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Discontinuous Galerkin FEM for Elliptic Problems in Polygonal Domains

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Im Dezember 2002 — Thomas P. Wihler

Abstract

The present work is concerned with the analysis of the Discontinuous Galerkin Finite Element Method (DGFEM) for linear

- diffusion problems,
- elasticity problems,
- Stokes problems.

The corresponding domains are assumed to be possibly non-convex polygons in two space dimensions. As it is well-known from the regularity theory of linear second order elliptic partial differential equations, the solutions of such problems may exhibit singularities in the corners of the polygons as well as in the points where the boundary conditions change (Dirichlet/Neumann). In order to describe this singular behaviour of solutions in a mathematical way, the theory of the so-called weighted Sobolev spaces is applied and extended to the class of problems under consideration.

In contrast to standard (conforming) finite element methods, the analysis of the DGFEM is, due to the occurrence of singularities, accompanied by some additional technical difficulties. In this thesis, these problems are treated with the aid of some newly developed tools. Furthermore, it is explained, how singularities can be resolved numerically by an appropriate choice of the finite element meshes and of the elemental polynomial degrees.

More precisely, for diffusion and elasticity problems it is proved that, in spite of the exact solutions being singular, the use of so-called graded meshes leads to optimal algebraic convergence rates for the h version of the DGFEM (fixed polynomial degree). In addition, the hp DGFEM for diffusion and Stokes problems achieves exponential convergence rates if geometric mesh refinement and judicious polynomial degree distribution strategies are applied.

Finally, it should be mentioned that the low-order h DGFEM for elasticity problems is found to be free of volume locking. This means that, in contrast to conforming finite element formulations in the primal variables, the DGFEM is completely robust with respect to nearly incompressible materials, without resorting to mixed formulations. This remarkable advantage of the DGFEM is a particularity which is very welcome in many practical applications.

Kurzfassung

In der vorliegenden Arbeit wird die Diskontinuierliche-Galerkin-Finit-Element-Methode (DGFEM) für lineare

- Diffusionsprobleme,
- Elastizitätsprobleme,
- Stokesprobleme

untersucht.

Die entsprechenden Gebiete sind allgemeine Polygone in zwei Ortsdimensionen. Wohlbekannte Aussagen der Regularitätstheorie für lineare elliptische partielle Differentialgleichungen zweiter Ordnung implizieren, dass die Lösungen solcher Probleme Singularitäten in den Ecken der Polygone, wie auch in den Punkten mit wechselnden Randbedingungen (Dirichlet/Neumann) aufweisen können. Eine Möglichkeit, dieses singuläre Verhalten von Lösungen mathematisch beschreiben zu können, ist die Theorie der sogenannten gewichteten Sobolevräume, die in dieser Arbeit hinzugezogen und für die vorliegende Problemklasse erweitert wird.

Das Auftreten von Singularitäten in den Lösungen führt bei der Analysis der DGFEM zu zusätzlichen technischen Schwierigkeiten, die bei konformen Finit-Element-Methoden nicht entstehen. Diese Probleme können mit Hilfe von neuen Aussagen, die in dieser Arbeit entwickelt werden, behandelt werden. Ferner ist es möglich, Singularitäten durch eine geeignete Wahl der Gitter und der elementweisen Polynomgrade optimal aufzulösen.

Genauer wird hier bewiesen, dass die h-Version der DGFEM (fester Polynomgrad) für Diffusions- und Elastizitätsprobleme auf sogenannten graduierten Gittern algebraisch optimal konvergiert, auch wenn die exakten Lösungen singulär sind. Ferner führen geometrische Gitterverfeinerungen und spezielle Polynomgradverteilungen zu exponentiellen Konvergenzraten der hp-DGFEM für Diffusions- und Stokesprobleme.

Schlussendlich sollte erwähnt werden, dass die h-DGFEM niedriger Ordnung für Elastizitätsprobleme vom Effekt des „Volume Locking“ nicht betroffen ist.

Dies bedeutet, dass die DGFEM – im Gegensatz zu konformen Finit-Element-Formulierungen in primären Variablen – völlig robust bezüglich (fast) inkompressiblen Materialen ist. Dieser bemerkenswerte Vorteil der DGFEM ist in vielen praktischen Anwendungen sehr gefragt.

Contents

Introduction

In this thesis, the discontinuous Galerkin finite element method (DGFEM) for several elliptic problems in polygons is analyzed. It is subdivided into three parts:

- The DGFEM for Diffusion Problems (Chapter 2).
- Locking-Free h DGFEM for Elasticity Problems (Chapter 3).
- Mixed hp DGFEM for Stokes Problems (Chapter 4).

Additionally, Chapter 1 presents a general setting for the analysis in this work. Finally, a few auxiliary results are collected in the appendix.

The DGFEM for Diffusion Problems. The discontinuous Galerkin FEM (DGFEM) was introduced in [49] for neutron transport problems. A numerical analysis for this case has been given in [42]. Later, DGFEM have been generalized to first order hyperbolic systems and to general convection-diffusion problems as one type of high order finite volume schemes; see e.g. [24, 25] and the recent survey [23]. Furthermore, an error analysis of the DGFEM has become available in [27, 38, 39, 45, 50, 67], for example. In addition, some implementational aspects are presented in [28].

It is noteworthy that discontinuous Galerkin approaches are extremely flexible with respect to mesh-design and the choice of boundary conditions—meshes with hanging nodes, elements of various types and shapes, local spaces of different orders, and even inhomogeneous boundary conditions can be easily dealt with. Therefore, since the eighties, the problem of treating second order elliptic, and especially diffusion problems, within the DGFEM context has attracted considerable interest. Today, it is well-known that there are several possibilities to formulate discontinuous Galerkin schemes for this class of problems: either resorting to an interior penalty method [1, 2, 3, 54, 64], or omitting stabilization completely, [43, 51]. An alternative approach is the use of local discontinuous Galerkin methods [16, 17, 44]. In all these works, error estimates of h- or of

p-type under strong regularity assumptions are given. For example, explicit estimates in h and p, which are based on global H^k regularity of the solutions, for pure diffusion problems were recently obtained in [36, 37, 43].

In polygons, however, singularities in the corners and in the points of changing boundary conditions (Dirichlet/Neumann) may arise. Hence, the solutions of the corresponding problems are typically only in $H^k(\Omega)$ for small $k > 1$, [5, 6]. A possible approach to describe such singular behaviour mathematically is given by the theory of the so-called weighted Sobolev spaces which were originally studied in [4, 5, 6, 31, 32, 33] for elasticity and potential problems. Furthermore, in order to resolve the singularities numerically with the DGFEM, some appropriate mesh refinement and polynomial degree distribution strategies have to be applied. Moreover, the reduced regularity imposes several technical difficulties and requires a careful treatment of the elements and the numerical fluxes near the singular vertices of the domain. In this thesis, it is shown that these problems may be overcome by applying some new trace theorems for functions with singularities.

The aim of Chapter 2 is to prove that optimal algebraic convergence rates for the h DGFEM (Theorem 2.5.7), and exponential convergence rates for the hp DGFEM (Theorem 2.6.6) for diffusion problems in convex as well as non-convex polygons (with mixed boundary conditions) may be achieved, even if the exact solutions are singular. To do so, so-called graded meshes for the h DGFEM and meshes which are geometrically refined towards the singularities for the hp DGFEM are used. In addition, for the hp DGFEM, the elemental polynomial degrees are chosen to be linearly increasing away from the singularities; see, e.g. [58] for the standard (conforming) hp FEM and [66] for the hp DGFEM.

Locking-Free h DGFEM for Elasticity Problems. In mechanical engineering, partial differential equations are often solved by low-order finite element methods. In many applications, the convergence of these schemes may strongly depend on various problem parameters. Unfortunately, this can result in non-robustness of the convergence, i.e. the asymptotic convergence regime of the method is reached only at such high numbers of degrees of freedom that the scheme is practically not feasible. In computational mechanics, this non-robustness of the FEM is termed *locking*.

There exist different kinds of locking: *Shear locking* typically appears if the corresponding domains are very thin and plate and shell theories, which include shear deformation, are used. In addition, in shell theories and their finite element models, there arises *membrane locking* which is caused by the interaction between bending and membrane energies. Finally, problems dealing with nearly incompressible materials are often accompanied by the so-called *volume locking*;

this type of locking is very typical for elasticity problems and will be studied in this thesis.

In order to overcome locking, a wide variety of alternative approaches have been suggested. For example, low-order mixed FEM, where an extra variable for the divergence term is introduced, yield adequate numerical results; cf. [18]. These methods are closely related to under-integration schemes. A further possibility is the use of non-conforming methods, where the global continuity of the numerical solutions is not anymore enforced; see [41], for example.

In 1983, M. Vogelius proved absence of volume locking for the p-version of the FEM on smooth domains [63]. Moreover, in 1992, I. Babuška, M. Suri [8] showed that, on polygonal domains, the h FEM is locking-free on regular triangular elements with $p \geq 4$. In addition, they proved that, for conforming methods, locking cannot be avoided on quadrilateral meshes for any $p \geq 1$. Recently, P. Hansbo and M. G. Larson [35] suggested the use of a DGFEM. Assuming at least H^2 regularity, they showed that the h-version of the DGFEM does not lock for all $p \geq 1$.

Following the classical approach [53, 64], Chapter 3 is devoted to the analysis of the DGFEM for linear elasticity problems in polygons (see also [65]). Based on a recent regularity result [34] it will be proved that, even if the exact solutions of the elasticity problems are singular (i.e. not H^2 anymore), the h-version of the DGFEM is locking-free. Additionally, the use of graded meshes leads to optimal algebraic convergence rates for the DGFEM (independent of the compressibility of the material) as in the diffusion case.

Mixed hp DGFEM for Stokes Problems. In recent years, several mixed DGFEM have been proposed for the discretization of incompressible fluid flow problems, see [9, 20, 21, 30, 35, 40, 62], for example. Some of the main motivations that lead to the above methods are the following: First of all, the discontinuous nature of the finite element spaces makes it possible to easily treat convective terms by suitable upwind fluxes, similarly to the original discontinuous Galerkin discretizations of (non-linear) hyperbolic equations (see [19, 22, 26] and the references therein). Thus, mixed DG methods provide robust and high-order accurate approximations particularly in transport-dominated regimes; see, e.g., [20, 30, 40] for mixed DGFEM for the Navier-Stokes and Oseen equations. Moreover, mixed DG methods are considerably flexible in the choice of velocity-pressure combinations, without extensive stabilization techniques. In the discontinuous Galerkin context, for example, no extra stabilization is required to use optimal mixed-order elements where the approximation degree for the pressure is of one order lower than that of the velocity; see [35, 62] for details.

The recent work in [56] presented a unifying framework for the analysis of mixed hp DGFEM for pure Stokes flow. For $\mathcal{Q}_p - \mathcal{Q}_{p-1}$ elements, the dependence of the discrete inf-sup constant on the polynomial degree p was shown to be of order $\mathcal{O}(1/p)$, for two- and three-dimensional domains. In three dimensions, this is exactly the same bound as that of [60] for conforming mixed hp FEM, however with an optimal gap of one order in the finite element spaces for the velocity and the pressure. The results in [56] then ensure (slightly suboptimal) error bounds for the p-version of the DGFEM where convergence is obtained by increasing the polynomial approximation order on a fixed (quasi-uniform) mesh. However, these bounds give algebraic rates of convergence and are restricted to piecewise smooth solutions; an assumption that is unrealistic in polygons, due to the presence of singularities. For conforming mixed methods, similar p-version results can be found in, e.g., [10, 11, 12, 59, 60] and the references therein.

In this work, the hp-approaches of [56] are extended to mixed hp DGFEM for Stokes flow in non-smooth polygonal domains where the exact solutions are piecewise analytic, however, exhibit singularities at the corners; cf. [57]. As in the elasticity case, the regularity of the exact solutions are described using the recent result from [34]. To prove exponential convergence for the mixed hp DGFEM, several ingredients from the analysis of conforming mixed hp FEM for Stokes flow on geometric meshes are used; see, e.g., [55, 58, 59]. Furthermore, combined with the techniques from Section 2.6, the setting [56] makes it possible to derive the exponential convergence result. Exemplarily, only the interior penalty DGFEM is considered here, however, the results hold true verbatim for all the DG methods studied in [56].

Chapter 1

Preliminaries

The aim of this preparatory part is to establish a functional setting for the analysis of the discontinuous Galerkin finite element methods (DGFEM) in the ensuing chapters. Moreover, a few definitions from the theory of standard (conforming) finite element methods will be recalled.

1.1 Polygonal Domains

Let $\Omega \subset \mathbb{R}^2$ be a bounded, polygonal domain. Suppose that its boundary $\Gamma = \partial\Omega$ is composed of a Dirichlet part Γ_D with $0 < \int_{\Gamma_D} ds < \infty$ and of a Neumann part Γ_N with $0 \leq \int_{\Gamma_N} ds < \infty$ (cf. Figure 1.1):

$$\overline{\Gamma} = \overline{\Gamma}_D \cup \overline{\Gamma}_N.$$

The corner vertices and the points of changing boundary conditions (Dirichlet/ Neumann) of Ω are called '*singular points*'. They are collected in the set

$$SP(\Omega, \Gamma_D, \Gamma_N) = \{A_i : i = 1, 2, \ldots, M\}.$$

Moreover, let $\boldsymbol{n}_\Omega$ be the unit outward vector on $\partial\Omega$.

1.2 Weighted Sobolev Spaces

The regularity of the elliptic problems considered in this work will be measured in terms of certain suitably chosen function spaces. To define them, to each singular point $A_i \in SP(\Omega, \Gamma_D, \Gamma_N)$ a *weight* $\beta_i \in [0, 1)$, $i = 1, 2, \ldots, M$, is associated. These numbers are stored in a *weight vector* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_M)$. Moreover, the

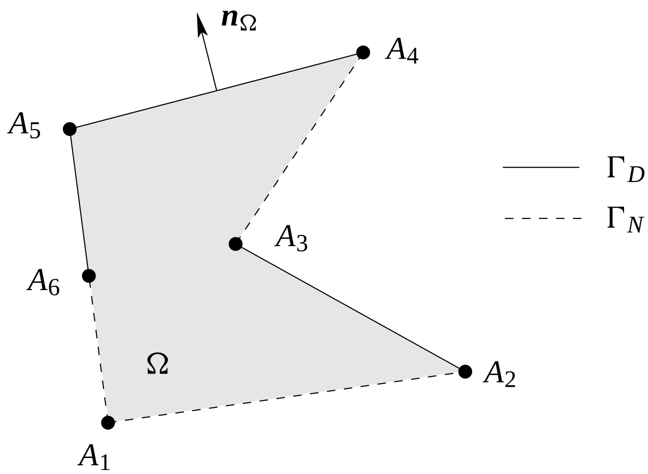

Figure 1.1: Polygon in $\mathbb{R}^2$ with Dirichlet and Neumann boundary conditions.

shorthand notation $C_1 \preccurlyeq \boldsymbol{\beta} \prec C_2$ is used to mean $C_1 \leq \beta_i < C_2$ for all $i = 1, 2, \ldots, M$. Additionally, for any number $k \in \mathbb{R}$, let

$$\boldsymbol{\beta} \pm k = (\beta_1 \pm k, \ldots, \beta_M \pm k).$$

Furthermore, by

$$\Phi_{\boldsymbol{\beta}}(\boldsymbol{x}) = \prod_{i=1}^{M} r_i(\boldsymbol{x})^{\beta_i}, \qquad r_i(\boldsymbol{x}) = |\boldsymbol{x} - A_i|$$

a *weight function* on Ω is introduced.

Then, for any integers $m \geq l \geq 0$, the *weighted Sobolev spaces* $H^{m,l}_{\boldsymbol{\beta}}(\Omega)$ are defined as the completion of the space $\mathcal{C}^\infty(\overline{\Omega})$ with respect to the *weighted Sobolev norms*

$$\|u\|^2_{H^{m,l}_{\boldsymbol{\beta}}(\Omega)} = \|u\|^2_{H^{l-1}(\Omega)} + |u|^2_{H^{m,l}_{\boldsymbol{\beta}}(\Omega)}, \qquad l \geq 1,$$

$$\|u\|^2_{H^{m}_{\boldsymbol{\beta}}(\Omega)} = \sum_{\substack{|\alpha|=k \\ k=0}}^{m} \||D^\alpha u|\Phi_{\boldsymbol{\beta}+k}\|^2_{L^2(\Omega)}, \qquad l = 0.$$

Here,

$$|u|^2_{H^{m,l}_{\boldsymbol{\beta}}(\Omega)} = \sum_{\substack{|\alpha|=k \\ k=l}}^{m} \||D^\alpha u|\Phi_{\boldsymbol{\beta}+k-l}\|^2_{L^2(\Omega)}$$

and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = u_{x_1^{\alpha_1} x_2^{\alpha_2}}$$

with $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $|\alpha| = \alpha_1 + \alpha_2$.

For non-integers $s = m+\theta,\ 0 < \theta < 1$, the weighted Sobolev spaces $H_{\boldsymbol{\beta}}^{s,l}(\Omega)$ are given by the K-method of interpolation:

$$H_{\boldsymbol{\beta}}^{m+\theta,l}(\Omega) = (H_{\boldsymbol{\beta}}^{m,l}(\Omega),\, H_{\boldsymbol{\beta}}^{m+1,l}(\Omega))_{\theta,\infty}.$$

Moreover, for a weight vector $\boldsymbol{\beta} = (\beta_1, \dots \beta_M)$ and $l \in \mathbb{N}$, the countably normed space $\mathcal{B}_{\boldsymbol{\beta}}^l(\Omega)$ consists of all functions u for which $u \in H_{\boldsymbol{\beta}}^{m,l}(\Omega)$ for all $m \geq l$ and

$$\||D^\alpha u|\Phi_{\boldsymbol{\beta}+k-l}\|_{L^2(\Omega)} \leq C d^{(k-l)}(k-l)!, \qquad |\alpha| = k = l, l+1, \dots \tag{1.1}$$

for some constants $C > 0,\ d \geq 1$ independent of k. It may be proved that, for any $u \in \mathcal{B}_{\boldsymbol{\beta}}^l(\Omega)$, $m \geq l$ and $0 < \theta < 1$,

$$\|u\|_{H_{\boldsymbol{\beta}}^{m+\theta,l}(\Omega)} \leq C d^{m+\theta-l}\Gamma(m+\theta-l+1),$$

where $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$ is the Gamma function satisfying $k! = \Gamma(k+1)$ for all $k \in \mathbb{N}_0$.

Finally, let Γ_i, $i = 1, 2, \dots, M$ denote the edges of Ω, and let

$$\mathcal{M} \subset \{1, 2, \dots, M\}$$

be an index set. Then, for

$$\gamma = \bigcup_{j \in \mathcal{M}} \overline{\Gamma}_j \subset \Gamma$$

the spaces $H_{\boldsymbol{\beta}}^{m-1/2,l-1/2}(\gamma)$ and $\mathcal{B}_{\boldsymbol{\beta}}^{l-1/2}(\gamma)$ are the trace spaces of $H_{\boldsymbol{\beta}}^{m,l}(\Omega)$ and $\mathcal{B}_{\boldsymbol{\beta}}^l(\Omega)$, and

$$\|g\|_{H_{\boldsymbol{\beta}}^{m-1/2,l-1/2}(\gamma)} = \inf_{\substack{G|_\gamma = g \\ G \in H_{\boldsymbol{\beta}}^{m,l}(\Omega)}} \|G\|_{H_{\boldsymbol{\beta}}^{m,l}(\Omega)}.$$

Remark 1.2.1 For a function space $X(D)$, where D is a polygonal domain in $\mathbb{R}^2$, and $d \in \mathbb{N}$, $d > 0$, let $X(D)^d$ and $X(D)^{d\times d}$ be the spaces of vector, respectively tensor fields whose components belong to $X(D)$. Without further specification, these spaces are equipped with the usual product norms (which are simply denoted by $\|\cdot\|_{X(D)}$). Furthermore, for $\boldsymbol{v}, \boldsymbol{w} \in X(D)^d$ and $\boldsymbol{\sigma}, \boldsymbol{\tau} \in X(D)^{d\times d}$ define the following scalar products:

$$\boldsymbol{v}\cdot\boldsymbol{w} = \sum_{i=1}^{d} v_i w_i, \qquad \boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i,j=1}^{d} \sigma_{ij}\tau_{ij},$$

and the norms

$$|\boldsymbol{v}| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}, \qquad |\boldsymbol{\sigma}| = \sqrt{\boldsymbol{\sigma} : \boldsymbol{\sigma}}.$$

Remark 1.2.2

a) If $u \in H_{\boldsymbol{\beta}}^{m,m}(\Omega)$, $m \geq 0$, (respectively $u \in \mathcal{B}_{\boldsymbol{\beta}}^2(\Omega)$) then $u \in H^m(\Omega_0)$ (respectively $u \in \mathcal{C}^\infty(\overline{\Omega}_0)$) for all domains $\Omega_0 \subset \Omega$ with

$$P \notin \overline{\Omega}_0 \qquad \forall P \in SP(\Omega, \Gamma_D, \Gamma_N).$$

b) Although $H_{\boldsymbol{\beta}}^{2,2}(\Omega) \not\subset H^2(\Omega)$, $\mathcal{B}_{\boldsymbol{\beta}}^2(\Omega) \not\subset H^2(\Omega)$, it was proved in [7] that $\mathcal{B}_{\boldsymbol{\beta}}^2(\Omega) \subset H_{\boldsymbol{\beta}}^{2,2}(\Omega) \subset \mathcal{C}^0(\overline{\Omega})$.

c) For $u \in H_{\boldsymbol{\beta}}^{2,2}(\Omega)$, there holds $\nabla u \in H_{\boldsymbol{\beta}}^{1,1}(\Omega)^2$.

1.3 Meshes and Trace Operators for the DGFEM

1.3.1 Finite Element Meshes

Consider a partition (FE mesh[1]) $\mathcal{T}$ of Ω into open elements K:

$$\mathcal{T} = \{K_i\}_i, \qquad \bigcup_{K\in\mathcal{T}} \overline{K} = \overline{\Omega}.$$

Henceforth, the $K \in \mathcal{T}$ are assumed to be images of the reference quadrilateral

$$\hat{Q} = (-1,1)^2 \tag{1.2}$$

or of the reference triangle

$$\hat{T} = \{(\hat{x},\hat{y}) : -1 \leq \hat{y} \leq -\hat{x}, \hat{x} \in (-1,1)\} \tag{1.3}$$

under *affine maps* $\boldsymbol{F}_K$, i.e. for all $K \in \mathcal{T}$ there exists a constant matrix $\mathbf{A}_K \in \mathbb{R}^{2\times 2}$ and a constant vector $\boldsymbol{b}_K \in \mathbb{R}^2$ such that with

$$\boldsymbol{F}_K(\boldsymbol{x}) = \mathbf{A}_K \boldsymbol{x} + \boldsymbol{b}_K \tag{1.4}$$

[1]Except for Chapter 3, the FE meshes may be irregular, i.e. hanging nodes are permitted for the DGFEM.

there holds

$$K = \boldsymbol{F}_K(\hat{K}), \tag{1.5}$$

where

$$\hat{K} = \hat{Q} \quad \text{or} \quad \hat{K} = \hat{T}. \tag{1.6}$$

With each element $K \in \mathcal{T}$ a polynomial degree (approximation order) $p_K \geq 1$ is associated. These numbers are stored in a *polynomial degree distribution vector*

$$\boldsymbol{p} = \{p_K : K \in \mathcal{T}\}$$

whose maximum entry is denoted by $|\boldsymbol{p}|$. Additionally, for each $K \in \mathcal{T}$, introduce

$$h_K = \text{diam}(K)$$

and

$$\rho_K = \sup\{\text{diam}(B) : B \text{ is a ball contained in } K\}.$$

Furthermore, let

$$\boldsymbol{h} = \{h_K : K \in \mathcal{T}\}.$$

Finally, the so-called *mesh width* of $\mathcal{T}$ is given by

$$h_{\mathcal{T}} = \sup_{K \in \mathcal{T}} h_K. \tag{1.7}$$

Throughout this work, all FE meshes are assumed to be shape regular and of bounded variation:

Definition 1.3.1 *Let $\mathcal{G} = \{\mathcal{T}_i\}_{i \in \mathbb{N}}$ be a family of FE meshes. Then,*

a) $\mathcal{G}$ is shape regular if there exists a constant $\mu > 0$ independent of i such that for all $i \in \mathbb{N}$

$$\mu \leq \min_{K \in \mathcal{T}_i} \frac{h_K}{\rho_K} \leq \max_{K \in \mathcal{T}_i} \frac{h_K}{\rho_K} \leq \mu^{-1}; \tag{1.8}$$

b) $\mathcal{G}$ is of bounded variation if there exists a constant $\kappa > 0$ independent of i such that for all $i \in \mathbb{N}$

$$\kappa \leq \frac{h_K}{h_{K'}} \leq \kappa^{-1}, \tag{1.9}$$

whenever K and K' share a common edge.

1.3.2 Averages and Jumps

Assume that there exists an index set $\mathcal{I} \subset \mathbb{N}$ such that the elements in the subdivision $\mathcal{T}$ are numbered in a certain way:

$$\mathcal{T} = \{K_i\}_{i \in \mathcal{I}}.$$

Furthermore, denote by $\mathcal{E}$ the set of element edges associated with the mesh $\mathcal{T}$. Since hanging nodes are permitted for the DGFEM, $\mathcal{E}$ will be understood to consist of the smallest edges in $\mathcal{T}$. Additionally, let Γ_{int} be the union of all edges $e \in \mathcal{E}$ not lying on $\partial\Omega$,

$$\Gamma_{\text{int}} = \bigcup_{\substack{e \in \mathcal{E}: \\ e \not\subset \Gamma}} e,$$

and

$$\mathcal{E}_{\text{int}} = \{e \in \mathcal{E} : e \subset \Gamma_{\text{int}}\}.$$

Moreover, define

$$\Gamma_{\text{int},D} = \Gamma_{\text{int}} \cup \bigcup_{\substack{e \in \mathcal{E}: \\ e \subset \Gamma_D}} e$$

and

$$\mathcal{E}_{\text{int},D} = \{e \in \mathcal{E} : e \subset \Gamma_{\text{int},D}\},$$

and for every $K \in \mathcal{T}$ set

$$\mathcal{E}_K = \{e \in \mathcal{E} : e \subset \partial K\}.$$

Obviously, for each $e \in \mathcal{E}_{\text{int}}$, there exist two indices i and j with $i < j$ such that K_i and K_j share the interface e:

$$e = \partial K_i \cap \partial K_j.$$

Thus, the following mapping is well-defined:

$$\begin{array}{rccc} \varphi_{\text{int}} : & \mathcal{E}_{\text{int}} & \longrightarrow & \mathbb{N}^2 \\ & e & \longmapsto & \begin{pmatrix} \varphi_{\text{int},1}(e) = i \\ \varphi_{\text{int},2}(e) = j \end{pmatrix}. \end{array}$$

If $e \in \mathcal{E}$ is a boundary edge, i.e. $e \subset \Gamma$, there is a unique element $K_i \in \mathcal{T}$ such that

$$e = \partial K_i \cap \Gamma.$$

Hence, the above definition may be extended as follows:

$$\begin{array}{rccc} \varphi_\Gamma : & \{e \in \mathcal{E} : e \subset \Gamma\} & \longrightarrow & \mathbb{N} \\ & e & \longmapsto & \varphi_\Gamma(e) = i. \end{array}$$

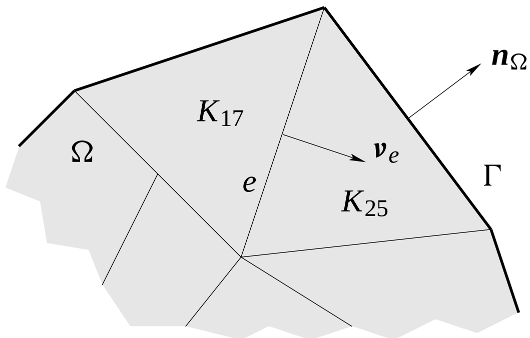

Figure 1.2: Numbering-dependent normal vectors in finite element meshes.

On $e \in \mathcal{E}_{\text{int}}$, let $\boldsymbol{\nu}_e$ be the normal vector which points from $K_{\varphi_1(e)}$ to $K_{\varphi_2(e)}$; for boundary edges $e \subset \Gamma$, set $\boldsymbol{\nu}_e = \boldsymbol{n}_\Omega$ (cf. Figure 1.2).

Since the DGFEM is based on functions in

$$H^{1,1}(\Omega, \mathcal{T}) = \{v \in L^2(\Omega) \,:\, v|_K \in W^{1,1}(K), K \in \mathcal{T}\} \not\subset \mathcal{C}^0(\Omega),$$

the discontinuities over element boundaries have to be controlled in a certain way. Consider therefore $u \in H^{1,1}(\Omega, \mathcal{T})$, $\boldsymbol{v} \in H^{1,1}(\Omega, \mathcal{T})^2$, $\mathbf{W} \in H^{1,1}(\Omega, \mathcal{T})^{2\times 2}$. Then, for $e \in \mathcal{E}_{\text{int}}$ and $\boldsymbol{x} \in e$, introduce the following *averages* at $\boldsymbol{x} \in e$,

$$\langle u \rangle = \frac{1}{2}(u^+ + u^-), \qquad \langle \boldsymbol{v} \rangle = \frac{1}{2}(\boldsymbol{v}^+ + \boldsymbol{v}^-), \qquad \langle \mathbf{W} \rangle = \frac{1}{2}(\mathbf{W}^+ + \mathbf{W}^-),$$

and the (numbering-dependent) *jumps* at $\boldsymbol{x} \in e$,

$$[u] = (u^+ - u^-)\boldsymbol{\nu}_e, \qquad [\boldsymbol{v}] = (\boldsymbol{v}^+ - \boldsymbol{v}^-) \cdot \boldsymbol{\nu}_e,$$
$$\underline{[\boldsymbol{v}]} = (\boldsymbol{v}^+ - \boldsymbol{v}^-) \otimes \boldsymbol{\nu}_e, \qquad [\mathbf{W}] = (\mathbf{W}^+ - \mathbf{W}^-)\boldsymbol{\nu}_e.$$

Here, $\boldsymbol{v} \otimes \boldsymbol{\nu}_e$ is the matrix whose ij-th component is $v_i \nu_{e,j}$, and v^+, v^- denote the traces of v onto e taken from within the interior of the elements $K_{\varphi_1(e)}$ and $K_{\varphi_2(e)}$, respectively (analogous for $\boldsymbol{v}^\pm$ and $\mathbf{W}^\pm$). For $e \subset \Gamma$, define $\langle u \rangle = u$, $\langle \boldsymbol{v} \rangle = \boldsymbol{v}$, $\langle \mathbf{W} \rangle = \mathbf{W}$, as well as $[u] = u\boldsymbol{n}_\Omega$, $[\boldsymbol{v}] = \boldsymbol{v} \cdot \boldsymbol{n}_\Omega$, $\underline{[\boldsymbol{v}]} = \boldsymbol{v} \otimes \boldsymbol{n}_\Omega$, and $[\mathbf{W}] = \mathbf{W}\boldsymbol{n}_\Omega$.

1.3.3 Elements Near Singular Points

In order to account for the singular behaviour of solutions near the singular points of the polygon Ω, the following sets have to be defined:

$$\mathcal{K}_0 = \{K \in \mathcal{T} \,:\, P \in \overline{K} \text{ for some } P \in SP(\Omega, \Gamma_D, \Gamma_N)\} \tag{1.10}$$

and
$$\partial\mathcal{K}_0 = \{e \in \mathcal{E} : \; P \in \overline{e} \text{ for some } P \in SP(\Omega, \Gamma_D, \Gamma_N)\}.$$

Let $K \in \mathcal{K}_0$. Henceforth, it will always be assumed that the finite element meshes are fine enough, i.e. that exactly one singular point belongs to $\overline{K}$. This vertex is denoted by A_K and the corresponding weight by β_K. Moreover, the spaces $H^{m,l}_{\beta_K}(K)$ are given as in Section 1.2, however, equipped with the weight function $\Phi_{\beta_K}(\boldsymbol{x}) = r_K^{\beta_K}$, $r_K = |\boldsymbol{x} - A_K|$.

The ensuing results are essential for the error analysis of the DGFEM.

