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Dedicated to Prof. Jindrich Necas on the occasion of his seventieth birthday

Abstract

The paper is concerned with the modelling of viscous incompressible flow in an unbounded exterior domain with the aid of the coupling of the nonlinear Navier–Stokes equations considered in a bounded domain with the linear Oseen system in an exterior domain. These systems are coupled on an artificial interface via suitable transmission conditions. The present paper is a continuation of the work [8], where the coupling of the Navier–Stokes problem with the Stokes problem is treated. However, the coupling "Navier–Stokes – Oseen" is physically more relevant. We give the formulation of this coupled problem and prove the existence of its weak solution for large data.

Keywords: viscous incompressible flow, Navier–Stokes equations, exterior Oseen problem, transmission conditions, coupled problem, weak solution.

Subject Classification: (MSC 1991): 31 A 05, 35 A 08, 45 A 05, 65, 76 D 05.

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0 Introduction

Very often the flow past bodies or obstacles is naturally formulated in exterior unbounded domains. However, the numerical solution of nonlinear exterior problems is difficult and, therefore, the unbounded domain is usually replaced by a bounded computational domain with an artificial boundary Γ. Then, of course, the problem of the choice of "nonreflecting" physically acceptable boundary conditions on Γ arises. Another possibility is to simulate the flow in the exterior of Γ with the aid of a suitable (preferable linear) approximation. This approach has become rather popular in various areas. Let us mention, e. g. [2, 7, 10].

In [8] we investigated the coupling of the incompressible Navier–Stokes system in the interior of Γ with the exterior Stokes problem. We proved in particular the existence of a solution of the coupled problem even for large data. This model can be used only in 3D and, moreover, the Stokes equations do not approximate sufficiently accurately the flow in the wake behind bodies. Here we will deal with a more relevant model using the coupling of the interior Navier–Stokes system with the exterior Oseen problem.

In comparison to [8], additional difficulties appear here. First, the transmission conditions used for the coupling "Navier–Stokes – Stokes" (inspired by considerations from [1]) are not suitable for the coupling between the Navier–Stokes and Oseen. In this case we have found that it is suitable to use the continuity of the normal stress augmented by the mean of the difference of the momentum flux transported from inside by the interior velocity $\boldsymbol{u}^-$ and from outside by a constant vector $\widetilde{\boldsymbol{u}}_\infty$ equal to the exterior farfield velocity $\boldsymbol{u}_\infty$. This condition is in agreement with one of the "natural" boundary conditions proposed in [3]. It can also be used for the coupling between the Navier–Stokes problem and the Stokes problem, putting $\widetilde{\boldsymbol{u}}_\infty = 0$. Then the analysis carried out in [8] would be completely analogous.

The second obstacle arises from the special form of the weak formulation of the exterior Oseen problem (cf., e. g., [9]). In contrast to the exterior Stokes problem, test functions cannot be considered as elements of the weighted Sobolev space where we seek a weak solution. This is the reason that the technique from [8] based on the properties of the Steklov–Poincaré operator is not used in the present paper.

Here we proceed in a quite different way than in [8]. Namely, we construct a monotone sequence of bounded domains covering the whole exterior domain and a sequence of corresponding approximate solutions converging to a solution of the coupled problem.

Since the exterior Oseen problem possesses a fundamental solution, see, e.g. [9], Vol. II, it is possible to reformulate this problem as a boundary integral equation on the artificial interface Γ. That is why our results represent a theoretical basis for the coupled finite element – boundary element procedures simulating numerically viscous incompressible flow.

1 Classical formulation of the problem

Let $\Omega \subset I\!\!R^N$ be an unbounded domain representing a two-dimensional ($N = 2$) or three dimensional ($N = 3$) region occupied by a fluid. We assume that its complement $I\!\!R^N - \overline{\Omega}$ ($\overline{M}$ denotes the closure of a set $M \subset I\!\!R^N$) consists of a finite number of components that are bounded domains $\Omega_i,\ i = 1, \dots, k$, with mutually disjoint closures $\overline{\Omega}_i$ and sufficiently regular boundaries $\partial\Omega_i$. Then $\Gamma_0 := \partial\Omega = \bigcup_{i=1}^k \partial\Omega_i$.

We consider stationary incompressible viscous flow in the exterior domain Ω past impermeable bodies $\Omega_i,\ i = 1, \dots, k$, and assume that the flow is homogeneous far from the bodies. We use the following **notation:** $x = (x_1, \dots, x_N) \in I\!\!R^N$ denotes a point of the N-dimensional Euclidean space, x_i $(i = 1, \dots, N)$ are the Cartesian coordinates of x, $\boldsymbol{u} = (u_1, \dots, u_N)$ is the velocity vector with components u_i in the directions x_i, $\boldsymbol{f} = (f_1, \dots, f_N)$ the density of outer volume force, p the kinematic static pressure, $\nu > 0$ the constant kinematic viscosity, $\boldsymbol{u}_\infty = (u_{1\infty}, \dots, z_{N\infty})$ the farfield velocity, $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_N)$ the nabla operator.

The **classical formulation** of the corresponding flow problem reads: Find $\boldsymbol{u} : \overline{\Omega} \to I\!\!R^N$ and $p : \overline{\Omega} \to I\!\!R$ such that

$$u_i \in C^2(\overline{\Omega}), \quad i = 1, \dots, N, \quad p \in C^1(\overline{\Omega}), \tag{1.1a}$$

$$-\nu \, \Delta \, \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \, \boldsymbol{u} + \nabla \, p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1b}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{u}|_{\Gamma_0} = 0 \tag{1.1d}$$

$$\lim_{|x| \to \infty} \boldsymbol{u}(x) = \boldsymbol{u}_\infty. \tag{1.1e}$$

This problem has been investigated in a number of works. A detailed treatment can be found, e. g., in [9]. For $N = 3$, see also [5]. However, this formulation in the unbounded domain Ω is not convenient for numerical simulation. That is why we introduce an artificial interface $\Gamma \subset \Omega$ dividing Ω into two subdomains: a bounded interior domain Ω^- with $\partial\Omega^- = \Gamma_0 \cup \Gamma$, in which we consider the Navier–Stokes system (1.1b - c), and an unbounded domain Ω^+ lying outside Γ with $\partial\Omega^+ = \Gamma$ and $\overline{\Omega}^+ = \Omega^+ \cup \Gamma$. In Ω^+ we approximate the nonlinear Navier–Stokes equations by the linear Oseen system. (For a detailed investigation of the exterior Oseen problem, see [9] and the references therein, or [5] for $N = 3$.)

Similarly as in the case of the coupling of the Navier–Stokes problem with the Stokes problem ([8]), an important question is the choice of **transmission conditions** on Γ. In [8] we proposed transmission conditions according to [1] augmenting the condition of the continuity of the normal stress on Γ by the kinetic energy from the interior side. However, this condition is not suitable in our case and, therefore, we propose its modification resembling a "natural" boundary condition from [3]. We arrive then at the following **classical formulation of the coupled problem:** Find $\boldsymbol{u}^\pm = (u_1^\pm, \dots, u_N^\pm) : \overline{\Omega}^\pm \to I\!\!R^N$, $p^\pm : \overline{\Omega}^\pm \to I\!\!R$ such that

$$u_i^\pm \in C^2(\overline{\Omega}^\pm), \; i = 1, \dots, N, \; p^\pm \in C^1(\overline{\Omega}^\pm), \tag{1.2a}$$

$$-\nu \, \Delta \, \boldsymbol{u}^- + (\boldsymbol{u}^- \cdot \nabla) \, \boldsymbol{u}^- + \nabla \, p^- = \boldsymbol{f} \quad \text{in } \Omega^-, \tag{1.2b}$$

