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Renormalization group, tree expansion, and non-renormalizable quantum field theories

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Summary

We study renormalizable and non-renormalizable Euclidean quantum field theories from the point of view of Wilson's renormalization group. We construct the tree expansion of Gallavotti and Nicolò, for the general case of a scalar field theory. It is an expansion of physical quantities in powers of the running coupling constants on all scales. These running coupling constants obey recursion relations (the flow equations) involving similar power series. We show that the tree expansion is finite to all orders, even if the theory is not renormalizable.

The ultraviolet problem is thus reduced (as far as perturbation theory is concerned) to the problem of finding a solution to the flow equations.

A naive way of finding a solution is to solve the flow equations in a power series in the running coupling constants on some fixed low energy scale (the renormalized coupling constants). This way of solving the flow equations yields for renormalizable theories the usual renormalized perturbation series: the BPH Theorem and the de Calan–Rivasseau $n!$-bounds can be proven with this method. For non-renormalizable theories, however, the naive way of solving the flow equations breaks down because of ultraviolet divergences. This does not mean that no solutions exist: There can exist a solution of the flow equations which does not depend in a C^∞ way on the renormalized coupling constants.

More can be said in a specific example of a non-renormalizable theory: a ϕ^4 theory in four dimensions with propagator $1/p^{2-\epsilon/2}$ (this model is similar to a $\phi^4_{4+\epsilon}$ theory, but is technically easier to handle). In this case we solve the flow equations with a fixed point ansatz, i.e., by setting all running coupling constants equal to each other. The result is that there does exist a non-trivial fixed point λ^* (at negative coupling constant) as predicted by a one-loop beta function calculation. We then re-write the flow equations in an expansion around this fixed point. In this form the flow equations can be solved in a finite expansion in powers of $\lambda - \lambda^*$, where λ are the renormalized coupling constants. This yields a two-parameter family of Euclidean quantum field theories expressed as finite expansions.

Results beyond perturbation theory are obtained in the planar limit (the $N \to \infty$ limit of a $\mathrm{tr}\phi^4$ theory, where ϕ is an $N \times N$ matrix). In this limit the tree expansion is not only finite to all orders, but also convergent, and the above-mentioned two-parameter family of theories can be rigorously constructed.

Riassunto

Studiamo teorie quantistiche euclidee dei campi, rinormalizzabili e non, dal punto di vista del gruppo di rinormalizzazione di Wilson. Costruiamo lo sviluppo in alberi proposto da Gallavotti e Nicolò, nel caso generale di una teoria di un campo scalare. Questo è uno sviluppo di quantità fisiche in potenze delle costanti d'accoppiamento effettive su tutte le scale. Queste costanti d'accoppiamento effettive soddisfano relazioni di ricorrenza (le equazioni di flusso) che pure contengono simili serie di potenze.

Mostriamo che lo sviluppo in alberi è finito a tutti gli ordini, anche se la teoria non è rinormalizzabile. Il problema ultravioletto è quindi ridotto (per quanto concerne la teoria delle perturbazioni) al problema di trovare una soluzione alle equazioni di flusso. Un modo ingenuo di trovarne una è di risolvere le equazioni di flusso in una serie di potenze nelle costanti d'accoppiamento effettive su una scala a bassa energia fissa (ovvero nelle costanti d'accoppiamento rinormalizzate). Questo modo di risolvere le equazioni di flusso genera la nota teoria delle perturbazioni rinormalizzata: il teorema BPH e i limiti $n!$ di de Calan e Rivasseau possono essere dimostrati con questo metodo. Per teorie non rinormalizzabili, invece, il detto modo di risolvere le equazioni di flusso fallisce a causa di divergenze ultraviolette. Ciò non significa però che non esistano soluzioni: può esistere una soluzione che non dipende in modo C^∞ dalle costanti d'accoppiamento rinormalizzate.

Si può dire di più in un esempio concreto di una teoria non rinormalizzabile: una teoria ϕ^4 in quattro dimensioni con propagatore $1/p^{2-\epsilon/2}$ (questo modello è simile a $\phi^4_{4+\epsilon}$, ma è tecnicamente più semplice da trattare). In questo caso risolviamo le equazioni di flusso supponendo l'esistenza di un punto fisso, cioè ponendo tutte le costanti d'accoppiamento effettive uguali l'una all'altra nelle equazioni. Il risultato è che un punto fisso λ^* esiste effettivamente (a costante d'accoppiamento negativa) ed è stabile nell'ultravioletto, come lo suggerisce un calcolo di funzione beta a un cappio. Indi riscriviamo le equazioni di flusso in uno sviluppo attorno a questo punto fisso. In questa forma le equazioni possono essere risolte in uno sviluppo finito in potenze di $\lambda - \lambda^*$, dove λ sono le costanti d'accoppiamento rinormalizzate. In questo modo si costruisce una famiglia a due parametri di teorie quantistiche euclidee espresse sotto forma di sviluppi in serie finiti a tutti gli ordini.

Possono essere ottenuti risultati al di là della teoria delle perturbazioni nel limite planare (il limite $N \to \infty$ di una teoria $\mathrm{tr}\phi^4$ dove ϕ è una matrice $N \times N$). In questo limite lo sviluppo in alberi è non solo finito a tutti gli ordini ma anche convergente, e la suddetta famiglia a due parametri di teorie può essere costruita rigorosamente.

1. Introduction

Wilson's renormalization group [23], [25] has brought new insight into renormalization theory: the concept of relevant and irrelevant interactions provided a new framework for the understanding of renormalisability. Instead of considering renormalization theory just as a recipe for eliminating unwanted infinities one thinks as follows: let a theory be defined on a very small distance scale Λ^{-1} as a perturbation of a (Gaussian) free field. Expanding about this free field one sees that only a finite number of interactions (the relevant ones) survives in the effective low energy theory on some scale μ, the others being suppressed by negative powers of $\frac{\Lambda}{\mu}$. Thus the theory has effectively only a few parameters as $\frac{\Lambda}{\mu} \to \infty$ and to parametrize the theory one usually chooses effective low energy coupling constants (the renormalized coupling constants) as parameters.

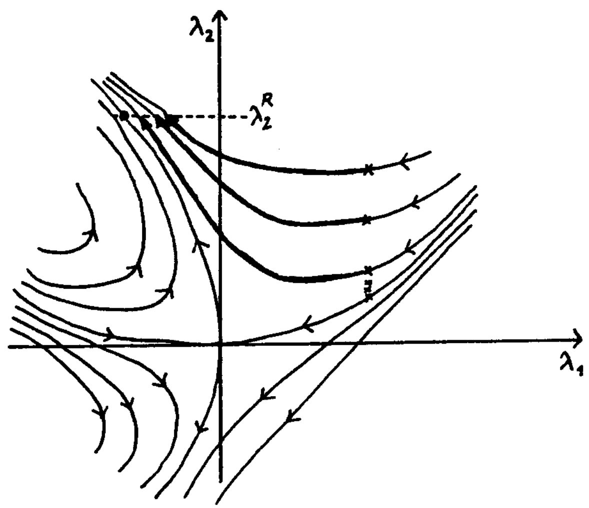


Fig. 1. Flow of coupling constants in a renormalizable theory

From this point of view one looks at non-renormalizable theories in the following way: in the limit $\frac{\Lambda}{\mu} \to \infty$ the theory has a finite number of parameters; in other words, once the renormalized coupling constants are given for the relevant interactions, everything is uniquely determined. In particular, the value of the irrelevant coupling constants on scale μ is determined. A non-renormalizable theory is a theory where one insists to give a value not only to the relevant coupling constants on scale μ but also to some irrelevant ones. Whereas this can be possible for finite Λ by tuning appropriately bare coupling constants, divergences appear as $\frac{\Lambda}{\mu} \to \infty$. This phenomenon can be understood by looking at Fig. 1 and Fig. 2. The picture of the situation given by perturbation theory of the renormalization group is that there exists in the vicinity of the Gaussian fixed point a finite dimensional manifold (tangential to the relevant directions) and an infinite dimensional stable manifold (tangential to the irrelevant directions). For sake of clarity consider only a two-dimensional picture: λ_1 is the irrelevant direction and λ_2 is the relevant one. In a renormalizable theory, one sets the bare irrelevant coupling constant $\lambda_1 = \lambda_1^0 = \text{const}$ (usually $= 0$), and chooses the bare relevant coupling constant $\lambda_2(\Lambda)$ in such a way that a renormalization condition $\lambda_2(\mu) = \lambda_2^R$ is satisfied, where $\lambda(\mu)$ is the effective coupling constant on some physical scale μ. To obtain $\lambda_2(\mu)$ from $\lambda_2(\Lambda)$ and λ_1^0 one iterates the renormalization group transformation as many times as needed to lower the cut-off from Λ to μ. For $\Lambda \to \infty$ (at μ fixed) we see that the bare coupling constant $\lambda_2(\Lambda)$ must be chosen closer and closer to the stable manifold (crosses in Fig. 1) in order for the renormalization condition to be satisfied. Moreover, we see that in the limit $\Lambda \to \infty$, the theory lies inevitably on the unstable finite dimensional manifold (Point A in Fig. 1).

For a non-renormalizable theory one imposes renormalization conditions also for irrelevant coupling constants. In our example one sets $\lambda_1(\mu) = \lambda_1^R$, $\lambda_2(\mu) = \lambda_2^R$. In this case the bare coupling constants have to be chosen farther and farther away (crosses in Fig. 2), giving rise to the ultraviolet divergences of non-renormalizable theories.

This interpretation of non-renormalizability provides a possible way of making sense of a non-renormalizable theory: the property for an interaction to be relevant depends on the fixed point one is expanding about. If one had another fixed point, some interaction which was irrelevant at the Gaussian fixed point might become relevant at the new fixed point (i.e. the other fixed point might be "ultraviolet stable") and one could fix the value of this interaction at the low energy scale μ.

A typical example where this general heuristic discussion can be made concrete is the ϕ^4 model in d dimensions, with Euclidean action

$$S = \int \left(\tfrac{1}{2} Z (\partial\phi)^2 + \tfrac{1}{2} m^2 \phi^2 + g :\phi^4: \right) d^d x. \tag{1.1}$$

The $:\phi^4:$ term is irrelevant in more than four dimensions and the renormalization group tells us that the theory tends to a free field as the cut-off Λ is removed (at least if g is small). This is in fact true for all $g \geq 0$ as was proven by Aizenman [1] and Fröhlich [7] using a lattice cut-off.

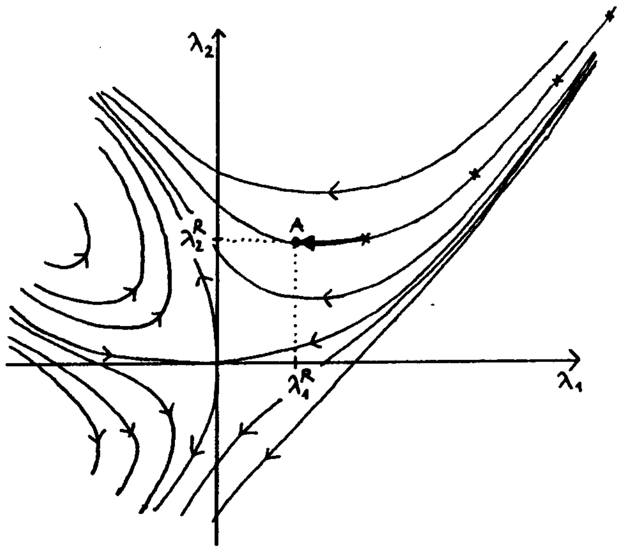


Fig. 2. Flow of coupling constants in a non-renormalizable theory

On the other hand, a one loop beta function calculation

$$\mu\frac{d}{d\mu}\lambda(\mu) = \epsilon\lambda(\mu) + \beta_2\lambda(\mu)^2, \qquad \epsilon = d-4, \qquad \beta_2 > 0, \tag{1.2}$$

where $\lambda(\mu)$ is the dimensionless running coupling constant, predicts the existence of a non-trivial fixed point solution of the flow equation $\lambda(\mu) = -\epsilon/\beta_2 = \lambda^*(\epsilon)$ at negative ("wrong sign") coupling. This fixed point is ultraviolet stable, i.e. the $:\phi^4:$ term becomes *relevant* at λ^* and the theory seems in fact to be renormalisable if expanded about this fixed point.