Lemma 1.3.2 *Let K be an element in $\mathcal{K}_0$. Then, there holds:*

a) $H^{0,0}_{\beta_K}(K) \subset L^1(K)$ *and*

$$\|u\|_{L^1(K)} \leq C h_K^{1-\beta_K} \|u\|_{H^{0,0}_{\beta_K}(K)}$$

for all $u \in H^{0,0}_{\beta_K}(K)$*;*

b) for $u \in H^{0,0}_{\beta_K}(K)$, $v \in L^\infty(K)$ *the integral* $\int_K uv\,d\boldsymbol{x}$ *is well-defined and*

$$\left| \int_K uv\,d\boldsymbol{x} \right| \leq C h_K^{1-\beta_K} \|v\|_{L^\infty(K)} \|u\|_{H^{0,0}_{\beta_K}(K)};$$

c) for all $u \in H^{1,1}_{\beta_K}(K)$ *the trace* $u|_{\partial K}$ *belongs to* $L^1(\partial K)$ *and satisfies*

$$\|u\|_{L^1(\partial K)} \leq C\big(\|u\|_{L^2(K)} + h_K^{1-\beta_K} |u|_{H^{1,1}_{\beta_K}(K)}\big).$$

All the constants $C > 0$ are independent of h_K and of p_K.

Proof: Let $u \in H^{0,0}_{\beta_K}(K)$. Then,

$$\int_K |u|\,d\boldsymbol{x} \leq \|r^{-\beta_K}\|_{L^2(K)} \|r^{\beta_K} u\|_{L^2(K)} = \|r^{-\beta_K}\|_{L^2(K)} \|u\|_{H^{0,0}_{\beta_K}(K)},$$

and since $\|r^{-\beta_K}\|_{L^2(K)} \leq C h_K^{1-\beta_K}$, a) is proved. The second assertion follows then straightforwardly from Hölder's inequality. For the proof of c), let $u \in H^{1,1}_{\beta_K}(K)$. Applying the standard trace theorem and using a scaling argument, yields

$$\|u\|_{L^1(\partial K)} \leq C\big(h_K^{-1} \|u\|_{L^1(K)} + \|\nabla u\|_{L^1(K)}\big).$$

Furthermore, since

$$h_K^{-1} \|u\|_{L^1(K)} \leq C \|u\|_{L^2(K)}$$

and $\nabla u \in H^{0,0}_{\beta_K}(K)^2$, the desired trace estimate follows similarly with the bound

$$\|\nabla u\|_{L^1(K)} \le C h_K^{1-\beta_K} |u|_{H^{1,1}_{\beta_K}(K)},$$

and the proof is complete. □

Remark 1.3.3 The trace of a function $u \in H^{1,1}_{\beta_K}(K)$ on an edge e of an element K is usually not in $L^2(e)$. To see this, consider the following example:

$$u(\boldsymbol{x}) = |\boldsymbol{x}|^{-1/2} \text{ on } K = (0,1)^2, \quad e = \{(x,0) : x \in (0,1)\}.$$

Obviously, $u \in H^{1,1}_{\beta_K}(K)$, $r_K = |\boldsymbol{x}|^{\beta_K}$, for all $\beta_K \in (1/2, 1)$, but $u|_e \notin L^2(e)$.

Lemma 1.3.4 *Let $u \in H^{1,1}_{\boldsymbol{\beta}}(\Omega)$ for a weight vector $0 \preccurlyeq \boldsymbol{\beta} \prec 1$. Then, for an interior edge $e \in \mathcal{E}_{int}$, there holds $[u] = 0$ a.e. on e.*

Proof: Remark 1.2.2 a) implies that $u \in H^1(\Omega_0)$ for all domains $\Omega_0 \subset \Omega$ with $P \notin \overline{\Omega}_0$, $\forall P \in SP(\Omega, \Gamma_D, \Gamma_N)$. Hence, $[u] = 0$ on all edges $e \in \mathcal{E}_{\text{int}}$ not including a singular point (cf. [48], Proposition 3.2.1).

Consider then the case where, for $e \in \mathcal{E}_{\text{int}}$, there exists a singular point $P \in SP(\Omega, \Gamma_D, \Gamma_N)$ with $P \in \overline{e}$. Assume that e is parameterized by $\overline{e} = \boldsymbol{x}(t)$, $t \in [0,1]$, with $\boldsymbol{x}(0) = P$. As above, it follows that $[u] = 0$ away from P, and therefore

$$\int_\varepsilon^1 |[u]||\dot{\boldsymbol{x}}(t)|\, dt = 0$$

for all $\varepsilon > 0$. Since e is a straight line, $|\dot{\boldsymbol{x}}|$ is constant. Thus,

$$\int_\varepsilon^1 |[u]|\, dt = 0.$$

By Lemma 1.3.2 c), $[u] \in L^1(e)$. Therefore, Lebesgue's dominated convergence theorem implies

$$\int_0^1 |[u]|\, dt = 0.$$

This finishes the proof. □

Chapter 2

The DGFEM for Diffusion Problems

2.1 The Diffusion Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded, polygonal domain as introduced in Section 1.1. Consider the following *diffusion problem*

$$\begin{aligned} -\nabla \cdot (\mathbf{A}\nabla u) + cu &= f && \text{in } \Omega \\ u &= g_D && \text{on } \Gamma_D \\ (\mathbf{A}\nabla u) \cdot \boldsymbol{n}_\Omega &= g_N && \text{on } \Gamma_N. \end{aligned} \tag{2.1}$$

Here,

$$\mathbf{A} = \left\{A_{ij}\right\}_{i,j=1}^{2} \in \mathcal{C}^\infty(\overline{\Omega})^{2\times 2}_{\text{sym}} \tag{2.2}$$

is the (symmetric) diffusivity,

$$c \in \mathcal{C}^\infty(\overline{\Omega}) \text{ with } c(\boldsymbol{x}) \geq 0 \quad \forall \boldsymbol{x} \in \overline{\Omega} \tag{2.3}$$

represents the reaction coefficient and $f \in H^{-1}(\Omega)$ is an external source. In addition, $g_D \in H^{1/2}(\Gamma_D)$ and $g_N \in H^{-1/2}(\Gamma_N)$ are prescribed Dirichlet and Neumann boundary conditions.

Henceforth, (2.1) is assumed to be strongly elliptic on $\overline{\Omega}$, i.e. there exist two constants $\overline{a}, \underline{a} > 0$ such that for all $\boldsymbol{x} \in \overline{\Omega}$ there holds

$$\underline{a}|\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi}^\top \mathbf{A}(\boldsymbol{x})\boldsymbol{\xi} \leq \overline{a}|\boldsymbol{\xi}|^2 \tag{2.4}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^2$.

2.2 Regularity

In [5, 6] the following regularity result was proved for the diffusion problem (2.1)–(2.4):

Theorem 2.2.1 (Regularity) *Let Ω be a polygon in $\mathbb{R}^2$ and $m \geq 0$ an integer. Then, there exists a weight vector $\boldsymbol{\beta}_{min}$ with $0 \preccurlyeq \boldsymbol{\beta}_{min} \prec 1$ depending on the diffusivity $\mathbf{A}$ and on the opening angles of Ω at the vertices $A_i \in SP(\Omega, \Gamma_D, \Gamma_N)$, $i = 1, 2, \ldots, M$, such that for weight vectors $\boldsymbol{\beta}$ with $\boldsymbol{\beta}_{min} \preccurlyeq \boldsymbol{\beta} \prec 1$ and for*

$$f \in H^{m,0}_{\boldsymbol{\beta}}(\Omega), \quad g_D \in H^{m+3/2,3/2}_{\boldsymbol{\beta}}(\Gamma_D), \quad g_N \in H^{m+1/2,1/2}_{\boldsymbol{\beta}}(\Gamma_N), \tag{2.5}$$

the diffusion problem (2.1)–(2.4) *has a unique solution $u \in H^{m+2,2}_{\boldsymbol{\beta}}(\Omega)$.*

Moreover, for piecewise analytic data

$$f \in \mathcal{B}^0_{\boldsymbol{\beta}}(\Omega), \quad g_D \in \mathcal{B}^{3/2}_{\boldsymbol{\beta}}(\Gamma_D), \quad g_N \in \mathcal{B}^{1/2}_{\boldsymbol{\beta}}(\Gamma_N), \tag{2.6}$$

the solution of (2.1)–(2.4) *belongs to $\mathcal{B}^2_{\boldsymbol{\beta}}(\Omega)$.*

2.3 Discontinuous Galerkin Discretization

2.3.1 Finite Element Spaces

In order to define an appropriate class of finite element spaces for the DGFEM, polynomial spaces have to be introduced. To do so, let $p \in \mathbb{N}$ be arbitrary and denote by K either a quadrilateral or a triangle. Then,

$$\mathcal{P}_p(K) = \Big\{ u = \sum_{\substack{0 \leq i,j \leq p \\ i+j \leq p}} c_{ij} x_1^i x_2^j \,:\, c_{ij} \in \mathbb{R},\ (x_1, x_2) \in K \Big\}$$

is the space of polynomials of *total degree* at most p on K, and

$$\mathcal{Q}_p(K) = \Big\{ u = \sum_{0 \leq i,j \leq p} c_{ij} x_1^i x_2^j \,:\, c_{ij} \in \mathbb{R},\ (x_1, x_2) \in K \Big\}$$

is the space of polynomials of degree at most p in *each variable* on K.

With these definitions, the (discontinuous) finite element spaces that will be used for the DGFEM are given by

$$V_h = \{ u \in L^2(\Omega) :\, u|_K \in \mathcal{V}_{p_K},\, K \in \mathcal{T} \}, \tag{2.7}$$

where $\mathcal{T}$ is a FE mesh on Ω, and $\mathcal{V}_{p_K} = \mathcal{P}_{p_K}(K)$ or $\mathcal{V}_{p_K} = \mathcal{Q}_{p_K}(K)$, $K \in \mathcal{T}$.

2.3.2 Variational Formulation

There is a wide variety of discontinuous Galerkin formulations for elliptic (and especially diffusion) problems. Most of them are closely related to each other (see [3, 46], for example). Following the approaches of Arnold [1], Rivière [52], Rivière, Wheeler, Girault [54], Süli, Schwab, Houston [61] and Wheeler [64], the symmetric (SIPG) and non-symmetric (NIPG) interior penalty Galerkin method will be considered here. For convenience, the non-symmetric formulation will be denoted by ’$+$’ and the symmetric formulation by ’$-$’, respectively.

Definition 2.3.1 (DGFEM) *Define two bilinear forms B_h^+ and B_h^- by*

$$\begin{aligned} B_h^{\pm}(u,v) &= \sum_{K\in\mathcal{T}} \int_K (\nabla v\cdot(\mathbf{A}\nabla u) + cuv)\,d\boldsymbol{x} \\ &\quad - \sum_{e\in\mathcal{E}_{int,D}} \int_e (\langle \mathbf{A}\nabla u\rangle\cdot[v] \mp [u]\cdot\langle\mathbf{A}\nabla v\rangle)\,ds \\ &\quad + \sum_{e\in\mathcal{E}_{int,D}} \int_e \mathsf{d}\,[u]\cdot[v]\,ds, \end{aligned}$$

and two corresponding linear functionals $L_h^{\pm}$ by

$$\begin{aligned} L_h^{\pm}(v) &= \sum_{K\in\mathcal{T}} \int_K f v\,d\boldsymbol{x} + \int_{\Gamma_N} g_N v\,ds \\ &\quad \pm \int_{\Gamma_D} ((\mathbf{A}\nabla v)\cdot\boldsymbol{n}_\Omega) g_D\,ds + \int_{\Gamma_D} \mathsf{d}\,g_D v\,ds. \end{aligned}$$

Here, $\mathsf{d}\in L^\infty(\mathcal{E}_{int,D})$ is the so-called discontinuity stabilization function which is given by

$$\mathsf{d} = \omega\frac{\mathsf{p}^2}{\mathsf{h}}, \tag{2.8}$$

where $\omega\in\mathbb{R}$, $\omega>0$ is a constant to be specified later and, for $e\in\mathcal{E}_{int,D}$,

$$\begin{aligned} \mathsf{p}|_e &= \begin{cases} \max\left\{p_{K_{\varphi_{int,1}(e)}}, p_{K_{\varphi_{int,2}(e)}}\right\}, & e\in\mathcal{E}_{int}, \\ p_{K_{\varphi_\Gamma(e)}}, & e\subset\Gamma_D, \end{cases} \\ \mathsf{h}|_e &= \begin{cases} \min\left\{h_{K_{\varphi_{int,1}(e)}}, h_{K_{\varphi_{int,2}(e)}}\right\}, & e\in\mathcal{E}_{int}, \\ h_{K_{\varphi_\Gamma(e)}}, & e\subset\Gamma_D. \end{cases} \end{aligned} \tag{2.9}$$

Then, the DGFEM for the diffusion problem (2.1)–(2.4) *reads as follows:*

Find $u_h^{\pm}\in V_h$ such that

$$B_h^{\pm}(u_h^{\pm},v) = L_h^{\pm}(v) \tag{2.10}$$

for all $v\in V_h$.

Proposition 2.3.2 (Consistency) *Let Ω be a polygon and $\boldsymbol{\beta}_{\min} \preccurlyeq \boldsymbol{\beta} \prec 1$ a weight vector. Then, for $f \in H^{0,0}_{\boldsymbol{\beta}}(\Omega)$, $g_D \in H^{3/2,3/2}_{\boldsymbol{\beta}}(\Gamma_D)$, $g_N \in H^{1/2,1/2}_{\boldsymbol{\beta}}(\Gamma_N)$, the bilinear forms and the linear functionals in Definition 2.3.1 are well-defined and, moreover, the DGFEM* (2.10) *is consistent:*

$$B_h^{\pm}(u,v) - L_h^{\pm}(v) = 0 \qquad \forall v \in V_h. \tag{2.11}$$

Here, u is the exact solution of the diffusion problem (2.1)–(2.4).

Proof: Due to Lemma 1.3.2, all terms in Definition 2.3.1 are well-defined. Furthermore, Theorem 2.2.1 implies that $u \in H^{2,2}_{\boldsymbol{\beta}}(\Omega)$. Thus, by Remark 1.2.2 b), u is continuous on $\overline{\Omega}$. Consequently,

$$[u]|_e = \mathbf{0} \quad \forall e \in \mathcal{E}_{\text{int}} \qquad \text{and} \qquad [u]|_e = g_D \boldsymbol{n}_\Omega \quad \forall e \in \mathcal{E}_{\text{int},D} \setminus \mathcal{E}_{\text{int}}.$$

In addition, using that, for all $e \in \mathcal{E} \setminus \mathcal{E}_{\text{int}}$,

$$[v] = v\boldsymbol{n}_\Omega \quad \text{and} \quad \boldsymbol{\nu}_e = \boldsymbol{n}_\Omega \quad \text{and} \quad \langle \mathbf{A}\nabla v\rangle = \mathbf{A}\nabla v,$$

reduces the left hand-side of (2.11) to

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \int_K \big(\nabla v \cdot (\mathbf{A}\nabla u) + cuv\big)\, d\boldsymbol{x} - \sum_{K\in\mathcal{T}} \int_K f v \, d\boldsymbol{x} \\ &\quad - \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e \langle \mathbf{A}\nabla u\rangle \cdot [v]\, ds - \int_{\Gamma_N} g_N v\, ds \end{aligned} \tag{2.12}$$

Since $\mathbf{A}\nabla u \in H^{1,1}_{\boldsymbol{\beta}}(\Omega)^2$, the integrals in the first sum of (2.12) may be integrated by parts (Lemma A.2.1). This yields

$$\begin{aligned} B_h^{\pm}(u,v) - L_h^{\pm}(v) &= \sum_{K\in\mathcal{T}} \int_{\partial K} v(\mathbf{A}\nabla u)\cdot \boldsymbol{n}_K\, ds - \int_{\Gamma_N} g_N v\, ds \\ &\quad - \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e \langle \mathbf{A}\nabla u\rangle \cdot [v]\, ds, \end{aligned}$$

where $\boldsymbol{n}_K$ is the unit outward vector on ∂K, $K \in \mathcal{T}$. Moreover, using that

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \int_{\partial K} v(\mathbf{A}\nabla u)\cdot \boldsymbol{n}_K\, ds \\ &= \sum_{e\in\mathcal{E}} \int_e [v(\mathbf{A}\nabla u)]\, ds \\ &= \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e [v]\cdot\langle \mathbf{A}\nabla u\rangle\; ds \\ &\quad + \sum_{e\in\mathcal{E}_{\text{int}}} \int_e \langle v\rangle\, [\mathbf{A}\nabla u]\, ds + \int_{\Gamma_N} g_N v\, ds \end{aligned}$$

implies that

$$B_h^{\pm}(u,v) - L^{\pm}(v) = \sum_{e\in\mathcal{E}_{\text{int}}} \int_e \langle v\rangle \, [\mathbf{A}\nabla u] \, ds.$$

Finally, applying Lemma 1.3.4 completes the proof. □

The following 'energy' norm will be associated with the DGFEM:

$$\begin{aligned}\|u\|_h^2 &= B_h^{+}(u,u)\\ &= \sum_{K\in\mathcal{T}} \left(\|\sqrt{\mathbf{A}}\nabla u\|_{L^2(K)}^2 + \|\sqrt{c}u\|_{L^2(K)}^2\right) + \sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[u]\|_{L^2(e)}^2.\end{aligned}$$

Remark 2.3.3 From Proposition A.2.13 it directly follows that

$$\|u\|_{L^2(\Omega)}^2 \leq C\left(\sum_{K\in\mathcal{T}} \|\sqrt{\mathbf{A}}\nabla u\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[u]\|_{L^2(e)}^2\right) \tag{2.13}$$

for a constant $C > 0$ which is independent of $\boldsymbol{h}$ and of $\boldsymbol{p}$. Therefore, $\|\cdot\|_h$ is a norm on V_h even if $c \equiv 0$.

2.3.3 Coercivity, Continuity, Existence and Uniqueness

In this section, some basic properties of the DGFEM defined above will be explored. To do so, the ensuing auxiliary result is required.

Lemma 2.3.4 *Let $u \in V_h$ for a given polynomial degree distribution $\boldsymbol{p}$ and a finite element mesh $\mathcal{T}$. Furthermore, let d be the discontinuity stabilization function from Definition 2.3.1 with $\omega > 0$, $\omega \in \mathbb{R}$. Then, there holds*

$$\sum_{e\in\mathcal{E}_{int,D}} \left\| \frac{1}{\sqrt{\mathsf{d}}} \langle \mathbf{A}\nabla u\rangle \right\|_{L^2(e)}^2 \leq C_{diff}\,\omega^{-1} \sum_{K\in\mathcal{T}} \|\sqrt{\mathbf{A}}\nabla u\|_{L^2(K)}^2. \tag{2.14}$$

Here, $\mathbf{A}$ is the diffusivity from (2.2) and $C_{diff} > 0$ is a constant independent of u, $\boldsymbol{p}$ and of $\boldsymbol{h}$.

Proof: The left-hand side of (2.14) is bounded as follows:

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_{\mathrm{int},D}} \left\| \frac{1}{\sqrt{\mathsf{d}}} \langle \mathbf{A}\nabla u\rangle \right\|_{L^2(e)}^2 \\
&\leq C \sum_{e\in\mathcal{E}_{\mathrm{int},D}} \left\| \frac{1}{\sqrt{\mathsf{d}}} \langle |\nabla u|\rangle \right\|_{L^2(e)}^2 \\
&\leq C\Big\{ \sum_{e\in\mathcal{E}_{\mathrm{int}}} \left(\left\| \frac{1}{\sqrt{\mathsf{d}}}\nabla u^+ \right\|_{L^2(e)}^2 + \left\| \frac{1}{\sqrt{\mathsf{d}}}\nabla u^- \right\|_{L^2(e)}^2 \right) \\
&\qquad + \sum_{\substack{e\in\mathcal{E}:\\ e\subset\Gamma_D}} \left\| \frac{1}{\sqrt{\mathsf{d}}}\nabla u \right\|_{L^2(e)}^2 \Big\} \qquad (2.15)\\
&\leq C \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_{\mathrm{int},D}:\\ e\subset\partial K}} \left\| \frac{1}{\sqrt{\mathsf{d}}}\nabla u \right\|_{L^2(e)}^2 \\
&\leq C\omega^{-1} \sum_{K\in\mathcal{T}} p_K^{-2} h_K \sum_{\substack{e\in\mathcal{E}_{\mathrm{int},D}:\\ e\subset\partial K}} \|\nabla u\|_{L^2(e)}^2 .
\end{aligned}
$$

The polynomial trace inequality (A.1) implies that

$$
\|\nabla u\|_{L^2(e)}^2 \leq C \frac{p_K^2}{h_K} \|\nabla u\|_{L^2(K)}^2
$$

for all $e \in \mathcal{E}_K$, $K \in \mathcal{T}$. Therefore,

$$
\sum_{e\in\mathcal{E}_{\mathrm{int},D}} \left\| \frac{1}{\sqrt{\mathsf{d}}} \langle \mathbf{A}\nabla u\rangle \right\|_{L^2(e)}^2 \leq C\omega^{-1} \sum_{K\in\mathcal{T}} \|\nabla u\|_{L^2(K)}^2 \sum_{\substack{e\in\mathcal{E}_{\mathrm{int},D}:\\ e\subset\partial K}} 1
$$

From (1.9) it follows that the second sum on the right-hand side is bounded for all $K \in \mathcal{T}$. Finally, referring to (2.4) completes the proof. □

Continuity of $B_h^{\pm}$

Proposition 2.3.5 *For $\omega > 0$ and d as in (2.8), the bilinear forms $B_h^{\pm}$ are continuous with respect to the $\|\cdot\|_h$-norm, i.e.*

$$
|B_h^{\pm}(u, v)| \leq \max\{2, 1 + C_{diff}\omega^{-1}\}\|u\|_h\|v\|_h
$$

for all $u, v \in V_h$. Here, C_{diff} is the constant from Lemma 2.3.4.

Proof: Let $u, v \in V_h$. Then, Hölder's inequality implies

$$
\begin{aligned}
|B_h^{\pm}(u,v)| \le & \sum_{K\in\mathcal{T}} \int_K (|\sqrt{\mathbf{A}}\nabla u||\sqrt{\mathbf{A}}\nabla v| + |\sqrt{c}u||\sqrt{c}v|)\,d\boldsymbol{x} \\
& + \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e (|[u]||\langle \mathbf{A}\nabla v\rangle| + |\langle \mathbf{A}\nabla u\rangle||[v]|)\,ds \\
& + \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e |\sqrt{\mathsf{d}}[u]||\sqrt{\mathsf{d}}[v]|\,ds \\
\le & \sum_{K\in\mathcal{T}} \left(\|\sqrt{\mathbf{A}}\nabla u\|_{L^2(K)}\|\sqrt{\mathbf{A}}\nabla v\|_{L^2(K)} + \|\sqrt{c}u\|_{L^2(K)}\|\sqrt{c}v\|_{L^2(K)}\right) \\
& + \sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[u]\|_{L^2(e)} \left\|\frac{1}{\sqrt{\mathsf{d}}}\langle \mathbf{A}\nabla v\rangle\right\|_{L^2(e)} \\
& + \sum_{e\in\mathcal{E}_{\text{int},D}} \left\|\frac{1}{\sqrt{\mathsf{d}}}\langle \mathbf{A}\nabla u\rangle\right\|_{L^2(e)} \|\sqrt{\mathsf{d}}[v]\|_{L^2(e)} \\
& + \sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[u]\|_{L^2(e)}\|\sqrt{\mathsf{d}}[v]\|_{L^2(e)}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|B_h^{\pm}(u,v)| \le & \Bigg\{ \sum_{K\in\mathcal{T}} \left(\|\sqrt{\mathbf{A}}\nabla u\|_{L^2(K)}^2 + \|\sqrt{c}u\|_{L^2(K)}^2\right) \\
& + \sum_{e\in\mathcal{E}_{\text{int},D}} \left(\left\|\frac{1}{\sqrt{\mathsf{d}}}\langle \mathbf{A}\nabla u\rangle\right\|_{L^2(e)}^2 + 2\|\sqrt{\mathsf{d}}[u]\|_{L^2(e)}^2\right)\Bigg\}^{1/2} \\
& \cdot \Bigg\{ \sum_{K\in\mathcal{T}} \left(\|\sqrt{\mathbf{A}}\nabla v\|_{L^2(K)}^2 + \|\sqrt{c}v\|_{L^2(K)}^2\right) \\
& + \sum_{e\in\mathcal{E}_{\text{int},D}} \left(\left\|\frac{1}{\sqrt{\mathsf{d}}}\langle \mathbf{A}\nabla v\rangle\right\|_{L^2(e)}^2 + 2\|\sqrt{\mathsf{d}}[v]\|_{L^2(e)}^2\right)\Bigg\}^{1/2}.
\end{aligned}
$$

Using (2.14) yields

$$
\begin{aligned}
|B_h^{\pm}(u,v)| \le & \Bigg(\sum_{K\in\mathcal{T}} \left((1 + C_{\text{diff}}\omega^{-1})\|\sqrt{\mathbf{A}}\nabla u\|_{L^2(K)}^2 + \|\sqrt{c}u\|_{L^2(K)}^2\right) \\
& + 2\sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[u]\|_{L^2(e)}^2\Bigg)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \Big(\sum_{K\in\mathcal{T}} \big((1 + C_{\text{diff}}\omega^{-1})\|\sqrt{\mathbf{A}}\nabla v\|^2_{L^2(K)} + \|\sqrt{c}v\|^2_{L^2(K)}\big) \\
&+ 2 \sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[v]\|^2_{L^2(e)}\Big)^{1/2} \\
\leq{}& \max\{2, 1 + C_{\text{diff}}\omega^{-1}\}\|u\|_h\|v\|_h.
\end{aligned}
$$

□

Coercivity of $B_h^\pm$

From the definition of the $\|\cdot\|_h$-norm it is already clear that B_h^+ is coercive (with respect to the $\|\cdot\|_h$-norm) with coercivity constant 1. The coercivity of B_h^- will be proved in the following

Proposition 2.3.6 *Let $\omega > C_{diff}$, where C_{diff} is the constant from Lemma 2.3.4, and d as in* (2.8)*. Then, for all $u \in V_h$, the ensuing estimate holds true:*

$$
B_h^-(u,u) \geq (1 - \sqrt{C_{diff}\omega^{-1}})\|u\|_h^2.
$$

Proof: The definitions of B_h^- and $\|\cdot\|_h$ imply

$$
\begin{aligned}
B_h^-(u,u) &= \|u\|_h^2 - 2\sum_{e\in\mathcal{E}_{\text{int},D}} \int_e \langle \mathbf{A}\nabla u\rangle \cdot [u]\, ds \\
&\geq \|u\|_h^2 - \sqrt{\frac{\omega}{C_{\text{diff}}}} \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e \frac{1}{\mathsf{d}} |\langle \mathbf{A}\nabla u\rangle|^2\, ds \\
&\quad - \sqrt{\frac{C_{\text{diff}}}{\omega}} \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e \mathsf{d}|[u]|^2\, ds
\end{aligned}
$$

Inserting (2.14) yields

$$
\begin{aligned}
B_h^-(u,u) &\geq \|u\|_h^2 - \sqrt{\frac{C_{\text{diff}}}{\omega}} \sum_{K\in\mathcal{T}} \|\sqrt{\mathbf{A}}\nabla u\|^2_{L^2(K)} \\
&\quad - \sqrt{\frac{C_{\text{diff}}}{\omega}} \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e \mathsf{d}|[u]|^2\, ds \\
&\geq (1 - \sqrt{C_{\text{diff}}\omega^{-1}})\|u\|_h^2.
\end{aligned}
$$

□

Existence and Uniqueness

Theorem 2.3.7 *Let* (2.5) *be satisfied for (at least)* $m = 0$*. Furthermore, let* $\omega > 0$ *in the '+'-formulation and* $\omega > C_{diff}$ *in the '−'-formulation of the DGFEM, where* C_{diff} *is the constant from Lemma 2.3.4. Then, under the assumptions* (2.2)–(2.4)*, and with* d *as in* (2.8)*, the DGFEM* (2.10) *have unique solutions* $u_h^{\pm} \in V_h$*.*

Proof: The Cauchy-Schwarz inequality and the broken Poincaré estimate (2.13) imply the continuity of the linear functionals $L_h^{\pm}$ with respect to the $\|\cdot\|_h$-norm (see also [44, Proposition 3.2]). Moreover, due to the continuity and the coercivity of the bilinear forms $B_h^{\pm}$, the Lax-Milgram Lemma is applicable and the proof is complete. □

2.4 Stability

Let $\pi_{\boldsymbol{p}} u \in V_h$ be an arbitrary interpolant of the exact solution u of the diffusion problem (2.1)–(2.4). Furthermore, let $u_h^{\pm}$ be the solutions of the DGFEM (2.10). Then, the finite element errors $e_h^{\pm}$ are decomposed in the following way:

$$e_h^{\pm} = u - u_h^{\pm} = \underbrace{u - \pi_{\boldsymbol{p}} u}_{=\eta} + \underbrace{\pi_{\boldsymbol{p}} u - u_h^{\pm}}_{=\xi^{\pm}}. \tag{2.16}$$

As the ensuing Proposition 2.4.1 shows, $\xi^{\pm}$ may be bounded by η.

