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Elliptic Isogenies and Slopes

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Abstract

In this thesis we give a detailed analysis of the methode of the slopes introduced by Bost in 1995 in a Bourbaki talk [3]. In particular we write down some proofs that are missing in his paper. In the first part of our dissertation we show how to modify the proof of the Subvariety Theorem by Bost in order to improve the bounds in a quantitative respect and to extend the Theorem to subspaces instead than hyperplanes. Given an abelian variety A defined over a number field K and a non-trivial period γ in a subspace $W \subset T_{A_K}$, the Subvariety Theorem (Theorem 2) shows the existence of an abelian subvariety B of A defined over $\overline{\mathbb{Q}}$, whose degree is bounded in terms of the height of W and of the norm of the period γ.

As a nice application of our Subvariety Theorem we deduce an upper bound for the degree of a minimal elliptic isogeny which improves the result of Masser and Wüstholz [20].

Riassunto

In questa tesi presentiamo una dettagliata analisi del metodo delle pendenze introdotto da Bost in un seminario Bourbaki nel 1995 [3]. In particolare diamo alcune dimostrazioni che non appaiono nell'articolo. Nella prima parte della dissertazione mostriamo come modificare la dimostrazione del Teorema della Sottovarieta' (Theorem 2) data da Bost, al fine di ottenere un miglioramento dei limiti ed estendiamo il risultato a sottospazi anziché considerare solamente iperpiani. Data una varietá abeliana A definita su un campo di numeri K e un periodo non nullo γ appartenente a un sottospazio $W \subset T_{A_K}$, il Teorema assicura l'esistenza di una sottovarietá abeliana B di A definita su $\overline{\mathbb{Q}}$, il cui grado e' limitato in funzione dell'altezza di W e della norma del periodo γ.

Come interessante applicazione del Teorema della Sottovarietá deduciamo un limite superiore per il grado di una isogenia minimale tra curve ellittiche che migliora il risultato ottenuto da Masser e Wüstholz [20].

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Introduction

In 1990 D. Masser and G. Wüstholz started a series of papers on period relations for abelian varieties [20]-[26]. As an application they obtained a new proof of the Tate Conjecture, which is very different from the proof originally given by G. Faltings in 1983. The Tate Conjecture is a crucial step in the proof of Faltings' theorem on the Mordell Conjecture. In Masser's and Wüstholz' work one of the central results states that, given an abelian variety A defined over a number field K, there exists only a finite number of K-isomorphism classes of abelian varieties defined over K which are isogenous to A. Their approach consists in giving a bound for the degree of a minimal abelian subvariety B of A whose tangent space at the origin contains a given period of the lattice associated to A. The proof is a completely effective version of the Analytic Subgroup Theorem by G. Wüstholz [39] in the special case where the group is an abelian variety and where torsion points are considered. No other known method gives such precise quantitative results.

As a test case, they started their research with elliptic curves rather than with abelian varieties. In [20] they showed that, given two isogenous elliptic curves E and E^* defined over a number field K, there exists an isogeny from E to E^* with degree bounded by $c(d) \cdot \max(1, h(E))^4$. Here $c(d)$ is a constant depending only on the degree d of the field K and $h(E)$ is the height of the Weierstrass equation defining the elliptic curve. The elliptic case turned out to be a model for the general case of an arbitrary abelian variety.

In 1995 J.-B. Bost [3] gave a Bourbaki talk on the work of Masser and Wüstholz. One of the interesting aspects of his approach is the intrinsic and geometric version of the argument originally given by Masser and Wüstholz. Several new tools were introduced. For example, the use of Arakelov geometry which had meanwhile become available. Arithmetic intersection theory allows, among others, to define the height of an algebraic variety in general. This height has nice functorial properties.
Other geometric ingredients are hermitian vector bundles on the spectrum of the ring of integers of a number field and the related concept of Arakelov degree and slopes. An interesting aspect in his work is the use of semistability in transcendence. This approach avoids theta functions, the study of the moduli space of polarized abelian varieties and the construction of auxiliary functions as they appear in the work of Masser and Wüstholz. As a consequence, proofs and effective calculations are more direct.

In this thesis we give a detailed analysis of Bost's approach and in particular we write down some proofs that are missing in his paper. In the first part of our dissertation we show how to modify the proof of the Subvariety Theorem by Bost in order to improve the bounds in a quantitative respect and to extend the theorem to subspaces instead than hyperplanes. Given an abelian variety A defined over a number field K and a non-trivial period γ in a subspace $W \subset T_{A_K}$, the Subvariety Theorem (Theorem 2) shows the existence of an abelian subvariety B of A, whose degree is bounded in terms of the height of W and of the norm of the period γ. Our result gives a bound which is linear in the height of W and polynomial of degree equal to the dimension of the subvariety B in the norm of the period γ. In [3] the bound is polynomial of degree equal to the dimension of A minus one in both variables.

As a nice application of our Subvariety Theorem we deduce, in §6, an upper bound for the degree of a minimal elliptic isogeny which improves the result of Maser and Wüstholz. Moreover we make the constant effective in the degree d of the field of definition of the elliptic curves E and E^*. We need some geometric modifications of their method in order to improve the bound to $c \cdot d^2 \max(1, h(E), \log d)^2$ for elliptic curves with complex multiplication and to $c \cdot d^2 \max(1, h(E), \log d)^3$ for elliptic curves without complex multiplication. Here $h(E)$ is the Faltings height of the curve E. We want to emphasize that the version of the Subvariety Theorem given by Bost, does not imply directly the result of Masser and Wüstholz in the special case of elliptic curves. Instead, it implies the existence of an isogeny with degree bounded by $c \cdot d^8 \cdot \max(1, h(A), \log d)^8$.

This simplest case has been a test for the more general case of an abelian variety. The modern techniques used here can possibly also be used to improve the result of Masser and Wüstholz for abelian varieties in a quantitative respect. However further technical difficulties are expected. For instance the many different types of complex multiplication for an abelian variety of dimension larger than 2, or the bigger range for dimensions of a proper abelian subvariety may cause problems.

A very ambitious conjecture is that the degree of the isogeny, at least in the case of elliptic curves, does not depend on the elliptic curve at all, but just on its field of definition. This would imply for instance that an elliptic curve defined over a number field K has only finitely many subgroups defined over K. This result was proven by Mazur in the case that the field of definition is the field of rational numbers, and was later generalized by Merel for number fields. However, how to extend the result to an arbitrary abelian variety seems to be unknown. We are convinced that further ingenious ideas are needed to prove this conjecture.

We shall now give some more details on the structure of this dissertation. The first two chapters are dedicated to Arakelov geometry, we introduce the degree and the slope of hermitian vector bundles on the spectrum of the ring of integers of a number field. We then explain how the degree behaves with respect to operations on hermitian vector bundles, like direct sum, tensor product, symmetric and exterior power. We determine the relation between the degree of a bundle and the degree of its image under a morphism. An important tool will be the slope inequality (7). This inequality relates the degree of a hermitian bundle to the slopes of a filtration of its image under an injective morphism. It will play a fundamental role in the proof of the Subvariety Theorem.
In the third chapter we recall the basic notions related to abelian varieties and we define their Moret-Bailly models.
The fourth chapter is dedicated to some properties of non-reduced subschemes of arithmetic varieties. We also define a filtration of sheaves associated to such a non-reduced scheme.
In chapter 5 we give the proof of the Subvariety Theorem. First we shall deal with an analytic problem, we have to bound the norm of operators associated to the filtration. In lemmas 2 and 5 we estimate the norm of the derivative of a trivialization of a section of a line bundle in some torsion points. As expected by the Cauchy inequality, the bound is given in terms of the norm of the section on a neighbourhood. The proofs are not difficult but involve tedious computations.
To prove lemma 7 we apply the Phragmen - Lindelöf Theorem to a certain entire periodic function. We get an estimate finer than the ones above. We consider a section s of a line bundle, with a zero of multiplicity $2gM$ at the origin. We proof that the norm of a trivialization f of s, as well as the norm of its derivatives up to order gM, are "very small" at a torsion sub-scheme. This last estimate plays a central role in the whole game. A good choice of the parameters combined with the slope inequality (7) and the above estimates, show that there exists a section of a line bundle which vanishes at a non reduced torsion sub-scheme of A. Our Subvariety Theorem is then a consequence of the Multiplicity Estimate Theorem 4.
In chapter 6 we give all details for estimating the variables appearing in the Subvariety Theorem, in the special case of a product of elliptic curves. Finally we show how to use these tools to improve the bound given by Masser and Wüstholz in [20].

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1 Metrics on Vector bundles

1.1 Notation

We want to fix notations about base change operations. Let A be a commutative ring, with a homomorphism to a field K. Let E be an A-module and B an A-algebra. We denote by E_K and B_K the tensor product $E \otimes_A K$ and $B \otimes_A K$ respectively.
If X is a scheme over $\operatorname{Spec} A$, we denote by X_K the fiber product of X and $\operatorname{Spec} K$ over $\operatorname{Spec} A$.
If $\mathcal{E}$ is a sheaf of $\mathcal{O}_X$-modules on X we denote by $\mathcal{E}_K$ the sheaf on X_K given by pulling-back $p_1^*\mathcal{E} = \mathcal{E}_K$, here p_1 is the canonical projection of the fiber product X_K on X.
We will not deal with the general situation where A is any ring, indeed we will only consider the ring of integers of a number field K. If L is a field extension of K then we will denote by X_K an algebraic variety defined over K and by X_L the algebraic variety got by base change. If $\sigma : K \to \mathbb{C}$ is an embedding we will write X_σ and $\mathcal{E}_\sigma$ instead of $X_{\mathbb{C}}$ and $\mathcal{E}_{\mathbb{C}}$.

1.2 Hermitian Vector Bundles

Definition 1 *Let X be a complex variety and $\mathcal{E}$ a holomorphic vector bundle on X. A hermitian metric h on $\mathcal{E}$ is a hermitian inner product on each fiber $\mathcal{E}_z$ of $\mathcal{E}$, varying smoothly with $z \in X$, that is such that the functions locally representing h are $\mathcal{C}^\infty$.*

The real part of a hermitian inner product gives a Riemannian metric called the induced Riemannian metric. When we speak of distance, area or volume on a complex manifold with hermitian metric, we always refer to the induced Riemannian metric.
We remark that to construct hermitian inner products it is enough to define them locally and then to glue the local definitions using a smooth partition of unity ([38] thm 1.11).

Definition 2 *A hermitian vector bundle $\overline{\mathcal{E}}$ on X is a pair $(\mathcal{E}, h)$, where $\mathcal{E}$ is a locally free sheaf of finite rank on X and h is a hermitian metric on $\mathcal{E}$.*

We denote by $\mathcal{E}^c$ the complex conjugate vector bundle of $\mathcal{E}$ whose $\mathbb{C}$-structure is given by the one of $\mathcal{E}$ composed with the complex conjugation $c : \mathbb{C} \to \mathbb{C}$.
The dual vector bundle $\mathcal{E}^\vee$ is the bundle of homorphisms from $\mathcal{E}$ to the trivial bundle.
We denote by $\mathcal{A}^0(X, \mathcal{E})$ the space of the smooth global sections of $\mathcal{E}$.

Remark :
the metric h is an element of $\mathcal{A}^0(X, \mathcal{E}^\vee \otimes \mathcal{E}^{c\vee})$. Indeed h is a sesquilinear form on each fiber, so we can see it as a linear form from the tensor product $\mathcal{E}_z \otimes {\mathcal{E}^c}_z$ to $\mathbb{C}$. That gives an element of the dual space $(\mathcal{E}_z \otimes \mathcal{E}^c_z)^\vee$.
The fiber $\mathcal{E}_z$ is of finite dimension, therefore $(\mathcal{E}_z \otimes \mathcal{E}^c_z)^\vee$ is equal to $\mathcal{E}^\vee_z \otimes {\mathcal{E}^{c\vee}}_z$.
By definition h varies smoothly with $z \in X$ so h is a global smooth section of $\mathcal{E}^\vee \otimes \mathcal{E}^{c\vee}$, (see [13] p. 27).
One says that an element of $h \in \mathcal{A}^0(X, \mathcal{E}^\vee \otimes \mathcal{E}^{c\vee})$ is positive if the induced quadratic form h_z is positive definite for all z.
We are going to define, in a canonical way, dual, direct sum, tensor product, n-th exterior power, n-th symmetric product and pull-back of a hermitian vector bundle $\overline{\mathcal{E}}$.

1.2.1 The Dual $\overline{\mathcal{E}}^\vee$

Let $\overline{\mathcal{E}} = (\mathcal{E}, h)$ be a hermitian vector bundle, we want to define the associated dual hermitian vector bundle $\overline{\mathcal{E}}^\vee = (\mathcal{E}^\vee, h^\vee)$ where $h^\vee$ is a hermitian metric on $\mathcal{E}^\vee$ induced canonically by h and h'. That is to define a positive element $h^\vee \in \mathcal{A}^0(X, \mathcal{E} \otimes \mathcal{E}^c)$.
The metric h induces the isomorphism $\Phi_z : \mathcal{E}^c_z \longrightarrow \mathcal{E}^\vee_z$ given by

$$\begin{array}{rcccl} \Phi_z & : & {\mathcal{E}^c}_z & \longrightarrow & \mathcal{E}^\vee_z \\ & & a & \longmapsto & h_z(\cdot, a) \end{array}$$

where

$$\begin{array}{rcccl} h_z(\cdot, a) & : & \mathcal{E}_z & \longrightarrow & \mathbb{C} \\ & & b & \longmapsto & h_z(b, a). \end{array}$$

We now define

$$\begin{array}{rcccl} h^\vee_z & : & \mathcal{E}^\vee_z \times {\mathcal{E}^{c\vee}}_z & \longrightarrow & \mathbb{C} \\ & & (v, v') & \longmapsto & h_z(\Phi_z^{-1}(v'), \Phi_z^{-1}(v)). \end{array} \tag{1}$$

All the previous maps are $\mathcal{C}^\infty$ thus $h^\vee$ is an element of $\mathcal{A}^0(X, \mathcal{E} \otimes \mathcal{E}^c)$.
Let us fix a point $z \in X$, if we choose an orthogonal basis of $\mathcal{E}_z$ we deduce from (1) that the dual basis is orthogonal in $\mathcal{E}^\vee_z$. This shows that $h^\vee$ is positive.

1.2.2 The Direct Sum $\overline{\mathcal{E}} \oplus \overline{\mathcal{E}}'$

Let $\overline{\mathcal{E}} = (\mathcal{E}, h)$ and $\overline{\mathcal{E}}' = (\mathcal{E}', h')$ be hermitian vector bundles on X. We shall define a hermitian metric $h \oplus h'$ on $\mathcal{E} \oplus \mathcal{E}'$ induced canonically by h, i.e. we

want to define a positive element $h \oplus h' \in \mathcal{A}^0(X, (\mathcal{E} \oplus \mathcal{E}')^\vee \otimes (\mathcal{E} \oplus \mathcal{E}')^{c\vee})$.
Since

$$(\mathcal{E} \oplus \mathcal{E}')^\vee \otimes (\mathcal{E} \oplus \mathcal{E}')^{c\vee} = (\mathcal{E}^\vee \oplus \mathcal{E}'^\vee) \otimes (\mathcal{E}^{c\vee} \oplus \mathcal{E}'^{c\vee}) =$$
$$(\mathcal{E}^\vee \otimes \mathcal{E}^{c\vee}) \oplus (\mathcal{E}^\vee \otimes \mathcal{E}'^{c\vee}) \oplus (\mathcal{E}'^\vee \otimes \mathcal{E}^{c\vee}) \oplus (\mathcal{E}'^\vee \otimes \mathcal{E}'^{c\vee}),$$

it follows that $(\mathcal{E}^\vee \otimes \mathcal{E}^{c\vee})$ and $(\mathcal{E}'^\vee \otimes \mathcal{E}'^{c\vee})$ are canonically embedded in $(\mathcal{E} \oplus \mathcal{E}')^\vee \otimes (\mathcal{E} \oplus \mathcal{E}')^{c\vee}$, as well as their smooth global sections

$$\mathcal{A}^0(X, \mathcal{E}^\vee \otimes \mathcal{E}^{c\vee}) \oplus \mathcal{A}^0(X, \mathcal{E}'^\vee \otimes \mathcal{E}'^{c\vee}) \to \mathcal{A}^0(X, (\mathcal{E} \oplus \mathcal{E}')^\vee \otimes (\mathcal{E} \oplus \mathcal{E}')^{c\vee}).$$

By abuse of notation we still call h and h' the image of h and h' under this embedding and set $h \oplus h' := h + h'$. Of course $h \oplus h'$ is positive.
We remark that:

$$\begin{array}{rcccl} (h \oplus h')_z & : & (\mathcal{E} \oplus \mathcal{E}')_z \times (\mathcal{E} \oplus \mathcal{E}')_z & \longrightarrow & \mathbb{C} \\ & & (v, v', w, w') & \longmapsto & h_z(v, w) + h'_z(v', w'). \end{array}$$

1.2.3 The Tensor Product $\overline{\mathcal{E}} \otimes \overline{\mathcal{E}}'$

Let $\overline{\mathcal{E}}$ and $\overline{\mathcal{E}}'$ be as before, we want to define a hermitian metric on $\mathcal{E} \otimes \mathcal{E}'$, i.e. a positive element in $\mathcal{A}^0(X, (\mathcal{E} \otimes \mathcal{E}')^\vee \otimes (\mathcal{E} \otimes \mathcal{E}')^{c\vee})$ canonically induced by h and h'.
Let us consider the natural embedding

$$\Phi : \mathcal{A}^0(X, \mathcal{E}^\vee \otimes \mathcal{E}^{c\vee}) \otimes \mathcal{A}^0(X, \mathcal{E}'^\vee \otimes \mathcal{E}'^{c\vee}) \longrightarrow \mathcal{A}^0(X, (\mathcal{E} \otimes \mathcal{E}')^\vee \otimes (\mathcal{E} \otimes \mathcal{E}')^{c\vee}).$$

The image of $h \otimes h'$ under the map Φ is the section we are looking for, by abuse of language we still call it $h \otimes h'$.
Let us fix a point $z \in X$. We give $(h \otimes h')_z$ explicitly as follows

$$\begin{array}{rcccl} (h \otimes h')_z & : & (\mathcal{E} \otimes \mathcal{E}')_z \times (\mathcal{E} \otimes \mathcal{E}')_z & \longrightarrow & \mathbb{C} \\ & & (\sum_i e_i \otimes e_i', \sum_j f_j \otimes f_j') & \longmapsto & \sum_{i,j} h_z(e_i, f_j) \cdot h'_z(e_i', f_j'). \end{array}$$

The map is bilinear and well defined. From this expression follows the positivity of $h \otimes h'$.
Remark:
a special case of the tensor product is the k-th power of a hermitian vector bundle $\overline{\mathcal{E}}^{\otimes k}$. The k-symmetric group $\mathfrak{S}_k$ acts on this bundle. From the explicit expression of the inner product it follows that $h^{\otimes k}$ is invariant under this action. We will use this remark in section (1.2.7).

1.2.4 Exact Sequences

Let $\overline{\mathcal{E}} = (\mathcal{E}, h)$ be a hermitian vector bundle. Given an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{E}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{E}'' \longrightarrow 0$$

we want to induce canonically hermitian inner products h' on $\mathcal{E}'$ and h'' on $\mathcal{E}''$. The exact sequence above induces the injective map

$$\alpha \otimes \alpha^c : \mathcal{E}' \otimes \mathcal{E}'^c \to \mathcal{E} \otimes \mathcal{E}^c \tag{2}$$

and the surjective map

$$\beta \otimes \beta^c : \mathcal{E} \otimes \mathcal{E}^c \to \mathcal{E}'' \otimes \mathcal{E}''^c. \tag{3}$$

Dualizing (2) we get a surjective map

$$(\alpha \otimes \alpha)^{c\vee} : (\mathcal{E} \otimes \mathcal{E}^c)^\vee \to (\mathcal{E}' \otimes \mathcal{E}')^{c\vee}.$$

We define h' to be the image of h under the map $(\alpha \otimes \alpha)^{c\vee}$.
We know from section 1.2.1 how to construct canonically the positive element $h^\vee$, which is a smooth global section of $\mathcal{E} \otimes \mathcal{E}^c$.
The image of $h^\vee$ under the map $\beta \otimes \beta$ is an element $h''^\vee \in \mathcal{A}^0(X, (\mathcal{E}'' \otimes \mathcal{E}''^c))$ whose dual $h'' := (h''^\vee)^\vee \in \mathcal{A}^0(X, (\mathcal{E}'' \otimes \mathcal{E}'')^{c\vee})$ defines the quotient hermitian inner product on $\mathcal{E}''$.
It turns out that h' is the natural restriction norm, and h'' is the restriction norm on the orthogonal complement of $\mathcal{E}'$ which is canonically isomorphic to $\mathcal{E}/\mathcal{E}'$. From this the positivity of h' and h'' follows .

1.2.5 The Pull-back $f^*\overline{\mathcal{E}}$

Let $f : Y \to X$ be a morphism of complex manifold and $\overline{\mathcal{E}}$ a hermitian vector bundle on X. We are going to define a hermitian metric f^*h on the sheaf $f^*\mathcal{E}$ on Y canonically induced by h.
We remark that

$$(f^*\mathcal{E} \otimes f^*\mathcal{E}^c)^\vee = (f^*\mathcal{E})^\vee \otimes (f^*\mathcal{E})^{c\vee} = f^*\mathcal{E}^\vee \otimes f^*\mathcal{E}^{c\vee} = f^*(\mathcal{E}^\vee \otimes \mathcal{E}^{c\vee}).$$

From the definition of f^*, the element f^*h is positive and so defines a hermitian inner product on $f^*\mathcal{E}$.

1.2.6 The k-th Exterior Product $\wedge^k \overline{\mathcal{E}}$

We want to define a hermitian metric $\wedge^k h$ on $\wedge^k \mathcal{E}$, canonically induced by h. The k-th exterior product $\wedge^k \overline{\mathcal{E}}$ is a quotient bundle of tensor product $\mathcal{E}^{\otimes k}$. On $\mathcal{E}^{\otimes k}$ we induce the inner product $h^{\otimes k}$ (see 1.2.3). We define $\wedge^k h$ as the quotient hermitian product (see 1.2.4) induced by $h^{\otimes k}$, and $\wedge^k h$ is clearly positive.
If we fix a point $z \in X$ and compute $(\wedge^k h)_z$ explicitly, we get

$$\begin{array}{rrcl}(\wedge^k h)_z : & \wedge^k \mathcal{E}_z \times \wedge^k \mathcal{E}_z & \longrightarrow & \mathbb{C} \\ & (v_1 \wedge ..... \wedge v_k, w_1 \wedge ..... \wedge w_k) & \longmapsto & \det(h_z(v_i, w_j)_{i,j}.\end{array}$$

1.2.7 The k-th Symmetric Product $\mathrm{Sym}^k \overline{\mathcal{E}}$

We want to define a hermitian metric $\mathrm{Sym}^k h$ of the vector bundle $\mathrm{Sym}^k \mathcal{E}$, that means a positive element of

$$\mathcal{A}^0(X, (\mathrm{Sym}^k \mathcal{E} \otimes \mathrm{Sym}^k \mathcal{E}^c)^\vee) = \mathcal{A}^0(X, \mathrm{Sym}^k \mathcal{E}^\vee \otimes \mathrm{Sym}^k \mathcal{E}^{c\vee}).$$

Let $\Gamma^k \mathcal{E}$ be the sub-vector bundle of $\mathcal{E}^{\otimes k}$ fixed under the action of the k-symmetric group $\mathfrak{S}_k$. We are going to show that the bundle $\mathrm{Sym}^k \mathcal{E}$ is isomorphic to $\Gamma^k \mathcal{E}$.
We consider the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}^{\otimes k} \xrightarrow{S} \Gamma^k \mathcal{E} \longrightarrow 0$$

where S is the projector $S(t) := \frac{1}{|\mathfrak{S}_k|} \sum_{\eta \in \mathfrak{S}_k} \eta(t)$. Then the kernel $\mathcal{K}$ can be identified with $(S - \mathrm{Id})\mathcal{E}^{\otimes k}$.
Let us consider the exact sequence

$$0 \longrightarrow \mathcal{R}^k \longrightarrow \mathcal{E}^{\otimes k} \xrightarrow{S} \mathrm{Sym}^k \mathcal{E} \longrightarrow 0$$

that defines $\mathrm{Sym}^k \mathcal{E}$.
The projector S is trivial on $\mathcal{R}^k$. In fact an element of $\mathcal{K}$ is of the form $u := \frac{1}{k!} \sum_{\eta \in \mathfrak{S}_k} \eta(t) - t$. By definition of $\mathcal{R}^k$ we have that $\pi(\eta(t)) = \pi(t)$ thus $\pi(u) = 0$. The projection π is trivial on $\mathcal{K}$. In fact a generator of $\mathcal{R}^k$ is of the form $\eta(t) - \tau(t)$ with $\eta, \tau \in \mathfrak{S}_k$. Thus $S(\eta(t) - \tau(t)) = 0$.
This implies that the map $S : \mathrm{Sym}^k \mathcal{E} \longrightarrow \Gamma^k \mathcal{E}$ given by $S(\overline{t}) := \frac{1}{|\mathfrak{S}_k|} \sum_{\eta \in \mathfrak{S}_k} \eta(t)$, where t is any representative of $\overline{t}$, is an isomorphism. It follows that the quotient hermitian product induced by $h^{\otimes k}$ via π on $\mathrm{Sym}^k \mathcal{E}$ coincides with the bull-back metric $S^* h_{\Gamma^k}$, where h_{Γ^k} is the restriction of $h^{\otimes k}$ to $\Gamma^k \mathcal{E}$. Hence we set $\mathrm{Sym}^k h := S^* h_{\Gamma^k}$.

We fix $z \in X$ and we compute $||e_I||$, where we use the following notations: $\{e_1, \cdots, e_n\}$ is an orthogonal basis of $\mathcal{E}_z$; $e_I = e_1^{i_1} \otimes \cdots \otimes e_n^{i_n}$ is an element of $\mathcal{E}_z^k$ and $I = (i_1, \cdots, i_n)$; moreover $|I| = \sum i_k$ and $I! = \prod_{j=1}^n i_j!$. So we have

$$||\overline{e_I}||^2_{\mathrm{Sym}^k} = \frac{1}{|\mathfrak{S}_k|^2} || \sum_{\eta \in \mathfrak{S}_k} \eta(e_I)||^2_{\Gamma^k} = \frac{1}{(k!)^2} \sum_{\eta,\tau} h_z(\eta(e_I), \tau(e_I)).$$

Note that

$$h_z(\eta(e_I), \tau(e_I)) = \begin{cases} 0 & : \quad \tau(e_I) \neq \eta(e_I) \\ 1 & : \quad \tau(e_I) = \eta(e_I) \end{cases}$$

If we set $F = \{(\tau, \eta) : (\tau(e_I) = \eta(e_I))\}$, it follows

$$||\overline{e_I}||^2_{\mathrm{Sym}^k} = \frac{1}{(k!)^2} \sum_{(\eta,\tau) \in F} 1 = \frac{1}{(k!)^2} \sharp F = \frac{1}{(k!)^2} k! I! = \frac{I!}{k!}. \qquad (4)$$

This shows that $\mathrm{Sym}^k h$ is positive.