$$\operatorname{div} \boldsymbol{u}^- = 0 \quad \text{in } \Omega^-, \tag{1.2c}$$

$$\boldsymbol{u}^-|_{\Gamma_0} = 0, \tag{1.2d}$$

$$-\nu \, \Delta \, \boldsymbol{u}^+ + (\boldsymbol{u}_\infty \cdot \nabla) \, \boldsymbol{u}^+ + \nabla \, p^+ = 0 \quad \text{in } \Omega^+, \tag{1.2e}$$

$$\operatorname{div} \boldsymbol{u}^+ = 0 \quad \text{in } \Omega^+, \tag{1.2f}$$

$$\lim_{|x| \to \infty} \boldsymbol{u}^+(x) = \boldsymbol{u}_\infty, \tag{1.2g}$$

$$\boldsymbol{u}^- = \boldsymbol{u}^+ \quad \text{on } \Gamma, \tag{1.2h}$$

$$\begin{aligned} -p^- \, \widehat{\boldsymbol{n}} + \nu \, \frac{\partial \boldsymbol{u}^-}{\partial \widehat{\boldsymbol{n}}} - \frac{1}{2} (\boldsymbol{u}^- \cdot \widehat{\boldsymbol{n}}) \, \boldsymbol{u}^- &= \\ = -p^+ \, \widehat{\boldsymbol{n}} + \nu \, \frac{\partial \boldsymbol{u}^+}{\partial \widehat{\boldsymbol{n}}} - \frac{1}{2} (\boldsymbol{u}_\infty \cdot \widehat{\boldsymbol{n}}) \, \boldsymbol{u}^+ & \text{ on } \Gamma. \end{aligned} \tag{1.2i}$$

Here $\boldsymbol{f}$, $\boldsymbol{u}_\infty$, $\nu > 0$ are given data. We assume that the support of $\boldsymbol{f}$, i.e., $\operatorname{supp} \boldsymbol{f} = \overline{\{x; \boldsymbol{f}(x) \neq 0\}} \subset \Omega^- \cup \Gamma_0$. (Hence, $\boldsymbol{f} = 0$ in $\overline{\Omega}^+$). By $\widehat{\boldsymbol{n}}$ we denote the unit outer normal to $\partial\Omega^-$ on Γ. This means that $\widehat{\boldsymbol{n}}$ points from Ω^- into Ω^+.

Remark 1.1. For simplicity we consider the terms $\partial \boldsymbol{u}^\pm / \partial \widehat{\boldsymbol{n}}$ in (1.2i), corresponding naturally to equations (1.2b and e). If we use the relations

$$\Delta u_i = \sum_{i=1}^{N} \frac{\partial D_{ij}(\boldsymbol{u})}{\partial x_j}, \qquad D_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

valid for $\boldsymbol{u} \in C^2(\Omega^\pm)$ with $\operatorname{div} \boldsymbol{u} = 0$, then $\partial \boldsymbol{u}^\pm / \partial \widehat{\boldsymbol{n}}$ can be replaced by $\sum_{j=1}^{N} D_{ij}(\boldsymbol{u}^\pm)\, \widehat{n}_j$ as in [8].

2 Weak formulation

In what follows **we will assume** that $\partial\Omega^- = \Gamma_0 \cup \Gamma$ is **Lipschitz-continuous**. First we introduce some function spaces. If $\widetilde{\Omega} \subset \Omega$ is a domain, then by $L^p(\widetilde{\Omega})$ and $W^{k,p}(\widetilde{\Omega})$ we denote the Lebesgue and Sobolev spaces, respectively, defined over $\widetilde{\Omega}$ (cf., [12]). For a bounded domain $\widetilde{\Omega}$ we set $W_0^{1,2}(\widetilde{\Omega}) = \{v \in W^{1,2}(\Omega);\ v|_{\partial\widetilde{\Omega}} = \text{the trace of } v \text{ on } \partial\widetilde{\Omega} = 0\}$. In $W_0^{1,2}(\widetilde{\Omega})$ we can use two equivalent norms

(2.1) $$\|w\|_{W_0^{1,2}(\widetilde{\Omega})} = \left(\int_{\widetilde{\Omega}} (|v|^2 + |\nabla v|^2)\,\mathrm{d}x\right)^{1/2}$$

and

(2.2) $$|v|_{W_0^{1,2}(\widetilde{\Omega})} = \left(\int_{\Omega} |\nabla v|^2\,\mathrm{d}x\right)^{1/2}.$$

It is well-known that

(2.3) $$W_0^{1,2}(\widetilde{\Omega}) = \text{closure of } C_0^\infty(\widetilde{\Omega}) \text{ in } W^{1,2}(\widetilde{\Omega}),$$

where $C_0^\infty(\widetilde{\Omega})$ is the space of all infinitely continuously differentiable functions with compact supports in $\widetilde{\Omega}$: $\operatorname{supp} v \subset \widetilde{\Omega}$ for $v \in C_0^\infty(\widetilde{\Omega})$.

For the unbounded domain Ω we define the weighted Sobolev space

(2.4) $$W^1(\Omega) = \left\{u; (1+|x|^2)^{-1/2}\,\sigma_N\, u \in L^2(\Omega),\ \frac{\partial u}{\partial x_i} \in L^2(\Omega)\right\},$$

where $\sigma_N(x) = 1$ for $N = 3$ and $\sigma_N(x) = |\ln(2+|x|)|^{-1}$ for $N = 2$, equipped with the norm

(2.5) $$\|u\|_{W^1(\Omega)} = \left\{\int_\Omega [(1+|x|^2)^{-1}\sigma_N^2\,|u|^2 + |\nabla u|^2]\,\mathrm{d}x\right\}^{1/2},$$

which is equivalent to the seminorm

(2.6) $$|u|_{W^1(\Omega)} = \left\{\int_\Omega |\nabla u|^2\,\mathrm{d}x\right\}^{1/2}.$$

(See, e.g., [4, Theorem 1, page 118] or [9, Vol. I, page 60].)

Further, we put

(2.7) $$W_0^1(\Omega) = \text{closure of } C_0^\infty(\Omega) \text{ in } W^1(\Omega).$$

Then

$$W_0^1(\Omega) = \left\{ v \in W^1(\Omega); v|_{\Gamma_0} = 0 \right\}. \tag{2.8}$$

We write $v \in W^{k,p}_{\mathrm{loc}}(\Omega)$, if $v|_{\widetilde{\Omega}} = W^{k,p}(\widetilde{\Omega})$ for every bounded domain $\widetilde{\Omega} \subset \Omega$.

In what follows we will work with N-dimensional vector valued functions. To this end, for a Banach space X with norm $\|\cdot\|_X$ we define the space $\boldsymbol{X} = X^N = \{\boldsymbol{u} : (u_1, \ldots, u_N);\ u_i \in X,\ i = 1, \ldots, N\}$, equipped with the norm

$$\|\boldsymbol{u}\|_{\boldsymbol{X}} = \left(\sum_{i=1}^{N} \|u_i\|_X^2 \right)^{1/2}, \qquad \boldsymbol{u} = (u_1, \ldots, u_N) \in \boldsymbol{X}. \tag{2.9}$$

In the same way we introduce the spaces $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{W}^1(\Omega)$, etc.

Now let us define subspaces of $\boldsymbol{W}^1(\Omega)$:

$$\begin{aligned} \boldsymbol{\mathcal{V}}(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}, \\ \boldsymbol{V}(\Omega) &= \text{closure of } \boldsymbol{\mathcal{V}}(\Omega) \text{ in } \boldsymbol{W}^1(\Omega). \end{aligned} \tag{2.10}$$

For functions $\boldsymbol{v}$ from subspaces of Sobolev spaces, the **restrictions** $\boldsymbol{v}|_\Gamma$, $\boldsymbol{v}|_{\Gamma_0}$ etc. will be understood in the sense of traces.