Proposition 2.4.1 (Stability) *Let* ω *be as in Theorem 2.3.7 and* $u \in H^{2,2}_{\boldsymbol{\beta}}(\Omega)$ *for a weight vector* $0 \preccurlyeq \boldsymbol{\beta} \prec 1$*. Additionally, assume that* $\eta = 0$ *in all element vertices of* $\mathcal{T}$*. Then, the a-priori estimate*

$$\|\xi^{\pm}\|_h^2 \le C C^{\pm} |\boldsymbol{p}|^2 (\omega + \omega^{-1})(E_1 + E_2 + E_3)$$

holds true, where

$$E_1 = \sum_{K \in \mathcal{T}} \|\eta\|^2_{H^1(K)}$$
$$E_2 = \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 |\eta|^2_{H^2(K)}$$
$$E_3 = \sum_{K \in \mathcal{K}_0} h_K^{2(1-\beta_K)} |\eta|^2_{H^{2,2}_{\beta_K}(K)},$$

and

$$C^{\pm} = \begin{cases} 1 & \text{for the '+'-formulation} \\ (1 - \sqrt{C_{diff}\omega^{-1}})^{-2} & \text{for the '−'-formulation} \end{cases}.$$

$C > 0$ *is a constant independent of* u*,* ω*,* $\boldsymbol{p}$ *and of* $\boldsymbol{h}$*.*

Proof: The consistency of the DGFEM (2.10) (Proposition 2.3.2) implies

$$B_h^{\pm}(\xi^{\pm},\xi^{\pm}) = B_h^{\pm}(u - u_h^{\pm} - \eta, \xi^{\pm}) = -B_h^{\pm}(\eta, \xi^{\pm}),$$

and hence, due to the coercivity of the bilinear forms $B_h^{\pm}$, it holds that

$$\|\xi^{\pm}\|_h^2 \leq \sqrt{C^{\pm}}|B_h^{\pm}(\eta, \xi^{\pm})|. \tag{2.17}$$

The right-hand side of (2.17) is bounded as follows:

$$|B_h^{\pm}(\eta, \xi^{\pm})| \leq I + IIa + IIb + III,$$

where

$$\begin{aligned}
I &= \sum_{K\in\mathcal{T}} \left(\|\sqrt{\mathbf{A}}\nabla\eta\|_{L^2(K)}\|\sqrt{\mathbf{A}}\nabla\xi^{\pm}\|_{L^2(K)} + \|\sqrt{c}\eta\|_{L^2(K)}\|\sqrt{c}\xi^{\pm}\|_{L^2(K)}\right),\\
IIa &= \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e |\langle \mathbf{A}\nabla\eta\rangle||[\xi^{\pm}]|\,ds,\\
IIb &= \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e |[\eta]||\left\langle \mathbf{A}\nabla\xi^{\pm}\right\rangle|\,ds,\\
III &= \sum_{e\in\mathcal{E}_{\text{int},D}} \int_e \mathsf{d}|[\eta]||[\xi^{\pm}]|\,ds.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&I + III\\
&\leq \left(\sum_{K\in\mathcal{T}} \left(\|\sqrt{\mathbf{A}}\nabla\eta\|_{L^2(K)}^2 + \|\sqrt{c}\eta\|_{L^2(K)}^2\right) + \sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[\eta]\|_{L^2(e)}^2\right)^{1/2}\\
&\quad\cdot\left(\sum_{K\in\mathcal{T}} \left(\|\sqrt{\mathbf{A}}\nabla\xi^{\pm}\|_{L^2(K)}^2 + \|\sqrt{c}\xi^{\pm}\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[\xi^{\pm}]\|_{L^2(e)}^2\right)^{1/2}\\
&\leq C\|\xi^{\pm}\|_h\left(\sum_{K\in\mathcal{T}} \|\eta\|_{H^1(K)}^2 + \sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[\eta]\|_{L^2(e)}^2\right)^{1/2}\\
&\leq C\|\xi^{\pm}\|_h\left(E_1 + \sum_{e\in\mathcal{E}_{\text{int},D}} \|\sqrt{\mathsf{d}}[\eta]\|_{L^2(e)}^2\right)^{1/2}.
\end{aligned}$$

Furthermore, by Hölder's inequality and Lemma A.1.1, it follows that

$$\begin{aligned}
IIa &\le C \sum_{e\in\mathcal{E}_{\mathrm{int},D}} \|[\xi^\pm]\|_{L^\infty(e)} \int_e \langle|\nabla\eta|\rangle\, ds \\
&\le C\omega^{-1/2} \sum_{e\in\mathcal{E}_{\mathrm{int},D}} \|\sqrt{\mathsf{d}}[\xi^\pm]\|_{L^2(e)} \int_e \langle|\nabla\eta|\rangle\, ds \\
&\le C\omega^{-1/2} \Big\{ \sum_{e\in\mathcal{E}_{\mathrm{int},D}} \|\sqrt{\mathsf{d}}[\xi^\pm]\|^2_{L^2(e)} \Big\}^{1/2} \Big\{ \sum_{e\in\mathcal{E}_{\mathrm{int},D}} \Big(\int_e \langle|\nabla\eta|\rangle\, ds \Big)^2 \Big\}^{1/2} \\
&\le C\omega^{-1/2} \|\xi^\pm\|_h \Big(\sum_{e\in\mathcal{E}_{\mathrm{int},D}} \|\,\langle|\nabla\eta|\rangle\,\|^2_{L^1(e)} \Big)^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{e\in\mathcal{E}_{\mathrm{int},D}} \|\,\langle|\nabla\eta|\rangle\,\|^2_{L^1(e)} &\le C \Big(\sum_{e\in\mathcal{E}_{\mathrm{int}}} \big(\|\nabla\eta^+\|^2_{L^1(e)} + \|\nabla\eta^-\|^2_{L^1(e)} \big) \\
&\qquad + \sum_{\substack{e\in\mathcal{E}:\\ e\subset\Gamma_D}} \|\nabla\eta\|^2_{L^1(e)} \Big) \\
&\le C \sum_{K\in\mathcal{T}} \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int},D}} \|\nabla\eta\|^2_{L^1(e)}.
\end{aligned}$$

Referring to Remark 1.2.2 and applying Lemma 1.3.2 c) results in

$$\begin{aligned}
\sum_{K\in\mathcal{T}} \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int},D}} \|\nabla\eta\|^2_{L^1(e)} &\le C \Big(\sum_{K\in\mathcal{T}} \|\nabla\eta\|_{L^2(K)} + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2 \|\nabla\eta\|^2_{H^1(K)} \\
&\qquad + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K} \|\nabla\eta\|^2_{H^{1,1}_{\beta_K}(K)} \Big) \\
&\le C(E_1 + E_2 + E_3).
\end{aligned}$$

Hence,

$$IIa \le C\omega^{-1/2} \|\xi^\pm\|_h \sqrt{E_1 + E_2 + E_3}.$$

An estimate for IIb is obtained by using Lemma 2.3.4:

$$\begin{aligned}
|IIb| &\le \Big(\sum_{e\in\mathcal{E}_{\mathrm{int},D}} \|\sqrt{\mathsf{d}}[\eta]\|^2_{L^2(e)} \Big)^{1/2} \Big(\sum_{e\in\mathcal{E}_{\mathrm{int},D}} \Big\| \frac{1}{\sqrt{\mathsf{d}}} \langle \mathbf{A}\nabla\xi^\pm \rangle \Big\|^2_{L^2(e)} \Big)^{1/2} \\
&\le C\omega^{-1/2} \Big(\sum_{e\in\mathcal{E}_{\mathrm{int},D}} \|\sqrt{\mathsf{d}}[\eta]\|^2_{L^2(e)} \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}} \|\sqrt{\mathbf{A}}\nabla\xi^\pm\|^2_{L^2(K)} \Big)^{1/2} \\
&\le C\omega^{-1/2} \|\xi^\pm\|_h \Big(\sum_{e\in\mathcal{E}_{\mathrm{int},D}} \|\sqrt{\mathsf{d}}[\eta]\|^2_{L^2(e)} \Big)^{1/2}.
\end{aligned}$$

Summing up all the bounds above leads to

$$|B_h^{\pm}(\eta,\xi^{\pm})| \le C\|\xi^{\pm}\|_h\Big\{\omega^{-1}(E_2+E_3) + (1+\omega^{-1})\Big(E_1+\sum_{e\in\mathcal{E}_{\mathrm{int},D}}\|\sqrt{\mathsf{d}}[\eta]\|^2_{L^2(e)}\Big)\Big\}^{1/2}.$$

It remains to estimate the last sum in the above inequality.

$$\begin{aligned}\sum_{e\in\mathcal{E}_{\mathrm{int},D}}\|\sqrt{\mathsf{d}}[\eta]\|^2_{L^2(e)} &\le C\sum_{e\in\mathcal{E}_{\mathrm{int}}}\big(\|\sqrt{\mathsf{d}}\eta^+\|^2_{L^2(e)}+\|\sqrt{\mathsf{d}}\eta^-\|^2_{L^2(e)}\big) + \sum_{\substack{e\in\mathcal{E}:\\ e\subset\Gamma_D}}\|\sqrt{\mathsf{d}}\eta\|^2_{L^2(e)}\\ &\le C\sum_{K\in\mathcal{T}}\sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int},D}}\|\sqrt{\mathsf{d}}\eta\|^2_{L^2(e)}\\ &\le C\omega|\boldsymbol{p}|^2\sum_{K\in\mathcal{T}}\sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int},D}}\Big\|\frac{1}{\sqrt{\mathsf{h}}}\eta\Big\|^2_{L^2(e)}.\end{aligned}$$

The trace inequalities from Lemma A.2.4 and the bounded variation property (1.9) imply that

$$\begin{aligned}&\sum_{K\in\mathcal{T}}\sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int},D}}\Big\|\frac{1}{\sqrt{\mathsf{h}}}\eta\Big\|^2_{L^2(e)}\\ &\qquad\le C\Big(\sum_{K\in\mathcal{T}\setminus\mathcal{K}_0}h_K^2|\eta|^2_{H^2(K)}+\sum_{K\in\mathcal{K}_0}h_K^{2-2\beta_K}|\eta|^2_{H^{2,2}_{\beta_K}(K)}\Big)\\ &\qquad\le C(E_2+E_3).\end{aligned}$$

Therefore,

$$|B_h^{\pm}(\eta,\xi^{\pm})| \le C|\boldsymbol{p}|\sqrt{\omega+\omega^{-1}}\sqrt{E_1+E_2+E_3}\|\xi^{\pm}\|_h.$$

Finally, inserting this estimate into (2.17) and dividing both sides by $\|\xi^{\pm}\|_h$ completes the proof. □

Corollary 2.4.2 *Let the assumptions of Proposition 2.4.1 be satisfied. Then, there holds*

$$\begin{aligned}&\|u-u_h^{\pm}\|_h^2\\ &\quad\le C|\boldsymbol{p}|^2\Big(\sum_{K\in\mathcal{T}}\|\eta\|^2_{H^1(K)}+\sum_{K\in\mathcal{T}\setminus\mathcal{K}_0}h_K^2|\eta|^2_{H^2(K)}+\sum_{K\in\mathcal{K}_0}h_K^{2(1-\beta_K)}|\eta|^2_{H^{2,2}_{\beta_K}(K)}\Big),\end{aligned} \tag{2.18}$$

where the constant $C>0$ is independent of u, $\boldsymbol{p}$ and of $\boldsymbol{h}$.

Proof: Since, by (2.16),

$$\|u - u_h^{\pm}\|_h \leq \|\eta\|_h + \|\xi\|_h,$$

the assertion follows directly from Proposition 2.4.1. □

2.5 Convergence of the h DGFEM

In the following section the h-version of the DGFEM, i.e. the DGFEM with fixed polynomial degree $p > 0$, $p \in \mathbb{N}$ on all elements, will be analyzed. It is well-known that, if $u \in H^{p+1}(\Omega)$, where u denotes the exact solution of (2.1)–(2.4), then the standard (conforming) finite element method (and also the DGFEM) converges at an optimal algebraic rate, i.e.

$$\|u - u_h\|_h \leq C N^{-p/2},$$

where $N = \dim(V_h)$ with $\boldsymbol{p} = \{p, p, \ldots, p\}$ is the number of degrees of freedom and $\mathcal{T}$ is a uniform mesh on Ω. Typically, this result is not anymore attainable if the assumption $u \in H^{p+1}(\Omega)$ is weakened, i.e. $u \in H^{p+1,2}_{\boldsymbol{\beta}}(\Omega)$ with $\boldsymbol{\beta} \succ \mathbf{0}$.

Although the convergence rate remains algebraic in this case, the optimal order $\mathcal{O}(N^{-p/2})$ is usually reduced to $\mathcal{O}(N^{-\alpha/2})$ with $\alpha \ll p$. This effect is especially pronounced at higher orders of approximation.

The aim of this section is to prove that the optimal convergence rate may be preserved even if the exact solution is singular, i.e. $u \notin H^{p+1}(\Omega)$. The main idea is to replace the uniform meshes by so-called 'graded meshes' which are able to resolve the singularities without the need of additional degrees of freedom.

2.5.1 h DGFEM Approximations on Polygons

By Corollary 2.4.2, the DG-error $\|u - u_h\|_h$ may be bounded by $\|\eta\|_h$, where $\eta = u - \pi_{\boldsymbol{p}} u$, for an arbitrary interpolant $\pi_{\boldsymbol{p}} u$ with $u = \pi_{\boldsymbol{p}} u$ in the element vertices of $\mathcal{T}$. Therefore, h DGFEM approximations of functions in $H^{p+1,2}_{\boldsymbol{\beta}}(\Omega)$ are of a main interest.

As mentioned above, graded meshes will be introduced (cf. [7]).

Graded Meshes on Ω

Definition 2.5.1 *Let $\boldsymbol{\gamma}$ be a weight vector as defined in Section 1.2 and $\Phi_{\boldsymbol{\gamma}}$ the corresponding weight function on Ω. Then, a mesh $\mathcal{T}_{\boldsymbol{\gamma}}$ on Ω is called a graded mesh with grading vector $\boldsymbol{\gamma}$ if there exists a constant $L > 0$ such that the following properties are satisfied:*

i) if $K \in \mathcal{T}_{\boldsymbol{\gamma}} \setminus \mathcal{K}_0$ *then*

$$L^{-1} h_{\mathcal{T}_{\boldsymbol{\gamma}}} \Phi_{\boldsymbol{\gamma}}(\boldsymbol{x}) \le h_K \le L h_{\mathcal{T}_{\boldsymbol{\gamma}}} \Phi_{\boldsymbol{\gamma}}(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in K;$$

ii) if $K \in \mathcal{K}_0$ *then*

$$L^{-1} h_{\mathcal{T}_{\boldsymbol{\gamma}}} \sup_{\boldsymbol{x}\in K} \Phi_{\boldsymbol{\gamma}}(\boldsymbol{x}) \le h_K \le L h_{\mathcal{T}_{\boldsymbol{\gamma}}} \sup_{\boldsymbol{x}\in K} \Phi_{\boldsymbol{\gamma}}(\boldsymbol{x}).$$

Here, $h_{\mathcal{T}_{\boldsymbol{\gamma}}}$ *is the mesh width of* $\mathcal{T}_{\boldsymbol{\gamma}}$*, cf.* (1.7).

Interpolants for the h DGFEM

The following results are required for the convergence analysis of the h DGFEM. The first statement shows, how $H^{2,2}_{\beta}$ functions may be approximated on elements abutting at a singular point. The second result is a well-known approximation statement on functions in H^{p+1}.

Lemma 2.5.2 *Let* K *be a triangle (or quadrilateral) with the vertices* A_1, A_2, A_3 *(and* A_4*). Further, let* $u \in H^{2,2}_{\beta_K}(K)$ *with* $\Phi_{\beta_K}(\boldsymbol{x}) = |\boldsymbol{x} - A_1|^{\beta_K}$, $0 < \beta_K < 1$. *Then, the linear (bilinear) interpolant of* u *in the vertices of* K*, denoted by* $I^0_K u$*, satisfies*

$$\|u - I^0_K u\|_{H^1(K)} \le C h_K^{1-\beta_K} |u|_{H^{2,2}_{\beta_K}(K)}, \tag{2.19}$$

$$\|u - I^0_K u\|_{H^{2,2}_{\beta_K}(K)} \le C |u|_{H^{2,2}_{\beta_K}(K)}. \tag{2.20}$$

Proof: See [58, Lemma 4.16 and Lemma 4.25]. □

Remark 2.5.3 The above Lemma 2.5.2 holds also true for $\beta_K = 0$, i.e. $u \in H^2(K)$. In this case, (2.19) and (2.20) simplify to

$$\|u - I^0_K u\|_{H^1(K)} \le C h_K |u|_{H^2(K)},$$
$$\|u - I^0_K u\|_{H^2(K)} \le C |u|_{H^2(K)}.$$

Lemma 2.5.4 *Let* $p \ge 1$ *be a fixed polynomial degree. Furthermore, let* K *be a triangle (or quadrilateral) and* $u \in H^{p+1}(K)$*. Then, there exists an interpolant* $I_K u \in \mathcal{P}_p(K)$ *of* u *with* $I_K u = u$ *in the vertices of* K *such that*

$$|u - I_K u|_{H^m(K)} \le C h_K^{p+1-m} |u|_{H^{p+1}(K)}, \qquad 0 \le m \le p+1,$$

with a constant $C > 0$ *independent of* u *and of* h_K.

Optimal h Approximations on Ω

Proposition 2.5.5 *Let $u \in H^{p+1,2}_{\boldsymbol{\beta}}(\Omega)$ for a weight vector $0 \preccurlyeq \boldsymbol{\beta} \prec 1$ and $p > 0$ be a fixed polynomial degree. Further, let $\mathcal{T}_{\boldsymbol{\gamma}}$ with $\boldsymbol{\gamma}_0 \preccurlyeq \boldsymbol{\gamma} \prec 1$, where*

$$\gamma_{0,i} = \begin{cases} 1 - \frac{1-\beta_i}{p} & \text{if } \beta_i > 0 \\ 0 & \text{if } \beta_i = 0 \end{cases}, \qquad i = 1, 2 \ldots, M, \tag{2.21}$$

be a graded mesh as in Definition 2.5.1. Moreover, the finite element spaces V_h from Section 2.3.1 are specified by the following choice of the degree vector

$$\boldsymbol{p} = \{p_K = p : \ K \in \mathcal{T}_{\boldsymbol{\gamma}}\} \tag{2.22}$$

and of the polynomial spaces $\mathcal{V}_p$,

$$\begin{aligned} &\text{for } p = 1: \qquad \mathcal{V}_1 = \begin{cases} \mathcal{P}_1(K) & \text{if } K \in \mathcal{T}_{\boldsymbol{\gamma}} \text{ is a triangle}, \\ \mathcal{Q}_1(K) & \text{if } K \in \mathcal{T}_{\boldsymbol{\gamma}} \text{ is a quadrilateral}, \end{cases} \\ &\text{for } p > 1: \qquad \mathcal{V}_p = \mathcal{P}_p(K) \text{ or } \mathcal{V}_p = \mathcal{Q}_p(K). \end{aligned} \tag{2.23}$$

Then, there exists an interpolant $\Pi_p u \in V_h$ of u with $\Pi_p u = u$ in all element vertices of $\mathcal{T}_{\boldsymbol{\gamma}}$, such that there holds the following estimate

$$\sum_{K \in \mathcal{T}_{\boldsymbol{\gamma}}} \|\eta\|^2_{H^1(K)} + \sum_{K \in \mathcal{T}_{\boldsymbol{\gamma}} \setminus \mathcal{K}_0} h_K^2 |\eta|^2_{H^2(K)} + \sum_{K \in \mathcal{K}_0} h_K^{2(1-\beta_K)} |\eta|^2_{H^{2,2}_{\beta_K}(K)} \leq C N^{-p/2},$$

where $\eta = u - \Pi_p u$ and $C > 0$ is a constant independent of $\boldsymbol{h}$ and of $N =$ $\dim(V_h)$.

In order to show this result, the ensuing lemma has to be proved:

Lemma 2.5.6 *Let $\mathcal{T}_{\boldsymbol{\gamma}}$ be a graded mesh as in Definition 2.5.1 and $\boldsymbol{p}$ be the polynomial degree vector from* (2.22)*. Then,*

$$N = \dim(V_h) \leq C p^2 h^{-2}_{\mathcal{T}_{\boldsymbol{\gamma}}},$$

where $C > 0$ is a constant independent of $\boldsymbol{p}$ and of $\boldsymbol{h}$.

Proof: Obviously, there holds

$$N \leq C p^2 \sum_{K \in \mathcal{T}_{\boldsymbol{\gamma}}} 1.$$

In addition, the shape regularity (1.8) implies that

$$h_K^2 \le C \int_K d\boldsymbol{x}.$$

Hence,

$$N \;\le\; Cp^2 \Big(\sum_{K \in \mathcal{T}_{\boldsymbol{\gamma}} \setminus \mathcal{K}_0} h_K^{-2} \int_K d\boldsymbol{x} + \sum_{K \in \mathcal{K}_0} 1 \Big).$$

From the definition of the graded meshes it follows

$$\begin{aligned} N \;&\le\; Cp^2 h_{\mathcal{T}_{\boldsymbol{\gamma}}}^{-2} \Big(\sum_{K \in \mathcal{T}_{\boldsymbol{\gamma}} \setminus \mathcal{K}_0} \int_K \Phi_{\mathcal{T}_{\boldsymbol{\gamma}}}^{-2} \, d\boldsymbol{x} + h_{\mathcal{T}_{\boldsymbol{\gamma}}}^2 \sum_{K \in \mathcal{K}_0} 1 \Big) \\ &\le\; Cp^2 h_{\mathcal{T}_{\boldsymbol{\gamma}}}^{-2} \Big(\int_\Omega \Phi_{\mathcal{T}_{\boldsymbol{\gamma}}}^{-2} \, d\boldsymbol{x} + h_{\mathcal{T}_{\boldsymbol{\gamma}}}^2 \sum_{K \in \mathcal{K}_0} 1 \Big) \end{aligned}$$

By definition, the components of $\boldsymbol{\gamma}$ are strictly smaller than 1, and thus the integral above is well-defined. Furthermore, since the number of elements in $\mathcal{K}_0$ is finite (due to the shape regularity of $\mathcal{T}_{\boldsymbol{\gamma}}$), $\sum_{K \in \mathcal{K}_0} 1$ is bounded, and hence, the proof is complete. □

Proof: (Proposition 2.5.5) Consider first the case, where $\beta_i > 0$ for all $i = 1, 2, \ldots, M$. Define an interpolant $\Pi_p u$ on $\mathcal{T}_{\boldsymbol{\gamma}}$ as follows:

$$\text{if } p = 1: \qquad \Pi_1 u|_K = I_K^0 u, \quad \forall K \in \mathcal{T},$$

$$\text{if } p > 1: \qquad \Pi_p u|_K = \begin{cases} I_K^0 u & \text{if } K \in \mathcal{K}_0, \\ I_K u & \text{if } K \in \mathcal{T} \setminus \mathcal{K}_0. \end{cases}$$

Here, I_K^0 and I_K are the interpolants from Lemma 2.5.2 and Lemma 2.5.4, respectively. Obviously, $\Pi_p u \in V_h$. Then,

$$\begin{aligned} &\sum_{K \in \mathcal{K}_0} \big(\|\Pi_p u - u\|_{H^1(K)}^2 + h_K^{2(1-\beta_K)} |\Pi_p u - u|_{H^{2,2}_{\beta_K}(K)}^2 \big) \\ &\qquad + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} \big(\|\Pi_p u - u\|_{H^1(K)}^2 + h_K^2 |\Pi_p u - u|_{H^2(K)}^2 \big) \\ &\quad \le C \Big(\sum_{K \in \mathcal{K}_0} h_K^{2(1-\beta_K)} |u|_{H^{2,2}_{\beta_K}(K)}^2 + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^{2p} |u|_{H^{p+1}(K)}^2 \Big) \\ &\quad \le C \Big(h_{\mathcal{T}_{\boldsymbol{\gamma}}}^{2(1-\beta_K)} \sum_{K \in \mathcal{K}_0} \big(\sup_{\boldsymbol{x} \in K} r_K^{\gamma_K} \big)^{2(1-\beta_K)} |u|_{H^{2,2}_{\beta_K}(K)}^2 \\ &\qquad + h_{\mathcal{T}_{\boldsymbol{\gamma}}}^{2p} \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} \int_K r_K^{2p\gamma_K} |D^{p+1} u|^2 \, d\boldsymbol{x} \Big). \end{aligned} \tag{2.24}$$

For all $K \in \mathcal{K}_0$ there holds $r_K(\boldsymbol{x}) \le h_K, \forall \boldsymbol{x} \in K$. Hence,

$$h_K \le C h_{\mathcal{T}_\gamma} \sup_{\boldsymbol{x} \in K} r_K^{\gamma_K} \le C h_{\mathcal{T}_\gamma} h_K^{\gamma_K},$$

and therefore

$$h_K \le C h_{\mathcal{T}_\gamma}^{\frac{1}{1-\gamma_K}}.$$

Thus,

$$\sup_{\boldsymbol{x} \in K} r_K^{\gamma_K} \le C h_K^{\gamma_K} \le C h_{\mathcal{T}_\gamma}^{\frac{\gamma_K}{1-\gamma_K}} \le C h_{\mathcal{T}_\gamma}^{\frac{\gamma_{0K}}{1-\gamma_{0K}}} \le C h_{\mathcal{T}_\gamma}^{\frac{p}{1-\beta_K}-1}.$$

Inserting these bounds into (2.24) results in

$$\begin{aligned}
&\sum_{K \in \mathcal{K}_0} \left(\|\Pi_p u - u\|_{H^1(K)}^2 + h_K^{2(1-\beta_K)} |\Pi_p u - u|_{H^{2,2}_{\beta_K}(K)}^2 \right) \\
&\qquad + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} \left(\|\Pi_p u - u\|_{H^1(K)}^2 + h_K^2 |\Pi_p u - u|_{H^2(K)}^2 \right) \\
&\qquad \le C h_{\mathcal{T}_\gamma}^{2p} \Bigg(\sum_{K \in \mathcal{K}_0} |u|_{H^{2,2}_{\beta_K}(K)}^2 + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} \int_K r^{2(p-(1-\beta_K))} |D^{p+1} u|^2 \, d\boldsymbol{x} \Bigg) \\
&\qquad \le C h_{\mathcal{T}_\gamma}^{2p} \Bigg(\sum_{K \in \mathcal{K}_0} |u|_{H^{2,2}_{\beta_K}(K)}^2 + \int_\Omega \Phi_{\boldsymbol{\beta}+p-1}^2 |D^{p+1} u|^2 \, d\boldsymbol{x} \Bigg) \\
&\qquad \le C h_{\mathcal{T}_\gamma}^{2p} |u|_{H^{p+1,2}(\Omega)}^2.
\end{aligned}$$

Finally, according to the previous Lemma 2.5.6,

$$h_{\mathcal{T}_\gamma}^{2p} \le C N^{-p}.$$

If $\beta_i = 0$ for some $i \in \{1, 2, \ldots, M\}$, the Proposition may be proved in a very similar way.

□

2.5.2 Optimal Convergence of the h DGFEM on Polygons

Inserting the interpolant from Proposition 2.5.5 into the error estimate (2.18) yields the following convergence result for the h DGFEM:

Theorem 2.5.7 *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $p \in \mathbb{N}$, $p \ge 1$. Moreover, let the exact solution u of* (2.1)–(2.4) *be in $H_{\boldsymbol{\beta}}^{p+1,2}(\Omega)$ for a weight vector $0 \preccurlyeq \boldsymbol{\beta} \prec 1$. Then, for ω as in Theorem 2.3.7, the solutions $u_h^\pm$ of the DGFEM* (2.10) *on a graded mesh $\mathcal{T}_{\boldsymbol{\gamma}}$ with $1 \succ \boldsymbol{\gamma} \succcurlyeq \boldsymbol{\gamma}_0$, where $\boldsymbol{\gamma}_0$ is defined as in* (2.21)*, satisfy the following optimal algebraic error estimate:*

$$\|u - u_h^{\pm}\|_h \le C N^{-p/2}. \tag{2.25}$$

Here, $\boldsymbol{p} = \{p, p, \ldots, p\}$ and $C > 0$ is a constant independent of $\boldsymbol{h}$ and of $N =$ $\dim(V_h)$.

Remark 2.5.8 Since $\mathcal{B}^2_{\boldsymbol{\beta}}(\Omega) \subset H^{p+1,2}_{\boldsymbol{\beta}}(\Omega)$, the above theorem holds also true for $u \in \mathcal{B}^2_{\boldsymbol{\beta}}(\Omega)$.

2.5.3 Numerical Results

Model Problems

The theory above will now be illustrated and confirmed with some numerical experiments. Consider therefore the following two model problems on the unit triangle (cf. Figure 2.1):

- *Problem 1.* The Laplace equation *with* absolute term

$$-\Delta u + u = f \quad \text{in } \Omega. \tag{P1}$$

- *Problem 2.* The Laplace equation *without* any absolute term

$$-\Delta u = 0 \quad \text{in } \Omega. \tag{P2}$$

Here, using polar coordinates, the right-hand side f in (P1) is set to be

$$f = \sqrt{r}\sin(\phi/2),$$

and the boundary conditions for both problems (P1) and (P2) are chosen as follows:

$$u = \sqrt{r}\sin(\phi/2) \quad \text{on } \partial\Omega.$$

A few calculations show that the exact solution of the above problems (P1) and (P2) is, in both cases, given by

$$u = \sqrt{r}\sin(\phi/2),$$

and belongs to $\mathcal{B}^2_{\boldsymbol{\beta}}(\Omega)$ for all $\boldsymbol{\beta} = (\beta_1, 0, 0)$, $\beta_1 > 1/2$.

Therefore, Theorem 2.5.7 implies that, for $p \ge 1$ and grading vectors

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}(\beta_1) = \Big(1 - \frac{1-\beta_1}{p}, 0, 0\Big), \tag{2.26}$$

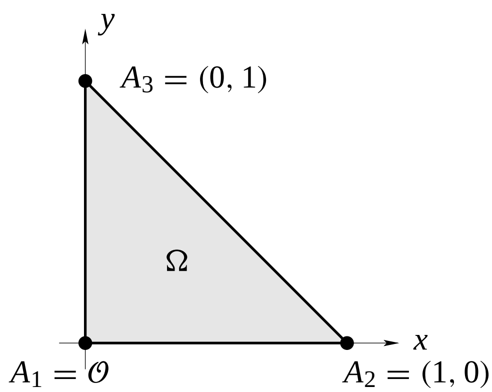

Figure 2.1: Boundary conditions on the computational domain.

the use of graded meshes $\mathcal{T}_{\boldsymbol{\gamma}}$ (cf. Definition 2.5.1) leads the DGFEM (2.10) for the model problems (P1) and (P2) to converge optimally as long as $\beta_1 > 1/2$ in (2.26).

In order to clarify the dependence of the h DGFEM's convergence regime on the regularity of the exact solution (i.e. on the choice of the weight vectors for the corresponding weighted Sobolev spaces), the gradings of the meshes in the following considerations are always expressed in terms of β_1 (and of p).

Variation in β_1

Referring to the explanations above, the h DGFEM for (P1) and (P2) is not expected to achieve optimal algebraic convergence rates on $\mathcal{T}_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\beta_1)$ (cf. (2.26)), if $\beta_1 \leq 1/2$. Indeed, as the results in Tables 2.1–2.3 and Figures 2.2–2.5 show, the algebraic convergence rates of the h DGFEM strongly depend on the choice of β_1 and clearly deteriorate as $\beta_1 \to 0$. Even in the borderline case $\beta_1 = 1/2$, a small loss of optimality is visible. However, in contrast to these findings, the choice $\beta_1 = 0.6 > 1/2$ leads the h DGFEM to show the desired convergence regime, and hence, the numerical experiments seem to correlate with the theory.

In these examples, the L^2 errors are found to converge twice as fast as the H^1 errors (for sufficiently large β_1); see also [3] for details.

Variation in p

Figures 2.6 and 2.7 show the performance of the ”+”- and of the ”−”-version (NIPG and SIPG) of the h DGFEM for different choices of p. As before, the

	$\beta_1 = 0.0$	$\beta_1 = 0.25$	$\beta_1 = 0.5$	$\beta_1 = 0.6$
H^1 error	0.25	0.32	0.44	0.48
L^2 error	0.73	0.93	1.00	1.00

Table 2.1: Algebraic convergence rates of the ”+”-version of the h DGFEM (NIPG) with $p = 1$ for the model problem (P1).

	$\beta_1 = 0.0$	$\beta_1 = 0.25$	$\beta_1 = 0.5$	$\beta_1 = 0.6$
H^1 error	0.25	0.32	0.44	0.48
L^2 error	0.73	0.91	0.99	0.99

Table 2.2: Algebraic convergence rates of the ”+”-version of the h DGFEM (NIPG) with $p = 1$ for the model problem (P2).

	$\beta_1 = 0.0$	$\beta_1 = 0.25$	$\beta_1 = 0.5$	$\beta_1 = 0.6$
H^1 error	0.25	0.32	0.43	0.47
L^2 error	0.73	0.92	0.99	0.99

Table 2.3: Algebraic convergence rates of the ”−”-version of the h DGFEM (SIPG) with $p = 1$ for the model problems (P1) and (P2) (results are similar for both problems).

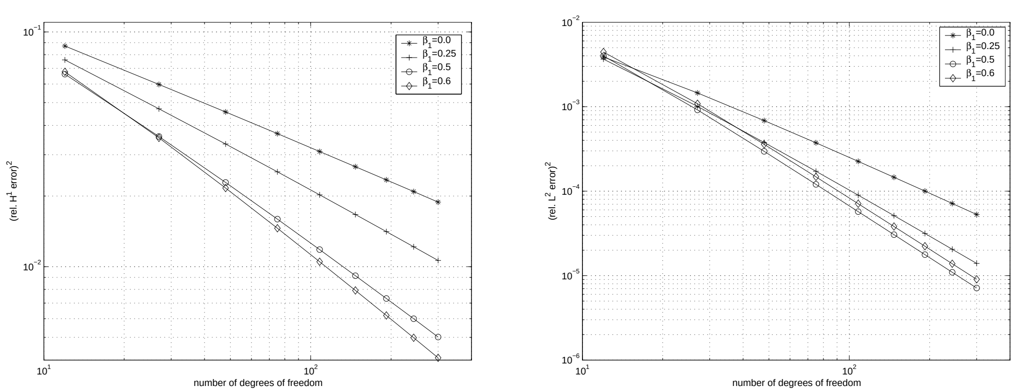

Figure 2.2: Performance of the NIPG for the model problem (P1).

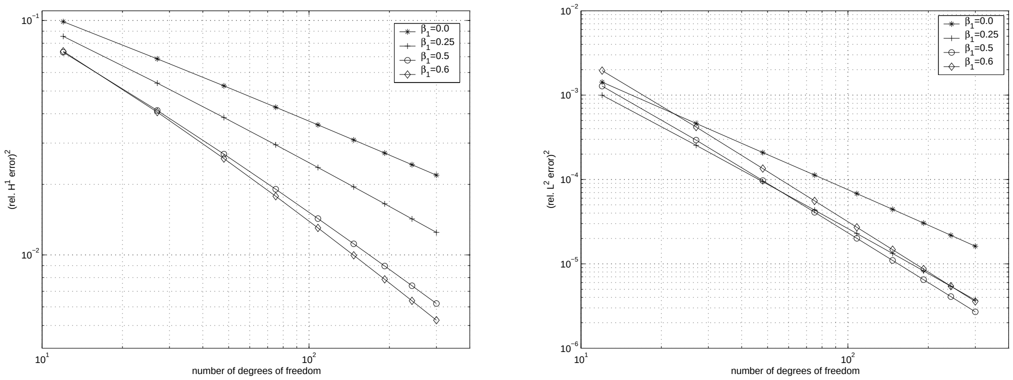

Figure 2.3: Performance of the SIPG for the model problem (P1).