1.3 Hermitian Vector Bundles on Arithmetic Varieties

Our next aim is to remove the hypothesis that X is a complex manifold and to extend the definition of paragraph 1.2 to an arithmetic variety.
We donote by $\mathcal{O}_K$ the ring of integers of a number field K.

Definition 3 *An arithmetic variety $\mathcal{X}$ over $\mathcal{O}_K$ is a scheme over* $\mathrm{Spec}\,\mathcal{O}_K$ *s.t.* $\pi : \mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_K$ *is a quasi-projective flat morphism of schemes. Moreover we require that there exists a section $\epsilon : \mathrm{Spec}\,\mathcal{O}_K \to \mathcal{X}$ and that the generic fibre is smooth and proper.*

The fiber product $\mathcal{X}_{\mathbb{C}} = \mathcal{X} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{C}$ is well defined.
The set of complex points $\mathcal{X}(\mathbb{C}) := \{\mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}, \mathcal{X})\}$ is the disjoint union of complex varieties $\mathcal{X}(\mathbb{C}) = \coprod_{\sigma : K \to \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$. In fact if $p : \mathrm{Spec}\,\mathbb{C} \to \mathcal{X}_K$ is a complex point, then the composition $\pi_K \circ p : \mathrm{Spec}\,\mathbb{C} \to \mathrm{Spec}\,K$ induces an embedding $\sigma := (\pi p)^\sharp$ from K to $\mathbb{C}$.

Definition 4 *Let $\mathcal{X}$ be an arithmetic variety over* $\mathrm{Spec}\,\mathcal{O}_K$. *A hermitian vector bundle $\overline{\mathcal{E}}$ on $\mathcal{X}$ is a pair $(\mathcal{E}, h)$ where $\mathcal{E}$ is a locally free sheaf of finite rank on $\mathcal{X}$, and $(\mathcal{E}(\mathbb{C}), h)$ is a hermitian vector bundle on $\mathcal{X}(\mathbb{C})$ invariant under conjugation. Here $\mathcal{E}(\mathbb{C})$ is the sheaf induced by $\mathcal{E}$ on $\mathcal{X}(\mathbb{C})$, as specified in 1.1. If $\mathcal{E}$ has rank 1 one says that $\overline{\mathcal{E}}$ is a hermitian line bundle.*

Notice that the hermitian metric is given just on the holomorphic vector bundle on the complex variety $\mathcal{X}(\mathbb{C})$.
Invariant under conjugation means that if σ and $\overline{\sigma}$ are conjugated embeddings of K, then for every open set U of $\mathcal{X}$ the map $id \otimes c : \mathcal{E}(U) \otimes_\sigma \mathbb{C} \to \mathcal{E}(U) \otimes_{\overline{\sigma}} \mathbb{C}$ is an isometry.
In order to apply the work done in the paragraph (1.2) to arithmetic varieties we have to check that the:

$$\begin{array}{rl} i) & \textit{dual}, \\ ii) & \textit{direct sum}, \\ ii) & \textit{tensor product}, \\ iv) & \textit{exterior power}, \\ v) & \textit{symmetric product}, \\ vi) & \textit{pull} - \textit{back} \end{array}$$

of locally free sheaves is still locally free. This follows from the general theory of coherent sheaves on an algebraic variety. (see [14] chap. II, prop.5.5, 5.7, ex. 5.1,5.16).
Finally we remark that given two hermitian vector bundles $\mathcal{E}$, $\mathcal{E}'$ on $\mathcal{X}$, the following relations hold:

$$\begin{array}{rl} i) & \mathcal{E}^\vee(\mathbb{C}) = (\mathcal{E}(\mathbb{C}))^\vee, \\ ii) & \mathcal{E}(\mathbb{C}) \oplus (\mathcal{E}'(\mathbb{C})) = (\mathcal{E} \oplus \mathcal{E}')(\mathbb{C}), \\ ii) & \mathcal{E}(\mathbb{C}) \otimes (\mathcal{E}'(\mathbb{C})) = (\mathcal{E} \otimes \mathcal{E}')(\mathbb{C},) \\ iv) & \wedge^r(\mathcal{E}(\mathbb{C})) = (\wedge^r \mathcal{E})(\mathbb{C}), \\ v) & \mathrm{Sym}^r(\mathcal{E}(\mathbb{C})) = (\mathrm{Sym}^r \mathcal{E})(\mathbb{C}), \\ vi) & f^*\mathcal{E}(\mathbb{C}) = (f^*\mathcal{E})(\mathbb{C}). \end{array}$$

1.4 An example: $\mathrm{Spec}\, \mathcal{O}_K$

In the special case of $\mathcal{S} := \mathrm{Spec}\, \mathcal{O}_K$, we have that $\mathcal{S}_\mathbb{C} = \coprod_{\{\sigma : K \to \mathbb{C}\}} \mathrm{Spec}\, \mathbb{C}$. In fact $\mathcal{S}_\mathbb{C} = \mathrm{Spec}(\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{C}) = \mathrm{Spec}(K \otimes_\mathbb{Q} \mathbb{C})$. Let $f(x)$ be an element of $\mathbb{Q}[x]$ such that $K \cong \mathbb{Q}[x]/(f(x))$. Then $(K \otimes_\mathbb{Q} \mathbb{C})$ is isomorphic to $\mathbb{C}[x]/(f(x))$.
In the field $\mathbb{C}$ the polynomial $f(x)$ splits in $d = [K : \mathbb{Q}]$ linear factors $f(x) = \prod_{i=1}^d (x - \alpha_i)$. By Galois theory there exists an isomorphism betwen the embeddings $\{\sigma : K \to \mathbb{C}\}$ and the roots of $f(x)$. Thus we can write $f(x) = \prod_\sigma (x - \alpha_\sigma)$. By the Chinese Reminder Theorem we get the isomorphism

$\mathbb{C}[x]/(\prod_\sigma(x-\alpha_\sigma)) \cong \prod_\sigma \mathbb{C}[x]/(x-\alpha_\sigma)$. Therefore $\mathcal{S}_\mathbb{C} = \operatorname{Spec}\prod_\sigma \mathbb{C}[x]/(x-\alpha_\sigma) = \coprod_{\sigma:K\to\mathbb{C}}\operatorname{Spec}\mathbb{C}$ and we get the following commutative diagram

$$\begin{array}{ccc}
\mathcal{S}_\mathbb{C} = \operatorname{Spec}(\mathbb{C}[x]/\left(f(x)\right)) & \longrightarrow & \operatorname{Spec}\mathbb{C} \\
\downarrow & & \downarrow \\
\mathcal{S}_\mathbb{Q} = \operatorname{Spec} K & \longrightarrow & \operatorname{Spec}\mathbb{Q} \\
\downarrow & & \downarrow \\
\mathcal{S} & \longrightarrow & \operatorname{Spec}\mathbb{Z}.
\end{array}$$

In an analogous way, if $\mathcal{X}$ is a $\mathcal{S}$-scheme then $\mathcal{X}_\mathbb{C} = \coprod_{\sigma:K\to\mathbb{C}}\mathcal{X}_\sigma$, where $\mathcal{X}_\sigma$ is the fiber product of $\mathcal{X}$ and $\operatorname{Spec}\mathbb{C}$ over $\operatorname{Spec} K$ through the embedding σ.
We remark that the invariance under conjugation of the inner product implies in particular that on $\operatorname{Spec}\mathcal{O}_K$ we get $||s||_\sigma = ||s||_{\overline{\sigma}}$.
A sheaf on an affine variety is locally free if and only if its global sections are a projective module, (see [14], chap. II, par. 5).
For a finitely generated module over a Dedekind-domain the notions of torsion free, flat and projective module coincide, (see [10], Thm. 13, p.95). These strong properties simplify a lot the situation we are dealing with. They tell us that there is an isomorphism of categories between the category of locally free sheaves of finite rank over $\operatorname{Spec}\mathcal{O}_K$ and the category of finitely generated torsion free $\mathcal{O}_K$-modules. For this reason we will often identify the objects of the two categories. The module we consider are finitely genrated module over a Dedekind-domain.

1.5 The push-forward of a Hermitian Vector Bundle

Let $\overline{\mathcal{E}}$ be a hermitian vector bundle on an arithmetic variety $\pi : \mathcal{X} \to \operatorname{Spec}\mathcal{O}_K$.
The work of Moret-Bailly [30] lem. 1.4.2 shows that the push forward $\pi_*\mathcal{E}$ of a locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ to $\operatorname{Spec}\mathcal{O}_K$ is still locally free.
We want to induce a metric on the vector bundle $E := H^0(\mathcal{S}, \pi_*\mathcal{E}) = H^0(\mathcal{X}, \mathcal{E})$ on $\operatorname{Spec}\mathcal{O}_K$. For each section $s \in E$ we define

$$||s||_\sigma^2 := \int_{\mathcal{X}_\sigma(\mathbb{C})} ||s_x||_{\mathcal{E}_\sigma}^2 d\mu_\sigma(x), \tag{5}$$

with $d\mu_\sigma$ a measure on $\mathcal{X}_\sigma$ and $||\cdot||_{\mathcal{E}_\sigma}^2 := h_{\sigma,x}(\cdot,\cdot)$. In the special case of an abelian variety we will choose $d\mu_\sigma$ to be the normalized Haar measure. In the case of a projective space we will use the Fubini-Study metric, (see 2.2.2).

2 The Arakelov Degree

2.1 The Arakelov Degree of a Hermitian Vector Bundle on $\operatorname{Spec} \mathcal{O}_K$

Definition 5 *Let $\overline{E}$ be a hermitian line bundle over* $\operatorname{Spec} \mathcal{O}_K$. *For any section s in E we define*

$$\widehat{\deg}\, \overline{E} := \log \sharp\, (E/s\mathcal{O}_K) - \sum_{\sigma: K \to \mathbb{C}} \log ||s||_\sigma. \tag{6}$$

If $\overline{E}$ is a hermitian vector bundle of rank r, we define

$$\widehat{\deg}\, \overline{E} := \widehat{\deg}\, \wedge^r \overline{E}.$$

The real number $\widehat{\deg}\, \overline{E}$ does not depend on the choice of the section, it is called the Arakelov degree of $\overline{E}$.

In order to prove that the definition does not depend on the choice of the section s we will give in lemma 1 an equivalent definition of the Arakelov degree. The independence will be an easy consequence.
Notations:
We denote by $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$ and by $v_\mathfrak{p}$ the associated non-archimedean valuation. Let E be a projective $\mathcal{O}_K$-module of rank 1. The isomorphism $j_\mathfrak{p} : E_\mathfrak{p} \to \mathcal{O}_{K\mathfrak{p}}$ between the localizations at $\mathfrak{p}$ is unique up to a unit of $\mathcal{O}_{K\mathfrak{p}}$, (see [2] II, 5.2 thm. 1). We extend the valuation at a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ to E as follows $v_\mathfrak{p}(s) := v_\mathfrak{p}(j_\mathfrak{p}(s))$ for any element $s \in E$.
The absolute value associated to $\mathfrak{p}$ is $||(s)||_{v_\mathfrak{p}} := N_\mathfrak{p}^{-v_\mathfrak{p}(s)}$ where $N_\mathfrak{p} := \sharp(\mathcal{O}_K/\mathfrak{p})$ is the absolute norm of an ideal.
We indicate by M_K the set of absolute valuations on K, by M_K^0 the set of the non-archimedean ones and by M_K^∞ the archimedean ones.

Lemma 1 *The following definition of Arakelov degree is equivalent to definition 5*

$$\widehat{\deg}\, \overline{E} := - \sum_{v \in M_K^0} \log ||(s)||_v - \sum_{v_\sigma \in M_K^\infty} \log ||s||_\sigma. \tag{7}$$

This formula is independent of the choice of $s \in E$.

Proof
We want to prove that $\log \sharp\, (E/s\mathcal{O}_K) = - \sum_{v \in M_K^0} \log ||(s)||_v$.

From [2] II.2.4 thm. 1,II.3.3 prop. 8 and the corollary of prop. 9, we get that for every projective $\mathcal{O}_K$-module of rank 1

$$(E/s\mathcal{O}_K) = \prod_{\mathfrak{p}} (E/s\mathcal{O}_K)_{\mathfrak{p}} = \prod_{\mathfrak{p}} (E_{\mathfrak{p}}/s\mathcal{O}_{K\mathfrak{p}}). \tag{8}$$

Using the isomorphism $j_{\mathfrak{p}} : E_{\mathfrak{p}} \to \mathcal{O}_{K\mathfrak{p}}$ and (8) we deduce that $(E/s\mathcal{O}_K) = \prod_{\mathfrak{p}}(\mathcal{O}_{K\mathfrak{p}}/j_{\mathfrak{p}}(s)\mathcal{O}_{K\mathfrak{p}}) \cong \prod_{\mathfrak{p}}(\mathcal{O}_K/\mathfrak{p})^{v_{\mathfrak{p}}(j_{\mathfrak{p}}(s))}$ And passing to the order we get $\sharp(E/s\mathcal{O}_K) = \prod_{v_{\mathfrak{p}}}(N_{\mathfrak{p}})^{v_{\mathfrak{p}}(j_{\mathfrak{p}}(s))} = \prod_{v\in M_K^0} ||s||_v^{-1}$.
To prove that $\widehat{\deg}\,\overline{E}$ does not depend on the choice of the section s, let t be another global section, then $t = ks$ for some $k \in K^*$. We deduce

$$\begin{aligned}
-\sum_{v\in M_K^0} \log||(t)||_v - \sum_{v\in M_K^\infty} \log||t||_\sigma &= \\
&= -\sum_{v\in M_K^0} \log||(ks)||_v - \sum_{v\in M_K^\infty} \log||ks||_\sigma \\
&= -\sum_{v\in M_K^0} \log||(s)||_v - \sum_{v\in M_K^\infty} \log||s||_\sigma - \sum_{v\in M_K} \log||k||_v \\
&= -\sum_{v\in M_K^0} \log||(s)||_v - \sum_{v\in M_K^\infty} \log||s||_\sigma,
\end{aligned}$$

the last equality because of the product formula $\prod_{v\in M_K} ||k||_v = 1$ for an element in K^* (see [10] III thm. 18). □

2.1.1 Some Properties of the Arakelov Degree

We are going to prove several properties related to the Arakelov degree. The final result, property 7, will play a crucial rule in the proof of lemma 8.

Property 1 *Let $\overline{E}$ and $\overline{F}$ be hermitian vector bundles over Spec $\mathcal{O}_K$ of rank n and m respectively. Let $\overline{L}$ be a hermitian line bundle over Spec $\mathcal{O}_K$ and $\overline{L}^\vee$ its dual.*
Then

1) $\widehat{\deg}\,(\overline{E}\otimes\overline{F}) = m\widehat{\deg}\,\overline{E} + n\widehat{\deg}\,\overline{F}$
2) $\widehat{\deg}\,(\overline{E}\oplus\overline{F}) = \widehat{\deg}\,\overline{E} + \widehat{\deg}\,\overline{F}$
3) $\widehat{\deg}\,\overline{L}^\vee = -\widehat{\deg}\,\overline{L}$.

Proof of 1)
As a first step we prove it supposing that E and F are line bundles. From the definition of induced metrics on the tensor product (see 1.2.3) it follows that

$$(i) \quad ||s \otimes t||_\sigma = ||s||_\sigma \cdot ||t||_\sigma.$$

The localization commutes with the tensor product, thus $(E \otimes F)_\mathfrak{p} = E_\mathfrak{p} \otimes F_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$. If $j_{E\mathfrak{p}} : E_\mathfrak{p} \to \mathcal{O}_{K\mathfrak{p}}$ and $j_{F\mathfrak{p}} : F_\mathfrak{p} \to \mathcal{O}_{K\mathfrak{p}}$ are the isomorphisms between the localizations, then $j_{(E\otimes F)\mathfrak{p}} : (E \otimes F)_\mathfrak{p} \to \mathcal{O}_{K\mathfrak{p}}$ is given by $j_{(E\otimes F)\mathfrak{p}}(s \otimes t) = j_{E\mathfrak{p}}(s) \cdot j_{F\mathfrak{p}}(t)$. Therefore

$$(ii) \quad v_\mathfrak{p}(s \otimes t) = v_\mathfrak{p}(s) + v_\mathfrak{p}(t).$$

We finish the case of line bundles by substituting (i) and (ii) in the definition (7) of the degree.
To reduce the general case to the case of line bundles we consider the isometric isomorphism

$$\wedge^{nm}(\overline{E} \otimes \overline{F}) \cong (\wedge^n \overline{E})^{\otimes m} \otimes (\wedge^m \overline{F})^{\otimes n},$$

(see [1] chap. III 1 prop.6).

□

Proof of 2) Let $k := \mathrm{rg}(E \oplus F) = n + m$. We get the result just using part (1) and the isometric isomorphism:

$$\wedge^k(\overline{E} \oplus \overline{F}) \cong \bigoplus_{i=0}^{k} \wedge^i \overline{E} \otimes \wedge^{k-i} \overline{F} = \wedge^n \overline{E} \otimes \wedge^m \overline{F}$$

(see [1] chap III 5 ex. 7, or [28] thm. C2). □

Proof of 3)
By the definition of the inverse sheaf we have a canonical isomorphism $I : L^\vee \otimes L \cong \mathcal{O}_K$. We endow $\mathcal{O}_K$ with the norms $||1||_\sigma = 1$ for every embedding σ. From the definition of the induced metric on the dual (see 1.2.1) it follows that I is an isometry, and so $\widehat{\deg}\,(\overline{L^\vee \otimes L}) = 0$.
Using point (1) we get $\widehat{\deg}\,\overline{L}^\vee + \widehat{\deg}\,\overline{L} = 0$. □

2.1.2 Normalized Degree and Slope

We want to determine the dependence of $\widehat{\deg}\,\overline{E}$ on extensions of scalars.
Let L be an extension of degree d of K, then $i^\sharp : \operatorname{Spec} \mathcal{O}_L \longrightarrow \operatorname{Spec} \mathcal{O}_K$ is finite, and by base change (see 1.1) we have $E_{\mathcal{O}_L} := E \otimes_{\mathcal{O}_K} \mathcal{O}_L$. If $s \in E$ we

still call s the global section of $E_{\mathcal{O}_L}$ given by $s \otimes 1$. The extension formula (see [10] III, 1.15) says that

$$\prod_{w \in M_L, w|v} ||s||_w = ||s||_v^{[L:K]}.$$

Here $w|v$ means that w equals v when restricted to K. Using the definition (7) of degree it immediately follows that $\widehat{\deg}\, \overline{E}_{\mathcal{O}_L} = [L : K]\widehat{\deg}\, \overline{E}$.
It is then natural to define the normalized Arakelov degree by

$$\widehat{\deg}_n \overline{E} := \frac{1}{[K : \mathbb{Q}]}\widehat{\deg}\, \overline{E},$$

and the normalized slope of $\overline{E}$ by

$$\hat{\mu}(\overline{E}) := \frac{1}{\operatorname{rg} E}\widehat{\deg}_n \overline{E}.$$

We have proven that $\widehat{\deg}_n \overline{E}$ and $\hat{\mu}(\overline{E})$ are invariant under extensions of scalars.

2.1.3 Saturated Submodules

Definition 6 *A submodule F of a module E is saturated if $F = (F \otimes_{\mathcal{O}_K} K) \cap E$. If F is not saturated we define its saturation as $F_s := (F \otimes_{\mathcal{O}_K} K) \cap E$.*

Remark:
If E is a finitely generated projective module over a dedekind domain then the saturation of a submodule F is torsion free and finitely generated thus it is projective. If $\overline{E}$ is a hermitian vector bundle over $\operatorname{Spec} \mathcal{O}_K$, then $\widehat{\deg}\, \overline{F}_s \geq \widehat{\deg}\, \overline{F}$ where we consider the induced inner products (see 1.2.4). Indeed, by definition, $\wedge^r F \subset \wedge^r F_s$. Using the definition (6) of Arakelov degree it trivially follows that $\widehat{\deg}\, \overline{F}_s \geq \widehat{\deg}\, \overline{F}$.

Property 2 *If F is a saturated submodule of E then E/F is torsion free, and the exact sequence*

$$0 \longrightarrow F \longrightarrow E \xrightarrow{\ p\ } E/F \longrightarrow 0$$

splits.

Proof
Suppose that E/F has torsion T, then $p^{-1}(T) \supset F$. Consider the exact sequence

$$0 \longrightarrow F \longrightarrow p^{-1}(T) \xrightarrow{p} T \longrightarrow 0.$$

Since K is flat over $\mathcal{O}_K$ we get that $F \otimes_{\mathcal{O}_K} K = p^{-1}(T) \otimes_{\mathcal{O}_K} K$ which contradicts the definition of a saturated module.
For the second claim just recall that torsion free means projective. □

Property 3 *Let $\overline{E}$ be an hermitian vector bundle on Spec $\mathcal{O}_K$ and F a saturated submodule of E. We endow F and E/F with the metrics canonically induced by E (see 1.2.4), then the following relation holds*

$$\widehat{\deg}\,\overline{E} = \widehat{\deg}\,\overline{F} + \widehat{\deg}\,\overline{E/F}$$

Proof From property (2) we know that F and E/F are direct summands of E. We define the canonical isomorphism

$$\begin{array}{rcll} I & : & \wedge^m \overline{F} \otimes \wedge^n \overline{E/F} & \longrightarrow \wedge^{m+n}\overline{E} \\ & & f_1 \wedge \cdots \wedge f_m \otimes \overline{e_1} \cdots \wedge \overline{e_n} & \longmapsto f_1 \wedge \cdots \wedge f_m \wedge e_1 \cdots \wedge e_m. \end{array}$$

where rank $F = m$ and rank $E/F = n$.
The isomorphism is canonical because for any representative of n classes $\overline{e_1}, \cdots, \overline{e_n} \in E/F$ and any m elements $f_1, \cdots, f_m \in F$ the exterior power $f_1 \wedge \cdots \wedge f_m \wedge e_1 \cdots \wedge e_m$ does not depend on the choice of the representative. From definitions (1.2.4) and (1.2.6) it follows that I is an isometry. Now apply property 1.

□

2.1.4 The Canonical Polygon

Let $\overline{E}$ be a hermitian vector bundle on Spec $\mathcal{O}_K$. We consider on a sub-bundle F the induced metric (see 1.2.4).

Definition 7 *In the Cartesian product* $[0\,, \text{rg } E] \times \mathbb{R}$ *we consider the set of points* (rg F, $\widehat{\deg}_n \overline{F}$) *where F is a sub-bundle of E. The convex hull of these points is a set bounded from above, (see below). Its upper boundary is a piecewise linear function* $P_E : [0\,, \text{rg } E] \longrightarrow \mathbb{R}$ *called the* canonical polygon *of E.*
We say that E is semi-stable if P_E is a linear function.

We remark that $P_E(0) = 0$ and $P_E(\mathrm{rg}\ E) = \widehat{\deg}_n \overline{E}$. For every $i \in [0, \mathrm{rg}\ E]$ we define

$$\hat{\mu}_i\,(E) := P_E(i) - P_E(i-1).$$

and we also define

$$\hat{\mu}_{\max}(\overline{E}) := \hat{\mu}_1(\overline{E})$$
$$\hat{\mu}_{\min}(\overline{E}) := \hat{\mu}_{\mathrm{rg}\ E}(\overline{E}).$$

Since the function P_E is convex, the $(\hat{\mu}_i)_{0 \leq i \leq \mathrm{rg}\, E}$ is a decreasing sequence of real numbers and $\sum_i \hat{\mu}_i = \widehat{\deg}_n E$.

Sketch of proof:

We are going to give a sketch of the proof that the set of points which we consider must be bounded from above. The trick is to use Grassmannians, which are the geometric analogous of the exterior product; and heights that turn out to be the analogue of the degree.

Let $\overline{E}$ be a hermitian vector bundle over $\mathrm{Spec}\, \mathcal{O}_K$, and let $E_{\overline{\mathbb{Q}}} := E \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}$. Let $X = G(d,\ E_{\overline{\mathbb{Q}}})$ be the Grassmannian variety representing the subspaces of dimension d in $E_{\overline{\mathbb{Q}}}$. We consider the natural projective Plucker embedding

$$\begin{array}{cccc} i & : & G(d,\ E_{\overline{\mathbb{Q}}}) & \longrightarrow & \mathbb{P}(\bigwedge^d E_{\overline{\mathbb{Q}}}) \\ & & V & \longmapsto & \bigwedge^d V. \end{array}$$

On $X = G(d,\ E_{\overline{\mathbb{Q}}})$ we consider the sheaf $i^*\mathcal{O}(1)$ endowed with the Fubini-Study metric (see 2.2.2), and we associate a model

$$i_L : (\mathcal{G}(d,\ E_{\overline{\mathbb{Q}}}), \overline{L}) \longrightarrow (\mathbb{P}(\wedge^d E), \overline{\mathcal{O}(1)})$$

By base change, every submodule F of E of rank d gives a d-dimensional subspace V of $E_{\overline{\mathbb{Q}}}$, V is a point P in $G(d,\ E_{\overline{\mathbb{Q}}})$ defined over $\mathcal{O}_K$, the point P on the generic fiber can be closed in the model.

Bost, Gillet and Soulé proved (see [5] prop.4.1.2) that $h_{\overline{L}}(P) = -\widehat{\deg}\, F$, where $h_{\overline{L}}(P)$ is the height of $i_L(P)$ in the projective space (see [5] 3.1.6).

By Northcott's theorem (see [19] II thm 2.2), the number of points in $\mathbb{P}^n_K$ of height bounded from above is finite, so the number of submodules of degree bounded from below is finite.

Stuhler [37] and Grayson [12] define a canonical filtration proving that if i_R is any point of discontinuity of P'_E then there exists a unique sub-module E_R of E of rank i_R such that $P_E(i_R) = \widehat{\deg}_n \overline{E_R}$. The chain $0 \subset E_1 ..... \subset E_R \subset ... \subset E$ is the canonical filtration. From the remark in 2.1.3 it follows that the modules E_R are saturated. The existence follows from the strict convexity of P_E.

Another characterization of the canonical filtration is to require that the E_R are semi-stable and their slopes are strictly decreasing.