In fact, as it stands, this argument is not very conclusive even at the perturbative level: the perturbation theory of the beta function for non-renormalizable theories *does not make sense*: higher order coefficients are ultraviolet divergent. To see this consider the L-loop contribution to the vertex function $\Gamma_4(p_1, p_2, p_3, p_4)$, say at the symmetric point $p_i^2 = p^2$, $p_ip_j = -\frac{1}{3}p^2$, and suppose that L is so large that $L(d-4) \geq 2$ but $L(d-4) < 4$. Then the infinite part of the corresponding Feynman graphs goes like

$$a_1\Lambda^{L(d-4)} + a_2p^2\Lambda^{L(d-4)-2}, \tag{1.3}$$

and while the first divergent term can be cancelled by a ϕ^4 counterterm, the second one would require a $\phi^3\Delta\phi$ counterterm, which is not present in the Lagrangean. Thus, in terms of the renormalized coupling constant with subtraction point μ, the L-loop contribution to Γ_4 becomes

$$\Gamma_4 = \cdots + \lambda^{L+1}\mu^{-(L+1)(d-4)}(a_2(p^2-\mu^2)\Lambda^{L(d-4)-2} + \text{finite}) + \cdots, \tag{1.4}$$

and the beta function can be computed by differentiating the running coupling constant $\Lambda(\mu') = \mu'^{(d-4)}\Gamma_4(p^2 = \mu'^2)$ with respect to $\ln\mu'$ at $\mu' = \mu$. Its L-loop contibution is

$$\beta(\lambda) = \cdots + \lambda^{(L+1)}(-2a_2(\tfrac{\Lambda}{\mu})^{L(d-4)-2} + \text{finite}) + \cdots \tag{1.5}$$

If L were even larger than $\frac{4}{d-4}$, more divergent terms would appear in (1.3), and also subleading divergences $(\frac{\Lambda}{\mu})^{L(d-4)-4}$, $(\frac{\Lambda}{\mu})^{L(d-4)-6}$, ..., would plague the beta function. Generically, the beta function in $d > 4$ dimensions has divergent coefficients starting from L-loop, where L is the smallest integer satisfying $L(d-4) - 2 \geq 0$. Optimistically [17], [21], one can conjecture that a non-C^∞ beta function might exist non-perturbatively, but it is only $[\frac{d-2}{d-4}]$ differentiable in the renormalised coupling constant.

Fortunately there is a way out from this problem, even in the framework of perturbation theory, the essential ingredient being the beta functional [9], [8] to be discussed later.

Although all the above arguments are heuristic, they can be made rigorous in some cases: first of all they can be made rigorous in perturbation theory: renormalizable theories can be constructed order by order in perturbation theory by parametrizing them by renormalized parameters. This is the BPHZ theorem which could be re-proved using the above arguments [19]. The fact that the theory has a finite number of parameters, i.e. whatever the value of the (dimensionless) bare coupling constants (within limits) of the irrelevant terms, one gets always the same theory (as $\Lambda \to \infty$), is contained in [6] (see also Section 5 of this thesis), in the perturbative framework. We called this phenomenon "perturbative triviality" because for ϕ^4_d theories in $d > 4$ dimensions it is a perturbative version of the results of [1], [7].

Another case where one can make these arguments rigorous is in the $\frac{1}{N}$ expansion of $(\vec{\phi}^2)^2$ theories. Here one can renormalize the theory recursively order by order in $\frac{1}{N}$. Parisi [17] treated $(\vec{\phi}^2)^2$ in $d > 4$ dimensions in $\frac{1}{N}$ expansion and discussed the construction of the ultraviolet stable fixed point at negative coupling. He obtained the result that, to all orders in $\frac{1}{N}$, the Green's functions are finite but not C^∞ in the renormalized coupling

Fig. 3. Leading 4-point graph in the $\frac{1}{N}$ expansion

constant. The essential features of this phenomenon can already be seen to second order in $\frac{1}{N}$: To first order in $\frac{1}{N}$ the four point function is essentially given by the graphs of the type represented in Fig. 3; these graphs can be summed explicitly (we do this in Section 4). The resulting renormalized four point function is then analytic in the renormalized coupling constant λ for all $\lambda \leq 0$, but the radius of convergence at $\lambda = 0$ shrinks to zero as the external momenta tend to infinity, due to singularities on the positive real axis. This implies that to order $\frac{1}{N^2}$, when the four point function renormalized to order $\frac{1}{N}$ is put into a bigger graph such as the one in Fig.4, the loop integration gives the Feynman amplitude a finite but singular behaviour at $\lambda = 0$, typically with an asymptotic expansion of the form

$$I(\lambda) = I_0(\lambda) + \lambda^{2/(d-4)} I_1(\lambda) + \lambda^{4/(d-4)} I_2(\lambda) + \cdots, \tag{1.6}$$

where I_0, I_1, I_2, ..., are C^∞ at $\lambda = 0$, and logarithms appear in rational dimensions. Whether expansions of the form (1.6) can be valid beyond the $\frac{1}{N}$ expansion is not clear.

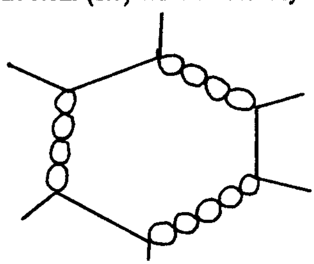


Fig. 4. Leading six-point graph in the $\frac{1}{N}$ expansion

At this stage we should point out that Symanzik [21] (see also [18]) proposed a method of renormalizing non-renormalizable theories which, although leading to expansions of the type (1.6), seems to be radically different from the ones discussed here. Unfortunately, his paper is still quite mysterious.

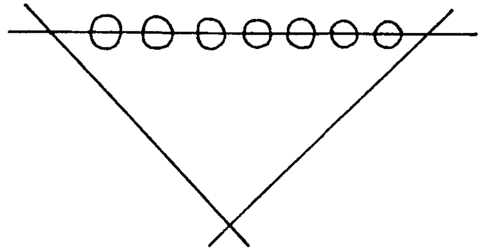


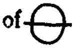
Fig. 5. A simple example of a graph with renormalons

Going back to the problem of constructing non-trivial fixed points avoiding ultraviolet divergencies, progress has been made in [6], [7], using the methods of Gallavotti and Nicolò [9], [8]. They introduced a beta functional

$$\mu\frac{d}{d\mu}\lambda(\mu)=\beta\big(\{\lambda(\mu')\}_{\mu'>\mu}\big), \tag{1.7}$$

for renormalizable theories (actually with a discrete renormalization step). This type of flow equation has many advantages over (1.2): the coefficients of β in perturbation theory can be constructed with renormalized Feynman graphs without overlapping divergences and without "useless" [3] counterterms. Moreover the perturbation series has only "instanton-" and no "renormalon-n factorials" [13], [14], [3]. These properties can be seen in a typical example: the graph of Fig. 5. In four dimensions, the renormalized amplitude of such a graph can be estimated noting that for large momentum the subgraph  goes like $p^2\ln\frac{p^2}{\mu^2}$, where μ is the subtraction point. The amplitude at zero momentum of a graph with n such subgraphs goes like

$$\int_{p^2>\mu^2}(\ln\frac{p^2}{\mu^2})^n p^{-6}\,d^4p\sim\mu^{-2}C^n(n-1)!, \tag{1.8}$$

This factorial is called "renormalon", in contrast to the "instanton" n factorial in the n^{th} order perturbation coefficient coming from the number of graphs with n vertices. Let us consider the situation more closely: the *unrenormalized* amplitude of  is

$$A(p^2)=C_1\Lambda^2+C_2p^2\ln\frac{\Lambda^2}{p^2}+\text{finite}, \tag{1.9}$$

(for simplicity we work in the massless theory). The divergent terms are compensated by mass and wave function renormalization counterterms

$$C_1\Lambda^2 + C_2 p^2 \ln \frac{\Lambda^2}{\mu^2}. \tag{1.10}$$

For each value of the loop momentum p in the graph of Fig. 5 one can decompose the wave function renormalisation counterterm in a "useful" part $C_2 p^2 \ln \frac{\Lambda^2}{p^2}$ and a "useless" part $C_2 p^2 \ln \frac{p^2}{\mu^2}$. The useful part exactly cancels the divergent part of $A(p^2)$ and is all we need to get a finite graph. The useless part is what we have to take along with the useful part in order to preserve locality, i.e., to keep the counterterm of the form (1.10). It is not only useless but also a nuisance since it is exactly this useless part that produces the renormalons.

In the tree expansion one expands in power of the running coupling constants instead of the renormalized coupling constants. The contribution of the subgraph to the running wave function renormalisation constant on scale μ is

$$\delta Z(\mu) = \frac{d}{dp^2}\bigg|_{p^2=\mu^2} A(p^2) = C_2 \ln \frac{\Lambda^2}{p^2} + \text{finite}. \tag{1.11}$$

The idea is to consider separately each loop momentum slice and to write, for $p^2 \approx \mu^2$, $A(p^2) = A_R(p^2) + p^2\delta Z(\mu)$; $A_R(p^2)$ is then the amplitude renormalized with useful subtraction only. The price to pay is that one gets additional graphs with vertex $p^2\delta Z(\mu)$, i.e., one has an expansion in power of the coupling constants on all scales. These coupling constants obey flow equations (involving all coupling constants), and the renormalons reappear if one solves the flow equations by expressing the running coupling constants in a power series in the renormalised coupling constants. The whole combinatorics of the decomposition of contributions coming from all scales and subgraphs comes out naturally if one looks at the problem from the point of view of the renormalization group transformation, as will be explained in Section 2.

For the planar theory (the $N \to \infty$ limit of a $\mathrm{tr}\phi^4$ theory where ϕ is an $N \times N$ matrix), the series defining (1.7) is *convergent* (for planar theories there are no instanton factorials, i.e., the number of (unlabelled) Feynman graphs with n vertices grows as $const^n$ [16], [2]). Thus the Gallavotti-Nicolò method provided a natural proof of BPHZ, of the $n!$–bounds of de Calan and Rivasseau [3], and of the existence of the wrong sign ϕ_4^4 planar thory [13], [20], which is asymptotically free.

It then turned out [6], [4], [5] that the beta functional is still finite (i.e. admits a perturbation series in the coupling constants around zero with finite coefficients) for non-renormalizable theories (in contrast to the beta *function*), and we come to the contents of this thesis.

The tree expansion (i.e., the expansion of the "effective potentials" or the Schwinger functions in powers of the running coupling constants on all scales) and the beta functional are introduced in Section 2. In Section 3 the relation between the effective potentials and the more standard Schwinger functions is explained.The formalism is illustrated by two simple examples, the Gaussian model and the $N \to \infty$ limit of $\left(\vec{\phi}^{\,2}\right)^2$ (the spherical model), in Section 4.

