The geometry of Markov diffusion generators

THE GEOMETRY OF MARKOV DIFFUSION GENERATORS

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These notes form a summary of a mini-course given at the Eidgenössische Technische Hochschule in Zürich in November 1998. They aim to present some of the basic ideas in the geometric investigation of Markov diffusion generators, as developed during the last decade by D. Bakry and his collaborators. In particular, abstract notions of curvature and dimension that extend the corresponding ones in Riemannian geometry are introduced and studied. Using functional tools, new analytic proofs of some classical comparison theorems in Riemannian geometry are presented together with their counterparts in infinite dimension. Particular emphasis is put on sharp constants, optimal inequalities and model spaces and generators. These notes are far from complete (in particular most proofs are only outlined) and only aim to give a flavour of the subject.

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INTRODUCTION

A basic source for several of the functional inequalities on Riemannian manifolds are isoperimetric inequalities. The classical isoperimetric inequality in $\mathbb{R}^n$ asserts that among all bounded open sets A in $\mathbb{R}^n$ with smooth boundary ∂A and with fixed volume, Euclidean balls are the ones with the minimal surface measure. In other words, if $\mathrm{vol}_n(A) = \mathrm{vol}_n(B)$ where B is a ball (and $n \geq 2$),

$$\mathrm{vol}_{n-1}(\partial A) \geq \mathrm{vol}_{n-1}(\partial B).$$

Similarly, on a sphere S^n in $\mathbb{R}^{n+1}$ equipped with its (normalized) invariant measure σ, geodesic balls (caps) are the extremal sets for isoperimetry. That is, if $\sigma(A) = \sigma(B)$ where B is a cap on S^n, then

$$\sigma_s(\partial A) \geq \sigma_s(\partial B)$$

where $\sigma_s(\partial A)$ denotes surface measure of the (smooth) boundary ∂A of A on S^n. One main interest in such inequalities is the explicit description of the extremal sets.

The isoperimetric inequality on spheres has been extended by M. Gromov, using ideas going back to P. Lévy, as a comparison theorem for Riemannian manifolds with strictly positive curvature. Let (M, g) be a compact connected smooth Riemannian manifold of dimension n (≥ 2) equipped with the normalized Riemanian volume element $d\mu = \frac{dv}{V}$ where V is the volume of M. Denote by $R = R(M)$ the infimum of the Ricci curvature tensor over all unit tangent vectors, and assume that $R > 0$. Note that the n-sphere S^n_r with radius $r > 0$ is of constant curvature with $R(S^n_r) = \frac{n-1}{r^2}$. Given M with dimension n and $R = R(M) > 0$, let S^n_r be the n-sphere with constant curvature $R(S^n_r) = \frac{n-1}{r^2} = R$. Then, for any open set A on M with smooth boundary such that $\mu(A) = \sigma(B)$ where B is a cap on S^n_r and σ is the normalized invariant measure on S^n_r,

$$\mu_s(\partial A) \geq \sigma_s(\partial B). \tag{1}$$

In other words, the isoperimetric function $\inf\{\mu_s(\partial A); \mu(A) = p\}$, $p \in [0, 1]$, of M is bounded below by the isoperimetric function of the sphere with the same dimension and constant curvature equal to the lower bound on the curvature of M. Such a comparison property strongly emphasizes the importance of a model space, here the canonical sphere, to which manifolds may be compared. Equality in (1) occurs only if M is a sphere and A a cap on this sphere. Notice furthermore that (1) applied to sets the diameter of which tends to zero implies the classical isoperimetric inequality in Euclidean space.

The distributional isoperimetric inequalities allow one to transfer optimal functional inequalities on the sphere to inequalities on manifolds with a strictly positive lower bound on the Ricci curvature. For example, the classical Sobolev inequality on the sphere S_r^n with radius r in $\mathbb{R}^{n+1}$, $n \geq 3$, indicates that for every smooth function f on S_r^n,

$$\left(\int_{S_r^n} |f|^p d\sigma \right)^{2/p} \leq \int_{S_r^n} f^2 d\sigma + \frac{4\, r^2}{n(n-2)} \int_{S_r^n} |\nabla f|^2 d\sigma \tag{2}$$

where $p = 2n/(n-2)$ and $|\nabla f|$ is the length of the gradient of f. By the comparison inequalities (1), one can then show that if M is as before a Riemannian manifold with dimension n (≥ 3) and $R = R(M) > 0$, for any smooth function f on M,

$$\left(\int_M |f|^p d\mu \right)^{2/p} \leq \int_M f^2 d\mu + \frac{4(n-1)}{n(n-2)R} \int_M |\nabla f|^2 d\mu \tag{3}$$

where now $|\nabla f|$ is the Riemannian length of the gradient of f on M.

It is an old observation in probability theory that uniform measures on spheres with dimension n and radius $\sqrt{n}$ converge (weakly) as $n \to \infty$ to the canonical Gaussian measure on $\mathbb{R}^{\mathbb{N}}$ (product measure on $\mathbb{R}^{\mathbb{N}}$ of standard Gaussian distributions on each coordinate). This limiting procedure may be performed on (2) to yield a Sobolev inequality of logarithmic type for Gaussian measures. Denote for example by γ the canonical Gaussian measure on $\mathbb{R}^k$. Then, for any smooth function f on $\mathbb{R}^k$,

$$\int_{\mathbb{R}^k} f^2 \log f^2 d\gamma - \int_{\mathbb{R}^k} f^2 d\gamma \log \int_{\mathbb{R}^k} f^2 d\gamma \leq 2 \int_{\mathbb{R}^k} |\nabla f|^2 d\gamma \tag{4}$$

where $|\nabla f|$ is the Euclidean length of the gradient of f on $\mathbb{R}^k$. The spherical Laplacian actually approaches in this limit the Ornstein-Uhlenbeck generator with γ as invariant measure. The left-hand side of (4), called the entropy of f (or rather f^2), has to be interpreted as a limit of L^p-norms as $p = 2 + \frac{4}{n-2} \to 2$ since, if $\| \cdot \|_p$ denotes the L^p-norm with respect to some measure ν,

$$\lim_{p \to 2} \frac{2}{p-2} \Big[\|f\|_p^2 - \|f\|_2^2 \Big] = \int f^2 \log f^2 d\nu - \int f^2 d\nu \log \int f^2 d\nu.$$

On the other hand, when $r = \sqrt{n}$ and since $p - 2 = \frac{4}{n-2}$, the constant in front of the energy in the right-hand side of (2) is stabilized so to yield (4). The main feature of inequality (4) is that it is dimension free, reflecting this infinite dimensional construction of Gaussian measures. In a geometric language, Gaussian measures appear in this limit as objects of constant curvature (one) and infinite dimension.

The preceding examples form the basis for an abstract analysis of functional inequalities of Sobolev type that would include in the same pattern examples such as the sphere or Gauss space by functional notions of curvature and dimension. Such a setting is provided to us by Markov diffusion generators and semigroups for which the notions of carré du champ and iterated carré du champ yield, in analogy with the classical Bochner formula in differential geometry, abstract definitions of curvature and dimension. In particular, we can reach in this way non-integer dimension as well as infinite dimensional examples with the models of one-dimensional diffusion generators. This framework suggests a functional approach to both isoperimetric and Sobolev type inequalities and leads to a functional analysis of various classical results in Riemannian geometry.

To illustrate some of the results that may be investigated in this way, let us recall for example Myers's theorem that asserts that a compact Riemannian manifold (M, g) with dimension n and Ricci curvature bounded below by $R = R(M) > 0$ has a diameter less than or equal to the diameter $\pi r = \pi\sqrt{\frac{n-1}{R}}$ of the constant curvature sphere S_r^n with dimension n and $R(S_r^n) = \frac{n-1}{r^2} = R$. This result may actually be seen as a consequence of the sharp Sobolev inequality (3). We will namely observe that a manifold satisfying the Sobolev inequality (3) with the sharp constant of the sphere has a diameter less than or equal to the diameter of the sphere. In the same way, when $R = R(M) > 0$, the spectral gap of the Laplace operator on M is bounded below by the spectral gap of the sphere with constant curvature R. Recent work by D. Bakry and Z. Qian describes optimal comparison theorems for eigenvalues using curvature, dimension and diameter by means of one-dimensional diffusion generators whatsoever the sign of curvature. We also provide an infinite dimensional interpretation of Gromov's comparison theorem. If μ is the invariant probability measure of a Markov diffusion generator with strictly positive curvature R and infinite dimension, then its isoperimetric function is bounded below by the isoperimetric function of the corresponding model space, or rather generator, here Gaussian measures as invariant measures of the Ornstein-Uhlenbeck generator. Extremal sets in Gauss space are half-spaces, the Gaussian measure of which is evaluated in dimension one. Therefore, if for example $R = 1$ and if $\mu(A) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} \mathrm{e}^{-x^2/2}dx$, then

$$\mu_s(\partial A) \geq \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-a^2/2}.$$

These results concern comparison theorems for manifolds with positive curvature and their extensions to infinite dimension. The model space for manifolds with non-negative Ricci curvature is simply the classical Euclidean space. Although the picture is less complete here, we present a sample of results on optimal heat kernel bounds under a Sobolev type inequality, as well as rigidity statements. For example, a Riemannian manifold with dimension n and non-negative Ricci curvature satisfying one of the classical Sobolev inequalities with the best constant of $\mathbb{R}^n$ must be isometric to $\mathbb{R}^n$. Negative curvature is still under study.

The first chapter presents the basic framework of Markov diffusion generators and the notions of curvature and dimension in this setting. We also discuss here the functional inequalities of interest, from Poincaré to Sobolev and logarithmic Sobolev inequalities. In Chapter 2, we investigate the case of infinite dimensional generators for which we describe a version of Gromov's comparison theorem and discuss some of its applications to logarithmic Sobolev inequalities. Chapter 3 is concerned with finite dimensional generators of strictly positive curvature. We present new analytic approaches to several classical results in Riemannian geometry such as volume, diameter and eigenvalue comparisons. In the last chapter, we study optimal heat kernel bounds and rigidity theorems in a non-compact setting. Complete references for the results only outlined in these notes are provided in the bibliography.

1. GEOMETRIC ASPECTS OF DIFFUSION GENERATORS

We review here the basic definitions on the geometric aspects of diffusion generators. The main reference is the St-Flour notes by D. Bakry [Ba3] from which we extract most of the material presented here and to which we refer for complete details.

1.1 Semigroups and generators

We consider a measurable space $(E, \mathcal{E})$ equipped with a σ-finite measure μ. When μ is finite, we always normalize it into a probability measure. We denote by $\mathrm{L}^p = \mathrm{L}^p(\mu)$, $1 \le p \le \infty$, the Lebesgue spaces with respect to μ, and sometimes set $\|\cdot\|_p$ for the norm in L^p.

Our fundamental object of interest is a family $(P_t)_{t\ge 0}$ of non-negative operators acting on the bounded mesurable functions f on E by

$$P_t f(x) = \int_E f(y) p_t(x, dy), \quad x \in E,$$

and satisfying the basic semigroup property: $P_s \circ P_t = P_{s+t}$, $s, t \ge 0$, $P_0 = \mathrm{Id}$. The non-negative kernels $p_t(x, dy)$ are called transition kernels. We will also assume that the operators P_t are bounded and continuous on $\mathrm{L}^2(\mu)$ in the sense that, for every f in $\mathrm{L}^2(\mu)$, $\|P_t f\|_2 \le C(t)\|f\|_2$ for every $t \ge 0$ and $\|P_t f - f\|_2 \to 0$ as $t \to 0$.

We say that $(P_t)_{t\ge 0}$ is Markov if $P_t 1 = 1$ for every t. Markov semigroups are naturally associated to Markov processes $(X_t)_{t\ge 0}$ with values in E by the relation

$$P_t f(x) = \mathbb{E}\big(f(X_t) \,|\, X_0 = x\big).$$

The prime example is of course Brownian motion $(B_t)_{t\ge 0}$ with values in $\mathbb{R}^n$ and starting from the origin with transition (heat) kernels

$$p_t(x, dy) = \frac{1}{(2\pi t)^{n/2}}\, \mathrm{e}^{-|x-y|^2/2t} dy, \quad t \ge 0, \quad x \in \mathbb{R}^n.$$

We denote by $\mathcal{D}_2(\mathrm{L})$ the domain in $\mathrm{L}^2(\mu)$ of the infinitesimal generator L of the semigroup $(P_t)_{t\ge 0}$. $\mathcal{D}_2(\mathrm{L})$ is defined as the set of all functions f in $\mathrm{L}^2(\mu)$ for which the limit

$$\mathrm{L} f = \lim_{t\to 0} \frac{1}{t}\,(P_t f - f)$$

exists. $\mathcal{D}_2(\mathrm{L})$ is dense in $\mathrm{L}^2(\mu)$ and may be equipped with the topology given by

$$\|f\|_{\mathcal{D}_2} = \|f\|_2 + \|\mathrm{L}f\|_2$$

(cf. [Yo]). The semigroup P_t leaves the domain $\mathcal{D}_2(\mathrm{L})$ stable, and for every f in $\mathcal{D}_2(\mathrm{L})$,

$$\frac{\partial}{\partial t} P_t f = P_t \mathrm{L} f = \mathrm{L} P_t f. \tag{1.1}$$

Conversely, the generator L and its domain $\mathcal{D}_2(\mathrm{L})$ completely determine $(P_t)_{t\geq 0}$: there exists a unique semigroup $(P_t)_{t\geq 0}$ of bounded operators on $\mathrm{L}^2(\mu)$ satisfying (1.1) for all functions of the domain $\mathcal{D}_2(\mathrm{L})$. For example, the Brownian semigroup has generator half of the usual Laplacian Δ on $\mathbb{R}^n$ and equation (1.1) is the classical heat equation.

The measure μ will be related to the semigroup $(P_t)_{t\geq 0}$ by two properties. The measure μ is said to be time reversible with respect to $(P_t)_{t\geq 0}$, or $(P_t)_{t\geq 0}$ is symmetric with respect to μ, if for every f, g in $\mathrm{L}^2(\mu)$,

$$\int f P_t g d\mu = \int g P_t f d\mu.$$

μ is invariant with respect to $(P_t)_{t\geq 0}$ if for every f in $\mathrm{L}^1(\mu)$,

$$\int P_t f d\mu = \int f d\mu.$$

When μ is finite, one may choose $g = 1$ to see that time reversible measures are invariant. Time reversible, resp. invariant, measures μ are described equivalently as those for which $\int f \mathrm{L} g d\mu = \int g \mathrm{L} f$, resp. $\int \mathrm{L} f d\mu = 0$, for every f, g in the domain of L.

When μ is a probability measure, we will also say that $(P_t)_{t\geq 0}$ is ergodic if $P_t f \to \int f d\mu$ μ-almost everywhere as $t \to \infty$.

In order to determine the semigroup $(P_t)_{t\geq 0}$ from its generator, it suffices to know L on some dense subspace of the domain. Indeed, in general only the generator is given and usually only a dense subset of the domain is known. Moreover, explicit formulas for the semigroup such as for Brownian motion are usually not available.

For simplicity, we will thus work with a nice algebra $\mathcal{A}$ of bounded functions on E, dense in $\mathcal{D}_2(\mathrm{L})$ and in all $\mathrm{L}^p(\mu)$-spaces, and stable by L. When μ is finite, we assume that $\mathcal{A}$ contains the constants and is stable by C^∞ functions of several variables. (In this case, it is easily seen that the semigroup is Markov if and only if $\mathrm{L}1 = 0$.) When μ is infinite, we replace this condition by the fact that $\mathcal{A}$ is stable by C^∞ functions which are zero at the origin.

Amongst Markov generators, a class of particular interest consists in the so-called diffusion generators. To this aim, we first introduce, following P.-A. Meyer, the "carré du champ" operator Γ as the symmetric bilinear operator on $\mathcal{A} \times \mathcal{A}$ defined by

$$2\Gamma(f,g) = \mathrm{L}(fg) - f\mathrm{L}g - g\mathrm{L}f, \quad f, g \in \mathcal{A}.$$

The operator Γ measures how far L is from a derivation. A fundamental property is the positivity of the carré du champ. Namely, since $p_t(x, dy)$ is a probability measure, by Jensen's inequality

$$(P_t f)^2 \leq P_t(f^2)$$

(at every point x). On the other hand, the definition of Γ shows that

$$\Gamma(f,g) = \lim_{t \to 0} \frac{1}{2t} \left[P_t(fg) - P_t f P_t g \right]$$

so that it immediately follows that

$$\Gamma(f,f) \geq 0 \tag{1.2}$$

(pointwise) for every $f \in \mathcal{A}$. As a consequence of (1.2), note that $\Gamma(f,g)^2 \leq \Gamma(f,f)\Gamma(g,g)$.

We then say that L is a diffusion if for every C^∞ function Ψ on $\mathbb{R}^k$, and every finite family $F = (f_1, \ldots, f_k)$ in $\mathcal{A}$,

$$\mathrm{L}\Psi(F) = \sum_{i=1}^{n} \partial_i \Psi(F)\, \mathrm{L} f_i + \sum_{i,j=1}^{n} \partial_{ij}^2 \Psi(F)\, \Gamma(f_i, f_j). \tag{1.3}$$

In particular, if ψ is C^∞ on $\mathbb{R}$, for every f in $\mathcal{A}$,

$$\mathrm{L}\psi(f) = \psi'(f)\, \mathrm{L} f + \psi''(f)\, \Gamma(f,f).$$

This hypothesis essentially expresses that L is a second order differential operator with no constant term and that we have a chain rule formula for Γ,

$$\Gamma\big(\psi(f), g\big) = \psi'(f)\, \Gamma(f,g), \quad f, g \in \mathcal{A}.$$

Applying (1.3) to $\Psi(f,g,h)$, $f,g,h \in \mathcal{A}$, moreover shows that Γ is a derivation in each argument:

$$\Gamma(fg, h) = f\Gamma(g,h) + g\Gamma(f,h).$$

The diffusion property is related to the regularity properties of the Markov process $(X_t)_{t \geq 0}$ associated to $(P_t)_{t \geq 0}$. If $(P_t)_{t \geq 0}$ is a diffusion semigroup on some algebra $\mathcal{A}$, the processes $(f(X_t))_{t \geq 0}$, $f \in \mathcal{A}$, have continuous paths.

When μ is invariant,the following basic integration by parts formula is satisfied

$$\int f(-\mathrm{L}g)\,d\mu = \int \Gamma(f,g)\,d\mu, \quad f,g \in \mathcal{A}. \tag{1.4}$$

In particular, $\int f(-\mathrm{L}f)\,d\mu \geq 0$.

To illustrate these abstract definitions, let us describe the main examples we wish to consider in these notes.

Let first E be finite with N elements, and μ charging all the points. A Markov generator L is described by an $N \times N$ matrix with non-negative entries (L_{ij}) so that

$$\mathrm{L}f(i) = \sum_{j=1}^{N} \mathrm{L}_{ij} f(j), \quad i = 1, \dots, N,$$

for every f on E. The operator P_t is the exponential of the matrix $t\mathrm{L}$. The carré du champ Γ may be written

$$\Gamma(f,g)(i) = \frac{1}{2} \sum_{i} \mathrm{L}_{ij} \big[f(i) - f(j)\big] \big[g(i) - g(j)\big].$$

The only diffusion operator is $\mathrm{L} = 0$. μ is invariant if $\sum_i \mathrm{L}_{ij} \mu_i = 0$ for every j. It is classical that the invariant measure is unique as soon as L is irreducible.

