# A sharp Trudinger - Moser type inequality for unbounded domains in $R^2$

# A sharp Trudinger - Moser type inequality for unbounded domains in $\mathbb{R}^2$

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**Abstract**

The classical Trudinger-Moser inequality says that for functions with Dirichlet norm smaller or equal to 1 in the Sobolev space $H_0^1(\Omega)$ (with $\Omega \subset \mathbb{R}^2$ a bounded domain), the integral $\int_\Omega e^{4\pi u^2} dx$ is uniformly bounded by a constant depending only on $\Omega$. If the volume $|\Omega|$ becomes unbounded then this bound tends to infinity, and hence the Trudinger-Moser inequality is not available for such domains (and in particular for $\mathbb{R}^2$).

In this paper we show that if the Dirichlet norm is replaced by the standard Sobolev norm, then the supremum of $\int_\Omega e^{4\pi u^2} dx$ over all such functions is uniformly bounded, *independently* of the domain $\Omega$. Furthermore, a sharp upper bound for the limits of *Sobolev normalized concentrating sequences* is proved for $\Omega = B_R$, the ball or radius $R$, and for $\Omega = \mathbb{R}^2$. Finally, the explicit construction of *optimal concentrating sequences* allows to prove that the above supremum is attained on balls $B_R \subset \mathbb{R}^2$ and on $\mathbb{R}^2$.

## 1 Introduction

Let $\Omega \subset \mathbb{R}^N$ denote a bounded domain. The Sobolev imbedding theorem states that $H_0^1(\Omega) \subset L^p(\Omega)$, for $1 \le p \le 2^* = \frac{2N}{N-2}$, or equivalently, using the Dirichlet norm $\|u\|_D = (\int_\Omega |\nabla u|^2 dx)^{1/2}$ on $H_0^1(\Omega)$,

$$\sup_{\|u\|_D \le 1} \int_\Omega |u|^p dx < +\infty \ , \quad \text{for} \quad 1 \le p \le 2^* \ ,$$

while this supremum is infinite for $p > 2^*$. The maximal growth $|u|^{2^*}$ is called "critical" Sobolev growth. In the case $N = 2$, every polynomial growth is admitted, but one knows by easy examples that $H_0^1(\Omega) \not\subseteq L^\infty(\Omega)$. Hence, one is led to look for a function $g(s) : \mathbb{R} \to \mathbb{R}^+$ with *maximal grwoth* such that

$$\sup_{\|u\|_D \le 1} \int_\Omega g(u) dx < +\infty.$$

It was shown by Pohozhaev [12], Trudinger [14] and Moser [11] that the maximal growth is of exponential type. More precisely, the Trudinger-Moser inequality states that for $\Omega \subset \mathbb{R}^2$ bounded

$$\sup_{\|u\|_D \le 1} \int_\Omega (e^{\alpha u^2} - 1) dx = c(\Omega) < +\infty \quad \text{for} \quad \alpha \le 4\pi \ , \tag{1.1}$$

The inequality is optimal: for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the corresponding supremum is $+\infty$.

The supremum (1.1) becomes infinite for domains $\Omega$ with $|\Omega| = \infty$, and therefore the Trudinger-Moser inequality is not available for unbounded domains. Related inequalities for unbounded domains have been proposed by Cao [5] and Tanaka [2], however they assume a growth $e^{\alpha u^2}$ with $\alpha < 4\pi$, i.e. with *subcritical* growth.

In this paper we show that replacing the *Dirichlet norm* $\|u\|_D = \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}$ by the standard *Sobolev norm* on $H_0^1(\Omega)$, namely

$$\|u\|_S = \left(\|u\|_D^2 + \|u\|_{L^2}^2\right)^{1/2} = \left(\int_\Omega (|\nabla u|^2 + |u|^2) dx\right)^{1/2} \tag{1.2}$$

yields a bound *independent* of $\Omega$. More precisely, we prove

**Theorem 1.1** *There exists a constant $d > 0$ such that for any domain $\Omega \subset \mathbb{R}^2$*

$$\sup_{\|u\|_S \leq 1} \int_\Omega (e^{4\pi u^2} - 1) dx \leq d \tag{1.3}$$

*The inequality is sharp: for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the supremum is $+\infty$.*

In an interesting paper, L. Carleson and A. Chang [6] proved that the supremum in (1.1) is attained if $\Omega = B_1(0)$, the unit ball in $\mathbb{R}^2$. This result was extended to arbitrary bounded domains in $\mathbb{R}^2$ by M. Flucher [9]. In their proof, Carleson and Chang used a "concentration-compactness" argument. They consider "normalized concentrating sequences", i.e. normalized (in the Dirichlet norm) sequences which converge weakly to 0 and (being radial) blow up at the origin. They showed that for any such sequence $\{u_n\}$ one has

$$\overline{\lim_{n\to\infty}} \int_{B_1(0)} (e^{4\pi u_n^2} - 1) dx \leq e\,|B_1| \tag{1.4}$$

Hence, one may say that $e\,|B_1|$ is the highest possible "concentration" or "non-compactness" level (see also P.L. Lions [10], and H. Brezis - L. Nirenberg [3] for the related situation for Sobolev embeddings). Carleson and Chang went on to show that

$$\sup_{\|u\|_D \leq 1} \int_{B_1} (e^{4\pi u^2} - 1) dx > e\,|B_1| \tag{1.5}$$

and hence, since no concentration can happen at a level above $e\,|B_1|$, they concluded that the supremum in (1.1) is attained.

Let us call the maximal limit in (1.4) the *Carleson-Chang limit*, in symbol: *cc-lim.* In [7] an *explicit* normalized concentrating sequence $\{y_n\}$ with

$$\lim_{n\to\infty} \int_{B_1} (e^{4\pi y_n^2} - 1) dx = \underset{\|u_n\|_D \leq 1}{\text{cc--lim}} \int_{B_1} (e^{4\pi u_n^2} - 1) dx = e\,|B_1| \tag{1.6}$$

was constructed.

In this paper we analyze the corresponding Carleson-Chang limit for concentrating sequences which are *normalized in the Sobolev norm.* We will show

**Theorem 1.2**
*1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $R > 0$ such that $|\Omega| = |B_R|$. Then*

$$\underset{\|u_n\|_S \le 1}{\text{cc}-\lim} \int_\Omega (e^{4\pi u_n^2} - 1)dx \le \pi \, e^{1-D(R)} \ , \tag{1.7}$$

*where*

$$D(R) = 2K_0(R)[2RK_1(R) - 1/I_0(R)] > 0 \ , \ \textit{with} \ \lim_{R\to+\infty} D(R) = 0 \ .$$

*Here, $I_k(x)$ and $K_k(x)$ denote the $k-th$ modified Bessel functions of the first and second kind, i.e. the solutions of the equation*

$$-x^2u''(x) - xu'(x) + (x^2 + k^2)u(x) = 0 \ , \ k = 0, 1, 2, ...$$

*2. Let $\Omega \subseteq \mathbb{R}^2$ be an arbitrary domain. Then*

$$\underset{\|u_n\|_S \le 1}{\text{cc}-\lim} \int_\Omega (e^{4\pi u_n^2} - 1)dx \le \pi \, e \ . \tag{1.8}$$

*3. The bound in (1.7) is sharp for $\Omega = B_R(0)$, and the bound in (1.8) is sharp for $\Omega = \mathbb{R}^2$.*

It is remarkable that for $\Omega = B_1(0)$ with Dirichlet normalization and for $\Omega = \mathbb{R}^2$ with Sobolev normalization the corresponding Carleson-Chang limits coincide, that is

$$\underset{\|u_n\|_D \le 1}{\text{cc}-\lim} \int_{B_1} (e^{4\pi u_n^2} - 1)dx \ = \ \underset{\|u_n\|_S \le 1}{\text{cc}-\lim} \int_{\mathbb{R}^2} (e^{4\pi u_n^2} - 1)dx = e \, \pi \ .$$

In the final result of the paper we prove

**Theorem 1.3** *For any ball $\Omega = B_R(0)$ and for $\Omega = \mathbb{R}^2$ holds*

$$\sup_{\|u\|_S \le 1} \int_\Omega (e^{4\pi u^2} - 1)dx \ > \ e^{1-D(R)} \, \pi \tag{1.9}$$

