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Characteristic Classes and Bounded Cohomology

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Contents

Abstract

This work is devoted to the study of characteristic classes of flat bundles from the point of view of bounded cohomology.

Our main result is a new proof of Gromov's boundedness of primary characteristic classes of flat bundles which, in contrast to Gromov's original proof, does not rely on Hironaka's resolution of singularities. Moreover, we point out that a representative for these classes can be found which in fact only takes a finite set of values (as opposed to merely being bounded) on singular simplices.

The conjectural generalization to secondary characteristic classes of flat bundles is discussed. In particular, we show that the well known conjecture stating that the simplicial volume of all locally symmetric spaces of noncompact type is strictly positive would follow from the boundedness of the secondary characteristic classes of flat bundles.

Résumé

Ce travail consiste en une étude des classes caractéristiques de fibrés plats du point de vue de la cohomologie bornée.

Notre résultat principal est une nouvelle preuve du théorème de Gromov stipulant que les classes caractéristiques de fibrés plats peuvent être représentées par des cocycles bornés. Notre preuve, contrairement à celle de Gromov, ne se base pas sur la résolution de singularités de Hironaka. De plus, nous montrons que ces classes peuvent être représentées par des cocycles ne prenant qu'un nombre fini de valeurs sur les simplex singuliers.

La généralisation conjecturale aux classes caractéristiques secondaires de fibrés plats est discutée. En particulier, nous montrons que la fameuse conjecture énonçant que le volume simplicial d'un espace compact localement symétrique de type non compact est strictement positif découlerait de la généralisation du théorème de Gromov aux classes charactéristiques secondaires.

Introduction

The first milestone in the history of bounded cohomology may very well be Milnor's characterization of flat oriented vector bundles over surfaces in terms of their Euler number ([Mi58]), later generalized to the unoriented case by Wood ([Wo71]).

Theorem 1 (Milnor-Wood inequality) *Let ξ be a $SL_2\mathbb{R}$-bundle over a surface Σ_g of genus $g \geq 1$. The bundle ξ is flat if and only if its Euler class $\varepsilon(\xi) \in H^2(\Sigma_g)$ satisfies*

$$|\varepsilon(\xi)[\Sigma_g]| \leq g - 1.$$

This result, or more precisely one of its implications, can in a natural way be put in the context of singular bounded cohomology. Indeed, the Euler class was proven to be bounded by Ivanov and Turaev in [IvTu82]. (A cohomology class is said to be bounded, if it can be represented by a cocycle whose set of values on singular simplices is bounded, or equivalently, if its norm $\|.\|_\infty$ is finite. Consult the Appendix for further details.)

Theorem 2 *If ξ is a flat $SL_n\mathbb{R}$-bundle then*

$$\|\varepsilon(\xi)\|_\infty \leq \frac{1}{2^n}.$$

This bound on the Euler class, together with the knowledge of the 1-norm of the fundamental class of a surface Σ_g (also called simplicial volume) implies half of the Milnor-Wood inequality, as pointed out by Ghys in [Ghys87] (see also [Ghys99]). It is a simple consequence of the duality of the two norms.

In his seminal paper [Gr82], Gromov generalized the boundedness of the Euler class of flat bundles to all characteristic classes:

Theorem 3 *Let G be an algebraic subgroup of $GL_n(\mathbb{R})$. Then every characteristic class of flat G-bundle can be represented by a bounded cocycle.*

An immediate corollary is that a topological space with amenable fundamental group does not possess any non trivial characteristic class of flat G-bundle, where G is, of course, an algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$.

Another consequence is the vanishing of all characteristic of flat G-bundles whenever G is an amenable algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$ admitting a cocompact lattice. This was well known for compact groups, for it follows from the fact that the Chern-Weil homomorphism is an isomorphism. For solvable groups, it can already be obtained from the following result of Goldman and Hirsch (see [GoHi81]): Every flat principal G-bundle is virtually trivial (meaning that there exists a finite covering of the base space, such that the pulled back bundle is trivial).

The hypothesis in the above Theorem of Gromov (Theorem 3) that G be algebraic can not be removed. Indeed, Goldman gives in [Go81] an example of a flat G-bundle over the 2-torus with nontrivial characteristic class in degree 2. This class can not be bounded since the bounded cohomology of the torus is trivial. The group G in question is the quotient of the Heisenberg group H of upper triangular unipotent 3 by 3 matrices with the normal subgroup generated by any central element, and the characteristic class in $H^2(BG)$ is the obstruction to the existence of a section of the projection $H \to G$.

We give here a new proof of Gromov's theorem with the advantage that a representative for every characteristic class of flat bundle can be found whose set of values on singular simplices is not only bounded, but furthermore finite. We thus prove:

Theorem 4 *Let G be an algebraic subgroup of $GL_n(\mathbb{R})$. Then every characteristic class of flat G-bundle can be represented by a cocycle whose set of value on singular simplices is finite.*

The first step of the proof, which is common to both Gromov's original proof of Theorem 3 and our Theorem 4, is to reduce to the following simplicial version of the statement:

Theorem 5 *Let G be an algebraic subgroup of $GL_n(\mathbb{R})$ and $\beta \in H^q(BG)$ a characteristic class. There exists a finite subset I of $\mathbb{R}$ such that for every flat G-bundle ξ over a simplicial complex K, the cohomology class $\beta(\xi) \in H^q(|K|)$ can be represented by a cocycle whose set of values on the q-simplices of K is contained in I.*

Again, the case of the Euler class was already well known: Sullivan proved in [Su76] that the Euler class of any flat $\mathrm{SL}_n(\mathbb{R})$-bundle over a simplicial complex can be represented by a simplicial cocycle taking values in $\{-1, 0, 1\}$ and Smillie improved this to $\{-1/2^n, 0, 1/2^n\}$ in [Sm81].

Let us point out, that both the proofs of the simplicial version of the theorem and the reduction to it are not only completely different from Gromov's but also much more elementary. It is only a technical artifice to show how one can reduce to the simplicial version of the theorem. The main difficulty thus really lies in the proof of this simplicial version. Our main tool is a bounded version of the existence of a finite triangulation of semi-algebraic sets as developed by Benedetti and Risler in [BeRi90], whereas Gromov needs Hironaka's deep resolution of singularities.

One possible generalization of Gromov's Theorem 3 (or more generally Theorem 4) is the following conjecture:

Conjecture 6 *Every secondary characteristic class in $H^{2q-1}(BG^\delta)$, for $q > 1$, can be represented by a bounded cocycle.*

In view of Dupont and Kamber's result that the continuous cohomology of a connected semisimple Lie group with finite center is generated by primary and secondary characteristic classes (see [DuKa90, Theorem 5.2]), Conjecture 6 immediately implies the following conjecture:

Conjecture 7 *Let G be a connected semisimple Lie group. For any $n \geq 2$, the comparison map*

$$H^n_{c,b}(G, \mathbb{R}) \longrightarrow H^n_c(G, \mathbb{R})$$

is surjective.

This question was already raised by Monod and further conjectured in the case of $\mathrm{SL}_n\mathbb{C}$ ([Mo01, Conjecture 9.3.8]). A straightforward consequence of the latter Conjecture is now a well known conjecture of Gromov:

Conjecture 8 *The simplicial volume of any compact locally symmetric space of non compact type is strictly positive.*

This conjecture is known to hold in the real rank one case, for Thurston proved that a uniform bound on the volume of geodesic simplices in the corresponding symmetric spaces exists, which implies both the validity of Conjectures 7 and 8 in this case (see [Th78] and [Gr82]). For locally symmetric spaces covered by $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n)$, Conjecture 8 was proven by Savage in [Sa82].

Our exposition is structured as follows: In Chapter 1 we review some elementary notions on principal bundles and classifying space. Simplicial complexes and their basic properties are exposed in Chapter 2. Primary and secondary characteristic classes are defined in Chapter 3. The case of the Euler class is examined there in detail, and the results and conjectures presented in this introduction are elaborated on. Finally in Chapter 4, after defining semi-algebraic sets and giving their first properties, we furnish the proof of Theorem 4. Note also that in the Appendix, a quick review on singular bounded cohomology is given.

Chapter 1

Bundles

1.1 Principal bundles and classifying spaces

1.1.1 Principal bundles

Let G be a topological group. A *topological principal G-bundle* ξ is a continuous surjective map

$$\xi = \begin{array}{c} P \\ \downarrow \pi \\ X \end{array}$$

between two topological spaces P and X together with a right continuous G-action $P \times G \to P$ satisfying

- for every x in X, the preimage $\pi^{-1}(x)$ of x by π is an orbit for the G-action on P,

- for every x in X, there exists a neighborhood U of x and a G-equivariant homeomorphism $\psi : \pi^{-1}(U) \to U \times G$, where the G-action on the product $U \times G$ is given by the trivial action on the first factor, and multiplication from the right on the second, such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times G \\ \pi \searrow & & \swarrow p_1 \\ & U & \end{array}$$

  commutes. Of course, the map p_1 denotes the projection on the first factor. This last condition is referred to as *local triviality* and the map ψ is a *local trivialization*.

The space P is called the *total space* and X the *base space* of the topological principal G-bundle ξ. Assuming that G is a Lie group we define a *smooth principal G-bundle* to be a topological principal G-bundle where all the spaces and maps in consideration are moreover assumed to be smooth, that is, so that the spaces P and X are smooth manifolds, the map π and the G-action are smooth, and the local trivializations ψ are diffeomorphisms. Note that by smooth we always mean infinitely differentiable. Whenever it will be clear from the context if we mean topological or smooth principal G-bundle we will simply speak about principal G-bundle.

The *right translation* R_g by any element g of the group G is defined as the map

$$\begin{array}{rccc} R_g : & P & \longrightarrow & P \\ & u & \longmapsto & ug. \end{array}$$

It is clear that the base space of any G-bundle is homeomorphic to the quotient of the total space by the action of the group G. Also, if a group G acts freely on a space P, then the quotient map $P \to P/G$ gives rise to a principal G-bundle.

A *bundle map* between two topological (smooth) principal G-bundles is a continuous (smooth) G-equivariant map between the two corresponding total spaces. A bundle map obviously induces a map between the corresponding base spaces as follows: Let $\xi_i = \{\pi_i : P_i \to X_i\}$, for $i = 0$ and 1, be a topological (smooth) principal G-bundle, and let $\overline{f} : P_0 \to P_1$ be a bundle map between ξ_0 and ξ_1. Define a continuous (smooth) map $f : X_0 \to X_1$ as

$$f(x) = \pi_1(\overline{f}(u)),$$

for every x in X_0 and u in $\pi_0^{-1}(x)$. As the map $\overline{f}$ is G-equivariant and the map π_1 is G-invariant, the map f is well defined. By definition, the diagram

$$\begin{array}{ccc} P_0 & \xrightarrow{\overline{f}} & P_1 \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} \\ X_0 & \xrightarrow{f} & X_1 \end{array}$$

commutes. We say that the map $\overline{f}$ *covers* the map f.

A bundle map is an *isomorphism* if it admits an inverse. Observe that an isomorphism necessarily covers a homeomorphism of the corresponding base spaces, and conversely, if a bundle map covers the identity, or more generally a homeomorphism, then it has to be an isomorphism.

The first example of a principal G-bundle is the product bundle

$$\begin{array}{c} X \times G \\ \downarrow \\ X, \end{array}$$

where the bundle map is given by the projection on the first factor, and the action of G on the product $X \times G$ is trivial on X and by right multiplication on G, that is,

$$(x, h) \cdot g = (x, hg),$$

for every (x, h) in $X \times G$ and g in G. A principal G-bundle over a base space X is said to be *trivial* if it is isomorphic to the product bundle $X \times G$.

Let $\xi = \{\pi : P \to X\}$ be a principal G-bundle. A *section* of ξ is a continuous (smooth) map $s : X \to P$ such that

$$\pi \circ s = \mathrm{Id}_X.$$

Lemma 9 *A principal G-bundle is trivial if and only if it admits a section.*

Proof. Let $\xi = \{\pi : P \to X\}$ be a trivial principal G-bundle. By definition, this means that there exists a G-equivariant invertible map

$$\overline{f} : X \times G \longrightarrow P$$

covering the identity. Now the product bundle surely admits a section, for example the trivial section

$$\begin{array}{rccc} s: & X & \longrightarrow & X \times G \\ & x & \longmapsto & (x, 1_G). \end{array}$$

Composing this section with $\overline{f}$ we obtain the desired section of ξ. Indeed, as $\pi \circ \overline{f}$ is the projection p_1 on the first factor of $X \times G$, we have

$$\pi \circ \left(\overline{f} \circ s\right)(x) = p_1 \circ s(x) = x.$$

Conversely, suppose that the principal G-bundle $\xi = \{\pi : P \to X\}$ admits a section $s : X \to P$. Define

$$\begin{array}{rccc} \overline{f}: & X \times G & \longrightarrow & P \\ & (x, g) & \longmapsto & s(x) \cdot g. \end{array}$$

The map $\overline{f}$ is clearly G-equivariant, and as it covers the identity, it is an isomorphism between the product bundle and the bundle ξ. ■

Transition functions

Let $\xi = \{\pi : P \to X\}$ be a principal G-bundle. Let $\{U_i\}_{i \in I}$ be a covering by open sets of the base space X so that the bundle ξ restricted to any U_i, for i in I, is trivial. For every i in I, let

$$\psi_i : \pi^{-1}(U_i) \to U_i \times G$$

be some local trivializations of the bundle ξ. For i, j in I, we can now consider the composition

$$U_i \cap U_j \times G \xrightarrow{\psi_j^{-1}|_{U_i \cap U_j \times G}} \pi^{-1}(U_i \cap U_j) \xrightarrow{\psi_i|_{\pi^{-1}(U_i \cap U_j)}} U_i \cap U_j \times G,$$

which surely is a G-equivariant homeomorphism and moreover is the identity on its first factor. Thus it has the form $(x, g) \longmapsto (x, f(x, g))$ for some continuous function $f : U_i \cap U_j \times G \to G$. But from the G-equivariance, it follows that $f(x, gh) = f(x, g)h$, so that the above composition of local trivializations actually has the form $(x, g) \longmapsto (x, f(x, 1_G)g)$. We can now define the so called *transition functions*

$$g_{ij} : U_i \cap U_j \longrightarrow G$$

for every i and j in I by $g_{ij}(x) = f(x, 1_G)$. They satisfy the relation

$$\psi_i \psi_j^{-1}(x, g) = (x, g_{ij}(x)g),$$

for every x in $U_i \cap U_j$ and g in G, and are moreover clearly completely determined by it. Those functions are of course dependent on the chosen local trivializations. They further fulfill the following *cocycle relations*:

- $g_{ii}(x) = \mathrm{Id}_G$, for every i in I and x in U_i,

- $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$, for every i, j, k in I and x in $U_i \cap U_j \cap U_k$.

Note that from the only knowledge of the covering $\{U_i\}_{i\in I}$ and the transition functions $\{g_{ij}\}_{i,j\in I}$ it is possible to recover the original bundle $\xi = \{\pi : P \to X\}$ up to isomorphism. Indeed, consider the quotient

$$P' = \left(\coprod_{i\in I} U_i \times G\right) / \sim$$

of the disjoint union of the products $U_i \times G$ by the equivalence relation

$$U_i \times G \ni (x, g_{ij}(x)g) \sim (x, g) \in U_j \times G,$$

for every x in $U_i \cap U_j$ and g in G. The space P' is endowed with the right action of G induced by the canonical action of G on the products $U_i \times G$. The projection $\pi' : P' \to X$ is induced from the projections $U_i \times G \to U_i$ on the first factor. The principal G-bundle $\xi' = \{\pi' : P' \to X\}$ is easily checked to be isomorphic to the original bundle ξ. Actually, letting

$$\psi_i : \pi^{-1}(U_i) \longrightarrow U_i \times G,$$

for every i in I, be the local trivializations defining the transition functions g_{ij}, an isomorphism between ξ and ξ' can for example be given by sending any element u in $\pi^{-1}(U_i)$ to the equivalence class represented by the element $\psi_i(u)$ in the product $U_i \times G$.

Lemma 10 *Let $\{g_{ij}\}_{i,j\in I}$ and $\{h_{ij}\}_{i,j\in I}$ be two families of transition functions relative to the same open covering $\{U_i\}_{i\in I}$ of some topological space X. Then the two corresponding G-bundles are isomorphic if and only if there exists maps $\lambda_i : U_i \to G$, for every i in I, such that*

$$g_{ij}(x) = \lambda_i(x)h_{ij}(x)(\lambda_j(x))^{-1}$$

for every i, j in I and x in $U_i \cap U_j$.

Proof. Let $\xi(g_{ij})$ and $\xi(h_{ij})$ be the two G-bundles obtained by the above procedure from the systems of transition functions $\{g_{ij}\}_{i,j\in I}$ and $\{h_{ij}\}_{i,j\in I}$ respectively.

Suppose that the two bundles are isomorphic and let the isomorphism be given by a map

$$\overline{f} : P(h_{ij}) \longrightarrow P(g_{ij}).$$

This map being a G-equivariant map covering the identity, it necessarily has the form

$$(x, g) \longmapsto (x, \lambda_i(x)g),$$

for (x, g) in $U_i \times G$ and some map $\lambda_i : U_i \to G$, when restricted to $U_i \times G$ and viewed as a map from $U_i \times G$ to itself. Let x belong to $U_i \cap U_j$ and let g be a group element. In $P(h_{ij})$ we then have

$$U_i \times G \ni (x, h_{ij}(x)g) \sim (x, g) \in U_j \times G.$$

Applying $\overline{f}$ we thus obtain in $P(g_{ij})$ that

$$U_i \times G \ni \overline{f}(x, h_{ij}(x)g) = (x, \lambda_i(x)h_{ij}(x)g) \sim \overline{f}(x, g) = (x, \lambda_j(x)g) \in U_j \times G.$$

But in $P(g_{ij})$ we also have that

$$U_i \times G \ni (x, g_{ij}(x)\lambda_j(x)g) \sim (x, \lambda_j(x)g) \in U_j \times G.$$

It thus follows that

$$\lambda_i(x)h_{ij}(x) = g_{ij}(x)\lambda_j(x).$$

Conversely, starting with two systems of transition functions for which there exists maps $\lambda_i : U_i \to G$ satisfying the above equality, we can define a map

$$\begin{array}{ccc} U_i \times G & \longrightarrow & U_i \times G \\ (x, g) & \longmapsto & (x, \lambda_i(x)g, \end{array}$$

which induces a well defined map between the two total spaces $P(h_{ij})$ and $P(g_{ij})$ because of the equality $g_{ij}(x) = \lambda_i(x)h_{ij}(x)(\lambda_j(x))^{-1}$. This map is further clearly G-equivariant and covers the identity, so that it lifers the desired isomorphism between the bundles $\xi(h_{ij})$ and $\xi(g_{ij})$. ■

Pull Backs

Given a principal G-bundle $\xi = \{\pi : P \to X\}$ and a continuous map $f : Y \to X$, where Y is a topological space we can consider the *pull back* of the bundle ξ to Y, denoted by $f^*(\xi)$. This is the G-bundle over Y with total space

$$f^*(P) = \{(y, u) \in Y \times P \mid f(y) = \pi(u)\},$$

and of course the bundle map is simply the projection on the first factor. Observe that a bundle map $\overline{f} : f^*(P) \to P$ is given by the projection on the second factor.

Lemma 11 *Let $\overline{f} : \xi_0 \to \xi_1$ be a bundle map between two principal G-bundles. Denote by $f : X_0 \to X_1$ the corresponding map of the base spaces. Then*

$$\xi_0 \cong f^*(\xi_1).$$

Proof. Denote, for $i = 0, 1$, by P_i and π_i the total space and bundle map of ξ_i. The isomorphism is given by

$$\begin{array}{ccc} P_0 & \longrightarrow & f^*(P_1) \\ u & \longmapsto & (\pi_0(u), \overline{f}(u)). \end{array}$$

Indeed, this map is well defined because the bundle map $\overline{f}$ commutes with the projections, it is G-equivariant, because $\overline{f}$ is, and it obviously covers the identity map. ■

A fundamental property of the pull back is its invariance under homotopy. More precisely, the pull backs of some G-bundle by two homotopic maps give rise to two isomorphic bundles. This is obtained at once as a corollary of the following theorem, the proof of which are, up to small adaptations, taken from [Hu66] (Chapter 3, Theorems 4.3 and 4.8).

Theorem 12 *Let ξ be a principal G-bundle over $X \times [0, 1]$, where X is a paracompact space, with total space P and bundle map π. Set*

$$\begin{array}{cccc} r : & X \times [0, 1] & \longrightarrow & X \times [0, 1] \\ & (x, t) & \longmapsto & (x, 1). \end{array}$$

Then there exists a bundle map $\overline{r}$ covering r:

$$\begin{array}{ccc} P & \xrightarrow{\overline{r}} & P \\ \downarrow & & \downarrow \\ B \times [0,1] & \xrightarrow{r} & B \times [0,1]. \end{array}$$

Proof. For simplicity, and since it will be all what we need in what follows, we restrict ourselves to the case when there exists a finite covering $\{U_i\}_{i=1}^q$ of X such that the bundle ξ restricted to $U_i \times [0,1]$ is trivial. (The proof of the general case, when the covering is only locally finite, is a straightforward generalization of this one and can be read in [Hu66].) For $i \in \{1, \dots q\}$, let

$$\phi_i : \pi^{-1}(U_i \times [0,1]) \longrightarrow U_i \times [0,1] \times G$$

be the local trivializations. For every $i \in \{1, \dots q\}$, choose functions

$$\nu_i : X \longrightarrow \mathbb{R}$$

such that

- $\mathrm{supp}(\nu_i) \subset U_i$,
- $\sum_{i=1}^{q} \nu_i(x) = 1$ for every x in X.

For every $i \in \{1, \dots q\}$, define

$$\begin{array}{rccl} u_i : & P & \longrightarrow & P \\ & u & \longmapsto & \left\{ \begin{array}{ll} u & \text{if } u \notin \pi^{-1}(U_i \times [0,1]), \\ \phi_i^{-1}(x, \min\{t + \nu_i(x), 1\}, g) & \text{if } u \in \pi^{-1}(U_i \times [0,1]) \text{ and} \\ & \quad u = \phi_i^{-1}(x,t,g). \end{array} \right. \end{array}$$

To see that this defines continuous maps, it is enough to realize, that, the support of ν_i being included in U_i, whenever x tends to the boundary of U_i, the value of ν_i on x tends to zero, and hence the minimum between $t + \nu_i(x)$ and 1 goes to the minimum between t and 1, which is equal to t since t belongs to the interval $[0,1]$. Since those maps are G-equivariant, they are bundle maps. Let

$$\begin{array}{rccl} r_i : & X \times [0,1] & \longrightarrow & X \times [0,1] \\ & (x,t) & \longmapsto & (x, \min\{t + \nu_i(x), 1\}) \end{array}$$

be the corresponding base space maps. We now need to prove that

$$r = r_q \circ r_{q-1} \circ \cdots \circ r_2 \circ r_1.$$

Let (x,t) be in $X \times [0,1]$ and let us compute the composition of the maps r_i on it:

$$\begin{aligned}
r_q \circ r_{q-1} \circ \cdots \circ r_2 \circ r_1(x,t) &= r_q \circ \cdots \circ r_3 \circ r_2(x, \min\{t+\nu_1(x),1\}) \\
&= r_q \circ \cdots \circ r_3(x, \min\{t+\nu_1(x)+\nu_2(x),1\}) \\
&= \cdots \\
&= r_q \circ \cdots \circ r_i(x, \min\{t+\sum_{j=1}^{i}\nu_j(x),1\}) \\
&= \cdots \\
&= (x, \min\{t+\nu_1(x)+...+\nu_q(x),1) \\
&= (x, \min\{t+1,1\}) \\
&= (x,1) \\
&= r(x,t).
\end{aligned}$$

The map

$$u = u_q \circ u_{q-1} \circ \cdots \circ u_2 \circ u_1$$

is thus a bundle map covering r. ■

Theorem 13 *Let X and Y be topological spaces, ξ a principal G-bundle over X and $f \simeq g : Y \to X$ two homotopic continuous maps, then*

$$f^*(\xi) \cong g^*(\xi).$$

Proof. Let $h : Y \times [0,1] \to X$ be the homotopy from f to g, so that $h_{|Y\times\{0\}} = f$ and $h_{|Y\times\{1\}} = g$. Then

$$f^*(\xi) \cong h^*(\xi)_{|Y\times\{0\}} \quad \text{and} \quad g^*(\xi) \cong h^*(\xi)_{|Y\times\{1\}}.$$

Apply Theorem 12 to the bundle $h^*(\xi)$ over $Y\times[0,1]$. Restricting the bundle map J to $\pi^{-1}(Y\times\{0\})$ we obtain a bundle map

$$\begin{array}{ccc}
h^*(P)_{\pi^{-1}(Y\times\{0\})} & \xrightarrow{\overline{r}_{\pi^{-1}(Y\times\{0\})}} & h^*(P)_{\pi^{-1}(Y\times\{1\})} \\
\downarrow & & \downarrow \\
B\times\{0\} & \xrightarrow{r_{\pi^{-1}(Y\times\{0\})}} & B\times\{1\}
\end{array}$$

covering the identity. It is thus an isomorphism. ■

Corollary 14 *If ξ is a principal G-bundle over a contractible base space, then the bundle is trivial.*

Proof. Let X be the contractible base space of ξ. Then the identity Id_X of X is homotopic to some constant map $c : X \to \{x_0\} \subset X$. By Theorem 13 the bundle ξ (being the pull back of itself via the identity map) is thus isomorphic to $c^*(\xi)$, which is the product bundle $X\times G$, since the condition that $c(x) = \pi(g)$, for $x \in X$, $g \in G$ is empty. (Of course, π denotes the projection map of ξ.) ■

Reduction of bundle

Let H and G be two groups and $i : H \to G$ a group homomorphism. Suppose $\eta = \{\pi_\eta : P_\eta \to X\}$ is a principal H-bundle and $\xi = \{\pi_\xi : P_\xi \to X\}$ is a principal G-bundle. The bundle ξ is said to be an *extension* of η (relative to i), or equivalently, η is a *reduction* of ξ (relative to i), if there exists a map $\overline{f} : P_\eta \to P_\xi$ such that

$$\begin{array}{ll} \overline{f}(\pi_\eta^{-1}(x)) \subset \pi_\xi^{-1}(x) & \text{for every } x \text{ in } X, \\ \overline{f}(uh) = \overline{f}(u)i(h) & \text{for every } u \text{ in } P \text{ and } h \text{ in } H. \end{array}$$

As a consequence of the fundamental result of Iwasawa ([Iw49]) and Mostow that every connected Lie group is topologically equivalent to product $K \times E$, where K is a maximal subgroup and E is contractible, one has:

Theorem 15 *Let G be a connected Lie group and K be a maximal compact subgroup. Every principal G-bundle admits a reduction to K.*

1.1.2 Classifying spaces

A principal G-bundle $\xi_G = \{\pi_G : PG \to BG\}$ is said to be *universal* if for every principal G-bundle $\xi = \{\pi : P \to B\}$ there exists a *classifying map* $f : B \to BG$, unique up to homotopy, such that the bundle ξ is isomorphic to the pull back $f^*(\xi_G)$. The base space BG of the universal bundle ξ_G is called the *classifying space*. Somehow the bundle ξ_G is the most complicated G-bundle possible: taking pull backs only simplifies the bundles.

Various constructions of classifying spaces exist. We will describe here a possible model for linear groups which we will need in our proof of our main theorem.

Space of frames

Let n and q be positive natural numbers and set $N = (q+1)n$. The *space of n-frames* in $\mathbb{R}^N$, which we denote by $\mathrm{Fr}_n(\mathbb{R}^N)$, consists of ordered n-tuples of linearly independent vectors in $\mathbb{R}^N$. It is naturally identified with the set of N times n matrices with linearly independent columns. There is a natural action of $\mathrm{GL}_n(\mathbb{R})$ from the right (and one of $\mathrm{GL}_N(\mathbb{R})$ from the left) simply given by matrix multiplication, which furnishes a right action of any closed subgroup G of $\mathrm{GL}_n\mathbb{R}$ on the space of n-frames $\mathrm{Fr}_n(\mathbb{R}^N)$. Define

$$PG_q = \mathrm{Fr}_n(\mathbb{R}^N) \quad \text{and} \quad BG_q = PG_q/G,$$

and let $\pi_G : PG_q \to BG_q$ denote the natural projection. We have thus obtained a principal G-bundle

$$\xi_G^q = \{\pi_G : PG_q \to BG_q\}\,.$$

For a frame A in $PG_q = \mathrm{Fr}_n(\mathbb{R}^N)$, that is, a N times n matrix with at least one of its maximal minor not zero, we denote by $[A]_G$ its equivalence class in the quotient $BG_q = PG_q/G$, so that $\pi_G(A) = [A]_G$.

Observe that for $G = \mathrm{GL}_n(\mathbb{R})$, the space BG_q is diffeomorphic to the Grassmanian manifold of n-dimensional vector subspaces of $\mathbb{R}^N$, and in general BG_q is a fiber bundle over the Grassmanian, with fiber diffeomorphic to $\mathrm{GL}_n(\mathbb{R})/G$.

The canonical inclusions $\mathbb{R}^q \hookrightarrow \mathbb{R}^q \times \{0\} \hookrightarrow \mathbb{R}^{q+1}$ also produce inclusions $\mathbb{R}^{nq} \hookrightarrow \mathbb{R}^{n(q+1)}$ which in turn induce canonical G-equivariant inclusions $PG_q \hookrightarrow PG_{q+1}$. Define PG as the limit

$$PG = \lim_{q\to\infty} PG_q.$$

The inclusions being G-equivariant, a right action of G is naturally given on the limit PG. Let BG be the quotient

$$BG = PG/G$$

and denote by $\pi_G : PG \to BG$ the natural projection.