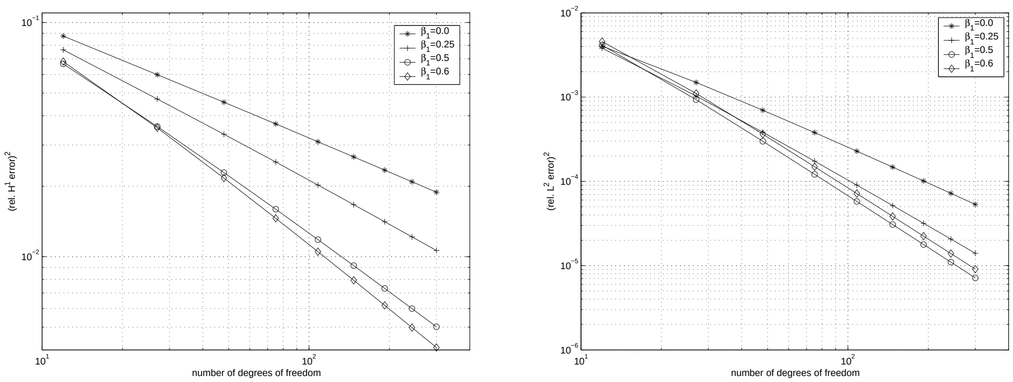

Figure 2.4: Performance of the NIPG for the model problem (P2).

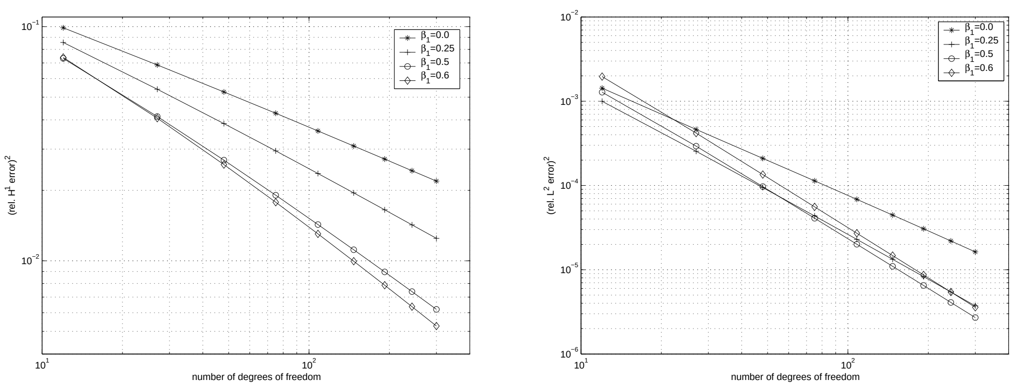

Figure 2.5: Performance of the SIPG for the model problem (P2).

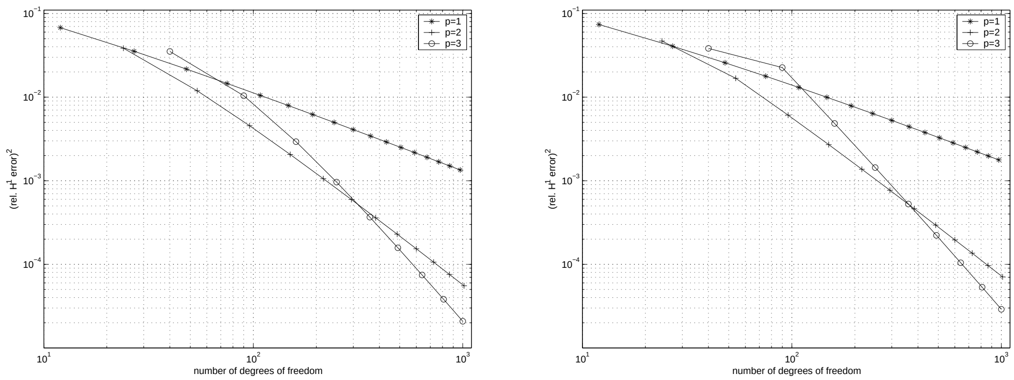

Figure 2.6: Performance of the NIPG and the SIPG with $\beta_1 = 0.6$ for the model problem (P1).

results for $\beta_1 = 0.5$ are slightly worse compared with those which were obtained using graded meshes with $\beta_1 = 0.6$. However, the expected convergence rates are acceptably achieved in both cases.

A detailed list of the algebraic convergence rates for the model problem (P2) is given in Tables 2.4 and 2.5. The corresponding numbers for the model problem (P1) are almost alike.

	$p=1$	$p=2$	$p=3$
NIPG	0.44	0.91	1.38
SIPG	0.43	0.90	1.38

Table 2.4: Algebraic convergence rates of the H^1 error of the h DGFEM with $\beta_1 = 0.5$ for the model problem (P2).

	$p=1$	$p=2$	$p=3$
NIPG	0.48	0.96	1.43
SIPG	0.47	0.96	1.44

Table 2.5: Algebraic convergence rates of the H^1 error of the h DGFEM with $\beta_1 = 0.6$ for the model problem (P2).

2.6 Convergence of the hp DGFEM

In this section it is proved that a judicious combination of mesh refinement and decrease of the polynomial degrees towards the singular points of the polygon (i.e., corner vertices and vertices of changing boundary condition type) leads the DGFEM to converge at an exponential rate.

Again, Corollary 2.4.2 implies that the errors $\|u - u_h^{\pm}\|_h$ of the DGFEM (2.10) may be estimated by the interpolation error $\eta = u - \pi_{\boldsymbol{p}} u$, where $\pi_{\boldsymbol{p}} u \in V_h$ is an arbitrary interpolant of the exact solution u. Thus, hp approximations of u in V_h have to be developed.

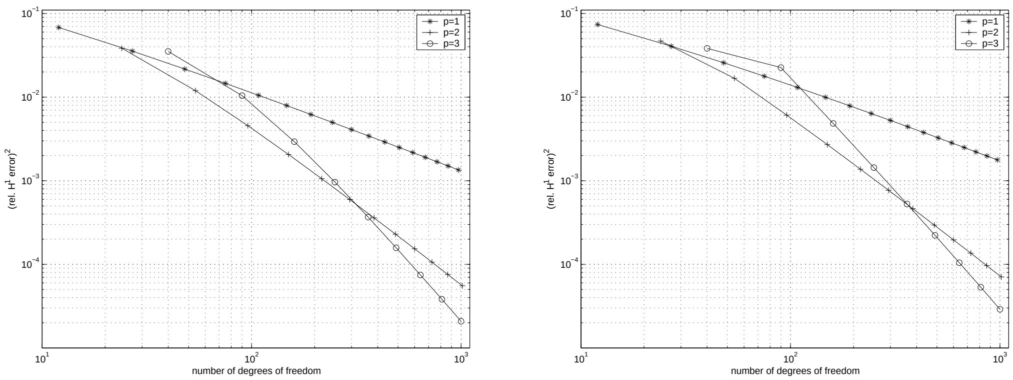

Figure 2.7: Performance of the NIPG and the SIPG with $\beta_1 = 0.6$ for the model problem (P2).

In order to resolve singular solution behaviour near singularities, *geometric meshes*, i.e. meshes that are geometrically refined towards the singular points of Ω will be introduced. Here, the main idea is to keep the ratio

$$\frac{\text{element diameter}}{\text{distance to the singularity}}$$

bounded as it is shown in Figures 2.8 and 2.9.

2.6.1 Geometric Meshes

First of all, basic geometric meshes on the unit square $\hat{\Omega} = (0, 1)^2$ with refinement towards the origin $\mathcal{O} = (0, 0)$ will be defined.

Definition 2.6.1 *Let $n \in \mathbb{N}_0$ and $\sigma \in (0, 1)$. On $\hat{\Omega}$, the basic geometric mesh $\Delta_{n,\sigma}$ with $n+1$ layers and grading factor σ is constructed recursively as follows: If $n = 0$, $\Delta_{0,\sigma} = \{\hat{\Omega}\}$. Given $\Delta_{n,\sigma}$ for $n \geq 0$, $\Delta_{n+1,\sigma}$ is generated by subdividing the square K with $\mathcal{O} = (0, 0) \in \overline{K}$ into four smaller rectangles by dividing its sides in a $\sigma : (1 - \sigma)$ ratio.*

An example of a basic geometric mesh is shown in Figure 2.8. The elements are denoted by $\{K_{ij}\}$ as indicated there. Moreover, the elements K_{1j}, K_{2j} and K_{3j} are said to constitute layer j, for $j \geq 2$. K_{11} is the element at the origin.

Remark 2.6.2 For simplicity, all elements in this section are supposed to be quadrilaterals. However, the following results (and especially the exponential convergence result, Theorem 2.6.6) may be extended to meshes consisting of triangles. Clearly, the geometric refinement property of the meshes has to be preserved.

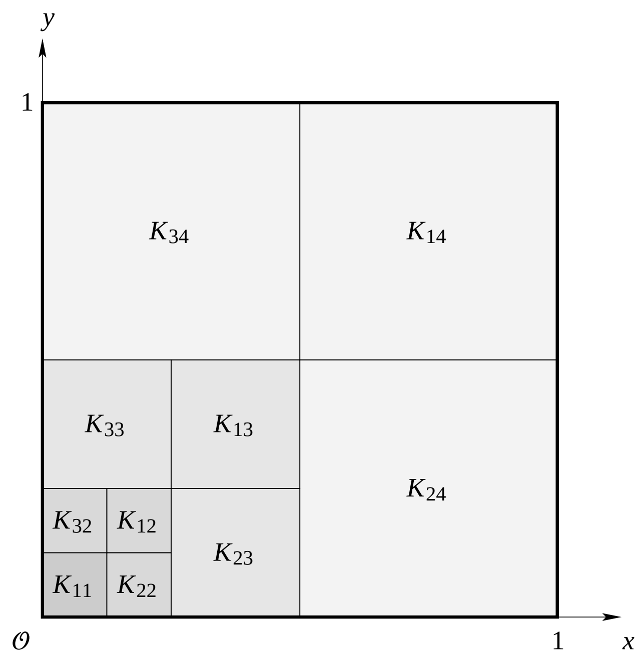

Figure 2.8: The basic geometric mesh $\Delta_{n,\sigma}$ with $n = 3$ and $\sigma = 0.5$.

Definition 2.6.3 *A geometric mesh $\mathcal{T}_{n,\sigma}$ in a polygon Ω is obtained by mapping the basic geometric meshes $\Delta_{n,\sigma}$ from Definition 2.6.1 affinely from $\hat{\Omega}$ to a vicinity of each convex corner of Ω. At vertices of changing boundary condition type two, and at reentrant corners three suitably scaled copies of $\Delta_{n,\sigma}$ are required. The remaining part of Ω is meshed with a fixed affine, quasi-uniform partition.*

Figure 2.9 shows an example of local geometric mesh refinement in a polygon.

Definition 2.6.4 *A polynomial degree distribution vector $\boldsymbol{p}$ on a geometric mesh $\mathcal{T}_{n,\sigma}$ is called linear with slope $\mu > 0$ if the elemental polynomial degrees are layer-wise constant in the geometric patches and given by $p_j = \max\{1, \lfloor \mu j \rfloor\}$ in layer j, $j = 1, 2, \ldots, n+1$. In the interior of the domain Ω the elemental polynomial degrees are set constant to* $\max\{1, \lfloor \mu(n+1) \rfloor\}$.

2.6.2 *hp* DG Approximations

Proposition 2.6.5 *Let $\Omega \subset \mathbb{R}^2$ be a polygon and $u \in \mathcal{B}^2_{\boldsymbol{\beta}}(\Omega)$ for a weight vector* $0 \preccurlyeq \boldsymbol{\beta} \prec 1$. *Then, there exists $\phi \in V_h$ and $\mu_0 = \mu_0(\sigma, \boldsymbol{\beta}) > 0$ such that for linear*

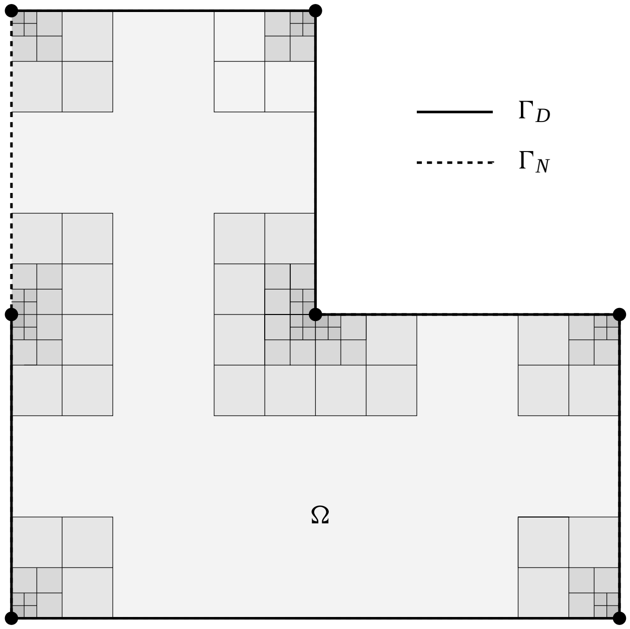

Figure 2.9: Example of geometric mesh refinement towards singular points in polygon.

polynomial degree distribution vectors $\mathbf{p}$ *with slope* $\mu \geq \mu_0$ *there holds*

$$\sum_{K \in \mathcal{T}_{n,\sigma}} \left(h_K^{-2} \|u - \phi\|_{L^2(K)}^2 + |u - \phi|_{H^1(K)}^2 \right) + \sum_{K \in \mathcal{T}_{n,\sigma} \setminus \mathcal{K}_0} h_K^2 |u - \phi|_{H^2(K)}^2 + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta_K} |u - \phi|_{H^{2,2}_{\beta_K}(K)}^2 \leq C e^{-b \sqrt[3]{N}}$$

with constants $C, b > 0$ *independent of* $N = \dim(V_h)$. *Furthermore,* $u = \phi$ *in all element vertices of* $\mathcal{T}_{n,\sigma}$.

Proof: The proof consists of two steps.

Step 1: Consider first the case where $\Omega = (0,1)^2$ and $\mathcal{T}_{n,\sigma} = \Delta_{n,\sigma}$ is the basic geometric mesh from Definition 2.6.1. Then, by Lemma 2.5.2, there exists $\phi_{11} \in \mathcal{Q}_1(K_{11})$ with $u = \phi_{11}$ in the vertices of K_{11} such that

$$h_{K_{11}}^{-2} \|u - \phi_{11}\|_{L^2(K_{11})}^2 + |u - \phi_{11}|_{H^1(K_{11})}^2 + h_{K_{11}}^{2-2\beta_{K_{11}}} |u - \phi_{11}|_{H^{2,2}_{\beta_{K_{11}}}(K_{11})}^2 \leq C \sigma^{2n(1-\beta_{K_{11}})} |u|_{H^{2,2}_{\beta_{K_{11}}}(K_{11})}^2.$$

Furthermore, from [58, Lemma 4.48] it follows that, for $K_{ij} \in \mathcal{T} \setminus \mathcal{K}_0$, there are $\phi_{ij} \in \mathcal{Q}_{p_{K_{ij}}}$ with $u = \phi_{ij}$ in the vertices of K_{ij} such that

$$h_{K_{ij}}^{-2}\|u-\phi_{ij}\|_{L^2(K_{ij})}^2 + |u-\phi_{ij}|_{H^1(K_{ij})}^2 + h_{K_{ij}}^2|u-\phi_{ij}|_{H^2(K_{ij})}^2 \\ \leq C\sigma^{2(n+2-j)(1-\beta_{K_{ij}})}\frac{\Gamma(p_{K_{ij}}-s_{ij}+1)}{\Gamma(p_{K_{ij}}+s_{ij}-1)}\left(\frac{\rho}{2}\right)^{2s_{ij}}\|u\|^2_{H^{s_{ij}+3,2}_{\beta_{K_{ij}}}(K_{ij})}$$

for any $1 \leq i \leq 3$, $2 \leq j \leq n+1$ and $s_{ij} \in [1, p_{K_{ij}}]$. Here, $\rho = \max\{1, {}^{(1-\sigma)}/\sigma\}$. Summing up, yields

$$\begin{aligned} &\sum_{K\in\Delta_{n,\sigma}}\left(h_K^{-2}\|u-\phi\|_{L^2(K)}^2 + |u-\phi|_{H^1(K)}^2\right) \\ &+ \sum_{K\in\Delta_{n,\sigma}\setminus\mathcal{K}_0} h_K^2|u-\phi|_{H^2(K)}^2 + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K}|u-\phi|^2_{H^{2,2}_{\beta_K}(K)} \\ &\leq C\sigma^{2n(1-\beta_{K_{11}})}\Bigg(\|u\|^2_{H^{2,2}_{\beta_{K_{11}}}(K_{11})} \\ &\quad + \sum_{i=1}^{3}\sum_{j=2}^{n+1}\sigma^{2(2-j)(1-\beta_{K_{ij}})}\frac{\Gamma(p_{K_{ij}}-s_{ij}+1)}{\Gamma(p_{K_{ij}}+s_{ij}-1)}\left(\frac{\rho}{2}\right)^{2s_{ij}}\|u\|^2_{H^{s_{ij}+3,2}_{\beta_{K_{ij}}}(K_{ij})}\Bigg). \end{aligned} \tag{2.27}$$

In [58, Section 4.5.3] it has been shown that there exist s_{ij}, $1 \leq i \leq 3$, $2 \leq j \leq n+1$ and $\mu_0 > 0$ such that, for linear polynomial degree distribution vectors as in Definition 2.6.4 with slope $\mu \geq \mu_0$, the right-hand side of (2.27) is exponentially small with respect to N. More precisely, there holds:

$$\sum_{K\in\Delta_{n,\sigma}}\left(h_K^{-2}\|u-\phi\|_{L^2(K)}^2 + |u-\phi|_{H^1(K)}^2\right) \\ + \sum_{K\in\Delta_{n,\sigma}\setminus\mathcal{K}_0} h_K^2|u-\phi|_{H^2(K)}^2 + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K}|u-\phi|^2_{H^{2,2}_{\beta_K}(K)} \leq Ce^{-b\sqrt[3]{N}}.$$

Step 2: Let now $\mathcal{T}_{n,\sigma}$ be a geometric mesh on an arbitrary polygon Ω, as in Definition 2.6.3. Since $\mathcal{T}_{n,\sigma}$ is obtained by mapping affinely up to three basic geometric mesh patches $\Delta_{n,\sigma}$ to a neighbourhood of each corner, it is possible to construct an interpolant ϕ as in Step 1, using a generalization of the result there to affinely mapped meshes. This may be established straightforwardly; see, e.g., [31, 32, 58] and the references therein. □

2.6.3 Exponential Convergence of the *hp* DGFEM

Combining the estimates from Corollary 2.4.2 and Proposition 2.6.5 leads to the main result of this section.

Theorem 2.6.6 *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. Moreover, assume that the exact solution u of* (2.1)–(2.4) *belongs to $\mathcal{B}^2_{\boldsymbol{\beta}}(\Omega)$ for a weight vector $0 \preccurlyeq \boldsymbol{\beta} \prec 1$. Then, for $n \in \mathbb{N}_0$ and $\sigma \in (0,1)$, there exists $\mu_0 = \mu_0(\sigma, \boldsymbol{\beta}) > 0$ such that for linear degree distribution vectors $\boldsymbol{p}$ with slope $\mu \geq \mu_0$ and ω as in Theorem 2.3.7, the solutions $u_h^{\pm}$ of the DGFEM* (2.10) *on a geometric mesh $\mathcal{T}_{n,\sigma}$ satisfy the following error estimate*

$$\|u - u_h^{\pm}\|_h \leq C e^{-b\sqrt[3]{N}}.$$

Here, $C,\ b > 0$ are independent of $N = \dim(V_h)$.

2.6.4 Numerical Results

The following numerical results are again based on the two model problems (P1) and (P2) from Section 2.5.3. However, here, the *hp*-version of the DGFEM is considered.

The computational domain is chosen to be an 'L-shaped' polygon with a reentrant corner at the origin $\mathcal{O}$ (cf. Figure 2.10). Furthermore, the right-hand side f in (P1) is given by

$$f = r^{2/3} \sin(2/3\phi),$$

where (r, ϕ) denote polar coordinates in $\mathbb{R}^2$. Moreover, the boundary conditions are set to be (cf. Figure 2.10):

$$u = 0 \quad \text{on } \Gamma_D$$

and

$$\nabla u \cdot \boldsymbol{n}_\Omega = \begin{cases} -2/3 r^{-1/3} \sin(\phi/3) & \text{on } \Gamma_{N_1} \\ 2/3 r^{-1/3} \cos(\phi/3) & \text{on } \Gamma_{N_2} \\ 2/3 r^{-1/3} \sin(\phi/3) & \text{on } \Gamma_{N_3} \\ 2/3 r^{-1/3} \cos(\phi/3) & \text{on } \Gamma_{N_4} \end{cases}.$$

Then, for both problems (P1) and (P2), the exact solution is

$$u = r^{2/3} \sin(2/3\phi),$$

and belongs to $\mathcal{B}^2_{\boldsymbol{\beta}}(\Omega)$ for all $\boldsymbol{\beta} = (\beta_1, 0, 0, 0, 0, 0)$ with $\beta_1 > 1/3$.

In order to obtain exponential convergence rates for the model problems (P1) and (P2), a geometric mesh with refinement towards the origin $\mathcal{O}$ (cf. Figure 2.11) has to be used for the *hp* DGFEM. The polynomial degree distribution vector is chosen as in 2.6.4, with slope $\mu = 1$.

Figures 2.12 and 2.13 show the performance of the *hp* DGFEM (NIPG and SIPG) for both problems (P1) and (P2). The asymptotic exponential convergence rates are clearly visible, and, in addition, they seem to be achieved already for a moderate number of degrees of freedom.

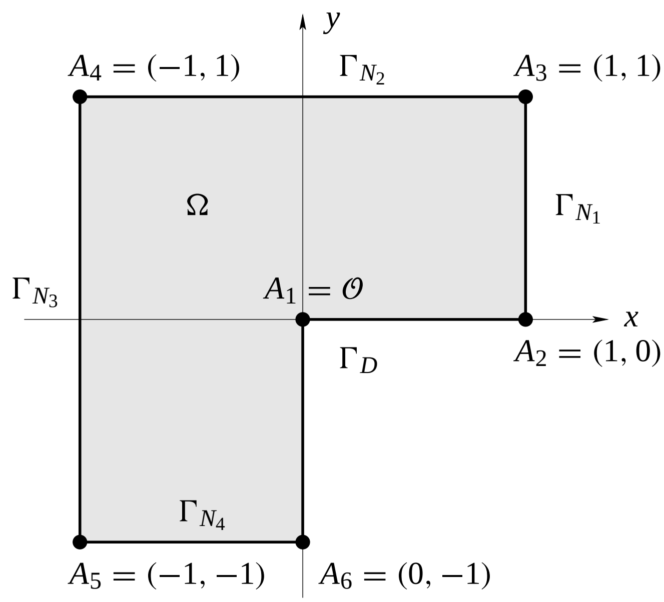

Figure 2.10: Boundary conditions on the computational domain.

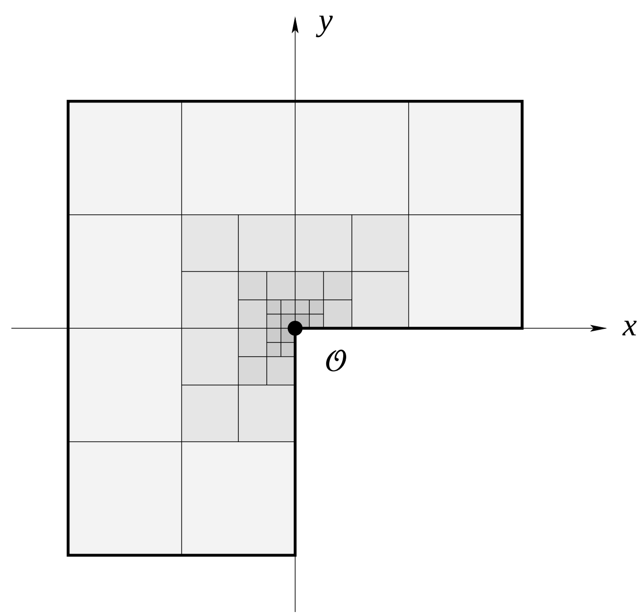

Figure 2.11: Geometric mesh refinement toward the origin $\mathcal{O}$.

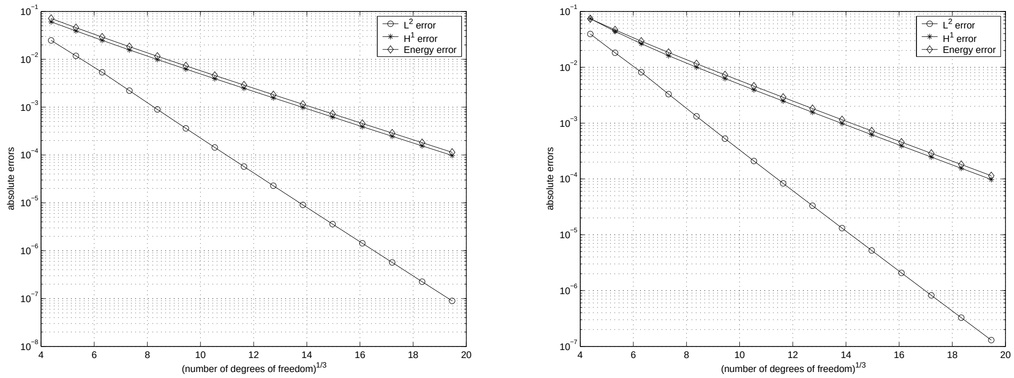

Figure 2.12: Performance of the hp NIPG for the model problems (P1) and (P2).

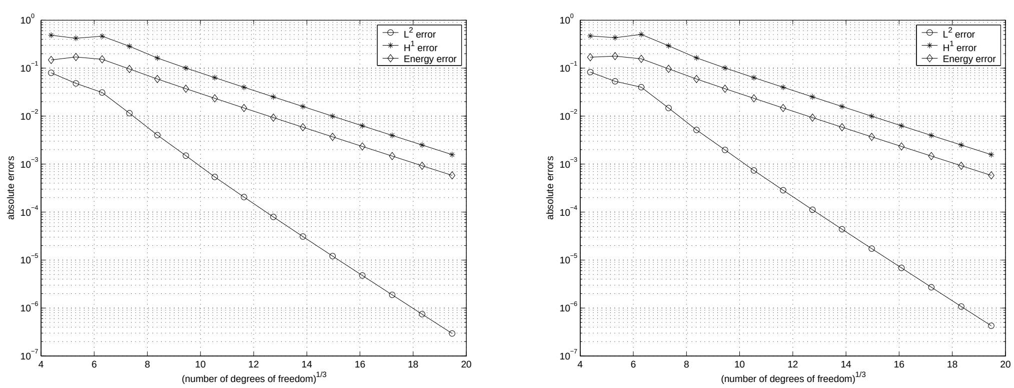

Figure 2.13: Performance of the hp SIPG for the model problems (P1) and (P2).

Chapter 3

Locking-Free h DGFEM for Elasticity Problems

3.1 Problem Formulation

Let $\Omega \subset \mathbb{R}^2$ be a polygon as in Section 1.1. Then, the linear elasticity problem reads as follows:

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} && \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{g}_D && \text{on } \Gamma_D \\ \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n}_\Omega &= \boldsymbol{g}_N && \text{on } \Gamma_N. \end{aligned} \tag{3.1}$$

Here, $\boldsymbol{u} = (u_1, u_2)$ is the *displacement* and $\boldsymbol{\sigma} = \{\sigma_{ij}\}_{i,j=1}^2$ is the *stress tensor* for homogeneous isotropic material given by

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\boldsymbol{\epsilon}(\boldsymbol{u}) + \lambda\nabla \cdot \boldsymbol{u}\, \mathbf{1}_{2\times 2},$$

where $\boldsymbol{\epsilon}(\boldsymbol{u}) = \{\epsilon_{ij}(\boldsymbol{u})\}_{i,j=1}^2$ with

$$\epsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_{x_i} u_j + \partial_{x_j} u_i) \tag{3.2}$$

is the *symmetric gradient* of $\boldsymbol{u}$. Furthermore, μ and λ are the so-called *Lam´e coeffi cients* satisfying

$$0 < \min\{\mu, \mu + \lambda\}.$$

3.2 Regularity

The functional setting in this chapter is again based on the theory of weighted Sobolev spaces (cf. Section 1.2).

3.2.1 Regularity of Generalized Stokes Problems

In order to obtain a regularity result for the elasticity problem (3.1), the following *generalized Stokes problem* in the polygon Ω is considered:

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) &= \boldsymbol{f} && \text{in } \Omega \\ -\nabla \cdot \boldsymbol{u} &= h && \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{g}_D && \text{on } \Gamma_D \\ \boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}_\Omega &= \boldsymbol{g}_N && \text{on } \Gamma_N. \end{aligned} \tag{3.3}$$

Here, $\boldsymbol{u}$ is the velocity field, p a Lagrange multiplier corresponding to the (hydrostatic) pressure in the incompressible limit and $\boldsymbol{\sigma}(\boldsymbol{u}, p)$ the hydrostatic stress tensor of $\boldsymbol{u}$ defined by

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = -p\, \mathbf{1}_{2\times 2} + 2\nu \boldsymbol{\epsilon}(\boldsymbol{u}),$$

where $\boldsymbol{\epsilon}(\boldsymbol{u})$ is given as in (3.2) and $\nu > 0$ is the (kinematic) viscosity. If $\Gamma_N = \emptyset$, the following compatibility condition has to be fulfilled:

$$\int_\Omega h\, d\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{g}_D \cdot \boldsymbol{n}_\Omega\, ds = 0 \tag{3.4}$$

In [34] the following regularity result was proved:

Theorem 3.2.1 *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $m \geq 0$. In addition, if $\Gamma_N = \emptyset$, let (3.4) be satisfied. Then, there exists a weight vector $0 \preccurlyeq \boldsymbol{\beta}_{min} \prec 1$ depending on the opening angles of Ω at the vertices $A_i \in SP(\Omega, \Gamma_D, \Gamma_N)$, $i = 1, 2, \ldots, M$ (cf. Section 1.2), such that for weight vectors $\boldsymbol{\beta}$ with $\boldsymbol{\beta}_{min} \preccurlyeq \boldsymbol{\beta} \prec 1$ and for $\boldsymbol{f} \in H^{m,0}_{\boldsymbol{\beta}}(\Omega)^2$, $h \in H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)$, $\boldsymbol{g}_D \in H^{m+3/2,3/2}_{\boldsymbol{\beta}}(\Gamma_D)^2$ and $\boldsymbol{g}_N \in H^{m+1/2,1/2}_{\boldsymbol{\beta}}(\Gamma_N)^2$ the generalized Stokes problem (3.3) admits a unique solution $(\boldsymbol{u}, p) \in H^{m+2,2}_{\boldsymbol{\beta}}(\Omega)^2 \times H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)$ and the a priori estimate*

$$\begin{aligned} &\|\boldsymbol{u}\|_{H^{m+2,2}_{\boldsymbol{\beta}}(\Omega)} + \|p\|_{H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)} \\ &\qquad \leq C\big(\|\boldsymbol{f}\|_{H^{m,0}_{\boldsymbol{\beta}}(\Omega)} + \|h\|_{H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)} \\ &\qquad\quad + \|\boldsymbol{g}_D\|_{H^{m+3/2,3/2}_{\boldsymbol{\beta}}(\Gamma_D)} + \|\boldsymbol{g}_N\|_{H^{m+1/2,1/2}_{\boldsymbol{\beta}}(\Gamma_N)}\big) \end{aligned} \tag{3.5}$$

holds true.

Moreover, if $\boldsymbol{f} \in \mathcal{B}^0_{\boldsymbol{\beta}}(\Omega)^2$, $h \in \mathcal{B}^1_{\boldsymbol{\beta}}(\Omega)$, $\boldsymbol{g}_D \in \mathcal{B}^{3/2}_{\boldsymbol{\beta}}(\Gamma_D)^2$, $\boldsymbol{g}_N \in \mathcal{B}^{1/2}_{\boldsymbol{\beta}}(\Gamma_N)^2$, then $(\boldsymbol{u}, p) \in \mathcal{B}^2_{\boldsymbol{\beta}}(\Omega)^2 \times \mathcal{B}^1_{\boldsymbol{\beta}}(\Omega)$.