*This implies in particular that the supremum (1.9) is attained in the cases of $\Omega = B_R(0)$ and $\Omega = \mathbb{R}^2$.*

## 2 A uniform bound

In this section we prove Theorem 1.1. We begin with

**Proposition 2.1** *Let $\Omega \subset \mathbb{R}^2$ denote a domain in $\mathbb{R}^2$, and let $H_0^1(\Omega)$ denote the standard Sobolev space equipped with the norm*

$$\|u\|_S = \left( \int_\Omega (|\nabla u|^2 + |u|^2)dx \right)^{1/2}$$

*Then there exists a constant $d$ (independent of $\Omega$) such that*

$$\sup_{\|u\|_S \le 1} \int_\Omega (e^{4\pi u^2} - 1)dx \le d \ . \tag{2.1}$$

**Proof.** It is clear that

$$\sup_{\|u\|_S\le 1}\int_\Omega (e^{4\pi u^2}-1)dx \le \sup_{\|u\|_S\le 1}\int_{\mathbb{R}^2}(e^{4\pi u^2}-1)dx \tag{2.2}$$

since any function $u\in H_0^1(\Omega)$ can be extended by zero outside of $\Omega$, obtaining a function in $(H^1(\mathbb{R}^2),\|\cdot\|_S)$. Hence, it is sufficient to show that

$$\sup_{\|u\|_S\le 1}\int_{\mathbb{R}^2}(e^{4\pi u^2}-1)dx \le d \tag{2.3}$$

We use symmetrization (see e.g. J. Moser [11]) by defining the radially symmetric function $u^*$ as follows:

$$\text{for every } \rho>0 \ \text{ let } \ m(\{x\in\mathbb{R}^2\ ;\ u^*(x)>\rho\}) = m(\{x\in\mathbb{R}^2\ ;\ u(x)>\rho\})\ .$$

Then $u^*$ is a non-increasing function in $|x|$. By construction

$$\int_{\mathbb{R}^2}(e^{4\pi|u^*|^2}-1)dx = \int_{\mathbb{R}^2}(e^{4\pi|u|^2}-1)dx \quad \text{and} \quad \int_{\mathbb{R}^2}|u^*|^2dx = \int_{\mathbb{R}^2}|u|^2dx$$

and it is known that

$$\int_{\mathbb{R}^2}|\nabla u^*|^2 \le \int_{\mathbb{R}^2}|\nabla u|^2dx\ .$$

It is therefore sufficient to prove (2.3) for radially symmetric functions $u(x)=u(|x|)$.

Thus, we may assume that $u$ in (2.3) is radially symmetric and non-increasing. We divide the integral (2.3) into two parts, with $r_0>0$ to be chosen:

$$\int_{\mathbb{R}^2}(e^{4\pi u^2}-1) = \int_{|x|\le r_0}(e^{4\pi u^2}-1) + \int_{|x|\ge r_0}(e^{4\pi u^2}-1) \tag{2.4}$$

We write the *second integral* as

$$\int_{|x|\ge r_0}(e^{4\pi u^2}-1) = \sum_{k=1}^{\infty}\int_{|x|\ge r_0}\frac{(4\pi)^k|u|^{2k}}{k!} \tag{2.5}$$

We estimate the single terms by the following "radial lemma" (see Berestycki - Lions, [4], Lemma A.IV):

$$|u(r)| \le \frac{1}{\sqrt{\pi}}\|u\|_{L^2}\frac{1}{r}\ , \quad \text{for all } r>0\ , \tag{2.6}$$

Hence we obtain for $k\ge 2$:

$$\int_{|x|\ge r_0}|u|^{2k} \le \|u\|_{L^2}^{2k}\,\frac{2}{\pi^{k-1}}\int_{r_0}^{\infty}\frac{1}{r^{2k}}\,rdr = \frac{1}{k-1}\,\|u\|_{L^2}^2\left(\frac{\|u\|_{L^2}^2}{\pi r_0^2}\right)^{k-1}\ . \tag{2.7}$$

This yields

$$\begin{aligned}\int_{|x|\ge r_0}(e^{4\pi u^2}-1) &\le 4\pi\|u\|_{L^2}^2 + 4\pi\|u\|_{L^2}^2\sum_{k=2}^{\infty}\frac{1}{k!}\left(\frac{4\|u\|_{L^2}^2}{r_0^2}\right)^{k-1}\\ &\le c(r_0)\ ,\end{aligned} \tag{2.8}$$

since $\|u\|_{L^2}\le 1$.

To estimate the *first integral* in (2.4), let

$$v(r) = \begin{cases} u(r) - u(r_0) & , \ 0 \le r \le r_0 \\ 0 & , \ r \ge r_0 \end{cases}$$

Then, by (2.6)

$$\begin{aligned} u^2(r) &= v^2(r) + 2v(r)u(r_0) + u^2(r_0) \\ &\le v^2(r) + v^2(r)\tfrac{1}{\pi r_0^2}\|u\|_{L^2}^2 + 1 + \tfrac{1}{\pi r_0^2}\|u\|_{L^2}^2 \\ &\le v^2(r)\left[1 + \tfrac{1}{\pi r_0^2}\|u\|_{L^2}^2\right] + d(r_0) \end{aligned} \tag{2.9}$$

hence

$$u(r) \le v(r)\left(1 + \frac{1}{\pi r_0^2}\|u\|_{L^2}^2\right)^{1/2} + d^{1/2}(r_0) =: w(r) + d^{1/2}(r_0)$$

By assumption

$$\int_{B_{r_0}} |\nabla v|^2 dx = \int_{B_{r_0}} |\nabla u|^2 dx \le 1 - \|u\|_{L^2}^2$$

and hence

$$\begin{aligned} \textstyle\int_{B_{r_0}} |\nabla w|^2 dx &= \textstyle\int_{B_{r_0}} |\nabla v(1 + \frac{1}{\pi r_0^2}\|u\|_{L^2}^2)^{1/2}|^2 \\ &= (1 + \tfrac{1}{\pi r_0^2}\|u\|_{L^2}^2)\textstyle\int_{B_{r_0}} |\nabla u|^2 dx \\ &\le (1 + \tfrac{1}{\pi r_0^2}\|u\|_{L^2}^2)(1 - \|u\|_{L^2}^2) \\ &= 1 + \tfrac{1}{\pi r_0^2}\|u\|_{L^2}^2 - \|u\|_{L^2}^2 - \tfrac{1}{\pi r_0^2}\|u\|_{L^2}^4 \le 1 \end{aligned} \tag{2.10}$$

provided that $r_0^2 \ge \frac{1}{\pi}$. Since by (2.9) $u^2(r) \le w^2(r) + d$ we get

$$\int_{|x|\le r_0} (e^{4\pi u^2} - 1)dx \le e^{4\pi d}\int_{B_{r_0}} e^{4\pi w^2} dx$$

The result follows by the Trudinger-Moser inequality, since $w \in H_0^1(B_{r_0})$ with $\|w\|_D^2 =$ $= \int_{B_{r_0}} |\nabla w|^2 dx \le 1$.

■

In the next proposition we show that the result is optimal (as in the Dirichlet-norm case), namely that the supremum in (2.1) becomes infinite if the exponent $4\pi$ is replaced by a number $\alpha > 4\pi$.