Theorem 16 *The bundle $\xi_G = \{\pi_G : PG \longrightarrow BG\}$ is a universal G-bundle.*

The classifying map

Theorem 17 *Let G be a subgroup of $GL_n\mathbb{R}$ and $\xi = \{\pi : P \to B\}$ be a principal G-bundle. Suppose that the base space B can be covered by $q+1$ open sets $U_0, ..., U_q$ relative to which there exists a partition of unity, and further that the bundle ξ is trivial when restricted to any of the U_i's. Then there exists a classifying map $f : B \to BG_q$.*

What we mean here by classifying map is really that the bundle ξ is isomorphic to the pull back through f of the approximation ξ_G^q of the universal bundle. We do not claim that the map f should be unique up to homotopy, which is actually false, so that the terminology of classifying map is presently slightly abusive. However, the composition of f with the canonical inclusion of BG_q in BG is a classifying map in the true sense of the word.
Proof. Let, for every i between 0 and q,

$$\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times G,$$

be some local trivialization of the bundle ξ and

$$g_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}_n(\mathbb{R}),$$

be the corresponding transition functions. Recall that those satisfy the defining equality

$$\phi_i \phi_j^{-1}(x, g) = (x, g_{ij}(x)g)$$

for every $x \in U_i \cap U_j$ and $g \in G$, and further fulfill the *cocycle relations*

$$g_{ii} \equiv Id_{U_i},$$
$$g_{ij}(x)g_{jk}(x) = g_{ik}(x) \quad \text{for every } x \in U_i \cap U_j \cap U_k.$$

Let $\{u_i\}_{i=0}^q$ be a partition of unity relative to the open covering $\{U_i\}_{i=0}^q$ of B. Thus, u_i is, for every i between 0 and q, a mapping

$$u_i : B \longrightarrow [0, 1]$$

whose support is strictly contained in U_i, and for each x in B we have

$$\sum_{i=0}^{q} u_i(x) = 1.$$

For every i between 0 and q, define a continuous G-equivariant map $\overline{f}_i : \pi^{-1}(U_i) \to PG_q$ as

$$\overline{f}_i(u) = \begin{pmatrix} u_0(\pi(u))g_{0i}(\pi(u))g \\ \vdots \\ u_i(\pi(u))g_{ii}(\pi(u))g \\ \vdots \\ u_q(\pi(u))g_{qi}(\pi(u))g \end{pmatrix},$$

where u belongs to $\pi^{-1}(U_i)$ and the image of u via ϕ_i is $\phi_i(u) = (\pi(u), g)$. Of course, the matrix is to be understood as an N times n matrix consisting of $q+1$ blocks of square matrices. If $g_{ji}(\pi(u))$ is not defined, it means that $\pi(u)$ does not belong to U_j, in which case $u_j(\pi(u))$ is zero, so that we consider $u_j(\pi(u))g_{ji}(\pi(u))$ as the n times n zero matrix. Observe that this N times n matrix really represents a frame, since the block $u_i(\pi(u))g_{ii}(\pi(u))g$ has non zero determinant.

We claim that it follows from the cocycle relations that $\overline{f}_i = \overline{f}_j$ on $\pi^{-1}(U_i \cap U_j)$. To see that, let u belong to $\pi^{-1}(U_i \cap U_j)$ and assume that $i < j$. We compute

$$\overline{f}_i(u) = \begin{pmatrix} u_0(\pi(u))g_{0i}(\pi(u))g \\ \vdots \\ u_i(\pi(u))g_{ii}(\pi(u))g \\ \vdots \\ u_j(\pi(u))g_{ji}(\pi(u))g \\ \vdots \\ u_q(\pi(u))g_{qi}(\pi(u))g \end{pmatrix} = \begin{pmatrix} u_0(\pi(u))g_{0j}(\pi(u))g_{ji}(\pi(u))g \\ \vdots \\ u_i(\pi(u))g_{ij}(\pi(u))g_{ji}(\pi(u))g \\ \vdots \\ u_j(\pi(u))g_{jj}(\pi(u))g_{ji}(\pi(u))g \\ \vdots \\ u_q(\pi(u))g_{qj}(\pi(u))g_{ji}(\pi(u))g \end{pmatrix},$$

which is precisely equal to $\overline{f}_j(u)$ since

$$\phi_j(u) = \phi_j\phi_i^{-1}(\pi(u), g) = (\pi(u), g_{ji}(\pi(u))g).$$

The maps $\overline{f}_i$ agreeing on their domain's intersection, they induce a continuous G-equivariant map

$$\overline{f} : P \longrightarrow PG_q.$$

Let $f : B \to BG_q$ be the corresponding map on the base spaces. By Lemma 11 it now follows that the pulled back bundle $f^*(\xi_G^q)$ is isomorphic to ξ. ∎

The classifying map for bundles over simplicial complexes

Let G be, as before, a subgroup of $\mathrm{GL}_n\mathbb{R}$ and suppose that $\xi = \{\pi : P \to |K|\}$ is a principal G-bundle over the geometric realization of some q-dimensional simplicial complex K. (Consult Chapter 2 for any reminder on the basics on simplicial complexes.) We would like to exhibit a finite covering of $|K|$ on which the bundle ξ can be trivialized. If we were ready to consider coverings with arbitrarily many subsets, we could consider the covering

$$\{\mathrm{star}(v)\}_{v\in K^0}.$$

Indeed, the stars being contractible (Lemma 44), the bundle ξ is trivial over them, as follows from Corollary 14. However, we would like to bound the number of sets in the covering independently of the simplicial complex (but depending on the dimension q), and here of course K^0 can get as big as one wants. To do so, we will consider the stars in the first barycentric subdivision of K and take union of stars of barycenters of simplices of K of same dimension.

More precisely, let K_{bar} denote the first barycentric subdivision of K, and observe that the stars in K_{bar} of two barycenters of simplices of K of same dimension are always disjoint. Defining S_i to be the open subset of $|K_{\text{bar}}|$ consisting of the union of the stars (in K_{bar}) of all barycenters of i-dimensional simplices of K,

$$S_i = \coprod_{\substack{s \in K, \\ \mathrm{Dim} s = i}} \mathrm{star}_{K_{\text{bar}}}(b^s),$$

we conclude that we get a finite covering $\{S_0, ..., S_q\}$ of $|K_{\text{bar}}| \simeq |K|$ such that the bundle ξ is trivial when restricted to any of the S_i's.

This covering of $|K_{\text{bar}}|$ is naturally endowed with a partition of unity. Indeed, every point x in $|K_{\text{bar}}|$ can uniquely be written as

$$x = \sum_{i=0}^{q} t_i b^{s_i},$$

where b^{s_i} is the barycenter of the i-dimensional simplex s_i of K, the t_i's are all non negative, and the sum $\sum t_i$ is equal to 1. We can thus define, for every i between 0 and q, functions $|K_{\text{bar}}| \to [0,1]$ by sending the point x to its coordinate t_i. This is not quite a partition of 1 subordinate to the covering $\{S_0, ..., S_q\}$ of $|K_{\text{bar}}|$ since the support of those functions is not strictly contained in the corresponding functions. However, since we are in the topological and not the differentiable setting, the classifying map constructed in the proof of Theorem 17 can be obtained analogously.

Let, for every i between 0 and q,

$$\phi_i : \pi^{-1}(S_i) \longrightarrow S_i \times G,$$

be some local trivialization of the bundle ξ and

$$g_{ij} : S_i \cap S_j \longrightarrow \mathrm{GL}_n(\mathbb{R}),$$

be the corresponding transition functions.

From the proof of Theorem 17 we now directly obtain an explicit classifying map for the bundle ξ.

Theorem 18 *Let ξ be a principal G-bundle over the geometric realization of some q-dimensional simplicial complex. Then the map*

$$\begin{array}{rccc} f: & |K| & \longrightarrow & BG_q \\ & x = \Sigma_{j=0}^{q} t_i b^{s_i} & \longmapsto & \begin{bmatrix} t_0 g_{0i}(x) \\ \vdots \\ t_i \mathrm{Id}_n \\ \vdots \\ t_q g_{qi}(x) \end{bmatrix}_G \end{array},$$

where i is chosen so that $t_i \neq 0$, is a classifying map for the bundle ξ.

1.2 Elements of Differential geometry

We review here the theory of connections and curvatures. Our exposition is strongly inspired from [KoNo63], where the reader is referred to for further details.

1.2.1 Connections

Let G be a Lie group and $\xi = \{\pi : P \to M\}$ be a smooth principal G-bundle over a manifold M. For every u in P, let $\mathfrak{g}_u$ be the subspace of the tangent space T_uP at u consisting of vectors tangent to the fibre through u. We call $\mathfrak{g}_u$ the *vertical subspace* of T_uP.

Definition 19 *A connection Γ in the principal G-bundle $\xi = \{\pi : P \to M\}$ is the choice, for each u in the total space P, of a horizontal subspace H_u of the tangent space T_uP at u such that*

1. $T_uP = \mathfrak{g}_u \oplus H_u$,
2. $H_{ug} = (R_g)_* H_u$, *for* $g \in G$,
3. *H_u depends differentially on u, that is, the assignment $u \longmapsto H_u$, viewed as a map $P \to Gr_r(TP)$, where $Gr_r(TP)$ is the Grassmanian bundle over P consisting of r-planes in T_uP (for every u in P) and r is equal to the dimension of P minus the dimension of G, is required to be smooth.*

Each vector X in T_uP has a unique decomposition $X = X_G + X_H$, where $X_G \in \mathfrak{g}_u$ and $X_H \in H_u$. We call X_G *vertical*, and X_H *horizontal*.

Let ξ and ξ' be two principal G-bundles and denote by P and P' their respective total spaces. Let $f : \xi' \to \xi$ be a bundle map. Any connection Γ on ξ pulls back, via f, to a connection on ξ'. Indeed, if H_u is, for u in P the horizontal space of the connection Γ, then define H'_v for ever v in P' as follows:

$$H'_v = \{X \in T_vP' \mid Tf(X) \in H_{f(u)}\}.$$

The so defined connection is denoted by $f^*(\Gamma)$.

To each connection Γ, one can now assign a connection 1-form ω in the following way:

Definition 20 *The connection form $\omega \in A^1(P, \mathfrak{g})$ is defined, for each u in P and for each X in T_uP as $\omega_u(X) = A \in \mathfrak{g}$, where A is the unique element in $\mathfrak{g}$ satisfying $(A^*)_u = X_G$.*

Note that for every u in P and X in T_uP we have that $\omega_u(X) = 0$ if and only if X is horizontal. When no confusion can occur, we sometimes omit the subscript u and write $\omega(X)$ instead of $\omega_u(X)$.

Proposition 21 *Let ξ be a principal G-bundle and ω a connection form on ξ. The following hold:*

1. $\omega(A^*) = A$, *for every A in $\mathfrak{g}$,*
2. $(R_g)^*\omega = \mathrm{Ad}(g^{-1})\omega$, *for every g in G.*

Conversely, any 1-form $\omega \in A^1(P, \mathfrak{g})$ satisfying the two above conditions uniquely determines a connection Γ whose connection form is ω.

For a proof, we refer the reader to Proposition 1.1 of Chapter 2 in [KoNo63]. We would just like to point out, that given a 1-form $\omega \in A^1(P, \mathfrak{g})$ satisfying the conditions of the proposition, the horizontal spaces H_u of the connection Γ are given, for every u in P, as

$$H_u = \{X \in T_u P \mid \omega(X) = 0\}.$$

We have thus established the equivalence between the knowledge of a connection and a 1-form in $A^1(P, \mathfrak{g})$ satisfying the conditions of Proposition 21. For this reason, we will later call connection forms simply connections. There is one further useful equivalent notion which we will now describe.

Consider the bundle

$$\operatorname{Proj}(\xi) = \begin{array}{c} \operatorname{Proj}(P) \\ \downarrow \\ P, \end{array}$$

where the total space $\operatorname{Proj}(P)$ over a point u of P is defined to be the space of projectors $T_u P \to T_u P$ with kernel equal to the vertical space $\mathfrak{g}_u$. Endow it with the following natural right action of G:

$$\begin{array}{ccc} \operatorname{Proj}(P) \times G & \longrightarrow & \operatorname{Proj}(P) \\ (h, g) & \longmapsto & hg, \end{array}$$

where hg is defined as follows: If h is a projector of $T_u P$, for some u in P, then hg is a projector of $T_{ug} P$ which is defined as

$$(hg)(X) = R_{g^{-1}*}(h_{ug^{-1}}(R_{g*}X)),$$

for every X in $T_{ug}P$. This indeed defines an action on $\operatorname{Proj}(P)$, since for every $h : T_u P \to T_u P$ in $\operatorname{Proj}(P)$ for some u in P, and for every g_1, g_2 in G, we have, for X in $T_{ug_1g_2}P$,

$$\begin{aligned} ((hg_1)g_2)_{ug_1g_2}(X) &= R_{g_2^{-1}*}((hg_1)_{ug_2^{-1}}(R_{g_2*}X)) \\ &= R_{g_2^{-1}*}R_{g_1^{-1}*}(h_{(ug_2^{-1})g_1^{-1}})(R_{g_1*}R_{g_2*}X) \\ &= R_{g_2^{-1}g_1^{-1}*}(h_{ug_2^{-1}g_1^{-1}})(R_{(g_1g_2)*}X) \\ &= R_{(g_1g_2)^{-1}*}(h_{u(g_1g_2)^{-1}})(R_{(g_1g_2)*}X). \end{aligned}$$

Observe that the projection map π :$\operatorname{Proj}(P) \to P$ of the bundle $\operatorname{Proj}(\xi)$ is G-equivariant.

Proposition 22 *There is a one-to-one correspondence between connections on ξ and smooth G-equivariant sections of the bundle* $\operatorname{Proj}(\xi)$.

Proof. We only indicate the correspondence, and leave the details to the reader. Given a connection Γ on ξ one defines the section of the bundle $\operatorname{Proj}(\xi)$ to be, on every point u of P the projector (along $\mathfrak{g}_u$) with image equal to the horizontal space H_u of the connection Γ. Conversely, starting with a section of the bundle $\operatorname{Proj}(\xi)$,

$$h : P \longrightarrow \operatorname{Proj}(P),$$

define a connection Γ to have horizontal space H_u, for every u in P, to be equal to the image of the projector $h(u) : T_u P \to T_u P$. ∎

It is now natural to wonder how one goes (directly) from the connection form to a smooth G-equivariant section of the bundle $\mathrm{Proj}(\xi)$ and conversely. Starting with a connection form ω, define a section

$$h : P \longrightarrow \mathrm{Proj}(P)$$

as

$$\begin{array}{rccc} h_u : & T_uP & \longrightarrow & T_uP \\ & X & \longmapsto & X - (\omega_u(X))^*_u, \end{array}$$

for every u in P. Conversely, starting with a section h of the bundle $\mathrm{Proj}(\xi)$, define a connection form ω, for every u in P and X in T_uP, as $\omega_u(X) = A$, where A is the unique element of $\mathfrak{g}$ for which the following equality holds:

$$(A^*)_u = X - h_u(X).$$

To summarize, we have thus established, for every principal G-bundles, the following correspondence:

$$\{\text{Connections } \Gamma\} \longleftrightarrow \left\{ \begin{array}{l} \text{Connection forms} \\ \omega \text{ satisfying} \\ \omega(A^*) = A \text{ and} \\ (R_g)^* \omega = \mathrm{Ad}\left(g^{-1}\right) \omega \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} G\text{-equivariant} \\ \text{sections} \\ \text{of } \mathrm{Proj}(\xi) \end{array} \right\}.$$

The horizontal spaces of a connection Γ are the kernel, respectively the image, of the corresponding connection form, respectively section of $\mathrm{Proj}(P)$. The relation between connection forms and section of $\mathrm{Proj}(P)$ is understood from the formula

$$h_u(X) = X - (\omega_u X)^*_u,$$

where u belongs to P.

With this correspondence in mind, it is now easy to prove the following useful lemma.

Lemma 23 *Let ω_0 and ω_1 be two connection forms with corresponding projectors h_0 and h_1. Let moreover ω be any connection form. Then*

$$\omega(h_0 - h_1) = \omega_1 - \omega_0.$$

Proof. We have

$$\begin{aligned} \left(\omega(h_0 - h_1)\right)_u(X) &= \omega_u(h_{0u}(X) - h_{1u}(X)) \\ &= \omega_u((X - \omega_{0u}(X)^*_u) - (X - \omega_{1u}(X)^*_u)) \\ &= \omega_u((\omega_{1u}(X) - \omega_{0u}(X))^*_u) \\ &= (\omega_{1u} - \omega_{0u})(X), \end{aligned}$$

where the last equality follows from the first assertion of Proposition 21. ■

The product bundle

Consider the trivial principal G-bundle $G \to \{*\}$ over a point. Since for every point g in G the tangent space T_gG is actually equal to the vertical space through g, there can exist only one connection Γ on the latter bundle, namely the one assigning the zero subspace of T_gG to any point g of G.

More generally, consider the product bundle

$$\begin{array}{c} M \times G \\ \downarrow \\ M. \end{array}$$

The *Maurer-Cartan connection* is defined to be the pull back of the unique connection on the trivial G-bundle over a point via the bundle map

$$\begin{array}{ccc} M \times G & \longrightarrow & G \\ \downarrow & & \downarrow \\ M & \longrightarrow & \{*\} \end{array}$$

given by the projection $M \times G \to G$ on the second factor. The horizontal spaces $H_{(x,g)}$ are then given, for every (x, g) in the total space $M \times G$, by the canonical direct sum decomposition of $T(M \times G) \cong TM \times TG$. More precisely, the vertical space $\mathfrak{g}_{(x,g)}$ corresponds to T_gG, and the horizontal space $H_{(x,g)}$ to T_xM.

Existence of connections

Let us recall some elementary facts on projectors sharing the same kernel.

Lemma 24 *Let V be a vector space. If $p, q : V \to V$ are two projectors with the same kernel, then*

$$p \circ q = p.$$

Proof. For any v in V the vector $q(v) - v$ belongs to $\mathrm{Ker}(q)$:

$$q(q(v) - v) = q^2(v) - q(v) = q(v) - q(v) = 0.$$

But since $\mathrm{Ker}(q) = \mathrm{Ker}(p)$, it follows that

$$p(q(v) - v) = 0,$$

and hence

$$p \circ q(v) = p(v)$$

as claimed. ■

Lemma 25 *Let V be a vector space and $p_0, p_1 : V \to V$ two projectors with the same kernel. For any t in $\mathbb{R}$, the convex linear combination of p_0 and p_1,*

$$(1 - t)p_0 + tp_1 : V \to V,$$

is again a projector.

Proof. We have

$$\begin{aligned}((1-t)p_0+tp_1)^2 &= ((1-t)p_0+tp_1)\,((1-t)p_0+tp_1)\\ &= (1-t)^2p_0^2+(1-t)tp_0p_1+t(1-t)p_1p_0+t^2p_1^2\\ &= (1-t)^2p_0+(1-t)tp_0+t(1-t)p_1+t^2p_1\\ &= (1-t)p_0+tp_1,\end{aligned}$$

where of course we have used Lemma 24 and the fact that p_0 and p_1 are projectors. ■

It now follows from Lemma 25, that we can more generally form the convex linear combination of any two sections

$$h_0, h_1 : P \longrightarrow \mathrm{Proj}(P)$$

of the bundle $\mathrm{Proj}(\xi)$ and obtain a section

$$(1-t)h_0 + th_1 : P \longrightarrow \mathrm{Proj}(P),$$

for every t in $\mathbb{R}$. Also, if h_0 and h_1 are chosen to be smooth and G-equivariant, the convex linear combination will enjoy the same properties. We now claim that the corresponding connection form is given as

$$(1-t)\omega_0 + t\omega_1,$$

where of course ω_0 and ω_1 are the connection forms corresponding to the smooth G-equivariant sections h_0 and h_1 respectively. To see that, denote by ω_t the connection form obtained from $(1-t)h_0+th_1$ and let A in $\mathfrak{g}$ be such that $(\omega_t)_u(X) = A$. Using the above correspondence between connections forms and sections of the bundle $\mathrm{Proj}(\xi)$, we have, for every u in P and X in T_uP,

$$\begin{aligned}A_u^* &= X - ((1-t)h_0+th_1)_u\,(X)\\ &= (1-t)(X-(h_0)_uX)+t(X-(h_1)_uX)\\ &= (1-t)(\omega_0)_u(X)_u^* + t(\omega_1)_u(X)_u^*\\ &= ((1-t)\omega_0+t\omega_1)_u(X)_u^*,\end{aligned}$$

and hence

$$(\omega_t)_u = ((1-t)\omega_0 + t\omega_1)_u$$

as claimed. We have thus proven the following proposition:

Proposition 26 *Any convex linear combination of connection forms is again a connection form.*

From the existence of a connection on the trivial bundle and Proposition 26, a standard argument using partition of unity (see for example Theorem 2.1 of Chapter II in [KoNo63]) now leads to:

Corollary 27 *Any principal G-bundle over a paracompact manifold admits a connection.*

1.2.2 Curvature

Let $\xi = \{\pi : P \to M\}$ be a principal G-bundle endowed with a connection form ω. Let h be the corresponding section of the bundle $\mathrm{Proj}(\xi) = \{\mathrm{Proj}(P) \to P\}$, so that $h_u : T_uP \to T_uP$ is, for every u in P, the projection along the vertical space $\mathfrak{g}_u$ onto the horizontal space H_u defined by the connection form ω. The *exterior covariant differentiation* $D : A^q(P,V) \to A^{q+1}(P,V)$, where V is any (real) vector space, is then defined as follows: for every q-form $\alpha \in A^q(P,V)$ define $D\alpha$ by

$$(D\alpha)_u (X_1, ..., X_{q+1}) = (d\alpha)_u(hX_1, ..., hX_{q+1}),$$

for every u in P and $X_1, ..., X_{q+1}$ in T_uP.

Definition 28 *The curvature $\Omega \in A^2(P, \mathfrak{g})$ of the connection form ω is defined as*

$$\Omega = D\omega.$$

Note that Ω is a so called *horizontal form*, which means that $\Omega(X,Y) = 0$ whenever X or Y is vertical.

Also observe that the pull back of a curvature is the curvature of the pull back of the starting connection.

Proposition 29 (Structure equation) *Let $\xi = \{\pi : P \to M\}$ be a principal bundle endowed with a connection form ω. Denote by Ω its corresponding curvature. Then for every $u \in P$ and every $X, Y \in T_uP$ the following equality holds:*

$$d\omega(X,Y) = -\frac{1}{2}[\omega(X), \omega(Y)] + \Omega(X,Y).$$

For a proof, see for example Theorem 5.2 of Chapter II in [KoNo63]. As an immediate consequence of the structure equation we obtain:

Corollary 30 *Let X, Y be two vectors in T_uP. If X and Y are horizontal, then*

$$\omega([X,Y]) = -2\Omega(X,Y),$$

and if X or Y is vertical, then

$$d\omega(X,Y) = -\frac{1}{2}[\omega(X), \omega(Y)].$$

Proposition 31 (Bianchi's identity) *Let $\xi = \{\pi : P \to M\}$ be a principal G-bundle endowed with a connection form ω, and corresponding curvature Ω. Then*

$$D\Omega = 0.$$

The proof of Bianchi's identity can be found in [KoNo63], Theorem 5.4 of Chapter II.

Lemma 32 *Let V be a (real) vector space and $\xi = P \to M$ be a principal G-bundle endowed with a connection Γ, so that covariant differentiation is defined. If $\alpha \in A^q(P,V)$ is in the image of*

$$\pi^* : A^*(M,V) \longrightarrow A^*(P,V)$$

then

$$d\alpha = D\alpha.$$

Proof. Let $\overline{\alpha} \in A^q(M, V)$ be such that $\pi^*(\overline{\alpha}) = \alpha$. Let u be a point in P and $X_1, \ldots, X_{q+1} \in T_uP$ be tangent vectors at u. The main point is that $d\alpha$ is horizontal, that is, $(d\alpha)_u(X_1, \ldots, X_{q+1})$ vanishes whenever one of the X_i's is vertical. Indeed, suppose that X_i is vertical, then using that $\pi_*(X_i) = 0$ we compute

$$\begin{aligned}
(d\alpha)_u(X_1, .., X_i, .., X_{q+1}) &= (d\pi^*(\overline{\alpha}))_u(X_1, .., X_i, .., X_{q+1}) \\
&= (\pi^* d(\overline{\alpha}))_u(X_1, .., X_i, .., X_{q+1}) \\
&= (d(\overline{\alpha}))_{\pi(u)}(\pi_*(X_1), .., 0, .., \pi_*(X_{q+1})) \\
&= 0.
\end{aligned}$$

It now follows by multilinearity of $d\alpha$, that for arbitrary $X_1, \ldots, X_{2q+1} \in T_uP$ we have

$$(d\alpha)_u(X_1, ..., X_{q+1}) = (d\alpha)_u(hX_1, \ldots, hX_{q+1}),$$

which proves the lemma, since the latter expression is the very definition of $(D\alpha)_u(X_1, \ldots, X_{q+1})$. ■

Lemma 33 *Let ω_0, ω_1 be two connection forms on some principal G-bundle ξ. For any $t \in \mathbb{R}$ denote by ω_t the following connection form:*

$$\omega_t = (1 - t)\omega_0 + t\omega_1.$$

Then the curvature Ω_t of ω_t is given as

$$\Omega_t = (1 - t)\Omega_0 + t\Omega_1 + \frac{1}{2}(t^2 - t)[\omega_1 - \omega_0, \omega_1 - \omega_0].$$

Proof. First note that we know from Proposition 26 that ω_t is indeed a connection form. Its corresponding projector is given as

$$h_t = (1 - t)h_0 + th_1,$$

where of course h_0 and h_1 are the projectors obtained from ω_0 and ω_1 respectively. Now let u be a point in P and X, Y vectors in T_uP. Let us compute the value of the curvature Ω_t on (X, Y):

$$\begin{aligned}
\Omega_t(X, Y) &= (d\omega_t)(h_tX, h_tY) \\
&= (1 - t)d\omega_0(h_tX, h_tY) + td\omega_1(h_tX, h_tY) \\
&= (1 - t)\Omega_0(h_tX, h_tY) + t\Omega_1(h_tX, h_tY) \\
&- \frac{1}{2}((1 - t)[\omega_0(h_tX), \omega_0(h_tY)] + t[\omega_1(h_tX), \omega_1(h_tX)]),
\end{aligned}$$

where the last equality follows from the structure equation (Proposition 29). From Lemma 24 we have $h_0h_t = h_0$ and $h_1h_t = h_1$ so that

$$\Omega_0(h_tX, h_tY) = \Omega_0(X, Y) \quad \text{and} \quad \Omega_1(h_tX, h_tY) = \Omega_1(X, Y).$$

Also, since ω_0 vanishes on horizontal vectors, one computes

$$\begin{aligned}
\omega_0(h_tX) &= \omega_0(h_tX - h_0h_tX) \\
&= \omega_0((1 - t)h_0X + th_1X - h_0X) \\
&= t\omega_0(h_1X - h_0X) \\
&= t(\omega_0 - \omega_1)X,
\end{aligned}$$

where the last equality holds by virtue of Lemma 23. Similarly, one obtains

$$\omega_1(h_t X) = (1-t)(\omega_1 - \omega_0)X.$$

We thus have

$$\begin{aligned} -\frac{1}{2}((1-t)[\omega_0(h_t X), \omega_0(h_t Y)] + t[\omega_1(h_t X), \omega_1(h_t X)]) \\ = -\frac{1}{2}\big((1-t)t^2[(\omega_0 - \omega_1)X, (\omega_0 - \omega_1)X] \\ + t(1-t)^2[(\omega_1 - \omega_0)X, (\omega_1 - \omega_0)X]\big) \\ = \frac{1}{2}(t-1)t[(\omega_0 - \omega_1)X, (\omega_0 - \omega_1)X], \end{aligned}$$

so that

$$\begin{aligned} \Omega_t(X,Y) &= (1-t)\Omega_0(X,Y) + t\Omega_1(X,Y) \\ &+ \frac{1}{2}(t^2 - t)[(\omega_0 - \omega_1)(X), (\omega_0 - \omega_1)(Y)], \end{aligned}$$

which finishes the proof of the lemma. ■

Lemma 34 *Let Ω_t be as in the previous lemma, then*

$$\frac{d}{dt}\Omega_t = D_t(\omega_1 - \omega_0).$$

Proof. From Lemma 33 above it follows that

$$\frac{d}{dt}\Omega_t = \Omega_1 - \Omega_0 + (t - \frac{1}{2})[\omega_1 - \omega_0, \omega_1 - \omega_0].$$

Also, we have

$$\begin{aligned} D_t(\omega_1 - \omega_0) &= d(\omega_1 - \omega_0)h_t \\ &= d(\omega_1 - \omega_0)((1-t)h_0 + th_1) \\ &= (1-t)d\omega_1 h_0 - (1-t)\Omega_0 + t\Omega_1 - td\omega_0 h_1. \end{aligned}$$

From the trivial relation $h_0 = h_1 + (h_0 - h_1)$ we obtain

$$\begin{aligned} d\omega_1 h_0 &= d\omega_1(h_1 + (h_0 - h_1)) \\ &= \Omega_1 - \frac{1}{2}[\omega_1(h_0 - h_1), \omega_1(h_0 - h_1)], \end{aligned}$$

where the last equality follows from the second assertion of Corollary 30. As from Lemma 23 we know that $\omega_1(h_0 - h_1) = \omega_1 - \omega_0$, we have

$$d\omega_1 h_0 = \Omega_1 - \frac{1}{2}[\omega_1 - \omega_0, \omega_1 - \omega_0],$$

and by symmetry also

$$\begin{aligned} d\omega_0 h_1 &= \Omega_0 - \frac{1}{2}[\omega_0 - \omega_1, \omega_0 - \omega_1] \\ &= \Omega_0 - \frac{1}{2}[\omega_1 - \omega_0, \omega_1 - \omega_0]. \end{aligned}$$

Putting all this together, we can finally conclude that

$$\begin{aligned} D_t(\omega_1 - \omega_0) = (1-t)\left(\Omega_1 - \frac{1}{2}[\omega_1 - \omega_0, \omega_1 - \omega_0]\right) - (1-t)\Omega_0 \\ + t\Omega_1 - t\left(\Omega_0 - \frac{1}{2}[\omega_1 - \omega_0, \omega_1 - \omega_0]\right) \\ = \Omega_1 - \Omega_0 + (t - \frac{1}{2})[\omega_1 - \omega_0, \omega_1 - \omega_0], \end{aligned}$$

which was to be proven. ■

1.3 Flat bundles

1.3.1 Definition

Let us start straightaway with the definition of flat bundle:

Definition 35 *Let ξ be a smooth principal G-bundle. A connection on ξ is said to be flat if its curvature form vanishes identically. A smooth G-principal bundle is called flat if it can be endowed with a flat connection.*

Since any convex linear combination of (non necessarily flat) connections is again a connection (see Proposition 26) it follows that the space of all connections is an affine subspace of $A^1(P, \mathfrak{g})$. The geometry of its subspace of flat connections is however a much more complicated. (As a simple example, the convex linear combination of two flat connections is in general not flat, as can easily be concluded from Lemma 33.) In order to understand the space of flat connections, we are going to give various equivalent definitions of flat bundles and translate the notion of being in the same path connected component in those new settings. Observe that those equivalent definitions will all make sense in the topological case, so that it will be possible to extend the definition of flat bundles to topological bundles. Before proceeding, let us give some trivial examples of flat bundles.

1. The Maurer-Cartan connection ω_G on the trivial G-bundle over a point,

$$\begin{array}{c} G \\ \downarrow \\ \{*\}, \end{array}$$

is flat since there are no non trivial horizontal vectors.

2. The Maurer-Cartan connection on the trivial product bundle

$$\begin{array}{c} M\times G \\ \downarrow \\ M \end{array}$$

is flat since it is the pull back of the Maurer-Cartan connection $\omega_G \in A^1(G,\mathfrak{g})$ through the projection map

$$M\times G \longrightarrow G,$$

and obviously, being flat is invariant under taking pull backs.

1.3.2 Transition functions

In this section we explain the interpretation of flat connection in terms of transition functions. The following theorem together with its corollary can be found in [Du78] (Theorem 3.21 and Corollary 3.22).

Theorem 36 *A connection ω in a principal G-bundle*

$$\xi = \begin{array}{c} P \\ \downarrow \pi \\ M \end{array}$$

is flat if and only if for every x in M there exists a neighborhood U of x and a trivialization of $P_{|U}$ such that the restriction of ω to $P_{|U}$ is induced by the Maurer-Cartan connection in $U\times G$.