The core of the work is Section 5 where the proof of finiteness (and convergence, in the planar thory) of the tree expansion and the beta functional is presented. The proof is for the general case of a scalar field theory, as in [6], but we adopt the method used in [4] which is probably simpler.

Section 6 is the summary of the perturbative results obtained in [6] ($n!$-bounds for the general case of a renormalizable theory, "perturbative triviality").

In Section 7 we construct a non-trivial fixed point for the ϕ^4_4-theory with propagator $1/p^{2-\epsilon/2}$ which has, as $\phi^4_{4+\epsilon}$, a non-renormalizable power counting, but is more tractable being in an integer number of number of dimensions.

Of course the resulting theory has no chance of being a relativistic quantum field theory for at least two reasons: first, in large N theories, scattering is suppressed by a factor $\frac{1}{N}$ and one gets a generalized free field, that can however have a rich mass spectrum. Second, even if a Euclidean theory could be constructed beyond this $\frac{1}{N}$ expansion, it could hardly be made unitary: Gawędzki and Kupiainen have constructed a ϕ^4_4 theory beyond the planar approximation [10] (at least in the hierarchical model) and seem to have evidence that Osterwalder-Schrader positivity is broken, and no continuation from Euclidean to Minkowski space is thus possible.

Nevertheless our result shows that one can cope with the difficulty mentioned above of a beta function which is not finite in perturbation theory for a non-trivial example (the $N \to \infty$ limit of $\left(\vec{\phi}^2\right)^2$) where this difficulty *is* present (unlike the spherical model).

Similar results have been obtained by Gawędzki and Kupiainen [11], [12]. They consider the two-dimensional Gross-Neveu model with $\not{p}/p^{2-\epsilon}$ propagator, and analyze the flow of the full effective hamiltonians rather than focussing on the coupling constants.

A probably more physical application of our methods is the ϵ-expansion [24], [25] in the statistical mechanics of critical phenomena. However, although the general idea will work, one has to modify the way one introduces the cut-off in order to handle theories with anomalous dimensions. In fact all models considered so far (the spherical model, the planar theory with singular propagator, the Gross-Neveu model with singular propagator, as well as the newly constructed [15] Wilson fixed point in infrared hierarchical ϕ_3^4) have ***trivial wave function renormalization***, and no anomalous dimensions.

A natural framework to treat rigorously (at least in the planar approximation) theories with non-trivial wave function renormalization has in fact already been provided by Wilson himself: it goes under the name "exact renormalization group" and can be found in Chapter 11 of [25], and was studied in more detail by Wegner [22].

2. The tree expansion and the beta functional

In this section we introduce the tree expansion and the beta functional for a general scalar field theory as in [6]. We work in perturbation theory and postpone the discussion of the planar theory, where the expansions are convergent, to a later section.

We consider a perturbation of a free field ϕ, a Gaussian random field with mean zero and covariance

$$C(x,y) = \frac{1}{(2\pi)^d} \int \frac{1}{p^2} e^{ip(x-y)} d^d p \tag{2.1}$$

in d-dimensional Euclidean space-time. We introduce a scale decomposition of ϕ by writing

$$\phi = \sum_{j=-\infty}^{\infty} \phi^{(j)}, \tag{2.2}$$

where $\phi^{(j)}$ are independent Gaussian fields with mean zero and covariance (in momentum space)

$$\hat{C}^j(p) = \frac{1}{p^2} \left\{ f(p^2/\gamma^{2j}) - f(p^2/\gamma^{2(j-1)}) \right\}, \tag{2.3}$$

where $f(p^2)$ is some cut-off function, e.g.,

$$f(p^2) = e^{-p^2}, \tag{2.4}$$

and $\gamma > 1$ is some fixed scale factor ($\gamma = 2$, say). We have the scaling relations

$$C^j(x,y) = \gamma^{(d-2)j} C^0(\gamma^j x, \gamma^j y), \tag{2.5}$$

$$\phi^{(j)} \stackrel{\text{distr}}{=} \phi^{(0)}(\gamma^j \cdot)\, \gamma^{\frac{d-2}{2} j}, \tag{2.6}$$

and the bounds $\left(\partial^l \equiv \prod_\mu \left(\frac{\partial}{\partial x^\mu}\right)^{l_\mu}, \quad |l| \equiv \sum_\mu l_\mu\right)$

$$|\partial^l C^j(0,x)| \le C_1(l) \gamma^{(d-2+|l|)j} e^{-C_2 \gamma^j |x|}. \tag{2.7}$$

The model is defined by introducing a cut-off on scale γ^N and taking as interaction a potential of the cut-off field $\phi^{\le N} = \sum_{j=-\infty}^{N} \phi^{(j)}$:

$$V^{(N)}(\phi^{\le N}) = \sum_\alpha \gamma^{-\sigma(\alpha)N} \lambda_\alpha(N) O_\alpha(\phi^{\le N}), \tag{2.8}$$

where α runs over a finite set. We choose as "interactions" $\mathcal{O}_\alpha$:

$$\begin{aligned} \mathcal{O}_{2'}(\phi) &= \int :(\partial\phi(x))^2: dx, & \sigma(2') &= 0, \\ \mathcal{O}_2(\phi) &= \int :\phi(x)^2: dx, & \sigma(2) &= -2, \\ \mathcal{O}_4(\phi) &= \int :\phi(x)^4: dx, & \sigma(4) &= d-4, \\ &\dots \\ \mathcal{O}_{2t}(\phi) &= \int :\phi(x)^{2t}: dx, & \sigma(2t) &= (d-2)t-d, \end{aligned} \tag{2.9}$$

where $\sigma(\alpha)$ is the dimension of $\mathcal{O}_\alpha$, so that $\lambda_\alpha(N)$ are the dimensionless bare coupling constants. One could also add any finite number of higher derivative interactions or uneven interactions, with only notational complication.

Following Wilson [23], we introduce "effective potentials" through the recursive definition

$$\begin{aligned} V^{(k)}(\phi^{\leq k}) &= -\ln E_{k+1} \exp\big(-V^{(k+1)}(\phi^{\leq k+1})\big), \\ k &= -\infty, \dots, N-1, \end{aligned} \tag{2.10}$$

where E_{k+1} denotes integration over the distribution of $\phi^{(k+1)}$. We expand $V^{(k)}$ in powers of the fields:

$$V^{(k)}(\phi^{\leq k}) = \sum_{m=0}^{\infty} \int V_m^{(k)}(x_1, \dots, x_m) :\phi^{\leq k}(x_1) \cdots \phi^{\leq k}(x_m):. \tag{2.11}$$

With our choice (2.9),the kernels $V_m^{(k)}$ in (2.11) will be Euclidean invariant distributions, non-vanishing only if m is even. In the next section we discuss how the effective potentials $V^{(k)}$ are related to the Schwinger functions. The result is that the kernels $V_m^{(k)}(x_1, \dots, x_m)$ are essentially Schwinger functions with an infrared cut-off on scale γ^k. Therefore solving the ultraviolet problem of the theory is equivalent to finding a sequence of bare coupling constants $\lambda_\alpha(N)$ such that the resulting $V^{(k)}$, with k fixed, remains finite when the cut-off is removed ($N \to \infty$).

The main idea in the renormalisation group program is that the full flow of $V^{(k)}$ (see (2.10)) is actually governed by a finite number of degrees of freedom, the relevant operators. We thus want to keep track of a finite number of running coupling constants $\lambda_\alpha(k)$ defined by

$$\sum_\alpha \gamma^{-\sigma(\alpha)k} \lambda_\alpha(k) \mathcal{O}_\alpha(\phi^{\leq k}) = L_k V^{(k)}(\phi^{\leq k}), \tag{2.12}$$

where L_k is a projector onto the space spanned by $\{O_\alpha\}_{\alpha=2',2,\dots,2t}$. Specifically, L_k acts as a linear operator on expressions of the form (2.11) according to (m even)

$$
\begin{aligned}
&L_k \int V(\mathbf{x}) : \prod_{i=1}^{m} \phi^{\leq k}(x_i) : d\mathbf{x} = \begin{cases} 0, & \text{if } m > 2t \\ \hat{V}(0)\, O_m, & \text{if } 2 < m \leq 2t \end{cases} \\
&L_k \int V(x_1, x_2) : \phi^{\leq k}(x_1) \phi^{\leq k}(x_2) : dx_1 dx_2 \\
&\qquad = \hat{V}(0)\, O_2(\phi^{\leq k}) + \frac{\partial}{\partial p^2} \hat{V}(p^2)\Big|_{p^2=0} O_{2'}(\phi^{\leq k}).
\end{aligned}
\tag{2.13}
$$

Not all coupling constants in (2.7) are relevant (relevant means $\sigma(\alpha) < 0$) with respect to the Gaussian fixed point, but we keep them all because they can become relevant with respect to some other fixed point or, more generally, to some other solution of the flow equation (2.10).

The next step is the tree expansion which is an expansion in powers of all λ_α's of the effective potentials. It is based on the cumulant expansion for the irrelevant part of $V^{(k)}$:

$$
\begin{aligned}
V^{(k)} &= \sum_\alpha \gamma^{-\sigma(\alpha)k} \lambda_\alpha(k)\, O_\alpha(\phi^{\leq k}) \\
&\quad + (1 - L_k) \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s!} E_{k+1}^T \left(V^{(k+1)}, \dots, V^{(k+1)} \right).
\end{aligned}
\tag{2.14}
$$

This expansion generates by iteration terms of the type ($k < h < j$)

$$
\begin{aligned}
&\gamma^{-2(d-4)j} \lambda_4(j)^2 \lambda_{2'}\, E_{k+1} E_{k+2} \cdots E_{h-1} (1 - L_{h-1}) \\
&E_h^T \left(E_{h+1} \cdots E_{j-1} (1 - L_{j-1}) E_j^T \left(\textstyle\int :\phi^4:, \int :\phi^4:\right), \int :(\partial\phi)^2 \right),
\end{aligned}
\tag{2.15}
$$

(we have used $E_k^T(\cdot) = E_k(\cdot)$, $E_k L_k = L_{k-1} E_k$, $L_k^2 = L_k$). These terms can be efficiently labelled by *trees* θ. For instance, the term (2.15) is labelled by the tree depicted in fig. 1. In general the tree expansion is given by the following expression:

$$
V^{(k)} = L_k V^{(k)} + \sum_{\substack{\theta, \mathbf{h}, \underline{\alpha} \\ h_{V_0} = k}}{}' \frac{1}{n(\theta)} V_{\text{ren}}(\theta, \mathbf{h}, \underline{\alpha}) \prod_{\substack{i \text{ endpoints} \\ \text{of } \theta}} \lambda_{\alpha_i}(h_i),
\tag{2.16}
$$

where, if θ is a tree, $\mathbf{h} = (h_V)_{V \in \theta}$ are integer labels of the branching points V of θ with $h_{V'} < h_{V''}$ if $V' < V''$ in the ordering of θ. The endpoints i of θ are given labels

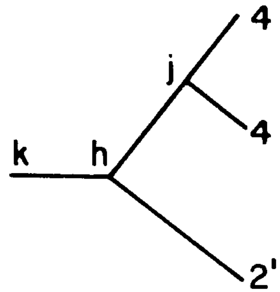


Fig. 6. A simple tree

$\alpha_i \in \{2', 2, 4, \ldots, 2t\}$, and h_i is the label assigned to the branching point to which the endpoint i is connected. The *root* V_0 is given the frequency label k of the effective potential, and the sum in (2.16) is over all non-trivial trees θ (the trivial tree is the one without branches), frequency assignments $\mathbf{h}$ and endpoint labels $\underline{\alpha}$. The combinatorial factor $n(\theta)$ is

$$n(\theta) = \prod_{V \in \theta} s_V!, \tag{2.17}$$

where s_V is the number of subtrees into which θ branches at V. Fig. 7 shows a general tree with root V_0 and first branching point V_1.