For $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, the limit at ∞ is zero and $\boldsymbol{v}|_{\Gamma_0} = 0$. In order to realize condition (1.2g) in the weak formulation, we introduce a function $\boldsymbol{\phi}_\infty$ defined in the following way. Let $\mathcal{B}$ be a sufficiently large ball with centre at the origin such that $\overline{\Omega}^- \subset \mathcal{B}$. Then $\Omega^* := (\mathcal{B} \cap \Omega) - \overline{\Omega}^- \subset \Omega^+$ and $\partial\Omega^* = \Gamma \cup \Gamma^*$, where Γ and Γ^* is the interior and exterior component of $\partial\Omega^*$, respectively, – see Fig. 2.1.

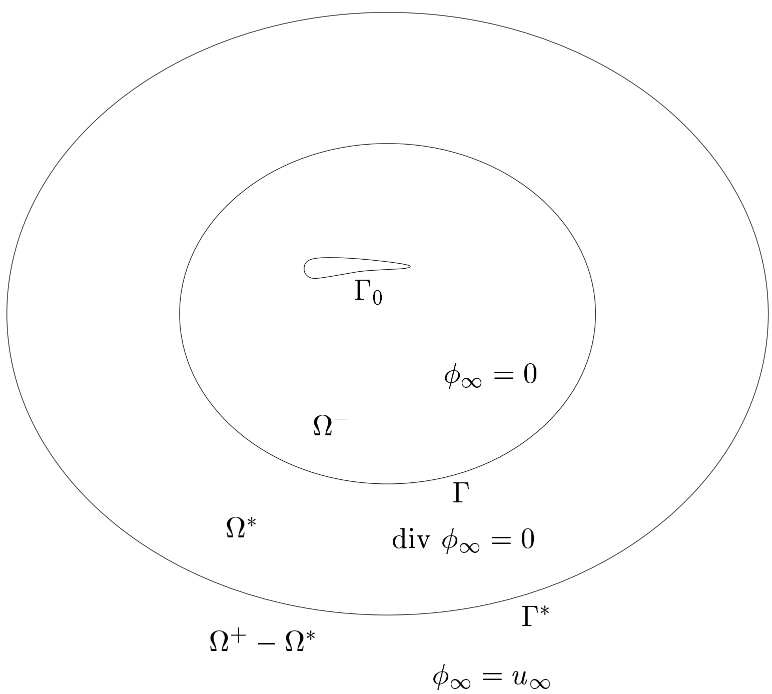

Figure 2.1.

Since $\int_\Gamma \boldsymbol{u}_\infty \cdot \boldsymbol{n} \, \mathrm{d}S = 0$, in virtue of [11, Lemma 2.2, page 24], there exists a function $\boldsymbol{\phi}^*$ such that

$$\boldsymbol{\phi}^* \in \boldsymbol{W}^{1,2}(\Omega^*), \quad \boldsymbol{\phi}^*|_\Gamma = 0, \quad \boldsymbol{\phi}^*|_{\Gamma^*} = \boldsymbol{u}_\infty, \quad \operatorname{div} \boldsymbol{\phi}^* = 0 \text{ in } \Omega^*. \tag{2.11}$$

Now we define $\boldsymbol{\phi}_\infty : \overline{\Omega} \to I\!R^N$:

$$\boldsymbol{\phi}_\infty = \begin{cases} 0 & \text{in } \overline{\Omega}^-, \\ \boldsymbol{\phi}^* & \text{in } \Omega^*, \\ \boldsymbol{u}_\infty & \text{in } \Omega^+ - \Omega^*. \end{cases} \tag{2.12}$$

Obviously, $\boldsymbol{\phi}_\infty \in \boldsymbol{W}^{1,2}_{\mathrm{loc}}(\Omega)$ and $\operatorname{div} \boldsymbol{\phi}_\infty = 0$ a. e. in Ω.

Let us assume that $\boldsymbol{u}^\pm$, $p^\pm$ form a classical solution of the coupled problem (1.2). Let $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega)$. Multiplying equation (1.2b) by $\boldsymbol{v}|_{\Omega^-}$ and (1.2e) by $\boldsymbol{v}|_{\Omega^+}$, integrating over Ω^- and Ω^+, respectively, summing these integrals, applying Green's theorem and using the fact that $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v}|_{\Gamma_0} = 0$, we obtain the identity

$$\begin{aligned} \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x &= \int_{\Omega^-} (-\nu \, \Delta \, \boldsymbol{u}^- + (\boldsymbol{u}^- \cdot \nabla) \, \boldsymbol{u}^- + \nabla \, p^-) \cdot \boldsymbol{v} \, \mathrm{d}x \\ &+ \int_{\Omega^+} (-\nu \, \Delta \, \boldsymbol{u}^+ + (\boldsymbol{u}_\infty \cdot \nabla) \, \boldsymbol{u}^+ + \nabla \, p^+) \cdot \boldsymbol{v} \, \mathrm{d}x \\ &= -\int_\Gamma \left(\nu \, \frac{\partial \boldsymbol{u}^-}{\partial \widehat{\boldsymbol{n}}} - p^- \, \widehat{\boldsymbol{n}} \right) \cdot \boldsymbol{v} \, \mathrm{d}S \\ &+ \int_{\Omega^-} \left\{ \nu \sum_{i,j=1}^N \frac{\partial u_i^-}{\partial x_j} \, \frac{\partial v_i}{\partial x_j} + \sum_{i,j=1}^N u_j^- \, \frac{\partial u_i^-}{\partial x_j} \, v_i \right\} \mathrm{d}x \\ &+ \int_\Gamma \left(\nu \frac{\partial \boldsymbol{u}^+}{\partial \widehat{\boldsymbol{n}}} - p^+ \, \widehat{\boldsymbol{n}} \right) \cdot \boldsymbol{v} \, \mathrm{d}S \\ &+ \int_{\Omega^+} \left\{ \nu \sum_{i,j=1}^N \frac{\partial u_i^+}{\partial x_j} \, \frac{\partial v_i}{\partial x_j} + \sum_{i,j=1}^N u_{\infty j} \, \frac{\partial u_i^+}{\partial x_j} \, v_j \right\} \mathrm{d}x. \end{aligned} \tag{2.13}$$

We define $\boldsymbol{u} : \overline{\Omega} \to I\!R^N$

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}^- & \text{in } \overline{\Omega}^-, \\ \boldsymbol{u}^+ & \text{in } \overline{\Omega}^+. \end{cases} \tag{2.14}$$

In view of (1.2h), $\boldsymbol{u}|_\Gamma = \boldsymbol{u}^-|_\Gamma = \boldsymbol{u}^+|_\Gamma$. Hence, $\boldsymbol{u} \in \boldsymbol{W}^{1,2}_{\mathrm{loc}}(\Omega)$. Moreover, $\operatorname{div} \boldsymbol{u} = 0$ a. e. in Ω. Now, using (1.2i), we get

$$\begin{aligned} &\nu \int_{\Omega^-} \sum_{i,j=1}^N \frac{\partial u_i}{\partial x_j} \, \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x + \nu \int_{\Omega^+} \sum_{i,j=1}^N \frac{\partial u_i}{\partial x_j} \, \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x + \int_{\Omega^-} \sum_{i,j=1}^N u_j \, \frac{\partial u_i}{\partial x_j} \, v_i \, \mathrm{d}x \\ &+ \int_{\Omega^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial u_i}{\partial x_j} \, v_i \, \mathrm{d}x - \frac{1}{2} \int_\Gamma [(\boldsymbol{u} - \boldsymbol{u}_\infty) \cdot \widehat{\boldsymbol{n}}] \, [\boldsymbol{u} \cdot \boldsymbol{v}] \, \mathrm{d}S = \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x. \end{aligned} \tag{2.15}$$