We already mentioned the heat or Brownian semigroup $(P_t)_{t \geq 0}$ on $\mathbb{R}^n$. In this case therefore, E is $\mathbb{R}^n$ equipped with its Borel σ-field and Lebesgue measure dx. The generator of the heat semigroup is half of the classical Laplacian Δ on $\mathbb{R}^n$ and the associated Markov process is Brownian motion $(B_t)_{t \geq 0}$. In order to simplify a number of analytical definitions later on, we will work throughout these notes with Δ rather than $\frac{1}{2}\Delta$ so that, for every sufficiently integrable function f on $\mathbb{R}^n$,

$$\begin{aligned} P_t f(x) &= \int_{\mathbb{R}^n} f(y)\, \mathrm{e}^{-|x-y|^2/4t} \frac{dy}{(4\pi t)^{n/2}} \\ &= \int_{\mathbb{R}^n} f\big(x + \sqrt{2t}\, y\big) d\gamma(y), \quad t \geq 0, \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.5}$$

where γ is the canonical Gaussian measure on $\mathbb{R}^n$ with density $(2\pi)^{-n/2} \mathrm{e}^{-|x|^2/2}$ with respect to Lebesgue measure. On the class $\mathcal{A}$, say, of all C^∞ compactly supported functions, we then have

$$\Gamma(f,g) = \nabla f \cdot \nabla g, \quad f, g \in \mathcal{A}.$$

Lebesgue measure is invariant and time reversible with respect to $(P_t)_{t \geq 0}$.

More generally, let us consider a smooth (C^1 say) function U on $\mathbb{R}^n$ and let $d\mu = \mathrm{e}^{-U(x)} dx$. As is well-known, μ may be described as the time reversible and invariant measure of the generator

$$\mathrm{L} = \Delta - \nabla U \cdot \nabla.$$

Alternatively, $\frac{1}{2}\mathrm{L}$ is the generator of the Markov semigroup $(P_t)_{t \geq 0}$ of the Kolmogorov process $X = (X_t)_{t \geq 0}$ solution of the Langevin stochastic differential equation

$$dX_t = dB_t - \tfrac{1}{2} \nabla U(X_t) dt.$$

Similarly, $\Gamma(f,g) = \nabla f \cdot \nabla g$ for smooth functions f and g. The choice of $U(x) = \frac{1}{2}|x|^2$ with invariant measure the canonical Gaussian measure γ corresponds to the Ornstein-Uhlenbeck generator

$$\mathrm{L}f(x) = \Delta f(x) - x \cdot \nabla f(x), \quad x \in \mathbb{R}^n.$$

Since in this case $X_t = \sqrt{2}\,\mathrm{e}^{-t}\int_0^t \mathrm{e}^s dB_s$, the Ornstein-Uhlenbeck semigroup $(P_t)_{t\ge 0}$ may be represented as

$$P_t f(x) = \int_{\mathbb{R}^n} f\big(\mathrm{e}^{-t}x + (1-\mathrm{e}^{-2t})^{1/2} y\big)\, d\gamma(y), \quad x \in \mathbb{R}^n, \quad t \ge 0. \tag{1.6}$$

Due to the integrability properties of Gaussian densities, one can choose here for $\mathcal{A}$ the class of C^∞ functions whose derivatives are rapidly decreasing.

Let further $E = M$ be an n-dimensional smooth connected manifold equipped with a measure μ on the Borel sets equivalent to Lebesgue measure. On the algebra $\mathcal{A}$ of, say, all C^∞ compactly supported functions, let L be a second order differential operator that may be written in a chart as

$$\mathrm{L} f(x) = \sum_{i,j=1}^n g^{ij}(x)\,\frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^n b^i(x)\,\frac{\partial f}{\partial x^i}(x)$$

where the functions g^{ij}, b^i are C^∞ and the matrix $(g^{ij}(x))$ is non-negative definite at each x. The generator L is then elliptic, and, when M is compact, each solution of (1.1) with f in $\mathcal{A}$ is such that $P_t f$ is in $\mathcal{A}$. Furthermore,

$$\Gamma(f,h) = \sum_{i,j=1}^n g^{ij}\,\frac{\partial f}{\partial x^i}\,\frac{\partial h}{\partial x^j}\,.$$

This is the origin of the terminology carré du champ. Indeed, the matrix $(g^{ij}(x))$ (when non-degenerate) defines a symmetric tensor field on M. The inverse tensor $(g_{ij}(x))$ then defines on M a Riemannian metric, and $\Gamma(f,f)$ is the square of the length of the gradient ∇f in this metric. This example covers the Laplace-Beltrami generator

$$\Delta f = \sum_{i,j=1}^n d^{-1/2}\partial_i\big(d^{1/2} g^{ij}\partial_j f\big)$$

with $d = \det(g_{ij})$ on a complete Riemannian manifold (M,g) with, under some mild geometric conditions such as Ricci curvature bounded below (cf. [Da]), associated heat semigroup $(P_t)_{t\ge 0}$.

Throughout these notes, we consider a Markov diffusion generator L *with semigroup* $(P_t)_{t\ge 0}$ *on a measure space* $(E,\mathcal{E},\mu)$*, acting on some algebra* $\mathcal{A}$ *of bounded functions on* E*, dense in* $\mathcal{D}_2(\mathrm{L})$ *and in all* $\mathrm{L}^p(\mu)$*-spaces. The algebra* $\mathcal{A}$ *is also assumed to be stable by* L*, and stable by* C^∞ *functions of several variables and containing the constants if* μ *is finite, and stable by* C^∞ *functions which are zero at the origin when* μ *is infinite. We assume* μ *to be time reversible and invariant with respect to* L *(or* $(P_t)_{t\ge 0}$*). When* μ *is finite, we assume that it is a probability measure and that* $(P_t)_{t\ge 0}$ *is ergodic with respect to* μ*.*

1.2 Curvature and dimension

Using the carré du champ, we define here functional notions of curvature and dimension. We start with the abstract setting described in Section 1.1 and consider thus a Markov generator L with semigroup $(P_t)_{t\geq 0}$ on $(E,\mathcal{E},\mu)$ and algebra $\mathcal{A}$. Reproducing the definition of the carré du champ by replacing the product by Γ, one may define the so-called iterated carré du champ as the symmetric bilinear operator on $\mathcal{A}\times\mathcal{A}$

$$2\Gamma_2(f,g) = \mathrm{L}\Gamma(f,g) - \Gamma(f,\mathrm{L}g) - \Gamma(g,\mathrm{L}f), \quad f,g\in\mathcal{A}.$$

For simplicity, we write $\Gamma f = \Gamma(f)$ for $\Gamma(f,f)$ and similarly with Γ_2. (One could define similarly the whole family of iterated gradient Γ_n, $n\geq 1$, by

$$2\Gamma_n(f,g) = \mathrm{L}\Gamma_{n-1}(f,g) - \Gamma_{n-1}(f,\mathrm{L}g) - \Gamma_{n-1}(g,\mathrm{L}f), \quad f,g\in\mathcal{A},$$

with $\Gamma_1 = \Gamma$ (see [Le1]). We will however not use them here.)

It is a classical exercise to check that for the usual Laplacian Δ on $\mathbb{R}^n$,

$$\Gamma_2(f) = \|\mathrm{Hess}\, f\|_2^2$$

where

$$\|\mathrm{Hess}\, f\|_2 = \bigg(\sum_{i,j=1}^n \Big(\frac{\partial^2 f}{\partial x^i\partial x^j}\Big)^2\bigg)^{1/2}$$

is the Hilbert-Schmidt norm of the tensor of the second derivatives of f. When $\mathrm{L} = \Delta - \nabla U\cdot\nabla$ with U of class C^2,

$$\Gamma_2(f) = \mathrm{Hess}\, U(\nabla f,\nabla f) + \|\mathrm{Hess}\, f\|_2^2.$$

In particular, for the Ornstein-Uhlenbeck generator $\mathrm{L} = \Delta - x\cdot\nabla$,

$$\Gamma_2(f) = |\nabla f|^2 + \|\mathrm{Hess}\, f\|_2^2 = \Gamma(f) + \|\mathrm{Hess}\, f\|_2^2. \tag{1.7}$$

In a Riemannian setting, for the Laplace-Beltrami operator Δ on a smooth manifold (M,g) with dimension n, Bochner's formula (cf. [Cha2], [G-H-L]) indicates that for any smooth function f on M,

$$\Gamma_2(f) = \mathrm{Ric}\,(\nabla f,\nabla f) + \|\mathrm{Hess}\, f\|_2^2$$

where Ric is the Ricci tensor on M. If we assume that the Ricci curvature of M is bounded below in the sense that $\mathrm{Ric}_x\,(u,v)\geq R g_x(u,v)$ for every tangent vectors $u,v\in T_x(M)$, and if we observe, by Cauchy-Schwarz, that $\|\mathrm{Hess}\, f\|_2^2 \geq \frac{1}{n}(\Delta f)^2$ (since Δf is the trace of $\mathrm{Hess}\, f$), we see that

$$\Gamma_2(f) \geq R\,|\nabla f|^2 + \frac{1}{n}\,(\Delta f)^2. \tag{1.8}$$

On the basis of these examples and observations, we introduce (functional) notions of curvature and dimension for abstract Markov generators.

Definition 1.2. *A generator* L *satifies a curvature-dimension inequality CD(R,n) of curvature $R \in \mathbb{R}$ and dimension $n \geq 1$ if, for all functions f in $\mathcal{A}$,*

$$\Gamma_2(f) \geq R\,\Gamma(f) + \frac{1}{n}\,(\mathrm{L}f)^2.$$

Inequalities in Definition 1.2 are understood either at every point or μ-almost everywhere.

This definition does not separate curvature and dimension. If L is of curvature-dimension $\mathrm{CD}(R,n)$, it is of curvature-dimension $\mathrm{CD}(R',n')$ for $R' \leq R$ and $n' \geq n$. (Moreover, it depends on the choice of the algebra $\mathcal{A}$.) According to (1.8), an n-dimensional complete Riemannian manifold (M,g) with Ricci curvature bounded below, or rather the Laplacian Δ on M, satisfies the inequality $\mathrm{CD}(R,n)$ with R the infimum of the Ricci tensor and n the geometric dimension. If $\mathrm{L} = \Delta + \nabla h$ for a smooth function h, and if (and only if), as symmetric tensors,

$$\nabla h \otimes \nabla h \leq (m-n)\big[\mathrm{Ric} - \nabla\nabla h - \rho g\big]$$

with $m \geq n$, then L satifies $\mathrm{CD}(\rho, m)$ (cf. [Ba3], Proposition 6.2). Conditions for more general differential operators of the form

$$\mathrm{L}f(x) = \sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}\,(x) + \sum_{i=1}^{n} b^i(x)\frac{\partial f}{\partial x_i}\,(x),$$

where $A(x) = (g^{ij}(x))_{1 \leq i,j \leq n}$ is symmetric positive definite at every point, to be of some curvature may be given in the same spirit.

The classical Laplacian Δ on $\mathbb{R}^n$ thus satisfies $\mathrm{CD}(0,n)$, whereas the Laplace-Beltrami operator on the unit sphere S^n is of curvature dimension $\mathrm{CD}(n-1,n)$ since curvature is constant and equal to $n-1$ in this case. (By homogeneity, the sphere S^n_r with dimension n and radius $r > 0$ has constant curvature equal to $\frac{n-1}{r^2}$.) But the preceding definition allows us to consider examples that do not enter a Riemannian setting. For example, by (1.7), the Ornstein-Uhlenbeck generator $\mathrm{L} = \Delta - x \cdot \nabla$ satifies $\mathrm{CD}(1,\infty)$. It does not satisfy any better condition with a finite dimension. Indeed, there is no $c > 0$ and $R \in \mathbb{R}$ such that

$$\|\mathrm{Hess}\, f\|_2^2 \geq (R-1)|\nabla f|^2 + c(\mathrm{L}f)^2$$

for every f as can be seen by choosing for example $f(x) = |x|^2$ and by letting $x \to \infty$. This observation connects with the description of Gaussian measures as limiting distributions of spherical measures as dimension goes to infinity. More generally, if $\mathrm{L} = \Delta - \nabla U \cdot \nabla$ on $\mathbb{R}^n$, L satifies $\mathrm{CD}(R,\infty)$ for some $R \in \mathbb{R}$, as soon as, at every point x, and as symmetric matrices, $\mathrm{Hess}\, U(x) \geq R\,\mathrm{Id}$. We will say more simply that L is of curvature $R \in \mathbb{R}$ if it satisfies $\mathrm{CD}(R,\infty)$ that is if, for all functions f in $\mathcal{A}$,

$$\Gamma_2(f) \geq R\,\Gamma(f) \tag{1.9}$$

(and also write sometimes $\Gamma_2 \geq R\Gamma$).

Besides allowing infinite dimension, Definition 1.2 also allows us to consider non-integer dimension through the family of (symmetric) Jacobi operators. Let namely, on $(-1,+1)$,

$$\mathrm{L}_n f(x) = (1-x^2)f''(x) - nxf'(x) \tag{1.10}$$

for every f smooth enough, where $n > 0$. In this example, $\Gamma(f) = (1-x^2)f'(x)^2$ and the invariant measure is given by $d\mu_n = c_n(1-x^2)^{(n/2)-1}dx$ on $(-1,+1)$. When n is an integer, L_n is known as the ultraspheric generator which is obtained as the projection of the Laplace operator of S^n on a diameter. It is easily checked that

$$\Gamma_2(f) = (n-1)\Gamma(f) + \frac{1}{n}\,(\mathrm{L}_n f)^2 + \Big(1 - \frac{1}{n}\Big)(1-x^2)^2 f''^2.$$

Therefore, L satisfies $\mathrm{CD}(n-1,n)$ for every $n \geq 1$. (Note that the curvature-dimension inequality is reversed when $n < 1$.) In particular, the dimension in $\mathrm{CD}(R,n)$ does not refer to any dimension of the underlying state space. Actually, by the change of variables $y = \sin^{-1} x$, L can be described as the differential operator

$$\mathrm{L}f = f'' - (n-1)\tan(y)f'$$

on the interval $(-\frac{\pi}{2}, +\frac{\pi}{2})$ (this choice will be explained in Section 3.3). More generally, when $\mathrm{L}f = f'' - a(x)f'$, $\mathrm{CD}(R,n)$ for L is equivalent to saying that

$$a' \geq R + \frac{a^2}{n-1}\,. \tag{1.11}$$

Note that the (one-dimensional) Ornstein-Uhlenbeck generator on the line $\mathrm{L}f = f'' - xf'$ enters this description with $a(x) = x$ and that (1.11) takes the form $a' \geq (=)\,1$ (that is $\mathrm{CD}(1,\infty)$). For a complete description of one-dimensional diffusion generators along these lines, cf. [Ma].

Together with the infinite dimensional Ornstein-Uhlenbeck generator, the Jacobi operators will be our models of generators with strictly positive curvature (extending thus the finite dimensional example of the sphere). We discuss more precisely the negative curvature in connection with eigenvalue comparison theorems in Section 3.3.

In the last part of this section, we turn to some equivalent descriptions of curvature of L as commutation properties of the associated semigroup $(P_t)_{t\geq 0}$. We thereby face quite an annoying question, namely the stability of our algebra $\mathcal{A}$ by P_t. It is clear that, in very basic examples such as the heat semigroup $(P_t)_{t\geq 0}$ on a non-compact manifold, compactly supported C^∞ functions are not stable by P_t. We however wish to work with expressions such as $\Gamma(P_t f)$ or $\Gamma_2(P_t f)$. In concrete examples, the latter may usually be defined without too many difficulties (see [Ba4] e.g. and the references therein). This stability is basically the only property really needed in the proofs below concerning $\mathcal{A}$, which we thus implicitely assume throughout these notes. We actually put emphasis in this work on the structure of the algebraic methods. In this regard, the stability of $\mathcal{A}$ by P_t removes all kind of analytic problems which are of a different nature. The question of extending the results to large classes of functions in the domain of given generators is thus another issue not adressed here.

As announced, the curvature assumption $\Gamma_2 \geq R\,\Gamma$ on the infinitesimal generator L may be translated equivalently on the semigroup $(P_t)_{t\geq 0}$. To better understand this relation, it might be important to notice that in the case of the classical heat semigroup $(P_t)_{t\geq 0}$ on $\mathbb{R}^n$ (with curvature $R=0$), for any smooth function f,

$$\nabla P_t f = P_t(\nabla f)$$

(which is immediate on the representation formula (1.5)). Similarly, for the Ornstein-Uhlenbeck semigroup (with curvature $R=1$),

$$\nabla P_t f = \mathrm{e}^{-t} P_t(\nabla f)$$

(by (1.6)). The general situation is the content of the following lemma. Since μ is invariant for P_t, the respective inequalities are understood either everywhere or μ-almost everywhere.

Lemma 1.2. $\Gamma_2 \geq R\,\Gamma$ *if and only if for every* f *in* $\mathcal{A}$ *and every* $t \geq 0$,

$$\Gamma(P_t f) \leq \mathrm{e}^{-2Rt} P_t(\Gamma f). \tag{1.12}$$

Proof. Let, for $f \in \mathcal{A}$ and $t>0$ fixed, $F(s) = \mathrm{e}^{-2Rs} P_s(\Gamma(P_{t-s}f))$, $0 \leq s \leq t$. Now, by definition of Γ_2,

$$F'(s) = 2\mathrm{e}^{-2Rs}\big[-RP_s\big(\Gamma(P_{t-s}f)\big) + P_s\big(\Gamma_2(P_{t-s}f)\big)\big].$$

Hence, by (1.9) applied to $P_{t-s}f$ for every s, F is non-decreasing and (1.12) follows. For the converse, note that (1.12) is an equality at $t=0$. Therefore,

$$0 \leq \lim_{t\to 0}\frac{1}{2t}\left[\mathrm{e}^{-2Rt}P_t(\Gamma f) - \Gamma(P_t f)\right] = \Gamma_2(f) - R\,\Gamma(f).$$

□

The preceding lemma does not make use of the diffusion property. It is a remarkable observation by D. Bakry [Ba1] that (1.12) may considerably be reinforced in this case.

Lemma 1.3. *When* L *is a diffusion,* $\Gamma_2 \geq R\,\Gamma$ *if and only if for every* f *in* $\mathcal{A}$ *and every* $t \geq 0$,

$$\sqrt{\Gamma(P_t f)} \leq \mathrm{e}^{-Rt}\sqrt{P_t(\Gamma f)}\,. \tag{1.13}$$

Inequality (1.13) is actually classical in Riemannian geometry but the proof of [Ba1] is completely algebraic. We take it from [Ba4].

Proof. Arguing as in the proof of Lemma 1.2 shows that the equivalent infinitesimal version of (1.13) is that for every f in $\mathcal{A}$,

$$\Gamma(f)\big(\Gamma_2(f) - R\,\Gamma(f)\big) \geq \frac{1}{4}\,\Gamma\big(\Gamma(f)\big). \tag{1.14}$$

We thus need to show that (1.14) follows from $\Gamma_2 \geq R\,\Gamma$ using the diffusion property. The proof is based on the change of variable formula for Γ_2. For f, g, h in $\mathcal{A}$, set

$$H(f)(g,h) = \frac{1}{2}\left[\Gamma\big(g, \Gamma(f,h)\big) + \Gamma\big(h, \Gamma(f,g)\big) - \Gamma\big(f, \Gamma(g,h)\big)\right].$$

This notation stands for the Hessian of f since, in a differentiable context, it is easily seen that $H(f)(g,h) = \operatorname{Hess} f(\nabla g, \nabla h)$. Let Ψ be a smooth function on $\mathbb{R}^k$ and let $F = (f_1, \dots, f_k)$ in $\mathcal{A}$. By the diffusion property (1.3) for L, we get (after cumbersome algebra!)

$$\begin{aligned}\Gamma_2\big(\Psi(F)\big) = \sum_{i,j} X_i X_j \Gamma_2(f_i, f_j) + 2\sum_{i,j,k} X_i X_{jk} H(f_i)(f_j, f_k)\\ + \sum_{i,j,k,\ell} X_{ij} X_{k\ell} \Gamma(f_i, f_k)\Gamma(f_j, f_\ell)\end{aligned}$$

where we use the shorthand notation

$$X_i = \frac{\partial \Psi(f)}{\partial x_i}\,, \quad X_{ij} = \frac{\partial^2 \Psi(f)}{\partial x_i \partial x_j}\,.$$

A similar change of variable formula for $Q = \Gamma_2 - R\,\Gamma$ easily follows. Now, when Ψ varies among all second order polynomials in k variables, $Q(\Psi(F))$ is, under (1.9), a positive quadratric expression in the variables X_i, X_{jk}. Let us specify this expression for two variables $f_1 = f$ and $f_2 = g$ but with $X_2 = X_{11} = X_{22} = 0$. We get

$$X_1^2 Q(f) + 4X_1 X_{12} H(f)(f,g) + 2X_{12}^2\left[\Gamma(f,g)^2 + \Gamma(f)\Gamma(g)\right] \geq 0.$$

Since $H(f)(f,g) = \frac{1}{2}\Gamma(g, \Gamma f)$, it follows that

$$\Gamma\big(g, \Gamma(f)\big)^2 \leq 2Q(f)\left[\Gamma(f,g)^2 + \Gamma(f)\Gamma(g)\right].$$

If we rewrite this inequality for $g = \Gamma(f)$, we see that

$$\Gamma(g)^2 \leq 2Q(f)\left[\Gamma(f,g)^2 + \Gamma(f)\Gamma(g)\right].$$

Now, $\Gamma(f,g)^2 \leq \Gamma(f)\Gamma(g)$ so that

$$\Gamma(g)^2 \leq 4Q(f)\Gamma(f)\Gamma(g)$$

from which the claim follows. The proof of Lemma 1.3 is complete. □

Lemmas 1.2 and 1.3 will be the key to various proofs of functional inequalities using curvature assumption. It is at this point one difficult question to understand these lemmas under some additional finite dimension, i.e. under a $\mathrm{CD}(R,n)$ condition.

