequality

$$\mathbf{U}(x) = -e^{-x} \le \sup_{x'>0} \big(\mathbf{U}(x') - x'y\big) + xy = y\log y - y + xy$$

with $y = \frac{Z_T^Q}{Z_t^Q}$ for some $Q \in \mathcal{M}^e(S)$ gives

$$E[\mathbf{U}(G)|\mathcal{F}_t] \le f_t^e(Q|P) - 1 + E_Q[G|\mathcal{F}_t] \le f_t^e(Q|P)$$

because $E_Q[G|\mathcal{F}_t] \le 0$ for any $G \in \mathcal{C}_t$, since $\int_t^\cdot H\,dS$ for $H \in \mathcal{H}_t$ is a Q-supermartingale for any $Q \in \mathcal{M}^e(S)$; see Lemma 2.6.15. Hence (2.7.6) holds as soon as

$$\operatorname*{ess\,inf}_{Q\in\mathcal{M}^e(S)} f_t^e(Q|P) \in \mathbf{L}^\infty.$$

One sufficient condition for this is that there exists some $Q \in \mathcal{M}^e(S)$ satisfying the reverse Hölder inequality $R_{L\log L}(P)$, i.e.,

$$f_t^e(Q|P) = E\left[\frac{Z_T^Q}{Z_t^Q}\log\frac{Z_T^Q}{Z_t^Q}\,\middle|\,\mathcal{F}_t\right] \le C$$

for all $t \in [0,T]$ with some constant C. This ends the example.

2.7.2 DMCUFs, indifference valuation and BSDEs

In this subsection we first recall and extend some known results about DMCUFs which are described by backward stochastic differential equations (BSDEs for short), since this provides us with a big class of time-consistent DMCUFs. Then we represent the preferences of our investor by such a DMCUF Φ and try to express the corresponding indifference valuation DMCUF in terms of BSDEs as well. As in Section 2.6, we apply the convolution to Φ and the market DMCUF given via the superhedging price to obtain an equivalent description for the indifference value respectively for the DMCUF Φ^{opt}. To this end, we first prove that the market DMCUF can also be described by a BSDE. Then we show that the DMCUF Φ^{opt} corresponds to a BSDE whose driver is given by the pointwise convolution of the drivers for Φ and for the market DMCUF. This extends results of Barrieu/El Karoui [BEK04] about the convolution of dynamic risk measures described by BSDEs.

We start by recalling a well-known existence result for solutions of BSDEs. To this end we introduce some notation and conventions. In particular, we require a very special structure of the filtration since the proof of the existence result relies on a martingale representation theorem.

Remark 2.7.1. An existence proof based on fixed point arguments instead of a martingale representation theorem can be found in [EKH97]. However, the integrability conditions there are too restrictive for our purposes. ◇

Let $W = (W_t)_{0\leq t\leq T}$ be a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ be the augmented filtration generated by W. As before, we assume that $\mathcal{F} = \mathcal{F}_T$. We introduce the notation $\mathbf{M}^2_{\mathbb{F}}(0, T; I\!R^n)$ for the space of all equivalence classes of $I\!R^n$-valued, $\mathbb{F}$-progressively measurable processes $(\vartheta_t)_{0\leq t\leq T}$ such that

$$E\left[\int_0^T \|\vartheta_t\|^2\,dt\right] < \infty,$$

where $\|\,.\,\|$ stands for the Euclidean norm. Hence two processes ϑ^1 and ϑ^2 are identified in $\mathbf{M}^2_{\mathbb{F}}(0, T; I\!R^n)$ if

$$E\left[\int_0^T \|\vartheta^1_t - \vartheta^2_t\|^2\,dt\right] = 0.$$

The drivers which appear in the BSDEs we consider are product-measurable functions $g : \Omega \times [0, T] \times I\!R \times I\!R^d \to I\!R$. We often write $g_t(y, z)$ instead of $g(\omega, t, y, z)$ and usually impose some of the following properties:

Definition 2.7.2. ($\mathcal{A}$) $(\omega, t) \mapsto g(\omega, t, y, z)$ is in $\mathbf{M}^2_{\mathbb{F}}(0, T; I\!R)$ for any $y \in I\!R$, $z \in I\!R^d$.

($\mathcal{B}$) g is Lipschitz in $(y, z) \in I\!R \times I\!R^d$, i.e., there exists a constant $C > 0$ such that $dP \otimes dt$-a.s. for all $(y_0, z_0), (y_1, z_1) \in I\!R \times I\!R^d$

$$|g_t(y_0, z_0) - g_t(y_1, z_1)| \le C(|y_0 - y_1| + \|z_0 - z_1\|).$$

($\mathcal{C}$) $dP \otimes dt$-a.s., g satisfies $g_t(y, 0) \equiv 0$ for any $y \in I\!R$.

($\mathcal{D}$) g does not depend on y.

($\mathcal{E}$) g is concave in (y, z), i.e. $dP \otimes dt$-a.s. for all (y_0, z_0), $(y_1, z_1) \in I\!R \times I\!R^d$ and $\alpha \in (0, 1)$

$$g_t\big(\alpha y_0 + (1-\alpha)y_1, \alpha z_0 + (1-\alpha)z_1\big) \ge \alpha g_t(y_0, z_0) + (1-\alpha)g_t(y_1, z_1).$$

($\mathcal{F}$) g is positively homogeneous in (y, z), i.e., $dP \otimes dt$-a.s. for all $\lambda \ge 0$ and $(y, z) \in I\!R \times I\!R^d$

$$g_t(\lambda y, \lambda z) = \lambda g_t(y, z).$$

The following result is taken from Peng [Pen97], Proposition 36.4; see also Pardoux/Peng [PP90], Theorem 4.1.

Theorem 2.7.3. *Let g satisfy ($\mathcal{A}$) and ($\mathcal{B}$) of Definition 2.7.2. For any (fixed) random variable $X \in \mathbf{L}^2 = \mathbf{L}^2(\Omega, \mathcal{F}_T, P)$ there exists a unique pair of processes (y, z) in $\mathbf{M}^2_{\mathbb{F}}(0, T; I\!R) \times \mathbf{M}^2_{\mathbb{F}}(0, T; I\!R^d)$ with y continuous, satisfying the BSDE*

$$y_t = X + \int_t^T g_s(y_s, z_s)\,ds - \int_t^T z_s^*\,dW_s, \quad t \in [0, T]. \tag{2.7.7}$$

The pair (y, z) is called g-solution with terminal value X *and satisfies $y_t \in \mathbf{L}^2(\mathcal{F}_t)$ for each t. If the driver g satisfies in addition property ($\mathcal{C}$), then $\mathcal{E}^g[X] := y_0$ is called* g-expectation of X *and for each $t \in [0, T]$ there exists a P-a.s. unique $\eta_t \in \mathbf{L}^2(\mathcal{F}_t)$ such that*

$$\mathcal{E}^g[\mathbf{1}_A X] = \mathcal{E}^g[\mathbf{1}_A \eta_t] \quad \text{for all } A \in \mathcal{F}_t.$$

Then $\eta_t = y_t$ and we call $\mathcal{E}^g_t[X] := y_t$ the conditional g-expectation of X under $\mathcal{F}_t$.

Remark 2.7.4. We can and do choose the process y in Theorem 2.7.3 continuous, since this will allow us to draw conclusions about the behavior of y_t which hold almost surely, simultaneously for all $t \in [0, T]$, instead of only almost surely almost everywhere. ◇

Next we recall some well-known properties of g-solutions from which we shall deduce conditions on the driver g under which a g-solution describes a time-consistent DMCUF.

Proposition 2.7.5. *Let g satisfy conditions $(\mathcal{A})$, $(\mathcal{B})$ of Definition 2.7.2 and denote for any $X \in \mathbf{L}^2$ by (y^X, z^X) the corresponding g-solution as defined in (2.7.7). Then the following assertions hold:*

a) $\mathcal{F}_t$-translation invariance: If g satisfies property $(\mathcal{D})$, then

$$y_t^{X+a_t} = y_t^X + a_t \quad \textit{for any } t \in [0,T] \textit{ and } a_t \in \mathbf{L}^2(\mathcal{F}_t).$$

b) Monotonicity: For any $X' \in \mathbf{L}^2$ such that $X' \geq X$, we have

$$y_t^{X'} \geq y_t^X \quad \textit{for any } t \in [0,T].$$

c) Concavity: If g satisfies property $(\mathcal{E})$, then we have for any $X_1, X_2 \in \mathbf{L}^2$ and any $\beta \in [0,1]$ that

$$y_t^{\beta X_1 + (1-\beta)X_2} \geq \beta y_t^{X_1} + (1-\beta) y_t^{X_2}.$$

d) $\mathcal{F}_t$-regularity: For any $X_1, X_2 \in \mathbf{L}^2$ and $A \in \mathcal{F}_t$,

$$y_t^{\mathbf{1}_A X_1 + \mathbf{1}_{A^c} X_2} = \mathbf{1}_A\, y_t^{X_1} + \mathbf{1}_{A^c}\, y_t^{X_2}.$$

e) Normalization: If g satisfies property $(\mathcal{C})$, then

$$y^0 \equiv 0.$$

f) Positive homogeneity: If g satisfies property $(\mathcal{F})$, then

$$y_t^{\lambda X} = \lambda y_t^X \quad \textit{for any } \lambda \geq 0.$$

g) Time-consistency: Let $0 \leq s \leq t \leq T$ and $X_1, X_2 \in \mathbf{L}^2$. Then

$$y_t^{X_1} = y_t^{X_2} \quad \textit{implies that also} \quad y_s^{X_1} = y_s^{X_2}.$$

Proof. For some parts of Proposition 2.7.5, proofs are available only for the special case that g satisfies in addition to $(\mathcal{A})$ and $(\mathcal{B})$ also

$(\mathcal{C}')$ $g_t(0,0) \equiv 0$.

Therefore we first show how the general case can be reduced to this situation. More precisely, we prove that $(\tilde{y}^X, \tilde{z}^X) := (y^X - y^0, z^X - z^0)$ is the g-solution for the driver

$$\tilde{g}_t(y,z) := g_t(y + y_t^0, z + z_t^0) - g_t(y_t^0, z_t^0), \quad t \in [0,T]$$

and terminal value X. In fact, it is easy to see that $\tilde{g}$ satisfies ($\mathcal{A}$), ($\mathcal{B}$) and ($\mathcal{C}'$). Hence by uniqueness, $(\tilde{y}^X, \tilde{z}^X)$ solves

$$\begin{aligned} -d\tilde{y}_t^X &= \left(g_t(y_t^X, z_t^X) - g_t(y_t^0, z_t^0)\right) dt - (z_t^X - z_t^0)^* \, dW_t & (2.7.8) \\ &= \left(g_t(\tilde{y}_t^X + y_t^0, \tilde{z}_t^X + z_t^0) - g_t(y_t^0, z_t^0)\right) dt - (\tilde{z}_t^X)^* \, dW_t & (2.7.9) \\ &= \tilde{g}_t(\tilde{y}_t^X, \tilde{z}_t^X) \, dt - (\tilde{z}_t^X)^* \, dW_t. & (2.7.10) \end{aligned}$$

Since $\tilde{y}_t^X = y_t^X - y_t^0$ for all $X \in \mathbf{L}^\infty$ and because the properties a) and d) are invariant under the translation by $-y_t^0$, we can thus assume for their proof that g satisfies ($\mathcal{C}'$) as well. After this preliminary step, the rest is easy:

a) For g satisfying ($\mathcal{C}'$), this can be found in Lemma 4.2 in [BCHMP00]; see also Example 11 in [Pen97].

b) See Proposition 3.5 in [EKPQ97].

c) See Proposition 3.5 in [EKPQ97].

d) If g satisfies ($\mathcal{C}'$), then

$$\mathbf{1}_A \, g_u(\,.\,,\,.\,) = g_u(\mathbf{1}_A \,.\,, \mathbf{1}_A \,.\,) \quad \text{for all } u \geq t \text{ and } A \in \mathcal{F}_t.$$

Hence the claim follows from 2) of the proof of Proposition 36.4 in [Pen97].

e) See Lemma 36.6 in [Pen97].

f) This is Proposition 8 in [RG06]; see also Example 10 in [Pen97].

g) This follows immediately from Proposition 2.5 in [EKPQ97] and the uniqueness of g-solutions.

□

Remark 2.7.6. To obtain normalization in e) it suffices to have ($\mathcal{C}'$) together with ($\mathcal{A}$) and ($\mathcal{B}$). However, the stronger condition ($\mathcal{C}$) yields in addition that the g-solution is independent of the time horizon. In fact, let us write $\mathcal{E}_{t,T}^g[X]$ instead of $\mathcal{E}_t^g[X]$ to emphasize the dependence on the time horizon T. Then property ($\mathcal{C}$) implies that

$$\mathcal{E}_{s,t}^g[X] = \mathcal{E}_{s,T}^g[X] \quad \text{for } s \leq t \leq T \text{ and } X \in \mathbf{L}^2(\mathcal{F}_t),$$

as described. Note also that ($\mathcal{D}$) implies the equivalence of ($\mathcal{C}$) and ($\mathcal{C}'$). ◇

Since BSDEs are typically defined on $\mathbf{L}^2$ spaces, it appears more natural in this context to define (dynamic) MCUFs on $\mathbf{L}^2$ instead of $\mathbf{L}^\infty$. Thus an MCUF at time t is a mapping from $\mathbf{L}^2$ into $\mathbf{L}^2(\mathcal{F}_t)$ which has all the properties of Definition 2.3.1 with $\mathbf{L}^\infty$ replaced by $\mathbf{L}^2$ everywhere and with its acceptance set defined as a subset of $\mathbf{L}^2$. In the same way, we extend Definition 2.3.23 of time-consistency by replacing $\mathbf{L}^\infty$ with $\mathbf{L}^2$. It is easy to check that Lemma 2.3.25 remains true for DMCUFs on $\mathbf{L}^2$ so that we can make use here of its equivalent conditions for time-consistency. In particular, the following result follows immediately from Proposition 2.7.5:

Corollary 2.7.7. *Let g satisfy properties $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{D})$ and $(\mathcal{E})$ of Definition 2.7.2 and denote by (y^X, z^X) the corresponding g-solution with terminal value $X \in \mathbf{L}^2$. Then*

$$\Phi_t(X) := y_t^X, \quad t \in [0, T]$$

defines a time-consistent DMCUF. It is normalized and therefore even strongly time-consistent if g satisfies in addition property $(\mathcal{C})$, and coherent if g also satisfies property $(\mathcal{F})$.

Remark 2.7.8. i) A similar result, stated for dynamic risk measures, is given in Proposition 19 of [RG06]. However, her definition of a dynamic risk measure (and hence of a DMCUF) differs from ours. For the convenience of the reader, we therefore showed here how Corollary 2.7.7 can be obtained. We also remark that in her Section 4.1.2, Rosazza Gianin states in addition conditions under which the converse holds true, i.e., for when a time-consistent DMCUF can be described by some g-solution.

ii) Note that DMCUFs described by g-solutions are in particular continuous in t. ◇

Now we consider an investor whose preferences can be expressed by a DMCUF Φ which is described by a g-solution, and we assume that this investor can trade in some financial market. As in Section 2.6, we want to obtain results for the indifference valuation DMCUF by convoluting Φ with the market DMCUF corresponding to the superhedging price process in the given market. To this end, we should like to express also the market DMCUF in terms of BSDEs. However, the superhedging price process (and hence the market DMCUF) is in general not a g-solution, but belongs to the bigger class of (constrained) g-supersolutions which we define next:

Definition 2.7.9. Let $X \in \mathbf{L}^2$ and let both $\psi : \Omega \times [0, T] \times I\!R \times I\!R^d \to I\!R_+$ and g satisfy $(\mathcal{A})$ and $(\mathcal{B})$ of Definition 2.7.2. We call a triple (y, z, A) *g-supersolution with terminal value X* if (y, z) is in $\mathbf{M}^2_{\mathbb{F}}(0, T; I\!R) \times \mathbf{M}^2_{\mathbb{F}}(0, T; I\!R^d)$ with y RCLL and $A = (A_t)_{0 \le t \le T}$ is an increasing $\mathbb{F}$-adapted RCLL process with $A_0 = 0$ and

$E\left[A_T^2\right] < \infty$ such that (y, z, A) satisfies

$$y_t = X + \int_t^T g_s(y_s, z_s)\,ds + (A_T - A_t) - \int_t^T z_s^* \,dW_s, \quad t \in [0, T]. \tag{2.7.11}$$

We call the triple a *ψ-constrained g-supersolution* if (y, z, A) satisfies in addition

$$\psi_t(y_t, z_t) = 0 \quad dP \otimes dt\text{-a.s.}$$

If y satisfies

$$y_t \le y'_t \quad \text{for all } t \in [0, T]\ P\text{-a.s.}$$

for any ψ-constrained g-supersolution (y', z', A') with terminal value X, we call (y, z, A) the *smallest ψ-constrained g-supersolution with terminal value X.*

Remark 2.7.10. i) Proposition 1.6 in [Pen99] implies uniqueness of the processes z and A in a g-supersolution (y, z, A) in the following sense: If (y, z', A') is also a g-supersolution with the same terminal value $X \in \mathbf{L}^2$, then z and z' respectively A and A' coincide.

ii) The original terminology in [Pen99] for a ψ-constrained g-supersolution is *g-supersolution under the constraint ψ*. We slightly change the terminology here in order to avoid confusion. In fact, we shall consider the indifference valuation for the case of *unconstrained* trading opportunities in the market. But the corresponding market DMCUF will be described by a *ψ-constrained g-*supersolution. The deeper reason for this mismatch is that the construction in terms of ψ-constrained g-supersolutions is somewhat artificial, as we describe the market DMCUF as a stochastic integral with respect to a Brownian motion W and as a process adapted to the filtration generated by W. It would be more natural to use stochastic integrals with respect to the price process S of the traded assets and work with the filtration generated by S.

◇

A fundamental result for BSDEs which we require later is the *comparison theorem*. The version we present in Theorem 2.7.11 can be found in [Pen99], Theorem 1.3.

Theorem 2.7.11. *Let $y, y', B, B', g' \in \mathbf{M}^2_{\mathbb{F}}(0, T; I\!R)$ where B and B' are RCLL processes with $B_0 = B'_0$, $E\left[\sup_{0\le t\le T} |B_t|\right] < \infty$ and $E\left[\sup_{0\le t\le T} |B'_t|\right] < \infty$. Moreover, let $z, z' \in \mathbf{M}^2_{\mathbb{F}}(0, T; I\!R^d)$, $X, X' \in \mathbf{L}^2$ and let g be a driver which satisfies $(\mathcal{A})$ and $(\mathcal{B})$. Assume that (y, z, B) solves*

$$y_t = X + \int_t^T g_t(y_t, z_t)\,dt + (B_T - B_t) - \int_t^T z_t^* \,dW_t, \quad t \in [0, T]$$

and (y', z', B') *solves*

$$y'_t = X' + \int_t^T g'_t\,dt + (B'_T - B'_t) - \int_t^T (z'_t)^*\,dW_t, \quad t \in [0, T].$$

If

$$X \geq X', \quad g_t(y'_t, z'_t) \geq g'_t \quad dP\otimes dt\text{—a.s.} \quad \text{and} \quad B \succeq B' \text{ (i.e., } B - B' \text{ is increasing),}$$

then we have P*-a.s.*

$$y_t \geq y'_t \quad \text{for all } t \in [0, T].$$

If in addition $P[X > X'] > 0$ *then* $P\big[y_t > y'_t$ for all $t \in [0, T]\big] > 0$.

In order to define the market functional, we now need to specify the financial market and the set of strategies we allow for trading in the present $\mathbf{L}^2$-setting. We retain the assumptions made at the beginning of this section with respect to the filtered probability space and the d-dimensional Brownian motion W. Our model consists of $n \leq d$ risky assets and one riskless asset which is constantly 1 so that the price processes of the n risky assets are already discounted. They are defined by

$$S_t^i = s_0^i \exp\left(\sum_{j=1}^d \int_0^t \sigma_s^{i,j}\,dW_s^j + \int_0^t \mu_s^i\,ds - \sum_{j=1}^d \frac{1}{2}\int_0^t \left|\sigma_s^{i,j}\right|^2 ds\right),$$

with $s_0^i > 0$, $i = 1, \ldots, n$, where μ and σ are uniformly bounded progressively measurable processes and such that the inverse of $\sigma\sigma^*$ exists and is uniformly bounded. Note that there exists an equivalent martingale measure for S so that there are no arbitrage opportunities in this market.

Definition 2.7.12. An *admissible portfolio* is a triple (x, π, K), where $x \in I\!R$, π is a progressively measurable $I\!R^n$-valued process and K is an adapted RCLL increasing process satisfying $K_0 = 0$ and

$$E\left[\int_0^T \|\pi_t^*\sigma_t\|^2\,dt + K_T^2\right] < \infty.$$

Here, x is the initial wealth, π_t^i is the amount of money invested in the i-th stock at time t, and K_t is the cumulative consumption up to time t. The corresponding *value process* is defined as the RCLL process $V = (V_t)_{0\leq t\leq T}$ given by

$$\begin{aligned} dV_t &= \pi_t^*\mu_t\,dt - dK_t + \pi_t^*\sigma_t\,dW_t, \\ V_0 &= x. \end{aligned} \tag{2.7.12}$$

An admissible portfolio (x, π, K) is a *hedging portfolio for* $X \in \mathbf{L}^2$ if $V_T = X$, and it is a *minimal hedging portfolio for* X if its value process V satisfies

$$V_t \leq V'_t \quad \text{for all } t \in [0, T] \ P\text{—a.s.}$$

for every hedging portfolio (x', π', K') for X with value process V'.

Remark 2.7.13. The definition of an admissible portfolio is here slightly different from the one given in Section 2.5. First of all, due to a different setting, we impose different integrability conditions. Moreover, the process H in an admissible portfolio (x, H, K) in Section 2.5 describes the portfolio by fixing the *numbers* of units of each asset held, whereas in this section the process π fixes the *amounts* invested in each of the assets. The relation between π and H is thus given by $\pi_t^i = H_t^i S_t^i$. ◇

For simplicity we only consider the case of unconstrained hedging in the sense of Sections 2.5 and 2.6. Thus the agent can use any admissible portfolio for hedging, and the set of payoffs she can superhedge by trading during $(t, T]$ is given by

$$\mathcal{C}_t := \left\{ \int_t^T \pi_u^* \, (\mu_u \, du + \sigma_u \, dW_u) - Y \,\middle|\, Y \in \mathbf{L}_+^2 \,,\, (0, \pi, 0) \text{ an admissible portfolio} \right\}.$$

Remark 2.7.14. In principle, the present approach via the results on minimal g-supersolutions can be extended to more general situations with constraints imposed on trading, i.e., when the set $\mathcal{C}_t'$ of payoffs which can be superhedged by trading during $(t, T]$ is a subset of the above $\mathcal{C}_t$. This idea goes back to Bender/Kohlmann [BK04] who also give many examples of general constraints. For the applications here, we need $\mathcal{C}_t'$ to be convex so that we can impose only convex constraints. ◇

Similarly to Section 2.6, the programme for describing the indifference valuation p with respect to Φ and the market corresponding to the family $(\mathcal{C}_t)$ now looks as follows:

1) Construct the market DMCUF corresponding to $(\mathcal{C}_t)$; compare (2.6.10).

2) Describe it via BSDEs.

3) Convolute it with Φ to obtain Φ^{opt}; compare (2.6.7) and (2.6.11).

4) Describe Φ^{opt} via BSDEs.

5) Express p via Φ^{opt}; compare (2.6.8).

Because we work here in $\mathbf{L}^2$ instead of $\mathbf{L}^\infty$, the above steps become technically slightly different. The main problem is that we cannot construct the market DMCUF on all of $\mathbf{L}^2$. But fortunately, the convolution with Φ can still be formed since it only needs the values of the market DMCUF on a suitable subset of $\mathbf{L}^2$; this essentially goes back to the last equality of (2.4.4) in Theorem 2.4.3. Let us explain this in more detail.

In analogy to (2.6.10), we should want to define the market DMCUF by

$$\Phi_t^{-\mathcal{C}_t}(X) := \operatorname{ess\,sup} \left\{ m_t \in \mathbf{L}^2(\mathcal{F}_t) \,\middle|\, X - m_t \in -\mathcal{C}_t \right\} \tag{2.7.13}$$

so that a simple reformulation would give

$$-\Phi_t^{-\mathcal{C}_t}(-X) = \operatorname{ess\,inf}\left\{ m_t \in \mathbf{L}^2(\mathcal{F}_t) \,\middle|\, X = m_t + G \text{ for some } G \in \mathcal{C}_t \right\}. \quad (2.7.14)$$

In other words, $-\Phi_t^{-\mathcal{C}_t}(-X)$ should correspond to the superhedging price of X at time t. But this does not work with every X in $\mathbf{L}^2$. It is well-known that in contrast to the $\mathbf{L}^\infty$-context, a hedging portfolio need not exist for every $X \in \mathbf{L}^2$ in general so that the set on the RHS of (2.7.14) can be empty and the essential infimum is possibly not well-defined. In particular, $\Phi_t^{-\mathcal{C}_t}$ in (2.7.13) is not an MCUF at time t because it is not defined on all of $\mathbf{L}^2$. We might try to save the situation by defining the essential infimum in (2.7.14) as ∞ when it is taken over an empty set. But in view of the desired interpretation as superhedging price, this is not appropriate either. In fact, there might exist a set $A \in \mathcal{F}_t$ with $P[A] > 0$ and such that there exists a hedging portfolio for X on A, i.e., for $X\mathbf{1}_A$, and then the superhedging price of X at time t should be finite on A. Hence the definition (2.7.13) cannot be used for every $X \in \mathbf{L}^2$; we must restrict X to some suitable subset of $\mathbf{L}^2$.

Now the reason why we consider the functional $\Phi_t^{-\mathcal{C}_t}$ is that we want to convolute it with the DMCUF Φ which expresses the agent's preferences. Fortunately, this operation does not need the values of $\Phi_t^{-\mathcal{C}_t}$ on all of $\mathbf{L}^2$; this can be seen from (2.4.4) which is easily extended from the $\mathbf{L}^\infty$- to the present $\mathbf{L}^2$-context. In more detail, (2.4.4) suggests that we should have

$$\Phi_t^{\mathrm{opt}}(X) = \text{“}\Phi_t \square \Phi_t^{-\mathcal{C}_t}(X)\text{”} = \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \left(\Phi_t(X+Y) + \Phi_t^{-\mathcal{C}_t}(-Y) \right) \quad (2.7.15)$$

for all $X \in \mathbf{L}^2$, where $\mathcal{B}$ is an arbitrary subset of $\mathbf{L}^2$ such that

$$\mathcal{B} \supseteq \left\{ Y \in \mathbf{L}^2 \,\middle|\, \Phi_t^{-\mathcal{C}_t}(Y) \text{ from (2.7.13) is well-defind in } \mathbf{L}^2 \text{ and } \geq 0 \right\}.$$

In other words, $\mathcal{B}$ should contain the “acceptance set of $\Phi_t^{-\mathcal{C}_t}$”. To prove that (2.7.15) is indeed true with $\mathcal{B} := -\mathcal{C}_0$, we shall first show that the superhedging price at time t for $X \in \mathcal{C}_0$ coincides with the RHS of (2.7.14), and is ≤ 0 if $X \in \mathcal{C}_t \subseteq \mathcal{C}_0$; hence $\Phi_t^{-\mathcal{C}_t}(X)$ is well-defined by (2.7.13) for $X \in -\mathcal{C}_0$ and ≥ 0 if $X \in -\mathcal{C}_t$. Then we prove that

$$\Phi_t^{\mathrm{opt}}(X) := \operatorname*{ess\,sup}_{G \in \mathcal{C}_t} \Phi_t(X+G) \quad (2.7.16)$$

coincides for every $X \in \mathbf{L}^2$ with the RHS of (2.7.15) for $\mathcal{B} := -\mathcal{C}_0$.

The next result achieves steps 1) and 2) in the above scheme. It shows that the superhedging price process for $X \in \mathbf{L}^2$ can be described via a constrained g-supersolution of a BSDE, and that this process is nonpositive at t if and only if $X \in \mathcal{C}_t$. Moreover, the superhedging price operator is shown to coincide with $-\Phi_t^{-\mathcal{C}_t}(-\,.)$

from (2.7.14) or (2.7.13) on $\mathcal{C}_0$. Note again that in contrast to the $\mathbf{L}^\infty$ case, these results do not hold on all of $\mathcal{C}_t + \mathbf{L}^2(\mathcal{F}_t) \supseteq \mathcal{C}_0$, because not every $X \in \mathbf{L}^2(\mathcal{F}_t)$ admits a superhedging portfolio.

Theorem 2.7.15. *a) Let $X \in \mathbf{L}^2$ be such that there exists a hedging portfolio for X. Then the minimal hedging portfolio $(\tilde{x}, \tilde{\pi}, \tilde{A})$ for X exists, and the corresponding value process $\tilde{V}$ coincides with the y-component from the smallest ψ-constrained g-supersolution of the BSDE*

$$-dy_t \quad = \quad g_t^m(z_t)\,dt + dA_t - z_t^*\,dW_t \tag{2.7.17}$$

with terminal value

$$y_T = X$$

and constraint

$$\psi_t(z_t) := \left\| z_t - \sigma_t^* \left(\sigma_t \sigma_t^*\right)^{-1} \sigma_t z_t \right\| = 0 \quad dP \otimes dt\text{-a.s.}, \tag{2.7.18}$$

where

$$g_t^m(z) := -z^* \sigma_t^* \left(\sigma_t \sigma_t^*\right)^{-1} \mu_t. \tag{2.7.19}$$

b) For $X \in \mathbf{L}^2$, the minimal hedging portfolio exists and has a value process $\tilde{V}$ which satisfies $\tilde{V}_t \le 0$ if and only if X belongs to $\mathcal{C}_t$.

c) For any $G^0 \in \mathcal{C}_0 \supseteq \mathcal{C}_t$, the minimal hedging portfolio $(\tilde{x}, \tilde{\pi}, \tilde{K})$ exists, and its value process $\tilde{V}$ coincides with $\big(-\Phi_t^{-\mathcal{C}_t}(-G^0)\big)_{0\le t\le T}$ with $-\Phi_t^{-\mathcal{C}_t}(-\,.\,)$ from (2.7.14): For each $t \in [0, T]$, we have

$$\tilde{V}_t = -\Phi_t^{-\mathcal{C}_t}(-G^0). \tag{2.7.20}$$

Remark 2.7.16. i) We denote the driver in (2.7.17) by $g^m_{\cdot}(\,.\,)$ to emphasize its connection to the market functional (respectively to $-\Phi_t^{-\mathcal{C}_t}(-\,.\,)$).

ii) Since $\sigma_t^* \left(\sigma_t \sigma_t^*\right)^{-1} \sigma_t$ is the projection onto the range of σ_t^*, the constraint (2.7.18) simply ensures that z_t is in the range of σ_t^*. As mentioned before, this is needed because our strategies ought to be expressed via S, not W.

◇

Proof of Theorem 2.7.15. a) We first show that (x, π, K) is a hedging portfolio for X with value process V if and only if $(V, \sigma^*\pi, K)$ is a ψ-constrained g-supersolution with terminal value X. To see this, note that (2.7.11) can equivalently be written as

$$-dy_t = g_t(y_t, z_t)\,dt + dA_t - z_t^*\,dW_t, \quad y_T = X$$

and that the constraint from ψ is always satisfied for $z = \sigma^*\pi$. Hence we only have to check the integrability conditions in Definitions 2.7.9 and 2.7.12, and show that for any ψ-constrained g-supersolution (y, z, A) we can write $z = \sigma^*\pi$ for a suitable process π. However, the latter holds since z satisfies the constraint from ψ so that we can take $\pi := (\sigma\sigma^*)^{-1}\sigma z$, and the former holds since μ, σ and $(\sigma\sigma^*)^{-1}$ are uniformly bounded processes. Now the assertion follows if we can prove the existence of a smallest ψ-constrained g-supersolution with terminal condition X. But this follows by Theorem 4.2 of [Pen99] already from the existence of a ψ-constrained g-supersolution with terminal value X or, equivalently, from the existence of a hedging portfolio for X.

b) Any $X \in \mathcal{C}_t$ is of the form $X = \int_t^T \pi_u^* (\mu_u\, du + \sigma_u\, dW_u) - Y$ where $Y \in \mathbf{L}^2_+$ and π is a progressively measurable $\mathbb{R}^d$-valued process such that

$$E\left[\int_0^T \|\pi_t^*\sigma_t\|^2\, dt\right] < \infty.$$

Hence $(0, \pi', K')$ with $K'_u := 0$ for $u < T$, $K'_T := Y$, $\pi' = 0$ on $[\![0, t]\!]$ and $\pi' = \pi$ on $]\!]t, T]\!]$ is a hedging portfolio for X so that by a) the minimal hedging portfolio for X exists. Moreover, since the value process V' of $(0, \pi', K')$ satisfies $V'_t = 0$, the value process $\tilde{V}$ of the minimal hedging portfolio for X satisfies $\tilde{V}_t \leq 0$. To finish the proof of b), it suffices to show that if for $X \in \mathbf{L}^2$ the minimal hedging portfolio (x, π, K) exists with value process V such that $V_t \leq 0$, then $X \in \mathcal{C}_t$. But this is easy since (2.7.12) implies that

$$X = V_T = (V_T - V_t) + V_t = \int_t^T \pi_u^* (\mu_u\, du + \sigma_u\, dW_u) - (K_T - K_t - V_t)$$

where $K_T - K_t - V_t \in \mathbf{L}^2_+$ so that $X \in \mathcal{C}_t$.

c) By part b), it suffices to prove (2.7.20). Also by b) the minimal hedging portfolio $(\tilde{x}, \tilde{\pi}, \tilde{K})$ for G^0 exists. If $\tilde{V}$ denotes the corresponding value process, we can write

$$G^0 = \tilde{V}_t + \int_t^T \tilde{\pi}_u^*(\mu_u\, du + \sigma_u\, dW_u) - (\tilde{K}_T - \tilde{K}_t) =: \tilde{V}_t + \tilde{G},$$

where $\tilde{G} \in \mathcal{C}_t$. This yields the estimate

$$\begin{aligned} -\Phi_t^{-\mathcal{C}_t}&(-G^0) \\ &= \operatorname{ess\,inf}\left\{ m_t \in \mathbf{L}^2(\mathcal{F}_t) \,\middle|\, G^0 = m_t + G' \text{ for some } G' \in \mathcal{C}_t \right\} \qquad (2.7.21) \\ &\leq \tilde{V}_t. \end{aligned}$$

The converse inequality is shown by contradiction. Suppose it does not hold. Then there exist $\varepsilon > 0$ and $A \in \mathcal{F}_t$ with $P[A] > 0$ such that

$$-\Phi_t^{-\mathcal{C}_t}(-G^0) + \varepsilon < \tilde{V}_t \quad \text{on } A.$$

By (2.7.21), we can then find $\overline{m}_t \in \mathbf{L}^2(\mathcal{F}_t)$ and $\overline{G} \in \mathcal{C}_t$ such that $G^0 = \overline{m}_t + \overline{G}$ and some set $B \subseteq A$ with $B \in \mathcal{F}_t$ and $P[B] > 0$ so that on B

$$\overline{m}_t \le -\Phi_t^{-\mathcal{C}_t}(-G^0) + \varepsilon < \tilde{V}_t = \tilde{V}_{t-} + \Delta\tilde{V}_t = \tilde{V}_{t-} - \Delta\tilde{K}_t \le \tilde{V}_{t-}. \quad (2.7.22)$$

By b) the minimal hedging portfolio $(\overline{x}, \overline{\pi}, \overline{K})$ for $\overline{G} \in \mathcal{C}_t$ exists and if we denote the corresponding value process by $\overline{V}$, we can write

$$\overline{G} = \overline{V}_t + \int_t^T \overline{\pi}_s^*(\mu_s\,ds + \sigma_s\,dW_s) - \overline{K}_T + \overline{K}_t. \quad (2.7.23)$$

Now we fix $t \in (0, T)$ and construct a new hedging portfolio $(\hat{x}, \hat{\pi}, \hat{K})$ for G^0 such that its value process $\hat{V}$ satisfies $\hat{V}_t = \overline{m}_t < \tilde{V}_t$ on B, which contradicts the minimality of $\tilde{V}$. To this end we define

$$\begin{aligned}
\hat{x} &:= \tilde{x}, \\
\hat{\pi} &:= \tilde{\pi}\mathbf{1}_{[\![0,t]\!]} + (\overline{\pi}\mathbf{1}_B + \tilde{\pi}\mathbf{1}_{B^c})\mathbf{1}_{]\!]t,T]\!]}, \\
\hat{K} &:= \tilde{K} \quad \text{on } [\![0,t[\![, \\
\hat{K}_u &:= \left(\tilde{V}_{t-} + \tilde{K}_{t-} - \overline{m}_t + \overline{K}_u - \overline{K}_t - \overline{V}_t\mathbf{1}_{\{u=T\}}\right)\mathbf{1}_B + \tilde{K}_u\mathbf{1}_{B^c} \\
&\quad\ \text{for } t \le u \le T.
\end{aligned}$$

We note that $\tilde{V}_{t-} + \tilde{K}_{t-} = \tilde{x} + \int_0^t \tilde{\pi}_s^*(\mu_s\,ds + \sigma_s\,dW_s)$ so that

$$\hat{V}_t = \hat{x} + \int_0^t \hat{\pi}_s^*\,(\mu_s\,ds + \sigma_s\,dW_s) - \hat{K}_t = \overline{m}_t \quad \text{on } B \quad (2.7.24)$$

and that by (2.7.23), we have on B from the definition of $\hat{K}_T$ that

$$\begin{aligned}
\hat{V}_T &= \tilde{V}_{t-} + \int_t^T \overline{\pi}_s^*(\mu_s\,ds + \sigma_s\,dW_s) - \hat{K}_T + \tilde{K}_{t-} \\
&= \overline{m}_t + \overline{G} \\
&= G^0.
\end{aligned}$$

Hence we are only left to show that $(\hat{x}, \hat{\pi}, \hat{K})$ is an admissible portfolio, which is obviously true if $\hat{K}$ is increasing. Because $\overline{G} \in \mathcal{C}_t$ implies that $\overline{V}_t \le 0$, this is obvious if $\Delta\hat{K}_t := \hat{K}_t - \hat{K}_{t-} = \hat{K}_t - \tilde{K}_{t-} \ge 0$ on B. However, the latter holds true since by (2.7.24) and (2.7.22) we have $\Delta\hat{K}_t = -\Delta\hat{V}_t = -(\overline{m}_t - \tilde{V}_{t-}) > 0$ on B. This establishes the contradiction and hence completes the proof.