Corollary 37 *Let ξ be a principal G-bundle over some manifold M. Are equivalent:*

1. *the bundle ξ can be endowed with a flat connection,*
2. *there exists a covering of M and a set of transition functions for ξ which are locally constant,*
3. *the bundle ξ has a reduction to G^δ.*

We will restrict to the proof of the equivalence (1) $\iff$ (2) and refer the reader to [Du78] for a complete proof.

Proof. (1) $\Longrightarrow$ (2): Let ξ be a G-bundle endowed with a flat connection ω. By Theorem 36, the manifold M has an open covering $\{U_i\}_{i\in I}$ for which there exists trivializations

$$\phi_i : P|_{U_i} \longrightarrow U_i\times G$$

such that the flat connection ω on $P|_{U_i}$ is induced from the Maurer-Cartan connection on the product bundle $U_i\times G$. The relation

$$\phi_j^{-1}(x, g_{ij}(x)g) = \phi_i^{-1}(x,g),$$

for every x in $U_i\cap U_j$ and g in G, defining the transition functions

$$g_{ij} : U_i\cap U_j \longrightarrow G,$$

is summarized in the diagram

$$\begin{array}{ccc} P|_{U_i\cup U_j} & \xrightarrow{\phi_i} & (U_i\cap U_j)\times G \\ \big\downarrow \mathrm{Id} & & \big\downarrow F \\ P|_{U_i\cap U_j} & \xrightarrow{\phi_j} & (U_i\cap U_j)\times G, \end{array}$$

where the map F is defined, for every x in $U_i \cap U_j$ and every g in G as

$$F(x,g) = F(x, g_{ij}(x)g).$$

Since the flat connection ω on $P|_{U_i\cap U_j}$ is induced from the flat Maurer-Cartan connection on $U_i \cap U_j \times G$ via ϕ_i, but also via ϕ_j, it follows that the map F must send the Maurer-Cartan connection to itself. Equivalently, this means that the induced map between the corresponding tangent bundles

$$TF : T\left(U_i \cap U_j\right) \times TG \longrightarrow T\left(U_i \cap U_j\right) \times TG,$$

as it need to preserve the horizontal spaces, sends $T(U_i \cap U_j) \times \{0\}$ to itself. Viewing the tangent space as equivalence classes of curves, we obtain that, for any curve $v : [-\varepsilon, \varepsilon] \to U_i \cap U_j$, passing through the point v_0 at time 0, and for any g in G, the image via TF of the curve $(v(t), g) \subset U_i \cap U_j \times G$ is equal to

$$(v(t), g_{ij}(v(t))g),$$

and thus the second coordinate must be constant. As this is valid for any curve $v(t)$ it follows that the g_{ij}'s must be locally constant.

(2) $\Longrightarrow$ (1): Let $\{U_i\}_{i\in I}$ be an open covering of M, and

$$\phi_i : P|_{U_i} \longrightarrow U_i \times G$$

local trivializations of the bundle ξ for which the corresponding transition functions

$$g_{ij} : U_i \cap U_j \longrightarrow G$$

are locally constant. Let ω_i be the flat connection on $P|_{U_i}$ which is the pull back by ϕ_i of the Maurer-Cartan connection on $U_i \times G$. We claim that the connections ω_i and ω_j agree on $P|_{U_i\cap U_j}$. This is equivalent to saying that the Maurer-Cartan flat connection on the product bundle over $U_i \cap U_j$ is equal to the pull back of the Maurer-Cartan connection via the map

$$\begin{array}{ccc} (U_i \cap U_j) \times G & \longrightarrow & (U_i \cap U_j) \times G \\ (x,g) & \longmapsto & (x, g_{ij}(x)g). \end{array}$$

Assume without loss of generality, that $U_i \cap U_j$ is connected, and since the transition functions are locally constant, we can define $g_{ij} := g_{ij}(x)$, for some x in $U_i \cap U_j$. Now, the Maurer-Cartan connection is induced by the projection on G from the Maurer-Cartan flat connection ω_G on the trivial bundle $G \to \{*\}$. By definition we have

$$\omega_G = L^*_{g_{ij}}\omega_G,$$

where $L_{g_{ij}} : G \to G$ stands for the left multiplication by g_{ij}. The claim now follows and we can thus define a global flat connection ω on ξ as $\omega = \omega_i$ on $\xi|_{U_i}$. ■

Note that conditions (2) and (3) of the above corollary also make sense for topological principal G-bundles. We can thus, as promised, extend the definition of flat bundles to them:

Definition 38 *Let G be a topological group. A topological principal G-bundle is said to be flat if it has a reduction to G^δ.*

Another useful consequence of Corollary 37 is the following:

Corollary 39 *Let ξ be a principal G-bundle over some manifold M endowed with a flat connection. Then any covering $\{U_i\}_{i\in I}$ of M satisfying $\pi_1(U_i) = 1$ for every i in I admits a set of transition functions for ξ which are locally constant.*

It follows from Corollary 37 that to any flat connection on a flat bundle, we can associate a set of locally constant transition functions, and conversely, given such a family of transition functions, it uniquely determines a flat connection. It is now easy to see that two families of locally constant transition functions $\{g_{ij}\}$ and $\{h_{ij}\}$ relative to the same open covering $\{U_i\}$ of the base space will determine the same flat connection if and only if there exists locally constant maps $\lambda_i : U_i \to G$ such that

$$g_{ij} = \lambda_i^{-1} h_{ij} \lambda_j.$$

Actually, this is Lemma 10 in the case where the topological group is G^δ: Indeed, a locally constant map $\lambda_i : U_i \to G$ is nothing else than a continuous map $\lambda_i : U_i \to G^\delta$.

Assuming for simplicity that the group G is connected, we can also conclude that the two flat connections obtained from the two families of locally constant transition functions $\{g_{ij}\}$ and $\{h_{ij}\}$ lie in the same path connected component of flat connections if and only if there exists a family of homotopies

$$H_{ij} : U_i \cap U_j \times [0,1] \longrightarrow G$$

between g_{ij} and h_{ij} such that for each fixed $t \in [0,1]$, the family $\{H_{ij}(.,t)\}$ is a system of locally constant transition functions.

1.3.3 The space of representations

Let X be a connected topological space for which the covering theory applies. The canonical example of flat bundle is the following (for the justification of the term "canonical" see Proposition 40 below): Let $h : \pi_1(X) \to G$ be a group homomorphism. There is a natural left diagonal action of the fundamental group of X on the product $\widetilde{X} \times G$ given by

$$\begin{array}{ccc} \pi_1(X) \times (\widetilde{X} \times G) & \longrightarrow & \widetilde{X} \times G \\ (\gamma, (x,g)) & \longmapsto & (\gamma \cdot x, h(\gamma) g). \end{array}$$

The group G still acts (from the right) on the quotient $\pi_1(X)\backslash(\widetilde{X} \times G)$, and it is easy to check that the G-bundle

$$\xi_h = \begin{array}{c} \pi_1(X)\backslash(\widetilde{X} \times G) \\ \downarrow \\ X \end{array}$$

is flat.

Proposition 40 *Every flat principal G-bundle over X is isomorphic to a bundle of the form ξ_h, for some homomorphism*

$$h : \pi_1(X) \longrightarrow G.$$

This is standard. We refer the reader to [Mi58] (Lemma 1) for a proof. Observe that the statement of the proposition can even be strenghtened to: Every principal G^δ-bundle over X is isomorphic, as a G^δ-bundle, to a bundle of the form ξ_h.

Note that it follows that if the fundamental group of X is trivial, then there can exist no non trivial flat bundles over X. This is the case for all spheres of dimension greater or equal to 2. For example the frame bundles associated to their tangent bundles (and thus the tangent bundles themselves) can not be flat.

Denote by $\mathrm{Rep}(\pi_1(X), G)$ the space of all homomorphisms from $\pi_1(X)$ to G and endow it with the compact-open topology. The following proposition is easy:

Proposition 41 *If h_0 and h_1 are in the same path connected component in $Rep(\pi_1(X), G)$, then the corresponding flat bundles ξ_{h_0} and ξ_{h_1} are isomorphic.*

The idea of the proof is that any representation $h : \pi_1(X) \to G$ gives rise to a canonical map $X \to BG$ classifying the bundle ξ_h. Now, a path between two representations in $\mathrm{Rep}(\pi_1(X), G)$ will automatically produce a homotopy between the corresponding classifying maps, so that the induced bundles will be isomorphic. Note that this isomorphism is really an isomorphism of G-bundles, and certainly not of G^δ-bundles in general.

If the fundamental group of X is finitely generated, then it admits a presentation of the form

$$\pi_1(X) = \langle s_1, ...s_k \mid r_i(s_1, ..., s_k) = 1,\ i \in I \rangle.$$

It is thus only natural to view the space of representations as

$$\mathrm{Rep}(\pi_1(X), G) = \{(g_1, ..., g_k) \in G^k \mid r_i(g_1, ..., g_k) = 1_G,\ i \in I\}.$$

Assuming further that G is an algebraic group, the space $\mathrm{Rep}(\pi_1(X), G)$ can naturally be endowed with the structure of an algebraic variety. It consequently only has finitely many path connected component and, as pointed out by Lusztig, we immediately obtain:

Corollary 42 *If $\pi_1(X)$ is finitely generated and G is algebraic, then there exists only finitely many isomorphism classes of flat bundles over X.*

Consider on the space of representations $\mathrm{Rep}(\pi_1(X), G)$ the natural equivalence relation given by conjugation. More precisely, two homomorphisms h_0 and h_1 are equivalent (denoted $h_0 \sim h_1$) if and only if there exists g in G such that

$$h_0(\gamma) = g h_1(\gamma) g^{-1}$$

for every γ in $\pi_1(X)$.

Now if the base space is a smooth connected manifold, say M, there is a one to one correspondence between flat connections ω and equivalence classes of homomorphisms $[h]$ in $\mathrm{Rep}(\pi_1(X), G)/\sim$. Assuming again for simplicity that G is connected, it is clear that two equivalent homomorphisms $h_0 \sim h_1$ are in the same path connected component of $\mathrm{Rep}(\pi_1(X), G)$. Observe further that two arbitrary homomorphisms h_0 and h_1 are in the same path connected component of $\mathrm{Rep}(\pi_1(X), G)$ if and only if the corresponding flat connections lie in the same path connected component in the space of flat connections.

Chapter 2

Simplicial complexes

2.1 Definitions

Let V be a set. A *simplicial complex* K consists of a family of non empty subsets of V, called the *simplices* of K, satisfying the two following properties:

- For every v in V, the set $\{v\}$ belongs to K.
- If k belongs to K, then so does any subset of k.

A *face* of a simplex k is a simplex k' which is contained in k, in which case we write $k' \leq k$. If k' is strictly contained in k, then it is said to be a proper face of k and we write $k' < k$.

The set V is denoted by $\mathrm{Vert}(K)$. Its elements are identified with the corresponding singletons of K and are called the *vertices* of K. A simplex of K containing precisely $q+1$ distinct vertices is a *q-simplex* and is said to have dimension q.

A *subcomplex* of a simplicial complex K is a subset of K which is itself a simplicial complex. The union of all simplices of a simplicial complex K of dimension smaller or equal to q forms a subcomplex of K, called the *q-skeleton* of K and denoted by K^q.

We say that a simplicial complex K is *finite* if its vertex set is finite. The *dimension* of K is equal to the maximal dimension of its simplices. The simplicial complex K is said to be *finite dimensional* if its dimension is finite. In this case, $K^q = K$ for some $q < \infty$. A simplicial complex is further *of finite type* if all its q-skeletons are finite.

A *simplicial map* between two simplicial complexes K and L is a map

$$\varphi : \mathrm{Vert}(K) \longrightarrow \mathrm{Vert}(L)$$

such that if the subset $k = \{v_0, ..., v_q\}$ of $\mathrm{Vert}(K)$ is a simplex of K, then the subset $\varphi(k) = \{\varphi(v_0), ..., \varphi(v_q)\}$ of $\mathrm{Vert}(L)$ is a simplex of L. Observe that if k is a q-simplex of K then $\varphi(k)$ is a simplex of L of possibly smaller dimension, for the set $\varphi(k)$ does not necessarily consists of $q+1$ distinct points.

Geometric realizations

Define $K_{\rm or}$ to be the set of all ordered simplices of K. For every simplex $k = \{v_0, ..., v_q\}$ of K denote by $[v_0, ..., v_q]$ the ordered simplex in $K_{\rm or}$ obtained from k and the ordering of the v_i's by their numbering.

To every simplicial complex K one can associate its *geometric realization* $|K|$ which is a topological space constructed as follows: For every ordered q-dimensional simplex $k = [v_0, ..., v_q]$, let Δ^q_k be a copy of the standard q-dimensional simplex

$$\Delta^q = \left\{ (t_0, ..., t_q) \in \mathbb{R}^{q+1} \;\middle|\; \sum_{i=0}^{q} t_i = 1,\ t_i \geq 0 \right\}.$$

The space $|K|$ is defined as the quotient

$$|K| = \left(\coprod_{\substack{k \in K_{\rm or} \\ k=[v_0,...,v_q] \\ {\rm Dim} k = q}} \Delta^q_k \right) / \sim,$$

where the equivalence relation $\sim$ is defined as follows: Let $k = [v_0, ..., v_q]$ and $k' = [w_0, ..., w_p]$ be ordered simplices of dimension q, respectively p, of K. Two points $(t_0, ..., t_q) \in \Delta^q_k$ and $(r_0, ..., r_p) \in \Delta^p_{k'}$ are equivalent if and only if the (unordered) simplex underlying k' is a face of the simplex underlying k, or in other words $\{w_0, ..., w_p\}$ is contained in $\{v_0, ..., v_q\}$, and moreover, letting $i_0, ..., i_p$ be the integers between 0 and n satisfying $v_{i_\ell} = w_\ell$, the requirements

$$\begin{array}{ll} t_i = 0 & \text{if } i \notin \{i_0, ..., i_p\}, \\ t_{i_j} = r_j & \text{if } i = i_j \text{ for } j \in \{0, ..., p\}, \end{array}$$

are fulfilled. Note that because the vertices of k and k' are all distinct, the assignment $\ell \mapsto i_\ell$ is a bijection between the sets $\{0, ..., p\}$ and $\{i_0, ..., i_p\} \subset \{0, ..., q\}$. Each copy of a standard q-simplex comes with the induced topology of $\mathbb{R}^{q+1}$, and the space $|K|$ is naturally endowed with the quotient topology. It is compact if the simplicial complex K is finite.

If $k = \{v_0, ..., v_q\}$ is a q-dimensional simplex of K, we write

$$\sum_{i=0}^{q} t_i v_i$$

for the image of the point $(t_0, ..., t_q) \in \Delta^q_k$ in $|K|$. For every q-dimensional simplex k of K, denote by $|k|$ the image of Δ^q_k in $|K|$. Recall that the interior ${\rm int}(\Delta^q)$ of the standard q-simplex is given as

$${\rm int}\,(\Delta^q) = \{(t_0, ..., t_q) \in \mathbb{R}^{q+1} \mid \sum_{i=0}^{q} t_i = 1,\ t_i > 0\}.$$

Define the *interior* ${\rm int}(k)$ of k as the image of ${\rm int}(\Delta^q_k)$ in $|K|$. Note that ${\rm int}(k)$ is in general not open in $|K|$: It is only open as a subset of $|k|$. The space $|K|$ is easily shown to be equal to the disjoint union of the interior of all its simplices.

Suppose now that K is a simplicial complex with countable vertex set $\text{Vert}(K) = \{v_1, v_2, ...\}$. It is then possible to visualize $|K|$ as a subspace of $\lim_{q\to\infty}\mathbb{R}^q$, where the limit is obtained from the canonical inclusion $\mathbb{R}^q \hookrightarrow \mathbb{R}^q \oplus \{0\} \hookrightarrow \mathbb{R}^{q+1}$. Let $\{e_1, e_2, ...\}$ be the canonical basis of $\lim_{q\to\infty}\mathbb{R}^q$. The geometric realization $|K|$ can naturally be identified with the union of the convex hull of all points $\{e_{i_0}, ..., e_{i_q}\}$ whenever $\{s_{i_0}, ..., s_{i_q}\}$ is a simplex of K. The topology on $|K|$ agrees with the induced topology of $\lim_{q\to\infty}\mathbb{R}^q$.

A simplicial map $\varphi : K \to L$ induces a continuous map

$$|\varphi| : |K| \longrightarrow |L|,$$

which is defined as follows: For every point $\sum_{i=0}^q t_i v_i$ of $|K|$, where $\{v_0, ..., v_q\}$ is a q-simplex of K , the sum $\sum_{i=0}^q t_i$ is equal to 1 and the t_i's are all greater or equal to zero, define

$$|\varphi|\left(\sum_{i=0}^q t_i v_i\right) = \sum_{i=0}^q t_i \varphi(v_i).$$

Note that this is well defined since φ being a simplicial map, the vertices $\varphi(v_0), \ldots, \varphi(v_q)$ of L span a simplex in L so that the right hand side of the previous equality makes sense.

The geometric realization of a simplicial complex is of course quite rigid since it consists of piecewise linear pieces. The following definition allows us to consider homeomorphy classes of geometric realization of simplicial complexes, so that for example smooth manifolds can then be considered.

Triangulations and refinements

A topological space X is said to be a *polyhedron* if there exists a simplicial complex K and a homeomorphism

$$h : X \longrightarrow |K|.$$

The pair (K, h) is called a *triangulation* of X.

Observe that it is obviously possible to glue triangulations in the following sense: Let X be a topological space which is the union of two subsets $X_1 \cup X_2$. Let (K_1, h_1) and (K_2, h_2) be triangulations of X_1 and X_2. Suppose that there exists subcomplexes L_1 of K_1 and L_2 of K_2 and an isomorphism $\varphi : L_1 \to L_2$ such that the diagram

$$\begin{array}{ccc} X_1 \cap X_2 & \longrightarrow & L_1 \\ & \searrow & \downarrow \\ & & L_2 \end{array}$$

commutes. A simplicial complex K is then obtained as follows: Define K as the quotient of the disjoint union $K_1 \coprod K_2$ by the equivalence relation $k_1 \sim k_2$ if k_1, respectively k_2, belongs to K_1, respectively K_2, and $k_2 = \varphi(k_1)$. The map $(h_1, h_2) : |K_1| \coprod |K_2| \to X$ factors through $|K|$, so that we obtain the desired triangulation of X.

It is often very useful to triangulate simplicial complexes themselves, or more precisely their geometric realizations. Of course, we do not want to triangulate them arbitrarily, for we would like the triangulation to restrict to a triangulation of each simplex of the simplicial complex. This is the purpose of the next definition.

A *refinement* of a simplicial complex K is a pair (L, r) consisting of a simplicial complex L and a homeomorphism

$$r : |K| \longrightarrow |L|$$

satisfying

$$r(K^q) \subset K^q$$

for every $q \geq 0$. The *index* of a refinement (L, r) of a simplicial complex K, which we denote by $[L : K]$, is the maximal number of simplices in the triangulation of L restricted to any simplex of K. More precisely, we have

$$[L : K] = \max_{k \in K} \sharp \left\{ \ell \;\middle|\; \begin{array}{l} r^{-1}(|\ell|) \subset |k|, \\ \text{the vertices of } \ell \text{ are distinct} \end{array} \right\}.$$

Similarly, we define the *index of degree* d of a refinement, to be

$$[L : K]_d = \max_{k \in K} \sharp \left\{ \ell \;\middle|\; \begin{array}{l} r^{-1}(|\ell|) \subset |k|, \\ \ell \text{ contains } d+1 \text{ distinct vertices} \end{array} \right\}.$$

The most important example of refinement is the *barycentric subdivision* K_{bar} of a simplicial complex: Let K be a simplicial complex. Define the simplicial complex K_{bar} to have vertices $\text{Vert}(K_{\text{bar}}) = \{k \in K \mid \text{the vertices of } k \text{ are all distinct}\}$, and $\{k_0, ..., k_q\}$ is a simplex of K_{bar} if and only if $k_0 \leq ... \leq k_q$. It is clear that we have thus defined a simplicial complex. The inverse of the homeomorphism $r : |K| \to |K_{\text{bar}}|$ is most easily defined as

$$\begin{array}{ccc} |K_{\text{bar}}| & \longrightarrow & |K| \\ \Sigma_{i=0}^q t_i k_i & \longmapsto & \Sigma_{i=0}^q t_i b^{k_i}, \end{array}$$

where if $k = \{v_0, ..., v_q\}$ is a q-dimensional simplex of K, its *barycenter* $b^k \in |k| \subset |K|$, is given as

$$b^k = \frac{1}{q+1} v_0 + \cdots + \frac{1}{q+1} v_q.$$

It was believed for quite some time that two triangulations of the same space always admitted a common refinement. This problem, named *Hauptvermutung* or *Principle Conjecture* can more precisely be formulated as follows: If (K, h) and (K', h') are two triangulations of some topological space X, then there exists a simplicial complex L and refinements (L, r) of K and (L, r') of K'. It was proven to be true for $n = 3$ by Moise ([Mo52]), but counterexamples were constructed by Milnor for every $n \geq 6$ ([Mi61]). Observe that in the semi-algebraic setting the Hauptvermutung holds (see the remark after Theorem 88).

Stars

Let K be a simplicial complex and k be a simplex of K. Define the star of k by

$$\text{star}(k) = \bigcup \{\text{Int}(t) \mid k \leq t\} \subset |K|.$$

The following lemmas on stars are obvious.

Lemma 43 *Let K be a simplicial complex, and k, ℓ two simplices of K. Then k is a face of k' if and only if $star(k) \supset star(k')$.*

Proof. The simplex k is a face of k' if and only if

$$\{\ell \mid k \leq \ell\} \subset \{\ell \mid k' \leq \ell\},$$

which is equivalent to

$$\mathrm{star}(k) = \bigcup_{\{\ell \ \mid \ k \leq \ell\}} \mathrm{Int}(\ell) \supset \bigcup_{\{\ell \ \mid \ k' \leq \ell\}} \mathrm{Int}(\ell) = \mathrm{star}(k').$$

■

Lemma 44 *Let K be a simplicial complex. The family*

$$\{star(v)\}_{v \in Vert(K)}$$

furnishes a covering of $|K|$ by open and contractible sets.

Lemma 45 *Let K be a simplicial complex and $v_0, ..., v_q$ in $Vert(K)$. The set $\{v_0, ..., v_q\}$ is a simplex of K if and only if*

$$\bigcap_{i=0}^{q} star(v_i) \neq \varnothing.$$

2.2 Examples

We give here three examples of triangulations in increasing difficulty. The last example will be important for the proof of our main theorem, or more precisely for the proof of the Technical Lemma 91.

1. Let $k = \{v_0, ..., v_q\}$ be a q-dimensional simplex and let us exhibit a simple triangulation of $|k| \times [0, 1]$. Define, for every i between 0 and q,

   $$a_i = (e_i, 0) \ \text{ and } \ b_i := (e_i, 1) \in \Delta^q \times [0, 1].$$

   Define a simplicial complex with vertex set $\{a_i, b_i\}_{i=0,...,q}$ and simplices

   $$\{a_0, \ldots, a_i, b_i, \ldots, b_q\}, \ \forall i = 0, \ldots, q,$$

   and all their subsets. The homeomorphism between the geometric realization of the just described simplicial complex and $|k|\times[0, 1]$ is given by sending the simplex $\{a_0, \ldots, a_i, b_i, \ldots, b_q\}$ to the convex hull of the corresponding points in $|k| \times [0, 1]$. This triangulation contains precisely $q + 1$ simplices of dimension $q + 1$.

2. Let now K be a q-dimensional simplicial complex with countable vertex set, and let us generalize the example above to a triangulation of $|K| \times [0, 1]$. Put an order $<$ on the vertices of K. Define a simplicial complex with vertex set

   $$\{(v, 0), (v, 1)\}_{v \in \mathrm{Vert} K}$$

and simplices

$$\{(v_0, 0), .., (v_i, 0), (v_i, 1), .., (v_{q+1}, 1)\},$$

where $\{v_0, .., v_i, .., v_{q+1}\}$ is a simplex of K with $v_0 < .. < v_i < ..v_{q+1}$, for some $i \in \{0, ..., q+1\}$, and all their subsets. This gives a triangulations of $|K| \times [0, 1]$ whose number of $(q+1)$-dimensional simplices is precisely

$$(q + 1) \cdot \sharp\{q\text{-dimensional simplices of } K\}.$$

3. Let finally (L, r) be a refinement of the q-dimensional simplicial K with countable vertex set and let us exhibit a triangulation (T, h) of $|K| \times [0, 1]$,

$$h : |K| \times [0, 1] \longrightarrow |T|,$$

having the properties that

- (T, h) restrict to a triangulation T_0, respectively T_1, of $|K| \times \{0\}$, resp. $|K| \times \{1\}$,
- there exists an isomorphism
$$\varphi_0 : T_0 \longrightarrow K$$
such that the composition of the maps
$$|K| \hookrightarrow |K| \times \{0\} \xrightarrow{h|_{T_0}} |T_0| \xrightarrow{|\varphi_0|} |K|$$
is the identity on $|K|$,
- there exists an isomorphism
$$\varphi_1 : T_1 \longrightarrow L$$
such that the composition of the maps
$$|L| \xrightarrow{r^{-1}} |K| \hookrightarrow |K| \times \{1\} \xrightarrow{h|_{T_1}} |T_1| \xrightarrow{|\varphi_1|} |L|$$
is the identity on $|L|$,
- the number of $(q + 1)$-dimensional simplices in T is bounded by
$$(q + 1) \cdot [L : K]_q \cdot \sharp\{q\text{-dimensional simplices of } K\}.$$

Let T be the simplicial complex with vertex set

$$\mathrm{Vert}T = \{(v, 0)\}_{v\in \mathrm{Vert}K} \cup \{(w, 1)\}_{w\in \mathrm{Vert}L}$$

and simplices

$$\{(v_0, 0), .., (v_i, 0), (w_0, 1), ..(w_{q-i+1}, 1)\}$$

whenever there exists a simplex $\{v_0, ..., v_i, v_{i+1}, ..., v_q\}$ of K with $v_i < v_j$ whenever $i < j$ and such that

$$r^{-1}(|\{w_0, ..., w_{q-i+1}\}|) \subset |\{v_i, ..., v_q\}|.$$

The inverse of the homeomorphism $h : |K| \times [0,1] \to |T|$ is obtained as follows: Let x be a point in the geometric realization of some simplex $\{(v_0, 0), .., (v_i, 0), (w_0, 1), ..(w_{q-i+1}, 1)\}$ of T. The point x can be written uniquely as

$$x = (1-t)(x_0, 0) + t(x_1, 1),$$

where x_0 belongs to $|\{v_0, ..., v_i\}|$ and x_1 belongs to $|\{w_0, ..., w_{q-i+1}\}|$. Define a map $h' : |T| \to |K| \times [0,1]$ by sending the point x to

$$((1-t)x_0 + tr^{-1}(x_1), t) \in |K| \times [0,1].$$

This is well defined since there exists a simplex of K containing both $(1-t)x_0$ and $tr^{-1}(x_1)$.

2.3 Simplicial approximation

Let K and L be simplicial complexes, $\varphi : K \to L$ a simplicial map, and $f : |K| \to |L|$ a continuous map. The map φ is said to be a *simplicial approximation* to f if

$$f(\mathrm{star}(v)) \subset \mathrm{star}(\varphi(v))$$

for every v in $\mathrm{Vert}(K)$.

Somehow, if a simplicial map φ is a simplicial approximation to some continuous map f, it means that the geometric realization $|\varphi|$ and the map f are not so far away from each other. We see for example from the next lemma that the image of any point by the map f always lies in the same simplex than its image by $|\varphi|$.

Lemma 46 *Let K and L be simplicial complexes, $\varphi : K \to L$ a simplicial map, and $f : |K| \to |L|$ a continuous map. Then φ is a simplicial approximation to f if and only if for every x in $|K|$, if x belongs to Int(ℓ), for some simplex ℓ of L, then $|\varphi|(x)$ lies in ℓ.*

Proof. Suppose that φ is a simplicial approximation to f. Let x be a point in $|K|$, belonging to $\mathrm{Int}k$, for some uniquely determined simplex $k = \{v_0, ..., v_q\}$ of K. The image $f(x)$ of x belongs to $\mathrm{Int}\ell$, for some uniquely determined simplex ℓ of L. Since φ is a simplicial approximation to f we have

$$f(x) \in f(\cap_{i=0}^{q} \mathrm{star} v_i) \subset \cap_{i=0}^{q} f(\mathrm{star} v_i) \subset \cap_{i=0}^{q} \mathrm{star}(\varphi(v_i)),$$

and as the latter intersection is a disjoint union of interior of simplices containing $f(x)$ which belongs to the interior of ℓ it follows that, for every i between 0 and q,

$$\mathrm{int}\ell \subset \mathrm{star}(\varphi(v_i)),$$

which is equivalent to $\varphi(v_i) \in \ell$. Now, since $\varphi(x)$ is a convex linear combination of the $\varphi(v_i)$'s, which all belong to ℓ, it follows that x also belongs to ℓ.

Conversely, suppose that for every x in $|K|$, if x belongs to $\mathrm{Int}(\ell)$, for some simplex ℓ of L, then $|\varphi|(x)$ lies in ℓ. Let v be a vertex of K. For every x in $\mathrm{star}v$, the image $f(x)$ of x belongs to the interior of a simplex $\varphi(x)$ belongs to, and thus $\varphi(v)$ belongs to. It follows that

$$f(\mathrm{star}v) = f(\cup_{v \in k} \mathrm{int}k) = \cup_{v \in k} f(\mathrm{int}k) \subset \cup_{\varphi(v) \in \ell} \mathrm{int}\ell = \mathrm{star}(\varphi(v)),$$

so that φ is a simplicial approximation to f. ■

Lemma 47 *Let K and L be simplicial complexes, $\varphi : K \to L$ a simplicial map, and $f : |K| \to |L|$ a continuous map. If φ is a simplicial approximation to f, then the (positive) linear convex combination of f and $|\varphi|$ is well defined and provides a homotopy between the maps f and $|\varphi|$.*

Proof. By Lemma 46 the image $|\varphi|(x)$ of any point x of $|K|$ lies in the smallest simplex $f(x)$ belongs to, say ℓ. It follows that for every $t \in [0,1]$, the point

$$(1-t)f(x) + t|\varphi|(x)$$

is well defined and belongs to ℓ. The map

$$\begin{array}{ccc} |K| \times [0,1] & \longrightarrow & |L| \\ (x,t) & \longmapsto & (1-t)f(x) + t|\varphi|(x) \end{array}$$

is the desired homotopy between f and $|\varphi|$. ∎

It follows from Lemma 47 that not every continuous map between the geometric realization of two simplicial complexes admits a simplicial approximation: there are in general infinitely many homotopy types of continuous maps $f : |K| \to |L|$ whereas the simplicial maps $\varphi : K \to L$ are in finite number as soon as K and L are finite. One useful criterion for a simplicial approximation to exist is the following easy consequence of Lemma 46:

Proposition 48 *Let K and L be two simplicial complexes and $f : |K| \to |L|$ a continuous map such that for every simplex k of K, there exists a simplex ℓ of L with $f(Int(k)) \subset Int(t)$. Then there exists a simplicial approximation to f.*

Proof. Define a map φ :Vert(K) $\to$Vert(L) as follows: Let v be a vertex of K, then by the assumption of the proposition there exists a simplex ℓ_v of L which contains $f(v)$. Let $\varphi(v)$ to be any of the vertex of ℓ_v.