1.3 Functional inequalities

The functional inequalities we will deal with are part of the family of Sobolev inequalities. Typically, on a Riemannian manifold (M, g) with its Riemannian volume element dv, a Sobolev inequality is of the type

$$\left(\int_M |f|^p dv \right)^{q/p} \leq A \int_M |f|^p dv + B \int_M |\nabla f|^q dv \tag{1.15}$$

for $p > q > 0$, constants $A, B \geq 0$ and all smooth f on M. These estimates describe quantitatively the Sobolev embeddings. When $A = 0$, we will speak of global inequalities (holding for compactly supported functions on the non-compact manifold M), whereas when $A > 0$, we speak of local inequalities. In the clasical case of $\mathbb{R}^n$, it was proved by S. Sobolev [So] that (1.15) holds for all smooth functions f whenever $1/p = 1/q - 1/n$, $q < n$. For simplicity, and since in our abstract setting of Markov generators $\Gamma(f) = |\nabla f|^2$, we reduce ourselves to the case $q = 2$ in (1.15).

The next definitions put a view on sharp constants in inequalities such as (1.15). Below, we reduce ourselves to the case where the reference measure is finite (probability measure). We come back to infinite measures in Chapter 4 through several examples. To illustrate the various constants we will investigate, let us start with the basic example of the unit sphere S^n equipped with its uniform normalized measure σ. It has been shown by Th. Aubin [Au] that for every smooth function f on S^n, $n \geq 3$,

$$\left(\int_{S^n} |f|^p d\sigma \right)^{2/p} \leq \int_{S^n} f^2 d\sigma + \frac{4}{n(n-2)} \int_{S^n} |\nabla f|^2 d\sigma \tag{1.16}$$

where $p = 2n/(n-2)$. As was shown later on by W. Beckner [Be], this inequality extends to all values of $1 \leq p \leq 2n/(n-2)$ in the form

$$\frac{n}{p-2} \left[\left(\int_{S^n} |f|^p d\sigma \right)^{2/p} - \int_{S^n} f^2 d\sigma \right] \leq \int_{S^n} |\nabla f|^2 d\sigma. \tag{1.17}$$

The latter inequality actually contains a number of limiting cases of interest. Namely, when $p = 1$, it reads as the Poincaré or spectral gap inequality

$$n \left[\int_{S^n} f^2 d\sigma - \left(\int_{S^n} f d\sigma \right)^2 \right] \leq \int_{S^n} |\nabla f|^2 d\sigma. \tag{1.18}$$

Since $\int |\nabla f|^2 d\sigma = \int f(-\Delta f) d\sigma$, (1.18) expresses by the min-max principle that the largest non-trivial eigenvalue λ_1 of the Laplacian Δ on the n-sphere S^n is larger than or equal to n (actually equal to n). Another limiting value is $p = 2$. Since it is easily seen that, for a probability measure ν,

$$\lim_{p \to 2} \frac{2}{(p-2)} \left[\left(\int |f|^p d\nu \right)^{2/p} - \int f^2 d\nu \right] = \int f^2 \log f^2 d\nu - \int f^2 d\nu \log \int f^2 d\nu,$$

we deduce from (1.17) a so-called logarithmic Sobolev inequality, or entropy-energy inequality, for σ,

$$n \left[\int_{S^n} f^2 \log f^2 d\sigma - \int_{S^n} f^2 d\sigma \log \int_{S^n} f^2 d\sigma \right] \leq 2 \int_{S^n} |\nabla f|^2 d\sigma. \tag{1.19}$$

Inequalities (1.16)–(1.19) will be the main inequalities that will be analyzed in Chapter 3, and for which we would like to provide sharp constants. Variations will be considered in the process of the notes. The following definitions extend to the general setting introduced in Section 1.1 the preceding example of the sphere. In the following, ν is a probability measure on $(E, \mathcal{E})$. Usually, ν will simply be the invariant measure μ, but the inequalities introduced in the definitions below may be considered similarly for the so-called heat-kernel measures. More precisely, for any $x \in E$ and $t \geq 0$, one may be interested in the previous inequalities and constants for the (probability) measure $\nu(dy) = p_t(x, dy)$. For example, inequalities for the classical (Brownian) heat kernel on $\mathbb{R}^n$ amount to inequalities for Gaussian measures. When the semigroup $(P_t)_{t \geq 0}$ is ergodic, one may recover inequalities for the invariant measure from the heat-kernel measures. This will be used in Chapter 2.

Definition 1.4. *The generator* L*, or rather its carré du champ* Γ*, is said to satisfy a Poincaré or spectral gap inequality on* $\mathcal{A}$ *with respect to a probability measure* ν *on* $(E, \mathcal{E})$*, if there exists* $c > 0$ *such that for any* $f \in \mathcal{A}$*,*

$$c \left[\int f^2 d\nu - \left(\int f d\nu \right)^2 \right] \leq \int \Gamma(f) d\nu.$$

The largest c *for which this inequality holds is denoted by* λ_1*.*

Definition 1.5. *The generator* L*, or rather its carré du champ* Γ*, is said to satisfy a logarithmic Sobolev inequality inequality on* $\mathcal{A}$ *with respect to a probability measure* ν *on* $(E, \mathcal{E})$*, if there exists* $c > 0$ *such that for any* $f \in \mathcal{A}$*,*

$$\frac{c}{2} \left[\int f^2 \log f^2 d\nu - \int f^2 d\nu \log \int f^2 d\nu \right] \leq \int \Gamma(f) d\nu.$$

The largest c *for which this inequality holds is denoted by* ρ_0*.*

Definition 1.6. *The generator* L*, or rather its carré du champ* Γ*, is said to satisfy a Sobolev inequality of order* $p > 2$ *on* $\mathcal{A}$ *with respect to a probability measure* ν *on* $(E, \mathcal{E})$*, if there exist constants* $A > 0$ *and* $B > 0$ *such that for any* $f \in \mathcal{A}$*,*

$$\left(\int |f|^p d\nu \right)^{2/p} \leq A \int f^2 d\nu + B \int \Gamma(f) d\nu.$$

The last definition has to be examined more carefully. Since it involves 2 constants, one has to emphasize which one is under investigation. In the example of the standard sphere S^n with $\nu = \sigma$, we have seen that for $p = 2n/(n-2)$, $n \geq 3$, one may take $A = 1$ and $B = 4/n(n-2)$. With B being equal to the latter value, Riemannian geometers concentrated their efforts on constant A (see [He1], [He2] and Chapter 3). We will be concerned here with constant B. It should be mentioned first of all that whenever $\lambda_1 > 0$, one may always take $A = 1$ in the preceding definition. This is a consequence of the following simple lemma ([Ba3], going back to [Ro2], [D-S]).

Lemma 1.7. *Let $p > 2$ and let f be a function in $\mathrm{L}^p(\nu)$ where ν is a probability measure. Then*

$$\left(\int |f|^p d\nu\right)^{2/p} \leq \left(\int f d\nu\right)^2 + (p-1)\left(\int \left|f - \textstyle\int f d\nu\right|^p d\nu\right)^{2/p}.$$

According to this lemma, if Γ satisfies a Sobolev inequality of order $p > 2$ with constants $A, B > 0$, and if $\lambda_1 > 0$, we may apply Lemma 1.7 and the Sobolev inequality of Definition 1.6 to $f - \int f d\nu$ to get

$$\begin{aligned} \left(\int |f|^p d\nu\right)^{2/p} &\leq \left(\int f d\nu\right)^2 + A(p-1)\int \left|f - \textstyle\int f d\nu\right|^2 d\nu \\ &\qquad\qquad + B(p-1)\int \Gamma\big(f - \textstyle\int f d\nu\big) d\nu \\ &\leq \int f^2 d\nu + \left[\frac{A}{\lambda_1} + B\right](p-1)\int \Gamma(f) d\nu \end{aligned}$$

from which the claims follows. On the basis of this observation, we concentrate below on Sobolev inequalities with $A = 1$. Furthermore, in order to compare the family of Sobolev inequalities we investigate, we introduce a new definition of the Sobolev constant.

Definition 1.8. *The generator* L, *or rather its carré du champ* Γ, *is said to satisfy a Sobolev inequality of order $p \geq 1$ on $\mathcal{A}$ with respect to a probability measure ν on $(E, \mathcal{E})$, if there exists $B > 0$ such that for any $f \in \mathcal{A}$,*

$$\frac{B}{p-2}\left[\left(\int |f|^p d\nu\right)^{2/p} - \int f^2 d\nu\right] \leq \int \Gamma(f) d\nu.$$

The largest B for which this inequality holds is denoted by s_p.

The last definition of course contains the two previous ones so that $s_1 = \lambda_1$ and, in the limit as $p \to 2$, $s_2 = \rho_0$. It is usually enough to consider non-negative functions f in $\mathcal{A}$ in the preceding definitions. The dependance of the constants λ_1, ρ_0, s_p on the measure ν will be clear from the context.

Definition 1.4 and the notation λ_1 is motivated by the analogy with Riemannian geometry and the equivalent description of λ_1 as the first non-trivial eigenvalue of the Laplacian on a compact manifold. There are moreover a number of relations between the preceding definitions and the constants λ_1, ρ_0, s_p. Some of these are summarized by the inequalities

$$s_p \leq s_1 = \lambda_1, \quad p \geq 1. \tag{1.20}$$

In particular, $\rho_0 \leq \lambda_1$. This may be seen by applying either the logarithmic Sobolev inequality or the Sobolev inequality to $1+\varepsilon f$ and by letting $\varepsilon \to 0$ by a simple Taylor expansion. They justify our conventions in the previous definitions. We would not know how to compare s_p and s_q for $1 < p < q$. Note finally that, in this terminology,

$s_p = n$ for every $1 \leq p \leq 2n/(n-2)$ on the standard sphere S^n, $n \geq 3$, equipped with its invariant probability measure σ.

The concept of logarithmic Sobolev inequality suggests a more general family of inequalities between entropy and energy as

$$\int f^2 \log f^2 d\nu - \int f^2 d\nu \log \int f^2 d\nu \leq \Phi\Big(\int \Gamma(f) d\nu \Big) \tag{1.21}$$

for every f in $\mathcal{A}$ with $\int f^2 d\nu = 1$ where Φ is a function on $\mathbb{R}^+$. Here ν need not be a probability measure. Logarithmic Sobolev inequalities correspond to linear Φ's. It is customary to only consider concave functions Φ. In this case, the preceding entropy-energy inequality amounts to the family of so-called defective logarithmic Sobolev inequalities

$$\int f^2 \log f^2 d\nu - \int f^2 d\nu \log \int f^2 d\nu \leq \Phi'(v) \int \Gamma(f) d\nu + \Psi(v) \int f^2 d\nu \tag{1.22}$$

where, for every $v > 0$, $\Psi(v) = \Phi(v) - v\Phi'(v)$. We investigate such families of logarithmic Sobolev inequalities in connection with heat kernel bounds in Chapter 4.

We conclude this section by mentioning the important stability property by products of both Poincaré and logarithmic Sobolev inequality. Somewhat informally, let two carré du champ operators Γ^1 and Γ^2 on two independent spaces E^1 and E^2, for which we have the inequalities

$$P^i(f^2) - (P^i f)^2 \leq P^i(\Gamma^i f)$$

and

$$P^i(f^2 \log f^2) - P^i(f^2) \log P^i(f^2) \leq 2P^i(\Gamma^i f),$$

$i = 1, 2$, f on E^i, where P^i are Markov operators (which essentially represent ν_i or P^i_t, $t \geq 0$). Then, in the product space $E^1 \times E^2$,

$$P^1 P^2(f^2) - (P^1 P^2 f)^2 \leq P^1 P^2(\Gamma^1 f + \Gamma^2 f) \tag{1.23}$$

and

$$P^1 P^2(f^2 \log f^2) - P^1 P^2(f^2) \log P^1 P^2(f^2) \leq 2P^1 P^2(\Gamma^1 f + \Gamma^2 f). \tag{1.24}$$

These inequalities may be shown to follow from simple convexity argument together with the necessary details about the underlying tensorization of the generators and the carrés du champ. As a consequence, to prove for example the Poincaré and logarithmic Sobolev inequalities for the Gaussian measure γ in $\mathbb{R}^n$, it is enough to work out the dimension one and then use the preceding. This feature emphasizes the infinite dimensional character of logarithmic Sobolev inequalities as opposed to the dimensional Sobolev inequalities (cf. Chapters 2,3).

2. INFINITE DIMENSIONAL GENERATORS

We present in this section, in the abstract diffusion generator setting, the infinite dimensional version of Gromov's comparison theorem for manifolds with positive curvature. The model space is provided here by the Ornstein-Uhlenbeck semigroup with Gaussian measure as invariant measure. We start our investigation by the similar questions for Poincaré and logarithmic Sobolev inequalities.

2.1 Logarithmic Sobolev inequalities

We consider here a diffusion Markov generator L acting on some algebra $\mathcal{A}$ with associated semigroup $(P_t)_{t\geq 0}$ as fixed in Section 1.1. As described in Section 1.2, we assume for simplicity that $\mathcal{A}$ is stable by P_t. In this chapter, we assume that L has curvature R, that is satisfies $\mathrm{CD}(R,\infty)$. In the first theorem, we establish both Poincaré and logarithmic Sobolev inequalities for the heat kernel measures $p_t(x,\cdot)$. We make essential use of the commutation properties (1.12) and (1.13). As we have seen, these relations are rather simple in case of the classical heat semigroup or the Ornstein-Uhlenbeck semigroup so that the arguments below in these particular cases provide minimal proofs of Poincaré and logarithmic Sobolev inequalities for Gaussian measures. Since the inequalities below are valid for $p_t(x,\cdot)$ for all or almost all x, we write for simplicity P_tf for $P_tf(x)$.

Theorem 2.1. *Let* L *satisfying CD(R,∞) for some $R\in\mathbb{R}$. Then, for every f in $\mathcal{A}$ and $t\geq 0$,*

$$P_t(f^2)-(P_tf)^2\leq c(t)P_t(\Gamma f) \tag{2.1}$$

and

$$P_t(f^2\log f^2)-P_t(f^2)\log P_t(f^2)\leq 2c(t)P_t(\Gamma f) \tag{2.2}$$

where

$$c(t)=\frac{1-\mathrm{e}^{-2Rt}}{R}\quad (=2t \text{ if } R=0).$$

We have seen in (1.20) that actually (2.1) follows from (2.2). However, to better illustrate the principle of the proof, we establish (2.1) and (2.2) separately. Before turning to the proof, let us illustrate the content of this result. If $(P_t)_{t\geq 0}$ is the heat semigroup on $\mathbb{R}^n$ (with generator Δ with our convention), the heat kernel measures are Gaussian measures. Since $R = 0$, Theorem 2.1 thus contains the classical Poincaré and logarithmic Sobolev inequalities with optimal constants for the canonical Gaussian measure γ on $\mathbb{R}^n$ in the form of

$$\int f^2 d\gamma - \left(\int f d\gamma\right)^2 \le \int |\nabla f|^2 d\gamma \tag{2.3}$$

and

$$\int f^2 \log f^2 d\gamma - \int f^2 d\gamma \log \int f^2 d\gamma \le 2 \int |\nabla f|^2 d\gamma. \tag{2.4}$$

Therefore, for the Ornstein-Uhlenbeck generator $\mathrm{L} = \Delta - x \cdot \nabla$ with invariant measure γ, $\lambda_1 = \rho_0 = 1$ in the notation of Section 1.3 ($\lambda_1 \le 1$ by choosing $f(x) = x$ in (2.3)).

If $(P_t)_{t \ge 0}$ is ergodic (μ probability) and $R > 0$, we can let $t \to \infty$ in Theorem 2.1 to see that, for the carré du champ Γ of L, $\lambda_1 \ge \rho_0 \ge R$. We also recover in this way the example of the Ornstein-Uhlenbeck generator the curvature of which is 1. More generally, if $\mathrm{L} = \Delta - \nabla U \cdot \nabla$ has invariant and ergodic probability measure $d\mu = Z^{-1} \mathrm{e}^{-U(x)} dx$, and if $\mathrm{Hess}\, U(x) \ge R\, \mathrm{Id}$ at every x for some $R > 0$, $\lambda_1 \ge \rho_0 \ge R$.

Proof. We use the same principle as the one used in the proofs of Lemmas 1.2 and 1.3 but at the level of Γ rather than Γ_2. (Setting the two arguments together, we are actually performing two derivations, which is one main aspect of the Γ_2-calculus.) Namely, fix f in $\mathcal{A}$ and $t > 0$. Write then

$$P_t(f^2) - (P_t f)^2 = \int_0^t \frac{d}{ds} P_s\big((P_{t-s} f)^2\big) ds.$$

By the definition of Γ,

$$P_t(f^2) - (P_t f)^2 = 2 \int_0^t P_s\big(\Gamma(P_{t-s} f)\big) ds.$$

Now, by Lemma 1.2, $\Gamma(P_{t-s} f) \le \mathrm{e}^{-2R(t-s)} P_{t-s}(\Gamma f)$ so that

$$P_t(f^2) - (P_t f)^2 \le 2 \int_0^t \mathrm{e}^{-2R(t-s)} P_t(\Gamma f) ds = c(t) P_t(\Gamma f)$$

where we used the semigroup property in the last step. This establishes the Poincaré inequality (2.1). The proof of the logarithmic Sobolev inequality (2.2) is similar but relies on the refined property (1.13) of Lemma 1.3. Namely, replace first f^2 by $f > 0$ to make the notation more simple. We then write

$$\begin{aligned} P_t(f \log f) - P_t f \log P_t f &= \int_0^t \frac{d}{ds} P_s\big(P_{t-s} f \log P_{t-s} f\big) ds \\ &= \int_0^t P_s\Big(\frac{\Gamma(P_{t-s} f)}{P_{t-s} f}\Big) ds. \end{aligned}$$

By (1.13), $\sqrt{\Gamma(P_{t-s} f)} \le \mathrm{e}^{-R(t-s)} P_{t-s}(\sqrt{\Gamma f})$, and by Cauchy-Schwarz,

$$P_{t-s}\big(\sqrt{\Gamma f}\big) \le P_{t-s} f \, P_{t-s}\Big(\frac{\Gamma f}{f}\Big).$$

Hence, as before,

$$P_t(f\log f) - P_t f \log P_t f \le \int_0^t \mathrm{e}^{-2R(t-s)} P_t\Big(\frac{\Gamma f}{f}\Big) ds = \frac{c(t)}{2}\, P_t(\Gamma f).$$

Changing back $f > 0$ into f^2 concludes the proof. □

It might be worthwhile noting that the preceding inequalities may be reversed. Namely, using that $P_s(\Gamma(P_{t-s}f)) \ge \mathrm{e}^{2Rs}\Gamma(P_t f)$ we get that

$$P_t(f^2) - (P_t f)^2 \ge 2\int_0^t \mathrm{e}^{2Rs}\Gamma(P_t f) ds = d(t)\Gamma(P_t f) \tag{2.3}$$

where

$$d(t) = \frac{\mathrm{e}^{2Rt} - 1}{R} \qquad (= 2t \text{ if } R = 0).$$

Similarly,

$$P_t(f^2\log f^2) - P_t(f^2)\log P_t(f^2) \ge 2d(t)\frac{\Gamma(P_t(f^2))}{P_t(f^2)}\,. \tag{2.4}$$

In particular, for the canonical Gaussian measure γ on $\mathbb{R}^n$ and f smooth enough,

$$\int f^2 d\gamma - \Big(\int f d\gamma\Big)^2 \ge \Big|\int \nabla f d\gamma\Big|^2$$

and, if $\int f^2 d\gamma = 1$,

$$\int f^2 \log f^2 d\gamma \ge 2\Big|\int \nabla(f^2) d\gamma\Big|^2.$$

Observe furthermore that the $\mathrm{CD}(R,\infty)$ condition in Theorem 2.1 is also necessary for (2.1) or (2.2) to hold (and similarly (2.3), (2.4)). It is enough to consider (2.1) (since (2.2) is a stronger inequality). For fixed f, the function of $t \ge 0$

$$\varphi(t) = c(t)P_t(\Gamma f) - P_t(f^2) + (P_t f)^2$$

is non-negative and is equal to 0 at $t = 0$. Therefore $\varphi'(0) \ge 0$ which amounts to $\Gamma_2(f) \ge R\Gamma(f)$.