□

We now pass on to steps 3) – 5) in our scheme. So let us fix a DMCUF Φ and define for each $X \in \mathbf{L}^2$ the indifference value $p_t(X)$ at time t implicitly by

$$\operatorname*{ess\,sup}_{G \in \mathcal{C}_t} \Phi_t(x_t + G) = \operatorname*{ess\,sup}_{G \in \mathcal{C}_t} \Phi_t(x_t - p_t(X) + X + G), \tag{2.7.25}$$

where $x_t \in \mathbf{L}^2(\mathcal{F}_t)$ is the initial endowment at time t. In addition, we define a functional $\Phi_t^{\mathrm{opt}}(X) = \operatorname*{ess\,sup}_{G \in \mathcal{C}_t} \Phi_t(X + G)$ as in (2.7.16). If

$$\Phi_t^{\mathrm{opt}}(0) \in \mathbf{L}^2(\mathcal{F}_t) \quad \text{and} \quad \Phi_t^{\mathrm{opt}}(X) \in \mathbf{L}^2(\mathcal{F}_t),$$

then we can use the translation invariance of Φ_t to solve (2.7.25) for $p_t(X)$ and get

$$p_t(X) = \Phi_t^{\mathrm{opt}}(X) - \Phi_t^{\mathrm{opt}}(0) \in \mathbf{L}^2(\mathcal{F}_t). \tag{2.7.26}$$

This last expression is a first answer to step 5). For steps 3) and 4), we assume that Φ is described by some g-solution and we should also like to express $\Phi^{\mathrm{opt}}(X)$ in terms of BSDEs. The idea to achieve this is as follows. Thanks to Theorem 2.7.15 and (2.7.15), we know that Φ_t^{opt} is "morally" the convolution of Φ_t with the market MCUF $\Phi_t^{-\mathcal{C}_t}$. Now Barrieu and El Karoui have proved in [BEK04] that the convolution of DMCUFs which are both described by g-solutions corresponds (under some technical assumptions) to the g-solution whose driver is the pointwise convolution (in the sense of Rockafellar as in (2.4.3)) at each time t of the drivers for the two original g-solutions. Since the market functional is not a g-solution but a constrained g-supersolution, we have here a slightly different setting. Nevertheless, we can extend the result of Barrieu and El Karoui to this more general setting by similar arguments.

Theorem 2.7.17. *a) Let the DMCUF Φ be described by a g-solution with a driver g which satisfies $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{D})$ and $(\mathcal{E})$. With g^m as in (2.7.19), define for $z \in \mathbb{R}^d$*

$$\begin{aligned} \hat{g}_t(z) &:= \sup_{v \in \mathbb{R}^n} \left\{ g_t(z + \sigma_t^* v) + g_t^m(-\sigma_t^* v) \right\} \\ &= \sup_{v \in \mathbb{R}^n} \left\{ g_t(z + \sigma_t^* v) + v^* \mu_t \right\} \end{aligned} \tag{2.7.27}$$

and fix $X \in \mathbf{L}^2$. If $\hat{g} : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ satisfies $(\mathcal{A})$ and $(\mathcal{B})$, then the g-solution $(\hat{y}, \hat{z})$ of

$$-d\hat{y}_t = \hat{g}_t(\hat{z}_t)\,dt - \hat{z}_t^*\,dW_t, \quad \hat{y}_T = X$$

exists. If in addition there exists $\bar{z} \in \mathbf{M}_{\mathbb{F}}^2(0, T; \mathbb{R}^d)$ satisfying the ψ-constraint (2.7.18) and such that

$$\hat{g}_t(\hat{z}_t) = g_t(\hat{z}_t + \bar{z}_t) + g_t^m(-\bar{z}_t) \quad dP \otimes dt\text{–a.s.}, \tag{2.7.28}$$

then

$$\Phi_t^{\mathrm{opt}}(X) = \operatorname*{ess\,sup}_{G^0 \in \mathcal{C}_0} \left\{ \Phi_t(X + G^0) + \Phi_t^{-\mathcal{C}_t}(-G^0) \right\} = \hat{y}_t. \tag{2.7.29}$$

In other words, $\Phi^{\mathrm{opt}}(X)$ *then equals the y-component of the g-solution with driver* $\hat{g}$ *and terminal value X.*

b) Suppose the assumptions of a) hold and denote by (y^0, z^0) *the g-solution with driver* $\hat{g}$ *and terminal value 0. If there exists* $\overline{z}^0 \in \mathbf{M}^2_{\mathbb{F}}(0, T; I\!R^d)$ *satisfying the* ψ*-constraint (2.7.18) and also*

$$\hat{g}_t(z_t^0) = g_t(z_t^0 + \overline{z}_t^0) + g_t^m(-\overline{z}_t^0),$$

then $\big(p_t(X)\big)_{0 \le t \le T}$ *is the g-expectation with driver*

$$\tilde{\hat{g}}_t(z) := \hat{g}_t(z + z_t^0) - \hat{g}_t(z_t^0) \tag{2.7.30}$$

and terminal value X.

Remark 2.7.18. i) It is easy to check that $\hat{g}$ always satisfies $(\mathcal{D})$ and $(\mathcal{E})$; the latter holds since g and g^m are both concave and for any $\beta \in [0, 1]$, we can replace the supremum over all $v \in I\!R^n$ in (2.7.27) by the supremum over all elements $\beta v^1 + (1 - \beta)v^2$, where $v^1, v^2 \in I\!R^n$. Moreover $\hat{g}$ can always be chosen product-measurable on $\Omega \times [0, T] \times I\!R^d$ and such that $(\omega, t) \mapsto \hat{g}(\omega, t, z)$ is $\mathbb{F}$-progressively measurable, so that $(\mathcal{A})$ is reduced to an integrability condition. In fact, we can fix a product-measurable $A \subseteq \Omega \times [0, T]$ such that A^c is a $dP \otimes dt$-nullset and $z \mapsto g(\omega, t, z)$ is continuous on $I\!R^d$ for all $(\omega, t) \in A$. Without loss of generality, $\mathbf{1}_A$ is $\mathbb{F}$-adapted; otherwise replace it by $A' := A \cap (B \times \Omega)$, where $B := \{t \in [0, T] \,|\, E[\mathbf{1}_A(\omega, t)] = 1\}$ is a Borel set. It follows from $E\left[\int_0^T \mathbf{1}_A(\omega, t)\, dt\right] = T$ and Fubini's theorem that $\int_0^T \mathbf{1}_B(t)\, dt = T$ P-a.s. so that $(A')^c$ is a $dP \otimes dt$-nullset. Adaptedness of $\mathbf{1}_{A'}$ is then implied by the usual conditions and since $E[\mathbf{1}_{A'}(\omega, t)] \in \{0, 1\}$ for each $t \in [0, T]$. Now, since $\mathbf{1}_A$ is product-measurable and adapted, it has a progressively measurable modification $Y = (Y_t)_{0 \le t \le T}$. Define

$$\overline{g}_t(z) := Y_t \sup_{v \in \mathbb{Q}^n} \{g_t(z + \sigma_t^* v) + v^* \mu_t\}$$

on $\Omega \times [0, T] \times I\!R^d$. Then $\overline{g}$ is product-measurable on $\Omega \times [0, T] \times I\!R^d$ and in addition $(\omega, t) \mapsto \overline{g}(\omega, t, z)$ is progressively measurable. Finally, we need to show that

$$\hat{g}_t(z) = \overline{g}_t(z) \quad \text{for all } z \in I\!R^d \quad dP \otimes dt\text{-a.s.}$$

To this end, note that by Fubini's theorem and since Y is a modification of $\mathbf{1}_A$ where A^c is a $dP \otimes dt$-nullset we have $Y\mathbf{1}_A = 1$ $dP \otimes dt$-a.s. Hence we can conclude that $dP \otimes dt$-a.s.

$$\hat{g}_t(z) = Y\mathbf{1}_A \hat{g}_t(z) = Y\mathbf{1}_A \sup_{v \in \mathbb{Q}^n} \{g_t(z + \sigma_t^* v) + v^* \mu_t\} = \overline{g}_t(z) \quad \text{for all } z \in \mathbb{R}^d,$$

where the second equality holds since g is continuous in z on A.

ii) If $\hat{g}$ satisfies $(\mathcal{A})$ and $(\mathcal{B})$, it is by Corollary 2.7.7 the driver of a g-solution which describes a time-consistent DMCUF. Note that the condition (2.7.28) on $\overline{z}$ depends on X via $\hat{z}$. If it does not hold for all $X \in \mathbf{L}^2$, steps 1) and 2) in the following proof still show that $\hat{y} = \hat{y}(X)$ is an upper bound for $\Phi^{\mathrm{opt}}(X)$. However, $\hat{y}(\,.\,)$ need not describe $\Phi^{\mathrm{opt}}(\,.\,)$ on all of $\mathbf{L}^2$ because the upper bound need not be attained.

iii) Suppose $p_{\cdot,T} = \mathcal{E}^{\tilde{\hat{g}}}_{\cdot,T}$ is described by the g-expectation with driver $\tilde{\hat{g}}$ on all of $\mathbf{L}^2$. Since $\tilde{\hat{g}}$ satisfies $(\mathcal{C})$, we know from Remark 2.7.6 that

$$\mathcal{E}^{\tilde{\hat{g}}}_{s,t}[X] = \mathcal{E}^{\tilde{\hat{g}}}_{s,T}[X] \quad \text{for all } s \le t \le T \text{ and } X \in \mathbf{L}^2(\mathcal{F}_t).$$

Since DMCUFs defined via BSDEs are always time-consistent, one might be tempted to conclude that the family p satisfies the recursiveness property

$$(\mathcal{R}) \quad p_{s,t}(p_{s,T}(X)) = p_{s,T}(X) \quad \text{for all } s \le t \le T \text{ and } X \in \mathbf{L}^2$$

introduced in Section 2.3. But how is $p_{\cdot,t}$ defined? In view of the desired interpretation, we should take $p_{\cdot,t} = \Phi^{\mathrm{opt}}_{\cdot,t} - \Phi^{\mathrm{opt}}_{\cdot,t}(0)$, where $\Phi^{\mathrm{opt}}_{\cdot,t}$ is described by the g-solution with driver $\hat{g}$ and time horizon t, and then ask if $p_{\cdot,t}$ coincides with $\mathcal{E}^{\tilde{\hat{g}}}_{\cdot,t}$. In general this is not true: Because $\tilde{\hat{g}}$ depends on z^0 which itself depends on the time horizon T, $p_{\cdot,t}$ will in general correspond to a g-expectation with a driver different from $\tilde{\hat{g}}$. However, if the driver $\hat{g}$ corresponding to Φ^{opt} is deterministic, one can show that

$$\Phi^{\mathrm{opt}}_{s,t}(X) = \Phi^{\mathrm{opt}}_{s,T}(X) - \Phi^{\mathrm{opt}}_{t,T}(0) \quad \text{for all } s \le t \le T \text{ and } X \in \mathbf{L}^2(\mathcal{F}_t).$$

This implies that

$$p_{s,t}(X) = \Phi^{\mathrm{opt}}_{s,t}(X) - \Phi^{\mathrm{opt}}_{s,t}(0) = \Phi^{\mathrm{opt}}_{s,T}(X) - \Phi^{\mathrm{opt}}_{s,T}(0) = p_{s,T}(X)$$

for $s \le t \le T$ and $X \in \mathbf{L}^2(\mathcal{F}_t)$ so that the time-consistency of $p_{\cdot,T}$ does imply $(\mathcal{R})$ after all. Example 2.7.19 below and the subsequent remark illustrate that p can satisfy $(\mathcal{R})$ even if $\hat{g}$ is not deterministic. It would be nice to have also an explicit example for p described by a g-solution where $(\mathcal{R})$ does not hold.

◇

Proof of Theorem 2.7.17. a) 1) We begin with the first equality in (2.7.29), i.e., we show that

$$\operatorname*{ess\,sup}_{G\in\mathcal{C}_t} \Phi_t(X+G) = \operatorname*{ess\,sup}_{G'\in\mathcal{C}_0} \left\{ \Phi_t(X+G') + \Phi_t^{-\mathcal{C}_t}(-G') \right\}. \quad (2.7.31)$$

Since $\mathcal{C}_t \subseteq \mathcal{C}_0$ and $\Phi_t^{-\mathcal{C}_t}$ is non-negative on $-\mathcal{C}_t$ by part b) of Theorem 2.7.15, the inequality "$\leq$" is trivial. The converse inequality follows if for any $G' \in \mathcal{C}_0$, we have $\Phi_t^{-\mathcal{C}_t}(-G') \in \mathbf{L}^2(\mathcal{F}_t)$ and $G := G' + \Phi_t^{-\mathcal{C}_t}(-G') \in \mathcal{C}_t$, since then $\mathcal{F}_t$-translation invariance of Φ_t implies that $\Phi_t(X+G') + \Phi_t^{-\mathcal{C}_t}(-G') = \Phi_t(X+G)$. To show that these two properties hold, we recall from Theorem 2.7.15 that the minimal hedging portfolio (x', π', K') exists for $G' \in \mathcal{C}_0$ and that its value V_t' at time t equals $-\Phi_t^{-\mathcal{C}_t}(-G')$ so that in particular $\Phi_t^{-\mathcal{C}_t}(-G') \in \mathbf{L}^2(\mathcal{F}_t)$. Again by Theorem 2.7.15, $G \in \mathcal{C}_t$ if and only if a hedging portfolio for G exists with a non-positive value at time t. But since $V_t' = -\Phi_t^{-\mathcal{C}_t}(-G')$ and (x', π', K') is a hedging portfolio for G', we have

$$G' = -\Phi_t^{-\mathcal{C}_t}(-G') + \int_t^T (\pi_s')^*(\mu_s\,ds + \sigma_s\,dW_s) - (K_T' - K_t').$$

Hence

$$G = \int_t^T (\pi_s')^*(\mu_s\,ds + \sigma_s\,dW_s) - (K_T' - K_t')$$

admits the hedging portfolio $(0, \pi, K)$ with $\pi := \pi'\mathbf{1}_{]\!]t,T]\!]}$ and $K := K'\mathbf{1}_{[\![t,T]\!]}$ which has value 0 at time t. This proves (2.7.31).

2) To show the second equality in (2.7.29) we take $G' \in \mathcal{C}_0$ and denote by (y, z) the g-solution for the driver g and terminal value $X + G'$. By Theorem 2.7.15 the process $(-\Phi_t^{-\mathcal{C}_t}(-G'))_{0\leq t\leq T}$ is the y-component of the smallest ψ-constrained g-supersolution (y', z', A') with ψ from (2.7.18), driver g^m and terminal value G'. Hence we get

$$\begin{aligned}
&-d\left(\Phi_t(X+G') + \Phi_t^{-\mathcal{C}_t}(-G')\right) \\
&\quad = \left(g_t(z_t) - g_t^m(z_t')\right) dt - dA_t' - \left(z_t - z_t'\right)^* dW_t \\
&\quad = \left(g_t(\tilde{z}_t + z_t') + g_t^m(-z_t')\right) dt - dA_t' - \tilde{z}_t^*\,dW_t, \quad (2.7.32)
\end{aligned}$$

with $\Phi_T(X+G') + \Phi_T^{-\mathcal{C}_T}(-G') = X$ and where we set $\tilde{z}_t := z_t - z_t'$ and use that $-g_t^m(\,.\,) = g_t^m(-\,.\,)$. Since

$$\hat{g}_t(\tilde{z}_t) \geq g_t(\tilde{z}_t + z_t') + g_t^m(-z_t')\ dP \otimes dt\text{-a.s. and } 0 \succeq -A',$$

the comparison result in Theorem 2.7.11 applied to the driver $\hat{g}_t(\,.\,)$ with solution $(\hat{y}, \hat{z}, 0)$ and the integrand $g_t(\tilde{z}_t + z'_t) + g^m_t(-z'_t)$ with solution

$$\left(\left(\Phi_t(X + G') + \Phi_t^{-\mathcal{C}_t}(-G')\right)_t, \tilde{z}, -A'\right)$$

yields

$$\hat{y}_t \geq \Phi_t(X + G') + \Phi_t^{-\mathcal{C}_t}(-G') \quad \text{for all } t \in [0, T] \ P\text{-a.s.} \tag{2.7.33}$$

Hence $\hat{y}$ is an upper bound for Φ^{opt}.

3) Next we construct an element $\check{G} \in \mathcal{C}_0$ for which this bound is attained to establish the second equality in (2.7.29). To this end set

$$\check{G} := \int_0^T g^m_t(-\bar{z}_t)\,dt + \int_0^T \bar{z}^*_t\,dW_t$$

and note that $\bar{z}$ by assumption satisfies the constraint ψ from (2.7.18). Hence $\pi_t := (\sigma_t\sigma^*_t)^{-1}\sigma_t\bar{z}_t$, $t \in [0, T]$, satisfies $\pi^*_t\sigma_t = \bar{z}^*_t$ so that

$$\check{G} = \int_0^T \pi^*_t(\mu_t\,dt + \sigma_t\,dW_t).$$

Thus $(0, \pi, 0)$ is a hedging portfolio for $\check{G}$ and so $\check{G} \in \mathcal{C}_0$ by Theorem 2.7.15. Next we define $(\check{y}_t)_{0 \leq t \leq T}$ as the continuous process

$$\check{y}_t := \int_0^t g^m_s(-\bar{z}_s)\,ds + \int_0^t \bar{z}^*_s\,dW_s.$$

Again since $g^m_t(-\,.\,) = -g^m_t(\,.\,)$, $(\check{y}, \bar{z})$ is the unique g-solution of

$$-d\check{y}_t = g^m_t(\bar{z}_t)\,dt - \bar{z}^*_t\,dW_t, \qquad \check{y}_T = \check{G}.$$

In particular, since $\bar{z}$ satisfies the constraint ψ from (2.7.18), the comparison result in Theorem 2.7.11 implies that the triple $(\check{y}, \bar{z}, 0)$ is the smallest ψ-constrained g-supersolution with terminal value $\check{G}$ and driver g^m. Thus and since $\check{G} \in \mathcal{C}_0$, parts c) and a) of Theorem 2.7.15 yield that $-\check{y}_t = \Phi_t^{-\mathcal{C}_t}(-\check{G})$. We know from (2.7.32) that for $G' := \check{G}$ we have

$$-d\left(\Phi_t(X + \check{G}) + \Phi_t^{-\mathcal{C}_t}(-\check{G})\right) = \left(g_t(\tilde{z}_t + \bar{z}_t) + g^m_t(-\bar{z}_t)\right)\,dt - \tilde{z}^*_t\,dW_t$$

for some $\tilde{z} \in \mathbf{M}^2_{\mathbb{F}}(0, T; I\!R^d)$. Since one can easily check that the driver

$$(g_t(\,.\, + \bar{z}_t) + g^m_t(-\bar{z}_t))_{0 \leq t \leq T}$$

satisfies ($\mathcal{A}$) and ($\mathcal{B}$) and since by assumption (2.7.28) we have

$$\hat{g}_t(\hat{z}_t) = g_t(\hat{z}_t + \bar{z}_t) + g_t^m(-\bar{z}_t) \quad dP \otimes dt\text{–a.s.},$$

uniqueness of g-solutions implies that

$$\Phi_t(X + \check{G}) + \Phi_t^{-\mathcal{C}_t}(-\check{G}) = \hat{y}_t \quad \text{for all } t \in [0, T] \ P\text{-a.s.}$$

Hence $\hat{y}$ is not only an upper bound for Φ^{opt}, but equal to it. This proves (2.7.29).

b) With a) and (2.7.26), this follows from the uniqueness of g-solutions and since

$$\begin{aligned} -d\left(y_t^X - y_t^0\right) &= \left(\hat{g}_t\left(z_t^X\right) - \hat{g}_t\left(z_t^0\right)\right) dt - \left(z_t^X - z_t^0\right)^* dW_t \\ &= \left(\hat{g}_t\left(\tilde{z}_t + z_t^0\right) - \hat{g}_t\left((z_t^0)\right)\right) dt - (\tilde{z}_t)^* \, dW_t, \\ y_T^X - y_T^0 &= X, \end{aligned}$$

where $\tilde{z} := z^X - z^0$.

□

We conclude this section with an explicit example where the DMCUF Φ is given by the conditional exponential certainty equivalent with risk aversion γ, i.e.,

$$\Phi_t(X) := -\frac{1}{\gamma} \log E[\exp(-\gamma X) \,|\, \mathcal{F}_t] \quad \text{for } X \text{ sufficiently integrable;}$$

see also Examples 2.3.3 and 2.7.1. Then Φ is described by the g-solution with driver $g_t(z) := -\frac{\gamma}{2}\|z\|^2$; see, e.g., Section 3.1 in [BEK04]. Although this driver obviously does not satisfy ($\mathcal{B}$), so that Theorem 2.7.17 cannot be applied, this is quite an illustrative example. In fact, the driver of the g-solution describing the indifference value process is known here explicitly, and we can show by formal calculations that it corresponds to $\tilde{\tilde{g}}$ from (2.7.30). Moreover, this example shows that if g has a nice structure, one can eliminate the dependence of $\tilde{\tilde{g}}$ on z^0 by expressing the value process as g-solution under an appropriate measure. Instead of successively solving two BSDEs (one for y^0, then one for y which depends on z^0), one can first do a measure change and then solve one BSDE (for y) under the new measure. While this usually does not reduce the difficulty of the problem, it still gives a conceptually clearer view.

Example 2.7.19. Let the DMCUF Φ be described by the g-solution with driver given by $g_t(z) := -\frac{\gamma}{2}\|z\|^2$. Then with $\theta_t := \sigma_t^*(\sigma_t\sigma_t^*)^{-1}\mu_t$, we have from (2.7.27)

$$\begin{aligned}\hat{g}_t(z) &= \sup_{v\in\mathbb{R}^n}\left\{g_t(z+\sigma_t^* v) + g_t^m(-\sigma_t^* v)\right\}\\ &= \sup_{v\in\mathbb{R}^n}\left\{-\frac{\gamma}{2}\|z+\sigma_t^* v\|^2 + v^*\mu_t\right\}\\ &= \sup_{v\in\mathbb{R}^n}\left\{-\frac{\gamma}{2}\|z+\sigma_t^* v\|^2 + (\sigma_t^* v)^*\theta_t\right\}.\end{aligned}$$

By completion of the square we can rewrite the term in brackets on the RHS as

$$-\frac{\gamma}{2}\left\|\sigma_t^* v - \left(-z + \frac{1}{\gamma}\theta_t\right)\right\|^2 - z^*\theta_t + \frac{1}{2\gamma}\|\theta_t\|^2$$

to get

$$\hat{g}_t(z) = -\frac{\gamma}{2}\left\|\Pi_t\left(-z + \frac{1}{\gamma}\theta_t\right)\right\|^2 - z^*\theta_t + \frac{1}{2\gamma}\|\theta_t\|^2,$$

where $\Pi_t(u)$ denotes the projection of u onto the orthogonal complement of $\sigma_t^*(\mathbb{R}^n)$. Denoting by $(\,.\,,\,.\,)$ the scalar product and using the properties

$$\|a\|^2 - \|b\|^2 = \|a-b\|^2 + 2(a-b\,,b)$$

and $\big(\Pi_t(a)\,,\Pi_t(b)\big) = \big(a\,,\Pi_t(b)\big)$ and linearity of the projection Π_t, we get

$$\begin{aligned}\tilde{\hat{g}}_t(z) &= \hat{g}_t(z+z_t^0) - \hat{g}_t(z_t^0)\\ &= -\frac{\gamma}{2}\left(\left\|\Pi_t\left(-z - z_t^0 + \frac{1}{\gamma}\theta_t\right)\right\|^2 - \left\|\Pi_t\left(-z_t^0 + \frac{1}{\gamma}\theta_t\right)\right\|^2\right) - z^*\theta_t\\ &= -\frac{\gamma}{2}\|\Pi_t(z)\|^2 - \gamma\left(-\Pi_t(z)\,,\frac{1}{\gamma}\Pi_t\left(-\gamma z_t^0 + \theta_t\right)\right) - z^*\theta_t\\ &= -\frac{\gamma}{2}\|\Pi_t(z)\|^2 + \left(z\,,\Pi_t\left(-\gamma z_t^0 + \theta_t\right)\right) - z^*\theta_t\\ &= -\frac{\gamma}{2}\|\Pi_t(z)\|^2 + \left(z\,,\Pi_t\left(-\gamma z_t^0 + \theta_t\right) - \theta_t\right).\end{aligned}$$

In particular, if the process $\mathcal{E}(\int \theta_s^0\,dW_s)$ for $\theta_t^0 := \Pi_t\left(-\gamma z_t^0 + \theta_t\right) - \theta_t$ is the density process of some equivalent martingale measure $Q^0 \in \mathcal{P}^e$, then we obtain

$$\begin{aligned}-dp_t(X) &= -\frac{\gamma}{2}\|\Pi_t(z_t)\|^2 + \left\langle z_t, \theta_t^0\right\rangle dt - z_t^*\,dW_t\\ &= -\frac{\gamma}{2}\|\Pi_t(z_t)\|^2 - z_t^*\,dW_t^0,\\ p_T(X) &= X,\end{aligned}$$

where $W^0 := W - \int \theta_s^0 \, ds$ is a Brownian motion under Q^0. This representation has the advantage that the driver does not depend on z^0; it was presented (in a more general setting) by Rouge and El Karoui in Theorem 5.1 in [REK00]. To see that their results agree with ours, note that the price/value process in [REK00] is the seller price process whereas we consider the value process for the buyer. Moreover, our process z^0 is associated to the g-solution which describes the process $\Phi^{\text{opt}}(0) = (-\frac{1}{\gamma} \operatorname{ess\,inf}_{G \in \mathcal{C}_t} E[\exp(-\gamma G)|\mathcal{F}_t])_{0 \le t \le T}$ whereas their process z^0 is associated to the g-solution which describes $-\Phi^{\text{opt}}(0)$. However, one can easily check that if (y, z) denotes the solution for a driver g and terminal condition $-X$ and if $(\tilde{y}, \tilde{z})$ denotes the solution for the driver $\tilde{g}_t(y, z) := -g_t(-y, -z)$ and terminal condition X, then $(\tilde{y}, \tilde{z}) = (-y, -z)$. Therefore the driver in [REK00] should be compared with $-\tilde{\hat{g}}_t(-\,.\,)$ where in addition z^0 is replaced by $-z^0$. The BSDE for Φ^{opt} can also be found in Theorem 7 of [HIM05]. For similar reasons as above, the driver there should be compared with $\hat{g}(-\,.\,)$.

Remark 2.7.20. Although the driver $\hat{g}$ in the above example does not satisfy $\hat{g}(0) = 0$ and is not deterministic, the corresponding indifference price satisfies $(\mathcal{R})$; see Proposition 15 in [MS05]. It would be interesting to see an explanation for why this happens. ◇

2.7.3 Extension to a dynamic MCUF

In this example we show that an MCohUF at time 0 cannot always be extended to a time-consistent normalized DMCUF; note that if there exists any time-consistent extension, then there also exists a normalized extension. More precisely, we consider the MCohUF

$$\Phi_0(X) := E[X] - a\|(X - E[X])^-\|_p \quad \text{for } X \in \mathbf{L}^\infty, \tag{2.7.34}$$

where $0 < a \le 1$ is a constant and $\|\,.\,\|_p$ is the $\mathbf{L}^p$-norm for some $1 \le p < \infty$. One straightforward extension to a DMCohUF can be obtained by setting

$$\Phi_t(X) := E[X|\mathcal{F}_t] - a\left(E\left[((X - E[X|\mathcal{F}_t])^-)^p \,|\mathcal{F}_t\right]\right)^{\frac{1}{p}}, \quad 0 \le t \le T.$$

Then for each time t we can specify a convex $\mathbf{L}^1$-closed set $\mathcal{Q}_t$ of measures representing Φ_t as in (2.3.8) of Theorem 2.3.16. However, we show by a counterexample that $\mathcal{Q}_0$ is in general not weakly m-stable so that by Lemma 2.3.29 Φ is not time-consistent. Moreover, we also show that it is even impossible to extend Φ_0 to any time-consistent DMCUF at all. The point of this example is to illustrate that time-consistency is a rather severe condition on a DMCUF.

The definition of Φ_0 is inspired by an example given in [Fis01] by Fischer who considers (static) coherent risk measures depending on one-sided moments. It is quite natural to define an MCohUF in this way, since it is just the expected value of the payoff minus a term which punishes the downside risk.

Let us first show that at each time $t \in [0, T]$, $\Phi_t(\,.\,)$ can be represented as

$$\Phi_t(X) \quad = \quad \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_t^e} E_Q[X|\mathcal{F}_t], \tag{2.7.35}$$

where for $p > 1$

$$\mathcal{Q}_t := \left\{ Q \in \mathcal{P}^a \,\middle|\, \frac{dQ}{dP} = 1 + a(Y - E[Y|\mathcal{F}_t]),\ Y \geq 0,\ E[Y^q|\mathcal{F}_t] \leq 1 \right\}$$

with q conjugate to p, and for $p = 1$

$$\mathcal{Q}_t := \left\{ Q \in \mathcal{P}^a \,\middle|\, \frac{dQ}{dP} = 1 + a(Y - E[Y|\mathcal{F}_t]),\ 0 \leq Y \leq 1 \right\}.$$

Note that by Example 2.3.3 b), this shows in particular that Φ is a DMCohUF. For $t = 0$ the proof of (2.7.35) can be found in [Del00], and for general $t \in [0, T]$, it works similarly as follows. Fix $X \in \mathbf{L}^\infty$ and $t \in [0, T]$. We start with the case when $p > 1$ and define

$$\tilde{Y} := \frac{\left((X - E[X|\mathcal{F}_t])^+\right)^{p-1}}{\left(E\left[\left((X - E[X|\mathcal{F}_t])^+\right)^p \middle|\, \mathcal{F}_t\right]\right)^{\frac{p-1}{p}}} \geq 0.$$

Then $E\big[\tilde{Y}^q \big| \mathcal{F}_t\big] = 1$ and hence also $E\big[\tilde{Y} \big| \mathcal{F}_t\big] \leq 1$ by the conditional Jensen inequality. Denote by $\tilde{Z}$ the density process of the corresponding measure $\tilde{Q}$ in $\mathcal{Q}_t$ so that $\tilde{Z}_s := E\left[1 + a\left(\tilde{Y} - E[\tilde{Y}|\mathcal{F}_t]\right) \middle|\, \mathcal{F}_s\right]$ for $s \in [0, T]$. Note that $\tilde{Z}_t = 1$. Since $\tilde{Y} - E[\tilde{Y}|\mathcal{F}_t] \geq -1$, $\tilde{Q}$ is equivalent to P for $a < 1$. If $a = 1$, then $\tilde{Q}$ need not be absolutely continuous with respect to P. However, we shall see that $\mathcal{Q}_t$ is convex and contains P, so that we can approximate $\tilde{Q}$ in $\mathbf{L}^1(P)$ by the sequence $(Q^n)_{n \in I\!N} \subseteq \mathcal{Q}_t^e$ associated to the sequence of densities $Z_T^\varepsilon := \varepsilon + (1 - \varepsilon)\tilde{Z}_T$, $0 < \varepsilon < 1$; see the proof of Lemma 2.3.29. Since $\tilde{Z}_t = 1$, we have

$$\begin{aligned}
E\left[\frac{\tilde{Z}_T}{\tilde{Z}_t} X \,\middle|\, \mathcal{F}_t\right] &= E[X|\mathcal{F}_t] + E\left[\tilde{Z}_T\,(X - E[X|\mathcal{F}_t]) \,\middle|\, \mathcal{F}_t\right] \\
&= E[X|\mathcal{F}_t] + E\left[(\tilde{Z}_T - 1 + aE[\tilde{Y}|\mathcal{F}_t])\,(X - E[X|\mathcal{F}_t]) \,\middle|\, \mathcal{F}_t\right] \\
&= E[X|\mathcal{F}_t] + E\left[a\tilde{Y}\,(X - E[X|\mathcal{F}_t]) \,\middle|\, \mathcal{F}_t\right] \qquad (2.7.36) \\
&= E[X|\mathcal{F}_t] + a\frac{E\left[\left((X - E[X|\mathcal{F}_t])^+\right)^{p-1}(X - E[X|\mathcal{F}_t]) \,\middle|\, \mathcal{F}_t\right]}{\left(E\left[\left((X - E[X|\mathcal{F}_t])^+\right)^p \,\middle|\, \mathcal{F}_t\right]\right)^{\frac{p-1}{p}}} \\
&= E[X|\mathcal{F}_t] + a\left(E\left[\left((X - E[X|\mathcal{F}_t])^+\right)^p \,\middle|\, \mathcal{F}_t\right]\right)^{\frac{1}{p}}. \qquad (2.7.37)
\end{aligned}$$

Now take $Q' \in \mathcal{Q}_t$ with corresponding Y' and denote the density process of Q' by Z'. As above we obtain

$$\begin{aligned}
E\left[\frac{Z'_T}{Z'_t}X \,\middle|\, \mathcal{F}_t\right] &= E[X|\mathcal{F}_t] + E\left[aY'(X - E[X|\mathcal{F}_t]) \mid \mathcal{F}_t\right] \\
&\le E[X|\mathcal{F}_t] + aE\left[Y'(X - E[X|\mathcal{F}_t])^+ \mid \mathcal{F}_t\right] \\
&\le E[X|\mathcal{F}_t] \\
&\quad + a\left(E\left[(Y')^q \mid \mathcal{F}_t\right]\right)^{\frac{1}{q}} \left(E\left[\left((X - E[X|\mathcal{F}_t])^+\right)^p \mid \mathcal{F}_t\right]\right)^{\frac{1}{p}} \\
&\le E[X|\mathcal{F}_t] + a\left(E\left[\left((X - E[X|\mathcal{F}_t])^+\right)^p \mid \mathcal{F}_t\right]\right)^{\frac{1}{p}}
\end{aligned}$$

by using Hölder's inequality and the definition of $\mathcal{Q}_t$. Replacing X by $\hat{X} := -X$ and using $(\hat{X} - E[\hat{X}|\mathcal{F}_t])^+ = (X - E[X|\mathcal{F}_t])^-$ gives after changing signs that

$$E\left[\frac{Z'_T}{Z'_t}X \,\middle|\, \mathcal{F}_t\right] \ge E[X|\mathcal{F}_t] - a\left(E\left[\left((X - E[X|\mathcal{F}_t])^-\right)^p \mid \mathcal{F}_t\right]\right)^{\frac{1}{p}} = \Phi_t(X).$$

Analogously, (2.7.37) can be transformed into

$$E\left[\frac{\tilde{Z}_T}{\tilde{Z}_t}X \,\middle|\, \mathcal{F}_t\right] = \Phi_t(X).$$

This proves (2.7.35) for $p > 1$. If $p = 1$ we take $\tilde{Y} := \mathbf{1}_{\{X < E[X|\mathcal{F}_t]\}}$ and obtain as in (2.7.36) that

$$\begin{aligned}
E\left[\frac{\tilde{Z}_T}{\tilde{Z}_t}X \,\middle|\, \mathcal{F}_t\right] &= E[X|\mathcal{F}_t] + aE\left[\tilde{Y}(X - E[X|\mathcal{F}_t]) \,\middle|\, \mathcal{F}_t\right] \\
&= E[X|\mathcal{F}_t] - aE\left[(X - E[X|\mathcal{F}_t])^- \mid \mathcal{F}_t\right] \\
&= \Phi_t(X)
\end{aligned}$$

and that for arbitrary $Q' \in \mathcal{Q}_t$ with corresponding Y' and density Z', we have

$$\begin{aligned}
E\left[\frac{Z'_T}{Z'_t}X \,\middle|\, \mathcal{F}_t\right] &\le E[X|\mathcal{F}_t] + aE\left[Y'(X - E[X|\mathcal{F}_t])^+ \mid \mathcal{F}_t\right] \\
&\le E[X|\mathcal{F}_t] + aE\left[(X - E[X|\mathcal{F}_t])^+ \mid \mathcal{F}_t\right].
\end{aligned}$$

The same arguments as above then again yield (2.7.35).