Let us check that φ actually is a simplicial map. Let $k = \{v_0, ..., v_q\}$ be a simplex of K and let ℓ_i, for every i between 0 and q, be the simplex of L for which $f(v_i) \in$intℓ_i, which is the smallest simplex of L containing $f(v_i)$. Now, let ℓ be the simplex of L satisfying $f(\text{int}k) \subset$intℓ. By continuity of f it follows that $f(|k|) \subset |\ell|$. As $f(v_i)$ belongs to $|\ell|$ for every i between 0 and q, it follows that $\ell_i \leq \ell$. For every i in $\{0, ..., 1\}$ the vertex $\varphi(v_i)$ belongs to ℓ_i and thus to ℓ, so that $\{\varphi(v_0), ..., \varphi(v_q)\}$ is a subset, and hence a face of ℓ.

By construction, the simplicial map φ satisfies the hypothesis of Lemma 46, so that it is a simplicial approximation to f. ∎

As pointed out earlier, simplicial approximation of continuous maps $f : |K| \to |L|$ do not always exist. However, it is in any case possible to find a refinement of K for which the map f admits a simplicial approximation (see Theorem 49 below). In certain cases, for example in the semi-algebraic setting, one can refine the simplicial complex K sufficiently for the hypothesis of Proposition 48 to be satisfied by pulling back, by the continuous map f, all the simplices of L, and finding a refinement of K, which restricts to a triangulation of all the $f^{-1}(\ell) \cap k$, for every ℓ in L and k in K. We will use this argument in the proof of our main theorem. In general this method does not work, but we nevertheless have:

Theorem 49 (Simplicial Approximation) *Let K and L be simplicial complexes and $f : |K| \to |L|$ a continuous map. Then there exists a refinement (K', r) of K and a simplicial approximation $\varphi : K' \to L$ to $f \circ r^{-1} : |K'| \to |L|$.*

As we do not need the Simplicial Approximation Theorem 49 in any of our proofs, we refer the interested reader to [Ro88, Theorem 7.3].

2.4 Simplicial cohomology

Definitions

Let K be a simplicial complex. Recall that K_{or} is defined as the set of all ordered simplices of K. We denote by $[v_0, ..., v_q]$ the ordered simplex in K_{or} given by the simplex $\{v_0, ..., v_q\}$ and the ordering obtained from the numbering of the v_i's. It is also convenient to define K^q_{or}, where q is any non negative integer, as the subset of K_{or} consisting of ordered simplices containing $q+1$ vertices (not necessarily distinct).

Define the space $C_q(K)$ of *simplicial q-chains* of K to be the (real) vector space generated by the family of oriented q-simplices K^q_{or} and satisfying the relations

$$[v_0, ..., v_q] = \mathrm{sign}(\sigma)[v_{\sigma(0)}, ..., v_{\sigma(q)}]$$

for every $[v_0, ..., v_q] \in K^q_{\mathrm{or}}$ and every permutation σ in S_{q+1}. Observe that if the vertices $v_0, ..., v_q$ are not all distinct, then $[v_0, ..., v_q] = 0$. In particular, we see that $C_q(K) = 0$ whenever q is strictly bigger than the dimension of K. The boundary operator

$$\partial : C_q(K) \longrightarrow C_{q-1}(K)$$

is defined on the generators of $C_q(K)$ as

$$\partial\left([v_0, ..., v_q]\right) = \sum_{i=0}^{q} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_q],$$

for every $[v_0, ..., v_q]$ in K^q_{or}, and extended linearly to the whole of $C_q(X)$. It is easy to check that $\partial^2 = 0$.

The space $C^q_{\mathrm{simpl}}(K)$ of *simplicial q-cochains* of K is defined to be the algebraic dual of the space $C_q(K)$ of simplicial q-chains, so that

$$C^q_{\mathrm{simpl}}(K) = \{c : C_q(K) \to \mathbb{R} \mid c \text{ is linear}\}.$$

The boundary operator ∂ has for dual the coboundary operator

$$\delta : C^q_{\mathrm{simpl}}(K) \longrightarrow C^{q+1}_{\mathrm{simpl}}(K)$$

given, for every c in $C^q_{\mathrm{simpl}}(K)$ and z in $C_{q+1}(K)$ as

$$\delta c(z) = \partial^* c(z) = c(\partial z).$$

Since $\partial^2 = 0$ it clearly follows that $\delta^2 = 0$.

We can now of course consider the homology and cohomology of the chain and cochain complexes $(C_q(K), \partial)$ and $(C^q_{\mathrm{simpl}}(K), \delta)$. As usual, we define $Z_q(K) = \mathrm{Ker}\partial \subset C_q(K)$ and $Z^q_{\mathrm{simpl}}(K) = \mathrm{Ker}\delta \subset C^q_{\mathrm{simpl}}(K)$ to be the vector spaces of *simplicial q-cycles*, respectively *q-cocycles*, and their subspaces $B_q(K) = \mathrm{Im}\partial$ and $B^q_{\mathrm{simpl}}(K) = \mathrm{Im}\delta$ to be the spaces of *simplicial q-boundaries*, respectively *q-coboundaries*. The *q-th simplicial homology* and *cohomology* of K are then defined as the quotients

$$H_q(K) = Z_q(K)/B_q(K)$$

and

$$H^q_{\text{simpl}}(K) = Z^q_{\text{simpl}}(K)/B^q_{\text{simpl}}(K).$$

Let K and L be two simplicial complexes, and let $\varphi : K \to L$ be a simplicial map between them. The map φ naturally induces a map $\varphi_* : C_q(K) \to C_q(L)$ between the corresponding spaces of q-chains: If $[v_0, ..., v_q]$ is an oriented simplex, then

$$\varphi([v_0, ..., v_q]) = [\varphi(v_0), ..., \varphi(v_q)].$$

The map φ_* is easily checked to be a chain map (i.e. $\partial\varphi_* = \varphi_*\partial$). Its dual we denote by $\varphi^* : C^q_{\text{simpl}}(L) \to C^q_{\text{simpl}}(K)$. It is then of course a cochain map, so that it induces a well defined map, which we still denote by φ^*, in the corresponding cohomologies:

$$\varphi^* : H^q_{\text{simpl}}(L) \longrightarrow H^q_{\text{simpl}}(K).$$

On the space of simplicial cochains K we can define a *simplicial norm* as follows:

$$\|c\|^K_{\text{simpl}} = \sup\{c(\widetilde{k}) \mid \widetilde{k} \text{ an ordered } q\text{-simplex of } K\},$$

for every simplicial cochain $c \in C^q_{\text{simpl}}(K)$. If K is finite, then the simplicial norm also is. In cohomology we then have

$$\|[c]\|^K_{\text{simpl}} = \inf\{\|c'\|^K_{\text{simpl}} \mid [c'] = [c]\}.$$

Simplicial versus singular cohomology

Let K be a simplicial complex. Every oriented q-dimensional simplex $k = [v_0, ..., v_q]$ in K^q_{or} clearly determines a singular simplex on $|K|$ as follows: Define $\sigma_k \in C_q(|K|)$ as

$$\begin{array}{cccc} \sigma_k : & \Delta^q & \longrightarrow & |K| \\ & (t_0, ..., t_q) & \longmapsto & (t_0, ..., t_q) \in \Delta^q_k. \end{array}$$

As the equality $k = [v_0, ..., v_q] = \text{sign}(\tau)[v_{\tau(0)}, ..., v_{\tau(q)}] = \text{sign}(\tau)\,(\tau \cdot k)$ holds in the space simplicial chains $C_q(K)$ on K, but the corresponding equality $\sigma_k = \text{sign}(\tau)\sigma_{\tau k}$ is in general false in the space of singular chains $C_q(|K|)$ on the geometric realization of K, we need to alternate over the possible σ_k to obtain a well defined linear map

$$\text{Id}^K_\flat : C_q(K) \longrightarrow C_q(|K|),$$

which sends the oriented q-dimensional simplex $k = [v_0, ..., v_q]$ in K^q_{or} to the alternating sum

$$\sum_{\tau \in S_{q+1}} \text{sign}(\tau)\sigma_{\tau k}.$$

The map $\text{Id}^K_\flat$ is easily checked to be a chain map. It induces a cochain map

$$\text{Id}^\flat_K : C^q_{\text{sing}}(|K|) \longrightarrow C^q_{\text{simpl}}(K),$$

which in turns determines a map

$$\text{Id}^\flat_K : H^q_{\text{sing}}(|K|) \to H^q_{\text{simpl}}(K),$$

which we still denote by $\text{Id}^\flat_K$ between the singular cohomology of the topological space $|K|$ and the simplicial cohomology of K.

Theorem 50 *The map* $\mathrm{Id}_K^\flat : H^q_{sing}(|K|) \to H^q_{simpl}(K)$ *is an isomorphism for all* $q \geq 0$.

The homological case of this theorem is theorem 7.22 of [Ro88].

More generally, a continuous map $h : |K| \to X$ between the realization of a simplicial complex K and a topological X induces a map between the alternating singular cochains of X and the simplicial cochains of K. Indeed, define

$$h^\flat : C^q_{\mathrm{sing}}(X) \longrightarrow C^q_{\mathrm{simpl}}(K)$$

as the composition of the map

$$h^* : C^q_{\mathrm{sing}}(X) \longrightarrow C^q_{\mathrm{sing}}(|K|)$$

and the map

$$\mathrm{Id}_K^\flat : C^q_{\mathrm{sing}}(|K|) \longrightarrow C^q_{\mathrm{simpl}}(K).$$

The induced map $h^\flat$ is a chain map, since both h^* and $\mathrm{Id}_K^\flat$ are, and hence induces a map, which we denote again by $h^\flat$, between the singular cohomology of X and the simplicial cohomology of K:

$$h^\flat : H^q_{\mathrm{sing}}(X) \longrightarrow H^q_{\mathrm{simpl}}(K).$$

The invariance by homotopy now is a straightforward consequence of the analogous result in the topological setting. More precisely:

Theorem 51 *Let* $h \simeq f : |K| \to X$ *be continuous homotopic maps, then the induced maps*

$$h^\flat = f^\flat : H^q_{sing}(X) \longrightarrow H^q_{simpl}(K)$$

are equal.

Proof. By definition we have $h^\flat = Id_K^\flat \circ h^*$ and $f^\flat = Id_K^\flat \circ f^*$, but as h and f are homotopic, it follows that the induced maps

$$h^* = f^* : H^q_{\mathrm{sing}}(X) \longrightarrow H^q_{\mathrm{sing}}(|K|)$$

are equal, so that $h^\flat = f^\flat$ as claimed. ■

From the very definition of the induced map $h^\flat$ we now obtain the following easy relation at the cochain level:

Lemma 52 *Let* K *be a simplicial complex,* X, Y *topological spaces, and* $h : |K| \to X$ *and* $f : X \to Y$ *continuous maps. Then*

$$h^\flat \circ f^* = (f \circ h)^\flat : C^q_{sing}(Y) \longrightarrow C^q_{simpl}(K).$$

Proof. We have

$$h^\flat \circ f^* = \left(Id_K^\flat \circ h^*\right) \circ f^* = Id_K^\flat \circ (h^* \circ f^*) = Id_K^\flat \circ (f \circ h)^* = (f \circ h)^\flat,$$

which finishes the proof of the lemma. ■

Lemma 53 *Let K and L be simplicial complexes, and $\varphi : K \to L$ a simplicial map. Then the diagram*

$$\begin{array}{ccc} C^q_{\text{sing}}(|L|) & \xrightarrow{\text{Id}^\flat_L} & C^q_{\text{simpl}}(L) \\ \downarrow{\scriptstyle |\varphi|^*} & & \downarrow{\scriptstyle \varphi^*} \\ C^q_{\text{sing}}(|K|) & \xrightarrow{\text{Id}^\flat_K} & C^q_{\text{simpl}}(K) \end{array}$$

commutes.

Proof. Let c be a singular cochain in $C^q_{\text{sing}}(|L|)$ and $[v_0, ..., v_q]$ an oriented simplex in K^q_{or}. We have

$$\begin{aligned} (\varphi^* \circ \text{Id}^\flat_L)(c)([v_0, ..., v_q]) &= (\text{Id}^\flat_L)(c)([\varphi(v_0), ..., \varphi(v_q)]) \\ &= \sum_{\tau \in S_{q+1}} \text{sign}(\tau) c([\varphi(v_{\tau(0)}), ..., \varphi(v_{\tau(q)})]) \\ &= \sum_{\tau \in S_{q+1}} \text{sign}(\tau) \varphi^*(c)([v_{\tau(0)}, ..., v_{\tau(q)}]) \\ &= \left(\text{Id}^\flat_K \circ \varphi^*\right)(c)([v_0, ..., v_q]), \end{aligned}$$

which proves the lemma. ■

Lemma 54 *Let K and L be simplicial complexes, X a topological space, $h : |L| \to X$ a continuous map, and $\varphi : K \to L$ a simplicial map. Then*

$$\varphi^* \circ h^\flat = |\varphi|^\flat \circ h^* : C^q_{sing}(X) \longrightarrow C^q_{simpl}(K).$$

Proof. By the very definition of $h^\flat$ we have

$$\varphi^* \circ h^\flat = \varphi^* \circ \text{Id}^\flat_L \circ h^*,$$

which by Lemma 53 is equal to

$$\text{Id}^\flat_K \circ |\varphi|^* \circ h^* = |\varphi|^\flat \circ h^*,$$

as desired. ■

Refinements

It is in general not possible to induce a canonical simplicial chain map starting from a continuous map $h : |K| \to |L|$ between the realizations of two simplicial complexes. However, if (L, h) happens to be a refinement of K, so that the map h maps the q-skeleton of K into the q-skeleton of L for every $q \geq 0$, we can define a map $h_\sharp : C_q(K) \to C_q(L)$ inductively as follows: For $q = 0$, define

$$h_\sharp([v]) = [h(v)]$$

for every vertex v of K. This is well defined since $h(\text{Vert}(K)) \subset\text{Vert}(L)$. Suppose now that the map $h_\sharp$ is defined on the $(q-1)$-chains. Let k be a q-simplex of K, and $\widetilde{k} \in C_q(K)$ the simplex

k together with the choice of an ordering of its vertices. Let $\ell_1, ..., \ell_r$ be distinct q-simplices of L such that

$$h(|k|) = \cup_{i=1}^r |\ell_i|.$$

For every i between 1 and r let $\widetilde{\ell_i} \in C_q(L)$ be the simplex ℓ_i together with a choice of ordering of the vertices of ℓ_i such that

$$\sum_{i=1}^r \partial \widetilde{\ell_i} = h_\sharp(\partial k).$$

Finally, define

$$h_\sharp(k) = \sum_{i=1}^r \widetilde{\ell_i}.$$

By definition, $h_\sharp$ is a chain map, so that its dual $h^\sharp$ is a cochain map and hence defines a map

$$h^\sharp : H^q_{\text{simpl}}(L) \longrightarrow H^q_{\text{simpl}}(K)$$

in cohomology which we still denote by $h^\sharp$.

Observe that arbitrary maps $h : |K| \to |L|$ can now be handled by passing to a refinement of K on which a simplicial approximation to h can be found.

Proposition 55 *Let K be a simplicial complex, and (L, r) a refinement of K. Then for any $c \in C^q_{simpl}(L)$*

$$\| r^\sharp(c) \|^K_{\text{simpl}} \leq [L : K]_q \, \| c \|^L_{simpl} \, .$$

Moreover, the same inequality holds in cohomology:

$$\| [r^\sharp(c)] \|^K_{\text{simpl}} \leq [L : K]_q \, \| [c] \|^L_{simpl} \, .$$

Proof. For the first inequality, we have

$$\begin{aligned}
\| r^\sharp(c) \|^K_{\text{simpl}} &= \max_{s \in K^q_{\text{or}}} |r^\sharp c(s)| && \text{by definition,} \\
&= |r^\sharp c(s_0)| && \text{for some } s_0 \in K^q_{\text{or}}, \\
&= |\textstyle\sum_{i=1}^r c(t_i)| && \text{where } h(s) = \cup_{i=0}^r t_i \\
& && \text{and } r \leq [L : K]_q, \\
&\leq r \max_{i \in \{1,\dots,q\}} |c(t_i)| \\
&\leq [L : K]_q \, \| c \|^L_{\text{simpl}} \, .
\end{aligned}$$

The second inequality then follows by a standard argument:

$$\begin{aligned}
\| [r^\sharp(c)] \|^K_{\text{simpl}} &= \inf \{ \| b \|^K_{\text{simpl}} | \; b \in Z^q_{\text{simpl}}(K), \; [b] = h^\sharp(c) \} \\
&\leq \inf \{ \| r^\sharp(c') \|^K_{\text{simpl}} | \; c' \in Z^q_{\text{simpl}}(L), \; [c] = [c'] \} \\
&\leq \inf \{ [L : K]_q \, \| c' \|^L | \; c' \in Z^q_{\text{simpl}}(L), \; [c] = [c'] \} \\
&\quad \text{by the first inequality,} \\
&= [L : K]_q \, \| c \|^L_{\text{simpl}} \, .
\end{aligned}$$

■

More generally, one can similarly show:

Proposition 56 *Let K be a simplicial complex, and (L,r) a refinement of K. Let c be a simplicial cochain in $C^q_{simpl}(L)$ and denote by I the subset of $\mathbb{R}$ consisting of the image by c on all oriented q-simplices of L,*

$$I = \{c(\ell) \mid \ell \in L^q_{or}\},$$

Then the cochain $r^\sharp(c)$ takes values in

$$\left\{ \sum_{i=1}^{r} n_i \,\middle|\, n_i \in I,\ r \leq [L:K] \right\}$$

on oriented q-simplices of K.

Let now (L,r) be a refinement of the simplicial complex K. It is in general not true that $h^\sharp \circ \mathrm{Id}^\flat_L = \mathrm{Id}^\flat_K \circ h^* : C^q_{\mathrm{sing}}(|L|) \to C^q_{\mathrm{simpl}}(K)$. However the equality holds at the cohomology level:

Proposition 57 *Let (L,r) be a refinement of the simplicial complex K. Then*

$$r^\sharp \circ Id^\flat_L = Id^\flat_K \circ r^* : H^q_{sing}(|L|) \to H^q_{simpl}(K).$$

Proof. Recall that two cochains are cohomologous if and only if they agree on all cycles. Let $c \in Z^q_{\mathrm{sing}}(|L|)$ be a singular cocycle and $z \in Z_q(K)$ a simplicial cycle. Let us compute, on one hand

$$r^\sharp \circ \mathrm{Id}^\flat_L(c)(z) = \mathrm{Id}^\flat_L(c)(r_\sharp(z)),$$

and on the other hand

$$\mathrm{Id}^\flat_K \circ r^*(c)(z) = r^*(c)(\sigma_z) = c(r_*(\sigma_z)).$$

The desired equality now follows from the fact that $r_\sharp(z) - r_*(\sigma_z)$ is a boundary in the singular chain complex $C_q(|L|)$ so that, as c is a cocycle, $c(r_\sharp(z) - r_*(\sigma_z)) = 0$. ■

Proposition 58 *Let X be a topological space, (L,r) a refinement of the simplicial complex K, and $f : |L| \to X$ a continuous map. Then*

$$(f \circ r)^\flat = r^\sharp \circ f^\flat : H^q_{sing}(X) \to H^q_{simpl}(K).$$

Proof. This is an immediate consequence of Proposition 57:

$$\begin{aligned}(f \circ r)^\flat &= \mathrm{Id}^\flat_K \circ (f \circ r)^* \\ &= \mathrm{Id}^\flat_K \circ r^* \circ f^* \\ &= r^\sharp \circ Id^\flat_L \circ f^* \\ &= r^\sharp \circ f^\flat.\end{aligned}$$

■

Chapter 3

Characteristic classes

3.1 Primary Characteristic classes

Let G be a topological group. A *characteristic class* c assigns to any principal G-bundle ξ over a topological space B a cohomology class $c(\xi) \in H^q(B)$ such that if $f : B' \to B$ is a continuous map then $c(f^*(\xi)) = f^*(c(\xi)) \in H^q(B')$.

Characteristic classes are easily seen to be in one to one correspondence with the cohomology of some (and hence any) classifying space BG. Indeed, a characteristic class c in particular assigns to the universal principal G-bundle ξ_G over the classifying BG the cohomology class $c(\xi_G) \in H^q(BG)$. Conversely, if c_G is a cohomology element in $H^q(BG)$ define a characteristic class as follows: For every principal G-bundle ξ over B, let $f : B \to BG$ be a classifying map for ξ, and define the value $c(\xi)$ of the characteristic class on ξ as

$$c(\xi) = f^*(c_G) \in H^q(B).$$

This is well defined since the classifying map f is uniquely defined up to homotopy, and two homotopic maps induce the same map in cohomology. Also, if furthermore $g : B' \to B$ is a continuous map, then $f \circ g : B' \to BG$ is a classifying map for the bundle $g^*(\xi)$, so that $c(g^*(\xi)) = g^*(c(\xi))$ as desired.

If G is a Lie group, a standard way to compute characteristic classes is via the Chern-Weil homomorphism whose description is the object of the next section.

3.1.1 The Chern-Weil homomorphism

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Define

$$I^q(G) = \{f : \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \to \mathbb{C} \mid f \ \text{ linear and } G\text{-invariant}\},$$

where the action of G is induced by the adjoint representation Ad: $G \to$GL$(\mathfrak{g})$. More precisely for every g in G, define

$$g \cdot f(v_1 \otimes \cdots \otimes v_q) = f(\mathrm{Ad}(g^{-1})v_1 \otimes \cdots \otimes \mathrm{Ad}(g^{-1})v_q)$$

for every $v_1, \ldots, v_q$ in $\mathfrak{g}$ and $f : \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \to \mathbb{C}$ linear. One can naturally define a multiplication

$$I^q(G) \otimes I^p(G) \longrightarrow I^{q+p}(G)$$

turning

$$I^*(G) = \cup_{q \geq 0} I^q(G)$$

into a graded algebra, the *algebra of invariant polynomials on* $\mathfrak{g}$.

The idea of the Chern-Weil theory is to assign to any differentiable principal G-bundle over some smooth manifold M a homomorphism from the algebra of invariant polynomials on $\mathfrak{g}$ to the cohomology of the base space M,

$$w_\xi : I^*(G) \longrightarrow H^*(M),$$

with all the desirable naturality properties.

Let $f \in I^q(G)$ be an invariant polynomial, P a smooth manifold, and $\alpha_1, ..., \alpha_q$ differentiable forms with coefficient in $\mathfrak{g}$ on P of degree $i_1, ..., i_q$ respectively (so that $\alpha_j \in A^{i_j}(P, \mathfrak{g})$ for every j in $\{1, ..., q\}$). A complex valued differential form $f(\alpha_1 \wedge ... \wedge \alpha_q)$ of degree $d = i_1 + ... + i_q$ on P is naturally defined as follows: For every u in P and $X_1, ..., X_d$ define

$$f(\alpha_1 \wedge ... \wedge \alpha_q)_u(X_1, ..., X_d) = f((\alpha_1 \wedge ... \wedge \alpha_q)_u(X_1, ..., X_d)) \in A^d(P, \mathbb{C}).$$

Let now $\xi = \{\pi : P \to M\}$ be a differentiable principal G-bundle endowed with a connection form ω. (Such a connection exists by virtue of corollary 27.) Denote by $\Omega \in A^2(P, \mathfrak{g})$ the corresponding curvature. By the above described procedure, we obtain from any invariant polynomial $f \in I^q(G)$ a complex valued differential form of degree $2q$ on P which we denote by $f(\Omega)$ as

$$f(\Omega) = f(\Omega^q) = f(\Omega \wedge ... \wedge \Omega) \in A^{2q}(P, \mathbb{C}).$$

We now want to show that

1. $f(\Omega)$ descends to a $2q$-form $\overline{f(\Omega)} \in A^{2q}(M, \mathbb{C})$ on M,
2. the form $\overline{f(\Omega)}$ is closed,
3. the cohomology class $[\overline{f(\Omega)}]$ is independent of the choice of the connection form.

From this we will obtain at once the desired conclusions. Let us thus prove those three assertions.

1. This is obvious. Indeed, since Ω is horizontal, $f(\Omega)$ surely also is, and Ω being equivariant and f invariant, $f(\Omega)$ is an invariant horizontal $2q$-form so that it is the lift of a $2q$-form on M, which is unique since π being surjective, the induced map $\pi^* : A^*(M) \to A^*(P)$ is injective. We denote it by $\overline{f(\Omega)} \in A^{2q}(M)$.
2. Since the map $\pi^* : A^*(M, \mathbb{C}) \to A^*(P, \mathbb{C})$ is injective, it is enough to show that $f(\Omega) \in A^*(P, \mathbb{C})$ is closed. Indeed, if $df(\Omega) = 0$, then

$$\pi^*(d\overline{f(\Omega)}) = d(\pi^*(\overline{f(\Omega)})) = df(\Omega) = 0,$$

and thus $d\overline{f(\Omega)} = 0$. By Lemma 32 we know that $df(\Omega) = Df(\Omega)$ so that

$$\begin{aligned} df(\Omega) &= Df(\Omega) \\ &= qf(D\Omega \wedge \Omega^{q-1}) \\ &= 0, \end{aligned}$$

where the last equality follows from Bianchi's identity $D\Omega = 0$ (Proposition 31).

3. Let ω_0 and ω_1 be two connection forms on ξ. Define ω_t to be the connection form consisting of the convex linear combination of ω_0 and ω_1. More precisely,

$$\omega_t = (1-t)\omega_0 + t\omega_1,$$

for every $t \in \mathbb{R}$. Denote by Ω_t the corresponding curvature. Define a $(2q-1)$-form $Tf(\omega_0, \omega_1)$ on P as

$$Tf(\omega_0, \omega_1) = q \int_0^1 f((\omega_1 - \omega_0) \wedge \Omega_t^{q-1}) dt \in A^{2q-1}(P, \mathbb{C}).$$

Being G-equivariant and horizontal, the form $Tf(\omega_0, \omega_1)$ descends to a unique form on M which we denote by $\overline{Tf}(\omega_0, \omega_1) \in A^{2q-1}(M, \mathbb{C})$. The assertion follows at once from the following proposition.

Proposition 59 *Let $f \in I^k(G)$ be an invariant polynomial and ω_0, ω_1 two connection forms on ξ with corresponding curvature Ω_0, Ω_1. Then $\overline{f(\Omega_0^k)}$ and $\overline{f(\Omega_1^k)}$ differ by a coboundary. More precisely,*

$$d\overline{Tf}(\omega_0, \omega_1) = \overline{f(\Omega_1^k)} - \overline{f(\Omega_0^k)}.$$

Proof. By the injectivity of $\pi^* : A^*(M, \mathbb{C}) \to A^*(P, \mathbb{C})$ it is enough to show that

$$dTf(\omega_0, \omega_1) = f(\Omega_1^k) - f(\Omega_0^k).$$

Consider the $(2q-1)$-form $f((\omega_1 - \omega_0) \wedge \Omega_t^{q-1})$ on P. It is G-equivariant and horizontal. Hence it descends to a $(2q-1)$-form on M so that by Lemma 32,

$$df((\omega_1 - \omega_0) \wedge \Omega_t^{q-1}) = D_t f((\omega_1 - \omega_0) \wedge \Omega_t^{q-1}).$$

Recalling Bianchi's identity $D_t \Omega_t = 0$ (proposition 31) and the equality $\frac{d}{dt}\Omega_t = D_t(\omega_1 - \omega_0)$ (Lemma 34) we compute

$$\begin{aligned} qdf((\omega_1 - \omega_0) \wedge \Omega_t^{q-1}) &= qD_t f((\omega_1 - \omega_0) \wedge \Omega_t^{q-1}) \\ &= qf(D_t(\omega_1 - \omega_0) \wedge \Omega_t^{q-1}) \\ &= qf(\frac{d}{dt}\Omega_t \wedge \Omega_t^{q-1}) \\ &= \frac{d}{dt} f(\Omega_t^q), \end{aligned}$$

and finally conclude

$$\begin{aligned} dTf(\omega_0, \omega_1) &= q \int_0^1 (df((\omega_1 - \omega_0) \wedge \Omega_t^{q-1}))dt \\ &= \int_0^1 \frac{d}{dt} f(\Omega_t^q) dt \\ &= f(\Omega_1) - f(\Omega_0). \end{aligned}$$

■

We have thus constructed a map

$$\begin{array}{rccc} w_\xi : & I^*(G) & \longrightarrow & H^*(M) \\ & f & \longmapsto & [f(\Omega)] \end{array}$$

which is easily checked to be a homomorphism (even an algebra homomorphism). This is called the *Chern-Weil homomorphism.*

Universal Chern-Weil homomorphism

The Chern-Weil homomorphism can be extended to the universal case, so as to obtain a map

$$I^q(G) \longrightarrow H^{2q}(BG).$$

Observe that some care is needed since the classifying space BG is not a manifold. This can be handled in various ways: One can consider BG as a limit of manifolds, in which case the existence of the Chern-Weil homomorphism in the universal case follows from the stability properties of the cohomology of the limit. Alternatively, the classifying space BG can be viewed as a simplicial manifold and as the Chern-Weil theory has been extended to simplicial manifolds by Dupont in [Du76], the existence of the universal Chern-Weil homomorphism follows.

Note that the problem of understanding characteristic classes does not reduce to a mere description of the algebra of invariant polynomials $I^*(G)$ since the Chern-Weil homomorphism is in general not an isomorphism: For example, it is not surjective for $\mathrm{SL}_n\mathbb{R}$ whenever n is even, for, as we shall see later, the Euler class $\varepsilon \in H^{2n}(B\mathrm{SL}_n\mathbb{R})$ is not in its image, and it is not injective for $\mathrm{GL}_n\mathbb{C}$. However, for compact groups it was proven by Cartan that:

Theorem 60 *If K is a compact Lie group, then the Chern-Weil homomorphism $I^*(K) \to H^{2*}(BK)$ is an isomorphism.*

Observe that since every principal G-bundle, where G is a Lie group with finitely many connected components, admits a reduction to any of its maximal compact subgroup (see Theorem 15), the classifying spaces BG and BK are homotopically equivalent, so that their corresponding cohomologies are isomorphic. The cohomology of BG is thus isomorphic to $I^*(K)$:

$$\begin{array}{ccc} I^q(G) & \longrightarrow & H^{2q}(BG) \\ \big\downarrow & & \big\downarrow \cong \\ I^q(K) & \xrightarrow{\cong} & H^{2q}(BK). \end{array}$$

3.1.2 Characteristic classes of flat bundles

If a principal G-bundle ξ happens to be flat, it can, by definition, be endowed with a connection with vanishing curvature, so that the image of the Chern-Weil homomorphism is trivial. This in turn implies that the composition

$$I^q(G) \longrightarrow H^{2q}(BG) \longrightarrow H^{2q}(BG^\delta)$$

is the zero map.