3.2.2 Regularity of Linear Elasticity Problems

A regularity result for linear elasticity problems in polygons has already been developed in [33, Theorem 5.2]. However, referring to the previous Theorem 3.2.1, a more specific statement, which clarifies the regularity of the linear elasticity problem (3.1) in dependence on the Lamé coefficient λ, can be proved.

Theorem 3.2.2 *Let $\Omega \subset \mathbb{R}^2$ be a polygon and $m \geq 0$. Then, there exists a weight vector $0 \preccurlyeq \boldsymbol{\beta}_{min} \prec 1$ depending on the opening angles of Ω at the vertices $A_i \in SP(\Omega, \Gamma_D, \Gamma_N)$, $i = 1, 2, \ldots, M$, such that for weight vectors $\boldsymbol{\beta}$ with $\boldsymbol{\beta}_{min} \preccurlyeq \boldsymbol{\beta} \prec 1$ and for*

$$\boldsymbol{f} \in H^{m,0}_{\boldsymbol{\beta}}(\Omega)^2, \quad \boldsymbol{g}_D \in H^{m+3/2,3/2}_{\boldsymbol{\beta}}(\Gamma_D)^2, \quad \boldsymbol{g}_N \in H^{m+1/2,1/2}_{\boldsymbol{\beta}}(\Gamma_N)^2, \tag{3.6}$$

the linear elasticity problem (3.1) *has a unique solution $\boldsymbol{u} \in H^{m+2,2}_{\boldsymbol{\beta}}(\Omega)^2$. In addition, there exists a constant $C > 0$ independent of λ such that the ensuing estimate holds true:*

$$\begin{aligned} \|\boldsymbol{u}\|_{H^{m+2,2}_{\boldsymbol{\beta}}(\Omega)} &+ |\lambda| \|\nabla \cdot \boldsymbol{u}\|_{H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)} \\ &\leq C\big(\|\boldsymbol{f}\|_{H^{m,0}_{\boldsymbol{\beta}}(\Omega)} + \|\boldsymbol{g}_D\|_{H^{m+3/2,3/2}_{\boldsymbol{\beta}}(\Gamma_D)} + \|\boldsymbol{g}_N\|_{H^{m+1/2,1/2}_{\boldsymbol{\beta}}(\Gamma_N)}\big). \end{aligned} \tag{3.7}$$

Proof: As already mentioned above, the unique solution $\boldsymbol{u}_{\text{elast}}$ of the linear elasticity problem (3.1) belongs to $H^{m+2,2}_{\boldsymbol{\beta}}(\Omega)^2$ ([33, Theorem 5.2]). Therefore, the choice

$$h = -\nabla \cdot \boldsymbol{u}_{\text{elast}} \in H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)$$

leads to the following solution $(\boldsymbol{u}, p)$ of the generalized Stokes problem (3.3):

$$p = -\lambda \nabla \cdot \boldsymbol{u}_{\text{elast}}$$

and

$$\boldsymbol{u} = \boldsymbol{u}_{\text{elast}}.$$

Hence, using (3.5) implies that

$$\begin{aligned} \|\boldsymbol{u}\|_{H^{m+2,2}_{\boldsymbol{\beta}}(\Omega)} &+ |\lambda| \|\nabla \cdot \boldsymbol{u}\|_{H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)} \\ &\leq C\big(\|\boldsymbol{f}\|_{H^{m,0}_{\boldsymbol{\beta}}(\Omega)} + \|\nabla \cdot \boldsymbol{u}\|_{H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)} \\ &\quad + \|\boldsymbol{g}_D\|_{H^{m+3/2,3/2}_{\boldsymbol{\beta}}(\Gamma_D)} + \|\boldsymbol{g}_N\|_{H^{m+1/2,1/2}_{\boldsymbol{\beta}}(\Gamma_N)}\big). \end{aligned} \tag{3.8}$$

Thus, if $|\lambda| < 2C$, it follows that

$$\begin{aligned}\|\boldsymbol{u}\|_{H^{m+2,2}_{\boldsymbol{\beta}}(\Omega)} + |\lambda| \|\nabla\cdot\boldsymbol{u}\|_{H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)} &\leq \tilde{C}\|\boldsymbol{u}\|_{H^{m+2,2}_{\boldsymbol{\beta}}(\Omega)} \\ &\leq \tilde{C}\big(\|\boldsymbol{f}\|_{H^{m,0}_{\boldsymbol{\beta}}(\Omega)} + \|\boldsymbol{g}_D\|_{H^{m+3/2,3/2}_{\boldsymbol{\beta}}(\Gamma_D)} + \|\boldsymbol{g}_N\|_{H^{m+1/2,1/2}_{\boldsymbol{\beta}}(\Gamma_N)}\big)\end{aligned}$$

for a constant $\tilde{C}$ independent of $|\lambda| \in (0, 2C)$. In the last step, Theorem 5.2 in [33] was applied.

Alternatively, if $|\lambda| \geq 2C$, the term $C\|\nabla\cdot\boldsymbol{u}\|_{H^{m+1,1}_{\boldsymbol{\beta}}(\Omega)}$ in the right-hand side of (3.8) may obviously be absorbed into the left-hand side. □

3.3 The Discontinuous Galerkin Method

3.3.1 Finite Element Spaces

In contrast to the previous Chapter 2, the finite element meshes considered here are assumed to be regular and to consist of triangles only. Furthermore, the ensuing analysis is restricted to the case of piecewise (discontinuous) linear functions, i.e. the finite element spaces are given by

$$\boldsymbol{V}_h = \{\boldsymbol{u} \in L^2(\Omega)^2 : \boldsymbol{u}|_K \in \mathcal{P}_1(K)^2, K \in \mathcal{T}\}, \tag{3.9}$$

where

$$\mathcal{P}_1(K) = \{u(x, y) = ax + by + c : a, b, c \in \mathbb{R}\}$$

is the space of all linear functions on the element K, $K \in \mathcal{T}$.

3.3.2 Variational Formulation

The non-symmetric interior penalty discontinuous Galerkin finite element method (NIPG) for the linear elasticity problem (3.1) is introduced.

Definition 3.3.1 (DGFEM) *Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that*

$$B_h(\boldsymbol{u}_h, \boldsymbol{v}) = L_h(\boldsymbol{v}) \tag{3.10}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_h$*, where the bilinear form* B_h *is given by*

$$\begin{aligned} B_h(\boldsymbol{u}, \boldsymbol{v}) = & \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\, d\boldsymbol{x} \\ & - \sum_{e \in \mathcal{E}_{int,D}} \int_e (\langle \boldsymbol{\sigma}(\boldsymbol{u}) \rangle : \underline{[\boldsymbol{v}]} - \underline{[\boldsymbol{u}]} : \langle \boldsymbol{\sigma}(\boldsymbol{v}) \rangle)\, ds \\ & + \mu \sum_{e \in \mathcal{E}_{int,D}} \frac{1}{|e|} \int_e \underline{[\boldsymbol{u}]} : \underline{[\boldsymbol{v}]}\, ds, \end{aligned}$$

and the corresponding linear functional L_h *is defined by*

$$\begin{aligned} L_h(\boldsymbol{v}) = & \sum_{K \in \mathcal{T}} \int_K \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{g}_N \cdot \boldsymbol{v}\, ds \\ & + \int_{\Gamma_D} (\boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{n}_\Omega) \cdot \boldsymbol{g}_D\, ds + \mu \sum_{\substack{e \in \mathcal{E}: \\ e \subset \Gamma_D}} \frac{1}{|e|} \int_e \boldsymbol{g}_D \cdot \boldsymbol{v}\, ds. \end{aligned}$$

Here, $|e|$ *denotes the length of* $e \in \mathcal{E}$*.*

The following norm is associated with the DGFEM:

$$\|\boldsymbol{u}\|_h^2 = \sum_{K \in \mathcal{T}} \|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L^2(K)}^2 + \frac{\mu}{m_{\text{elast}}} \sum_{e \in \mathcal{E}_{\text{int},D}} \frac{1}{|e|} \int_e |\underline{[\boldsymbol{u}]}|^2\, ds, \tag{3.11}$$

where

$$m_{\text{elast}} = 2 \min\{\mu, \mu + \lambda\}.$$

Remark 3.3.2 For all $\boldsymbol{u} \in \boldsymbol{V}_h$ there holds that $\left(\sum_{K \in \mathcal{T}} |\boldsymbol{u}|_{H^1(K)}^2\right)^{1/2} \leq C \|\boldsymbol{u}\|_h$, where $C > 0$ is a constant independent of $\boldsymbol{u}$ and of $\boldsymbol{h}$. A corresponding result may be found in [13], where a discrete Korn inequality was proved.

3.3.3 Basic Properties

Proposition 3.3.3 (Consistency) *If, for a weight vector* $0 \preccurlyeq \boldsymbol{\beta} \prec 1$*, the exact solution* $\boldsymbol{u}$ *of the linear elasticity problem* (3.1) *belongs to* $H_{\boldsymbol{\beta}}^{2,2}(\Omega)^2$*, then the DGFEM* (3.10) *is consistent:*

$$B_h(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{3.12}$$

Proof: Using the integration by parts formula from Lemma A.2.2, the proof is very similar as in the diffusion case (Proposition 2.3.2). □

Proposition 3.3.4 (Coercivity) *The bilinear form B_h is coercive on $\boldsymbol{V}_h$. More precisely,*

$$B_h(\boldsymbol{u},\boldsymbol{u}) \geq m_{elast} \|\boldsymbol{u}\|_h^2$$

for all $\boldsymbol{u} \in \boldsymbol{V}_h$.

Proof: For $K \in \mathcal{T}$, let

$$\boldsymbol{\epsilon}_0(\boldsymbol{u}) = \boldsymbol{\epsilon}(\boldsymbol{u}) - \frac{1}{2}\nabla\cdot\boldsymbol{u}\,\mathbf{1}_{2\times 2}.$$

Then, there holds that

$$\begin{aligned}
&\int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{u})\, d\boldsymbol{x}\\
&\qquad = 2\mu \int_K \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{u})\, d\boldsymbol{x} + \lambda \int_K |\nabla\cdot\boldsymbol{u}|^2\, d\boldsymbol{x}\\
&\qquad = 2\mu \int_K \left(\boldsymbol{\epsilon}_0(\boldsymbol{u}) + \frac{1}{2}\nabla\cdot\boldsymbol{u}\,\mathbf{1}_{2\times 2}\right) : \left(\boldsymbol{\epsilon}_0(\boldsymbol{u}) + \frac{1}{2}\nabla\cdot\boldsymbol{u}\,\mathbf{1}_{2\times 2}\right) d\boldsymbol{x}\\
&\qquad\quad + \lambda \int_K |\nabla\cdot\boldsymbol{u}|^2\, d\boldsymbol{x}\\
&\qquad = 2\mu \int_K \left(\boldsymbol{\epsilon}_0(\boldsymbol{u}) : \boldsymbol{\epsilon}_0(\boldsymbol{u}) + \frac{1}{2}|\nabla\cdot\boldsymbol{u}|^2\right) d\boldsymbol{x} + \lambda \int_K |\nabla\cdot\boldsymbol{u}|^2\, d\boldsymbol{x}\\
&\qquad = 2\mu \int_K \boldsymbol{\epsilon}_0(\boldsymbol{u}) : \boldsymbol{\epsilon}_0(\boldsymbol{u})\, d\boldsymbol{x} + (\mu+\lambda) \int_K |\nabla\cdot\boldsymbol{u}|^2\, d\boldsymbol{x}.
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\int_K \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{u})\, d\boldsymbol{x} &= \int_K \left(\boldsymbol{\epsilon}_0(\boldsymbol{u}) + \frac{1}{2}\nabla\cdot\boldsymbol{u}\,\mathbf{1}_{2\times 2}\right) : \left(\boldsymbol{\epsilon}_0(\boldsymbol{u}) + \frac{1}{2}\nabla\cdot\boldsymbol{u}\,\mathbf{1}_{2\times 2}\right) d\boldsymbol{x}\\
&= \int_K \left(\boldsymbol{\epsilon}_0(\boldsymbol{u}) : \boldsymbol{\epsilon}_0(\boldsymbol{u}) + \frac{1}{2}|\nabla\cdot\boldsymbol{u}|^2\right) d\boldsymbol{x},
\end{aligned}$$

it follows that

$$\int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{u})\, d\boldsymbol{x} \geq m_{\text{elast}} \int_K \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{u})\, d\boldsymbol{x}.$$

Thus,

$$\begin{aligned}
B_h(\boldsymbol{u},\boldsymbol{u}) &\geq m_{\text{elast}} \sum_{K\in\mathcal{T}} \int_K \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{u})\, d\boldsymbol{x} + \mu \sum_{e\in\mathcal{E}_{\text{int},D}} |e|^{-1} \int_e \underline{[\boldsymbol{u}]} : \underline{[\boldsymbol{u}]}\, ds\\
&\geq m_{\text{elast}} \|\boldsymbol{u}\|_h^2.
\end{aligned}$$

□

The above result immediately implies:

Theorem 3.3.5 *Let* (3.6) *hold for (at least) $m = 0$. Then, the DGFEM* (3.10) *has a unique solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$.*

3.4 Error Analysis

3.4.1 Interpolants

Proposition 3.4.1 *Let $K \in \mathcal{T}$ be a triangle with vertices A_1, A_2, A_3. Then, for each $\beta_K \in [0, 1)$ and for $\Phi_{\beta_K}(\boldsymbol{x}) = r_K^{\beta_K} = |\boldsymbol{x} - A_1|^{\beta_K}$, there exists an interpolant*

$$\pi_K : H^{2,2}_{\beta_K}(K)^2 \longrightarrow \mathcal{P}_1(K)^2$$

such that the following properties are satisfied:

a) $\displaystyle\int_e (\boldsymbol{u} - \pi_K \boldsymbol{u})\, ds = \boldsymbol{0}, \quad \forall e \in \mathcal{E}_K;$

b) $\displaystyle\int_e (\boldsymbol{u} - \pi_K \boldsymbol{u}) \cdot \boldsymbol{n}_e\, ds = 0, \quad \forall e \in \mathcal{E}_K;$

c) $\displaystyle\int_K \nabla \cdot (\boldsymbol{u} - \pi_K \boldsymbol{u})\, d\boldsymbol{x} = 0.$

Here, for $e \in \mathcal{E}_K$, $\boldsymbol{n}_e$ denotes the unit outward vector of K on e.

Proof: For $\boldsymbol{u} \in H^{2,2}_{\beta_K}(K)^2$ the interpolant $\pi_K \boldsymbol{u} \in \mathcal{P}_1(K)^2$ is uniquely defined by

$$\pi_K \boldsymbol{u}(\overline{x}_e) = \frac{1}{|e|} \int_e \boldsymbol{u}\, ds, \quad \forall e \in \mathcal{E}_K,$$

where $\overline{x}_e$ denotes the midpoint of $e \in \mathcal{E}_K$. Then, a) and b) follow directly from this definition. c) results from b) and from Green's formula:

$$\int_K \nabla \cdot (\boldsymbol{u} - \pi_K \boldsymbol{u})\, d\boldsymbol{x} = \int_{\partial K} (\boldsymbol{u} - \pi_K \boldsymbol{u}) \cdot \boldsymbol{n}_{\partial K}\, ds = 0.$$

□

Proposition 3.4.2 *For $\boldsymbol{u} \in H^{2,2}_{\beta_K}(K)^2$, $K \in \mathcal{T}$, the interpolant $\pi_K \boldsymbol{u}$ from Proposition 3.4.1 satisfies the following estimates:*

$$\|\boldsymbol{u} - \pi_K \boldsymbol{u}\|_{L^2(K)} + h_K |\boldsymbol{u} - \pi_K \boldsymbol{u}|_{H^1(K)} \leq C h_K^{2-\beta_K} |\boldsymbol{u}|_{H^{2,2}_{\beta_K}(K)} \quad (3.13)$$

$$|\boldsymbol{u} - \pi_K \boldsymbol{u}|_{H^{2,2}_{\beta_K}(K)} \leq |\boldsymbol{u}|_{H^{2,2}_{\beta_K}(K)} \quad (3.14)$$

and

$$\|\nabla \cdot (\boldsymbol{u} - \pi_K \boldsymbol{u})\|_{L^2(K)} \leq C h_K^{1-\beta_K} |\nabla \cdot \boldsymbol{u}|_{H^{1,1}_{\beta_K}(K)} \quad (3.15)$$

$$|\nabla \cdot (\boldsymbol{u} - \pi_K \boldsymbol{u})|_{H^{1,1}_{\beta_K}(K)} \leq |\nabla \cdot \boldsymbol{u}|_{H^{1,1}_{\beta_K}(K)}. \quad (3.16)$$

$C > 0$ is a constant independent of h_K and of $\boldsymbol{u}$.

Proof: Set $\boldsymbol{U} = \boldsymbol{u} - \pi_K \boldsymbol{u}$. Then, since $\pi_K \boldsymbol{u} \in \mathcal{P}_1(K)^2$, there holds:

$$|\boldsymbol{U}|_{H^{2,2}_{\beta_K}(K)} = |\boldsymbol{u}|_{H^{2,2}_{\beta_K}(K)} \quad \text{and} \quad |\nabla \cdot \boldsymbol{U}|_{H^{1,1}_{\beta_K}(K)} = |\nabla \cdot \boldsymbol{u}|_{H^{1,1}_{\beta_K}(K)}.$$

Thus, applying Proposition A.2.12 to $\boldsymbol{U}$ and Corollary A.2.11 to $\nabla \cdot \boldsymbol{U}$, completes the proof. □

3.4.2 Stability

In a polygon Ω consider a finite element mesh $\mathcal{T}$ satisfying the conditions from Section 1.3.1. Moreover, let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_M)$ be a weight vector and $\Phi_{\boldsymbol{\beta}}$ the corresponding weight function as described in Section 1.2. Then, on $\boldsymbol{V}_h$, define the interpolant

$$\Pi_{\mathcal{T}} : H^{2,2}_{\boldsymbol{\beta}}(\Omega)^2 \longrightarrow \boldsymbol{V}_h$$

by

$$\Pi_{\mathcal{T}}|_K \boldsymbol{u} = \pi_K \boldsymbol{u}, \quad \forall K \in \mathcal{T},$$

where π_K, $K \in \mathcal{T}$ is the interpolant from Proposition 3.4.1.

Then, the DG-error $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$, where $\boldsymbol{u}$ is the exact solution of the linear elasticity problem (3.1) and $\boldsymbol{u}_h$ is the solution of the DGFEM (3.10), can be represented as follows:

$$\boldsymbol{e}_h = \underbrace{\boldsymbol{u} - \Pi_{\mathcal{T}} \boldsymbol{u}}_{=\boldsymbol{\eta}} + \underbrace{\Pi_{\mathcal{T}} \boldsymbol{u} - \boldsymbol{u}_h}_{=\boldsymbol{\xi}}. \tag{3.17}$$

Remark 3.4.3 Since $H^{2,2}_{\boldsymbol{\beta}}(\Omega)^2 \subset \mathcal{C}^0(\overline{\Omega})^2$ (cf. Remark 1.2.2), $\boldsymbol{u} \in H^{2,2}_{\boldsymbol{\beta}}(\Omega)^2$ implies that

$$\int_e \underline{[\boldsymbol{\eta}]}\, ds = \boldsymbol{0}$$

for all edges $e \in \mathcal{E}_{\text{int}}$.

Proposition 3.4.5 shows that $\|\boldsymbol{\xi}\|_h$ is bounded by $\|\boldsymbol{\eta}\|_h$. Therefore, the error $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$ of the DGFEM may be controlled by $\boldsymbol{\eta}$ only (as in the diffusion case). In order to prove this, consider the following Lemma which will be useful for the error analysis of the DGFEM.

Lemma 3.4.4 *Let $\boldsymbol{u} \in H^{2,2}_{\boldsymbol{\beta}}(\Omega)^2$. Then,*

$$
\begin{aligned}
&\mu^2 \sum_{K\in\mathcal{T}} \|\boldsymbol{\epsilon}(\boldsymbol{\eta})\|^2_{L^2(K)} + \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\mathcal{E}_{int,D}}} \|\boldsymbol{\sigma}(\boldsymbol{\eta})\|^2_{L^1(e)} + \mu^2 \sum_{e\in\mathcal{E}_{int,D}} |e|^{-1}\|[\boldsymbol{\eta}]\|^2_{L^2(e)} \\
&\quad\le C\Big\{\mu^2\Big(\sum_{K\in\mathcal{T}} \big(h_K^{-2}\|\boldsymbol{\eta}\|^2_{L^2(K)} + |\boldsymbol{\eta}|^2_{H^1(K)}\big) + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2 |\boldsymbol{\eta}|^2_{H^2(K)} \\
&\qquad + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K} |\boldsymbol{\eta}|^2_{H^{2,2}_{\beta_K}(K)}\Big) + \lambda^2\Big(\sum_{K\in\mathcal{T}} \|\nabla\cdot\boldsymbol{\eta}\|^2_{L^2(K)} \\
&\qquad + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2 |\nabla\cdot\boldsymbol{\eta}|^2_{H^1(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K} |\nabla\cdot\boldsymbol{\eta}|^2_{H^{1,1}_{\beta_K}(K)}\Big)\Big\},
\end{aligned}
$$

where $\boldsymbol{\eta} = \boldsymbol{u} - \Pi_{\mathcal{T}}\boldsymbol{u}$.

Proof: Obviously,

$$
\sum_{K\in\mathcal{T}} \|\boldsymbol{\epsilon}(\boldsymbol{\eta})\|^2_{L^2(K)} \le C \sum_{K\in\mathcal{T}} |\boldsymbol{\eta}|^2_{H^1(K)}.
$$

Furthermore, Lemma 1.3.2 c) and Remark 1.2.2 a) imply that

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\mathcal{E}_{\mathrm{int},D}}} \|\boldsymbol{\sigma}(\boldsymbol{\eta})\|^2_{L^1(e)} \\
&\le C\Big(\mu^2 \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\mathcal{E}_{\mathrm{int},D}}} \|\boldsymbol{\epsilon}(\boldsymbol{\eta})\|^2_{L^1(e)} + \lambda^2 \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\mathcal{E}_{\mathrm{int},D}}} \|\nabla\cdot\boldsymbol{\eta}\|^2_{L^1(e)}\Big) \\
&\le C\mu^2\Big(\sum_{K\in\mathcal{T}} \|\nabla\boldsymbol{\eta}\|^2_{L^2(K)} + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2|\boldsymbol{\eta}|^2_{H^2(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K}|\boldsymbol{\eta}|^2_{H^{2,2}_{\beta_K}(K)}\Big) \\
&\quad + C\lambda^2\Big(\sum_{K\in\mathcal{T}} \|\nabla\cdot\boldsymbol{\eta}\|^2_{L^2(K)} + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2|\nabla\cdot\boldsymbol{\eta}|^2_{H^1(K)} \\
&\quad + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K}|\nabla\cdot\boldsymbol{\eta}|^2_{H^{1,1}_{\beta_K}(K)}\Big).
\end{aligned}
$$

Additionally, by the standard trace theorem (cf. [58, Theorem A.11], for example), there holds

$$
\begin{aligned}
\sum_{e\in\mathcal{E}_{\mathrm{int},D}} |e|^{-1}\|[\boldsymbol{\eta}]\|^2_{L^2(e)} &\le C \sum_{K\in\mathcal{T}} \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int},D}} |e|^{-1}\|\boldsymbol{\eta}\|^2_{L^2(e)} \\
&\le C \sum_{e\in\mathcal{E}_{\mathrm{int},D}} \big(|e|^{-2}\|\boldsymbol{\eta}\|^2_{L^2(K)} + |\nabla\boldsymbol{\eta}|^2_{L^2(K)}\big)
\end{aligned}
$$

$$\leq C \sum_{e \in \mathcal{E}_{\mathrm{int},D}} \left(h_K^{-2} \|\boldsymbol{\eta}\|^2_{L^2(K)} + |\nabla \boldsymbol{\eta}|^2_{L^2(K)}\right).$$

□

Proposition 3.4.5 (Stability) *Let the exact solution $\boldsymbol{u}$ of the linear elasticity problem* (3.1) *be in $H^{2,2}_{\boldsymbol{\beta}}(\Omega)^2$ for a weight vector $0 \preccurlyeq \boldsymbol{\beta} \prec 1$. Then, there holds the following stability estimate for the DGFEM* (3.10)

$$\|\boldsymbol{\xi}\|_h^2 \leq C C_{\mu,\lambda} (\mu^2 (E_1^\mu + E_2^\mu + E_3^\mu) + \lambda^2 (E_1^\lambda + E_2^\lambda + E_3^\lambda))$$

with

$$E_1^\mu = \sum_{K \in \mathcal{T}} \left(h_K^{-2} \|\boldsymbol{\eta}\|^2_{L^2(K)} + |\boldsymbol{\eta}|^2_{H^1(K)}\right) \qquad E_1^\lambda = \sum_{K \in \mathcal{T}} \|\nabla \cdot \boldsymbol{\eta}\|^2_{L^2(K)}$$

$$E_2^\mu = \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 |\boldsymbol{\eta}|^2_{H^2(K)} \qquad E_2^\lambda = \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 |\nabla \cdot \boldsymbol{\eta}|^2_{H^1(K)}$$

$$E_3^\mu = \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta_K} |\boldsymbol{\eta}|^2_{H^{2,2}_{\beta_K}(K)} \qquad E_3^\lambda = \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta_K} |\nabla \cdot \boldsymbol{\eta}|^2_{H^{1,1}_{\beta_K}(K)},$$

where $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are defined in (3.17)*, and where*

$$C_{\mu,\lambda} = \max\left\{1, \sqrt{\frac{2\min\{\mu, \mu + \lambda\}}{\mu}}\right\}$$

is bounded independently of λ and μ as $\lambda \to \infty$. Moreover, $C > 0$ is a constant independent of μ, λ and of $\boldsymbol{h}$.

Proof: Due to the consistency of the DGFEM (cf. Proposition 3.3.3), it holds that

$$B_h(\boldsymbol{\xi}, \boldsymbol{\xi}) = B_h(\boldsymbol{e} - \boldsymbol{\eta}, \boldsymbol{\xi}) = -B_h(\boldsymbol{\eta}, \boldsymbol{\xi}).$$

Therefore, by Proposition 3.3.4,

$$m_{\mathrm{elast}} \|\boldsymbol{\xi}\|_h^2 \leq -B_h(\boldsymbol{\eta}, \boldsymbol{\xi}). \tag{3.18}$$

Furthermore, using that $\nabla \cdot \boldsymbol{\xi}$ and $\boldsymbol{\sigma}(\boldsymbol{\xi})$ are element-wise constant leads to

$$\begin{aligned}
B_h(\boldsymbol{\eta}, \boldsymbol{\xi}) = &\sum_{K \in \mathcal{T}} \int_K \boldsymbol{\sigma}(\boldsymbol{\eta}) : \boldsymbol{\epsilon}(\boldsymbol{\xi})\, d\boldsymbol{x} \\
&- \sum_{e \in \mathcal{E}_{\text{int},D}} \int_e (\langle \boldsymbol{\sigma}(\boldsymbol{\eta}) \rangle : \underline{[\boldsymbol{\xi}]} - \underline{[\boldsymbol{\eta}]} : \langle \boldsymbol{\sigma}(\boldsymbol{\xi}) \rangle)\, ds \\
&+ \mu \sum_{e \in \mathcal{E}_{\text{int},D}} |e|^{-1} \int_e \underline{[\boldsymbol{\eta}]} : \underline{[\boldsymbol{\xi}]}\, ds \\
= &\, 2\mu \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\epsilon}(\boldsymbol{\eta}) : \boldsymbol{\epsilon}(\boldsymbol{\xi})\, d\boldsymbol{x} + \lambda \sum_{K \in \mathcal{T}} \nabla \cdot \boldsymbol{\xi} \int_K \nabla \cdot \boldsymbol{\eta}\, d\boldsymbol{x} \\
&- \sum_{e \in \mathcal{E}_{\text{int},D}} \Big(\int_e \langle \boldsymbol{\sigma}(\boldsymbol{\eta}) \rangle : \underline{[\boldsymbol{\xi}]}\, ds - \langle \boldsymbol{\sigma}(\boldsymbol{\xi}) \rangle : \int_e \underline{[\boldsymbol{\eta}]}\, ds \Big) \\
&+ \mu \sum_{e \in \mathcal{E}_{\text{int},D}} |e|^{-1} \int_e \underline{[\boldsymbol{\eta}]} : \underline{[\boldsymbol{\xi}]}\, ds.
\end{aligned}$$

Applying Proposition 3.4.1 and Remark 3.4.3 results in

$$\begin{aligned}
B_h(\boldsymbol{\eta}, \boldsymbol{\xi}) = &\, 2\mu \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\epsilon}(\boldsymbol{\eta}) : \boldsymbol{\epsilon}(\boldsymbol{\xi})\, d\boldsymbol{x} - \sum_{e \in \mathcal{E}_{\text{int},D}} \int_e \langle \boldsymbol{\sigma}(\boldsymbol{\eta}) \rangle : \underline{[\boldsymbol{\xi}]}\, ds \\
&+ \mu \sum_{e \in \mathcal{E}_{\text{int},D}} |e|^{-1} \int_e \underline{[\boldsymbol{\eta}]} : \underline{[\boldsymbol{\xi}]}\, ds \\
= &\, I - II + III.
\end{aligned}$$

By Hölder's inequality, there holds that

$$\begin{aligned}
|I| &= \Big| 2\mu \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\epsilon}(\boldsymbol{\eta}) : \boldsymbol{\epsilon}(\boldsymbol{\xi})\, d\boldsymbol{x} \Big| \\
&\le \Big(4\mu^2 \sum_{K \in \mathcal{T}} \|\boldsymbol{\epsilon}(\boldsymbol{\eta})\|^2_{L^2(K)} \Big)^{1/2} \Big(\sum_{K \in \mathcal{T}} \|\boldsymbol{\epsilon}(\boldsymbol{\xi})\|^2_{L^2(K)} \Big)^{1/2}.
\end{aligned}$$

A bound for II is obtained as follows:

$$\begin{aligned}
|II| &\le \sum_{e \in \mathcal{E}_{\text{int},D}} \int_e |\langle \boldsymbol{\sigma}(\boldsymbol{\eta}) \rangle|\,|[\boldsymbol{\xi}]|\, ds \\
&\le \sum_{e \in \mathcal{E}_{\text{int},D}} \|[\boldsymbol{\xi}]\|_{L^\infty(e)} \|\langle \boldsymbol{\sigma}(\boldsymbol{\eta}) \rangle\|_{L^1(e)} \\
&\le C \sum_{K \in \mathcal{T}} \sum_{e \in \mathcal{E}_K \cap \mathcal{E}_{\text{int},D}} \|[\boldsymbol{\xi}]\|_{L^\infty(e)} \|\boldsymbol{\sigma}(\boldsymbol{\eta})\|_{L^1(e)}.
\end{aligned}$$

Furthermore, Lemma A.1.1 implies that

$$\begin{aligned}
|II| &\leq C \sum_{K\in\mathcal{T}} \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\text{int},D}} |e|^{-1/2}\|[\boldsymbol{\xi}]\|_{L^2(e)}\|\boldsymbol{\sigma}(\boldsymbol{\eta})\|_{L^1(e)} \\
&\leq C\Big(\sum_{K\in\mathcal{T}} \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\text{int},D}} |e|^{-1}\|[\boldsymbol{\xi}]\|^2_{L^2(e)}\Big)^{1/2} \\
&\quad \cdot \Big(\sum_{K\in\mathcal{T}} \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\text{int},D}} \|\boldsymbol{\sigma}(\boldsymbol{\eta})\|^2_{L^1(e)}\Big)^{1/2} \\
&= C\sqrt{\frac{m_{\text{elast}}}{\mu}}\Big(\frac{\mu}{m_{\text{elast}}} \sum_{e\in\mathcal{E}_{\text{int},D}} |e|^{-1}\|[\boldsymbol{\xi}]\|^2_{L^2(e)}\Big)^{1/2} \\
&\quad \cdot \Big(\sum_{K\in\mathcal{T}} \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\text{int},D}} \|\boldsymbol{\sigma}(\boldsymbol{\eta})\|^2_{L^1(e)}\Big)^{1/2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
|III| \leq \sqrt{\frac{m_{\text{elast}}}{\mu}}\Big(\mu^2 \sum_{e\in\mathcal{E}_{\text{int},D}} |e|^{-1}\|[\boldsymbol{\eta}]\|^2_{L^2(e)}\Big)^{1/2} \\
\Big(\frac{\mu}{m_{\text{elast}}} \sum_{e\in\mathcal{E}_{\text{int},D}} |e|^{-1}\|[\boldsymbol{\xi}]\|^2_{L^2(e)}\Big)^{1/2}.
\end{aligned}$$

Summing up and using (3.18) yields

$$\begin{aligned}
\|\boldsymbol{\xi}\|_h^2 &\leq \frac{1}{m_{\text{elast}}}|B_h(\boldsymbol{\eta},\boldsymbol{\xi})| \\
&\leq \frac{1}{m_{\text{elast}}}(|I| + |II| + |III|) \\
&\leq C\max\Big\{1, \sqrt{\frac{m_{\text{elast}}}{\mu}}\Big\}\|\boldsymbol{\xi}\|_h \cdot \Big(\mu^2 \sum_{K\in\mathcal{T}} \|\boldsymbol{\epsilon}(\boldsymbol{\eta})\|^2_{L^2(K)} \\
&\quad + \sum_{K\in\mathcal{T}} \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_{\text{int},D}} \|\boldsymbol{\sigma}(\boldsymbol{\eta})\|^2_{L^1(e)} + \mu^2 \sum_{e\in\mathcal{E}_{\text{int},D}} |e|^{-1}\|[\boldsymbol{\eta}]\|^2_{L^2(e)}\Big)^{1/2}.
\end{aligned}$$

Applying Lemma 3.4.4 completes the proof immediately. □

A direct consequence of the above statement is the ensuing

Corollary 3.4.6 *Let the assumptions of Proposition 3.4.5 be satisfied. Moreover, let* E_i^μ *and* E_i^λ, $i = 1, 2, 3$, *be defined as before. Then, the following a priori error estimate holds true*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h^2 \leq C\widetilde{C}_{\mu,\lambda}(\mu^2(E_1^\mu + E_2^\mu + E_3^\mu) + \lambda^2(E_1^\lambda + E_2^\lambda + E_3^\lambda)).$$

Here, $\boldsymbol{u}$ *is the exact solution of* (3.1), $\boldsymbol{u}_h$ *is the solution of the DGFEM* (3.10) *and*

$$\widetilde{C}_{\mu,\lambda} = \max\{\mu^{-2}, \mu^{-1} m_{elast}^{-1}, C_{\mu,\lambda}\},$$

where $C_{\mu,\lambda}$ *is the constant from Proposition 3.4.5.*

Remark 3.4.7 Obviously, the constant $\widetilde{C}_{\mu,\lambda}$ from the above Corollary 3.4.6 loses its dependence on λ if λ is sufficiently large, i.e.:

$$\exists \lambda_0(\mu) : \widetilde{C}_{\mu,\lambda} \leq \widetilde{C}_{\mu} \quad \forall \lambda > \lambda_0,$$

where $\widetilde{C}_{\mu}$ is a constant independent of λ.