In a second step, we now prove that $\mathcal{Q}_t$ is convex and closed in $\mathbf{L}^1$. Convexity is easy since for $p > 1$, the boundedness by 1 of $E\left[\left(\alpha Y + (1-\alpha)Y'\right)^q \mid \mathcal{F}_t\right]$ follows from the conditional Minkowski inequality. To show closedness, we fix a sequence $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{Q}_t$ which converges in $\mathbf{L}^1$ to some $\overline{Q}$ and denote by Z_T^n and $\overline{Z}_T$ their respective densities and by $(Y_n)_{n\in\mathbb{N}}$ and $\overline{Y}$ the associated random variables from the definition of $\mathcal{Q}_t$. Since each $f_n := E[Y_n|\mathcal{F}_t]$ satisfies $0 \le f_n \le 1$, Lemma 3.2 of [Sch92] ensures the existence of a sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ of convex combinations $\hat{f}_n \in \mathrm{conv}\{f_n, f_{n+1}, \ldots\}$ which converges to some $\hat{f}$ almost surely and hence also in $\mathbf{L}^1$. Denote for each $n \in \mathbb{N}$ by $\hat{Z}_T^n \in \mathrm{conv}\{Z_T^n, Z_T^{n+1}, \ldots\}$ and $\hat{Y}_n \in \mathrm{conv}\{Y_n, Y_{n+1}, \ldots\}$ the convex combinations with the same weights as $\hat{f}_n$. Then

$$\begin{aligned} E[\,|\hat{Y}_m - \hat{Y}_n|\,] &\le E\left[\left|(\hat{Y}_m - \hat{f}_m) - (\hat{Y}_n - \hat{f}_n)\right|\right] + E\left[\left|\hat{f}_m - \hat{f}_n\right|\right] \\ &= E\left[\left|\frac{1}{a}(\hat{Z}_T^m - 1) - \frac{1}{a}(\hat{Z}_T^n - 1)\right|\right] + E\left[\left|\hat{f}_m - \hat{f}_n\right|\right], \end{aligned}$$

and the RHS converges to 0 for $m, n \to \infty$ since $(\hat{Z}_T^n)_{n\in\mathbb{N}}$ converges like $(Z_T^n)_{n\in\mathbb{N}}$ to $\overline{Z}_T$ in $\mathbf{L}^1$; this uses that $\hat{Z}_T^n \in \mathrm{conv}\{Z_T^n, Z_t^{n+1}, \ldots\}$. Thus the Cauchy sequence $(\hat{Y}_n)_{n\in\mathbb{N}}$ converges to some $\hat{Y} \ge 0$ in $\mathbf{L}^1$. If $p > 1$, the conditional Minkowski inequality implies that $E[\hat{Y}_n^q|\mathcal{F}_t] \le 1$ for each $n \in \mathbb{N}$ and hence by Fatou's lemma also $E[\hat{Y}^q|\mathcal{F}_t] \le 1$; for $p = 1$, we have $0 \le \hat{Y} \le 1$. Moreover,

$$\hat{Z}_T^n = 1 + a\left(\hat{Y}_n - E\left[\hat{Y}_n\middle|\mathcal{F}_t\right]\right) \ge 0$$

converges for $n \to \infty$ in $\mathbf{L}^1$ to

$$\hat{Z}_T := 1 + a\left(\hat{Y} - E\left[\hat{Y}\middle|\mathcal{F}_t\right]\right)$$

since $\hat{Y}_n \to \hat{Y}$ in $\mathbf{L}^1$. So $\hat{Z}_T$ is ≥ 0 and the density of an element of $\mathcal{Q}_t$. But we already know that $\hat{Z}_T^n \to \overline{Z}_T$ in $\mathbf{L}^1$; hence $\overline{Z}_T = \hat{Z}_T$ which implies that $\mathcal{Q}_t$ is closed.

Finally we provide a counterexample which shows that Φ is in general not a time-consistent DMCohUF and that it is even impossible to redefine it for $t \in (0, T]$ such that Φ becomes a time-consistent DMCUF. In fact, the counterexample shows that $\mathcal{Q}_0$ is not weakly m-stable in general, which is by Lemma 2.3.29 a necessary condition for time-consistency if Φ_0 is positively homogeneous; note that we showed in the proof of Lemma 2.3.29 that the $\mathbf{L}^1$-closed convex st $\mathcal{Q}_0$ representing a DMCohUF at time 0 is unique.

Counterexample: Let $\Omega = \{\omega_1, \ldots, \omega_6\}$, $\mathcal{F}$ the power set of Ω, $T = 2$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\})$, $\mathcal{F}_2 = \mathcal{F}$, $a = p = 1$, $p_i := P[\{w_i\}]$, $i = 1, \ldots, 6$, $p_1 = 100p_2$, $p_6 = 100p_5$. Define two densities Z_T^1 and Z_T^2 of elements of $\mathcal{Q}_0$ by their associated random variables Y^1 and Y^2, where $E[Y^i] = \frac{1}{2}$, $0 \le Y^i \le 1$, $i = 1, 2$, and

$$Y^1(\omega_1) = \frac{1}{100}, \quad Y^1(\omega_2) = 1, \quad Y^1(\omega_5) = 1, \quad Y^1(\omega_6) = 1,$$

$$Y^2(\omega_1) = \frac{1}{100}, \quad Y^2(\omega_2) = 1, \quad Y^2(\omega_5) = 1, \quad Y^2(\omega_6) = \frac{1}{100}.$$

If $\mathcal{Q}_0$ is weakly m-stable, then

$$\begin{aligned}
\overline{Z}_T &:= Z_1^1 \frac{Z_T^2}{Z_1^2} \\
&= \left(1 + (E[Y^1|\mathcal{F}_1] - E[Y^1])\right) \frac{1 + (Y^2 - E[Y^2])}{1 + (E[Y^2|\mathcal{F}_1] - E[Y^2])} \\
&= 1 + \left(-1 + \left(1 + (E[Y^1|\mathcal{F}_1] - E[Y^1])\right) \frac{1 + (Y^2 - E[Y^2])}{1 + (E[Y^2|\mathcal{F}_1] - E[Y^2])}\right)
\end{aligned}$$

must be the density of some element of $\mathcal{Q}_0$. Since

$$\tilde{Y} := \left(1 + (E[Y^1|\mathcal{F}_1] - E[Y^1])\right) \frac{1 + (Y^2 - E[Y^2])}{1 + (E[Y^2|\mathcal{F}_1] - E[Y^2])}$$

has $E[\tilde{Y}] = 1$ we can write

$$\overline{Z}_T = 1 + \left((\tilde{Y} + c) - E[\tilde{Y} + c]\right)$$

for any $c \in I\!R$, and this is the unique decomposition of the form "$1 + (Y - E[Y])$", where Y is a random variable, except for the constant c. $\overline{Z}_T$ is an element of $\mathcal{Q}_0$ if and only if there exists $c \in I\!R$ such that $0 \leq \tilde{Y} + c \leq 1$. Since $\tilde{Y} \geq 0$ this is equivalent to

$$\max_{i \in \{1,\ldots,6\}} \tilde{Y}(\omega_i) - \min_{i \in \{1,\ldots,6\}} \tilde{Y}(\omega_i) \leq 1. \tag{2.7.38}$$

However, $E[Y^1|\mathcal{F}_1](\omega_1) = E[Y^2|\mathcal{F}_1](\omega_1)$ so that

$$\tilde{Y}(\omega_1) = 1 + Y^2(\omega_1) - E\left[Y^2\right] = 0.51$$

and $E[Y^1|\mathcal{F}_1](\omega_5) = 1$ and $E[Y^2|\mathcal{F}_1](\omega_5) = \frac{2}{101}$ imply that

$$\tilde{Y}(\omega_5) = \left(1 + 1 - \frac{1}{2}\right) \frac{1 + 1 - \frac{1}{2}}{1 + \frac{2}{101} - \frac{1}{2}} = \frac{303}{70}.$$

Therefore (2.7.38) is not satisfied and $\mathcal{Q}_0$ is not weakly m-stable. ◇

Chapter 3

Utility based good deal bounds

3.1 Introduction

In this chapter we study good deal value bounds in a dynamic setting. We model the discounted price processes of the traded assets in an incomplete market by a multi-dimensional semimartingale S and denote the set of equivalent local martingale measures for S by $\mathcal{M}^e(S)$; the latter is assumed to be non-empty. For a (possibly untraded) discounted random payoff X, the expectation under any $Q \in \mathcal{M}^e(S)$ is an arbitrage free value for X and the interval of arbitrage-free values is given by

$$\left(\inf_{Q \in \mathcal{M}^e(S)} E_Q[X], \sup_{Q \in \mathcal{M}^e(S)} E_Q[X] \right).$$

But this interval is usually too big to be useful in practice. On the other hand, introducing subjective criteria to single out one pricing measure can be very restrictive. Therefore we pursue a middle course. $\mathcal{M}^e(S)$ contains many martingale measures which are not reasonable for pricing because they are too "good" in some way, or, more technically, too far away from the reference measure P in an appropriate sense. Consider for instance the following simple example of a finite model with only one time step and two traded assets. Their payoff structure is given in Figure 3.1 below and we assume that the subjective measure P assigns the same probability to each of the three states. The set of equivalent martingale measures is given by

$$\mathcal{M}^e(S) = \left\{ (2q, 1-3q, q) \,\middle|\, q \in \left(0, \frac{1}{3}\right) \right\}.$$

Thus, for the payoff $X = (0, 1, 0)$, the interval of arbitrage-free values is the whole interval $(0, 1)$. The values close to the boundaries of this interval are attained by probability measures which either have hardly any mass in the second state or concentrate

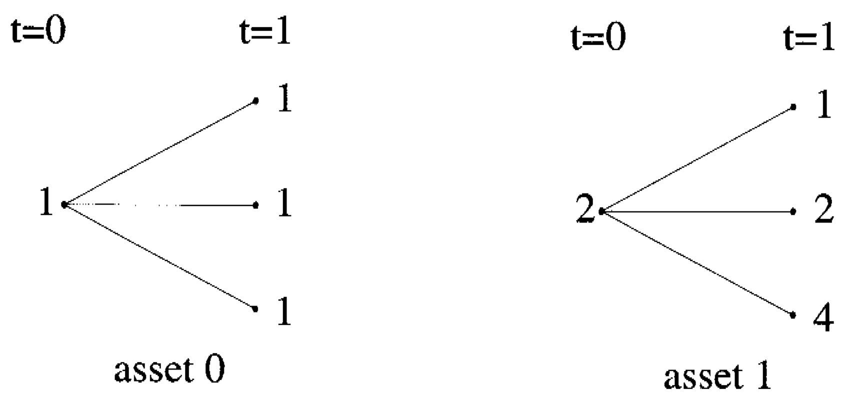

Figure 3.1:

most of the mass there. Since these measures are very different from the subjective measure P, they are not very reasonable pricing measures. To exclude such extreme values and degenerate measures, one might for instance impose a bound on the variance of the density, i.e., allow only those measures for pricing which are contained in

$$\mathcal{N} := \left\{ Q \in \mathcal{M}^e(S) \,\middle|\, \frac{dQ}{dP} = Z_T, \quad \text{Var}[Z_T] \leq A \right\},$$

where A is some constant. This gives for X the value interval

$$\left[\inf_{Q \in \mathcal{N}} E_Q[X], \ \sup_{Q \in \mathcal{N}} E_Q[X] \right]$$

which is (depending on the choice of A) much smaller than the no-arbitrage value interval.

The first study of this approach is due to Cochrane and Saà-Requejo in [CSR00]. They use a performance measurement (the Sharpe ratio, to be precise) to quantify the attractiveness of some payoff priced with respect to some $Q \in \mathcal{M}^e(S)$, and want to exclude those measures from pricing which yield a *good deal*, i.e., an investment opportunity which is too attractive in comparison with those traded in the market. Using an inequality of Hansen/Jagannathan [HJ91], they find that a restriction on the variance of the density of $Q \in \mathcal{M}^e(S)$ with respect to P yields an upper bound for the attractiveness of all payoffs priced with respect to Q. Therefore they take for pricing only measures which are contained in $\mathcal{N}$ and thus do not yield good deals.

We show below that the upper bound for the Sharpe ratio in [CSR00] is just the maximal attainable Sharpe ratio in an extended market, where the extension depends on the respective pricing measure. Moreover, we prove that one can obtain the same set $\mathcal{N}$ of *no-good-deal measures* by imposing a bound on the maximal attainable quadratic utility in the extended market. This gives rise to a more general approach where we replace the quadratic by other utility functions. To obtain a set of no-good-deal measures $\mathcal{N}$, i.e., of reasonable pricing measures, we then restrict the maximal attainable utility in the extended market. Our main goal is to study the no-good-deal values and value bounds as processes. Because their computability and dynamic properties depend on the set $\mathcal{N}$, the main difficulty is to find an appropriate and yet workable definition for this set in a dynamic context. In a Lévy setting, we define $\mathcal{N}$ by a pointwise restriction on an appropriate integrand. This allows us to apply dynamic programming techniques. We show that this "local" restriction implies a bound on the corresponding "global" criterion and clarify the connection between the pointwise and the global restriction. The corresponding lower value bound is a dynamic monetary coherent utility functional, i.e., up to sign a dynamic coherent risk measure and in particular time-consistent. Any such functional Φ is fully described by a set $\mathcal{Q}$ of probability measures $Q \approx P$ via $\Phi_t(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} E_Q[X|\mathcal{F}_t]$; see [Del06] or Lemma 2.3.29. Although $\mathcal{Q}$ has a clear economic interpretation as the set of all possible scenarios, it is often not clear which measures one should choose. Thus a byproduct of our approach is that it yields a very intuitive way to specify with $\mathcal{Q} := \mathcal{N}$ a set of scenarios in the context of valuation in incomplete markets.

Good deal value or price bounds have been studied in some recent papers. Similarly to Cochrane/Saà-Requejo, Björk/Slinko [BS06] use the Sharpe ratio and impose a bound on the variance of the densities of the pricing measures. Ross [Ros05] also works with this set of pricing measures, but obtains it in the Capital Asset Pricing Model from a different reasoning. Černý [Cer03] obtains good deal bounds for several utility functions via a constraint on the indifference value in an extended market. Bernardo/Ledoit [BL00] use the *gain-loss ratio* as a measure for the attractiveness; for a payoff X priced with respect to $Q \in \mathcal{M}^e(S)$, this is the ratio of the expectations (with respect to P) of the positive and the negative parts of the excess return $X - E_Q[X]$. Pinar/Salih [PS05] use a similar gain-loss trade-off. Longarela [Lon01] and Bondarenko/Longarela [BL04] suggest to take only those measures which are not too far away from some benchmark valuation measure and propose several definitions for the distance between valuation measures. A different type of approach is pursued by Černý/Hodges [CH02], Jaschke/Küchler [JK01], Staum [Sta04], Carr et al. [CGM01] and Cherny [Che06,05a,05b]. They start with a set of *desirable payoffs*, and a good deal is a desirable claim which is available for free. This latter approach is strongly related with monetary risk measures; see the discussion at the end of Section 2.6. Of all these works on good deal price bounds, only [CSR00], [Cer03] and [BS06] consider also a dynamic setting; in Section 2.6 we also work in a dynamic framework, however our results are formulated in terms of indifference valuation.

The chapter is structured as follows. In Section 3.2 we recall the original definition of good deal bounds from [CSR00] and explain how it can be generalized. This is done in a static setting. Section 3.3 explains the link between good deal value bounds and monetary risk measures. Moreover, we present in more detail the connections between the different existing approaches on good deal bounds. But our main goal is to study good deal values and value bounds as processes. In order to have a nice parametrization for the set of all equivalent local martingale measures, we choose to work in a Lévy framework. Section 3.4 provides some auxiliary results on such parameterizations and on Lévy processes. Section 3.5 deals with the extension of no-good-deal valuation to a dynamic setting. The main difficulty is to find a reasonable definition for the set of no-good-deal measures which still leads to mathematically tractable problems. Our definition will be obtained from a pointwise restriction on an appropriate integrand. This "local" restriction is motivated by a "global" criterion, and we explain how the two are linked. Section 3.6 discusses the properties of the resulting good deal values and value bounds as processes. Finally, we present two explicit examples in Section 3.7.

3.2 Static good deal bounds

In this section we introduce some notation and the concept of good deal value bounds. We generalize the original approach by Cochrane/Saà-Requejo [CSR00] and introduce several criteria for good deals.

To describe a financial market, we start with a probability space $(\Omega, \mathcal{F}, P)$ with a right-continuous and complete filtration $(\mathcal{F}_t)_{0\le t\le T}$ where $T < \infty$ is a finite time horizon, $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$. There are one riskless and d risky assets with discounted price processes modelled by an $I\!R^d$-valued semimartingale S. In this section we assume that S is locally bounded. We can and do choose RCLL versions for all semimartingales. By $\mathcal{P}^a$ we denote the set of all probability measures $Q \ll P$ and by $\mathcal{P}^e$ the subset of equivalent measures. $E[\,.\,]$ denotes the expectation with respect to P.

For any $Q \in \mathcal{P}^a$, its density process $Z = (Z_t)_{0\le t\le T}$ and density Z_T are defined with respect to P. The problem we investigate is the following. For an agent who can dynamically trade in S, what is a reasonable value for an untraded (discounted) payoff $\hat{X}$ contained in $\mathbf{L}^\infty(P)$ or in $\mathbf{L}^0_+(P)$? Let us fix a payoff $\hat{X}$ for the moment. A first reasonable requirement is to have absence of arbitrage. An arbitrage-free value for $\hat{X}$ is typically given by the expectation of $\hat{X}$ under some equivalent local martingale measure for S. We denote by $\mathcal{M}^e(S)$ the set of all $Q \in \mathcal{P}^e$ such that S is a local martingale under Q and exclude arbitrage opportunities by the standing assumption

$$\mathcal{M}^e(S) \neq \emptyset.$$

But for an incomplete market, $\mathcal{M}^e(S)$ contains infinitely many elements. Thus no-arbitrage arguments imply only that the value of $\hat{X}$ should lie in the interval

$$\left(\inf_{Q\in\mathcal{M}^e(S)} E_Q[\hat{X}],\ \sup_{Q\in\mathcal{M}^e(S)} E_Q[\hat{X}] \right).$$

As illustrated in the introduction, $\mathcal{M}^e(S)$ contains in general many martingale measures which are not reasonable for pricing because they are too "good" respectively too far away from the reference measure P in an appropriate sense. By omitting those measures, we define a set $\mathcal{N} \subseteq \mathcal{M}^e(S)$ of *no-good-deal measures* and obtain the value interval

$$\left[\inf_{Q\in\mathcal{N}} E_Q[\hat{X}],\ \sup_{Q\in\mathcal{N}} E_Q[\hat{X}]\right] \tag{3.2.1}$$

which is smaller than the no arbitrage interval. This is the abstract concept of *good deal bounds.* To make things more concrete, one has to specify what too "good" measures are. In the pioneering work by Cochrane and Saà-Requejo in [CSR00] this is done as follows. If $Q \in \mathcal{M}^e(S)$ is chosen as pricing measure, the *excess return* of some payoff $\hat{X} \in \mathbf{L}^2(P)$ is $\hat{X} - E_Q[\hat{X}]$ and the corresponding *Sharpe ratio* is defined as

$$\mathrm{SR}(\hat{X}, Q) := \frac{E\left[\hat{X} - E_Q[\hat{X}]\right]}{\sqrt{\mathrm{Var}\left[\hat{X} - E_Q[\hat{X}]\right]}}.$$

This is a widely used performance measure. Cochrane and Saà-Requejo now argue that Sharpe ratios observed in the market tend to be rather low. Therefore they define *good deals* as excess returns with high Sharpe ratios. To obtain a mathematically better tractable problem, they use an inequality due to Hansen/Jagannathan [HJ91]. For $Q \in \mathcal{M}^e(S)$ with density $Z_T \in \mathbf{L}^2(P)$ and $\hat{X} \in \mathbf{L}^2(P)$, this inequality yields

$$\mathrm{SR}(\hat{X}, Q) \leq \frac{\sqrt{\mathrm{Var}[Z_T]}}{E[Z_T]} = \sqrt{E[Z_T^2] - 1}. \tag{3.2.2}$$

Thus, a bound on $\mathrm{Var}[Z_T]$ or, equivalently, on $E[Z_T^2]$ implies a bound on the Sharpe ratios of all payoffs valued by Q. Therefore, [CSR00] define the set of no-good-deal measures by

$$\mathcal{N}^q := \left\{Q \in \mathcal{M}^e(S)\ \middle|\ \frac{dQ}{dP} = Z_T,\ E[Z_T^2] \leq A^q\right\}$$

for some constant A^q; here q stands for *quadratic.*

A first and important question is in which sense the inequality (3.2.2) is sharp. To discuss this, we need to clarify for which payoffs we should like to have a bound on the Sharpe ratio. Our agent can dynamically trade in S. If we assume that also the terminal payoff $\hat{X}$ is dynamically traded for the price $E_Q[\hat{X}|\mathcal{F}_t]$, we want the Sharpe ratio obtainable by dynamically trading in $S^Q := \left(S, (E_Q[\hat{X}|\mathcal{F}_t])_{0\leq t\leq T}\right)$ to be restricted. We define the set of all wealth processes obtainable by trading in S^Q

with initial capital $x \in I\!R$ by

$$\mathcal{X}(x, S^Q) := \Big\{ V = (V_t)_{0\le t\le T} \,\Big|\, V_t = x + (H \cdot S^Q)_t \text{ for some predictable, } S^Q\text{-integrable } H \text{ such that } V \text{ is uniformly bounded from below} \Big\}.$$

The corresponding set of payoffs which are bounded from below and dominated by the terminal values of these wealth processes is

$$\mathcal{C}(x, S^Q) := \Big\{ X \in \mathbf{L}^0(P) \,\Big|\, X^- \in \mathbf{L}^\infty(P) \text{ and } X \le V_T \text{ for some } V \in \mathcal{X}(x, S^Q) \Big\}. \tag{3.2.3}$$

This is the set of payoffs whose Sharpe ratios we want to be bounded. Since S^Q is a local Q-martingale, each $V \in \mathcal{X}(x, S^Q)$ is a local Q-martingale and a Q-supermartingale. Hence $\mathcal{C}(x, S^Q)$ is contained in

$$\mathcal{C}(x, Q) := \Big\{ X \in \mathbf{L}^1(Q) \,\Big|\, E_Q[X] \le x,\ X^- \in \mathbf{L}^\infty(P) \Big\}. \tag{3.2.4}$$

Recall that by duality theory, in a *complete* market with traded assets S' modelled by a locally bounded semimartingale and unique martingale measure Q', we have for any $X \in \mathbf{L}^\infty(P)$

$$E_{Q'}[X] \le x \iff X \in \mathcal{C}(x, S').$$

In this sense the payoffs in $\mathcal{C}(x, Q)$ constitute the natural analogue of those obtainable by dynamically trading in an extended market with unique martingale measure Q. The following lemma shows that the RHS of (3.2.2) is the maximal Sharpe ratio obtainable in this Q-extended market.

Lemma 3.2.1. *For $x \in I\!R$ and $Q \in \mathcal{P}^e$ with density $Z_T \in \mathbf{L}^2(P)$ we have*

$$\sup_{\substack{X\in\mathcal{C}(0,Q)\\ E[X]<\infty}} \mathrm{SR}(X, Q) = \sup_{\substack{X\in\mathcal{C}(x,Q)\\ E[X]<\infty}} \mathrm{SR}(X, Q) = \sqrt{E[Z_T^2] - 1}. \tag{3.2.5}$$

Remark 3.2.2. Note that if $E[X] = \infty$, the variance and hence the Sharpe ratio of X is not well-defined. If $E[X] < \infty$ and $\mathrm{Var}[X] = \infty$, we simply set the Sharpe ratio equal to zero. ◇

Proof. Note first that $\mathcal{C}(x, Q) = x + \mathcal{C}(0, Q)$ and $\mathrm{SR}(x+X, Q) = \mathrm{SR}(X; Q)$. Hence the first equality in (3.2.5) is clear and we may without loss of generality take $x = 1$. Now, let $X \in \mathcal{C}(1, Q)$ with $E[X] < \infty$ and define $\hat{X} := X - E[X]$ so that $E[\hat{X}] = 0$. The Cauchy-Schwarz inequality implies that

$$E_Q^2[\hat{X}] = E^2[Z_T\hat{X}] = E^2[(Z_T - 1)\hat{X}] \le E[(Z_T - 1)^2]E[\hat{X}^2]$$

so that

$$\mathrm{SR}(X, Q) = \mathrm{SR}(\hat{X}, Q) = \frac{-E_Q[\hat{X}]}{\sqrt{E[\hat{X}^2]}} \leq \sqrt{E[(Z_T - 1)^2]} = \sqrt{E[Z_T^2] - 1}.$$

This proves "$\leq$" in (3.2.5). To show equality, first assume that $Z_T \in \mathbf{L}^\infty(P)$ and define $X := 1 - Z_T$ so that $E[X] = 0$ and $X = \hat{X}$. Since then $\mathrm{SR}(X, Q) = \sqrt{E[Z_T^2] - 1}$ and $X \in \mathcal{C}(1, Q)$, this establishes equality in (3.2.5). If $Z_T \notin \mathbf{L}^\infty(P)$, then approximate X by

$$X_n := 1 - Z_T \mathbf{1}_{\{Z_T \leq n\}}$$

which is clearly in $\mathcal{C}(x, Q)$. Computing the Sharpe ratio $\mathrm{SR}(X_n, Q)$ explicitly and then using monotone convergence both for its numerator and denominator directly gives

$$\lim_{n\to\infty} \mathrm{SR}(X_n, Q) = \sqrt{E[Z_T^2] - 1}$$

and thus completes the proof. □

Exploiting the same estimate via the extended market, one can get the same good deal bounds via a different criterion. In fact, it can be obtained from maximizing expected utility in the extended market for the quadratic "utility" function which is defined for fixed $a \in I\!R$ by

$$\mathbf{U}^q(x) := -(a - x)^2, \quad x \in I\!R.$$

Proposition 3.2.3. *Let $Q \in \mathcal{P}^e$ with density $Z_T \in \mathbf{L}^2(P)$ and $x < a$. Then*

$$\sup_{X \in \mathcal{C}(x, Q)} E[\mathbf{U}^q(X)] = \mathbf{U}^q(x) \frac{1}{E[Z_T^2]}. \tag{3.2.6}$$

Proof. "$\leq$" holds by the Cauchy-Schwarz inequality since for any $X \in \mathcal{C}(x, Q)$

$$\begin{aligned} E[\mathbf{U}^q(X)]E[Z_T^2] &= -E[(a - X)^2]E[Z_T^2] \leq -(E[Z_T(a - X)])^2 \\ &\leq -(a - x)^2 = \mathbf{U}^q(x). \end{aligned}$$

If $Z_T \in \mathbf{L}^\infty(P)$, equality follows since $X := a - \frac{a-x}{E[Z_T^2]} Z_T \in \mathcal{C}(x, Q)$ and

$$E[(a - X)^2] = \frac{(a - x)^2}{E[Z_T^2]}.$$

For $Z_T \notin \mathbf{L}^\infty(P)$ approximate X by

$$X_n := a - \frac{a - x}{E[Z_T^2]} \left(Z_T \mathbf{1}_{\{Z_T \leq n\}} + c_n \right)$$

with $c_n = E[Z_T^2 \mathbf{1}_{\{Z_T > n\}}] \searrow 0$. Then $X_n \in \mathcal{C}(x, Q)$, and combining an explicit computation with the dominated convergence theorem yields

$$\lim_{n \to \infty} E\left[(a - X_n)^2\right] = \frac{(a-x)^2}{E[Z_T^2]}.$$

□

Proposition 3.2.3 shows that the maximal expected utility in the extended market is separable into the utility of the initial capital x and a term depending on Q only. Our aim is to find a set $\mathcal{N}$ of no-good-deal pricing measures. We want to deduce the criterion for good deals from a restriction on the maximal quadratic utility, and we should like it to be independent of the initial capital x. Therefore, we work with the term depending on Q only and thus choose the value of the variance or, equivalently, of the second moment of Z_T as criterion for no-good-deal measures. Since $\mathbf{U}^q(\,.\,) \leq 0$, the set of no-good-deal measures is then like for the restriction of the Sharpe ratio given by

$$\mathcal{N}^q = \left\{ Q \in \mathcal{M}^e(S) \,\middle|\, Z_T := \frac{dQ}{dP},\ E[Z_T^2] \leq A^q \right\}$$

for some constant A^q.

Remark 3.2.4. a) Both Černý [Cer03] and Hodges [Hod98] purport to illustrate with an example that the Sharpe ratio is not a good performance measure. In a finite state model, they specify the excess returns of two payoffs in such a way that except for one state, the excess returns of both payoffs are equal, but the Sharpe ratio of the payoff with the higher excess return in the remaining state is smaller. To describe this mathematically, suppose there are n different states. Denote by x_i and y_i the respective payoffs in state $i \in \{1, \ldots, n\}$, by q_i the probability of state i under the pricing measure and by $\varepsilon > 0$ the difference of the excess returns in state n. By assumption, the excess returns satisfy

$$\begin{aligned}
x_1 - \sum_{i=1}^n x_i q_i &= y_1 - \sum_{i=1}^n y_i q_i, \\
&\vdots \\
x_{n-1} - \sum_{i=1}^n x_i q_i &= y_{n-1} - \sum_{i=1}^n y_i q_i, \\
x_n - \sum_{i=1}^n x_i q_i &= \varepsilon + y_n - \sum_{i=1}^n y_i q_i.
\end{aligned}$$

Since $\sum q_i = 1$, multiplying the i-th equation with q_i and summing up both sides of the system of equations gives

$$\sum_{i=1}^{n} x_i q_i - \sum_{i=1}^{n} x_i q_i = \varepsilon q_n + \sum_{i=1}^{n} y_i q_i - \sum_{i=1}^{n} y_i q_i .$$

This can only be true if $q_n = 0$, i.e., if the pricing measure Q is only absolutely continuous, but not equivalent to P. Hence this animadversion against the Sharpe ratio is not justified.

b) The quadratic "utility" function $\mathbf{U}^q$ is not increasing and thus not an economically reasonable utility function. Note, however, that this does not affect the criterion derived from the restriction of the maximal attainable quadratic utility since this criterion is based only on the term depending on the measure Q.

◇

Up to now, we have examined the original approach of Cochrane and Saà-Requejo which is based on a restriction for the Sharpe ratio. We have shown that restricting the maximal attainable quadratic utility yields the same set of no-good-deal measures. An obvious generalization of this approach is thus to introduce and study no-good-deal criteria from maximizing expected utility for more general utility functions like

$$\begin{aligned}
\mathbf{U}^e(x) &:= -\frac{1}{\beta} e^{-\beta x}, \quad \beta > 0; \\
\mathbf{U}^p(x) &:= \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & : \ x > 0 \\ \lim\limits_{x \searrow 0} \frac{x^{1-\gamma}}{1-\gamma} & : \ x = 0 \\ -\infty & : \ x < 0 \end{cases} \qquad 0 < \gamma \neq 1; \\
\mathbf{U}^\ell(x) &:= \begin{cases} \log x & : \ x > 0 \\ -\infty & : \ x \leq 0. \end{cases}
\end{aligned}$$

We set $\mathbf{U}^p$ and $\mathbf{U}^\ell$ equal to $-\infty$ on $\mathbb{R}_-$ to avoid having to distinguish between utility functions on $\mathbb{R}$ and $\mathbb{R}_+$ and define $\operatorname{dom}(\mathbf{U}^i) := \{x \in \mathbb{R} \mid \mathbf{U}^i(x) > -\infty\}$. The well-known approach to calculate maximal expected utility is to apply duality theory. We are interested in the solution to the *primal* problem of maximizing expected utility over some set of payoffs $\mathcal{C}(x)$, e.g., $\mathcal{C}(x) = \mathcal{C}(x, S)$ from (3.2.3). That is, for $\mathbf{U}^i : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and $i \in \{e, p, \ell\}$ we want to find

$$u^i(x) := \sup_{X \in \mathcal{C}(x)} E[\mathbf{U}^i(X)].$$

Here we make the convention that $E[\mathbf{U}^i(X)] := -\infty$ whenever $E\left[\left(\mathbf{U}^i(X)\right)^-\right] = \infty$.

The *convex conjugate* of the concave function $\mathbf{U}^i$ is defined by

$$V^i(y) := \sup_{x \in I\!R} \{\mathbf{U}^i(x) - xy\}, \quad y > 0 \tag{3.2.7}$$

and satisfies

$$\mathbf{U}^i(x) = \inf_{y>0} \{V^i(y) + xy\}, \quad x \in \mathrm{dom}(\mathbf{U}^i). \tag{3.2.8}$$

In particular, we have

$$V^e(y) = \frac{y}{\beta}(\log y - 1), \quad V^p(y) = \frac{\gamma}{1-\gamma} y^{\frac{\gamma-1}{\gamma}}, \quad V^\ell(y) = -\log y - 1. \tag{3.2.9}$$

If the market is complete with a unique martingale measure Q' with density Z'_T, the *dual* formulation is

$$v^i(y) := v^{i,Q'}(y) := E[V^i(yZ'_T)]. \tag{3.2.10}$$

Under appropriate assumptions on $\mathcal{C}(x)$, u^i and v^i are conjugate so that

$$u^i(x) = \inf_{y>0} \{v^i(y) + xy\}.$$

If the market is incomplete, then the RHS in (3.2.10) involves in addition an infimum over an (extended) set of equivalent martingale measures; see [KS99] and [Sch01] for precise statements. We apply these duality results to the above utility functions to obtain the maximal expected utility in the Q-extended market with payoffs $\mathcal{C}(x, Q)$ from (3.2.4).

Definition 3.2.5. For $i \in \{e, p, \ell\}$ and $Q \in \mathcal{M}^e(S)$, the *maximal* $(\mathbf{U}^i, Q)$*-utility* from $x \in \mathrm{dom}(\mathbf{U}^i)$ is defined as

$$u^{i,Q}(x) := \sup_{X \in \mathcal{C}(x,Q)} E[\mathbf{U}^i(X)]. \tag{3.2.11}$$

We also introduce the set

$$\mathcal{Q}^i_u := \left\{ Q \in \mathcal{M}^e(S) \;\middle|\; \exists\, x \in \mathrm{dom}(\mathbf{U}^i) \text{ such that } u^{i,Q}(x) < \mathbf{U}^i(\infty) \right\}.$$

Remark 3.2.6. For $i \in \{p, \ell\}$ we have $\mathbf{U}^i(\infty) = \infty$, so that $u^{i,Q}(x) < \mathbf{U}^i(\infty)$ holds by concavity for all $x \in \mathrm{dom}(\mathbf{U}^i)$ if $Q \in \mathcal{Q}^i_u$. For $i = e$, this holds because of the multiplicative dependence of $u^{i,Q}$ on x. ◇

Proposition 3.2.7. *For* $i \in \{e, p, \ell\}$, $x \in \mathrm{dom}(\mathbf{U}^i)$ *and* $Q \in \mathcal{Q}^i_u$ *with density* Z_T, *we have*

$$u^{i,Q}(x) = \inf_{y>0} \{v^{i,Q}(y) + xy\} \tag{3.2.12}$$

where $v^{i,Q}(y) := E[V^i(yZ_T)]$ *so that*

$$u^{e,Q}(x) = \mathbf{U}^e(x)e^{-E[Z_T \log Z_T]}; \tag{3.2.13}$$

$$u^{p,Q}(x) = \mathbf{U}^p(x)E\left[Z_T^{\frac{\gamma-1}{\gamma}}\right]^\gamma; \tag{3.2.14}$$

$$u^{\ell,Q}(x) = \mathbf{U}^\ell(x) - E\left[\log Z_T\right]. \tag{3.2.15}$$

Proof. (3.2.13) – (3.2.15) follow from easy calculations as soon as we prove (3.2.12). However, it is easy to check that $\mathcal{C} := \mathcal{C}(1, Q) \cap \mathbf{L}^0_+$ and $\mathcal{D} := \left\{ Y \in \mathbf{L}^0_+(P) \middle| Y \leq Z_T \right\}$ satisfy conditions (i) – (iii) of Proposition 3.1 in [KS99] and that for $i \in \{p, \ell\}$ we can replace in (3.2.11) $\mathcal{C}(x, Q)$ by $x\mathcal{C}$ since $\mathbf{U}^i(x) = -\infty$ for $x < 0$. Hence for $i \in \{p, \ell\}$ the claim follows from Theorem 3.1 there. For $i = e$, it can be shown as in the proof of Theorem 2.1 in [Sch01], approximating $\mathbf{U}^e$ by a sequence of functions $\mathbf{U}^e_n$ to which one can apply the results from the first part of this proof. □

Proposition 3.2.7 shows that like for $\mathbf{U}^e$, the maximal attainable utility for $\mathbf{U}^e$, $\mathbf{U}^p$ and $\mathbf{U}^\ell$ is separable into the utility of the initial capital and a term depending on Q only. We thus propose the following criteria for no-good-deal measures.

Definition 3.2.8. Define the *f-divergences* of $Q \in \mathcal{P}^e$ with respect to P by

$$f^e(Q|P) := E[Z_T \log Z_T],$$

$$f^p(Q|P) := E\left[\operatorname{sign}(1-\gamma) Z_T^{\frac{\gamma-1}{\gamma}}\right] \quad \textit{with } 0 < \gamma \neq 1,$$

$$f^\ell(Q|P) := E\left[-\log Z_T\right],$$

and for $i \in \{e, p, \ell\}$ the corresponding subsets $\mathcal{N}^i \subseteq \mathcal{M}^e(S)$ of *no-good-deal measures* by

$$\mathcal{N}^i := \left\{ Q \in \mathcal{M}^e(S) \middle| f^i(Q|P) \leq A^i \right\} \tag{3.2.16}$$

for some constants A^i.

Remark 3.2.9. a) Any functional like $f^e(Q|P)$, $f^p(Q|P)$ or $f^\ell(Q|P)$ of the form $E[f(Z_T)]$ for a convex function f is called *f-divergence*; see [LV87]. It is a measure for the distance between Q and P. Therefore another interpretation of the set of no-good-deal measures is that it is the set of all measures which are close enough to P with respect to the f-divergence associated with the utility under consideration.

b) In the definition of $f^p(Q|P)$, the term $\operatorname{sign}(1-\gamma)$ is introduced to ensure that f in $E[f(Z_T)]$ is convex.

◇

In order to have the set $\mathcal{N}^i$ in (3.2.16) non-empty, the smallest choice for the bound A^i is $\inf_{Q\in\mathcal{M}^e(S)} f^i(Q|P)$. Since we obtained $f^i(Q|P)$ via Proposition 3.2.7 from $u^{i,Q}$, this lower bound is linked to the infimum over all $(\mathbf{U}^i, Q)$-utilities where Q runs through $\mathcal{M}^e(S)$. The following proposition shows that this infimum has a very intuitive meaning. It is the maximal expected utility attainable from dynamically trading in the basis assets S only.