2.1.5 Some Properties of Slopes of Hermitian Vector Bundles

Property 4 *Let $\overline{E}_1, ..., \overline{E}_N$ be hermitian vector bundles over* Spec $\mathcal{O}_K$ *then*

$$\hat{\mu}_{\max}(\oplus_{i=1}^N \overline{E}_i) = \max_{1\leq i\leq N} \hat{\mu}_{\max}(\overline{E}_i).$$

Proof
Of course $\hat{\mu}_{\max}(\oplus_{i=1}^N \overline{E}_i) \geq \max_{1\leq i\leq N} \hat{\mu}_{\max}(\overline{E}_i)$, so it is sufficient to prove $\hat{\mu}_{\max}(\oplus_{i=1}^N \overline{E}_i) \leq \max_{1\leq i\leq N} \hat{\mu}_{\max}(\overline{E}_i)$.
It is enough to prove it for $\overline{E}_1 \oplus \overline{E}_2$, then use induction.
There exists a saturated submodule F of $\overline{E}_1 \oplus \overline{E}_2$ of rank r such that $\hat{\mu}_{\max}(\overline{E}_1 \oplus \overline{E}_2) = \frac{1}{r}\widehat{\deg}_n \overline{F}$. Note that r is the first point of discontinuity of the first derivative of P_E.
Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{F}_1 := \overline{F} \cap \overline{E}_1 & \longrightarrow & \overline{F} & \longrightarrow & \overline{F}_2 := \overline{F}/\overline{F}_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \overline{E}_1 & \longrightarrow & \overline{E}_1 \oplus \overline{E}_2 & \longrightarrow & \overline{E}_2 & \longrightarrow & 0
\end{array}$$

Let r_1 and r_2 be the rank of F_1 and F_2 respectively. The metrics are the induced ones, so we get

$$\begin{aligned}
\frac{\widehat{\deg}_n \overline{F}}{r} &= \frac{\widehat{\deg}_n \overline{F}_1 + \widehat{\deg}_n \overline{F}_2}{r} \\
&\leq \frac{r_1 \hat{\mu}_{\max}(\overline{E}_1) + r_2 \hat{\mu}_{\max}(\overline{E}_2)}{r_1 + r_2} \\
&\leq \max\left(\hat{\mu}_{\max}(\overline{E}_1), \hat{\mu}_{\max}(\overline{E}_2)\right).
\end{aligned}$$

□

Property 5 *Let $\overline{E}$ be a hermitian vector bundle and $\overline{L}$ a hermitian line bundle over* Spec $\mathcal{O}_K$*, then*

$$\hat{\mu}_{\max}(\overline{E} \otimes \overline{L}) = \hat{\mu}_{\max}(\overline{E}) + \widehat{\deg}_n \overline{L}.$$

Proof First we prove that $\hat{\mu}_{\max}(\overline{E} \otimes \overline{L}) \leq \hat{\mu}_{\max}(\overline{E}) + \widehat{\deg}_n \overline{L}$. Let F be a submodule of $E \otimes L$ of rank r such that $\hat{\mu}_{\max}(\overline{E} \otimes \overline{L}) = \frac{1}{r}\widehat{\deg}_n \overline{F}$.
Consider the submodule $F_1 := F \otimes L^{-1} \subset E \otimes L \otimes L^{-1} = E$. We have

$$\hat{\mu}_{\max}(\overline{E} \otimes \overline{L}) = \frac{\widehat{\deg}_n \overline{F}}{r} = \frac{\widehat{\deg}_n (\overline{F}_1 \otimes \overline{L})}{r}.$$

From property 1 we get

$$\frac{\widehat{\deg}_n \overline{F}}{r} = \frac{\widehat{\deg}_n \overline{F}_1 + r\widehat{\deg}_n \overline{L}}{r} \leq \hat{\mu}_{\max}\overline{E} + \widehat{\deg}_n \overline{L}.$$

It remains to prove $\hat{\mu}_{\max}(\overline{E} \otimes \overline{L}) \geq \hat{\mu}_{\max}\overline{E} + \widehat{\deg}_n \overline{L}$. For this just consider a submodule F_r of E of rank r such that $\hat{\mu}_{\max}\overline{E} = \frac{1}{r}\widehat{\deg}_n \overline{F}_r$. The module $F_r \otimes L$ is a submodule of $E \otimes L$ and $\widehat{\deg}_n (\overline{F}_r \otimes \overline{L}) = \widehat{\deg}_n \overline{F}_r + r\widehat{\deg}_n \overline{L}$ hence $\hat{\mu}_{\max}(\overline{E} \otimes \overline{L}) \geq \hat{\mu}(\overline{F}_r \otimes \overline{L}) = \hat{\mu}_{\max}\overline{E} + \widehat{\deg}_n \overline{L}$.

□

2.2 The Arakelov Degree and Morphisms

Let $\phi : \overline{E} \to \overline{F}$ be a morphism of hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$. We define the norm of ϕ to be the operator norm

$$||\phi||_\sigma := \sup_{0 \neq s \in E} \frac{||\phi(s)||_\sigma}{||s||_\sigma}.$$

Property 6 *Let $\phi : \overline{E} \to \overline{F}$ be a non trivial injective morphism of hermitian vector bundles over $\operatorname{spec} \mathcal{O}_K$. Then*

$$\widehat{\deg}_n \overline{E} \leq \sum_{i=1}^{\operatorname{rg} E} \hat{\mu}_i(F) + \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \log || \wedge^{\operatorname{rg} E} \phi||_\sigma$$

Proof
We first consider the case of a line bundle E, in this case $\widehat{\deg}\, \phi(\overline{E}) := \log \sharp\, (\phi(E)/\phi(s)\mathcal{O}_K) - \sum_{\sigma : K \to \mathbb{C}} \log ||\phi(s)||_\sigma$
Since ϕ is injective, $\sharp\, (\phi(E)/\phi(s)\mathcal{O}_K) = \sharp\, (E/s\mathcal{O}_K)$.
Moreover $\log ||\phi(s)||_\sigma = \log \frac{||\phi(s)||_\sigma}{||(s)||_\sigma} + \log ||(s)||_\sigma$ and therefore

$$\begin{aligned} \widehat{\deg}_n \overline{E} &= \widehat{\deg}_n \phi(E) + \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \log \frac{||\phi(s)||_\sigma}{||(s)||_\sigma} \\ &\leq \hat{\mu}_{\max}(F) + \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \log \frac{||\phi(s)||_\sigma}{||(s)||_\sigma} \qquad (9) \\ &\leq \hat{\mu}_1(F) + \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \log ||\phi||_\sigma. \end{aligned}$$

Now we pass to the case $\mathrm{rg}\, E \geq 1$. Since ϕ is injective $\mathrm{rg}\, \phi(E) = \mathrm{rg}\, E$. We consider the injective map $\wedge^{\mathrm{rg}\, E}\phi : \wedge^{\mathrm{rg}\, E} E \to \wedge^{\mathrm{rg}\, E}\phi(E)$. From the formula (9) we get

$$\widehat{\deg}_n \,(\bigwedge^{\mathrm{rg}\ E} \overline{E}) = \widehat{\deg}_n \bigwedge^{\mathrm{rg}\ E} \overline{\phi(E)} + \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \log \frac{|| \wedge^{\mathrm{rg}\ E} \phi(s)||_\sigma}{|| \wedge^{\mathrm{rg}\ E} (s)||_\sigma}. \tag{10}$$

We remark that $\hat{\mu}_i(\overline{\phi(E)}) \leq \hat{\mu}_i(\overline{F})$ for every $i \leq \mathrm{rg}\ E$. This implies

$$\widehat{\deg}_n \overline{\phi(E)} = \sum_i^{\mathrm{rg}\, E} \hat{\mu}_i(\overline{\phi(E)}) \leq \sum_i^{\mathrm{rg}\, E} \hat{\mu}_i(\overline{F}).$$

Therefore

$$\widehat{\deg}_n \overline{E} \leq \sum_i^{\mathrm{rg}\, E} \hat{\mu}_i(\overline{F}) + \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \log || \wedge^{\mathrm{rg}\ E} \phi||_\sigma.$$

$\square$

Remark: Since $|| \wedge^r \Phi|| \leq ||\Phi||^r$, we deduce from Property 6 that

$$\widehat{\deg}_n E \leq \sum_i^{\mathrm{rg}\, E} \hat{\mu}_i(\overline{F}) + \frac{\mathrm{rg}\, E}{[K:\mathbb{Q}]} \sum_\sigma \log ||\phi||_\sigma.$$

Corollary 1 *If $\phi : E \to F$ is injective and non trivial then*

$$\hat{\mu}_{\max}(\overline{E}) \leq \hat{\mu}_{\max}(\overline{F}) + \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \log ||\phi||_\sigma.$$

Proof Let $\overline{E}_r$ be a sub-vector bundle of $\overline{E}$ of rank r such that $\widehat{\deg}_n \overline{E}_r/r = \hat{\mu}_{\max}(\overline{E})$. The restriction map $\phi|_{E_r} : E_r \to \phi(E_r)$ is still injective and $\phi(E_r)$ is a submodule of F of rank r.
We apply property 6 to $\phi|_{E_r} : E_r \to \phi(E_r)$. It follows

$$\hat{\mu}_{\max}(\overline{E}) = \frac{\widehat{\deg}_n \overline{E}_r}{r} \leq \frac{\widehat{\deg}_n \overline{\phi(E_r)}}{r} + \frac{\sum_\sigma \log || \wedge^r \phi|_{E_r}||_\sigma}{[K:\mathbb{Q}]r}$$

and so

$$\hat{\mu}_{\max}(\overline{E}) \leq \hat{\mu}_{\max}(\overline{F}) + \frac{1}{[K:\mathbb{Q}]}\sum_\sigma \log ||\phi|_{E_r}||.$$

$\square$

2.2.1 A Key Property of the Arakelov Degree

We give a generalization of the property 6 where we replace F by a filtration. Let $\overline{E}$ be a hermitian vector bundle over $\operatorname{Spec}\mathcal{O}_K$ and let F be a vector bundle over $\operatorname{Spec}\mathcal{O}_K$. Let

$$F = F_N \supset F_{N-1} \supset .... \supset F_1 \supset F_0 = 0$$

be a filtration of F such that the quotients $G_i = F_i/F_{i-1}$ are torsion free for every i. Let $\phi : E \to F$ be a map of $\mathcal{O}_K$ modules. We endow the vector bundles G_i with hermitian metrics. The map $\phi_i : \phi^{-1}(F_i) \to G_i$ is the composition of ϕ and the projection. On $\phi^{-1}(F_i)$ we consider the metric induced by E, so the norms $||\phi_i||$ are defined.

Property 7 *We use the notations above. If $\phi : E \to F$ is injective and non trivial then*

$$\widehat{\deg}_n \overline{E} \leq \sum_{i=1}^{N} \left(\operatorname{rg}(\phi^{-1}(F_i)/\phi^{-1}(F_{i-1}))\right) \left(\hat{\mu}_{\max}(\overline{G}_i) + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma:K\to\mathbb{C}} \log ||\phi_i||_\sigma\right).$$

Proof From the injectivity of ϕ and the "3-lemma" we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \phi^{-1}(F_{i-1}) \arrow[r] \arrow[d, hook, "\phi"] & \phi^{-1}(F_i) \arrow[r, "\pi_{E_i}"] \arrow[d, hook, "\phi"] & \phi^{-1}(F_i)/\phi^{-1}(F_{i-1}) \arrow[r] \arrow[d, hook, "\varphi_i"] & 0 \\
0 \arrow[r] & F_{i-1} \arrow[r] & F_i \arrow[r, "\pi_{F_i}"] & G_i \arrow[r] & 0
\end{tikzcd}$$

Notice that the norms on $\phi^{-1}(F_i)$ and $\phi^{-1}(F_i)/\phi^{-1}(F_{i-1})$ are the ones canonically induced by $\overline{E}$ on sub-vector bundles and quotient bundles. However G_i has its own norm, independent of any other.
Since F_i/F_{i-1} is torsion free and φ_i is an inclusion then also $\phi^{-1}(F_i)/\phi^{-1}(F_{i-1})$ is torsion free. This means that $\phi^{-1}(F_{i-1})$ is a direct summand of $\phi^{-1}(F_i)$. We can apply property 3 to get

$$\widehat{\deg}_n \left(\overline{\phi^{-1}(F_i)/\phi^{-1}(F_{i-1})}\right) = \widehat{\deg}_n \overline{\phi^{-1}(F_i)} - \widehat{\deg}\, \overline{\phi^{-1}(F_{i-1})}.$$

Recall that $\sum_{i=1}^N \widehat{\deg}_n \overline{\phi^{-1}(F_i)} - \widehat{\deg}_n \overline{\phi^{-1}(F_{i-1})} = \widehat{\deg}_n \overline{E}$.
It follows

$$\begin{aligned}
\widehat{\deg}_n \overline{E} &= \sum_{i=1}^{N} \widehat{\deg}_n \left(\overline{\phi^{-1}(F_i)/\phi^{-1}(F_{i-1})}\right) \\
&\leq \sum_{i=1}^{N} \operatorname{rg}(\phi^{-1}(F_i)/\phi^{-1}(F_{i-1}))\hat{\mu}_{\max}(\overline{\phi^{-1}(F_i)/\phi^{-1}(F_{i-1})}).
\end{aligned} \tag{11}$$

If $\Phi_i = 0$ then $\mathrm{rg}(\phi^{-1}(F_i)/\phi^{-1}(F_{i-1})) = 0$ and so there is no contribution to the right-hand side of (11). If $\Phi_i \neq 0$ then we apply corollary 1 and we find

$$\widehat{\deg}_n \overline{E} \leq \sum_{i=1}^{N} \big(\mathrm{rg}(\phi^{-1}(F_i)/\phi^{-1}(F_{i-1})) \big) \big(\hat{\mu}_{\max}(\overline{G}_i) + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma : K \to \mathbb{C}} \log ||\varphi_i||_\sigma \big).$$

From the above diagram we have $\pi_{E_i}\varphi_i = \phi_i$ with $\phi_i := \phi\pi_{F_i}$, and so $||\pi_{E_i}||||\varphi_i|| = ||\phi_i||$. Since the norms on the first line are the induced ones, $||\pi_{E_i}|| = 1$. Therefore $||\varphi_i|| = ||\phi_i||$ and we get the result.

□

2.2.2 An Example: the Arakelov Degree of $\mathcal{O}_{\mathbb{P}^1}(1)$

The arithmetic n-projective space over $\mathcal{O}_K$ is defined to be

$$\mathbb{P}^n_{\mathcal{O}_K} := \operatorname{Proj} \mathcal{O}_K[x_0, ..., x_n].$$

Let σ be any embedding of K in $\mathbb{C}$, then $\mathbb{P}^n_\sigma = \operatorname{Proj} \mathcal{S}(V_\sigma)$ with $V_\sigma := (\mathcal{O}_K x_0 \oplus ... \oplus \mathcal{O}_K x_n) \otimes_\sigma \mathbb{C}$ and $\mathcal{S}(\cdot)$ the simmetric algebra. To avoid heavy notations we leave out the index σ. A point of $\mathbb{P}^n$ is an embedding $i_1 : \operatorname{Proj} \mathcal{S}(V_1) \to \operatorname{Proj} \mathcal{S}(V)$ with V_1 a 1-dimensional vector space. Equivalently, a point is a quotient given by a surjection $i_1^\sharp : \mathcal{S}(V) \to \mathcal{S}(V_1)$ up to a multiplication for an element of $\mathbb{C}^*$. The surjection $i_1^\sharp$ is completely determined by a surjection $i_1^\sharp : V \to V_1$, thus a point of $\mathbb{P}^n$ is just a 1-dimensional quotient V_1 of V. We denote by $V \times \mathbb{P}^n$ the trivial bundle of rank n on $\mathbb{P}^n$. Let H be the sub-bundle of $V \times \mathbb{P}^n$ such that the stalk at a point V_1 is $\ker i_1^\sharp$, and the transition functions related to the standard affine open sets $U_i := \{x_i \neq 0\}$ are given by $g_{ij} := x_j/x_i$. We denote $V \otimes \mathcal{O}_{\mathbb{P}^n}$ and $\mathcal{H}$ the sheaves associated to $V \times \mathbb{P}^n$ and H respectively. We define $\mathcal{O}(1)$ to be the cokernel sheaf of the exact sequence $0 \to \mathcal{H} \to V \otimes \mathcal{O}_{\mathbb{P}^n}$.
Let h_V be a hermitian inner product on the vector space V, this defines naturally a hermitian metric on the sheaf $V \otimes \mathcal{O}_{\mathbb{P}^n}$. We call Fubini-Study the induced quotient metric h_F on $\mathcal{O}(1)$. We recall that the induced Riemannian metric is the real part of h_F.
Now we consider the special case of $\mathbb{P}^1_{\mathcal{O}_K}$.
Let ω be the Chern form of the line bundle $\mathcal{O}(1)$. A representative of the Chern class is a $(1,1)$- form and the following relation holds

$$\int_{\mathbb{P}^1} \omega = 1.$$

In fact this is the integral of the Poincaré dual of a line on $\mathbb{P}^1$, that means the intersection number of a hyperplane and a line (see [13] p. 122). One can prove that this is equivalent to say that ω is the only normalized measure on $\mathbb{P}^1$ invariant under the action of $\mathbb{P}GL_2$.
We endow $\pi_* \mathcal{O}(1)$ with the norm

$$||s||^2_{\mathcal{O}(1),\sigma} = \int_{\mathbb{P}^1_\sigma(\mathbb{C})} ||s(p)||^2_{\delta F}\ \omega(p)$$

where δ is a real number and F the Fubini-Study metric..
The curvature form of $\mathcal{O}(1)$ is given by $\Theta = -\frac{1}{2}\partial\overline{\partial} \log h_{[z]}(s,s)$ for any hermitian inner product $h_{[z]}$ (see [13] p.77) and the Chern form is $\omega = \frac{i}{2\pi}\Theta$ (see [13] p.141).

We recall that $H^0(\mathbb{P}^1, \mathcal{O}(1)) = V$. If we choose a orthonormal basis x_0, x_1 for the global sections of $\mathcal{O}(1)$, the matrix representation of the Fubini-Study metric takes the form $h_{F,[z]} = \frac{1}{\langle z,z\rangle} = \frac{1}{x_0\overline{x_0}+x_1\overline{x_1}}$ on $\mathcal{O}(1)_z \otimes \overline{\mathcal{O}(1)}_z$.
We want to determine δ so that the sections x_0 and x_1 have norm 1.
Since the affine open sets U_i are dense in the Riemannian topology of $\mathbb{P}^1$, the above integral can be restricted to any open set. The curvature form on the open set $U_0 = \{p \in \mathbb{P}^1_\sigma : x_0 \neq 0\}$ becomes $\Theta = \frac{1}{2}\frac{1}{(1+|\omega_1|^2)^2}\, d\omega_1 \wedge d\overline{\omega_1}$ where $\omega_1 = \frac{x_1}{x_0}$, (see [13] p.30).
It follows that

$$||x_0||^2_{\mathcal{O}(1),\sigma} = \int_{U_{0,\sigma}} \frac{|x_0(p)|^2}{\sum |x_j(p)|^2}\, \omega =$$

$$\int_{U_{0,\sigma}} \frac{i}{2\pi}\frac{1}{(1+|\omega_1|^2)^3}\, d\omega_1 \wedge d\overline{\omega_1}.$$

If we change the variable of integration

$$\begin{cases} \omega_1 & = & \rho\cos\theta + i\rho\sin\theta \\ \overline{\omega_1} & = & \rho\cos\theta - i\rho\sin\theta \end{cases}$$

we get

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{2\rho}{(1+\rho^2)^3}\, d\rho \wedge d\theta = \frac{1}{2}.$$

The same we get for the section x_1 integrating on $U_1 = \{p \in \mathbb{P}^1_\sigma : x_1 \neq 0\}$.
Hence in order to normalize the two sections x_0 and x_1 it is enough to set $\delta := 2$. That means to define on $\mathcal{O}(1)$ the hermitian inner product given by the function $h_{2F} := 2 \cdot h_F$. Let g be a non zero holomorphic function, then any hermitian inner product $h' := g \cdot h_f$, induces the same curvature and Chern-form. In fact $\partial\overline{\partial}\log g \cdot h_F = \partial(\frac{\overline{\partial} g}{g} + \overline{\partial}\log h_F)$ and $\overline{\partial}$ of a holomorphic function is zero.
Now we show that the sections x_0 and x_1 are orthogonal.

$$< x_0, x_1 >_{\mathcal{O}(1)} = \int_{U_0\cap U_1} \frac{Re(x_0(p)\cdot \overline{x_1(p)})}{\sum |x_j(p)|^2}\, \omega =$$

$$= \int_{U_0\cap U_1} \frac{i}{2\pi}\frac{Re(\overline{\omega_1})}{(1+|\omega_1|^2)^3}\, d\omega_1 \wedge d\overline{\omega_1} =$$

Using the same change of variable as above we get

$$= \frac{1}{2\pi}\int_0^{\infty} \frac{2\rho^2}{(1+\rho^2)^3}\int_0^{2\pi} Re(\cos\theta - i\sin\theta) d\theta d\rho = 0.$$

Property 8 *The hermitian vector bundle* $(\pi_* \mathcal{O}(1), || \ ||_{2F,\mathcal{O}(1)})$ *is semi-stable of Arakelov degree zero.*

Proof
Using [14] ex.5.16.3v we know that $\pi_*\mathcal{O}(1) = H^0(\mathbb{P}^1_{\mathcal{O}_K}, \ \mathcal{O}(1)) = \mathcal{O}_K\, x_o \oplus \mathcal{O}_K\, x_1$. From the computation above the two sections x_0 , x_1 are orthogonal and so
$\widehat{\deg}\ \overline{\pi_*\mathcal{O}(1)} = \widehat{\deg}\,(\overline{\mathcal{O}_K\, x_o} \oplus \overline{\mathcal{O}_K\, x_1}) = \widehat{\deg}\,\overline{\mathcal{O}_K\, x_o} + \widehat{\deg}\,\overline{\mathcal{O}_K\, x_1}$.
Since the sections x_0 and x_1 have norm 1 $\widehat{\deg}\,\overline{\mathcal{O}_K\, x_0} := \log \sharp\,(\mathcal{O}_K\, x_0/x_0\mathcal{O}_K) - \sum_\sigma \log ||x_0||_\sigma = 0$ and in the same way $\widehat{\deg}\,\overline{\mathcal{O}_K\, x_1} = 0$, thus $\widehat{\deg}\ \pi_*\mathcal{O}(1) = 0$. The semi-stability follows from the fact that $\pi_*\mathcal{O}(1)$ is the direct sum of line bundles of the same slope.

$\square$

3 Abelian Varieties and MB-Models

3.1 Recall about Abelian Varieties

Let A be a complex abelian variety and $\mathcal{L}$ an ample line bundle on A. The Euler-Poincaré characteristic $\chi(A, \mathcal{L})$ is defined as the alternating sum of the $H^i(A, \mathcal{L})$ dimensions. The degree of A with respect to $\mathcal{L}$ is defined as the intersection number of the first Chern class $c_1(\mathcal{L})$ with itself g-times. The Riemann-Roch theorem gives the relation

$$\chi(A, \mathcal{L}) = \frac{1}{g!} \deg_{\mathcal{L}} A. \tag{12}$$

(see [18] thm 3.10).
If $\mathcal{L}$ is an ample line bundle then the i-th cohomology group vanishes for every $i \neq 0$, so $\chi(A, \mathcal{L}) = \dim H^0(A, \mathcal{L})$, (see [18] cor. 3.11). One says that $\mathcal{L}$ is a principal polarization for A if $\chi(A, \mathcal{L}) = 1$. We denote the tangent bundle of the variety A by $\mathcal{T}_A$ and its stalk at zero by T_A. We indicate the sheaf of differential by Ω_A and its stalk at zero by $\check{T}_A$.
We denote the tangent bundle of the variety A by $\mathcal{T}_A$ and its stalk at zero by T_A. We indicate the sheaf of differentials by Ω_A and its stalk at zero by $\check{T}_A$.
On an abelian variety the global forms are translation invariant, this is a consequence of the fact that the translation maps are isomorphisms. ([31] par. 11 prop. p98.)
Let $\exp : \mathrm{Lie}(A) = T_A \to A$ be the exponential map of A, we denote by Λ_A its kernel.
The first Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$ is an element of $H^2(A, \mathbb{Z}) = H^2(\Lambda_A, \mathbb{Z})$. There esists a unique translation invariant representative of $c_1(\mathcal{L})$. It defines an alternating 2-form $E(\gamma_1, \gamma_2)$ on Λ_A with values in $\mathbb{Z}$ such that $E(ix, iy) = E(x, y)$ and $E = \mathrm{Im} H$ where H is a hermitian form on $\mathcal{T}_A \times \mathcal{T}_A$ ([18] Lem. 3.1, 3.4, [31] 1.4). Since $\mathcal{L}$ is ample H turns out to be positive defined, thus it can be seen as a positive element ω_H in $\Omega^{1,1}(\check{T}_A)$, we define $d\lambda := \underbrace{\omega_H \wedge \ldots \wedge \omega_H}_{g-times}$,
([18] 3.3). The determinant of a matrix representation of E does not depends on the choice of the basis, we denote it by $\det E$. We have the following relation

$$\sqrt{\det E} = \chi(A, \mathcal{L}) = \int_{\mathfrak{F}_A} d\lambda \tag{13}$$

(see [18] thm. 2.3).
The radius of injectivity of A with respect to the metric on T_A induced by $\mathcal{L}$ is the largest real number $\rho_i(A, \mathcal{L})$ such that the restriction of the

exponential map to the open ball with center in zero and radius $\rho_i(A,\mathcal{L})$ is a homeomorphism. By definition

$$\rho_i(A,\mathcal{L}) = \frac{1}{2}\min_{\lambda\in\Lambda^*}||\lambda||_{T_A}. \tag{14}$$

Minkowski's theorem (see [6] VIII.4.3.) yields

$$\rho_i(A,\mathcal{L}) \leq \pi^{-\frac{1}{2}}(\deg_A\mathcal{L})^{\frac{1}{2g}}. \tag{15}$$

An important estimate for the radius of injectivity is given in [21] lem. 8.6. If A is an abelian variety of dimension g defined over K, for every line bundle $\mathcal{L}$ on A one has

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\sigma:K\to\mathbb{C}}\rho_i(A_\sigma,\mathcal{L}_\sigma)^{-2} \leq C(g)\max\left(1, h(A)+\frac{1}{2}\log\chi(A,\mathcal{L})\right) \tag{16}$$

where $C(g)$ is a constant depending only on g.
The radius of surjectivity of A is the smallest real number $\rho_s(A,\mathcal{L})$ such that the restriction of the exponential map to the closed ball with center in zero and radius $\rho_s(A,\mathcal{L})$ is surjective. If d is the degree of the number field K and A has principal polarization with respect to $\mathcal{L}$ then from Minkowski's theorem and (16) it follows that

$$\rho_s(A_\sigma,\mathcal{L}_\sigma) \leq C'(g)(d\max(1,h(A))^{g-1/2} \tag{17}$$

where $C'(g)$ is a constant depending only on g.