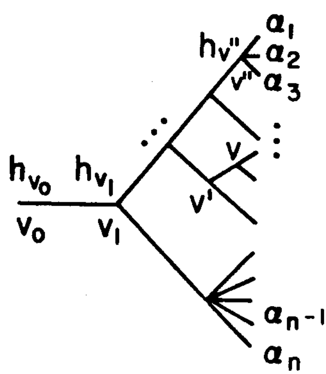


Fig. 7. A general tree

The tree coefficients $V_{\text{ren}}(\theta, \mathbf{h}, \underline{\alpha})$ can be computed by the following recursive formulae: if $\theta = \theta_0$ is the trivial tree,

$$V_{\text{ren}}(\theta_0, h_{V_0}, \alpha) = \gamma^{-\sigma(\alpha) h_{V_0}} O_\alpha\left(\phi^{\leq h_{V_0}}\right), \tag{2.18}$$

and if θ branches at the first branching point V_1 into trees $\theta_1, \ldots, \theta_s$,

$$\begin{aligned} V_{\text{ren}}(\theta, \mathbf{h}, \underline{\alpha}) = & E_{h_{V_0}+1} \cdots E_{h_{V_1}-1}(I - L_{h_{V_1}-1}) \\ & E^T_{h_{V_1}}\left(V_{\text{ren}}(\theta_1, \mathbf{h}_1, \underline{\alpha}_1), \ldots, V_{\text{ren}}(\theta_s, \mathbf{h}_s, \underline{\alpha}_s)\right), \end{aligned} \tag{2.19}$$

$\mathbf{h}_r, \underline{\alpha}_r$ being the restriction of $\mathbf{h}, \underline{\alpha}$ to θ_r, $r = 1, \ldots, s$.

Note that (2.16) is an expansion in powers of all running coupling constants $\lambda_\alpha(h)$ and that the only cut-off dependence arises from the fact that the sum in (2.16) is restricted to $h_V \leq N$, $V \in \theta$, so that the cut-off can be removed simply by ignoring this restriction. The crucial observation will be that all sums over momentum scales $\mathbf{h}$ are *convergent* order by order in $\lambda_\alpha(h)$!

If we act with L_k on (2.10) in cumulant expansion

$$V^{(k)} = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s!} E^T_{k+1}(V^{(k+1)}, \ldots, V^{(k+1)}), \tag{2.20}$$

and insert (2.16) on the right-hand side, we get a recursion relation between the running coupling constants (the flow equation of the coupling constants)

$$\lambda_\alpha(k) = \gamma^{-\sigma(\alpha)} \lambda_\alpha(k+1) - \sum_{\substack{\theta, \mathbf{h}, \underline{\alpha} \\ h_{V_0} = k \\ h_{V_1} = k+1}}{}' \frac{1}{n(\theta)} \beta_\alpha(\theta, \mathbf{h}, \underline{\alpha}) \prod_i \lambda_{\alpha_i}(h_i), \tag{2.21}$$

where, again, the prime means that we sum over all non-trivial trees. The *beta functional* coefficients $\beta_\alpha(\theta, \mathbf{h}, \underline{\alpha})$ are defined by

$$\begin{aligned} & \sum_\alpha \gamma^{-\sigma(\alpha) k} \beta_\alpha(\theta, \mathbf{h}, \underline{\alpha})\, O_\alpha(\phi^{\leq k}) \\ & \quad = L_k E_{k+1}\left(V_{\text{ren}}(\theta_1, \mathbf{h}_1, \underline{\alpha}_1), \ldots, V_{\text{ren}}(\theta_s, \mathbf{h}_s, \underline{\alpha}_s)\right). \end{aligned} \tag{2.22}$$

Again, the cut-off is removed by summing over $\mathbf{h}$ in (2.21) without the restriction $h_V \leq N$.

If we expand $V_{\rm ren}(\theta, \mathbf{h}, \underline{\alpha})$ in power of the fields

$$V_{\rm ren}(\theta,\mathbf{h},\underline{\alpha}) = \sum_m \int V_{\rm ren}(\theta,\mathbf{h},\underline{\alpha};x_1,\dots,x_m) : \prod_{i=1}^m \phi^{\le h_{v_0}}(x_i):, \qquad (2.23)$$

we can compute the kernels $V_{\rm ren}$ as sums over Feynman graphs:

$$V_{\rm ren}(\theta,\mathbf{h},\underline{\alpha};x_1,\dots,x_m) = \sum_{\substack{G \text{ compatible} \\ \text{with } \theta,\underline{\alpha}}} V_{{\rm ren},G}(\theta,\mathbf{h},\underline{\alpha};x_1,\dots,x_m), \qquad (2.24)$$

where the sum is over all Feynman graphs with m external lines and n vertices, if θ has n endpoints, which are compatible with $\theta,\underline{\alpha}$. A graph G is said to be compatible with $\theta,\underline{\alpha}$ if the following two conditions are satisfied:

(a) For $i = 1, \dots, n$, the $i^{\rm th}$ vertex is an O_{α_i} insertion, i.e., it has coordination number 2 if $\alpha_i = 2, 2'$, and $r = 2, 4, \dots, 2t$ if $\alpha_i = r$.

(b) G has two types of internal lines: *hard* lines (coming from the truncated expectations E_h^T) and *soft* lines (coming from Wick reordering). If we draw a bubble around each set of vertices corresponding to subtrees of θ, then the hard lines should connect all bubbles contained in any bubble.

Fig. 8 shows a graph which is compatible with the tree of Fig. 6. The ∂'s recall that the third vertex is an $O_{2'}$ insertion.

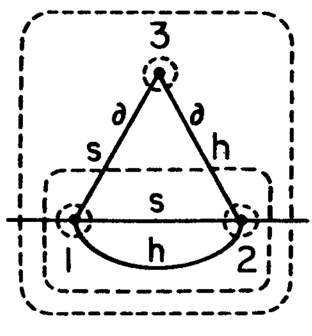


Fig. 8. A graph compatible with the tree of Fig. 6

3. Schwinger functions

There are three ways of calculating the correlation (Schwinger) functions given the effective potentials. The first one is the most straightforward and is described in [8]: the Schwinger functions are expanded in a sum of Schwinger functions calculated in the ensemble generated by the effective potentials. This method is not very convenient, since it requires some extra work to show the convergence of these expansions. In the second method one expresses the Schwinger functions in terms of the limit $k \to -\infty$ of $V^{(k)}$. Thus uniform (in k) bounds on $V^{(k)}$ are sufficient to control the Schwinger functions. Moreover Schwinger functions with an *infrared* cut-off on scale γ^k are obtainable directly from $V^{(k)}$, so that the construction of $V^{(k)}$, with k fixed, provides a solution of the ultraviolet problem. The third method is more conceptually important. It shows that one can recover all Schwinger functions from the Schwinger functions of the $V^{(k)}$ ensemble for any single k.

Let us now describe the three methods in more detail:

1) The idea is to write

$$< \phi(x_1)\cdots\phi(x_m) >= \sum_{k_1\cdots k_m} < \phi^{(k_1)}(x_1)\cdots\phi^{(k_m)}(x_m) >, \tag{3.1}$$

and the expectations on the right-hand side can be computed taking as interaction $V^{(k)}$, where $k = \max_{i=1,\dots,m} k_i$.

2) The second way of computing Schwinger functions is based on the observation that $V^{(k)}$ is the generating functional of the amputated connected Feynman graphs with infrared cut-off propagators. To prove this we start from the definition of $V^{(k)}$

$$V^{(k)}(\phi^{\leq k}) = -\ln\int \exp\left\{-V^{(N)}(\phi^{\leq k}+\bar\phi) - \tfrac{1}{2}(\bar\phi, Q^{-1}\bar\phi)\right\} D\bar\phi \tag{3.2}$$

where $Q = C^{\leq N} - C^{\leq k} = \frac{1}{p^2}\left(f(p^2/\gamma^{2N}) - f(p^2/\gamma^{2k})\right)$ and we have written out explicitly the functional integral for clarity, neglecting ϕ independent contributions to $V^{(k)}$. We change variables:

$$\begin{aligned} \phi &= \phi^{\leq k} + \bar\phi \\ V^{(k)}(\phi^{\leq k}) &= \tfrac{1}{2}(\phi^{\leq k}, Q^{-1}\phi^{\leq k}) \\ &\quad -\ln\int \exp\left\{-V^{(N)}(\phi) - \tfrac{1}{2}(\phi, Q^{-1}\phi) + (Q^{-1}\phi^{\leq k}, \phi)\right\} D\phi, \end{aligned} \tag{3.3}$$

and get the generating functional of the connected, Q-amputated Schwinger functions and infrared cut-off provided by the propagator Q on scale γ^k.

The connected Schwinger functions are thus generated by the functional

$$W(J) = \lim_{k \to -\infty} (-V^{(k)}(QJ) + \tfrac{1}{2}(J, QJ)). \tag{3.4}$$

3) The third method shows that the whole information is actually contained in every single $V^{(k)}$. Consider the generating functional

$$W(J) = \ln \int \exp\left\{-V^{(N)}(\phi^{\leq N}) - \tfrac{1}{2}(\phi^{\leq N}, C^{\leq N}\phi^{\leq N}) + (j, \phi^{\leq N})\right\} D\phi, \tag{3.5}$$

and introduce the momentum decomposition

$$C^{\leq N} = C^{\leq k} + Q.$$

Writing again the functional integral explicitly

$$\begin{aligned} W(J) = \ln \int \exp\Big\{&-V^{(N)}(\phi^{\leq k} + \bar{\phi}) + (j, \phi^{\leq k} + \bar{\phi}) \\ &- \tfrac{1}{2}(\phi^{\leq k}, (C^{\leq k})^{-1}\phi^{\leq k}) - \frac{1}{2}(\bar{\phi}, Q^{-1}\bar{\phi})\Big\} D\phi^{\leq k} D\bar{\phi}, \end{aligned} \tag{3.6}$$

we look for a change of variables which eliminates the coupling of the source J to the high energy field $\bar{\phi}$. The right choice is

$$\begin{aligned} \phi_1 &= \phi^{\leq k} + QJ \\ \phi_2 &= \bar{\phi} - QJ, \end{aligned} \tag{3.7}$$

so that, after some algebra,

$$W(J) = -\tfrac{1}{2}(J, Q\tilde{J}) + \ln \int \exp\left\{-V^{(k)}(\phi^{\leq k}) + \tfrac{1}{2}(\phi^{\leq k}, (C^{\leq k})^{-1}\phi^{\leq k}) + (\tilde{J}, \phi^{\leq k})\right\} D\phi^{\leq k}, \tag{3.8}$$

where $\tilde{J} = C^{\leq N}(C^{\leq k})^{-1}J$. What this says, graphically, is that the Schwinger functions of $V^{(k)}$ are given by all Feynman graphs, except for the fact that the external lines are low frequency propagators $C^{\leq k}$ instead of full propagators $C^{\leq N}$. The action of the factor

$C^{\leq N}(C^{\leq k})^{-1}$ in the definition of $\tilde{J}$ is just to amputate the $C^{\leq k}$ propagators and replace them by $C^{\leq N}$ propagators. For the graph with no vertices (a single line) the discussion has to be done separetely: the first term on the right-hand side of (3.8) compensates the error in the zeroth order contribution of the functional integral:

$$-\tfrac{1}{2}(J, Q\tilde{J}) + \tfrac{1}{2}(\tilde{J}, C^{\leq k}\tilde{J}) = \tfrac{1}{2}(J, C^{\leq N}J). \tag{3.9}$$

4. Illustrative examples

(a) The Gaussian case. In the Gaussian model $V^{(N)} = \int V_2^{(N)}(x-y)\phi^{\leq N}(x)\phi^{\leq N}(y)\,dx\,dy$ one can compute $V^{(k)}$ explicitly from the definition

$$e^{-V^{(k)}(\phi^{\leq k})} = E_{k+1}\cdots E_N e^{-V^{(N)}(\phi^{\leq N})}, \tag{4.1}$$

since all integrations are Gaussian. The result is that then $V^{(k)}$ is again Gaussian:

$$V^{(k)}(\phi^{\leq k}) = \int V_2^{(k)}(x-y)\phi^{\leq k}(x)\phi^{\leq k}(y), \tag{4.2}$$

and, if $V_2^{(N)}$ is Euclidean invariant, $V_2^{(k)}$ is given in Fourier space by

$$V_2^{(k)}(p^2)^{-1} = V_2^{(N)}(p^2)^{-1} + C^{\leq N}(p^2) - C^{\leq k}(p^2), \tag{4.3}$$

where, again,

$$C^{\leq k}(p^2) = \frac{1}{p^2} f(p^2/\gamma^{2k}) \tag{4.4}$$

is the cut-off propagator.