Let us introduce the forms

$$
\begin{aligned}
(2.16)\qquad a_0(\boldsymbol{u},\boldsymbol{v}) &= \nu\int_{\Omega^-}\sum_{i,j=1}^{N}\frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x,\\
a_1(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) &= \int_{\Omega^-}\sum_{i,j=1}^{N}u_j\,\frac{\partial w_i}{\partial x_j}\,v_i\,\mathrm{d}x,\\
a_2(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) &= -\frac{1}{2}\int_{\Gamma}[(\boldsymbol{u}-\boldsymbol{u}_\infty)\cdot\hat{\boldsymbol{n}}]\,[\boldsymbol{w}\cdot\boldsymbol{v}]\,\mathrm{d}S,\\
b_0(\boldsymbol{u},\boldsymbol{v}) &= \nu\int_{\Omega^+}\sum_{i,j=1}^{N}\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x,\\
b_1(\boldsymbol{u},\boldsymbol{v}) &= \int_{\Omega^+}\sum_{i,j=1}^{N}u_{\infty j}\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}x,\\
L(\boldsymbol{v}) &= \int_{\Omega^-}\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x,\\
a(\boldsymbol{u},\boldsymbol{v}) &= a_0(\boldsymbol{u},\boldsymbol{v})+a_1(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})+a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})+b_0(\boldsymbol{u},\boldsymbol{v})+b_1(\boldsymbol{u},\boldsymbol{v}),
\end{aligned}
$$

$$\text{for } \boldsymbol{u},\,\boldsymbol{v}:\Omega\to I\!\!R^N,\ \ \boldsymbol{u},\,\boldsymbol{w}\in\boldsymbol{W}^{1,2}_{\mathrm{loc}}(\Omega),\ \ \boldsymbol{v}\in\boldsymbol{C}_0^\infty(\Omega).$$

On the basis of the above considerations we come to the following concept:

Definition 2.1. We call a vector valued function $\boldsymbol{u}:\Omega\to I\!\!R^N$ a **weak solution** of the coupled problem (1.2), if the following conditions are satisfied:

$$(2.17a)\qquad \text{a)}\qquad \boldsymbol{u}-\boldsymbol{\phi}_\infty\in\boldsymbol{V}(\Omega),$$

$$(2.17b)\qquad \text{b)}\qquad a(\boldsymbol{u},\boldsymbol{v})=L(\boldsymbol{v})\quad\forall\,\boldsymbol{v}\in\boldsymbol{\mathcal{V}}(\Omega).$$

Remark 2.1. From (2.13) – (2.16) it follows that the function $\boldsymbol{u}$ defined on the basis of a classical solution $\boldsymbol{u}^\pm$ by (2.14) satisfies identity (2.17b). In (2.17a), conditions (1.2c, d, f, g) are hidden and $\boldsymbol{u}\in\boldsymbol{W}^{1,2}_{\mathrm{loc}}(\Omega)$. Since $\boldsymbol{v}\in\boldsymbol{\mathcal{V}}(\Omega)$ has compact support, all integrals over Ω in (2.16) have sense. Moreover, also the form a_2 is well defined as follows from the trace theorem for functions from $\boldsymbol{W}^{1,2}(\widetilde{\Omega})$, where $\widetilde{\Omega}\subset\Omega$ is a bounded domain with $\Gamma\subset\partial\widetilde{\Omega}$. However, it is not possible to use $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$ as test functions in (2.17b), because the form $b_1(\boldsymbol{u},\boldsymbol{v})$ is not defined for $\boldsymbol{u}\in\boldsymbol{W}^{1,2}_{\mathrm{loc}}(\Omega)$ and $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$ in general (cf. [9]). This is the reason that we cannot carry out the existence proof as in [8]. We will develop a completely different approach for proving the existence of a solution of problem (2.17a -b). In fact, this new technique can also be applied to the coupling of the interior Navier–Stokes problem with the exterior Stokes problem from [8].

Remark 2.2. On the basis of results from [9], Chap. VII, the weak solution $\boldsymbol{u}$ of problem (2.17a-b) can be associated with the pressure $p\in L^2_{\mathrm{loc}}(\Omega)$ such that

$$(2.18)\qquad a(\boldsymbol{u},\boldsymbol{v})-(p,\mathrm{div}\boldsymbol{v})=L(\boldsymbol{v})\quad\forall\boldsymbol{v}\in\boldsymbol{C}_0^\infty(\Omega).$$

3 Existence of a weak solution

First we prove some important properties of the forms a_0, a_1, a_2 defined in (2.16). These forms have sense, of course, also for functions from the space $\boldsymbol{W}^{1,2}(\Omega^-)$, as follows from (2.16) and the continuous imbedding $\boldsymbol{W}^{1,2}(\Omega^-) \hookrightarrow \boldsymbol{L}^4(\Omega^-)$ and the continuity of the trace operator from the space $\boldsymbol{W}^{1,2}(\Omega^-)$ into $\boldsymbol{L}^3(\Gamma)$. (We simply write $\boldsymbol{W}^{1,2}(\Omega^-) \hookrightarrow \boldsymbol{L}^3(\Gamma)$.)

Lemma 3.1. *a_0 is a continuous bilinear form on $\boldsymbol{W}^{1,2}(\Omega^-)$. Further, a_1 and a_2 are continuous trilinear forms on $\boldsymbol{W}^{1,2}(\Omega^-)$.* $\square$

Let us set

$$\begin{aligned} \boldsymbol{V}_0(\Omega^-) &= \left\{\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega^-); \boldsymbol{v}|_{\Gamma_0} = 0, \operatorname{div} \boldsymbol{v} = 0 \ \text{ a.e. in } \Omega^-\right\}, \\ \boldsymbol{\mathcal{V}}_0(\Omega^-) &= \left\{\boldsymbol{v} \in C^\infty(\overline{\Omega^-}); \operatorname{supp} \boldsymbol{v} \subset \Omega^- \cup \Gamma, \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega^-\right\}, \end{aligned} \tag{3.1}$$

$$\tilde{a}(\boldsymbol{u}, \boldsymbol{v}) = -\frac{1}{2}\int_\Gamma (\boldsymbol{u} \cdot \hat{\boldsymbol{n}})\, |\boldsymbol{v}|^2 \,\mathrm{d}S, \quad \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega^-). \tag{3.2}$$

In virtue of [8], Lemma 2.1,

$$\boldsymbol{V}_0(\Omega^-) = \text{closure of } \boldsymbol{\mathcal{V}}_0(\Omega^-) \text{ in } \boldsymbol{W}^{1,2}(\Omega^-). \tag{3.3}$$

Similarly as in [8], Lemma 4.1, Corollary 4.2 and Lemma 4.3, we can prove the following results.