3.2 Semiabelian Schemes and Moret-Bailly Models

Definition 8 *A semiabelian scheme $\pi : \mathcal{A} \to \operatorname{Spec}\mathcal{O}_K$ is a smooth group scheme (separated and of finite type), such that the components of its fibers are extensions of abelian varieties by tori (semiabelian group), and its generic fiber is an abelian variety.*

A semiabelian scheme $\mathcal{A}$ is in particular an arithmetic variety, thus for any line bundle $\mathfrak{L}$ on $\mathcal{A}$ the direct image $\pi_*\mathfrak{L}$ is locally free sheaf on $\operatorname{Spec}\mathcal{O}_K$, (par. 1.3).
If $\overline{\mathfrak{L}}$ is a hermitian ample line bundle, we can endow the vector bundle $\pi_*\mathfrak{L}$ with the L^2-metric

$$||s||^2_{\mathfrak{L},\sigma} := \int_{\mathcal{A}_\sigma(\mathbb{C})}||s_x||^2_{\mathcal{L}}\, d\mu_\sigma(x)$$

where $d\mu_\sigma(x)$ is the Haar-measure on $\mathcal{A}_\sigma$, i.e. the only normalized measure, invariant under the group law. The bundle $\pi_*\overline{\mathfrak{L}} := (\pi_*\mathfrak{L}, ||s||^2_{\mathfrak{L}})$ is an hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$ of rank equal to $\dim H^0(\mathcal{A}, \mathfrak{L})$.
We denote by $\Omega_{\mathcal{A}/\mathcal{S}}$ the sheaf of relative differentials of $\mathcal{A}$ with respect to $\mathcal{S} = \operatorname{Spec}\mathcal{O}_K$ and by $\Omega^g_{\mathcal{A}/\mathcal{S}} = \bigwedge^g \Omega_{\mathcal{A}/\mathcal{S}}$ the sheaf of relative g-forms.
The sheaf $\Omega^g_{\mathcal{A}/\mathcal{S}}$ admits a natural hermitian structure $||\cdot||$ defined by

$$||\alpha||^2_\sigma = \frac{i^{g^2}}{(2\pi)^g}\int_{A_\sigma} \alpha_\sigma \wedge \overline{\alpha}_\sigma$$

for any embedding $\sigma : K \to \mathbb{C}$.
Since the global forms are translation invariant, it follows that

$$\omega_{\mathcal{A}/\mathcal{S}=\pi_*\Omega^g_{\mathcal{A}/\mathcal{S}}} \simeq 0^*_{\mathcal{A}}\Omega^g_{\mathcal{A}/\mathcal{S}}$$

with $0_{\mathcal{A}}$ the neutral element of $\mathcal{A}$.
The normalized Arakelov degree of $(\omega_{\mathcal{A}/\mathcal{S}}, ||\cdot||)$ does not depend on the choice of $\mathcal{A}$ and K and is called the Faltings height of A

$$h(A) := \widehat{\deg}_n\, \overline{\omega}_{\mathcal{A}/\mathcal{S}}. \tag{18}$$

We are going to recall the definition of MB-models given in [4] 4.3.1.
Let A be an abelian variety over $\overline{\mathbb{Q}}$, $\mathcal{L}$ an ample symmetric line bundle over A, and Σ a finite subset of $A(\overline{\mathbb{Q}})$. A MB-model of $(A, \mathcal{L}, \Sigma)$ over a number field K is defined as the data

- a semiabelian scheme $\pi : \mathcal{A} \to \operatorname{Spec}\mathcal{O}_K$,
- an isomorphism $i : A \cong \mathcal{A}_{\overline{\mathbb{Q}}}$ of abelian varieties over $\overline{\mathbb{Q}}$,
- a hermitian line bundle $\overline{\mathfrak{L}}$ on $\mathcal{A}$ that satisfies the theorem of the cube,
- an isomorphism $\mathcal{L} \cong \mathfrak{L}_{\overline{\mathbb{Q}}}$,
- for each point $P \in \Sigma$ a section $\epsilon_P : \operatorname{Spec}\mathcal{O}_K \to \mathcal{A}$ of π such that the geometric point $\epsilon_{P,\overline{\mathbb{Q}}}$ coincides with $i(P)$,

which satisfy the following condition: there exists a subscheme $\mathcal{K}$ of $\mathcal{A}$ flat and finite over $\operatorname{Spec}\mathcal{O}_K$ such that $i^{-1}(\mathcal{K}_{\overline{\mathbb{Q}}})$ coinciding with the Mumford group $K(\mathcal{L}^2)$ (see [18] 4.1).
The properties of MB-model we are going to use are summarized in the following theorem (see [4] thm 4.10 and [3] 4.2. for the semistability in (iii)).

Theorem 1 *i) Existence. There exists a MB-model for the data* $(A, \mathcal{L}, \Sigma)$ *defined over a finite number field extension of the field of definition of* A *and* Σ*, whose relative degree depends only on the dimension* g *of* A.
ii) Néron-Tate height. For any MB-model and any $P \in \Sigma$ *the Arakelov degree* $\widehat{\deg}_n \, \epsilon^*{}_{\mathcal{P}}\overline{\mathfrak{L}}$ *coincides with the normalized Néron-Tate height of* p *associated to* $\mathcal{L}$.
iii) The vector bundle $\pi_*\overline{\mathfrak{L}}$ *is semi-stable and its slope is*

$$\hat{\mu}(\pi_*\overline{\mathfrak{L}}) = -\frac{1}{2}h(A) + \frac{1}{4}\log\frac{\chi(A, \mathcal{L})}{(2\pi)^g}.$$

iv) Compatibility of a MB-model with scalar extension. If we have a MB-model of $(A, \mathcal{L}, \Sigma)$ *over some number field* K *and* L *is a finite extension of* K*, then the model got by extension of scalar from* $\mathcal{O}_K$ *to* $\mathcal{O}_L$ *is a MB-model of* $(A, \mathcal{L}, \Sigma)$ *over the number field* L.

4 Non-Reduced Subschemes and Filtrations

4.1 Statement of the Subvariety Theorem

The aim of the two chapters 4 and 5 is to prove the Subvariety Theorem.

Theorem 2 *Let A be an abelian variety defined over K and $\mathcal{L}$ a symmetric ample line bundle on A. Let W be a subspace of $T_{A_{K'}}$ defined over a finite extension K' of K. Let $\sigma_0 : K' \to \mathbb{C}$ be an embedding and γ a non-trivial period of $A_{\sigma_0}(\mathbb{C})$ such that $\gamma \in W_{\sigma_0}$. Then there exists a proper abelian subvariety B of A_{σ_0} defined over $\overline{\mathbb{Q}}$ such that*

$$\gamma \in T_B \subset W_{\sigma_0}$$

and

$$\deg_{\mathcal{L}_{\sigma_0}} B \leq C(g) \max(\deg_{\mathcal{L}} A, dhr, dr \log(dr)) \tag{19}$$

where

$$\begin{aligned} d &= [K : \mathbb{Q}], \\ h &= \max(1, h(A), \log \deg_{\mathcal{L}} A, h(W)), \\ r &= \max(1, ||\gamma||_{\sigma_0}^{2\dim B}) \end{aligned}$$

and $C(g)$ a constant depending only on g.

A Zero Lemma 4 ensures the existence of a subvariety B any times we can produce a section s of an invertible sheaf $\mathcal{L}$ of A with good order of zero at a torsion subscheme Σ of A.
In Lemma 8 we determine the relations that the parameters of the problem must satisfy in order to deduce that such a section exists.
The parameters of the problem are the dimension of the space of global sections of a D tensor power of $\mathcal{L}$ (parameter D), the number of points in Σ (parameter N), the multiplicity of the section at the subscheme Σ along a sub-space W of $T_{A_{K'}}$ (parameter M).
The existence of the section s is equivalent to the non injectivity of the restriction map $\phi : E \to F$ between vector bundles on $\operatorname{Spec} \mathcal{O}_K$, where E is the push-forward of $\mathfrak{L}^D$ to $\operatorname{Spec} \mathcal{O}_K$ and F the push-forward of the restriction of $\mathfrak{L}^D$ to $\Sigma_{W,gM}$, (see 4.3).
The idea of the proof of lemma 8 is to suppose that ϕ is injective and to deduce the slope inequality (46) from property 7. On one side we use theorem 1 to find a lower bound for the left-hand side of (46). On the other side we use an analitic method (lemmas 2 and 7) to give an upper bound for the right-hand side of (46). We then choose the parameters N, M and D so that lower and

upper bound are sharp enough to contradict the slope inequality (46). We can conclude that ϕ is non injective.
To choose the parameters so that the operator norm estimates (see will contradict the slope inequality. This implies that, under this choice of parameters, the map $\phi : E \to F$ can not be injective and we are done.
In order to apply property 7 we must endow E with a hermitian inner product (see 1.5). Moreover we have to define a filtration of the vector bundle F (see (24)) and hermitian inner products on the corresponding quotients (see (5.1.1)).

4.2 Non-reduced Subschemes of Abelian Varieties and their MB-models

4.2.1 General Notions and Notations

If X is a scheme we denote by X^{top} its underlying topological space and by $\mathcal{O}_X$ its structural sheaf.

Definition 9 *Let $j : Y \to X$ be a closed immersion. We denote by $\mathcal{I}_Y :=$ $\ker j^\sharp$ the ideal sheaf of Y.*
The t-th power of $\mathcal{I}_Y$ is a sheaf of ideals, we indicate it by $\mathcal{I}_Y^t$. The cokernel $\mathcal{O}_{Y,t}$ of the natural inclusion $\mathcal{I}_Y^t \to \mathcal{O}_X$ defines a scheme $Y_t := (Y^{\text{top}}, j^{-1}\mathcal{O}_{Y,t})$, (see [14] ex 3.2.5). We call Y_t the subscheme of X of multiplicity t at Y.

The exact sequence

$$0 \longrightarrow \mathcal{I}_Y/\mathcal{I}_Y^t \longrightarrow \mathcal{O}_X/\mathcal{I}_Y^t \longrightarrow \mathcal{O}_X/\mathcal{I}_Y \longrightarrow 0.$$

induce a closed immersion $i_t : Y \to Y_t$.

We recall the notion of schematic image and closure of a scheme, (see [14] ex 3.11.d).

Definition 10 *Let $f : Z \to X$ be a morphism of schemes. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors trough Y, and if Y' is any other closed subscheme of X trough which f factors, then $Y \to X$ factors trough Y' too. The scheme Y is called the schematic image of Z in X. If f is an immersion then the scheme Y is called the schematic closure of Z in X.*

If Z is a reduced subscheme then Y is the topological closure of the image $f(Z)$ with the restricted structural sheaf. Equivalently we can say that Y is the projective limit of all closed subschemes Y' that contains $f(Z)$.

Notations:
In the following paragraphs we denote by A an abelian variety of dimension g defined over a number field, by $\overline{\mathcal{L}}$ an ample symmetric hermitian line bundle over A, by Σ_K a finite subset of $A(K)$ and by S the scheme $\operatorname{Spec} K$. We denote by T_A the tangent space at zero of $A(\mathbb{C})$ and by $\check{T}_A$ its dual the space of differential at zero.
Finally we denote by $\mathcal{S}$ the arithmetic variety $\operatorname{Spec} \mathcal{O}_K$ and by $(\pi : \mathcal{A} \to \mathcal{S}, \mathcal{L}, \Sigma)$ a MB-model of $(A, \mathcal{L}, \Sigma_K)$, (see 3.2).

4.2.2 An Example: Non-Reduced Points on a Semiabelian Scheme

This example will be of fundamental importance. We consider the case of a point $\mathcal{P}$ on the semiabelian scheme $\mathcal{A}$ over $\mathcal{S}$.
A point $\mathcal{P}$ of $\mathcal{A}$ with value in $\mathcal{O}_K$ is defined as a morphism of schemes

$$\mathcal{P} : \mathcal{S} \to \mathcal{A}.$$

Since $\mathcal{A}$ is a scheme over $\mathcal{S}$, the morphism $\mathcal{P}$ is a closed immersion.
We denote by $\mathcal{P}_t$, meaning $\mathcal{P}_t : \mathcal{S}_t \to \mathcal{A}$, the subscheme of $\mathcal{A}$ of multiplicity t at $\mathcal{P}$, (see def. 9).
The scheme $\mathcal{A}$ is smooth over $\mathcal{S}$ thus the push forward via $\pi \circ \mathcal{P}_t$ of the structural sheaf $\mathcal{O}_{\mathcal{P},t}$ of $\mathcal{P}_t$ is a locally free sheaf over $\mathcal{O}_{\mathcal{P}}$.
Remark:
If we would have worked with an abelian variety over $S = \operatorname{Spec} K$ and a point $P : S \to A$ defined over K , we would have gotten the subscheme $P_t : S_t \to A$ of multiplicity t at P. If $\mathcal{A}$ is a MB-model of A then the subscheme $\mathcal{P}_t$ is the schematic closure in $\mathcal{A}$ of the subscheme P_t.

4.2.3 Non Reduced Subschemes of Dimension Zero

We consider an immersion $\Sigma_K : (\coprod S) \to A$ with values in $A(K)$, i.e. a disjoint union of points $P : S \to A$ defined over K. We can extend the definitions of multiplicity in the following way.
Let $t : {\Sigma_K}^{\mathrm{top}} \to \mathbb{N}^+$ be a map that associates to any $P \in \Sigma_K$ an integer number $t(P)$ that we call multiplicity at P.
For each point $P \in \Sigma_K$ we consider the subscheme $P_{t(P)}$ of multiplicity $t(P)$ at P. We call the scheme $\Sigma_{K,t} := \coprod_{P \in \Sigma_K} P_{t(P)}$ subscheme of multiplicity t at Σ_K .
Remark:
the ideals $\mathcal{I}_{P_i}^{t(P_i)}$ are pairwise coprime, i.e. $\mathcal{I}_{P_i}^{t(P_i)} + \mathcal{I}_{P_j}^{t(P_j)} = \mathcal{O}_A$. Therefore the structural sheaf $\mathcal{O}_{\Sigma_{K,t}} := \bigoplus_{P \in \Sigma_K} \mathcal{O}_{P,t(P)}$ is isomorphic to ${\Sigma_K}^{-1}(\mathcal{O}_A / \cap \mathcal{I}_P^{t(P)})$ (see [2] II.1 prop.6).

Now we extend the definition to the semi-abelian scheme $\mathcal{A}$.
We have seen in the example 4.2.2 that the scheme $\mathcal{P}_t$ is the schematic closure of the scheme P_t. We define the scheme associated to $\Sigma_{K,t}$ as $\Sigma_t := \coprod_{P\in\Sigma_K} \mathcal{P}_{t(P)}$.
Remark:
The scheme Σ_t is not always a subscheme of $\mathcal{A}$. We recall that, from the definition of MB-model (§3.2), for every point $P \in \Sigma_K$ there exists a section $\epsilon_P : \operatorname{Spec} \mathcal{O}_K \to \mathcal{A}$. Therefore there is a natural epimorphism from Σ_t to the schematic closure of $\Sigma_{K,t}$. It is an isomorphism only if the ideals $(\mathcal{I}_{\mathcal{P}})_{\mathcal{P}\in\Sigma}$ are pairwise coprime.

4.2.4 Subschemes of Multiplicity t along a Sub-Bundle of the Tangent Bundle

We denote by $0_A : \operatorname{Spec} K \to A$ the origin of the abelian variety A. Let W be a sub-space of the tangent space T_A defined over K. Since A is smooth we have the isomorphism $\mathcal{O}_A/\mathcal{I}_{0_A}^2 \cong K \oplus \mathcal{I}_{0_A}/\mathcal{I}_{0_A}^2 = K \oplus \check{T}_A$. We denote by $S_{W,1}$ the spectrum of $K \oplus \check{W}$. The inclusion $W \hookrightarrow T_A$ induces a surjection of algebras $\mathcal{O}_{A_0}/\mathcal{I}_{0_A}^2 \to K \oplus \check{W}$ and hence a closed embedding of schemes

$$0_{A,W,1} : S_{W,1} \hookrightarrow A.$$

We consider the schematic image $S_{W,t}$ of the scheme $S_{W,1} \times \cdots \times S_{W,1}$ under the addition morphism

$$+_t : \underbrace{A \times ..... \times A}_{t-\text{times}} \longrightarrow A.$$

We define $0_{A,W,t} : S_{W,t} \to A$ to be the subscheme of A of multiplicity t at 0_A along W.

Let us consider the semi-abelian scheme $\mathcal{A}$.
We denote by $\mathcal{S}_{W,t}$ the schematic closure of $S_{W,t}$ in $\mathcal{A}$ and we call

$$0_{\mathcal{A},W,t} : \mathcal{S}_{W,t} \to \mathcal{A}$$

the subscheme of $\mathcal{A}$ of multiplicity t at $0_{\mathcal{A}}$ along W.
Since $\mathcal{A}$ is smooth, the scheme $0_{\mathcal{A},W,t}$ is a flat finite subscheme of $\mathcal{A}$ hence affine.

If P is any point of A different from the origin we consider the translation isomorphism $t_{-P} : A \to A$. We define the subscheme $P_{W,t}$ of multiplicity t at P along W as the pull-back via t_P of the scheme $0_{A,W,t}$.

Finally we define the scheme $\mathcal{P}_{W,t}$ of multiplicity t at $\mathcal{P}$ along W as the schematic closure of $P_{W,t}$ in $\mathcal{A}$ or equivalently as the pull-back via the translation map $t_{-\mathcal{P}}$ of the immersion $0_{\mathcal{A},W,t}$.

4.2.5 Ideals Sheaves Associated to a Non-Reduced Subscheme

Let k be a positive integer. Let $0_{A,W,k}$ be the subscheme of A of multiplicity k at 0_A along W and let $0_{\mathcal{A},W,k}$ be the scheme associated to $0_{A,W,k}$, (see 4.2.4).

The schemes $0_{A,W,k}$ and $0_{\mathcal{A},W,k}$ are affine schemes, hence we identify a sheaf on $0_{A,W,k}$ or $0_{\mathcal{A},W,k}$ with the module of its global sections.
From the definition of push-forward the module of global sections of a sheaf on $0_{\mathcal{A},W,k}$ is an $\mathcal{O}_K$-module that coincides with the module of global sections of the push-forward of the sheaf on $\operatorname{Spec}\mathcal{O}_K$. Thus we will identify locally free sheaves on $0_{\mathcal{A},W,k}$ and their push-forward to $\operatorname{Spec}\mathcal{O}_K$.

By definition $0_{A,W,k}$ is the generic fiber of $0_{\mathcal{A},W,k}$ thus $\mathcal{O}_{0_A,W,k} = \mathcal{O}_{0_{\mathcal{A}},W,k} \otimes_{\mathcal{O}_K} K$ and since K is flat over $\mathcal{O}_K$ we have an embedding of algebras $\mathcal{O}_{0_{\mathcal{A}},W,k} \hookrightarrow \mathcal{O}_{0_A,W,k}$.
Let's choose any positive integer M and let g be the dimension of the abelian variety A. We want to define a filtration of $\mathcal{O}_K$-modules associated to the subscheme $0_{A,W,2gM}$. Let Σ_K be a reduced non-connected subscheme of A containing 0_A. Let $\Sigma_{K,W,gM} := 0_{A,W,2gM} \coprod_{0_A \neq P \in \Sigma_K} P_{W,gM}$ be the associated non-reduced subscheme. Then there exist closed immersions $0_A = 0_{A,W,0} \hookrightarrow 0_{A,W,1} \hookrightarrow 0_{A,W,2} \cdots \hookrightarrow \Sigma_{K,W,gM}$. For any integer $1 \leq k \leq 2gM$ let us denote by $\mathcal{I}_{0_A,W,k}$ the sheaf of ideals of $0_{A,W,k-1}$ in $0_{A,W,2gM}$.
We define $\mathcal{O}_K$-modules associated to the above ideals as follows

$$\mathcal{I}_{0_{\mathcal{A}},W,k} := \mathcal{I}_{0_A,W,k} \cap \mathcal{O}_{0_{\mathcal{A}},W,2gM}.$$

The ascending chain

$$0 = \mathcal{I}_{0_A,W,2gM} \subset \cdots \subset \mathcal{I}_{0_A,W,k} \subset \cdots \subset \mathcal{I}_{0_A,W,1} \subset \mathcal{O}_{\Sigma_K,W,gM}$$

defines the filtration

$$0 = \mathcal{I}_{0_{\mathcal{A}},W,2gM} \subset \cdots \subset \mathcal{I}_{0_{\mathcal{A}},W,k} \subset \cdots \subset \mathcal{I}_{0_{\mathcal{A}},W,1} \subset \mathcal{O}_{\Sigma,W,gM}$$

of saturated submodules of $\mathcal{O}_{\Sigma,W,gM}$, hence the quotients $\mathcal{I}_{0_{\mathcal{A}},W,k}/\mathcal{I}_{0_{\mathcal{A}},W,k+1}$ are torsion free.
There exists a natural map

$$I : \operatorname{Sym}^k(\check{W}) \to \mathcal{I}_{0_A,W,k}/\mathcal{I}_{0_A,W,k+1}. \tag{20}$$

Since any formal group over a field of characteristic zero is an additive formal group the map I is an isomorphism, (see [15] thm. 1).
We recall that $T_A = \mathcal{T}_A \otimes_{\mathcal{O}_K} K$. We define the saturated submodule $\mathcal{W} := W \cap \mathcal{T}_A$ of $\mathcal{T}_A$.
Intersecting with $\check{\mathcal{T}}_A$ the exact sequence

$$0 \longrightarrow \mathrm{Ker} \longrightarrow \check{T}_A \longrightarrow \check{W} \longrightarrow 0$$

we deduce that $\check{\mathcal{W}} = \check{W} \cap (\check{\mathcal{T}}_A/(\mathrm{Ker} \cap \check{\mathcal{T}}_A))$. We have that $\check{W} \cong \mathcal{I}_{0_A,W,1}/\mathcal{I}_{0_A,W,2}$, hence $\check{\mathcal{W}} = (\mathcal{I}_{0_A,W,1}/\mathcal{I}_{0_A,W,2}) \cap (\check{\mathcal{T}}_A/(\mathrm{Ker} \cap \check{\mathcal{T}}_A) = \mathcal{I}_{0_A,W,1}/\mathcal{I}_{0_A,W,2}$. Moreover $\mathcal{I}_{0_A,W,k}/\mathcal{I}_{0_A,W,k+1} = (\mathcal{I}_{0_A,W,1}/\mathcal{I}_{0_A,W,2})^k$.
It follows that the map I restricts to a morphism of $\mathcal{O}_K$-modules

$$J : \mathrm{Sym}^k(\check{\mathcal{W}}) \longrightarrow \mathcal{I}_{0_A,W,k}/\mathcal{I}_{0_A,W,k+1}. \tag{21}$$

Both modules are torsion free hence this map is injective. From [11] lem. 2.4 it follows that the cokernel of the morphism J is a torsion module annihilated by $k!$.

If Σ_K is a disjoint union of points of A, we reproduce the previous construction for each $P \in \Sigma_K$ and we define $\mathcal{O}_K$-modules $\mathcal{I}_{\mathcal{P},W,k}$ for which the quotients $(\mathcal{I}_{\mathcal{P},W,k}/\mathcal{I}_{\mathcal{P},W,k+1})$ are torsion free. Then there is a natural isomorphism

$$\mathrm{Sym}^k(\check{W}) \longrightarrow \mathcal{I}_{P,W,k}/\mathcal{I}_{P,W,k+1} \tag{22}$$

and there exists an injection of modules

$$\mathrm{Sym}^k(\check{\mathcal{W}}) \longrightarrow \mathcal{I}_{\mathcal{P},W,k}/\mathcal{I}_{\mathcal{P},W,k+1} \tag{23}$$

whose cokernel is annihilated by $k!$.

4.3 Filtration of a Locally free Sheaf on $\mathrm{Spec}\,\mathcal{O}_K$

Let K be the field of definition of the abelian variety A of dimension g. Let W be a subspace of T_{A_L}, defined over a finite extension K' of K. Let $\sigma_0 : K' \longrightarrow \mathbb{C}$ be an embedding and let $\gamma \in W_{\sigma_0}$ be a non zero period of A_{σ_0}. For any given integer M and N we do the following construction.
Let $P_i : S \longrightarrow A$ be the reduced N-torsion point of A such that $P_{i,\sigma_0}^{\mathrm{top}} = \exp_{\sigma_0} \frac{i\gamma}{N}$. We suppose that the points P_i are defined over the field K'.

Definition 11 *We call γ-linear N-torsion subscheme of multiplicity gM along W, the subscheme of A defined by the disjoint union $\Sigma_{K',W,gM} := 0_{A,W,2gM} \coprod_{i=1}^{N-1} P_{i,W,gM}$.*
We also define the $\mathcal{O}_K$-scheme associated to $\Sigma_{K',W,gM}$ as the disjoint union $\Sigma_{W,gM} := 0_{\mathcal{A},W,2gM} \coprod_{i=1}^{N-1} \{\mathcal{P}_i\}_{W,gM}$.

We want to emphasize that these two schemes have multiplicity $2gM$ at zero but multiplicity gM at the all other points.
We consider a symmetric ample line bundle $\mathcal{L}$ on A. Let $(\mathcal{A}, \mathfrak{L}, \Sigma)$ be a MB-model of $(A, \mathcal{L}, \Sigma_{K'})$. For any integer D we consider the D tensor power $\mathfrak{L}^D$ of the invertible sheaf $\mathfrak{L}$ on $\mathcal{A}$.
We denote the push forward via π of the line bundle $\mathfrak{L}^D$ by

$$E := \pi_* \mathfrak{L}^D$$

and the the pull-back via $\Sigma_{W,gM}$ of the line bundle $\mathfrak{L}^D$ by

$$F := \Sigma_{W,gM}{}^* \mathfrak{L}^D.$$

Since $\pi : \mathcal{A} \to S$ is an arithmetic variety E is a locally free sheaf on $\operatorname{Spec} \mathcal{O}_K$ of rank equal to the dimension of $H^0(A, \mathcal{L}^D)$, (see 1.5). From the definition F is also a locally free sheaves over $\operatorname{Spec} \mathcal{O}_K$ (see 1.3).
We denote the restriction map that sends a global section $s \in \mathfrak{L}^D$ to its pull-back via $\Sigma_{W,gM}$ by

$$\phi : E \to F.$$

We remark that the precise definition of F is $F := (\pi \Sigma_{K',W,gM})_* \Sigma^*_{W,gM} \mathfrak{L}^D$ but we identify the $\mathcal{O}_K$-module of the global sections of the push-forward $(\pi \Sigma_{K',W,gM})_* \Sigma^*_{W,gM} \mathfrak{L}^D$ with the $\mathcal{O}_K$-module of the global section of $\Sigma^*_{W,gM} \mathfrak{L}^D$.
We have $E \otimes K' = H^0(A, \mathcal{L}^D)$ and $F \otimes K' = H^0(\Sigma_{K',W,gM}, \Sigma^*_{K',W,gM} \mathcal{L}^D)$.
By flatness of K over $\mathcal{O}_K$ the restriction map $\phi : E \to F$ is injective if and only if $\Phi : H^0_{\mathbb{C}}(A, \mathcal{L}^D) \to H^0(\Sigma_{K',W,gM}, \Sigma^*_{K',W,gM} \mathcal{L}^D)$ is injective.