In dimensionless notation

$$U_2^{(k)}(p^2) \equiv \gamma^{-2k} V_2^{(k)}(p^2\gamma^{2k}), \tag{4.5}$$

we get

$$U_2^{(k)}(p^2)^{-1} = \gamma^{2(k-N)} U_2^{(N)}(\gamma^{2(k-N)}p^2)^{-1} + \frac{1}{p^2}(f(p^2\gamma^{2(k-N)}) - f(p^2)), \tag{4.6}$$

with convergence to the "high temperature" fixed point

$$U_2^{(-\infty)}(p^2) = \frac{p^2}{1-f(p^2)} = f'(0)^{-1} + O(p^2), \qquad (p^2 \to 0), \tag{4.7}$$

if $U_2^{(N)} \neq 0$, and to the massless Gaussian

$$U_2^{(-\infty)}(p^2) = \frac{p^2}{\alpha^{-1} + 1 - f(p^2)} = \alpha p^2 + O(p^4), \qquad (p^2 \to 0), \tag{4.8}$$

if $U^{(N)}(0) = \alpha p^2 + O(p^4)$, $\alpha > 0$.

This behaviour can of course already be seen in the space of the running coupling constants. The flow equations can be deduced from (4.6) at $p^2 = 0$:

$$\lambda_2(k)^{-1} = \gamma^{-2}\lambda_2(k+1)^{-1} + f'(0)(\gamma^{-2} - 1), \tag{4.9}$$

$$\lambda_2(k)^{-2}\lambda_{2'}(k) = \gamma^{-4}\lambda_2(k+1)^{-2}\lambda_{2'}(k+1) + f''(0)(\gamma^{-4} - 1), \tag{4.10}$$

with coupling of scale k only to scale $k+1$ (this is a consequence of the fact that only trees with one branching point contribute in the Gaussian case). The behaviour of the solutions of (4.9), (4.10) is shown in Fig. 9. The axis $\lambda_2 = 0$ is the line of massless fixed points.

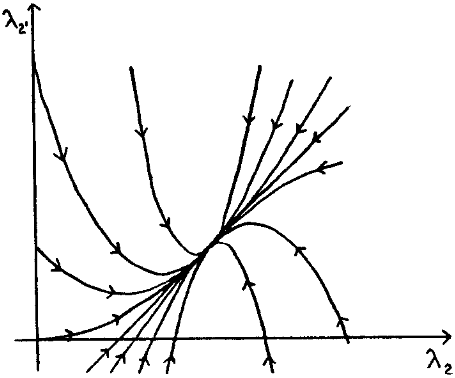


Fig. 9. The flow of the coupling constants in the Gaussian case

(b) The spherical model. Let $\vec{\phi}(x) = (\phi_1(x), \ldots, \phi_n(x))$ be a real n-component scalar field. The theory is given by the action

$$S = n \int \left(\tfrac{1}{2}(\partial\vec{\phi})^2 + \tfrac{m_0^2}{2}\vec{\phi}^2 + \tfrac{\lambda_0}{4!}(\vec{\phi}^2)^2 \right) d^d x, \tag{4.11}$$

and we are interested in the limit $n \to \infty$ of this model. The momentum decomposition is written for the free propagator

$$\frac{1}{n} f(p^2/\gamma^{2N})/p^2 \tag{4.12}$$

and the relation between effective potentials is given formally in terms of the finite (as $n \to \infty$) quantities $v^{(k)} \equiv \frac{1}{n}V^{(k)}$ by

$$v^{(k)}(\phi^{\leq k}) = -\lim_{n\to\infty} \tfrac{1}{n} E_{k+1}\cdots E_h \exp -nv^{(h)}(\phi^{\leq h}), \tag{4.13}$$

where E_j is the integration over the distribution of $\phi^{(j)}$, a Gaussian with covariance

$$\frac{1}{n}\big(f(p^2/\gamma^{2j}) - f(p^2/\gamma^{2(j-1)})\big)/p^2. \tag{4.14}$$

The Feynman rules to construct the graphs are the same as the ones of the $n = 1$ theory, except for factors of n to some power: a factor of n^{-1} for each propagator, a factor n for each vertex, and a factor n for each trace over $O(n)$ indices in closed loops.

We can thus evaluate the kernels in the expansion of $v^{(k)}$ in the fields:

$$\begin{aligned} v^{(k)}(\vec{\phi}) = \sum_{m=1}^{\infty} \tfrac{1}{(2m)!} \int v_{2m}^{(k)}(x_1,\ldots,x_{2m}) \\ (\vec{\phi}(x_1)\cdot\vec{\phi}(x_2))\cdots(\vec{\phi}(x_{2m-1})\cdot\vec{\phi}(x_{2m}))\,dx_1\cdots dx_{2m} \end{aligned} \tag{4.15}$$

summing the Feynman graphs which do not vanish in the $n \to \infty$ limit. We do this for the two and four point functions, which can be computed with the graphs of Fig. 10.

From now to the end of this section we switch to the more standard notation $\mu = \gamma^k$, $\Lambda = \gamma^N$ and write $v_2(\cdot,\mu)$ instead of $v_2^{(k)}(\cdot)$. The lines in Fig. 10 represent "hard" propagators

$$Q_{\mu\Lambda}(p^2) = \frac{1}{p^2}\big(f(p^2/\Lambda^2) - f(p^2/\mu^2)\big) \tag{4.17}$$

(we have no soft propagators here because we do not Wick order in this explicit calculation). It is convenient to pass to one-particle irreducible (1PI) functions on scale μ, denoted $\Gamma_{2m}(\cdot,\mu)$, given by the graphs of Fig. 11. The kernels $v_2(\cdot,\mu)$ can be computed from the 1PI functions on scale μ by the following relations: In momentum space we have for the two point function

$$\begin{aligned} v_2(p^2,\mu) &= \tilde{\Gamma}_2(p^2,\mu^2) - \tilde{\Gamma}_2(p^2,\mu^2)Q_{\mu\Lambda}(p^2)\tilde{\Gamma}_2(p^2,\mu^2) \\ &\quad + \tilde{\Gamma}_2(p^2,\mu^2)Q_{\mu\Lambda}(p^2)\tilde{\Gamma}_2(p^2,\mu^2)Q_{\mu\Lambda}(p^2)\tilde{\Gamma}_2(p^2,\mu^2) + \cdots \\ &= \tilde{\Gamma}_2(p^2,\mu^2)\frac{1}{1+Q_{\mu\Lambda}(p^2)\tilde{\Gamma}_2(p^2,\mu^2)} \\ &= Q_{\mu\Lambda}(p^2)^{-1}\Gamma_2(p^2,\mu)^{-1}Q_{\mu\Lambda}(p^2)^{-1} - Q_{\mu\Lambda}(p^2)^{-1}, \\ \Gamma_2(p^2,\mu) &= \tilde{\Gamma}_2(p^2,\mu^2) + Q_{\mu\Lambda}(p^2)^{-1}, \end{aligned} \tag{4.18}$$

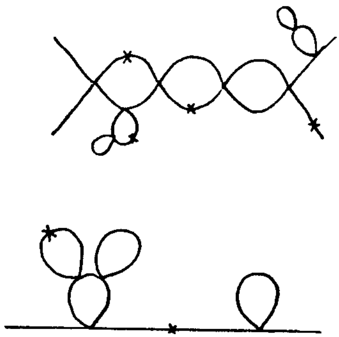


Fig. 10. Graphs contributing to the 2- and 4-point functions

where $\tilde{\Gamma}_2$ is the sum over all non-trivial 1PI graphs with two (amputated) external lines. Traditionally one defines Γ_2 by adding to $\tilde{\Gamma}_2$ the free inverse propagator $Q_{\mu\Lambda}^{-1}$ (the bare 2-point vertex). The four-point function can be recovered from its 1PI part by noting that the external lines in the four point graphs in Fig. 10 can be resummed to

$$= v_2(p^2,\mu)Q_{\mu\Lambda}(p^2), \tag{4.19}$$

yielding the result

$$v_4(p_1,\dots,p_4,\mu) = \prod_{i=1}^{4}(1+v_2(p_i^2,\mu)Q_{\mu\Lambda}(p_i^2))\Gamma_4(p_1,\dots,p_4,\mu), \qquad p_1+\cdots+p_4=0, \tag{4.20}$$

and we see from the graphs of Fig. 11 that Γ_4 depends on the external momenta only in the combination $(p_1+p_2)^2=(p_3+p_4)^2$.

By the analysis of Section 3, the 1PI functions converge as $\mu\to 0$ ($k\to-\infty$) to the (amputated) 1PI Schwinger functions (Euclidean vertex functions) of the theory. In this context we can see this simply diagrammatically, noting that the graphs of Fig. 6 are exactly the graphs contributing to the vertex functions and that, for $\mu \to 0$, $Q_{\mu\Lambda} \to \frac{1}{p^2} f(p^2/\Lambda^2)$, the (UV cut-off) propagator.

The 1PI functions on scale μ, $\Gamma_2(p^2,\mu)$, $\Gamma_4(p^2,\mu)$, can be evaluated quite explicitly by summing all the graphs or, equivalently, by solving the integral equations (Fig. 12)

$$\begin{aligned}\Gamma_2(p^2,\mu) &= m_0^2 + \tfrac{\lambda_0}{6}\int \Gamma_2^{-1}(q^2,\mu)\frac{d^dq}{(2\pi)^d} + Q_{\mu\Lambda}^{-1}(p^2),\\ \Gamma_4(p^2,\mu) &= \lambda_0 - \tfrac{\lambda_0}{6}\int \Gamma_2^{-1}(q^2,\mu)\Gamma_2^{-1}((p-q)^2,\mu)\frac{d^dq}{(2\pi)^d}\Gamma_4(p^2,\mu),\end{aligned} \tag{4.21}$$

where the "full propagator" (the sum of all two-point bubbles in Fig. 12) is equal to Γ_2^{-1}, as one can easily seen by summing a geometric series as in (4.18).