Proof: From the error splitting (3.17) it follows that

$$\begin{aligned}
\|\boldsymbol{e}\|_h^2 &\leq C(\|\boldsymbol{\eta}\|_h^2 + \|\boldsymbol{\xi}\|_h^2) \\
&\leq C\Big(\sum_{K\in\mathcal{T}} \|\boldsymbol{\epsilon}(\boldsymbol{\eta})\|_{L^2(K)}^2 + \frac{\mu}{m_{\text{elast}}} \sum_{e\in\mathcal{E}_{\text{int},D}} |e|^{-1} \int_e |\underline{[\boldsymbol{\eta}]}|^2\, ds + \|\boldsymbol{\xi}\|_h^2 \Big) \\
&\leq C \max\{\mu^{-2}, \mu^{-1} m_{\text{elast}}^{-1}\} \Big(C \|\boldsymbol{\xi}\|_{L^2(K)}^2 \\
&\quad + \mu^2 \sum_{K\in\mathcal{T}} \|\boldsymbol{\epsilon}(\boldsymbol{\eta})\|_{L^2(K)}^2 + \mu^2 \sum_{e\in\mathcal{E}_{\text{int},D}} |e|^{-1} \int_e |\underline{[\boldsymbol{\eta}]}|^2\, ds \Big).
\end{aligned}$$

Thus, using Lemma 3.4.4 and inserting the stability bound from Proposition 3.4.5 completes the proof. ☐

3.4.3 Optimal Convergence of the DGFEM

The mesh refinement strategies from Section 2.5 (graded meshes), which lead to optimal algebraic convergence rates for the DGFEM for diffusion problems, are also applicable to the DGFEM presented in this chapter. More precisely, it will be proved here that for $\boldsymbol{u} \in H_{\boldsymbol{\beta}}^{2,2}(\Omega)^2$, where $\boldsymbol{u}$ denotes the exact solution of the linear elasticity problem (3.1), the following error bound may be obtained:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h \leq C N^{-1/2}. \tag{3.19}$$

In addition, the convergence is robust, i.e. the constant C in (3.19) is independent of the Lamé coefficient λ as $\lambda \to \infty$. This is typically not true for conforming finite element methods whose convergence regime usually deteriorates substantially for nearly incompressible materials (cf. Section 3.5 and the numerical experiments there).

Theorem 3.4.8 (Robust Optimal Convergence) *Let the exact solution* $\boldsymbol{u}$ *of* (3.1) *belong to* $H^{2,2}_{\boldsymbol{\beta}}(\Omega)^2$ *for a weight vector* $0 \preccurlyeq \boldsymbol{\beta} \prec 1$*. Moreover, let* $\mathcal{T}_{\boldsymbol{\gamma}}$ *with* $1 \succ \boldsymbol{\gamma} \succeq \boldsymbol{\beta}$ *be a graded mesh as introduced in Definition 2.5.1 (for* $p = 1$*). Then, for the solution* $\boldsymbol{u}_h$ *of the h DGFEM* (3.10)*, there holds the following optimal algebraic error estimate:*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h \leq C\widetilde{C}_{\mu,\lambda} N^{-1/2}.$$

Here, $N = \dim(\boldsymbol{V}_h)$*,* $\widetilde{C}_{\mu,\lambda}$ *is the constant from Corollary 3.4.6 (independent of* λ *as* $\lambda \to \infty$*) and* $C > 0$ *is a constant independent of* N *and of the Lamé coefficients* μ *and* λ*.*

Proof: Let $\Pi_{\mathcal{T}_{\boldsymbol{\gamma}}}$ be the global interpolant from Section 3.4.2, i.e.

$$\Pi_{\mathcal{T}_{\boldsymbol{\gamma}}}|_K = \pi_K, \quad K \in \mathcal{T}_{\boldsymbol{\gamma}},$$

where π_K is the interpolant from Proposition 3.4.1. Referring to Corollary 3.4.6 yields the following error bound for the DGFEM:

$$\begin{aligned}
&\|\boldsymbol{u} - \boldsymbol{u}_h\|_h^2 \\
&\quad \leq C\widetilde{C}_{\mu,\lambda}\Big\{\mu^2\Big(\sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}}\big(h_K^{-2}\|\boldsymbol{u} - \pi_K\boldsymbol{u}\|^2_{L^2(K)} + |\boldsymbol{u} - \pi_K\boldsymbol{u}|^2_{H^1(K)}\big) \\
&\qquad + \sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}\setminus\mathcal{K}_0} h_K^2|\boldsymbol{u} - \pi_K\boldsymbol{u}|^2_{H^2(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K}|\boldsymbol{u} - \pi_K\boldsymbol{u}|^2_{H^{2,2}_{\beta_K}(K)}\Big) \\
&\qquad + \lambda^2\Big(\sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}}\|\nabla\cdot(\boldsymbol{u} - \pi_K\boldsymbol{u})\|^2_{L^2(K)} \\
&\qquad + \sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}\setminus\mathcal{K}_0} h_K^2|\nabla\cdot(\boldsymbol{u} - \pi_K\boldsymbol{u})|^2_{H^1(K)} \\
&\qquad + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K}|\nabla\cdot(\boldsymbol{u} - \pi_K\boldsymbol{u})|^2_{H^{1,1}_{\beta_K}(K)}\Big)\Big\}.
\end{aligned}$$

Moreover, inserting the estimates from Proposition 3.4.2 results in

$$\begin{aligned}
&\|\boldsymbol{u} - \boldsymbol{u}_h\|_h^2 \\
&\quad \leq C\widetilde{C}_{\mu,\lambda}\Big\{\mu^2\Big(\sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}\setminus\mathcal{K}_0} h_K^2|\boldsymbol{u}|^2_{H^2(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K}|\boldsymbol{u}|^2_{H^{2,2}_{\beta_K}(K)}\Big) \\
&\qquad + \lambda^2\Big(\sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}\setminus\mathcal{K}_0} h_K^2|\nabla\cdot\boldsymbol{u}|^2_{H^1(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K}|\nabla\cdot\boldsymbol{u}|^2_{H^{1,1}_{\beta_K}(K)}\Big)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= C\widetilde{C}_{\mu,\lambda}\Big(\sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}\setminus\mathcal{K}_0} h_K^2 \big(\mu^2|\boldsymbol{u}|^2_{H^2(K)} + \lambda^2|\nabla\cdot\boldsymbol{u}|^2_{H^1(K)}\big) \\
&\quad + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K}\big(\mu^2|\boldsymbol{u}|^2_{H^{2,2}_{\beta_K}(K)} + \lambda^2|\nabla\cdot\boldsymbol{u}|^2_{H^{1,1}_{\beta_K}(K)}\big)\Big).
\end{aligned}
\tag{3.20}
$$

Furthermore, from the definition of the graded meshes (Definition 2.5.1) it follows that

$$
\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}_h\|_h^2 \\
&\quad\le C\widetilde{C}_{\mu,\lambda}\Big(h^2_{\mathcal{T}_{\boldsymbol{\gamma}}} \sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}\setminus\mathcal{K}_0} \int_K r_K^{2\gamma_K}(\mu^2|D^2\boldsymbol{u}|^2 + \lambda^2|D^1(\nabla\cdot\boldsymbol{u})|^2)\,d\boldsymbol{x} \\
&\qquad + \sum_{K\in\mathcal{K}_0} h^{2-2\beta_K}_{\mathcal{T}_{\boldsymbol{\gamma}}}\big(\sup_{\boldsymbol{x}\in K} r_K^{\gamma_K}\big)^{2-2\beta_K}\big(\mu^2|\boldsymbol{u}|^2_{H^{2,2}_{\beta_K}(K)} + \lambda^2|\nabla\cdot\boldsymbol{u}|^2_{H^{1,1}_{\beta_K}(K)}\big)\Big).
\end{aligned}
$$

For all $K \in \mathcal{K}_0$ there holds $r_K(\boldsymbol{x}) \le h_K \ \forall \boldsymbol{x} \in K$. Hence,

$$
h_K \le Ch_{\mathcal{T}_{\boldsymbol{\gamma}}} \sup_{\boldsymbol{x}\in K} r_K^{\gamma_K} \le Ch_{\mathcal{T}_{\boldsymbol{\gamma}}} h_K^{\gamma_K},
$$

and therefore

$$
h_K \le Ch_{\mathcal{T}_{\boldsymbol{\gamma}}}^{\frac{1}{1-\gamma_K}}.
$$

This implies that

$$
\sup_{\boldsymbol{x}\in K} r_K^{\gamma_K} \le Ch_K^{\gamma_K} \le Ch_{\mathcal{T}_{\boldsymbol{\gamma}}}^{\frac{\gamma_K}{1-\gamma_K}} \le Ch_{\mathcal{T}_{\boldsymbol{\gamma}}}^{\frac{\beta_K}{1-\beta_K}}.
$$

Thus,

$$
\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}_h\|_h^2 \\
&\quad\le C\widetilde{C}_{\mu,\lambda} h^2_{\mathcal{T}_{\boldsymbol{\gamma}}}\Big(\sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}\setminus\mathcal{K}_0} \int_K r_K^{2\gamma_K}(\mu^2|D^2\boldsymbol{u}|^2 + \lambda^2|D^1(\nabla\cdot\boldsymbol{u})|^2)\,d\boldsymbol{x} \\
&\qquad + \sum_{K\in\mathcal{K}_0} \big(\mu^2|\boldsymbol{u}|^2_{H^{2,2}_{\beta_K}(K)} + \lambda^2|\nabla\cdot\boldsymbol{u}|^2_{H^{1,1}_{\beta_K}(K)}\big)\Big) \\
&\quad\le C\widetilde{C}_{\mu,\lambda} h^2_{\mathcal{T}_{\boldsymbol{\gamma}}}\Big(\sum_{K\in\mathcal{T}_{\boldsymbol{\gamma}}\setminus\mathcal{K}_0} \int_K \Phi^2_{\boldsymbol{\beta}}(\mu^2|D^2\boldsymbol{u}|^2 + \lambda^2|D^1(\nabla\cdot\boldsymbol{u})|^2)\,d\boldsymbol{x} \\
&\qquad + \sum_{K\in\mathcal{K}_0} \big(\mu^2|\boldsymbol{u}|^2_{H^{2,2}_{\beta_K}(K)} + \lambda^2|\nabla\cdot\boldsymbol{u}|^2_{H^{1,1}_{\beta_K}(K)}\big)\Big) \\
&\quad\le C\widetilde{C}_{\mu,\lambda} h^2_{\mathcal{T}_{\boldsymbol{\gamma}}}\Big(\int_\Omega \Phi^2_{\boldsymbol{\beta}}(\mu^2|D^2\boldsymbol{u}|^2 + \lambda^2|D^1(\nabla\cdot\boldsymbol{u})|^2)\,d\boldsymbol{x} \\
&\qquad + \sum_{K\in\mathcal{K}_0} \big(\mu^2|\boldsymbol{u}|^2_{H^{2,2}_{\beta_K}(K)} + \lambda^2|\nabla\cdot\boldsymbol{u}|^2_{H^{1,1}_{\beta_K}(K)}\big)\Big)
\end{aligned}
$$

$$\leq C\widetilde{C}_{\mu,\lambda} h^2_{\mathcal{T}_{\boldsymbol{\gamma}}} \left(\mu^2 |\boldsymbol{u}|^2_{H^{2,2}_{\beta_K}(\Omega)} + \lambda^2 |\nabla \cdot \boldsymbol{u}|^2_{H^{1,1}_{\beta_K}(\Omega)}\right).$$

Finally, by Lemma 2.5.6, i.e.

$$h_{\mathcal{T}_{\boldsymbol{\gamma}}} \leq C N^{-1/2},$$

and with the aid of Theorem 3.2.2, the proof is complete. □

Remark 3.4.9 On uniform meshes $\mathcal{T}_{\boldsymbol{\gamma}}$ it holds:

$$h_{\mathcal{T}_{\boldsymbol{\gamma}}} \sim h_K \sim \frac{1}{\sqrt{N}} \qquad \forall K \in \mathcal{T}_{\boldsymbol{\gamma}}.$$

Therefore, (3.20) directly implies that, even if $\boldsymbol{\gamma} = \mathbf{0}$, the DGFEM still converges independently of μ and λ. However, due to the occurrence of the term $h_K^{2-2\beta_K}$, the rate of convergence is not anymore optimal for $\boldsymbol{\beta} \succ \mathbf{0}$.

Remark 3.4.10 The DGFEM above is closely related to non-conforming methods of Crouzeix-Raviart type. In 1992, S. C. Brenner, L. Sung [14] already showed that these schemes are locking-free even for $p = 1$. However, their results are based on the assumption that the displacements are H^2 regular, and therefore, the case of non-convex polygons is in general not covered by that work. Nevertheless, applying the regularity results and the mesh refinement strategies presented in this chapter (Theorem 3.2.2, Theorem 3.4.8), it can be proved that the convergence statements in [14] are extensible to the case where the exact solutions of the elasticity problems exhibit corner singularities.

3.5 Numerical Results

The aim of this section is to confirm the theoretical results with some numerical examples. More precisely, it will be shown that, even if the exact solutions of the corresponding problems are singular, the convergence rate of the DGFEM on graded meshes remains of order $\mathcal{O}(N^{-1/2})$, as expected. Moreover, the robustness of the method against volume locking will be illustrated.

3.5.1 L-shaped Domain

Model Problem

Let Ω be the polygonal domain with vertices

$$A_1 = (0,0),\ A_2 = (-1,-1),\ A_3 = (1,-1),\ A_4 = (1,1),\ A_5 = (-1,1).$$

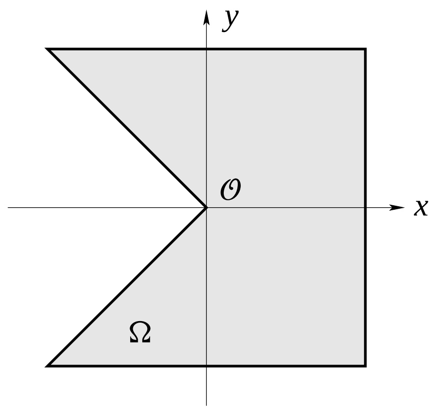

Figure 3.1: Polygonal domain Ω.

Note, that the origin $\mathcal{O} = (0,0)$ is a reentrant corner of Ω (cf. Figure 3.1). Then, consider the following model problem

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{0} \quad \text{in} \quad \Omega \\ \boldsymbol{u} &= \boldsymbol{g}_D \quad \text{on} \quad \Gamma_D = \partial\Omega. \end{aligned} \tag{3.21}$$

Here, $\boldsymbol{g}_D := \boldsymbol{u}|_{\Gamma_D}$, where $\boldsymbol{u}$ is the exact solution of (3.21) given by its polar coordinates

$$\begin{aligned} u_r(r,\theta) &= \frac{1}{2\mu} r^\alpha(-(\alpha+1)\cos((\alpha+1)\theta) + (C_2 - (\alpha+1))C_1\cos((\alpha-1)\theta)), \\ u_\theta(r,\theta) &= \frac{1}{2\mu} r^\alpha((\alpha+1)\sin((\alpha+1)\theta) + (C_2 + \alpha - 1)C_1\sin((\alpha-1)\theta)). \end{aligned}$$

Above, $\alpha \approx 0.544484$ is the solution of the equation

$$\alpha \sin(2\omega) + \sin(2\omega\alpha) = 0$$

with $\omega = {}^{3\pi}/_{4}$, and

$$C_1 = -\frac{\cos((\alpha+1)\omega)}{\cos((\alpha-1)\omega)}, \qquad C_2 = \frac{2(\lambda+2\mu)}{\lambda+\mu}.$$

Robust Optimal Convergence Rates on Graded Meshes

A few calculations show that the exact solution $\boldsymbol{u}$ of the model problem (3.21) is in $H^{2,2}_{\boldsymbol{\beta}}(\Omega)^2$ with $\boldsymbol{\beta} = (\beta_1, 0, 0, 0, 0)$ for all $1 > \beta_1 > 1 - \alpha \approx 0.455516$. Thus,

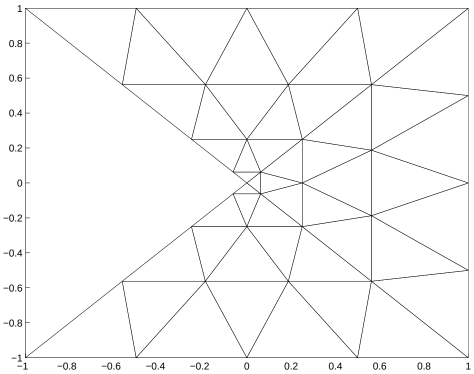

Figure 3.2: Graded mesh with refinement towards the origin ($\boldsymbol{\gamma} = (1/2, 0, 0, 0, 0)$).

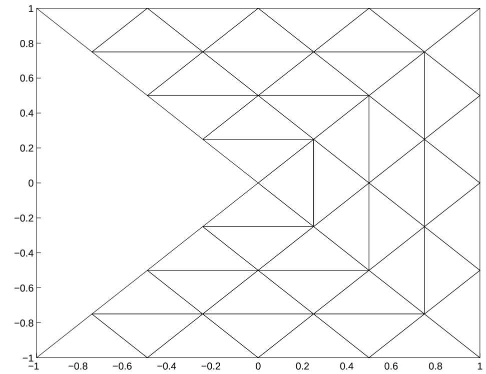

Figure 3.3: Uniform mesh (i.e. graded mesh with $\boldsymbol{\gamma} = (0, 0, 0, 0, 0)$).

in order to obtain the optimal convergence rate, a graded mesh with refinement towards the origin must be used for the numerical simulations.

The first picture of Figure 3.4 shows the errors of the DGFEM for

$$\lambda \in \{1, 100, 500, 1000, 5000\} \quad (\mu = 1)$$

in the energy norm

$$\|\boldsymbol{u}\|_h^2 = \sum_{K \in \mathcal{T}} \|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L^2(K)}^2 + \frac{1}{m_{\text{elast}}} \sum_{e \in \mathcal{E}_{\text{int},D}} |e|^{-1} \int_e |\underline{[\boldsymbol{u}]}|^2 \, ds$$

on a graded mesh with grading vector $\boldsymbol{\gamma} = (1/2, 0, 0, 0, 0)$ (cf. Figure 3.2). Obviously, the convergence rate of the DGFEM is already almost optimal for approximately 5000 degrees of freedom ($\sim$ 800 elements). Moreover, the expected robustness of the DGFEM with respect to the Lamé coefficient λ is clearly visible (the lines for $\lambda \geq 100$ almost coincide).

In the second picture of Figure 3.4 the energy error of the DGFEM on a uniform mesh (i.e. $\boldsymbol{\gamma} = (0, 0, 0, 0, 0)$) is presented. Although the DGFEM still converges robustly, the optimal convergence rate is not anymore achieved (cf. Remark 3.4.9) and the use of graded meshes is justified.

In addition, the L^2 errors for the computations above are shown in Figure 3.5. Again, the performance of the DGFEM on a uniform mesh is notably worse. However, the convergence rate of the L^2 error seems to be twice as high as of the energy error.

Volume Locking

Figure 3.6 shows that the standard (i.e. conforming) finite element method does not converge independently of λ. Although the asymptotic rate of convergence is

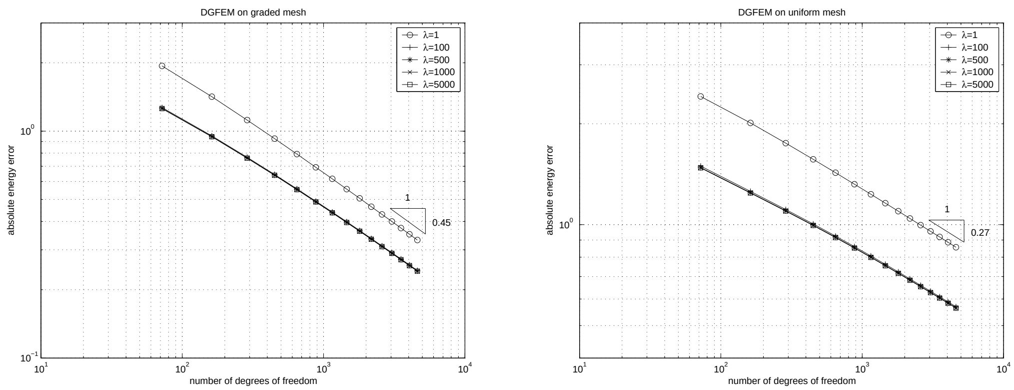

Figure 3.4: Performance of the DGFEM on the L-shaped domain with $\boldsymbol{\gamma} = (1/2, 0, 0, 0, 0)$ (graded mesh) and with $\boldsymbol{\gamma} = \mathbf{0}$ (uniform mesh).

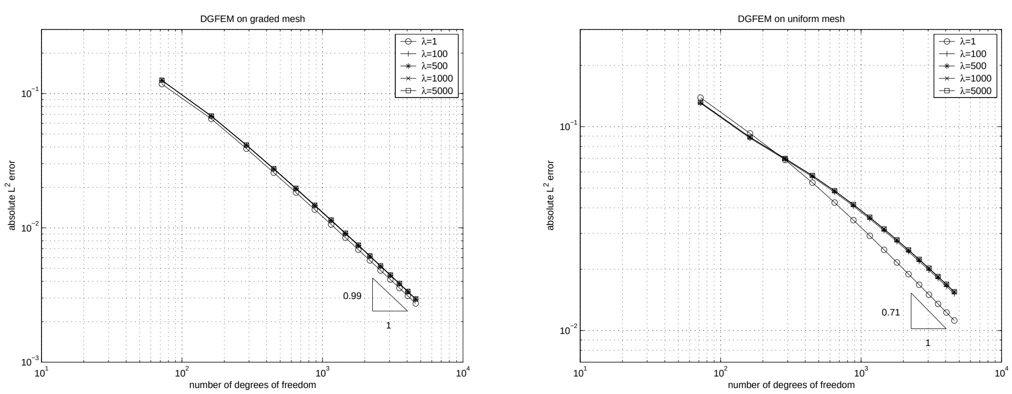

Figure 3.5: Performance of the DGFEM on the L-shaped domain with $\boldsymbol{\gamma} = (1/2, 0, 0, 0, 0)$ (graded mesh) and with $\boldsymbol{\gamma} = \mathbf{0}$ (uniform mesh).

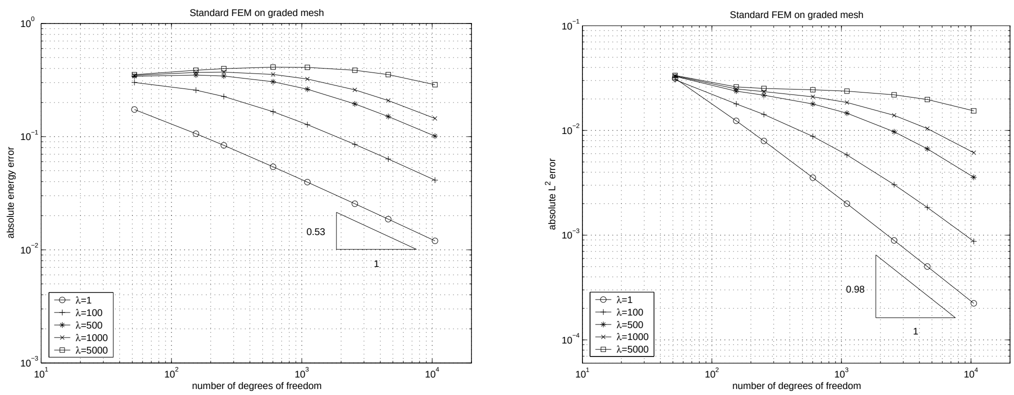

Figure 3.6: Performance of the conforming FEM on the L-shaped domain with $\boldsymbol{\gamma} = (1/2, 0, 0, 0, 0)$ (graded mesh).

optimal on graded meshes, the onset of the errors' decay is remarkably retarded for $\lambda \to \infty$. This non-robustness of the convergence rate with respect to λ is widely known as 'volume locking' which, in contrast to the DGFEM, seems to be unavoidable for low-order standard h FEM in the primal variables.

3.5.2 Unit Square

Consider the following problem on $\Omega = (0, 1)^2$:

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{0} && \text{in} \quad \Omega \\ \boldsymbol{u} &= \binom{g_D^{(1)}}{0} && \text{on} \quad \Gamma_D = \partial\Omega \end{aligned} \tag{3.22}$$

with

$$g_D^{(1)}(x, y) = \begin{cases} 1 - 4(x - 1/2)^2 & \text{if } (x, y) \in (0, 1) \times \{1\}, \\ 0 & \text{else.} \end{cases}$$

Due to Theorem 3.2.2, the exact solution of this problem belongs to $H^2(\Omega)^2$. Therefore, referring to the numerical analysis above, no mesh refinement is required for the DGFEM to converge optimally. The computational (uniform) mesh is shown in Figure 3.7. Additionally, the results for different choices of λ are presented (Figures 3.8–3.11). In contrast to the DGFEM, the standard FEM shows clear evidence of locking.

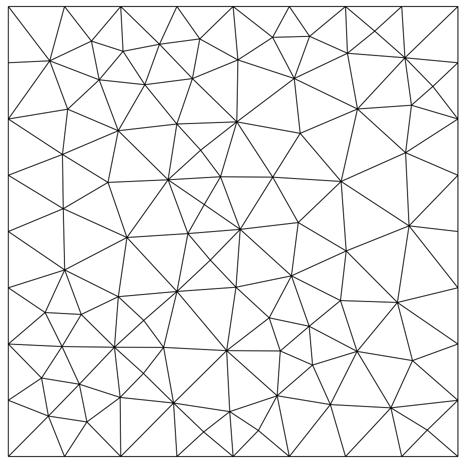

Figure 3.7: Computational mesh.

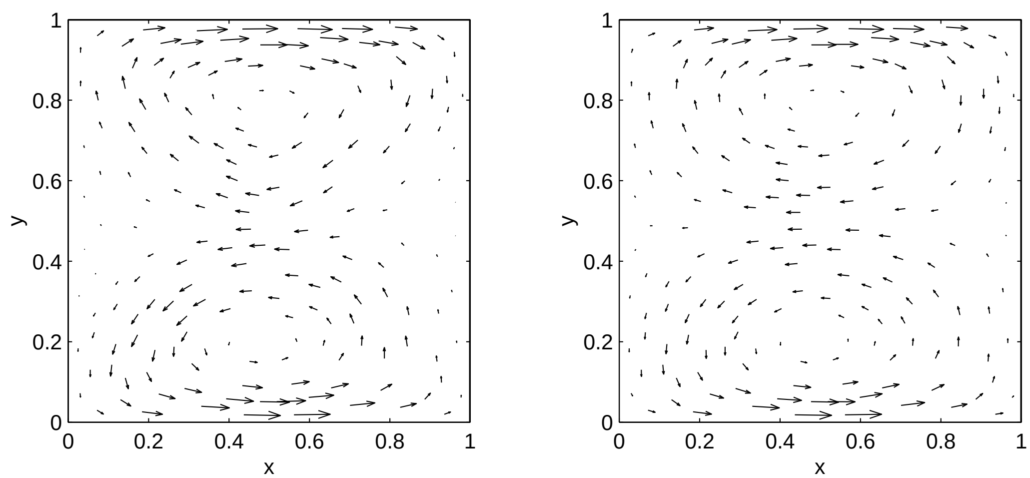

Figure 3.8: Standard FEM / DGFEM for $\lambda = 100$.

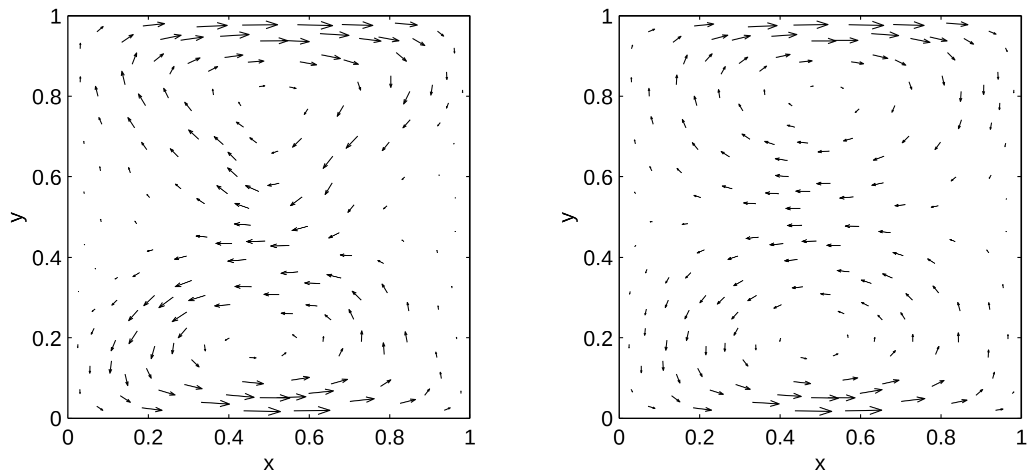

Figure 3.9: Standard FEM / DGFEM for $\lambda = 500$.