In order to apply property 7 we define

$$\begin{array}{lll} F_{0,k} & := \mathcal{I}_{0_{\mathcal{A}},W,2gM-k} \otimes \Sigma^* \mathfrak{L}^D & \text{for } 0 \leq k \leq 2gM - 1, \\ F_k & := (\mathcal{I}_{0_{\mathcal{A}},W,2gM} \oplus_{0_{\mathcal{A}} \neq \mathcal{P} \in \Sigma} \mathcal{I}_{\mathcal{P},W,gM-k}) \otimes \Sigma^* \mathfrak{L}^D & \text{for } 0 \leq k \leq gM - 1, \\ F_{gM} & := F_{0,0} & \end{array} \tag{24}$$

and

$$\begin{array}{lll} G_{0,2gM-k} & := F_{0,2gM-k}/F_{0,2gM-k-1} & \\ & = (\mathcal{I}_{0_{\mathcal{A}},W,k}/\mathcal{I}_{0_{\mathcal{A}},W,k+1}) \otimes \Sigma^* \mathfrak{L}^D & \text{for } 0 \leq k \leq 2gM, \\ G_{gM-k} & := F_{2gM-k}/F_{2gM-k-1} & \\ & = (\oplus_{0_{\mathcal{A}} \neq \mathcal{P} \in \Sigma} \mathcal{I}_{\mathcal{P},W,k}/\mathcal{I}_{\mathcal{P},W,k+1}) \otimes \Sigma^* \mathfrak{L}^D & \text{for } \ 0 \leq k \leq gM. \end{array} \tag{25}$$

We denote by $\mathcal{I}_{\Sigma,W,k} := \mathcal{I}_{0_{\mathcal{A}},W,2gM} \oplus_{\mathcal{P} \in \Sigma} \mathcal{I}_{\mathcal{P},W,k}$.
The descending chain of $\mathcal{O}_K$-modules

$$\mathcal{I}_{0_{\mathcal{A}}} \supset .... \supset \mathcal{I}_{0_{\mathcal{A}},W,2gM} \supset \mathcal{I}_{\Sigma} \supset \mathcal{I}_{\Sigma,W,1} \supset ..... \supset \mathcal{I}_{\Sigma,W,gM}$$

induces the filtration of the sheaf F on $\operatorname{Spec} \mathcal{O}_K$

$$0 = F_0 \subset F_1 \subset \ldots\ldots \subset F_{gM} = F_{0,0} \subset \ldots\ldots \subset F_{0,2gM-1} \subset F_{0,2gM} := F.$$

Remark
We kept the case of the point zero separated from the case of the non-trivial N-torsion points because the first case will represent the construction part and the second the extrapolation for the proof of Lemma 8.

5 The Proof of the Subvariety Theorem

5.1 Estimates for Operators Norms

5.1.1 Hermitian Metrics and Morphisms on a Filtration

We want to define a metric on the quotients F_k/F_{k-1} and $F_{0,k}/F_{0,k-1}$. We emphasize that we do not define a metric on F to induce then the quotient metric on F_k/F_{k-1}, but we keep the freedom to give a local definition on the quotients of the filtration.
From the isomorphism (20), (22) and the definition of G_k and $G_{0,k}$ we deduce that for all $1 \leq k \leq gM$ there exist isomorphisms

$$G_{gM-k,\sigma} \cong \bigoplus_{0_A \neq P \in \Sigma_K} \mathrm{Sym}^k \check{W}_\sigma \otimes P^* \mathcal{L}_\sigma^D. \tag{26}$$

For all $1 \leq k \leq 2gM$ there exist isomorphisms

$$G_{0,2gM-k,\sigma} \cong \mathrm{Sym}^k \check{W}_\sigma \otimes {0_A}^* \mathcal{L}_\sigma^D. \tag{27}$$

We assumed to have an hermitian line bundle $\overline{\mathcal{L}}$ on A and we endowed the tangent bundle T_A with the hermitian inner product defined by the first Chern-class of $\mathcal{L}$, (see 3.1). We consider on $\check{T}_A$ the dual hermitian product of T_A (see 1.2.1) and on $\check{W}$ its restriction (see 1.2.4). We endow $P^*\mathcal{L}^D$ with the pull-back metric given by $\overline{\mathcal{L}}^D$, (see 1.2.1).
We define on G_{gM-k} and $G_{0,2gM-k}$ a hermitian inner product using the isomorphisms (26) and (27) and the hermitian product canonically induced on the symmetric product, tensor product and direct sum as explained in 1.2.7, 1.2.3 and 1.2.2..

Definition 12 *We define morphisms $\phi_k : \phi^{-1}(F_k) \to G_k$ to be the composition of the restriction map ϕ and the natural projection $F_k \to G_k = F_k/F_{k-1}$; and morphisms $\phi_{0,k} : \phi^{-1}(F_{0,k}) \to G_{0,k}$ to be the composition of the restriction map ϕ and the natural projections $F_{0,k} \to G_{0,k} = F_{0,k}/F_{0,k-1}$*

5.1.2 Trivialization on the Tangent Space

Let $\exp : \mathrm{Lie}(A) \to A$ be the exponential map of an abelian variety A of dimension g and let $\mathcal{L}$ be an ample symmetric line bundle on A.
We consider on T_A, the tangent space at zero, the hermitian metric induced by the first Chern class of $\mathcal{L}$. We denote the related norm $||\cdot||$. We denote by $|\cdot|$ the standard Euclidean norm on $\mathbb{C}$. We endow the trivial bundle $\mathcal{O}_{T_A}$ on T_A with the norm

$$||f(z)||_{z^*\mathcal{L}} := |f(z)|e^{m(z)}$$

with f any section of $\mathcal{O}_{T_A}$ and $m(z) := -\frac{\pi}{2}||z||^2$.
The line bundle $\exp^* \mathcal{L}$ is trivial on T_A. We can choose a trivialization such that the isomorphism $\zeta : \exp^* \overline{\mathcal{L}} \to \overline{\mathcal{O}}_{T_A}$ is an isometry, (see [3] 5.3.3).
In particular if Ω is an open set of T_A on which exp is an homeomorphism and s is a section of $\mathcal{L}^D(\exp(\Omega))$ then for every $z \in \Omega$ we have

$$||s(\exp z)||_{\mathcal{L}^D} = |f(z)|e^{m(z)} \tag{28}$$

with $f := \zeta(\exp^* s)$ and $m(z) := -\frac{\pi}{2}D||z||^2$.
Let $\mathfrak{F}_A$ be a fundamental domain for the lattice $\Lambda_A := \ker(\exp)$. Then (13) gives $\chi(A, \mathcal{L}) = \int_{\mathfrak{F}_A} d\lambda$ with $d\lambda := \underbrace{\omega_H \wedge ... \wedge \omega_H}_{g-times}$ and ω_H the translation invariant representative of $c_1(\mathcal{L})$.
By definition the normalized Haar-measure on A satisfies

$$1 = \int_A d\mu = \int_{\mathfrak{F}_A} \exp^* d\mu. \tag{29}$$

Then we can write

$$\frac{d\lambda}{\exp^* d\mu} = \chi(A, \mathcal{L}). \tag{30}$$

5.1.3 Bound for the Norm of the Operators $\phi_{0,k}$

We are now ready to give an estimate for the norm of the operator $\phi_{0,k}$, see def. 12. We recall that we choose the norm of a morphism between hermitian vector bundles over Spec $\mathcal{O}_K$ to be the operator norm. We denote by $\epsilon_\sigma = \min(1, \rho(A_\sigma, \mathcal{L}_\sigma))$ where $\rho(A_\sigma, \mathcal{L}_\sigma)$ is the radius of injectivity of $\mathcal{L}_\sigma$, (see 15).

Lemma 2 *For any integer $1 \leq k \leq 2gM$ and any embedding $\sigma : K' \to \mathbb{C}$ the operator norm of the restriction map $\phi_{0,k}$ satisfies the inequality*

$$||\phi_{0,k}||_\sigma^2 \leq D^g \chi(A, \mathcal{L}) \pi^{-g} e^{\pi D \epsilon_\sigma^2} \frac{(k+g)!}{k!} \epsilon_\sigma^{-2(k+g)}.$$

Proof We have to estimate $||\phi_{0,2gM-k}||_\sigma^2 = \sup_{s \neq 0} \frac{||\phi_{0,2gM-k}(s)||^2_{G_{0,2gM-k},\sigma}}{||s||^2_{\mathcal{L}_\sigma^D}}$. The proof will be the same for any embedding σ, for a matter of easier notation we forget the index σ.
As first step we find a lower bound for $||s||^2_{\mathcal{L}^D}$.
We recall that the norm of a section of a line bundle on an abelian variety is by definition

$$||s||^2_{\mathcal{L}^D} := \int_A ||s_z||^2_{\mathcal{L}^D} d\mu$$

with $d\mu_\sigma$ the normalized Haar-measure. From the relations (29) and (30) we get

$$||s||^2_{\mathcal{L}^D} := \int_{\mathfrak{T}_A} ||s_z||^2_{\mathcal{L}^D} \exp^* d\mu \geq \int_\Omega ||s_z||^2_{\mathcal{L}^D} \chi(A, \mathcal{L}^D)^{-1} d\lambda$$

where Ω is an open of T_A on which exp is a homeomorphism. In particular we can chose Ω to be the open ball $B(0, \epsilon)$ with center in zero and radius $\epsilon = \min(1, \rho(A, \mathcal{L}))$.
Replacing formula 28 in the integral above we get

$$||s||^2_{\mathcal{L}^D} \geq \chi(A, \mathcal{L}^D)^{-1} \int_{B(0,\epsilon)} |f(z)|^2 e^{-\pi D ||z||^2} d\lambda. \tag{31}$$

We choose an orthonormal basis $e_1, ...e_g$ of $\check{T}_A$ with respect to the inner product induced by $c_1(\mathcal{L}^D)$. Let $z_1, ...z_g$ denote the corresponding coordinates. The holomorphic function $f(z)$ can be developed in Taylor expansion

$$f(z) = \sum_I a_I z^I$$

where I is a multi-index $(i_1, ...i_g)$, z^I is the monomial $z_1^{i_1} \cdot \ldots \cdot z_g^{i_g} \in \mathrm{Sym}^{|I|}\left(\check{T}_A\right)$ and $a_I \in \mathbb{C}$.
Substituting this development in the last integral we get

$$||s||^2_{\mathcal{L}^D} \geq \chi(A, \mathcal{L}^D)^{-1} \int_{B(0,\epsilon)} |\sum_I a_I z^I|^2 e^{-\pi D ||z||^2} d\lambda.$$

Parseval's Formula yields

$$\int_{B(0,\epsilon)} |\sum_I a_I z^I|^2 e^{-\pi D ||z||^2} d\lambda = \sum_I |a_I|^2 \int_{B(0,\epsilon)} |z^I|^2 e^{-\pi D ||z||^2} d\lambda.$$

Lemma 4 below gives the relation

$$\int_{B(0,\epsilon)} |z^I|^2 e^{-\pi D ||z||^2} d\lambda = C\left(g, |I|, \epsilon\right) ||z^I||^2_{S^k}$$

with

$$C\left(g, k, \epsilon\right)^{-1} \leq \pi^{-g} e^{\pi D \epsilon^2} \frac{(k+g)!}{k!} \epsilon^{-2(k+g)}.$$

Therefore

$$||s||^2_{\mathcal{L}^D} \geq \chi(A, \mathcal{L}^D)^{-1} \sum_I |a_I|^2 C\left(g, |I|, \epsilon\right) ||z^I||^2_{S^k}. \tag{32}$$

As second step we compute the norm of the image of the section s

$$\begin{aligned}
&||\phi_{0,2gM-k}\left(s\right)||^2_{G_{0,2gM-k}} \\
&\qquad = || \sum_{|I|=k, i_g=0} a_I \otimes z^I ||^2_{\epsilon_0^* \mathcal{L}^D \otimes \mathrm{Sym}^k \check{W}} \\
&\qquad = \sum_{|I|=k, i_g=0} ||a_I||^2_{\epsilon_0^* \mathcal{L}^D} \cdot ||z^I||^2_{S^k} \\
&\qquad = \sum_{|I|=k, i_g=0} |a_I|^2 e^{-\pi D ||0||^2} \cdot ||z^I||^2_{S^k}.
\end{aligned}$$

As third step we estimate the norm of the operator $\phi_{0,2gM-k}$ using step 1 and step 2.

$$\begin{aligned}
||\phi_{0,2gM-k}||^2 &= \sup_{s \neq 0} \frac{||\phi_{0,2gM-k}\left(s\right)||^2_{G_{0,2gM-k}}}{||s||^2_{\mathcal{L}^D}} \\
&\leq \sup_{s \neq 0} \chi(A, \mathcal{L}^D) \frac{1}{C\left(g, k, \epsilon\right)} \frac{\sum_{|I|=k} |a_I|^2 \cdot ||z^I||^2_{S^k}}{\sum_{|I|} |a_I|^2 \cdot ||z^I||^2_{S^k}} \\
&\leq \chi(A, \mathcal{L}^D) C\left(g, k, \epsilon\right)^{-1}.
\end{aligned}$$

Recall that $\chi\big(A, \mathcal{L}^D\big) = D^g \chi(A, \mathcal{L})$ and so the result follows.

□

We now recall a classical lemma on metrics that we are going to use in lemma 4.

Lemma 3 *Let $\varphi : G \to \mathrm{GL}\left(W\right)$ be an irreducible representation of a group G on a vector space W. Let W be endowed with the two G-invariant scalar products $\langle\, , \rangle_1$ and $\langle\, , \rangle_2$. Suppose moreover that $\langle \cdot, \cdot \rangle_1 = 0$ if and only if $\langle \cdot, \cdot \rangle_2 = 0$. Then there exist a constant $\lambda \in \mathbb{R}$ such that*

$$|| \cdot ||_2 = \lambda || \cdot ||_1.$$

Proof
We define $\lambda_1 := \inf_{w \in W} \frac{||w||_2}{||w||_1}$, the set $E := \{w \in W : ||w||_2 = \lambda_1 ||w||_1\}$ is a sub-space of W. The two norms are G-invariant hence the space E is G-invariant. The irreducibility of the representation implies $E = W$. □

It remains to prove the following

Lemma 4 *Let $z_1, ..., z_g$ be an orthonormal basis of the space of differential $\check{T}_A$ with respect to the hermitian inner product h_E defined by the Riemann-form of the line bundle $\mathcal{L}^D$. Let I, J be two multiindeces of norm k and let z^I, z^J be the related monomial of* $\mathrm{Sym}^k\left(\check{T}_A\right)$.
We consider on $\mathrm{Sym}^k\left(\check{T}_A\right)$ *two different inner products. The first one is defined by*

$$\langle z^I, z^J\rangle_{\mathcal{L}^D} := \int_{B(0,\epsilon)} z^I \overline{z}^J e^{-\pi D||z||^2} d\lambda.$$

The second one is the induced quotient metric as we have described in 1.2.7

$$\langle z^I, z^J\rangle_{S^k} := S^k h_E\left(z^I, z^J\right).$$

Then
i) there exists a constant $C(g,k,\epsilon)$ such that

$$||\cdot||^2_{\mathcal{L}^D} = C(g,k,\epsilon)\,||\cdot||^2_{S^k}.$$

ii) The following estimate holds

$$\pi^g e^{-\pi D\epsilon^2}\frac{k!}{(k+g)!}\epsilon^{2(k+g)} \le C(g,k,\epsilon) \le \pi^g \frac{k!}{(k+g)!}\epsilon^{2(k+g)}.$$

Proof i) From Parseval's formula it follows that $\langle\cdot,\cdot\rangle_{\mathcal{L}^D} = 0$ if and only if $\langle\cdot,\cdot\rangle_{S^k} = 0$. Moreover the action of the unitary group is irreducible on $\mathrm{Sym}^k\left(\check{T}_A\right)$ (see [16] I.4). Applying lemma 3 we deduce i).
ii)From the choice of the basis e_i of T_A we have that the Riemann form of $\mathcal{L}^D$ has a diagonal representation, thus $||z||^2 = \sum |z_i|^2$. From i) we know that $C(g,k,\epsilon)$ is the same for any element in $\mathrm{Sym}^k(\Omega_A)$, it is enough to compute it for the element

$$\left(\sum |z_i|^2\right)^k = r^{2k}.$$

First we estimate the integral

$$\int_{B(0,\epsilon)} \left(\sum |z_i|^2\right)^k e^{-\pi D\sum |z_i|^2} d\lambda.$$

Passing to polar coordinates we get

$$\begin{aligned}\int_{B(0,\epsilon)} \left(\sum |z_i|^2\right)^k e^{-\pi D\sum |z_i|^2} d\lambda \\ = \mathrm{vol}\left(S^{2g-1}\right)\int_0^\epsilon r^{2k+2g-1} e^{-\pi D r^2} dr\end{aligned} \qquad (33)$$

where S^{2g-1} is the unitary sphere and vol indicates its volume. The minimum of the function $e^{-\pi D r^2}$ for $r \in (0, \epsilon)$ is attained in $r = \epsilon$ thus

$$\begin{aligned}\int_{B(0,\epsilon)} \left(\sum |z_i|^2\right)^k e^{-\pi D \sum |z_i|^2} d\lambda \\ \geq \mathrm{vol}\left(S^{2g-1}\right) e^{-\pi D \epsilon^2} \frac{\epsilon^{2(k+g)}}{2(k+g)}.\end{aligned} \tag{34}$$

The maximum of the function $e^{-\pi D r^2}$ for $r \in (0, \epsilon)$ is attained in $r = 0$ thus

$$\begin{aligned}\int_{B(0,\epsilon)} \left(\sum |z_i|^2\right)^k e^{-\pi D \sum |z_i|^2} d\lambda \\ \leq \mathrm{vol}\left(S^{2g-1}\right) \frac{\epsilon^{2(k+g)}}{2(k+g)}.\end{aligned} \tag{35}$$

From (4) we know that

$$||z_I||_{S^k} = \frac{I!}{k!} \qquad \text{(where } I! = i_1! \cdot \ \cdot i_g!).$$

Moreover from the generalized Binomial formula we have $\left(\sum |z_i|^2\right)^k = \sum \frac{k!}{I!} |z^I|^2$ and so

$$\begin{aligned}\int_{B(0,\epsilon)} \left(\sum |z_i|^2\right)^k e^{-\pi D \sum |z_i|^2} d\lambda \\ = \sum_{|I|=k} \frac{k!}{I!} \int_{B(0,\epsilon)} |z^I|^2 e^{-\pi ||z||^2} d\lambda \\ = C(g, k, \epsilon) \binom{g+k-1}{k}.\end{aligned} \tag{36}$$

We recall that the volume of the $(2g-1)$-dimensional sphere is

$$\mathrm{vol}\left(S^{2g-1}\right) = \frac{2\pi^g}{(g-1)!}.$$

Comparing (34) and (36) we deduce

$$\pi^g e^{-\pi D \epsilon^2} \frac{k!}{(k+g)!} \epsilon^{2(k+g)} \leq C(g, k, \epsilon).$$

Comparing (35) and (36) we deduce

$$C(g, k, \epsilon) \leq \pi^g \frac{k!}{(k+g)!} \epsilon^{2(k+g)}.$$

□

5.1.4 Bound for the Norm of the Operators ϕ_k

First we are going to state the analogous of lemma 2 for the operators ϕ_k.

Lemma 5 *For any integer $1 \le k \le gM$ and any embedding $\sigma : K \to \mathbb{C}$ the operator norm of the restriction map $||\phi_{gM-k}||_\sigma$ satisfies the inequality*

$$||\phi_{gM-k}||_\sigma^2 \le (N-1)\, D^g \chi(A,\mathcal{L})\, \pi^{-g} e^{\pi D \epsilon_\sigma^2} \frac{(k+g)!}{k!} \epsilon_\sigma^{-2(k+g)}.$$

Proof The proof follows the proof of lemma 2. We recall that the function $|f(z)|^2 e^{-\pi D||z||^2}$ is periodic with respect to $\mathfrak{F}_A$. From formula (31) we deduce that for each point P_i

$$||s||^2_{\mathcal{L}^D} \ge \chi(A, \mathcal{L}^D)^{-1} \int_{B(p_i,\epsilon)} |f(z)|^2 e^{-\pi D||z||^2} d\lambda.$$

Let $f(z) = \sum_I c_I z^I$ be the Taylor expansion of the function $f(z)$ centered in p_i with $P_i = \exp p_i$.
From Parseval's Formula and lemma 4 we deduce
Therefore

$$||s||^2_{\mathcal{L}^D} \ge \chi(A, \mathcal{L}^D)^{-1} \sum_I |c_I|^2 C(g, |I|, \epsilon)\, ||z^I||^2_{S^k}. \tag{37}$$

Now we compute the norm of the image of the section s

$$\begin{aligned} ||\phi_{2gM-k}(s)||^2_{G_{0,2gM-k}} & \\ &= || \sum_{|I|=k, i_g=0} c_I \otimes z^I ||^2_{P_i^*\mathcal{L}^D \otimes \mathrm{Sym}^k \check{W}} \\ &= \sum_{|I|=k, i_g=0} ||c_I||^2_{p_i^*\mathcal{L}^D} \cdot ||z^I||^2_{S^k} \\ &= \sum_{|I|=k, i_g=0} |c_I|^2 e^{-\pi D||p_i||^2_{\mathcal{L}^D}} \cdot ||z^I||^2_{S^k}. \end{aligned} \tag{38}$$

Finally we estimate the norm of the operator ϕ_{2gM-k} using (37) and (38).

$$\begin{aligned} ||\phi_{2gM-k}||^2 &= \sup_{s\neq 0} \frac{||\phi_{2gM-k}(s)||^2_{G_{2gM-k}}}{||s||^2_{\mathcal{L}^D}} \\ &\le \sup_{s\neq 0} \chi(A, \mathcal{L}^D) \frac{1}{C(g,k,\epsilon)} \frac{\sum_{|I|=k} |c_I|^2 \cdot ||z^I||^2_{S^k}}{\sum_{|I|} |c_I|^2 \cdot ||z^I||^2_{S^k}} \\ &\le \chi(A, \mathcal{L}^D) C(g,k,\epsilon)^{-1}. \end{aligned}$$

This conclude the proof.

□

We want to find a better bound for the norm of the operators $||\phi_k||$. The idea is to use the fact that every section in the domain of ϕ_k has a zero of multiplicity at least $2gM$ in zero and to apply a special form of the Schwarz lemma.

Let consider the spaces

$$\Omega_z := \{f : \mathbb{C} \to \mathbb{C} \text{ holomorphic } : \exists\, C \in \mathbb{R} \; |f(z)| \leq Ce^{\pi D|z|^2}\}$$

$$\Omega_y := \{f : \mathbb{C} \to \mathbb{C} \text{ holomorphic } : \exists\, C \in \mathbb{R} \; |f(z)| \leq Ce^{2\pi D y^2}\}$$

with $z = x + iy$.

We endow Ω_z with the norm $||f||_{\Omega_z} := \sup_{z\in\mathbb{C}} |f(z)|e^{-\pi D|z|^2}$ and Ω_y with the norm $||f||_{\Omega_y} := \sup_{z\in\mathbb{C}} |f(z)|e^{-2\pi D y^2}$.

Lemma 6 *The map*

$$\begin{array}{rcccc} I & : & (\Omega_z, ||\cdot||_{\Omega_z}) & \longrightarrow & (\Omega_y, ||\cdot||_{\Omega_y}) \\ & : & f(z) & \longmapsto & f(z)\, e^{-\pi D z^2} \end{array}$$

is an isometric isomorphism.

Proof

We first proof that the norm of I is 1. This follows from the fact that

$$\begin{aligned} |f(z)\, e^{-\pi D z^2}|e^{-2\pi D y^2} &= |f(z)|\,|e^{-\pi D\left(x^2-y^2+2ixy\right)}|e^{-2\pi D y^2} = \\ &= |f(z)|e^{-\pi D\left(x^2-y^2\right)}e^{-2\pi D y^2} = |f(z)|e^{-\pi D|z|^2}. \end{aligned}$$

Finally I is an isomorphism because $e^{\pi D z^2} \neq 0$. □

We will need to apply the Phragmen-Lindelöf Theorem that we write for simplicity. This theorem gives conditions under which the theorem of the maximum for bounded domains can be extended to unbounded domains.

Theorem 3 (see [34] 12.9) *Suppose*

$$\Delta := \{z = x + iy \; : \; |y| < R\} \quad \text{and} \quad \overline{\Delta} := \{z = x + iy \; : \; |y| = R\}.$$

Let f be continues on $\overline{\Delta}$ and holomorphic on Δ. Suppose that there are constants $\alpha < 1$ and $A < \infty$ such that

$$|f(z)| < \exp\{A\exp(\alpha|x|)\} \qquad z = x + iy \in \Delta,$$

then

$$|f(z)| \leq \sup_{w\in\partial\overline{\Delta}} |f(w)| \qquad \textit{for all } z \in \Delta.$$

We are going to give here a special form of the Schwarz Lemma, that we will need in order to get the "good" bound of lemma 7.

Property 9 (Schwarz Lemma) *Let $f(z)$ be a function in the space Ω_y. We suppose that $f(z)$ is $L \cdot \mathbb{Z}$-periodic with $L \in \mathbb{R}^+$ and that has a zero of order at lest T at zero. Let us consider the strip*

$$\overline{\Delta} := \{z = x + iy \ : \ |y| \leq \frac{T}{2\pi DL^2}\}.$$

If $2\pi DL^2 \geq T$ then for every $z \in \overline{\Delta}$ we have

$$\left| \frac{f(z)}{(\sin \pi z)^T} \right| \leq ||f||_{\Omega_y} e^{\frac{-T^2}{2\pi DL^2}}.$$

Proof The function

$$g(z) := \frac{f(L \cdot z)}{(\sin \pi z)^T}$$

is entire because of the zero multiplicity of $f(z)$ at $L \cdot \mathbb{Z}$. Let $z = x + iy$, we recall that $|\sin(x+iy)|^2 = |\sin x \cosh y + i \cos x \sinh y|^2 = (\sin x \cosh y)^2 + (\cos x \sinh y)^2$. Since $(\cosh y)^2 > (\sinh y)^2$ we have that $|\sinh y| \leq |\sin(z)| \leq \cosh y$.
We defined above the norm on the space Ω_y as $||f||_{\Omega_y} := \sup_{z \in \mathbb{C}} |f(z)| e^{-2\pi D y^2}$, thus

$$|g(z)| \leq ||f||_{\Omega_y} \frac{e^{2\pi DL^2 y^2}}{|\sinh \pi y|^T} \qquad \text{for all } z = x + iy. \tag{39}$$

We want to verify that $g(z)$ satisfies the hypothesis of the Phragmen-Lindelöf Theorem above. We have to control that the function $g(z)$ does not grow too fast when $|x|$ goes to ∞. Since both $f(L \cdot z)$ and $sin(\pi z)$ are $2\mathbb{Z}$-periodic on the x-axis also $g(z)$ is. Moreover $g(z)$ is entire so it is bounded on a compact set, thus

$$|g(z)| \leq C \qquad \text{for } z \in \Delta$$

with C a constant.
By the Phragmen-Lindelöf Theorem we conclude that the maximum of the entire function $g(z)$ on the strip $\overline{\Delta}$ is attained on its boundary $\partial\overline{\Delta}$, i.e.

$$|g(z)| \leq \sup_{w \in \partial\overline{\Delta}} |g(w)| \qquad \text{for all } z \in \Delta. \tag{40}$$

Now we want to estimate $g(z)$ on this boundary $\partial\overline{\Delta}$. Because of (39) it is enough to estimate $\frac{e^{2\pi DL^2 y^2}}{|\sinh \pi y|^T}$ for $|y| = T/2\pi DL^2$. The function is symmetric hence we shall study it only for positive values of y. Since $e^{-1} \leq \frac{1}{3}e$ then, for any $y \geq 1$, we have that $\sinh \pi y = \frac{1}{2}(e^{\pi y} - e^{-\pi y}) \geq \frac{1}{3}e^{\pi y}$ which in turn is estimate by $\frac{1}{3}e^{\pi y} \geq e^{2y}$. For this last estimate just remark that $\frac{1}{3}e^{\pi y} \geq e^{\pi y - 1.1}$. It follows that for $y \geq 1$ the relation $\frac{e^{2\pi DL^2 y^2}}{|\sinh \pi y|^T} \leq \frac{e^{2\pi DL^2 y^2}}{e^{2Ty}}$ holds. In particular if we set $y = \frac{T}{2\pi DL^2} \geq 1$ we have $e^{2\pi DL^2 y^2 - 2Ty} = e^{-T^2/2\pi DL^2}$. Now by (39) we deduce

$$|g(z)| \leq ||f||_{\Omega_y} e^{\frac{-T^2}{2\pi DL^2}} \qquad \text{for } z \in \partial\overline{\Delta},$$

and from (40) follows the desired inequality.