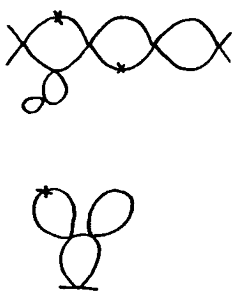


Fig. 11. One-particle irreducible 2- and 4-point graphs

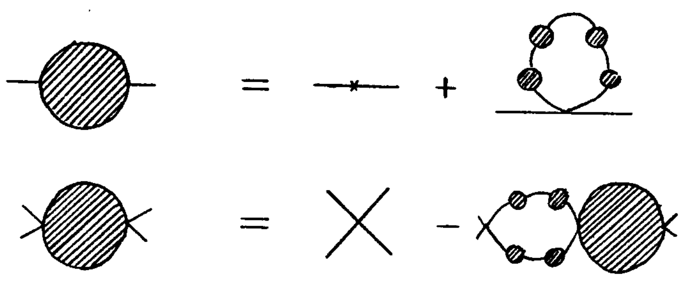


Fig. 12. Graphical representation of the integral equations (4.21)

To construct a massive theory we impose renormalization conditions at $\mu = 0$ (i.e., on the vertex functions) and at zero momentum

$$\begin{aligned} \Gamma_2(0,0) &= m^2, \\ \Gamma_4(0,0) &= \lambda. \end{aligned} \tag{4.22}$$

Then the relation between renormalized and bare quantities can be computed by taking (4.21) at zero momentum and setting $\mu = 0$, and noting that by the first of (4.21), Γ_2 is of the form $Q^{-1}_{\mu\Lambda}$ + const:

$$\begin{aligned} m^2 &= m_0^2 + \tfrac{\lambda_0}{6} \int \frac{1}{q^2 f^{-1}(q^2/\Lambda^2) + m^2} d^d q/(2\pi)^d, \\ \lambda^{-1} &= \lambda_0^{-1} + \tfrac{1}{6} \int \left(\frac{1}{q^2 f^{-1}(q^2/\Lambda^2) + m^2} \right)^2 d^d q/(2\pi)^d, \end{aligned} \tag{4.23}$$

and the vertex functions are given by

$$\begin{aligned} \Gamma_2(p^2,0) &= m^2 + p^2 f^{-1}(p^2/\Lambda^2), \\ \Gamma_4(p^2,0)^{-1} &= \lambda^{-1} + \tfrac{1}{6} \int \left(\Gamma_2(q^2,0)^{-1} \Gamma_2((p-q)^2,0)^{-1} - \Gamma_2(q^2,0)^{-2} \right) \frac{d^d q}{(2\pi)^d}. \end{aligned} \tag{4.24}$$

In the renormalizable case $4 < d < 6$ the integrals in (4.24) diverge as $\Lambda \to \infty$ as $C_1 \Lambda^{d-2}$ and $C_2 \Lambda^{d-4}$ respectively, where C_1, C_2 are (f-dependent) positive constants, but choosing $\lambda_0 \Lambda^{d-4} \underset{\Lambda\to\infty}{=} \lambda^* \equiv -6C_2^{-1} < 0$, $m_0 \Lambda^{-2} \underset{\Lambda\to\infty}{=} m^{*2} \equiv C_2^{-1} C_1$ one obtains finite λ, m^2, and vertex functions [17]

$$\begin{aligned} \Gamma_2(p^2,0) &= m^2 + p^2, \\ \Gamma_4(p^2,0)^{-1} &= \lambda^{-1} + \tfrac{1}{6}(4\pi)^{-d/2} [F(\tfrac{4-d}{2}, 1; \tfrac{3}{2}; \tfrac{-p^2}{4m^2}) - 1] m^{d-4} \Gamma(\tfrac{4-d}{2}), \end{aligned} \tag{4.25}$$

where F is Gauss's hypergeometric function, which is analytic in the cut p^2 plane $\mathbb{C} \setminus \{p^2 < -4m^2\}$ and whose large p^2 asymptotics can be computed from the analytic continuation formula

$$\begin{aligned} F(\tfrac{4-d}{2}, 1; \tfrac{3}{2}; \tfrac{-p^2}{4m^2}) = \tfrac{1}{2}\sqrt{\pi} \frac{\Gamma(\frac{d-2}{2})}{\Gamma(\frac{d-1}{2})} \left(\tfrac{p^2}{4m^2}\right)^{\frac{d-4}{2}} \left(1 + \tfrac{4m^2}{p^2}\right)^{\frac{d-3}{2}} \\ - \tfrac{1}{d-2} \tfrac{m^2}{p^2} F(\tfrac{1}{2}, 1; \tfrac{d}{2}; -\tfrac{4m^2}{p^2}). \end{aligned} \tag{4.26}$$

In the same way one can compute all effective potentials starting from $\Gamma_2(p^2,\mu)$, $\Gamma_4(p^2,\mu)$ given by (4.18), (4.20) with $f(p^2/\Lambda^2)$ replaced by $f(p^2/\Lambda^2) - f(p^2/\mu^2)$ and m^2, λ replaced by the (dimensional) running coupling constants $m^2(\mu)$, $\lambda(\mu)$. These running coupling constants are related to the ones defined in Section 2 by a finite renormalization (at each fixed μ), by (4.22).

Thus we have seen that, due to existence of an ultraviolet fixed point, one can construct a two-parameter family of theories. That this can be done also in the planar theory is the content of the next sections.

5. Finiteness of the tree expansion

The effective potentials, expressed as power series in the fields can be brought to the form (2.11)

$$V^{(k)} = \sum_{\substack{m=0 \\ m \text{ even}}} \int V_m^{(k)}(x_1, \ldots, x_m) : \phi^{\leq k}(x_1) \cdots \phi^{\leq k}(x_m) : dx_1 \cdots dx_m \qquad (5.1)$$

by a formal partial integration of a sum of Wick monomials in $\phi^{\leq k}$ and $\partial\phi^{\leq k}$. We therefore assume that $V_m^{(k)} \in \mathcal{V}_2(\mathbb{R}^{dm})$, where

$$\mathcal{V}_l(\mathbb{R}^{dm}) = \Bigg\{ V \in S'(\mathbb{R}^{dm}) \Bigg| V(x_1, \ldots, x_m) = \sum_{\{l_i, k_i, j_i\}_{i=1}^{s}} \tilde{V}_{\{l_i, k_i, j_i\}}(x_1, \ldots, x_m) \prod_{i=1}^{s} \partial^{l_i} \delta(x_{j_i} - x_{k_i}), \tilde{V}_{\{l_i, k_i, j_i\}} \in L^1_{\text{loc}}(\mathbb{R}^{dm}) \Bigg\}. \qquad (5.2)$$

Moreover $V_m^{(k)}$ will always be Euclidean invariant.

The kernels $V_m^{(k)}$ can be computed in tree expansion (2.16) as power series in the running coupling constants $\lambda_\alpha(k)$, which obey the recursion relation (2.21). The aim of this section is to prove that (2.16),(2.21) are *finite* order by order in $\underline{\lambda}$, and *convergent* for the planar theory.

The *dimension* (degree of convergence) of a Wick monomial of degree m with l gradients is defined by

$$\sigma(m, l) = -d + \frac{d-2}{2} m + l \qquad (5.3)$$

The main result of this section is the following

Theorem 5.1. *If $d > 2$ and $t \geq [\frac{d-2}{2}]$, i.e. the theory is barely renormalizable or not renormalizable, then*
(i) Let $f_1, \ldots, f_m$ be $S(\mathbb{R}^d)$-test functions, and θ a non-trivial tree. Then

$$V_{\text{ren}}(\theta, \mathbf{h}, \underline{\alpha}; f_1, \ldots, f_m) = \sum_{\substack{\{l_i\}_{i=1}^{m} \\ 0 \leq l_i \leq 3 \\ \sigma(m, \sum_i |l_i|) \geq 1}} \tilde{V}_{\text{ren}.\{l_i\}}(\theta, \mathbf{h}, \underline{\alpha}; \partial^{l_1} f_1, \ldots, \partial^{l_m} f_m) \qquad (5.4)$$

where $\bar{V}_{\text{ren.}\{l_i\}} \in \mathcal{V}_0(\mathbb{R}^{dm})$ *and*

$$\int_{\Delta_2\times\cdots\times\Delta_m} \left|\bar{V}_{\text{ren.}\{l_i\}}(0,x_2,\ldots,x_m)\right| dx_2\cdots dx_m \\ \le C_0^n n(\theta) e^{-\frac{C_2}{2}\gamma^{h_{V_0}}\operatorname{dist}(0,\Delta_2,\ldots,\Delta_m)} \gamma^{-\sigma(m\sum_i |l_i|)h_{V_0}} \\ \prod_{\substack{V\in\theta\\ V>V_0}} \gamma^{-\frac{1}{2}(h_V-h_{V'})}((t-1)n)! \tag{5.5}$$

(ii) The beta functional coefficients are bounded by

$$|\beta(\theta,\mathbf{h},\underline{\alpha})| \le C_0^n n(\theta) \prod_{\substack{V\in\theta\\ V>V_1}} \gamma^{-\frac{1}{2}(h_V-h_{V'})}((t-1)n)! \tag{5.6}$$

where $n \ge 2$ *is the number of endpoints of* θ*;* $\Delta_2,\ldots,\Delta_m$ *are cubes in* $\mathbb{R}^d$ *with side size* $\gamma^{-h_{V_0}}$*;* $\operatorname{dist}(0,\Delta_2,\ldots,\Delta_m)$ *is the length of the shortest graph connecting* $0,\Delta_2,\ldots,\Delta_m$*;* V_0 *is the root of* θ*,* V_1 *is the first branching point, and, for* $V\in\theta$*,* V' *is the branching point immediately preceding* V *(see Fig. 2);* $C_0 = C_0(d,t,\gamma)$ *is a positive constant, and* C_2 *is the constant appearing in (2.7);* $n(\theta)$ *is defined in (2.17). The factor* $((t-1)n)!$ *is not present in the planar theory.*

Discussion. By a barely renormalizable theory we mean a theory such that any additional term to the Lagrangean makes it not renormalizable. Examples of barely renormalizable theories are ϕ_4^4, ϕ_3^6, but also $\phi_{d>4}^4$. Not covered by the theorem are superrenormalizable theories such as ϕ_3^4. The exponential decay factors $\gamma^{-\frac{1}{2}(h_V-h_{V'})}$ (recall that $h_V < h_{V'}$) allow us to sum over all $\mathbf{h}$. There are at most const^n trees θ with n endpoints with given labels $\underline{\alpha}$. We have thus the following result: The n^{th} order term in the expansions

$$V_m^{(k)} = \sum_{\mathbf{n}} V_{m,\mathbf{n}}^{(k)} \underline{\lambda}^{\mathbf{n}} \tag{5.6}$$

$$\lambda_\alpha(k) = \gamma^{-\sigma(\alpha)}\lambda_\alpha(k+1) + \sum_{\substack{\mathbf{n}\\ |\mathbf{n}|\ge 2}} \beta_{\alpha.\mathbf{n}}(k)\underline{\lambda}^{\mathbf{n}} \tag{5.7}$$

(in the notation $\underline{\lambda}^{\mathbf{n}} = \prod_{\alpha,k} n_\alpha(k)$) is bounded uniformly in the cut-off:

$$V_{m,\mathbf{n}}^{(k)}(f_1,\ldots,f_m) = \sum_{\{l_i\}} \tilde{V}_{m,\mathbf{n},\{l_i\}}^{(k)}(\partial^{l_1} f_1,\ldots,\partial^{l_m} f_m) \tag{5.8}$$

$$\sum_{|\mathbf{n}|=n} \int_{\Delta_2\times\cdots\times\Delta_m} |V_{m,\mathbf{n},\{l_i\}}^{(k)}(0,x_2,\ldots,x_m)|\underline{\lambda}^{\mathbf{n}} dx_2\cdots dx_m$$
$$\leq C^n((t-1)n)!\|\underline{\lambda}\|_\infty^n e^{-\frac{c_2}{2}\mathrm{dist}(0,\Delta_2,\ldots,\Delta_m)}\gamma^{-\sigma(m,\{l_i\})k} \tag{5.9}$$

$$\sum_{|\mathbf{n}|=n} \beta_{\alpha,\mathbf{n}}(k)\underline{\lambda}^{\mathbf{n}} \leq C^n\|\lambda\|_\infty^n((t-1)n)! \tag{5.10}$$

In the planar theory, where the factor $((t-1)n)!$ (coming from the counting of Feynman graphs) does not appear, the series (5.6), (5.7) converge absolutely for small $\|\underline{\lambda}\|_\infty$.