In particular, the well known Chern and Pontrjagin classes are trivial for flat bundles. Also, if the Chern-Weil homomorphism is surjective, there can be no nontrivial characteristic classes of flat bundles. This is the case for compact groups (Theorem 60), for $\mathrm{GL}_n(\mathbb{R})$ when n odd (which is generated by the Pontrjagin classes), $\mathrm{GL}_n(\mathbb{C})$ (generated by the Chern classes), etc.

Observe that even though the triviality of the Chern and Pontrjagin classes is almost a tautology in the differential setting, it is nevertheless a difficult result when using the topological definition of the Chern and Pontrjagin classes (see [KaTo68] and [KaTo75]).

Fortunately, there are nontrivial characteristic classes of flat bundles. The first examples of both nontrivial flat bundles and non trivial characteristic classes were given by Milnor in [Mi58], where flat bundles over surfaces are characterized in terms of their Euler class. We will examine the example of the Euler class carefully below.

More characteristic classes of flat bundles were exhibited by Dupont in [Du78, Chapter 9]. The author considers the commutative diagram

$$\begin{array}{ccccc} I^q(G) & \longrightarrow & H^{2q}(BG) & \longrightarrow & H^{2q}(BG^\delta) \\ \downarrow & & \downarrow & & \\ I^q(K) & \longrightarrow & H^{2q}(BK), & & \end{array}$$

where K is a maximal compact subgroup of G. As mentioned earlier the lower horizontal arrow is an isomorphism (because K is compact), and also the cohomologies of BG and BK are isomorphic. The map $H^{2q}(BG) \to H^{2q}(BG^\delta)$ is thus completely equivalent to the homomorphism $I^q(K) \to H^{2q}(BG^\delta)$, which is explicitly described by Dupont, and provides us with concrete non trivial characteristic classes of flat bundles. For example if G is the real symplectic group

$$G = \left\{ g \in \mathrm{GL}_{2n}\mathbb{R} \;\middle|\; g^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

then any maximal compact subgroup K is isomorphic to the unitary group $U(n)$, and the Chern polynomials in $I^*(K)$ lead to non trivial characteristic classes in $H^{2q}(BG^\delta)$. In particular for $n = 1$, the symplectic group is isomorphic to $\mathrm{SL}_2\mathbb{R}$, the first Chern polynomial in $I^1(U(1))$ is sent to the Euler class in $H^2(B\mathrm{SL}_2\mathbb{R}^\delta)$ and the very description of the homomorphism $I^q(K) \to H^{2q}(BG^\delta)$ allows Dupont to give a new proof of Milnor's inequality.

3.1.3 The Euler class

The Euler class is a cohomology class ε in $H^n(B\mathrm{SL}_n\mathbb{R})$. Of course it is only non trivial when n is even, since characteristic classes always live in even degree. Let thus $n = 2m$. The Euler class is most easily described as the image, via the isomorphism $I^m(\mathrm{SO}(2m)) \to H^{2m}(B\mathrm{SL}_{2m}\mathbb{R})$ described above, of the ad(SO($2m$))-invariant *Pfaffian polynomial* Pf $\in I^m(\mathrm{SO}(2m))$ defined as

$$\mathrm{Pf}(A) = \frac{1}{2^{2m}\pi^m m!} \sum_{\sigma \in S_{2m}} \mathrm{sign}(\sigma) a_{\sigma(1)\sigma(2)} \cdot \ldots \cdot a_{\sigma(2m-1)\sigma(2m)},$$

were $A = (a_{ij})$ belongs to $\mathfrak{so}(2m) = \{A \in M_{2m}\mathbb{R} \mid \mathrm{Tr}A = 0,\ A = A^t\}$.

In the topological setting, the Euler class can be defined as follows: Let ξ be a $\mathrm{SL}_n\mathbb{R}$-bundle over the geometric realization $|K|$ of a simplicial complex K. Then the Euler class is the obstruction to the existence of a nowhere vanishing section on the n-skeleton of K in the associated vector bundle $\xi_{\mathbb{R}}$.

To see still another definition of the Euler class, and proofs that those definitions are all equivalent, the reader is invited to consult the excellent [MiSt79, §9 and Appendix C].

Using the definition of the Euler class as an obstruction class, Sullivan could easily show in [Su76] our simplicial version (Theorem 5) for the Euler class:

Theorem 61 *Let ξ be a flat $SL_n\mathbb{R}$-bundle over a finite simplicial complex K. Then the (simplicial) Euler class $\varepsilon(\xi) \in H^n_{simpl}(K)$ can be represented by a cocycle whose evaluation on the n-simplices of K takes value in $\{-1, 0, 1\}$.*

Proof. First observe that we can without loss of generality assume that the dimension of the simplicial complex K is equal to n. Consider the covering of $|K|$ given by the sets U_k defined for every n-dimensional simplex k of K to be a small neighborhood of $|k|$. Since the U_k's are contractible there exists local trivializations $\varphi_k : \xi_{\mathbb{R}}|_{U_k} \to U_k \times \mathbb{R}^n$ of the associated vector bundle $\xi_{\mathbb{R}}$ such that the corresponding transition functions $g_{kk'}$, relative to this open covering, are locally constant and thus constant.

Choose for every vertex v of K a point $s(v) \in \xi_{\mathbb{R}}$ in the fiber over v in such a way that if $v_0, ..., v_{n-1}$ generate an $(n-1)$-dimensional simplex of K then the convex hull of the projection to $\mathbb{R}^n$ of the points $\varphi_k(s(v_0)), ..., \varphi_k(s(v_{n-1}))$, where k is any n-dimensional simplex of K containing $v_0, ..., v_{n-1}$, does not contain 0. This is, from dimension considerations, always possible. Define now local sections $s_\ell : \ell \to \xi_{\mathbb{R}}|_\ell$ for each $(n-1)$-dimensional simplex ℓ of K as the composition of φ_k^{-1}, where k is an n-dimensional simplex containing ℓ, and the obvious convex linear combination of the points $\varphi_k(s(v_0)), ..., \varphi_k(s(v_{n-1}))$. Since the transition functions correspond to linear transformations this defines a global section $s : K^{n-1} \to \xi_{\mathbb{R}}|_{K^{n-1}}$.

By definition of the Euler class as an obstruction class, we now obtain a (simplicial) cocycle e representing the class $\varepsilon(\xi) \in H^n_{\mathrm{simpl}}(K)$ as follows: the evaluation of e on an n-dimensional simplex k of K is the homotopy class in $\pi_{n-1}(\mathbb{R}^n \backslash \{0\}) \cong \mathbb{Z}$ of the map $\varphi_k \circ s|_{\partial k} \to \partial k \times \mathbb{R}^n$ composed with the second projection. Since it is linear on each face of ∂k, it is either trivial, or one of the generators of $\pi_{n-1}(\mathbb{R}^n \backslash \{0\}) \cong \mathbb{Z}$, thus proving the Theorem. ■

By the following slight modification of Sullivan's argument, Smillie was capable to improve the bound to $1/2^n$ for n even, and 0 for n odd (see [Sm81]): Consider the 2^{n+1} sections on an n-dimensional simplex of K constructed as above from their vertices value

$$\pm\varphi_k(s(v_0)), ..., \pm\varphi_k(s(v_n)).$$

Exactly two of those sections will give a non trivial value for the above given corresponding representative for the Euler class. In odd dimension, respectively even dimension, they will have opposite sign, resp. identical sign. Averaging over all such possible sections, Smillie's result follows.

The singular version of the Theorem for the Euler class was then obtained in [IvTu82] by Ivanov and Turaev.

Theorem 62 *Let ξ be a flat $SL_n\mathbb{R}$-bundle over a topological space. Then*

$$\|\varepsilon(\xi)\|_\infty \leq \frac{1}{2^n}.$$

The idea of the proof is to average, not only on a finite number of possible sections as was done by Smillie, but on all admissible sections. This leads to the cocycle $E \in Z^n(\mathrm{SL}_n\mathbb{R}^\delta)$ representing the Euler class in $H^n(B\mathrm{SL}_n\mathbb{R}^\delta) \cong H^n(\mathrm{SL}_n\mathbb{R}^\delta)$ defined as

$$\begin{array}{rccc} E: & (\mathrm{SL}_n\mathbb{R})^n & \longrightarrow & [-1/2^n, 1/2^n] \subset \mathbb{R} \\ & (g_1, ..., g_n) & \longmapsto & \int_{(\mathbb{D}^n)^{n+1}} t(v_0, g_1 v_1, ..., g_n v_n) dv_0 ... dv_n, \end{array}$$

where $t(v_0, ..., v_n)$ is equal to 1 if the convex hull of the vectors $v_0, ..., v_n$ contain 0 and $\{v_1, ..., v_n\}$ are positively oriented, -1 if the convex hull of the vectors $v_0, ..., v_n$ contain 0 and $\{v_1, ..., v_n\}$ are negatively oriented, and 0 otherwise. The cocycle E is easily checked to be bounded, but note that it is by no means finite, for it takes, when n is even, all possible values in the interval $[-1/2^n, 1/2^n]$.

As observed by Ghys in [Ghys87] (see also [Ghys99]), it is now easy to show one part of Milnor-Wood inequality, namely

Corollary 63 *Let ξ be a flat $SL_n\mathbb{R}$-bundle over a surface Σ_g of genus $g \geq 1$. Then*

$$|\varepsilon(\xi)[\Sigma_g]| \leq g - 1.$$

Proof. Let X be a topological space, $z \in Z_q(X)$ a q-cycle and $c \in Z^q(X)$ a q-cocycle. By the very definition of the 1-norm and ∞-norm on the space of chains and cochains one has the inequality

$$|c(z)| \leq \|c\|_\infty \|z\|_1,$$

which induces the corresponding inequality

$$|c(z)| \leq \|[c]\|_\infty \|[z]\|_1$$

in cohomology whenever $\|[z]\|_1$ is not zero.

The simplicial volume of surfaces is easily computed (see for example [Gr82]), and is equal to

$$\|[\Sigma_g]\|_1 = 4g - 4.$$

Together with Ivanov and Turaev's Theorem 62 we can now conclude that

$$|\varepsilon(\xi)[\Sigma_g]| \leq \|[\varepsilon(\xi)]\|_\infty \|[\Sigma_g]\|_1 \leq \frac{1}{4}(4g-4) = g - 1.$$

■

The other implication of Milnor-Wood's inequality is proven by exhibiting, for every claimed possible value of the Euler number, a flat bundle with this Euler class. This proves the assertion since isomorphy classes of bundles over surfaces are completely determined by their Euler class. Also, this gives the first examples of non trivial flat bundles and non trivial characteristic classes of flat bundles. We refer the reader to the original paper [Mi58] for more details.

3.1.4 Characteristic classes in degree 2 when $\pi_1(G) \cong \mathbb{Z}$

Let G be a topological group whose fundamental group $\pi_1(G)$ is isomorphic to $\mathbb{Z}$. Denoting by $\widetilde{G}$ the universal cover of G, we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } \tilde{G} \xrightarrow{\ p\ } G \longrightarrow 1$$

which is such that the image of $\mathbb{Z}$ by i is contained in the center of $\widetilde{G}$. It is a standard fact, that there is a one to one correspondence between isomorphy classes of central extensions of G by $\mathbb{Z}$ and cohomology classes in $H^2(G,\mathbb{Z})$. Recall that any cocycle representing the cohomology class corresponding to the above central extension can be obtained in the following way: Pick a fundamental domain $D \subset \widetilde{G}$ for G and let $s : G \to \widetilde{G}$ be the unique set theoretic section of $p : \widetilde{G} \to G$ satisfying $\mathrm{Im} s = D$. Define

$$c : G \times G \longrightarrow \mathbb{Z}$$

by

$$i(c(g,h)) = s(g)s(h)s(gh)^{-1},$$

for every g, h in G.

The cohomology class $[c] \in H^2(G,\mathbb{Z}) \cong H^2(BG^\delta,\mathbb{Z})$ is in fact a primary characteristic class, by which we mean that it is in the image of the natural map

$$H^2(BG,\mathbb{Z}) \longrightarrow H^2(BG^\delta,\mathbb{Z}) \cong H^2(G,\mathbb{Z}).$$

In the simplicial case, this characteristic class is most easily described as the obstruction to the existence of a section of the G-bundle restricted on the 2-skeleton.

It is clear that if we can choose the fundamental domain D such that $D \cdot D$ is contained in a finite union of translates of D, then the cocycle $c(D)$ is a bounded cocycle. Let us examine a few examples:

- If G is compact, then this is always the case. However nothing new is gained since we already know that there exists no non trivial characteristic class for flat G-bundles whenever G is compact.

- Let $G = \mathrm{SL}_2\mathbb{R}$ (or more generally $Homeo_+(S^1)$, the group of orientation preserving homeomorphism of the circle) the characteristic class corresponding to the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{SL}_2(\mathbb{R})} \longrightarrow \mathrm{SL}_2(\mathbb{R}) \longrightarrow 1$$

  is precisely the Euler class $\varepsilon \in H^2(SL_2(\mathbb{R})) \cong H^2(B(SL_2\mathbb{R})^\delta)$. In this case, Ghys exhibited in [Ghys87] and [Ghys99] a canonical fundamental domain D with the property that $D \cdot D$ is the union of D and the translate of D by the positive generator of $\mathbb{Z}$. It follows that the Euler class can be represented by a cocycle taking values in $\{0, 1\}$.

- This example is due to Golman [Go81]. Let G be the quotient of the Heisenberg group

$$H = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| \, x, y, z \in \mathbb{R} \right\}$$

of upper triangular unipotent 3 by 3 matrices by the normal subgroup generated by the central element

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Of course, H can be taken as the universal cover of G. Consider the fundamental domain

$$D = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y \in \mathbb{R}, \ 0 \leq z < 1 \right\}.$$

It is easy to show that $D \cdot D$ is not contained in a finite union of translates of D: For example, let n be an arbitrary integer. Then

$$\begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n & n \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

belongs to $T^n D$, even though the two matrices on the left hand side of the equality belong to D. The corresponding cocycle

$$c : G \times G \longrightarrow \mathbb{Z}$$

is thus unbounded.

Let n be an arbitrary integer. Define a representation

$$h_n : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow G$$

by sending the two canonical generators a and b of $\mathbb{Z} \oplus \mathbb{Z}$ to the respective projections onto G of the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of H. This is well defined since $ABA^{-1}B^{-1}$ is equal to T^n. Now, because the evaluation of the 2-cycle $[a, b] - [b, a]$ on c is equal to n, it follows that the representations h_n induce infinitely many non isomorphic flat G-bundles over the 2-torus. It thus follows from Lusztig's Corollary 42 that G can not be algebraic.

Furthermore, because the class $h_n^*([c]) \in H^2(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{R})$ is non trivial for n different from 0, it can not belong to the image of

$$\{0\} = H_b^2(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{R}) \longrightarrow H^2(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{R}),$$

so that the primary characteristic class $[c]$ itself can impossibly belong to the image of

$$H_b^2(G, \mathbb{R}) \longrightarrow H^2(G, \mathbb{R}).$$

3.1.5 Finiteness properties

Let us recall our main result:

Theorem 64 *Let G be an algebraic subgroup of $GL_n(\mathbb{R})$. Then every characteristic class of flat G-bundle can be represented by a cocycle whose set of value on singular simplices is finite.*

This generalizes Gromov's Theorem 3, which admits the following reformulation:

Theorem 65 *Let G be an algebraic subgroup of $GL_n(\mathbb{R})$. Then the image of the map $H^*(BG) \to H^*(BG^\delta)$ is contained in the image of the comparison map $H_b^*(BG^\delta) \to H^*(BG^\delta)$.*

As mentioned in the introduction, an immediate corollary is now:

Corollary 66 *Let G be an algebraic subgroup of $GL_n(\mathbb{R})$ and X a topological space with amenable fundamental group. Then X does not possess any non trivial characteristic class of flat G-bundle.*

Proof. It is well known that the bounded cohomology of topological spaces with amenable fundamental group is trivial. The corollary thus follows from Theorem 65 and the commutativity of the following diagram:

$$\begin{array}{ccccc} H^*(BG) & \longrightarrow & H^*(BG^\delta) & \longleftarrow & H_b^*(BG^\delta) \\ & \searrow & \downarrow & & \downarrow \\ & & H^*(X) & \longleftarrow & H_b^*(X) = \{0\}. \end{array}$$

■

Observe that it is necessary to assume, in Gromov's Theorem 65 (and also in our Theorem 64), that the group G is algebraic. For example, if G is the quotient of the Heisenberg group by any central element, it was pointed out by Goldman in [Go81] that the primary characteristic class in $H^2(G,\mathbb{R})$ obtained from the central extension given from the universal covering of G is not in the image of

$$H_b^2(G,\mathbb{R}) \longrightarrow H^2(G,\mathbb{R}).$$

The details of this counter-example are given above.

Let us point out that our Theorem 64 is really a strengthening of Gromov's Theorem 65. More information is gained from knowing that a cohomology class can be represented by a finite cocycle. Indeed, if we define finite group cohomology $H_f^*(G,\mathbb{R})$ analogously to bounded group cohomology from the subcomplex consisting of cochains taking only finitely many values, then it is so that

$$H_f^*(G,\mathbb{R}) \neq H_b^*(G,\mathbb{R})$$

in general. For example $H_f^2(\mathbb{Z},\mathbb{R})$ is not zero, whereas $H_b^2(\mathbb{Z},\mathbb{R})$ is trivial since $\mathbb{Z}$ is amenable. To see that, consider the following diagram where the horizontal lines are short exact sequences

$$\begin{array}{ccccc} \mathbb{Q}/\mathbb{Z} \cong H_f^1(\mathbb{Z},\mathbb{R}/\mathbb{Z}) & \longrightarrow & H_f^2(\mathbb{Z},\mathbb{Z}) & \longrightarrow & H_f^2(\mathbb{Z},\mathbb{R}) \neq 0 \\ \downarrow & & \downarrow = & & \downarrow \\ \mathbb{R}/\mathbb{Z} \cong H_b^1(\mathbb{Z},\mathbb{R}/\mathbb{Z}) & \xrightarrow{\cong} & H_b^2(\mathbb{Z},\mathbb{Z}) & \longrightarrow & H_b^2(\mathbb{Z},\mathbb{R}) = 0. \end{array}$$

3.2 Secondary characteristic classes

In this chapter we define the secondary invariants of Cheeger-Simons following the original paper of Cheeger and Simons ([ChSi85]). Those depend on an invariant polynomial $f \in I^q(G, \mathbb{F})$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, and a cohomology class $[u] \in H^{2q}(BG, \Lambda)$ for some discrete subring $\Lambda < \mathbb{F}$, satisfying $w_G(f) = r([u])$, where $r : H^{2q}(BG, \Lambda) \to H^{2q}(BG, \mathbb{F})$ is induced from the inclusion of coefficients. In the introduction we asserted there existence in the case of bundles endowed with a flat connection. However, it is both natural and convenient to define them more generally for any connection. We define below the ring of differential characters, which contains the usual cohomology ring, where the secondary invariants $S_{(f,u)}$ find there natural receptacle.

3.2.1 Differential characters

Let M be a smooth connected manifold. Recall that by smooth we really mean infinitely differentiable. Denote by $C_*(M, \mathbb{Z})$ the complex of smooth singular chains on M (with integer coefficients), and let $Z_*(M, \mathbb{Z})$ be its subcomplex of smooth singular cycles. Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$, and write $C^*(M, \mathbb{F})$ for the complex of smooth singular cochains on M with coefficients in $\mathbb{F}$, that is,

$$C^*(M, \mathbb{F}) = \mathrm{Hom}_Z(C_*(M, \mathbb{Z}), \mathbb{F}).$$

Let $A^*(M, \mathbb{F})$ denote the complex of smooth differential forms on M with coefficients in $\mathbb{F}$. There is a natural inclusion of $A^*(M, \mathbb{F})$ in $C^*(M, \mathbb{F})$ given by integrating differential forms over smooth chains. Observe that Stoke's theorem is equivalent to saying that this inclusion is a chain map. The induced map in cohomology is the de Rham isomorphism between the de Rham cohomology and the singular cohomology of the manifold M. Let Λ be a discrete subring of $\mathbb{F}$ (typically, $\Lambda = \{0\}$ or $\mathbb{Z}$). Composing the integration with the reduction modulo Λ we obtain a map

$$\iota_\Lambda : A^q(M, \mathbb{F}) \longrightarrow C^q(M, \mathbb{F}/\Lambda)$$

defined by

$$\iota_\Lambda(\alpha)(\sigma) = \int_\sigma \alpha \mathrm{mod} \Lambda,$$

where α is a differential q-form, and $\sigma : \Delta^q \to M$ is a smooth simplex.

Lemma 67 *The map ι_Λ is injective.*

Proof. Put on the set of smooth singular q-chains $S^q(M) = \{\sigma : \Delta^q \to M \mid \sigma \text{ is smooth}\}$ the compact-open topology. Since M is connected, it is also path connected (recall that M is a manifold). We claim that $S^q(M)$ is also path connected. To see that, we show that any $\sigma : \Delta^q \to M$ can be connected to some constant map $\sigma_0(t) = x_0$, for some fixed $x_0 \in M$. Since Δ^q is contractible, there exists a homotopy between the identity of Δ^q and some constant map, say

$$H : \Delta^q \times [0,1] \longrightarrow \Delta^q,$$

with $H(t, 0) = \mathrm{Id}_{\Delta^q}$ and $H(t, 1) = t_0$ for some t_0 in Δ^q. Now, as M is path connected, there exists a path $\gamma : [0,1] \to M$ with $\gamma(0) = \sigma(t_0)$ and $\gamma(1) = x_0$. Finally define

$$\begin{array}{rccc} F: & \Delta^q \times [0,1] & \longrightarrow & M \\ & (t,s) & \longmapsto & \left\{ \begin{array}{cl} \sigma(H(t,2s)) & \text{if } s \leq \frac{1}{2}, \\ \gamma(2s-1) & \text{if } s \geq \frac{1}{2}. \end{array} \right. \end{array}$$

Now any differential form α in $A^q(M,\mathbb{F})$ defines, by integration, a continuous map $\alpha : S^q(M) \to \mathbb{F}$. But if $\iota_\Lambda(\alpha) = 0$, we have that the image of α is equal to Λ. Since $S^q(M)$ is connected per arc, its image by α is also, and as it contains zero and Λ is discrete, it must be equal to zero. ■

Definition 68 *Let M be a smooth manifold. The group of differential characters of degree q on M is defined by*

$$\hat{H}^q(M,\mathbb{F}/\Lambda) = \{f \in \operatorname{Hom}(Z_q(M,\mathbb{Z}),\mathbb{F}/\Lambda) \mid \delta f \in \iota_\Lambda(A^{q+1}(M,\mathbb{F}))\}.$$

The first examples of elements of $\hat{H}^q(M,\mathbb{F}/\Lambda)$ are cohomology classes in $H^q(M,\mathbb{F}/\Lambda)$. Indeed, let $[c]$ belong to $H^q(M,\mathbb{F}/\Lambda)$, and let c be a cocycle representing it. Then of course δc lies in the image of ι_Λ since it is zero, so that we obtain a differential character by restricting c to the cycles of M. Observe that c being a cocycle, its value on any cycle of M only depends on the cohomology class $[c]$, so that we have defined a homomorphism

$$\begin{array}{ccc} H^q(M,\mathbb{F}/\Lambda) & \longrightarrow & \hat{H}^q(M,\mathbb{F}/\Lambda) \\ [c] & \longmapsto & c_{|Z_q(M)}, \end{array}$$

which will easily be checked, in Lemma 69, to be an injection.

Define now $A_0^{q+1}(M)$ to be the set of closed differential $(q+1)$-forms on M with periods in Λ, that is,

$$A_0^{q+1} = \{\alpha \in A^{q+1}(M,\mathbb{F}) \mid d\alpha = 0 \text{ and } \int_z \alpha \in \Lambda \ \forall z \in Z_{q+1}(M)\}.$$

We can define a map, which we denote by δ thereby slightly abusing notation, by

$$\begin{array}{ccc} \delta : \hat{H}^q(M,\mathbb{F}/\Lambda) & \longrightarrow & A_0^{q+1}(M) \\ f & \longmapsto & \alpha, \end{array}$$

where α is the (unique) differential form such that $\delta f = \iota_\Lambda \alpha$. Let us check that this is well defined. That such a differential form exists follows from the definition of differential characters. It is unique by injectivity of ι_Λ. Using Stoke's theorem and the injectivity of ι_Λ it is easy to see that it is closed:

$$\iota_\Lambda(d\alpha)(\sigma) = \int_\sigma d\alpha \mathrm{mod}\Lambda = \int_{\partial\sigma} \alpha \mathrm{mod}\Lambda = \iota_\Lambda \alpha(\partial\sigma) = \delta^2 f(\sigma) = 0,$$

for every smooth singular simplex $\sigma : \Delta^{q+2} \to M$. It remains to check that it has its periods in Λ. Let z be a smooth singular $q+1$-cycle on M, then

$$\int_z \alpha \mathrm{mod}\Lambda = \iota_\Lambda(\alpha)(z) = \delta f(z) = f(\partial z) = 0.$$

Putting those two maps together, we obtain the nice following lemma (Theorem 1.1 of [ChSi85]):

Lemma 69 *There is a short exact sequence*

$$0 \longrightarrow H^q(M,\mathbb{F}/\Lambda) \longrightarrow \hat{H}^q(M,\mathbb{F}/\Lambda) \longrightarrow A_0^{q+1}(M) \longrightarrow 0.$$

Proof. We have:

- Injectivity of $H^q(M,\mathbb{F}/\Lambda) \to \hat{H}^q(M,\mathbb{F}/\Lambda)$: a cocycle vanishing on all cycles represents the zero class.
- $\mathrm{Im}(H^q(M,\mathbb{F}/\Lambda) \to \hat{H}^q(M,\mathbb{F}/\Lambda)) \subset \mathrm{Ker}(\hat{H}^q(M,\mathbb{F}/\Lambda) \to A_0^{q+1}(M)$: for a differential character f defined as the restriction of a cocycle, we surely have $\delta f = 0 = \iota_\Lambda(0)$.
- $\mathrm{Im}(H^q(M,\mathbb{F}/\Lambda) \to \hat{H}^q(M,\mathbb{F}/\Lambda)) \supset \mathrm{Ker}(\hat{H}^q(M,\mathbb{F}/\Lambda) \to A_0^{q+1}(M)$: if a differential character f is such that $\delta f = \iota_\Lambda(0) = 0$, we claim that we can extend it to a cocycle on M. To see that observe that the quotient $C_q(M,\mathbb{Z})/Z_q(M,\mathbb{Z})$ is a free $\mathbb{Z}$-module since it is isomorphic to its image in $C_{q-1}(M,\mathbb{Z})$ by the boundary operator ∂, and thus a submodule of a free $\mathbb{Z}$-module. It follows that
$$C_q(M,\mathbb{Z}) = Z_q(M,\mathbb{Z}) \oplus F,$$
for some free $\mathbb{Z}$-module F. We can hence define a cochain $\overline{f}$ on M as $\overline{f}(c) = f(c)$ if c is a cycle, $\overline{f}(c) = 0$ for c in F, and extend it Λ-linearly to $C_q(M,\mathbb{Z})$. Of course, the cochain $\overline{f}$ is actually a cocycle.
- Surjectivity of $\hat{H}^q(M,\mathbb{F}/\Lambda) \to A_0^{q+1}(M)$: Let α be a closed differential form with periods in Λ. The form α in particular defines, by integration, a map a from the smooth cycles on M to Λ. By the same argument as above, we can extend a to a cocycle in $C^{q+1}(M,\Lambda)$, which we still denote by a. Let $r(a)$ be the image of a by the injection $C^{q+1}(M,\Lambda) \to C^{q+1}(M,\mathbb{F})$ induced by the inclusion of coefficients $\Lambda \hookrightarrow \mathbb{F}$. Observe that $\alpha - r(a)$ is zero when evaluated on cycles, and thus is a coboundary, so that
$$\alpha - r(a) = \delta\overline{f},$$
for some $\overline{f}$ in $C^q(M,\mathbb{F})$. Finally define f to be the restriction of $\overline{f}$ modulo Λ to the cycles of M, and conclude that
$$\delta f = \iota_\Lambda(\alpha),$$
since $r(a)$ modulo Λ vanishes.

■

We see in particular, that cohomology with coefficients in $\mathbb{F}/\Lambda$ can indeed be considered as a subgroup (actually even a subring) of the differential characters.

Let us now investigate the relation of the ring of differential characters with the Bockstein map $b : H^q(M,\mathbb{F}/\Lambda) \to H^{q+1}(M,\Lambda)$ associated to the short exact sequence of coefficients $\Lambda \hookrightarrow \mathbb{F} \twoheadrightarrow \mathbb{F}/\Lambda$. Let $f \in \widehat{H}^q(M,\mathbb{F}/\Lambda)$ be a differential character with $\delta f = \iota_\Lambda(\alpha)$. As in the proof of the above lemma, we can extend f to a $\mathbb{F}/\Lambda$-valued singular cochain $\overline{f}$. There clearly exists a cochain $c \in C^q(M,\mathbb{F})$ with $\overline{f} = c \mathrm{mod}\Lambda$. Since the reduction modulo Λ of the cocycle $\alpha - \delta c \in C^{q+1}(M,\mathbb{F})$ is zero, it follows from the exactness of the sequence
$$0 \longrightarrow C^{q+1}(M,\Lambda) \longrightarrow C^{q+1}(M,\mathbb{F}) \longrightarrow C^{q+1}(M,\mathbb{F}/\Lambda) \longrightarrow 0$$
that there exists a Λ-valued cochain $u \in C^{q+1}(M,\Lambda)$ such that $r(u) = \alpha - \delta c$, where r stands for the inclusion $r : C^{q+1}(M,\Lambda) \to C^{q+1}(M,\mathbb{F})$. The cochain u necessarily is a cocycle, again by exactness. The only choice involved in this construction is the choice of cochain $c \in C^q(M,\mathbb{F})$ with $\overline{f} = c \mathrm{mod}\Lambda$. Suppose thus that $c' \in C^q(M,\mathbb{F})$ is another cochain satisfying $\overline{f} = c' \mathrm{mod}\Lambda$ and let

$u' \in C^{q+1}(M,\Lambda)$ be obtained as above. Since the reduction modulo Λ of $c-c'$ is zero it follows that there exists $v \in C^q(M,\Lambda)$ with $r(v) = c - c'$. But then surely

$$r(\delta v) = \delta c - \delta c' = \alpha - r(u) - (\alpha - r(u')) = r(u') - r(u),$$

so that by injectivity of r the cocycles u and u' differ by a coboundary. It follows that we have defined a map

$$\widehat{H}^q(M,\mathbb{F}/\Lambda) \longrightarrow H^{q+1}(M,\Lambda),$$

and as by construction it agrees with $-b$ on $H^q(M,\mathbb{F}/\Lambda)$, we abuse notation and denote it by $-b$.

Lemma 70 *There is a short exact sequence*

$$0 \longrightarrow A^q(M,\mathbb{F})/A^q_0(M) \longrightarrow \widehat{H}^q(M,\mathbb{F}/\Lambda) \longrightarrow H^{q+1}(M,\Lambda) \longrightarrow 0.$$

3.2.2 The definition of the secondary invariants

From now on, and until the end of the chapter, let $f \in I^q(G,\mathbb{F})$ be an invariant polynomial, where we recall that $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, and $[u] \in H^{2q}(BG,\Lambda)$ a cohomology class satisfying $w_G(f) = r([u])$, with $r : H^{2q}(BG,\Lambda) \to H^{2q}(BG,\mathbb{F})$ induced from the inclusion of coefficients.