□

Remark 1 Since $|\sinh \pi y| \leq e^{\pi y}$ the real function $R(y) := \frac{e^{2\pi DL^2 y^2}}{|\sinh \pi y|^T}$ is bounded from below by $e^{2\pi DL^2 y^2 - \pi T y}$. Thus $\min_y e^{2\pi DL^2 y^2 - \pi T y} \leq \min_y R(y)$. The minimum of $e^{2\pi DL^2 y^2 - \pi T y}$ is attained for $y = \frac{T}{4DL^2}$ and its value is $e^{\frac{-\pi T^2}{4DL^2}}$. This means that another choice of the strip $\overline{\Delta}$ or a better approximation of $|sinh\pi y|^T$ would not have essentially given any better result but just a slightly better constant.

Remark 2 The points of minimum of $R(y)$ are the same of $r(y) := \log R(y)$. The first derivative of $r(y)$ is

$$r'(y) := 4\pi DL^2 y - T\pi \coth \pi y. \tag{41}$$

The function $r'(y)$ has just one zero given by the intersection of the line $\frac{4DL^2}{T}y$ and the function $\coth \pi y$. This zero is a point of minimum because $r(y)$ goes to infinity for y that goes to zero or to infinity. If $\frac{4DL^2}{T} < 1$ then the line $\frac{4DL^2}{T}y$ intersects $\coth \pi y$ for a value y_0 such that $\coth y_0$ is 'about' 1. By the relation (41) it follows that y_0 approaches $\frac{T}{4DL^2}$. This explains why we shall suppose $T > 4DL^2$, moreover it gives an approximation of the value of y for which $R(y)$ attains its minimum.

We denote by $\epsilon_\sigma = \min((1, \rho_s(A,\sigma)^{-1}, \rho_i(A_\sigma, \mathcal{L}_\sigma))$ where $\rho_i(A_\sigma, \mathcal{L}_\sigma)$ is the radius of injectivity (see 15) and $\rho_s(A_\sigma, \mathcal{L}_\sigma)$ is the radius of surjectivity of $\mathcal{L}_\sigma$ (see 17) .

Lemma 7 *If $gM > 2D||\gamma||^2$ then for any $0 \leq k \leq gM$ and any embedding $\sigma : K' \to \mathbb{C}$ that coincide with σ_0 on the field of definition of $P_i \in \Sigma$, the operator norm of the non-trivial restriction map ϕ_{gM-k} satisfies the inequality*

$$||\phi_{gM-k}||_\sigma^2 \leq (N-1)D^g\chi(A,\mathcal{L})(2g^2\log 3)^g\pi^{-g}\binom{k+g-1}{k}e^{2\pi D - \frac{(gM)^2}{\pi D||\gamma||^2}}\epsilon_\sigma^{-2g}.$$

Proof
We define $p_i = \frac{i}{N}\gamma$ and we recall that $\exp_{\sigma_0} p_i = P_i$.
Let $s \in \phi^{-1}(F_{gM})$ be a global section of $\mathfrak{L}^D$ in the domain of ϕ_{gM}, then $\exp^* s = f(\underline{t})$ is an entire function. From (28) it follows that the restriction $f(z) := f(z\gamma/||\gamma||)$ belongs to Ω_y. The function $f(z)$ has a zero of order $2gM$ in zero and it is thus $\mathbb{Z}||\gamma||$-periodic. We are in the hypothesis of the lemma 9, where

$$g(z) := \frac{f(z||\gamma||)}{(\sin \pi z)^{2gM}}$$

It follows that if $gM \geq 2D$ then

$$|g(z)| \leq 3||f||_{\Omega_y} e^{\frac{-\pi(gM)^2}{2D||\gamma||^2}}$$

for any z in the strip $\overline{\Delta} := \{z = x + iy \ : \ |y| \leq \frac{gM}{2D||\gamma||^2}\}$.
In particular we get that $|f(p_i)| \leq ||f||_{\Omega_y} e^{\frac{-\pi(gM)^2}{2D||\gamma||^2}}$, where $p_i = \frac{i}{N}\gamma$.
From Cauchy's estimate ([17] cor. 4.3) we deduce that the function $\frac{1}{I!} f^I(\underline{t})$ belongs to Ω_y. Moreover, by assumption, this function has a zero of order at least $2gM - |I|$ at $\mathbb{Z}$. Applying lemma 9, If $2gM - |I| \geq 2\pi D||\gamma||^2$ then

$$\left|\frac{f^I(p_i)}{I!}\right|^2 \leq ||f||^2_{\Omega_y} e^{\frac{-(2gM-k)^2}{\pi D||\gamma||^2}}$$

We get the worst bound when $|I| = gM$, namely for $gM \geq 2\pi D||\gamma||^2$ and $|\frac{1}{I!} f^I(p_i)|^2 \leq ||f||^2_{\Omega_y} e^{\frac{-(gM)^2}{\pi D||\gamma||^2}}$, where $|I| = gM$.
Now we are ready to estimate $||\phi_{gM-k}(s)||^2$ as follows:

$$\begin{aligned}
||\phi_{gM-k}(s)||^2_{G_{gM-k}} &= \left|\left| \sum_{i=1}^{N-1} \sum_{|I|=k, i_g=0} \frac{1}{I!} f^I(\underline{t}) \right|\right|^2_{\oplus P_i^* \mathcal{L}^D \otimes Sym^k \check{W}} \\
&= \sum^{N} -1_{i=1} \sum_{|I|=k, i_g=0} \left|\frac{1}{I!} f^I(p_i)\right|^2_{p_i^* \mathcal{L}^D} \otimes ||t^I||^2_{S^k} = \\
&= \sum_{i=1}^{N-1} \sum_{|I|=k, i_g=0} \left|\frac{1}{I!} f^I(p_i)\right|^2 e^{-\pi D||p_i||^2} \frac{I!}{k!} = \\
&\leq (N-1)\binom{k+g-1}{k} e^{-\pi D \frac{||\gamma||^2}{N^2}} ||f||^2_{\Omega_y} e^{\frac{-(2gM-k)^2}{\pi D||\gamma||^2}}.
\end{aligned} \tag{42}$$

We still need to relate $||f||^2_{\Omega_y}$ and $||s||^2_{\mathcal{L}^D}$. Since $f(\underline{t})$ is the pull-back of a global section of $\mathcal{L}^D$ on A there exists t_0 in the fundamental domain $\mathfrak{F}_A$ such that

$\sup_{\underline{t}\in\mathbb{C}^g} |f(\underline{t})|e^{-\pi D||\underline{t}||^2} = |f(t_0)|e^{-\pi D||t_0||^2}$. From the Cauchy formula we have $f(\underline{t}) = \frac{1}{2\pi i}\int_T \frac{f(\zeta)}{\prod(\zeta_i - t_i)} d\zeta_1 \wedge \cdot \wedge \zeta_g$, where T is the torus $T = S^1(\delta_1) \times ... \times S^1(\delta_g)$ with $S^1(\delta_i)$ is a circle of center t_i and radius δ_i. We denote by R the real interval $[\frac{\epsilon}{4g}, \frac{3\epsilon}{4g}]$. We remark that the Cauchy formula is true for any radius δ_i so we can integrate over R^g and we deduce

$$\begin{aligned} f(\underline{t}) &= \frac{1}{2\pi i}\left(\frac{2g}{\epsilon}\right)^g \int_{R^g}\int_{T(\delta)} \frac{f(\zeta)}{\prod(\zeta_i - t_i)} d\zeta_1 \wedge \cdot \wedge d\zeta_g \wedge d\delta_1 \wedge \cdot \wedge d\delta_g \\ &= \left(\frac{1}{2\pi i}\right)^g \left(\frac{2g}{\epsilon}\right)^g \int_{\mathfrak{A}} \frac{f(\zeta)}{\prod(\zeta_i - t_i)} d\zeta_1 \wedge \cdot \wedge d\zeta_g \wedge d\delta_1 \wedge \cdot \wedge d\delta_g, \end{aligned}$$

where $\mathfrak{A}$ is the annulus given by the Cartesian product $T \times R^g$. Since $\delta_i d\zeta_i \wedge d\delta_i = (\zeta_i - t_i) d\zeta_i \wedge d\bar{\zeta}_i$ we have

$$|f(\underline{t})|^2 \leq \left(\frac{1}{2\pi}\right)^{2g} \left(\frac{2g}{\epsilon}\right)^g \left(\int_{\mathfrak{A}} |f(\zeta)| \frac{\prod(\zeta_i - t_i)}{\prod|\zeta_i - t_i|^2} d\lambda\right)^2.$$

From the Cauchy-Schwarz inequality we deduce

$$|f(\underline{t})|^2 \leq \left(\frac{1}{2\pi}\right)^{2g} \left(\frac{2g}{\epsilon}\right)^{2g} \int_{\mathfrak{A}} |f(\zeta)|^2 d\lambda \int_{\mathfrak{A}} \frac{1}{\prod|\zeta_i - t_i|^2} d\lambda.$$

Computing the integral $\int_{\mathfrak{A}} \frac{1}{\prod|\zeta_i - t_i|^2} d\lambda = (2\pi)^g (\log 3)^g$ we find the bound

$$|f(\underline{t})|^2 \leq \left(\frac{2}{\pi}\right)^g \left(\frac{g}{\epsilon}\right)^{2g} (\log 3)^g \int_{\mathfrak{A}} |f(\zeta)|^2 d\lambda. \tag{43}$$

We remark that from the definition of ϵ the exponential map is injective on the annulus $\mathfrak{A}$. From the periodicity of f we have

$$\begin{aligned} \int_{\mathfrak{A}} |f(\zeta)|^2 e^{-\pi D||\zeta||^2} d\lambda &= \int_{\mathfrak{A} \mathrm{mod} \mathfrak{F}_A} |f(\zeta)|^2 e^{-\pi D||\zeta||^2} d\lambda \\ &\leq \int_{\mathfrak{F}_A} |f(\zeta)|^2 e^{-\pi D||\zeta||^2} d\lambda \end{aligned}$$

Since ϵ is smaller than the inverse of the surjectivity radius we see that $\epsilon \leq 1/||\underline{t}||$ and so

$$\int_{\mathfrak{A}} |f(\zeta)|^2 e^{-\pi D||\zeta||^2} d\lambda \geq e^{-\pi D(||\underline{t}||^2 + \epsilon^2 + 1)} \int_{\mathfrak{A}} |f(\zeta)|^2 d\lambda. \tag{44}$$

Moreover from (43) and (44) we deduce

$$||f||^2_{\Omega_y} e^{-\pi D(\epsilon+1)} \left(\frac{\pi}{2}\right)^g \left(\frac{\epsilon}{g}\right)^{2g} (\log 3)^{-g} \leq D^g \int_{\mathfrak{F}_A} |f(\zeta)|^2 e^{-\pi D||\zeta||^2} d\lambda = ||s||^2_{\mathcal{L}^D}$$

and from the definition of $||s||^2_{\mathcal{L}^D}$ we have

$$||f||^2_{\Omega_y}\chi(A,\mathcal{L}^D)e^{-\pi D(\epsilon+1)}(\log 3)^{-g}\left(\frac{\pi}{2}\right)^g\left(\frac{\epsilon}{g}\right)^{2g}\leq ||s||^2_{\mathcal{L}^D} \tag{45}$$

Comparing the upper bound (42) for $||\Phi_{gM-k}(s)||^2$ with the lower bound (45) for $||s||^2$ we get the result.

□

5.2 Choice of the Parameters and Slope inequality

If we suppose that the restriction map $\phi : E \to F$ is injective then we are in the hypothesis of property 7 for the map $\phi : E \to F$ and the filtration $\{F_{0,i},\ F_i\}$, where the corresponding quotients $G_{0,i},\ G_i$ are hermitian vector bundles as specified in 4.3 and 5.1.1. Property 7 gives the inequality

$$\begin{aligned}\widehat{\deg}_n \overline{E} \leq & \sum_{i=1}^{gM}\left(\operatorname{rg}\phi^{-1}(F_i/F_{i-1})\right)\left(\hat{\mu}_{\max}(\overline{G}_i)+\frac{1}{[K':\mathbb{Q}]}\sum_{\sigma:K'\to\mathbb{C}}\log||\phi_i||_\sigma\right)+\\ & \sum_{i=1}^{2gM}\left(\operatorname{rg}\phi^{-1}(F_{0,i}/F_{0,i-1})\right)\left(\hat{\mu}_{\max}(\overline{G}_{0,i})+\frac{1}{[K':\mathbb{Q}]}\sum_{\sigma:K'\to\mathbb{C}}\log||\phi_{0,i}||_\sigma\right).\end{aligned} \tag{46}$$

The idea of this proof is the following. On one side we use Theorem 1 to find a lower bound for $\widehat{\deg}_n \overline{E}$, (49). On the other side we use an analytic method to give an upper bound for the norm of the operators appearing in (46). We then choose the parameters N, M and D so that the lower and upper bounds are sharp enough to contradict the slope inequality (46). In this way we can conclude that ϕ is non injective.

Lemma 8 *Let A be an abelian variety of dimension g defined over a number field K. Let W be a subspace of $T_{A_{K'}}$ with K' a finite extension of K. Let $\sigma_0 : K' \to \mathbb{C}$ be an embedding and $\gamma \in W_{\sigma_0}$ be a non trivial period of A_{σ_0}. Let Σ be the reduced γ-linear N-torsion subscheme of A defined over K', (see definition 11).*
There exist integers M, N and D satisfying

$$\begin{aligned} D^g &\geq C_1(g)(\deg_{\mathcal{L}} A)^{-1}M^{g-s}hd\\ M &\geq C_2(g)Dr\\ M^2hd &\geq C_3(g)N^{2g}Drd(D+Mh+M\log M+\log N)\end{aligned} \tag{47}$$

such that

$$\Phi : H^0(A, \mathcal{L}^{\otimes D}) \to H^0(\Sigma, \mathcal{L}^{\otimes D}_{\Sigma,W,gM})$$

is not injective. Here $h := \max(1, h(A), \log \deg_{\mathcal{L}} A, h(W))$, $r := \max(1, ||\gamma||^2_{\sigma_0})$, $d = [K : \mathbb{Q}]$, *s is the codimension of* W *and* $C_i(g)$ *are constants depending only on* g.

Proof By contradiction we assume that Φ is injective. Then we prove that

$$\widehat{\deg}_n \overline{E} > \sum_{i=1}^{gM+1} \left(\operatorname{rg} \phi^{-1}(F_i/F_{i-1})\right) \left(\hat{\mu}_{\max}(\overline{G}_i) + \frac{1}{[K':\mathbb{Q}]} \sum_{\sigma : K' \to \mathbb{C}} \log ||\phi_i||_\sigma\right) + \sum_{i=1}^{2gM} \left(\operatorname{rg} \phi^{-1}(F_{0,i}/F_{0,i-1})\right) \left(\hat{\mu}_{\max}(\overline{G}_{0,i}) + \frac{1}{[K':\mathbb{Q}]} \sum_{\sigma : K' \to \mathbb{C}} \log ||\phi_{0,i}||_\sigma\right). \tag{48}$$

This contradict property 7 for an injective map. Thus ϕ can not be injective. We will denote the right hand-side of the above inequality by RHS and the left hand-side by LHS.
Using Theorem 1, we get

$$\widehat{\deg}_n \overline{E} = D^g \chi(A, \mathcal{L}) \left(-\frac{1}{2} h(A) + \frac{1}{4} \log \frac{D^g \chi(A, \mathcal{L})}{(2\pi)^g}\right)$$

whence

$$LHS \geq -\frac{1}{2} h D^g \chi(A, \mathcal{L}) \tag{49}$$

Now we estimate the right hand-side. We use the isomorphisms (26) and (27) to bound $\hat{\mu}_{\max}(\overline{G}_{0,2gM-k})$ and $\hat{\mu}_{\max}(\overline{G}_{gM-k})$. Since P_i are torsion points $\widehat{\deg}_n P_i^* \mathcal{L}^{\otimes D} = 0$.
From [11] prop. 4.1 we get the estimates

$$\hat{\mu}_{\max}(\overline{G}_{0,2gM-k}) \leq c_1(g) k h + k \log k \qquad \text{for } 1 \leq k \leq 2gM$$

and

$$\hat{\mu}_{\max}(\overline{G}_{gM-k}) \leq c_2(g) k h + k \log k \qquad \text{for } 1 \leq k \leq gM,$$

where

$$h := \max(1, h(A), \log \deg_{\mathcal{L}} A, h(W)).$$

We supposed that ϕ is injective thus

$$(\operatorname{rg} \phi^{-1}(F_{0,k}) - \operatorname{rg} \phi^{-1}(F_{0,k-1})) \leq \operatorname{rg} G_{0,k} \qquad \text{for } k \leq 2gM$$

and

$$(\mathrm{rg}\,\phi^{-1}(F_k) - \mathrm{rg}\,\phi^{-1}(F_{k-1})) \leq \mathrm{rg}\, G_k \qquad \text{for } \ k \leq gM.$$

Using the isomorphisms (20) and (22) we can compute

$$\mathrm{rg}\, G_{0,2gM-k} = \binom{k+g-s-1}{g-s-1} \qquad \text{for } \ k \leq 2gM$$

$$\mathrm{rg}\, G_{gM-k} = (N-1)\binom{k+g-s-1}{g-s-1} \qquad \text{for } \ k \leq gM.$$

Since the logarithm is a convex function, relations (16) and (17) imply

$$\frac{1}{[K':\mathbb{Q}]}\sum_{\sigma:K'\to\mathbb{C}} \log \epsilon_\sigma^{-2} \leq c(g)\log h \tag{50}$$

Lemma 2 and relation (50) yield

$$\frac{1}{[K':\mathbb{Q}]}\sum_{\sigma:K'\to\mathbb{C}} \log ||\phi_{0,2gM-k}||_\sigma \leq C_4(g)(D+k+k\log h).$$

We recall that the degree of the field of definition of a N-torsion point of A is at most $N^{2g}d$ with $d=[K:\mathbb{Q}]$ and K the field of definition of A. We suppose that $gM > 2D||\gamma||^2$. Using lemma 5 and lemma 7 we deduce

$$\frac{1}{[K':\mathbb{Q}]}\sum_{\sigma:K'\to\mathbb{C}} \log ||\phi_{gM-k}||_\sigma \leq C_5(g)(D+k+k\log h+\log N) - \frac{C_6(g)}{2N^{2g}d}\frac{M^2}{D||\gamma||^2}.$$

We can give a first bound for RHS

$$\begin{aligned} RHS \leq\ & C_4(g)\sum_{k=0}^{2gM}\binom{k+g-s-1}{g-s-1}(D+kh+k\log h+k\log(k+1)) \\ & + C_5(g)(N-1) \\ & \qquad \sum_{k=0}^{gM}\binom{k+g-s-1}{g-s-1}(D+kh+k\log h+k\log(k+1)+\log N) \\ & + C_6(g)(N-1)\sum_{k=1}^{gM+1}\left(\mathrm{rg}\,\phi^{-1}(F_i/F_{i-1}\right)\left(-\frac{M^2}{2N^{2g}dD||\gamma||^2}\right) \end{aligned}$$

and so

RHS

$$\leq C_7(g)\sum_{k=0}^{2gM}\binom{k+g-s-1}{g-s-1}(D+(k+1)h+k\log(k+1))$$
$$+C_8(g)(N-1)\sum_{k=0}^{gM}\binom{k+g-s-1}{g-s-1}(D+(k+1)h+k\log(k+1)+\log N)$$
$$-C_6(g)(N-1)\sum_{k=1}^{gM+1}\left(\mathrm{rg}\,\phi^{-1}(F_i/F_{i-1})\right)\left(\frac{M^2}{2N^{2g}dD||\gamma||^2}\right)$$

The negative term does not depend on k so it remains to find a lower bound for $\sum_{k=1}^{gM+1}(\mathrm{rg}\,\phi^{-1}(F_i/F_{i-1})) = \mathrm{rg}\,F_{gM}$. By definition $F_{gM} = \mathcal{I}_{0_{\mathcal{A}},W,2gM} \otimes \Sigma^*\mathfrak{L}^D$, i.e. it is the module of sections of $\mathfrak{L}^D$ with a zero of multiplicity at least $2gM$ along W at $0_{\mathcal{A}}$, therefore from (22) we have

$$\mathrm{rg}\,F_{gM} \geq D^g\chi(A,\mathcal{L}) - \binom{2gM+g-s}{g-s}.$$

We deduce the bound

RHS

$$\leq C_7(g)\sum_{k=0}^{2gM}\binom{k+g-s-1}{g-s-1}(D+(k+1)h+k\log(k+1))$$
$$+C_8(g)(N-1)\sum_{k=0}^{gM}\binom{k+g-s-1}{g-s-1}(D+(k+1)h+k\log(k+1)+\log N)$$
$$-C_6(g)(N-1)\sum_{k=0}^{gM}\left[D^g\chi(A,\mathcal{L})-\binom{2gM+g-s}{g-s}\right]\frac{M^2}{N^{2g}dD||\gamma||^2}$$

that implies

$$\begin{aligned}RHS \leq\ & C_9(g)NM^{g-s}(D+Mh+M\log M+\log N)\\ & - C_{10}(g)(N-1)[D^g\chi(A,\mathcal{L})-C_{11}(g)M^{g-s}]\frac{M^2}{N^{2g}dD||\gamma||^2}.\end{aligned} \tag{51}$$

We suppose that $N \geq 2$ and that the following inequalities hold

$$\begin{aligned}D^g\chi(A,\mathcal{L}) &\geq 2C_{11}(g)M^{g-s}hd\\ M^2hd &\geq C_{12}(g)N^{2g}Dd||\gamma||^2(D+Mh+M\log M+\log N).\end{aligned}$$

Under these conditions we deduce from (51)

$$RHS \leq -hD^g\chi(A,\mathcal{L}).$$

This contradicts (49) and proves (48). $\square$

5.3 The Multiplicity Estimate and Conclusion

Let A be an abelian variety of dimension g defined over a number field K and $\mathcal{L}$ a symmetric ample line bundle on A. Let $\sigma : K \to \overline{\mathbb{Q}}$ be an embedding. We consider the addition morphism

$$\begin{array}{ccccc} + & : & \underbrace{A \times \ldots\ldots \times A}_{g-\text{times}} & \longrightarrow & A \\ & : & (p_1, \ldots, p_g) & \longmapsto & p_1 + .. + p_g. \end{array}$$

Let S be a subscheme of A of dimension zero. We denote by $\Sigma : S_g \to A$ the schematic image of $\underbrace{S \times \ldots \times S}_{g-\text{times}}$ under the addition morphism.

If B is an abelian subvariety of $A_{\overline{\mathbb{Q}}}$ we denote by $r : A_{\overline{\mathbb{Q}}} \to A_{\overline{\mathbb{Q}}}/B$ the natural projection.

Theorem 4 (Multiplicity Estimate) *If the restriction map*

$$\Phi : H^0(A,\mathcal{L}) \longrightarrow H^0(A, \Sigma_*\mathcal{O}_{S_g} \otimes \mathcal{L})$$

is non injective then there exists an abelian subvariety B of $A_{\overline{\mathbb{Q}}}$, different from $A_{\overline{\mathbb{Q}}}$, such that

$$\mathrm{length}(r(S)) \cdot deg_{\mathcal{L}_\sigma} B \leq deg_{\mathcal{L}} A$$

with $\mathrm{length}(r(S))$ *the length of the scheme $r(S)$.*

We recall that the length of a module M is the length of a chain $0 = M_0 \subset \cdots \subset M_r = M$ with M_i/M_{i-1} simple (see [28] p. 12). And the length of an affine scheme is the length of the module of global sections of its structural sheaf.
Wüstholz has proven as first results of this type (see [40]). One can find other formulations and refinement of his result, where the effective constants are improved, see for example [33], [32] or [7].

5.3.1 The Proof of The Subvariety Theorem

Let us consider a power $\mathcal{L}^{\otimes D}$ of a line bundle $\mathcal{L}$ on A. We choose the scheme $S := \Sigma_{L,W,M}$ to be the γ-linear N-torsion subscheme of A of multiplicity M along W, where M is an integer, W the subspace of $T_{A_{K'}}$ of codimension s and γ a non trivial period (see 4.2.3). Note that $[L : K] \leq N^{2g}$ with L the minimal field of definition of S. We denote by $d := [K : \mathbb{Q}]$ the degree of the field K of definition of A. Let K' be a field of definition for W and S. The schematic image of $\Sigma_{L,W,M} \times .... \times \Sigma_{L,W,M}$ under the addition morphism turns out to be $\Sigma_{L,W,gM}$.

Lemma 8 tell us that if we choose N, D and M satisfying the conditions (47) then the hypothesis of the Zero Lemma are satisfied for the subscheme S and the sheaf $\mathcal{L}^{\otimes D}$. It follows that there exists a abelian subvariety B of $A_{\overline{\mathbb{Q}}}$ different from A, such that

$$\mathrm{length}(r(S)) \deg_{\mathcal{L}_\sigma^{\otimes D}} B \leq \deg_{\mathcal{L}^{\otimes D}} A$$

i.e.

$$\mathrm{length}(r(S)) \deg_{\mathcal{L}_\sigma} B \leq D^c \deg_{\mathcal{L}} A \tag{52}$$

with c the codimension of B in A.