**Proposition 2.2** *Suppose that $\alpha > 4\pi$. Then, for any domain $\Omega \subseteq \mathbb{R}^2$*

$$\sup_{\|u\|_S \le 1} \int_\Omega (e^{\alpha u^2} - 1)dx = +\infty \ . \tag{2.11}$$

**Proof.**

We may suppose that $0 \in \Omega$, and that for some $\rho > 0$ the ball $B_\rho(0) \subset \Omega$. We use a modified "Moser-sequence", see [11], defined in $B_\rho(0)$ and continued by zero in $\Omega \setminus B_\rho(0)$, and with Sobolev-norm $\leq 1$ :

$$m_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \frac{\log(\rho/|x|)}{(\log n)^{1/2}}(1 - \frac{\rho^2}{4\log n})^{1/2} & , \quad \frac{\rho}{n} \leq |x| \leq \rho \\ (\log n)^{1/2}(1 - \frac{\rho^2}{4\log n})^{1/2} & , \quad 0 \leq |x| \leq \rho/n \end{cases} \tag{2.12}$$

One checks that $\|m_n\|^2_{H_0^1(\Omega)} \leq 1$, for $n$ large. Hence one has

$$\begin{aligned} \sup_{\|u\|_S \leq 1} \int_\Omega (e^{\alpha u^2} - 1)dx &\geq \lim_{n\to\infty} \int_{B_\rho} (e^{\alpha m_n^2} - 1)dx \\ &\geq 2\pi \int_0^{\rho/n} \left( e^{\frac{\alpha}{2\pi} \log n[1-\rho^2/(4\log n)]} - 1 \right) rdr \\ &= 2\pi \left( n^{\frac{\alpha}{2\pi}} \; e^{-\frac{\alpha\rho^2}{8\pi}} - 1 \right) \frac{r^2}{2}\Big|_0^{\rho/n} \to +\infty \ , \text{ as } \ n \to \infty \end{aligned} \tag{2.13}$$

■

## 3 Critical growth and concentration

Numerous studies in recent years have shown the close connection of critical growth with concentration phenomena, see e.g. the pioneering work of H. Brezis - L. Nirenberg [3].

As pointed out in the introduction, it is of particular interest to study the "highest level of noncompactness" for the functional $\int_\Omega (e^{4\pi u_n^2} - 1)dx$, under the restriction $\|u\|_S \leq 1$. In view of this, we make the following definition:

**Definition 3.1** *A sequence $\{u_n\} \subset H_0^1(\Omega)$ is a Sobolev-normalized concentrating sequence (for short, SNC-sequence), if*

*a)* $\|u_n\|_S = 1$

*b)* $u_n \rightharpoonup 0$ , *weakly in* $H_0^1(\Omega)$

*c)* $\exists\ x_0 \in \Omega$ *such that* $\forall \rho > 0$ : $\int_{\Omega\setminus B_\rho(x_0)} (|\nabla u_n|^2 + |u_n|^2)dx \to 0$

Next, we define the **Carleson-Chang limit** as the maximal limit of SNS-sequences:

**Definition 3.2** *Let*

$$\Sigma := \big\{ \{u_n\} \subset H_0^1(\Omega) \mid \{u_n\} \textit{ is a SNC-sequence} \big\} \ ,$$

*and define the Carleson-Chang limit as*

$$\underset{\|u_n\|_S \leq 1}{\text{cc--lim}} \int_\Omega (e^{4\pi u_n^2} - 1)dx := \sup_\Sigma \limsup_{n\to\infty} \int_\Omega (e^{4\pi u_n^2} - 1)dx \ .$$

The following "concentration-compactness alternative" by P.L. Lions (restated in our notation) is relevant for our purposes:

**Proposition** (P.L. Lions, [10], Theorem I.6). *Let $\{u_n\} \subset H_0^1(\Omega)$ satisfy $\|u_n\|_S \leq 1$; we may assume that $u_n \rightharpoonup u$. Then either*

$\{u_n\}$ *is a SNC-sequence*

*or*

$\int_\Omega (e^{4\pi u_n^2}-1)dx \to \int_\Omega (e^{4\pi u^2}-1)dx$*; this holds in particular if* $u \neq 0$.

Then one has

**Proposition 3.3** *Suppose that*

$$S := \sup_{\|u\|_S \le 1} \int_\Omega (e^{4\pi u^2}-1)dx \quad > \quad \underset{\|u_n\|_S\le 1}{\text{cc}-\lim} \int_\Omega (e^{4\pi u_n^2}-1)dx \ .$$

*Then the supremum* $S$ *is attained.*

**Proof.** Let $\{y_n\}$ denote a maximizing sequence for $S$, and assume that $S$ is not attained. We may assume that $y_n \rightharpoonup y$. By the alternative of P.L. Lions we get $y=0$, and $\{y_n\}$ is a SNC-sequence. Hence

$$S = \lim_{n\to\infty} \int_\Omega (e^{4\pi y_n^2}-1)dx \ \le \ \underset{\|u_n\|_S\le 1}{\text{cc}-\lim} \int_\Omega (e^{4\pi u_n^2}-1)dx < S$$

Contradiction! ∎

# 4 Upper bound for the Carleson-Chang limit

In this section we prove an explicit upper bound for the Carleson-Chang limit. In particular, we prove the estimates (1.7) and (1.8) of Theorem 1.2. In section 7 we will show that the bound in (1.7) is sharp for $\Omega = B_R$, with any radius $R>0$, and the bound in (1.8) is sharp for $\Omega = \mathbb{R}^2$.

**Proof.**
**1.** Using symmetrization as in section 2, we see that it is sufficient to prove (1.7) for radial functions in $B_R(0)$. Following J. Moser [11] we perform the change of variables

$$r = e^{-t/2} \ , \text{ and setting } \ w_n(t) = (4\pi)^{1/2} y_n(r) \ , \tag{4.1}$$

we transform the radial integrals on $[0,R]$ into integrals on the half-line $[-2\log R, +\infty)$. We will write throughout the paper: $\alpha_R = -2\log R$, with $\alpha_R = -\infty$ if $R = +\infty$. One checks that

$$\int_{B_R} |\nabla y_n(x)|^2 dx = 2\pi \int_0^R |\frac{d}{dr} y_n(r)|^2 r dr = \int_{\alpha_R}^\infty |w_n'(t)|^2 dt$$

and

$$\int_{B_R} (e^{4\pi y_n^2(x)}-1)dx = 2\pi \int_0^R (e^{4\pi y_n^2(r)}-1) r dr = \pi \int_{\alpha_R}^\infty (e^{w_n^2(t)}-1)e^{-t}dt \tag{4.2}$$

and similarly

$$\int_{B_R} |y_n(x)|^2 dx = 2\pi \int_0^R |y_n(r)|^2 r dr = \frac{1}{4}\int_{\alpha_R}^\infty |w_n(t)|^2 e^{-t} dt \ . \tag{4.3}$$

The SNC-sequences in this new setting are characterized by:

a) $\|w_n\|_S^2 := \int_{\alpha_R}^\infty (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t})dt = 1 \ , \ w_n(\alpha_R) = 0$

b) $w_n \rightharpoonup 0$ , weakly in $H^1([\alpha_R, +\infty))$

c) $\displaystyle\int_{\alpha_R}^{A}(|w_n'|^2+\frac{1}{4}|w_n|^2e^{-t})dt \to 0$ for any fixed $A>0$,

and the estimate (1.7) (which we seek to prove) becomes

$$\text{cc-}\lim_{\|w_n\|_S\le 1}\ \pi\int_{\alpha_R}^{\infty}(e^{w_n^2(t)}-1)e^{-t}dt \le \pi e^{1-D(R)} \tag{4.4}$$

for SNC-sequences $\{w_n\}\subset H^1([\alpha_R,+\infty))$.

Let now denote $\{w_n\}$ a maximizing SNC-sequence for the Carleson-Chang limit (1.7). We may assume that the sequence $\{w_n\}$ satisfies

$$\lim_{n\to\infty}\pi\int_{\alpha_R}^{\infty}(e^{w_n^2}-1)e^{-t}dt > 2\,\pi e^{-D(R)}\ , \tag{4.5}$$

since otherwise the theorem is proved. Note that we may assume that $w_n(t)$ is an increasing function on $[\alpha_R,+\infty)$. Fix $A_R\ge 1$ such that

$$t-2\log t - D(R) > 1\ ,\ \forall\ t\ge A_R\ . \tag{4.6}$$

**Claim 1:** There exists a number $n_1$ such that

$$w_n(t)<1\ ,\ \forall\ t\le A_R\ ,\ \forall\ n\ge n_1$$

Indeed, for $0<R<+\infty$ we can estimate

$$\begin{aligned} w_n(t)\ &\le (A_R+2\log R)^{1/2}\ \left(\int_{\alpha_R}^{A_R}|u_n'|^2dt\right)^{1/2} \\ &=: (A_R+2\log R)^{1/2}\ \delta_n\ ,\quad \text{for}\ \ t\le A_R\ , \end{aligned} \tag{4.7}$$

with $\delta_n\to 0$ as $n\to 0$, by c).
For $R=+\infty$ and $0<t\le A_R$ we estimate

$$w_n(t)=w_n(0)+\int_0^t w'(t)dt \le w_n(0)+t^{1/2}(\int_0^t|w_n'|^2)^{1/2}dt$$

The second term goes to zero, as above. For the estimate of $w_n(0)$ we use the following Radial Lemma (see W. Strauss, [13]), valid for radial functions $v(r)$ in $H^1(\mathbb{R}^2)$ and for $r\ge 1$:

$$(r+\frac{1}{2})v^2(r)\le \frac{5}{4}\int_r^{\infty}(|v'|^2+|v|^2)\rho d\rho$$

We transform this inequality (as before) by the change of variables $r=e^{-t/2}$ and $w(t)=(4\pi)^{1/2}v(r)$ and get, for $t\le 0$ :

$$(e^{-t/2}+\frac{1}{2})w^2(t)\le \frac{5}{2}\int_{-\infty}^{e^{-t/2}}(|w'(t)|^2+\frac{1}{4}|w(t)|^2e^{-t})dt\ . \tag{4.8}$$