Lemma 3.2. *For $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{V}_0(\Omega^-)$ we have*

$$a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -a_1(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) - \tilde{a}(\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}) + \tilde{a}(\boldsymbol{u}, \boldsymbol{v}) + \tilde{a}(\boldsymbol{u}, \boldsymbol{w}). \tag{3.4}$$

Proof. In virtue of Lemma 3.1 and (3.3) we can consider $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}_0(\Omega^-)$. Using Green's theorem in Ω^-, we get

$$a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = -\tilde{a}(\boldsymbol{u}, \boldsymbol{v}). \tag{3.5}$$

Now, setting $\boldsymbol{v} := \boldsymbol{v} + \boldsymbol{w}$ and using the trilinearity of the form a_1, we get

$$-\tilde{a}(\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}) = a_1(\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}, \boldsymbol{v} + \boldsymbol{w}) = a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) + a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) + a_1(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) + a_1(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{w}).$$

This and (3.5) yield (3.4). $\square$

Lemma 3.3. *Let us define the form*

$$d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) + a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \quad \boldsymbol{u},\, \boldsymbol{v},\, \boldsymbol{w} \in \boldsymbol{W}^{1,2}(\Omega^-). \tag{3.6}$$

Then it holds: If $\boldsymbol{z}$, $\boldsymbol{v}$, $\boldsymbol{z}_n \in \boldsymbol{V}_0(\Omega^-)$, $n = 1, 2, \dots$, and *if*

$$|\boldsymbol{z}_n|_{\boldsymbol{W}^{1,2}(\Omega^-)} \le C, \quad n = 1, 2, \dots, \tag{3.7a}$$

$$\boldsymbol{z}_n \longrightarrow \boldsymbol{z} \quad \textit{strongly in} \quad \boldsymbol{L}^2(\Omega^-) \tag{3.7b}$$

$$\boldsymbol{z}_n|_\Gamma \longrightarrow \boldsymbol{z}|_\Gamma \quad \textit{strongly in} \quad \boldsymbol{L}^3(\Gamma) \tag{3.7c}$$

$$\textit{as } n \to \infty,$$

then

$$d(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \longrightarrow d(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \quad \textit{as } n \to \infty. \tag{3.8}$$

Proof. Since d is a continuous trilinear form and (3.7a) together with (3.3) hold, we can suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}_0(\Omega^-)$. By Lemma 3.2,

$$d(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) = -a_1(\boldsymbol{z}_n, \boldsymbol{v}, \boldsymbol{z}_n) - \tilde{a}(\boldsymbol{z}_n, \boldsymbol{z}_n + \boldsymbol{v}) + \tilde{a}(\boldsymbol{z}_n, \boldsymbol{z}_n) + \tilde{a}(\boldsymbol{z}_n, \boldsymbol{v}) + a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}).$$

Similar relation holds, if $\boldsymbol{z}_n$ is replaced by $\boldsymbol{z}$. In view of (2.16),

$$|a_1(\boldsymbol{z}_n, \boldsymbol{v}, \boldsymbol{z}_n) - a_1(\boldsymbol{z}, \boldsymbol{v}, \boldsymbol{z})| = \left| \int_{\Omega^-} \sum_{i,j=1}^{N} (z_{nj} z_{ni} - z_j z_i) \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x \right| \leq$$

$$\leq \ c(\boldsymbol{v}) \int_{\Omega^-} \sum_{i,j=1}^{N} |z_{nj} z_{ni} - z_j z_i| \, \mathrm{d}x \longrightarrow 0$$

due to (3.7b). The limit process in the terms with the form $\tilde{a}$ can be easily carried out on the basis of (3.7c) and the Hölder inequality over Γ. (Cf. [8], Lemma 4.3.) $\square$

For any positive integer n we denote by $\mathcal{B}_n$ the ball with radius n and centre at the origin. We will consider $n \geq n_0$ with fixed n_0 such that $\mathcal{B} \subset \mathcal{B}_{n_0}$ ($\subset \mathcal{B}_n$), where $\mathcal{B}$ is the ball used in the definition of the function $\boldsymbol{\phi}_\infty$. Hence, $\partial\mathcal{B}_n \subset \Omega$ and $\boldsymbol{\phi}_\infty|_{\partial\mathcal{B}_n} = \boldsymbol{u}_\infty$ for $n \geq n_0$. We set $\Omega_n = \Omega \cap \mathcal{B}_n$ and $\Omega_n^+ = \Omega^+ \cap \mathcal{B}_n$. Then for $n \geq n_0$, we have $\Omega^- \subset \Omega_n$, $\Omega_n = \Omega^- \cup \Gamma \cup \Omega_n^+$, $\partial\Omega_n = \Gamma_0 \cup \Gamma_n$ and $\partial\Omega_n^+ = \Gamma \cup \Gamma_n$. Moreover, $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n=n_0}^\infty \Omega_n = \Omega$. Γ_n is the exterior component of $\partial\Omega_n$ and $\partial\Omega_n^+$.

For $n \geq n_0$ we define the forms

$$\begin{aligned} b_0^n(\boldsymbol{u}, \boldsymbol{v}) &= \nu \int_{\Omega_n^+} \sum_{i,j=1}^{N} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x, \\ b_1^n(\boldsymbol{u}, \boldsymbol{v}) &= \int_{\Omega_n^+} \sum_{i,j=1}^{N} \phi_{\infty j} \frac{\partial u_i}{\partial x_j} v_i \, \mathrm{d}x, \\ a^n(\boldsymbol{u}, \boldsymbol{v}) &= a_0(\boldsymbol{u}, \boldsymbol{v}) + a_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b_0^n(\boldsymbol{u}, \boldsymbol{v}) + b_1^n(\boldsymbol{u}, \boldsymbol{v}), \\ &\quad \boldsymbol{u}, \, \boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega_n). \end{aligned} \tag{3.9}$$

For every $n \geq n_0$ we introduce the spaces

$$\begin{aligned} \boldsymbol{\mathcal{V}}(\Omega_n) &= \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega_n); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega_n\}, \\ \boldsymbol{V}(\Omega_n) &= \text{closure of } \boldsymbol{\mathcal{V}}(\Omega_n) \text{ in } \boldsymbol{W}^{1,2}(\Omega_n) \\ &= \left\{\boldsymbol{v} \in \boldsymbol{W}_0^{1,2}(\Omega_n); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega_n\right\} \end{aligned} \tag{3.10}$$

and consider the following **auxiliary problem** in Ω_n: Find $\boldsymbol{u}_n : \Omega_n \to I\!R^N$ such that

$$\boldsymbol{u}_n - \boldsymbol{\phi}_\infty|_{\Omega_n} \in \boldsymbol{V}(\Omega_n), \tag{3.11a}$$

$$a^n(\boldsymbol{u}_n, \boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}(\Omega_n) \tag{3.11b}$$

(the form $L(\boldsymbol{v})$ has sense for $\boldsymbol{v} \in \boldsymbol{V}(\Omega_n)$ extended by zero on Ω). Condition (3.11a) implies that $\boldsymbol{u}_n|_{\Gamma_0} = 0$, $\boldsymbol{u}_n|_{\Gamma_n} = \Phi_\infty$ and $\operatorname{div} \boldsymbol{u}_n = 0$ a.e. in Ω_n. Conditions (3.11a–b) represent the weak formulation of a coupled "Navier–Stokes – Oseen" problem in the bounded domain $\Omega_n = \Omega^- \cup \Gamma \cup \Omega_n^+$.

The solution of problem (2.17) can be written in the form

$$\boldsymbol{u} = \boldsymbol{\phi}_\infty + \boldsymbol{z}, \quad \boldsymbol{z} \in \boldsymbol{V}(\Omega). \tag{3.12}$$

Hence, (2.17) is equivalent to finding $\boldsymbol{z} : \Omega \to I\!R^N$ such that

$$\boldsymbol{z} \in \boldsymbol{V}(\Omega), \tag{3.13a}$$

$$a(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega). \tag{3.13b}$$

Similarly we can reformulate problem (3.11): Find $\boldsymbol{z}_n : \Omega_n \to I\!R^N$ such that

$$\boldsymbol{z}_n \in \boldsymbol{V}(\Omega_n), \tag{3.14a}$$

$$a^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}(\Omega_n). \tag{3.14b}$$

Then $\boldsymbol{u}_n = \boldsymbol{\phi}_\infty + \boldsymbol{z}_n$. By (2.12), $\boldsymbol{u}_n = \boldsymbol{z}_n$ in $\overline{\Omega}^-$.