Using (22) we compute the length of the subscheme $r(S)$. If $B = 0$ then

$$\mathrm{length}(r(S)) = N\binom{g-s+M}{g-s} \geq N\frac{M^{g-s}}{(g-s)!}. \tag{53}$$

If $T_B \not\subset W_\sigma$ then $r(S)$ contains $0_{A/B,M}$ and so

$$\mathrm{length}(r(S)) \geq \natural(S+B/B)\binom{c-s+1+M}{c-s+1} \geq \natural(S+B/B)\frac{M^{c-s+1}}{(c-s+1)!}. \tag{54}$$

Finally if $T_B \subset W_\sigma$ then $r(S)$ contains $0_{A/B,W',M}$ where W' is a subspace of $T_{A/B}$ of codimension s, and so

$$\mathrm{length}(r(S)) \geq \natural(S+B/B)\binom{c-s+M}{c-s} \geq \natural(S+B/B)\frac{M^{c-s}}{(c-s)!}. \tag{55}$$

If N, D and M satisfy the conditions

$$\begin{aligned} NM^{g-s} &\geq C_1(g)D^c \deg_{\mathcal{L}} A \\ M &\geq C_1(g)D \end{aligned} \tag{56}$$

then, from (53) and (54), we exclude the cases $B = 0$ and $T_B \not\subset W_\sigma$. It follows that $T_B \subset W_\sigma$. Moreover we can suppose $\natural(S + B/B) = 1$, this implies that S is a subgroup of B therefore the period γ is an element of the tangent space T_B. In this case by (55) and (52) we deduce

$$\deg_{\mathcal{L}} B \leq C(g)\frac{D^c}{M^{c-s}}\deg_{\mathcal{L}} A, \tag{57}$$

The conditions (47), (56) are compatible. They are satisfied if we choose N depending only on g, and D and M to be the integer parts of

$$D^* = C_D(g)(\deg_{\mathcal{L}} A)^{-1}\max\left(1, hdr^{\frac{g-s}{s}}, d(r\log(rd))^{\frac{g-s}{s}}\right)$$

$$M^* = C_M(g)(\deg_{\mathcal{L}} A)^{-1}\max\left(1, hdr^{\frac{g}{s}}, d(r\log(rd))^{\frac{g}{s}}\right),$$

where $d = [K : \mathbb{Q}]$, $h := \max(1, h(A), \log\deg_{\mathcal{L}} A, h(W))$, $r := \max(1, ||\gamma||^2_{\sigma_0})$, s the codimension of W and $C_D(g)$ and $C_M(g)$ constants depending only on g. Substituting those values in (57) we deduce the Subvariety Theorem 2.

6 Bounded Degree for Elliptic Isogenies

6.1 Preliminaries

Masser and Wüstholz proved in [20] that, given two isogenous elliptic curves E and E^* defined over a number field K, there exists an isogeny from E to E^* with degree bounded by $c(d) \cdot \max(1, h(E))^4$. Here $c(d)$ is a constant depending only on the degree d of the field K and $h(E)$ is the "naive" height, i.e. the height of the Weierstrass equation defining the elliptic curve. In this chapter we use the Subvariety Theorem 2 in order to improve this results in a quantitative respect and to make explicit in d the constant $c(d)$. We need some geometric trick in order to improve the bound to $c \cdot d^2 \max(1, h(E), \log d)^2$ for elliptic curves with complex multiplication and to $c \cdot d^2 \max(1, h(E), \log d)^3$ for elliptic curves without complex multiplication. Here $h(E)$ is the Faltings height of E. Faltings and näive height are equivalent (see (64)).
Since now on we fix an embedding $\sigma : K \to \mathbb{C}$ and we identify E (resp. E^*) with E_σ (resp. E^*_σ).

6.1.1 The Isogeny Lemma

From the "Isogeny Lemma" we learn how a non-split abelian subvariety of a product variety gives rise to an isogeny. The degree of this isogeny is bounded in terms of the degree of the subvariety. This theorem is the link between the Subvariety Theorem and Theorems 6 and 7.
One says that a connected algebraic subgroup H of a product group $E^{n_1} \times E^{*n_2}$ is split if it has the form $H = H_1 \times H_2$ for algebraic subgroup H_1 of E^{n_1} and H_2 of E^{*n_2}.

Theorem 5 (Isogeny Lemma [20]) *For positive integers n_1 and n_2 suppose $E^{n_1} \times E^{*n_2}$ has a non-split connected algebraic subgroup of dimension d and degree Δ. Then there is an isogeny between E and E^* of degree at most $3^{2d}\Delta^2$.*

Using the Subvariety Theorem we construct an abelian subvariety B of $A := E^{n_1} \times E^{*n_2}$. The degree of B is bounded in terms of the height of A, of the height of a given subspace $W \subset T_{A_{K'}}$ and of the norm of a non trivial period $\gamma \in W$ of A. If we can give good bounds for $h(W)$ and $||\gamma||$ then we can obtain interesting results on the degree of a minimal isogeny between the elliptic curves E and E^*.

6.1.2 The Successive Minima

Let E be an elliptic curve defined over a number field K. Let $\mathcal{L}$ be a symmetric ample line bundle on E that gives principal polarization, for example $\mathcal{L}(0_E)$.
The exponential map of the elliptic curve E defines the exact sequence

$$0 \longrightarrow \Lambda \longrightarrow T_E \xrightarrow{\exp} E \longrightarrow 0$$

where T_E is the tangent space of E at zero and Λ the kernel of exp.
We endow the tangent space T_E with the metric induced by $\mathcal{L}$, (see 3.1). We denote the successive minima of the Euclidean lattice $\langle \Lambda, ||\cdot||_{T_E}\rangle$ by λ_1, λ_2. We fix elements ω_1, $\omega_2 \in \Lambda_1$ such that

$$\lambda_i = ||\omega_i||_{T_E} \quad i = 1, 2. \tag{58}$$

We call ω_1 and ω_2 the minimal periods of E_1. We remark that in dimension two the minimal periods are an integral basis for the lattice Λ. Indeed if $\omega \in \Lambda$ then $\omega = q_1\omega_1 + q_2\omega_2$ with $q_i \in \mathbb{Q}$, without loss of generality we can assume that $-1/2 \leq q_i < 1/2$ and that $\langle \omega_1, \omega_2\rangle \leq 0$. If $q_2 = 0$ we have $||\omega|| \leq 1/2||\omega_1||$ contradicting the minimality of ω_1. If $q_2 \neq 0$ Then $||\omega|| \leq 1/2(||\omega_1||^2 + ||\omega_2||^2)^{1/2} < ||\omega_2||$ contradicting the minimality of ω_2.

6.1.3 The Injectivity Radius

The radius of injectivity of E with respect to the metric on T_E induced by a symmetric ample line bundle $\mathcal{L}$ (see 3.1) is the largest real number $\rho(E, \mathcal{L})$ such that the restriction of the exponential map to the open ball with center in zero and radius $\rho(E, \mathcal{L})$ is a homeomorphism. Then

$$\rho(E, \mathcal{L}) = \frac{1}{2}||\omega_1||.$$

Indeed it is clear that $\rho(E, \mathcal{L}) \leq \frac{1}{2}||\omega_1||$. Let's now prove that the exponential map is a homeomorphism when restricted to the open ball $B(0, \frac{1}{2}||\omega_1||)$. Suppose that there exist z_1 and $z_2 \in B(0, \frac{1}{2}||\omega_1||)$ such that $\exp(z_1) = \exp(z_2)$, i.e. such that $z_1 - z_2$ is an element ω of the lattice Λ_1. Recall that ω_1 is the minimal period of E and $||z_1 - z_2|| < ||\omega_1||$; this gives a contradiction. By definition $B(0, \rho(E, \mathcal{L})) \leq \chi(E, \mathcal{L})$ and therefore

$$||\omega_1|| \leq \pi^{-\frac{1}{2}}\chi(E, \mathcal{L})^{\frac{1}{2}}. \tag{59}$$

In the case of a principal polarization we get

$$||\omega_1|| \leq \pi^{-\frac{1}{2}}. \tag{60}$$

6.1.4 The Product Variety and its Model

Let E and E^* be elliptic curves defined over a number field K. We choose symmetric ample line bundles $\mathcal{L}_1$ on E and $\mathcal{L}_2$ on E^* that give principal polarizations. We denote by p_i the natural projection on the i-th factor. We consider on $E \times E^*$ the symmetric ample line bundle $\mathcal{L} := p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2$. From the Künneth formula, $\mathcal{L}$ is a principal polarization for $E \times E^*$.
We denote by $T_{E\times E^*}$ the tangent space of $E \times E^*$ at zero and we recall that $T_{E\times E^*} = T_E \times T_{E^*}$. We define on $T_{E\times E^*}$ the metric induced by $\mathcal{L}$. Since the first Chern class of the tensor product is the sum of the first Chern-classes we have that for any $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 \in T_E$ and $\gamma_2 \in T_{E^*}$ the following relation holds

$$||\gamma||^2_{\mathcal{L}} = ||\gamma_1||^2_{\mathcal{L}_1} + ||\gamma_2||^2_{\mathcal{L}_2}. \tag{61}$$

We denote by $(\mathcal{E}, \pi, \mathfrak{L}_1)$ (respectively $(\mathcal{E}^*, \pi, \mathfrak{L}_2)$) a MB-model for $(E, \mathcal{L}_1)$ (respectively $(E^*, \mathcal{L}_2)$), (see 3.2). We indicate by $\mathcal{T}_{\mathcal{E}}$ (respectively $\mathcal{T}_{\mathcal{E}^*}$) the corresponding tangent bundles and since the MB-model commutes with the product we have $\mathcal{T}_{\mathcal{E}\times\mathcal{E}^*} = \mathcal{T}_{\mathcal{E}} \times \mathcal{T}_{\mathcal{E}^*}$.

6.1.5 The Height of a Subspace

Any subspaces W of the Lie-algebra Lie(A) of an abelian variety defines a hermitian $\mathcal{O}_K$-module $\mathcal{W} := \mathcal{T}_{\mathcal{A}} \cap W$ endowed with the norm induced by $\mathcal{L}$. We define the height of the subspace W as

$$h(W) := -\widehat{\deg}_n \overline{\mathcal{W}}.$$

6.1.6 The Faltings Height

We recall that the Faltings height is defined as $h(E) := \widehat{\deg}_n \overline{\Omega}_{\mathcal{E}/\mathcal{S}}$ where $\Omega_{\mathcal{E}/\mathcal{S}}$ is the sheaf of relative differentials of $\mathcal{E}$ with respect to $\mathcal{S} = \operatorname{Spec}\mathcal{O}_K$ (see 18). From the definition we deduce that $h(E \times E^*) = -\widehat{\deg}_n \bigwedge^2(\overline{T_E \oplus T_{E^*}}) = -\widehat{\deg}_n (\overline{T_E \otimes T_{E^*}})$ and from property 1 we have $h(E \times E^*) = -\widehat{\deg}_n \overline{T}_E - \widehat{\deg}_n \overline{T}_{E^*}$. Since E is a curve its canonical bundle is the dual of $\mathcal{T}_{E^*}$, the same holds for E^* and so

$$h(E \times E^*) = h(E) + h(E^*).$$

In [36] prop 2.1 we can read how the Faltings height of an elliptic curve is related to the imaginary part of $\tau := \omega_2/\omega_1$, namely

$$\operatorname{Im}(\tau) \leq c \cdot \max(1, h(E)) \tag{62}$$

with c an absolute constant. This constant can be explicitly computed after using the estimates $\mathrm{Im}(\tau) \leq (2\pi)^{-1} \log(|j(\tau)| + 1193)$ (see [8] p.187) and the estimates in [36] prop. 1.1, ex. p. 256 and 2.(11). We deduce

$$\begin{aligned} \mathrm{Im}(\tau) &\leq \pi^{-1}(120 + 24.3 \max(1, h(E)) \\ &\leq 7^2 \max(1, h(E)). \end{aligned} \tag{63}$$

We denote by $h_N(E)$ the näive height of the curve E. From [36] proposition 2.1 we have

$$h_N(E) = 6h(E) + O(1 + \log(1 + h(E))). \tag{64}$$

6.2 Technical Results

We now report [20] lem. 4.1 because we need the relation appearing in the proof in order to prove lemma 10. We will use lemma 10 to estimate the height of a sub-bundle, (see 6.3).

Let E and E^* be elliptic curves defined over K and $\varphi : E \longrightarrow E^*$ an isogeny. The corresponding differential map on the tangent spaces satisfies $d\varphi\Lambda \subset \Lambda^*$. Let ω_1, ω_2, (respectively ω_1^*, ω_2^*) minimal basis of Λ (respectively Λ^*), (see 6.1.2). Then there are integers m_{ij} such that

$$d\varphi(\omega_1) = m_{11}\omega_1^* + m_{12}\omega_2^*, \quad d\varphi(\omega_2) = m_{21}\omega_1^* + m_{22}\omega_2^* \tag{65}$$

and

$$\deg \varphi = det(m_{ij}) = N.$$

We set the following notations $\tau := \omega_2/\omega_1$ and $\tau^* := \omega_2^*/\omega_1^*$; $y := \mathrm{Im}(\tau)$ and $y^* := \mathrm{Im}(\tau^*)$.

Lemma 9 [20] lem. 4.1 *With the above notations, we have*

$$|m_{ij}| \leq 20 \cdot N^{\frac{1}{2}}(yy^*)^{\frac{1}{2}}.$$

Proof The differential map on the tangent spaces is the multiplication by a number α. The above relations yield

$$\tau = \frac{m_{21} + m_{22}\tau^*}{m_{11} + m_{12}\tau^*} \tag{66}$$

and by taking imaginary parts we deduce

$$y = (m_{11}m_{22} - m_{12}m_{21})y^*|m_{11} + m_{12}\tau^*|^{-2}.$$

Hence

$$|m_{11} + m_{12}\tau^*|^2 = Ny^*/y \tag{67}$$

where $\tau = x + iy$ and $\tau^* = x^* + iy^*$.
Using this last relation we get

$$|m_{11} + m_{12}\tau^*|^2 = (m_{12}x^* + m_{11})^2 + (m_{12}y^*)^2 = Ny^*/y$$

which implies

$$|m_{12}| \leq (N/yy^*)^{\frac{1}{2}} \tag{68}$$

and

$$|m_{11}| \leq |m_{12}x^*| + (|m_{12}y^*|^2 + Ny^*/y)^{\frac{1}{2}}.$$

This two inequalities, together with $|s| \leq \frac{1}{2}$, give

$$|m_{11}| \leq 3(Ny^*/y)^{\frac{1}{2}}. \tag{69}$$

From (66) we get

$$|m_{21} + m_{22}\tau^*|^2 = |\tau|^2|m_{11} + m_{12}\tau^*|^2.$$

Since $|x| \leq \frac{1}{2}$ and $|y| \geq \frac{\sqrt{3}}{2}$, we see that $|\tau| \leq 2y^*$ and it follows by (67)

$$|m_{21} + m_{22}\tau^*|^2 \leq 4Nyy^*.$$

We play the same game as before to get

$$|m_{22}| \leq 4(Ny/y^*)^{\frac{1}{2}} \tag{70}$$

and

$$|m_{21}| \leq 20(Nyy^*)^{\frac{1}{2}}. \tag{71}$$

□

Lemma 10 *In the above notations we have*

$$|m_{ij}| \leq 21N. \tag{72}$$

Proof First we give the proof in the case $m_{ij} \neq 0$ for every i, j. We know that $N = m_{11}m_{22} - m_{12}m_{21}$. Since any $|m_{ij}|$ is bigger or equal than 1 it follows that

$$|m_{11}| \leq N + |m_{12}m_{21}|$$
$$|m_{22}| \leq N + |m_{12}m_{21}|$$
$$|m_{21}| \leq N + |m_{11}m_{22}|.$$

From relations (68), (69), (70) and (71) we deduce the claim.

We remark that $d\varphi\omega_1$, $d\varphi\omega_2$ are a minimal basis for $d\varphi\Lambda$, in fact the differential is a linear transformation between 1-dimensional vector spaces.
Let us now suppose that one of m_{ij} is zero.
If $m_{21} = 0$ then $m_{11}m_{22} = N$ and so $|m_{11}|$, $|m_{22}| \leq N$. We already know from (68) that

$$|m_{12}| \leq \frac{4}{3}N^{\frac{1}{2}}. \tag{73}$$

If $m_{22} = 0$ then $N = m_{12}m_{21}$ thus $|m_{12}|$, $|m_{21}| \leq N$. Since $d\varphi\omega_1$, $d\varphi\omega_2$ are a minimal basis, we have that $|m_{11}\omega_1^* + m_{12}\omega_2^*| \leq |m_{21}\omega_1^*|$. Dividing by $|\omega_1^*|$ we deduce $(m_{12}x^* + m_{11})^2 + (m_{12}y^*)^2 \leq (m_{21})^2$, which implies $y^* \leq |m_{21}|/|m_{12}|$. Using (69) we get $|m_{11}| \leq 3N$.
If $m_{11} = 0$ then $|m_{21}| \leq N$. From relation (67) we deduce $(\frac{y}{y^*})^{\frac{1}{2}} \leq N^{\frac{1}{2}}$. Using (70) we get $|m_{22}| \leq 4N$.
If $m_{12} = 0$ then $|m_{11}|$, $|m_{22}| \leq N$. We shall prove that $|m_{21}| \leq \frac{1}{2}|m_{11}| + |m_{22}|$. We consider the element $\omega := d\varphi(\omega_2 + j\omega_1)$ with $j = \pm 1$. The norm of ω is given by

$$|\omega|^2 = |\omega_1|^2((m_{21} + jm_{11} + m_{22}x^*)^2 + (m_{22}y^*)^2).$$

On the other hand we have

$$|d\varphi\omega_2|^2 \leq |\omega_1|^2((m_{21} + m_{22}x^*)^2 + (m_{22}y^*)^2).$$

We have already remarked that $d\varphi$ preserves the inequality of norms, the fact that $|\omega_2 + j\omega_1| \geq |\omega_2|$ implies that

$$|d\varphi\omega_2|^2 \leq |\omega|^2$$

or

$$(m_{21} + m_{22}x^*)^2 \leq (m_{21} + jm_{11} + m_{22}x^*)^2.$$

choosing j so that $jm_{11}(m_{21} + m_{22}x^*)$ is negative, we deduce that the relation $|m_{21}| \leq (|m_{11}| + |m_{22}|)/2$ must hold. □

6.3 The Height of a Sub-Bundle of the Tangent Bundle

The main idea to compute the height of a subspace W of the tangent space $T_E^{n_1} \times T_{E^*}^{n_2}$ is to define W as the image of $T_E^{n'_1} \times T_{E^*}^{n'_2}$ under an injective morphism of bounded norm and to apply property 6. The next property will be useful to bound the norm of a linear operator. However this property implies that the metric induced by $\mathcal{L}_\sigma$ is controlled by $\mathcal{L}$ independently of σ.

Proposition 1 *Let E be an elliptic curve defined over K and $\mathcal{L}$ an ample symmetric line bundle. Then for every embedding $\sigma : K \to \mathbb{C}$ we have*

$$||\omega_{1,\sigma}||^2_{\mathcal{L},\sigma} \geq \frac{\chi(E,\mathcal{L})}{7^2 \max(1, h(E))}$$

with $\omega_{1,\sigma}$ the minimal period of E_σ.

Proof The translation invariant representative of the first Chern class of $\mathcal{L}_\sigma$ is an alternating form $R_\sigma(\gamma_1, \gamma_2)$ which takes integer values on $\Lambda_\sigma \times \Lambda_\sigma$. Let $H_\sigma(z_1, z_2) := R_\sigma(iz_1, z_2) + iR_\sigma(z_1, z_2)$ be the associated hermitian metric on $T_{E,\sigma}$, which is by definition the metric induced by $\mathcal{L}_\sigma$, (see 3.1).
Let ω_1, ω_2 be a $\mathbb{Z}$-basis for Λ_σ such that $R_\sigma(\omega_2, \omega_1) \geq 0$. If we take a matrix representation of $R_\sigma(\gamma_1, \gamma_2)$ with respect to this basis we get

$$R_\sigma = \begin{pmatrix} 0 & R_\sigma(\omega_2, \omega_1) \\ -R_\sigma(\omega_2, \omega_1) & 0 \end{pmatrix}$$

and

$$\chi(E_\sigma, \mathcal{L}_\sigma) = R_\sigma(\omega_2, \omega_1). \tag{74}$$

We want to calculate

$$||\omega_1||^2_{\mathcal{L},\sigma} := R_\sigma(i\omega_1, \omega_1).$$

Let $\omega_1 = x_1 + iy_1$ and $\omega_2 = x_2 + iy_2$. In order to use the matrix representation R_σ we must express $i\omega_1$ as a linear combination of the basis ω_1, ω_2, i.e. $i\omega_1 = \lambda_1\omega_1 + \lambda_2\omega_2$. This gives the relations

$$\begin{aligned} x_1 &= \lambda_1 y_1 + \lambda_2 y_2 \\ -y_1 &= \lambda_1 x_1 + \lambda_2 x_2 \end{aligned}$$

which imply

$$\lambda_2 = \frac{x_1^2 + y_1^2}{x_1 y_2 - y_1 x_2}.$$

Since R_σ is alternating $R_\sigma(\lambda\omega_1 + \lambda_2\omega_2, \omega_1) = \lambda_2 R_\sigma(\omega_2, \omega_1)$.
Choose the isomorphism of T_E with $\mathbb{C}$ such that $\omega_1 = 1$ and ω_2 belongs to the upper half plane. We deduce

$$||\omega_1||^2_{\mathcal{L},\sigma} = \frac{1}{y_2} R_\sigma(\omega_2, \omega_1).$$

Using the relations (74) and (62) we conclude the proof. $\square$

Corollary 2 *Let E be an elliptic curve defined over K and $\mathcal{L}$ an ample symmetric line bundle. Then for every embedding $\sigma : K \to \mathbb{C}$ we have*

$$||\omega_{2,\sigma}||^2_{\mathcal{L},\sigma} \leq 14^2 \chi(E,\mathcal{L}) \max(1, h(E))$$

with $\omega_{2,\sigma}$ the biggest period of a minimal basis of E_σ.

Proof
We have proven in paragraph 6.1 that $||\omega_1||_{\mathcal{L},\sigma} = \lambda_{1,\sigma}$ and $||\omega_2||_{\mathcal{L},\sigma} = \lambda_{2,\sigma}$ with $\lambda_{i,\sigma}$ the minimal successive of Λ_σ.
From Minkowski's second Theorem (see [6] VIII.4.3.) we deduce

$$||\omega_1||_{\mathcal{L},\sigma} ||\omega_2||_{\mathcal{L},\sigma} \leq 4\chi(E,\mathcal{L}).$$

Using the lower bound of proposition 1 we deduce the corollary. □

Now we are going to estimate the height of a subspace W of a tangent space $E_1^{n_1} \times E_2^{n_2}$.

Lemma 11 *Let $l : T_E^{n_1} \to T_{E^*}^{n_2}$ be a linear map defined over K and let Γ_l be the graph of l. Then*

$$h(\Gamma_l) \leq n_1 \big(h(E) + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma : K \to \mathbb{C}} \log(1 + ||l||_\sigma) \big). \tag{75}$$

Proof
The metric on $T_{E^{n_1} \times E^{*n_2}}$ is the one induced by $\mathcal{L} \cong (p_1^* \mathcal{L}_1)^{n_1} \otimes (p_2^* \mathcal{L}_2)^{n_2}$ and on the subspaces we consider the restriction metric, (see 1.2.4). The height of a subspace W (see 6.1.5) is defined as

$$h(W) := -\widehat{\deg}_n \overline{\mathcal{W}}.$$

The linear map $L := id \oplus l : \; T_E^{n_1} \to T_E^{n_1} \times T_{E^*}^{n_2}$ is injective because it is the identity on the first factor and $L(\; T_E^{n_1}) = \Gamma_l$.
Applying property 6 we get

$$\widehat{\deg}_n \overline{T_E^{n_1}} \leq \widehat{\deg}_n \overline{L(T_E^{n_1})} + \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \log || \wedge^{n_1} (L) ||_\sigma. \tag{76}$$

We recall that $|| \wedge^r L|| \leq ||L||^r$. By the definition of L we deduce that

$$|| \wedge^{n_1} L||_\sigma \leq (1 + ||l||^2_\sigma)^{\frac{n_1}{2}} \leq (1 + ||l||_\sigma)^{n_1}.$$

Since E is a curve, its canonical bundle is the dual of $\mathcal{T}_E$. Thus $h(E) = -\widehat{\deg}_n \overline{T}_E$. Moreover $\widehat{\deg}_n \overline{T_E^{n_1}} = n_1 \widehat{\deg}_n \overline{T}_E$ (see property 1) and the proposition follows.

□

We deduce two corollaries that we need in the proof of Theorems 6 and 7.

Corollary 3 *Let $\varphi : E \to E^*$ be an isogeny between elliptic curves defined over K. Let $d\varphi : T_E \to T_{E^*}$ be the differential map on the tangent spaces. For any integer m let $m : T_E \to T_E$ be the multiplication by m. We consider the subspace image* $\mathrm{Im}(m \oplus d\varphi)$. *Then*

$$h(\mathrm{Im}(m \oplus d\varphi)) \leq h(E) + \log\left(14^2 \max(m, h(E)^{1/2} \deg \varphi)\right). \qquad (77)$$

Proof The map m is injective hence the map $m \oplus d\varphi$ is injective, too. From lemma 11 relation (76) we deduce

$$h(\mathrm{Im}(m \oplus d\varphi)) \leq h(E) + \frac{1}{[K : \mathbb{Q}]} \sum_{\sigma : K \to \mathbb{C}} \log(||m \oplus d\varphi||_\sigma). \qquad (78)$$

Since $||m \oplus d\varphi||_\sigma = (m^2 + ||d\varphi||_\sigma^2)^{1/2} \leq m + ||d\varphi||_\sigma$ it remains to estimate $||d\varphi||_\sigma$. First we want to find a lower bound for $||d\hat{\varphi}||_\sigma$ with $\hat{\varphi}$ the dual isogeny of φ. Since $d\hat{\varphi}$ is a linear operator on a 1 dimensional vector space its norm is given by $\frac{||d\hat{\varphi}(x)||_\sigma}{||x||_\sigma}$ for any $x \in T_E$. We recall that from (60) we get $||\omega_1^*||_\sigma \leq \pi^{-\frac{1}{2}}$ and from proposition 1 we have $||\omega_1||_\sigma^2 \geq 1/7^2 \max(1, h(E))$. We have chosen minimal bases and $d\hat{\varphi}\Lambda^* \subset \Lambda$ hence

$$||d\hat{\varphi}||_\sigma = \frac{||d\hat{\varphi}(\omega_1^*)||_\sigma}{||\omega_1^*||_\sigma} \geq \frac{||\omega_1||_\sigma}{||\omega_1^*||_\sigma} \geq \frac{\pi^{\frac{1}{2}}}{7^2 \max(1, h(E))^{\frac{1}{2}}}. \qquad (79)$$

Let N be the degree of φ, we know that $||d\varphi||_\sigma ||d\hat{\varphi}||_\sigma = N$. Using the lower bound (79) for $||d\hat{\varphi}||_\sigma$ we get

$$||d\varphi||_\sigma \leq 7^2 \max(1, h(E))^{\frac{1}{2}} N. \qquad (80)$$

We can conclude that

$$\prod_\sigma (m + ||d\varphi||_\sigma) \leq (14)^d \max(m, h^{\frac{1}{2}}(E) \deg \varphi)^d.$$

Substituting this in the formula (78), we deduce the result.