2.2 Lévy-Gromov isoperimetric inequality.

The Lévy-Gromov isoperimetric inequality [Lé], [Gro] (cf. e.g. [G-H-L]) indicates that if M is a (compact) connected Riemannian manifold of dimension n (≥ 2) and of Ricci curvature bounded below by $R > 0$, then its isoperimetric function is larger than or equal to the isoperimetric function of the sphere S^n_r of dimension n and constant curvature $R(S^n_r) = \frac{n-1}{r^2} = R$. In other words, if we denote by $\sigma(r)$ the normalized volume of a geodesic ball of radius $r \ge 0$ on the n-sphere with curvature R, for every open set A in M with smooth boundary ∂A,

$$\sigma' \circ \sigma^{-1}\big(\mu(A)\big) \le \mu_s(\partial A) \tag{2.5}$$

where μ denotes the normalized Riemannian measure on M and $\mu_s(\partial A)$ stands for the surface measure of the boundary ∂A of A (see below). This holds in particular for $\mu = \sigma$ itself ([Lé], [Sc]).

As we have seen in the introduction, spherical measures on spheres with dimension n and radius $\sqrt{n}$ converge, as $n \to \infty$, to Gaussian distributions. This limiting procedure, known as Poincaré's limit (cf. [MK], although it seems to go back to Maxwell and Mehler!) may be used to yield an isoperimetric inequality for Gaussian measures [Bor], [S-T]. Let indeed denote by γ the canonical Gaussian measure on $\mathbb{R}^n$. Then, for every Borel set A in $\mathbb{R}^n$ with smooth boundary,

$$\varphi \circ \Phi^{-1}\big(\gamma(A)\big) \leq \gamma_s(\partial A) \tag{2.6}$$

where $\Phi(r) = (2\pi)^{-1/2}\int_{-\infty}^{r} \mathrm{e}^{-x^2/2}dx$, $r \in \mathbb{R}$, is the distribution function of the canonical Gaussian measure in dimension one and $\varphi = \Phi'$. In particular, half-spaces satisfy the equality in (2.6) and are the extremal sets of the Gaussian isoperimetric inequality. It is known also that the infinitesimal versions (2.5) or (2.6) of the isoperimetric statement may easily be integrated. It yields, in the Gaussian case for example, that if A is a Borel set in $\mathbb{R}^k$ with $\gamma(A) \geq \Phi(a)$, then, for every $r \geq 0$, $\gamma(A_r) \geq \Phi(a+r)$ where A_r is the Euclidean (or Hilbertian in case of an abstract Wiener measure [Le4]) neighborhood of order r of A. (This was actually established directly in [F-L-M], [Bor], [S-T].)

In this section, we present a version of the Lévy-Gromov comparison theorem for infinite dimensional generator with isoperimetric model the Gaussian isoperimetric function $\mathcal{U} = \varphi \circ \Phi^{-1}$. That is, in our framework, we consider L with strictly positive curvature and infinite dimension, and compare the isoperimetric function of the invariant (probability) μ of L to the isoperimetric function of the Gaussian measure, invariant measure of the Ornstein-Uhlenbeck generator. For further purposes, note that the function $\mathcal{U} = \varphi \circ \Phi^{-1}$ defined on $[0,1]$ is non-negative, concave, symmetric with respect to the vertical line going through $\frac{1}{2}$ with a maximum there equal to $(2\pi)^{-1/2}$ and such that $\mathcal{U}(0) = \mathcal{U}(1) = 0$. Its behavior at 0, or at 1 by symmetry, is given by the equivalence

$$\lim_{x\to 0} \frac{\mathcal{U}(x)}{x\sqrt{2\log\frac{1}{x}}} = 1. \tag{2.7}$$

This is easily seen by noting that the derivative of $\mathcal{U}(x)$ is $-\Phi^{-1}(x)$ which is of the order of $\sqrt{2\log\frac{1}{x}}$ as $x \to 0$. Of basic importance for the subsequent developments, observe furthermore that $\mathcal{U}$ satisfies the differential equation $\mathcal{U}\mathcal{U}'' = -1$.

Isoperimetric inequalities on spheres or in Gauss space are usually established through delicate symmetrization arguments ([Sc], [F-L-M] – cf. e.g. [B-Z], [Os] and the references therein –, and [Eh] for the Gaussian case). We will work here in our Markov generator setting and will deal with functional inequalities rather than inequalities on sets. The appropriate functional inequality is provided by a remarkable observation of S. Bobkov [Bob] who showed that, for the canonical Gaussian γ measure on $\mathbb{R}^k$ and every smooth function f with values in $[0,1]$,

$$\mathcal{U}\bigg(\int f d\gamma\bigg) \leq \int \sqrt{\mathcal{U}^2(f) + |\nabla f|^2}\, d\gamma. \tag{2.8}$$

When restricted to the characteristic function of some open set A with smooth boundary ∂A, this inequality amounts to the Gaussian isoperimetric inequality (2.6).

Inequality (2.8) may defined for some probability measure ν on $(E, \mathcal{E})$ with respect to a carré du champ Γ, and may actually be included in the family of inequalities discussed in Section 1.3. In particular, the inequality

$$\mathcal{U}\bigg(\int f d\nu\bigg) \leq \int \sqrt{\mathcal{U}^2(f) + \frac{1}{c}\,\Gamma(f)}\, d\nu \tag{2.9}$$

for some $c > 0$ and all f in $\mathcal{A}$ with values in $[0,1]$ shares a number of properties similar to spectral gap and logarithmic Sobolev inequalities. Denote by is the best c in (2.9). First of all,

$$\text{is} \leq \rho_0 \leq \lambda_1. \tag{2.10}$$

The second inequality has been pointed out in (1.20). We owe the first one to W. Beckner [Be2] who observed that (2.9) applied to εf^2 as $\varepsilon \to 0$ together with (2.7) yields a logarithmic Sobolev inequality with constant c. Moreover, the preceding isoperimetric inequalities (2.9) are stable by products. Indeed, as in (1.23) and (1.24), if for two carré du champ operators Γ^1 and Γ^2 on two independent spaces E^1 and E^2, we have the inequalities

$$\mathcal{U}(P^i f) \leq P^i\Big(\sqrt{\mathcal{U}^2(f) + \Gamma^i(f)}\Big), \quad i = 1, 2,$$

where P^1, P^2 are Markov operators, then

$$\mathcal{U}(P^1 P^2 f) \leq P^1 P^2\Big(\sqrt{\mathcal{U}(f) + \Gamma^1(f) + \Gamma^2(f)}\Big). \tag{2.11}$$

As we have seen in Section 2.1, a Markov diffusion generator L of curvature $R > 0$ with finite (normalized) invariant measure μ satisfies the logarithmic Sobolev inequality

$$R\bigg(\int f^2 \log f^2 d\mu - \int f^2 d\mu \log \int f^2 d\mu\bigg) \leq 2\int \Gamma(f) d\mu$$

(for every f in $\mathcal{A}$), that is $\rho_0 \geq R$. Our purpose here will actually be to prove that under the same curvature assumption we also have an isoperimetric inequality for the measure μ in the form of the functional inequalities (2.8) and (2.9) with, more precisely, is $\geq R$. More generally, we deal with heat kernel measures as in Section 2.1. Recall the Gaussian isoperimetric function $\mathcal{U} = \varphi \circ \Phi^{-1}$.

Theorem 2.2. *Let* L *be a Markov diffusion generator satisfying $CD(R, \infty)$ for some $R \in \mathbb{R}$. Then, for every f in $\mathcal{A}$ with values in $[0,1]$ and every $t \geq 0$,*

$$\mathcal{U}(P_t f) \leq P_t\Big(\sqrt{\mathcal{U}^2(f) + c(t)\Gamma(f)}\Big)$$

where

$$c(t) = \frac{1 - \mathrm{e}^{-2Rt}}{R} \quad t \geq 0.$$

Theorem 2.2 admits several corollaries. In particular, when the invariant measure μ is finite and normalized to a probability measure, and when $R > 0$, we may let $t \to \infty$ to get the following corollary.

Corollary 2.3. *Let* L *be a Markov diffusion generator of curvature $R > 0$ with invariant probability measure μ. Then, for every f in $\mathcal{A}$ with values in* $[0,1]$,

$$\mathcal{U}\Big(\int f\,d\mu\Big) \le \int \sqrt{\mathcal{U}^2(f) + \frac{1}{R}\,\Gamma(f)}\,d\mu. \tag{2.12}$$

In other words, is $\ge R$.

We recover in this way the Gaussian inequality (1.8) that also follows directly from Theorem 2.2 applied to the heat kernel on $\mathbb{R}^n$.

Let us now comment about the isoperimetric aspects of the preceding inequalities, especially (2.12) with say $R = 1$ for simplicity. It namely gives rise to a geometric Lévy-Gromov isoperimetric inequality in this infinite dimensional setting. On more concrete spaces, (2.12) indeed really turn into a geometric inequality. We may define a pseudo-metric d on E by setting

$$d(x,y) = \text{ess sup}\big|f(x) - f(y)\big|, \quad x, y \in E,$$

the supremum being running over all f's in $\mathcal{A}$ with $\Gamma(f) \le 1$ almost surely. Assume actually for what follows that functions f in $\mathcal{A}$ are true functions (rather than classes), and that d is a true metric and μ a separable non-atomic Borel probability measure on (E,d). Assume furthermore that $\sqrt{\Gamma(f)}$ may be identified to a modulus of gradient as

$$\sqrt{\Gamma(f)}(x) = \limsup_{d(x,y)\to 0} \frac{\big|f(x) - f(y)\big|}{d(x,y)}, \quad x \in E.$$

These requirements are in particular fulfilled in differentiable structures such as Riemannian manifolds with Riemannian measures. Then, when f approximates the indicator function of some closed set A in E, $\int \sqrt{\Gamma(f)}\,d\mu$ approaches the lower-outer Minkowski content of the boundary of A

$$\mu_s(\partial A) = \liminf_{r\to 0} \frac{1}{r}\big[\mu(A_r) - \mu(A)\big]$$

where $A_r = \{x \in E; d(x,A) < r\}$. Since $\mathcal{U}(0) = \mathcal{U}(1) = 0$, (2.12) (with thus $R = 1$) therefore read on sets as

$$\mathcal{U}\big(\mu(A)\big) \le \mu_s(\partial A). \tag{2.13}$$

Hence, the isoperimetric function of μ is larger than or equal to the Gaussian isoperimetric function $\mathcal{U}$, which is the analogue of Lévy-Gromov's result. In particular, we recover the full strength of the Gaussian isoperimetric inequality with half-spaces as extremal sets [Bor], [S-T]. We refer to [Bob] and [B-H] for a proof of the equivalence between (2.12) and (2.13) and for further general comments and results on the geometric aspects of these functional inequalities.

The differential inequality (2.13) may also be integrated to yield that whenever A is a Borel set in (E,d) with $\mu(A) \geq \Phi(a)$ for some real number a, for every $r \geq 0$,

$$\mu(A_r) \geq \Phi(a+r). \tag{2.14}$$

For example, if f is such that $\Gamma(f) \leq 1$ and if $\mu(\{f \leq m\}) \geq \frac{1}{2} = \Phi(0)$, for every $r \geq 0$,

$$\mu\big(\{f \leq m+r\}\big) \geq \Phi(r) \geq 1 - \tfrac{1}{2}\,\mathrm{e}^{-r^2/2}. \tag{2.15}$$

Such a result is part of the concentration of measure phenomenon, of powerful importance in applications (cf. [Le2]), and whose connections with logarithmic Sobolev inequalities are presented in [Le4]. The property (2.15) may also be seen as an infinite dimensional analogue of the Riemannian comparison theorems of volumes of balls (cf. [Cha2]).

We refer to [B-L2] for the proof of Theorem 2.2. It is actually similar in its basic principle to the proof of Theorem 2.1, although more involved at the technical level. If $f \in \mathcal{A}$ with values in $[0,1]$ and $t \geq 0$ are fixed, let for every $0 \leq s \leq t$,

$$F(s) = P_s\Big(\sqrt{\mathcal{U}^2(P_{t-s}f) + c(s)\Gamma(P_{t-s}f)}\,\Big).$$

Since $c(0) = 0$, it is enough to show that F is non-decreasing. Making basis use of the relation $\mathcal{U}\mathcal{U}'' = -1$, one actually proves that $F'(s) \geq 0$ from which the result follows. This is established from the curvature condition (1.14) after a number of changes of variables in Γ_2.

As for (2.3) and (2.4), there is a reverse form of Theorem 2.2. Under the $\mathrm{CD}(R,\infty)$ condition, for every f in $\mathcal{A}$ with values in $[0,1]$, and every $t \geq 0$,

$$\mathcal{U}(P_tf) \geq \sqrt{\big(P_t\big(\mathcal{U}(f)\big)\big)^2 + d(t)\,\Gamma(P_tf)} \tag{2.16}$$

where we recall that $d(t) = (\mathrm{e}^{2Rt} - 1)/t$ (cf. [B-L2]).

To conclude this section, let us mention that we of course would like to follow a similar procedure in case of the original Lévy-Gromov inequality involving the dimension parameter n of the Markov generator. This however turns out to be much more involved since, as we have seen, there is no equivalent formulation at this point of the $\mathrm{CD}(R,n)$ hypothesis on the semigroup $(P_t)_{t\geq 0}$ similar to (1.12) and (1.13). In particular, we cannot reach in this way (dimensional) Sobolev type inequalities.

3. SHARP SOBOLEV INEQUALITIES AND COMPARISON THEOREMS

In this chapter, we concentrate on finite dimensional curvature-dimension hypotheses for which we would like to provide comparison theorems especially with the family of Jacobi generators. Starting with the example of the sphere, we examine in this way volume, diameter and eigenvalue comparisons. Since we cannot reach in this setting the full dimensional Lévy-Gromov isoperimetric inequality, we rather concentrate on related statements that may be established by alternate arguments.

3.1 Sobolev inequalities

The first step in our program will be to establish the Sobolev inequalities (1.16) for a generator L satisfying the curvature-dimension condition $\mathrm{CD}(R,n)$ with $R>0$. It should be first mentioned that, in a Riemannian setting, these inequalities were established by S. Ilias [Il] on the basis of the Aubin-Beckner result on the sphere and the Lévy-Gromov comparison theorem (2.5) of the isoperimetric functions together with standard rearrangement inequalities. While we do not know whether (2.5) can be extended to our setting, we however show here how to reach directly these Sobolev inequalities for an abstract Markov generator. We should mention that the proof we suggest is actually much simpler that the combination of the preceding deep results.

Let us start to recall that by Sobolev inequality (with dimension n) we understand an inequality of the type

$$\left(\int |f|^p d\nu\right)^{2/p} \le A\int f^2 d\nu + B\int \Gamma(f)d\nu, \tag{3.1}$$

for all f in some nice class $\mathcal{A}$ and $p=2n/(n-2)$. We have seen that in the case of the classical unit sphere S^n, $n\ge 3$, equipped with its normalized invariant measure, the optimal constants are $A=1$ and $B=4/n(n-2)$. In an inequality such as (3.1), one may be interested in either A or B as a function of one another. It has been shown by Th. Aubin (cf. [Au]) that on a compact n-dimensional Riemannian manifold (M,g) with μ the normalized Riemannian measure $\frac{dv}{V}$ and $\Gamma(f)=|\nabla f|^2$, for every $A\ge 1$,

$$B=B(A)\ge \frac{4}{n(n-2)}\left(\frac{V(M)}{V(S^n)}\right)^{2/n} \tag{3.2}$$

where $V(M) = V$, resp. $V(S^n)$, is the Riemannian volume of M, resp. of S^n. Moreover, as a deep important contribution, E. Hebey and M. Vaughon (see [He1], [He2]) proved that, on any manifold M, there is A_0 such that $B(A_0)$ achieves equality in (3.2).

On the other hand, we have seen with Lemma 1.7 that one can always choose $A = 1$ in (3.1) so that one may ask for the best possible value of B in this case. This is the question that we examine in this section, and answer under curvature assumptions. We take again the setting described in Section 1.1. In all the chapter, μ is the invariant probability measure of our Markov diffusion generator L.

Theorem 3.1. *Let* L *be a Markov generator satisfying* $CD(R, n)$ *for some* $R > 0$ *and* $n > 2$. *Then, for every* $1 \leq p \leq 2n/(n-2)$ *and* $f \in \mathcal{A}$,

$$\frac{c}{p-2}\left[\left(\int |f|^p d\mu\right)^{2/p} - \int f^2 d\mu\right] \leq \int \Gamma(f) d\mu$$

with $c = nR/(n-1)$. *In other words, with the notation of Definition 1.8,*

$$s_p \geq \frac{nR}{n-1}.$$

This result extends the case of spheres due to Th. Aubin [Au] and W. Beckner [Be]. In particular, since $s_1 = \lambda_1$, Theorem 3.1 contains the famous Lichnerowicz minoration

$$\lambda_1 \geq \frac{nR}{n-1} \tag{3.3}$$

of the first non-trivial eigenvalue λ_1 of the Laplacian on a compact Riemannian manifold with Ricci curvature bounded below by $R > 0$. Similarly, since $s_2 = \rho_0$, we recover the important result of D. Bakry and M. Emery [B-E] about the logarithmic Sobolev constant

$$\rho_0 \geq \frac{nR}{n-1}. \tag{3.4}$$

Note in particular that since $\lambda_1 = n$ on the unit sphere S^n, we have on S^n, $s_p = n$ for every $1 \leq p \leq 2n/(n-2)$, and in particular $\lambda_1 = \rho_0 = n$, a result first obtained in [M-W]. We will see in Section 3.3 that the equality $\lambda_1 = \rho_0$ has no reason to hold in general.