Proof of the Theorem. The proof is in three steps. First, we prove that the operator $1 - L_k$ generates irrelevant terms with "good" bounds. The second step is an estimate on Feynman amplitudes and the third is an estimate on the sum over all Feynman graphs compatible with a given tree.

1st step. For $\tilde{V} \in \mathcal{V}_0(\mathbb{R}^{dm})$ (i.e., an Euclidean invariant function times δ functions) we define the weighted norm

$$\|\tilde{V}\|_\rho = \int |\tilde{V}(x_1,\ldots,x_m)|e^{\rho\,\mathrm{dist}(x_1,\ldots,x_m)}dx_2\cdots dx_m \tag{5.11}$$

where $\rho \geq 0$.

Proposition 5.2. *Let $V \in \mathcal{V}_2(\mathbb{R}^{dm})$ and*

$$\begin{aligned}(1-L_k)\int V(x_1,\ldots,x_m) :\phi^{\leq k}(x_1)\cdots\phi^{\leq k}(x_m): \\ = \int V_{\mathrm{ren}}(x_1,\ldots,x_m) :\phi^{\leq k}(x_1)\cdots\phi^{\leq k}(x_m):\end{aligned} \tag{5.12}$$

Let also $d > 2$, $t \geq [\frac{d}{d-2}]$. If V has a zero of order l in momentum space, i.e., if for some

$\{l_i\}_{i=1}^m$ with $\sum_i |l_i| = l$,

$$V(f_1,\dots,f_m) = \tilde{V}(\partial^{l_1} f_1,\dots,\partial^{l_m} f_m), \qquad \tilde{V} \in \mathcal{V}_0(\mathbb{R}^{dm}), \tag{5.13}$$

then

(i) If $(m,l) = (2,0)$, (2.1), $(2,2)$, $(4,0)$, $(6,0)$, $\dots$, $(2t,0)$,

$$V_{\text{ren}}(f_1,\dots,f_m) = \sum_{\substack{\{l_i\} \\ |l_i|\le 3 \\ \sum_i l_i = l_{\text{ren}}}} \tilde{V}_{\text{ren},\{l_i\}}(\partial^{l_1} f_1,\dots,\partial^{l_m} f_m), \qquad \tilde{V}_{\text{ren},\{l_i\}} \in \mathcal{V}_0(\mathbb{R}^{dm}), \tag{5.14}$$

$$\|\tilde{V}_{\text{ren},\{l_i\}}\|_\rho \le \left(\frac{1}{\rho}\right)^{l_{\text{ren}}-l} \|\tilde{V}\|_\rho, \tag{5.15}$$

where $l_{\text{ren}} = 1$ if $m \ge 4$, and $l_{\text{ren}} = 3$ if $m = 2$.

(ii) If (m,l) is not one of the pairs mentioned in (i), then $V_{\text{ren}} = V$.

Remark. $(1 - L_k)$ transforms Wick monomials of dimension $\sigma(m,l)$ into Wick monomials of dimension $\sigma(m,l_{\text{ren}})$. The condition $t \ge [\frac{2}{d-2}]$ implies that all *relevant* operators are included in the list of (i) and thus $\sigma(m,l_{\text{ren}}) \ge 1$, i.e, $(1 - L_k)$ generates only *irrelevant* operators.

Proof. (ii) follows from the definition of L_k. To prove (i), we discuss separately the case

$m = 2$: we prove (λ, μ, ν are Lorentz indices)

$$
\begin{aligned}
&a) 4 \le m \le 2t,\ l = 0,\ V = \tilde{V}\\
&\quad V_{\mathrm{ren}}(f_1, \ldots, f_m) = \sum_{i=2}^{m} \tilde{V}^{(i)}_{\mathrm{ren},\lambda}(f_1, \ldots, \partial_\lambda f_i, \ldots, f_m), \quad \|\tilde{V}^{(i)}_{\mathrm{ren},\lambda}\|_\rho \le \frac{1}{\rho}\|V\|_\rho.\\
&b) m = 2,\ l = 0,\ V = \tilde{V}\\
&\quad V_{\mathrm{ren}}(f_1, f_2) = \tilde{V}_{\mathrm{ren},\lambda\mu\nu}(f_1, \partial_\lambda\partial_\mu\partial_\nu f_2), \qquad \|\tilde{V}_{\mathrm{ren},\lambda\mu\nu}\|_\rho \le \left(\frac{1}{\rho}\right)^3\|V\|_\rho.\\
&c) m = 2,\ l = 1,\ V(f_1, f_2) = \tilde{V}(f_1, \partial_\lambda f_2)\\
&\quad V_{\mathrm{ren}}(f_1, f_2) = \tilde{V}_{\mathrm{ren},\mu\nu}(f_1, \partial_\lambda\partial_\mu\partial_\nu f_2), \qquad \|\tilde{V}_{\mathrm{ren},\mu\nu}\|_\rho \le \left(\frac{1}{\rho}\right)^2\|\tilde{V}\|_\rho.\\
&d) m = 2,\ l = 2,\ V(f_1, f_2) = \tilde{V}(f_1, \partial_\lambda\partial_\mu f_2)\\
&\quad V_{\mathrm{ren}}(f_1, f_2) = \tilde{V}_{\mathrm{ren},\nu}(f_1, \partial_\lambda\partial_\mu\partial_\nu f_2), \qquad \|\tilde{V}_{\mathrm{ren},\nu}\|_\rho \le \frac{1}{\rho}\|\tilde{V}\|_\rho.
\end{aligned}
$$

The most complicated case is b), which we prove explicitly. The other cases are proven exactly the same way, but are simpler. By the Taylor remainder theorem (with $x_{21} \equiv x_2 - x_1$),

$$
\begin{aligned}
V_{\mathrm{ren}}(f_1, f_2) &= \int V(x_1, x_2) f(x_1)\{f_2(x_2) - f_2(x_1) - x_{21}^\mu \partial_\mu f_2(x_1) - \tfrac{1}{2} x_{21}^\mu x_{21}^\nu \partial_\mu\partial_\nu f_2(x_1)\}\\
&= \int_0^1 dt \frac{(1-t)^2}{2!} \left(\frac{d}{dt}\right)^3 \int V(x_1, x_2) f_1(x_1) f_2(tx_2 + (1-t)x_1)\, dx_1 dx_2\\
&= \int_0^1 dt \frac{(1-t)^2}{2!} t^{-7} \int V\left(\frac{x_1}{t}, \frac{x_2}{t}\right) x_{21}^\lambda x_{21}^\mu x_{21}^\nu f_1(x_1) \partial_\lambda\partial_\mu\partial_\nu f_2(x_2)\, dx_1 dx_2
\end{aligned}
\tag{5.16}
$$

This proves b) with

$$
\tilde{V}_{\mathrm{ren},\lambda\mu\nu} = \int_0^1 dt \frac{(1-t)^2}{2!} t^{-7} \int V\left(\frac{x_1}{t}, \frac{x_2}{t}\right) x_{21}^\lambda x_{21}^\mu x_{21}^\nu. \tag{5.17}
$$

The bound is

$$
\begin{aligned}
\|\tilde{V}_{\mathrm{ren},\lambda\mu,\nu}\|_\rho &\le \int dx \int_0^1 dt \frac{(1-t)^2}{2!} |V(0, x) x^\lambda x^\mu x^\nu| e^{\rho|x|t}\\
&\le \left(\frac{1}{\rho}\right)^3 \|V\|_\rho,
\end{aligned}
\tag{5.18}
$$

where we have used

$$\int_0^1 dt(1-t)^2|x|^3 e^{\rho|x|t} = e^{\rho|x|}\rho^{-3}\int_0^{\rho|x|} du\, u^2 e^{-u} \le \frac{2!}{\rho^3}e^{\rho|x|}. \tag{5.19}$$

2nd step. We want now to estimate inductively tree coefficients $V(\theta,\mathbf{h},\underline{\alpha})$ using the recursion relation (2.19). Recalling that $E_k L_k = L_{k-1}E_k$, we can write this relation as

$$\begin{aligned} &V(\theta,\mathbf{h},\underline{\alpha}) = E_{h_{V_0}}\cdots E_{h_{V_1}-1}E^T_{h_{V_1}}\Big(V_{\rm ren}(\theta_1,\mathbf{h}_1,\underline{\alpha}_1),\ldots,V_{\rm ren}(\theta_s,\mathbf{h}_s,\underline{\alpha}_s)\Big),\\ &V_{\rm ren}(\theta,\mathbf{h},\underline{\alpha}) = (1-L_{h_{V_0}})V(\theta,\mathbf{h},\underline{\alpha}), \end{aligned} \tag{5.20}$$

and we focus on the contribution V_G of a single graph G compatible with θ, $\underline{\alpha}$, with subgraphs $G_1,\ldots,G_s$ corresponding to $\theta_1,\ldots,\theta_s$. We estimate V_G by simple power counting and then use Proposition 5.2 to estimate $V_{{\rm ren},G}$.

Proposition 5.3. *Let G be a graph compatible with θ, $\underline{\alpha}$, and $d>2$, $t\ge[\frac{d}{d-2}]$. Then*

$$V_{{\rm ren},G}(\theta,\mathbf{h},\underline{\alpha};f_1,\ldots,f_m) = \sum_{\substack{\{l_i\}\\ |l_i|\le 3\\ \sigma(m\sum_i|l_i|)\ge 1}} \tilde V_{{\rm ren},G,\{l_i\}}(\theta,\mathbf{h},\underline{\alpha};\partial^{l_1}f_1,\ldots,\partial^{l_m}f_m), \tag{5.21}$$

where $\tilde V_{{\rm ren},G,\{l_i\}}\in\mathcal{V}_0(\mathbb{R}^{dm})$ and

$$\left\|\tilde V_{{\rm ren},G,\{l_i\}}(\theta,\mathbf{h},\underline{\alpha};\cdot)\right\|_{\frac{C_2}{2}\gamma^{h_{V_0}}} \le C^{2tn-m}\gamma^{h_{V_0}\sigma(m,\sum|l_i|)}\prod_{\substack{V\in\theta\\ V>V_0}}\gamma^{-(\frac12+c'm_V)(h_V-h_{V'})}, \tag{5.22}$$

where n is the number of endpoints of θ (and of vertices of G) and m_V is the number of external lines of the subgraph G_V of G corresponding to the subtree θ_V of θ with first branching point V. C and c' are positive constants.