Theorem 71 *For every smooth principal G-bundle ξ over a smooth manifold M endowed with a connection form ω, there exists a unique differential character*

$$S_{(f,u)}(\xi,\omega) \in \hat{H}^{2q-1}(M,\mathbb{F}/\Lambda)$$

satisfying

- $\delta S_{(f,u)}(\xi,\omega) = \iota_\Lambda(C_q(\Omega))$,
- $-bS_{(f,u)}(\xi,\omega) = [u(\xi)]$,
- the assignment $(\xi,\omega) \mapsto S_{(f,u)}(\xi,\omega)$ is natural in the sense that if $\phi : N \to M$ is a smooth map, then $S_{(f,u)}(\phi^*(\xi),\phi^*(\omega)) = \phi^*(S_{(f,u)}(\xi,\omega))$.

The original proof of Cheeger and Simons (Theorem 2.2 in [ChSi85]), as most of their proofs, goes via the universal bundle classifying both bundle and connections, which was shown to exist by Narasimhan and Ramanan in [NaRa61] and [NaRa63]. We present in the next section an alternative constructive proof in the case of the Chern class which was already sketched in [ChSi85] and is very well explained in [DuHaZu00].

3.2.3 A direct proof of the existence and uniqueness of the Cheeger-Chern-Simons classes

Recall that the Chern polynomials $C_q \in I^*(\mathrm{GL}_n(\mathbb{C}))$ are defined by the relation

$$\det(\lambda \mathrm{Id}_n - \frac{1}{2i\pi}A) = \sum_{q=0}^{n} C_q(A^q)\lambda^{n-q}.$$

Also, there exists for every q, a unique class $[c_q] \in H^{2q}(B\mathrm{GL}_n(\mathbb{C}),\mathbb{Z})$ satisfying

$$r([c_q]) = w_{\mathrm{GL}_n(\mathbb{C})}(C_q) \in H^{2q}(BGL_n(\mathbb{C}),\mathbb{R}).$$

Define for any principal $\mathrm{GL}_n(\mathbb{C})$-bundle ξ endowed with a connection ω the *Cheeger-Chern-Simons class* $\widehat{c}_q \in \widehat{H}^{2q-1}(M,\mathbb{C}/\mathbb{Z})$ to be the secondary invariant associated to the couple $(C_q,[c_q])$, that is, $\widehat{c}_q(\xi,\omega) = S_{(C_q,[c_q])}$. Observe that from the unicity of $[c_q]$ it follows that the condition that $-b(\widehat{c}_q(\xi,\omega)) = [c_q(\xi)]$ is superfluous in Theorem 71.

There exists an explicit construction for the Cheeger-Chern-Simons classes, which we will describe in this section, thereby giving a direct proof of Theorem 71 in this particular case. (See the original version in [ChSi85, § 4] or a more detailed one in [DuHaZu00, § 3.5].)

Consider the Stiefel manifold $V_{n-q+1}(\mathbb{C}^n)$ consisting of $(n-q+1)$-tuples of linearly independent vectors in $\mathbb{C}^n$. It is elementary to observe that $V_{n-q+1}(\mathbb{C}^n)$ is homotopically equivalent to its submanifold $\mathrm{U}(n)/\mathrm{U}(q-1)$. One now obtains the homology of the Stiefel manifold by classical means of computation. (See for example [St51], § 25.7, for the analogous statement for the homotopy groups of the Stiefel manifold.)

Proposition 72 *One has*

$$H_i(V_{n-q+1}(\mathbb{C}^n)) \cong \begin{cases} 0 & \textit{if } i < 2q-1, \\ \mathbb{Z} & \textit{if } i = 2q-1. \end{cases}$$

Moreover, the $(2q-1)$-th homology group $H_{2q-1}(V_{n-q+1}(\mathbb{C}^n))$ is generated by the cycle $S^{2q-1} \cong U(q)/U(q-1)$.

Let $\xi = P \to M$ be a principal G-bundle over a smooth manifold M. Suppose the bundle ξ is endowed with a connection form ω and denote by Ω the corresponding curvature form. Let

$$\begin{array}{c} V_{n-q+1}(P) \\ \downarrow \pi \\ M \end{array}$$

be the corresponding Stiefel bundle with fiber $V_{n-q+1}(\mathbb{C}^n)$. (The total space is $V_{n-q+1}(P) = V_{n-q+1}(\mathbb{C}^n)\times_G P$.) We can now compute the Serre exact sequence of this fibration and in particular obtain that the following sequence is exact (see [McCl01], example 5.D):

$$H_{2q-1}(V_{n-q+1}(\mathbb{C}^n)) \longrightarrow H_{2q-1}(V_{n-q+1}(P) \underset{\longrightarrow}{\pi_*} H_{2q-1}(M) \longrightarrow 0. \qquad (*)$$

Consider now the pull back of ξ by the bundle map π of the Stiefel bundle:

$$\begin{array}{ccc} \pi^*(P) & & P \\ \downarrow & & \downarrow \\ V_{n-q+1}(P) & \longrightarrow & M. \end{array}$$

Lemma 73 *The bundle $\pi^*(\xi)$ admits a reduction to $GL_{q-1}(\mathbb{C})$.*

We hence have that

$$[\pi^*(C_q(\Omega))] = [C_q(\pi^*(\Omega))] = 0 \in H^{2q}(V_{n-q+1}(P)),$$

and it thus follows that the form $C_q(\pi^*(\Omega)) \in A^{2q}(V_{n-q+1}(P))$ is exact. Of course, we would now like to choose a form whose differential is $C_q(\pi^*(\Omega))$ in a natural and canonical way. As we will soon prove this choice is possible, however only up to exact reminder.

We obtain a connection ω_0 on $\pi^*(\xi)$ by extending to it the connection $\pi^*(\omega)$ restricted to the $\mathrm{GL}_{q-1}(\mathbb{C})$ reduction. Note that ω_0 depends on the decomposition $\pi^*(P) \cong \eta \oplus \varepsilon^{n-q+1}$. By Proposition 59 we have

$$d\overline{TC_q}(\pi^*(\omega), \omega_0) = \overline{C_q(\pi^*(\Omega))} - \overline{C_q(\Omega_0)}.$$

But by construction of ω_0, the form $\overline{C_q(\Omega_0)} \in A^{2q}(V_{n-q+1}(P))$ vanishes identically.

The following proposition (which is Proposition 3.8 in [DuHaZu00]) shows that, up to exact reminder, we had no choice for the form $\overline{TC_q}(\pi^*(\omega), \omega_0)$.

Proposition 74 *The form $\overline{TC_q}(\pi^*(\omega), \omega_0) \in A^{2q-1}(V_{n-q+1}(P))$ is uniquely determined, up to exact reminder, by naturality and the relation*

$$d\overline{TC_q}(\pi^*(\omega), \omega_0) = \overline{C_q(\pi^*(\Omega))}.$$

It follows that the only way to define the Cheeger-Chern-Simons class on the bundle $\pi^*(\xi)$ endowed with the connection $\pi^*(\omega)$ is:

$$\widehat{c}_q(\pi^*(\xi), \pi^*(\omega)) = \overline{TC_q}(\pi^*(\omega), \omega_0) \mathrm{mod}\mathbb{Z}|_{Z_{2q-1}(V_{n-q+1}(P))}.$$

This is clear from the requirement that $\delta\widehat{c}_q(\pi^*(\xi), \pi^*(\omega)) = \iota_{\mathbb{Z}}(C_q(\Omega^q))$ and Proposition 74.

Theorem 75 *With the notation as above, the Cheeger-Chern-Simons class $\widehat{c}_q(\xi, \omega)$ is defined, on every cycle $z \in Z_{2q-1}(M, \mathbb{Z})$, as*

$$\widehat{c}_q(\xi, \omega)(z) = \widehat{c}_q(\pi^*(\xi), \pi^*(\omega))(\widetilde{z}) + C_q(\Omega^q)(y) mod\mathbb{Z},$$

where $z = \pi_*(\widetilde{z}) + \partial y$, for some cycle $\widetilde{z} \in Z_{2q-1}(V_{n-q+1}(P))$ and chain $y \in C_{2q}(M)$.

Proof. We need to prove that this expression is well defined and that it satisfies the desired properties of Theorem 71 for the Cheeger-Chern-Simons class.

To see that it is well defined, first observe that since the map

$$H_{2q-1}(V_{n-q+1}(P)) \xrightarrow{\pi_*} H_{2q-1}(M)$$

is surjective, it follows that for any cycle $z \in Z_{2q-1}(M)$ there exists a cycle $\widetilde{z}$ in $Z_{2q-1}(V_{n-q+1}(P))$ such that $[z] = [\pi_*(\widetilde{z})] \in H_{2q-1}(M)$. There thus exists a chain $y \in C_{2q}(M)$ with $z = \pi_*(\widetilde{z}) + \partial y$. Secondly, if $z = \pi_*(\widetilde{z}) + \partial y = \pi_*(\widetilde{z}') + \partial y'$ for some cycles $\widetilde{z}, \widetilde{z}' \in Z_{2q-1}(V_{n-q+1}(P))$ and chains $y, y' \in C_{2q}(M)$ we need to show that

$$\begin{aligned}\widehat{c}_q(\pi^*(\xi), \pi^*(\omega))(\widetilde{z}) + C_q(\Omega^q)(y)\mathrm{mod}\mathbb{Z} = &\widehat{c}_q(\pi^*(\xi), \pi^*(\omega))(\widetilde{z}') \\ &+ C_q(\Omega^q)(y')\mathrm{mod}\mathbb{Z}.\end{aligned} \qquad (**)$$

It follows from the exact sequence (*) above, that the class $[\widetilde{z} - \widetilde{z}']$ belonging to $H_{2q-1}(V_{n-q+1}(P))$ is in the image of the map $H_{2q-1}(V_{n-q+1}(\mathbb{C}^n)) \xrightarrow{i_*} H_{2q-1}(V_{n-q+1}(P))$. There thus exists a cycle $v \in Z_{2q-1}(V_{n-q+1}(\mathbb{C}^n))$ and a chain $w \in C_{2q}(V_{n-q+1}(P))$ such that

$$\widetilde{z} - \widetilde{z}' = i_* v + \partial w.$$

We claim that $\widehat{c}_q(\pi^*(\xi),\pi^*(\omega))(i_*v)$ is equal to zero. Assuming this for a moment, we see that the desired equation (**) reduces to

$$\begin{aligned} C_q(\Omega^q)(y')\mathrm{mod}\mathbb{Z}-C_q(\Omega^q)(y)\mathrm{mod}\mathbb{Z} &= \widehat{c}_q(\pi^*(\xi),\pi^*(\omega))(\partial w) \\ &= \delta\widehat{c}_q(\pi^*(\xi),\pi^*(\omega))(w) \\ &= C_q(\pi^*(\Omega)^q)(w)\mathrm{mod}\mathbb{Z} \\ &= C_q(\Omega^q)(\pi_* w)\mathrm{mod}\mathbb{Z}, \end{aligned}$$

where the last equality follows from the naturality of the Chern form C_q. Observe that $\partial(y'-y) = \pi_*(\widetilde{z}-\widetilde{z}') = \pi_*(i_*v+\partial w) = \partial(\pi_* w)$, so that $\pi_* w + y - y'$ is a $2q$-cycle on M and therefore $C_q(\Omega^q)(\pi_* w + y - y')$ belongs to $\mathbb{Z}$ thus proving the equality (**).

As for the claim that

$$\widehat{c}_q(\pi^*(\xi),\pi^*(\omega))(i_*v) = 0,$$

we do not know how to prove this without using the universal bundle. Let $[v] = S^{2q-1}$ be a generator of $H_{2q-1}(V_{n-q+1}(\mathbb{C}^n))$. Its image i_*v is then a cycle which necessarily is a boundary since $H_{2q-1}(V_{n-q+1}(U)) \cong H_{2q-1}(B) = 0$. Let $b \in C_{2q}(V_{n-q+1}(U))$ be such that $\partial b = i_*v$. Then $\pi_*(b)$ is a cycle on the base space of the universal bundle since $\pi_* i_* v = 0$. But now we have

$$\int_v \overline{TC_q}(\pi^*(\omega),\omega_0) = \int_b \overline{C_q(\pi^*(\Omega)^q)} = \int_{\pi_*(b)} \overline{C_q(\Omega)} \in \mathbb{Z},$$

which finishes the proof of the claim.

Finally, let us check that the so defined $\widehat{c}_q$ satisfies the properties of Theorem 71:

- Let $c \in C_{2q}(M,\mathbb{Z})$ be a chain on M. Then

  $$\delta\widehat{c}_q(\xi,\omega)(c) = \widehat{c}_q(\xi,\omega)(\partial c) = C_q(\Omega^q)(c),$$

  since the cycle ∂c has the form $\partial c = \pi^*(0) + \partial c$.

- Let $f : N \to M$ be a smooth map. We need to show that

  $$\widehat{c}_q(f^*(\xi), f^*(\omega))(z) = f^*\widehat{c}_q(\xi,\omega)(z)$$

  for every cycle $z \in Z_{2q-1}(N,\mathbb{Z})$. We have the commutative diagram

  $$\begin{array}{ccc} V_{n-q+1}(f^*(\xi)) & \xrightarrow{\pi_N} & N \\ \Big\downarrow{\scriptstyle \overline{f}} & & \Big\downarrow{\scriptstyle f} \\ V_{n-q+1}(\xi) & \xrightarrow{\pi_M} & M \end{array}$$

  Let z be a $(2q-1)$-cycle on N. There exists a cycle $\widetilde{z} \in Z_{2q-1}(V_{n-q+1}(f^*(\xi)))$ and a chain $y \in C_{2q}(N)$ such that

  $$z = (\pi_N)_*\widetilde{z} + \partial y.$$

  By definition of $\widehat{c}_q$ we have on one hand

  $$\begin{aligned} \widehat{c}_q(f^*(\xi), f^*(\omega))(z) &= \widehat{c}_q(\pi_N^*(f^*(\xi)),\pi_N^*(f^*(\omega)))(\widetilde{z}) + C_q(f^*(\Omega)^q)(y)\mathrm{mod}\mathbb{Z} \\ &= \widehat{c}_q(\overline{f}^*(\pi_N^*(\xi)),\overline{f}^*(\pi_N^*(\omega)))(\widetilde{z}) + C_q(f^*(\Omega)^q)(y)\mathrm{mod}\mathbb{Z}, \end{aligned}$$

and on the other hand,

$$\begin{aligned} f^*(\widehat{c}_q(\xi,\omega))(z) &= \widehat{c}_q(\xi,\omega)(f_*(z)) \\ &= \widehat{c}_q(\pi_M^*(\xi),\pi_M^*(\omega))(\overline{f}_*\widetilde{z}) + C_q(\Omega^q)(f_*y)\mathrm{mod}\mathbb{Z}, \end{aligned}$$

since $f_*z = f_*((\pi_N)_*\widetilde{z} + \partial y) = (\pi_M)_*(\overline{f}_*\widetilde{z}) + \partial(f_*y)$. The naturality of the Chern form C_q implies that

$$C_q(f^*(\Omega)^q)(y) = C_q(\Omega^q)(f_*y).$$

It remains to show that

$$\widehat{c}_q(\overline{f}^*(\pi_N^*(\xi)), \overline{f}^*(\pi_N^*(\omega)))(\widetilde{z}) = \widehat{c}_q(\pi_M^*(\xi),\pi_M^*(\omega))(\overline{f}_*\widetilde{z}).$$

This is clear from the naturality of the form TC_q and the fact that if ω_0^M is some connection chosen as above, then we can take $\omega_0^N = \overline{f}^*\omega_0^M$.

■

It is now easy to give a direct and constructive proof of Theorem 71 for the Cheeger-Chern-Simons class.

Proof of Theorem 71 for the Cheeger-Chern-Simons class. The existence of the Cheeger-Chern-Simons class follows from Theorem 75. We thus just need to prove the uniqueness. It follows from Lemma 69, that the horizontal sequences of the following diagram are exact:

$$\begin{array}{ccccc} H^{2q-1}(M,\mathbb{C}/\mathbb{Z}) & \hookrightarrow & \widehat{H}^{2q-1}(M,\mathbb{C}/\mathbb{Z}) & \twoheadrightarrow & A_0^{2q}(M) \\ \downarrow & & \downarrow & & \downarrow \\ H^{2q-1}(V(P),\mathbb{C}/\mathbb{Z}) & \hookrightarrow & \widehat{H}^{2q-1}(V(P),\mathbb{C}/\mathbb{Z}) & \twoheadrightarrow & A_0^{2q}(V(P)), \end{array}$$

where we have written $V(P)$ for the frame bundle $V_{n-q+1}(P)$. The right and left vertical arrows are injective, so that, by the five lemma, the middle one is also injective. Now, we have seen from the above discussion, that the requirement that $\delta\widehat{c}_q(\pi^*(\xi),\pi^*(\omega)) = -\iota_{\mathbb{Z}}C_q(\pi^*(\Omega)^q)$ gave no choice for the Cheeger-Chern-Simons class $\widehat{c}_q(\pi^*(\xi),\pi^*(\omega)) \in \widehat{H}^{2q-1}(V(P),\mathbb{C}/\mathbb{Z})$, which in turn proves the uniqueness of the Cheeger-Chern-Simons class since by naturality we have

$$\widehat{c}_q(\pi^*(\xi),\pi^*(\omega)) = \pi^*(\widehat{c}_q(\xi,\omega)).$$

■

3.2.4 Dependency on the connection

Let us now return to the general setting. It is essential to understand how the secondary invariants of Cheeger-Simons vary when we change the connection. Let ξ be a principal G-bundle endowed with two connections ω_0 and ω_1. We then clearly have

$$-b(S_{(f,u)}(\xi,\omega_1)) + b(S_{(f,u)}(\xi,\omega_0)) = u(\xi) - u(\xi) = 0,$$

and it thus follows from the short exact sequence of Lemma 70 that the difference of the two secondary invariants can be given by a differential form modulo Λ, that is

$$S_{(f,u)}(\xi,\omega_1) - S_{(f,u)}(\xi,\omega_0) = \iota_\Lambda(\alpha)$$

for some form $\alpha \in A^*(M,\mathbb{F})$, uniquely determined up to an element of $A_0^*(M)$. Keeping in mind that any two connections can be joined by a smooth path of connections, we can even exhibit such a differential form.

Proposition 76 *Let $\xi = P \longrightarrow M$ be a principal G-bundle and ω_t be a smooth 1-parameter family of connection on ξ, with $t \in [0,1]$, then*

$$S_{(f,u)}(\xi,\omega_1) - S_{(f,u)}(\xi,\omega_0) = \iota_\Lambda \left(q \int_0^1 \overline{f(\frac{d}{dt}\omega_t \wedge \Omega_t^{q-1})dt} \right) |_{Z_{2q-1}(M)}.$$

Recall our notational convention that if $\alpha \in A^*(P,\mathbb{F})$ is in the image of $\pi^* : A^*(M,\mathbb{F}) \to A^*(P,\mathbb{F})$, where $\pi : P \to M$ is the bundle map, we write $\overline{\alpha} \in A^*(M,\mathbb{F})$ for the unique form on M satisfying $\pi^*(\overline{\alpha}) = \alpha$.

Proof. Let thus $z \in Z_{2q-1}(M,\mathbb{Z})$ be a smooth cycle on M. It should be clear that there exists a $(2q)$-chain $\tilde{z}$ on $M \times [0,1]$ such that

$$\partial\tilde{z} = (i_1)_*(z) - (i_0)_*(z),$$

where i_k, for $k = 0,1$, stands for the canonical inclusion

$$i_k : M \hookrightarrow M \times \{k\} \subset M \times [0,1].$$

Consider now the principal G-bundle

$$\tilde{\xi} = \begin{array}{c} P \times [0,1] \\ \downarrow \tilde{\pi} \\ M \times [0,1], \end{array}$$

where of course, $\tilde{\pi}(u,t) = (\pi(u),t)$. Notice that, for $k = 0,1$,

$$\xi \cong i_k^*(\tilde{\xi}).$$

At any point $(u,t) \in P \times [0,1]$ in the total space of $\tilde{\xi}$, the tangent space at (u,t) naturally decomposes in the direct sum of the tangent space of u in P and the one of t in $[0,1] \subset \mathbb{R}$, that is,

$$T_{(u,t)}(P \times [0,1]) = T_u P \oplus \mathbb{R}.$$

For any X in $T_{(u,t)}(P \times [0,1])$, let us write X_P for the orthogonal projection of X onto $T_u P$. Let ω_t be a smooth path of flat connections. Define on $\tilde{\xi}$ a form $\tilde{\omega} \in A^1(P \times \mathbb{R}, \mathfrak{g})$ by

$$\tilde{\omega}_{(u,t)}(X) = \omega(t)_u(X_P),$$

for every $(u,t) \in P \times [0,1]$ and $X \in T_{(u,t)}\mathbb{R}$. It is straightforward to check that the form $\tilde{\omega}$ actually defines a connection on $\tilde{\xi}$. Notice that, for $k = 0,1$, we have

$$i_k^*(\widetilde{\omega}) = \omega_k \in A^1(P,\mathfrak{g}).$$

Using the naturality of the secondary invariants, we can now compute the difference of the two differential characters in consideration evaluated on the cycle z:

$$\begin{aligned}
S_{(f,u)}(\xi,\omega_1)(z) - S_{(f,u)}(\xi,\omega_0)(z) &= S_{(f,u)}(\xi, i_1^*(\tilde{\omega}))(z) - S_{(f,u)}(\xi, i_0^*(\tilde{\omega}))(z) \\
&= i_1^*(S_{(f,u)}(\widetilde{\xi},\widetilde{\omega}))(z) - i_0^*(S_{(f,u)}(\widetilde{\xi},\widetilde{\omega}))(z) \\
&= S_{(f,u)}(\widetilde{\xi},\tilde{\omega})((i_1)_*(z) - (i_0)_*(z)) \\
&= S_{(f,u)}(\widetilde{\xi},\tilde{\omega})(\partial\tilde{z}) \\
&= \delta S_{(f,u)}(\widetilde{\xi},\tilde{\omega})(\tilde{z}) \\
&= \int_{\tilde{z}} \overline{f(\tilde{\Omega}^q)} \bmod \Lambda,
\end{aligned}$$

where $\tilde{\Omega}$ is the curvature form associated to $\tilde{\omega}$. Since $\pi^* : A^*(M) \to A^*(P)$ is injective, the proposition will now follow from

$$\int_{\tilde{z}} f(\tilde{\Omega}^q) = \int_z q \int_0^1 f(\frac{d}{dt}\omega(t) \wedge \Omega(t)^{q-1})dt.$$

To show this, we start by computing the curvature $\widetilde{\Omega}$ and its powers: Let (u,t) be in $P \times [0,1]$ and X, Y in $T_{(u,t)}(P \times [0,1])$. We have

$$\widetilde{\Omega} = dt \wedge \frac{\partial\tilde{\omega}}{\partial t} + \Omega_t.$$

We now claim that

$$\widetilde{\Omega}^q = q(dt \wedge \frac{\partial\tilde{\omega}}{\partial t} \wedge \Omega_t^{q-1}) + \Omega_t^q.$$

To see that, assume by induction that it is true for $q-1$, and compute

$$\begin{aligned}
\widetilde{\Omega}^q &= \widetilde{\Omega}^{q-1} \wedge \widetilde{\Omega} \\
&= \left((q-1)(dt \wedge \frac{\partial\tilde{\omega}}{\partial t} \wedge \Omega_t^{q-2}) + \Omega_t^{q-1}\right) \wedge (dt \wedge \frac{\partial\tilde{\omega}}{\partial t} + \Omega_t) \\
&= (q-1)(dt \wedge dt \wedge \frac{\partial\tilde{\omega}}{\partial t} \wedge \frac{\partial\tilde{\omega}}{\partial t} \wedge \Omega_t^{q-2}) + q(dt \wedge \frac{\partial\tilde{\omega}}{\partial t} \wedge \Omega_t^{q-1}) + \Omega_t^q,
\end{aligned}$$

which proves the claim since the first summand vanishes as $dt \wedge dt$ does. We now have

$$\int_{\tilde{z}} \overline{f(\tilde{\Omega}^q)} = \int_{\tilde{z}} \overline{f(q(dt \wedge \frac{\partial\tilde{\omega}}{\partial t} \wedge \Omega_t^{q-1})} + \int_{\tilde{z}} \overline{f(\Omega_t^q)}.$$

Observe that since there is no dt in $f(\Omega_t^q)$ the last summand must necessarily vanish. As for the first, we have

$$\int_{\tilde{z}} \overline{f(q(dt \wedge \frac{\partial\tilde{\omega}}{\partial t} \wedge \Omega_t^{q-1})} = \int_z q \int_0^1 \overline{f(\frac{\partial\omega_t}{\partial t} \wedge \Omega_t^{q-1})}dt,$$

which finishes the proof of the proposition. ∎

Taking the path joining the two connections ω_0 and ω_1 to be the convex linear combination $\omega_t = (1-t)\omega_0 + t\omega_1$ and recalling how the curvature of ω_t is computed (Lemma 33) we obtain the following proposition as an immediate corollary of Proposition 76.

Proposition 77 *Let $\xi = P \to M$ be a principal G-bundle and ω_0, ω_1 two connections on ξ. Then*

$$\hat{c}_q(\xi,\omega_1) - \hat{c}_q(\xi,\omega_0) = \iota_\Lambda \left(q \int_0^1 \overline{f(\omega_1 - \omega_0 \wedge ((1-t)\Omega_0 + t\Omega_1 + \frac{1}{2}(t^2 - t)[\omega_1 - \omega_0, \omega_1 - \omega_0])^{q-1})dt} \right) |_{Z_{2q-1}(M)}.$$

Proposition 76 is useful whenever the two connections considered are flat and can be joined by a path consisting of flat connections (whereas the convex linear combination of two flat connections is in general not flat as can be seen from Lemma 33), since in this case the right hand side of the conclusion of the proposition is zero as soon as q is strictly greater than one, as the curvature $\Omega(t)$ vanishes identically for every t. Sometimes however, it is more convenient to have a concrete description of the difference, as in Proposition 77, which only depends on the connections ω_0 and ω_1, but not on any choice of path.

3.2.5 Flat bundles

Let ω be a flat connection form on some principal G-bundle $\xi = P \to M$. Then the associated curvature form Ω of course vanishes identically, so that in particular $f(\Omega) = 0$ in $A^{2q}(M,\mathbb{F})$. It follows thus from the short exact sequence of Lemma 69 that any secondary invariant $S_{(f,u)}$ is in the image of the injective map

$$H^{2q-1}(M,\mathbb{F}/\Lambda) \hookrightarrow \widehat{H}^{2q-1}(M,\mathbb{F}/\Lambda).$$

Let us abuse notation, and write $S_{(f,u)}$ for the corresponding cohomology class in $H^{2q-1}(M,\mathbb{F}/\Lambda)$.

Theorem 78 (Rigidity) *Let ξ be a principal G-bundle over M, a smooth manifold, ω_0, ω_1 two flat connections on ξ in the same path connected component of flat connection and $q > 1$ a positive number. Then*

$$S_{(f,u)}(\xi,\omega_0) = S_{(f,u)}(\xi,\omega_1).$$

Proof. From Proposition 76 we obtain that the difference of the two differential characters is equal to

$$\iota_\Lambda \left(q \int_0^1 \overline{f(\frac{d}{dt}\omega(t) \wedge \Omega(t)^{q-1})dt} \right) |_{Z_{2q-1}(M)},$$

where $\omega(t)$ is any path of connections joining ω_0 to ω_1. But since the two flat connections are in the same path connected component (in the space of flat connections) we can take $\omega(t)$ to be flat for every t, so that $\Omega(t)$ vanishes for every t, and thus $\Omega(t)^{q-1} = 0$ whenever q is strictly greater than 1. ■

3.2.6 Boundedness properties

Conjecture 79 *If $q > 1$, then every secondary characteristic class $S_{(f,u)}$ in $H^{2q-1}(BG^\delta)$ can be represented by a bounded cocycle.*

Note that the assumption $q > 1$ is necessary, since for $q = 1$ the secondary characteristic class $\widehat{c}_1 \in H^1(BG^\delta, \mathbb{C}/\mathbb{Z}) \cong H^1(G^\delta, \mathbb{C}/\mathbb{Z})$ can be represented in the Eilenberg-MacLane group cohomology as

$$g \longmapsto \frac{1}{2\pi i}\log(\mathrm{tr}(g)) \in \mathbb{C}/\mathbb{Z}.$$

Dupont showed in [Du76] that primary and secondary classes admit explicit representatives by continuous cocycles in the Eilenberg-MacLane group cohomology $H^*(G^\delta) \cong H^*(BG^\delta)$ and asked in [Du78] whether those cocycles are moreover bounded. On the other hand, from Theorem 3 and the hypothetical Conjecture 79 it would follow that primary and secondary can be represented by bounded cocycles. It is only natural to ask if those classes can further be represented by cocycles which are both continuous and bounded. This is surely the case if Conjecture 79 holds and G admits a cocompact lattice, as we see from the argument in the proof that Conjecture 79 implies Conjecture 80. More generally, in view of Dupont and Kamber's result that the continuous cohomology of a connected semisimple Lie group with finite center is generated by primary and secondary characteristic classes (see [DuKa90, Theorem 5.2]), Conjecture 79 immediately implies Conjecture 80 below. This question was raised by Monod in [Mo01, p.126].

Conjecture 80 *If G is a connected semisimple real algebraic Lie group, then the comparison map*

$$H^n_{c,b}(G, \mathbb{R}) \longrightarrow H^n_c(G, \mathbb{R})$$

is surjective.

In degree 2, it was proven by Guichardet and Wigner that $H^2_c(G, \mathbb{R})$ is either trivial or one dimensional, according to the associated symmetric space being of Hermitian type or not. In the case where $H^2_c(G, \mathbb{R}) \cong \mathbb{R}$, an explicit bounded generator can be exhibited, so that the comparison map

$$H^2_{c,b}(G, \mathbb{R}) \longrightarrow H^2_c(G, \mathbb{R})$$

is surjective in degree 2.

In the case where G has real rank one, the surjectivity of the comparison map follows from the existence of a uniform bound on the volume of geodesic simplices in the corresponding symmetric spaces (see [Th78] or [Gr82]).

Proof that Conjecture 79 implies Conjecture 80. As mentioned in the introduction, Dupont and Kamber showed that the continuous cohomology of a connected semisimple Lie group is generated by primary and secondary characteristic classes. Together with Conjecture 79, this means that any cohomology element in the continuous cohomology of G can be represented by a (not necessarily continuous) bounded cocycle in the Eilenberg-MacLane cohomology of the group. It thus only remains to show that this cohomology element moreover admits a continuous and bounded representative.