Hence, we get for $w_n(0)$, using the concentration property of $w_n$

$$w_n^2(0)\le\frac{5}{3}\int_{-\infty}^{0}(|w'(t)|^2+\frac{1}{4}|w(t)|^2e^{-t})dt =: \sigma_n^2\to 0\ ,\quad\text{as}\ \ n\to\infty\ .$$

Thus the claim is proved.

By claim 1 we conclude that for $n$ sufficiently large ($0 < R \le +\infty$)

$$w_n^2(t) < 1 < A_R - 2\log A_R - D(R)\ ,\ \alpha_R \le t \le A_R\ .$$

Let now $a_n > A_R$ denote the first $t > A_R$ with

$$w_n^2(a_n) = a_n - 2\log a_n - D(R)\ . \tag{4.9}$$

Such an $a_n$ exists (for $n$ sufficiently large), since otherwise

$$w_n^2(t) < t - 2\log t - D(R)\ , \forall\ t \ge A_R \ge 1\ ,\quad \text{as}\quad n \to \infty\ ,$$

and thus

$$\pi \int_{\alpha_R}^{\infty} (e^{w_n^2} - 1)e^{-t} \le \pi \int_{\alpha_R}^{A_R} (e^{w_n^2} - 1)e^{-t} + \pi \int_{A_R}^{\infty} e^{t-2\log t - D(R) - t}$$

The second term on the right is bounded by $\pi e^{-D(R)}$, and in the following claim 2 we prove that the first term goes to 0, for $n \to \infty$, and thus we have a contradiction to assumption (4.5).

**Claim 2:** $\quad \pi \int_{\alpha_R}^{A_R} (e^{w_n^2} - 1)e^{-t} \to 0$ as $n \to \infty$.

This is immediate for $0 < R < +\infty$, since then this term can be estimated, using (4.7), by

$$\pi(R^2 - e^{-A_R})(e^{\delta_n^2(A_R + \alpha_R)} - 1) \to 0\quad \text{as}\quad n \to \infty\ .$$

If $R = +\infty$ we write

$$\int_{-\infty}^{0} (e^{w_n^2} - 1)e^{-t}dt + \int_{0}^{A_R} (e^{w_n^2} - 1)e^{-t}dt$$

The second term is now estimated as before, while for the first term we use a series expansion:

$$\int_{-\infty}^{0} (e^{w_n^2} - 1)e^{-t}dt = \int_{-\infty}^{0} \sum_{k=1}^{\infty} \frac{|w_n(t)|^{2k}}{k!} e^{-t}dt$$

$$= \int_{-\infty}^{0} |w_n(t)|^2 e^{-t}dt + \int_{-\infty}^{0} \frac{1}{2}|w_n(t)|^4 e^{-t}dt + \sum_{k=3}^{\infty} \int_{-\infty}^{0} \frac{|w_n(t)|^{2k}}{k!} e^{-t}dt$$

The first term goes to zero by concentration, the second term can be estimated by Sobolev (by returning to the variable $r$ and back to $t$)

$$\int_{-\infty}^{0} w_n^4 e^{-t}dt \le c_0 \left( \int_{-\infty}^{0} (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t})dt \right)^2$$

and hence also goes to zero by concentration. For the third term, observe that by (4.8) we get for $t \le 0$

$$w_n^2(t) \le \frac{5}{4}\ \frac{1}{e^{-t/2} + 1/2}\ \sigma_n^2 \le c\ e^{t/2}\ \sigma_n^2$$

Hence we can estimate the series as

$$\sum_{k=3}^{\infty} \int_{-\infty}^{0} \frac{c^k}{k!}\ \sigma_n^{2k} e^{k\ t/2} e^{-t}dt \le \sum_{k=3}^{\infty} c^k \sigma_n^{2k} \int_{-\infty}^{0} e^{t/2}dt \le c_1\ \sigma_n^6\ 2\ ,$$

and thus claim 2 is proved.

Thus we have proved the existence of a number $a_n > A_R$ as claimed in (4.9).

We now prove, for $0 < R \leq +\infty$

i) $\pi \int_{\alpha_R}^{a_n} (e^{w_n^2} - 1)e^{-t} dt \to 0$, as $n \to \infty$.

ii) $\lim_{n\to\infty} \pi \int_{a_n}^{\infty} (e^{w_n^2} - 1)e^{-t} dt \leq \pi e^{1-D(R)}$

Proof of i): Note that the argument above shows that $a_n \to +\infty$ as $n \to \infty$, since for an arbitrarily large number $A_R$ there exists $n_0(A_R)$ such that $a_n > A_R$ for $n \geq n_0$. By (4.9) we have

$$\pi \int_{\alpha_R}^{a_n} (e^{w_n^2} - 1)e^{-t} dt \leq \int_{\alpha_R}^{A} (e^{w_n^2} - 1)e^{-t} dt + \pi \int_{A}^{a_n} e^{-2\log t - D(R)} dt$$

Let $\epsilon > 0$: for the second term we get $\pi e^{-D(R)}(\frac{1}{A} - \frac{1}{a_n}) < \epsilon/2$, for $A$ sufficiently large, and then the first term becomes $\leq \epsilon/2$, for $n \geq n_0(A, \epsilon)$, proceeding as in Claim 2.

Proof of ii): We apply the following basic estimate which was proved in [6] (we cite it here in the form given in [7], Proposition 2.2):

**Lemma** (Carleson-Chang): *For $a > 0$ and $\delta > 0$ given, suppose that $\int_a^\infty |w'(t)|^2 dt \leq \delta$. Then*

$$\int_a^\infty e^{w^2 - t} dt \leq e\frac{1}{1-\delta} e^K \ , \qquad \text{with } \ K = w^2(a)(1 + \frac{\delta}{1-\delta}) - a \ .$$

We apply this Lemma to our sequence $\{w_n\}$, with $a = a_n$ given in (4.9), and $\delta = \delta_n = \int_{a_n}^\infty (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t}) dt$. Furthermore, in the following section 5, (5.1) and section 6, Proposition 6.4, it is shown that:

*For $a > 0$ and $b > 0$ given, let*

$$S_{a,b} = \{u \in H^1(\alpha_R, a),\ u(\alpha_R) = 0\,,\ \int_{\alpha_R}^a (|u'|^2 + \frac{1}{4}|u|^2 e^{-t}) dt = b\} \ .$$

*Then the supremum*

$$\sup\{\|u\|_\infty^2 \ : \ u \in S_{a,b}\}$$

*is attained by a function $y$, with*

$$\|y\|_\infty^2 = y^2(a) = b(a - D(R)) + O(\frac{1}{a}).$$

Thus, choosing $a = a_n$ and $b = b_n = 1 - \delta_n$ we get for $w_n \in S_{a_n, b_n}$

$$w_n^2(a_n) \leq a_n - a_n\delta_n - D(R) + O(\delta_n) + O(\frac{1}{a_n}) \ ,$$

which implies together with (4.9)

$$\delta_n \leq \frac{2\log a_n}{a_n} + O(\frac{\log a_n}{a_n^2}) \tag{4.10}$$

Thus we have for $K = K_n$ in the Lemma of Carleson and Chang

$$
\begin{aligned}
K_n &= w_n^2(a_n)(1 + \frac{\delta_n}{1-\delta_n}) - a_n \\
&\leq \left( a_n - a_n\delta_n - D(R) + O(\frac{\log a_n}{a_n}) \right) (1 + \delta_n + O(\delta_n^2)) - a_n \\
&= -D(R) - \delta_n D(R) + O(\frac{\log a_n}{a_n}) + a_n O(\delta_n^2) \\
&= -D(R) + O(\frac{(\log a_n)^2}{a_n})
\end{aligned} \tag{4.11}
$$

Hence we obtain by the Lemma of Carleson and Chang for any maximizing SNC-sequence $\{w_n\}$

$$
\lim_{n\to\infty} \pi \int_{a_n}^{\infty} (e^{w_n^2} - 1)e^{-t}dt \leq \lim_{n\to\infty} \pi\, e\, \frac{1}{1-\delta_n}\, e^{K_n} \leq \pi\, e^{1-D(R)} \ ;
$$

thus ii) is proved.