First let us prove the existence and estimates of the solution $\boldsymbol{z}_n$ of problem (3.14). Similarly as in Lemma 3.1, we can establish some properties of the forms a^n.

Lemma 3.4. *Let $n \geq n_0$. Then a_0, b_0^n and b_1^n are continuous bilinear forms on $\boldsymbol{W}^{1,2}(\Omega_n)$. The forms a_1 and a_2 are continuous trilinear forms on $\boldsymbol{W}^{1,2}(\Omega_n)$.* □

Lemma 3.5. *We have*

$$\begin{aligned} & a_1(\boldsymbol{z},\boldsymbol{z},\boldsymbol{z}) + a_2(\boldsymbol{z},\boldsymbol{z},\boldsymbol{z}) + b_1^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{z}) \\ = {} & \int_{\Omega_n^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial \phi_{\infty i}}{\partial x_j} z_i \,\mathrm{d}x \quad \forall\, \boldsymbol{z} \in \boldsymbol{V}(\Omega_n). \end{aligned} \tag{3.15}$$

Proof. In virtue of Lemma 3.4 and the density of the space $\boldsymbol{\mathcal{V}}(\Omega_n)$ in $\boldsymbol{V}(\Omega_n)$, it is sufficient to prove (3.15) for $\boldsymbol{z} \in \boldsymbol{\mathcal{V}}(\Omega_n)$. By (2.16) and (3.9) we have for such a function

$$\begin{aligned} & a_1(\boldsymbol{z},\boldsymbol{z},\boldsymbol{z}) + a_2(\boldsymbol{z},\boldsymbol{z},\boldsymbol{z}) + b_1^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{z}) \\ = {} & \int_{\Omega^-} \sum_{i,j=1}^N z_j \frac{\partial z_i}{\partial x_j} z_i \,\mathrm{d}x - \frac{1}{2}\int_\Gamma [(\boldsymbol{z} - \boldsymbol{u}_\infty)\cdot\widehat{\boldsymbol{n}}]\, |\boldsymbol{z}|^2 \,\mathrm{d}S \\ & + \int_{\Omega_n^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial (z_i + \phi_{\infty i})}{\partial x_j} z_i \,\mathrm{d}x. \end{aligned} \tag{3.16}$$

Since $\boldsymbol{z}|_{\Gamma_0} = 0$ and $\operatorname{div} \boldsymbol{z} = 0$, by Green's theorem we find that

$$\int_{\Omega^-} \sum_{i,j=1}^N z_j \frac{\partial z_i}{\partial x_j} z_i \,\mathrm{d}x = \frac{1}{2}\int_{\Omega^-} \sum_{i,j=1}^N z_j \frac{\partial z_i^2}{\partial x_j} \,\mathrm{d}x = \frac{1}{2}\int_\Gamma (\boldsymbol{z}\cdot\widehat{\boldsymbol{n}})\, |\boldsymbol{z}|^2 \,\mathrm{d}S. \tag{3.17}$$

Similarly, taking into account that $\boldsymbol{z}|_{\Gamma_n} = 0$, $\boldsymbol{\phi}_\infty|_{\Gamma_n} = \boldsymbol{u}_\infty$, $\operatorname{div} \boldsymbol{z} = 0$, $\operatorname{div} \boldsymbol{u}_\infty = 0$ and $\operatorname{div} \boldsymbol{\phi}_\infty = 0$, we find that

$$\begin{aligned} & \int_{\Omega_n^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial (z_i + \phi_{\infty i})}{\partial x_j} z_i \,\mathrm{d}x \\ = {} & \frac{1}{2}\int_{\Omega_n^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial z_i^2}{\partial x_j} \,\mathrm{d}x + \int_{\Omega_n^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial \phi_{\infty i}}{\partial x_j} z_i \,\mathrm{d}x \\ = {} & -\frac{1}{2}\int_\Gamma (\boldsymbol{u}_\infty\cdot\widehat{\boldsymbol{n}})\, |\boldsymbol{z}|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_n} (\boldsymbol{u}_\infty\cdot\boldsymbol{n})\, |\boldsymbol{z}|^2 \,\mathrm{d}S \\ & -\frac{1}{2}\int_{\Omega_n^+} (\operatorname{div} \boldsymbol{u}_\infty)\, |\boldsymbol{z}|^2 \,\mathrm{d}x + \int_{\Omega_n^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial \phi_{\infty i}}{\partial x_j} z_i \,\mathrm{d}x \\ = {} & -\frac{1}{2}\int_\Gamma (\boldsymbol{u}_\infty\cdot\widehat{\boldsymbol{n}})\, |\boldsymbol{z}|^2 + \int_{\Omega_n^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial \phi_{\infty i}}{\partial x_j} z_i \,\mathrm{d}x. \end{aligned} \tag{3.18}$$

Now (3.16) – (3.18) already yield (3.15). □

Lemma 3.6. *There exists a constant $c > 0$ such that*

$$\left| \int_{\Omega_n^+} \sum_{i,j=1}^{N} u_{\infty j} \, \frac{\partial \phi_{\infty i}}{\partial x_j} \, z_i \, \mathrm{d}x \right| \le c |\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega_n)} \tag{3.19}$$

for every $\boldsymbol{z} \in \boldsymbol{V}(\Omega_n)$ and every $n \ge n_0$.

Proof. Taking into account that, by (2.12), $\partial \phi_{\infty i}/\partial x_j = 0$ in $\Omega^+ - \Omega^*$ and that $\Omega^* \subset \Omega^+$, we have

$$\int_{\Omega_n^+} \sum_{i,j=1}^{N} u_{\infty j} \, \frac{\partial \phi_{\infty i}}{\partial x_j} \, z_i \, \mathrm{d}x = \int_{\Omega^*} \sum_{i,j=1}^{N} u_{\infty j} \, \frac{\partial \phi_{\infty i}}{\partial x_j} \, z_i \, \mathrm{d}x.$$

This and the Cauchy inequality imply that

$$\left| \int_{\Omega_n^+} \sum_{i,j=1}^{N} u_{\infty j} \frac{\partial \phi_{\infty i}}{\partial x_j} \, z_i \, \mathrm{d}x \right| \le c(\boldsymbol{u}_\infty) \, |\boldsymbol{\phi}_\infty|_{\boldsymbol{W}^{1,2}(\Omega^*)} \|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega^*)}. \tag{3.20}$$

Furthermore,

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega^*)} \le \|\boldsymbol{z}\|_{\boldsymbol{L}^2((\Omega^*)^-)}, \tag{3.21}$$

where $(\Omega^*)^- = \Omega^* \cup \Gamma \cup \Omega^-$. Since $\Gamma_0 \subset \partial(\Omega^*)^-$ and $\boldsymbol{z}|_{\Gamma_0} = 0$, we can use the Friedrichs inequality ([12]):

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^2((\Omega^*)^-)} \le c^* |\boldsymbol{z}|_{\boldsymbol{W}^{1,2}((\Omega^*)^-)} \tag{3.22}$$

with a constant c^* independent of $\boldsymbol{z}$. Since $(\Omega^*)^- \subset \Omega_n$ for $n \ge n_0$, we have

$$|\boldsymbol{z}|_{\boldsymbol{W}^{1,2}((\Omega^*)^-)} \le |\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega_n)}. \tag{3.23}$$

Now from (3.20) – (3.22) we immediately get (3.19) with a constant c independent of $\boldsymbol{z}$ and $n \ge n_0$. $\square$