In [Fo], a somewhat sharper bound than Theorem 3.1 is obtained involving the spectral gap λ_1, R and n as a convex combination. Namely, for every $1 \leq p \leq 2n/(n-2)$,

$$s_p \geq \alpha \, \frac{nR}{n-1} + (1-\alpha)\lambda_1 \tag{3.5}$$

where

$$\alpha = \alpha(p) = \frac{(p-1)(n-1)^2}{(p-2)+(n+1)^2}.$$

Note that $\alpha(p) = 1$ for the critical exponent $p = 2n/n-2$ while $\alpha(0) = 0$ when $p = 1$. When $p = 2$, the lower bound (3.5) of the logarithmic Sobolev constant

$$\rho_0 = s_2 \geq \frac{(n-1)^2}{(n+1)^2} \cdot \frac{nR}{n-1} + \frac{4n}{(n+1)^2}\,\lambda_1 \tag{3.6}$$

goes back to O. Rothaus [Ro2], and contains both (3.3) and (3.4) (since $\lambda_1 \geq \rho_0$). Furthermore, (3.5) is still of interest when the curvature R is somewhat negative (depending on n). In (3.5), it is assumed more precisely that we already know that some Sobolev inequality (of dimension n) holds for L (as for example on compact Riemannian manifolds). Theorem 3.1 also has a version for $n = 1, 2$ and in fact any value of $p \geq 1$ is then allowed. As $p \to \infty$, the power type Sobolev inequalities of Theorem 3.1 then turn into exponential type inequalities studied by N. Trudinger, J. Moser and E. Onofri. We refer to [Fo] for more details in this respect. Inequality (3.5) will be used below to bound above λ_1 in terms of n, R and the diameter D. Note also that, together with Obata's theorem [Ob], a compact n-dimensional Riemannian manifold with Ricci curvature bounded below and with $s_p = n$ for some $p < 2n/(n-2)$ is isometric to the standard n-sphere S^n.

Furthermore, with (3.2), we can already state a geometric consequence of Theorem 3.1 known as Bishop's volume comparison in Riemannian geometry.

Corollary 3.2. *Let (M, g) be an n-dimensional Riemannian manifold with Ricci curvature bounded below by $R > 0$. Then $V(M) \leq V(S_r^n)$ where S_r^n is the n-sphere with constant curvature $R(S_r^n) = \frac{n-1}{r^2} = R$.*

Proof. By homogeneity of the Riemannian metric, it is enough to deal with the case $R = n-1$ (and thus $r = 1$). Theorem 3.1 then shows that

$$B(A) \leq B(1) \leq \frac{4}{n(n-2)}\,.$$

Together with (3.2), the result follows. □

The proof of Theorem 3.1 moreover shows that the optimal A_0 in (3.2) satisfies $A_0 \geq R/(n-1)$ (cf. [Fo]).

The proof of Theorem 3.1 relies on the study of the non-linear equation

$$c\big(f^{p-1} - f\big) = -\mathrm{L}f \tag{3.7}$$

and existence of a non-constant minimizer f. This is a standard procedure in non-linear analysis. Starting from the best constant c in the Sobolev inequality, one construct an approximating sequence of solutions from which one can extract, by compactness, a non-trivial solution f of (3.7). We refer to [Ba3] and [Fo] for details about the existence of a non-trivial solution $f > 0$ of (3.7) and briefly discuss the subsequent algebraic argument. (Actually, the arguments require to work with $p < 2n/(n-2)$ and $c(f^{p-1} - (1+\varepsilon)f) = -\mathrm{L}f$.) Lower bounds on $c > 0$ are then obtained by a comparison with the $\mathrm{CD}(R, n)$ inequality following ideas introduced by O. S. Rothaus [Ro2] in the context of logarithmic Sobolev inequalities (see also

[BV-V]). The method consists in performing the change of variables $f \to f^r$, $r \neq 0$ ($f > 0$), in this non-linear equation as well as another change $f \to f^s$, $s \neq 0$, into the $\mathrm{CD}(R,n)$ inequality. Then, by the diffusion property and integration by parts, the result may be shown to follow from optimal choices of the parameters r and s. Let thus $f > 0$ be a non-constant solution of (3.7) and set $u = f^r$. We get, by the change of variable formula (1.3) for L,

$$c\big(u^{r(p-1)} - u^r\big) = -ru^{r-1}\mathrm{L}u - r(r-1)u^{r-2}\Gamma(u). \tag{3.8}$$

Multiplying (3.8) by $u^{-r}\Gamma(u)$ and integrating with respect to μ yields

$$c\bigg(\int u^{r(p-2)}\Gamma(u)d\mu - \int \Gamma(u)d\mu\bigg) = -r\int \frac{1}{u}\,\mathrm{L}u\Gamma(u)d\mu - r(r-1)\int \frac{\Gamma^2(u)}{u^2}\,d\mu.$$

Now, by integration by parts,

$$\int u^{r(p-2)}\Gamma(u)d\mu = -\frac{1}{r(p-2)+1}\int u^{r(p-2)+1}\mathrm{L}ud\mu.$$

Multiplying (3.8) by $u^{1-r}\mathrm{L}u$ and integrating with respect to μ yields

$$c\bigg(\int u^{r(p-2)+1}\mathrm{L}ud\mu - \int u\mathrm{L}u\bigg) = -r\int (\mathrm{L}u)^2 d\mu - r(r-1)\int \frac{1}{u}\,\mathrm{L}u\Gamma(u)d\mu.$$

Combining the latter,

$$\begin{aligned} c\big[r(p-2)+1\big]&\int u^{r(p-2)}\Gamma(u)d\mu \\ &= r\int (\mathrm{L}u)^2 d\mu + r(r-1)\int \frac{1}{u}\,\mathrm{L}u\Gamma(u)d\mu + c\int \Gamma(u)d\mu. \end{aligned}$$

We thus get

$$\begin{aligned} c(p-2)\int \Gamma(u)d\mu = \int (\mathrm{L}u)^2 d\mu &+ r(p-1)\int \frac{1}{u}\,\mathrm{L}u\Gamma(u)d\mu \\ &+ (r-1)\big[r(p-2)+1\big]\int \frac{\Gamma^2(u)}{u^2}\,d\mu. \end{aligned} \tag{3.9}$$

Now, we perform a similar change of functions on the $\mathrm{CD}(R,n)$ condition applied to $f = u^s$. We get

$$\begin{aligned} \Gamma_2(u) &+ (s-1)\frac{1}{u}\,\Gamma\big(u,\Gamma(u)\big) + (s-1)^2\frac{1}{u^2}\,\Gamma^2(u) \\ &\geq R\,\Gamma(u) + \frac{1}{n}\,(\mathrm{L}u)^2 + \frac{2}{n}\,(s-1)\frac{1}{u}\,\mathrm{L}u\Gamma(u) + \frac{1}{n}\,(s-1)^2\frac{1}{u^2}\,\Gamma^2(u). \end{aligned}$$

After integration, and integration by parts, we see that

$$\begin{aligned} R\int \Gamma(u)d\mu \leq \Big(1-\frac{1}{n}\Big)\int (\mathrm{L}u)^2 d\mu &+ s'\Big(1+\frac{2}{n}\Big)\int \frac{1}{u}\,\mathrm{L}u\Gamma(u)d\mu \\ &- s'\Big[1+s'\Big(1-\frac{1}{n}\Big)\Big]\int \frac{\Gamma^2(u)}{u^2}\,d\mu \end{aligned} \tag{3.10}$$

where $s' = s - 1$. Combining (3.9) and (3.10), we eliminate the term $\int \frac{1}{u} \mathrm{L}u\Gamma(u)d\mu$. Setting

$$\frac{s'}{r} = (p-1)\,\frac{n-1}{n-2}\,,$$

the coefficient in front of $\int (\mathrm{L}u)^2 d\mu$ is 0 and we are left with

$$\Big(\frac{c(p-2)(n-1)}{n} - R\Big) \int \Gamma(u)d\mu \geq K(s', r) \int \frac{\Gamma^2(u)}{u^2}\, d\mu.$$

The constant $K(s', r)$ is easily seen to be non-negative as soon as $1 \leq p \leq 2n/(n-2)$ which thus yields

$$\frac{c(p-2)(n-1)}{n} - R \geq 0$$

since f, and thus u, is assumed to be non-constant. Therefore, Theorem 3.1 is established in this way.

3.2 Myers's diameter theorem

The classical theorem of (Bonnet-) Myers on the diameter [My] (see [Cha2], [G-H-L]) states that if (M, g) is a complete connected Riemannian manifold of dimension n (≥ 2) such that $\mathrm{Ric} \geq (n-1)g$, then its diameter $D = D(M)$ is less than or equal to π (and in particular M is compact). Equivalently, after a change of scale, if $\mathrm{Ric} \geq Rg$ with $R > 0$, and if S_r^n is the sphere of dimension n and constant curvature $R = \frac{n-1}{r^2}$ where $r > 0$ is the radius of S_r^n, then the diameter of M is less than or equal to the diameter of S_r^n, that is

$$D \leq \pi r = \pi \sqrt{\frac{n-1}{R}}\,. \tag{3.11}$$

Our aim in this section is to provide a functional proof of this geometric statement. That is, rather than to work with curvature, we will deduce (3.11) from a Sobolev inequality. According to the sharp constants in Sobolev inequalities described by Theorem 3.1, we then recover the geometric statement (3.11) about the diameter. The approach consists in an analysis of extremal functions. In our abstract context, we may define the "diameter" of a carré du champ Γ as

$$D = D(\Gamma) = \sup\big\{ \|\widetilde{f}\|_{\mathrm{L}^\infty(\mu\otimes\mu)} ; f \in \mathcal{A}, \|\Gamma(f,f)\|_\infty \leq 1\big\} \tag{3.12}$$

where $\widetilde{f}(x,y) = f(x) - f(y)$, $x, y \in E$. Of course, $D(\Gamma)$ is relative to $\mathcal{A}$ (and is as smaller as $\mathcal{A}$ is small). When L is the Laplace-Beltrami operator on a compact manifold M and $\mathcal{A}$ the algebra of C^∞ functions on M, it is easily seen that the functional definition (3.12) of the diameter coincides with the geometric definition of the diameter. Note that $D(\Gamma) = \infty$ for the Ornstein-Uhlenbeck generator.

Using a simple iteration procedure (in the spirit of Moser's iteration [Mo]), it is not difficult to see that the existence of a Sobolev inequality (3.1) with $A = 1$ forces the diameter D to be finite. The next statement produces the optimal bound.

Theorem 3.3. *Let Γ be a carré du champ satisfying the Sobolev inequality*

$$\left(\int |f|^p d\mu\right)^{2/p} \leq \int f^2 d\mu + \frac{4}{n(n-2)} \int \Gamma(f) d\mu, \quad f \in \mathcal{A}, \tag{3.13}$$

for some $n > 2$ where $p = 2n/(n-2)$. Then

$$D = D(\Gamma) \leq \pi.$$

If Γ is changed in $a\Gamma$ for some $a > 0$, then $D(a\Gamma) = a^{-1/2} D(\Gamma)$. Therefore, if Γ satifies the inequality

$$\left(\int |f|^p d\mu\right)^{2/p} \leq \int f^2 d\mu + B \int \Gamma(f) d\mu, \quad f \in \mathcal{A},$$

for some $p > 2$, then

$$D = D(\Gamma) \leq \pi \, \frac{\sqrt{2pB}}{p-2} \, . \tag{3.14}$$

Together with Theorem 3.1, Theorem 3.3 answers our initial question.

Corollary 3.4. *Let* L *be a Markov generator satisfying the curvature-dimension inequality CD(R, n) for some $R > 0$ and $n \geq 2$. Then*

$$D = D(\mathrm{L}) \leq \pi \sqrt{\frac{n-1}{R}} \, .$$

The case $n = 2$, and by extension $1 \leq n \leq 2$, is somewhat particular. Since Theorem 3.1 holds for every $p \geq 1$ in this case, we get together with (3.14)

$$D \leq \pi \left(\frac{n-1}{nR} \cdot \frac{2p}{p-2}\right)^{1/2}$$

for every $p > 2$. When $p \to \infty$,

$$D \leq \pi \left(\frac{2(n-1)}{nR}\right)^{1/2} .$$

However, this is optimal only for $n = 2$.

When (3.5) holds, we also see that

$$D^2 \leq \pi^2 \, \frac{2p}{p-2} \cdot \frac{1}{\alpha(p)(nR/n-1) + (1-\alpha(p))\lambda_1}$$

for every $2 < p \leq 2n/n - 2$. Optimizing over p, we can obtain upper bounds on λ_1. For example, if $R = 0$, we get

$$\lambda_1 \leq \frac{\pi^2 n^2 (n+2)}{D^2}$$

(the constant is not sharp). If $R \geq -K$, $K \geq 0$, then

$$\lambda_1 \leq \frac{(n-1)K}{4} + \frac{C(n)}{D^2}$$

for some $C(n) > 0$ only depending on n. Thus, we recover with these functional tools geometric bounds first established in [Ch], [Che] (see [Cha1]).

The proof of Theorem 3.2 is based again on non-linear analysis. Actually, we make advantage of the form of the extremal functions of the Sobolev inequality on the (unit) sphere S^n. As was shown by Th. Aubin [Au], the functions $f_\lambda = (1 + \lambda \sin(d))^{1-(n/2)}$, $-1 < \lambda < +1$, where d is the distance to a fixed point, are solutions of the non-linear equation

$$f_\lambda^{p-1} - f_\lambda = -\frac{4}{n(n-2)} \Delta f_\lambda,$$

$p - 1 = (n+2)/(n-2)$, and satisfy the equality in the Sobolev inequality (3.13). In general, we let f be a function in $\mathcal{A}$ such that $\|\Gamma(f)\|_\infty \leq 1$ and apply Sobolev's inequality (3.13) to the family of functions $f_\lambda = (1+\lambda \sin(f))^{1-(n/2)}$, $-1 < \lambda < +1$, to deduce a (non-linear) differential inequality on

$$F(\lambda) = \int \big(1 + \lambda \sin(f)\big)^{2-n} d\mu, \quad -1 < \lambda < +1.$$

This differential inequality takes a nice form due to the optimal constant in the Sobolev inequality and the miracle is that it may be integrated to exactly bound the diameter of L by its Sobolev constant. The crucial argument of the proof consists in showing that when $\int \sin(f) d\mu > 0$ (resp. < 0), then (essentially)

$$F(1) = \int \big(1 + \sin(f)\big)^{2-n} d\mu < \infty$$

(resp. $F(-1) < \infty$). Iterating the result on the basis again of the Sobolev inequality, we actually have that

$$\big\| \big(1 \pm \sin(f)\big)^{-1} \big\|_\infty < \infty$$

from which the conclusion then easily follows. We refer to [B-L1] for the complete proof.

If (M, g) is a Riemannian manifold with dimension n and $\mathrm{Ric} \geq (n-1)g$ and if the diameter of M is equal to π, S.-Y. Cheng [Che] showed that M is isometric to the unit sphere S^n, generalizing the Topogonov theorem [To] that was dealing with the sectional curvature (cf. [Cha2] for a modern geometric proof of the Topogonov-Cheng result). Our next theorem is an analogue of this result. It is again formulated in terms of the Sobolev constant and shows that if $D(\Gamma) = \pi$, the constant in (3.13) is reached on functions of the form $f_\lambda = (1 + \lambda \sin(f))^{1-(n/2)}$, $-1 < \lambda < +1$, for some non-constant function f with $\int \sin(f) d\mu = 0$. In particular, we include in this way the example of the spheres themselves.

Theorem 3.5. *Let Γ be a carré du champ satisfying the Sobolev inequality (3.13) for some $n > 2$. If there is a function f in $\mathcal{A}$ such that $\|\Gamma(f)\|_\infty \leq 1$ and $\|\widetilde{f}\|_\infty = \pi$, then there exist non-constant extremal functions of (3.13). More precisely, if we translate f such that $\int \sin(f) d\mu = 0$, for every $-1 < \lambda < +1$,*

$$\left(\int |f_\lambda|^p d\mu\right)^{2/p} = \int f_\lambda^2 d\mu + \frac{4}{n(n-2)} \int \Gamma(f_\lambda) d\mu,$$

$p = 2n/(n-2)$, where $f_\lambda = (1 + \lambda \sin(f))^{1-(n/2)}$. Furthermore, if we set $X = \sin(f)$, L agrees on the functions of X with the ultraspheric generator of dimension n, that is, for every smooth function φ on $\mathbb{R}$,

$$\mathrm{L}\varphi(X) = (1 - X^2)\varphi''(X) - nX\varphi'(X).$$

In particular, if $\mathrm{L} = \Delta$ is the Laplace-Beltrami operator on a n-dimensional compact manifold (M, g) with $\mathrm{Ric} \geq (n-1)g$ and with diameter equal to π, then M is isometric to the sphere S^n.

As we have seen in (1.20), applying the Sobolev inequality (3.13) to $f = 1 + \varepsilon\varphi$ where $\int \varphi d\mu = 0$, and using a Taylor expansion at $\varepsilon = 0$ shows that $\lambda_1 \geq n$ (i.e. (3.3)). In the same way, a Taylor expansion on the functions f_λ at $\lambda = 0$ in Theorem 3.5 shows that $\lambda_1 = n$. Therefore, if L is the Laplace-Beltrami operator on a n-dimensional Riemannian manifold (M, g) with $\mathrm{Ric} \geq (n-1)g$ and if $D = \pi$, then $\lambda_1 = n$ and therefore, by Obata's theorem [Ob] ([G-H-L]), M is isometric to the unit sphere S^n, proving the last assertion of Theorem 3.5. This functional approach thus provides a new proof of the Topogonov-Cheng theorem.

We may also note to conclude that since h_a is extremal in the Sobolev inequality (3.13),

$$h_a^{p-1} - h_a = -\frac{4}{n(n-2)} \mathrm{L} h_a.$$

After a change of variables, we get, with $X = \sin(f)$,

$$-2\sqrt{1+a^2}\, X - a(1 + X^2) = \frac{2}{n}\left[\left(\sqrt{1+a^2} + aX\right)\mathrm{L}X - \frac{n}{2}\, a\Gamma(X)\right].$$

When $a = 0$, we recover that $-\mathrm{L}X = nX$, and if we replace then $\mathrm{L}X$ by $-nX$ and symplify by a, we see that $\Gamma(X, X) = 1 - X^2$. These observations thus indicate that on the functions of X, L coincide with the Jacobi generator L_n of dimension n (1.10). In a Riemannian setting, we used Obata's theorem to conclude that L is "isometric" to the Laplacian of a sphere. In general however, we do not know exactly what kind of rigidity can be expected.

3.3 Eigenvalue comparison theorems

In this section, we present some recent results of D. Bakry and Z. Qian on comparison theorems for spectral gap using curvature, dimension and diameter. Let (M, g) be a compact connected Riemannian manifold with dimension n. Denote by $\lambda_1 = \lambda_1(M)$ the first non-trivial eigenvalue of the Laplacian Δ on M. A huge literature (see e.g. [Bé] and the references therein) has been devoted to both upper and lower bounds of λ_1 in terms of the geometry of the manifold. The modern analysis has demonstrated in particular that λ_1 may be estimated by the dimension, an upper bound on the diameter and a lower bound on the Ricci curvature. Instances of particular interest are the following. As we have seen, if $\mathrm{Ric} \geq (n-1)g$, Lichnerowicz's minoration (3.3) shows that

$$\lambda_1 \geq n\,, \tag{3.16}$$

that is, $\lambda_1(M) \geq \lambda_1(S^n)$. Thus (3.16) is a comparison theorem. When $\mathrm{Ric} = 0$, it has been shown by P. Li [Li1] and H. C. Yang and J. Q. Zhong [Y-Z] that

$$\lambda_1 \geq \frac{\pi^2}{D^2} \tag{3.17}$$

where D is the diameter of M. This lower bound is optimal since achieved on the one-dimensional torus. It is important to realize that both (3.16) and (3.17) may also be seen as comparison theorems at the level of generators rather than manifolds. To this aim, recall the family of one-dimensional generators $\mathrm{L}f = f'' - a(x)f'$ of Section 1.2 for which $\mathrm{CD}(R,n)$ is equivalent to saying that

$$a' \geq R + \frac{a^2}{n-1}\,. \tag{3.18}$$

Then (3.16) amounts equivalently to the fact that

$$\lambda_1 \geq \lambda_1(n-1, n, \pi) \tag{3.19}$$

where $\lambda_1(n-1,n,\pi)$ is the first non-zero eigenvalue of the Neumann problem

$$v'' - (n-1)\tan(x)v' = -\lambda v$$

on the interval $(-\frac{\pi}{2}, +\frac{\pi}{2})$. Similarly, (3.17) is equivalent to

$$\lambda_1 \geq \lambda_1(0, \infty, D) \tag{3.20}$$

where $\lambda_1(0,\infty,D)$ is the first non-zero eigenvalue of the Neumann problem $v'' = -\lambda v$ on $(-\frac{D}{2}, +\frac{D}{2})$. Two remarks are here in order. The diameter π appears naturally in (3.19) since when $\mathrm{Ric} \geq (n-1)g$, $D \leq \pi$ by Myers's theorem. The natural family of operators for the comparison (3.20) is $\mathrm{L}f = f'' + \frac{n-1}{x}f'$. It remains to determine the appropriate interval to consider. The right choice turn out to be the interval with length D symmetric with respect to the center. Since the center may be chosen to be either 0 or ∞, the dimension vanishes as $x \to \infty$ and the operator of interest becomes $\mathrm{L}f = f''$. Notice then that since the latter is invariant under translation, one may take any interval of length D.