With i) and ii) we now easily complete the proof of the first statement of Theorem 1.2

**2.** It is clear that for $\Omega_0 \subset \Omega_1$ the corresponding cc-limits are increasing. Thus, it is sufficient to prove 2) for $\Omega = \mathbb{R}^2$; this corresponds to setting $R = +\infty$, which was included in the proof of 1).

■

# 5 An auxiliary variational problem

In this section we consider the following variational problem: Determine

$$
\sup \{\|u\|_\infty^2 \mid u \in S_{a,b}\} \ , \tag{5.1}
$$

where

$$
S_{a,b} = \left\{ u \in H^1(\alpha_R, a) \mid u(\alpha_R) = 0, \int_{\alpha_R}^{a} \left( |u'|^2 + \frac{R^2}{4}|u|^2 e^{-t} \right) dt = b > 0 \right\}
$$

Note that $S_{a,b} \subset L^\infty(\alpha_R, a)$, with compact embedding, and hence it is easily seen that the supremum in (5.1) is attained: let $y_a \in S_{a,b}$ such that

$$
\|y_a\|_\infty^2 = \sup \{\|u\|_\infty^2 \mid u \in S_{a,b}\} \ . \tag{5.2}
$$

In order to determine the value of (5.2) we need to identify the maximizing function $y_a \in S_{a,b}$. The natural way to do this consists in deriving the Euler-Lagrange equation associated to (5.1), but we encounter the difficulty that the functional $y \mapsto \|y\|_\infty^2$ is not differentiable. However, this functional is convex, and hence its *subdifferential* exists. We briefly recall this notion, and then derive the Euler-Lagrange equation for (5.1). For the proofs of some of the results we refer to [8].

**Definition 5.1** *Let $E$ be a Banach space, and $\psi : E \to \mathbb{R}$ continuous and convex. Then we denote by $\partial\psi(u) \subset E'$ the* subdifferential *of $\psi$ in $u \in E$, given by*

$$
\mu_u \in \partial\psi(u) \Leftrightarrow \psi(u+v) - \psi(u) \geq \langle \mu_u, v \rangle \ , \ \forall\, v \in E \ ;
$$

*here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $E$ and $E'$. An element $\mu_u \in \partial\psi(u)$ is called a subgradient of $\psi$ at $u$.*

In [8], Lemma 2.2, it is proved that

**Lemma:** *If $\psi$ satisfies in addition*

$$\psi(x) \geq 0 \ , \ \forall x \in E \ , \ \text{and} \ \ \psi(tx) = t^2 \psi(x) \ , \ \forall \ t \geq 0 \ , \tag{5.3}$$

*then*

$$\mu \in \partial\psi(u) \quad \Leftrightarrow \quad \begin{cases} \langle \mu, u \rangle = 2\psi(u) \\ \langle \mu, x \rangle \leq \langle \mu, u \rangle \ , \ \forall \ x \in \psi^u = \{x \in E; \psi(x) \leq \psi(u)\} \ . \end{cases}$$

Furthermore, by an easy variation of [8], Lemma 2.3 and Corollary 2.4, one has:

**Lemma 5.2** *Suppose that $\psi : E \to \mathbb{R}$ satisfies (5.3), and $\phi \in C^1(E, \mathbb{R})$ satisfies $\langle \phi'(x), x \rangle = 2\phi(x)$, $\forall \, x \in E$. If $y \in E$ is such that*

$$\psi(y) = \sup_{\{u \in E, \ \phi(u) = b\}} \psi(u) \ ,$$

*then*

$$\phi'(u) \in \frac{b}{\psi(u)} \partial\psi(u)$$

**Proof.** The Euler-Lagrange equation

$$\phi'(u) \in \lambda \partial\psi(u) \ \ \text{for some} \ \ \lambda > 0 \tag{5.4}$$

is obtained as in [8], Lemma 2.3 and Corollary 2.4. The value

$$\lambda = \frac{b}{\psi(u)}$$

is found by testing (5.4) with $u$:

$$2b = 2\phi(u) = \langle \phi'(u), u \rangle = \lambda \langle \mu_u, u \rangle = \lambda 2\psi(u) \ .$$

■

We now apply Lemma 5.2 to our situation, and obtain

**Theorem 5.3** *Let $E = \{v \in H^1(\alpha_R, a); v(\alpha_R) = 0\}$, and consider*

$$\psi(u) = \|u\|_\infty^2 : E \to \mathbb{R}$$

*and*

$$\phi(u) = \int_{\alpha_R}^{a} (|u'(x)|^2 + \frac{1}{4}|u(x)|^2 e^{-x}) dx \ .$$

*Suppose that $y \in E$ satisfies*

$$\psi(y) = \sup\{\psi(u) \mid u \in E \ , \ \phi(u) = b\} \ ;$$

*then $y$ satisfies (weakly) the equation*

$$-y''(x) + \frac{1}{4} y(x) e^{-x} = \frac{b}{\|y\|_\infty^2} \mu_y \ , \quad \text{where} \ \ \mu_y \in \partial\psi(y) \subset E' \tag{5.5}$$

# 6 The auxiliary Euler-Lagrange equation

It remains to determine the subgradient $\mu_y$ in equation (5.5). Again following [8], Lemma 2.6, 2.7 and 2.8 we find:

**Proposition 6.1** *Let $K_y = \{x \in [\alpha_R, a]; |y(x) = \|y\|_\infty\}$. Then*
*i) supp $\mu_y \subset K_y$*
*ii) $K_y = \{a\}$*
*iii) $\mu_y = \|y\|_\infty \delta_a$, the Dirac delta-function concentrated in the point $a$.*

Thus, equation (5.5) becomes

$$\left\{\begin{array}{rcl} -y'' + \frac{1}{4} y e^{-t} & = & \frac{b}{\|y\|_\infty}\, \delta_a \ , \ \alpha_R \le t \le a \\ y(\alpha_R) & = & 0 \end{array}\right. \tag{6.1}$$

From this one now concludes easily that equation (5.5) is equivalent to solving the equation

$$\left\{\begin{array}{rl} -w'' + \frac{1}{4} w e^{-t} = & 0 \\ w(\alpha_R) = & 0 \end{array}\right. \quad , \ \alpha_R \le t < a \ , \tag{6.2}$$

with the condition that

$$\int_{\alpha_R}^{a} (|w'(t)|^2 + \frac{1}{4}|w(t)|^2 e^{-t}) dt = b \ ; \tag{6.3}$$

the last condition is obtained by multiplying equation (6.1) by $y$ and integrating.

We now determine the explicit solution of equation (6.2).

**Theorem 6.2** *The solution of equation (6.2) is given by*
• *for $0 < R < +\infty$:*

$$w(t) = \gamma \left( K_0(e^{-t/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-t/2}) \right) =: \gamma\, z(t) \tag{6.4}$$

• *for $R = +\infty$:*

$$w(t) = \gamma K_0(e^{-t/2}) \ , \tag{6.5}$$

*with unique coefficients $\gamma = \gamma(R, a, b) \in \mathbb{R}^+$.*

*Here $I_k(x)$ and $K_k(x)$ are the $k-th$ modified Bessel functions of first and second kind, i.e. the solutions of the equation*

$$-x^2 u''(x) - x u'(x) + (x^2 + k^2) u(x) = 0 \ , \ k = 1, 2, ...$$

**Proof.** By inspection. ∎

It is crucial to dermine with precision the value of the coefficient $\gamma = \gamma(R, a, b)$ of $w(t)$. This requires some lengthy calculations.