Theorem 3.1. *For each $n \ge n_0$ problem (3.14) has at least one solution $\boldsymbol{z}_n$. There exists a constant $K > 0$ independent of n such that*

$$|\boldsymbol{z}_n|_{\boldsymbol{W}^{1,2}(\Omega_n)} \le K, \quad n \ge n_0. \tag{3.24}$$

Proof. We use the Galerkin method in a standard way as, e. g., in [11], Theorem 1.2, page 280, [14], Chap. II, or [6], Par. 8.4.20. Cf. also [8], Section 4. For every n, there exists a sequence $\{\boldsymbol{w}^i\}_{i=1}^\infty \subset \boldsymbol{\mathcal{V}}(\Omega_n)$ of linearly independent elements such that

$$\boldsymbol{V}(\Omega_n) = \text{closure of } \textstyle\bigcup_{k=1}^\infty \boldsymbol{X}_k \text{ in } \boldsymbol{W}^{1,2}(\Omega_n), \tag{3.25}$$

where $\boldsymbol{X}_k$ is the linear space spanned by the set $\{\boldsymbol{w}^1, \ldots, \boldsymbol{w}^k\}$. $\boldsymbol{X}_k$ can be considered as a Hilbert space with the scalar product

$$((\boldsymbol{u}, \boldsymbol{v})) = \int_{\Omega_n} \sum_{i,j=1}^{N} \frac{\partial u_i}{\partial x_j} \, \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x. \tag{3.26}$$

For any $k = 1, 2, \ldots$ we define the Galerkin approximation $\boldsymbol{z}_n^k \in \boldsymbol{X}_k$ satisfying the condition

$$a^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n^k, \boldsymbol{w}^i) = L(\boldsymbol{w}^i), \quad i = 1, \ldots, k. \tag{3.27}$$

By the Riesz representation theorem, for each $\boldsymbol{z} \in \boldsymbol{X}_k$ there exists $\boldsymbol{P}_k(\boldsymbol{z}) \in \boldsymbol{X}_k$ such that

$$((\boldsymbol{P}_k(\boldsymbol{z}), \boldsymbol{v})) = a^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}) - L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}(\Omega_n). \tag{3.28}$$

(Of course, $\boldsymbol{P}_k$ depends on n.) Similarly as in [11, 14, 6, 8], it can be shown that $\boldsymbol{P}_k : \boldsymbol{X}_k \to \boldsymbol{X}_k$ is a continuous mapping. Let us show that it is coercive. For any $\boldsymbol{z} \in \boldsymbol{X}_k$, by (3.9) we have

$$((\boldsymbol{P}_k(\boldsymbol{z}), \boldsymbol{z})) = a_0(\boldsymbol{z}, \boldsymbol{z}) + a_1(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) + b_0^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{z}) + b_1^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{z}) - L(\boldsymbol{z}).$$

From (3.9), the relation $\Omega_n = \Omega^- \cup \Gamma \cup \Omega_n^+$ Lemmas 3.4, 3.5, 3.6, the Cauchy inequality and the fact that $\partial\phi_{\infty i}/\partial x_j = 0$ outside $\Omega^* \subset \Omega_n$ it follows that

$$\begin{aligned} ((P_k(\boldsymbol{z}), \boldsymbol{z})) &= \nu \int_{\Omega_n} \sum_{i,j=1}^N \left| \frac{\partial z_i}{\partial x_j} \right|^2 \mathrm{d}x + \nu \int_{\Omega_n^+} \sum_{i,j=1}^N \frac{\partial \phi_{\infty i}}{\partial x_j} \frac{\partial z_i}{\partial x_j} \mathrm{d}x \\ &+ \int_{\Omega_n^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial \phi_{\infty i}}{\partial x_j} z_i \,\mathrm{d}x - \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{z} \,\mathrm{d}x \\ &\geq \nu |\boldsymbol{z}|^2_{\boldsymbol{W}^{1,2}(\Omega_n)} - c|\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega_n)} \\ &- \nu |\boldsymbol{\phi}_\infty|_{\boldsymbol{W}^{1,2}(\Omega^*)} |\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega^*)} - \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega^-)} \|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega^-)} \end{aligned} \tag{3.29}$$

(the constant c is independent of $\boldsymbol{z}, k$ and n). Since $\boldsymbol{z}|_{\Omega^-} \in \boldsymbol{W}^{1,2}(\Omega^-)$ and $\boldsymbol{z}|_{\Gamma_0} = 0$, in virtue of the Friedrichs inequality and the inclusion $\Omega^- \subset \Omega_n$,

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega^-)} \leq \tilde{c} |\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega^-)} \leq \tilde{c} |\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega_n)}, \tag{3.30}$$

where the constant $\tilde{c}$ is independent of $\boldsymbol{z}$ and n. Now (3.29), (3.30) and the inequality $|\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega^*)} \leq |\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega_n)}$, imply the existence of a constant $c^* > 0$ (independent of $\boldsymbol{z}, k$ and n) such that

$$((P_k(\boldsymbol{z}), \boldsymbol{z})) \geq \nu |\boldsymbol{z}|^2_{\boldsymbol{W}^{1,2}(\Omega_n)} - c^* |\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega_n)}, \quad \boldsymbol{z} \in \boldsymbol{X}_k,\ n \geq n_0. \tag{3.31}$$

Hence, because $\nu > 0$, there exists $K > 0$ (independent of $\boldsymbol{z}, k$ and n) such that $((\boldsymbol{P}_k(\boldsymbol{z}), \boldsymbol{z})) \geq 0$ for all $\boldsymbol{z} \in \boldsymbol{X}_k$ with $|\boldsymbol{z}|_{\boldsymbol{W}^{1,2}(\Omega_n)} = K$. Since $|\cdot|_{\boldsymbol{W}^{1,2}(\Omega_n)}$ is a norm in $\boldsymbol{X}_k$, in virtue of [6], Lemma 4.1.53 or [13], Chap. I, Par. 4.3, Lemma 4.3, there exists a solution

$$\boldsymbol{z}_n^k \in \boldsymbol{X}_k, \quad \text{with} \quad |\boldsymbol{z}_n^k|_{\boldsymbol{W}^{1,2}(\Omega_n)} \leq K \tag{3.32}$$

of the equation $\boldsymbol{P}_k(\boldsymbol{z}_n^k) = 0$, equivalent to (3.27).

Hence, we get a sequence $\{\boldsymbol{z}_n^k\}_{k=1}^\infty$ of solutions of (3.27) bounded in $\boldsymbol{V}(\Omega_n)$. Then there exists a subsequence (for simplicity again denoted by $\{\boldsymbol{z}_n^k\}_{k=1}^\infty$) and a function $\boldsymbol{z}_n \in \boldsymbol{V}(\Omega_n)$ such that

$$\boldsymbol{z}_n^k \longrightarrow \boldsymbol{z}_n \quad \text{weakly in } \boldsymbol{V}(\Omega_n) \text{ as } k \to \infty. \tag{3.33}$$

From the compact imbedding $\boldsymbol{W}^{1,2}(\Omega_n) \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega_n)$ and the compactness of the trace operator from $\boldsymbol{W}^{1,2}(\Omega_n)$ into $\boldsymbol{L}^3(\Gamma)$ (we write $\boldsymbol{W}^{1,2}(\Omega_n) \hookrightarrow\hookrightarrow \boldsymbol{L}^3(\Gamma)$) it follows that

$$\boldsymbol{z}_n^k \longrightarrow \boldsymbol{z}_n \quad \text{strongly in } \boldsymbol{L}^2(\Omega_n), \tag{3.34a}$$

$$\boldsymbol{z}_n^k|_\Gamma \longrightarrow \boldsymbol{z}_n|_\Gamma \quad \text{strongly in } \boldsymbol{L}^3(\Gamma), \tag{3.34b}$$

$$\text{as } k \to \infty.$$

Now we carry out the limit process in (3.27) for $k \to \infty$. In virtue of the bilinearity and continuity of the forms a_0, b_0^n and b_1^n and (3.33), we get

$$\begin{aligned} &a_0(\boldsymbol{z}_n^k, \boldsymbol{w}_i) \longrightarrow a_0(\boldsymbol{z}_n, \boldsymbol{w}_i), \\ &b_\alpha^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n^k, \boldsymbol{w}_i) \longrightarrow b_\alpha^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{w}_i) \\ &\qquad \text{as } k \to \infty,\ i = 1, 2, \dots,\ \alpha = 0, 1. \end{aligned} \tag{3.35}$$