Proof. The proof is inductive. We use (5.19) and compute the contribution of G to $V(\theta,\mathbf{h},\underline{\alpha})$ from the contributions of the subgraphs $G_1,\ldots,G_s$ of G to $V(\theta_1,\mathbf{h}_1,\underline{\alpha}_1),\ldots,$ $V(\theta_s,\mathbf{h}_s,\underline{\alpha}_s)$ (see Fig. 8).

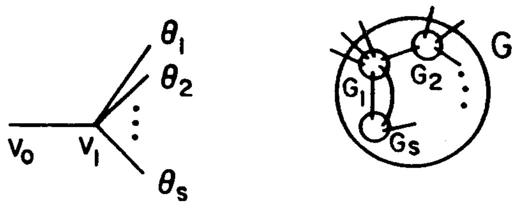


Fig. 8. The setting of the inductive proof: the graph G contributing to the tree θ has subgraphs $G_1, \ldots, G_s$ corresponding to the subtrees $\theta_1, \ldots, \theta_s$

Some of the θ_i's are trivial trees. They contribute to the computation of V_G a factor $\gamma^{-\sigma(\alpha_i)h_{V_1}}$, by (2.18). Assume now inductively that the proposition holds for the non-trivial trees in the list $\theta_1, \ldots, \theta_s$. Then V_G is given by a generalized convolution of the kernels V_{G_r} with propagators on scale h_{V_1}:

$$V_G(\theta, \mathbf{h}, \underline{\alpha}; f_1, \ldots, f_m) = \int V_{\text{ren},G_1} \cdots V_{\text{ren},G_s} \prod_{\substack{<i,j> \\ \text{hard}}} C^{h_{V_1}}(x_i, x_j) \prod_{\substack{<i,j> \\ \text{soft}}} C^{<h_{V_1}}(x_i, x_j) \prod_{i=1}^{m} f_i(x_i) \prod_{r=1}^{s} d^d x_r. \tag{5.23}$$

For the trivial trees θ_r, $V_{\text{ren},G}$ is replaced by $\gamma^{-\sigma(\alpha_r)h_{V_1}}$ in (5.23). After replacing V_{ren,G_r}, $r = 1, \ldots, s$, by $\tilde{V}_{\text{ren},G_r,\{l_i\}}$ (using (5.21)), and integrating over the internal vertices, V_G becomes an expression of the form (we suppress the $\theta, \mathbf{h}, \underline{\alpha}$-dependence):

$$V_G(f_1, \ldots, f_m) = \sum_{\substack{\{l_i\} \\ |l_i| \le 3}} \tilde{V}_{G,\{l_i\}}(\partial^{l_1} f_1, \ldots, \partial^{l_m} f_m) \tag{5.24}$$

and, to estimate $\|V_G\|_{\frac{1}{2}C_s\gamma^{h_{V_1}}}$, we note that

$$\gamma^{h_{V_1}} \operatorname{dist} x_G \le \gamma^{h_{V_1}} \sum_{r=1}^{s} \operatorname{dist} x_{G_r} + \gamma^{h_{V_1}} \sum_{<i,j> \in T} |x_i - x_j|, \tag{5.25}$$

where x_G is the set of points associated with those vertices of G which are connected with external lines, and T is some tree subgraph of hard lines of G connecting $G_1, \ldots, G_s$. For the propagators and their gradients we insert the bounds (2.7) (note that there are at most 6 derivatives acting on each propagator):

$$\begin{aligned} \left|\partial^l C^h(x_i, x_j)\right| &\le C_1(l)\gamma^{(d-2+|l|)h} e^{C_2\gamma^h|x_i-x_j|} \\ \left|\partial^l C^{<h}(x_i, x_j)\right| &\le \frac{C_1(l)}{1-\gamma^{-2}}\gamma^{(d-2+|l|)h}. \end{aligned} \tag{5.26}$$

We keep the exponential decay factor $e^{C_2\gamma^h|x_i-x_j|}$ only for $(i,j) \in T$. This allows us to integrate over the internal vertices. The result is (with $C_1 \equiv \max_{0\le l\le 6} \frac{C_1(l)}{1-\gamma^{-2}}$)

$$\begin{aligned} \|\mathcal{V}_{G,\{l_i\}}\|_{\frac{1}{2}C_2\gamma^{h}v_1} \le & \sum_{\{l_i^{(1)}\},\ldots,\{l_i^{(s)}\}} \gamma^{\left[(d-2)\left(\sum_{r=1}^s m_r - m\right)+\sum_{r=1}^s |l^{(r)}|-|l|\right]h_{V_1}} \\ & C_1^{\Sigma m_r - m} B_1^{s-1}\gamma^{d(s-1)h_{V_1}} \prod_{r=1}^s \|\mathcal{V}_{\text{ren},G_r,\{l_i^{(r)}\}}\|_{\frac{1}{2}C_2\gamma^{h}v_1} \\ \le & B_2^{\Sigma m_r - m} C^{2tn-\Sigma m_r}\gamma^{-\sigma(m,|l|)h_{V_1}} \prod_{V>V_1} \gamma^{(\frac{1}{2}+c'm_V)(h_V-h_{V'})}, \end{aligned} \tag{5.27}$$

where $|l| = \sum_i |l_i|$, m_r is the number of external lines of G_r, and B_1, B_2 are positive constants. For θ_r trivial, $\|\mathcal{V}_{\text{ren},G,\{l_i^{(r)}\}}\|_{\frac{1}{2}C_2\gamma^h v_1}$ is replaced by $\gamma^{-\sigma(\alpha_r)h_{V_1}}$, with same bound. We then apply Proposition 5.2 to estimate $\mathcal{V}_{\text{ren},G,\{l_i\}}$:

$$\begin{aligned} \|\mathcal{V}_{\text{ren},G,\{l_i'\}}\|_{\frac{1}{2}C_2\gamma^h v_0} &\le \sum_{\{l_i\}} \gamma^{-h_{V_1}(|l'|-|l|)} \|\mathcal{V}_{G,\{l_i\}}\|_{\frac{1}{2}C_2\gamma^h v_1} \\ &\le C^{2tn-m}\gamma^{-h_{V_1}\sigma(m,|l'|)} \prod_{V>V_1} \gamma^{-(\frac{1}{2}+c'm_V)(h_V-h_{V'})} \\ &\le C^{2tn-m}\gamma^{-h_{V_0}\sigma(m,|l'|)} \prod_{V>V_0} \gamma^{-(\frac{1}{2}+c'm_V)(h_V-h_{V'})} \end{aligned} \tag{5.28}$$

Here $\mathcal{V}_{G,\{l_i\}}$ are the terms in (5.24) giving $\mathcal{V}_{\text{ren},G;\{l_i'\}}$ under the action of $(1-L_{h_{V_0}})$. In the last step of (5.28) we used

$$\sigma(m,|l^{(r)}|) \ge \max(1, -d + \frac{d-2}{2}m_r) \ge \tfrac{1}{2} + c'm_r, \qquad c' = \frac{d-2}{4(d+1)} > 0, \tag{5.29}$$

and choosed $C \geq B_2$. Proposition 5.3 is thus proven.

3rd step. There are less than $\text{const}^n((t-1)n)!$ unlabelled Feynman graphs with n vertices having coordination number less than or equal to $2t$. or the planar theory [16], [2], this number is reduced to const^n.A simple combinatorial argument (see [8], Appendix F) shows that an unlabelled graph with n vertices can be labelled in at most $C_\delta^n n(\theta) \exp \delta \sum_{V \in \theta} m_V$ ways to be compatible with a tree θ such that for all $V \in \theta$, the subgraph G_V corresponding to V has m_V external lines; δ can be chosen as small as desired. We can thus sum over all graphs the bounds of Proposition 5.3:

$$\|V_{\text{ren},\{l_i\}}(\theta, \mathbf{h}, \underline{\alpha}; \cdot)\|_{\frac{1}{2}C_2\gamma^{h_{V_0}}} \leq C_0^n n(\theta) \gamma^{h_{V_0} \sigma(m, \sum |l_i|)} \prod_{V > V_0} \gamma^{-\frac{1}{2}(h_V - h_{V'})}, \tag{5.30}$$

with $\delta = c' \ln \gamma$, $C_0 = \text{const} C_\delta C^{2t}$. The claim of the theorem follows by restricting the integration in $\|\cdot\|$ to the domain $\Delta_2 \times \cdots \times \Delta_m$.

6. Perturbative solutions of the flow equation

In the previous sections we have seen how to formulate a renormalizable or non-renormalizable theory with the ultraviolet cut-off removed: the effective potentials (and the Schwinger functions) are expressed in terms of the running coupling constants on all scales, and these obey a recursion relation. What remains to do is to solve this recursion relation.

In this section we shortly discuss a "perturbative" solution of the recursion relation. The result is the usual renormalized perturbation theory (with a finite number of counterterms) which is divergent for non-renormalizable theories. For renormalizable theories one gets $n!$–bounds as a by-product. The discussion is rather sketchy. For details see [6].

It is natural to solve the recursion relation (2.21), which we write as

$$\lambda_\alpha(k) = \gamma^{-\sigma(\alpha)}\lambda_\alpha(k+1) - \beta_\alpha^{(k)}(\underline{\lambda}), \tag{6.1}$$

in a power series in $\lambda_\alpha(0)$, the "renormalised coupling constants". This is accomplished by using the Duhamel formula

$$\lambda_\alpha(k) = \gamma^{\sigma(\alpha)k}\lambda_\alpha(0) + \sum_{h=0}^{k-1}\gamma^{\sigma(\alpha)(k-h)}\beta_\alpha^{(h)}(\underline{\lambda}) \tag{6.2}$$

and iterating ($\beta_\alpha^{(h)}(\underline{\lambda})$ is second order in $\underline{\lambda}$) — we insert (6.2) for the $\lambda_\alpha(k)$ appearing in $\beta_\alpha^{(h)}(\underline{\lambda})$. This recursive solution of the flow equations ("renormalised perturbation theory") is finite to all orders for renormalisable theories, i.e., if all $\sigma(\alpha) \leq 0$. If however, for some α, $\sigma(\alpha) > 0$, this perturbative solution is divergent at all sufficiently large order, as the dangerous factor $\gamma^{\sigma(\alpha)}$ suggests.

Another way of finding a perturbative solution to (6.1) is standard in the theory of dynamical system with hyperbolic fixed points: for the irrelevant directions $\sigma(\alpha) > 0$ one should give initial conditions at the scale of the cut-off $k = N$, i.e., the natural parameters of the theory should be $\{\lambda_\alpha(0)\}_{\sigma(\alpha)\leq 0}$, $\{\lambda_\alpha(N)\}_{\sigma(\alpha)>0}$. Thus, instead of (6.2), we iterate

$$\begin{aligned}
\lambda_\alpha(k) &= \gamma^{\sigma(\alpha)k}\lambda_\alpha(0) + \sum_{h=0}^{k-1}\gamma^{\sigma(\alpha)(k-h)}\beta_{\alpha,N}^{(h)}(\underline{\lambda}), \qquad \sigma(\alpha) \leq 0 \\
\lambda_\alpha(k) &= \gamma^{-\sigma(\alpha)(N-k)}\lambda_\alpha(N) - \sum_{h=k}^{N-1}\gamma^{-\sigma(\alpha)(h-k)}\beta_{\alpha,N}^{(h)}(\underline{\lambda}), \qquad \sigma(\alpha) > 0,
\end{aligned} \tag{6.3}$$

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