We begin by recalling the following relations for the modified Bessel functions (see e.g. [1], 9.6.27,28):

$$\frac{d}{dx} I_0(x) = I_1(x) \, , \ \frac{d}{dx} K_0(x) = -K_1(x) \, , \ \frac{d}{dx}(x\, K_1(x)) = -x\, K_0(x) \, , \tag{6.6}$$

and the following integral relations

$$\int_a^b |K_0(r)|^2 r dr = \left[\tfrac{1}{2} r^2 (K_0^2(r) - K_1^2(r))\right]_a^b$$

$$\int_a^b |K_1(r)|^2 r dr = \left[\tfrac{1}{2} r^2 (K_1^2(r) - K_0(r)K_2(r))\right]_a^b$$

$$\int_a^b |I_0(r)|^2 r dr = \left[\tfrac{1}{2} r^2 (I_0^2(r) - I_1^2(r))\right]_a^b \tag{6.7}$$

$$\int_a^b |I_1(r)|^2 r dr = \left[\tfrac{1}{2} r^2 (I_1^2(r) - I_0(r)I_2(r))\right]_a^b$$

$$\int_a^b [I_1(r)K_1(r) - I_0(r)K_0(r)] r dr = [I_0(r)K_1(r) r]_a^b$$

see [1]; for the last relation use integration by parts and (6.6).

Using these relations we will prove:

**Theorem 6.3**
*1) Condition (6.3) yields for the coefficient $\gamma = \gamma(R, a, b)$ in (6.4)*

$$\gamma^2 = 4 \, \frac{b}{a} \, \left[1 - \frac{4}{a} \, C(R)\right] + O(\frac{1}{a^3}) \ ,$$

*for a large, with*

$$\begin{aligned} C(R) \ &= \ \tfrac{1}{4} R^2 \left(K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \tfrac{I_2(R)}{I_0(R)})\right) \\ &+ \ 2RK_0(R)K_1(R) - 2\tfrac{K_0(R)}{I_0(R)} \end{aligned} \tag{6.8}$$

*and* $\quad C(+\infty) = 0.$

*2) The solution $w(t)$, $\alpha_R \leq t \leq a$, of equation (6.2) is given by*

• *for $0 < R < +\infty$:*

$$w(t) = 2 \, \sqrt{\frac{b}{a}} \, \left(1 - \frac{4}{a} \, C(R) + O(\frac{1}{a^2})\right)^{1/2} \left(K_0(e^{-t/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-t/2})\right) \tag{6.9}$$

• *for $R = +\infty$:*

$$w(t) = 2\sqrt{\frac{b}{a}} \left(1 + O(\frac{1}{a^2})\right)^{1/2} \ K_0(e^{-t/2}) \tag{6.10}$$

**Proof.** Recall the definition of $w(t)$ given in (6.4). We begin by evaluating the expression

$$W^2(a) := \int_{\alpha_R}^{a} (|w'(x)|^2 + \frac{1}{4}|w^2(x)|^2 e^{-x}) dx$$

Using the explicit form of $w(t)$ in (6.4), the change of variable $r = e^{-x/2}$, and the relations (6.6), we get

$$
\begin{aligned}
W^2(a) &= \frac{1}{4}\int_{\alpha_R}^{a}\left\{\left|K_0'(e^{-x/2})-\frac{K_0(R)}{I_0(R)}I_0'(e^{-x/2})\right|^2+\left|K_0(e^{-x/2})-\frac{K_0(R)}{I_0(R)}I_0(e^{-x/2})\right|^2\right\}e^{-x}dx\\
&= \frac{1}{2}\int_{e^{-a/2}}^{R}\left\{\left|-K_1(r)-\frac{K_0(R)}{I_0(R)}I_1(r)\right|^2+\left|K_0(r)-\frac{K_0(R)}{I_0(R)}I_0(r)\right|^2\right\}rdr\\
&= \frac{1}{2}\int_{e^{-a/2}}^{R}\left\{|K_1(r)|^2+\frac{K_0^2(R)}{I_0^2(R)}|I_1(r)|^2+|K_0(r)|^2+\frac{K_0^2(R)}{I_0^2(R)}|I_0(r)|^2\right.\\
&\left.+\,2\frac{K_0(R)}{I_0(R)}\left(K_1(r)I_1(r)-K_0(r)I_0(r)\right)\right\}rdr
\end{aligned}
\tag{6.11}
$$

Using the relations (6.7) we get

$$
\begin{aligned}
&\frac{1}{2}\left\{\left[\frac{1}{2}r^2(K_1^2(r)-K_0(r)K_2(r))\right]_{e^{-a/2}}^{R}+\frac{K_0^2(R)}{I_0^2(R)}\left[\frac{1}{2}r^2(I_1^2(r)-I_0(r)I_2(r))\right]_{e^{-a/2}}^{R}\right.\\
&\quad+\left[\frac{1}{2}r^2(K_0^2(r)-K_1^2(r))\right]_{e^{-a/2}}^{R}+\frac{K_0^2(R)}{I_0^2(R)}\left[\frac{1}{2}r^2(I_0^2(r)-I_1^2(r))\right]_{e^{-a/2}}^{R}\\
&\quad\left.+\,2\frac{K_0(R)}{I_0(R)}\left[I_0(r)K_1(r)r\right]_{e^{-a/2}}^{R}\right\}
\end{aligned}
\tag{6.12}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{\left[\frac{1}{2}r^2\left(K_0^2(r)-K_0(r)K_2(r)+\frac{K_0^2(R)}{I_0^2(R)}(I_0^2(r)-I_0(r)I_2(r))\right)\right]_{e^{-a/2}}^{R}\right.\\
&\quad\left.+\,2\frac{K_0(R)}{I_0(R)}\left[I_0(r)K_1(r)\ r\right]_{e^{-a/2}}^{R}\right\}
\end{aligned}
$$

Evaluating at the boundaries we obtain

$$
\begin{aligned}
&\tfrac{1}{4}R^2\left(K_0^2(R)-K_0(R)K_2(R)+K_0^2(R)(1-\tfrac{I_2(R)}{I_0(R)})\right)+2RK_0(R)K_1(R)\\
&\quad-\tfrac{1}{4}e^{-a}\{\,K_0^2(e^{-a/2})-K_0(e^{-a/2})K_2(e^{-a/2})\\
&\quad+\tfrac{K_0^2(R)}{I_0^2(R)}\,[\,I_0^2(e^{-a/2})-I_0(e^{-a/2})I_2(e^{-a/2})\,]\,\}\\
&\quad-2e^{-a/2}\tfrac{K_0(R)}{I_0(R)}I_0(e^{-a/2})K_1(e^{-a/2})
\end{aligned}
\tag{6.13}
$$

For the terms with argument $e^{-a/2}$, $a$ large, we now use the following behavior of the Bessel functions for $x>0$ small, see [1],9.6.7-9: :

$$
\begin{array}{lll}
K_0(x)\sim-\log x & K_1(x)\sim\frac{1}{x} & K_2(x)\sim\frac{2}{x^2}\\
I_0(x)\sim 1 & I_1(x)\sim\frac{1}{2}x & I_2(x)\sim\frac{1}{8}x^2
\end{array}
\tag{6.14}
$$