Proposition 3.2.10. *Let $i \in \{e, p, \ell\}$, $\mathcal{Q}_u^i \neq \emptyset$ and $x \in \mathrm{dom}(\mathbf{U}^i)$. Then*

$$\inf_{Q\in\mathcal{M}^e(S)} u^{i,Q}(x) = \sup_{X\in\mathcal{C}(x,S)} E[\mathbf{U}^i(X)].$$

Proof. We first show for $Q \in \mathcal{M}^e(S)$ the equivalence

$$Q \in \mathcal{Q}_u^i \quad \Longleftrightarrow \quad \inf_{y>0}\{v^{i,Q}(y) + xy\} < \mathbf{U}^i(\infty) \quad \text{for all } x \in \mathrm{dom}(\mathbf{U}^i). \tag{3.2.17}$$

If $Q \in \mathcal{Q}_u^i$, then Remark 3.2.6 and (3.2.12) imply that for $x \in \mathrm{dom}(\mathbf{U}^i)$

$$\mathbf{U}^i(\infty) > u^{i,Q}(x) = \inf_{y>0}\{v^{i,Q}(y) + xy\}.$$

Now assume that $Q \notin \mathcal{Q}_u^i$ and denote its density by Z_T. Then relation (3.2.8) implies that for all $x \in \mathrm{dom}(\mathbf{U}^i)$ and all $y > 0$

$$\begin{aligned} \mathbf{U}^i(\infty) &\leq u^{i,Q}(x) = \sup_{X\in\mathcal{C}(x,Q)} E[\mathbf{U}^i(X)] \\ &\leq E[V^i(yZ_T)] + \sup_{X\in\mathcal{C}(x,Q)} E_Q[X]y \\ &\leq E[V^i(yZ_T)] + xy. \end{aligned} \tag{3.2.18}$$

Since $v^{i,Q}(y) = E[V^i(yZ_T)]$, this proves (3.2.17). Next we claim that

$$\begin{aligned} \sup_{X\in\mathcal{C}(x,S)} E[\mathbf{U}^i(X)] &= \inf_{y>0}\{\inf_{Q\in\mathcal{M}^e(S)} v^{i,Q}(y) + xy\} \\ &= \inf_{Q\in\mathcal{M}^e(S)} \inf_{y>0}\{v^{i,Q}(y) + xy\}. \end{aligned} \tag{3.2.19}$$

For $i \in \{p, \ell\}$ this is implied by [KS99] Theorem 2.1 (i) and Theorem 2.2 (iv). For $i = e$, it follows from the first part of this proof and [Sch01] Theorem 2.1 (i), Remark 2.3 and the discussion on page 697 in [Sch01] that $\mathcal{C}_U^b(x)$ there can be replaced by $\mathcal{C}(x, S)$; the latter uses in addition that $\mathbf{U}^e$ is bounded from above so that it suffices to consider for the maximal attainable utility those elements of $\mathcal{C}_U^b(x)$ which are bounded from below. Finally, for $Q \in \mathcal{Q}_u^i$ and $x \in \mathrm{dom}(\mathbf{U}^i)$,

$$\mathbf{U}^i(\infty) > u^{i,Q}(x) \geq \sup_{X\in\mathcal{C}(x,S)} [\mathbf{U}^i(X)]. \tag{3.2.20}$$

Thus $\mathcal{Q}_u^i \neq \emptyset$, the first part of this proof, (3.2.19) and (3.2.12) imply that for $x \in \text{dom}(\mathbf{U}^i)$

$$\begin{aligned} \mathbf{U}^i(\infty) &> \sup_{X \in \mathcal{C}(x,S)} [\mathbf{U}^i(X)] = \inf_{Q \in \mathcal{M}^e(S)} \inf_{y>0} \{v^{i,Q}(y) + xy\} \\ &= \inf_{Q \in \mathcal{Q}_u^i} \inf_{y>0} \{v^{i,Q}(y) + xy\} = \inf_{Q \in \mathcal{Q}_u^i} u^{i,Q}(x) \geq \inf_{Q \in \mathcal{M}^e(S)} u^{i,Q}(x) \end{aligned}$$

so that the claim follows from (3.2.20). □

Remark 3.2.11. For later reference we remark that that for $i \in \{e, \ell\}$ and $i = p$, i.e., $\mathbf{U}^p(x) = \frac{x^{1-\gamma}}{1-\gamma}$, with $\gamma \in (0, 1)$ we have

$$\mathcal{Q}_u^i = \mathcal{Q}^i := \left\{Q \in \mathcal{M}^e(S) \,\middle|\, f^i(Q|P) < \infty\right\}.$$

In fact, "$\subseteq$" holds by Proposition 3.2.7 and the definition of $\mathcal{Q}_u^i$, and for "$\supseteq$" note that by (3.2.18) with $y = 1$ we have

$$u^{i,Q}(x) \leq E[V^i(Z_T)] + x.$$

Thus from Remark 3.2.6, (3.2.9) and Definition 3.2.8 we have for $Q \in \mathcal{M}^e(S)$

$$\begin{aligned} Q \notin \mathcal{Q}_u^i \quad &\Rightarrow \quad u^{i,Q}(x) = \mathbf{U}^i(\infty) \quad \text{for all } x \in \text{dom}(\mathbf{U}^i) \\ &\Rightarrow \quad E[V^i(Z_T)] = \infty \\ &\Rightarrow \quad f^i(Q|P) = \infty. \end{aligned}$$

Similarly one can show for $i = p$ with $\gamma > 1$ that

$$\mathcal{Q}_u^p = \mathcal{Q}^p := \{Q \in \mathcal{M}^e(S) \mid f^p(Q|P) < 0\}.$$

◇

In Definition 3.2.8 we defined for $\mathbf{U}^i$ and $i \in \{e, p, \ell\}$ a set of no-good-deal measures. However, we did not say what a good deal exactly is. Consider an agent with preferences corresponding to $\mathbf{U}^i$ and initial capital $x \in \text{dom}(\mathbf{U}^i)$. Suppose she gets offered a future payoff X for the price x. For her, this is a good deal if $E\left[\mathbf{U}^i(X)\right]$ is bigger than the maximal utility attainable by trading with initial capital x in the basic assets S; the latter utility is known from Propositions 3.2.10 and 3.2.7. However, we want to define good deals independently of any initial capital. Therefore we suggest the following

Definition 3.2.12. Let $i \in \{e, p, \ell\}$, $Q \in \mathcal{Q}_u^i$ and $X \in \bigcup_{x \in \text{dom}(\mathbf{U}^i)} \mathcal{C}(x, Q)$. We call (X, Q) a good deal of level δ if

a) $i = e$ and $E\left[\mathbf{U}^e(X)\right] \geq \frac{1}{\delta}\mathbf{U}^e\left(E_Q[X]\right)$;

b) $i = p$, i.e., $\mathbf{U}^p(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma \in (0, 1)$ and $E\left[\mathbf{U}^p(X)\right] \geq \delta\mathbf{U}^p\left(E_Q[X]\right)$,

or $i = p$, $\gamma > 1$ and $E\left[\mathbf{U}^p(X)\right] \geq \frac{1}{\delta}\mathbf{U}^p\left(E_Q[X]\right)$;

c) $i = \ell$ and $E\left[\mathbf{U}^\ell(X)\right] \geq \delta + \mathbf{U}^\ell\left(E_Q[X]\right)$.

This definition deserves some comments. First of all, note that it is such that an increase of δ corresponds to good deals defined with respect to a higher utility level; this is because $\mathbf{U}^e$ and $\mathbf{U}^p$ with $\gamma > 1$ are non-positive. Moreover, by Proposition 3.2.7, if $i \in \{e, p, \ell\}$, $Q \in \mathcal{Q}_u^i$ and $f^i(Q|P) \leq A$, then choosing Q as pricing measures excludes good deals (X, Q) of some level δ for any initial capital $x \in \text{dom}(\mathbf{U}^i)$. More precisely, if $i = e$, then good deals of all levels $\delta \geq e^A$ are excluded; if $i = \ell$ of all levels $\delta \geq A$; if $i = p$ and $\gamma \in (0, 1)$, of all levels $\delta \geq A^\gamma$; and if $i = p$ and $\gamma > 1$, of all levels $\delta > (-\frac{1}{A})^\gamma$; note that in the last case any reasonable A is negative. Of course, the question arises how Definition 3.2.12 is related to good deals defined as excess returns with a high Sharpe ratio. An important difference is that for the Sharpe ratio criterion we consider excess returns instead of payoffs. Therefore the bound specifying a good deal there does not depend on $E_Q[X]$. In addition, for the agent, the attractiveness of a payoff does not depend on the initial capital she has, provided there is a riskless asset in the market. In fact, suppose she has initial capital x and the opportunity to buy a payoff X for a price p such that the Sharpe ratio $(E[X] - p)/\sqrt{\text{Var}[X - p]}$ is higher than that in the market. Then the payoff $X - p + x$ can be obtained from initial capital x and has the same, attractive, Sharpe ratio. In contrast, for a good deal defined via Definition 3.2.12, it is not clear that adding the constant $x - p$ to some good deal X still results in a good deal.

We might define good deals for $\mathbf{U}^q(x) = -(a - x)^2$ analogously as for $\mathbf{U}^e$. However, relation (3.2.5) holds only for $x < a$ since $\mathbf{U}^q$ is decreasing for $x \geq a$. Therefore we suggest for preferences corresponding to quadratic utility to stick to the original definition via a high Sharpe ratio.

3.3 Monetary utility functionals

In this section we give a review of the existing literature on good deal bounds and explain where our approach fits in. It has been noticed quite early that good deal bounds are closely connected with risk measures. To clarify the relations between the different types of existing approaches on good deal bounds, we recall some key results concerning risk measures; see Section 2.3 for a more detailed discussion of the latter. We formulate these in terms of the recently very popular monetary utility functions; these are defined as $-\rho(\cdot)$ for a risk measure ρ. In particular, this is more convenient to explain the connection with von Neumann-Morgenstern expected utility.

Definition 3.3.1. A *monetary concave utility functional* on $\mathbf{L}^\infty(P)$ is a mapping $\Phi : \mathbf{L}^\infty(P) \to I\!R$ satisfying

A) *monotonicity*: $X_1 \leq X_2$ implies $\Phi(X_1) \leq \Phi(X_2)$;

B) *translation invariance*: $\Phi(X + a) = \Phi(X) + a$ for $a \in I\!R$;

C) *concavity*: $\Phi\big(\alpha X_1 + (1-\alpha)X_2\big) \geq \alpha\Phi(X_1) + (1-\alpha)\Phi(X_2)$ for $\alpha \in [0, 1]$.

The *acceptance set* of Φ is defined as $\mathcal{A} := \{X \in \mathbf{L}^\infty(P) | \Phi(0) \geq 0\}$. Φ is a *monetary coherent utility functional* if it satisfies in addition

D) *positive homogeneity*: $\Phi(\lambda X) = \lambda\Phi(X)$ for $\lambda \geq 0$.

Remark 3.3.2. Translation invariance distinguishes monetary concave utility functionals from von Neumann-Morgenstern expected utilities, most of which do not have this property. Presuming as usual that all payoffs are already discounted, translation invariance implies that utility is measured in monetary units; see the book of Föllmer/Schied [FS04] for an overview of the theory of risk measures. $\diamond$

For the convenience of the reader and to keep this chapter self-contained, we recall from Section 2.3 some important results on monetary concave utility functionals.

Proposition 3.3.3. (Proposition 4.6 of [FS04]) *Let Φ be a monetary concave utility functional with acceptance set $\mathcal{A}$. Define a functional on $\mathbf{L}^\infty(P)$ by*

$$\Phi^{\mathcal{A}}(X) := \sup\{m \in I\!R \mid X - m\mathbf{1}_\Omega \in \mathcal{A}\} = \sup\big((X - \mathcal{A}) \cap I\!R\big).$$

Then $\Phi = \Phi^{\mathcal{A}}$, i.e., Φ can be recovered from its acceptance set.

The following theorem gives a dual representation for monetary utility functionals which is of great importance for mathematical calculations. In addition, this result shows that for any non-empty set of no-good-deal measures $\mathcal{N}$, the lower good deal value bound

$$X \mapsto \inf_{Q \in \mathcal{N}} E_Q[X]$$

as a function on $\mathbf{L}^\infty(P)$ is a monetary coherent utility functional; see Remark 3.3.5 below.

Theorem 3.3.4. *For a functional $\Phi : \mathbf{L}^\infty(P) \to I\!R$ the following are equivalent:*

a) Φ is a monetary concave utility functional which is continuous from above, i.e., for any sequence $(X_n)_{n \in I\!N} \subseteq \mathbf{L}^\infty(P)$ decreasing to some $X \in \mathbf{L}^\infty(P)$ we have $\lim_{n\to\infty} \Phi(X_n) = \Phi(X)$.

b) Φ *can be represented as*

$$\Phi(X) = \inf_{Q \in \mathcal{P}^a} \{E_Q[X] - \alpha^0(Q)\} \tag{3.3.1}$$

for a mapping $\alpha^0 : \mathcal{P}^a \to [-\infty, +\infty)$.

If Φ *satisfies one of these equivalent conditions and is in addition positively homogeneous, then it can be represented as*

$$\Phi(X) = \inf_{Q \in \hat{\mathcal{P}}} E_Q[X],$$

where $\hat{\mathcal{P}}$ *is a subset of* $\mathcal{P}^a$.

Proof. See Theorem 4.3 and Corollary 4.34 in [FS04]. □

Remark 3.3.5. a) To see that the lower good deal bound has a representation as in (3.3.1), take $\alpha^0(Q) := 0$ if $Q \in \mathcal{N}$ and $\alpha^0(Q) := -\infty$ otherwise.

b) If Φ has a representation as in (3.3.1) with some α^0, then it can also be represented with

$$\alpha(Q) := \inf_{Y \in \mathbf{L}^\infty(P)} \{E_Q[Y] - \Phi(Y)\},$$

i.e., α^0 can be chosen as the concave conjugate α of Φ. Moreover, if there exists $Q \in \mathcal{P}^e$ such that $\alpha(Q) > -\infty$, then we can replace $\mathcal{P}^a$ by $\mathcal{P}^e$ in (3.3.1) if we take $\alpha^0 = \alpha$.

c) The relation between upper and lower good deal bound is given by

$$\sup_{Q \in \mathcal{N}} E_Q[X] = -\inf_{Q \in \mathcal{N}} E_Q[-X].$$

◇

Now consider as in Section 3.2 a financial market with the process S describing the discounted prices of the basic assets. In the seminal work on good deal bounds by Cochrane/Saà-Requejo [CSR00] and also in the more recent work by Björk/Slinko [BS06], good deals are defined as excess returns with high Sharpe ratio. Using an inequality from Hansen/Jagannathan [HJ91], these authors define the set $\mathcal{N}^q \subseteq \mathcal{M}^e(S)$ of no-good-deal measures Q by stipulating that the variance of the density dQ/dP remains below some threshold. We have shown in Section 3.2 that the estimate which is induced by the Hansen-Jagannathan inequality for the Sharpe ratio of payoffs priced with respect to $Q \in \mathcal{M}^e(S)$ corresponds to the maximal attainable Sharpe ratio in an extended market with payoffs from $\mathcal{C}(x, Q)$. Ross [Ros05] also obtains an upper bound on the variance of the density; however, he does not argue with the Sharpe ratio, but works with the Capital Asset Pricing Model. Without stating this link clearly and generally, Černý [Cer03] proposes to define no-good-deal measures from several utility functions via the associated indifference price in the extended market. The same approach is used in the early work by Bernardo/Ledoit [BL00], but for good deals defined via the *gain-loss ratio*. This is the ratio of the expectations, under some benchmark measure, of the positive and the negative parts of the excess return. Pinar/Salih [PS05] use a similar gain-loss trade-off to define good deals. The formulation suggested by Longarela [Lon01] and Bondarenko/Longarela [BL04] is also very similar to those above. They define no-good-deal measures by the condition that the distance between these measures and a benchmark valuation measure is not too big and propose several definitions for this distance. Our approach is inspired by [Cer03], and one might say that our subjective initial ingredient is a (von Neumann-Morgenstern) utility function.

A somewhat different line is taken by Černý/Hodges [CH02], Jaschke/Küchler [JK01] and also by Cherny [Che05a,06]. They first define a set of *desirable claims* as the (abstract) acceptance set of a monetary coherent utility functional and a good deal as a desirable claim with zero or negative price. Their aim is to find linear pricing functionals which price the basic assets correctly and do not yield good deals. A similar approach is pursued by Cherny [Che05b] who allows the prices of the basic assets to have bid-ask spreads. Instead of good deal bounds, he specifies no-good-deal bid and ask prices in such a way that it is not possible to construct a good deal by trading. Staum [Sta04] permits in addition for acceptance sets of monetary *concave* utility functionals. Carr et al. [CGM01] define their set of desirable claims via valuation and stress test measures. By requiring positive prices for these claims, they obtain that the pricing functional must be a convex combination of the valuation test measures. The common feature of this line of work is that desirability and hence good deals are defined via a fairly abstract set only satisfying certain properties.

Although the two methods explained above look rather different at first sight, they are actually closely related. Utility maximization also comes up in the second approach, but with respect to monetary utility functionals instead of von Neumann-Morgenstern expected utility. The key observation behind this is that the abstract set used to define desirability induces by its properties in a natural way a monetary utility functional; see [JK01], [Sta04] and also the discussion at the end of Section 2.6. More precisely, let Φ be a monetary coherent utility functional with acceptance set $\mathcal{A}$. Since this functional is defined on $\mathbf{L}^\infty(P)$, we restrict to $\mathbf{L}^\infty(P)$ the payoffs which can be dominated by trading in the basic assets from initial capital zero and write $\mathcal{C}^b(x,S) := \mathcal{C}(x,S) \cap \mathbf{L}^\infty(P)$. Jaschke/Küchler [JK01] define a good deal as an element $X \in \mathcal{C}^b(0,S)$ such that there exists $\varepsilon > 0$ with $X - \varepsilon \mathbf{1}_\Omega \in \mathcal{A}$. Hence a good deal is a payoff which can be superreplicated by trading from zero initial capital and from which one can even subtract ε monetary units and still have a payoff which is desirable, i.e., an element of the (acceptance) set $\mathcal{A}$. The lower bound $\pi^{\mathcal{A}}$ for prices of

$\hat{X} \in \mathbf{L}^\infty(P)$ is then obtained as the biggest monetary amount which can be subtracted from $\hat{X}$ so that it is not possible to turn the resulting payoff into a good deal by trading in the basic assets. Formally, it is defined by

$$\pi^{\mathcal{A}}(\hat{X}) := \sup\left\{m \in I\!R \,\middle|\, \hat{X} - m\mathbf{1}_\Omega + X \in \mathcal{A} \quad \text{for some } X \in \mathcal{C}^b(0, S)\right\}.$$

In fact, if the agent could buy the random payoff $\hat{X}$ for a price $\pi(\hat{X}) < \pi^{\mathcal{A}}(\hat{X})$, then there exist $X \in \mathcal{C}^b(0, S)$ and $\varepsilon > 0$ with $\pi(\hat{X}) + \varepsilon \leq \pi^{\mathcal{A}}(\hat{X}) - \varepsilon$ and such that

$$\hat{X} - \big(\pi(\hat{X}) + \varepsilon\big)\mathbf{1}_\Omega + X$$

is contained in the acceptance set $\mathcal{A}$. Hence the agent could buy $\hat{X}$ for $\pi(\hat{X})$, use the superhedging strategy corresponding to X and obtain the payoff $\hat{X} - \pi(\hat{X}) + X$ which is a good deal. By Proposition 3.3.3, the maximal utility (with respect to Φ) attainable from the random endowment $\hat{X} \in \mathbf{L}^\infty(P)$ by trading in the basic assets is given by

$$\begin{aligned}
&\sup_{X \in \mathcal{C}^b(0,S)} \Phi(\hat{X} + X) \\
&\quad = \sup_{X \in \mathcal{C}^b(0,S)} \sup\left\{m \in I\!R \middle| \hat{X} + X - m\mathbf{1}_\Omega \in \mathcal{A}\right\} \\
&\quad = \sup\left\{m \in I\!R \,\middle|\, \hat{X} - m\mathbf{1}_\Omega + X \in \mathcal{A} \quad \text{for some } X \in \mathcal{C}^b(0, S)\right\} \\
&\quad = \pi^{\mathcal{A}}(\hat{X}). \qquad (3.3.2)
\end{aligned}$$

Note that $\mathcal{C}^b(x, S) = x + \mathcal{C}^b(0, S)$ and that Φ is translation invariant. Thus an initial capital x would show up in the above equations as an additive term. Hence the lower good deal price bound of [JK01] is just the term which is independent of the initial capital x in the maximal monetary utility attainable from trading in S with random endowment $\hat{X}$. The generalization from monetary coherent utility functions to monetary concave utility functions then corresponds to the approach suggested by Staum [Sta04] (if the prices of the basic assets are linear); see also the discussion at the end of Section 2.6.

Instead of linking good deals from abstract (acceptance) sets to good deal bounds from utility maximization as above, one can relate the former directly to our approach here via martingale measures. The key insight behind this is that a given set of measures naturally induces a monetary utility functional, which in principle brings us back to the situation just discussed. In more detail, let $\mathcal{N} \subseteq \mathcal{M}^e(S)$ be a set of no-good-deal measures. Assume that $\mathcal{N}$, identified with the corresponding set of densities with respect to P, is weakly relatively compact; for $\mathcal{N}^q$ and $\mathcal{N}^e$ this holds by the la Vallée-Poussin theorem in [DM75], Theorem II.22. Moreover, let $\mathcal{Q}' \subseteq \mathcal{P}^a$ be any weakly relatively compact set such that $\mathcal{N} \subseteq \mathcal{Q}'$ and $\mathcal{N} = \mathcal{Q}' \cap \mathcal{M}^e(S)$. By Theorem 3.3.4 we can define a monetary coherent utility functional Φ by

$$\Phi(X) := \inf_{Q \in \mathcal{Q}'} E_Q[X]$$

with acceptance set $\mathcal{A} := \{X \in \mathbf{L}^\infty(P) \mid \Phi(X) \geq 0\}$. Lemma 3.3.6 below yields that

$$\inf_{Q \in \mathcal{N}} E_Q[\hat{X}] = \sup_{X \in \mathcal{C}^b(0,S)} \Phi(\hat{X} + X).$$

If the acceptance set $\mathcal{A}$ specifies a set of desirable claims, then (3.3.2) implies that

$$\pi^{\mathcal{A}}(\hat{X}) = \inf_{Q \in \mathcal{N}} E_Q[\hat{X}].$$

Thus the lower good deal price bound of Jaschke/Küchler [JK01] defined with respect to $\mathcal{A}$ is the same then that obtained from the set of no-good-deal measures $\mathcal{N}$ with our approach.

Lemma 3.3.6. *Let $\mathcal{N} \subseteq \mathcal{M}^e(S)$ be nonempty and $\mathcal{Q}' \subseteq \mathcal{P}^a$ such that $\mathcal{N} \subseteq \mathcal{Q}'$ and $\mathcal{N} = \mathcal{Q}' \cap \mathcal{M}^e(S)$. Define $\Phi(\,.\,) := \inf_{Q \in \mathcal{Q}'} E_Q[\,.\,]$. If $\mathcal{Q}'$ is weakly relatively compact, then*

$$\inf_{Q \in \mathcal{N}} E_Q[\hat{X}] = \sup_{X \in \mathcal{C}^b(0,S)} \Phi(\hat{X} + X) \quad \textit{for all } \hat{X} \in \mathbf{L}^\infty(P).$$

Proof. Denote by $\overline{\mathcal{N}}$ and $\overline{\mathcal{Q}'}$ the $\mathbf{L}^1(P)$-closed convex hulls of $\mathcal{N}$ and $\mathcal{Q}'$, identified with the corresponding set of densities with respect to P. Note that

$$\inf_{Q \in \mathcal{N}} E_Q[X] = \inf_{Q \in \overline{\mathcal{N}}} E_Q[X] \quad \text{for all } X \in \mathbf{L}^\infty(P) \tag{3.3.3}$$

and that an analogous statement holds for $\mathcal{Q}'$. Since $\overline{\mathcal{Q}'}$ is a Hausdorff compact space and $\mathcal{C}^b(0, S)$ is convex, the minimax theorem in [Sim98] thus implies that

$$\begin{aligned} \sup_{X \in \mathcal{C}^b(0,S)} \Phi(\hat{X} + X) &= \sup_{X \in \mathcal{C}^b(0,S)} \inf_{Q \in \overline{\mathcal{Q}'}} E_Q[\hat{X} + X] \\ &= \inf_{Q \in \overline{\mathcal{Q}'}} \sup_{X \in \mathcal{C}^b(0,S)} E_Q[\hat{X} + X]. \end{aligned} \tag{3.3.4}$$

It is well known that $Q \in \mathcal{P}^e$ is contained in $\mathcal{M}^e(S)$ if and only if $E_Q[X] \leq 0$ for all $X \in \mathcal{C}^b(0, S)$; see, e.g., Lemma 2.6.15. Since $\mathcal{C}^b(0, S)$ is a cone and contains 0, we have for $Q \in \mathcal{P}^e$

$$\sup_{X \in \mathcal{C}^b(0,S)} E_Q[\hat{X} + X] = \begin{cases} E_Q[\hat{X}] & \text{if } Q \in \mathcal{M}^e(S), \\ +\infty & \text{otherwise.} \end{cases}$$

Thus (3.3.4), $\mathcal{N} = \mathcal{Q}' \cap \mathcal{M}^e(S)$, $\overline{\mathcal{Q}'} \cap \mathcal{M}^e(S) \subseteq \overline{\mathcal{N}}$ and (3.3.3) imply

$$\sup_{X \in \mathcal{C}^b(0,S)} \Phi(\hat{X} + X) = \inf_{Q \in \overline{\mathcal{Q}'} \cap \mathcal{M}^e(S)} E_Q[\hat{X}] = \inf_{Q \in \mathcal{N}} E_Q[\hat{X}].$$

□

Remark 3.3.7. a) The exact choice of $\mathcal{Q}'$ does not matter. In particular $\mathcal{Q}' = \mathcal{N}$ is always possible, provided that $\mathcal{N}$ is weakly relatively compact.

b) Weak relative compactness of a set $\mathcal{Q} \subseteq \mathcal{P}^a$ is equivalent to a continuity property of the corresponding monetary coherent utility functional $\inf_{Q\in\mathcal{Q}} E_Q[\,.\,]$; see Corollary 4.35 in [FS04].

◇

3.4 Auxiliary results on Lévy processes

As mentioned in the introduction, we are mainly interested in the good deal value bounds as processes. To obtain results in a dynamic context, we need a nice representation of the set of all equivalent local martingale measures. Therefore we choose to work in a Lévy framework. In this section we introduce some terminology, provide some auxiliary results about Lévy processes, descriptions of probability measures via Girsanov parameters and relative entropy. The proofs or references for proofs can all be found in Esche/Schweizer [ES05]. Their main reference is the book by Jacod/Shiryaev [JS87].

We first fix some notation. As before we work on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions. $\mathbf{P}$ denotes the predictable σ-field on $\Omega\times[0,T]$ and $\mathcal{B}^d$ the Borel σ-field on $I\!R^d$. For a d-dimensional semimartingale X we denote by μ^X the random measure associated with its jumps and by ν^P the predictable P-compensator of μ^X; only in this subsection we denote by X a process and not a payoff. Moreover, we work throughout with a fixed but arbitrary truncation function $h : I\!R^d \to I\!R^d$. By (B, C, ν) we denote the P-characteristics of the semimartingale X with respect to h. We can and do always choose a version of the form

$$B = \int b\,dA, \quad C = \int c\,dA \quad \text{and} \quad \nu(\omega; dx, dt) = K_{\omega,t}(dx)\,dA_t(\omega), \tag{3.4.1}$$

where A is a real-valued, predictable, increasing and locally integrable process, b is an $I\!R^d$-valued predictable process, c a predictable process with values in the set of symmetric non-negative definite $d\times d$-matrices, and $K_{\omega,t}(dx)$ a transition kernel from $(\Omega\times[0,T],\mathbf{P})$ into $(I\!R^d, \mathcal{B}^d)$ with $K_{\omega,t}(\{0\}) = 0$ and $\int_{I\!R^d}(1\wedge\|x\|^2)\,K_{\omega,t}(dx) \leq 1$. Let $Q \in \mathcal{P}^a$ and $L = (L_t)_{0\leq t\leq T}$ be an adapted stochastic process null at 0 with RCLL paths. We call L a Q-Lévy process if for all $s \leq t$, the random variables $L_t - L_s$ are independent of $\mathcal{F}_s$ under Q and have a distribution depending only on $t-s$. If $Q = P$ we sometimes omit the mention of P. Every Lévy process is a semimartingale, and a P-semimartingale L null at 0 is a P-Lévy process if and only if its P-characteristics are of the form

$$B_t = bt, \quad C_t = ct \quad \text{and} \quad \nu^P(dx, dt) = K(dx)\,dt, \tag{3.4.2}$$

where $b \in I\!R^d$, c is a symmetric non-negative definite $d \times d$-matrix and K is a σ-finite measure on $(I\!R^d, \mathcal{B}^d)$ with $K(\{0\}) = 0$ and $\int_{I\!R^d}(1 \wedge \|x\|^2)\, K(dx) < \infty$.

Next we recall Girsanov's theorem to introduce the Girsanov parameters (β, Y) of some $Q \in \mathcal{P}^a$.

Theorem 3.4.1. ([JS87], Theorem III 3.24) *Let X be a semimartingale with P-characteristics (B^P, C^P, ν^P) and denote by c and A the corresponding processes from (3.4.1). For any $Q \in \mathcal{P}^a$, there exist a $\mathbf{P} \otimes \mathcal{B}^d$-measurable function $Y \geq 0$ on $\Omega \times [0, T] \times I\!R^d$ and a predictable $I\!R^d$-valued process β satisfying Q-a.s.*

$$\int_0^t \int_{I\!R^d} \|(Y(s, x) - 1)h(x)\|\, \nu^P(dx, ds) + \int_0^t \|c_s \beta_s\|\, dA_s + \int_0^t \beta_s^* c_s \beta_s\, dA_s < \infty$$

for all $t \in [0, T]$ and such that the Q-characteristics (B^Q, C^Q, ν^Q) of X are given by

$$\begin{aligned} B_t^Q &= B_t^P + \int_0^t c_s \beta_s\, dA_s + \int_0^t \int_{I\!R^d} \big((Y(s, x) - 1)h(x)\big)\, \nu^P(dx, ds), \\ C_t^Q &= C_t^P, \\ \nu^Q(dx, dt) &= Y(t, x)\, \nu^P(dx, dt). \end{aligned}$$

We call β and Y the Girsanov parameters *of Q (with respect to P, relative to X).*

Remark 3.4.2. Note that the Girsanov parameters are not unique. In fact, $Y(\omega, t, x)$ is unique only ν^P-a.e., and for fixed c and A we have A-a.e. uniqueness only for $c\beta$. In what follows we fix a Lévy process L and express the Girsanov parameters of any $Q \in \mathcal{P}^a$ relative to L. We then identify all versions of Girsanov parameters (β, Y) which describe the same Q. In particular, if we say that the Girsanov parameters (β, Y) of Q are time-independent, we mean that there exists one version with this property. $\diamond$

In order to obtain nice parametrizations for the set of probability measures, we make the following assumption for the rest of this chapter.

The filtration $(\mathcal{F}_t)_{0 \leq t \leq T} = (\mathcal{F}_t^L)_{0 \leq t \leq T}$ is the P-augmentation of the filtration generated by a d-dimensional Lévy process L with semimartingale characteristics described by a triplet (b, c, K) as in (3.4.2).

The following result expresses the density process of any $Q \in \mathcal{P}^e$ in terms of its Girsanov parameters (β, Y) and the Lévy process L. As usual $\mathcal{E}(\,.\,)$ denotes the stochastic exponential.

Proposition 3.4.3. (Proposition 3 in [ES05]) *If $Q \in \mathcal{P}^e$ has Girsanov parameters (β, Y), the density process of Q with respect to P is given by $Z^Q = \mathcal{E}(N^Q)$ with*

$$N_t^Q := \int_0^t \beta_s^*\, dL_s^c + \int_0^t \int_{I\!R^d} (Y(s, x) - 1)\left(\mu^L(dx, ds) - K(dx)\, ds\right), \qquad (3.4.3)$$

for $t \in [0, T]$, where L^c denotes the continuous local martingale part of L.

We can also go the other way round, i.e., start with some processes $\overline{\beta}$ and $\overline{Y}(\,.\,)$ and identify them with the Girsanov parameters of some probability measure. Let us first introduce the convex function

$$g(y) := y \log y - y + 1 \quad \text{for } y \in [0, \infty), \tag{3.4.4}$$

where we set $0 \log 0 := 0$. This function is denoted by f in [ES05]. However, in order to preserve the variable f for the f-divergence, we use the notation g here. The following result is a combination of Propositions 5 and 7 from [ES05].

Proposition 3.4.4. *If $\overline{\beta}$ is a predictable process and $\overline{Y} > 0$ a predictable function such that*

$$E\left[\exp\left(\int_0^T \left(\frac{1}{2}\overline{\beta}_s^* c \overline{\beta}_s + \int_{\mathbb{R}^d} g\left(\overline{Y}(s,x)\right) K(dx)\right) ds\right)\right] < \infty,$$

then $\overline{Y} - 1$ is integrable with respect to $\mu^L(dx, dt) - K(dx)\,dt$, and $\overline{Z} := \mathcal{E}(\overline{N})$ with

$$\overline{N}_t := \int_0^t \overline{\beta}_s^* \, dL_s^c + \int_0^t \int_{\mathbb{R}^d} \left(\overline{Y}(s,x) - 1\right) \left(\mu^L(dx, ds) - K(dx)\,ds\right), \quad t \in [0, T],$$

is a strictly positive P-martingale. In particular, $\overline{Z}$ is the density process of some $Q \in \mathcal{P}^e$ with Girsanov parameters $(\overline{\beta}, \overline{Y})$.

Let $\mathbf{M}$ be a fixed $d \times d$-matrix and denote by $\mathcal{M}^a(\mathbf{M}L)$ the set of all absolutely continuous local martingale measures for $\mathbf{M}L$ and by $\mathcal{M}^e(\mathbf{M}L) \subseteq \mathcal{M}^a(\mathbf{M}L)$ those which are equivalent to P. The following result describes the elements of $\mathcal{M}^a(\mathbf{M}L)$.

Proposition 3.4.5. (Proposition 10 of [ES05]) *Let $Q \in \mathcal{P}^a$ with Girsanov parameters (β, Y) and such that $E_Q\big[\int_0^T \int_{\mathbb{R}^d} g\big(Y(s,x)\big) K(dx)\,ds\big] < \infty$. Then $\mathbf{M}L$ is a local Q-martingale if and only if Q-a.s. both $\int_0^T \int_{\mathbb{R}^d} \|\mathbf{M}(xY(s,x) - h(x))\| K(dx)\,ds < \infty$ and for all $t \in [0, T]$*

$$\mathbf{M}\left(b + c\beta_t + \int_{\mathbb{R}^d} \left(xY(t,x) - h(x)\right) K(dx)\right) = 0. \tag{3.4.5}$$

Condition (3.4.5) is called the martingale condition *for $\mathbf{M}L$.*

As we have illustrated in Section 3.2, for $Q \in \mathcal{P}^e$ with density process Z, the relative entropy $f^e(Q|P) := E[Z_T \log Z_T]$ can be used as a criterion for the definition of a set of no-good-deal measures. Analogously we define the *$\mathcal{F}_t$-relative entropy* of $Q \in \mathcal{P}^e$ with respect to P by

$$f_t^e(Q|P) := E_Q\left[\frac{Z_T}{Z_t} \log \frac{Z_T}{Z_t} \,\middle|\, \mathcal{F}_t\right]. \tag{3.4.6}$$

Esche/Schweizer [ES05] give in their Lemma 12 a formula for $f_0^e(Q|P)$ in terms of the Girsanov parameters of Q. This can immediately be generalized to a formula for $f_t^e(Q|P)$.

Proposition 3.4.6. *If $Q \in \mathcal{P}^e$ with Girsanov parameters (β, Y) and $f_0^e(Q|P) < \infty$, then for all $t \in [0,T]$*

$$f_t^e(Q|P) = E_Q\left[\int_t^T \frac{1}{2}\beta_s^* c\beta_s\, ds + \int_t^T \int_{\mathbb{R}^d} g\left(Y(s,x)\right)\, K(dx)\, ds \,\middle|\, \mathcal{F}_t\right]. \quad (3.4.7)$$

If it exists, we denote by $Q^e(\mathbf{M}L)$ that probability measure $Q \in \mathcal{M}^e(\mathbf{M}L)$ which minimizes the relative entropy $f^e(Q|P) = f_0^e(Q|P)$ over $\mathcal{M}^e(\mathbf{M}L)$. $Q^e(\mathbf{M}L)$ is called the *minimal entropy martingale measure* for $\mathbf{M}L$. The following is one of the main results from Esche/Schweizer [ES05]. It shows that $Q^e(\mathbf{M}L)$ preserves the Lévy property of L; conditions for the existence of $Q^e(\mathbf{M}L)$ can also be found in [ES05].

Theorem 3.4.7. (Theorem A of [ES05]) *If $Q^e(\mathbf{M}L)$ exists, and if there exists some $Q \in \mathcal{M}^e(\mathbf{M}L)$ such that both $f^e(Q|P) < \infty$ and L is a Q-Lévy process, then L is also a $Q^e(\mathbf{M}L)$-Lévy process. In particular, the Girsanov parameters of $Q^e(\mathbf{M}L)$ are time-independent and deterministic.*

3.5 Dynamic good deal bounds

Our main goal is to study the good deal value bounds as processes. Since their computability and dynamic properties depend on the set of no-good-deal measures, the main difficulty is to find an appropriate definition for this set in a dynamic context. This is the subject of this section. The motivation for our way to proceed comes from a restriction on the maximal attainable exponential utility, i.e., from the utility function $\mathbf{U}^e(x) = -\frac{1}{\beta}e^{-\beta x}$ with $\beta > 0$. Results for more general utility functions can be deduced from Chapter 4 below.