In order to deal with arbitrary curvature-dimension $\mathrm{CD}(R,n)$ conditions, D. Bakry and Z. Qian introduced in [B-Q] general one-dimensional models $\mathrm{L}_{R,n}$ described in the following way.

When $R > 0$, $n > 1$, $\mathrm{L}_{R,n}$ is the operator on the interval $\left(-\sqrt{\frac{R}{n-1}}\frac{\pi}{2}, +\sqrt{\frac{R}{n-1}}\frac{\pi}{2}\right)$ defined by

$$\mathrm{L}_{R,n}f(x) = f''(x) - \sqrt{(n-1)R}\,\tan\left(\sqrt{\frac{R}{n-1}}\,x\right)f'(x).$$

When $R < 0$, $n > 1$, $\mathrm{L}_{R,n}$ is the operator on an extended line

$$(0, +\infty) \cup (-\infty, +\infty) \cup (-\infty, 0) = I_1 \cup I_2 \cup I_3$$

defined by

$$\mathrm{L}_{R,n}f(x) = f''(x) - \sqrt{-(n-1)R}\,\coth\left(\sqrt{-\frac{R}{n-1}}\,x\right)f'(x) \quad \text{on} \quad I_1 \cup I_3$$

and

$$\mathrm{L}_{R,n}f(x) = f''(x) - \sqrt{-(n-1)R}\,\tanh\left(\sqrt{-\frac{R}{n-1}}\,x\right)f'(x) \quad \text{on} \quad I_2.$$

When $R = 0$ and $n > 1$, the operator $\mathrm{L}_{0,n}$ is defined by

$$\mathrm{L}_{R,n}f(x) = f''(x) + \frac{n-1}{x}\,f'(x) \quad \text{on} \quad I_1 \cup I_3$$

and

$$\mathrm{L}_{R,n}f(x) = f''(x) \quad \text{on} \quad I_2.$$

Finally, for $R \neq 0$ and $n = \infty$, $\mathrm{L}_{R,\infty}$ is defined on the real line by

$$\mathrm{L}_{R,\infty}f(x) = f''(x) - Rxf'(x)$$

while the special case $\mathrm{L}_{0,\infty}$ consists of the operators on the real line defined by

$$\mathrm{L}_{0,\infty}f(x) = f''(x) - af'(x)$$

where a is a constant.

When the curvature parameter R is strictly positive, the operators $\mathrm{L}_{r,n}$ describe the family of Jacobi operators (1.10) when n is finite, or the Ornstein-Uhlenbeck and Laguerre generators when $n = \infty$. The negative curvature is obtained by a continuous extension to the negative numbers.

The main result of [B-Q] is the following statement.

Theorem 3.6. *Let (M, g) be an n-dimensional compact Riemannian manifold such that* $\mathrm{Ric} \geq Rg$ *where $R \in \mathbb{R}$. Denote moreover by D an upper bound on the diameter of M (recall that $D \leq \pi/\sqrt{\frac{n-1}{R}}$ when $K > 0$). Then $\lambda_1(M) \geq \lambda_1(R, n, D)$ where $\lambda_1(R, n, D)$ is the first non-zero eigenvalue of the Neumann problem*

$$\mathrm{L}_{R,n}v = -\lambda v$$

on the interval $(-\frac{D}{2}, +\frac{D}{2})$.

The proof of Theorem 3.6 is based on a comparison on gradients of eigenfunctions going back to [Kr]. Although Theorem 3.6 is stated for Riemannian manifolds, many ideas of its proof are inspired by the abstract Markov generator setting. In particular, the comparison with one-dimensional operators is one main aspect of this investigation. We refer to [B-Q] for complete details.

The logarithmic Sobolev constant of a Riemannian manifold (with finite volume) also shares some relations to dimension, diameter and curvature. Let (M, g) be a Riemannian manifold with dimension n and finite volume V. Denote by $d\mu = \frac{dv}{V}$ the normalized Riemannian volume element on M. According to Definition 1.4, denote by ρ_0 the logarithmic Sobolev constant of the Laplacian Δ on M.

As we have seen in Section 3.2, the existence of Sobolev inequality on M forces M to be compact, and the diameter to be bounded above by the Sobolev constant (and the dimension). On the other hand, there exist compact manifolds of constant negative sectional curvature with spectral gaps uniformly bounded away from zero, and arbitrarily large diameters (cf. [SC]). It is one main observation by L. Saloff-Coste [SC] that the existence of a logarithmic Sobolev inequality on a manifold with Ricci curvature bounded below forces the manifold to be compact. This yield examples for which the ratio ρ_0/λ_1 can be made arbitrarily small. Quantitatively, it was shown in [Le4] that if $\mathrm{Ric} \geq -Kg$, $K \geq 0$,

$$\rho_0 \leq C\sqrt{n}\,\max\Big(\frac{1}{\sqrt{\rho_0}}\,,\,\frac{\sqrt{K}}{D}\Big)$$

where $C > 0$ is a numerical constant. (It is known from the theory of hypercontractive semigroups (cf. [D-S]) that conversely there exists $C(n, K, \varepsilon)$ such that

$$\rho_0 \geq \frac{C(n, K, \varepsilon)}{D}$$

when $\lambda_1 \geq \varepsilon > 0$.) In particular, if $\mathrm{Ric} \geq 0$,

$$\rho_0 \leq \frac{Cn}{D^2} \tag{3.21}$$

for some numerical constant $C > 0$. (3.21) has to be compared to Cheng's upper bound on the spectral gap [Che] of compact manifolds with non-negative Ricci curvature

$$\lambda_1 \leq \frac{2n(n+4)}{D^2}$$

so that, generically, the difference between the upper bound on λ_1 and ρ_0 seems to be of the order of n.

As another application, assume that $\mathrm{Ric} \geq Rg > 0$. As we have seen in Theorem 3.1, $\rho_0 \geq \frac{nR}{n-1}$. Therefore, by (3.21)

$$D \leq \sqrt{C}\sqrt{\frac{n-1}{R}}\,.$$

Up to the numerical constant, this is just Myers' theorem on the diameter of a compact manifold $D \leq \pi\sqrt{\frac{n-1}{R}}$. This could suggest that the best numerical constant in (3.21) is π^2.

Dimension free lower bounds on the logarithmic Sobolev constant in manifolds with non-negative Ricci curvature, similar to the lower bound (3.17) on the spectral gap, are also available. It has been shown by F.-Y. Wang [Wa] (see also [B-L-Q] and [Le4] for slightly improved quantitative estimates) that, if $\mathrm{Ric} \geq 0$,

$$\rho_0 \geq \frac{\lambda_1}{1 + 2D\sqrt{\lambda_1}}\,.$$

In particular, together with (3.17),

$$\rho_0 \geq \frac{\pi^2}{(1 + 2\pi)D^2}\,.$$

The proof is based on a variation of the semigroup techniques developed in Section 2.1 with differentiation along a path. We refer to the previous references for further details.

4. SOBOLEV INEQUALITIES AND HEAT KERNEL BOUNDS

In this chapter, we present some of the connections between the functional Sobolev type inequalities and heat kernel bounds. In particular, logarithmic Sobolev inequalities will be shown to usually produce sharp bounds. Finally, we mention some rigidity theorems for manifolds satisfying optimal Sobolev type inequalities.

4.1 Equivalent Sobolev inequalities

It has been one important feature of the analysis of abstract symmetric semigroups in the eighties to show that functional Sobolev inequalities are equivalent to heat kernel bounds in a rather wide setting. A basic theorem of N. Varopoulos [Va1], [Va2] (cf. [Va3]) indeed shows that the Sobolev inequality

$$\|f\|_p^2 \leq A\|f\|_2^2 + B\int \Gamma(f)d\mu, \quad f \in \mathcal{A}, \tag{4.1}$$

with $p = 2n/(n-2)$, $n > 2$, holding for some constants A, B is equivalent to saying that

$$\|P_t\|_{1\to\infty} \leq C\,t^{-n/2} \tag{4.2}$$

for every $t > 0$ or $t_0 \geq t > 0$ according as $A = 0$ or not, where $(P_t)_{t\geq 0}$ is the semigroup with generator L and associated carré du champ Γ. Note that when $\|P_t\|_{1\to\infty} < \infty$, the heat kernel measures $p_t(x, dy)$ have densities $p_t(x, y)$ with respect to the reference measure μ so that $\|P_t\|_{1\to\infty} = \sup_{x,y} p_t(x, y)$. Such equivalences have been produced similarly with Nash type inequalities in [C-K-S] and with families of logarithmic Sobolev inequalities in [Da] and [Ba3]. The uniform heat kernel bound (4.2) has been completed by off-diagonal bounds

$$p_t(x, y) \leq \frac{C_\varepsilon}{t^{n/2}} \exp\left(-\frac{d(x, y)^2}{(4+\varepsilon)t}\right) \tag{4.3}$$

in [Va], [C-K-S], [Da], [Ba3] and others. Of particular efficient use to establish (4.3) is the Davies method the idea of which is to perform the uniform bound for the semigroup $P_t f = \mathrm{e}^{-h} P_t(\mathrm{e}^h f)$ for Lipschitz functions h. Lower bounds may be obtained under further geometric conditions (cf. [Da]). As a main contribution of the works of N. Varopoulos and P. Li and S.-T. Yau, heat kernels are actually controled by volume growths (see [Va3], [L-Y] for details). We refer to the previous references for further aspects on this geometric investigation and restrict ourselves here to the functional equivalence between (4.1) and (4.2).

Sobolev inequalities such as (4.1) are actually parts of more general families of inequalities considered by E. Gagliardo and J. Nirenberg in the late fifties that include a number of limiting cases of interest. Consider namely the inequalities

$$\|f\|_r \le \big(A\|f\|_2^2 + B{\textstyle\int}\Gamma(f)d\mu\big)^{\theta/2}\|f\|_s^{1-\theta} \tag{4.4}$$

for every $f \in \mathcal{A}$, where $0 < r, s \le \infty$, $\theta \in (0,1]$. Various cases have to be distinguished according to the value of the parameter $p \neq 0$ defined by

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{s}.$$

According to the example of $\mathbb{R}^n$ for which $1/p = 1/2 - 1/n$, $n > 2$, the parameter p should be considered as a dimensional parameter. One interest for the family (4.4) is that it describes in an unified way several inequalities that appear in the literature. For example, in $\mathbb{R}^n$, $n > 2$, with $1/p = 1/2 - 1/n$ fixed, the choice of $r = 2$, $s = 1$, $\theta = n/(n+2)$, yields the Nash inequality

$$\|f\|_2^{2+4/n} \le \big(A\|f\|_2^2 + B{\textstyle\int}\Gamma(f)d\mu\big)\|f\|_1^{4/n}, \quad f \in \mathcal{A}, \tag{4.5}$$

which is one of the main tools used by J. Nash in his celebrated 1958 paper on the Hölder regularity of solutions of divergence form uniformly elliptic equations [Na]. In the subsequent work [Mo] on the subject, J. Moser considers $r = 2 + 4/n$, $s = 2$ and $\theta = n/(n+2)$. One limiting case appears as $\theta \to 0$, with $r = 2$ for example, in which case (4.4) takes the form

$$\int f^2 \log f^2 d\mu \le \frac{n}{2}\log\Big(A + +B{\textstyle\int}\Gamma(f)d\mu\Big) \tag{4.6}$$

for all f with $\|f\|_2 = 1$, as parts of the entropy-energy inequalities (1.21).

Let us be somewhat more precise in these implications. Set for simplicity

$$W(f) = A\|f\|_2^2 + B{\textstyle\int}\Gamma(f)d\mu.$$

Hölder's inequality first shows that the Sobolev inequality

$$\|f\|_p^2 \le W(f)$$

implies all the inequalities of the family (4.4) with fixed p (and with the same constants). It also shows that, when r is fixed, the inequalities (4.4) are stronger as θ decreases. Let us discuss more carefully the limiting case $\theta = 0$. Fix $r = 2$ for simplicity. Taking logarithms in (4.4) shows that

$$\log\|f\|_2 - \log\|f\|_s \le \theta\log\Big(\frac{W(f)^{1/2}}{\|f\|_s}\Big)$$

with $1/2 = \theta/p + (1-\theta)/s$. Since

$$\frac{1}{s} - \frac{1}{2} = \frac{p-2}{2p} \cdot \frac{\theta}{1-\theta},$$

we get that

$$-\frac{\log \|f\|_s - \log \|f\|_2}{1/s - 1/2} \le \frac{p(1-\theta)}{p-2} \log\Big(\frac{W(f)^{1/2}}{\|f\|_s}\Big).$$

Consider $\phi(u) = \log \|f\|_{1/u}$, $u > 0$. ϕ is convex by Hölder's inequality, so its slope

$$\psi(u) = \frac{\phi(u) - \phi(1/2)}{u - 1/2}$$

is non-decreasing and equal to $\phi'(1/2)$ at $u = 1/2$. Moreover,

$$-\|f\|_2^2 \phi'(1/2) = \int f^2 \log \frac{f^2}{\|f\|_2^2} \, d\mu.$$

Hence, as $\theta \to 0$ $(s \to 2)$, if $\|f\|_2 = 1$,

$$\int f^2 \log f^2 d\mu \le \frac{p}{p-2} \log W(f).$$

Note that $p/(p-2) = n/2$ if $p = 2n/(n-2)$. Conversely, since ψ is non-decreasing, if we start from a logarithmic Sobolev inequality as above

$$\int f^2 \log f^2 d\mu \le \frac{p}{p-2} \log W(f), \quad \|f\|_2 = 1,$$

we see that for every $s < 2$,

$$-\psi(1/s) \le \frac{p}{p-2} \log W(f)$$

that amounts (by homogeneity) to (4.4) with $r = 2$ and

$$\theta = \frac{2-s}{p-s} \cdot \frac{p}{2}.$$

In particular, if $p = 2n/(n-2)$, $s = 1$ and $\theta = n/(n+2)$, we see that Nash's inequality (4.5) follows from the logarithmic Sobolev inequality (4.6).

As mentioned above, Nash's inequality (4.5) and the logarithmic Sobolev inequality (4.6) have been considered in connection with regularity of heat equation solutions, and, once the dimensional parameter $p = 2n/(n-2)$, $n > 2$, is fixed, both produce the same dimensional heat kernel bound (4.2) (cf. [C-K-S], [Da], [Ba3]). In particular, inequalities (4.1), (4.5) and (4.6) are equivalent up to the constants A and B.

Actually, it may be shown directly that all the inequalities of the family (4.4) are equivalent. Whenever $0 < p \leq \infty$ is fixed, an inequality such as (4.4) holding for some constants A, B and all f in $\mathcal{A}$ is equivalent to the Sobolev inequality (4.1) holding for some possibly different (but explicit) constants A, B. We refer to the work [B-C-L-SC] for a self-contained proof of this claim, as well as for a careful examination of the other cases $-\infty < p < 0$ and $p = \infty$. In order to give the spirit of the idea, we however would like to outline the proof of the equivalence (up to constants) between the Sobolev inequality

$$\|f\|_p^2 \leq W(f), \quad f \in \mathcal{A},$$

where $p = 2n/(n-2)$, and the Nash inequality

$$\|f\|_2^{2+4/n} \leq W(f)\|f\|_1^{4/n}, \quad f \in \mathcal{A},$$

where, as before, we write, for simplicity, $W(f) = A\|f\|_2^2 + B\int \Gamma(f)d\mu$. As discussed before, the proof of this implication also shows that the logarithmic Sobolev inequality (4.6) is equivalent to the Sobolev inequality (4.1). Let $f \geq 0$ be fixed in some nice class $\mathcal{A}$. For each $k \in \mathbb{Z}$, set $f_k = (f - 2^k)^+ \wedge 2^k$ which we assume also in $\mathcal{A}$. Note that

$$2^k \mathrm{I}_{\{f \geq 2^{k+1}\}} \leq f_k \leq 2^k \mathrm{I}_{\{f \geq 2^k\}}$$

so that

$$\|f_k\|_2^{2+4/n} \geq \left(2^{2k}\mu(\{f \geq 2^{k+1}\})\right)^{1+2/n} \quad \text{and} \quad \|f_k\|_1^{4/n} \leq \left(2^k\mu(\{f \geq 2^k\})\right)^{4/n}.$$

If we let $a_k = 2^{pk}\mu(\{f \geq 2^k\})$, $k \in \mathbb{Z}$, and if we apply Nash's inequality to f_k, we get, for every k,

$$a_{k+1} \leq 2^p W(f_k)^\theta a_k^{2(1-\theta)} \tag{4.7}$$

where we recall that $\theta = n/(n+2)$ in case of Nash's inequality. Summing up over $k \in \mathbb{Z}$ the inequalities (4.7) and applying Hölder's inequality, we get

$$\sum_k a_k \leq 2^p \Big(\sum_k W(f_k)\Big)^\theta \Big(\sum_k a_k^2\Big)^{1-\theta} \leq 2^p \Big(\sum_k W(f_k)\Big)^\theta \Big(\sum_k a_k\Big)^{2(1-\theta)}.$$

Therefore,

$$\sum_k a_k \leq 2^{p/(2\theta-1)} \Big(\sum_k W(f_k)\Big)^{\theta/(2\theta-1)}. \tag{4.8}$$

Now, since $\Gamma(f_k) = \Gamma(f)\mathrm{I}_{B_k}$ where $B_k = \{2^k \leq f \leq 2^{k+1}\}$, $k \in \mathbb{Z}$,

$$\begin{aligned}\sum_k W(f_k) &= A\sum_k \|f_k\|_2^2 + B\sum_k \int_{B_k} \Gamma(f)d\mu \\ &\leq A\sum_k 2^{2k}\mu(\{f \geq 2^k\}) + B\int \Gamma(f)d\mu.\end{aligned} \tag{4.9}$$

By a standard argument,

$$\sum_k 2^{2k}\mu\big(\{f \geq 2^k\}\big) \leq \frac{4}{3}\,\|f\|_2^2 \quad \text{and} \quad \sum_k a_k \geq 2^{-p}\,\|f\|_p^p.$$

Hence, as a consequence of (4.8) and (4.9),

$$\|f\|_p^p \leq 2^p 2^{p/(2\theta-1)}\Big(\frac{4}{3}\Big)^{\theta/(2\theta-1)} W(f)^{\theta/(2\theta-1)}.$$

Hence,

$$\|f\|_p^2 \leq \frac{4}{3}\,2^{2p}W(f)$$

since $2\theta-1 = 1/(p-1)$ and $\theta/(2\theta-1) = p/2$. The argument is thus complete. Note that $\frac{4}{3}\,2^{2p}$ is bounded above uniformly in $2 < p \leq p_0$, that is uniformly in $n \geq n_0$.

4.2 Logarithmic Sobolev inequalities and hypercontractivity

In this section, we present the famous equivalence between logarithmic Sobolev inequalities and hypercontractivity due to L. Gross [Gr1]. Hypercontractivity is a smoothing property introduced in quantum field theory that roughly expresses that P_t maps L^2 in to L^4 for some $t > 0$. It actually gives rise to bounds on the operator norm $\|P_t\|_{p,q}$ of P_t from L^p into L^q, $1 \leq p \leq q \leq \infty$. Below, we deal with more general inequalities between entropy and energy. In particular, the approach will imply the heat kernel bound (4.2) under the logarithmic Sobolev inequality (4.6) with optimal constants as shown in the next section.