We get

$$
\begin{aligned}
&\tfrac{1}{4}R^2\left(K_0^2(R)-K_0(R)K_2(R)+K_0^2(R)(1-\tfrac{I_2(R)}{I_0(R)})\right)+2RK_0(R)K_1(R)\\
&\quad -\tfrac{1}{4}e^{-a}\{\,(-\log(e^{-a/2}))^2-(-\log(e^{-a/2}))\tfrac{2}{e^{-a}}\\
&\quad +\tfrac{K_0^2(R)}{I_0^2(R)}\,[\,1-\tfrac{1}{8}e^{-a}\,]\,\}-2e^{-a/2}\tfrac{K_0(R)}{I_0(R)}\tfrac{1}{e^{-a/2}}
\end{aligned}
$$

$$
\begin{aligned}
&=\tfrac{1}{4}R^2\left(K_0^2(R)-K_0(R)K_2(R)+K_0^2(R)(1-\tfrac{I_2(R)}{I_0(R)})\right)+2RK_0(R)K_1(R)\\
&\quad -\tfrac{1}{4}e^{-a}\{\,(\tfrac{a}{2})^2-\tfrac{a}{2}2e^{a}+\tfrac{K_0^2(R)}{I_0^2(R)}\,[\,1-\tfrac{1}{8}e^{-a})\,]\,\}-2\tfrac{K_0(R)}{I_0(R)}\\
&=\tfrac{1}{4}R^2\left(K_0^2(R)-K_0(R)K_2(R)+K_0^2(R)(1-\tfrac{I_2(R)}{I_0(R)})\right)+2RK_0(R)K_1(R)\\
&\quad +\tfrac{1}{4}\,a-2\tfrac{K_0(R)}{I_0(R)}+O(a^2e^{-a})\\
&=\tfrac{1}{4}a+C(R)+O(a^2e^{-a})\ ,
\end{aligned}
\tag{6.15}
$$

with $C(R)$ as in (6.8). Conditions (6.3) and (6.4) yield now

$$
b=\gamma^2W^2(a)=\gamma^2\left(\frac{1}{4}a+C(R)+O(a^2e^{-a})\right)
\tag{6.16}
$$

We rewrite (6.16) as

$$
\gamma^2\frac{a}{4}\left(1+\frac{4}{a}C(R)+O(ae^{-a})\right)=b
\tag{6.17}
$$

which yields for $\gamma=\gamma(a,b)$

$$
\gamma^2=4\ \frac{b}{a}\ \left[1-\frac{4}{a}C(R)\right]+O(\frac{1}{a^3})
\tag{6.18}
$$

This proves 1). Assertion 2) follows now from (6.4). Formula (6.10) follows from (6.9), noting that $C(+\infty)=0$ and $K_0(+\infty)/I_0(+\infty)=0$. ∎

With this information we can now calculate the value $\|w\|_\infty^2=w^2(a)$:

**Proposition 6.4** *Let $w(t)$ denote the solution of (6.2), (6.3) and hence of (5.1). Then*

$$
\|w\|_\infty^2=w^2(a)=b\ \,[a-D(R)]+O(\frac{1}{a})\ .
$$

**Proof.** By (6.4) we have, using (6.14)

$$
\begin{aligned}
w^2(a)\ &=\gamma^2\left(K_0(e^{-a/2})-\tfrac{K_0(R)}{I_0(R)}I_0(e^{-a/2})\right)^2\\
&=4\ \tfrac{b}{a}\ \left[(1-\tfrac{4}{a}C(R))+O(\tfrac{1}{a^2})\right]\ \left(K_0(e^{-a/2})-\tfrac{K_0(R)}{I_0(R)}I_0(e^{-a/2})\right)^2\\
&=4\ \tfrac{b}{a}\ \left[(1-\tfrac{4}{a}C(R))\right]\left(\tfrac{a}{2}-\tfrac{K_0(R)}{I_0(R)}\right)^2+O(\tfrac{\log a}{a^3})\\
&=b\ \left[a-4C(R)-4\tfrac{K_0(R)}{I_0(R)}\right]+O(\tfrac{1}{a})
\end{aligned}
\tag{6.19}
$$

Set

$$D(R) = 4C(R) + 4\frac{K_0(R)}{I_0(R)} \ ; \tag{6.20}$$

then (6.19) becomes

$$w^2(a) = b\ [a - D(R)] + O(\frac{1}{a}) \tag{6.21}$$

■

# 7 Construction of optimal concentrating sequences

In this section we show that the upper bounds for the Carleson-Chang limit

$$\underset{\|u_n\|_S \le 1}{\text{cc−lim}} \int_\Omega (e^{4\pi u^2} - 1)dx \le \pi e^{1-D(R)} \ , \tag{7.1}$$

given in Theorem 1.2 are sharp for $\Omega = B_R$ and $\Omega = \mathbb{R}^2$. We do this by constructing explicit optimal SNC-sequences $\{w_n\}$ for (7.1) for which the Carleson-Chang limit is equal to the bound on the right.

The construction of this sequence follows closely the proof of the upper bound for the Carleson-Chang limit, section 4, in combination with information on the optimal sequence for the corresponding Dirichlet-norm problem, see [7].

We begin by defining the sequence $\{w_n(t)\}$ on $[\alpha_R, n]$: in Theorem 6.3, set $a = n$ and $b = 1 - \frac{2\log n}{n}$. Then, for $0 < R \le +\infty$, let $w_n(t)$ be given by (6.9) or (6.10), respectively. Thus, $w_n(t)$ satisfies equation (6.2) with $a = n$, and condition (6.3) with $b = 1 - \frac{2\log n}{n}$. Furthermore, we have by Proposition 6.4

$$w_n^2(n) = \sup\{\|w_n\|_\infty^2 \mid w_n \in S_n\} = n - 2\log n - D(R) + O(\frac{1}{n}) \ , \tag{7.2}$$

where $S_n = \{u \in H^1(\alpha_R, n) \mid u(\alpha_R) = 0, \int_{\alpha_R}^n (|u'|^2 + \frac{1}{4}|u|^2 e^{-t})dt = 1 - \frac{2\log n}{n}\}$. We remark that formula (7.2) constitutes a (late) motivation for the choice of $a_n$ in (4.9).

It remains to define $\{w_n(t)\}$ in $[n, +\infty)$. Here we can follow [7] where an optimal Dirichlet normalized concentrating sequence was constructed by analyzing carefully the proof of Carleson-Chang [6].

The complete definition of the *optimal SNC-sequence* $\{w_n(t)\}$ is:

**Definition 7.1** *Let $w_n(t)$ be given by:*

$$w_n(t) = \begin{cases} w_n(t)\ , & \text{given by (6.9) or (6.10), respectively,} \quad \alpha_R \le t \le n \\ & \text{with } \ a = n \ \text{ and } \ b = 1 - \frac{2\log n}{n} \\ w_n(n) + \dfrac{1}{w_n(n)} \log \dfrac{1 + A_n}{A_n + e^{-(t-n)}} & t \ge n \end{cases} \tag{7.3}$$

*where $A_n \in \mathbb{R}^+$ is such that*

$$\int_{\alpha_R}^\infty (|w_n'(t)|^2 + \frac{1}{4}|w_n(t)|^2 e^{-t})dt = 1 \ . \tag{7.4}$$

We show that $A_n \in \mathbb{R}^+$ can be chosen as in Definition 7.1, i.e. satisfying (7.4), with the estimate

**Lemma 7.2**

$$A_n = \frac{1}{n^2\, e} + O(\frac{1}{n^4}) \tag{7.5}$$

**Proof.** First note that by condition (6.3)

$$\int_{\alpha_R}^{n} (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t}) dt = 1 - \frac{2\log n}{n} \tag{7.6}$$

Thus, we look for a constant $A_n$ such that

$$\int_{n}^{\infty} (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t}) dt = \frac{2\log n}{n} \tag{7.7}$$

Assume that $A_n \geq \frac{1}{3n^2}$, then one has

$$\log(\frac{1+A_n}{A_n + e^{-(t-n)}}) \leq \log(1 + \frac{1}{A_n}) \leq \log(1+3n^2)$$

and then by (7.3) and using that $w_n(n) = n + O(\log n)$ (by Proposition 6.4)

$$w_n(t) \leq w_n(n) + \frac{1}{w_n(n)} \log(1+3n^2) \leq 2\,n\ , \quad \text{for } t \geq n\ ,\ n \ \text{ large} \quad ,$$

and hence

$$\int_{n}^{\infty} |w_n|^2 e^{-t} dt \leq 4\, n^2 e^{-n}$$

Therefore, condition (7.7) becomes

$$\int_{n}^{\infty} |w_n'|^2 = \frac{2\log n}{n} + O(n^2 e^{-n}) \tag{7.8}$$

One proves as in [7] that this yields

$$A_n = \frac{1}{n^2\, e} + O(\frac{1}{n^4})$$

∎

We now give an asymptotic lower bound for $\pi \int_{\alpha_R}^{\infty} (e^{w_n^2} - 1) e^{-t} dt$, as $n \to \infty$:

**Theorem 7.3** *Let $\{w_n\}$ denote the sequence (7.3), and let $D(R)$ be given by (6.20). Then*

$$\pi \int_{\alpha_R}^{\infty} (e^{w_n^2} - 1) e^{-t} \geq e\, \pi e^{-D(R)}\, (1 + 2D(R)\, \frac{\log n}{n}) + O(\frac{1}{n})\ .$$

**Proof.**
a) First note that

$$\pi \int_{\alpha_R}^{n} (e^{w_n^2} - 1) e^{-t} dt \geq 0\ , \quad \text{for all} \ \ n \tag{7.9}$$

b) Consider now

$$\pi \int_n^\infty (e^{w_n^2} - 1)e^{-t} = \pi \int_n^\infty e^{w_n^2 - t} + O(e^{-n}) \ .$$

Performing the change of variables $s = t - n$, setting

$$v_n(s) = \frac{1}{w_n(n)} \log \frac{A_n + 1}{A_n + e^{-s}}$$

and using that by Proposition 6.4

$$\begin{aligned} w_n^2(n) &= (1 - \frac{2\log n}{n})[n - D(R)] + O(\frac{1}{n}) \\ &= n - D(R) - 2\log n + \frac{2\log n}{n} D(R) + O(\frac{1}{n}) \end{aligned}$$

we obtain

$$\begin{aligned} &\pi \int_{\alpha_R}^\infty \exp\big([w_n(n) + v_n(s)]^2 - s - n\big)\, ds \\ &\geq \pi \int_{\alpha_R}^\infty \exp\big(w_n^2(n) + 2w_n(n)v_n(s) - s - n\big)\, ds \\ &\geq \pi \int_{\alpha_R}^\infty \exp\left(n - 2\log n - D(R) + 2D(R)\frac{\log n}{n} + O(\frac{1}{n}) + 2\log\frac{A_n + 1}{A_n + e^{-s}} - s - n\right) \\ &= \pi \int_0^\infty \exp(-2\log n - D(R) + 2\log\frac{A_n + 1}{A_n + e^{-s}} - s + 2D(R)\frac{\log n}{n} + O(\frac{1}{n})) \qquad (7.10) \\ &= \pi e^{-D(R)} \frac{1}{n^2} \int_0^\infty \left(\frac{1 + A_n}{A_n + e^{-s}}\right)^2 e^{-s} ds\ (1 + 2D(R)\frac{\log n}{n} + O(\frac{1}{n})) \\ &= \pi e^{-D(R)}\ \frac{1}{n^2}\frac{1 + A_n}{A_n}\ (1 + 2D(R)\frac{\log n}{n} + O(\frac{1}{n})) \\ &= e\,\pi e^{-D(R)}\ (1 + 2D(R)\frac{\log n}{n}) + O(\frac{1}{n})\ , \quad \text{as} \quad n \to \infty\ . \end{aligned}$$

Joining (7.9) and (7.10) we get

$$\pi \int_{\alpha_R}^\infty (e^{w_n^2} - 1)e^{-t} dt \geq e\,\pi\, e^{-D(R)}(1 + 2D(R)\frac{\log n}{n}) + O(\frac{1}{n})\ ,$$

and hence the theorem is proved. ■

We conclude this section by proving some properties of the function $D(R)$:

**Lemma 7.4** *Let $D(R)$ given by (6.20). Then*

$$D(R) = 4R\, K_0(R)K_1(R) - 2\,\frac{K_0(R)}{I_0(R)}\ . \qquad (7.11)$$

*Furthermore, $D(R) > 0$, for all $R \in \mathbb{R}^+$, and*

$$D(R) \sim -2\log R\ , \quad as \quad R \to 0$$

*and*

$$D(R) \sim \frac{\pi}{R} e^{-2R}\ , \quad as \quad R \to +\infty\ .$$

**Proof.** The explicit form of $D(R)$ is

$$\begin{aligned} D(R) &= 4C(R) + 4\tfrac{K_0(R)}{I_0(R)} \\ &= R^2\left(K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1-\tfrac{I_2(R)}{I_0(R)})\right) + 8RK_0(R)K_1(R) - 4\tfrac{K_0(R)}{I_0(R)} \end{aligned}$$

Using the relations (see [1], 9.6.26)

$$K_2(x) - K_0(x) = \frac{2}{x}K_1(x) \quad \text{and} \quad I_0(x) - I_2(x) = \frac{2}{x}I_1(x)$$

we get

$$D(R) = 6RK_0(R)K_1(R) + (2RK_0(R)I_1(R) - 4)\,\frac{K_0(R)}{I_0(R)}\ . \tag{7.12}$$

which simplifies, using (see [1], 9.6.15)

$$K_1(x)I_0(x) + K_0(x)I_1(x) = \frac{1}{x} \tag{7.13}$$

to (7.11).

We prove that $D(R) > 0$, for all $R > 0$: by (7.11) we get, using again (7.13)

$$\begin{aligned} D(R) &= 2\frac{K_0(R)}{I_0(R)}[RK_1(R)I_0(R) - 1 + RK_1(R)I_0(R)] \\ &= 2\frac{K_0(R)}{I_0(R)}[RK_1(R)I_0(R) - 1 + 1 - RK_0(R)I_1(R)] > 0\ , \end{aligned}$$

since $K_1(x) > K_0(x)$ and $I_0(x) > I_1(x)$, for all $x > 0$.

Next, using the behavior of the Bessel functions (6.14), for $R > 0$ small, we have

$$D(R) \sim -4\log R - 2(-\log R) = -2\log R\ , \quad \text{for} \quad R > 0 \ \ \text{small}\ .$$

For the behavior of $D(R)$ at $+\infty$ we use the asymptotic behavior of the Bessel functions at $+\infty$, see [1], 9.7.1-2:

$$\begin{aligned} I_i(x) &\sim \tfrac{1}{\sqrt{2\pi x}}e^{x}(1-\tfrac{4i^2-1}{8x}) \\ K_i(x) &\sim \tfrac{\pi}{\sqrt{2\pi x}}e^{-x}(1+\tfrac{4i^2-1}{8x}) \end{aligned} \tag{7.14}$$

Hence, we obtain by (7.11)

$$\begin{aligned} D(R) \ &\sim \ 4R\frac{\pi}{\sqrt{2\pi R}}\,e^{-R}\left(1-\frac{1}{8R}\right)\frac{\pi}{\sqrt{2\pi R}}\,e^{-R}\left(1+\frac{3}{8R}\right) \\ &\quad - 2\frac{\pi}{\sqrt{2\pi R}}\,e^{-R}\left(1+\frac{-1}{8R}\right)\sqrt{2\pi R}\,e^{-R}\left(1-\frac{1}{8R}+O(\frac{1}{R^2})\right) \\ &\sim \ 2\pi e^{-2R}\left(1+\frac{1}{4R}\right) - 2\pi e^{-2R}\left(1-\frac{1}{4R}\right) = \frac{\pi}{R}e^{-2R}\ . \end{aligned} \tag{7.15}$$

■

## 8 The Supremum is attained

In this section we show that the supremum

$$\sup_{\|u\|_S \le 1} \int_\Omega (e^{4\pi u^2} - 1)dx$$

is attained for any ball $\Omega = B_R(0)$, as well as for $\Omega = \mathbb{R}^2$.

By Proposition 3.3 it suffices to prove

**Theorem 8.1** *Let* $0 < R \le +\infty$. *Then*

$$\sup_{\|u\|_S \le 1} \pi \int_{\alpha_R}^{\infty} (e^{u^2} - 1)e^{-t}dt \quad > \quad \underset{\|u_n\|_S \le 1}{\text{cc–lim}} \ \pi \int_{\alpha_R}^{\infty} (e^{u_n^2} - 1)e^{-t}dt$$

**Proof.** This follows immediately by Theorem 7.3: Choose an element of the maximizing sequence $\{w_n\}$, with $n$ sufficiently large. Then

$$\sup_{\|u\|_S = 1} \pi \int_{\alpha_R}^{\infty} (e^{u^2} - 1)e^{-t} \ge \pi \int_{\alpha_R}^{\infty} (e^{w_n^2} - 1)e^{-t} > \pi e^{1-D(R)} = \underset{\|u_n\|_S \le 1}{\text{cc–lim}} \int_{\alpha_R}^{\infty} (e^{u_n^2} - 1)dx \ .$$

■

This completes the proof of Theorem 1.3.

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