For dynamic considerations it is important to have a nice parametrization for the set of probability measures in a model which is still as general as possible. Therefore we use the same approach as Esche and Schweizer in [ES05]. Let $\mathbf{M}$ be a fixed $d \times d$ matrix and $L = (L_t)_{0\le t\le T}$ a d-dimensional Lévy process with characteristics (b, c, K) as in (3.4.2). The filtration $(\mathcal{F}_t)_{0\le t\le T} = (\mathcal{F}_t^L)_{0\le t\le T}$ is the P-augmentation of that generated by L. We consider the set $\mathcal{M}^e(\mathbf{M}L)$ of all equivalent local martingale measures for $\mathbf{M}L$ because this allows for several possibilities to model the discounted price processes of the basic assets S. For instance, the Lévy process L is a local Q-martingale for $Q \in \mathcal{P}^e$ if and only if $S := \mathcal{E}(L)$ is a local Q-martingale. One can also model S as a process with stochastic volatility; see [ES05]. To exclude arbitrage opportunities we assume that

$$\mathcal{M}^e(\mathbf{M}L) \neq \emptyset.$$

In analogy to the static case, we now consider for each time $t \in [0, T]$ and each $Q \in \mathcal{M}^e(\mathbf{M}L)$ the maximal attainable utility from trading in $(t, T]$ with initial capital $x_t \in \mathbf{L}^\infty(P, \mathcal{F}_t)$ in the Q-extended market. More precisely, we consider the maximal attainable utility over the set of payoffs

$$\mathcal{C}_t(x_t, Q) := \left\{X \in \mathbf{L}^1(Q) \,\middle|\, E_Q[X|\mathcal{F}_t] \leq x_t \text{ and } X^- \in \mathbf{L}^\infty(P)\right\}.$$

Definition 3.5.1. For $Q \in \mathcal{M}^e(\mathbf{M}L)$ we define the *maximal* $(\mathbf{U}^e, Q)$*-utility at time* t from $x_t \in \mathbf{L}^\infty(P, \mathcal{F}_t)$ by

$$u_t^{e,Q}(x_t) := \operatorname*{ess\,sup}_{X \in \mathcal{C}_t(x_t, Q)} E[\mathbf{U}^e(X)|\mathcal{F}_t].$$

We recall from Remark 3.2.11 the set

$$\mathcal{Q}^e := \left\{Q \in \mathcal{M}^e(\mathbf{M}L) \,\middle|\, f^e(Q|P) < \infty\right\}.$$

By Proposition 3.4.6 we have for $Q \in \mathcal{Q}^e$ that also $f_t^e(Q|P)$ is finite for all $t \in (0, T]$.

Proposition 3.5.2. *For* $x \in \operatorname{dom}(\mathbf{U}^e)$ *and* $Q \in \mathcal{Q}^e$ *with density process* $Z = (Z_t)_{0 \leq t \leq T}$ *we have*

$$u_t^{e,Q}(x_t) = \mathbf{U}^e(x_t) e^{-E\left[\frac{Z_T}{Z_t} \log \frac{Z_T}{Z_t} \middle| \mathcal{F}_t\right]} = \mathbf{U}^e(x_t) e^{-f_t^e(Q|P)} \tag{3.5.1}$$

where $f_t^e(Q|P)$ *is the* $\mathcal{F}_t$*-relative entropy introduced in (3.4.6).*

Proof. The conjugate function of $\mathbf{U}^e$ from (3.2.7) is $V^e(y) = -\frac{y}{\beta} + \frac{y}{\beta} \log y$ for $y > 0$, and the duality relation from (3.2.8) implies that

$$\mathbf{U}^e(x) = \inf_{y>0} \left\{V^e(y) + xy\right\} \leq V^e(y') + xy'$$

for any $y' > 0$. If we set $y' := y'(\omega) := \frac{Z_T}{Z_t} \exp\left(-\beta x_t - f_t^e(Q|P)\right)$ and take $\mathcal{F}_t$-conditional expectations, we obtain for $X \in \mathcal{C}_t(x_t, Q)$ that

$$E[\mathbf{U}^e(X)|\mathcal{F}_t] \leq \mathbf{U}^e(x_t) e^{-f_t^e(Q|P)}.$$

This proves "$\leq$" in (3.5.1). To prove equality, choose

$$X := -\frac{1}{\beta}\left(\log\left(\frac{Z_T}{Z_t}\right) - \beta x_t - f_t^e(Q|P)\right).$$

If $X^- \in \mathbf{L}^\infty(P)$, then we have $X \in \mathcal{C}_t(x_t, Q)$, and the result follows. Otherwise let $A_n := \left\{\frac{Z_T}{Z_t} \le n\right\}$ and

$$X_n := -\frac{1}{\beta}\left(\mathbf{1}_{A_n}\log\frac{Z_T}{Z_t} - \beta x_t - E\left[\mathbf{1}_{A_n}\frac{Z_T}{Z_t}\log\frac{Z_T}{Z_t}\,\middle|\,\mathcal{F}_t\right]\right)$$

so that $X_n^- \in \mathbf{L}^\infty(P)$ and $E_Q[X_n|\mathcal{F}_t] = x_t$. Finally, the conditional monotone convergence theorem for a uniformly bounded from below sequence of random variables implies that $\lim_{n\to\infty} E_Q\left[\mathbf{1}_{A_n}\frac{Z_T}{Z_t}\log\frac{Z_T}{Z_t}\,\middle|\,\mathcal{F}_t\right] = f_t^e(Q|P)$ and

$$\lim_{n\to\infty} E[\mathbf{U}^e(X_n)|\mathcal{F}_t] = E[\mathbf{U}^e(X)|\mathcal{F}_t].$$

This completes the proof. □

Proposition 3.5.2 suggests to use the $\mathcal{F}_t$-relative entropy as a measurement for the attractiveness of any $Q \in \mathcal{M}^e(\mathbf{M}L)$ at time t. As in the static case $t = 0$, this is an *$\mathcal{F}_t$-conditional divergence* and thus a measure for the distance between Q and P. Therefore we define a set of no-good-deal measures at time t by imposing an upper bound on $f_t^e(Q|P)$. In order to have this set non-empty, we introduce a *benchmark measure* $\hat{Q} \in \mathcal{M}^e(\mathbf{M}L)$ to obtain a lower bound for $f_t^e(Q|P)$. We take the same benchmark measure for all $t \in [0, T]$ because the t-benchmark will usually be the measure which minimizes the $\mathcal{F}_t$-conditional divergence over $\mathcal{M}^e(\mathbf{M}L)$ and the following result from Kabanov/Stricker [KS02] shows that this is achieved by the same $\hat{Q}$ for all t.

Proposition 3.5.3. (Proposition 4.1 of [KS02]) *Let there exist $Q^e(\mathbf{M}L) \in \mathcal{M}^e(\mathbf{M}L)$ with density process Z^e such that for any $Q \in \mathcal{M}^e(\mathbf{M}L)$ with density process Z,*

$$E[Z_T^e \log Z_T^e] \le E[Z_T \log Z_T].$$

Then also for any stopping time $0 \le \tau \le T$

$$E\left[\frac{Z_T^e}{Z_\tau^e}\log\frac{Z_T^e}{Z_\tau^e}\,\middle|\,\mathcal{F}_\tau\right] \le E\left[\frac{Z_T}{Z_\tau}\log\frac{Z_T}{Z_\tau}\,\middle|\,\mathcal{F}_\tau\right].$$

Remark 3.5.4. An analogous result holds for $\mathbf{U}^q$, $\mathbf{U}^p$ and $\mathbf{U}^\ell$; see [KS02] and Lemma 5.1.4 in the Appendix. ◇

For dynamic considerations, the choice of the bound for the $\mathcal{F}_t$-conditional divergences over time is very important. In principle, we should like to specify at each time t the set of no-good-deal measures $\mathcal{N}_t$ as follows. Fix two adapted processes

$\eta' = (\eta'_t)_{0\le t\le T}$ and $\theta' = (\theta'_t)_{0\le t\le T}$ with $\eta'_t \ge 1$ and $\theta'_t \ge 0$, and choose a benchmark measure $\hat{Q}$. Then define $\mathcal{N}_t$ as the set of all $Q \in \mathcal{M}^e(\mathbf{M}L)$ such that

$$f_t^e(Q|P) \le \eta'_t f_t^e(\hat{Q}|P) + \theta'_t \tag{3.5.2}$$

or equivalently

$$-e^{-f_t^e(Q|P)} \le -\theta''_t e^{-\eta'_t f_t^e(\hat{Q}|P)} \quad \text{with} \quad \theta''_t := e^{-\theta'_t}.$$

Let $g(y) := y \log y - y + 1$. If $Q \in \mathcal{P}^e$ has Girsanov parameters (β, Y) and satisfies $f_0^e(Q|P) < \infty$, then Proposition 3.4.6 implies for all $t < T$ that

$$\begin{aligned} f_t^e(Q|P) &= E_Q\left[\int_t^T \frac{1}{2}\beta_s^* c\beta_s\,ds + \int_t^T \int_{\mathbb{R}^d} g\Big(Y(s,x)\Big)\,K(dx)\,ds \,\Big|\, \mathcal{F}_t\right] \\ &=: E_Q\left[\int_t^T k^e\Big(\beta_s, Y(s,\,.)\Big)\,ds \,\Big|\, \mathcal{F}_t\right]. \end{aligned} \tag{3.5.3}$$

Despite this fairly explicit expression, calculating the good deal value bounds

$$\left[\operatorname*{ess\,inf}_{Q\in\mathcal{N}_t} E_Q[\hat{X}|\mathcal{F}_t],\ \operatorname*{ess\,sup}_{Q\in\mathcal{N}_t} E_Q[\hat{X}|\mathcal{F}_t]\right]$$

as processes for some random payoff $\hat{X} \in \mathbf{L}^\infty(P)$ is mathematically intractable with the above general definition of $\mathcal{N}_t$. Therefore we want to replace the *global* constraint (3.5.2) on Q by a *local* restriction on the integrand $k^e\big(\beta_s, Y(s,\,.)\big)$ in (3.5.3). It will turn out to be very helpful that if we choose the minimal entropy measure $Q^e(\mathbf{M}L)$ as benchmark measure $\hat{Q}$, this integrand is very simple. In fact, Theorem 3.4.7 implies that under appropriate assumptions, the Girsanov parameters (β^e, Y^e) of $Q^e(\mathbf{M}L)$ are time-independent and deterministic. So if we write $\hat{k}^e := k^e\big(\beta^e, Y^e(\,.\,)\big)$ for this constant integrand, then

$$f_t^e(\hat{Q}|P) = f_t^e(Q^e|P) = (T-t)\hat{k}^e.$$

Hence imposing a bound on $f^e(\hat{Q}|P)$ is clearly equivalent to imposing a bound on $\hat{k}^e$. Generalizing the latter to other Q still remains tractable, in contrast to (3.5.2).

The above discussion, motivated by exponential utility, leads us to the following general problem. In order to emphasize the dependence of $Q \in \mathcal{P}^e$ on its Girsanov parameters (β, Y) (with respect to L), we write $Q^{(\beta,Y)}$.

Problem: *Let k be a deterministic function on $\mathbb{R}^d \times \mathbb{R}_+$ and $f_t(Q|P) \ge 0$ an $\mathcal{F}_t$-conditional divergence. For the latter, assume that for all $Q^{(\beta,Y)}$ contained in*

$$\mathcal{Q}_t^f := \{Q \approx P \mid f_t(Q|P) < \infty\},$$

it has the form

$$f_t(Q^{(\beta,Y)}|P) = E_{Q^{(\beta,Y)}}\left[\int_t^T k\Big(\beta_s, Y(s,\ .)\Big)\,ds\,\Big|\,\mathcal{F}_t\right]. \tag{3.5.4}$$

For a benchmark measure $\hat{Q} = Q^{(\hat{\beta},\hat{Y})} \in \mathcal{Q}_t^f$ *and* global *restrictions*

$$f_t(Q^{(\beta,Y)}|P) \le \eta_t' f_t(Q^{(\hat{\beta},\hat{Y})}|P) + \theta_t' \quad \text{for all } t \in [0,T], \tag{3.5.5}$$

find more tractable but economically still reasonable local *restrictions on the integrand* $k\big(\beta_s, Y(s,\ .)\big)$*, which imply the global restrictions.*

Remark 3.5.5. a) The conditional expectation in (3.5.4) could also be with respect to some other measure $R(\beta, Y)$ depending on the same (β, Y) as $Q^{(\beta,Y)}$ and the integral $\int k(\beta, Y)\,ds$ could be replaced by $e^{\int k(\beta,Y)\,ds}$. This will actually be required for $\mathbf{U}^q$, $\mathbf{U}^p$ and $\mathbf{U}^\ell$; see Chapter 4. The same arguments as below then still work. In order to keep notation simple and to concentrate on the main ideas of our approach, we do not introduce $R(\beta, Y)$ here.

b) Similarly, the assumption $f_t(Q|P) \ge 0$ is made only for simplicity. For more general $f_t(Q|P)$ the process η' might have to be chosen differently.

c) In this section we do not assume that the basic assets S are locally bounded. Therefore it would be natural to consider instead of $\mathcal{M}^e(S) = \mathcal{M}^e(\mathbf{M}L)$ the set

$$\mathcal{Q}^\sigma := \{Q \in \mathcal{P}^e \mid S \text{ is a } \sigma\text{-martingale under } Q\};$$

see [DS98]. However, if S is uniformly bounded from below, then $Q \in \mathcal{Q}^\sigma$ if and only if $Q \in \mathcal{M}^e(S)$; this holds since by Theorem 88 in [Pro04] we can write any σ-martingale under Q as the integral with respect to some Q-martingale and this integral is a local martingale if it is uniformly bounded from below. Moreover, it was pointed out to us by F. Delbaen that the assumption $\mathcal{M}^e(S) \ne \emptyset$ implies by Theorem 1.1 in [KS01] that $\mathcal{M}^e(S)$ is dense in $\mathcal{Q}^\sigma$ with respect to the total variation norm. In fact, this holds since one can replace P in Theorem 1.1 of [KS01] by some $Q \in \mathcal{M}^e(S)$ so that as remarked there, the set $\mathcal{Q}_b^\sigma = \left\{\tilde{Q} \in \mathcal{Q}^\sigma \,\middle|\, \frac{d\tilde{Q}}{dQ} \in \mathbf{L}^\infty\right\}$ is contained in $\mathcal{M}^e(S)$ and dense in $\mathcal{Q}^\sigma$. Thus, for the purpose of good deal bounds it suffices to consider $\mathcal{M}^e(S)$.

◇

To tackle the above problem, we fix a benchmark measure $\hat{Q} = Q^{(\hat{\beta},\hat{Y})} \in \mathcal{M}^e(\mathbf{M}L)$ and processes $\eta = (\eta_s)_{0\le s\le T}$ and $\theta = (\theta_s)_{0\le s\le T}$ with $\eta_s \ge 1$ and $\theta_s \ge 0$. Then we define for each time t a set of no-good-deal measures by local restrictions on the integrand $k\big(\beta_s, Y(s,\ .)\big)$. In order to obtain meaningful results we always assume that

$$\hat{Q} \in \mathcal{Q}_t^f \quad \text{for all } t \in [0,T).$$

Definition 3.5.6. The set of *no-good-deal measures* at time $t \in [0, T)$ is defined by

$$\begin{aligned}\mathcal{NL}_t &:= \mathcal{NL}_t^{\eta,\theta,\hat{Q}} \\ &:= \Big\{ Q^{(\beta,Y)} \in \mathcal{M}^e(\mathbf{M}L) \,\Big|\, k(\beta_s, Y(s, \,.)) \le \eta_s k(\hat{\beta}_s, \hat{Y}(s, \,.)) + \theta_s \\ &\qquad\qquad dP \otimes dt\text{-a.e. on } \Omega \times [t, T] \Big\}. \end{aligned} \tag{3.5.6}$$

Note that $\hat{Q} \in \mathcal{Q}_t^f$ implies $\mathcal{NL}_t \neq \emptyset$ for all $t \in [0, T)$, and that the restriction in (3.5.6) is completely analogous to the one in (3.5.5); the only difference is that it is formulated at the level of the integrands k instead of the integrals $f(Q|P)$, see (3.5.4). Moreover, the value of $f_t(Q^{(\beta,Y)}|P)$ depends on $Q^{(\beta,Y)}$ not only via the integrand $k\big(\beta_s, Y(s, \,.)\big)$ but also via the $Q^{(\beta,Y)}$-conditional expectation. Therefore, even for deterministic η and θ, the local restriction in (3.5.6) does not necessarily imply the global restriction (3.5.5). However, suppose that the benchmark measure $\hat{Q}$ has deterministic and time-independent Girsanov parameters $(\hat{\beta}, \hat{Y})$ (as it is the case for the minimal entropy measure). Then it seems reasonable to choose η and θ as deterministic functions as well. These two assumptions then imply that the local restrictions induce the global restrictions.

Theorem 3.5.7. *Let $\eta = \eta(\cdot)$ and $\theta = \theta(\cdot)$ be deterministic functions on $[0, T]$ with $\eta \ge 1$ and $\theta \ge 0$ and let the benchmark measure $\hat{Q}$ have time-independent and deterministic Girsanov parameters $(\hat{\beta}, \hat{Y})$. Suppose also that (3.5.4) holds. Then for $\eta'(t) := \frac{1}{T-t}\int_t^T \eta(s)\,ds$ and $\theta'(t) := \int_t^T \theta(s)\,ds$, we have*

$$\mathcal{NL}_t^{\eta,\theta,\hat{Q}} \subseteq \big\{ Q \in \mathcal{M}^e(\mathbf{M}L) \,\big|\, f_t(Q|P) \le \eta'(t) f_t(\hat{Q}|P) + \theta'(t) \big\}.$$

Proof. This is obvious from (3.5.4), (3.5.6) and the assumptions. □

Remark 3.5.8. Why do we introduce the process θ? If $\theta \equiv 0$, the upper bound for the $\mathcal{F}_t$-conditional divergence of Q is proportional to the $\mathcal{F}_t$-conditional divergence of the benchmark measure $\hat{Q}$. A convenient choice for $\hat{Q}$ is that measure which minimizes the $\mathcal{F}_t$-conditional divergence over $\mathcal{M}^e(\mathbf{M}L)$. Then $f_t(\hat{Q}|P)$ quantifies how far P is away from being a local martingale measure. If P itself is already a local martingale measure, then one should choose $\hat{Q} = P$, and if then $f_t(Q|P) = E\big[f(dQ/dP)\big|\mathcal{F}_t\big]$ with $f(1) = 0$, we would get $f_t(\hat{Q}|P) \equiv 0$. Hence in this case, the set of no-good-deal measures would (for a reasonable f) contain only P. If one starts with a local martingale measure for P, one can of course argue that this is a valid pricing measure; but it is still a matter of taste whether or not one is willing to say that it is the only reasonable one. If one is, then $\theta \equiv 0$ is a very convenient choice. However, for greater generality we allow also for $\theta \not\equiv 0$. ◇

Having defined $\mathcal{NL}_t^{\eta,\theta,\hat{Q}}$, we next have to ask which dynamics the processes η and θ should have. As argued above, they should be deterministic functions in order that for a benchmark measure with time-independent and deterministic Girsanov parameters, the local restrictions imply the global ones. Given our Lévy framework and the above desired properties of the benchmark measure, it seems convenient to let also η and θ or η' and θ' be time-independent in an appropriate sense. This can be made more precise in two ways.

A first possibility is time-independence with respect to the global restriction (3.5.5), i.e., such that η' and θ' are independent of t in (3.5.5). However, for fixed t, this yields for η and θ from Definition 3.5.6 that $\eta_s \equiv \eta'$ and $\theta_s = \theta_s^{(t)} \equiv \frac{\theta'}{T-t}$ for all $s \in [t, T)$. In other words, the local restriction which specifies $\mathcal{NL}_t^{\eta,\theta,\hat{Q}}$ in (3.5.6) then depends on t. This is very inconvenient if we want to apply dynamic programming techniques.

A much more convenient choice is time-independence with respect to the local restrictions. One then chooses $\eta_s \equiv \eta$ and $\theta_s \equiv \theta$ for all $s \in [0, T]$, and if $\theta \equiv 0$, one gets the same result as for the first possibility. This second choice allows to apply dynamic programming techniques; moreover, the set of no-good-deal measures $\mathcal{NL}_t = \mathcal{NL}_t^{\eta,\theta,\hat{Q}}$ can be chosen independent of t in the following sense.

Lemma 3.5.9. *If $\eta_s \equiv \eta \geq 1$, $\theta_s \equiv \theta \geq 0$ for all $s \in [0, T]$ and the Girsanov parameters of $\hat{Q}$ are time-independent and deterministic, then for any $X \in \mathbf{L}^\infty(P)$*

$$\operatorname*{ess\,inf}_{Q \in \mathcal{NL}_t^{\eta,\theta,\hat{Q}}} E_Q[X|\mathcal{F}_t] = \operatorname*{ess\,inf}_{Q \in \mathcal{NL}_0^{\eta,\theta,\hat{Q}}} E_Q[X|\mathcal{F}_t].$$

Proof. Due to the Bayes rule, the $\mathcal{F}_t$-conditional expectations under Q depend only on $(N_s^Q)_{t \leq s \leq T}$ with N^Q from (3.4.3). Hence restricting β and Y via k has the same effect if it is done on $[t, T]$ or on $[0, T]$. □

Now assume that η, θ and the benchmark measure $\hat{Q}$ are as in Lemma 3.5.9. Then we can skip the mention of t for the set of no-good-deal measures and set

$$\mathcal{NL} := \mathcal{NL}_0 = \mathcal{NL}_0^{\eta,\theta,\hat{Q}}.$$

The next result illustrates how the local and global restrictions are linked if we impose in addition that any no-good-deal measure must preserve the Lévy property of L or, equivalently, must have time-independent and deterministic Girsanov parameters. Under this condition, the global and local restrictions on the set of no-good-deal measures are equivalent.

Proposition 3.5.10. *Suppose $\eta_s \equiv \eta \geq 1$, $\theta_s \equiv \theta \geq 0$ and the Girsanov parameters of $\hat{Q}$ are time-independent and deterministic. Define*

$$\begin{aligned} \mathcal{M}^e_{\text{Lévy}}(\mathbf{M}L) &:= \{Q \in \mathcal{M}^e(\mathbf{M}L) \mid L \text{ is a } Q\text{-Lévy process}\}, \\ \mathcal{NL}^L &:= \mathcal{NL} \cap \mathcal{M}^e_{\text{Lévy}}(\mathbf{M}L) \end{aligned}$$

and for each $t \in [0, T]$ set

$$\mathcal{N}_t^L := \{Q \in \mathcal{M}^e(\mathbf{M}L) \mid f_t(Q|P) \leq \eta f_t(\hat{Q}|P) + \theta(T-t)\} \cap \mathcal{M}^e_{\text{Lévy}}(\mathbf{M}L).$$

Then

$$\operatorname*{ess\,inf}_{Q \in \mathcal{NL}^L} E_Q[X|\mathcal{F}_t] = \operatorname*{ess\,inf}_{Q \in \mathcal{N}_t^L} E_Q[X|\mathcal{F}_t] \quad \textit{for all } \hat{X} \in \mathbf{L}^\infty(P).$$

Proof. Girsanov's theorem and (3.4.2) imply that $Q^{(\beta,Y)} \in \mathcal{M}^e(\mathbf{M}L)$ preserves the Lévy property of L, i.e., is in $\mathcal{M}^e_{\text{Lévy}}(\mathbf{M}L)$, if and only if (β, Y) is time-independent. This implies that instead of Theorem 3.5.7, we have $\mathcal{NL}_t \cap \mathcal{M}^e_{\text{Lévy}}(\mathbf{M}L) = \mathcal{N}_t^L$. The rest then goes as for Lemma 3.5.9. □

To finish this section, let us relate Theorem 3.5.7 and Proposition 3.5.10 to the articles [CSR00], [Cer03] and [BS06] which also deal with good deal bounds obtained from local restrictions. [CSR00] and [Cer03] work in a Brownian setting and obtain a connection between the local and global restrictions by taking limits. [BS06] extend that model by adding a marked point process, but they do not study the relation between the local and global restrictions. In contrast, Theorem 3.5.7 proves in a general setting that the local implies the global restriction, and Proposition 3.5.10 provides a precise description of a situation when the local and global restrictions coincide for the choice of no-good-deal pricing measures. Moreover, none of the above articles gives a justification why a constant or deterministic local restriction is reasonable, nor in particular why it induces a non-empty set of no-good-deal measures.

3.6 Valuation processes induced by good deal bounds

In this section we discuss properties of the processes of no-good-deal values and valuation bounds. As in the previous section, we work with the P-augmentation of the filtration generated by a d-dimensional Lévy process L and assume that $\mathcal{M}^e(\mathbf{M}L) \neq \emptyset$. Proceeding in the abstract setting motivated by the exponential good deal bounds, we fix a deterministic function k which describes some non-negative f-divergence, constants $\eta \geq 1$ and $\theta \geq 0$ and a benchmark measure $\hat{Q}$ with time-independent and deterministic Girsanov parameters $(\hat{\beta}, \hat{Y})$. The set of no-good-deal measures is

$$\mathcal{NL} = \Big\{ Q^{(\beta,Y)} \in \mathcal{M}^e(\mathbf{M}L) \;\Big|\; k\big(\beta_s, Y(s,\,.)\big) \leq \eta k\big(\hat{\beta}, \hat{Y}(\,.\,)\big) + \theta \quad dP \otimes dt\text{-a.e. on } \Omega \times [0,T] \Big\}. \tag{3.6.1}$$

Definition 3.6.1. For any $X \in \mathbf{L}^\infty(P)$ we define the *lower* and *upper good deal value process* by

$$\begin{aligned}\pi_t^\ell(X) &:= \operatorname*{ess\,inf}_{Q\in\mathcal{NL}} E_Q[X|\mathcal{F}_t] \quad \text{for } t \in [0,T],\\ \pi_t^u(X) &:= -\pi_t^\ell(-X) = \operatorname*{ess\,sup}_{Q\in\mathcal{NL}} E_Q[X|\mathcal{F}_t] \quad \text{for } t \in [0,T].\end{aligned}$$

In analogy to the static case, π^ℓ is a dynamic monetary coherent utility functional in the sense of the following definition; see Section 2.3 for more details.

Definition 3.6.2. Fix $t \in [0,T]$. A mapping $\Phi_t : \mathbf{L}^\infty(\mathcal{F}_T) \to \mathbf{L}^\infty(\mathcal{F}_t)$ is a *monetary coherent utility functional at time* t if it satisfies

A) *monotonicity:* $\Phi_t(X_1) \le \Phi_t(X_2)$ for $X_1 \le X_2$;

B) *$\mathcal{F}_t$-translation invariance:* $\Phi_t(X + a_t) = \Phi_t(X) + a_t$ for $a_t \in \mathbf{L}^\infty(\mathcal{F}_t)$;

C) *concavity:* $\Phi_t\big(\alpha X_1 + (1-\alpha)X_2\big) \ge \alpha\Phi_t(X_1) + (1-\alpha)\Phi_t(X_2)$ for $\alpha \in [0,1]$;

D) *positive homogeneity:* $\Phi_t(\lambda X) = \lambda\Phi_t(X)$ for $\lambda \ge 0$.

If each Φ_t is a monetary coherent utility functional at time $t \in [0,T]$, we call the family $\Phi = \big(\Phi_t(\,.\,)\big)_{0\le t\le T}$ a *dynamic monetary coherent utility functional.*

Lemma 3.6.3. *π^ℓ is a dynamic monetary coherent utility functional.*

Proof. Easy to check. □

The set $\mathcal{NL}$ of no-good-deal measures has the following property which is very important for the existence of a regular version and a nice dynamic behaviour of π^ℓ as well as for the application of dynamic programming techniques.

Definition 3.6.4. A set $\mathcal{S} \subseteq \mathcal{P}^a$ such that $\mathcal{S} \cap \mathcal{P}^e \neq \emptyset$ is called *m-stable* if it has the following property: If we take $Q^1, Q^2 \in \mathcal{S}$ with associated density processes Z^1, Z^2 (with respect to P), fix a stopping time $\tau \le T$, impose that $Q^2 \approx P$ and define

$$Z_T := Z_\tau^1 \frac{Z_T^2}{Z_\tau^2},$$

then Z_T is the density of some element in $\mathcal{S}$.

Remark 3.6.5. a) Although in the Definition 2.3.27 of weak m-stability there occurs an additional set A, the only difference between weak m-stability and m-stability is that the latter is defined with respect to stopping times instead of deterministic times; see Remark 2.3.28.

b) The set $\mathcal{NL}^L$ from Proposition 3.5.10 is in general not m-stable; the reason is that the Girsanov parameters of the probability measure defined by the concatenation operation in Definition 3.6.4 change their value at time τ, i.e., are time-dependent.

$\diamond$

Proposition 3.6.6. *a) The set $\mathcal{NL}$ from (3.6.1) is m-stable.*

b) For each $X \in \mathbf{L}^\infty(P)$ there exists an RCLL version of $\pi^\ell = \left(\pi_t^\ell(X)\right)_{0\leq t\leq T}$, again denoted by π^ℓ, such that

$$\pi_\tau^\ell(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{NL}} E[X|\mathcal{F}_\tau]$$

for any stopping time $\tau \leq T$.

Proof. Part a) holds since $\mathcal{M}^e(S)$ is m-stable by Proposition 5 in [Del06] and because $\mathcal{NL}$ is defined by a pointwise restriction and b) holds by Lemmata 22 and 23 in [Del06]. $\square$

In the sequel, we choose an RCLL version for every $\pi^\ell(X)$. The following two properties are of interest for the dynamic behaviour of any dynamic monetary coherent utility functional and in particular for π^ℓ.

Definition 3.6.7. Let $\mathcal{S} \subseteq \mathcal{P}^e$ be non-empty and define for each stopping time $\tau \leq T$ and $X \in \mathbf{L}^\infty(P)$

$$\Phi_\tau(X) := \operatorname*{ess\,inf}_{Q\in\mathcal{S}} E_Q[X|\mathcal{F}_\tau].$$

Φ is called *stopping-time-consistent* if $\Phi_\tau(X^1) \leq \Phi_\tau(X^2)$ implies $\Phi_\sigma(X^1) \leq \Phi_\sigma(X^2)$ for any pair of stopping times $\sigma \leq \tau \leq T$. We call Φ *recursive with respect to stopping times* if $\Phi_\sigma\left(\Phi_\tau(X)\right) = \Phi_\sigma(X)$ for any pair of stopping times $\sigma \leq \tau \leq T$.

Remark 3.6.8. a) The difference between stopping-time-consistency here and time-consistency as defined in Definition 2.3.23 is that we allow here for stopping times instead of deterministic times. The literature usually does not distinguish between these two properties and refers to both as time-consistency.

b) In economic terms, stopping-time-consistency preserves over time the ordering induced by Φ. Recursiveness means that if we want to value the time T payoff X at time σ, we can either do this directly or first value it at time $\tau \geq \sigma$ and then value that result at time σ.

◇

One can show that the two properties are in fact equivalent, and that they automatically hold if $\mathcal{S}$ is m-stable. If $\mathcal{S} = \mathcal{S}' \cap \mathcal{P}^e$ for some set $\mathcal{S}' \subseteq \mathcal{P}^a$ such that the set of densities corresponding to $\mathcal{S}'$ is convex and norm closed in $\mathbf{L}^1(P)$, then they are even equivalent to m-stability of $\mathcal{S}$; see Theorem 12 in [Del06].

Proposition 3.6.9. *π^ℓ is time-consistent and recursive with respect to stopping times.*

Proof. This follows as in the proof of Theorem 12 in [Del06]. □

We next turn to the question of nice representations for the value bound processes. Let us assume that the basic assets S are locally bounded and that $\mathcal{M}^e(\mathbf{M}L) = \mathcal{M}^e(S)$. For any $X \in \mathbf{L}^\infty$, recall that its *superhedging value process* is given by

$$\left(\operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(\mathbf{M}L)} E_Q[X|\mathcal{F}_t] \right)_{0 \leq t \leq T}.$$

At time t, this corresponds to the smallest amount of money which allows one to obtain, by trading in S during $(t, T]$, a payoff which dominates X. It is well known that this process has the *optional decomposition*

$$\operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(\mathbf{M}L)} E_Q[X|\mathcal{F}_t] = x_0 + (\vartheta \cdot S)_t - C_t, \tag{3.6.2}$$

where $x_0 \in I\!R$, ϑ is predictable, S-integrable and such that $\vartheta \cdot S$ is locally bounded from below, and C is an increasing adapted RCLL process; see [Kra96] or [FK97]. But even if we replace C by some adapted RCLL process $\tilde{C}$ of finite variation which is not necessarily increasing, we cannot hope in general to obtain such a representation for the upper good deal value bound process π^u, except for the special cases when $\mathcal{NL} = \mathcal{M}^e(\mathbf{M}L)$ or when X is attainable by trading in the basic assets. In fact, suppose we could write

$$\pi_t^u(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{NL}} E_Q[X|\mathcal{F}_t] = \tilde{x}_0 + (\tilde{\vartheta} \cdot S)_t - \tilde{C}_t.$$

For simplicity, assume that the filtration is Brownian so that any local martingale of finite variation is constant. Since for any $Q \in \mathcal{NL}$ the process $\pi^u(X)$ is a Q-supermartingale and $\tilde{\vartheta} \cdot S$ is a Q-local martingale, we see that $\tilde{C}$ must be an increasing process. Hence if a representation exists, it must be of the form (3.6.2). However, by Theorem 3.1 in [FK97], an optional decomposition exists if and only if

$\pi^u(X)$ is a local supermartingale for all $Q \in \mathcal{M}^e(\mathbf{M}L)$. This is not true in general; we can get the supermartingale property for $Q \in \mathcal{NL}$, but have no information for $Q \in \mathcal{M}^e(\mathbf{M}L) \setminus \mathcal{NL}$.

Another way to obtain a representation might be to add a finite number of semimartingales to the basic assets S such that $\mathcal{NL}$ becomes the set of equivalent local martingale measures for this enlarged family of processes. Unfortunately, this also does not help in general. In fact, assume that the Lévy process L is continuous, so that every strictly positive martingale is of the form $\mathcal{E}(\int \beta^* \, dL^c)$ and

$$\mathcal{NL} = \left\{ Q^\beta \in \mathcal{M}^e(\mathbf{M}L) \,\middle|\, k(\beta_t) \le \eta k(\hat{\beta}) + \theta \quad dP \otimes dt\text{-a.e.} \right\}.$$

By Theorem 19 of [Del06], $\mathcal{NL}$ corresponds to a set of equivalent local martingale measures if and only if for each $t \in [0, T]$ and each $\omega \in \Omega$,

$$\left\{ \beta_t(\omega) \in I\!R^d \,\middle|\, k\big(\beta_t(\omega)\big) \le \eta k(\hat{\beta}) + \theta \right\} - \hat{\beta}$$

is a subspace of $I\!R^d$. But this is obviously not true in general. In view of the above negative results, it would be interesting to know whether there are other useful decompositions for the good deal value bound processes π^ℓ and π^u.

We finish this section with a brief consideration concerning dynamic no-arbitrage properties of no-good-deal value processes. Fix $X \in \mathbf{L}^\infty(P)$. Every $Q^1 \in \mathcal{NL}$ induces an arbitrage-free value process $(E_{Q^1}[X|\mathcal{F}_t])_{0 \le t \le T}$ for X. This is a subjective no-good-deal value process of some agent, and so the agent might want to switch from Q^1 to some other pricing measure $Q^2 \in \mathcal{NL}$ at some stopping time τ. This raises the question when such a change of the pricing measure does not yield arbitrage opportunities. For simplicity, assume that the basic assets S with $\mathcal{M}^e(\mathbf{M}L) = \mathcal{M}^e(S)$ are locally bounded. Then there exist no arbitrage opportunities if and only if there exists $Q \in \mathcal{M}^e(\mathbf{M}L)$ such that

$$\overline{p}_t(X) := \begin{cases} E_{Q^1}[X|\mathcal{F}_t] & \text{on } [\![0, \tau[\![\\ E_{Q^2}[X|\mathcal{F}_t] & \text{on } [\![\tau, T]\!] \end{cases} \tag{3.6.3}$$

is a (true) Q-martingale; see [DS94] for locally bounded and [DS98] for unbounded processes, and note that $\overline{p}(X)$, like X, is bounded. Since the filtration is generated by a Lévy process, it is quasi-left continuous. If the stopping time τ is predictable, no jumps can occur for $\overline{p}_\tau(X)$ and we can give a condition which is necessary and sufficient for the existence of an appropriate $Q \in \mathcal{M}^e(\mathbf{M}L)$. For a totally inaccessible stopping time, we have found so far only a sufficient condition.

Proposition 3.6.10. *Denote by Z^1, Z^2 the density processes of $Q^1, Q^2 \in \mathcal{NL}$ from (3.6.3). Define $\overline{Q} \in \mathcal{NL}$ by the density $\overline{Z}_T := Z^1_\tau \frac{Z^2_T}{Z^2_\tau}$ and set*

$$\Delta^\tau := E_{Q^2}[X|\mathcal{F}_\tau] - E_{Q^1}[X|\mathcal{F}_\tau].$$

If $\overline{M} := \Delta^\tau \mathbf{1}_{[\![\tau,T]\!]}$ *is a* $\overline{Q}$*-martingale, then* $\overline{p}(X)$ *is a* $\overline{Q}$*-martingale. If* τ *is a predictable stopping time, then the following are equivalent:*

a) *There exists* $Q \in \mathcal{M}^e(\mathbf{M}L)$ *such that* $\overline{p}(X)$ *is a* Q*-martingale.*

b) $\overline{p}(X)$ *is a* $\overline{Q}$*-martingale.*

c) $\overline{p}_{\tau-}(X) = \overline{p}_\tau(X)$.