Furthermore, (3.32) and (3.34a) imply that

$$\tag{3.36} \begin{gathered} |\boldsymbol{z}_n^k|_{\boldsymbol{W}^{1,2}(\Omega^-)} \le K, \\ \boldsymbol{z}_n^k \longrightarrow \boldsymbol{z}_n \quad \text{strongly in } \boldsymbol{L}^2(\Omega^-) \text{ as } k \to \infty. \end{gathered}$$

We see from (3.36) and (3.34b) that the sequence $\{\boldsymbol{z}_n^k\}_{k=1}^\infty$ satisfies the assumptions of Lemma 3.3. Hence,

$$\tag{3.37} \begin{gathered} a_1(\boldsymbol{z}_n^k, \boldsymbol{z}_n^k, \boldsymbol{w}_i) + a_2(\boldsymbol{z}_n^k, \boldsymbol{z}_n^k, \boldsymbol{w}_i) \longrightarrow a_1(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{w}_i) + a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{w}_i) \\ \text{as } k \to \infty,\ i = 1, 2, \dots . \end{gathered}$$

From (3.35) and (3.37) we conclude that $\boldsymbol{z}_n$ is a solution of problem (3.14). Moreover, in virtue of (3.32) and (3.33), estimate (3.24) is valid. □

Finally, we come to the main result of this paper.

Theorem 3.2. *There exists at least one solution $\boldsymbol{u}$ of problem* (2.17). *This $\boldsymbol{u}$ is a weak solution of the coupled problem* (1.2).

Proof. As was stated above, problem (2.17) is equivalent to problem (3.13). In order to prove the solvability of problem (3.13), we extend the solution $\boldsymbol{z}_n$ of problem (3.14) ($n \ge n_0$) by zero from the domain Ω_n onto Ω. For simplicity, we will denote this extension again by $\boldsymbol{z}_n$. Hence, we have a sequence $\{\boldsymbol{z}_n\}_{n=n_0}^\infty$ such that

$$\tag{3.38} \begin{gathered} \boldsymbol{z}_n \in \boldsymbol{V}(\Omega), \quad n \ge n_0, \\ |\boldsymbol{z}_n|_{\boldsymbol{W}^1(\Omega)} = |\boldsymbol{z}_n|_{\boldsymbol{W}^{1,2}(\Omega_n)} \le K, \quad n \ge n_0. \end{gathered}$$

Since the space $\boldsymbol{V}(\Omega)$ is reflexive and the sequence $\{\boldsymbol{z}_n\}_{n=n_0}^\infty$ is bounded in $\boldsymbol{V}(\Omega)$, there exists $\boldsymbol{z} \in \boldsymbol{V}(\Omega)$ and a subsequence of $\{\boldsymbol{z}_n\}_{n=n_0}^\infty$ (let us denote it again by $\{\boldsymbol{z}_n\}$) such that

$$\tag{3.39} \boldsymbol{z}_n \longrightarrow \boldsymbol{z} \quad \text{weakly in } \boldsymbol{V}(\Omega) \text{ as } n \to \infty.$$

Our goal is to show that $\boldsymbol{z}$ is a solution of problem (3.13).

Let $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega)$. Then there exists $n^* \ge n_0$ such that $\operatorname{supp} \boldsymbol{v} \subset \Omega_{n^*}$ and, in virtue of (3.14), (2.16) and (3.9) we have $\boldsymbol{v}|_{\Omega_n} \in \boldsymbol{V}(\Omega_n)$ for $n \ge n^*$ and

$$\tag{3.40} a(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) = a^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) = a^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) = L(\boldsymbol{v}), \quad n \ge n^*.$$

Taking into account that $|\boldsymbol{z}_n|_{\boldsymbol{W}^{1,2}(\Omega_{n^*})} \le |\boldsymbol{z}_n|_{\boldsymbol{W}^1(\Omega)}$, from (3.38) we see that the sequence $\{\boldsymbol{z}_n|_{\Omega_{n^*}}\}$ is bounded in $\boldsymbol{W}^{1,2}(\Omega_{n^*})$. Thus, we can suppose that

$$\tag{3.41} \boldsymbol{z}_n|_{\Omega_{n^*}} \longrightarrow \boldsymbol{z}|_{\Omega_{n^*}} \quad \text{weakly in } \boldsymbol{W}^{1,2}(\Omega_{n^*}) \quad \text{as } n \to \infty.$$

This and the compact imbeddings $\boldsymbol{W}^{1,2}(\Omega_{n^*}) \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega_{n^*})$ and $\boldsymbol{W}^{1,2}(\Omega_{n^*}) \hookrightarrow\hookrightarrow \boldsymbol{L}^3(\Gamma)$ imply that

$$\tag{3.42} \begin{gathered} \boldsymbol{z}_n|_{\Omega_{n^*}} \longrightarrow \boldsymbol{z}|_{\Omega_{n^*}} \qquad \text{strongly in } \boldsymbol{L}^2(\Omega_{n^*}), \\ \boldsymbol{z}_n|_\Gamma \longrightarrow \boldsymbol{z}|_\Gamma \quad \text{strongly in } \boldsymbol{L}^3(\Gamma), \\ \text{as } n \to \infty. \end{gathered}$$

Now we are ready to carry out the limit process in (3.40) for $n \to \infty$. Linearity and continuity of the forms $a_0(\boldsymbol{\phi}_\infty + \cdot, \boldsymbol{v}) = a_0(\cdot, \boldsymbol{v})$, $b_0(\boldsymbol{\phi}_\infty + \cdot, \boldsymbol{v})$ and $b_1^{n^*}(\boldsymbol{\phi}_\infty + \cdot, \boldsymbol{v})$ (let us remind that $\boldsymbol{\phi}_\infty = 0$ in Ω^-) imply that

$$
\begin{aligned}
a_0(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) = a_0(\boldsymbol{z}_n, \boldsymbol{v}) &\longrightarrow a_0(\boldsymbol{z}, \boldsymbol{v}) = a_0(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}), \\
b_0^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) &\longrightarrow b_0^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}), \\
b_1^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) &\longrightarrow b_1^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}) \\
&\qquad\qquad \text{as } n \to \infty.
\end{aligned} \tag{3.43}
$$

It remains to prove that

$$
a_1(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) + a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \longrightarrow a_1(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \quad \text{as } n \to \infty. \tag{3.44}
$$

Concluding from (3.38) and (3.42) that the sequence $\{z_n\}_{n=n_0}^{\infty}$ satisfies conditions (3.7a–c), we see that (3.44) is a consequence of Lemma 3.3.

Now, (3.40), (3.43) and (3.44) imply that the function $\boldsymbol{z} \in \boldsymbol{V}(\Omega)$ satisfies the identity

$$
a(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}) = L(\boldsymbol{v}) \quad \text{for all}\, \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega).
$$

This means that $\boldsymbol{z}$ is a solution of problem (3.13) and $\boldsymbol{u} = \boldsymbol{\phi}_\infty + \boldsymbol{z}$ is a solution of problem (2.17), which we wanted to prove. $\square$

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