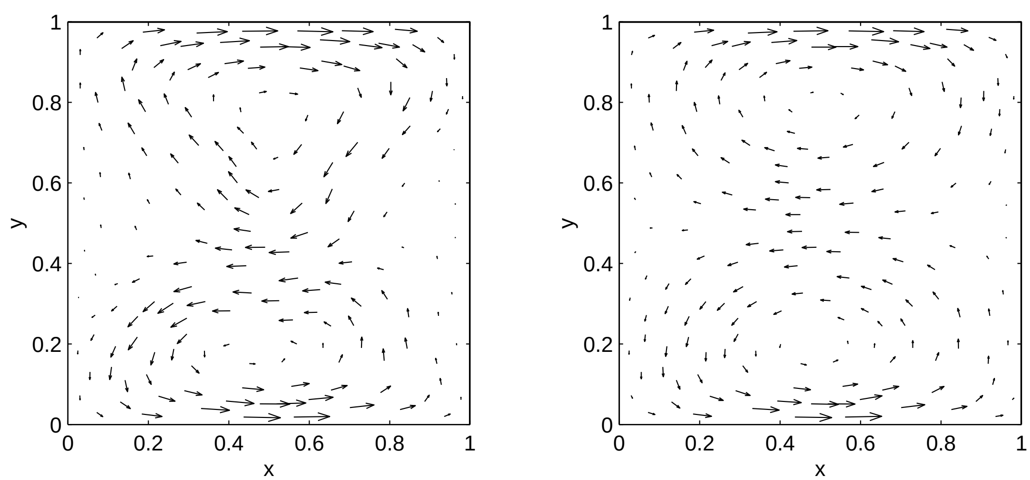

Figure 3.10: Standard FEM / DGFEM for $\lambda = 1000$.

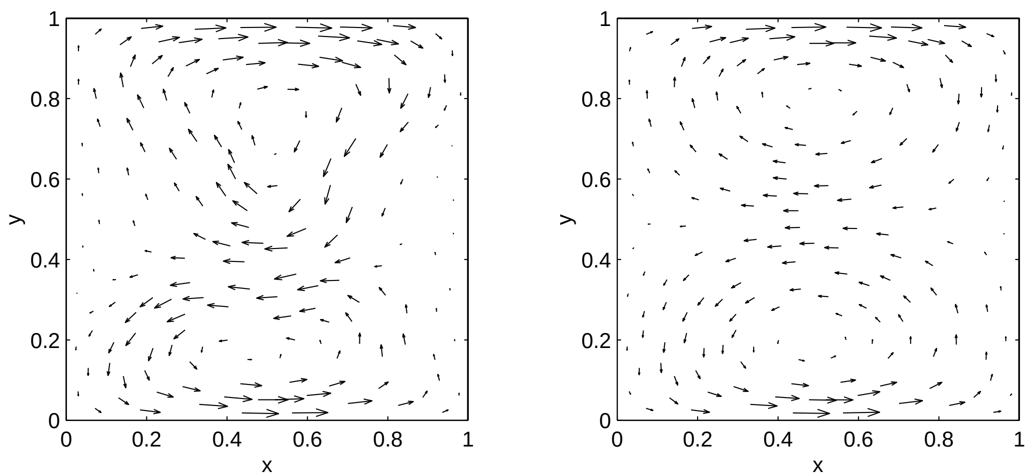

Figure 3.11: Standard FEM / DGFEM for $\lambda = 5000$.

Chapter 4

Mixed *hp* DGFEM for Stokes Problems

4.1 Problem Formulation

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with $\Gamma_D = \Gamma = \partial\Omega$ (cf. Section 1.1). The Stokes problem is to find a velocity field $\boldsymbol{u}$ and a pressure p such that

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

Here, the right-hand side $\boldsymbol{f} \in H^{-1}(\Omega)^2$ is an exterior body force, and $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^2$ a prescribed Dirichlet datum satisfying the compatibility condition

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n}_\Omega \, ds = 0. \tag{4.2}$$

Due to the continuous inf-sup condition, the Stokes system (4.1) has a unique solution $(\boldsymbol{u}, p)$ in $H_0^1(\Omega)^2 \times L_0^2(\Omega)$ (see, e.g., [15, 29] for details). Here,

$$L_0^2(\Omega) = \Big\{ u \in L^2(\Omega) : \int_\Omega u \, d\boldsymbol{x} = 0 \Big\}.$$

4.2 Regularity

A regularity result for (4.1), (4.2) follows directly from Theorem 3.2.1.

Theorem 4.2.1 *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. Then, there exists a weight vector $0 \preccurlyeq \boldsymbol{\beta}_{min} \prec 1$ depending on the opening angles at the vertices $A_i \in$*

$SP(\Omega, \Gamma_D, \Gamma_N)$, $i = 1, 2, \ldots, M$ (cf. Section 1.2), such that for weight vectors $\boldsymbol{\beta}$ with $\boldsymbol{\beta}_{min} \preccurlyeq \boldsymbol{\beta} \prec 1$ and for

$$\boldsymbol{f} \in \mathcal{B}^0_{\boldsymbol{\beta}}(\Omega)^2, \qquad \boldsymbol{g} \in \mathcal{B}^{3/2}_{\boldsymbol{\beta}}(\partial\Omega)^2 \tag{4.3}$$

the Stokes problem (4.1) *has a unique solution*

$$(\boldsymbol{u}, p) \in \mathcal{B}^2_{\boldsymbol{\beta}}(\Omega)^2 \times \mathcal{B}^1_{\boldsymbol{\beta}}(\Omega). \tag{4.4}$$

4.3 Discontinuous Galerkin Discretization

In this section, a mixed discontinuous Galerkin finite element method for the Stokes problem is introduced, and, using the recent results in [56], the well-posedness of the scheme is recalled.

4.3.1 Mixed DGFEM

Given a mesh $\mathcal{T}$ and a degree vector $\boldsymbol{p} = \{p_K\}$, $p_K \geq 1$, $K \in \mathcal{T}$, the Stokes problem is approximated by finite element functions $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$, where

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in L^2(\Omega)^2 : \boldsymbol{v}|_K \in \mathcal{Q}_{p_K}(K)^2,\ K \in \mathcal{T} \,\}, \\ Q_h &= \{\, q \in L^2_0(\Omega) : q|_K \in \mathcal{Q}_{p_K-1}(K),\ K \in \mathcal{T} \,\}. \end{aligned}$$

Here, $\mathcal{Q}_p(K)$ denotes the space of all polynomials of degree at most $p \geq 0$ in each variable on K. In addition, for further reference, the following space is introduced:

$$\widetilde{Q}_h = \{\, q \in L^2(\Omega) : q|_K \in \mathcal{Q}_{p_K-1}(K),\ K \in \mathcal{T} \,\}.$$

Definition 4.3.1 (Mixed DGFEM) *Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that*

$$\begin{array}{rcccl} A_h(\boldsymbol{u}_h, \boldsymbol{v}) & + & B_h(\boldsymbol{v}, p_h) & = & F_h(\boldsymbol{v}) \\ -B_h(\boldsymbol{u}_h, q) & & & = & G_h(q) \end{array} \tag{4.5}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h$. The forms A_h and B_h are discontinuous Galerkin forms that discretize the Laplace operator and the incompressibility constraint, respectively, with corresponding right-hand sides F_h and G_h. These forms are *given by*

$$\begin{aligned}
A_h(\boldsymbol{u},\boldsymbol{v}) &= \sum_{K\in\mathcal{T}}\int_K \nabla\boldsymbol{u}:\nabla\boldsymbol{v}\,d\boldsymbol{x} - \int_{\mathcal{E}}(\langle\nabla\boldsymbol{v}\rangle : \underline{[\boldsymbol{u}]} + \langle\nabla\boldsymbol{u}\rangle : \underline{[\boldsymbol{v}]})\,ds \\
&\quad + \int_{\mathcal{E}} \mathsf{d}\,\underline{[\boldsymbol{u}]} : \underline{[\boldsymbol{v}]}\,ds, \\
B_h(\boldsymbol{v},q) &= -\sum_{K\in\mathcal{T}}\int_K q\,\nabla\cdot\boldsymbol{v}\,d\boldsymbol{x} + \int_{\mathcal{E}}\langle q\rangle\,[\boldsymbol{v}]\,ds, \qquad (4.6)\\
F_h(\boldsymbol{v}) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\boldsymbol{x} - \int_{\mathcal{E}}(\boldsymbol{g}\otimes\boldsymbol{n}):\nabla\boldsymbol{v}\,ds + \int_\Gamma \mathsf{d}\,\boldsymbol{g}\cdot\boldsymbol{v}\,ds, \\
G_h(q) &= -\int_\Gamma q\,\boldsymbol{g}\cdot\boldsymbol{n}\,ds.
\end{aligned}$$

Here, $\mathsf{d}\in L^\infty(\mathcal{E})$ *is the discontinuity stabilization function from* (2.8) *with* $\omega > 0$.

Remark 4.3.2 Lemma 1.3.2 implies that the forms A_h and B_h are well-defined when inserting the exact solution $(\boldsymbol{u}, p)$ satisfying (4.4). Similarly, F_h and G_h are well-defined due to (4.3).

Remark 4.3.3 The form A_h corresponds to the symmetric interior penalty discretization (SIPG) of the Laplace operator; see [3] and [56], where the presentation and analysis of several different DG methods were unified for diffusion problems and the Stokes system, respectively.

The results presented in this chapter hold true verbatim for all the mixed discontinuous Galerkin methods investigated in [56].

4.3.2 Well-posedness

Well-posedness of the discrete system (4.5) was established in [56]. Indeed, by introducing the space $\boldsymbol{V}(h) = \boldsymbol{V}_h + H^1(\Omega)^2$, endowed with the broken norm

$$\|\boldsymbol{v}\|_h^2 = \sum_{K\in\mathcal{T}}\|\nabla\boldsymbol{v}\|_{L^2(K)}^2 + \int_{\mathcal{E}}\frac{\mathsf{p}^2}{\mathsf{h}}|\underline{[\boldsymbol{v}]}|^2\,ds, \qquad \boldsymbol{v}\in\boldsymbol{V}(h),$$

the forms A_h and B_h are continuous on $\boldsymbol{V}_h$ and Q_h, that is

$$\begin{aligned}
|A_h(\boldsymbol{v},\boldsymbol{w})| &\le C\|\boldsymbol{v}\|_h\|\boldsymbol{w}\|_h, &&\forall \boldsymbol{v},\boldsymbol{w}\in\boldsymbol{V}_h\\
|B_h(\boldsymbol{v},q)| &\le C\|\boldsymbol{v}\|_h\|q\|_{L^2(\Omega)}, &&\forall\boldsymbol{v}\in\boldsymbol{V}_h,\ \forall q\in Q_h,
\end{aligned}$$

with continuity constants $C > 0$ independent of $\boldsymbol{h}$ and $\boldsymbol{p}$. Furthermore, there exists a parameter $\omega_{min} > 0$ independent of $\boldsymbol{h}$ and $\boldsymbol{p}$ such that for any $\omega \ge \omega_{min}$ there is a coercivity constant $C > 0$ independent of $\boldsymbol{h}$ and $\boldsymbol{p}$ with

$$A_h(\boldsymbol{v},\boldsymbol{v}) \ge C\|\boldsymbol{v}\|_h^2, \qquad \forall\boldsymbol{v}\in\boldsymbol{V}_h.$$

Henceforth, assume that $\omega \geq \omega_{min}$.

Finally, for $p_K \geq 2$, the following discrete inf-sup condition for the finite element spaces $\boldsymbol{V}_h$ and Q_h holds true:

$$\inf_{0\neq q\in Q_h} \sup_{\mathbf{0}\neq \boldsymbol{v}\in \boldsymbol{V}_h} \frac{B_h(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_h \|q\|_{L^2(\Omega)}} \geq C|\boldsymbol{p}|^{-1} > 0,$$

with a constant $C > 0$ that is independent of $\boldsymbol{h}$ and $\boldsymbol{p}$.

The above properties of the forms A_h and B_h imply the well-posedness of the system (4.5).

4.3.3 Basic Error Estimates

The following abstract error bounds were obtained in [56, Sect. 3 and 4]: Let $(\boldsymbol{u}, p)$ be the exact solution of the Stokes system and $(\boldsymbol{u}_h, p_h)$ the discontinuous Galerkin approximation (4.5). Then there holds

$$\begin{aligned}
&\|\boldsymbol{u} - \boldsymbol{u}_h\|_h \\
&\quad \leq C|\boldsymbol{p}|\Big(\inf_{\boldsymbol{w}\in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{w}\|_h + \inf_{q\in Q_h} \|p - q\|_{L^2(\Omega)} + \sup_{\boldsymbol{v}\in \boldsymbol{V}_h} \frac{|R_h(\boldsymbol{u}, p; \boldsymbol{v})|}{\|\boldsymbol{v}\|_h}\Big),
\end{aligned} \tag{4.7}$$

as well as

$$\begin{aligned}
&\|p - p_h\|_{L^2(\Omega)} \\
&\quad \leq C|\boldsymbol{p}|^2\Big(\inf_{q\in Q_h} \|p - q\|_{L^2(\Omega)} + \inf_{\boldsymbol{w}\in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{w}\|_h + \sup_{\boldsymbol{v}\in \boldsymbol{V}_h} \frac{|R_h(\boldsymbol{u}, p; \boldsymbol{v})|}{\|\boldsymbol{v}\|_h}\Big),
\end{aligned} \tag{4.8}$$

where the constants $C > 0$ are independent of $\boldsymbol{h}$ and $\boldsymbol{p}$. In the above estimates (4.7) and (4.8), the term $R_h(\boldsymbol{u}, p; \boldsymbol{v})$ is a residual term which results from the nonconformity of the DG method. It will be defined and investigated next. To do so, consider the auxiliary space

$$\boldsymbol{\Sigma}_h := \{\, \boldsymbol{\tau} \in L^2(\Omega)^{2\times 2} : \boldsymbol{\tau}|_K \in \mathcal{Q}_{p_K}(K)^{2\times 2},\ K \in \mathcal{T}\,\}.$$

Moreover, introduce the lifting operators $\boldsymbol{\mathcal{L}} : \boldsymbol{V}(h) \to \boldsymbol{\Sigma}_h$, as well as $\mathcal{M} : \boldsymbol{V}(h) \to Q_h$ given by

$$\begin{aligned}
\int_\Omega \boldsymbol{\mathcal{L}}(\boldsymbol{v}) : \boldsymbol{\tau}\, d\boldsymbol{x} &= \int_{\mathcal{E}} \underline{[\boldsymbol{v}]} : \langle \boldsymbol{\tau} \rangle\, ds, \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\
\int_\Omega \mathcal{M}(\boldsymbol{v}) q\, d\boldsymbol{x} &= \int_{\mathcal{E}} [\boldsymbol{v}]\, \langle q \rangle\, ds, \qquad \forall q \in Q_h.
\end{aligned}$$

The residual can be expressed as follows:

Lemma 4.3.4 *Let $\boldsymbol{f} \in B^0_{\boldsymbol{\beta}}(\Omega)^2$. For all test functions $\boldsymbol{v} \in \boldsymbol{V}_h$, there holds*

$$R_h(\boldsymbol{u}, p; \boldsymbol{v}) = \sum_{K \in \mathcal{T}} \int_K (\nabla \boldsymbol{u} - p\,\boldsymbol{I}) : \nabla \boldsymbol{v}\, d\boldsymbol{x} - \sum_{K \in \mathcal{T}} \int_K \nabla \boldsymbol{u} : \boldsymbol{\mathcal{L}}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega p\mathcal{M}(\boldsymbol{v})\, d\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x}.$$

4.4 Error Analysis

This section is dedicated to an error analysis of the DGFEM valid for piecewise analytic solutions.

4.4.1 The Residual

For smooth solutions, the residual expression in Lemma 4.3.4 has been shown to be optimally convergent in [56]. For solutions satisfying the regularity assumption (4.4) a more careful investigation is required.

Lemma 4.4.1 *Assume* (4.3) *and* (4.4). *Let $\boldsymbol{P} : L^2(\Omega)^{2\times 2} \to \Sigma_h$ and $P : L^2_0(\Omega) \to Q_h$ denote the L^2-projections onto $\boldsymbol{\Sigma}_h$ and Q_h, respectively. Then, there holds*

$$R_h(\boldsymbol{u}, p; \boldsymbol{v}) = \int_{\mathcal{E}} \langle \nabla \boldsymbol{u} - \boldsymbol{P}(\nabla \boldsymbol{u}) \rangle : \underline{[\boldsymbol{v}]}\, ds - \int_{\mathcal{E}} \langle p - P(p) \rangle\, [\boldsymbol{v}]\, ds$$

for all $\boldsymbol{v} \in \boldsymbol{V}_h$.

Proof: First note that, by definition of the lifting operators,

$$\sum_{K \in \mathcal{T}} \int_K \nabla \boldsymbol{u} : \boldsymbol{\mathcal{L}}(\boldsymbol{v})\, d\boldsymbol{x} = \sum_{K \in \mathcal{T}} \int_K \boldsymbol{P}(\nabla \boldsymbol{u}) : \boldsymbol{\mathcal{L}}(\boldsymbol{v})\, d\boldsymbol{x} = \int_{\mathcal{E}} \langle \boldsymbol{P}(\nabla \boldsymbol{u}) \rangle : \underline{[\boldsymbol{v}]}\, ds$$

and

$$\int_\Omega p\mathcal{M}(\boldsymbol{v})\, d\boldsymbol{x} = \int_\Omega P(p)\mathcal{M}(\boldsymbol{v})\, d\boldsymbol{x} = \int_{\mathcal{E}} \langle P(p) \rangle\, [\boldsymbol{v}]\, ds.$$

Furthermore, integrating by parts (cf. Lemma A.2.3) the expression in Lemma 4.3.4 over each element $K \in \mathcal{T}$ results in

$$\begin{aligned} R_h(\boldsymbol{u}, p; \boldsymbol{v}) &= \int_\Omega (-\Delta \boldsymbol{u} + \nabla p - \boldsymbol{f}) \cdot \boldsymbol{v}\, d\boldsymbol{x} + \sum_{K \in \mathcal{T}} \int_{\partial K} (\nabla \boldsymbol{u} - p\boldsymbol{I}) : (\boldsymbol{v} \otimes \boldsymbol{n}_K)\, ds \\ &\quad - \int_{\mathcal{E}} \langle \boldsymbol{P}(\nabla \boldsymbol{u}) \rangle : \underline{[\boldsymbol{v}]}\, ds + \int_{\mathcal{E}} \langle P(p) \rangle\, [\boldsymbol{v}]\, ds. \end{aligned}$$

Note that all the integrals above are well-defined thanks to Lemma 1.3.2, (4.3) and (4.4). Elementary manipulations then show that

$$\sum_{K\in\mathcal{T}}\int_{\partial K}(\nabla\boldsymbol{u}-p\,\boldsymbol{I}):(\boldsymbol{v}\otimes\boldsymbol{n}_K)\,ds$$
$$=\int_{\mathcal{E}_{\text{int}}}[\nabla\boldsymbol{u}-p\,\boldsymbol{I}]\cdot\langle\boldsymbol{v}\rangle\;ds+\int_{\mathcal{E}}\langle\nabla\boldsymbol{u}-p\,\boldsymbol{I}\rangle:\underline{[\boldsymbol{v}]}\,ds.$$

Application of Lemma 1.3.4 implies

$$\sum_{K\in\mathcal{T}}\int_{\partial K}(\nabla\boldsymbol{u}-p\,\boldsymbol{I}):(\boldsymbol{v}\otimes\boldsymbol{n}_K)\,ds=\int_{\mathcal{E}}\langle\nabla\boldsymbol{u}\rangle:\underline{[\boldsymbol{v}]}\,ds-\int_{\mathcal{E}}\langle p\rangle\,[\boldsymbol{v}]\,ds.$$

Combining the above results and observing that $-\Delta\boldsymbol{u}+\nabla p=\boldsymbol{f}$ in $H^{0,0}_{\underline{\beta}}(\Omega)^2$, yields the assertion. □

R_h may be bounded in the following way:

Lemma 4.4.2 *Assume* (4.3) *and* (4.4). *For* $\boldsymbol{v}\in\boldsymbol{V}_h$, *the following estimate is satisfied,*

$$|R_h(\boldsymbol{u},p;\boldsymbol{v})|\le C\|\boldsymbol{v}\|_h(\|\boldsymbol{u}-\boldsymbol{w}\|_h+\|p-q\|_{L^2(\Omega)})$$
$$+\left|\int_{\mathcal{E}}\langle\nabla\boldsymbol{u}-\nabla\boldsymbol{w}\rangle:\underline{[\boldsymbol{v}]}\,ds-\int_{\mathcal{E}}\langle p-q\rangle\,[\boldsymbol{v}]\,ds\right|,$$

for any $(\boldsymbol{w},q)\in\boldsymbol{V}_h\times Q_h$.

Proof: Let $(\boldsymbol{w},q)\in\boldsymbol{V}_h\times Q_h$ be arbitrary. From the result in Lemma 4.4.1 and since the L^2-projections reproduce polynomials in $\boldsymbol{\Sigma}_h$ and Q_h, respectively, it follows that

$$R_h(\boldsymbol{u},p;\boldsymbol{v})=\int_{\mathcal{E}}\langle\nabla\boldsymbol{u}-\nabla\boldsymbol{w}-\boldsymbol{P}(\nabla\boldsymbol{u}-\nabla\boldsymbol{w})\rangle:\underline{[\boldsymbol{v}]}\,ds$$
$$-\int_{\mathcal{E}}\langle p-q-P(p-q)\rangle\,[\boldsymbol{v}]\,ds.$$

The term T with the L^2-projections may be bounded by

$$\begin{aligned}|T| &= \left|\int_{\mathcal{E}}\langle\boldsymbol{P}(\nabla\boldsymbol{u}-\nabla\boldsymbol{w})\rangle:\underline{[\boldsymbol{v}]}\,ds-\int_{\mathcal{E}}\langle P(p-q)\rangle\,[\boldsymbol{v}]\,ds\right|\\ &\le C\|\boldsymbol{v}\|_h\sum_{K\in\mathcal{T}}\left(\frac{h_K}{p_K^2}\|\boldsymbol{P}(\nabla\boldsymbol{u}-\nabla\boldsymbol{w})\|^2_{L^2(\partial K)}+\frac{h_K}{p_K^2}\|P(p-q)\|^2_{L^2(\partial K)}\right)^{1/2}\\ &\le C\|\boldsymbol{v}\|_h\big(\|\boldsymbol{P}(\nabla\boldsymbol{u}-\nabla\boldsymbol{w})\|_{L^2(\Omega)}+\|P(p-q)\|_{L^2(\Omega)}\big)\\ &\le C\|\boldsymbol{v}\|_h\big(\|\boldsymbol{u}-\boldsymbol{w}\|_h+\|p-q\|_{L^2(\Omega)}\big).\end{aligned}$$

Here, the Cauchy-Schwarz inequality, the definition of h and of p, the fact that $|[\boldsymbol{v}]|^2 \leq |\underline{[\boldsymbol{v}]}|^2$, the discrete trace inequality Lemma A.1.2 and the stability of the L^2-projections were used. The triangle inequality completes the proof. □

4.4.2 Error Estimates

To obtain the ensuing result, the bounds (4.7) and (4.8) have to be combined with the estimates in Lemma 4.4.2.

Theorem 4.4.3 *Let the exact solution* $(\boldsymbol{u}, p)$ *of the Stokes system satisfy* (4.4). *In addition, let* $(\boldsymbol{u}_h, p_h)$ *be the discontinuous Galerkin approximation* (4.5) *with* $p_K \geq 2$, *for all* $K \in \mathcal{T}$. *Then, for any* $(\boldsymbol{w}, \widetilde{q}) \in \boldsymbol{V}_h \times \widetilde{Q}_h$, *there holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h + \|p - p_h\|_{L^2(\Omega)} \leq C\,|\boldsymbol{p}|^3 \left(E_1 + E_2 + E_3\right),$$

where

$$\begin{aligned} E_1^2 &= \sum_{K\in\mathcal{T}} \left(|\boldsymbol{u} - \boldsymbol{w}|^2_{H^1(K)} + h_K^{-2}\|\boldsymbol{u} - \boldsymbol{w}\|^2_{L^2(K)} + \|p - \widetilde{q}\|^2_{L^2(K)}\right), \\ E_2^2 &= \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2 \left(|\boldsymbol{u} - \boldsymbol{w}|^2_{H^2(K)} + |p - \widetilde{q}|^2_{H^1(K)}\right), \\ E_3^2 &= \sum_{K\in\mathcal{K}_0} h_K^{2(1-\beta_K)} \left(|\boldsymbol{u} - \boldsymbol{w}|^2_{H^{2,2}_{\beta_K}(K)} + |p - \widetilde{q}|^2_{H^{1,1}_{\beta_K}(K)}\right). \end{aligned}$$

The constant $C > 0$ *is independent of* $\boldsymbol{h}$ *and of* $\boldsymbol{p}$.

Proof: Let $\boldsymbol{w} \in \boldsymbol{V}_h$, $\widetilde{q} \in \widetilde{Q}_h$ be arbitrary. Set $q = \widetilde{q} - \frac{1}{|\Omega|}\int_\Omega \widetilde{q}\, d\boldsymbol{x} \in Q_h$. Then, the bounds from (4.7), (4.8) and Lemma 4.4.2 yield

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_h\|_h + \|p - p_h\|_{L^2(\Omega)} \\ &\qquad \leq C|\boldsymbol{p}|^2\Big(\|\boldsymbol{u} - \boldsymbol{w}\|_h + \|p - q\|_{L^2(\Omega)} + \sup_{\boldsymbol{v}\in\boldsymbol{V}_h} \frac{|E_h(\boldsymbol{u} - \boldsymbol{w}, p - q; \boldsymbol{v})|}{\|\boldsymbol{v}\|_h}\Big), \end{aligned} \tag{4.9}$$

with E_h given by

$$E_h(\boldsymbol{u} - \boldsymbol{w}, p - q; \boldsymbol{v}) = \int_{\mathcal{E}} \langle \nabla\boldsymbol{u} - \nabla\boldsymbol{w}\rangle : \underline{[\boldsymbol{v}]}\, ds - \int_{\mathcal{E}} \langle p - q\rangle\, [\boldsymbol{v}]\, ds.$$

In the following, the right-hand side of (4.9) is estimated in terms of $\{E_i\}_{i=1}^3$.

First, using the mesh properties (1.8), (1.9), and the standard trace inequality

$$\|\varphi\|^2_{L^2(\partial K)} \leq C\left(h_K^{-1}\|\varphi\|^2_{L^2(K)} + h_K|\varphi|^2_{H^1(K)}\right), \qquad \forall \varphi \in H^1(K),$$

yields

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{w}\|_h^2 &= \sum_{K\in\mathcal{T}} |\boldsymbol{u}-\boldsymbol{w}|_{H^1(K)}^2 + \int_{\mathcal{E}} \frac{\mathsf{p}^2}{\mathsf{h}} |\underline{[\boldsymbol{u}-\boldsymbol{w}]}|^2 \, ds \\
&\le C|\boldsymbol{p}|^2 \Big(\sum_{K\in\mathcal{T}} |\boldsymbol{u}-\boldsymbol{w}|_{H^1(K)}^2 + C \sum_{K\in\mathcal{T}} h_K^{-1} \|\boldsymbol{u}-\boldsymbol{w}\|_{L^2(\partial K)}^2 \Big) \\
&\le C|\boldsymbol{p}|^2 \sum_{K\in\mathcal{T}} \big(h_K^{-2} \|\boldsymbol{u}-\boldsymbol{w}\|_{L^2(K)}^2 + |\boldsymbol{u}-\boldsymbol{w}|_{H^1(K)}^2 \big) \\
&\le C|\boldsymbol{p}|^2 E_1^2.
\end{aligned} \tag{4.10}$$

Next, since $\int_\Omega p \, d\boldsymbol{x} = \int_\Omega q \, d\boldsymbol{x} = 0$, it follows

$$\begin{aligned}
\|p-q\|_{L^2(\Omega)} &= \Big\| p - \widetilde{q} - |\Omega|^{-1} \int_\Omega (p-\widetilde{q}) \, d\boldsymbol{x} \Big\|_{L^2(\Omega)} \\
&\le \|p-\widetilde{q}\|_{L^2(\Omega)} + |\Omega|^{-1/2} \int_\Omega |p-\widetilde{q}| \, d\boldsymbol{x} \\
&\le 2\|p-\widetilde{q}\|_{L^2(\Omega)} \\
&\le 2E_1.
\end{aligned} \tag{4.11}$$

Moreover,

$$\begin{aligned}
|E_h(\boldsymbol{u}-\boldsymbol{w}, p-q; \boldsymbol{v})| &\le \sum_{e\in\mathcal{E}} \int_e (|\langle \nabla\boldsymbol{u} - \nabla\boldsymbol{w} \rangle : \underline{[\boldsymbol{v}]}| + |\langle p-q \rangle [\boldsymbol{v}]|) \, ds \\
&\le \sum_{e\in\mathcal{E}} \int_e (|\langle \nabla\boldsymbol{u} - \nabla\boldsymbol{w} \rangle| + |\langle p-q \rangle|) |\underline{[v]}| \, ds \\
&\le \sum_{e\in\mathcal{E}} \|\underline{[v]}\|_{L^\infty(e)} \int_e (|\langle \nabla\boldsymbol{u} - \nabla\boldsymbol{w} \rangle| + |\langle p-q \rangle|) \, ds.
\end{aligned}$$

Applying the inverse inequality from Lemma A.1.1 yields

$$\|\underline{[v]}\|_{L^\infty(e)} = \||\underline{[v]}|^2\|_{L^\infty(e)}^{1/2} \le C \frac{\mathsf{p}|_e}{\sqrt{\mathsf{h}|_e}} \||\underline{[v]}|^2\|_{L^1(e)}^{1/2} \le C \frac{\mathsf{p}|_e}{\sqrt{\mathsf{h}|_e}} \|\underline{[v]}\|_{L^2(e)}.$$

Therefore, using the shape regularity of the mesh it follows that

$$\begin{aligned}
&|E_h(\boldsymbol{u}-\boldsymbol{w}, p-q; \boldsymbol{v})| \\
&\le C \sum_{e\in\mathcal{E}} \Big\| \frac{\mathsf{p}}{\sqrt{\mathsf{h}}} \underline{[v]} \Big\|_{L^2(e)} \int_e (|\langle \nabla\boldsymbol{u} - \nabla\boldsymbol{w} \rangle| + |\langle p-q \rangle|) \, ds \\
&\le C \Big(\int_{\mathcal{E}} \frac{\mathsf{p}^2}{\mathsf{h}} |\underline{[v]}|^2 \, ds \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}} \|\nabla\boldsymbol{u} - \nabla\boldsymbol{w}\|_{L^1(\partial K)}^2 + \|p-q\|_{L^1(\partial K)}^2 \Big)^{1/2} \\
&\le C \|\boldsymbol{v}\|_h \Big(\sum_{K\in\mathcal{T}} \|\nabla\boldsymbol{u} - \nabla\boldsymbol{w}\|_{L^1(\partial K)}^2 + \|p-q\|_{L^1(\partial K)}^2 \Big)^{1/2}.
\end{aligned}$$

In addition, the third assertion in Lemma 1.3.2 implies that

$$
\begin{aligned}
\frac{|E_h(\boldsymbol{u}-\boldsymbol{w},\, p-q;\, \boldsymbol{v})|}{\|\boldsymbol{v}\|_h}
&\leq C\Big(\sum_{K\in\mathcal{T}} \big(|\boldsymbol{u}-\boldsymbol{w}|^2_{H^1(K)} + \|p-q\|^2_{L^2(K)}\big) \\
&\quad + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2 \big(|\boldsymbol{u}-\boldsymbol{w}|^2_{H^2(K)} + |p-q|^2_{H^1(K)}\big) \\
&\quad + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K} \big(|\boldsymbol{u}-\boldsymbol{w}|^2_{H^{2,2}_{\beta_K}(K)} + |p-q|^2_{H^{1,1}_{\beta_K}(K)}\big)\Big)^{1/2}.
\end{aligned}
$$

Finally, applying (4.11) and using the fact $\nabla(q-\widetilde{q}) \equiv 0$ results in

$$
\begin{aligned}
\frac{|E_h(\boldsymbol{u}-\boldsymbol{w},\, p-q;\, \boldsymbol{v})|}{\|\boldsymbol{v}\|_h}
&\leq C\Big(E_1^2 + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2 \big(|\boldsymbol{u}-\boldsymbol{w}|^2_{H^2(K)} + |p-\widetilde{q}|^2_{H^1(K)}\big) \\
&\quad + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta_K} \big(|\boldsymbol{u}-\boldsymbol{w}|^2_{H^{2,2}_{\beta_K}(K)} + |p-\widetilde{q}|^2_{H^{1,1}_{\beta_K}(K)}\big)\Big)^{1/2} \\
&\leq C(E_1+E_2+E_3),
\end{aligned}
\tag{4.12}
$$

for all $\boldsymbol{v} \in \boldsymbol{V}_h$. Combining (4.10)–(4.12) with (4.9) completes the proof. □

4.5 Exponential Rates of Convergence

The aim of this section is to show that the error estimates in Theorem 4.4.3 are exponentially convergent on geometric meshes.