Let us start again with our abstract setting of a Markov diffusion generator L with semigroup $(P_t)_{t\geq 0}$ as presented in Chapter 1. Following (1.21), consider a general inequality

$$\int f^2\log f^2 d\mu \leq \Phi\Big(\int \Gamma(f)d\mu\Big) \tag{4.10}$$

for every f in $\mathcal{A}$ with $\int f^2 d\mu = 1$. μ need not be a probability measure. In most examples $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is concave, strictly increasing, and of class C^1, which we assume throughout the argument below. Therefore, for every u and $v > 0$,

$$\Phi(u) \leq \Phi(v) + \Phi'(u)(u-v),$$

so that (4.10) reads, for every $v > 0$ and every f (together with homogeneity),

$$\int f^2\log f^2 d\mu \leq \Phi'(v)\int \Gamma(f)d\mu + \Psi(v)\int f^2 d\mu \tag{4.11}$$

where $\Psi(v) = \Phi(v) - v\Phi'(v)$. We thus deal equivalently with a family of logarithmic Sobolev inequalities. Now, by the diffusion property of L, changing $f > 0$ into $f^{s/2}$ shows that, for every $f > 0$, $v > 0$, $s > 1$,

$$\begin{aligned}\int f^s\log f^s d\mu &- \int f^s d\mu\log\int f^s d\mu \\ &\leq -\,\Phi'(v)\,\frac{s^2}{4(s-1)}\int f^{s-1}\mathrm{L}f\,d\mu + \Psi(v)\int f^s d\mu.\end{aligned} \tag{4.12}$$

Choose now in this inequality a function $v \to v(s) > 0$, $s > 1$. Then, we make use of the fundamental argument of L. Gross [Gr1]. Namely, if

$$V(t) = \mathrm{e}^{-m(t)} \|P_t f\|_{s(t)}, \quad t \geq 0,$$

the preceding inequality will show that $V' \leq 0$ and V is non-increasing as soon as s and m are chosen so that

$$\frac{s^2(t)}{s'(t)} = \Phi'\big(v\big(s(t)\big)\big)\,\frac{s^2(t)}{4(s(t)-1)} \quad \text{and} \quad m'(t) = \Psi\big(v\big(s(t)\big)\big)\,\frac{s'(t)}{s(t)^2}\,.$$

Fix then $1 \leq p < q \leq \infty$ and consider the differential system

$$\begin{cases} dt = \dfrac{\Phi'(v(s))}{4(s-1)}\,ds, \quad s(0) = p, \\[2mm] dm = \dfrac{\Psi(v(s))}{s^2}\,ds. \end{cases}$$

Thus, we may conclude to the following statement (see [Ba3] for the detailed proof).

Theorem 4.1. *Under the entropy-energy inequality (4.10), for every* $1 \leq p < q \leq \infty$,

$$\|P_t\|_{p,q} \leq \mathrm{e}^m \tag{4.13a}$$

where

$$t = t_{p,q} = \int_p^q \Phi'\big(v(s)\big)\frac{ds}{4(s-1)} \quad \text{and} \quad m = m_{p,q} = \int_p^q \Psi\big(v(s)\big)\frac{ds}{s^2} \tag{4.13b}$$

provided we can find a function v for which these two integrals are finite.

The optimal choice for the function v that will be used throughout this work is given by $v(s) = \lambda s^2/(s-1)$, where $\lambda > 0$ is a parameter.

It might be worthwhile noting that, conversely, the previous bounds on $\|P_t\|_{p,q}$ imply that the corresponding entropy-energy inequality (4.12) holds. This is a consequence of the following proposition.

Proposition 4.2. *Under the previous notation, assume that, for some $1 \leq p < \infty$ and every q in some neighborhood of p, $\|P_t\|_{p,q} \leq \mathrm{e}^m$ where t and m are defined with (4.13b) through some function Φ. Then, for every non-negative f in $\mathcal{A}$,*

$$\int f^p \log f^p d\mu - \int f^p d\mu \log \int f^p d\mu \leq -\,\Phi'(v)\,\frac{p^2}{4(p-1)}\int f^{p-1}\mathrm{L}f\,d\mu + \Psi(v)\int f^p d\mu.$$

The proof reduces to check that if, for $f > 0$ in $\mathcal{A}$,

$$U(\varepsilon) = \mathrm{e}^{-m(\varepsilon)} \|P_{t(\varepsilon)} f\|_{p+\varepsilon}$$

where $t(\varepsilon) = t_{p,p+\varepsilon}$, $m(\varepsilon) = m_{p,p+\varepsilon}$, then $U'(0) \leq 0$ amounts to the inequality of the proposition.

Now, we describe with some examples the range of application of Theorem 4.1. We start with the famous hypercontractivity property. Further examples will be examined in the next section.

Assume that Φ is linear, more precisely, in accordance with Definition 1.5, that $\Phi(x) = \frac{2}{\rho_0}\, x$, $x \geq 0$. Since $\Phi' = \frac{2}{\rho_0}$ and $\Psi = 0$, it follows from (4.13) that $\|P_t\|_{p,q} \leq 1$ with

$$t = t_{p,q} = \frac{1}{2\rho_0}\, \log\Big(\frac{q-1}{p-1}\Big).$$

As a consequence of Theorem 4.1 and Proposition 4.2, we may state.

Corollary 4.3. L *satisfies a logarithmic Sobolev inequality with constant* ρ_0 *if and only if, for (some) every* $1 < p < q < \infty$ *and every* $t \geq 0$ *such that* $\mathrm{e}^{2\rho_0 t} \geq \frac{q-1}{p-1}$,

$$\|P_t\|_{p,q} \leq 1.$$

The typical and first example of hypercontractive semigroup is the Ornstein-Uhlenbeck semigroup. As $\rho_0 = 1$, $\|P_t\|_{p,q} \leq 1$ when $\mathrm{e}^{2t} \geq \frac{q-1}{p-1}$. It is not difficult to see that $\|P_t\|_{p,q} = \infty$ when $\mathrm{e}^{2t} < \frac{q-1}{p-1}$ in this example.

The equivalence between logarithmic Sobolev inequality and hypercontractivity is an important issue when studying rates of convergence to the equilibrium (cf. [Gr2] and the references therein).

4.3 Optimal heat kernel bounds

As we have seen, one essential feature of the equivalence between the Sobolev, Nash and logarithmic Sobolev inequalities (4.1), (4.5) and (4.6) is that they all yield the heat kernel bound (4.2) with the same dimensional parameter n (although n need not be an integer here). (Nash's inequality perhaps provides the simplest proof of this heat kernel bound [Da].) Constants are however not preserved in general in this procedure. One may however ask, in simple cases such as $\mathbb{R}^n$, whether or not any of the Sobolev inequalities (4.4) (with $A = 0$) could imply the optimal Euclidean heat kernel bound

$$\sup_{x,y \in \mathbb{R}^n} p_t(x,y) \leq \frac{1}{(4\pi t)^{n/2}}\,, \quad t > 0. \tag{4.14}$$

To expect (4.14) to hold, it seems necessary to already start with a Sobolev inequality with optimal constant. Amongst (4.4), and in $\mathbb{R}^n$, only few examples of optimal constants are known. Namely [Au]

$$C = \frac{\big(\frac{n!}{2}\,\omega_n\big)^{2/n}}{\pi^2 n(n-2)}\,,$$

where ω_n denotes the volume of the unit ball B^n in $\mathbb{R}^n$, in case of the Sobolev inequality

$$\|f\|_p^2 \leq C \textstyle\int |\nabla f|^2 dx, \quad f \in C_c^\infty(\mathbb{R}^n) \tag{4.15}$$

$(p = 2n/(n-2))$. As was proved in [C-L],

$$C = \frac{2((n+2)/2)^{(n+2)/n}}{n\omega_n^{2/n}\lambda_1^N},$$

where λ_1^N denotes the first non-zero Neumann eigenvalue of the Laplacian on radial functions on B^n, in case of the Nash inequality

$$\|f\|_2^{2+4/n} \leq C \int |\nabla f|^2 dx \|f\|_1^{4/n}, \quad f \in C_c^\infty(\mathbb{R}^n) \tag{4.16}$$

(that corresponds to $r = 2$, $s = 1$, $\theta = n/(n+2)$). In case of the logarithmic Sobolev inequality that corresponds to the limiting case $r = 2$ and $\theta \to 0$, the task is easier [Ca]. One may simply start with the logarithmic Sobolev inequality for the canonical Gaussian measure γ

$$\int g^2 \log g^2 d\gamma \leq 2 \int |\nabla g|^2 d\gamma$$

for every smooth function g on $\mathbb{R}^n$ with $\int g^2 d\gamma = 1$. Set

$$f^2(x) = (2\pi)^{-n/2} \mathrm{e}^{-|x|^2/2} g^2(x), \quad x \in \mathbb{R}^n,$$

so that $\int f^2 dx = 1$. Then

$$\int f^2 \log\big(f^2 (2\pi)^{n/2} \mathrm{e}^{|x|^2/2}\big) dx \leq 2 \int \Big|\nabla f + \frac{x}{2} f\Big|^2 dx.$$

An integration by parts easily yields

$$\int f^2 \log f^2 dx \leq 2 \int |\nabla f|^2 dx - \frac{n}{2} \log(2\pi) - n.$$

Changing f into $\lambda^{n/2} f(\lambda x)$, $\lambda > 0$, which still satisfies the normalization $\int f^2 dx = 1$, shows that, for every $\lambda > 0$ thus,

$$\int f^2 \log f^2 dx \leq 2\lambda^2 \int |\nabla f|^2 dx - \frac{n}{2} \log(2\pi) - n - n \log \lambda.$$

Optimizing in λ, we get that for every smooth f on $\mathbb{R}^n$ with $\int f^2 dx = 1$,

$$\int f^2 \log f^2 dx \leq \frac{n}{2} \log\bigg(\frac{2}{n\pi \mathrm{e}} \int |\nabla f|^2 dx\bigg). \tag{4.17}$$

Since we started from the logarithmic Sobolev inequality for γ with its best constant for which exponential functions are extremal,

$$C = \frac{2}{n\pi\mathrm{e}}$$

is the best constant in (4.17) on $\mathbb{R}^n$. To further convince ourselves that this constant is optimal, one may note that among all functions $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for every smooth function f on $\mathbb{R}^n$ with $\int f^2 dx = 1$,

$$\int f^2 \log f^2 dx \le \Phi\Big(\int |\nabla f|^2 dx\Big), \tag{4.18}$$

the function

$$\Phi(u) = \frac{n}{2} \log\Big(\frac{2u}{n\pi \mathrm{e}}\Big)$$

is actually best possible. Indeed, apply (4.18) to $f^2(x) = \lambda^n (2\pi)^{-n/2} \mathrm{e}^{-\lambda^2 |x|^2/2}$, $\lambda > 0$. Since $\int f^2 dx = 1$, we get

$$\frac{n}{2} \log\Big(\frac{\lambda^2}{2\pi}\Big) - \frac{n}{2} \le \Phi\Big(\frac{n\lambda^2}{4}\Big).$$

The claim follows by setting $u = n\lambda^2/4$.

Our main observation here is that the general Theorem 4.1 actually produces the optimal heat kernel bound (4.14) if we start from a logarithmic Sobolev inequality (4.10) for the generator L (or rather the carré du champ Γ) with the best entropy-energy function of $\mathbb{R}^n$. In particular, Theorem 4.4 provides a proof of the heat kernel bound (4.2) under one of the (equivalent) Sobolev type inequalities (4.4) (with $p = 2n/(n-2)$, $n \ge 3$).

Theorem 4.4. *Assume that for every f in $\mathcal{A}$ with $\int f^2 d\mu = 1$,*

$$\int f^2 \log f^2 d\mu \le \frac{n}{2} \log\Big(\frac{2}{n\pi \mathrm{e}} \int \Gamma(f) d\mu\Big). \tag{4.19}$$

Then,

$$\sup p_t(x,y) \le \frac{1}{(4\pi t)^{n/2}}, \quad t > 0$$

(where the supremum is understood in the esssup sense).

If

$$\int f^2 \log f^2 d\mu \le \frac{n}{2} \log\Big(C \int \Gamma(f) d\mu\Big), \quad \int f^2 d\mu = 1,$$

for some $C > 0$, by a simple change of variables,

$$\sup p_t(x,y) \le \Big(\frac{n\mathrm{e}C}{8t}\Big)^{n/2}, \quad t > 0.$$

It is worthwhile mentioning that the logarithmic Sobolev inequality (4.19) of Theorem 4.4 behaves correctly under tensor product. Namely, if two carrés du champ Γ_1 and Γ_2 satisfies such an inequality with dimensions n_1 and n_2 respectively, then the product operator $\Gamma_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \Gamma_2$ will satisfy this inequality with dimension $n_1 + n_2$. This immediately follows from the classic product property of entropy

(Section 1.3) together with the linearized version (4.11) of (4.18). This stability property is reflected similarly on the heat kernel bound as can be seen from the example of the Euclidean spaces.

Proof of Theorem 4.4. We simply use (4.13) with

$$\Phi(u) = \frac{n}{2} \log\Big(\frac{2u}{n\pi \mathrm{e}}\Big), \quad u > 0,$$

and $v(s) = \lambda s^2/(s-1)$, $\lambda > 0$, $s > 1$. Hence $\Phi'(u) = n/2u$ and

$$\Psi(u) = \frac{n}{2} \log\Big(\frac{2u}{n\pi \mathrm{e}^2}\Big).$$

Then $\|P_t\|_{1,\infty} \leq \mathrm{e}^m$ with

$$t = t(\lambda) = \frac{n}{8\lambda} \int_1^\infty \frac{ds}{s^2} = \frac{n}{8\lambda}$$

and

$$m = m(\lambda) = \frac{n}{2} \int_1^\infty \log\bigg(\frac{2\lambda}{n\pi \mathrm{e}^2} \cdot \frac{s^2}{s-1}\bigg) \frac{ds}{s^2}\,.$$

¿From the first equality, $\lambda = n/8t$. The second yields

$$\begin{aligned} m &= \frac{n}{2} \log\bigg(\frac{2\lambda}{n\pi \mathrm{e}^2}\bigg) + \frac{n}{2} \int_1^\infty \log\Big(\frac{s^2}{s-1}\Big) \frac{ds}{s^2} \\ &= \frac{n}{2} \log\bigg(\frac{2\lambda}{n\pi \mathrm{e}^2}\bigg) - \frac{n}{2} \int_0^1 \log\big(r(1-r)\big)\,dr \\ &= \frac{n}{2} \log\bigg(\frac{2\lambda}{n\pi \mathrm{e}^2}\bigg) + n \end{aligned}$$

with the change of variable $r = 1/s$. Since $\lambda = n/8t$,

$$m = m(\lambda) = \frac{n}{2} \log\Big(\frac{1}{4\pi t}\Big)$$

which yields

$$\|P_t\|_{1,\infty} \leq \mathrm{e}^m = \frac{1}{(4\pi t)^{n/2}}$$

and the result. The proof is complete. □

It is clear that the same proof yields upper bounds for $\|P_t\|_{p,q}$ for every $1 \leq p < q \leq \infty$. Namely $\|P_t\|_{p,q} \leq \mathrm{e}^m$ where

$$t = t_{p,q}(\lambda) = \frac{n}{8\lambda} \int_p^q \frac{ds}{s^2} = \frac{n}{8\lambda}\Big(\frac{1}{p} - \frac{1}{q}\Big)$$

and

$$m = m_{p,q}(\lambda) = \frac{n}{2} \int_p^q \log\bigg(\frac{2\lambda}{n\pi \mathrm{e}^2} \cdot \frac{s^2}{s-1}\bigg) \frac{ds}{s^2}\,.$$

After some calculations very similar to the previous ones, we find that

$$\|P_t f\|_{p,q} \le \left[\frac{p^{\frac{1}{p}}\left(1-\frac{1}{p}\right)^{1-\frac{1}{p}}}{q^{\frac{1}{q}}\left(1-\frac{1}{q}\right)^{1-\frac{1}{q}}}\right]^{\frac{n}{2}} \left[\frac{1}{4\pi t}\left(\frac{1}{p}-\frac{1}{q}\right)\right]^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}. \tag{4.20}$$

It is less clear however why these bounds should be optimal on $\mathbb{R}^n$. The next proposition answers this question positively. These bounds (4.20) and their optimality may also be shown to follow from the work of E. Lieb [Lie] on Gaussian maximizers of Gaussian kernels.

Proposition 4.5. *With the preceding notation, let Φ be such that (4.10) holds and $\|P_t\|_{1,\infty} = \mathrm{e}^m$ where*

$$t = \int_1^\infty \Phi'\Big(\frac{\lambda s^2}{s-1}\Big)\frac{ds}{4(s-1)}$$

and

$$m = \int_1^\infty \Psi\Big(\frac{\lambda s^2}{s-1}\Big)\frac{ds}{s^2}$$

for some $\lambda > 0$. Then, for every $1 \le p < q \le \infty$,

$$\|P_{t_{p,q}}\|_{p,q} = \mathrm{e}^{m_{p,q}}$$

where

$$t_{p,q} = \int_p^q \Phi'\Big(\frac{\lambda s^2}{s-1}\Big)\frac{ds}{4(s-1)}$$

and

$$m_{p,q} = \int_p^q \Psi\Big(\frac{\lambda s^2}{s-1}\Big)\frac{ds}{s^2}.$$

The proof of the proposition is easy. By the hypothesis,

$$\|P_{t_{1,p}+t_{p,q}+t_{q,\infty}}\|_{1,\infty} = \|P_t\|_{1,\infty} = \mathrm{e}^m = \mathrm{e}^{m_{1,p}+m_{p,q}+m_{q,\infty}}.$$

By the semigroup property,

$$\|P_{t_{1,p}+t_{p,q}+t_{q,\infty}}\|_{1,\infty} \le \|P_{t_{1,q}}\|_{1,q}\|P_{t_{p,q}}\|_{p,q}\|P_{t_{q,\infty}}\|_{q,\infty} \le \mathrm{e}^{m_{1,p}+m_{p,q}+m_{q,\infty}}$$

so that it is impossible that $\|P_{t_{p,q}}\|_{p,q} < \mathrm{e}^{m_{p,q}}$ for some $p < q$. Clearly, this result applies in the Euclidean case to prove that (4.20) are actually equalities in this case. Moreover, it implies the somewhat surprising following observation. For every $1 \le p < q < r \le \infty$, and every $t \ge 0$, there is an unique t' such that

$$\|P_t\|_{p,r} = \|P'_t\|_{p,q}\|P_{t-t'}\|_{q,r}.$$

Indeed, since $\Phi'(u) = n/2u$ in this case, one may define $\lambda > 0$ by

$$t = t_{p,r}(\lambda) = \int_p^r \Phi'\Big(\frac{\lambda s^2}{s-1}\Big)\frac{ds}{4(s-1)} = \frac{n}{8\lambda}\int_p^r \frac{ds}{s^2}.$$

Then set $t' = t_{q,r}(\lambda)$ and the claim follows from the preceding argument.

As further explained in the paper [B-C-L] the optimal Euclidean logarithmic Sobolev inequality may also be used to yield sharp off-diagonal bounds that are optimal for a special class of metrics (given by harmonic functions). More precisely, under the logarithmic Sobolev inequality (4.19),

$$p_t(x,y) \leq \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{d^H(x,y)^2}{4t}\right) \tag{4.21}$$

where

$$d^H(x,y) = \sup_{\Gamma(f,f)\leq 1,\, \mathrm{L}f=0} [f(x) - f(y)].$$

Note that $d^H = d$ on $\mathbb{R}^n$ that thus improves in this case upon (4.3) (while $d^H = 0$ on a compact Riemannian manifold).