Proof. The process $p = (p_t)_{0\le t\le T}$ defined by

$$p_t := E_{Q^1}[X|\mathcal{F}_{t\wedge\tau}] + \mathbf{1}_{[\![\tau,T]\!]}(E_{Q^2}[X|\mathcal{F}_t] - E_{Q^2}[X|\mathcal{F}_\tau])$$

is a $\overline{Q}$-martingale. Since $\overline{M} = \overline{p}(X) - p$ we see that $\overline{p}(X)$ is a $\overline{Q}$-martingale if and only if $\overline{M}$ is. The m-stability of $\mathcal{NL}$ implies that $\overline{Q} \in \mathcal{NL} \subseteq \mathcal{M}^e(\mathbf{M}L)$. This proves the sufficient condition. Now suppose that τ is a predictable stopping time. The filtration is quasi-left-continuous so that $\mathcal{F}_\tau = \mathcal{F}_{\tau-}$. This together with Theorem VI.14 of [DM82] implies that if $\overline{p}(X)$ is a martingale for some $Q \in \mathcal{M}^e(\mathbf{M}L)$, then c) must hold. Moreover, if c) holds then $\overline{p}(X)$ is a martingale for $\overline{Q} \in \mathcal{NL}$. This finishes the proof. □

3.7 Examples

In this section we study two examples. In the first we obtain a partial integro-differential equation for the value bound via dynamic programming techniques. The second examines good deal value bounds under the additional assumption that the pricing measure preserves the Lévy property of the process L which generates the filtration.

3.7.1 Stochastic control

In this example we derive a stochastic control problem for the good deal value process. We fix the truncation function $h(x) := x\mathbf{1}_{\{\|x\|\le 1\}}$ on $\mathbb{R}^d$. The filtration is generated by a d-dimensional Lévy process $L = (L_t)_{0\le t\le T}$ with dynamics

$$dL_t = \alpha dt + \sigma\, dB_t + \int_{\|x\|\le 1} x \left(\mu(dx, dt) - K(dx)dt\right) + \int_{\|x\|>1} x\, \mu(dx, dt),$$

where B is a d-dimensional Brownian motion, $K(dx)dt$ is the compensator of the random measure μ, α is a d-dimensional vector and σ is a $d \times d$ nonnegative definite matrix. We introduce the notations

$$\sigma = \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{d1} & \dots & \sigma_{dd} \end{pmatrix} = \begin{pmatrix} \sigma_1^* \\ \vdots \\ \sigma_d^* \end{pmatrix} \qquad \text{and} \qquad \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix}.$$

The triplet (b, c, K) in (3.4.2) which describes the characteristics of L is given by

$$b = \alpha \quad \text{and} \quad c = \sigma\sigma^* =: \begin{pmatrix} c_{11} & \dots & c_{1d} \\ \vdots & \ddots & \vdots \\ c_{d1} & \dots & c_{dd} \end{pmatrix} = \begin{pmatrix} c_1^* \\ \vdots \\ c_d^* \end{pmatrix}.$$

We define $\overline{S} = (\overline{S}_t)_{0 \leq t \leq T}$ as the stochastic exponential of L, i.e.,

$$\overline{S} = (\overline{S}^1, \dots, \overline{S}^d)^* = \mathcal{E}(L),$$

and for a fixed $d \times d$-matrix $\mathbf{M}$ we denote by $\mathcal{M}^e(\mathbf{M}L)$ the set of all equivalent local martingale measures for $\mathbf{M}L$. The interpretation is as follows. $\overline{S}$ describes the discounted price processes of some assets. If we assume, e.g., that trading is possible only in $\overline{S}^1$ and $\overline{S}^2$ and set

$$\mathbf{M} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

then $\mathcal{M}^e(\mathbf{M}L) = \mathcal{M}^e\big(\mathcal{E}(\mathbf{M}L)\big) = \mathcal{M}^e\Big(\overline{S}^1, \overline{S}^2\Big)$ corresponds to the set of equivalent local martingale measures for the *traded* assets. For a payoff $X = \Psi(\overline{S}_T)$ which is sufficiently integrable (e.g., $X \in \mathbf{L}^\infty(P)$), we want to find the solution to the following optimal control problem which describes the upper good deal value process for X.

Problem: Find

$$\operatorname*{ess\,sup}_{\text{allowed } (\beta, Y)} E_{Q^{(\beta,Y)}}[\Psi(\overline{S}_T)|\mathcal{F}_t],$$

where the allowed Girsanov parameters (β, Y) *satisfy for all* $t \in [0, T]$ *the conditions*

$$Y(t,x) \;>\; 0, \tag{3.7.1}$$

$$\int_0^T \int_{\mathbb{R}^d} \big\|\mathbf{M}\big(xY(t,x) - h(x)\big)\big\| \, K(dx)\, dt \;<\; \infty, \tag{3.7.2}$$

$$\mathbf{M}\left(b + c\beta_t + \int_{\mathbb{R}^d} \Big(xY(t,x) - h(x)\Big)\, K(dx)\right) \;=\; 0, \tag{3.7.3}$$

$$k^e\big(\beta_t, Y(t, \,.)\big) := \frac{1}{2}\beta_t^* c\beta_t + \int_{\mathbb{R}^d} g\big(Y(t,x)\big)\, K(dx) \;\leq\; \hat{\eta}, \tag{3.7.4}$$

and where $g(y) = y \log y - y + 1$ *and* $\hat{\eta}$ *is a fixed constant.*

Remark 3.7.1. Recall from Proposition 3.4.5 that the conditions (3.7.2) and (3.7.3) ensure that $Q^{(\beta,Y)} \in \mathcal{M}^e(\mathbf{M}L)$.

In order to apply dynamic programming techniques, we make the following

Assumptions:

A) $f_0^e(Q|P) < \infty$ for some $Q \in \mathcal{M}^e(\mathbf{M}L)$.

B) For every allowed $Q = Q^{(\beta,Y)}$, the Girsanov parameters (β, Y) are of the form

$$\beta_t = \beta(t, \overline{S}_{t-}) \quad \text{and} \quad Y(t,x) = Y(t, \overline{S}_{t-}, x)$$

for deterministic functions $\beta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $Y : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.

Assumption **B)** implies that the Markov property of L is preserved under $Q^{(\beta,Y)}$ so that

$$\operatorname*{ess\,sup}_{\text{allowed }(\beta,Y)} E_{Q^{(\beta,Y)}}[\Psi(\overline{S}_T)|\mathcal{F}_t] = V(t, \overline{S}_t)$$

for a deterministic function $V : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$. Note that $B - \int \sigma^* \beta_s \, ds$ is a $Q^{(\beta,Y)}$-Brownian motion and that the compensator of the random measure μ under $Q^{(\beta,Y)}$ is given by $Y(t,x)\,K(dx)dt$. If V is in $C^{1,2}(\mathbb{R}_+, \mathbb{R}^d)$ and $\hat{\eta}$ is big enough, then formally Itô's formula implies that V should solve the following problem:

$$\begin{aligned}
\frac{\partial}{\partial t} V(t,s) + \sup_{\text{allowed }(\beta,Y)} \mathcal{A}^{(\beta,Y)} V(t,s) &= 0, \\
V(T,s) &= \Psi(s), \\
Y(t,s,x) &> 0, \\
\mathbf{M}\left(b + c\beta_t + \int_{\mathbb{R}^d} \Big(xY(t,x) - h(x) \Big) K(dx) \right) &= 0, \\
\frac{1}{2} \beta^*(t,s)\, c\, \beta(t,s) + \int_{\mathbb{R}^d} g\big(Y(t,s,x)\big)\, K(dx) &\le \hat{\eta},
\end{aligned}$$

where $\mathcal{A}^{(\beta,Y)}$ denotes the $Q^{(\beta,Y)}$ generator of S. Even if V solves the above partial integro-differential equation, it remains to check that Y satisfies (3.7.2) and that V actually yields a solution to the original stochastic control problem; see [OS05] for a more detailed discussion and Theorem 3.1 there for a verification problem in a similar setting.

Remark 3.7.2. In [BS06] there is a detailed discussion of a control problem which differs from this example only with respect to the condition $k^e\big(\beta(t,s), Y(t,s,\,.)\big) \le \hat{\eta}$ which is replaced there by

$$\beta^*(t,s)\, c\, \beta(t,s) + \int_{\mathbb{R}^d} \big(Y(t,s,x) - 1\big)^2 K(dx) \le \hat{\eta}.$$

◇

3.7.2 Good deal bounds and preservation of the Lévy property

In this example we consider good deal value bounds under the additional condition that the pricing measure has to preserve the Lévy property of L or, equivalently, has deterministic time-independent Girsanov parameters. In this case one cannot apply the methods of dynamic programming. However, the restriction to time-independent Girsanov parameters often yields relatively simple optimization problems for the value bounds. To illustrate this, let the filtration be generated by the 2-dimensional Lévy process

$$L_t = \alpha t + \sigma B_t + \int_0^t \int_{\mathbb{R}^2} x\, \mu(dx, dt),$$

where $B = (B^1, B^2)^*$ is a 2-dimensional Brownian motion, $\alpha = (\alpha_1, \alpha_2)^*$ is a vector in $\mathbb{R}^2$ and $\sigma = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}$ with $\sigma_1, \sigma_2 > 0$ and $\rho \in [0,1]$. We denote by $\delta_{\{x\}}$ the Dirac measure in $x \in \mathbb{R}^2$ and assume that the compensator of μ is of the form $K(dx)dt$ with

$$K(dx) = \sum_{j=1}^n \lambda_j\, \delta_{\{x_j\}}(dx)$$

for fixed $\lambda_1, \ldots, \lambda_n > 0$ and $x_1, \ldots, x_n \in \mathbb{R}^2$ such that with $x_j =: (x'_j, x''_j)^*$ we have $x'_j > -1$ and $x''_j > -1$ for $j \in \{1, \ldots, n\}$. This means that under P the sum of jumps $\int \int_{\mathbb{R}^2} x\, \mu(dx, dt)$ is the sum of n independent Poisson processes with respective jump size x_j and intensity λ_j. We introduce the notation $\big(\overline{S}', \overline{S}''\big)^* := \mathcal{E}(L)$. As before, it is the choice of the $\mathbb{R}^{2\times 2}$-matrix $\mathbf{M}$ which decides which processes are traded assets. By $\mathcal{M}^e(\mathbf{M}L)$, we again denote the set of equivalent local martingale measures for $\mathbf{M}L$. If we impose that any pricing measure $Q^{(\beta,Y)}$ has to preserve the Lévy property of $L = (L^1, L^2)^*$, the corresponding Girsanov parameters (β, Y) are fully described by a constant vector $\beta = (\beta_1, \beta_2)^* \in \mathbb{R}^2$ and by the values $Y(x_j) =: y_j \in (0, \infty)$ for $j \in \{1, \ldots, n\}$.

Assumption: There exists $Q^{(\beta,Y)} \in \mathcal{M}^e(\mathbf{M}L)$ with (β, Y) deterministic and time-independent and with $f_0^e(Q^{(\beta,Y)}|P) < \infty$.

Let us consider the payoff $X = (\overline{S}''_T)^2$ and define $W_t := \rho B_t^1 + \sqrt{1-\rho^2} B_t^2$. Denoting $x = (x', x'')^*$, Itô's formula implies that $\overline{S}''_t = \mathcal{E}(L_t^2) = e^{\tilde{L}_t}$, where

$$\begin{aligned}
\tilde{L}_t &= L_t^2 + \int_0^t \int_{I\!R^2} \left(\log(x''+1) - x'' \right) \mu(dx, dt) - \frac{1}{2}\sigma_2^2 t \\
&= \alpha_2 t + \sigma_2 W_t + \int_0^t \int_{I\!R^2} \log(x''+1)\, \mu(dx, dt) - \frac{1}{2}\sigma_2^2 t \\
&= \left(\alpha_2 - \frac{1}{2}\sigma_2^2 + \sigma_2 \left(\rho\beta_1 + \sqrt{1-\rho^2}\beta_2 \right) \right) t \\
&\quad + \sigma_2 W_t^{(\beta,Y)} + \int_0^t \int_{I\!R^2} \log(x''+1)\, \mu(dx, dt)
\end{aligned}$$

and $W^{(\beta,Y)}$ is the $Q^{(\beta,Y)}$-Brownian motion $W_t^{(\beta,Y)} := W_t - \left(\rho\beta_1 + \sqrt{1-\rho^2}\beta_2 \right) t$. Note that $\int \int_{I\!R^2} 2\log(x''+1)\, \mu(dx, dt)$ is under $Q^{(\beta,Y)}$ the sum of n independent Poisson processes with respective jump sizes $2\log(x_j''+1)$ and intensities $\lambda_j y_j$ so that

$$\begin{aligned}
& E_{Q^{(\beta,Y)}} \left[\exp\left(\int_t^T \int_{I\!R^2} 2\log(x''+1)\, \mu(dx, dt) \right) \middle| \mathcal{F}_t \right] \\
&= \prod_{j=1}^n \left(\sum_{k=0}^\infty e^{2\log(x_j''+1)k} \frac{\left(\lambda_j y_j (T-t)\right)^k}{k!} e^{-\lambda_j y_j (T-t)} \right) \\
&= \prod_{j=1}^n \left(\sum_{k=0}^\infty \frac{1}{k!} \left((x_j''+1)^2 \lambda_j y_j (T-t) \right)^k e^{-\lambda_j y_j (T-t)} \right) \\
&= \exp\left(\sum_{j=1}^n \lambda_j y_j (T-t) \left(2x_j'' + (x_j'')^2 \right) \right).
\end{aligned}$$

Hence $(s, \omega) \mapsto \exp\left(\int_0^s \int_{I\!R^2} 2\log(x''+1)\, \mu(dx, dt) - s \sum_{j=1}^n \lambda_j y_j \left(2x_j'' + (x_j'')^2 \right) \right)$ is a $Q^{(\beta,Y)}$-martingale. Since continuous and purely discontinuous martingales are orthogonal we thus have that

$$
\begin{aligned}
&E_{Q^{(\beta,Y)}}\left[(\overline{S}''_T)^2 \,\middle|\, \mathcal{F}_t\right] \\
&= (\overline{S}''_t)^2 E_{Q^{(\beta,Y)}}\left[\exp\left(2(\tilde{L}_T - \tilde{L}_t)\right) \,\middle|\, \mathcal{F}_t\right] \\
&= (\overline{S}''_t)^2 \exp\Bigg(\Big(2\alpha_2 + \sigma_2^2 + 2\sigma_2\big(\rho\beta_1 + \sqrt{1-\rho^2}\beta_2\big) \\
&\qquad + \sum_{j=1}^n \lambda_j y_j \big(2x''_j + (x''_j)^2\big)\Big)(T-t)\Bigg) \\
&\quad \times E_{Q^{(\beta,Y)}}\Bigg[\exp\left(2\sigma_2\big(W_T^{(\beta,Y)} - W_t^{(\beta,Y)}\big) - 2\sigma_2^2(T-t)\right) \\
&\qquad \times \exp\Bigg(\int_t^T \int_{\mathbb{R}^2} 2\log(x''+1)\,\mu(dx,dt) \\
&\qquad - \sum_{j=1}^n \lambda_j y_j (T-t)\big(2x''_j + (x''_j)^2\big)\Bigg) \,\Bigg|\, \mathcal{F}_t\Bigg] \\
&= (\overline{S}''_t)^2 \exp\Bigg(\Big(2\alpha_2 + \sigma_2^2 + 2\sigma_2\big(\rho\beta_1 + \sqrt{1-\rho^2}\beta_2\big) \\
&\qquad + \sum_{j=1}^n \lambda_j y_j \big(2x''_j + (x''_j)^2\big)\Big)(T-t)\Bigg).
\end{aligned}
$$

Since $(\overline{S}''_t)^2 > 0$, calculating the upper good deal value process thus reduces to the following optimization problem (which is independent of t):

maximize $\quad 2\alpha_2 + \sigma_2^2 + 2\sigma_2\left(\rho\beta_1 + \sqrt{1-\rho^2}\beta_2\right) + \sum_{j=1}^n \lambda_j y_j (2x''_j + (x''_j)^2)$
over $\beta \in \mathbb{R}^2$ *and* $y \in \mathbb{R}^n$, *subject to*

$$
\begin{aligned}
y_1, y_2, \ldots, y_n &> 0, \\
\mathbf{M}\left(\alpha + \sigma\sigma^*\beta + \sum_{j=1}^n x_j y_j \lambda_j\right) &= 0, \\
\frac{1}{2}\beta^*\sigma\sigma^*\beta + \sum_{j=1}^n (y_j \log y_j - y_j + 1)\lambda_j &\le \hat{\eta}
\end{aligned}
$$

for some constant $\hat{\eta}$.

It seems unlikely that this problem can be solved explicitly, but at least one can resort to numerical methods. For other payoffs, analogous computations could be done.

Chapter 4

Preservation of the Lévy property under an optimal change of measure

4.1 Introduction

Lévy models are very popular in finance due to their tractability and their good fitting properties. However, Lévy models typically yield incomplete markets. This raises the question of which measure one should choose for valuation or pricing of untraded payoffs. Very often, a measure is chosen which minimizes a particular functional over the set $\mathcal{M}^e(S)$ of equivalent local martingale measures for the underlying assets S. This choice can be motivated by a dual formulation of a primal utility maximization problem. If P denotes the subjective measure, then the functional on $\mathcal{M}^e(S)$ is typically of the form $Q \mapsto E_P\big[f\big(\frac{dQ}{dP}\big)\big]$ where f is a convex function on $(0, \infty)$. Then $f(Q|P) := E_P\big[f\big(\frac{dQ}{dP}\big)\big]$, known as the *$f$-divergence* of Q with respect to P, is a measure for the distance between Q and P; see [LV87] for a textbook account. Hence one chooses as pricing measure the martingale measure which is closest to P with respect to some f-divergence.

In this chapter, we consider $f(Q|P)$ corresponding to

$$f^{\ell}(z) := -\log z, \quad f^{p}(z) := z^{-\delta} \text{ for } \delta \in (0, \infty) \quad \text{and} \quad f^{q}(z) := z^2; \qquad (4.1.1)$$

they are (strictly) convex. More precisely, we work on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration which is the P-augmentation of that generated by a d-dimensional Lévy process L. Instead of determining the basic assets S explicitly, we only assume that they are such that $\mathcal{M}^e(S) = \mathcal{M}^e(\mathbf{M}L)$ where $\mathbf{M}$ is a fixed $d \times d$-matrix; this allows several choices for S. Our main result then is that for a fixed

$f \in \{f^\ell, f^p, f^q\}$, the process L is still a Lévy process under the *f-minimal martingale measure* (for $\mathbf{M}L$) Q^f which minimizes $f(Q|P)$ over $\mathcal{M}^e(\mathbf{M}L)$. Put differently, we show that the Lévy property is preserved under a change to Q^f. This is a very pleasant result. Firstly, it simplifies the determination of Q^f significantly. Secondly, for values of payoffs calculated with respect to this measure one obtains relatively simple formulas and often even closed-form expressions.

We obtain our main result as follows. Since the filtration is generated by the P-Lévy process L, any probability measure $Q \approx P$ can be fully described via two parameters β and Y, called *Girsanov parameters* of Q with respect to L. The f-divergence $f(Q|P)$ is a convex functional of the Girsanov parameters of Q. Thus one can apply Jensen's inequality to show that $f(Q|P)$ can be reduced by averaging β and Y. More precisely, the new parameters obtained by averaging define a measure $\overline{Q}$ with $f(\overline{Q}|P) \leq f(Q|P)$. Since we are interested in the measure Q^f which minimizes $f(Q|P)$ over $Q \in \mathcal{M}^e(\mathbf{M}L)$, we need to ensure in addition that $\overline{Q}$ is a local martingale measure if Q is. This is not true in general, but it does hold if we take Q from a suitable subset of $\mathcal{M}^e(\mathbf{M}L)$, which is specified via an additional integrability condition for L. We then show that this subset is dense in $\mathcal{M}^e(\mathbf{M}L)$ in an appropriate sense, and this allows us to prove that Q^f has time-independent and deterministic Girsanov parameters. Because this characterizes those measures which preserve the Lévy property of L, our main result follows.

The chapter is structured as follows. In Section 4.2, we motivate our results and relate them to the existing literature. In Section 4.3 we fix some notation and recall some important facts about Lévy processes and changes of measure. In particular, we explain how equivalent measures can be described by their Girsanov parameters and give conditions for the latter to describe a measure in $\mathcal{M}^e(\mathbf{M}L)$; for convenience of the reader and to keep this chapter self-contained, we also recall results which were already presented in Section 3.4. Section 4.4 then contains the main results. In Subsection 4.4.1 we explicitly define the averaging procedure for the Girsanov parameters and show how it reduces the f-divergence. In the following Subsection 4.4.2 we specify a dense subset of $\mathcal{M}^e(\mathbf{M}L)$ consisting of measures for which the averaging procedure results in measures again contained in $\mathcal{M}^e(\mathbf{M}L)$. This is then exploited to prove our main result, i.e., that L is a Lévy process under Q^f. In Section 4.5 we restrict to $f(z) = f^q(z) = z^2$, where $f(Q|P)$ is basically the variance of the density dQ/dP. There is a direct connection between Q^q and the *variance-optimal signed martingale measure*. We show that if these two coincide, then one can show directly that Q^q preserves the Lévy property. This uses that in a Lévy setting the variance-optimal signed martingale measure agrees with the *minimal signed martingale measure* for which an explicit formula is known.

For better reading we omit long proofs from the main body of the chapter. They are collected in the appendix together with some auxiliary results.

4.2 Motivation and background

In this section we motivate the results contained in this chapter and connect them to those of Esche and Schweizer in [ES05], abbreviated ES in the sequel. Our approach is mainly inspired by ES.

The study of f-minimal martingale measures naturally arises in valuation in incomplete markets. One way to specify a value for a nontraded payoff is to fix a measure $Q \in \mathcal{M}^e(S)$ and to take the Q-expectation of the payoff. A common choice for Q is to take the martingale measure which is closest to the subjective measure P, or, more precisely, to take the measure which minimizes the f-divergence $f(Q|P)$ over $\mathcal{M}^e(S)$ for some convex function f. Typical choices are, e.g., $f(z) = z^p$ with $p \geq 1$ or $f(z) = z \log z$. The first choice results in minimizing the $\mathbf{L}^p$-norm of the density $\frac{dQ}{dP}$, the second in minimizing the relative entropy. The same pricing measures can also be obtained from an approach based on utility maximization. In fact, each (primal) utility maximization problem has a corresponding (dual) f-divergence minimization problem; see [KS99] and [Sch01a] for precise results. The f-divergences introduced in (4.1.1) are associated to the following utility functions $\mathbf{U}$. The function f^ℓ corresponds to $\mathbf{U}^\ell(x) := -\log x$ with $x \in (0, \infty)$, f^p to $\mathbf{U}^p(x) := \frac{\delta+1}{\delta} x^{\frac{\delta}{\delta+1}} = \frac{1}{\hat{\delta}} x^{\hat{\delta}}$ with $\hat{\delta} := \frac{\delta}{\delta+1} \in (0, 1)$ and $x \in (0, \infty)$ and, finally, f^q to $\mathbf{U}^q(x) = -(1-x)^2$ with $x \in I\!R$; see Chapter 3 for precise results. Loosely speaking, minimizing $f(Q|P)$ over $Q \in \mathcal{M}^e(S)$ is dual to maximizing expected $\mathbf{U}$-utility from terminal wealth, and f is essentially the Legendre transform of $\mathbf{U}$. We remark that the superscript ℓ stands for log, p for power and q for quadratic.

Instead of specifying a unique measure in $\mathcal{M}^e(S)$ and thus a unique value for any payoff, one can relax this approach to obtain a whole interval of possible values. For instance, one can allow all those measures in $\mathcal{M}^e(S)$ for pricing which do not yield *good deals*. The latter are investment opportunities which are too good in an appropriate sense, e.g., because they have a very high Sharpe ratio; see [CSR00] and [BS06]. Such good deals can also be defined via an upper bound on the maximal attainable utility (in some extended market); this is done, e.g., in [Cer03] and Chapter 3 of this thesis. In order to define a good deal one then has to calculate as benchmark the maximal utility attainable from trading in the basic assets, which again leads via duality theory to the problem of finding the f-minimal martingale measure Q^f. In a model where the filtration is generated by a multi-dimensional process L which is a Lévy process under the subjective measure P, it is then very convenient for interpretation as well as calculation purposes if Q^f preserves the Lévy property of L; see Chapter 3 for a precise statement. This has been the original motivation for this chapter.

In Chapter 3, we exploit the result of [ES05] that in a Lévy model, the minimal entropy martingale measure preserves the Lévy property. It turns out that this approach can actually be extended to the whole class of f-divergences defined with respect to the functions in (4.1.1). In contrast to ES, we consider only *equivalent* local martingale measures here to avoid complications with the definitions of $f^\ell(Q|P)$ and $f^p(Q|P)$.

We remark that for the purposes of Chapter 3, we need the f-minimal martingale measure not only at time $t = 0$, but also at any time t between 0 and the finite time horizon T, i.e., we seek at time t a measure Q which minimizes $E_P\big[f\big(Z_T^Q/Z_t^Q\big)\big|\mathcal{F}_t\big]$. However, Lemma 5.1.4 in the appendix shows that the measure which is f-minimal at time $t = 0$ is also optimal at any later time $t \leq T$.

4.3 Change of measure and Lévy processes

4.3.1 Notation and conventions

In this subsection we fix some notation and conventions, assumed to hold for the whole chapter; for unexplained notation we refer to [JS87], abbreviated JS in the sequel. We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions under P, with a finite time horizon $T < \infty$ and $\mathcal{F}_0$ trivial. A localizing sequence is a sequence of stopping times $\tau_n \leq T$ such that $\lim_{n\to\infty} P[\tau_n < T] = 0$. We denote by $E_Q[\,.\,]$ the expectation with respect to Q. For $Q = P$ we often drop the mention of Q and write $E[\,.\,]$. For any $Q \ll P$, its real-valued density process $Z^Q = (Z_t^Q)_{0\leq t\leq T}$ with $Z_t^Q := E[dQ/dP|\mathcal{F}_t]$ and its density $Z_T^Q := E[dQ/dP|\mathcal{F}_T]$ are defined with respect to P. If not mentioned otherwise, processes are assumed to be $\mathbb{R}^d$-valued and if it exists, we choose a right-continuous version. For a semimartingale X and a probability measure $Q \approx P$, we denote by μ^X the random measure associated with the jumps of X and by ν^Q (or $\nu^{X,Q}$ if confusion is possible) the predictable Q-compensator of μ^X. If μ is a random measure and W is sufficiently integrable, we denote by $W * \mu$ the integral process of W with respect to μ. Moreover, we work throughout with a fixed but arbitrary truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$; the results do not depend on the particular choice of h. By $\mathbf{P}$ we denote the predictable σ-field on $\Omega \times [0, T]$ and by (B, C, ν) the P-characteristics of the semimartingale X with respect to h. As in Proposition II.2.9 of JS, we can and do always choose a version of the form

$$B = \int b\,dA, \qquad C = \int c\,dA, \qquad \nu(\omega;\, dt, dx) = dA_t(\omega)\, K_{\omega,t}(dx), \quad (4.3.1)$$

where A is a real-valued predictable increasing locally integrable process, b an $\mathbb{R}^d$-valued predictable process, c a predictable process with values in the set of all symmetric nonnegative definite $d \times d$-matrices, and $K_{\omega,t}(dx)$ a transition kernel from $(\Omega \times [0, T], \mathbf{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ with $K_{\omega,t}(0) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge \|x\|^2)K_{\omega,t}(dx) \leq 1$ for all $t \leq T$. We denote by $\|\,.\,\|$ the usual norm on $\mathbb{R}^d$.

We now turn to the description of absolutely continuous probability measures. As mentioned in the introduction, any $Q \ll P$ can be described by two parameters β and Y. To introduce them, we recall Girsanov's theorem.

Proposition 4.3.1. (Theorem III.3.24 of JS) *Let X be a semimartingale with P-characteristics (B^P, C^P, ν^P) and denote by c, A the processes from (4.3.1). For any probability measure $Q \ll P$, there exist a $\mathbf{P} \otimes \mathcal{B}^d$-measurable function $Y \geq 0$ and a predictable $\mathbb{R}^d$-valued process β satisfying*

$$\|(Y-1)h\| * \nu_T^P + \int_0^T \|c_s\beta_s\| \, dA_s + \int_0^T \beta_s^* c_s \beta_s \, dA_s < \infty \quad Q\text{-a.s.}$$

and such that the Q-characteristics (B^Q, c^Q, ν^Q) of X are given by

$$\begin{aligned} B_t^Q &= B_t^P + \textstyle\int_0^t c_s\beta_s \, dA_s + ((Y-1)h) * \nu_t^P, \\ C_t^Q &= C_t^P, \\ \nu^Q(dt, dx) &= Y_t(x)\, \nu^P(dt, dx). \end{aligned}$$

We call β and Y the Girsanov parameters *of Q (with respect to P, relative to X) and write $Q = Q^{(\beta,Y)}$ to emphasize the dependence.*

Remark 4.3.2. a) In Proposition 4.3.1 we have $Y(x) > 0 \; dP \otimes dt$-a.e. for K-a.e. x if and only if $Q \approx P$.

b) Note that β and Y are not unique. For Y and β we identify all versions which are ν^P-a.e. respectively A-a.e. equal. Thus β and Y are called, e.g., time-independent, if there exists one such version. In addition, for fixed c and A only $c\beta$ is A-a.e. unique. To have β A-a.e. unique, we can and do define β as suggested in Remark NV in ES. When defining quantities depending on β and Y directly, we always fix one version.

◇

4.3.2 Lévy processes

We are concerned with models where the underlying filtration is generated by a Lévy process. Therefore, we recall some facts about Lévy processes.

Let $Q \approx P$ and $L = (L_t)_{0 \leq t \leq T}$ be an $\mathbb{F}$-adapted stochastic process with RCLL paths and $L_0 = 0$. Then L is called a $(Q, \mathbb{F})$-*Lévy process* if for all $s \leq t \leq T$, the increment $L_t - L_s$ is independent of $\mathcal{F}_s$ under Q and has a distribution which depends on $t - s$ only. If there is only a process L with $L_0 = 0$ and independent, stationary increments under Q, we call L a Q-*Lévy process*, take as $\mathbb{F}$ the Q-augmentation of the filtration generated by L and denote this by $\mathbb{F}^{L,Q}$. Recall that a Lévy process is a Feller process, so that $\mathbb{F}^{L,Q}$ automatically satisfies the usual conditions under Q. If $Q = P$, we even sometimes drop the mention of P, i.e., refer to L simply as a Lévy process and write $\mathbb{F}^L$. In particular, if $Q = P$ and $\mathbb{F} = \mathbb{F}^L$ for quantities depending on P and L we often do not write this dependence explicitly; this is done, e.g., for Girsanov parameters. We will use frequently that for $Q \approx P$, every $(Q, \mathbb{F})$-Lévy process is an $\mathbb{F}$-semimartingale and a $(Q, \mathbb{F})$-martingale if and only if it is a $(Q, \mathbb{F})$-local martingale; see [HWY92], Theorem 11.46.

Another important fact is that Lévy processes have the weak predictable representation property; see JS, Theorem III.4.34. That is, if $\mathbb{F} = \mathbb{F}^L$, then every local P-martingale (starting in zero) is the sum of an integral with respect to the continuous martingale part L^c and an integral with respect to the compensated jump measure. This allows to give an explicit formula for the density process of any probability measure $Q \approx P$ with Girsanov parameters β and Y. We quote this from from [ES05], abbreviated ES in the sequel.

Proposition 4.3.3. (Proposition 3 of ES) *Let L be a P-Lévy process and $\mathbb{F} = \mathbb{F}^L$. If $Q \approx P$ with Girsanov parameters β, Y, the density process of Q with respect to P is given by $Z^Q = \mathcal{E}(N^Q)$ with*

$$N_t^Q = \int_0^t \beta_s^* \, dL_s^c + (Y - 1) * (\mu^L - \nu^P)_t.$$

Remark 4.3.4. We frequently use that in the setting of Proposition 4.3.3, for a function $k : (-1, \infty) \to I\!R$ we have $\sum_{s \le t} k(\Delta N_s^Q) = k(Y - 1) * \mu_t^L$. ◇

In (4.3.1) we have introduced an integral version for the characteristics of a semimartingale. The main result of this chapter is based on the fact that the characteristics of a Lévy process have a very particular structure.

Lemma 4.3.5. (Corollary II.4.19 of JS) *Let $Q \approx P$ and L be an $(Q, \mathbb{F})$-semimartingale. Then L is a $(Q, \mathbb{F})$-Lévy process if and only if there exists a version of its Q-characteristics such that*

$$B_t^Q(\omega) = b^Q t, \quad C_t^Q(\omega) = c^Q t, \quad \nu^Q(\omega; \, dt, dx) = dt \, K^Q(dx) \tag{4.3.2}$$

where $b^Q \in I\!R^d$, c^Q is a symmetric non-negative definite $d \times d$-matrix, K^Q is a positive measure on $I\!R^d$ that integrates $(\|x\|^2 \wedge 1)$ and satisfies $K^Q(\{0\}) = 0$. We call (b^Q, c^Q, K^Q) the Lévy characteristics *of L (with respect to Q).*

Remark 4.3.6. For a P-Lévy process we drop the mention of P and denote the Lévy characteristics by (b, c, K). Moreover, without further mention we always use the notation

$$\nu^P(dx, dt) = K(dx)\, dt.$$

◇

As an immediate consequence of Girsanov's theorem and Lemma 4.3.5, for any $(P, \mathbb{F})$-Lévy process L, we can characterize the set of all probability measures $Q \approx P$ under which L is a $(Q, \mathbb{F})$-Lévy process. We denote this set by

$$\overline{\mathcal{Q}} := \overline{\mathcal{Q}}(L) := \{Q \approx P \mid L \text{ is a } (Q, \mathbb{F})\text{-Lévy process}\}.$$

Corollary 4.3.7. *Let L be an $(P, \mathbb{F})$-Lévy process and $Q \approx P$ with Girsanov parameters β and Y. Then L is a $(Q, \mathbb{F})$-Lévy process if and only if β and $Y(x)$ are $dP \otimes dt$-a.e. time-independent and deterministic for K-a.e. $x \in \mathbb{R}^d$.*

4.3.3 Change of measure

In the previous subsections, we have described for any $Q \approx P$ the corresponding Girsanov parameters. Now we want to start with arbitrary predictable processes β and Y and give conditions under which they define a probability measure $Q \approx P$ and can be identified as the Girsanov parameters of Q. Obviously, β and Y have to satisfy some integrability conditions. In view of our aim to minimize for $f \in \{f^\ell, f^p, f^q\}$ the f-divergence $f(Q|P)$ over some set of probability measures $Q \approx P$, we impose integrability conditions on β and Y which naturally arise in the computation of $f(Q|P)$; see Subsection 4.4.1 below. In order to formulate these conditions, we need to introduce the following functions g on $(0, \infty)$:

$$\begin{aligned} g^\ell(y) &:= -\log y + y - 1, \\ g^p(y) &:= y^{-\delta} - 1 + \delta(y-1) \quad \text{for } \delta \in (0, \infty), \\ g^q(y) &:= (y-1)^2. \end{aligned}$$

As shown in Lemma 5.1.2 below, each of these functions is strictly convex and nonnegative.

Remark 4.3.8. a) For $f = f^i$ with $i \in \{\ell, p, q\}$, we refer to g^i as the *corresponding* function g. Moreover, for $f^p(z) = z^{-\delta}$, the corresponding $g = g^p$ is defined with respect to the same $\delta \in (0, \infty)$.

b) ES consider the relative entropy, i.e., the $z \log z$-divergence. The corresponding function g there is $y \log y - (y-1)$ and is (unfortunately) denoted by f. However, in order to preserve the variable f for the f-divergence, we use the notation g here.

◇

Proposition 4.3.9. *Let L be a P-Lévy process with Lévy characteristics (b, c, K), $\mathbb{F} = \mathbb{F}^L$, $g \in \{g^\ell, g^p, g^q\}$, β a predictable process and $Y > 0$ a $\mathbf{P} \otimes \mathcal{B}^d$-measurable function. If*

$$\int_{\mathbb{R}^d} g(Y_s(x))\, K(dx) \le \text{const.} \qquad dP \otimes dt\text{-a.e.,} \tag{4.3.3}$$

then $Y - 1$ *is integrable with respect to* $\mu^L - \nu^P$. *If in addition*

$$\beta_s^* c \beta_s \leq \text{const.} \quad dP \otimes dt\text{-a.e.,} \tag{4.3.4}$$

then $Z := \mathcal{E}(N)$ *with*

$$N_t = \int_0^t \beta_s^* \, dL_s^c + (Y-1) * (\mu^L - \nu^P)_t \tag{4.3.5}$$

is a strictly positive P*-martingale.*

Proof. See appendix. □

Remark 4.3.10. a) Note that $\int \beta_s^* c \beta_s \, ds = \langle N^c \rangle$.

b) Proposition 4.3.9 is very similar to Proposition 5 in ES. However, their integrability condition on β and Y is of a weaker form. The proof then works because there the function g considered (and denoted by f) is $g^e(y) := y \log y - (y-1)$; g^e naturally arises when minimizing the relative entropy. More precisely, for integrability conditions with respect to g^e there exist rather general results which imply that $\mathcal{E}(N)$ from Proposition 4.3.9 is a (true) P-martingale; see Theorems III.1 and IV.3 of [LM78]. However, this is not (yet) the case for $g \in \{g^\ell, g^p, g^q\}$ considered here. On the other hand, in order to obtain the main results of this chapter, it would already be sufficient to prove Proposition 4.3.9 for time-independent and thus deterministic β and Y. From this point of view, Proposition 4.3.9 is even more general than required. But the conditions imposed on β and Y in Proposition 4.3.9 are exactly those which appear in connection with the computation of dynamic good deal bounds. Thus the generality chosen for Proposition 4.3.9 is exactly what is required to apply the results of this chapter to calculate good deal bounds; see Section 3.7 for such an application.

◇

The following result now allows to identify a priori given β and Y with the Girsanov parameters of the measure Q defined via $Z = \mathcal{E}(N)$, where N is constructed from β and Y via (4.3.5).