□

Let $\varphi : E \to E^*$ be an isogeny between elliptic curves defined over K. Let $d\varphi : T_E \to T_{E^*}$ be the corresponding differential map and let $d\varphi(\omega_1) = m_{11}\omega_1^* + m_{12}\omega_2^*$ and $d\varphi(\omega_2) = m_{21}\omega_1^* + m_{22}\omega_2^*$ (see (65)).
Let $M := (m_{ij})_{i,j=1,2}$ be the associated matrix. We consider M as a linear map $M : T_{E^*} \times T_{E^*} \to T_{E^*} \times T_{E^*}$. We consider the composition map

$$l = d\varphi^{-1} \times d\varphi^{-1} \circ M : T_{E^*} \times T_{E^*} \to T_E \times T_E \qquad (81)$$

Corollary 4 *Let $\varphi : E \to E^*$ be an isogeny between elliptic curves defined over K. Let l be the composition map defined in (81) and Γ_l its graph. Then*

$$h(\Gamma_l) \leq 2\left(h(E) + \log\left(2 \cdot 3^2 7^3 \max(1, h(E^*))^{\frac{1}{2}} \deg\varphi\right)\right). \tag{82}$$

Proof
From lemma 11 we deduce

$$h(W) \leq 2\big(h(E) + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma:K\to\mathbb{C}} \log(1 + ||l||_\sigma)\big). \tag{83}$$

Then it is enough to estimate $(1 + ||l||_\sigma)$.
We remark that $||l||_\sigma = 2||d\varphi||_\sigma^{-1}||M||_\sigma$. By the estimate (79) for φ we get

$$||d\varphi||_\sigma^{-1} \leq \pi^{-\frac{1}{2}} 7^2 \max(1, h(E^*))^{\frac{1}{2}}. \tag{84}$$

Since $||z_1^* + z_2^*||_\sigma^2 \leq 3||z_1^*||_\sigma^2 + ||z_2^*||_\sigma^2$ it follows

$$||M||_\sigma \leq 3 \max_{ij}(|m_{ij}|).$$

By lemma 10 we deduce

$$||M||_\sigma \leq 63 \deg\varphi \qquad\qquad i = 1, 2.$$

We conclude that

$$\prod_\sigma (1 + ||l||_\sigma) \leq (2 \cdot 3^2 7^3 \deg\varphi)^d \max(1, h(E^*))^{\frac{d}{2}}$$

that implies (82). □

6.4 Bounded Degree for the Minimal Isogeny

We want to see how to estimates the degree of a minimal isogeny between elliptic curves using our Subvariety Theorem. We consider separately the complex multiplication case and the non complex multiplication case.

Remark 3 *If φ is an isogeny between two elliptic curves both defined over a number field K, then the isogeny φ is defined over an extension of K of relative degree at most* 12, (see [20] lem. 6.1).

First we want to relate the height of two isogenous elliptic curves. This is a result which we will use in both cases.

Lemma 12 *Let E and E^* be elliptic curves defined over a number field K. Let $\varphi : E \longrightarrow E^*$ be an isogeny. Then*

$$h(E^*) \leq 2h(E) + \log N \tag{85}$$

with N the degree of φ.

Proof
Let L be the field of definition of φ and let d its degree. We know from remark 3 that L is an extension of K of relative degree at most 12. Moreover $\prod_{\sigma:L\to\mathbb{C}} ||d\varphi||_\sigma \cdot ||d\hat{\varphi}||_\sigma = N^d$. Thus one of the following relations holds:

$$\prod_\sigma ||d\varphi||_\sigma \leq N^{\frac{d}{2}} \tag{86}$$

or

$$\prod_\sigma ||d\hat{\varphi}||_\sigma \leq N^{\frac{d}{2}}. \tag{87}$$

We consider the restriction of the injective map $d\varphi : \mathcal{T}_{E_1} \longrightarrow \mathcal{T}_{E_2}$ of hermitian $\mathcal{O}_L$-modules, (see 6.1.5). From 2.1.2 we know that the normalized degree is invariant under finite extensions of scalars. We apply property 6, relation (10) to get

$$\widehat{\deg}_n \overline{T}_E \leq \widehat{\deg}_n \overline{T}_{E^*} + \frac{1}{d} \log \prod_\sigma ||d\varphi||_\sigma \tag{88}$$

If (86) holds then we deduce

$$h(E^*) \leq h(E) + \frac{1}{2} \log N.$$

If (87) holds, from relation (79) we get $\prod_\sigma ||d\hat{\varphi}||_\sigma \geq (\pi/7^4 \max(1, h(E)))^{d/2}$. This implies immediately $\prod_\sigma ||d\varphi||_\sigma \leq (7^2 N)^d \max(1, h(E))^{d/2}$. Substituting in (88) we have

$$h(E^*) \leq (1+\varepsilon)h(E) + \log N.$$

□

For a more precise estimate see [9] 4.3.1, one has

$$h(E^*) \leq h(E) + \frac{1}{2} \log N. \tag{89}$$

We gave here this easy proof to show how the Arakelov theory can simplify things.

6.4.1 The Complex Multiplication Case

We say that an isogeny is cyclic if its kernel is a cyclic group.

Lemma 13 *Let $\varphi : E \longrightarrow E^*$ be an isogeny. We consider the corresponding differential map $d\varphi$ on the tangent spaces whose Betti-representation on the lattices gives*

$$d\varphi(\omega_1) = m_{11}\omega_1^* + m_{12}\omega_2^*, \quad d\varphi(\omega_2) = m_{21}\omega_1^* + m_{22}\omega_2^*. \tag{90}$$

The greatest common divisor of all m_{ij} for $i, j = 1, 2$ is one if and only if the isogeny φ is cyclic. Moreover if φ is minimal then it is cyclic.

Proof If we suppose that $\ker\varphi$ is non cyclic then it must contain a product of two cyclic groups of order p, for a certain prime p. In particular $\ker\varphi$ contains the kernel of the multiplication by p. Therefore all m_{ij} are divisible by p. This contradicts the assumption and proves that φ is cyclic. Vice versa if $p|m_{ij}$ for all $i, j = 1, 2$ then φ factors trough the multiplication by p thus its kernel contains a copy of $\mathbb{Z}_p \times \mathbb{Z}_p$ which is not cyclic.
Suppose now that φ is minimal. If it is not cyclic then, as we have just seen, there exists a positive number p such that $p|m_{ij}$ for all $i, j = 1, 2$. Therefore φ/p is also an isogeny. This contradicts the minimality of φ and proves the lemma. □

Definition 13 *We say that the isogeny $\varphi : E \longrightarrow E^*$ is lower triangular if the corresponding Betti-representation has the following form:*

$$\begin{aligned} d\varphi(\omega_1) &= m_{11}\omega_1^* \\ d\varphi(\omega_2) &= m_{21}\omega_1^* + m_{22}\omega_2^* \end{aligned}$$

Theorem 6 *Let E and E^* be isogenous elliptic curves defined over a number field K. We suppose that E or E^* have complex multiplication.*
Then there exist an isogeny $\varphi : E \longrightarrow E^*$ such that

$$\deg\varphi \leq Cd^2 \max(1, h(E), \log d)^2$$

with $d := [K : \mathbb{Q}]$ the degree of the field K, C an absolute constant and $h(E)$ the Faltings height of E.

Proof
Since E and E^* have complex multiplication the module $\mathrm{Hom}(E, E^*) = \mathbb{Z} + \alpha\mathbb{Z}$ is a free $\mathbb{Z}$-module of rank 2. Let $\varphi : E \longrightarrow E^*$ be a minimal isogeny,

the isogenies φ and $\varphi' := \alpha\varphi$ are $\mathbb{Q}$-linear independent. The corresponding differential maps $d\varphi$ and $d\varphi'$ on the tangent spaces satisfy

$$d\varphi(\Lambda) \subset \Lambda^*$$

and

$$d\varphi'(\Lambda) \subset \Lambda^*.$$

These inclusions can be expressed in the form

$$d\varphi(\omega_1) = m_{11}\omega_1^* + m_{12}\omega_2^*, \quad d\varphi(\omega_2) = m_{21}\omega_1^* + m_{22}\omega_2^* \tag{91}$$

and

$$d\varphi'(\omega_1) = m'_{11}\omega_1^* + m'_{12}\omega_2^*, \quad d\varphi'(\omega_2) = m'_{21}\omega_1^* + m'_{22}\omega_2^*. \tag{92}$$

We want to proof that there exists a lower triangular isogeny $\phi \in \mathrm{Hom}(E, E^*)$.
If $m_{12} = 0$, then φ is lower triangular thus we can set $\phi = \varphi$.
If $m_{12} \neq 0$, we consider the linear combination

$$\phi := m'_{12}\varphi + m_{12}\varphi'.$$

From the relations (91) and (92) it follows

$$\begin{aligned} d\phi(\omega_1) &= M_{11}\omega_1^* \\ d\phi(\omega_2) &= M_{21}\omega_1^* + M_{22}\omega_2^* \end{aligned}$$

where $M_{11} := (m'_{12}m_{11} - m_{12}m'_{11})$. Since φ and φ' are $\mathbb{Q}$-linear independent and $m_{12} \neq 0$, then ϕ is non-trivial. If $p|M_{ij}$ for $i, j = 1, 2$ then ϕ/p is an isogeny as well, we can then suppose that the M_{ij} have no common factors. We consider the linear map

$$T_E \xrightarrow{M_{11} \times d\phi} T_E \times T_{E^*}$$

where $M_{11} : T_E \to T_E$ is the multiplication by M_{11}.
We define the subspace W to be the image of $M_{11} \times d\phi$. The differential map $d\phi$ and so W are defined over K', with K' the field of definition of ϕ. From remark 3 we have $[K' : K] \leq 12$.
Note that $\gamma := (\omega_1, \omega_1^*)$ is an element of the vector space W. In fact $(M_{11} \times d\phi)({M_{11}}^{-1}\omega_1) = (\omega_1, d\phi({M_{11}}^{-1}\omega_1)) = (\omega_1, \omega_1^*)$ is an element of W.
We consider on the product variety $A = E \times E^*$ the line bundle $\mathcal{L} := p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2$ and on the tangent space at zero the metric induced by $\mathcal{L}$ (see 6.1.4).
We are now in the condition to apply the Subvariety Theorem where the abelian variety A, the line bundle $\mathcal{L}$, the space W and the period $\gamma := (\omega_1, \omega_1^*)$ are the ones described above. The Theorem ensures the existence of an

abelian subvariety B non-trivial and different from A (i.e. $\dim B = 1$), such that $T_B \subset W$ (in this case $T_B = W$) and

$$\deg_{\mathcal{L}} B \leq C_1 \max(\deg_{\mathcal{L}} A, hrd, rd\log(rd))) \tag{93}$$

with $h := \max(1, h(A), \log\deg_{\mathcal{L}} A, h(W))$ and $r := \max(1, ||\gamma||^2)$. We have assumed principal polarizations so the Riemann-Roch formula (12) gives $\deg_{\mathcal{L}} A = 2$. We want to bound the height of W and the norm of γ in order to estimate the maximum appearing in (93) with the height of E.
Corollary 3 gives

$$h(W) \leq h(E) + \log\left(14^2 \max\left(M_{11}, h^{\frac{1}{2}}(E)\deg\phi\right)\right).$$

Since $N := \deg\phi = M_{11} \cdot M_{22}$ we see that

$$h(W) \leq h(E) + \log\left(14^2 N \max(1, h(E))^{\frac{1}{2}}\right). \tag{94}$$

Now we have to bound the norm of the period γ. From (60) we know that $||\omega_1||^2_{\mathcal{L}_1} \leq \pi^{-1}$ and $||\omega_1^*||^2_{\mathcal{L}_2} \leq \pi^{-1}$. The norm induced by $\mathcal{L}$ (see 61) gives

$$||\gamma||^2_{\mathcal{L}} = ||\omega_1||^2_{\mathcal{L}_1} + ||\omega_1^*||^2_{\mathcal{L}_2} \leq 2\pi^{-1}. \tag{95}$$

Substituting the estimates (94) and (95) in (93) and using $h(A) = h(E) + h(E^*)$ we get

$$\deg_{\mathcal{L}} B \leq C_2 d \max(1, h(E) + h(E^*) + \log N, \log d). \tag{96}$$

From lemma 12 we deduce

$$\deg_{\mathcal{L}} B \leq C_3 d \max(1, h(E) + \log N, \log d).$$

We remark that B is non-split because $T_B = (\omega_1, \omega_1^*) \cdot \mathbb{C}$.
If $\phi = \varphi$ we apply the "Isogenies Lemma" of Masser-Wüstholz (see 6.1-1) and we find an isogeny of degree $N_1 \leq Cd^2 \max(1, h(E) + \log N, \log d)^2$. We assumed that φ is minimal then

$$\deg\varphi \leq C_4 d^2 \max(1, h(E), \log d)^2.$$

If $\phi \neq \varphi$ we consider the intersections $B \cap 0_E \times E^*$ and $B \cap E_1 \times 0_{E^*}$. Since B is non-split these intersections are finite. By lemma 13 we know that ϕ is cyclic. Thus there exists an element $\omega^* \in \Lambda^*$ whose class generates $\Lambda^*/d\phi\Lambda$. Then $P = \exp(d\phi^{-1}M_{11}\omega^*, \omega^*)$ is a point of $B \cap E_1 \times 0_{E^*}$ of exact order M_{22}. On the other hand, any element $\gamma^* \in \Lambda^*$ is equivalent to $t\omega^*$

modulo $d\phi\Lambda$ thus the point P generates the whole intersection. It follows that $\sharp(B \cap E \times 0_{E^*}) = M_{22}$. Since $\deg E \times 0_{E^*} = 3$, by Bézout Theorem and relation (96) we have

$$M_{22} \leq C_5 d \max(1, h(E) + \log N, \log d).$$

The dual isogeny $\hat{\phi}$ is cyclic as well. Thus the group $d\phi\Lambda/N\Lambda^*$ is generated by an element ω of exact order N. By isomorphism the group $(d\phi\Lambda/N)/\Lambda^*$ is generated by ω/N. It follows that $Q = \exp(\omega, d\phi\omega/M_{11})$ is a point of $B \cap 0_E \times E^*$ and has exact order M_{11}. On the other hand, for any $\gamma \in \Lambda$, the element $d\phi\gamma$ is equivalent to $t\omega$ modulo $N\Lambda^*$. Thus the point Q generates the whole intersection. We conclude that $\sharp(B \cap 0_E \times E^*) = M_{11}$. Then, by Bézout Theorem and relation (96), we have

$$M_{11} \leq C_5 d \max(1, h(E) + \log N, \log d).$$

Since the isogeny ϕ is lower triangular we have $N = \deg \phi = M_{11} \cdot M_{22}$, we deduce

$$N \leq C_6 d^2 \max(1, h(E) + \log N, \log d)^2$$

whence

$$N \leq C_7 d^2 \max(1, h(E), \log d)^2.$$

By the minimality of φ we deduce

$$\deg \varphi \leq C_7 d^2 \max(1, h(E), \log d)^2$$

which conclude the proof.

$\square$

Remark:
The proof of Theorem 6 works also in the case of two elliptic curves without complex multiplication related by a lower triangular isogeny.

6.4.2 The Non-Complex Multiplication Case

Let E and E^* be isogenous elliptic curves defined over a number field K. Let $\varphi : E \to E^*$ be a minimal isogeny of degree N. Let exp be the usual exponential map from $T_{E^2} \times T_{E^{*2}} \to E^2 \times E^{*2}$. We consider independent complex variables z_1, z_2, z_1^*, z_2^* and define the subspace

$$W = \left\{ \begin{array}{lcl} d\varphi z_1 & = & m_{11} z_1^* + m_{12} z_2^* \\ d\varphi z_2 & = & m_{21} z_1^* + m_{22} z_2^* \end{array} \right. \qquad (97)$$

corresponding to (65).

Lemma 14 *The intersection of* $\exp(W)$ *with* $0_{E^2} \times E^{*2}$ *is a cyclic group of cardinality* $N = \deg\varphi$.

Proof Let J be the intersection. Since φ is a minimal isogeny, by lemma 13, it follows that φ is cyclic. This means that the quotient group $\Lambda^*/d\varphi\Lambda$ is isomorphic to $\mathbb{Z}/N\mathbb{Z}$. Let $\omega^* \in \Lambda^*$ be any representative of a class generating $\Lambda^*/d\varphi\Lambda$. Then J is the set of points

$$\exp\left((m_{11}a_1 + m_{1,2}a_2)d\varphi^{-1}\omega^*, (m_{11}a_1 + m_{1,2}a_2)d\varphi^{-1}\omega^*, a_1\omega^*, a_2\omega^*\right)$$

where a_1 and a_2 varies in $\mathbb{Z}$. Since ω^* has exact order N the group $\mathbb{Z}d\varphi^{-1}\omega^* \times \mathbb{Z}d\varphi^{-1}\omega^* \mathrm{mod}\Lambda \times \Lambda$ is isomorphic to $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.
We consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\times\mathbb{Z} \arrow[r, "M"] \arrow[d, hook, "N"] & \mathbb{Z}\times\mathbb{Z} \arrow[r] \arrow[d, hook, "N"] & \mathrm{coker}M \arrow[r] \arrow[d, "N"] & 0 \\
0 \arrow[r] & \mathbb{Z}\times\mathbb{Z} \arrow[r, "M"] \arrow[d, "\pi_N"] & \mathbb{Z}\times\mathbb{Z} \arrow[r] \arrow[d, "\pi_N"] & \mathrm{coker}M \arrow[r] & 0 \\
\mathrm{ker}\overline{M} \arrow[r, hook] & \mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z} \arrow[r, "\overline{M}"] & \mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z} \arrow[r] & \mathrm{coker}\overline{M} \arrow[r] & 0
\end{tikzcd}$$

where M is the linear map induced by the matrix $\left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right)$, N is the multiplication by $N = \det M$ and rows and columns are exact. In this notations we have $J \cong \pi_N \circ M(\mathbb{Z}\times\mathbb{Z})$. Since φ is cyclic $\mathrm{coker}M \cong \mathbb{Z}/N\mathbb{Z}$, so the multiplication by N is the zero map. In view of the Snake-Lemma we have a long exact sequence of kernels and cokernels. It follows at once that $\ker\overline{M} \cong \mathbb{Z}/N\mathbb{Z}$ and $\mathrm{coker}\overline{M} \cong \mathbb{Z}/N\mathbb{Z}$. Therefore $J \cong (\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}) \big/ \mathbb{Z}/N\mathbb{Z}$ which in turn is isomorphic to $\mathbb{Z}/N\mathbb{Z}$.
For this last isomorphism let $l : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}, 1 \mapsto (\lambda_1, \lambda_2)$ be the inclusion of $\ker\overline{M}$. Since $\ker\overline{M}$ is cyclic of order N then $\lambda_1$and λ_2 must be coprime numbers thus there exist integers l_1 and l_2 such that $\lambda_1 l_1 + \lambda_2 l_2 = 1$.
We consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/N\mathbb{Z} \arrow[r, "\mathrm{id}\times 0"] \arrow[d, "\mathrm{id}"] & \mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z} \arrow[r, "\pi_2"] \arrow[d, "L"] & \mathbb{Z}/N\mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}/N\mathbb{Z} \arrow[r, "l"] & \mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z} \arrow[r] & \mathrm{coker}l \arrow[r] & 0
\end{tikzcd}$$

where $L = \left(\begin{smallmatrix} \lambda_1 & l_2 \\ \lambda_2 & l_1 \end{smallmatrix}\right)$. Since $\det L = 1$ the map L is an isomorphism. By the 5-Lemma we conclude that $J \cong \mathrm{coker}l \cong \mathbb{Z}/N\mathbb{Z}$.

$\square$

Remark 4 *Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be an irreducible variety and π the natural projection on one of the factors $\mathbb{P}^n$ or $\mathbb{P}^m$. If X and $\pi(X)$ have the same dimension then* $\deg \pi(X) \leq \deg X$. ([20] lemma 2.1).

Theorem 7 *Let E and E^* be isogenous elliptic curves defined over K. Then there exists an isogeny $\varphi : E \to E^*$ such that*

$$\deg \varphi \leq Cd^2 \max(1, h(E), \log d)^3$$

where $d := [K : \mathbb{Q}]$ is the degree of K, C is an absolute constant and $h(E)$ is the Faltings height of E.

Proof We consider the linear map $l : T_{E^*} \times T_{E^*} \to T_E \times T_E$ given by $l(z_1^*, z_2^*) = (d\varphi^{-1}(m_{11}z_1^* + m_{12}z_2^*), d\varphi^{-1}(m_{22}z_1 - m_{12}z_2))$ and the abelian variety $A = E \times E \times E^* \times E^*$. Let W be the graph of l in T_A, by remark 3, W is defined over a field extension of K of degree at most 12. Note that the period $\gamma := (\omega_1, \omega_2, \omega_1^*, \omega_2^*)$ is an element of the vector space W.
On the product variety A we consider the line bundle $\mathcal{L} := p_1^*\mathcal{L}_1 \otimes p_2\mathcal{L}_1 \otimes p_3^*\mathcal{L}_2 \otimes p_4^*\mathcal{L}_2$ and on the tangent bundle the metric induced by $\mathcal{L}$ (see 6.1.4). We are now in the condition to apply the Subvariety Theorem 2 where the abelian variety A, the line bundle $\mathcal{L}$, the space W and the period γ are defined above. The Theorem ensures the existence of an abelian subvariety B non-trivial and different from A, such that $\gamma \in T_B \subset W$ and

$$\deg_{\mathcal{L}} B \leq C(g) \max(\deg_{\mathcal{L}} A, dhr, dr \log(dr)) \tag{98}$$

with $h := \max(1, h(A), \log \deg_{\mathcal{L}} A, h(W_i))$, $r := \max(1, ||\gamma_i||^{2\delta})$ and δ the dimension of B. We have assumed principal polarizations, so the Riemann-Roch formula (12) gives $\deg_{\mathcal{L}} A = 4!$.
We want to bound the height of W and the norm of γ in order to estimate the maximum appearing in (98) with the height of E.
Corollary 4 gives

$$h(W) \leq 2\left(h(E) + \log\left(2 \cdot 3^2 7^3 N \max(1, h(E^*))^{\frac{1}{2}}\right)\right) \tag{99}$$

where $N = \deg \varphi$ and from lemma 12 we deduce

$$h(W) \leq 3 \max(1, h(E) + \log N) \tag{100}$$

Now we have to bound the norm of the period γ. From the principal polarization assumption and relation (60) we know that $||\omega_1||^2_{\mathcal{L}_1} \leq \pi^{-1}$ and $||\omega_1^*||^2_{\mathcal{L}_2} \leq \pi^{-1}$. Using corollary 2 we have $||\omega_2||^2_{\mathcal{L},\sigma} \leq 7^2 \max(1, h(E))$ and $||\omega_2^*||^2_{\mathcal{L},\sigma} \leq 7^2 \max(1, h(E^*))$. The norm induced by $\mathcal{L}$ (see 61) gives $||\gamma||^2_{\mathcal{L}} =$

$||\omega_1||^2_{\mathcal{L}_1} + ||\omega_2||^2_{\mathcal{L}_1} + ||\omega_1^*||^2_{\mathcal{L}_2} + ||\omega_2^*||^2_{\mathcal{L}_2} \leq 2(\pi + 7^2)\max(1, h(E^*) + h(E))$, and so by lemma 12 we have

$$||\gamma_i||^2_{\mathcal{L}} \leq 3 \cdot 7^2 \max(1, h(E) + \log N). \tag{101}$$

Substituting the estimates (100) and (101) in (98) and using $h(A) = 2h(E) + 2h(E^*)$ we get

$$\deg_{\mathcal{L}} B \leq C_1 d \max\left(1, (h(E) + \log N)^{1+\delta}, (h(E) + \log N))^{\delta} \log d\right). \tag{102}$$

In relation to the dimension δ of B we have to consider two cases.
First case: the dimension of B is 1.
From relation (102) we deduce

$$\deg_{\mathcal{L}} B \leq C_2 d \max\left(1, h(E) + \log N, \log d\right)^2.$$

We consider the projection map $\pi : A \to E \times E^*$ defined by $(P_1, P_2, P_1^*, P_2^*) \mapsto (P_1, P_1^*)$. We remark that the tangent space of $\overline{B} := \pi(B)$ is defined over a finite extension of K. Note that the period $\overline{\gamma} = d\pi(\gamma) = (\omega_1, \omega_1^*)$ is an element of $T_{\overline{B}}$ thus the dimension of $T_{\overline{B}}$ is one. By Lemma 4 we deduce

$$\deg_{\mathcal{L}} \overline{B} \leq C_2 d \max\left(1, h(E) + \log N, \log d\right)^2.$$

In view of the decomposition Theorem [31] cor 19.1 there exists an isogeny $\phi : E \to \overline{B}$ of degree at most $\deg_{\mathcal{L}} \overline{B}$. From Lemma 12 we deduce

$$h(\overline{B}) \leq 2h(E) + \log(\deg_{\mathcal{L}} \overline{B})$$

whence

$$h(T_{\overline{B}}) \leq C_3 d \max\left(1, h(E) + \log N, \log d\right)^2. \tag{103}$$

Moreover, by relation (60), we have that

$$||\overline{\gamma}_i||^2 = ||(\omega_1, \omega_1^*)||^2 \leq 2\pi^{-1}. \tag{104}$$

Now we can apply, once more, the Subvariety Theorem 2 to the abelian variety $A = E \times E^*$, the subspace $T_{\overline{B}}$ and the period $\overline{\gamma}$. Since the dimension of $T_{\overline{B}}$ is one we deduce

$$\deg_{\mathcal{L}} \overline{B} \leq C_4 \max\left(\deg_{\mathcal{L}} A, hrd, rd \log(rd)\right) \tag{105}$$

with $h := \max(1, h(A), \log \deg_{\mathcal{L}} A, h(T_{\overline{B}}))$ and $r := \max(1, ||\gamma||^2)$. The Riemann-Roch formula gives $\deg_{\mathcal{L}} A = 2$. Substituting the estimates (103) and (104) in (105) we deduce that

$$\deg_{\mathcal{L}} \overline{B} \leq C_5 d \max(1, h(E) + \log N, \log d). \tag{106}$$

Clearly $\overline{B}$ is non-split because $(\omega_1, \omega_1^*) \in T_{\overline{B}}$. Applying the "Isogenies Lemma" of Masser and Wüstholz (see 6.1-1) we get the existence of an isogeny of degree $N_1 \leq Cd^2 \max(1, h(E) + \log N, \log d)^2$. If we suppose that φ is minimal then

$$\deg \varphi \leq C_6 d^2 \max(1, h(E), \log d)^2.$$

Case II: If B has dimension 2.
By relation (102) we deduce

$$\deg_{\mathcal{L}} B \leq C_7 d \max\big(1, h(E) + \log N, \log d\big)^3.$$

In this case $B = \exp(W)$. By Lemma 14 we know that $\sharp(B \cap 0 \times 0 \times E^* \times E^*) = N = \deg \varphi$. The abelian variety $0 \times 0 \times E^* \times E^*$ has degree 9, by Bézout Theorem we deduce that

$$N \leq C_8 d \max\big(1, h(E) + \log N, \log d\big)^3.$$

Which implies

$$N \leq C_8 d \max\big(1, h(E), \log d\big)^3.$$

This conclude the proof. $\square$

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