Theorem 4.5.1 *Assume that the exact solution* $(\boldsymbol{u}, p)$ *of the Stokes equations satisfies* (4.4) *with* $\boldsymbol{\beta}_{\min} \preccurlyeq \boldsymbol{\beta} \prec 1$. *Let* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ *be the hp DGFEM approximation* (4.5) *on geometric meshes* $\mathcal{T}_{n,\sigma}$ *(cf. Definition 2.6.3). Then there exists* $\mu_0 = \mu_0(\sigma, \boldsymbol{\beta}) > 0$ *such that for linear polynomial degree vectors* $\boldsymbol{p}$ *with slope* $\mu \geq \mu_0$ *(cf. Definition 2.6.4) there holds the error estimate*

$$
\|\boldsymbol{u}-\boldsymbol{u}_h\|_h + \|p-p_h\|_{L^2(\Omega)} \leq C e^{-b\sqrt[3]{N}}
$$

with constants $C, b > 0$ *independent of* $N = \dim(\boldsymbol{V}_h) \approx \dim(Q_h)$.

Proof: The proof consists of two steps.

Step 1: Consider first the case where $\Omega = (0,1)^2$ and $\mathcal{T}_{n,\sigma} = \Delta_{n,\sigma}$ is the basic geometric mesh from Definition 2.6.1. From [55, Proposition 27] and [32] or Lemma 2.5.2, there exist $\widetilde{q}_{11} \in \mathcal{Q}_0(K_{11})$ and $\boldsymbol{w}_{11} \in \mathcal{Q}_1(K_{11})^2$ such that

$$\|p - \widetilde{q}_{11}\|^2_{L^2(K_{11})} + h_{K_{11}}^{2-2\beta_{K_{11}}} |p - \widetilde{q}_{11}|^2_{H^{1,1}_{\beta_{K_{11}}}(K_{11})} \le C\sigma^{2n(1-\beta_{K_{11}})} |p|^2_{H^{1,1}_{\beta_{K_{11}}}(K_{11})}$$

and

$$h_{K_{11}}^{-2} \|\boldsymbol{u} - \boldsymbol{w}_{11}\|^2_{L^2(K_{11})} + |\boldsymbol{u} - \boldsymbol{w}_{11}|^2_{H^1(K_{11})} + h_{K_{11}}^{2-2\beta_{K_{11}}} |\boldsymbol{u} - \boldsymbol{w}_{11}|^2_{H^{2,2}_{\beta_{K_{11}}}(K_{11})} \le C\sigma^{2n(1-\beta_{K_{11}})} |\boldsymbol{u}|^2_{H^{2,2}_{\beta_{K_{11}}}(K_{11})}.$$

Moreover, for $K_{ij} \in \mathcal{K}_{\text{int}}$ there are $\widetilde{q}_{ij} \in \mathcal{Q}_{p_{K_{ij}}-1}(K_{ij})$ and $\boldsymbol{w}_{ij} \in \mathcal{Q}_{p_{K_{ij}}}(K_{ij})^2$ such that

$$\|p - \widetilde{q}_{ij}\|^2_{L^2(K_{ij})} + h^2_{K_{ij}} |p - \widetilde{q}_{ij}|^2_{H^1(K_{ij})} \le C\sigma^{2(n+2-j)(1-\beta_{K_{ij}})} \frac{\Gamma(k_{K_{ij}} - s_{ij} + 1)}{\Gamma(k_{K_{ij}} + s_{ij} - 1)} \left(\frac{\varrho}{2}\right)^{2s_{ij}} \|p\|^2_{H^{s_{ij}+3,1}_{\beta_{K_{ij}}}(K_{ij})}$$

and

$$h_{K_{ij}}^{-2} \|\boldsymbol{u} - \boldsymbol{w}_{ij}\|^2_{L^2(K_{ij})} + |\boldsymbol{u} - \boldsymbol{w}_{ij}|^2_{H^1(K_{ij})} + h^2_{K_{ij}} |\boldsymbol{u} - \boldsymbol{w}_{ij}|^2_{H^2(K_{ij})} \le C\sigma^{2(n+2-j)(1-\beta_{K_{ij}})} \frac{\Gamma(k_{K_{ij}} - s_{ij} + 1)}{\Gamma(k_{K_{ij}} + s_{ij} - 1)} \left(\frac{\varrho}{2}\right)^{2s_{ij}} \|\boldsymbol{u}\|^2_{H^{s_{ij}+3,2}_{\beta_{K_{ij}}}(K_{ij})}$$

for any $1 \le i \le 3$, $2 \le j \le n+1$ and $s_{ij} \in [1, k_{K_{ij}}]$. Here, $\varrho = \max(1, {}^{(1-\sigma)}/_{\sigma})$. This was proved, e.g., in [55, Sect. 5.2] in all details. Referring to Theorem 4.4.3 implies that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_h + \|p - p_h\|^2_{L^2(\Omega)} \le C\sigma^{2n(1-\beta_{K_{11}})} \Big(\Psi^{2,1}_{\beta_{K_{11}}}(\boldsymbol{u}, p) + \sum_{i=1}^{3} \sum_{j=2}^{n+1} \sigma^{2(2-j)(1-\beta_{K_{ij}})} \frac{\Gamma(k_{K_{ij}} - s_{ij} + 1)}{\Gamma(k_{K_{ij}} + s_{ij} - 1)} \left(\frac{\varrho}{2}\right)^{2s_{ij}} \Psi^{s_{ij}+3, s_{ij}+3}_{\beta_{K_{ij}}}(\boldsymbol{u}, p) \Big), \tag{4.13}$$

where

$$\Psi^{m,l}_{\beta_K}(\boldsymbol{u}, p) = \|\boldsymbol{u}\|^2_{H^{m,2}_{\beta_K}(K)} + \|p\|^2_{H^{l,1}_{\beta_K}(K)}.$$

In [5, 31] or [58, Sect. 4.5.3] it was shown that there exist s_{ij}, $1 \le i \le 3$, $2 \le j \le n+1$ and $\mu_0 > 0$ such that, for linear polynomial degree distributions as in

Definition 2.6.4 with slope $\mu \geq \mu_0$, the right-hand side of (4.13) is exponentially small with respect to N. More precisely, there holds:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h + \|p - p_h\|_{L^2(\hat{Q})} \leq Ce^{-b\sqrt[3]{N}}.$$

Step 2: A generalization of the above result to arbitrary polygon domains Ω is obtained as described in the proof of Proposition 2.6.5. □

Appendix

A.1 Inverse Inequalities

Lemma A.1.1 *Let* $I = (a, b)$ *be a bounded interval and* $h = b - a$. *Then, for every* $u \in \mathcal{P}_p(I) = \{u(x) = \sum_{i=0}^{p} a_i x^i : a_i \in \mathbb{R}, i = 1, 2 \ldots, p\}$ *it holds that*

$$\|u\|_{L^\infty(I)} \leq 2\left(\frac{8}{h}\right)^{1/q} p^{2/q} \|u\|_{L^q(I)}, \qquad 1 \leq q \leq \infty.$$

A proof of this result may be found in [47].

Lemma A.1.2 *Let* $K \in \mathcal{T}$ *be an element in a finite element mesh* $\mathcal{T}$, *i.e. there exists an affine mapping* $\boldsymbol{F}_K$ *with* $K = \boldsymbol{F}_K(\hat{K})$, *where* $\hat{K}$ *is either the reference triangle or the reference square (cf. Section 1.3.1). Furthermore, let* $u \in \mathcal{P}_{p_K}(K)$, $p_K \geq 1$. *Then, there exists a constant* $C > 0$ *independent of* u, p_K *and and of* h_K *such that the trace inequality*

$$\|u\|_{L^2(\partial K)} \leq C \frac{p_K}{\sqrt{h_K}} \|u\|_{L^2(K)} \tag{A.1}$$

holds true.

Proof: The statement follows directly from [58, Theorem 4.76] and a standard scaling argument. □

A.2 Auxiliary Results in Weighted Sobolev Spaces

Without further specifications, all elements in the present Section are denoted by K and are assumed to be triangles with vertices A_1, A_2, A_3 satisfying the properties from Section 1.3.1. Additionally, suppose that the weight function from Section 1.2 is given by

$$\Phi_\beta(\boldsymbol{x}) = |\boldsymbol{x} - A_1|^\beta = r^\beta,$$

with $\beta \in [0, 1)$.

A.2.1 Integration by Parts Formulas

In the following, n_K denotes the unit outward vector on ∂K. Exemplarily, the proof of the first lemma will be given in full details. The other statements may then be proved in a completely similar way.

Lemma A.2.1 *Let $\boldsymbol{u} \in H^{1,1}_{\beta}(K)^2$ and $v \in \mathcal{C}^1(\overline{K})$. Then, the following integration by parts formula holds true*

$$\int_K v\,\nabla\cdot\boldsymbol{u}\,d\boldsymbol{x} = \int_{\partial K} (\boldsymbol{u}\cdot\boldsymbol{n}_K)v\,ds - \int_K \boldsymbol{u}\cdot\nabla v\,d\boldsymbol{x}.$$

Proof: First of all, note that all the integrals above are well-defined due to Lemma 1.3.2 and the fact that $\nabla\cdot\boldsymbol{u} \in H^{0,0}_{\beta}(K)$. Furthermore, since $\mathcal{C}^\infty(\overline{K})$ is dense in $H^{1,1}_{\beta}(K)$, there exists a sequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}} \subset \mathcal{C}^\infty(\overline{K})^2$ with $\boldsymbol{u}_n \to \boldsymbol{u}$ in $H^{1,1}_{\beta}(K)^2$. Clearly,

$$\int_K v\,\nabla\cdot\boldsymbol{u}_n\,d\boldsymbol{x} = \int_{\partial K} (\boldsymbol{u}_n\cdot\boldsymbol{n}_K)v\,ds - \int_K \boldsymbol{u}_n\cdot\nabla v\,d\boldsymbol{x}$$

for all $n \in \mathbb{N}$. Lemma 1.3.2 implies that

$$\begin{aligned}\left|\int_K v\,\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_n)\,d\boldsymbol{x}\right| &\le C\|v\|_{L^\infty(K)}\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_n)\|_{H^{0,0}_{\beta}(K)} \\ &\le C\|v\|_{L^\infty(K)}\|\boldsymbol{u}-\boldsymbol{u}_n\|_{H^{1,1}_{\beta}(K)},\end{aligned}$$

and

$$\begin{aligned}\left|\int_K (\boldsymbol{u}-\boldsymbol{u}_n)\cdot\nabla v\,d\boldsymbol{x}\right| &\le \|\nabla v\|_{L^2(K)}\|\boldsymbol{u}-\boldsymbol{u}_n\|_{L^2(K)} \\ &\le \|\nabla v\|_{L^2(K)}\|\boldsymbol{u}-\boldsymbol{u}_n\|_{H^{1,1}_{\beta}(K)}.\end{aligned}$$

Furthermore, again with Lemma 1.3.2,

$$\begin{aligned}\int_{\partial K} (\boldsymbol{u}-\boldsymbol{u}_n)\cdot\boldsymbol{n}_\Omega v\,ds &\le \|v\|_{L^\infty(\partial K)}\|\boldsymbol{u}-\boldsymbol{u}_n\|_{L^1(K)} \\ &\le C\|v\|_{L^\infty(\partial K)}\|\boldsymbol{u}-\boldsymbol{u}_n\|_{H^{1,1}_{\beta}(K)}.\end{aligned}$$

Passing to the limits finishes the proof. □

Lemma A.2.2 *Let $\boldsymbol{u} \in H^{1,1}_{\beta}(K)^2$ and $\boldsymbol{v} \in \mathcal{C}^1(\overline{K})^2$. Then, there holds*

$$\int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\,d\boldsymbol{x} = \int_{\partial K} (\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}_K)\cdot\boldsymbol{v}\,ds - \int_K (\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}))\cdot\boldsymbol{v}\,d\boldsymbol{x},$$

where $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are the tensor fields from Section 3.1.

Lemma A.2.3 *Let $\boldsymbol{\tau} \in H^{1,1}_{\beta}(K)^{2\times 2}$ and $\boldsymbol{v} \in C^1(\overline{K})^2$. Then, it holds that*

$$\int_K (\nabla\cdot\boldsymbol{\tau})\cdot\boldsymbol{v}\,d\boldsymbol{x} = \int_{\partial K} \boldsymbol{\tau} : (\boldsymbol{v}\otimes\boldsymbol{n}_K)\,ds - \int_K \boldsymbol{\tau} : \nabla\boldsymbol{v}\,d\boldsymbol{x}.$$

A.2.2 Trace Inequalities

The following two Lemmas hold true for triangles as well as for quadrilaterals.

Lemma A.2.4 *Let $e \in \mathcal{E}_K$ be an edge of K. Then, there holds:*

i) if $u \in H^2(K)$ and $u = 0$ at the vertices of K, then the trace $u|_e \in H^1(e)$ and

$$\|u\|^2_{H^1(e)} \leq C h_K |u|^2_{H^2(K)}, \tag{A.2}$$

$$\|u\|^2_{L^2(e)} \leq C h_K^3 |u|^2_{H^2(K)}; \tag{A.3}$$

ii) if $u \in H^{2,2}_\beta(K)$, $0 < \beta < 1$, and $u = 0$ at the vertices of K, then

$$\|u\|^2_{H^1(e)} \leq C h_K^{1-2\beta} |u|^2_{H^{2,2}_\beta(K)}, \tag{A.4}$$

$$\|u\|^2_{L^2(e)} \leq C h_K^{3-2\beta} |u|^2_{H^{2,2}_\beta(K)}, \tag{A.5}$$

if $\overline{e}$ does not contain the vertex A_1.

Proof: The proof of this lemma may be found in [58, Lemma 4.55]. □

A.2.3 Compactness

Proposition A.2.5 *The embedding $H^{1,1}_\beta(K) \subset L^2(K)$ is compact.*

In order to prove this Proposition, the following two lemmas are required.

Lemma A.2.6 *Let $u \in H^{1,1}_\beta(K)$. Then, there exists a constant $C = C(\beta) > 0$ such that*

$$\int_K |\boldsymbol{x}|^{2\beta-2} u^2 \, d\boldsymbol{x} \leq C \|u\|^2_{H^{1,1}_\beta(K)}.$$

Proof: The proof follows directly from [58, Lemma 4.18]. □

Lemma A.2.7 *Let $u \in H^{1,1}_\beta(K)$. Then, $r^\beta u \in H^1(K)$.*

Proof: Set $v = r^\beta u$. Then,

$$D^1 v = \beta \begin{pmatrix} x \\ y \end{pmatrix} r^{\beta-2} u + r^\beta D^1 u,$$

and thus

$$|D^1 v|^2 \le C(r^{2\beta-2}u^2 + r^{2\beta}|D^1 u|^2).$$

With the aid of Lemma A.2.6, this results in

$$\|v\|^2_{H^1(K)} = \|r^\beta u\|^2_{L^2(K)} + \|D^1 v\|^2_{L^2(K)} \le C\|u\|^2_{H^{1,1}_\beta(K)}.$$

□

Remark A.2.8 In general, $u \in H^1(K)$ does not imply $r^{-\beta}u \in H^{1,1}_\beta(K)$. A counter-example is given by $u(\boldsymbol{x}) \equiv 1$. However, it may be proved easily that for all $u \in H^1(K)$, $r^{-\beta}u \in H^{1,1}_{\beta+\varepsilon}(K)$ for all $\varepsilon \in (0, 1-\beta)$.

Proof: (Proposition A.2.5) The case $\beta = 0$ is already covered by Rellich's Theorem. Therefore, suppose that $\beta > 0$.

Let $\{u_j\}_{j=1}^\infty$ be a bounded sequence in $H^{1,1}_\beta(K)$. Then, due to the previous Lemma A.2.7, $\{r^\beta u_j\}_{j=1}^\infty$ is a bounded sequence in $H^1(K)$. Since $H^1(K)$ is compactly embedded in $L^t(K)$ for all $t \in [1,\infty)$, there exists $\overline{v} \in L^t(K)$ and a convergent subsequence $\{r^\beta u_{j'}\}_{j'}$ such that

$$\|r^\beta u_{j'} - \overline{v}\|_{L^t(K)} \xrightarrow{j'\to\infty} 0.$$

Now, putting $\overline{u} = r^{-\beta}\overline{v}$ implies that

$$\begin{aligned}
\|\overline{u}\|_{L^2(K)} &= \|r^{-\beta}\overline{v}\|_{L^2(K)} \\
&= \|r^{-2\beta}\overline{v}^2\|^{1/2}_{L^1(K)} \\
&\le \|r^{-2\beta}\|^{1/2}_{L^s(K)}\|\overline{v}^2\|^{1/2}_{L^{s'}(K)} \\
&\le \|r^{-\beta s}\|^{1/s}_{L^2(K)}\|\overline{v}^{s'}\|^{1/s'}_{L^2(K)} \\
&\le C\|\overline{v}\|_{L^{2s'}(K)},
\end{aligned}$$

where

$$1/s + 1/s' = 1 \quad \text{and} \quad \beta < s\beta < 1.$$

Thus, $\overline{u} \in L^2(K)$. In the same way it follows that

$$\|u_{j'} - \overline{u}\|_{L^2(K)} \le C\|r^\beta u_{j'} - \overline{v}\|_{L^{2s'}(K)} \xrightarrow{j'\to\infty} 0.$$

Hence, $\{u_{j'}\}_{j'}$ converges in $L^2(K)$. □

A.2.4 Poincaré Inequalities

Theorem A.2.9 *Let*

$$\mathcal{A}: H^{1,1}_\beta(K) \longrightarrow \mathbb{R}$$

be a continuous, linear operator with

$$\ker \mathcal{A} \cap \{u \equiv constant\} = \{0\}. \tag{A.6}$$

Then, the inequality

$$\|u\|_{L^2(K)} \le C\left(|u|_{H^{1,1}_\beta(K)} + |\mathcal{A}u|\right) \tag{A.7}$$

holds true.

Proof: (By contradiction) If (A.7) was false, there would exist a sequence

$$\{u_j\}_{j=1}^\infty \subset H^{1,1}_\beta(K)$$

such that

$$\|u_j\|_{L^2(K)} = 1 \qquad \forall j,$$

and

$$|u_j|_{H^{1,1}_\beta(K)} + |\mathcal{A}u_j| \overset{j\to\infty}{\longrightarrow} 0. \tag{A.8}$$

Due to the previous Proposition A.2.5, there exists a subsequence $\{u_{j'}\}_{j'}$ and $\overline{u} \in L^2(K)$ such that

$$\|u_{j'} - \overline{u}\|_{L^2(K)} \overset{j'\to\infty}{\longrightarrow} 0.$$

By (A.8) it follows that $\{u_{j'}\}$ is a Cauchy sequence in $H^{1,1}_\beta(K)$ and hence $\{r^\beta u_{j'}\}$ is Cauchy in $H^1(K)$:

$$\|r^\beta(u_{j'} - u_{k'})\|_{H^1(K)} \le C\|u_{j'} - u_{k'}\|_{H^{1,1}_\beta(K)} \qquad \text{(cf. Lemma A.2.7).}$$

Therefore, there exists $\overline{v} \in H^1(K)$ with

$$\|r^\beta u_{j'} - \overline{v}\|_{H^1(K)} \overset{j'\to\infty}{\longrightarrow} 0.$$

Thus, $\overline{u} = r^{-\beta}\overline{v}$. Moreover, for all $\varepsilon \in (0, 1-\beta)$, there holds

$$\begin{aligned}
|u_{j'} - \overline{u}|^2_{H^{1,1}_{\beta+\varepsilon}(K)} &= \|r^{\beta+\varepsilon}|D^1(r^{-\beta}(r^\beta u_{j'} - \overline{v}))|\|^2_{L^2(K)} \\
&\le C\|r^{\beta+\varepsilon}(r^{-\beta-1}|r^\beta u_{j'} - \overline{v}| + r^{-\beta}|D^1(r^\beta u_{j'} - \overline{v})|)\|^2_{L^2(K)} \\
&\le C\|r^{\varepsilon-1}|r^\beta u_{j'} - \overline{v}| + r^\varepsilon|D^1(r^\beta u_{j'} - \overline{v})|\|^2_{L^2(K)} \\
&\le C\left(\int_K r^{2\varepsilon-2}|r^\beta u_{j'} - \overline{v}|^2\,d\boldsymbol{x} + \|D^1(r^\beta u_{j'} - \overline{v})\|^2_{L^2(K)}\right).
\end{aligned}$$

Applying Lemma A.2.6 yields

$$|u_{j'} - \overline{u}|^2_{H^{1,1}_{\beta+\varepsilon}(K)} \leq C(\|r^\beta u_{j'} - \overline{v}\|^2_{H^{1,1}_{\varepsilon}(K)} + \|D^1(r^\beta u_{j'} - \overline{v})\|^2_{L^2(K)})$$
$$\leq C\|r^\beta u_{j'} - \overline{v}\|^2_{H^1(K)} \overset{j'\to\infty}{\longrightarrow} 0.$$

Hence,

$$|\overline{u}|_{H^{1,1}_{\beta+\varepsilon}(K)} \leq |\overline{u} - u_{j'}|_{H^{1,1}_{\beta+\varepsilon}(K)} + C|u_{j'}|_{H^{1,1}_{\beta}(K)} \overset{j'\to\infty}{\longrightarrow} 0,$$

and therefore, $\overline{u}$ is constant on K. Furthermore,

$$|\mathcal{A}\overline{u}| \leq |\mathcal{A}u_{j'}| + |\mathcal{A}(\overline{u} - u_{j'})|$$
$$\leq C(|\mathcal{A}u_{j'}| + \|\overline{u} - u_{j'}\|_{H^{1,1}_{\beta}(K)}) \overset{j'\to\infty}{\longrightarrow} 0,$$

and thus, by (A.6), $\overline{u} \equiv 0$ on K, in contrast to

$$\|\overline{u}\|_{L^2(K)} = \lim_{j'\to\infty} \|u_{j'}\|_{L^2(K)} = 1.$$

□

Corollary A.2.10 (1st Poincaré Inequality) *Let $e \in \mathcal{E}_K$ be an edge of K. Then, there holds that*

$$\left\| u - \frac{1}{|e|}\int_e u\,ds \right\|_{L^2(K)} \leq Ch_K^{1-\beta}|u|_{H^{1,1}_{\beta}(K)} \tag{A.9}$$

for all $u \in H^{1,1}_{\beta}(K)$.

Proof: For $u \in H^{1,1}_{\beta}(K)$, let

$$\mathcal{A}u = \int_e u\,ds.$$

Then, by Lemma 1.3.2, $\mathcal{A} : H^{1,1}_{\beta}(K) \to \mathbb{R}$ is continuous, and inserting $u - \frac{1}{|e|}\int_e u\,ds$ into (A.7) completes the proof. □

Corollary A.2.11 (2nd Poincaré Inequality) *For all $u \in H^{1,1}_{\beta}(K)$ and for all $\widetilde{K} \subset K$ with $\int_{\widetilde{K}} d\boldsymbol{x} > 0$ the following inequality holds true:*

$$\left\| u - \frac{1}{|\widetilde{K}|}\int_{\widetilde{K}} u\,d\boldsymbol{x} \right\|_{L^2(K)} \leq Ch_K^{1-\beta}|u|_{H^{1,1}_{\beta}(K)}. \tag{A.10}$$

Proof: Set

$$\mathcal{A}u := \frac{1}{|\widetilde{K}|}\int_{\widetilde{K}} u\,d\boldsymbol{x}.$$

Then,

$$|\mathcal{A}u| \le \|u\|_{L^1(K)} \le C\|u\|_{L^2(K)} \le C\|u\|_{H^{1,1}_\beta(K)}.$$

Hence, $\mathcal{A} : H^{1,1}_\beta(K) \to \mathbb{R}$ is continuous and, inserting $u - \frac{1}{|\widetilde{K}|}\int_{\widetilde{K}} u\,d\boldsymbol{x}$ into (A.7) finishes the proof immediately. □

Proposition A.2.12 *Let $\boldsymbol{u} \in H^{2,2}_\beta(K)^2$. Then, there exists a constant $C > 0$ independent of $\boldsymbol{u}$ and of h_K such that*

$$\|\boldsymbol{u}\|^2_{H^{2,2}_\beta(K)} \le C\Big(|\boldsymbol{u}|^2_{H^{2,2}_\beta(K)} + \sum_{e\in\mathcal{E}_K}\Big|\int_e \boldsymbol{u}\,ds\Big|^2\Big)$$

holds true.

Proof: The proof is very similar to the proof of [58, Lemma 4.16]. □

Proposition A.2.13 *Let $\mathcal{T}$ be a finite element mesh on a polygon Ω (with $\Gamma_D \subset \partial\Omega$, $\int_{\Gamma_D} ds > 0$) as in Section 1.3.1, and $u_h \in V_h$, where V_h is a finite element space as in* (2.7)*. Then, there holds the following inequality*

$$\|u_h\|^2_{L^2(\Omega)} \le C\Big(\sum_{K\in\mathcal{T}}\|\nabla u_h\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_{int,D}}\|\sqrt{\mathsf{d}}[u_h]\|^2_{L^2(e)}\Big),$$

where $C > 0$ is a constant independent of $\boldsymbol{h}$ and of $\boldsymbol{p}$, and d is the discontinuity stabilization function from (2.8)*.*

Remark A.2.14 The above Proposition is a generalization of [1, Lemma 2.1], where a broken Poincaré inequality on convex polygons was proved.

Proof: (Proposition A.2.13) Due to Theorem [5, Theorem 2.1], there exists $w \in H^{2,2}_{\boldsymbol{\beta}}(\Omega)$, where $\boldsymbol{\beta}$ depends on the opening angles at the vertices of Ω, such that

$$\begin{aligned} -\Delta w &= u_h && \text{in } \Omega \\ w &= 0 && \text{on } \Gamma_D \\ \nabla w\cdot\boldsymbol{n}_\Omega &= 0 && \text{on } \Gamma_N, \end{aligned}$$

and

$$\|w\|_{H^{2,2}_{\boldsymbol{\beta}}(\Omega)} \le C\|u_h\|_{L^2(\Omega)}. \tag{A.11}$$

Therefore,

$$\begin{aligned}\|u_h\|^2_{L^2(\Omega)} &= -\sum_{K\in\mathcal{T}}\int_K u_h \Delta w \, d\boldsymbol{x} \\ &= \sum_{K\in\mathcal{T}}\int_K \nabla w\cdot\nabla u_h \, d\boldsymbol{x} - \sum_{K\in\mathcal{T}}\int_{\partial K}(\nabla w\cdot\boldsymbol{n}_K)u_h \, ds \\ &= \sum_{K\in\mathcal{T}}\int_K \nabla w\cdot\nabla u_h \, d\boldsymbol{x} - \sum_{e\in\mathcal{E}_{\mathrm{int},D}}\int_e \nabla w\cdot[u_h] \, ds.\end{aligned}$$

Applying Lemma 1.3.2 c) and Lemma A.1.1 results in

$$\begin{aligned}\|u_h\|^2 &\le C\Big(\sum_{K\in\mathcal{T}}\|\nabla u_h\|_{L^2(K)}\|\nabla w\|_{L^2(K)} + \sum_{e\in\mathcal{E}_{\mathrm{int},D}}\|[u_h]\|_{L^\infty(e)}\int_e|\nabla w| \, ds\Big) \\ &\le C\Big(\sum_{K\in\mathcal{T}}\|\nabla u_h\|_{L^2(K)}\|\nabla w\|_{L^2(K)} \\ &\quad + \sum_{K\in\mathcal{T}}\sum_{e\in\mathcal{E}_{\mathrm{int},D}\cap\mathcal{E}_K}\|\sqrt{\mathsf{d}}[u_h]\|_{L^2(e)}\|\nabla w\|_{L^1(e)}\Big) \\ &\le C\Big(\sum_{K\in\mathcal{T}}\|\nabla u_h\|_{L^2(K)}\|\nabla w\|_{L^2(K)} \\ &\quad + \sum_{K\in\mathcal{T}}\|w\|_{H^{2,2}_{\beta_K}(K)}\sum_{e\in\mathcal{E}_{\mathrm{int},D}\cap\mathcal{E}_K}\|\sqrt{\mathsf{d}}[u_h]\|_{L^2(e)}\Big) \\ &\le C\|w\|_{H^{2,2}_{\boldsymbol{\beta}}(\Omega)}\Big(\sum_{K\in\mathcal{T}}\|\nabla u_h\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_{\mathrm{int},D}}\|\sqrt{\mathsf{d}}[u_h]\|^2_{L^2(e)}\Big)^{1/2}\end{aligned}$$

Using (A.11) and dividing both sides by $\|u_h\|_{L^2(\Omega)}$ completes the proof. □

List of Symbols

$\boldsymbol{\nu}_e$	inter-element normal vector	11
$\mathbb{N}$	set of possitive integers, $\{1, 2, 3, \ldots\}$	7
$\mathbb{R}^n$	n-dimensional Euclidean space	5
$\mathbf{A}$	diffusivity	15
$\boldsymbol{\epsilon}$	symmetric gradient	45
μ, λ	Lamé coefficients	45
∇	gradient	8
$\nabla \cdot$	divergence operator	15
Ω	polygonal domain in $\mathbb{R}^2$	5
$\otimes$	tensor product	11
$\partial \mathcal{K}_0$	edges abutting at a singular point	12
Φ_{β_K}	weight function corresponding to K	12
$\Phi_{\boldsymbol{\beta}}$	weight function on Ω	6
$\preccurlyeq, \prec$	comparison operators for vectors	6
$\boldsymbol{h}$	vector of element diameters	9
$\boldsymbol{\beta}$	weight vector on Ω	5
$\boldsymbol{n}_K$	unit outward vector on ∂K	18
$\boldsymbol{p}$	polynomial degree (distribution) vector	9
$\boldsymbol{n}_\Omega$	unit outward vector on $\partial\Omega$	5
$\lVert \cdot \rVert_{H^{m,l}_{\boldsymbol{\beta}}(\Omega)}$	weighted Sobolev norm	6
$\lvert e \rvert$	length of edge e	49
D^α	multi-derivative	7
e	element edge or Euler's number	10
$H^m(\Omega)$	standard Sobolev space on Ω	8

$H_{\boldsymbol{\beta}}^{m,l}(\Omega)$	weighted Sobolev space on Ω	6
$H_{\boldsymbol{\beta}}^{m-1/2,l-1/2}(\gamma)$	weighted Sobolev trace space on γ	7
h_K	diameter of K	9
$h_{\mathcal{T}}$	mesh width of $\mathcal{T}$	9
K	an element in a finite element mesh	8
$L^2(\Omega)$, $L^2(K)$	square integrable functions on Ω and K	11
$L^\infty(\mathcal{E}_{\mathrm{int},D})$	bounded functions a.e. on $\mathcal{E}_{\mathrm{int},D}$	17
p_K	elemental polynomial degree	9
R_h	residual term	72
r_K	distance function corresponding to K	12
$SP(\Omega, \Gamma_D, \Gamma_N)$	singular points of the polygon Ω	5
$\mathcal{B}_{\boldsymbol{\beta}}^{l-1/2}(\gamma)$	weighted Sobolev trace space on γ	7
$\mathcal{B}_{\boldsymbol{\beta}}^{l}(\Omega)$	weighted Sobolev space on Ω	7
$\mathcal{C}^0(\overline{\Omega})$	continuous functions on $\overline{\Omega}$	8
$\mathcal{C}^\infty(\overline{\Omega})$	infinitely differentiable functions on $\overline{\Omega}$	6
$\mathcal{C}^\infty(\overline{\Omega})^{2\times 2}_{\mathrm{sym}}$	symmetric 2×2-matrices with entries in $\mathcal{C}^\infty(\overline{\Omega})$	15
$\mathcal{E}_K$	set of edges of K	10
$\mathcal{K}_0$	elements abutting at a singular point	11
$\mathcal{P}_p$	polynomials of total degree at most p	16
$\mathcal{Q}_p$	polynomials of degree at most p in each variable	16
$\mathcal{T}_{\boldsymbol{\gamma}}$	graded mesh	27
$\mathcal{T}_{n,\sigma}$	geometric mesh	38
$\mathcal{V}_p$	either $\mathcal{P}_p$ or $\mathcal{Q}_p$	16
$\mathcal{L}$, $\mathcal{M}$	lifting operators	72

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Curriculum Vitae

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