Proposition 4.3.11. (Proposition 7 of ES) *Let* L *be a* P*-Lévy process with Lévy characteristics* (b, c, K) *and* $\mathbb{F} = \mathbb{F}^L$. *Let* β *be a predictable process, integrable with respect to* L^c, *and* $Y > 0$ *a predictable function such that* $Y - 1$ *is integrable with respect to* $\mu^L - \nu^P$, *and define* $N := \int \beta_s^* \, dL_s^c + (Y-1) * (\mu^L - \nu^P)$. *If there exists a probability measure* $Q \approx P$ *with density process* $Z^Q = Z := \mathcal{E}(N)$, *then* β *and* Y *are the Girsanov parameters of* Q.

4.3.4 Martingale measures

We have seen that a probability measure $Q \ll P$ can be described via its Girsanov parameters. In particular, we haven given conditions under which two processes β and Y are the Girsanov parameters of some equivalent probability measure. For a fixed P-Lévy process L and a matrix $\mathbf{M} \in I\!R^{d\times d}$ we now look for conditions on the Girsanov parameters of $Q \approx P$ which ensure that $\mathbf{M}L$ is a local martingale under Q, i.e, that Q is a local martingale measure for $\mathbf{M}L$. We denote the set of all equivalent local martingale measures for a semimartingale S by

$$\mathcal{M}^e(S) := \{Q \approx P \mid S \text{ is a local martingale under } Q\}.$$

Remark 4.3.12. We consider equivalent local martingale measures for $\mathbf{M}L$ because this allows for several possibilities to model the price processes in a financial market model; see Section 1 of ES for a detailed discussion. ◇

Proposition 4.3.13. *Let L be a P-Lévy process with Lévy characteristics (b, c, K), $\mathbb{F} = \mathbb{F}^L$, $\mathbf{M}$ a $d \times d$-matrix and $Q \approx P$ with Girsanov parameters β and Y. Then $Q \in \mathcal{M}^e(\mathbf{M}L)$ if and only if we have both*

$$\|\mathbf{M}x - h(\mathbf{M}x)\| * \nu_T^{L,Q} < \infty \quad Q\text{-a.s.} \tag{4.3.6}$$

and

$$\mathbf{M}\left(b + c\beta_t + \int_{I\!R^d} (xY_t(x) - h(x))\, K(dx)\right) = 0 \quad dQ \otimes dt\text{-a.e.} \tag{4.3.7}$$

Condition (4.3.7) is called the martingale condition *for* $\mathbf{M}L$.

Proof. This is Proposition 10 of ES, except that we skip the proof of the equivalence of (4.3.6) with the integrability condition $\|\mathbf{M}(xY - h)\| * \nu_T^P < \infty$ there. Therefore we do not require their additional assumption on the integrability of Y. □

Property (4.3.6) is an integrability condition on the big jumps of $\mathbf{M}L$. As shown below in Subsection 4.4.2, if $Q^{(\beta,Y)} \approx P$ has finite f-divergence for some $f \in \{f^\ell, f^p, f^q\}$, then for the corresponding $g \in \{g^\ell, g^p, g^q\}$, $g(Y)$ has certain integrability properties. In that case, condition (4.3.6) is equivalent to $\|\mathbf{M}(xY - h)\| * \nu_T^P < \infty$ which is technically more convenient. This is a consequence of the following result.

Lemma 4.3.14. *With the notation of Proposition 4.3.13 and for $g \in \{g^\ell, g^p, g^q\}$*

$$\|\mathbf{M}x - h(\mathbf{M}x)\| * \nu_T^{L,Q} \le \|\mathbf{M}(xY - h)\| * \nu_T^P + \text{const.} \left(g(Y_T(x)) + (1 \wedge \|x\|^2)\right) * \nu_T^P(dx),$$

$$\|\mathbf{M}(xY - h)\| * \nu_T^P \le \|\mathbf{M}x - h(\mathbf{M}x)\| * \nu_T^{L,Q} + \text{const.} \left(g(Y_T(x)) + (1 \wedge \|x\|^2)\right) * \nu_T^P(dx).$$

Proof. See Lemma 5.1.10 d) from the appendix. □

4.4 Preservation of the Lévy property by an f-minimal change of measure

For all of Section 4.4, we fix a matrix $\mathbf{M} \in \mathbb{R}^{d\times d}$, a P-Lévy process L with Lévy characteristics (b, c, K) and assume that $\mathbb{F} = \mathbb{F}^L$. For $f \in \{f^\ell, f^p, f^q\}$ we call $Q^f \in \mathcal{M}^e(\mathbf{M}L)$ the *f-minimal martingale measure* if it minimizes the f-divergence over $\mathcal{M}^e(\mathbf{M}L)$, i.e., if $f(Q^f|P) \leq f(Q|P)$ for all $Q \in \mathcal{M}^e(\mathbf{M}L)$, and denote it by Q^ℓ, Q^p and Q^q, respectively. Note that since f is strictly convex, Q^f is unique if it exists. The aim is to show that Q^f is contained in $\overline{\mathcal{Q}} = \overline{\mathcal{Q}}(L)$, i.e., that Q^f preserves the Lévy property of L.

We proceed as follows. From $Q \in \mathcal{M}^e(\mathbf{M}L)$ with Girsanov parameters β and Y, we define a new pair of Girsanov parameters $\overline{\beta}$ and $\overline{Y}$ by averaging β and Y over t and ω. Then we show that for $\overline{Q} = Q^{(\overline{\beta},\overline{Y})}$ we have $f(\overline{Q}|P) \leq f(Q|P)$, i.e., the f-divergence is reduced by this averaging procedure. We refer to $\overline{Q}$ as obtained by "averaging Q". Since we want to minimize $f(Q|P)$ over $Q \in \mathcal{M}^e(\mathbf{M}L)$, we have to ensure that averaging preserves in addition the integrability condition (4.3.6) and the martingale condition (4.3.7). This requires some extra care.

4.4.1 Reducing the f-divergence by averaging

Averaging β and $Y(x)$ means that we consider them (for fixed x) as random variables on $\Omega \times [0, T]$ and take their expectations. Thus, we need to find a candidate for the measure which is used to define the expectation on $\Omega \times [0, T]$. This candidate will naturally arise from the formula we derive next for the f-divergence of Q in terms of its Girsanov parameters. We can restrict the averaging procedure to measures with finite f-divergence, i.e., which are contained in

$$\mathcal{Q}^f := \{Q \approx P \mid f(Z^Q) \text{ is a } P\text{-submartingale}\}.$$

Note that each $f \in \{f^\ell, f^p, f^q\}$ defines a different set $\mathcal{Q}^f$ denoted by $\mathcal{Q}^{f^\ell}$, $\mathcal{Q}^{f^p}$ and $\mathcal{Q}^{f^q}$, respectively, and that $Q \approx P$ is contained in $\mathcal{Q}^f$ if and only if $E\big[f(Z_T^Q)\big] < \infty$; see Corollary 5.1.3. In particular, for $i \in \{\ell, p, q\}$ we have for $\mathcal{Q}^i$ from Chapter 3 that

$$\mathcal{Q}^i = \mathcal{Q}^{f^i} \cap \mathcal{M}^e(\mathbf{M}L).$$

We state the results separately for each $f \in \{f^\ell, f^p, f^q\}$.

Proposition 4.4.1. *Let $Q = Q^{(\beta,Y)} \in \mathcal{Q}^{f^\ell}$ with density process $Z = Z^Q = \mathcal{E}(N)$. The canonical P-decomposition of $f^\ell(Z) = -\log Z = M^\ell + A^\ell$ is*

$$\begin{aligned}
M^\ell &= -N + g^\ell(Y) * (\mu^L - \nu^P) = -N^c - \log(Y) * (\mu^L - \nu^P),\\
A^\ell &= \frac{1}{2}\langle N^c\rangle + g^\ell(Y) * \nu^P =: \tilde{A}^\ell + \tilde{\tilde{A}}^\ell,
\end{aligned}$$

where $g^\ell(y) = -\log y + y - 1$, M^ℓ *is a uniformly integrable* P*-martingale and* $\tilde{A}^\ell$ *and* $\tilde{\tilde{A}}^\ell$ *are predictable, increasing and* P*-integrable. Thus,*

$$f^\ell(Q|P) = E\left[f^\ell(Z_T)\right] = E[A^\ell_T] = E\left[\frac{1}{2}\int_0^T \beta_s^* c\beta_s\, ds + g^\ell(Y) * \nu_T^P\right]. \quad (4.4.1)$$

Proof. See appendix. □

Proposition 4.4.2. *Let* $Q = Q^{(\beta,Y)} \in \mathcal{Q}^{f^p}$ *with density process* $Z = Z^Q = \mathcal{E}(N)$. *The canonical* P*-decomposition of* $f^p(Z) = Z^{-\delta} = M^p + A^p$ *is*

$$\begin{aligned} M^p &= \textstyle\int Z_-^{-\delta}\, d\hat{M}^p = \int f^p(Z_-)\, d\hat{M}^p \\ \textit{with } \hat{M}^p &:= -\delta\, N + g^p(Y) * (\mu^L - \nu^P) = -\delta\, N^c + (Y^{-\delta} - 1) * (\mu^L - \nu^P) \\ \textit{and } \quad A^p &= \textstyle\int Z_-^{-\delta}\, d\hat{A}^p = \int f^p(Z_-)\, d\hat{A}^p \\ \textit{with } \hat{A}^p &:= \tfrac{\delta(\delta+1)}{2}\langle N^c\rangle + g^p(Y) * \nu^P, \end{aligned}$$

where $g^p(y) = y^{-\delta} - 1 + \delta(y-1)$. *In particular,* $f^p(Z) = \mathcal{E}(\hat{M}^p)\mathcal{E}(\hat{A}^p)$ *where* $\mathcal{E}(\hat{M}^p)$ *is a strictly positive uniformly integrable* P*-martingale. With* $\frac{dR^p}{dP} := \mathcal{E}(\hat{M}^p)_T$ *the process* $\mathcal{E}(\hat{A}^p) = e^{\hat{A}^p}$ *is increasing and* R^p*-integrable and*

$$\begin{aligned} f^p(Q|P) &= E\left[f^p(Z_T)\right] = E_{R^p}\left[\mathcal{E}(\hat{A}^p)_T\right] \\ &= E_{R^p}\left[\exp\left(\int_0^T\left(\frac{\delta(\delta+1)}{2}\beta_t^* c\beta_t + \int_{\mathbb{R}^d} g^p(Y_t(x))\, K(dx)\right)dt\right)\right]. \end{aligned} \quad (4.4.2)$$

Proof. See appendix. □

For $f^q(z) = z^2$ the results are (formally) exactly those of Proposition 4.4.2 for $\delta = -2$. However, to introduce the notation and because of the importance of the case f^q, we decided to state the result for f^q separately; see Section 4.5 below.

Proposition 4.4.3. *Let* $Q = Q^{(\beta,Y)} \in \mathcal{Q}^{f^q}$ *with density process* $Z = Z^Q = \mathcal{E}(N)$. *With* $g^q(y) = (y-1)^2$, *the canonical* P*-decomposition of* $f^q(Z) = Z^2 = M^q + A^q$ *is*

$$\begin{aligned} M^q &= \textstyle\int Z_-^2\, d\hat{M}^q = \int f^q(Z_-)\, d\hat{M}^q \\ \textit{with } \hat{M}^q &:= 2\, N + g^q(Y) * (\mu^L - \nu^P) = 2\, N^c + (Y^2 - 1) * (\mu^L - \nu^P) \\ \textit{and } \quad A^q &= \textstyle\int Z_-^2\, d\hat{A}^q = \int f^q(Z_-)\, d\hat{A}^q \\ \textit{with } \hat{A}^p &:= \langle N^c\rangle + g^q(Y) * \nu^P. \end{aligned}$$

In particular, $f^q(Z) = \mathcal{E}(\hat{M}^q)\mathcal{E}(\hat{A}^q)$ *where* $\mathcal{E}(\hat{M}^q)$ *is a strictly positive uniformly integrable* P*-martingale. With* $\frac{dR^q}{dP} := \mathcal{E}(\hat{M}^q)_T$ *the process* $\mathcal{E}(\hat{A}^q) = e^{\hat{A}^q}$ *is increasing and* R^q*-integrable and*

$$\begin{aligned} f^q(Q|P) &= E\left[f^q(Z_T)\right] = E_{R^q}\left[\mathcal{E}(\hat{A}^q)_T\right] \\ &= E_{R^q}\left[\exp\left(\int_0^T \left(\beta_t^* c\beta_t + \int_{\mathbb{R}^d} g^q(Y_t(x))\, K(dx)\right) dt\right)\right]. \end{aligned} \tag{4.4.3}$$

Proof. See appendix. □

In Subsection 4.3.3 we have associated to each function $f \in \{f^\ell, f^p, f^q\}$ a corresponding $g \in \{g^\ell, g^p, g^q\}$; (the proofs of) Propositions 4.4.1, 4.4.2 and 4.4.3 show where the definitions for g^ℓ, g^p and g^q come from. From now on, we associate in addition to each f and each $Q = Q^{(\beta,Y)}$ a probability measure $R = R(f;\beta,Y)$. For $f \in \{f^p, f^q\}$, the corresponding $R(f^p;\beta,Y) := R^p$ and $R(f^q;\beta,Y) := R^q$ are defined in Propositions 4.4.2 and 4.4.3. For $f = f^\ell$ we define $R(f^\ell;\beta,Y) := P$; this is actually independent of β, Y. The introduction of R^ℓ allows us to treat the different f-divergences simultaneously and highlights the analogy in their treatment. The measures $R \in \{R^\ell, R^p, R^q\}$ or, more precisely, the product measures $R \otimes \lambda$ are the candidate measures for the averaging procedure to reduce the f-divergence. The next lemma provides us with some important integrability properties, with respect to $R = R(f;\beta,Y)$, of functionals depending on the Girsanov parameters β and Y.

Remark 4.4.4. For $f(z) = z \log z$ as considered in ES, the candidate measure is $Q \otimes \lambda$. This can be seen from Lemma 12 of ES. ◇

Lemma 4.4.5. *Let* $f \in \{f^\ell, f^p, f^q\}$, $Q = Q^{(\beta,Y)} \in \mathcal{Q}^f$ *with correspondinge* $g \in \{g^\ell, g^p, g^q\}$ *and* $R = R(f;\beta,Y)$. *The following random variables are* R*-integrable:*

a) $\int_0^T \beta_t^* c\beta_t\, dt$,

b) $\int_0^T \|\beta_t\|\, dt$,

c) $g(Y) * \nu_T^P$,

d) $\int_0^T Y_t(x)\, dt$ *for* K*-a.e.* $x \in \mathbb{R}^d$.

Proof. Parts a) and c) follow from Propositions 4.4.1, 4.4.2 and 4.4.3; note that $g \geq 0$, $\beta^* c\beta \geq 0$ and $e^x \geq x$. Finally, b) and d) can be deduced from a), c) and Lemma 5.1.1 c) analogously to Lemma 12 in ES with Q-expectations and f there replaced by R-expectations and g. □

The following theorem shows that for $f \in \{f^\ell, f^p, f^q\}$, the quantities $\overline{\beta}$ and $\overline{Y}(x)$ obtained from $Q = Q^{(\beta,Y)} \in \mathcal{Q}^f$ via averaging with respect to $R(f; \beta, Y) \otimes \lambda$ define some $\overline{Q} = Q^{(\overline{\beta},\overline{Y})}$ with $f(\overline{Q}|P) \le f(Q|P)$. Since $\overline{\beta}$ and $\overline{Y}$ are time-independent and thus also deterministic, $\overline{Q}$ is contained in $\overline{\mathcal{Q}}(L)$ by Corollary 4.3.7, i.e., preserves the Lévy property of L.

Theorem 4.4.6. *Let $f \in \{f^\ell, f^p, f^q\}$, $Q = Q^{(\beta,Y)} \in \mathcal{Q}^f$ with corresponding measure $R = R(f; \beta, Y)$, and define $\overline{\beta} \in \mathbb{R}^d$ and a measurable function $\overline{Y}$ from $\mathbb{R}^d$ into $(0, \infty)$ by*

$$\begin{aligned}
\overline{\beta} &:= E_R\left[\frac{1}{T}\int_0^T \beta_t\, dt\right], \\
\overline{Y}(x) &:= E_R\left[\frac{1}{T}\int_0^T Y_t(x)\, dt\right] \quad \textit{if this is finite (which holds } K\textit{-a.e.),} \\
\overline{Y}(x) &:= 1 \quad \textit{otherwise.}
\end{aligned}$$

Then there exists $\overline{Q} \approx P$ with Girsanov parameters $\overline{\beta}$, $\overline{Y}$ such that $\overline{Q} \in \mathcal{Q}^f \cap \overline{\mathcal{Q}}$ and

$$f(\overline{Q}|P) \le f(Q|P),$$

with equality iff $dP \otimes dt$-a.e. both $\beta = \overline{\beta}$ and $Y(x) = \overline{Y}(x)$ for K-a.e. x, i.e., iff $Q \in \overline{\mathcal{Q}}$.

Proof. See appendix. □

Remark 4.4.7. The measure $\overline{Q}$ does not depend on the version of β and Y. ◇

4.4.2 Preservation of the martingale property

In the previous subsection we have shown how for $f \in \{f^\ell, f^p, f^q\}$, the f-divergence $f(Q|P)$ of some $Q^{(\beta,Y)} \in \mathcal{Q}^f$ can be reduced by averaging the Girsanov parameters β and Y. Since the obtained $\overline{\beta}$ and $\overline{Y}$ are time-independent, the corresponding measure $\overline{Q} = Q^{(\overline{\beta},\overline{Y})}$ is in $\overline{\mathcal{Q}}(L)$ and hence preserves the Lévy property of L. If in addition the original Q is in $\mathcal{M}^e(\mathbf{M}L)$ we should like this to hold for $\overline{Q}$ as well; in other words, we hope that also (4.3.6) and the martingale condition (4.3.7) are preserved under averaging. Since β and Y enter (4.3.7) linearly, we naturally expect this to be true. However, to actually prove this, we need to apply Fubini's theorem, which requires an additional integrability condition. This condition, which will also be used to show that (4.3.6) is preserved under averaging, holds for all Q contained in

$$\mathcal{Q}^f_{\text{int}} := \left\{ Q^{(\beta,Y)} \in \mathcal{Q}^f \,\middle|\, E_R\left[\|\mathbf{M}(xY - h)\| * \nu^P_T\right] < \infty \ \text{ for } R = R(f; \beta, Y) \right\}.$$

Therefore we prove that for any $Q \in \mathcal{Q}^f$ there exists a sequence $(Q^n)_{n\in\mathbb{N}} \subseteq \mathcal{Q}^f_{\text{int}}$ with $\lim_{n\to\infty} f(Q^n|P) = f(Q|P)$. Moreover, we show that for $\overline{Q}$ and $\overline{Q}^n$ obtained from averaging Q and Q^n, we also have $\lim_{n\to\infty} f\left(\overline{Q}^n|P\right) = f\left(\overline{Q}|P\right)$. These results then imply that the f-minimal martingale measure Q^f preserves the Lévy property of L and that it suffices to look for Q^f in the set $\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f \cap \overline{\mathcal{Q}}$. Before we show that measures from $\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f_{\text{int}}$ preserve the Lévy property under averaging, we remark that

$$\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f \cap \overline{\mathcal{Q}} \subseteq \mathcal{Q}^f_{\text{int}}.$$

In fact, Lemma 4.3.14 yields

$$\|\mathbf{M}(xY-h)\|*\nu_T^P \leq \|\mathbf{M}x-h(\mathbf{M}x)\|*\nu_T^{L,Q}+\text{const.}\left(g(Y_T(x)) + (1 \wedge \|x\|^2)\right)*\nu_T^P(dx).$$

Moreover, if $Q \in \mathcal{M}^e(\mathbf{M}L)$, then the first summand on the RHS is finite by Proposition 4.3.13 and if $Q \in \mathcal{Q}^f$ so is the second by Lemma 4.4.5 c). Finally, if $Q \in \overline{\mathcal{Q}}$, then Y is deterministic, hence so is $\|\mathbf{M}(xY - h)\| * \nu_T^P$, and then finiteness is the same as R-integrability.

Proposition 4.4.8. *Let $f \in \{f^\ell, f^p, f^q\}$ and $Q = Q^{(\beta,Y)} \in \mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f_{\text{int}}$. Then $\overline{Q}$ from Theorem 4.4.6 is in $\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f \cap \overline{\mathcal{Q}}$ so that* $\mathbf{M}L$ *is a local $\overline{Q}$-martingale.*

Proof. See appendix. □

We next show that $\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f_{\text{int}}$ is dense in $\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f$ in a suitable sense.

Proposition 4.4.9. *Let $Q \in \mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f$ for $f \in \{f^\ell, f^p, f^q\}$ and suppose that $\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f \cap \overline{\mathcal{Q}} \neq \emptyset$. Then there exists a sequence $(Q^n)_{n\in\mathbb{N}}$ in $\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f_{\text{int}}$ with*

$$\lim_{n\to\infty} f(Q^n|P) = f(Q|P).$$

Proof. See appendix. □

In addition, the f-divergences of the corresponding averaged measures converge as well.

Proposition 4.4.10. *In the setting of Proposition 4.4.9 denote by $\overline{Q}$ and $\overline{Q}^n$ the corresponding averaged measures as defined in Theorem 4.4.6. Then*

$$\lim_{n\to\infty} f\left(\overline{Q}^n|P\right) = f\left(\overline{Q}|P\right).$$

Proof. See appendix. □

Putting all this together we obtain our main result.

Theorem 4.4.11. *Let $f \in \{f^\ell, f^p, f^q\}$ and suppose that $\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f \cap \overline{\mathcal{Q}} \neq \emptyset$. If there exists $Q^f \in \mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f$ such that*

$$f(Q^f|P) \leq f(Q|P) \quad \text{for all } Q \in \mathcal{M}^e(\mathbf{M}L),$$

then $Q^f \in \overline{\mathcal{Q}}$, i.e., L is a Lévy process under Q^f.

Proof. Suppose $Q^f \notin \overline{\mathcal{Q}}$. By Theorem 4.4.6 we can obtain from averaging Q^f some $\overline{Q}^f \in \overline{\mathcal{Q}} \cap \mathcal{Q}^f$ such $f(\overline{Q}^f|P) < f(Q^f|P)$. This is not yet a contradiction, since $\overline{Q}^f$ need not be contained in $\mathcal{M}^e(\mathbf{M}L)$. However, Proposition 4.4.9 ensures the existence of a sequence $(Q^n)_{n \in \mathbb{N}} \subseteq \mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f_{\text{int}}$ such that $\lim_{n\to\infty} f(Q^n|P) = f(Q^f|P)$. In addition, Proposition 4.4.10 implies that also the f-divergences of the measures $\overline{Q}^n$ constructed from Q^n as in Theorem 4.4.6 satisfy

$$\lim_{n\to\infty} f(\overline{Q}^n|P) = f(\overline{Q}^f|P) < f(Q^f|P).$$

Thus there exists $n \in \mathbb{N}$ such that $f(\overline{Q}^n|P) < f(Q^f|P)$ and

$$\overline{Q}^n \in \mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f \cap \overline{\mathcal{Q}}$$

by Proposition 4.4.8. This yields a contradiction. Hence $Q^f \in \overline{\mathcal{Q}}$. □

Remark 4.4.12. We assumed in Theorem 4.4.11 the existence of the f-minimal martingale measure Q^f. This is a non-trivial assumption if we do not impose that $\mathbf{M}L$ is locally bounded, i.e., that it has bounded jumps; see Theorem 1.1 in Bellini/Frittelli [BF02]. ◇

Theorem 4.4.11 suggests that it is enough to look for the f-minimal martingale measure Q^f in $\overline{\mathcal{Q}}$. The following corollary show that this is indeed true.

Corollary 4.4.13. *Let $f \in \{f^\ell, f^p, f^q\}$. If there exists $Q' \in \mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f \cap \overline{\mathcal{Q}}$ such that $f(Q'|P) \leq f(Q|P)$ for all $Q \in \mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f \cap \overline{\mathcal{Q}}$, then we also have that $f(Q'|P) \leq f(Q|P)$ for all $Q \in \mathcal{M}^e(\mathbf{M}L)$, i.e., $Q' = Q^f$ is the f-minimal martingale measure.*

Proof. Suppose there exists $\tilde{Q} \in (\mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f) \setminus \overline{\mathcal{Q}}$ such that $f(\tilde{Q}|P) < f(Q'|P)$. By Theorem 4.4.6 we can average $\tilde{Q}$ to obtain some $\overline{Q} \in \mathcal{Q}^f \cap \overline{\mathcal{Q}}$ such that

$$f(\overline{Q}|P) < f(\tilde{Q}|P) < f(Q'|P).$$

As in the proof of Theorem 4.4.11 one then applies Propositions 4.4.9 and 4.4.10 to obtain a contradiction. □

4.5 Connections to the variance-optimal measure

In this section we relate the f^q-minimal martingale measure Q^q to the *variance-optimal signed martingale measure* $\tilde{P}$. An intensively studied pricing and hedging approach in incomplete markets is *mean-variance hedging*; see [Sch01] for an overview and terminology not explained here. In that approach, the value of a payoff is defined as the initial capital of the strategy which minimizes the $\mathbf{L}^2$-norm of the hedging error over all self-financing $\mathbf{L}^2$-strategies. This value is equal to the expectation of the payoff under $\tilde{P}$, which minimizes the P-variance of the density $\frac{dQ}{dP}$ (or, equivalently, $f^q(Q|P)$) over all *signed* local martingale measures Q for $\mathbf{M}L$. If $\tilde{P}$ is a probability measure *equivalent* to P, then it coincides with the f^q-minimal martingale measure Q^q which we have studied in the previous sections. In a Lévy setting, $\tilde{P}$ is (under some mild additional assumptions) equal to the *minimal signed martingale measure* $\hat{P}$ which occurs in the local risk minimizing hedging approach. Since there is an explicit formula for the density of $\hat{P}$, it is then very easy to show that $\tilde{P} = \hat{P}$ preserves the Lévy property of L. Thus in the special case when Q^q is equal to $\tilde{P}$, i.e., if the latter is an equivalent probability measure, one can show almost directly that $Q^q = \tilde{P} = \hat{P}$ preserves the Lévy property. We explain this in more detail in this section.

Let L be a P-Lévy process with Lévy characteristics (b, c, K), $\mathbb{F} = \mathbb{F}^L$ and $\mathbf{M} \in I\!R^{d\times d}$ a fixed matrix such that $\mathbf{M}L$ is a special semimartingale. By Corollary II.2.38 and Proposition II.2.29 of JS we have

$$\begin{aligned}
\mathbf{M}L_t &= \Big(\mathbf{M}L_t^c + \mathbf{M}x * (\mu^L - \nu^P)_t\Big) + \Big(\mathbf{M}bt + \big(\mathbf{M}x - h(\mathbf{M}x)\big) * \nu_t^P\Big) \\
&= \Big(\mathbf{M}L_t^c + \mathbf{M}x * (\mu^L - \nu^P)_t\Big) + \Big(\mathbf{M}b + \int_{I\!R^d} \big(\mathbf{M}x - h(\mathbf{M}x)\big)\, K(dx)\Big)t \\
&=: M_t + \gamma t =: M_t + A_t .
\end{aligned}$$

Let the local martingale M be locally square integrable (and thus square integrable) with respect to P. Then we have

$$\begin{aligned}
\langle M\rangle_t &= \mathbf{M}c\mathbf{M}^* t + (\mathbf{M}xx^*\mathbf{M}^*) * \nu_t^P \\
&= \mathbf{M}\Big(c + \int_{I\!R^d} xx^*\, K(dx)\Big)\mathbf{M}^*\, t \\
&=: \sigma t .
\end{aligned}$$

We assume in addition that $\mathbf{M}L$ satisfies the structure condition **(SC)**, i.e., that there exists a d-dimensional predictable process $\hat{\lambda}$ such that

$$A = \int d\langle M\rangle \hat{\lambda} \quad \text{and} \quad \hat{K}_T := \int \hat{\lambda}^*\, d\langle M\rangle \hat{\lambda} < \infty\,; \tag{4.5.1}$$

see Definition 1.1 in [CS96] and Subsection 12.3 in [DS06] for a related discussion.

Since $A_t = \gamma t$ and $\langle M \rangle_t = \sigma t$, (4.5.1) is satisfied if and only if $\hat{\lambda}$ satisfies

$$\gamma = \sigma \hat{\lambda},$$

i.e., if $\gamma \in \text{range}(\sigma)$. In particular, we then can and do choose $\hat{\lambda}$ constant. We can now define

$$\hat{N} := -\int \hat{\lambda}^* \, dM.$$

If $\hat{Z} := \mathcal{E}(\hat{N})$ is a P-martingale, then $\frac{d\hat{P}}{dP} := \hat{Z}_T$ defines a signed measure called the *minimal signed martingale measure* for $\mathbf{M}L$. By Proposition 2 of [Sch95], it is a local martingale measure for $\mathbf{M}L$ in the sense that $\hat{Z}\mathbf{M}L$ is a local P-martingale. Note that if $\hat{Z} > 0$ (i.e., if $-\hat{\lambda}^* \Delta M > -1$), then $\hat{Z} = \mathcal{E}(\hat{N})$ is a local martingale and as in the proof of Proposition 4.3.9 an application of Theorem II.5 of [LM78] yields that it is automatically a P-martingale. In particular, it is then in $\mathcal{M}^e(\mathbf{M}L)$.

Under the above assumptions, the *mean-variance tradeoff process*

$$\hat{K}_t := \int_0^t \hat{\lambda}^* \, dA_s = \left\langle \int \hat{\lambda}^* \, dM \right\rangle_t = \hat{\lambda}^* \sigma \hat{\lambda} \, t$$

is deterministic. This implies by Theorem 8 of [Sch95] that $\hat{P}$ is equal to the *variance-optimal signed martingale measure* $\tilde{P}$. If we denote the density of $\tilde{P}$ by $\tilde{Z}_T$, then $\tilde{P}$ is defined by the property that

$$E[f^q(\tilde{Z}_T)] \leq E[f^q(Z_T)]$$

for all P-martingales Z with $Z_0 = 1$ such that $Z\mathbf{M}L$ is a local P-martingale; the corresponding measures Q with $\frac{dQ}{dP} = Z_T$ are called *signed local martingale measures* for $\mathbf{M}L$. Hence, if $\hat{Z} = \tilde{Z} > 0$ so that $\hat{P} \in \mathcal{M}^e(\mathbf{M}L)$, then $\hat{P}$ coincides with the f^q-minimal martingale measure Q^q.

Since we have an explicit formula for $\hat{P}$, it is very easy to check that it preserves the Lévy property of L. Indeed, by Corollary 4.3.7 we only need to identify the Girsanov parameters of $\hat{P}$ and show that they are time-independent and deterministic. But

$$\hat{N} \quad = \quad -\hat{\lambda}^* M = -\hat{\lambda}^* \mathbf{M} L^c - (\hat{\lambda}^* \mathbf{M} x) * (\mu^L - \nu^P)$$

so that by Proposition 4.3.11 the Girsanov parameters of $\hat{P}$ are

$$\beta := -\mathbf{M}^* \hat{\lambda} \quad \text{and} \quad Y(x) = -\hat{\lambda}^* \mathbf{M} x + 1$$

which are obviously time-independent and deterministic.

In conclusion, if $\mathbf{M}L$ is sufficiently integrable and $\gamma \in \text{range}(\sigma)$, then $\hat{P} = \tilde{P}$, and if this is an equivalent probability measure, then $Q^q = \hat{P}$, i.e., the f^q-minimal martingale measure is just the minimal martingale measure and preserves the Lévy property of L.

Chapter 5

Appendix

In Section 5.1 we collect a number of auxiliary results for Chapter 4, and in Section 5.2 we give the proofs omitted from the main body of Chapter 4.

5.1 Auxiliary results

Lemma 5.1.1. *Let* $t \leq T$, $k : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *a convex function and* Y *a random variable such that* $E[|k(Y)|] < \infty$.

a) If $E[|Y|] < \infty$ *then*

$$k\left(E[Y|\mathcal{F}_t]\right) \leq E\left[k(Y)|\,\mathcal{F}_t\right]. \tag{5.1.1}$$

b) If k *is strictly convex on* $\mathrm{dom}(k) := \{x \in \mathbb{R} \,|\, k(x) < \infty\}$, $E[|Y|] < \infty$ *and* $t = 0$, *then equality holds in (5.1.1) if and only if* $Y = E[Y]$ P*-a.s.*

c) If Y *is bounded from below,* $\lim_{x\to\infty} k(x) = +\infty$ *and if there exists an affine function* $\ell(x) := mx + b$ *with* $b \in \mathbb{R}$ *and* $m > 0$ *such that* $k(x) \geq \ell(x)$ *on* $\mathbb{R}$, *then* $E[|Y|] < \infty$.

Proof. Part a) is just the conditional Jensen inequality and part b) can be shown like Lemma C.6 in [Esc04]. For c), note that since Y is bounded from below, $E[Y]$ is well defined (possibly $+\infty$) and it suffices to show that $E[Y] < \infty$. Suppose $E[Y] = \infty$. Then

$$\infty = \ell\left(E[Y]\right) = E[\ell(Y)] \leq E[k(Y)] < \infty$$

which is a contradiction. Thus $E[Y] < \infty$. □

Lemma 5.1.2. *The following functions are all strictly convex on* $(0,\infty)$*:*

$$\begin{aligned} f^\ell(z) &:= -\log z \quad \textit{and} \quad g^\ell(y) := -\log y + y - 1, \\ f^p(z) &:= z^{-\delta} \quad \textit{and} \quad g^p(y) := y^{-\delta} - 1 + \delta(y-1) \quad \text{for } \delta \in (0,\infty), \\ f^q(z) &:= z^2 \quad \textit{and} \quad g^q(y) := (y-1)^2. \end{aligned}$$

In addition, each $g \in \{g^\ell, g^p, g^q\}$ *is nonnegative and attains its unique minimum in* $y = 1$ *where* $g(y) = 0$.

Proof. Strict convexity is obvious. Hence it suffices to show that each $g \in \{g^\ell, g^p, g^q\}$ has its derivative zero in $y = 1$ where $g(1) = 0$. So we compute

$$\frac{d}{dy} g^\ell(y) = -\frac{1}{y} + 1, \quad \frac{d}{dy} g^p(y) = -\frac{\delta}{y^{\delta+1}} + \delta \quad \text{and} \quad \frac{d}{dy} g^q(y) = 2(y-1).$$

□

The following result is an immediate consequence of Lemmas 5.1.1 and 5.1.2.