It is well known, that G contains a cocompact lattice, say Γ. It is standard that there exists a map $H^*(\Gamma) \to H^*_c(G)$ such that the composition

$$H^*_c(G) \longrightarrow H^*(\Gamma) \longrightarrow H^*_c(G)$$

is the identity, and that the same holds in bounded cohomology. Chasing in the diagram

$$\begin{array}{ccccc} H^*_c(G) & \leftrightarrows & H^*(\Gamma) & \longleftarrow & H^*(G^\delta) \\ \uparrow & & \uparrow & & \uparrow \\ H^*_{c,b}(G) & \leftrightarrows & H^*_b(\Gamma) & \longleftarrow & H^*_b(G^\delta) \end{array}$$

leads to the desired conclusion. ■

A straightforward consequence of Conjecture 80 is the following conjecture of Gromov:

Conjecture 81 *The simplicial volume of any compact locally symmetric space of non compact type is strictly positive.*

The real rank one case is again due to Thurston, and simply follows from Conjecture 80 being valid for real rank one semi-simple Lie groups.

Savage proved in [Sa82] the existence of a uniform bound on the volume of certain top dimensional geometric simplices in $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n)$, which in turn proves Conjecture 81 for locally symmetric spaces covered by $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n)$.

Proof that Conjecture 80 implies Conjecture 81. Let $M = \Gamma\backslash G/K$ be a compact locally symmetric space of dimension n. Upon replacing Γ by a finite index subgroup, we can suppose that G is equal to the connected component of the identity $\mathrm{Isom}(\widetilde{M})^0$ of the isometries group of the universal cover of M. Note that by doing so, we replace M by a finite covering of itself, which has no effect on the non vanishing of the simplicial volume, since the two seminorms differ by the index of the covering. We claim that the simplicial volume of M is strictly positive if and only if the comparison map

$$H_b^n(\Gamma) \longrightarrow H^n(\Gamma)$$

is surjective. To see that, firstly observe that since M is a $K(\Gamma, 1)$, both the usual and the bounded cohomologies of M are canonically isomorphic to the corresponding group cohomologies of the fundamental group Γ of M, so that by the commutativity of the following diagram

$$\begin{array}{ccc} H_b^n(\Gamma) & \longrightarrow & H^n(\Gamma) \\ \cong\downarrow & & \downarrow\cong \\ H_b^n(M) & \longrightarrow & H^n(M) \end{array}$$

the surjectivity of the above given comparison map amounts to the surjectivity of the comparison map

$$H_b^n(M) \longrightarrow H^n(M).$$

Secondly, let $\beta \in H^n(M)$ be the dual of the fundamental class $[M] \in H_n(M)$ of the compact manifold M. The following easy relation was first proven by Gromov ([Gr82, page 17])

$$\|M\| = \frac{1}{\|\beta\|_\infty}.$$

Thus, the simplicial volume of M is strictly positive if and only if the cohomology class β admits a bounded representative, which is equivalent to saying that β is in the image of the comparison map $H_b^n(M) \longrightarrow H^n(M)$. The claim is hence proven since $H^n(M)$ is generated by β.

Consider now the commutative diagram

$$\begin{array}{ccc} H_b^n(\Gamma) & \longrightarrow & H^n(\Gamma) \\ \uparrow & & \uparrow \\ H_{c,b}^n(G) & \longrightarrow & H_c^n(G), \end{array}$$

and note that the theorem follows from the surjectivity of the lower horizontal and of the right vertical maps. Let us thus finish by proving this.

- Surjectivity of $H^n_{c,b}(G) \to H^n_c(G)$: The group $G = \mathrm{Isom}(\widetilde{M})^0$ is connected, and M being of non-compact type, it is moreover semisimple and has finite center, so that it is real algebraic and hence satisfies the assumptions of Conjecture 80. Note that if M were of Euclidean type this would not be so since the group G would then not be semisimple.

- Surjectivity of $H^n_c(G) \to H^n(\Gamma)$: As Γ is a cocompact lattice in G, by integrating over G/Γ, one obtains a map $H^*(\Gamma) \to H^*_c(G)$, and it is well known that the composition

$$H^*_c(G) \longrightarrow H^*(\Gamma) \longrightarrow H^*_c(G)$$

  is equal to the identity. Now, since $H^n(\Gamma)$ is one dimensional, the surjectivity of the map $H^n_c(G) \to H^n(\Gamma)$ reduces to the non triviality of $H^n(G)$. Since G is semi-simple it admits a compact form U, and the continuous cohomology is computed as

$$H^*_c(G) \cong H^*(U/K),$$

  where the latter cohomology is the de Rham cohomology of the manifold U/K. Note that the compact group K can in our case be chosen to be the same as the original group K (from $\Gamma\backslash G/K$), since it needs to be a maximal (non necessarily proper) compact subgroup of G, and as the latter is non compact K is equal to the maximal proper compact subgroup of G. This is not the case when M is of compact type. The dimension of M is equal to the dimension of its universal cover G/K so that

$$n = \mathrm{Dim}\mathfrak{g} - \mathrm{Dim}\mathfrak{k}.$$

  Since the dimension of the Lie algebra $\mathfrak{g}$ of G and the Lie algebra $\mathfrak{u}$ of U agree, it is also true that

$$\mathrm{Dim}(U/K) = \mathrm{Dim}\mathfrak{u} - \mathrm{Dim}\mathfrak{k} = \mathrm{Dim}\mathfrak{g} - \mathrm{Dim}\mathfrak{k} = n,$$

  so that $H^n(U/K)$ is one dimensional and thus

$$H^n_c(G) \neq \{0\},$$

  which was to be proven.

■

Chapter 4

The proof of the main Theorem

4.1 Semi-algebraic sets

The aim of this section is to introduce all standard results on semi-algebraic sets which we will need for our proof of Theorem 5. For the sake of conciseness, we omit most proofs, and invite the interested reader to consult Chapter 2 of the book [BeRi90] by Benedetti and Risler.

Definitions and first properties

A subset X of $\mathbb{R}^n$ is said to be *semi-algebraic* if it admits a representation of the form

$$X = \cap_{i=1}^{s} \cup_{j=1}^{r_i} \{x = (x_1, ..., x_n) \in \mathbb{R}^n \mid P_{i,j}(x) \;\geq\; 0\},$$

where $P_{i,j}(T_1, ..., T_n)$ is a polynomial in n variables belonging to $\mathbb{R}[T_1, ..., T_n]$ for every i and j. Such a representation is by no means unique as will soon be clear.

Note that semi-algebraic sets are closed under finite unions, finite intersections and complementation.

We can surely measure the complexity of a semi-algebraic set X in terms of the dimension of the affine space X belongs to, and the minimal number and degree of the polynomials involved in a representation of X. More precisely, let R be a representation as above of some semi-algebraic set. Define

$$C(R) = \sum_{i=1}^{s} r_i \;\text{ and }\; D(R) = \max_{i,j}\{\deg(P_{i,j})\}.$$

Let $n, c, d \in \mathbb{N}$ and set

$$S(n, c, d) := \left\{ X \subset \mathbb{R}^n \;\middle|\; \begin{array}{l} X \text{ is semi-algebraic and admits} \\ \text{a representation } R \text{ with} \\ C(R) \leq c \;\text{ and }\; D(R) \leq d \end{array} \right\}.$$

We say that a semi-algebraic set X *is of complexity* $S(n, c, d)$ if X belongs to $S(n, c, d)$. Some examples are in order:

1. Algebraic sets are surely semi-algebraic. In particular, the affine space $\mathbb{R}^n$ is semi-algebraic, and belongs to $S(n,0,0)$.

2. The standard q-simplex
$$\Delta^q = \{(t_1,...,t_q) \in \mathbb{R}^q \mid t_i \geq 0,\ 1-\Sigma_{i=1}^q t_i \geq 0\}$$
belongs to $S(q,q+1,1)$,

3. More generally, any finite simplicial complex K is semi-algebraic of complexity $S(n,c,1)$, where n and c depend on the number of simplices of K.

4. Observe that the minimal complexity of a semi-algebraic set is not well defined: the semi-algebraic set
$$\{x \in \mathbb{R} \mid x^2 \geq 1\} \ = \ \{x \in \mathbb{R} \mid x \leq -1\} \cup \{x \in \mathbb{R} \mid x \geq 1\}$$
is both of complexity $S(1,1,2)$ and $S(1,2,1)$.

Lemma 82 *If $X_1,...,X_\ell$ are semi-algebraic sets of complexity $S(n,c,d)$, then the intersection $\cap_{i=1}^d X_i$ is semi-algebraic of complexity $S(n,\ell c,d)$.*

Lemma 83 *Let X and Y be two algebraic subsets of $\mathbb{R}^n$. If X and Y are of complexity $S(n,c,d)$ then their join*
$$X \star Y = \{t(x,0)+(1-t)(y,1) \mid 0 \leq t \leq 1,\ x \in X,\ y \in Y\} \subset \mathbb{R}^n \times \mathbb{R}$$
is semi-algebraic of complexity $S(n+1,C,D)$, where C and D depend only on n,c and d.

Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be semi-algebraic. A map $f: X \to Y$ is called *semi-algebraic* if it is continuous and its graph is a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$. It is moreover called *semi-algebraic of complexity* $S(n,c,d)$ if its graph is semi-algebraic of complexity $S(n,c,d)$.

Before enumerating some further useful properties of semi-algebraic sets and maps which we will need in the proof of our Theorem 5, let us introduce some convenient notation. Let $n_1,...,n_q$ and n be natural numbers (or more generally functions or various objects). We write $n \lhd (n_1,...,n_q)$ if the number n is bounded by a number depending only on $n_1,...,n_q$. As an example, given a polynomial $f \in \mathbb{R}[T]$, denote by $r(f)$ the number of roots of f, and by $\deg(f)$ the degree of f, then $r(f) \lhd \deg(f)$.

Theorem 84 (Tarski-Seidenberg) *Let n,m,c,d be natural numbers. Then there exists $C,D \lhd n+m,c,d$ such that for every semi-algebraic sets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and for every semi-algebraic map $f: X \to Y$, if $A \subset X$ is a semi-algebraic set of complexity $S(n,c,d)$ and f is of complexity $S(n+m,c,d)$, then $f(X) \subset Y$ is a semi-algebraic subset of $\mathbb{R}^m$ of complexity $S(m,C,D)$.*

Corollary 85 *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be semi-algebraic sets, $f: X \to Y$ a semi-algebraic map of complexity $S(n+m,c,d)$. Suppose that $A \subset Y$ is a semi-algebraic subset of complexity $S(m,c,d)$, then $f^{-1}(A) \subset X$ is semi-algebraic of complexity $S(n,C,D)$, where $C,D \lhd n,m,c,d$.*

Corollary 86 *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^p$ be semi-algebraic sets, $f: X \to Y$ and $g: Y \to Z$ semi-algebraic maps. Suppose that f is of complexity $S(n+m,c,d)$ and g of complexity $S(m+p,c,d)$. Then the map $g \circ f: X \to Z$ is semi-algebraic of complexity $S(n+p,C,D)$, where $c,d \lhd n,m,p,c,d$.*

Another maybe less obvious consequence of Tarski-Seidenberg's Theorem 84 is that closures, interiors and boundaries of semi-algebraic sets are semi-algebraic:

Proposition 87 *Let X be a semi-algebraic set. Then its closure $\overline{X}$, its interior $Int(X)$ and its boundary $\partial X = \overline{X}\backslash Int(X)$ are semi-algebraic.*

Proof. Let X be a semi-algebraic subset of $\mathbb{R}^n$. Since $\partial X = \overline{X}\backslash \mathrm{Int}(X)$ and semi-algebraic sets are closed under taking complements, it is enough to show that $\overline{X}$ and $\mathrm{Int}(X)$ are semi-algebraic.

Consider the two following semi-algebraic subsets of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$:

$$\begin{aligned} X' = \ & \{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid r \leq 0,\ y \in X\} \\ & \cup \{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid \ |x-y|^2 - r < 0,\ y \in X\}. \end{aligned}$$

and

$$\begin{aligned} X'' = \ & \{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid r > 0\} \\ & \cup \{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid \ |x-y|^2 - r \geq 0\} \\ & \cup \{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid y \in X\}. \end{aligned}$$

Let p' and p'' be the two following canonical projections

$$\begin{array}{rccc} p' : & \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \\ & (x,y,t) & \longmapsto & (x,y) \end{array}$$

and

$$\begin{array}{rccc} p'' : & \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (x,y) & \longmapsto & x. \end{array}$$

Note that

$$\overline{X} = p''(\mathbb{R}^n \times \mathbb{R}^n \backslash p'(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \backslash X'))$$

and

$$\mathrm{Int}(X) = p''(\mathbb{R}^n \times \mathbb{R}^n \backslash p'(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \backslash X''))$$

so that it follows from 84 that $\overline{X}$ and $\mathrm{Int}(X)$ are semi-algebraic. ■

Triangulations of semi-algebraic sets

Theorem 88 below is the most technical tool which we need for our proof of Theorem 5. It is a bounded version of the existence of semi-algebraic triangulations of semi-algebraic sets. The unbounded version (that is, the existence of a semi-algebraic triangulation with no bound on the number or on the complexity of the simplices) was proven by Hironaka in [Hi74] following the analogous result by Lojasiewicz for semi-analytic sets. It was then observed by Benedetti and Risler, that one straightforwardly obtains the corresponding bounded version, by bounding every step of the constructive proof of Hironaka, as detailed in [BeRi90, Theorem 2.9.4].

Let X be a semi-algebraic set. A triangulation $h : X \to |K|$ of X is said to be a *semi-algebraic triangulation* if the homeomorphism h between X and the geometric realization of the simplicial complex K is semi-algebraic.

Theorem 88 *For every compact semi-algebraic set X and every semi-algebraic subsets $X_1, \ldots, X_\ell \subset X$, if $X_1, ..., X_\ell$ and X are of complexity $S(n, c, d)$ then there exists a semi-algebraic triangulation*

$$h : X \longrightarrow |K|$$

such that

1. *X_i is a finite union of $h^{-1}(s)$ for some simplices s of K, for every i between 1 and ℓ;*
2. *the number of simplices of K is bounded by k, where $k \lhd (n, c, d, \ell)$;*
3. *for every simplex s of K the set $h^{-1}(s)$ is semi-algebraic of complexity $S(n, C, D)$, for some $C, D \lhd (n, c, d, \ell)$.*

Observe that it follows from Theorem 88 that the Hauptvermutung holds in the semi-algebraic setting. Indeed, given two finite triangulations of some semi-algebraic set, apply Theorem 88 to all the simplices appearing in both triangulations in order to obtain a common refinement.

Corollary 89 *Let X be a semi-algebraic set. Then every connected component of X is semi-algebraic.*

Proof. Upon successively embedding X in a projective space and affine space of appropriate dimensions we can without loss of generality assume that $\overline{X}$ is compact. By Theorem 88, $\overline{X}$ admits a triangulation by semi-algebraic simplices, in such a way that X is a finite union of semi-algebraic simplices. In particular, every connected component of X is a finite union of semi-algebraic simplices, and thus is semi-algebraic. ■

Semi-algebraicity of the classifying space and classifying map

In the sequel we examine the question of semi-algebraicity for the classifying space BG_q and the classifying map f, which we defined in Section 1.1.2, in the case where the bundle in consideration is flat.

The classifying space Let us first examine the case of the space of n-frames $\mathrm{Fr}_n(\mathbb{R}^N)$. Recall that it is naturally identified with the set of all N times n matrices with linearly independent columns. The latter condition being equivalent to the non vanishing of at least one of the maximal minor, it is immediate that the space $\mathrm{Fr}_n(\mathbb{R}^N)$ can be viewed as a semi-algebraic subset of $\mathbb{R}^{Nn}$.

Let G be an algebraic subgroup of $\mathrm{GL}_n\mathbb{R}$ and let us show that BG_q is semi-algebraic. The main point is that BG_q can in a natural way be viewed as a homogeneous space. Indeed, consider the action of $\mathrm{GL}_N\mathbb{R}$ on BG_q (where, as in Section 1.1.2, $N = (q + 1)n$) given by left matrix multiplication

$$\begin{aligned} \mathrm{GL}_N\mathbb{R} \times BG_q &\longrightarrow BG_q \\ (A, [X]_G) &\longmapsto [AX]_G. \end{aligned}$$

The stabilizer of the point $\begin{bmatrix} 1_n \\ 0 \end{bmatrix}_G \in BG_q$ is easily checked to be

$$H(\mathbb{R}) = \left\{ \begin{pmatrix} g & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_N(\mathbb{R}) \middle| g \in G \right\}.$$

Our space BG_q is thus diffeomorphic to the homogeneous space

$$\mathrm{GL}_N(\mathbb{R})/H(\mathbb{R}).$$

Since G is algebraic, it is clear that $H(\mathbb{R})$ is a real algebraic subgroup of $\mathrm{GL}_N\mathbb{R}$. It is a consequence of a well known theorem of Chevalley that the homogeneous space

$$Y(\mathbb{C}) = \mathrm{GL}_N(\mathbb{C})/H(\mathbb{C})$$

of the corresponding complex algebraic groups is a complex quasi-projective variety (see [Bo91], §6 or more precisely Theorem 6.8). However, it is in general false that the real points $Y(\mathbb{R})$ of $Y(\mathbb{C})$ form the homogeneous space $\mathrm{GL}_N(\mathbb{R})/H(\mathbb{R})$. To see that let us consider the following examples:

- The quotient of $\mathrm{GL}_1(\mathbb{C})$ by its finite subgroup $\{+1, -1\}$ can naturally be identified with $\mathrm{GL}_1(\mathbb{C})$ in such a way that the quotient mapping is given by

$$\begin{array}{ccc} \mathrm{GL}_1(\mathbb{C}) & \longrightarrow & \mathrm{GL}_1(\mathbb{C}) \\ z & \longmapsto & z^2. \end{array}$$

  But in the real case, the quotient $\mathrm{GL}_1(\mathbb{R})/\{+1, -1\}$ is of course not diffeomorphic to $\mathrm{GL}_1(\mathbb{R})$. Actually, it is diffeomorphic to one connected component of $\mathrm{GL}_1(\mathbb{R})$.

- More generally, the quotient of $\mathrm{GL}_n(\mathbb{C})$ by its orthogonal subgroup $\mathrm{O}(n, \mathbb{C})$ is naturally identified with the space of non degenerated quadratic forms over $\mathbb{C}$, or equivalently, the space of symmetric non degenerated complex valued $(n \times n)$-matrices. But the non degenerated quadratic forms over $\mathbb{R}$, contrarily to the complex case, are not all equivalent, so that the action of $\mathrm{GL}_n(\mathbb{R})$ is not transitive: it has precisely $n + 1$ orbits corresponding to the signature of the non degenerated symmetric matrices. The homogeneous space $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n, \mathbb{R})$ actually is diffeomorphic to the orbit of the identity, that is the set of symmetric real valued $(n \times n)$-matrices for which all eigenvalues are strictly positive. It can thus be viewed as a semi-algebraic set.

The problem in the two above examples is that the projection map

$$\mathrm{GL}_N(\mathbb{C}) \longrightarrow \mathrm{GL}_N(\mathbb{C})/H(\mathbb{C}) = Y(\mathbb{C}),$$

which is defined over $\mathbb{R}$, is not surjective anymore when restricted to the underlying real varieties:

$$\mathrm{GL}_N(\mathbb{R}) \longrightarrow Y(\mathbb{R}).$$

Equivalently, the action of $\mathrm{GL}_N\mathbb{R}$ on $Y(\mathbb{R})$ is not transitive.

Let $\overline{1}$ denote the image of the identity via the projection map $\mathrm{GL}_N(\mathbb{C}) \longrightarrow Y(\mathbb{C})$ and let $X(\mathbb{R})$ be its orbit in $Y(\mathbb{R})$ under the action of $\mathrm{GL}_N(\mathbb{R})$. The stabilizer of $\overline{1}$ is then clearly

$$H(\mathbb{C}) \cap GL_N(\mathbb{R}) = H(\mathbb{R}),$$

so that

$$BG_q \cong \mathrm{GL}_N(\mathbb{R})/H(\mathbb{R}) \cong X(\mathbb{R}).$$

Because $X(\mathbb{R})$ is a finite union of connected components of $Y(\mathbb{R})$, it is semi-algebraic by Corollary 89.

Observe that we recover that the space of n-frames is semi-algebraic since $\mathrm{Fr}_n\mathbb{R}^N$ is nothing else than $B\{1_n\}_N$, where $\{1_n\}$ of course stands for the trivial group (in $\mathrm{GL}_n\mathbb{R}$). In this case the homogeneous space $Y(\mathbb{C})$ is equal to the space $\mathrm{Fr}_n(\mathbb{C}^N)$ of complex n-frames, so that its space of real points $Y(\mathbb{R})$ is precisely $\mathrm{Fr}_n\mathbb{R}^N$. The space $X(\mathbb{R})$ is here thus the whole $Y(\mathbb{R})$.

For further use, define $n(BG_q)$ to be equal to the dimension of the affine space that BG_q belongs to. (In particular, BG_q then belongs to $S(n(BG_q), c, d)$ for some c, d.)

The projection map Let us now show that the natural projection $\pi_G : PG_q \to BG_q$ is a semi-algebraic map. Because of the universal property of the quotient (see [Bo91], §6), there exists a unique algebraic map π :$\mathrm{Fr}_n(\mathbb{C}^N) \to Y(\mathbb{C})$ defined over $\mathbb{R}$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GL}_N(\mathbb{C}) & \longrightarrow & Y(\mathbb{C}) \\ \downarrow & \nearrow \pi & \\ \mathrm{Fr}_n(\mathbb{C}^N) & & \end{array}$$

Restricting the map π to the real point of the corresponding varieties, we obtain an algebraic map π :$\mathrm{Fr}_n(\mathbb{R}^N) \to Y(\mathbb{R})$. But the commutativity of the diagram implies that the image of π is equal to $X(\mathbb{R})$. The map π, viewed as a map from $\mathrm{Fr}_n(\mathbb{R}^N) = PG_q$ to $X(\mathbb{R}) = BG_q$ is thus semi-algebraic, and by unicity it is the natural projection.

The classifying map Let K be a q-dimensional simplicial complex. Recall (see Section 1.1.2) that any flat principal G-bundle over $|K|$ can be obtained as the pull back of the classifying map

$$\begin{array}{cccc} f: & |K| & \longrightarrow & BG_q \\ & x = \Sigma_{j=0}^{q} t_i b^{s_i} & \longmapsto & \begin{bmatrix} t_0 g_{0i}(\underline{t}) \\ \vdots \\ t_i \mathrm{Id}_n \\ \vdots \\ t_q g_{qi}(\underline{t}) \end{bmatrix}_G \end{array},$$

where i is chosen so that $t_i \neq 0$, and the g_{ij}'s are constant on $\mathrm{star}(b^{s_i}) \cap \mathrm{star}(b^{s_j})$. As in Section 1.1.2, s_i always denotes an i-dimensional simplex of K, and b^{s_i} is its barycenter in K_{bar}. Let $k = \{b^{s_0}, ..., b^{s_q}\}$ be a q-dimensional simplex of K_{bar} and, slightly abusing notation, let us denote by $g_{i0} \in G$ the value of the transition function g_{i0} on $\mathrm{star}(b^{s_i}) \cap \mathrm{star}(b^{s_0})$. The classifying map restricted to $|k|$ can then be given as

$$\begin{array}{cccc} f|_{|k|}: & |k| & \longrightarrow & BG_q \\ & x = \Sigma_{j=0}^{q} t_i b^{s_i} & \longmapsto & \begin{bmatrix} t_0 \\ t_1 g_{10}(x) \\ \vdots \\ t_q g_{q0}(x) \end{bmatrix}_G \end{array}.$$

Note that $f|_{|k|}$ is really well defined on the whole of $|k|$ and not only those points for which $t_0 \neq 0$. The map $f|_{|k|}$ is now clearly equal to the composition of the map

$$\begin{array}{cccc} \overline{f}: & |k| & \longrightarrow & PG_q = \mathrm{Fr}_n(\mathbb{R}^N) \\ & x = \Sigma_{j=0}^q t_i b^{s_i} & \longmapsto & \begin{pmatrix} t_0 \\ t_1 g_{10}(x) \\ \vdots \\ t_q g_{q0}(x) \end{pmatrix} \end{array}$$

and the natural projection

$$\pi_G : PG_q \longrightarrow BG_q.$$

We have already seen that the latter map is semi-algebraic. Its complexity depends of G and q.

We claim that the map $\overline{f}$ is also semi-algebraic, of complexity depending only on n and q. Indeed, its graph admits the representation

$$\begin{aligned} \mathrm{Graph}(\overline{f}) = & \Big\{((t_0, ..., t_q), ((x_{ij}^0), ..., (x_{ij}^q))) \in \mathbb{R}^{q+1} \times \left(\mathbb{R}^{n^2}\right)^{q+1} \\ & \Sigma_{k=0}^q t_k = 1,\ t_k \geq 0\ \forall k,\ x_{ij}^k - t_k (g_{k0})_{ij} = 0\ \forall\ i, j, k\Big\}, \end{aligned}$$

so that it belongs to

$$S\left((q+1)\left(1+n^2\right), 1 + (q+1) + (q+1)\, n^2, 1\right).$$

It follows by Corollary 86 that the classifying map is, when restricted to any simplex of K, of complexity depending only on G and q.

4.2 The simplicial version

In this section, we will start by proving the simplicial version of our main theorem. However, to easily deduce the main theorem from its simplicial version we will need a slightly stronger form of the latter, which we will state and prove below.

Theorem 90 *Let G be an algebraic subgroup of $GL_n(\mathbb{R})$ and $\beta \in H^q(BG)$ be a characteristic class. There exists a finite subset $I \subset \mathbb{R}$ such that for every flat principal G-bundle ξ over a finite simplicial complex K the cohomology class $\beta(\xi) \in H^q_{simpl}(K)$ can be represented by a cocycle whose set of value on the q-simplices of K is contained in I.*

Note that this is exactly Sullivan's Theorem (the Theorem 61 here) for the Euler class. The finite subset obtained by Sullivan in this case is $I = \{-1, 0, 1\}$ and is improved by Smillie to $I = \{-1/2^n, 0, 1/2^n\}$. Our method will not produce such accurate bounds.

Proof. First observe that it is enough to prove the theorem for simplicial complexes of dimension smaller or equal to q. Indeed, a simplicial q-cocycle is defined on the q-dimensional simplices and two q-cocycles represent the same cohomology class if they differ by a coboundary, which also only depends on the q-skeleton.

Now, any principal G-bundle over a q-dimensional simplicial complex, can be obtained as the pull back of the approximation to the universal bundle BG_q, where BG_q is as in Section 1.1.2.

The space BG_q was shown to be semi-algebraic in Section 4.1, so that in particular its closure $\overline{BG}_q$ admits, by Theorem 88, a finite semi-algebraic triangulation

$$h : \overline{BG}_q \longrightarrow |T|.$$

Observe that whenever the space BG_q is non compact (which is the general case), its triangulation T will have simplices in the boundary of BG_q. Upon replacing T by its first barycentric subdivision we can however require that any open simplex contained in BG_q has at least one of its vertex in BG_q. Also the classifying map $f : |K| \to BG_q$ exhibited in Section 1.1.2 was proven in Section 4.1 to be semi-algebraic, and furthermore of complexity bounded independently of the bundle ξ or even the multi-simplicial complex K, when restricted to any simplex of K. Indeed the complexity of the classifying map was then shown to only depend on the dimension q and the group G.

Our next aim is to find a simplicial approximation of the classifying map $h \circ f : |K| \to BG_q \to |T|$ (or to be very accurate, actually an approximation to the map $h \circ \overline{f} : |K| \to BG_q \to |T|$, where $\overline{f} : |K| \to BG_q$ is a map homotopic to f). Of course it is a well known fact (see Theorem 49) that upon passing to an arbitrarily fine subdivision of K this is always possible. Our main point is now precisely that we only need to refine K in a uniformly bounded way. This will follow at once from the following Technical Lemma.

Lemma 91 (Technical Lemma) *There exists a refinement (L, r) of K and a continuous map $\overline{f} : |K| \to BG_q$ homotopic (in BG_q) to f such that*

- *the index of the refinement satisfies the inequality*

$$[L : K]_q \leq m,$$

where m depends only on q and G.

- *the interior of every simplex ℓ of L is mapped by $\overline{f}$ inside the interior of some simplex t of T whose interior is contained in BG_q, or more precisely, for every simplex ℓ of L there exists a simplex t of T such that $h^{-1}(Int(t)) \subset BG_q$ and*

$$\overline{f}(r^{-1}(Int(\ell))) \subset h_T^{-1}(Int(t)).$$

We postpone for the time being the proof of the Technical Lemma and show how the theorem is now easily proved. We exhibit a simplicial approximation φ to the continuous map $h \circ \overline{f} \circ r^{-1} : |L| \to |T|$ by an argument almost identical to that of Proposition 48. The difference lies in the fact that some care is needed in order for our simplicial approximation not to land in the boundary of BG_q. Let T_0 be the biggest subcomplex of T such that $h^{-1}(|T_0|) \subset BG_q$. For every vertex v of L, define $\varphi(v) \in T^0$ to be any vertex of the only open simplex of T containing $\overline{f}(v)$. By the assumption made on T that any open simplex contained in BG_q has at least one vertex inside BG_q, we can moreover assume that $\varphi(v)$ is contained in T_0. That this indeed defines a simplicial map

$$\varphi : L \longrightarrow T$$

which is a simplicial approximation to $h \circ \overline{f} \circ r^{-1}$ now follows exactly as in Proposition 48.

Let $b \in Z^q(BG_q)$ be an alternating cocycle representing the cohomology class corresponding to the characteristic class β. We have

$$\beta(\xi) = \left[f^\flat(b)\right] = \left[\overline{f}^\flat(b)\right] \in H^q_{\mathrm{simpl}}(K),$$

since the maps f and $\overline{f}$ are homotopic. Let

$$h_0 : |T_0| \longrightarrow BG_q$$

denote the restriction of h^{-1} to $|T_0|$. Observe that the diagram

$$\begin{array}{ccc} |K| & \xrightarrow{f \sim \overline{f}} & BGq \\ \downarrow r & & \uparrow h_0 \\ |L| & \xrightarrow{|\varphi|} & |T| \end{array}$$

commutes up to homotopy. We thus have

$$\beta(\xi) = [(h_0 \circ |\varphi| \circ r)^\flat(b)].$$

But by Proposition 58 the simplicial cocycle $(h_0 \circ |\varphi| \circ r)^\flat(b)$ is cohomologous to $r^\sharp \circ (h_0 \circ |\varphi|)^\flat(b)$. Applying successively Lemma 52 and Lemma 54 we obtain

$$r^\sharp \circ (h_0 \circ |\varphi|)^\flat(b) = r^\sharp \circ |\varphi|^\flat \circ h_0^*(b) = r^\sharp \circ \varphi^* \circ h_0^\flat(b).$$

Define $b_T = h_0^\flat(b) \in Z^q_{\text{simpl}}(T_0)$. Since the simplicial cocycle T_0 is finite, the simplicial cocycle b_T takes, when evaluated on q-dimensional simplices of T_0, a finite number of values. Let J be the finite subset of $\mathbb{R}$ consisting of all those possible values. The cocycle $\varphi^*(b_T)$ now surely also takes its values in J when evaluated on q-dimensional simplices. Finally, it follows from Proposition 56 that the cocycle $r^\sharp \circ \varphi^* \circ h_0^\flat(b)$ takes its values in the following finite subset of $\mathbb{R}$:

$$\left\{ \sum_{i=1}^{r} n_i \,\middle|\, n_i \in J,\, r \leq [L : K]_q \right\},$$

which finishes the proof of the theorem. ■

Of course, the so obtained bound is absolutely out of proportions. Observe that it is composed of two parts: the possible values of a cocycle on $BG_q = |T|$ representing the characteristic class β evaluated on the fixed triangulation T, and the amount of simplices (the m from the Technical Lemma 91) needed to refine the simplicial complex K so as to have a simplicial approximation of the classifying map. The latter bound is effective and could actually be computed, even though not accurately.