It is easy to see finally that Theorem 4.4 has a local version. Under a entropy-energy inequality

$$\int f^2 \log f^2 d\mu \leq \frac{n}{2} \log\left(A + B\int \Gamma(f) d\mu\right),$$

one concludes to the semigroup bound for the small values of t

$$\|P_t\|_{1,\infty} \leq C\, t^{-n/2}, \quad 0 < t \leq 1.$$

4.4 Rigidity properties

In order to efficiently use Theorem 4.4, it would be worthwhile knowing how to establish (4.19) with sharp constant in a Riemannian or abstract setting using curvature-dimension hypotheses. Let us consider the case of a (non-compact) Riemannian manifold M with dimension n and volume element dv. Denote by $(P_t)_{t\geq 0}$ the heat semigroup on M. We conjecture that if

$$\lim_{t\to\infty} (4\pi t)^{n/2} P_t f = c \int f dv$$

for every smooth f on a manifold M with dimension n and non-negative Ricci curvature, then the logarithmic Sobolev inequality (4.19) holds with its sharp constant from $\mathbb{R}^n$, i.e.

$$\int f^2 \log f^2 dv \leq \frac{n}{2} \log\left(\frac{2\, c^{2/n}}{n\pi \mathrm{e}} \int |\nabla f|^2 dv\right), \quad \int f^2 dv = 1.$$

Here the constant $c > 0$ refers to a possible normalization of the volume that is not fixed in our setting. We have not been able to prove such a result although we strongly conjecture that it must be true. So far, we have only been able to prove the inequality with a constant that misses the optimal one by a factor $1/\log 2$ (see [B-C-L]).

Now we turn to the question of identifying manifolds with non-negative Ricci curvature satisfying a global Sobolev inequality with the optimal Euclidean constant. Two results have been obtained in this direction. The first one concerns Euclidean type logarithmic Sobolev inequalities and relies on heat kernel bounds. The second one concerns classical Sobolev inequalities and makes use of extremal functions and non-linear analysis. We briefly conclude these notes with these results. For rigidity theorems in the compact case, we refer to [Dr] and [He2].

The following statement has been observed in [B-C-L]. We sketch the proof.

Theorem 4.6. *Let (M,g) be a complete Riemannian manifold of dimension n with non-negative Ricci curvature satisfying a logarithmic Sobolev inequality with the best constant of $\mathbb{R}^n$, i.e.*

$$\int f^2 \log f^2 dv \leq \frac{n}{2} \log\left(\frac{2}{n\pi\mathrm{e}} \int |\nabla f|^2 dv\right), \quad \int f^2 dv = 1.$$

Then M is isometric to $\mathbb{R}^n$.

Proof. Denote by $V(x,r)$ the volume of the ball with center $x \in M$ and radius $r > 0$ in M. By Theorem 4.4, for every $x, y \in M$, $t > 0$,

$$p_t(x,y) \leq \frac{1}{(4\pi t)^{n/2}}. \tag{4.22}$$

In particular, by the results of [L-Y], M has maximal volume growth, that is

$$\liminf_{r\to\infty} \frac{V(x,r)}{r^n} \geq \theta > 0.$$

Li's result [L2] then indicates that, for every $x, y, z \in M$,

$$\lim_{t\to\infty} V\big(z, \sqrt{t}\,\big) p_t(x,y) = \frac{\omega_n}{(4\pi)^{n/2}}$$

where we recall that ω_n is the volume of the Euclidean unit ball. Together with (4.22), we thus see that

$$\liminf_{r\to\infty} \frac{V(x,r)}{\omega_n r^n} \geq 1. \tag{4.23}$$

Now, by Gromov's comparison theorem (cf. [Cha2]), for $s < r$,

$$\frac{V(x,r)}{\omega_n r^n} \leq \frac{V(x,s)}{\omega_n s^n}$$

and, in particular, $V(x,r) \leq \omega_n r^n$ as $s \to 0$. But now, by (4.23), we also get $V(x,s) \geq \omega_n s^n$ as $r \to \infty$. Therefore $V(x,r) = \omega_n r^n$ for every $x \in M$ and $r > 0$. By the case of equality in the volume comparison theorem [Cha2], M is isometric to $\mathbb{R}^n$. The theorem is established. □

Now, let us consider the family of Sobolev inequalities

$$\left(\int |f|^p dv\right)^{q/p} \le C \int |\nabla f|^q dv, \tag{4.24}$$

$1 \le q < n$, $1/p = 1/q - 1/n$, f C^∞ and compactly supported on M. The best constants $C = K(n,q)$ for which (4.24) holds in $\mathbb{R}^n$ are known and were described by Th. Aubin [Au] and G. Talenti [Ta]. Namely, $K(n,1) = n^{-1}\omega_n^{-1/n}$ where ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$, while

$$K(n,q) = \frac{1}{n}\left(\frac{q-1}{n-q}\right)^{q-1}\left(\frac{\Gamma(n+1)}{n\omega_n\Gamma(n/q)\Gamma(n+1-n/q)}\right)^{q/n}$$

if $q > 1$. Moreover, for $q > 1$, the equality in (4.24) is attained by the functions $(\lambda + |x|^{q/(q-1)})^{1-(n/q)}$, $\lambda > 0$, where $|x|$ is the Euclidean length of the vector x in $\mathbb{R}^n$. The following is the analogue of Theorem 4.6 for Sobolev inequalities.

Theorem 4.7. *Let M be a complete n-dimensional Riemannian manifold with non-negative Ricci curvature. If one of the Sobolev inequalities (4.24) is satisfied with $C = K(n,q)$, then M is isometric to $\mathbb{R}^n$.*

The particular case $q = 1$ ($p = n/(n-1)$) is of course well-known. In this case indeed, the Sobolev inequality is equivalent to the isoperimetric inequality

$$\big(\mathrm{vol}_n(A)\big)^{(n-1)/n} \le K(n,1)\mathrm{vol}_{n-1}(\partial A)$$

where ∂A is the boundary of a smooth bounded open set A in M. If we let $V(x,r) = V(r)$ be the volume of the geodesic ball $B(x,r) = B(r)$ with center x and radius $r > 0$ in M, we have

$$\frac{d}{dr}\mathrm{vol}_n\big(B(r)\big) = \mathrm{vol}_{n-1}\big(\partial B(r)\big).$$

Hence, setting $A = B(x,r)$ in the isoperimetric inequality, we get

$$V(r)^{(n-1)/n} \le K(n,1)V'(r)$$

for all $r > 0$. Integrating yields $V(r) \ge (nK(n,1))^{-n}r^n$, and since $K(n,1) = n^{-1}\omega_n^{-1/n}$, for every $r > 0$,

$$V(r) \ge \omega_n r^n \tag{4.25}$$

If M has non-negative Ricci curvature, by Bishop's comparison theorem (cf. e.g. [Cha2]) $V(x,r) \le \omega_n r^n$ for every r, and by (4.25) and the case of equality, M is isometric to $\mathbb{R}^n$. The main interest of Theorem 4.7 therefore lies in the case $q > 1$. As usual, the classical value $q = 2$ (and $p = 2n/(n-2)$) is of particular interest. It should be noticed that known results already imply that the scalar curvature of M is zero in this case (cf. [He1], Prop. 4.10).

The proof of Theorem 4.7 is inspired by the technique developed in Chapter 3 for Myers's theorem on the diameter of a compact Riemannian manifold satisfying a Sobolev inequality. We assume that the Sobolev inequality (4.24) is satisfied with

$C = K(n,q)$ for some $q > 1$. Recall first that the extremal functions of this inequality in $\mathbb{R}^n$ are the functions $(\lambda + |x|^{q/(q-1)})^{1-(n/q)}$, $\lambda > 0$. Let now x be a fixed point in M and let $\theta > 1$. Set $f = \theta^{-1} d(\cdot, x)$ where d is the distance function on M. The idea is then to apply the Sobolev inequality (4.24), with $C = K(n,q)$, to $(\lambda + f^{q/(q-1)})^{1-(n/q)}$, for every $\lambda > 0$ to deduce a differential inequality whose solutions may be compared to the extremal Euclidean case. We refer to [Le3] or [He2] for the proof of this result.

It is natural to conjecture that Theorem 4.7 may actually be turned into a volume comparison statement as it is the case for $q = 1$. That is, in a Riemannian manifold (M,g) satisfying the Sobolev inequality (4.24) with the constant $K(n,q)$ for some $q > 1$, and without any curvature assumption, for every x and every $r > 0$,

$$V(x,r) \geq \omega_n r^n.$$

This is well-known ([Va3], [Ba3]) up to a constant (depending only on n and q) but the preceding proof does not seem to be able to yield such a conclusion.

Recently, the analogue of Theorems 4.6 and 4.7 for the Nash inequality (4.5) has been proved, however for simply connected manifolds, by O. Druet, E. Hebey and M. Vaugon [D-H-V].

References

[Au] Th. Aubin. Nonlinear analysis on manifolds. Monge-Ampère equations. Springer (1982).

[Ba1] D. Bakry. Transformations de Riesz pour les semigroupes symétriques. Séminaire de Probabilités XIX. Lecture Notes in Math. 1123, 2130–174 (1985). Springer.

[Ba2] D. Bakry. Inégalités de Sobolev faibles : un critère Γ_2. Séminaire de Probabilités XXV. Lecture Notes in Math. 1485, 234–261 (1991). Springer.

[Ba3] D. Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. Ecole d'Eté de Probabilités de St-Flour. Lecture Notes in Math. 1581, 1–114 (1994). Springer-Verlag.

[Ba4] D. Bakry. On Sobolev and logarithmic Sobolev inequalities for Markov semigroups. New trends in Stochastic Analysis. 43–75 (1997). World Scientific.

[Ba-E] D. Bakry, M. Emery. Diffusions hypercontractives. Séminaire de Probabilités XIX. Lecture Notes in Math. 1123, 177–206 (1985). Springer-Verlag.

[B-C-L] D. Bakry, D. Concordet, M. Ledoux. Optimal heat kernel bounds under logarithmic Sobolev inequalities. ESAIM: Probability and Statistics 1, 391–407 (1997).

[B-C-L-SC] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste. Sobolev inequalities in disguise. Indiana J. Math. 44, 1033–1074 (1995).

[B-L1] D. Bakry, M. Ledoux. Sobolev inequalities and Myers's diameter theorem for an abstract Markov generator. Duke Math. J. 85, 253–270 (1996).

[B-L2] D. Bakry, M. Ledoux. Lévy-Gromov's isoperimetric inequality for an infinite dimensional diffusion generator. Invent. math. 123, 259–281 (1996).

[B-L-Q] D. Bakry, M. Ledoux, Z. Qian. Preprint (1997).

[B-Q] D. Bakry, Z. Qian. Comparison theorem for spectral gap via dimension, diameter and Ricci curvature. Preprint (1998).

[Be1] W. Beckner. Sobolev inequalities, the Poisson semigroup and analysis on the sphere S^n. Proc. Nat. Acad. Sci. 89, 4816–4819 (1992).

[Be2] W. Beckner. Personal communication (1998).

[Bé] P. H. Bérard. Spectral geometry: Direct and inverse problems. Lecture Notes in Math. 1207 (1986). Springer.

[BV-V] M.-F. Bidaut-Veron, L. Veron. Nonlinear elliptic equations on compact manifolds and asymptotics of Emden equations. Invent. math. 106, 489–539 (1991).

[Bob] S. Bobkov. An isoperimetric inequality on the discrete cube and an elementary proof of the isoperimetric inequality in Gauss space. Ann. Probability 25, 206–214 (1997).

[B-H] S. Bobkov, Ch. Houdré. Some connections between Sobolev-type inequalities and isoperimetry. Memoirs of the A.M.S. 616 (1997).

[Bor] C. Borell. The Brunn-Minkowski inequality in Gauss space. Invent. math. 30, 207–216 (1975).

[B-Z] Y. D. Burago, V. A. Zalgaller. Geometric inequalities. Springer-Verlag (1988). First Edition (russian): Nauka (1980).

[Ca] E. Carlen. Superadditivity of Fisher's information and logarithmic Sobolev inequalities. J. Funct. Anal. 101, 194–211 (1991).

[C-L] E. Carlen, M. Loss. Sharp constant in Nash's inequality. Duke Math. J., International Math. Research Notices 7, 213-215 (1993).

[C-K-S] E. Carlen, S. Kusuoka, D. Stroock. Upperbounds for symmetric Markov transition functions. Ann. Inst. H. Poincaré 23, 245–287 (1987).

[Cha1] I. Chavel. Eigenvalues in Riemannian geometry. Academic Press (1984).

[Cha2] I. Chavel. Riemannian geometry - A modern introduction. Cambridge Univ. Press (1993).

[Ch] J. Cheeger. The relation between the Laplacian and the diameter for manifolds of non-negative curvature. Arch. der Math. 19, 558–560 (1968).

[Che] S.-Y. Cheng. Eigenvalue comparison theorems and its geometric applications. Math. Z. 143, 289–297 (1975).

[Da] E. B. Davies. Heat kernel and spectral theory. Cambridge Univ. Press (1989).

[D-S] J.-D. Deuschel, D. Stroock. Large deviations. Academic Press (1989).

[Dr] O. Druet. Optimal Sobolev inequalities of arbitrary order on compact Riemannian manifolds (1998). To appear in J. Funct. Anal.

[D-H-V] O. Druet, E. Hebey, M. Vaugon. Optimal Nash's inequalities on Riemannian manifolds: the influence of geometry. Preprint (1998).

[Eh] A. Ehrhard. Symétrisation dans l'espace de Gauss. Math. Scand. 53, 281–301 (1983).

[F-L-M] T. Figiel, J. Lindenstrauss, V. D. Milman. The dimensions of almost spherical sections of convex bodies. Acta Math. 139, 52–94 (1977).

[Fo] E. Fontenas. Sur les constantes de Sobolev des variétés riemanniennes compactes et les fonctions extrémales des sphères. Bull. Sci. math. 121, 71–96 (1997).

[G-H-L] S. Gallot, D. Hulin, J. Lafontaine. Riemannian Geometry. Second Edition. Springer (1990).

[Gro] M. Gromov. Paul Lévy's isoperimetric inequality. Preprint I.H.E.S. (1980).

[G-M] M. Gromov, V. D. Milman. A topological application of the isoperimetric inequality. Amer. J. Math. 105, 843–854 (1983).

[Gr1] L. Gross. Logarithmic Sobolev inequalities. Amer. J. Math. 97, 1061–1083 (1975).

[Gr2] L. Gross. Logarithmic Sobolev inequalities and contractive properties of semigroups. Dirichlet Forms, Varenna 1992. Lect. Notes in Math. 1563, 54–88 (1993). Springer-Verlag.

[He1] E. Hebey. Sobolev spaces on Riemannian manifolds. Lecture Notes in Math. 1635. Springer (1996).

[He2] E. Hebey. Nonlinear analysis on manifolds: Sobolev spaces and inequalities. CIMS Lecture Notes 1998.

[Il] S. Ilias. Constantes explicites pour les inégalités de Sobolev sur les variétés riemanniennnes compactes. Ann. Inst. Fourier 33, Fasc. 2, 151–165 (1983).

[Kr] P. Kröger. On the spectral gap for compact manifolds. J. Differential Geometry 36, 315–330 (1992).

[Le1] M. Ledoux. L'algèbre de Lie des gradients itérés d'un générateur markovien – Développements de moyennes et entropies. Ann. scient. Éc. Norm. Sup. 28, 435–460 (1995).

[Le2] M. Ledoux. Isoperimetry and Gaussian Analysis. Ecole d'Eté de Probabilités de St-Flour 1994. Lecture Notes in Math. 1648, 165–294 (1996). Springer-Verlag.

[Le3] M. Ledoux. On manifolds with non-negative Ricci curvature and Sobolev inequalities (1997). To appear in Comm. in Analysis and Geometry.

[Le4] M. Ledoux. Concentration of measure and logarithmic Sobolev inequalities (1998). Séminaire de Probabilités XXXIII. Lecture Notes in Math., to appear. Springer.

[Lé] P. Lévy. Problèmes concrets d'analyse fonctionnelle. Gauthier-Villars (1951).

[Li1] P. Li. A lower bound for the first eigenvalue of the Laplacian on a compact manifold. Indiana Univ. Math. J. 28, 1013–1019 (1979).

[Li2] P. Li. Large time behavior of the heat equation on complete manifolds with non-negative Ricci curvature. Ann. Math. 124, 1–21 (1986).

[L-Y] P. Li, S.-T. Yau. On the parabolic kernel of the Schrödinger operator. Acta Math. 156, 153–201 (1986).

[Lie] E. Lieb. Gaussian kernels have only Gaussian maximizers. Invent. math. 102, 179–208 (1990).

[MK] H. P. McKean. Geometry of differential space. Ann. Probability 1, 197–206 (1973).

[Ma] O. Mazet. Classification des semigroupes de diffusion sur $\mathbb{R}^n$ associés à une famille de polynômes orthogonaux. Séminaire de Probabilités XXXI. Lecture Notes in Math. 1655, 40-53 (1997). Springer.

[Mo] J. Moser. A Harnack inequality for parabolic differential equations. Comm. Pure Appl. Math. 17, 101-134 (1964).

[M-W] C. Muller, F. Weissler. Hypercontractivity of the heat semigroup for ultraspherical polynomials and on the n-sphere. J. Funct. Anal. 48, 252–283 (1982).

[My] S. B. Myers. Connections between differential geometry and topology. Duke Math. J. 1, 376–391 (1935).

[Na] J. Nash. Continuity of solutions of parbolic and elliptic equations. Amer. J. Math. 80, 931–954 (1958).

[Ob] M. Obata. Certain conditions for a Riemannian manifold to be isometric with a sphere. J. Math. Soc. Japan 14, 333–340 (1962).

[Os] R. Osserman. The isoperimetric inequality. Bull. Amer. Math. Soc. 84, 1182–1238 (1978).

[Ro1] O. Rothaus. Diffusion on compact Riemannian manifolds and logarithmic Sobolev inequalities. J. Funct. Anal. 42, 358–367 (1981).

[Ro2] O. Rothaus. Hypercontractivity and the Bakry-Emery criterion for compact Lie groups. J. Funct. Anal. 65, 358–367 (1986).

[SC] L. Saloff-Coste. Convergence to equilibrium and logarithmic Sobolev constant on manifolds with Ricci curvature bounded below. Colloquium Math. 67, 109–121 (1994).

[Sc] E. Schmidt. Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie. Math. Nach. 1, 81–157 (1948).

[So] S. L. Sobolev. On a theorem in functional analysis. Amer. Math. Soc. Translations (2) 34, 39–68 (1963); translated from Mat. Sb. (N.S.) 4 (46), 471–497 (1938).

[S-T] V. N. Sudakov, B. S. Tsirel'son. Extremal properties of half-spaces for spherically invariant measures. J. Soviet. Math. 9, 9–18 (1978); translated from Zap. Nauch. Sem. L.O.M.I. 41, 14–24 (1974).

[Ta] G. Talenti. Best constants in Sobolev inequality. Ann. di Matem. Pura ed Appl. 110, 353–372 (1976).

[To] V. A. Topogonov. Riemannian spaces having their curvature bounded below by a positive number. Usphei Math. Nauk. 14, 87–135 (1959). Transl. Amer. Math. Soc. 37, 291–336 (1964).

[Va1] N. Varopoulos. Une généralisation du théorème de Hardy-Littlewood-Sobolev pour les espaces de Dirichlet. C. R. Acad. Sci. Paris 299, 651–654 (1984).

[Va2] N. Varopoulos. Hardy-Littlewood theory for semigroups. J. Funct. Anal. 63, 240–260 (1985).

[Va3] N. Varopoulos. Analysis and geometry on groups. Proceedings of the International Congress of Mathematicians, Kyoto (1990), vol. II, 951–957 (1991). Springer-Verlag.

[Y-Z] H.C. Yang, J. Q. Zhong. On the estimate of the first eigenvalue of a compact Riemannian manifold. Sci. Sinica Ser.A 27, 1265–1273 (1984).

[Wa] F.-Y. Wang. Logarithmic Sobolev inequalities on noncompact Riemannian manifolds. Proab. Theory Relat. Fields 109, 417–424 (1997).

[Yo] K. Yoshida. Functional Analysis. Sixth Edition. Springer (1995).