Corollary 5.1.3. *Let* $f \in \{f^\ell, f^p, f^q\}$ *and let* Z *be a strictly positive martingale. If* $E[f(Z_T)] < \infty$*, then* $f(Z)$ *is a submartingale.*

Proof. For f^p and f^q this is obvious, since they are bounded from below. For f^ℓ denote by $\ell(x) := mx + b$ some linear function with $m < 0$ and $f^\ell(x) \geq \ell(x)$ for all $x > 0$. Similarly to Lemma 5.1.1 c) one can then prove that $E[f^\ell(Z_T)] > -\infty$. □

Lemma 5.1.4. *Let* $f \in \{f^\ell, f^p, f^q\}$ *and let* $\mathcal{D}$ *be a set of strictly positive density processes such that for all* $t \leq T$ *and* $Z^1, Z^2 \in \mathcal{D}$*, also* $\tilde{Z}_. := Z^1_{.\wedge t} \frac{Z^2_{.\wedge T}}{Z^2_{.\wedge t}}$ *defines an element of* $\mathcal{D}$*. If* $Z^0 \in \mathcal{D}$ *satisfies* $E[f(Z^0_T)] \leq E[f(Z_T)]$ *for all* $Z \in \mathcal{D}$*, then also*

$$E\left[f\left(\frac{Z^0_T}{Z^0_t}\right) \middle| \mathcal{F}_t \right] \leq E\left[f\left(\frac{Z_T}{Z_t}\right) \middle| \mathcal{F}_t \right] \quad \text{for all } Z \in \mathcal{D} \text{ and for each } t \in [0,T].$$

Remark 5.1.5. This pasting property is in particular satisfied for the set of equivalent local martingale measures for any semimartingale; see Proposition 5 in [Del06] and note that we do not require there that the semimartingale is locally bounded. ◇

Proof of Lemma 5.1.4. For $f \in \{f^p, f^q\}$, this is Proposition 4.1 in [KS02], and for f^ℓ the result can be shown analogously. Indeed, suppose there exists $Z^1 \in \mathcal{D}$ such that $E\left[f^\ell(Z_T^1/Z_t^1)\middle|\mathcal{F}_t\right] \le E\left[f^\ell(Z_T^0/Z_t^0)\middle|\mathcal{F}_t\right]$ and such that "<" holds with positive probability. Define $\tilde{Z}_. = Z^0_{.\wedge t}\frac{Z^1_{.\wedge T}}{Z^1_{.\wedge t}} \in \mathcal{D}$. Then

$$E\left[-\log \tilde{Z}_T\right] = E\left[-\log Z_t^0 + E\left[-\log \frac{Z_T^1}{Z_t^1}\middle| \mathcal{F}_t\right]\right]$$
$$< \quad E\left[-\log Z_t^0 + E\left[-\log \frac{Z_T^0}{Z_t^0}\middle| \mathcal{F}_t\right]\right] = E\left[-\log Z_T^0\right],$$

which contradicts the optimality of Z^0 at time $t = 0$. □

Lemma 5.1.6. *For $g \in \{g^\ell, g^p\}$ we have*

$$\left(1-\sqrt{y}\right)^2 \le \text{const.}\, g(y) \qquad \text{for all } \; y > 0.$$

Proof. For $g^p(y) = y^{-\delta} - 1 + \delta(y-1)$ we can take const. $= \frac{1}{\delta}$. In fact, define for $y > 0$ the function $k^p(y) := \frac{g^p(y)}{\delta} - (1-\sqrt{y})^2$. Since $\frac{d}{dy}k^p(y) = -\frac{1}{y^{\delta+1}} + \frac{1}{\sqrt{y}} < 0$ for $0 < y < 1$ and $\frac{d}{dy}k^p(y) > 0$ for $y > 1$, k^p attains its unique minimum in $y = 1$. Since $k^p(1) = 0$, this proves the claim. Analogously, for g^ℓ we can take const. $= 1$. Indeed, let $k^\ell(y) := g^\ell(y) - (1-\sqrt{y})^2$ for $y > 0$. Then $\frac{d}{dy}k^\ell(y) = -\frac{1}{y} + \frac{1}{\sqrt{y}}$ and the same arguments hold true. □

Lemma 5.1.7. *Let $g \in \{g^\ell, g^p, g^q\}$ and fix $\overline{y} \in (1,\infty)$. There exists a constant $C = C(\overline{y}) > 0$ such that for all $c \ge C$*

$$(y-1)^2 \le c g(y) \qquad \text{for all } \; y \in (0,\overline{y}].$$

Proof. For g^q this is trivial. We first show that for g^ℓ we can take $C = C^\ell := 2\overline{y}^2$. To see this, let $k^\ell(y) := cg^\ell(y) - (y-1)^2$ with $c \ge C^\ell$. Then $\frac{d}{dy}k^\ell(y) = c\left(1-\frac{1}{y}\right) - 2(y-1)$ and for $0 < y \le \overline{y}$ we have $\frac{d^2}{dy^2}k^\ell(y) = \frac{c}{y^2} - 2 \ge \frac{2\overline{y}^2}{y^2} - 2 \ge 0$. Thus k^ℓ is nonnegative on $(0,\overline{y}]$ since it is a convex function there with unique minimum in $y = 1$ where $k^\ell(1) = 0$. Analogously, we can show that for $g^p(y) = y^{-\delta} - 1 + \delta(y-1)$ we can take $C = C^p := \frac{2}{\delta(\delta+1)}\overline{y}^{\delta+2}$. In fact, let $c \ge C^p$ and $k^p(y) := cg^p(y) - (y-1)^2$. Then $\frac{d}{dy}k^p(y) = c\delta\left(1-\frac{1}{y^{\delta+1}}\right) - 2(y-1)$ and for $0 < y \le \overline{y}$ we have $\frac{d^2}{dy^2}k^p(y) = c\frac{\delta(\delta+1)}{y^{\delta+2}} - 2 \ge 2\frac{\overline{y}^{\delta+2}}{y^{\delta+2}} - 2 \ge 0$. Again, the unique minimum is in $y = 1$ where $k^p(1) = 0$. This finishes the proof. □

Lemma 5.1.8. *Let $g \in \{g^\ell, g^p, g^q\}$. There exists $C > 0$ such that for all $c \geq C$*

$$|y-1|b \leq c\left(g(y)+b^2\right) \qquad \text{for all } \; y>0 \; \text{ and all } b \in [0,1].$$

Proof. By Lemma 5.1.2, there exists a unique $\overline{y} > 1$ with $g(\overline{y}) = 1$. By $C_1 = C_1(\overline{y})$ denote the corresponding constant from Lemma 5.1.7. We claim that we can then take $C = C^\ell := \max\left\{\frac{1}{2}\sqrt{C_1}, \frac{\overline{y}}{\overline{y}-1}, \frac{\overline{y}-1}{2}\right\}$ for g^ℓ, $C = C^p := \max\left\{\frac{1}{2}\sqrt{C_1}, \frac{1}{\delta}\right\}$ for g^p and $C = C^q := \max\left\{\frac{1}{2}\sqrt{C_1}, 1\right\}$ for g^q. To see this, fix $y > 0$, $c \geq C$ and define

$$q^{c,y}(b) := c(g(y)+b^2) - |y-1|b = cb^2 - |y-1|b + cg(y)$$

for $b \in I\!R$. We claim that $q^{c,y}(b)$ considered as quadratic function of b is nonnegative on $[0, 1]$. Because $c > 0$, this obviously holds true if we can show that either $q^{c,y}$ has at most one zero, or all zeros occur for $b \geq 1$. The latter holds true if and only if

$$\frac{|y-1| - \sqrt{(y-1)^2 - 4c^2 g(y)}}{2c} \geq 1, \tag{5.1.2}$$

and the first if the expression under the square root is less or equal to zero. Thus we may assume that

$$(y-1)^2 - 4c^2 g(y) > 0 \tag{5.1.3}$$

and show that (5.1.2) then holds. However, since $c \geq \frac{1}{2}\sqrt{C_1}$, (5.1.3) and $g(y) > 0$ imply that $(y-1)^2 > C_1 g(y)$, so that by the definition of C_1 and of $\overline{y}$ we have $y > \overline{y} > 1$ and $g(y) > 1$. We deduce that $(y-1)^2 > 4c^2 g(y) > 4c^2$ and thus that $|y-1| > 2c$, i.e., that $|y-1| - 2c > 0$. As a consequence and since $|y-1| = y-1$, we have

$$\begin{aligned} (5.1.2) \quad &\Longleftrightarrow \quad (y-1) - 2c \geq \sqrt{(y-1)^2 - 4c^2 g(y))} \\ &\Longleftrightarrow \quad y-1 \leq c(g(y)+1). \end{aligned} \tag{5.1.4}$$

In order to prove that the last inequality holds, we consider each $g \in \{g^\ell, g^p, g^q\}$ separately. For $g^\ell(x) = -\log x + x - 1$ and $x > 0$ we define

$$k^\ell(x) := c(g^\ell(x)+1) - (x-1).$$

Since $c \geq \frac{\overline{y}}{\overline{y}-1}$ the function k^ℓ is increasing on $[\overline{y}, \infty)$ because there

$$\frac{d}{dx}k^\ell(x) = c\left(1 - \frac{1}{x}\right) - 1 \geq c\frac{\overline{y}-1}{\overline{y}} - 1 \geq 0.$$

Thus (5.1.4) holds if $k^\ell(\overline{y}) \geq 0$. However the definition of $\overline{y}$ implies that we have $k^\ell(\overline{y}) = c(1+1) - \overline{y} + 1$ and this is non-negative since $c \geq \frac{\overline{y}-1}{2}$.

For $g^p(y) = y^{-\delta} - 1 + \delta(y-1)$ we define $k^p(y) := c(g^p(y)+1) - (y-1)$. Then $k^p(y) \geq 0$ iff $\frac{c}{y^\delta} \geq (y-1)(1-c\delta)$ and the latter holds true since the RHS is non-positive because $y > \overline{y} > 1$ and $c \geq \frac{1}{\delta}$.

It remains to consider $g^q(y) = (y-1)^2$ for which we have $\overline{y} = 2$. Now $y > \overline{y} = 2$ implies that $(y-1)^2 \geq y-1$ and thus $c(g^q(y)+1) > cg^q(y) = c(y-1)^2 \geq c(y-1)$ and hence (5.1.4) holds since $c \geq 1$.

□

Lemma 5.1.9. *For $g \in \{g^\ell, g^p, g^q\}$ there exists $C > 0$ such that*

$$(y-1)^2 \wedge |y-1| \leq Cg(y) \quad \text{for all } y > 0.$$

Proof. For $g = g^q$ this is trivial and it suffices to consider $g \in \{g^\ell, g^p\}$. Lemma 5.1.7 with $\overline{y} = 2$ implies the claim for $0 \leq y \leq 2$. To see it also for $y > 2$, not that there $(y-1)^2 \geq |y-1| = y-1$, and define $k(y) := Cg(y) - (y-1)$. For $k(y) = k^\ell(y)$ note that the tangent of $\log y$ in $y = 2$ yields with $b := 1 - \log 2 > 0$ the estimate $-\log y \geq -\frac{y}{2} + b$. Thus $k^\ell(y) \geq y\left(\frac{C}{2} - 1\right) + C(b-1) + 1$ and if in addition $C \geq 2$ so that $\left(\frac{C}{2} - 1\right) \geq 0$ we have for $y \geq 2$ that $k^\ell(y) \geq 2\left(\frac{C}{2} - 1\right) + C(b-1) + 1 = Cb - 1$. The last expression is clearly nonnegative if C is big enough. For $k(y) = k^p(y)$ note that $y \geq 2$ implies that $y^{\delta+1} \geq 2$ so that $\frac{d}{dy}k^p(y) = C\delta\left(1 - \frac{1}{y^{\delta+1}}\right) - 1 \geq C\delta\frac{1}{2} - 1$. Thus for $C \geq \frac{2}{\delta}$, k^p is increasing on $[2,\infty)$. If in addition $C \geq \frac{1}{g^p(2)}$, then $k^p(2) \geq 0$. This finishes the proof. □

For the next result, we recall that $h : \mathbb{R}^d \to \mathbb{R}^d$ is an arbitrary but fixed truncation function, i.e., a bounded function with compact support such that $h(x) = x$ in a neighbourhood of 0. The canonical choice is $h(x) := x\mathbf{1}_{\{\|x\| \leq 1\}}$.

Lemma 5.1.10. *Let $g \in \{g^\ell, g^p, g^q\}$, $\mathbf{M}$ a $d \times d$-matrix, $Y > 0$ a measurable function on $\mathbb{R}^d$, K a σ-finite measure on $\mathbb{R}^d$ and $0 < \varepsilon \leq 1$. Then the following estimates hold:*

a) $\int_{\mathbb{R}^d} \mathbf{1}_{\{\|x\| \geq \varepsilon\}} Y(x)\, K(dx) \leq \text{const.} \int_{\mathbb{R}^d} \left(g(Y(x)) + (1 \wedge \|x\|^2)\right) K(dx)$.

b) $\int_{\mathbb{R}^d} \|\mathbf{M}h(x) - h(\mathbf{M}x)\| Y(x)\, K(dx) \leq \text{const.} \int_{\mathbb{R}^d} \left(g(Y(x)) + (1 \wedge \|x\|^2)\right) K(dx)$.

c) $\int_{\mathbb{R}^d} \|(Y(x) - 1)h(x)\|\, K(dx) \leq \text{const.} \int_{\mathbb{R}^d} \left(g(Y(x)) + (1 \wedge \|x\|^2)\right) K(dx)$.

d) $\int \|\mathbf{M}x - h(\mathbf{M}x)\| Y(x)\, K(dx)$
$\leq \int \|\mathbf{M}\,(xY(x) - h(x))\|\, K(dx) + \text{const.} \int_{\mathbb{R}^d} \left(g(Y(x)) + (1 \wedge \|x\|^2)\right) K(dx),$

$\int \|\mathbf{M}\,(xY(x) - h(x))\|\, K(dx)$
$\leq \int \|\mathbf{M}x - h(\mathbf{M}x)\| Y(x)\, K(dx) + \text{const.} \int_{\mathbb{R}^d} \left(g(Y(x)) + (1 \wedge \|x\|^2)\right) K(dx).$

Proof. a) We first show that from Lemma 5.1.8 one can deduce that there exists a constant $C > 0$ such that for all $y > 0$ we have $y \leq C(g(y) + 1)$. In fact, Lemma 5.1.8 with $b = 1$ yields the existence of some constant $c > 1$ such that

$$y - 1 \leq |y - 1| \leq c(g(y) + 1)$$

for all $y > 0$. Since $g(y) \geq 0$, we then get for $C := c - 1$ that

$$y \leq c(g(y) + 1) + 1 \leq C(g(y) + 1)$$

as required. From this and since g is nonnegative, we deduce that

$$\mathbf{1}_{\{\|x\| \geq \varepsilon\}} Y(x) \leq C\left(g(Y(x)) + \mathbf{1}_{\{\|x\| \geq \varepsilon\}}\right).$$

Since $\mathbf{1}_{\{\|x\| \geq \varepsilon\}} \leq \frac{1}{\varepsilon^2}\left(1 \wedge \|x\|^2\right)$, it suffices to take const. $= \frac{C}{\varepsilon^2}$.

b) Since by Lemma C.3 of [Esc04] there exists $0 < \tilde{\varepsilon} \leq 1$ such that

$$\|\mathbf{M}h(x) - h(\mathbf{M}x)\| \leq \text{const.} \mathbf{1}_{\{\|x\| \geq \tilde{\varepsilon}\}},$$

this follows immediately from a).

c) Note that $\|h(x)\| \leq \text{const.}\left(\|x\| \mathbf{1}_{\{\|x\| \leq 1\}} + \mathbf{1}_{\{\|x\| > 1\}}\right)$. Thus

$$\begin{aligned} &\int_{\mathbb{R}^d} \|(Y(x) - 1)h(x)\|\, K(dx) \\ &\leq \text{const.} \int_{\mathbb{R}^d} \left(|Y(x) - 1|\|x\| \mathbf{1}_{\{\|x\| \leq 1\}} + |Y(x) - 1| \mathbf{1}_{\{\|x\| > 1\}}\right) K(dx) \end{aligned}$$

and the result now follows from Lemma 5.1.8 applied to each summand separately with $b = \|x\| \mathbf{1}_{\{\|x\| \leq 1\}}$ respectively $b = \mathbf{1}_{\{\|x\| > 1\}}$.

d) This follows from b) and c) as in the proof of Lemma 9 in ES.

□

5.2 Omitted proofs

Proof of Proposition 4.3.9. The integrability of $Y-1$ with respect to $\mu^L - \nu^P$ follows for g^q from Theorem II.1.33 a) in JS and for g^ℓ and g^p from Lemma 5.1.6 together with Theorem II.1.33 d) in JS. Thus by (4.3.4) N is a local martingale and in addition quasi-left-continuous, so that by Theorem II.5 in [LM78] $\mathcal{E}(N)$ is a martingale if

$$\text{the predictable compensator of } \langle N^c\rangle_\cdot + \sum_{s\leq \cdot}\Big((\Delta N_s)^2 \wedge |\Delta N|_s\Big) \text{ is bounded;} \quad (5.2.1)$$

note that for Theorem II.5 of [LM78] it suffices if N is a local martingale. In addition, $\mathcal{E}(N)$ is strictly positive since $Y > 0$ implies that $\Delta N > -1$ so that it only remains to show (5.2.1). For $\langle N^c\rangle = \int \beta_t^* c \beta_t \, dt$ which is already the predictable compensator of itself, the claim is trivial by (4.3.4). The jump term can be rewritten as

$$\sum_{s\leq t}\Big((\Delta N_s)^2 \wedge |\Delta N_s|\Big) = \Big((Y-1)^2 \wedge |Y-1|\Big) * \mu_t^L. \qquad (5.2.2)$$

Since N is in particular a special semimartingale, (5.2.2) defines by Propositions II.1.28 and II.2.29 a) of JS a locally integrable process. Also by Proposition II.1.28, the latter has $(Y-1)^2 \wedge |Y-1| * \nu^P$ as predictable P-compensator. This compensator is then bounded thanks to Lemma 5.1.9 and assumption (4.3.3). This finishes the proof. □

Proof of Proposition 4.4.1. Itô's formula applied to $Z = Z^Q = \mathcal{E}(N)$ yields

$$-\log Z_t = -N_t + \frac{1}{2}\langle N^c\rangle_t - \sum_{s\leq t}\big(\log(1+\Delta N_s) - \Delta N_s\big) =: -N_t + \frac{1}{2}\langle N^c\rangle_t - D_t.$$

Recall from Proposition 4.3.3 the expression for N and note that $\langle N^c\rangle = \int \beta_t^* c \beta_t \, dt$ and N are locally P-integrable and, since $Q \in \mathcal{Q}^{f^\ell}$, that so is $\log Z$. This implies that also

$$-D = (-\log Y + Y - 1) * \mu^L = g^\ell(Y) * \mu^L$$

is locally P-integrable. Since g^ℓ is nonnegative, Proposition II.1.28 of JS then implies that the predictable compensator of $-D$ is $g^\ell(Y) * \nu^P$. Moreover,

$$-(Y-1) * (\mu^L - \nu^P) + (-\log Y + Y - 1) * (\mu^L - \nu^P) = -\log Y * (\mu^L - \nu^P)$$

since both sides are local martingales having the same jumps; see Definition II.1.27 in JS. This and the formula for N from Proposition 4.3.3 yield the canonical decomposition

$$\begin{aligned}
-\log Z &= -N + (-\log Y + Y - 1) * (\mu^L - \nu^P) + \frac{1}{2}\langle N^c\rangle \\
&\quad + (-\log Y + Y - 1) * \nu^P \\
&= -N^c - \log(Y) * (\mu^L - \nu^P) + \frac{1}{2}\langle N^c\rangle + g^\ell(Y) * \nu^P \\
&= M^\ell + \tilde{A}^\ell + \tilde{\tilde{A}}^\ell.
\end{aligned}$$

Since $\tilde{A}^\ell$ and $\tilde{\tilde{A}}^\ell$ are increasing and nonnegative, they are both P-integrable if and only if $A^\ell = \tilde{A}^\ell + \tilde{\tilde{A}}^\ell$ is. But $Q \in \mathcal{Q}^{f^\ell}$ so that $-\log Z$ is a P-submartingale and of class (D) since

$$-Z_\tau \leq -\log Z_\tau \leq -E[\log Z_T | \mathcal{F}_\tau] \quad \text{for all stopping times } \tau \leq T.$$

Theorem III.7 of [Pro04] then implies that the process M^ℓ of the unique Doob-Meyer decomposition above is a martingale. Then A^ℓ must be P-integrable since $\log Z$ is, (4.4.1) holds trivially true, and the proof is completed. □

Proof of Propositions 4.4.2 and 4.4.3. We only prove Proposition 4.4.2. Setting $\delta = -2$ and changing the notation from p to q then gives the proof for Proposition 4.4.3. An application of Itô's formula to $Z = Z^Q = \mathcal{E}(N)$ yields

$$\begin{aligned}
Z_t^{-\delta} &= \int_0^t Z_{s-}^{-\delta} \left(-\delta\, dN_s + \frac{\delta(\delta+1)}{2}\, d\langle N^c\rangle_s \right) \\
&\quad + \sum_{s \leq t} Z_{s-}^{-\delta} \left((\Delta N_s + 1)^{-\delta} + \delta(\Delta N_s + 1) - 1 - \delta \right).
\end{aligned}$$

Recall from Proposition 4.3.3 the expression for N and note that $\langle N^c\rangle = \int \beta_t^* c \beta_t\, dt$ and N are locally P-integrable and, since $Q \in \mathcal{Q}^{f^p}$, so is $Z^{-\delta}$. Thus we have local P-integrability also for $\sum_{s \leq t} Z_{s-}^{-\delta} \left((\Delta N_s + 1)^{-\delta} + \delta(\Delta N_s + 1) - 1 - \delta \right)$ and for

$$\begin{aligned}
\sum_{s \leq t} \left((\Delta N_s + 1)^{-\delta} + \delta(\Delta N_s + 1) - 1 - \delta \right) &= \left(Y^{-\delta} - 1 + \delta(Y - 1) \right) * \mu^L \\
&= g^p(Y) * \mu^L.
\end{aligned}$$

Since g^p is nonnegative, Proposition II.1.28 of JS then implies that the predictable compensator of $g^p(Y) * \mu^L$ is $g^p(Y) * \nu^P$. Moreover,

$$\left(Y^{-\delta} - 1 + \delta(Y - 1) \right) * (\mu^L - \nu^P) - \delta(Y - 1) * (\mu^L - \nu^P) = \left(Y^{-\delta} - 1 \right) * (\mu^L - \nu^P)$$

since both sides are local martingales having the same jumps; see Definition II.1.27 in JS. From this and the formula for N from Proposition 4.3.3 we obtain the canonical decomposition

$$\begin{aligned} dZ^{-\delta} &= Z_-^{-\delta}\Big(-\delta\, dN^c + d\left((Y^{-\delta}-1)*(\mu^L-\nu^P)\right) \\ &\qquad + \frac{\delta(\delta+1)}{2} d\langle N^c\rangle + d\left(\left(Y^{-\delta}-1+\delta(y-1)\right)*\nu^P\right)\Big) \\ &= Z_-^{-\delta}\left(d\hat{M}^p + d\hat{A}^p\right) \\ &= d\mathcal{E}\left(\hat{M}^p+\hat{A}^p\right) \\ &= d\left(\mathcal{E}(\hat{M}^p)\mathcal{E}(\hat{A}^p)\right), \end{aligned} \tag{5.2.3}$$

where the last equality holds by Yor's formula since $\hat{A}^p$ is of finite variation and continuous so that $[\hat{M}^p,\hat{A}^p]\equiv 0$. Moreover, $Q\in\mathcal{Q}^{f^p}$ implies that $Z^{-\delta}$ is a positive submartingale and thus of class (D) since $0\le Z_\tau^{-\delta}\le E\left[Z_T^{-\delta}\,\middle|\,\mathcal{F}_\tau\right]$ for all stopping times $\tau\le T$. Since $\hat{A}^p\ge 0$ we have $\mathcal{E}(\hat{A}^p)=e^{\hat{A}^p}\ge 1$ so that (5.2.3) implies that $\mathcal{E}(\hat{M}^p)$ is a local P-martingale of class (D) and thus a martingale; this uses that $\mathcal{E}(\hat{M}^p)$ is positive since $\Delta\hat{M}^p = Y^{-\delta}-1>-1$ because $Y>0$. Moreover, (5.2.3) then implies the R^p-integrability of $\mathcal{E}(\hat{A}^p)$ and the strict positivity of $\mathcal{E}(\hat{M}^p)$. This completes the proof. □

Proof of Theorem 4.4.6. Note that $\overline{\beta}$ and $\overline{Y}$ are well-defined thanks to Lemma 4.4.5 b) and d) and since Y is positive. Moreover, since Y is $\mathbf{P}\otimes\mathcal{B}^d$-measurable, $\overline{Y}$ is a measurable function on $\mathbb{R}^d$ and thus can be considered as a $\mathbf{P}\otimes\mathcal{B}^d$-measurable function on the product space $\Omega\times[0,T]\times\mathbb{R}^d$. By Jensen's inequality (with Lemma 5.1.1 c)), Fubini's theorem and Lemma 4.4.5,

$$\begin{aligned} \int_{\mathbb{R}^d} g\left(\overline{Y}(x)\right)\,K(dx) &\le \int_{\mathbb{R}^d} E_R\left[\frac{1}{T}\int_0^T g(Y_s(x))\,ds\right]K(dx) \\ &= \frac{1}{T}E_R\left[g(Y)*\nu_T^P\right]<\infty. \end{aligned} \tag{5.2.4}$$

Thus Propositions 4.3.9 and 4.3.11 yield the existence of $\overline{Q}=Q^{(\overline{\beta},\overline{Y})}\approx P$ with $\overline{Q}\in\overline{\mathcal{Q}}$ by Corollary 4.3.7. We prove Theorem 4.4.6 only for $f=f^p$ and explain the differences occurring for $f\in\{f^\ell,f^q\}$ briefly at the end of the proof. Since $Q\in\mathcal{Q}^{f^p}$, Proposition 4.4.2 yields

$$f^p(Q|P)=E_{R^p}\left[e^{\hat{A}_T^p}\right] \tag{5.2.5}$$

where $\frac{dR^p}{dP} = \mathcal{E}(\hat{M}_T^p)$. Both $\hat{A}^p$ and R^p depend on Q. In order to apply Proposition 4.4.2 also to $\overline{Q}$, we show below that $\overline{Q} \in \mathcal{Q}^{f^p}$. Then we get

$$f^p(\overline{Q}|P) = E_{R^p(\overline{Q})}\left[e^{\hat{A}_T^p(\overline{Q})}\right],$$

where we emphasize the dependence on $\overline{Q}$ by adding it in brackets. Note that

$$\hat{A}^p(\overline{Q}) = \frac{\delta(\delta+1)}{2}\langle N^c(\overline{Q})\rangle + g^p(\overline{Y}) * \nu^P \tag{5.2.6}$$

is deterministic since $\langle N^c(\overline{Q})\rangle_t = \overline{\beta}^* c\overline{\beta}$ and both $\overline{\beta}$ and $\overline{Y}$ are deterministic; hence $f^p(\overline{Q}|P) = e^{\hat{A}_T^p(\overline{Q})}$. But by (5.2.5) and Jensen's inequality $f^p(Q|P) \geq e^{E_{R^p}\left[\hat{A}_T^p\right]}$, and so $f^p(\overline{Q}|P) \leq f^p(Q|P)$ if

$$\hat{A}_T^p(\overline{Q}) \leq E_{R^p}\left[\hat{A}_T^p\right].$$

This can be done as in ES, by showing the inequality for both summands of $\hat{A}_T^p$ and $\hat{A}_T^p(\overline{Q})$ separately, using Jensen's inequality. Moreover, since all above inequalities go back to Jensen's inequality, we have $f^p(Q|P) = f\left(\overline{Q}\middle|P\right)$ iff all involved variables are deterministic and time-independent, i.e., iff β and Y are; see Lemma 5.1.1 b).

It remains to show that $\overline{Q} \in \mathcal{Q}^{f^p}$, i.e., that $f^p(\overline{Q}|P) < \infty$. This can be shown by an application of Itô's formula as in the proof of Proposition 4.4.2 and has not been considered in ES. Similarly as for Proposition 4.4.2, one then obtains the canonical decomposition and in particular that $f^p\left(\frac{d\overline{Q}}{dP}\right) = e^{\hat{A}_T^p(\overline{Q})_T}\, \mathcal{E}\big(\hat{M}^p(\overline{Q})\big)_T$. The only difference in the proof is the way one obtains that $g(\overline{Y}) * \mu^L$ is locally P-integrable; this cannot be done as before since we do not know if $\overline{Q} \in \mathcal{Q}^{f^p}$. However, since g is nonnegative we obtain immediately from (5.2.4) that $g(\overline{Y}) * \nu^P$ is locally P-integrable and this is by Proposition II.1.28 of JS equivalent to the local P-integrability of $g(\overline{Y}) * \mu^L$. Thus it only remains to show that $f^p(\overline{Q}|P) = E\left[e^{\hat{A}_T^p(\overline{Q})_T}\, \mathcal{E}\big(\hat{M}_T^p(\overline{Q})\big)_T\right]$ is finite. This is true since $\Delta\hat{M}^p(\overline{Q}) > -1$ implies that $\mathcal{E}\big(\hat{M}^p(\overline{Q})\big)$ is a P-supermartingale, and since $\hat{A}^p(\overline{Q})$ is deterministic and finite. This completes the proof for $f = f^p$.

For $f = f^q$ the proof is exactly the same, except that instead of Proposition 4.4.2 one applies Proposition 4.4.3. For $f = f^\ell$ one takes Proposition 4.4.1. There, the canonical decomposition is of a simpler form so that no stochastic exponentials occur. This simplifies the arguments slightly. In particular, for the proof that $\overline{Q} \in \mathcal{Q}^{f^\ell}$ it then suffices to note that a local P-martingale which is in addition an P-Lévy process is a (true) P-martingale; see Theorem 11.46 of [HWY92]. □

Proof of Proposition 4.4.8. In Theorem 4.4.6 we have already proved that $\overline{Q} = Q^{(\overline{\beta},\overline{Y})}$ is in $\mathcal{Q}^f \cap \overline{\mathcal{Q}}$. In order to show that $\overline{Q} \in \mathcal{M}^e(\mathbf{M}L)$, we have to show that it satisfies

(4.3.6) and (4.3.7). We exploit that since $\overline{Q} \in \mathcal{Q}^f$, condition (4.3.6) for $\overline{Q}$ is by Lemmatas 4.3.14 and 4.4.5 c) equivalent to $\int_{\mathbb{R}^d} \left\| \mathbf{M}\left(x\overline{Y}(x) - h(x)\right)\right\| \; K(dx) < \infty$. But the latter now follows from the definition of $\overline{Y}$, Fubini's theorem and the fact that $Q \in \mathcal{Q}^f_{\text{int}}$, since

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left\|\mathbf{M}\left(x\overline{Y}(x) - h(x)\right)\right\| \; K(dx) \\
&\qquad \le \int_{\mathbb{R}^d} E_R\left[\frac{1}{T}\int_0^T \|\mathbf{M}\left(xY_s(x) - h(x)\right)\| \; ds\right] K(dx) \\
&\qquad = \frac{1}{T} E_R\left[\|\mathbf{M}(xY - h)\| * \nu_T^P\right] < \infty.
\end{aligned}$$

In particular, this allows us to apply Fubini's theorem for $\mathbf{M}(xY_s(x) - h(x))$ in order to obtain that the martingale condition (4.3.7) is satisfied by $\overline{\beta}$, $\overline{Y}$. Indeed,

$$\begin{aligned}
&\mathbf{M}\left(b + c\overline{\beta} + \int_{\mathbb{R}^d} \left(x\overline{Y}(x) - h(x)\right) \; K(dx)\right) \\
&\quad = \frac{1}{T} E_R\left[\int_0^T \mathbf{M}\left(b + c\beta_s + \int_{\mathbb{R}^d} \left(xY_s(x) - h(x)\right) \; K(dx)\right) ds\right] = 0,
\end{aligned}$$

where we have also used Lemma 4.4.5 b). □

Proof of Proposition 4.4.9. Let $\tilde{Q} \in \mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f \cap \overline{\mathcal{Q}}$ so that $\tilde{Q} \in \mathcal{Q}^f_{\text{int}}$ as pointed out before Proposition 4.4.8. Denote by β, Y the Girsanov parameters of Q and apply Proposition 4.3.3 to write the density process as $Z^Q = \mathcal{E}(N)$ with

$$N = \int \beta_s^* \, dL_s^c + (Y - 1) * (\mu^L - \nu^P).$$

Analogous quantities with a superscript $\tilde{}$ refer to $\tilde{Q}$. Since $\mathbf{M}L$ is a local Q-martingale, $\|\mathbf{M}x - h(\mathbf{M}x)\| * \nu_T^{L,Q}$ is finite by Proposition 4.3.13 and so is $\|\mathbf{M}(xY - h)\| * \nu_T^P$ by Lemma 4.3.14 and Lemma 4.4.5 c) since $Q \in \mathcal{Q}^f$. Hence the continuous process $\|\mathbf{M}(xY - h)\| * \nu^P$ is even locally R-integrable for the corresponding $R = R(f; \beta, Y)$ with localizing sequence $(\tau_n)_{n \in \mathbb{N}}$. As in Proposition 18 in ES, for $n \in \mathbb{N}$ we construct Q^n which coincides with Q until τ_n and with $\tilde{Q}$ afterwards by setting

$$\begin{aligned}
\beta_s^n &:= \beta_s \mathbf{1}_{[\![0,\tau_n]\!]} + \tilde{\beta}\mathbf{1}_{]\!]\tau_n,T]\!]}, \\
Y_s^n(x) &:= Y_s(x)\mathbf{1}_{[\![0,\tau_n]\!]} + \tilde{Y}(x)\mathbf{1}_{]\!]\tau_n,T]\!]},
\end{aligned}$$

$N^n := \int (\beta_s^n)^* \, dL_s^c + (Y^n - 1) * (\mu^L - \nu^P)$ and $Z^n := \mathcal{E}(N^n)$. We hence obtain that $Z^n = Z\mathbf{1}_{[\![0,\tau_n]\!]} + \frac{Z_{\tau_n}}{\tilde{Z}_{\tau_n}} \tilde{Z}\mathbf{1}_{]\!]\tau_n,T]\!]}$ is a strictly positive martingale and thus the density process of some $Q^n \approx P$.

Now we show that $Q^n \in \mathcal{M}^e(\mathbf{M}L) \cap \mathcal{Q}^f_{\text{int}}$. The definition of Y^n, Proposition 4.3.13 and $Q, \tilde{Q} \in \mathcal{Q}^f$ yield that P-a.s.

$$\begin{aligned} \|\mathbf{M}x - h(\mathbf{M}x)\| * \nu_T^{Q^n} &= \|\mathbf{M}x - h(\mathbf{M}x)\| Y^n * \nu_T^P \\ &\leq \|\mathbf{M}x - h(\mathbf{M}x)\| * \nu_T^Q + \|\mathbf{M}x - h(\mathbf{M}x)\| * \nu_T^{\tilde{Q}} < \infty. \end{aligned}$$

Since β^n and Y^n satisfy the martingale condition (4.3.7) by construction, we thus by Proposition 4.3.13 we have $Q^n \in \mathcal{M}^e(\mathbf{M}L)$. Let $R = R(f; \beta, Y)$, $\tilde{R} = R(f; \tilde{\beta}, \tilde{Y})$ and $R^n = R(f; \beta^n, Y^n)$. Then $R^n = R$ on $\mathcal{F}_{\tau_n}$ and $\tilde{Y}$ is deterministic so that

$$\begin{aligned} & E_{R^n}\left[\left\|\mathbf{M}(xY^n - h)\right\| * \nu_T^P\right] \\ & \quad \leq E_R\left[\|\mathbf{M}(xY - h)\| * \nu_{\tau_n}^P\right] + T \int_{\mathbb{R}^d} \|\mathbf{M}(x\tilde{Y} - h)\| \, K(dx) \\ & \quad < \infty. \end{aligned}$$

Next we use that Q and $\tilde{Q}$ are in $\mathcal{Q}^f$ to deduce from Corollary 5.1.3 that $Q^n \in \mathcal{Q}^f$; this has not been considered in ES. For $f = f^\ell$ this follows immediately from the definition of Q^n and since $\log(ab) = \log a + \log b$. For $f \in \{f^p, f^q\}$, note that $\tilde{N}$ is a $(P, \mathbb{F}^L)$-Lévy process since $\tilde{\beta}, \tilde{Y}$ are deterministic and time-independent. Therefore $\tilde{Z}_T / \tilde{Z}_{\tau_n} = \int_{\tau_n}^T d\mathcal{E}(\tilde{N})_s$ is independent of $\tilde{Z}_{\tau_n}$ and of $Z^Q_{\tau_n}$. Thus

$$E\left[f\left(\tilde{Z}_{\tau_n}\right)\right] E\left[f\left(\tilde{Z}_T / \tilde{Z}_{\tau_n}\right)\right] = E[f(\tilde{Z}_T)] < \infty$$

and $Q, \tilde{Q} \in \mathcal{Q}^f$ imply that

$$0 \leq f(Q^n | P) = E[f(Z^n_T)] = E\left[f\left(Z_{\tau_n}\right)\right] E\left[f\left(\tilde{Z}_T / \tilde{Z}_{\tau_n}\right)\right] < \infty.$$

Hence $Q^n \in \mathcal{Q}^f$ and therefore $Q^n \in \mathcal{Q}^f_{\text{int}}$.

It remains to show that $\lim_{n\to\infty} f\left(Q^n \middle| P\right) = f(Q|P)$. First let $f \in \{f^p, f^q\}$. Then by applying Proposition 4.4.2 (for f^p) respectively 4.4.3 (for f^q) to Q^n and Q, we have $f\left(Q^n \middle| P\right) = E_{R^n}[\exp(\hat{A}^n_T)]$ and that $\hat{\tilde{A}} = \hat{A}(\tilde{Q})$ is deterministic and given by $\hat{\tilde{A}}_t = \hat{\tilde{A}}_1 t$ because $\tilde{Q} \in \overline{\mathcal{Q}}$ has deterministic time-independent Girsanov parameters. Hence $\hat{A}^n_T = \hat{A}_{\tau_n} + \hat{\tilde{A}}_1 (T - \tau_n)$ is $\mathcal{F}_{\tau_n}$-measurable and therefore

$$E_{R^n}[\exp(\hat{A}^n_T)] = E_R[\exp(\hat{A}^n_T)] = E_R\left[\exp\left(\hat{A}_{\tau_n}\right) \exp\left((T - \tau_n)\hat{\tilde{A}}_1\right)\right].$$

In addition, $\exp\left(\hat{A}^n_T\right) \leq \exp\left(\hat{A}_T\right) \exp\left(T \hat{\tilde{A}}_1\right)$ where the first factor is R-integrable since $Q \in \mathcal{Q}^f$ and the second is deterministic. Then the assertion follows by dominated convergence. For $f = f^\ell$ one uses formula (4.4.1) for the f^ℓ-divergence and proves analogously that $E\left[A^{n,\ell}_T\right]$ converges to $E\left[A^\ell_T\right]$. This ends the proof. □

Proof of Proposition 4.4.10. Since $\overline{Q}$ and $\overline{Q}^n$ have time-independent and deterministic Girsanov parameters and since for any sequence (a_n) we have $\lim_{n\to\infty} e^{a_n} = e^a$ if and only if $\lim_{n\to\infty} a_n = a$, the formulae (4.4.1) (for f^ℓ), (4.4.2) (for f^p) and (4.4.3) (for f^q) imply that it suffices to show that

$$\begin{aligned} \lim_{n\to\infty} (\overline{\beta}^n)^* c \overline{\beta}^n &= \overline{\beta}^* c \overline{\beta}, \\ \lim_{n\to\infty} \int g\left(\overline{Y}^n(x)\right) K(dx) &= \int g\left(\overline{Y}(x)\right) K(dx). \end{aligned}$$

This can be shown with the same arguments as in the proof of Proposition 19 in ES with f there replaced by g and by plugging in our definitions for $\overline{\beta}^n$, $\overline{\beta}$, $\overline{Y}^n$ and $\overline{Y}$. This changes the Q- and Q^n-expectations there to expectations under $R = R(f; \beta, Y)$ and $R^n = R(f; \beta^n, Y^n)$. □

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Curriculum vitae

Personal Data

Name	Susanne Klöppel
Date of Birth	April 18, 1978
Nationality	German

Education

2003–2006	PhD in Mathematics at ETH Zurich, Switzerland Supervisor: Prof. Martin Schweizer
2002–2003	PhD student in Mathematics at LMU Munich, Germany Supervisor: Prof. Martin Schweizer
2001–2002	M.Sc. in Mathematics at NUI Galway, Ireland
1997–2001	Diploma with distinction in Economical Mathematics, University of Ulm, Germany
1988–1997	Gymnasium am Deutenberg, VS-Schwenningen, Germany

Work Experience

2003–2006	Assistent at ETH Zurich, Switzerland
2002–2003	Assistant at LMU Munich, Germany
2001–2002	Student assistant at NUIG, Ireland
2000–2001	Student trainee at Visual Analysis AG, Munich, Germany
1998–2000	Student assistant at University of Ulm, Germany