Proof of the Technical Lemma 91. To simplify the notation we will identify the classifying space BG_q with the geometric realization of T. Also, we will systematically identify the geometric realization of any simplicial complex with that of its refinements.

We will prove the Lemma inductively by showing that for every $0 \leq i \leq q$ there exists constants c_i, d_i and m_i depending only on i, the group G and the dimension q of the simplicial complex, a refinement L_i of the i-skeleton K^i of K and a continuous map $f_i : |K| \to |T| = BG_q$ homotopic (in BG_q) to f such that

1. $[L_i : K^i]_i \leq m_i$,
2. the image by f_i of the interior of every simplex of L_i is contained in the interior of some simplex t of T,

3. every simplex ℓ of L_i is semi-algebraic of complexity $S(i, c_i, d_i)$,
4. the map f_i restricted to any simplex of K^i is semi-algebraic of complexity $S(i+n(BG_q), c_i, d_i)$.

The two first properties are exactly the conclusion of the Technical Lemma for $i = q$, and the two last ones are added for inductive purposes. For $i = 0$, there is nothing to prove: Take $f_0 = f$ and $L_0 = K^0$ (so that $c_0 = n(BG_q), d_0 = 1$ and $m_0 = 1$). Let us thus assume that a refinement L_{i-1} of the $(i-1)$-skeleton of K and a continuous map $f_{i-1} : |K| \to |T|$ satisfying the above properties are given.

The strategy of the proof is the following: We are going to triangulate each i-dimensional simplex k of K in such a way that the triangulation on the boundary ∂k of k is precisely the first barycentric subdivision of L_{i-1}, so that we obtain a triangulation of the i-skeleton of K. To do so, we subdivide every i-dimensional simplex k in two subsets k_{int} and k_{ext}. After defining the map f_i and checking that it satisfies the above property 4 we prove that there exists triangulations of k_{int} and k_{ext} which agree on $k_{\text{int}} \cap k_{\text{ext}}$ and correspond to the first barycentric subdivision of L_{i-1} on ∂k. We show that both the triangulation of k_{int} and k_{ext} satisfy the above properties 1, 2 and 3, thus proving the Technical Lemma.

The subsets k_{int} and k_{ext}. Let k be an i-th dimensional simplex of K and consider the two following subsets of its geometric realization: Choose ε with $0 < \varepsilon < 1$ and define

$$k_{\text{int}} = \left\{ \sum_{j=0}^{i} t_j v_j \;\middle|\; t_i \geq \frac{\varepsilon}{1+i} \; \forall\, j = 0, \ldots, i, \right\}$$

and

$$k_{\text{ext}} = \left\{ \sum_{j=0}^{i} t_j v_j \;\middle|\; \exists\, j \in \{0, ..., i\} \text{ with } t_i \leq \frac{\varepsilon}{1+i} \right\},$$

where of course $v_0, ..., v_i$ are the vertices of k. The subset k_{ext} is the closure of some sufficiently small neighborhood of the boundary of k so that k_{ext} is homotopically equivalent to ∂k. The subset k_{int} is the closure of $k \backslash k_{\text{ext}}$, that is, a homothetic copy of k centered at the barycenter of k and contraction factor strictly smaller than 1.

The map f_i. Define a continuous map $\alpha_k : |k| \to |k|$ to be, on k_{int} the natural affine homothety between k_{int} and k, and on k_{ext} the projection from the barycenter of k onto the boundary ∂k. More precisely, we have

$$\alpha_k \left(\sum_{j=0}^{i} t_j v_j \right) = \begin{cases} \sum_{j=0}^{i} \frac{1}{1-\varepsilon} \left(t_j - \frac{\varepsilon}{i+1} \right) v_j & \text{if } \sum_{j=0}^{i} t_j v_j \in k_{\text{int}}, \\ \sum_{j=0}^{i} \frac{t_j - \min_{0 \leq j \leq m} t_j}{1-(i+1) \min_{0 \leq j \leq m} t_j} v_j & \text{if } \sum_{j=0}^{i} t_j v_j \in k_{\text{ext}}. \end{cases}$$

Clearly α_k is well-defined, continuous and semi-algebraic. Also, since for every i-dimensional simplex k, the map α_k is the identity on ∂k, it defines a continuous map $\alpha : |K^i| \to |K^i|$. Furthermore, it is obvious that it extends to a continuous map $|K| \to |K|$, still denoted by α, which we can moreover assume to map every simplex of K to itself and to be semi-algebraic of complexity $S(2q, c_\alpha, d_\alpha)$, when restricted to any simplex of K, where the constants c_α and d_α do not depend on anything else than i and q. Such a map α is clearly homotopic to the identity.

Define

$$f_i = f_{i-1} \circ \alpha : |K| \longrightarrow |T| = BG_q.$$

Since f_{i-1} is homotopic (in BG_q) to f, the same is true for f_i and by Corollary 86, the map f_i is, when restricted to any simplex of K, semi-algebraic of complexity $S(q+n(BG_q), c_i, d_i)$, where $c_i, d_i \lhd q, n(BG_q), c_\alpha, d_\alpha, c_{i-1}, d_{i-1}$, and thus $c_i, d_i \lhd q, G, i$.

The triangulation of k_{int}. The map α is, when restricted to k_{int} a homothety from k_{int} to k. Thus the first barycentric subdivision $(L_{i-1})_{\text{bar}}$ of the triangulation L_{i-1} restricted to the boundary of k naturally induces, via α, a triangulation by semi-algebraic simplices of complexity $S(n_{i-1}, c_{i-1}, d_{i-1})$ of the boundary of k_{int}. We would now like to have a semi-algebraic triangulation of k_{int} agreeing with the following two families of semi-algebraic subsets:

- The simplices of the triangulation of ∂k_{int} induced by $(L_{i-1})_{\text{bar}}$.
- The pull back by f_i of the simplices of T.

We are of course going to apply Theorem 88 to k_{int} and those two families of semi-algebraic subsets, so let us first check that the above sets all are of uniformly bounded complexity, and in uniformly bounded quantity. Note that k_{int} is of complexity $S(i, i+1, 1)$.

- Since each simplex of L_{i-1} is, by induction, of complexity $S(i-1, c_{i-1}, d_{i-1})$, it follows that each simplex of $(L_{i-1})_{\text{bar}}$ is of complexity $S(i-1, c_{i-1}, d_{i-1})$, and the same is true for the corresponding simplices in ∂k_{int}.

  There are at most $(i+1) \cdot m_{i-1} \cdot i!$ such simplices.

- Since the semi-algebraic triangulation T of BG_q is finite, any simplex t of T is of complexity $S(n(BG_q), c_T, d_T)$, for some c_T, d_T depending only on G and q. By Corollary 85 it follows that $f_i^{-1}(t)$ is semi-algebraic of complexity $S(q, C, D)$, where $C, D \lhd q, n(BG_q), c_i, d_i, c_T, d_T$, thus $C, D \lhd q, G, i$. By Lemma 82 we now obtain that $f_i^{-1}(t) \cap k_{\text{int}}$ is semi-algebraic of complexity $S(i, 2\max\{i+1, C\}, \max\{1, D\})$ for every simplex t of T.

  Of course, the number of such sets is majorized by the number of simplices of T, which only depends on q and G.

Let us now apply Theorem 88 to k_{int} and its two above given families of semi-algebraic subsets. We thus obtain a semi-algebraic triangulation L_{int} of k_{int} fulfilling the following properties:

1. - The triangulation L_{int} restricted to the boundary of k_{int} is a refinement of the triangulation corresponding to the first barycentric subdivision of the triangulation L_{i-1} restricted to ∂k.
   - For every simplex t of T, the semi-algebraic set $f_i^{-1}(t) \cap k_{\text{int}}$ is a finite union of simplices of L_{int}, so that the image by f_i of the interior of any simplex of L_{int} is contained in the interior of some simplex of T.
2. The number of simplices of L_{int} is bounded by m_{int}, where m_{int} is a constant depending only on q, G and i.
3. each simplex of L_{int} is semi-algebraic of complexity $S(n, c_{\text{int}}, d_{\text{int}})$, where $c_{\text{int}}, d_{\text{int}}$ are constants depending only on q, G and i.

The triangulation of k_{ext}. It now remains to triangulate k_{ext} in such a way that the triangulation agrees with the first barycentric subdivision of L_{i-1} on ∂k and with the triangulation L_{int} on $k_{\text{ext}} \cap k_{\text{int}} = \partial k_{\text{int}}$. This triangulation should of course also enjoy the desired properties. To do so, we consider the homeomorphism between k_{ext} and $\partial k \times [0,1]$ given by

$$\begin{array}{rccc} \beta: & k_{\text{ext}} & \longrightarrow & \partial k \times [0,1] \\ & x = \sum_{j=0}^{i} t_j v_j & \longmapsto & (\alpha(x), \frac{i+1}{\varepsilon} \min\{t_0, ..., t_i\}). \end{array}$$

The boundary $\partial k \subset k_{\text{ext}}$ is thus mapped by β to $\partial k \times \{0\}$, and $k_{\text{ext}} \cap k_{\text{int}} = \partial k_{\text{int}}$ to $\partial k \times \{1\}$.

We are now exactly in the situation of the last example in 2.2: We have a refinement $L_{\text{int}}|_{\partial k_{\text{int}} \cong \partial k}$ of the triangulation $L_{i-1}|_{\partial k}$ of the $(i-1)$-dimensional simplicial complex ∂k, and we can, by the example, find a triangulation L_{ext} of $|\partial k| \times [0,1]$ having the property that it agrees on $|\partial k| \times \{0\}$, respectively $|\partial k| \times \{1\}$, with the triangulation $L_{i-1}|_{\partial k}$ of $|\partial k|$, resp. $L_{\text{int}}|_{\partial k_{\text{int}} \cong \partial k}$ of $|\partial k_{\text{int}}| \cong |\partial k|$. Moreover we have:

1. A bound for the number of i-dimensional simplices of L_{ext} is

$$i \cdot [L_{\text{int}} : L_{i-1}]_{i-1} \cdot \sharp\{(i-1)\text{-dimensional simplices of } L_{i-1}\}.$$

Note that as L_{i-1} is a triangulation of ∂k we have

$$\begin{array}{rcl} [L_{\text{int}} : L_{i-1}]_{i-1} & \leq & [L_{\text{int}} : \partial k]_{i-1} \\ & \leq & [L_{\text{int}} : k]_i \\ & \leq & m_{\text{int}}. \end{array}$$

As for the number of $(i-1)$-dimensional simplices in the triangulation L_{i-1} restricted to ∂k, it is clearly bounded by the number of faces of ∂k times the index $[L_{i-1} : K^{i-1}]_{i-1}$, the latter number being, by induction hypothesis bounded by m_{i-1}. We thus obtain that the amount of simplices of L_{ext} is bounded by

$$m_{\text{ext}} = i^2 \cdot m_{\text{int}} \cdot m_{i-1}.$$

2. Observe that the diagram

$$\begin{array}{ccc} k_{\text{ext}} & \xrightarrow{\beta} & \partial k \times [0,1] \\ \downarrow f_i & & \downarrow \text{proj}_1 \\ |T| & \xleftarrow{f_{i-1}} & \partial k, \end{array}$$

where of course proj_1 stands for the projection on the first factor, is commutative. The interior of any simplex of L_{ext} is by construction mapped inside the interior of some simplex of L_{i-1} and as by induction the image by f_{i-1} of the interior of any simplex of L_{i-1} and hence also of $(L_{i-1})_{\text{bar}}$ is contained in the interior of some simplex of T the conclusion follows.

3. By induction hypothesis, the simplices of L_{i-1} are all semi-algebraic of complexity $S(i-1, c_{i-1}, d_{i-1})$. Also, the simplices of the triangulation L_{int} on $\partial k_{\text{int}} \cong \partial k$ are semi-algebraic of complexity $S(i-1, c_{\text{int}}, d_{\text{int}})$. By Lemma 83, the join of any simplex of $L_{\text{int}}|_{\partial k_{\text{int}} \cong \partial k}$ and L_{i-1}, and thus any simplex of L_{ext} is semi-algebraic of complexity $S(i, c_{\text{ext}}, d_{\text{ext}})$, where $c_{\text{ext}}, d_{\text{ext}} \lhd c_{i-1}, d_{i-1}, c_{\text{int}}, d_{\text{int}}$ and thus $c_{\text{ext}}, d_{\text{ext}} \lhd i, G, q$.

■

Before stating the slight generalization of Theorem 5 from which it will be easy to obtain at once our main Theorem, let us recall that a cohomology class in $H^*(BG^\delta)$ is said to be a primary characteristic class if it is contained in the image of the natural map $H^*(BG) \to H^*(BG^\delta)$.

Theorem 92 *Let G be an algebraic subgroup of $GL_n\mathbb{R}$ and $\beta \in H^q(BG^\delta)$ a primary characteristic class. Then there exists a finite subset $I \subset \mathbb{R}$ such that for every finite simplicial complex K and every continuous map $\sigma : |K| \to BG^\delta$, there exists a cochain $b \in C^q_{sing}(BG^\delta)$ such that the simplicial cochain $\sigma^\flat(b) \in C^q_{simpl}(K)$ is a cocycle representing $\sigma^\flat(\beta)$ and taking values in I when evaluated on q-simplices.*

The only difference with Theorem 5 is that given any classifying map $\sigma : |K| \to BG^\delta$ we require the cocycle representing the desired characteristic class to be in the image of the induced map $\sigma^\flat : C^q_{\text{sing}}(BG^\delta) \to C^q_{\text{simpl}}(K)$. This will greatly simplify our life when taking inverse limit, in the next section, over all couples of the form (K, σ) as above.

Proof. The proof really relies on the proof of Theorem 5. The idea is quite simple: If σ is injective on q-simplices in the sense that for any two simplicial isomorphisms $\tau_i : \Delta^q \to k_i$, for $i = 1, 2$, where the k_i's are oriented q-dimensional simplices of K, if $\sigma \circ \tau_1 = \sigma \circ \tau_2$ then $k_1 = k_2$ and $\tau_1 = \tau_2$, then there is not much to do. Indeed, letting $b \in Z^q_{\text{simpl}}(K)$ be the simplicial cocycle obtained in Theorem 5, define a singular cochain $b' \in C^q_{\text{singl}}(BG^\delta)$ as

$$b'(\sigma') = \begin{cases} b(\tau(\Delta^q)) & \text{if } \sigma' = \sigma \circ \tau \text{ for some map } \tau : \Delta^q \longrightarrow K, \\ \text{arbitrarily} & \text{otherwise,} \end{cases}$$

for every singular simplex $\sigma' : \Delta^q \to BG^\delta$. Observe that thanks to our injectivity condition, b' is well defined. Also, it is alternating and surely $\sigma^\flat(b') = b$ as desired.

In the case where our injectivity condition is not fulfilled, we will show that the cocycle $b \in Z^q_{\text{simpl}}(K)$ constructed in the proof of Theorem 5 can actually be chosen such that $b(k_1) = \text{sign}(\tau) b(k_2)$, whenever there exists a simplicial isomorphism $\tau : k_1 \to k_2$ between the two oriented q-simplices k_1 and k_2 such that $\sigma \circ \tau = \sigma|_{k_1}$. From such a cocycle, one can then define a well defined singular cochain $b' \in C^q_{\text{singl}}(BG^\delta)$ as above and obtain, once again, the desired conclusion.

Consider the covering of $|K|$ by the sets $\{S_0, ..., S_q\}$ exhibited in Section 1.1.2. We claim that locally constant transition functions relative to the covering $\{S_0, ..., S_q\}$ can be found such that for every $x \in S_i \cap S_j$ belonging to the connected component $\text{star}(b^{s_i}) \cap \text{star}(b^{s_j})$ for some i-simplex s_i and j-simplex s_j of K, the value of the transition function g_{ij} on x (and hence on $\text{star}(b^{s_i}) \cap \text{star}(b^{s_j})$) only depends on the image by σ of the one dimensional simplex (b^{s_i}, b^{s_j}) of K_{bar}. To see that, choose, for every vertex b^s of K_{bar}, a point $u(b^s)$ in the fiber over b^s or equivalently, in the fiber over $\sigma(b^s)$, and define $g_{ij}(x)$, for x in $\text{star}(b^{s_i}) \cap \text{star}(b^{s_j})$, as the difference between the parallel transport of the point $u(b^{s_i})$ along the simplex (b^{s_i}, b^{s_j}) and the point $u(b^{s_j})$. It is readily seen that this defines transition functions with the desired property.

It now follows that if there exists a simplicial isomorphism $\tau : k_1 \to k_2$ such that $\sigma \circ \tau = \sigma|_{k_1}$ then the classifying map obtained in the proof of Theorem 5 also satisfies

$$f \circ \tau = f|_{k_1}.$$

But the refinement L of K of Theorem 5 being defined inductively on the skeleton of K in such a way that it depends only on the classifying map f, we can choose L and the simplicial approximation

$\varphi : L \to T$ of Theorem 5 so that the class $\varphi^*(b_T)$ is such that if there exists a simplicial isomorphism $\tau : k_1 \to k_2$ such that $\sigma \circ \tau = \sigma|_{k_1}$ then

$$\varphi^*(b_T)(k_1) = \text{sign}(\tau)\varphi^*(b_T)(k_2).$$

■

4.3 Proof of the singular version

We are now almost ready to give a proof of Theorem 4 stating that primary characteristic classes of flat bundles can be represented by cocycles taking only finitely many values on singular simplices. The theorem will be a simple consequence of its simplicial version (Theorem 92) by an argument of inverse limit. Before attacking the proof, we recall the elementary definitions of inverse systems and limits.

4.3.1 Inverse limits

- A *directed set* is a non-empty, partially ordered set $(\Lambda, \geq)$ such that

$$\forall\ \lambda, \mu \in \Lambda,\ \exists\ \nu \in \Lambda \ \text{ with } \nu \geq \lambda,\ \nu \geq \mu.$$

- An *inverse system* $(X_\lambda, \pi_{\mu\lambda})$ of sets over a directed set Λ is a family of sets $(X_\lambda)_{\lambda\in\Lambda}$ together with maps $\pi_{\mu\lambda} : X_\lambda \to X_\mu$ whenever $\lambda \geq \mu$ satisfying the two following conditions:
  - $\pi_{\lambda\lambda} = \text{Id}_{X_\lambda}$,
  - $\pi_{\nu\mu}\pi_{\mu\lambda} = \pi_{\nu\lambda}$, for $\lambda \geq \mu \geq \nu$.
- The *inverse limit* of the inverse system $(X_\lambda, \pi_{\mu\lambda})$ is defined as

$$\lim_{\leftarrow} X_\lambda = \{(g_\lambda) \in \prod_{\lambda\in\Lambda} X_\lambda \mid \pi_{\mu\lambda}(g_\lambda) = g_\mu\ \forall\ \lambda \geq \mu\}.$$

Proposition 93 *If $(X_\lambda, \pi_{\mu\lambda})$ is an inverse system of non empty compact spaces over a directed set Λ, then*

$$\lim_{\leftarrow} X_\lambda \neq \emptyset.$$

Proof. For every finite subset $S \subset \Lambda$, define

$$L(S) := \{(x_\lambda) \in \prod_{\lambda\in\Lambda} X_\lambda \mid \pi_{\mu\lambda}x_\lambda = x_\mu\ \forall\lambda, \mu \in S, \lambda \geq \mu\},$$

and write $P := \prod_{\lambda\in\Lambda} X_\lambda$.

The set $L(S)$ is closed in P and non empty. It is non empty since as S is a finite subset of a directed set $\exists\ \nu \in \Lambda$ such that $\nu \geq \lambda$ for every $\lambda \in S$. Now choose some $x_\nu \in X_\nu \neq \emptyset$, and define $(x_\lambda) \in L(S)$ by

$$x_\lambda := \begin{cases} x_\nu \in X_\nu & \text{if } \lambda = \nu \\ \pi_{\lambda\nu}x_\nu \in X_\lambda & \forall\lambda \in S \\ x_\lambda \in X_\lambda \ \text{ arbitrary} & \forall\lambda \notin S \cup \{\nu\} \end{cases}$$

By Tychonov's theorem P is compact since it is a product of compact spaces. Set $L = \cap\{L(S) \mid S \subset \Lambda, S \text{ finite}\}$. If L were empty, it would mean that the intersection over only a finite set of finite subsets of Λ is already empty. But clearly,

$$\cap_{i=1}^r L(S_i) = L(\cup_{i=1}^r S_i),$$

for S_i finite subsets of Λ. But as $L(\cup_{i=1}^r S_i)$ is non empty, as we have just seen, it follows that L is non empty. But now, L is exactly $\lim_{\leftarrow} X_\lambda$. ■

Our main example of directed set. Let X be a non empty topological space and let L be the full simplicial complex over the canonical basis of $\mathbb{R}^\infty$, that is $L^0 = \{e_1, e_2, \dots\}$ and every set of vertices in L^0 generates a simplex of L. Set

$$\Lambda = \left\{ (K, \sigma) \;\middle|\; \begin{array}{l} K \subset L \text{ finite simplicial complex,} \\ \sigma : |K| \to X \text{ continuous} \end{array} \right\}.$$

It is non empty since X is non empty. Put the following partial order on Λ: Let $(K_1, \sigma_1), (K_2, \sigma_2) \in \Lambda$, then

$$(K_2, \sigma_2) \geq (K_1, \sigma_1) \text{ if } \quad \begin{array}{l} \exists \text{ a simplicial injection } i : K_1 \to K_2 \\ \text{such that } \sigma_2 \circ |i| = \sigma_1. \end{array}$$

It is readily seen that Λ is a directed set. Indeed, let $(K_1, \sigma_1), (K_2, \sigma_2)$ be in Λ. As K_1 is a finite simplicial complex, $K_1 \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. Define a simplicial map $t_n : L \to L$ by $t_n(e_i) = e_{i+n}$. It is clear that t_n is an injection. Observe that K_1 and $t_n(K_2)$ are disjoint subcomplexes of L. Define $K = K_1 \amalg t_n(K_2)$ and $\sigma : |K| \to X$ by

$$\sigma\big|_{|K_1|} = \sigma_1 \text{ and } \sigma\big|_{|t_n(K_2)|} = \sigma_2 \circ |t_n|^{-1}.$$

Notice that the last expression makes sense, since as $|t_n|$ is injective, it is bijective on its image. Obviously, taking i to be the canonical inclusion of K_1 in K we get $(K, \sigma) \geq (K_1, \sigma_1)$, and $(K, \sigma) \geq (K_2, \sigma_2)$ since $t_n : K_2 \to K$ is a simplicial inclusion and $\sigma \circ |t_n| = \sigma_2$ by definition of σ.

Our main example of inverse system. Let X be a topological space, $\beta \in H^q_{\text{sing}}(X)$ a singular cohomology class on X and I a compact subset of $\mathbb{R}$. For every (K, σ) in Λ, define

$$Y_{(K,\sigma)} = \left\{ b \in Z^q_{\text{simpl}}(K) \;\middle|\; \begin{array}{l} [b] = \sigma^\flat(\beta),\ b \in \sigma^\flat(C^q_{\text{sing}}(X)), \\ b(k) \in I \ \forall\ q\text{-simplex } k \in K \end{array} \right\}.$$

If $(K_1, \sigma_1) \leq (K_2, \sigma_2)$, the simplicial inclusion $i : K_1 \to K_2$ induces a map

$$i^* : Y_{(K_2,\sigma_2)} \longrightarrow Y_{(K_1,\sigma_1)}.$$

Note that from the requirement that any cocycle of $Y_{(K,\sigma)}$ belongs to the image of σ^* it follows that the map i^* does not depend on the choice of simplicial injection i. Indeed, suppose $j : K_1 \to K_2$ is another simplicial injection with $\sigma_2 \circ |j| = \sigma_1 = \sigma_2 \circ |i|$, then

$$i^* \circ \sigma_2^* = (\sigma_2 \circ |i|)^* = \sigma_1^* = (\sigma_2 \circ |j|)^* = j^* \circ \sigma_2^*,$$

so that i^* and j^* agree on the image of σ_2^* in which $Y_{(K_2,\sigma_2)}$ is contained. Observe moreover that

- for every (K, σ) in Λ, the map $Y_{(K,\sigma)} \to Y_{(K,\sigma)}$ is the identity since it is induced by the identity on K,

- if $(K_1, \sigma_1) \leq (K_2, \sigma_2) \leq (K_3, \sigma_3)$ with simplicial injections $i : K_1 \to K_2$, $j : K_2 \to K_3$, then $(j \circ i)^*_{|Y_{(K_3,\sigma_3)}} = (i^* j^*)_{|Y_{(K_3,\sigma_3)}} = i^*_{|Y_{(K_2,\sigma_2)}} \circ j^*_{|Y_{(K_3,\sigma_3)}}$.

We have thus proven that $\{Y_{(K,\sigma)}\}$ forms an inverse system over Λ.

4.3.2 Proof of Theorem 4

Let G be an algebraic subgroup of $\mathrm{GL}_n\mathbb{R}$ and $\beta \in H^q(BG^\delta)$ a primary characteristic class. Let Λ be the directed set constructed above for $X = BG^\delta$, and $\{Y_{(K,\sigma)}\}$ the inverse system obtained from $X = BG^\delta$, $\beta \in H^q(BG^\delta)$ and the compact subset I of $\mathbb{R}$ from Theorem 92. The conclusion of Theorem 92 is exactly equivalent to that $Y_{(K,\sigma)}$ is non empty for every (K, σ) in Λ. Moreover, the $Y_{(K,\sigma)}$'s are compact: Indeed, for every (K, σ) in Λ, the space $Y_{(K,\sigma)}$ is the subspace of the finite dimensional vector space $Z^q_{\text{simpl}}(K)$ formed of the intersection of an affine subspace (the image of the coboundary δ), a linear subspace (the image of σ^*), and a compact subset (from that b takes its values in the compact set I). It now follows from Proposition 93, that the inverse limit of the inverse system $\{Y_{(K,\sigma)}\}$ is non empty:

$$\varnothing \neq \varprojlim Y_{(K,\sigma)}.$$

Let thus $(b_{(K,\sigma)})$ be an element in the inverse limit, and define a singular cochain $b \in C^q_{\text{sing}}(X)$ by $b(\sigma) = b_{(\Delta^q,\sigma)}(\Delta^q)$, for every singular simplex $\sigma : \Delta^q \to X$. It is clear from the definition of b, that the cochain b takes its values in I on singular simplices.

It remains to show that the cochain b is a cocycle representing β. Let thus c be an arbitrary cocycle representing β. It is now enough to show that b and c agree on singular cycles. Let thus $z = \Sigma a_i \sigma_i \in Z_q(X)$ be a singular cycle on X. Up to rescaling z we can suppose that the coefficients a_i lie in $\mathbb{Z}$. It is clear that to this cycle corresponds a continuous map

$$\sigma : |K| \longrightarrow X,$$

whose restriction to any q-dimensional simplex of the simplicial complex K is either degenerated or corresponds to one of the singular simplices σ_i appearing in the decomposition of z. Of course, the simplicial complex K can be chosen to be closed. Let us abuse notation and write K for the corresponding simplicial cycle. Then

$$c(z) = \sigma^*(c)(K)$$

and

$$b(z) = \sigma^*(b)(K) = b_{(K,\sigma)}(K).$$

Comparing those two equalities, we conclude from the fact that K is a cycle, and that by definition of $Y_{(K,\sigma)}$ the simplicial cocycles $\sigma^*(c)$ and $b_{(K,\sigma)}$ are cohomologous, the equality

$$c(z) = b(z)$$

holds as desired.

Appendix A

Bounded cohomology

Let X be a topological space. The space $C_q(X)$ of singular q-chains on X is defined to be the (real) vector space over the basis of singular simplices $S_q(X) = \{\sigma : \Delta^q \to X \mid \sigma \text{ is continuous}\}$, where the standard simplex Δ^q is the convex hull of the canonical basis of $\mathbb{R}^{q+1}$. The space of chains is endowed with a natural boundary operator $\partial : C_q(X) \to C_{q-1}(X)$ defined as $\partial\sigma = \Sigma_{i=0}^q (-1)^i \sigma_i$, where $\sigma_i : \Delta^{q-1} \to X$ is the composition of the inclusion of Δ^{q-1} in the i-th face of Δ^q and σ. As $\partial^2 = 0$ we can consider the homology of the complex $(C_*(X), \partial)$, which is called the singular homology of X. The space $C^q(X)$ of singular q-cochains is defined to be the algebraic dual of $C_q(X)$. It is endowed with the adjoint operator $\delta = \partial^* : C^q(X) \to C^{q+1}(X)$. The homology of the complex $(C^*(X), \delta)$ now gives the singular cohomology of X.

The 1-norm with respect to the canonical basis $S_q(X)$ of $C_q(X)$ can be considered:

$$\|z\|_1 = \sum_\sigma |z_\sigma|, \quad \text{for } z = \sum_\sigma z_\sigma \sigma \in C_q(X).$$

This norm induces a semi-norm on the homology of X. If X is a compact manifold, the 1-norm of its fundamental class $[X] \in H_{\mathrm{Dim} X}(X)$ is called the simplicial volume of X. We can now of course consider the topological dual of the normed space $C_q(X)$ which we denote by $C_b^q(X)$ and name the space of (singular) bounded cochains on X, so that

$$C_b^q(X) = \{c \in C^q(X) \mid \|c\|_\infty < \infty\},$$

where

$$\begin{aligned} \|c\|_\infty &= \sup\{|c(z)| \mid z \in C_q(X),\ \|z\|_1 = 1\} \\ &= \sup\{|c(\sigma)| \mid \sigma \in S_q(X)\}. \end{aligned}$$

The boundary operator δ restricts to bounded cochains, so that we can define the (singular) bounded cohomology $H_b^*(X)$ of the space X to be the homology of the complex $(C_b^*(X), \delta)$. Note however, that this is not a cohomology theory: the excision axiom does not hold.

We will say that a cohomology class $[c] \in H^q(X)$ is bounded if it can be represented by a bounded cocycle, or equivalently, if it is contained in the image of the comparison map

$$H_b^q(X) \longrightarrow H^q(X),$$

where the latter map is of course induced by the inclusion $C_b^q(X) \hookrightarrow C^q(X)$. Note that the comparison map is in general neither injective nor surjective.

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Curriculum Vitae

I was born in Geneva on the 25th of July 1976. In this town, I attended primary and secondary school. I obtained the *Maturité Scientifique* and was awarded the Marc Birgikt Price in June 1995. In October 1995 I began my studies in mathematics at the University of Geneva. I obtained the *Diplôme de Mathematiques* in September 1999. My Diploma thesis was written under the direction of Prof. Pierre de la Harpe. In parallel, my studies at the Conservatory of Geneva led me to obtain in June 1997 the *Certificat de Perfectionnement* for the recorder, and in June 1999 the *Certificat de 5ème terminale, mention excellent*, for the piano.

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